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Mahran Rihani

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# Maxwell's equations in presence of negative materials

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à l'École Nationale Supérieure de Techniques Avancées

École doctorale n°574 Mathématiques Hadamard (EDMH)  
Spécialité de doctorat: mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, France, le 17/02/2022, par

**Mahran Rihani**

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To Kamel and Latifa.

## Synthèse(en français)

Le sujet principal de cette thèse est l'étude de la propagation des ondes électromagnétiques, en régime harmonique, dans un milieu hétérogène (en 3D) composé d'un diélectrique et d'un matériau négatif (c'est-à-dire avec une permittivité diélectrique négative  $\varepsilon$  et/ou une perméabilité magnétique négative  $\mu$ ) qui sont séparés par une interface avec une pointe conique.

En raison du changement de signe de la permittivité  $\varepsilon$  et/ou la perméabilité  $\mu$ , les équations de Maxwell peuvent être mal posées dans les cadres classiques (basés sur l'espace  $L^2$ ). Classiquement, il est connu que l'étude des équations de Maxwell nécessite l'étude de deux problèmes scalaires qui sont associés à  $\varepsilon$  et  $\mu$ . Dans la littérature, le seul travail qui traite de ce lien entre ces problèmes, dans le cas où  $\varepsilon$  et/ou  $\mu$  change(nt) de signe(s) est présenté dans [1]. Il a été démontré que lorsque les deux problèmes scalaires associés, impliquant respectivement  $\varepsilon$  et  $\mu$ , sont bien posés dans l'espace  $H^1$ , les équations de Maxwell sont également bien posées dans les espaces classiques. La contribution principale présentée dans cette thèse est de proposer une nouvelle théorie pour l'étude des équations de Maxwell lorsque l'un des /les problème(s) scalaire(s) n'est/ne sont pas bien posé(s) dans l'espace  $H^1$ . La thèse est composée de quatre parties.

Dans la première partie (Chapitre 2,3), en combinant la méthode de la T-coercivité et l'analyse de Mellin dans les espaces de Sobolev à poids (i.e. la théorie de Kondratièv) nous présentons une étude détaillée de ces problèmes scalaires. En particulier, nous prouvons que pour chacun d'entre eux, le caractère bien posé dans  $H^1$  est perdu si et seulement si le contraste associé appartient à un ensemble critique appelé intervalle critique. Ces intervalles critiques correspondent aux ensembles de contrastes négatifs pour lesquels des singularités propagatives, aussi appelées ondes de trou noir, apparaissent à l'extrémité de la pointe. Ces singularités se comportent comme  $r^{-1/2+i\eta}$  ( $\eta \in \mathbb{R}$ ) au voisinage de la pointe (avec  $r$  est la distance à la pointe). Elles peuvent être interprétées comme des ondes qui se propagent vers/depuis la pointe conique. Contrairement au cas 2D d'une interface avec coin, pour une pointe 3D, plusieurs ondes de trou noir peuvent exister. Des expressions explicites de ces intervalles critiques sont obtenues pour le cas particulier des pointes coniques circulaires. Pour les contrastes critiques, en utilisant le principe de radiation de Mandelstam, nous construisons une infinité des cadres fonctionnels dans lesquels le caractère bien posé des problèmes scalaires est restauré. Pour choisir, parmi ces cadres fonctionnels, le cadre qui est physiquement pertinent nous avons utilisé le principe d'absorption limite.

Au passage, dans la deuxième partie de ce travail (Chapitre 4), nous présentons une nouvelle méthode numérique pour approcher les solutions des problèmes scalaires dans le cas des contrastes non-critiques. Cette nouvelle méthode est basée sur une reformulation des problèmes scalaires en problèmes de contrôle optimal. Contrairement aux techniques existantes, la convergence cette approche, ne nécessite pas d'hypothèses supplémentaires ni sur le maillage au voisinage de l'interface ni sur la régularité de la solution.

La troisième partie de la thèse (Chapitre 8) concerne l'étude des équations de Maxwell avec un ou deux coefficients critiques. En utilisant de nouveaux résultats de potentiels vecteurs dans des espaces de Sobolev à poids et de nouveaux résultats de régularité, nous expliquons comment construire de nouveaux cadres fonctionnels dans lesquels les problèmes électrique et magnétique sont à nouveau bien posés. Ces cadres sont directement liés à ceux obtenus pour les deux problèmes scalaires associés. En outre, nous avons prouvé que si nous utilisons le cadre qui respecte le principe d'absorption limite pour les problèmes scalaires, alors les cadres fournis, par notre approche, pour les problèmes électrique et magnétique sont également cohérents avec le principe d'absorption limite.

Enfin, dans la dernière partie de ce travail (Chapitre 8), nous sommes intéressés à l'étude des processus d'homogénéisation des équations de Maxwell (en régime harmonique) et les problèmes

scalaires associés dans un domaine 3D qui contient une distribution périodique d'inclusions dans un matériau négatif. En utilisant l'approche de T-coercivité et un nouveau résultat de compacité uniforme, nous obtenons des conditions sur les contrastes (associés aux problèmes scalaires dans les cellules) qui assurent que le processus d'homogénéisation est possible pour les problèmes scalaires et vectoriels (Maxwell). D'une manière non intuitive, nous montrons que les matrices homogénéisées associées aux problèmes de limites sont soit définies positives, soit définies négatives.

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# Chapter 1

## Introduction

For the past two decades, the scientific community has been particularly interested in the analysis of Maxwell's equations in unusual situations involving real valued dielectric permittivity  $\varepsilon$  and magnetic permeability  $\mu$  whose sign changes on the domain of interest. The main motivation for the study of these problems comes from spectacular progress made in the field of plasmonics and from a more prospective point of view, from the development of the so-called metamaterials.

**Plasmonics** is the branch of physics that consists in studying the propagation of electromagnetic waves, or more precisely, of the collective oscillations of electrons, on the surface of a metal at optical frequencies. These waves are called plasmonic waves or plasmonic resonances. They are exploited in many interesting realizations such as the Lygurgus cup (see Figure 1.1). This cup looks green when illuminated from outside but appears red when illuminated from inside. The explanation of this change of color lies in the fact that it is composed by an alloy of gold and silver nanoparticles. In particular, when one illuminates the cup from inside, the red color results from the strong enhancement of the scattered field associated to some particular wavelengths due to the excitation of plasmonic resonances. Recently these waves have been used in new applications concerning the design of biosensors, cancer therapies, the production of efficient photovoltaic cells and many others (see [106]). From a mathematical point of view, the existence of these waves is mainly due to the fact that at optical frequencies, some metals like silver or gold have a dielectric permittivity  $\varepsilon$  with a small imaginary part and a negative real part (see [45, Chapter 1] for a more rigorous explanation). Neglecting the imaginary part, for these ranges of frequencies, we are led to consider a real-valued  $\varepsilon$  which is negative in the metal and positive in the air around the metal. This gives us a first simple configuration in which the dielectric permittivity has a change of sign.

**Metamaterials** are artificial materials with physical properties that can not be found in nature. Usually they are made of a periodic assembly of a large number of resonant micro-structures (see Figure 1.1). For these materials, all the game consists in choosing cleverly the structure as well as the resonators so that the effective medium, after an homogenization process, presents interesting properties. These materials have been intensively studied in the past two decades due to their potential very exciting applications such as, among others, sub-wavelength imaging and focusing, cloaking, sensing or data storage (see [135]). Let us mention that concrete realizations of these materials are still in progress. Mathematically it was proved (see [132]) that it is possible to design materials modelled by some effective  $\varepsilon$  and  $\mu$  that have, in some range of frequencies, negative real values and small imaginary parts.

We emphasize that all the interesting phenomena related to these negative materials (i.e. metals at optical frequencies or negative metamaterials for well-chosen ranges of frequencies) arise only when these materials are associated with classical (positive) ones and importantly when dissipation is very small. Therefore we will focus our attention on the propagation of electromagnetic waves

inside media where  $\varepsilon$  and  $\mu$  are real valued and where one or the two constants change(s) sign in the physical domain. Note that this is also the most interesting case for the mathematical analysis.

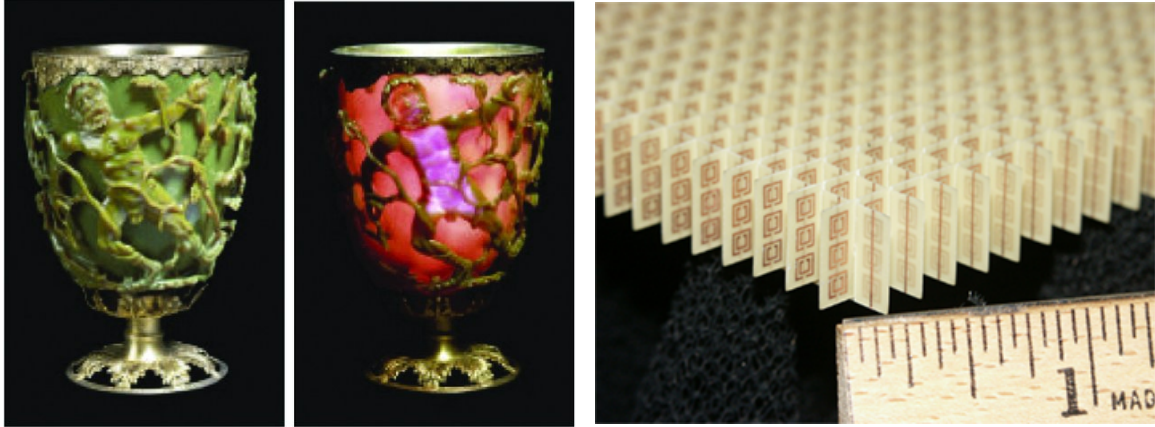


Figure 1.1: On the left: the Lygurgus cup [9]. On the right: an example of metamaterial (NASA Glenn Research/Wikimedia Commons).

In what follows, we will be particularly interested in the study of the time harmonic Maxwell's equations in a bounded domain<sup>1</sup>  $\Omega$  of  $\mathbb{R}^3$  made of an inclusion of negative material  $\Omega_2$  surrounded by some positive material  $\Omega_1$ . We denote by  $\Sigma$  the interface between the two regions so that finally we have  $\overline{\Omega_2} \subset \Omega$ ,  $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$  and  $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$  (see Figure 1.2 for an illustration).

We assume that  $\partial\Omega$  (the boundary of  $\Omega$ ) is Lipschitz-continuous and connected. Moreover we denote by  $n$  the unit normal vector to  $\partial\Omega$  oriented to the exterior of  $\Omega$ . To set the ideas, in this introduction we focus our attention on the problem satisfied by the electric field  $\mathbf{E}$  when  $\Omega$  is surrounded by a perfect conductor. This problem writes

$$\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = i\omega \mathbf{J} \text{ in } \Omega \quad \text{and} \quad \mathbf{E} \times n = 0 \text{ on } \partial\Omega. \quad (1.1)$$

Above  $\omega \in \mathbb{R}$  is the frequency,  $\mathbf{J}$  is the injected current density which is assumed to satisfy  $\text{div}(\mathbf{J}) = 0$  in  $\Omega$  while  $\varepsilon$  (resp.  $\mu$ ) is a piecewise constant function such that  $\varepsilon = \varepsilon_1 \in \mathbb{R}_+^*$  (resp.  $\mu = \mu_1 \in \mathbb{R}_+^*$ ) in  $\Omega_1$  and  $\varepsilon = \varepsilon_2 \in \mathbb{R}_-^*$  (resp.  $\mu = \mu_2 \in \mathbb{R}_-^*$ ) in  $\Omega_2$ . Because of the change of sign of the functions  $\varepsilon$  and  $\mu$ , the study of Problem (1.1) can not be made as in the classical case. In order to identify the difficulties raised by the sign-changing  $\varepsilon$ ,  $\mu$ , let us start by recalling in a brief way how one shows the well-posedness of (1.1) in the standard situation when  $\varepsilon_2$  and  $\mu_2$  are positive.

**The classical configuration.** It is well-known that when  $\varepsilon$  and  $\mu$  are positive, Problem (1.1) is not elliptic (see [63]). This makes the study of its well-posedness a little bit different from the analysis of strongly elliptic problems. The choice of the functional framework in which we can set Problem (1.1) is not unique [63]. The most natural setting is the one which reflects the fact that the electromagnetic energy contained in  $\Omega$  is finite. This boils down to impose that both  $\mathbf{E}$  and  $\mathbf{curl} \mathbf{E}$  belong to the space  $\mathbf{L}^2(\Omega) := (\mathbf{L}^2(\Omega))^3$ . This leads us to work in

$$\mathbf{H}_N(\mathbf{curl}, \Omega) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega) \text{ and } \mathbf{u} \times n \text{ in } \Omega\}.$$

Endowed with its natural norm

$$\|\mathbf{u}\|_{\mathbf{H}_N(\mathbf{curl}, \Omega)} = (\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2)^{1/2},$$

$\mathbf{H}_N(\mathbf{curl}, \Omega)$  is a Hilbert space. Furthermore, it can be shown that  $(\mathcal{D}(\Omega))^3$ , the space of infinitely differentiable functions with compact support in  $\Omega$ , is dense in  $\mathbf{H}_N(\mathbf{curl}, \Omega)$  (see [81]). With this

<sup>1</sup>i.e. an open connected subset.

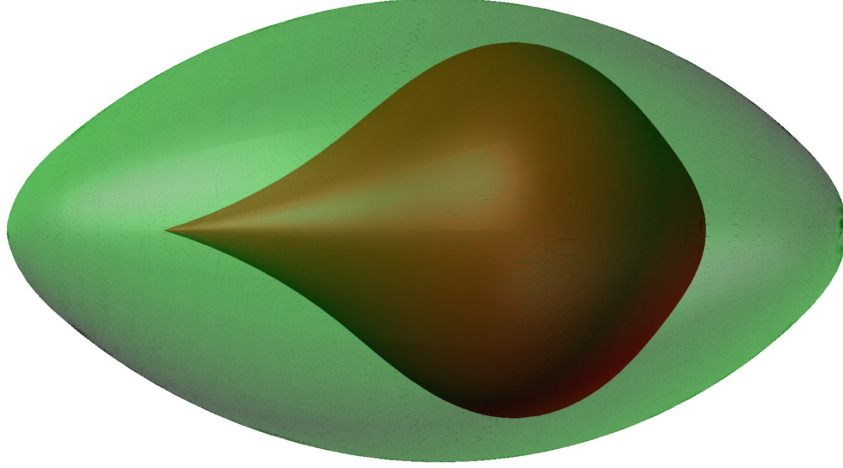


Figure 1.2: An example of considered geometry where the green (resp. red) part is occupied by  $\Omega_1$  (resp.  $\Omega_2$ ).

in mind, one can show that when  $\mathbf{J}$  belongs to  $\mathbf{L}^2(\Omega)$ , Problem (1.1) set in  $\mathbf{H}_N(\mathbf{curl}, \Omega)$  is equivalent to the following variational formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}_N(\mathbf{curl}, \Omega) \text{ such that} \\ \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{H}_N(\mathbf{curl}, \Omega). \end{array} \right. \quad (1.2)$$

By observing that for all  $\varphi \in H_0^1(\Omega)$  the vector field  $\nabla\varphi$  belongs to the space  $\mathbf{H}_N(\mathbf{curl}, \Omega)$  and that  $\|\nabla\varphi\|_{\mathbf{H}_N(\mathbf{curl}, \Omega)} = \|\nabla\varphi\|_{L^2(\Omega)}$ , one can prove that the embedding of  $\mathbf{H}_N(\mathbf{curl}, \Omega)$  into  $\mathbf{L}^2(\Omega)$  is not compact (see the end of this introduction). Moreover, for the same reason (the fact that  $\nabla H_0^1(\Omega) \subset \mathbf{H}_N(\mathbf{curl}, \Omega)$ ), what seems the “principal” part of (1.2) is not coercive. All this to say that one can not apply the “coercive +compact” theory to prove the well-posedness of (1.2). One way to solve this difficulty is to exploit the fact that  $\text{div}(\mathbf{J}) = 0$  in  $\Omega$ , which, according to (1.1), gives  $\text{div}(\varepsilon\mathbf{E}) = 0$  in  $\Omega$  for all  $\omega \neq 0$ . Imposing this constraint leads us to work in the space

$$\mathbf{X}_N(\varepsilon, \Omega) := \{\mathbf{u} \in \mathbf{H}_N(\mathbf{curl}, \Omega) \mid \text{div}(\varepsilon\mathbf{u}) = 0\}.$$

Then we introduce the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_N(\varepsilon, \Omega) \text{ such that} \\ \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{X}_N(\varepsilon, \Omega). \end{array} \right. \quad (1.3)$$

It has been proved in [65] that when  $\varepsilon$  is positive, the embedding of  $\mathbf{X}_N(\varepsilon, \Omega)$  into  $\mathbf{L}^2(\Omega)$  is compact. Furthermore, using that  $\mu$  is positive, one can prove that the principal part of (1.3) is coercive. As a result Problem (1.3) is well posed in the Fredholm sense for all  $\omega \in \mathbb{R}$  and in the Hadamard sense except for a discrete subset of frequencies of  $\mathbb{R}$ . To complete the analysis and to prove in particular that a solution to (1.3) yields a solution to (1.1), we need to show the equivalence between formulations (1.3) and (1.2). It is obvious that any solution of problem (1.2) is a solution of (1.3). Let us establish the converse statement. For all  $\mathbf{v} \in \mathbf{H}_N(\mathbf{curl}, \Omega)$ , introduce  $\varphi_{\mathbf{v}} \in H_0^1(\Omega)$  the unique function which solves the problem

$$\text{Find } \varphi_{\mathbf{v}} \in H_0^1(\Omega) \text{ such that } \text{div}(\varepsilon\nabla\varphi_{\mathbf{v}}) = \text{div}(\varepsilon\mathbf{v}). \quad (1.4)$$

Then we can write  $\mathbf{v} = \nabla\varphi_{\mathbf{v}} + \tilde{\mathbf{v}}$  with  $\varphi_{\mathbf{v}} \in H_0^1(\Omega)$  and  $\tilde{\mathbf{v}} \in \mathbf{X}_N(\varepsilon, \Omega)$ . Taking  $\tilde{\mathbf{v}}$  as a test function in (1.3) and using the fact that for all  $\mathbf{u} \in \mathbf{X}_N(\varepsilon, \Omega)$  we have

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \tilde{\mathbf{v}} = \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}}, \quad \int_{\Omega} \varepsilon \mathbf{u} \cdot \tilde{\mathbf{v}} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \text{ and } \int_{\Omega} \mathbf{J} \cdot \tilde{\mathbf{v}} = \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}},$$

we deduce that any function satisfying (1.3) solves (1.2). This ensures that Problem (1.2) is well-posed in the Fredholm sense for all  $\omega \in \mathbb{R}^*$  and in the Hadamard sense except for a discrete subset of frequencies of  $\mathbb{R}^*$ . Now, let us go back to the case of sign-changing coefficients.

**The case of sign-changing coefficients.** In the reminder above, the positivity of  $\varepsilon$  was used twice, first to show that the embedding of  $\mathbf{X}_N(\varepsilon, \Omega)$  into  $\mathbf{L}^2(\Omega)$  is compact and second to prove the equivalence between (1.3) and (1.2). In addition to that, the positivity of  $\mu$  was the key argument to show the coercivity of the principal part of the formulation (1.3). When  $\varepsilon$  and/or  $\mu$  change/changes sign, these arguments must be reconsidered.

However, if we focus our attention on the proof of equivalence between formulations (1.2) and (1.3), we notice that what is needed is not the positivity of  $\varepsilon$  but rather that the problem (1.4) is well-posed in the Hadamard sense. Indeed it was proved in [22] that when the problems

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } -\operatorname{div}(\varepsilon \nabla u) = f \in (H_0^1(\Omega))^*, \quad (1.5)$$

$$\text{Find } u \in H_{\#}^1(\Omega)^2 \text{ such that } -\operatorname{div}(\mu \nabla u) = g \in (H_{\#}^1(\Omega))^*, \quad (1.6)$$

are well-posed (for all  $f \in (H_0^1(\Omega))^*$  and  $g \in (H_{\#}^1(\Omega))^*$ ) in the Hadamard sense, then Problem (1.3) (resp. (1.2)) is well-posed in the Fredholm sense for all  $\omega \in \mathbb{R}$  (resp.  $\omega \in \mathbb{R}^*$ ) and in the Hadamard sense for all  $\omega \in \mathbb{R} \setminus \Lambda$  (resp.  $\omega \in \mathbb{R}^* \setminus \Lambda$ ) where  $\Lambda$  is a discrete subset of  $\mathbb{R}$  (resp.  $\mathbb{R}^*$ ). Naturally, this brings us to the following question:

Under which condition(s) on  $\varepsilon$  (resp. on  $\mu$ ) the problem (1.5) (resp. (1.6)) is well-posed (in the Fredholm sense) for all  $f \in (H_0^1(\Omega))^*$  (resp.  $g \in (H_{\#}^1(\Omega))^*$ )?

The previous question was the subject of several contributions in the literature [147, 49, 45] (especially in 2D configurations). Let us summarize, in a brief way, the principle conclusions of these works.

**State of the art about the scalar problems.** Interestingly, in the literature, two main approaches have been proposed.

- The first one is based on a reformulation of the problem into an integral equation [92, 92, 32, 67] posed on the interface  $\Sigma$ . Then desired conditions concerning  $\varepsilon$  or  $\mu$  to ensure the well-posedness of the problems can be expressed in terms of the spectrum of the so-called Neumann-Poincaré operator (this will be detailed in §2.2).
- The second one is variational [147, 49]. It is based on a reformulation of the classical inf-sup theory called the T-coercivity approach. For example for Problem (1.5), it consists in finding an operator  $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that the sesquilinear form  $(u, v) \mapsto \int_{\Omega} \varepsilon \nabla u \cdot \nabla(\overline{Tv})$  becomes coercive on  $H_0^1(\Omega) \times H_0^1(\Omega)$ .

When the interface  $\Sigma$  is smooth (of class  $\mathcal{C}^{1,\gamma}$  with  $\gamma \in (0, 1]$ ), the two approaches lead to the same conclusion: Problem (1.5) (resp. (1.6)) is well-posed in the Fredholm sense as soon as the contrast  $\kappa_{\varepsilon} := \varepsilon_2/\varepsilon_1$  (resp.  $\kappa_{\mu} := \mu_2/\mu_1$ ) is such that  $\kappa_{\varepsilon} \neq -1$  (resp.  $\kappa_{\mu} \neq -1$ ). We will show later in Chapter 2 that this is also the case for the general case of interfaces of class  $\mathcal{C}^1$ . It is worth to note that in some particular situations, for example for symmetric domains (i.e.  $\Omega_1$  is the symmetric of  $\Omega_2$  with respect to  $\Sigma$ ) in 2D/ 3D, one can show [49] that for  $\kappa_{\varepsilon} = -1$  (resp.  $\kappa_{\mu} = -1$ ), Problem (1.5) (resp. (1.6)) has a kernel of infinite dimension. For more details concerning the study of the particular case  $\kappa_{\varepsilon} = -1$  (resp.  $\kappa_{\mu} = -1$ ), see [117].

As soon as the interface  $\Sigma$  has geometric singularities (corners, conical points, edges, ...), the situation is totally different. As we shall see in Chapter 2 of this thesis, one can show that for

$${}^2H_{\#}^1(\Omega) := \{u \in H^1(\Omega) \mid \int_{\partial\Omega} u = 0\}.$$

the general case of Lipschitz-interfaces, the Fredholmness of (1.5) (resp. (1.6)) can be lost when the contrast  $\kappa_\varepsilon$  (resp.  $\kappa_\mu$ ) belongs to some set  $I_\Sigma^\varepsilon$  (resp.  $I_\Sigma^\mu$ ) known as the critical interval. The expression of  $I_\Sigma^\varepsilon$  (resp.  $I_\Sigma^\mu$ ) is not known in general.

For 2D polygonal interfaces,  $I_\Sigma^\varepsilon$  and  $I_\Sigma^\mu$  have been obtained explicitly. They are intervals whose bounds are functions of the sharpest opening angle of  $\Sigma$  [49, 25].

The approach based on the use of the Neumann-Poincaré operator leads to the same result. This was done in [125]. Let us mention that the expression of the critical interval can be deduced from the results of the Phd work of Carleman [44] dating from 1916!

In 3D, the situation is much more complicated, even in the simple case of an interface with a circular conical tip. Actually, for this particular configuration, the T-coercivity approach allows us to get an estimation of the bounds of the critical interval [49], but there is no guarantee about the optimality of these bounds. The approach relying on the use of the Neumann-Poincaré operator was considered in [104], but it seems that there is no clear result about the exact expression of the critical interval in this configuration. One of the objectives of this thesis is to find a characterization of the critical intervals  $I_\Sigma^\varepsilon$  and  $I_\Sigma^\mu$  in the case when  $\Sigma$  has a smooth conical tip (see Figure 1.2). Furthermore, we will show how to combine the T-coercivity approach and the approach based on the Neumann-Poincaré in order to obtain an explicit expression of the critical interval.

By applying the results of [22], we can then conclude about the well-posedness of the Maxwell's problem when  $\kappa_\varepsilon$  and  $\kappa_\mu$  do not belong respectively to  $I_\Sigma^\varepsilon$  and  $I_\Sigma^\mu$ . In Chapter 7, we will explain how to use these results in order to study the homogenization of the scalar problems and the time-harmonic Maxwell's equations in a composite medium with periodically distributed small inclusions of a negative material.

When one of the contrasts  $\kappa_\mu$  or  $\kappa_\varepsilon$  is critical, i.e. when  $\kappa_\varepsilon \in I_\Sigma^\varepsilon$  or  $\kappa_\mu \in I_\Sigma^\mu$ , the well-posedness of the Maxwell's problem in the classical frameworks  $\mathbf{H}_N(\mathbf{curl}, \Omega)$  and  $\mathbf{X}_N(\varepsilon, \Omega)$  is not guaranteed. This leads us to the following questions:

What happens to the Maxwell's problem (1.1) when  $\varepsilon$  and/or  $\mu$  are/is critical? Is it well-posed in the classical framework? If yes, how to prove this? If the answer is no, what would be the appropriate framework (from the physical point of view) in which we can set the problem?

The answer to these questions is the main motivation of this thesis. To address them, one first needs to study what happens to the scalar problems (1.5) and (1.6) when  $\varepsilon$  or  $\mu$  becomes critical. To set ideas, let us focus our attention on the problem (1.5). In the literature, to the best of our knowledge, the only existing work in this direction is [25]. In this article, the authors have considered the particular case where the interface  $\Sigma$  has a right corner. They showed, by adapting the Kondratiev theory [100], that the loss of Fredholmness for (1.5) is due to the appearance of two strongly oscillating functions  $\mathfrak{s}^\pm$  (called propagating singularities or black-hole singularities) that behave like  $r^{\pm i\eta}$  ( $\eta \in \mathbb{R}_+^*$ ) near the corner where  $r$  is the distance to the corner vertex. One can check that these functions do not belong to the space  $H^1(\Omega)$ .

In order to restore well-posedness of (1.5) when  $\kappa_\varepsilon \in I_\Sigma^\varepsilon$ , the authors of [25] used the analogy with the propagation of waves in waveguides (in this analogy the corner plays the role of infinity) to propose a new functional framework, that replaces  $H^1(\Omega)$ , in which the scalar problem (1.5) becomes well-posed. This functional framework is obtained by adding one of these two propagating singularities (the outgoing one) to a well-chosen weighted space (composed by more regular functions). The selection of the outgoing behavior is done thanks to the limiting absorption principle: the physical solution of the problem must be the limit (in some space to define) as  $\delta \rightarrow 0^+$  of  $u_\delta$  where  $u_\delta$  is the unique solution of  $-\operatorname{div}((\sigma + i\delta)\nabla u_\delta) = f$ . The extension of this approach to the case of 3D interfaces with a conical point is one of the main results of this



thesis. This will be the subject of Chapters 2-3.

Now, let us go back to the study of the Maxwell's problem. If, for example, the dielectric permittivity  $\varepsilon$  is such that  $\kappa_\varepsilon \in I_\Sigma$ , the proof of equivalence between (1.2) and (1.3) can not be done because the scalar problem (1.4) is ill-posed. This suggests that the classical  $\mathbf{L}^2$  setting is not adapted to the study of the Maxwell's problem in this configuration. As a result one needs to propose a new functional framework in order to restore Fredholmness. Intuitively, to ensure that this new functional framework leads to the physical solution of the problem, it must contain the gradient of the outgoing singularity(ies) (we shall see later that in 3D several outgoing singularities can exist). This leads us to study the Maxwell's problem in a non- $\mathbf{L}^2$  framework. From a mathematical point of view, this will prevent us from using many of the classical tools for the analysis of Maxwell's equations, such as results of existence of vector potentials, Helmholtz decomposition, compact embedding, ... For this reason, a new theory has to be constructed. This new theory can be seen as an adaptation of Kondratiev approach [100] to Maxwell's equations. It is worth to note that our technique is conceptually different from the one used in [65] (for the classical configuration) where the Kondratiev theory is used to characterize the singular behaviour of the classical solutions. Our results in this direction will be presented in Chapters 5-6.

Once the theory will be developed, we will consider the question of the approximation of these problems by finite elements methods. Unfortunately, our contributions to this question concern only the scalar case. Because of the change of sign of  $\varepsilon$  (resp.  $\mu$ ), the convergence of the numerical approximation to the exact solution as one refines the mesh in general is not clear. This leads us to the following questions:

How to design convergent FEM-based numerical method to approximate the solutions of scalar problems when they are well-posed?

In the literature, several convergent approaches have been proposed for the non-critical case. Some of the strategies are based on the use of so-called T-conforming meshes (see [49, 45]). Unfortunately, the construction of such meshes seems to be not easy (see [45]), especially when the interface has corners or in 3D. For general meshes (that respect the interface), other techniques have been designed. Some of them suffer from the fact that their convergence can not be guaranteed for all contrasts for which the (continuous) problem is well-posed. This the case in particular of the method developed in [147, 51, 119]. In 2017, a new method based on the use of an optimal control reformulation has been proposed in [1]. It is proved to be convergent on general meshes as soon as the exact solution belongs to the space  $\text{PH}^s(\Omega) := \{u \mid u|_{\Omega_1} \in \text{H}^s(\Omega_1) \text{ and } u|_{\Omega_2} \in \text{H}^s(\Omega_2)\}$  with  $s < 3/2$ . Unfortunately, this regularity condition is not always satisfied, especially when  $\Sigma$  has corners in 2D or conical points in 3D. In Chapter 4, we will present a new strategy which relies on the use of a different optimal control reformulation and which converges without any restriction neither on the mesh (the interface simply needs to coincide with edges of the mesh) nor on the regularity of the exact solution.

Now, it is time to summarize the outline of our work. This thesis will be divided into four parts.

**Part 1: Study of the scalar problems with sign-changing coefficients.** It contains two chapters (Chapter 2 and Chapter 3) and is devoted to the analysis of the scalar problems when the interface  $\Sigma$  has a conical point. In particular, we will give a characterization of the critical intervals  $I_\Sigma^\mu$  and  $I_\Sigma^\mu$  and, more importantly, we will explain how to use the Mandelstam radiation principle and the limiting absorption principle in order to derive a new (physical) functional framework in which Fredholmness is restored.

**Part 2: Numerical approximation of the scalar problems with sign-changing coefficients.** This part is made of Chapter 4 in which we will present a new numerical method to approximate the solution of the 2D/3D scalar transmission problems. It is based on a finite elements approximation and we will show that it converges without any restrictive condition on the mesh near the interface.

**Part 3: Time harmonic Maxwell's equations with sign-changing coefficients.** Here we turn our attention to the study of Maxwell's equations in a situation where the interface has a conical point and where the contrasts take critical values. In Chapter 5, we study the configuration when just one of the electromagnetic parameters is critical. In Chapter 6, we propose an analysis when both parameters  $\varepsilon$  and  $\mu$  are critical.

**Part 4: Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients.** In this part, we consider the question of the homogenization of the scalar problems and of the time-harmonic Maxwell's equations in a composite material with periodically distributed small inclusions of a negative medium. We explain why the homogenization process is possible as soon as the contrast associated to the cell problem is small or large enough. Our results will be presented in Chapter 7.

As promised above, we finish this introduction by proving that the embedding of  $\mathbf{H}_N(\mathbf{curl}, \Omega)$  into  $\mathbf{L}^2(\Omega)$  is not compact. Let  $(\varphi_i)_{i \in \mathbb{N}}$  be an orthonormal sequence of  $H_0^1(\Omega)$ . The sequence of vector fields  $(\nabla \varphi_i)_{i \in \mathbb{N}}$  is then orthonormal in  $\mathbf{H}_N(\mathbf{curl}, \Omega)$ . If the embedding  $\mathbf{H}_N(\mathbf{curl}, \Omega) \subset \mathbf{L}^2(\Omega)$  was compact, then one could find a sub-sequence, that will be indexed by  $i$ , of  $(\nabla \varphi_i)_{i \in \mathbb{N}}$  that converges in  $\mathbf{L}^2(\Omega)$  to some  $u \in \mathbf{L}^2(\Omega)$ . From the fact that  $\|\nabla \varphi_i - \nabla \varphi_j\|_{\mathbf{H}_N(\mathbf{curl}, \Omega)} = \|\nabla \varphi_i - \nabla \varphi_j\|_{\mathbf{L}^2(\Omega)} = \sqrt{2}$  for  $i \neq j$ , we conclude that this is not possible.



## Chapter 2

# Study of the scalar transmission problem in presence of a conical tip of negative material

### 2.1 Introduction

In this chapter, we investigate the scalar transmission problem between two domains<sup>1</sup> of  $\mathbb{R}^3$  filled with materials modelled by physical coefficients of different signs. We assume that  $\Omega_1$  (resp.  $\Omega_2$ ) corresponds to the positive (resp. negative) material and more specifically, we consider situations where the interface  $\Sigma$  separating the two regions is smooth (of class  $\mathcal{C}^1$ ) everywhere except near some point  $O$ , where it has a conical tip. We set  $\Omega := \Omega_1 \cup \Omega_2 \cup \Sigma$  and to simplify a little bit the analysis below, we suppose that  $\Omega$  is connected with a Lipschitz-continuous boundary  $\partial\Omega$ . In addition to that, we make the assumption that  $\overline{\Omega_2} \subset \Omega$ . This simply means that the domain  $\Omega_1$  surrounds  $\Omega_2$  and ensures that  $\Sigma \cap \partial\Omega = \emptyset$ . Without loss of generality, we suppose that  $O = (0, 0, 0)$ . A full description of the conical singularity at  $O$  can be done via the description of the domain  $\Omega_2$  near  $O$ . For this purpose, let us describe the intersection between  $\Omega_2$  and  $B(O, \rho)$  the open ball of  $\mathbb{R}^3$  of center  $O$  and of radius  $\rho$  sufficiently small. We consider the following configuration:

$$\Omega_2 \cap B(O, \rho) = \{x \in \mathbb{R}^3, |x| < \rho, x/|x| \in \mathcal{A}\} \quad (2.1)$$

where  $\mathcal{A}$  is a smooth (of class  $\mathcal{C}^2$ ) sub-domain of  $\mathbb{S}^2$  the unit sphere of  $\mathbb{R}^3$ . To simplify notations, we shall assume that  $\rho = 1$  in (2.1) (in particular this means that  $B(O, 1) \subset \Omega$ ). A more precise description of  $\mathcal{A}$  will be given below. An example of geometry for which all these assumptions are satisfied is given in Figure 2.1. Note that the class of conical tips described by (2.1) contains the particular case of circular (rotationally symmetric) conical tips obtained by revolution of a half-line around a fixed axis, say the  $z$  axis, in  $\mathbb{R}^3$ . Even though the primary goal of this chapter is to treat the class of general conical tips of the form (2.1), a particular interest will be devoted to the case of circular ones because, in such case, explicit calculus can be done.

In the sequel, we denote by  $\mathcal{K}$  the cone  $\mathcal{K} := \{x \in \mathbb{R}^3, |x| < \rho, x/|x| \in \mathcal{A}\}$ . In order to make the presentation of our results as clear as possible, we limit ourselves to the case where  $\partial\mathcal{A}$  can be parameterized by a function  $g \in \mathcal{C}_{per}^2([0, 2\pi])$ . In other words, we assume that

$$\mathcal{A} = \{(r, \theta, \varphi) \mid r = 1 \text{ and } \theta < g(\varphi)\} \quad \text{and} \quad \partial\mathcal{K} = \{(r, \theta, \varphi) \mid r \in \mathbb{R}_+ \text{ and } \theta = g(\varphi)\}.$$

Here  $(r, \theta, \varphi) \in (0; +\infty) \times (0; \pi) \times (0; 2\pi)$  are the classical spherical coordinates such that for  $x \in \mathbb{R}^3$ , we have  $x = (r \sin(\theta) \cos(\varphi), r \sin(\theta) \sin(\varphi), r \cos(\theta))$  with  $\theta \in (0; \pi)$  and  $\varphi \in (0; 2\pi)$ . In Figure 2.2, we display two examples of geometries that fit into the class of the domains described

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<sup>1</sup>Here domain means an open connected subset of  $\mathbb{R}^3$ .

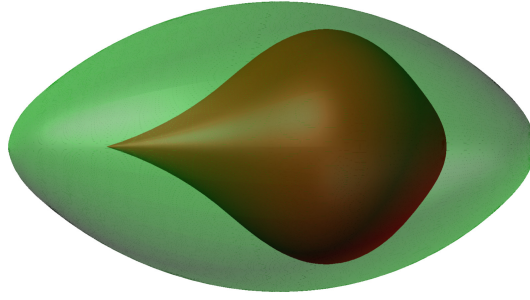


Figure 2.1: An example in which the red part ( $\Omega_2$ ) is filled with a negative material and the green one ( $\Omega_1$ ) is filled with a positive one.

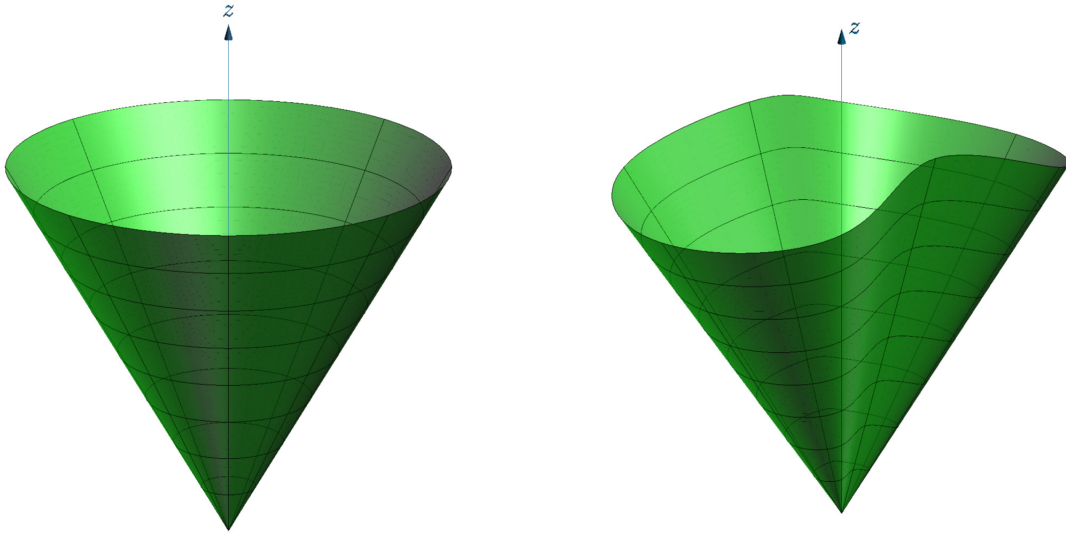


Figure 2.2: Shape of the domain  $\partial\mathcal{K}$  for a circular conical tip (left) and for a more general conical tip (right).

previously. It is important to note that all the results that we are going to present below can be easily extended to the two following situations:

- $\partial\mathcal{A}$  is of class  $\mathcal{C}^2$  but cannot be parameterized by a single function  $g \in \mathcal{C}_{per}^2([0, 2\pi])$ .
- $\partial\mathcal{A}$  is of class  $\mathcal{C}^1$  and piecewise  $\mathcal{C}^2$ .

To complete the description of our transmission problem, we need to introduce a physical parameter  $\sigma \in L^\infty(\Omega)$  such that  $\sigma|_{\Omega_1} = \sigma_1 \in \mathbb{R}_+^* := (0; +\infty)$  and  $\sigma|_{\Omega_2} = \sigma_2 \in \mathbb{R}_-^* := (-\infty; 0)$ . We denote by  $\kappa_\sigma := \sigma_2/\sigma_1 \in (-\infty; 0)$  the contrast associated with  $\sigma$ . Now the transmission problem that we want to study writes:

$$\text{Find } u \in \mathbf{H}_0^1(\Omega) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f \in (\mathbf{H}_0^1(\Omega))^*. \quad (2.2)$$

The properties of the above problem depend on the features of the bounded operator  $A_\sigma : \mathbf{H}_0^1(\Omega) \rightarrow (\mathbf{H}_0^1(\Omega))^*$  defined with the Riesz representation theorem such that

$$\langle A_\sigma u, v \rangle = \int_\Omega \sigma \nabla u \cdot \nabla \bar{v}, \quad u, v \in \mathbf{H}_0^1(\Omega).$$

Since  $\sigma$  changes sign, Problem (2.2) is not elliptic and its well-posedness (for an arbitrary  $f \in (\mathbf{H}_0^1(\Omega))^*$ ) is not guaranteed even in the Fredholm sense (i.e the operator  $A_\sigma$  may not be of Fredholm type<sup>2</sup>). By dividing  $A_\sigma$  by  $\sigma_1$ , one observes that the Fredholmness of  $A_\sigma$  depends only

<sup>2</sup>An operator  $B : X \rightarrow Y$  is said to be of Fredholm type if  $\ker(B)$  and  $\operatorname{coker}(B)$  are of finite dimensions and its range is closed. The index of  $B$  is defined by  $\operatorname{index}(B) := \dim(\operatorname{Ker}(B)) - \dim(\operatorname{coker}(B))$ .

on the contrast  $\kappa_\sigma$ . To be coherent with the vocabulary used in the literature [50, 25], the set of contrasts  $\kappa_\sigma$  for which the operator  $A_\sigma$  is not of Fredholm type is called the critical interval and is denoted by  $I_\Sigma$ . However, it is important to note that even though  $I_\Sigma$  is called the critical interval, there is no result that allows us to say that  $I_\Sigma$  is indeed an interval of the form  $[a; b]$  ( $\subset \mathbb{R}_-$ ). In particular,  $I_\Sigma$  could be an union of disjoint intervals. All we can say is that  $I_\Sigma$  is a closed subset of  $\mathbb{R}_-$  (see Proposition 2.2.1). More information about  $I_\Sigma$  are given in the next section.

In the 2D configuration, the study of the scalar transmission problem between a positive and a negative material with an interface having a corner has been clarified in [25]. It was shown that the critical interval is an interval of  $\mathbb{R}_-$  whose bounds are given explicitly as functions of the opening angle of the corner. Furthermore, when the contrast belongs to the critical interval, the loss of the Fredholmness of the operator is caused by the appearance of two strongly oscillating functions  $\mathfrak{s}^\pm$  (also known as propagating singularities or black hole singularities) that do not belong to the space  $H^1$  near the corner (these functions behave like  $r^{i\eta}$  with  $\eta \in \mathbb{R}^*$  and  $r$  is the distance to the corner vertex). Using these functions  $\mathfrak{s}^\pm$ , one can construct a Weyl sequence for the operator  $A_\sigma$  and show that the range of  $A_\sigma$  is not closed. In order to restore Fredholmness of the problem, the authors of [25] propose a new functional framework that takes into account these singular functions. They prove that by adding the space spanned by one of these two singular functions to a well-chosen weighted space, one obtains a functional framework in which the problem is again well-posed. Since the physical solution must be outgoing, they used the limiting absorption principle in order to choose the outgoing singular function (the one that propagates energy toward the corner).

The main goal of this chapter is to extend the results and the techniques used in [25] to the 3D configuration where the interface has a smooth conical tip. More precisely, we want to understand what are the propagating singularities in 3D and how to use them in order to characterize the critical interval  $I_\Sigma$ . More importantly, we shall explain how to make use of some of them in order to define a new functional framework in which the scalar problem is again well-posed and that is coherent with the classical physical principles: the Mandelstam radiation condition [112, 103] and the limiting absorption principle.

This chapter is organized as follows. In Section 2.2, we present some results concerning the critical interval  $I_\Sigma$ . In the process we underline the relation between  $I_\Sigma$  and the spectrum of the so-called Neumann-Poincaré operator. In order to study Problem (2.2), we will use localization techniques. This will lead us to consider two different localized versions of the problem. The first one is related to what happens far from the origin. We call this problem the far problem and we study it in Section 2.3. The second one is related to the analysis of the well-posedness of the problem near the origin. We call it the near problem and its study is the subject of Section 2.5. The main results concerning the features of Problem (2.2) are summarized in Section 2.6. In particular, in §2.6.2, we explain how to use the Mandelstam radiation principle in order to construct an infinite number of functional frameworks in which Fredholmness of the problem is recovered when the function  $\sigma$  is critical. The selection of the relevant physical framework will be done via the limiting absorption principle in §2.6.3. The last section is devoted to present some conclusions, some possible extensions, the remaining open questions and to give a few words concerning the numerical approximation of the solution.

## 2.2 General properties of the critical interval

As mentioned above, the critical interval is defined as the set of contrasts  $\kappa_\sigma$  such that the operator  $A_\sigma$  is not of Fredholm type. Along this chapter, when  $\kappa_\sigma$  belongs to  $I_\Sigma$ , we shall say that  $\sigma$  is critical. The main objective of this section is to present some general results concerning

the set  $I_\Sigma$ . It will be useful to introduce the operator  $I' : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  such that

$$\langle I'u, v \rangle = \int_{\Omega} u\bar{v}.$$

**Proposition 2.2.1.** *The set  $I_\Sigma$  is a closed bounded subset of  $(-\infty; 0)$ .*

**Proof.** First, let us explain why  $I_\Sigma$  is closed. By dividing  $A_\sigma$  by  $\sigma_1$ , one can suppose that  $\sigma_1 = 1$  and  $\sigma_2 = \kappa_\sigma$ . Furthermore, from the compactness of the embedding  $H_0^1(\Omega) \subset L^2(\Omega)$ , we obtain the equivalence:  $\kappa_\sigma \in (-\infty; 0] \setminus I_\Sigma$  if and only if  $A_\sigma + iI'$  is an isomorphism. The continuity of  $\kappa_\sigma \mapsto A_\sigma + iI'$  implies that  $(-\infty; 0] \setminus I_\Sigma$  is open. Thus  $I_\Sigma$  is a closed subset of  $(-\infty; 0]$ . The second step is to show that  $I_\Sigma$  is bounded. For this, we are going to use the T-coercivity approach. To do so, we first start by defining the spaces  $V_1(\Omega)$ ,  $V_2^\#(\Omega_2)$ , such that

$$V_1(\Omega_1) := \{u \in H^1(\Omega_1) \mid u = 0 \text{ on } \partial\Omega_1 \setminus \Sigma\}, \quad V_2^\#(\Omega_2) := \{u \in H^1(\Omega_2) \mid \int_{\Sigma} u = 0\},$$

$$H_{\#}^{1/2}(\Sigma) := \{u \in H^{1/2}(\Sigma) \mid \int_{\Sigma} u = 0\}.$$

Next we introduce the operators  $R_{2 \rightarrow 1} : H^{1/2}(\Sigma) \rightarrow V_1(\Omega_1)$  and  $R_{1 \rightarrow 2} : H_{\#}^{1/2}(\Sigma) \rightarrow V_2^\#(\Omega_2)$  that are defined as follows: for all  $\varphi \in H^{1/2}(\Sigma)$  and  $\varphi' \in H_{\#}^{1/2}(\Sigma)$  we have

$$R_{2 \rightarrow 1}(\varphi) \in V_1(\Omega_1) \text{ s.t. } \begin{cases} \Delta R_{2 \rightarrow 1}(\varphi) = 0 & \text{in } \Omega_1 \\ R_{2 \rightarrow 1}(\varphi) = \varphi & \text{on } \Sigma \end{cases}, \quad R_{1 \rightarrow 2}(\varphi') \in V_2^\#(\Omega_2) \text{ s.t. } \begin{cases} \Delta R_{1 \rightarrow 2}(\varphi') = 0 & \text{in } \Omega_2 \\ R_{1 \rightarrow 2}(\varphi') = \varphi' & \text{on } \Sigma. \end{cases}$$

Without any difficulty, one shows that there exists  $0 < C$  such that for all  $\varphi \in H^{1/2}(\Sigma)$  and all  $\varphi' \in H_{\#}^{1/2}(\Sigma)$  we have

$$\|\nabla R_{1 \rightarrow 2}(\varphi)\|_{L^2(\Omega_1)} \leq C\|\varphi\|_{H^{1/2}(\Sigma)} \quad \text{and} \quad \|\nabla R_{2 \rightarrow 1}(\varphi')\|_{L^2(\Omega_2)} \leq C\|\varphi'\|_{H^{1/2}(\Sigma)}. \quad (2.3)$$

To obtain the previous estimate, we have used the fact that in  $V_1(\Omega)$  (resp.  $V_2^\#(\Omega_2)$ ) the application  $u \mapsto \|\nabla u\|_{L^2(\Omega_1)}$  (resp.  $u \mapsto \|\nabla u\|_{L^2(\Omega_2)}$ ) is a norm that is equivalent to the classical one. For all  $u \in H_0^1(\Omega)$ , we denote by  $u_1$  and by  $u_2$  its restriction to  $\Omega_1$  and  $\Omega_2$  respectively. We define the operators  $T_1 : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  and  $T_2 : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that for all  $u \in H_0^1(\Omega)$ , we have

$$T_1(u) = \begin{cases} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_{1 \rightarrow 2}(u|_{\Sigma} - M_{\Sigma}(u)) + 2M_{\Sigma}(u) & \text{on } \Omega_2, \end{cases}$$

$$T_2(u) = \begin{cases} -u_1 + 2R_{2 \rightarrow 1}(u|_{\Sigma} - M_{\Sigma}(u)) & \text{in } \Omega_1 \\ u_2 - 2M_{\Sigma}(u) & \text{on } \Omega_2 \end{cases}$$

in which  $M_{\Sigma}(u)$  is defined by

$$M_{\Sigma}(u) = \frac{1}{|\Sigma|} \int_{\Sigma} u \, ds.$$

One can easily check that  $T_1$  and  $T_2$  are continuous and bijective (we have  $T_i \circ T_i = \text{Id}$  for  $i = 1, 2$ ). Thanks to Estimate (2.3) and to the continuity of the trace operator (because  $\Sigma$  is Lipschitz), we conclude that the numbers

$$\|R_{1 \rightarrow 2}\| = \sup_{u \in H_0^1(\Omega), u \neq 0} \frac{\|\nabla R_{1 \rightarrow 2}(u|_{\Sigma} - M_{\Sigma}(u))\|_{L^2(\Omega_2)}}{\|\nabla u_1\|_{L^2(\Omega_1)}}$$

$$\|R_{2 \rightarrow 1}\| = \sup_{u \in H_0^1(\Omega), u \neq 0} \frac{\|\nabla R_{2 \rightarrow 1}(u|_{\Sigma} - M_{\Sigma}(u))\|_{L^2(\Omega_1)}}{\|\nabla u_2\|_{L^2(\Omega_2)}}$$

are positive and finite. By applying the results of [50, Theorem 1.1.1] or by working as in [42], we conclude that  $A_\sigma$  is an isomorphism if

$$1/|\kappa_\sigma| > \|R_{1 \rightarrow 2}\|^2 \quad \text{or} \quad |\kappa_\sigma| > \|R_{2 \rightarrow 1}\|^2.$$

The result is then proved. ■

**Remark 2.2.1.** *The previous result holds if one replaces  $\Sigma$  by any Lipschitz interface. The proof is also based on the use of the harmonic extension operators. To the best of our knowledge when the interface  $\Sigma$  is not Lipschitz, the question whether  $I_\Sigma$  is bounded or not is still open. In 2D, there are configurations where the critical interval is equal to  $\mathbb{R}_-$  (see [32] for more details). We conjecture that when the interface  $\Sigma$  is not Lipschitz, we have  $I_\Sigma = (-\infty; 0)$ . For an example of non-Lipschitz interface, think to the surface of two touching conical tips.*

**Remark 2.2.2.** *In 2D with corners, the critical interval is known explicitly (see [25]). In 3D however the situation is much more complicated. In Section 2.6.1, we shall give an explicit expression of  $I_\Sigma$  for the case of circular conical tips.*

The remaining part of this section is devoted to clarify the link that exists between the set  $I_\Sigma$  and the essential spectrum of the so-called Neumann-Poincaré operator. Recently, the study of the spectral properties of this operator was the subject of many contributions such as those of M. Putinar et al. [97], those of H. Ammari et al. [7] for the case of smooth interfaces, those of E. Bonnetier et al. for the case of interfaces with corners [32] and those of K. M. Perfekt et al. [92] for the case of 2D curved interfaces as well as 3D interfaces with conical tips. Our goal is to explain how the spectrum of the Neumann-Poincaré operator is related to  $I_\Sigma$ .

### 2.2.1 Relation between the critical interval and the spectrum of the Neumann-Poincaré operator

Most of results of this paragraph are inspired by the ones developed in [35]. The starting point is to define the operator  $T_{\Omega_2} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that for all  $u \in H_0^1(\Omega)$  we have

$$\int_{\Omega} \nabla(T_{\Omega_2}(u)) \cdot \nabla \bar{v} = \int_{\Omega_2} \nabla u \cdot \nabla \bar{v}, \quad \forall v \in H_0^1(\Omega).$$

The existence and continuity of the operator  $T_{\Omega_2}$  are consequences of the Riesz representation theorem. In the literature,  $T_{\Omega_2}$  is called the Poincaré variational operator (see [35]). Since  $T_{\Omega_2}$  is symmetric, it is then a self-adjoint positive operator. In the sequel, we denote by  $\sigma(T_{\Omega_2})$  the spectrum of  $T_{\Omega_2}$  and by  $\sigma_{ess}(T_{\Omega_2})$  its essential spectrum which is defined as the set of  $\lambda \in \mathbb{R}$  for which the operator  $T_{\Omega_2} - \lambda I$  is not of Fredholm type (here  $I$  stands for the identity operator of  $H_0^1(\Omega)$ ). To proceed, let us denote by  $\tilde{A}_\sigma : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  the operator that is defined by

$$\int_{\Omega} \nabla(\tilde{A}_\sigma(u)) \cdot \nabla \bar{v} = \int_{\Omega} \sigma \nabla u \cdot \nabla \bar{v} = \langle A_\sigma u, v \rangle, \quad \forall u, v \in H_0^1(\Omega).$$

Clearly we have an equivalence between the Fredholmness of  $A_\sigma$  and  $\tilde{A}_\sigma$ . Furthermore, one can write

$$\tilde{A}_\sigma = \sigma_1 I + (\sigma_2 - \sigma_1) T_{\Omega_2} = (\sigma_2 - \sigma_1) \left( T_{\Omega_2} - \frac{1}{1 - \kappa_\sigma} I \right).$$

This leads us to the following

**Lemma 2.2.1.** *We have  $\kappa_\sigma \in I_\Sigma$  if and only if  $[\kappa_\sigma < 0 \text{ and } 1/(1 - \kappa_\sigma) \in \sigma_{ess}(T_{\Omega_2})]$ .*

Without particular difficulty one can prove the following statement.

**Proposition 2.2.2.** *The operator  $T_{\Omega_2}$  satisfies the following properties:*

1.  $\sigma(T_{\Omega_2}) \subset [0; 1]$  and  $0, 1 \in \sigma_{ess}(T_{\Omega_2})$ .
2.  $0, 1$  are eigenvalues of infinite geometric multiplicity of  $T_{\Omega_2}$ . More precisely, the spaces  $\ker(T_{\Omega_2})$  and  $\ker(T_{\Omega_2} - I)$  are given by

$$\ker(T_{\Omega_2}) = \{u \in H_0^1(\Omega) \mid u|_{\Omega_2} = \text{constant}\} \text{ and } \ker(T_{\Omega_2} - I) = \{u \in H_0^1(\Omega) \mid u|_{\Omega_1} = 0\}.$$

3. We have the decomposition

$$H_0^1(\Omega) = \ker(T_{\Omega_2}) \overset{\perp}{\oplus} \ker(T_{\Omega_2} - I) \overset{\perp}{\oplus} N \quad (2.4)$$

where  $N$  is given by

$$N := \{u \in H_0^1(\Omega) \mid \Delta u = 0 \text{ in } \Omega_1 \cup \Omega_2 \text{ and } \langle \partial_n(u|_{\Omega_2}), 1 \rangle_{H^{-1/2}(\Sigma), H^{1/2}(\Sigma)} = 0\}. \quad (2.5)$$

Since in our study the contrast  $\kappa_\sigma$  belongs to  $(-\infty; 0)$ , the real number  $1/(1 - \kappa_\sigma)$  can not be equal neither to 0 nor to 1. This means that we have the equivalence:  $\kappa_\sigma \in I_\Sigma$  if and only if  $1/(1 - \kappa_\sigma) \in \sigma_{ess}(T_{\Omega_2}) \setminus \{0, 1\}$ . Now, let us introduce the space

$$S := \{u \in H_0^1(\Omega) \mid \Delta u = 0 \text{ in } \Omega_1 \cup \Omega_2\}.$$

Starting from the decomposition (2.4), we can easily show that we have the decomposition

$$S = \text{span}(u_0) \overset{\perp}{\oplus} N.$$

where the function  $u_0 \in H_0^1(\Omega)$  is the harmonic extension of the function  $1|_{\Omega_2}$  to the whole domain  $\Omega$  (obviously, one has  $u_0 \in \text{Ker}(T_{\Omega_2})$ ). This implies that  $S$  is a closed sub-space of  $H_0^1(\Omega)$  and that it is an invariant sub-space for the operator  $T_{\Omega_2}$ . As a consequence,  $T_{\Omega_2}$  induces a linear operator from  $S$  to  $S$  that will be denoted by  $T_{\Omega_2}^S$ . Without any difficulty one can show that 0 is a simple eigenvalue of  $T_{\Omega_2}^S$  and that  $\sigma_{ess}(T_{\Omega_2}^S) = \sigma_{ess}(T_{\Omega_2}) \setminus \{1, 0\}$ . Thus, we obtain the equivalence

$$\kappa_\sigma \in I_\Sigma \quad \text{if and only if} \quad 1/(\kappa_\sigma - 1) \text{ belongs to } \sigma_{ess}(T_{\Omega_2}^S).$$

The goal of the next paragraph is to explain how the spectrum of  $T_{\Omega_2}^S$  is related to the spectrum of the Neumann-Poincaré operator.

### Definition of the Neumann-Poincaré operator

Let  $G_3 : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{C}$  be the Green function of the Laplace operator in the free space  $\mathbb{R}^3$ . Classically, this function is given by

$$G_3(x, y) = \frac{1}{4\pi|x - y|} \text{ for all } x \neq y \in \mathbb{R}^3$$

and satisfies the equation  $\Delta_y G_3(x, \cdot) = \delta_x$  (where  $\delta_x$  is the Dirac distribution at  $x$ ). We also need to introduce  $P$  the Laplace kernel that is a correction of the function  $G_3$  in order to take into account the homogeneous Dirichlet boundary condition. The function  $P$  is defined in  $\Omega \times \Omega$  by the relation  $P(x, y) = G_3(x, y) + C_x(y)$  where  $C_x$  is the solution of the problem  $\Delta_y C_x = 0$  in  $\Omega$  and  $C_x(y) = -G(x, y)$  on  $\partial\Omega$ . This means that the function  $P$  satisfies  $\Delta_y P(x, \cdot) = \delta_x$  and  $P(x, \cdot) = 0$  on  $\partial\Omega$ . We introduce the single layer potential  $S_\Sigma : H^{1/2}(\Sigma) \rightarrow H_0^1(\Omega)$  associated with  $\Sigma$  such that for all  $\varphi \in H^{1/2}(\Sigma)$ , we set

$$S_\Sigma(\varphi)(x) = \int_\Sigma P(x, y)\varphi(y)d\sigma(y) \text{ for almost all } x \in \Omega.$$



By observing that for all  $x \in \Omega_1 \cup \Omega_2$  the function  $P(x, \cdot)|_{\Sigma} \in H^{1/2}(\Sigma)$ , we deduce that the operator  $S_{\Sigma}$  can be extended to an operator  $S_{\Sigma} : H^{-1/2}(\Sigma) \rightarrow H_0^1(\Omega)$ . Classically (see [32]), one can prove that for all  $\varphi \in H^{-1/2}(\Sigma)$  the function  $S_{\Sigma}(\varphi)$  belongs to the space  $S$ , i.e.  $S_{\Sigma}(\varphi)$  is harmonic in  $\Omega_1 \cup \Omega_2$ . Conversely, it is a classical result that any function  $u$  of the space  $S$  admits the representation

$$u = S_{\Sigma}([\partial_n u|_{\Sigma}]) \quad (2.6)$$

in which  $[\partial_n u|_{\Sigma}] = \partial_n u_1|_{\Sigma} - \partial_n u_2|_{\Sigma}$  where  $n$  is the outward unit normal vector to  $\partial\Omega_2$ ,  $u_1 = u|_{\Omega_1}$  and  $u_2 = u|_{\Omega_2}$ . This means that  $S_{\Sigma}$  realizes a bijection (and then it is an isomorphism) between  $H^{-1/2}(\Sigma)$  and  $S$ . The normal derivative of  $S_{\Sigma}(\varphi)$  is generally discontinuous across  $\Sigma$ . This discontinuity can be described by the Plemelj jump relations:

$$\partial_n S_{\Sigma}(\varphi)_1 = \varphi/2 + K_{\Sigma}^{np}(\varphi) \quad \text{and} \quad \partial_n S_{\Sigma}(\varphi)_2 = -\varphi/2 + K_{\Sigma}^{np}(\varphi) \quad (2.7)$$

where again  $n$  is the unit normal vector to  $\partial\Omega_2$  oriented to the exterior of  $\Omega_2$ , the functions  $S_{\Sigma}(\varphi)_1$  and  $S_{\Sigma}(\varphi)_2$  are, respectively, the restriction of  $S_{\Sigma}(\varphi)$  to  $\Omega_1$  and to  $\Omega_2$ . The Neumann-Poincaré operator is denoted by  $K_{\Sigma}^{np} : H^{-1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$  and is defined as the extension of the operator  $\tilde{K}_{\Sigma}^{np} : L^2(\Sigma) \rightarrow L^2(\Sigma)$  such that

$$\tilde{K}_{\Sigma}^{np}(\varphi)(x) = \int_{\Sigma} \partial_{n_y} P(x, y) \varphi(y) d\sigma(y) \text{ for almost all } x \in \Sigma$$

in which  $n_y$  stands for the unit outward normal vector to  $\Omega_2$  at  $y \in \Sigma$ . The operator  $K_{\Sigma}^{np}$  is not self-adjoint because it is not symmetric with respect to the classical inner product of  $H^{-1/2}(\Sigma)$ . To circumvent this difficulty, we introduce the sesquilinear form  $\langle \cdot, \cdot \rangle_{S_{\Sigma}} : H^{-1/2}(\Sigma) \times H^{-1/2}(\Sigma) \rightarrow \mathbb{C}$  such that

$$\langle \varphi, \psi \rangle_{S_{\Sigma}} =: -\langle \varphi, S_{\Sigma} \psi \rangle_{H^{-1/2}(\Sigma), H^{1/2}(\Sigma)}, \quad \forall \varphi, \psi \in H^{-1/2}(\Sigma).$$

Thanks to an integration by parts and by using the jump relations (2.7), one can show that

$$\langle \varphi, \psi \rangle_{S_{\Sigma}} = \int_{\Omega} \nabla S_{\Sigma}(\varphi) \cdot \nabla \overline{S_{\Sigma}(\psi)}. \quad (2.8)$$

As a result, we infer that  $\langle \cdot, \cdot \rangle_{S_{\Sigma}}$  is an inner product in  $H^{-1/2}(\Sigma)$ . We denote by  $\| \cdot \|_{S_{\Sigma}}$  the norm associated to this inner product. It is equivalent to the classical one  $\| \cdot \|_{H^{-1/2}(\Sigma)}$  (see [35]). As a result,  $(H^{-1/2}(\Sigma), \| \cdot \|_{S_{\Sigma}})$  is a Hilbert space. Note that one can easily see that we have the identity

$$\| \varphi \|_{S_{\Sigma}} = \| S_{\Sigma}(\varphi) \|_{H_0^1(\Omega)}, \quad \varphi \in H^{-1/2}(\Sigma).$$

By endowing the space  $H^{-1/2}(\Sigma)$  with this inner product  $\langle \cdot, \cdot \rangle_{S_{\Sigma}}$ , one can show that  $K_{\Sigma}^{np}$  becomes self-adjoint. Furthermore, we also have the following

**Lemma 2.2.2.** *Let  $\lambda \in \mathbb{R}$  and define  $\lambda' = 1/2 - \lambda$ . Then for all  $\varphi \in H^{-1/2}(\Sigma)$  the function  $u = S_{\Sigma}(\varphi) \in S$  satisfies*

$$\| T_{\Omega_2}^S u - \lambda u \|_{H_0^1(\Omega)} = \| K_{\Sigma}^{np} \varphi - \lambda' \varphi \|_{S_{\Sigma}}.$$

**Proof.** We denote by  $u_1 = u|_{\Omega_1}$  and  $u_2 = u|_{\Omega_2}$ . The first step is to compute explicitly the quantity

$$\int_{\Omega} \nabla (T_{\Omega_2}^S u - \lambda u) \cdot \nabla \overline{S_{\Sigma}(\psi)}$$

for an arbitrary  $\psi \in H^{-1/2}(\Sigma)$ . Thanks to an integration by parts and by using the fact that  $u, T_{\Omega_2}^S u \in S$ , we obtain

$$\begin{aligned} \int_{\Omega} \nabla (T_{\Omega_2}^S u - \lambda u) \cdot \nabla \overline{S_{\Sigma}(\psi)} &= -\lambda \int_{\Omega_1} \nabla u \cdot \nabla \overline{S_{\Sigma}(\psi)} + (1 - \lambda) \int_{\Omega_2} \nabla u \cdot \nabla \overline{S_{\Sigma}(\psi)} \\ &= \langle (\lambda \partial_n u_1 + (1 - \lambda) \partial_n u_2), S_{\Sigma}(\psi) \rangle_{H^{-1/2}(\Sigma), H^{1/2}(\Sigma)} \end{aligned}$$

$$\begin{aligned} \text{By using (2.7)-(2.8)} \quad &= \langle \lambda (K_{\Sigma}^{np} \varphi_n + \varphi_n/2) + (1 - \lambda) (K_{\Sigma}^{np} \varphi_n - \varphi_n/2), \psi \rangle_{S_{\Sigma}} \\ &= \langle K_{\Sigma}^{np} \varphi_n - \lambda' \varphi_n, \psi \rangle_{S_{\Sigma}}. \end{aligned}$$

Since for all  $\psi \in \mathbf{H}^{-1/2}(\Sigma)$  we have  $\|S_\Sigma(\psi)\|_{\mathbf{H}_0^1(\Omega)} = \|\psi\|_{S_\Sigma}$ , one deduces that we have

$$\frac{\int_\Omega \nabla(T_{\Omega_2}^S u - \lambda u) \cdot \nabla \overline{S_\Sigma(\psi)}}{\|S_\Sigma(\psi)\|_{\mathbf{H}_0^1(\Omega)}} = \frac{\langle K_\Sigma^{np} \varphi - \lambda' \varphi, \psi \rangle_{S_\Sigma}}{\|\psi\|_{S_\Sigma}}, \quad \forall \psi \in \mathbf{H}^{-1/2}(\Sigma) \setminus \{0\}.$$

By taking the supremum over all  $\psi \in \mathbf{H}^{-1/2}(\Sigma) \setminus \{0\}$ , and by recalling that  $S_\Sigma : \mathbf{H}^{-1/2}(\Sigma) \rightarrow \mathbf{S}$  is an isomorphism (and also the fact that  $(\mathbf{S}, \|\cdot\|_{\mathbf{H}_0^1(\Omega)})$  is a Hilbert space), we obtain the wanted result:

$$\|T_{\Omega_2}^S u - \lambda u\|_{\mathbf{H}_0^1(\Omega)} = \|K_\Sigma^{np} \varphi - \lambda' \varphi\|_{S_\Sigma}. \quad \blacksquare$$

### Final result

Now, we have all the tools to show the

**Theorem 2.2.1.** *The essential spectra of  $T_{\Omega_2}^S : \mathbf{S} \rightarrow \mathbf{S}$  and  $K_\Sigma^{np} : \mathbf{H}^{-1/2}(\Sigma) \rightarrow \mathbf{H}^{-1/2}(\Sigma)$  are linked by the relation*

$$\sigma_{ess}(T_{\Omega_2}^S) = 1/2 - \sigma_{ess}(K_\Sigma^{np}).$$

**Proof.** Since both operators are self-adjoint (the space  $\mathbf{H}^{-1/2}(\Sigma)$  is endowed with  $\langle \cdot, \cdot \rangle_{S_\Sigma}$ ), we can use the characterization of the essential spectrum by means of singular Weyl sequences. The fact that  $\lambda \in \sigma_{ess}(T_{\Omega_2}^S)$  implies that there exists a sequence  $(u_n)_{n \in \mathbb{N}}$  of elements of  $\mathbf{S}$  such that

$$\begin{cases} \|u_n\|_{\mathbf{H}_0^1(\Omega)} = 1 & \text{for all } n \in \mathbb{N}, \\ u_n \rightharpoonup 0 & \text{weakly in } \mathbf{S}, \\ T_{\Omega_2}^S u_n - \lambda u_n \rightarrow 0 & \text{strongly in } \mathbf{S}. \end{cases}$$

Since  $S_\Sigma : (\mathbf{H}^{-1/2}(\Sigma), \|\cdot\|_{S_\Sigma}) \rightarrow (\mathbf{S}, \|\cdot\|_{\mathbf{H}_0^1(\Omega)})$  is an isomorphism, we introduce  $(\varphi_n)_{n \in \mathbb{N}}$  the sequence of elements of  $\mathbf{H}^{-1/2}(\Sigma)$  such that  $S_\Sigma(\varphi_n) = u_n$  for all  $n \in \mathbb{N}$ . Easily, one can see that  $\|\varphi_n\|_{\mathbf{H}^{-1/2}(\Sigma)} = 1$  for all  $n \in \mathbb{N}$ . Moreover, since for all  $\psi \in \mathbf{H}^{-1/2}(\Sigma)$ , we have

$$\langle \varphi_n, \psi \rangle_{S_\Sigma} = \int_\Omega \nabla u_n \cdot \nabla \overline{S_\Sigma(\psi)}$$

and since  $S_\Sigma(\psi) \in \mathbf{S}$ , we infer that  $(\varphi_n)_{n \in \mathbb{N}}$  converges weakly to zero in  $\mathbf{H}^{-1/2}(\Sigma)$ . According to Lemma 2.4, we know that for  $\lambda' = \lambda - 1/2$  and all  $n \in \mathbb{N}$ , we have

$$\|T_{\Omega_2}^S u_n - \lambda u_n\|_{\mathbf{H}_0^1(\Omega)} = \|K_\Sigma^{np} \varphi_n - \lambda' \varphi_n\|_{S_\Sigma}.$$

This shows that  $K_\Sigma^{np} \varphi_n - \lambda' \varphi_n$  converges to zero as  $n$  tends to  $+\infty$ . Consequently,  $\lambda' \in \sigma_{ess}(K_\Sigma^{np})$ . The converse statement can be proved in the same way.  $\blacksquare$

As a consequence of the previous theorem, we obtain

**Theorem 2.2.2.** *There holds  $\kappa_\sigma \in I_\Sigma$  if and only if  $\frac{\kappa_\sigma + 1}{2(\kappa_\sigma - 1)} \in \sigma_{ess}(K_\Sigma^{np})$ . And we have*

$$\sigma_{ess}(K_\Sigma^{np}) = \left\{ \frac{\kappa_\sigma + 1}{2(\kappa_\sigma - 1)}, \kappa_\sigma \in I_\Sigma \right\}, \quad I_\Sigma = \left\{ \frac{a + 1/2}{a - 1/2}, a \in \sigma_{ess}(K_\Sigma^{np}) \right\}.$$

**Proof.** We already know that  $\kappa_\sigma \in I_\Sigma$  if and only if  $1/(1 - \kappa_\sigma) \in \sigma_{ess}(T_{\Omega_2}^S)$ . According to the previous theorem, we can say that  $\kappa_\sigma \in I_\Sigma$  if and only if  $1/2 - 1/(1 - \kappa_\sigma) = (\kappa_\sigma + 1)/2(\kappa_\sigma - 1) \in \sigma_{ess}(K_\Sigma^{np})$ . The second part of the proof is a simple consequence of the fact that the function  $x \mapsto (x + 1)/(2(x - 1))$  is bijective from  $\mathbb{R}_-^*$  to  $(-1/2; 1/2)$  and that its inverse coincides with the function  $x \mapsto (x + 1/2)/(x - 1/2)$ .  $\blacksquare$



The previous theorem tells us how the critical interval is related to the essential spectrum of the Neumann-Poincaré operator. When the interface  $\Sigma$  is smooth (of class  $\mathcal{C}^{1,\alpha}$  with  $0 < \alpha < 1$ ), one can prove that  $K_{np}^\Sigma$  is a Hilbert-Schmidt operator and, then, it is compact. Consequently, its spectrum is composed by a sequence of real eigenvalues that has zero as only possible point of accumulation. For a general interface of class  $\mathcal{C}^1$ ,  $K_{np}^\Sigma$  is not necessarily a Hilbert-Schmidt operator. As a result, its compactness is not guaranteed. In Section 2.3, we are going to show that when  $\Sigma$  is of class  $\mathcal{C}^1$  the critical interval reduces to  $\{-1\}$ . This implies, using the previous theorem, that  $\sigma_{ess}(K_{np}^\Sigma) = \{0\}$ . With this in mind, we can show the

**Lemma 2.2.3.** *When the interface  $\Sigma$  is of class  $\mathcal{C}^1$ , we have  $\sigma_{ess}(K_{np}^\Sigma) = \{0\}$  and  $K_{np}^\Sigma : H^{-1/2}(\Sigma) \rightarrow H^{-1/2}(\Sigma)$  is compact.*

**Proof.** The fact that  $\sigma_{ess}(K_{np}^\Sigma) = \{0\}$  is a consequence of the fact that when  $\Sigma$  is of class  $\mathcal{C}^1$  then  $I_\Sigma = \{-1\}$  (see §2.3). It remains to explain why  $K_{np}^\Sigma$  is compact. From the fact that  $K_{np}^\Sigma$  is self-adjoint, we deduce that  $\sigma_{disc}(K_{np}^\Sigma)$  (the discrete spectrum of  $K_{np}^\Sigma$ ) is either composed by a finite number of real eigenvalues or by a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  of real eigenvalues that tends to zero as  $n$  goes to  $+\infty$ . In both cases, we define the space

$$F := \left( \bigoplus_{\lambda \in \sigma_{disc}(K_{np}^\Sigma)} E(\lambda) \right)^\perp$$

in which  $E(\lambda)$  stands for the eigenspace associated to  $\lambda \in \sigma_{disc}(K_{np}^\Sigma)$  (note that by definition of the discrete spectrum,  $E(\lambda)$  is finite dimensional). The space  $F$  is then a closed subspace of the Hilbert space  $(H^{-1/2}, \|\cdot\|_{S_\Sigma})$ . This implies that  $(F, \|\cdot\|_{S_\Sigma})$  is also a Hilbert space. Furthermore, without any difficulty one shows that  $F$  is stable by  $K_{np}^\Sigma$  and that  $K_{np}^\Sigma : F \rightarrow F$  is self-adjoint with a spectrum that is reduced to  $\{0\}$ . Consequently, the spectral radius of  $K_{np}^\Sigma : F \rightarrow F$  is equal to 0 and then  $K_{np}^\Sigma$  vanishes in  $F$ . Using this result, we are going to show that  $K_{np}^\Sigma$  is the limit of finite rank operators and then it is a compact operator. The proof in the case when  $\sigma_{disc}(K_{np}^\Sigma)$  is finite is obvious. It remains to study the case when  $\sigma_{disc}(K_{np}^\Sigma) = \{\lambda_n; n \in \mathbb{N}\}$  where  $(\lambda_n)$  is a sequence of real number that converges to zero as  $n$  tends to  $\infty$ . Denote by  $F_n$  the space  $F_n := F \oplus_{i=1}^n E(\lambda_i)$ . Clearly, for all  $n \in \mathbb{N}$  the space  $F_n$  is stable by  $K_{np}^\Sigma$ . Moreover, the restriction of  $K_{np}^\Sigma$  to  $F_n$  has a finite range. Let  $P_n : H^{-1/2}(\Sigma) \rightarrow F_n$  be the orthogonal projector of  $H^{-1/2}(\Sigma)$  into  $F_n$  (with respect to  $\langle \cdot, \cdot \rangle_{S_\Sigma}$ ) and define the finite rank operator  $K_n := K_{np}^\Sigma \circ P_n$ . One can easily see that

$$|\langle (K_{np}^\Sigma - K_n)u, u \rangle_{S_\Sigma}| \leq \max_{n < m} (|\lambda_m|) \|u\|_{S_\Sigma}^2.$$

By letting  $n$  tend to  $+\infty$  and using the fact that  $(\lambda_n)$  tends to 0 as  $n$  tends to infinity, we can say that  $K_{np}^\Sigma$  is the limit of  $(K_n)_{n \in \mathbb{N}}$  and then it is compact. ■

In the literature, the compactness of  $K_{np}^\Sigma$  for  $\mathcal{C}^1$  interfaces (in 2D) is established in [78]<sup>3</sup> by using technical tools related to the study of integral operator. The proof of the above lemma can be seen as an alternative (more simple) to the one presented in [78].

Let us finish this section by mentioning that by using the same localization techniques as in the works of K.M. Perfeket et al [92, 104], one can show the following statement

**Lemma 2.2.4.** *Assume that the interface  $\Sigma$  is as in (2.1). Then we have  $\sigma_{ess}(K_{np}^\Sigma) = \sigma_{ess}(K_{\partial\mathcal{K}}^{np})$ . Recall that  $\mathcal{K} = \{x \in \mathbb{R}^3, |x| < \rho, x/|x| \in \mathcal{A}\}$ .*

## 2.3 Study of the far problem

This section aims at studying the well-posedness of the far problem. Let us detail this a bit. For  $\tau$  small enough (e.g. for  $\tau < 1/2$ ) we define  $\Omega^\tau = \Omega \setminus B(O, \tau)$ . Our goal is to study the

<sup>3</sup>The author would like to thank Charles Dapogny for suggesting this reference.

well-posedness (in the Fredholm sense) of the problem

$$\text{Find } u \in \mathbf{H}_0^1(\Omega^\tau) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f \quad (2.9)$$

for an arbitrary  $f \in (\mathbf{H}_0^1(\Omega^\tau))^*$ . We set  $\Omega_1^\tau := \Omega_1 \cap \Omega^\tau$ ,  $\Omega_2^\tau := \Omega_2 \cap \Omega^\tau$ . The interface between  $\Omega_1^\tau$  and  $\Omega_2^\tau$  is denoted by  $\Sigma^\tau$ . It is smooth and meets the boundary of  $\Omega^\tau$  orthogonally at  $\partial B(O, \tau)$ . As previously, the analysis of the well-posedness (in the Fredholm sense) of (2.9) is equivalent to study the Fredholmness of the operator  $F_\sigma^\tau : \mathbf{H}_0^1(\Omega^\tau) \rightarrow (\mathbf{H}_0^1(\Omega^\tau))^*$  such that

$$\langle F_\sigma^\tau u, v \rangle := \int_{\Omega^\tau} \sigma \nabla u \cdot \nabla \bar{v} dx, \quad \forall u, v \in \mathbf{H}_0^1(\Omega^\tau).$$

The main result of this section is given by the following

**Theorem 2.3.1.** *Assume that  $\tau \leq 1/2$ . If  $\kappa_\sigma \neq -1$ , then the operator  $F_\sigma^\tau$  is a Fredholm operator of index 0. In particular, we have the estimate*

$$\|u\|_{\mathbf{H}_0^1(\Omega^\tau)} \leq C(\|F_\sigma^\tau u\|_{(\mathbf{H}_0^1(\Omega^\tau))^*} + \|u\|_{L^2(\Omega^\tau)}), \quad \forall u \in \mathbf{H}_0^1(\Omega^\tau)$$

with  $C$  independent of  $u$ .

To prove the previous theorem, we will use localization techniques. For this, we need to study two different versions of the problem. The first one is related to the problem near any point  $x \in \Sigma^\tau$ , the second one is related to the problem near some point  $x \in \bar{\Sigma}^\tau \cap \partial B(O, \tau)$ . A complete proof of the previous theorem will be given in §2.3.3.

### 2.3.1 Preliminaries

Let  $g : [0; 1]^2 \rightarrow \mathbb{R}$  be a bounded function of class  $\mathcal{C}^1$  and let  $0 < L$ . We define  $\Omega^L \subset \mathbb{R}^3$  as  $\Omega^L = \Omega_1^L \cup \Omega_2^L \cup \Sigma^L$  where  $\Omega_1^L$ ,  $\Omega_2^L$  and  $\Sigma^L$  are defined as follows:

$$\Omega_1^L := \{(x, y, z) \in \mathbb{R}^3 \text{ such that } (x, y) \in (0; 1)^2 \text{ and } g(x, y) - L < z < g(x, y)\},$$

$$\Omega_2^L := \{(x, y, z) \in \mathbb{R}^3 \text{ such that } (x, y) \in (0; 1)^2 \text{ and } g(x, y) < z < g(x, y) + L\},$$

$$\Sigma^L := \{(x, y, z) \in \mathbb{R}^3 \text{ such that } (x, y) \in (0; 1)^2 \text{ and } z = g(x, y)\}.$$

We consider the operator  $A_\sigma^L : \mathbf{H}_0^1(\Omega^L) \rightarrow (\mathbf{H}_0^1(\Omega^L))^*$  such that for all  $u, v \in \mathbf{H}_0^1(\Omega^L)$  we have

$$\langle A_\sigma^L u, v \rangle := \int_{\Omega^L} \sigma \nabla u \cdot \nabla \bar{v} dx dy dz$$

in which the function  $\sigma$  is such that  $\sigma|_{\Omega_1^L} = \sigma_1$  and  $\sigma|_{\Omega_2^L} = \sigma_2$ , where  $0 < \sigma_1$  and  $\sigma_2 < 0$ . Our goal is to find an explicit condition on  $\kappa_\sigma := \sigma_2/\sigma_1$  in order to ensure that  $A_\sigma$  is an isomorphism. For this purpose, we are going to use the  $T$ -coercivity method.

**Lemma 2.3.1.** *Assume that  $\sigma$  is such that  $\max(|\kappa_\sigma|, 1/|\kappa_\sigma|) > (1 + 2\|\nabla g\|_{L^\infty(\Sigma)} + 4\|\nabla g\|_{L^\infty(\Sigma)}^2)$ . Then the operator  $A_\sigma^L$  is an isomorphism.*

**Proof.** The proof is a generalization of the one given in [50, Theorem 1.2.10] for the 2D case. For all  $u \in \mathbf{H}_0^1(\Omega)$ , we define the functions  $u_1$  and  $u_2$  such that  $u_1 = u|_{\Omega_1^L} \in \mathbf{H}^1(\Omega_1^L)$  and  $u_2 = u|_{\Omega_2^L} \in \mathbf{H}^1(\Omega_2^L)$ . We introduce the operators  $T_1, T_2 : \mathbf{H}_0^1(\Omega^L) \rightarrow \mathbf{H}_0^1(\Omega^L)$  such that

$$T_1(u) = \begin{cases} u_1 & \text{in } \Omega_1^L \\ -u_2 + 2R_1(u_1) & \text{in } \Omega_2^L \end{cases} \quad \text{and} \quad T_2(u) = \begin{cases} -u_1 + 2R_2(u_2) & \text{in } \Omega_1^L \\ u_2 & \text{in } \Omega_2^L \end{cases}$$

where  $R_1$  and  $R_2$  are the linear operators defined as follows

$$\begin{cases} R_1(u_1)(x, y, z) = u_1(x, y, 2g(x, y) - z) & \text{for } (x, y, z) \in \Omega_2^L \\ R_2(u_2)(x, y, z) = u_2(x, y, 2g(x, y) - z) & \text{for } (x, y, z) \in \Omega_1^L. \end{cases}$$

One can check that for all  $u \in H_0^1(\Omega^L)$ , we have  $R_1(u_1)|_{\Sigma^L} = R_2(u_2)|_{\Sigma^L} = u|_{\Sigma^L}$ . Moreover, one can also see that  $R_1(u_1)|_{\partial\Omega^L \cap \partial\Omega_1^L} = 0$  and  $R_2(u_2)|_{\partial\Omega^L \cap \partial\Omega_2^L} = 0$ . This leads us to conclude that for all  $u \in H_0^1(\Omega)$ , the functions  $T_1(u)$  and  $T_2(u)$  belong also to  $H_0^1(\Omega^L)$ . As a result  $T_1$  (resp.  $T_2$ ) defines a linear operators from  $H_0^1(\Omega^L)$  into itself. We define

$$\|R_1\| = \sup_{u \in H_0^1(\Omega^L), u \neq 0} \frac{\|\nabla R_1(u_1)\|_{L^2(\Omega_2^L)}}{\|\nabla u_1\|_{L^2(\Omega_1^L)}} \quad \text{and} \quad \|R_2\| = \sup_{u \in H_0^1(\Omega^L), u \neq 0} \frac{\|\nabla R_2(u_2)\|_{L^2(\Omega_1^L)}}{\|\nabla u_2\|_{L^2(\Omega_2^L)}}.$$

Now, by applying [50, Theorem 1.1.1], one can say that if

$$1/|\kappa_\sigma| > \|R_1\|^2 \quad \text{or} \quad |\kappa_\sigma| > \|R_2\|^2,$$

then  $A_\sigma^L$  is an isomorphism. Therefore, we need to find upper bounds of  $\|R_1\|$  and  $\|R_2\|$ . Let us start with  $\|R_1\|$ . First, observe that for all  $u \in H_0^1(\Omega^L) \setminus \{0\}$  we have

$$\int_{\Omega_2^L} |\nabla R_1(u_1)|^2(\hat{x}, \hat{y}, \hat{z}) d\hat{x}d\hat{y}d\hat{z} = \int_{\Omega_2^L} (\partial_{\hat{x}}(R_1(u_1)))^2 + (\partial_{\hat{y}}(R_1(u_1)))^2 + (\partial_{\hat{z}}(R_1(u_1)))^2 d\hat{x}d\hat{y}d\hat{z}.$$

By performing the change of variables  $(x, y, z) = J(\hat{x}, \hat{y}, \hat{z})$  (this is possible since  $J$  is of class  $\mathcal{C}^1$  and  $J \circ J = I$ ) and by observing that

$$\begin{cases} \partial_{\hat{x}} R_1(u_1)(\hat{x}, \hat{y}, \hat{z}) = \partial_x u_1(x, y, z) + 2\partial_x g(x, y) \partial_z u_1(x, y, z) \\ \partial_{\hat{y}} R_1(u_1)(\hat{x}, \hat{y}, \hat{z}) = \partial_y u_1(x, y, z) + 2\partial_y g(x, y) \partial_z u_1(x, y, z), \\ \partial_{\hat{x}} R_1(u_1)(\hat{x}, \hat{y}, \hat{z}) = -\partial_z u_1(x, y, z), \end{cases}$$

we obtain the estimate

$$\begin{aligned} \int_{\Omega_2^L} |\nabla R_1(u_1)|^2(\hat{x}, \hat{y}, \hat{z}) d\hat{x}d\hat{y}d\hat{z} &= \int_{\Omega_1^L} (\partial_x u_1(x, y, z) + 2\partial_x g(x, y) \partial_z u_1(x, y, z))^2 dx dy dz \\ &+ \int_{\Omega_1^L} (\partial_y u_1(x, y, z) + 2\partial_y g(x, y) \partial_z u_1(x, y, z))^2 dx dy dz \\ &+ \int_{\Omega_1^L} (\partial_z u_1(x, y, z))^2 dx dy dz \\ &\leq A \|\nabla u_1\|_{L^2(\Omega_1^L)}^2 \end{aligned}$$

with  $A = (1 + 2\|\nabla g\|_{L^\infty(\Sigma^L)} + 4\|\nabla g\|_{L^\infty(\Sigma^L)}^2)$ . This means that  $\|R_1\|^2 \leq (1 + 2\|\nabla g\|_{L^\infty(\Sigma^L)} + 4\|\nabla g\|_{L^\infty(\Sigma^L)}^2)$ . Working in a similar way (by exchanging the role of  $\Omega_1^L$  and  $\Omega_2^L$ ), we find that  $\|R_2\|^2 \leq (1 + 2\|\nabla g\|_{L^\infty(\Sigma^L)} + 4\|\nabla g\|_{L^\infty(\Sigma^L)}^2)$ . The lemma is then proved.  $\blacksquare$

**Proposition 2.3.1.** *Assume that  $g \in \mathcal{C}^1([0; 1]^2)$  is such that the function  $g^1 : (x, y, z) \mapsto z - g(x, y)$  satisfies  $\partial_n g^1 = 0$  (here  $n$  is the outward normal vector to  $\Omega^L$ ) on  $\partial\Omega^L \cap \overline{\Sigma^L}$ . Then  $A_\sigma^L$  is a Fredholm operator of index zero for all  $\kappa_\sigma \neq -1$ .*

The assumption  $\partial_n g^1 = 0$  on  $\partial\Omega^L \cap \overline{\Sigma^L}$  is equivalent to say that the normal vector to  $\Sigma^L$  (which coincides with  $\nabla g^1$ ) is tangential to  $\partial\Omega^L$  in  $\partial\Omega^L \cap \overline{\Sigma^L}$ . This means that  $\Sigma^L$  meets  $\partial\Omega^L$  orthogonally.

**Proof.** The proof is inspired by the proof of the a priori estimate obtained in [50, §1.3.4] and will be based on the use of localization techniques. Near each  $\mathbf{x} \in \Sigma^L$ , we denote by  $(s_x, t_x, w_x)$  a system of local coordinates of origin  $\mathbf{x}$  (in such way that the plane  $w_x = 0$  is tangent to  $\Sigma^L$  at

$\mathbf{x}$ ). The existence of such system of coordinates is due to the fact that  $g$  is of class  $\mathcal{C}^1$ . Near, any  $\mathbf{x} \in \Sigma^L$  the interface  $\Sigma^L$  can be seen as the graph of a function  $(s_x, t_x) \mapsto \tilde{g}^x(s_x, t_x)$ . Furthermore, for all  $\mathbf{x} \in \Sigma^L$ , we can find three positive numbers  $a_x, b_x$  and  $\delta_x < L$  such that the domain

$$\Omega^x(a_x, b_x, \delta_x) := \{(s_x, t_x, w_x) \mid s_x \in (-a_x; a_x), t_x \in (-b_x; b_x), w_x \in (-\delta_x + \tilde{g}^x(s_x, t_x); \delta_x + \tilde{g}^x(s_x, t_x))\}$$

is a subset of  $\Omega^L$ . With this in mind, we can define the domains  $\Omega_1^x$  and  $\Omega_2^x$  such that  $\Omega_1^x(a_x, b_x, \delta_x) := \Omega_1^L \cap \Omega^x(a_x, b_x, \delta_x)$  and  $\Omega_2^x(a_x, b_x, \delta_x) := \Omega_2^L \cap \Omega^x(a_x, b_x, \delta_x)$ . Regarding the definition of  $\Omega_1^L$  and  $\Omega_2^L$ , one deduces that the domains  $\Omega_1^x$  and  $\Omega_2^x$  admit the representation

$$\begin{aligned} \Omega_2^x(a_x, b_x, \delta_x) &:= \{(s_x, t_x, w_x) \mid s_x \in (-a_x; a_x), t_x \in (-b_x; b_x), w_x \in (\tilde{g}^x(s_x, t_x); \delta_x + \tilde{g}^x(s_x, t_x))\} \\ \Omega_1^x(a_x, b_x, \delta_x) &:= \{(s_x, t_x, w_x) \mid s_x \in (-a_x; -a_x), t_x \in (-b_x; b_x), w_x \in (-\delta_x + \tilde{g}^x(s_x, t_x); \tilde{g}^x(s_x, t_x))\}. \end{aligned}$$

When  $\mathbf{x} \in \overline{\Sigma^L} \cap \partial\Omega^L$ , thanks to the assumption made on the function  $g$ , we can find a new system of coordinates  $(s_x, t_x, w_x)$  that is obtained by rotating the initial system of coordinates (in which the plane  $w_x = 0$  is tangential to  $\overline{\Sigma^L}$  at  $\mathbf{x}$ ) and three positive numbers  $a_x, b_x$  and  $\delta_x < L$  such that the domain

$$\Omega^x(a_x, b_x, \delta_x) := \{(s_x, t_x, w_x) \mid s_x \in (0; a_x), t_x \in (0; b_x) \text{ and } w_x \in (-\delta_x + \tilde{g}^x(s_x, t_x); \delta_x + \tilde{g}^x(s_x, t_x))\}$$

is a subset of  $\Omega^L$  in which  $(s_x, t_x) \mapsto \tilde{g}^x$  is a function whose graph coincides with  $\Sigma_L$  near  $\mathbf{x}$ . We define the domains  $\Omega_1^x$  and  $\Omega_2^x$  as in the case of  $\mathbf{x} \in \Sigma^L$ . To simplify notations, we shall denote by  $\tilde{\nabla}$  the gradient operator with respect to  $(s_x, t_x, w_x)$ . Since  $(s_x, t_x, w_x)$  is obtained by rotating the original system of coordinates, it follows that for all  $\mathbf{x} \in \overline{\Sigma}$  and all  $u, v \in H^1(\Omega^x)$ , we have

$$\int_{\Omega^x(a_x, b_x, \delta_x)} \sigma \nabla u \cdot \nabla v \, dx dy dz = \int_{\tilde{\Omega}^x} \sigma \tilde{\nabla} u \cdot \tilde{\nabla} v \, ds^x dt^x dw^x$$

where  $\tilde{\Omega}^x = (-a_x; a_x) \times (-b_x; b_x) \times (-\delta_x; \delta_x)$  when  $\mathbf{x} \in \Sigma^L$  and  $\tilde{\Omega}^x = (0; a_x) \times (0; b_x) \times (-\delta_x; \delta_x)$  when  $\mathbf{x} \in \overline{\Sigma^L} \cap \partial\Omega^L$ . Given that for all  $\mathbf{x} \in \overline{\Sigma^L}$ , the plane  $w_x = 0$  is tangential to  $\overline{\Sigma^L}$  at  $\mathbf{x}$ , we then have  $\tilde{\nabla} \tilde{g}^x(0, 0) = 0$  and since the function  $\tilde{g}^x$  is of class  $\mathcal{C}^1$ , we can say, using the fact that  $\kappa_\sigma \neq -1$ , that for all  $\mathbf{x} \in \overline{\Sigma^L}$ , we can find  $a_x^*, b_x^*$  small enough so that

$$\max(|\kappa_\sigma|, 1/|\kappa_\sigma|) > (1 + 2\|\tilde{\nabla} \tilde{g}^x\|_{L^\infty(\Sigma^x)} + 4\|\tilde{\nabla} \tilde{g}^x\|_{L^\infty(\Sigma^x)}^2)$$

where  $\Sigma^x := \Omega_1^x(a_x^*, b_x^*, \delta_x) \cap \Omega_2^x(a_x^*, b_x^*, \delta_x)$ . As a consequence, by applying the results of the previous lemma, we infer that for all  $\mathbf{x} \in \Sigma^L$  the operator  $A_\sigma^x : H_0^1(\Omega^x(a_x^*, b_x^*, \delta_x^*)) \rightarrow (H_0^1(\Omega^x(a_x^*, b_x^*, \delta_x^*)))^*$  that is defined by

$$\langle A_\sigma^x u, v \rangle := \int_{\Omega^x(a_x^*, b_x^*, \delta_x)} \sigma \tilde{\nabla} u \cdot \tilde{\nabla} v \, ds^x dt^x dw^x, \quad \forall u, v \in H_0^1(\Omega^x(a_x^*, b_x^*, \delta_x^*))$$

is an isomorphism. For all  $\mathbf{x} \in \overline{\Sigma^L}$ , we define  $\chi^x \in \mathcal{D}(\overline{\Omega^L}, [0; 1])$  that is equal to 1 in  $\Omega^x(a_x^*/2, b_x^*/2, \delta_x/2)$  and that vanishes in  $\Omega^L \setminus \Omega^x(3a_x^*/4, 3b_x^*/4, 3\delta_x/4)$ .

By noticing that  $\bigcup_{\mathbf{x} \in \overline{\Sigma}} \Omega^x(a_x^*/2, b_x^*/2, \delta_x/2)$  covers  $\overline{\Sigma^L}$  and since the latter is compact, one deduces

that there exist  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \overline{\Sigma^L}$  with  $n \in \mathbb{N}$  such that

$$\overline{\Sigma^L} \subset \Omega_n := \bigcup_{\mathbf{x} \in \{\mathbf{x}_1, \dots, \mathbf{x}_n\}} \Omega^x(a_x^*/2, b_x^*/2, \delta_x/2).$$

To simplify, for all  $\mathbf{x} \in \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  the domain  $\Omega^x(a_x^*, b_x^*, \delta_x)$  will be denoted by  $\Omega^x$ . To proceed, denote by  $\Omega^0$  the domain  $\Omega^0 := \Omega \setminus \overline{\Omega_n}$  and let  $\chi^0 \in \mathcal{D}(\overline{\Omega^L}, [0; 1])$  such that  $\chi^0|_{\Omega^0} = 1$  and that

vanishes near  $\Sigma$ . Starting from the fact that for all  $\mathbf{x} \in \Omega$ , we have  $1 \leq \chi^0(\mathbf{x}) + \sum_{i=1}^n \chi^{x_i}(\mathbf{x})$ , we

deduce that for all  $u \in H_0^1(\Omega^L) \setminus \{0\}$ , we have the estimate

$$\|u\|_{H_0^1(\Omega^L)} \leq \|\chi^0 u\|_{H_0^1(\text{supp}(\chi^0))} + \sum_{i=1}^n \|\chi^{x_i} u\|_{H_0^1(\Omega^{x_i})}. \quad (2.10)$$

For all  $\mathbf{x} = \mathbf{x}_1, \dots, \mathbf{x}_n$ , we define the operator  $T^{\mathbf{x}} : H_0^1(\Omega^{\mathbf{x}}) \rightarrow H_0^1(\Omega^{\mathbf{x}})$  as in the proof of the previous lemma. We also need to define the operator  $T^0 : H_0^1(\text{supp}(\chi^0)) \rightarrow H_0^1(\text{supp}(\chi^0))$  such that for all  $v \in H_0^1(\text{supp}(\chi^0))$ , we have

$$T^0(v) = \begin{cases} v & \text{in } \Omega_1 \cap \text{supp}(\chi^0) \\ -v & \text{in } \Omega_2 \cap \text{supp}(\chi^0). \end{cases}$$

The local ellipticity of the problem far from  $\Sigma$  and the continuity of  $T^0 : L^2(\text{supp}(\chi^0)) \rightarrow L^2(\text{supp}(\chi^0))$  combined with the relation (2.10) lead us to the estimate

$$\begin{aligned} \|\chi^0 u\|_{H_0^1(\text{supp}(\chi^0))}^2 &\leq C |\langle \sigma \nabla(\chi^0 u), \nabla T^0(\chi^0 u) \rangle| \\ &\leq C |\langle \text{div}(\sigma \nabla u), \chi^0 T^0(\chi^0 u) \rangle| + C |\langle \sigma \nabla u \cdot \nabla \chi^0, T^0(\chi^0 u) \rangle| \\ &\quad + C |(\sigma u \nabla \chi^0, \nabla T^0(\chi^0 u))_{L^2(\text{supp}(\chi^0))}| \\ &\leq C (\|A_\sigma^L u\|_{(H_0^1(\Omega^L))^*} + \|u\|_{L^2(\Omega^L)}) \|u\|_{H_0^1(\Omega^L)}. \end{aligned}$$

Above and in the rest of the proof,  $C$  denotes a constant whose value may change from line to line but that is independent of  $u$ . By replacing the operator  $T^0$  by  $T^{\mathbf{x}}$  and  $\text{supp}(\chi^0)$  by  $\Omega^{\mathbf{x}}$  in the above calculi, we conclude that for all  $\mathbf{x} = \mathbf{x}_1, \dots, \mathbf{x}_n$ , we have the estimate

$$\|\chi^{\mathbf{x}} u\|_{H_0^1(\Omega^{\mathbf{x}})}^2 \leq C (\|A_\sigma^L u\|_{(H_0^1(\Omega^L))^*} + \|u\|_{L^2(\Omega^L)}) \|u\|_{H_0^1(\Omega^L)}.$$

With the help of (2.10), we infer that we have

$$\|u\|_{H_0^1(\Omega^L)} \leq C (\|A_\sigma^L u\|_{(H_0^1(\Omega^L))^*} + \|u\|_{L^2(\Omega^L)}).$$

By using that the embedding of  $H_0^1(\Omega^L)$  into  $L^2(\Omega^L)$  is compact and that  $A_\sigma^L$  is symmetric we deduce, by applying Proposition 2.8.2, that  $A_\sigma^L$  is a Fredholm operator of index zero.  $\blacksquare$

### 2.3.2 Study of the problem in the vicinity the boundary

In this paragraph, we turn our attention to the study of the scalar problem near  $\partial B(0, \tau)$ . To do that, we start by defining the domain  $\omega^\tau = \Omega^\tau \cap B(O, 2\tau) = B(O, 2\tau) \setminus \overline{B(O, \tau)}$  and then we introduce the operator  $C_\sigma^\tau : H_0^1(\omega^\tau) \rightarrow (H_0^1(\omega^\tau))^*$  that is defined by the relation

$$\langle C_\sigma^\tau u, v \rangle := \int_{\omega^\tau} \sigma \nabla u \cdot \nabla \bar{v} \, dx, \quad u, v \in H_0^1(\omega^\tau)$$

where  $\sigma = \sigma_1$  in  $\omega_1^\tau := \Omega_1 \cap \omega^\tau$  and  $\sigma = \sigma_2$  in  $\omega_2^\tau := \Omega_2 \cap \omega^\tau$ . Since by assumption we have  $\tau < 1/2$ , the interface  $\Sigma$  meets the boundary of  $\omega^\tau$  orthogonally at  $\partial B(0, \tau)$  and at  $\partial B(0, 2\tau)$ . Furthermore, one can easily see that thanks to the assumptions made on  $\Sigma$  near the origin (see the introduction of this chapter), we have

$$\begin{aligned} \omega_1^\tau &= \{(r, \theta, \varphi) \mid r \in (\tau; 2\tau), g(\varphi) < \theta, \varphi \in [0, 2\pi]\} \\ \omega_2^\tau &= \{(r, \theta, \varphi) \mid r \in (\tau; 2\tau), \theta < g(\varphi), \varphi \in [0, 2\pi]\} \end{aligned}$$

where  $g : [0, 2\pi] \rightarrow [0, \pi]$  is a periodic function of  $\mathcal{C}^2$  (see the introduction of this chapter).

**Proposition 2.3.2.** *Assume that  $\kappa_\sigma \neq -1$ . Then the operator  $C_\sigma^\tau$  is a Fredholm operator of index zero. In particular, we have the estimate*

$$\|u\|_{H_0^1(\omega^\tau)} \leq C (\|C_\sigma^\tau u\|_{(H_0^1(\omega^\tau))^*} + \|u\|_{L^2(\omega^\tau)})$$

where  $C$  is a constant that does not depend in  $u$ .

**Proof.** By working in spherical coordinates, one can easily see that for all  $u, v \in H_0^1(\omega^\tau)$  we have

$$\langle C_\sigma^\tau u, v \rangle = \int_{\Omega^\tau} \sigma \nabla u \cdot \nabla \bar{v} \, dx = \int_\tau^{2\tau} \int_{\mathbb{S}^2} \sigma(\omega) ((r \partial_r u)(r \partial_r \bar{v}) + \nabla_S u \cdot \nabla_S \bar{v}) \, dr d\omega.$$

Above  $\nabla_S$  stands for the surface gradient operator on  $\mathbb{S}^2$ . By performing the Euler change of variables  $(r, \omega) \mapsto (t, \omega) = (\log(r), \omega)$ , we transform the domain  $\omega^\tau$  into the domain  $\tilde{\omega}^\tau := (a; b) \times \mathbb{S}^2$  where the constants  $a$  and  $b$  are given by  $a = \log(\tau)$  and  $b = \log(2\tau)$ . Furthermore, by using the classical angular coordinates  $(\theta, \varphi) \in (0; \pi) \times (0; 2\pi)$  to parameterize the sphere  $\mathbb{S}^2$ , the domain  $\tilde{\omega}^\tau$  can be also defined as follows:

$$\tilde{\omega}^\tau := \{(t, \theta, \varphi) \mid t \in (a; b), \theta \in (0; \pi) \text{ and } \varphi \in (0; 2\pi)\}.$$

To proceed, we define the domains  $\tilde{\omega}_1^\tau$  and  $\tilde{\omega}_2^\tau$ , respectively, as the images of the domains  $\omega_1^\tau$  and  $\omega_2^\tau$  by the Euler change of variable. We also denote by  $\tilde{\Sigma}^\tau := \tilde{\omega}_1^\tau \cap \tilde{\omega}_2^\tau$ .

For all  $u \in H_0^1(\omega^\tau)$ , we denote by  $\tilde{u}$  the function that is defined in  $\tilde{\omega}^\tau$  by the relation  $\tilde{u}(t, \omega) = u(e^t, \omega)$  for almost all  $(t, \omega) \in \tilde{\omega}^\tau$ . By observing that  $\partial_t \tilde{u}(t, \omega) = \partial_r u(r, \omega) \partial_t r = r \partial_r u(r, \omega)$ , we deduce that the Euler change of variables induces an isomorphism between the spaces  $H_0^1(\omega^\tau)$  and the space

$$W_0^1(\tilde{\omega}^\tau) = \{(t, \omega) \mapsto \tilde{u}(t, \omega) \mid \int_{\tilde{\omega}^\tau} e^t ((\partial_t \tilde{u})^2 + |\nabla_S \tilde{u}|^2) \, dt d\omega < +\infty \text{ and } \tilde{u} = 0 \text{ on } \partial \tilde{\omega}^\tau\}.$$

Given that the function  $(t, \omega) \mapsto e^t$  is bounded in  $\tilde{\omega}^\tau$ , we infer that the space  $W_0^1(\tilde{\omega}^\tau)$  coincides with  $H_0^1((a; b) \times \mathbb{S}^2)$ . With this in mind, we can say that the Fredholmness of the operator  $C_\sigma^\tau$  is equivalent to the Fredholmness of the operator  $\tilde{C}_\sigma^\tau : H_0^1((a; b) \times \mathbb{S}^2) \rightarrow (H_0^1((a; b) \times \mathbb{S}^2))^*$  that is defined as follows:

$$\langle \tilde{C}_\sigma^\tau \tilde{u}, \tilde{v} \rangle = \int_a^b \int_{\mathcal{A}} e^t \sigma(\omega) (\partial_t \tilde{u} \partial_t \tilde{v} + \nabla_S \tilde{u} \cdot \nabla_S \tilde{v}) \, dt d\omega, \quad u, v \in W_0^1(\tilde{\omega}^\tau).$$

By observing that the function  $\sigma$  depends only in  $\theta$  and not in the other two variables  $t$  and  $\varphi$ , and by noticing that  $\tilde{\Sigma}^\tau$  meets  $\partial \tilde{\omega}^\tau$  orthogonally, one can adapt<sup>4</sup> the proofs of the Lemma 2.3.1 and the Proposition 2.3.1 in order to deduce that  $\tilde{C}_\sigma^\tau$  is a Fredholm operator of index zero as soon as  $\kappa_\sigma \neq -1$ . This leads us to the wanted result.  $\blacksquare$

### 2.3.3 Final proof

Here, we shall present a proof of the Theorem 2.3.1.

**Proof.** The idea is to use localization techniques as in the proof of Proposition 2.3.1. We start by defining the domains  $A^\tau = \Omega^\tau \cap B(O, 2\tau)$  and  $B^\tau = \Omega^\tau \setminus \overline{B(O, 3\tau/2)}$ . For  $i = 1, 2$ , we also define the domains  $A_i^\tau = A^\tau \cap \Omega_i$  and  $B_i^\tau = B^\tau \cap \Omega_i$ . Next, we introduce the interfaces  $\Sigma_A^\tau = \overline{A_1^\tau} \cap \overline{A_2^\tau}$  and  $\Sigma_B^\tau = \overline{B_1^\tau} \cap \overline{B_2^\tau}$ . It is worth to note that  $\Sigma_B^\tau \cap \partial \Omega^\tau = \emptyset$  and that  $\Sigma^\tau \subset \Sigma_A^\tau \cup \Sigma_B^\tau$ .

Given that  $\overline{\Sigma_B^\tau} \subset \Omega^\tau$  is of class  $\mathcal{C}^1$  and by working as in the proof of the previous proposition (this is possible since  $\kappa_\sigma \neq -1$ ), one can show, that for all  $\mathbf{x} \in \overline{\Sigma_B^\tau}$  we can find  $\Omega^{\mathbf{x}} \subset \Omega$  a neighborhood of  $\mathbf{x}$  such that the operator  $A_\sigma^{\mathbf{x}} : H_0^1(\Omega^{\mathbf{x}}) \rightarrow (H_0^1(\Omega^{\mathbf{x}}))^*$  that is defined by

$$\langle A_\sigma^{\mathbf{x}} u, v \rangle := \int_{\Omega^{\mathbf{x}}} \sigma \nabla u \cdot \nabla \bar{v}, \quad \forall u, v \in H_0^1(\Omega^{\mathbf{x}})$$

<sup>4</sup>The proof is based on the use of local reflections with respect to  $\tilde{\Sigma}^\tau$  and local rotation of the system of coordinates but this time the difficulty comes from the fact that we are working in spherical coordinates. Details about these techniques can be found in §3.6.1.

is an isomorphism (T-coercive). Denote by  $\Omega^{\Sigma_B^\tau}$  the open set

$$\Omega^{\Sigma_B^\tau} := \bigcup_{x \in \overline{\Sigma_B^\tau}} \Omega^x.$$

By observing that  $\Omega^{\Sigma_B^\tau}$  covers the compact set  $B_0^\tau := \overline{\Omega^{\Sigma_B^\tau}} \cap \overline{B^\tau}$ , we can then find  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \overline{\Sigma_B^\tau}$  (with  $n \in \mathbb{N}$ ) such that

$$\overline{\Sigma_B^\tau} \subset B_0^\tau \subset \bigcup_{i=1}^n \Omega^{\mathbf{x}_i}.$$

By applying the partition of unity theorem, we can find  $n$  smooth functions  $\chi_1, \dots, \chi_n \in \mathcal{D}(\overline{\Omega^\tau})$  such that

- $\mathbf{x} \mapsto \chi_i(\mathbf{x}) \in [0; 1]$  for all all  $i = 1, \dots, n$  and for all  $\mathbf{x} \in \Omega$ ,
- $\chi_i$  is compactly supported in  $\Omega^{\mathbf{x}_i}$ ,
- $\sum_{i=1}^n \chi_i = 1$  in  $B_0^\tau$ .

The next step, is to introduce  $\chi^0 \in \mathcal{D}(\overline{\Omega^\tau}, [0; 1])$  that depends only in  $r = |x|$  and that is equal to 1 for  $|x| < 3/2\tau$  and that vanishes for  $7\tau/4 < |x|$ . This means that  $\chi^0$  is supported in  $A^\tau$ . We finish the series of notations by introducing  $\tilde{\chi} \in \mathcal{D}(\overline{\Omega^\tau}, [0; 1])$  that is equal to 1 in  $\Omega^\tau \setminus (\overline{B(O, 3\tau/2)} \cup B_0^\tau)$  and that vanishes near  $\Sigma^\tau$ . Observe that we have

$$1 \leq \tilde{\chi}(\mathbf{x}) + \chi^0(\mathbf{x}) + \sum_{i=1}^n \chi^i(\mathbf{x}), \quad \mathbf{x} \in \Omega^\tau.$$

As a result, there exists  $0 < C$  such that

$$\|u\|_{\mathbf{H}_0^1(\Omega^\tau)} \leq C(\|\tilde{\chi}u\|_{\mathbf{H}_0^1(\text{supp}(\tilde{\chi}))} + \|\chi^0u\|_{\mathbf{H}_0^1(A^\tau)} + \sum_{i=1}^n \|\chi^i u\|_{\mathbf{H}_0^1(\Omega^{\mathbf{x}_i})}) \quad \forall u \in \mathbf{H}_0^1(\Omega^\tau).$$

Before getting into details, along the rest of this proof, we denote by  $C$  a positive constant whose value may change from line to line but that is independent of  $u$ . Given that the function  $\tilde{\chi}$  vanishes near  $\Sigma^\tau$  and by means of the T-coercivity approach (see the proof of Proposition 2.3.1), we obtain the estimate

$$\|\tilde{\chi}u\|_{\mathbf{H}_0^1(\text{supp}(\tilde{\chi}))}^2 \leq C(\|\text{div}(\sigma \nabla u)\|_{\mathbf{H}_0^1(\Omega^\tau)^*} + \|u\|_{\mathbf{L}^2(\Omega^\tau)})\|u\|_{\mathbf{H}_0^1(\Omega^\tau)}.$$

In the same way, given that the operators  $A_\sigma^{\mathbf{x}_i}$  are T-coercive and working as in the proof of Proposition 2.3.1 we arrive to the estimate

$$\|\chi^i u\|_{\mathbf{H}_0^1(\Omega^{\mathbf{x}_i})}^2 \leq C(\|\text{div}(\sigma \nabla u)\|_{\mathbf{H}_0^1(\Omega^\tau)^*} + \|u\|_{\mathbf{L}^2(\Omega^\tau)})\|u\|_{\mathbf{H}_0^1(\Omega^\tau)} \quad \text{for } i = 1, \dots, n.$$

The next step is to deal with the term  $\|\chi^0 u\|_{\mathbf{H}_0^1(A^\tau)}$ . Unfortunately, this time, we can not use T-coercivity approach. However, starting from the result of Proposition 2.3.2, we conclude that we have the estimate

$$\|\chi^0 u\|_{\mathbf{H}_0^1(A^\tau)} \leq C(\|\text{div}(\sigma \nabla(\chi^0 u))\|_{\mathbf{H}_0^1(A^\tau)^*} + \|\chi^0 u\|_{\mathbf{L}^2(A^\tau)}).$$

Given that the function  $\chi^0$  is independent of  $r = |x|$ , one can easily prove that for all  $v \in \mathbf{H}_0^1(A^\tau)$ , there holds  $\text{div}(\sigma v \nabla \chi^0) \in \mathbf{L}^2(A^\tau)$ . Moreover, we have the estimate

$$\|\text{div}(\sigma v \nabla \chi)\|_{\mathbf{L}^2(A^\tau)} \leq C\|v\|_{\mathbf{H}_0^1(A^\tau)}$$

where  $C$  that does not depend in  $v$ . Combining this with the identity

$$(\sigma \nabla(\chi^0 u), \nabla v)_{\mathbf{L}^2(A^\tau)} = -(\text{div}(\sigma \nabla u), \chi^0 v)_{\mathbf{L}^2(A^\tau)} + (u, \text{div}(\sigma v \nabla \chi^0))_{\mathbf{L}^2(A^\tau)} + (\sigma u \nabla \chi^0, \nabla v)_{\mathbf{L}^2(A^\tau)}$$



for all  $u, v \in H_0^1(A^\tau)$ , we get

$$\|\operatorname{div}(\sigma \nabla(\chi^0 u))\|_{H_0^1(A^\tau)^*} \leq C(\|\operatorname{div}(\sigma \nabla u)\|_{H_0^1(\Omega^\tau)^*} + \|u\|_{L^2(\Omega^\tau)}).$$

This leads us to

$$\|\chi^0 u\|_{H_0^1(A^\tau)} \leq C(\|\operatorname{div}(\sigma \nabla u)\|_{H_0^1(\Omega^\tau)^*} + \|u\|_{L^2(\Omega^\tau)}).$$

By combining all these estimates, we obtain

$$\|u\|_{H_0^1(\Omega^\tau)} \leq C(\|F_\sigma^\tau u\|_{H_0^1(\Omega^\tau)^*} + \|u\|_{L^2(\Omega^\tau)})$$

where  $C$  is independent of  $u \in H_0^1(\Omega^\tau)$ . Since the operator  $F_\sigma^\tau$  is symmetric and since the embedding of  $H_0^1(\Omega^\tau)$  into  $L^2(\Omega^\tau)$  is compact, Proposition 2.8.2 guarantees that  $F_\sigma^\tau$  is a Fredholm operator of index zero.  $\blacksquare$

## 2.4 Study of the problem in the whole space

In the previous section, we studied the behavior of Problem (2.2) far from the origin. We proved that it is well-posed as soon as  $\kappa_\sigma \neq -1$ . Here, we want to get a closer look on the situation near the origin. Naturally, this leads us to study the well-posedness of the problem

$$\text{Find } u \in H_{\text{loc}}^1(\mathbb{R}^3) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f.$$

Here the function  $\sigma$  is defined as follows:  $\sigma = \sigma_2$  in  $\mathcal{K}$  and  $\sigma = \sigma_1$  in  $\mathbb{R}^3 \setminus \overline{\mathcal{K}}$ . The classical way to study the well-posedness of the previous problem is to use the Fourier transform but since the function  $\sigma$  does not have a constant behavior at infinity, this approach cannot be used. To cope with this difficulty, and because the  $\sigma$  function is independent of  $r = |x|$ , we will use the so-called Mellin transformation. The use of this transformation will allow us to study the well-posedness of the problem in weighted Sobolev (Kondratiev) spaces [100, 101]. The analysis conducted in this section will be of great importance since it will allow, on the one hand to determine a “simple” condition ensuring the well-posedness (in the Fredholm sense) of (2.2) and on the other hand it will help us constructing an alternative functional framework in which the scalar problem is again well-posed when the original problem is ill-posed in the usual setting.

### 2.4.1 Weighted Sobolev (Kondratiev) spaces

For  $\beta \in \mathbb{R}$  and  $m \in \mathbb{N}$ , we introduce the (homogeneous)<sup>5</sup> weighted Sobolev (Kondratiev) space (see [100]) associated to the punctured domain  $\mathbb{R}^3 \setminus \{O\}$  denoted by  $V_\beta^m(\mathbb{R}^3)$  and defined as the closure of  $\mathcal{D}(\mathbb{R}^3 \setminus \{O\})$  for the norm

$$\|\varphi\|_{V_\beta^m(\mathbb{R}^3)} := \left( \sum_{|\alpha| \leq m} \|r^{|\alpha| - m + \beta} \partial_x^\alpha \varphi\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2}.$$

Here  $r = |x|$  and  $\mathcal{D}(\mathbb{R}^3 \setminus \{O\})$  denotes the space of infinitely differentiable functions which are compactly supported in  $\mathbb{R}^3 \setminus \{O\}$ . Clearly we have  $V_0^0(\mathbb{R}^3) = L^2(\mathbb{R}^3)$ . Moreover, one observes that for all  $m \in \mathbb{N}^*$  and  $\beta \in \mathbb{R}$ , we have the inclusion  $V_\beta^m(\mathbb{R}^3) \subset V_{\beta-1}^{m-1}(\mathbb{R}^3)$ . It is worth to note that for a given  $m \in \mathbb{N}$  and  $\beta_1, \beta_2 \in \mathbb{R}$  such that  $\beta_1 < \beta_2$ , we have  $V_{\beta_1}^m(\mathbb{R}^3) \not\subset V_{\beta_2}^m(\mathbb{R}^3)$  and  $V_{\beta_2}^m(\mathbb{R}^3) \not\subset V_{\beta_1}^m(\mathbb{R}^3)$ . It is also interesting to mention that thanks to the classical Hardy inequalities, one can show (see [101, Theorem 7.1.1]) that  $V_0^1(\mathbb{R}^3) = H^1(\mathbb{R}^3)$ .<sup>6</sup>

<sup>5</sup>For the case of nonhomogeneous ones see [101, Chapter 7].

<sup>6</sup>Note that this is wrong in 2D.



### 2.4.2 The Mellin transform

The Mellin transform is one of the classical integral transformations which is a useful tool to study the properties of partial differential equations. Compared to other classical transformations such as Fourier or Laplace transform, the Mellin transformation has the particularity to be adapted to the study linear PDE in weighted Sobolev spaces and more generally to the study of linear PDE near point singularities. Formally, by using this transformation, the study of the well-posedness of a linear PDE near a singular point and the study of the asymptotic expansion of its solution (when it is well-defined) near these points, reduces to the analysis of the spectral properties of its Mellin transform (also known as the Mellin symbol). The goal of this paragraph is to recall some of the basic properties of this transformation that will be used in the next sections.

Let  $r \mapsto f(r)$  be a smooth function that is compactly supported in  $\mathbb{R}_+^*$ . The Mellin transform of  $f$ , denoted by  $\hat{f}(\lambda)$ , is the function defined for all  $\lambda \in \mathbb{C}$  by the formula:

$$\hat{f}(\lambda) = \int_0^\infty r^{-\lambda-1} f(r) dr.$$

Note that since  $f$  has a compact support in  $(0; \infty)$ , one can show that  $\lambda \mapsto \hat{f}(\lambda)$  is analytic. When  $f$  does not have compact support in  $(0; \infty)$ ,  $\hat{f}(\lambda)$  is no longer defined for all  $\lambda \in \mathbb{C}$ . As we shall see later, the set of  $\lambda \in \mathbb{C}$  for which  $\hat{f}(\lambda)$  exists depends on the regularity of  $f$  in weighted Sobolev spaces. To simplify notations, for all  $\gamma \in \mathbb{R}$ , we denote by  $\ell_\gamma$  the vertical line  $\ell_\gamma := \{\lambda \in \mathbb{C} \mid \Re(\lambda) = \gamma\}$ .

**Lemma 2.4.1.** [102, Theorem 6.1.3] *The Mellin transformation satisfies the following properties.*

- For all  $u \in \mathcal{D}(\mathbb{R}_+^*)$ , we have  $\widehat{(r\partial_r u)}(\lambda) = \lambda \hat{u}(\lambda)$  for all  $\lambda \in \mathbb{C}$ .
- For all  $u, v \in \mathcal{D}(\mathbb{R}_+^*)$ , we have the Parseval equality

$$\int_0^\infty r^{2\beta-1} u(r)v(r) dr = \frac{1}{2i\pi} \int_{\ell_{-\beta}} \hat{u}(\lambda)\hat{v}(\lambda) d\lambda.$$

As a result the Mellin transformation can be continuously extended as an isomorphism between the weighted space  $V_{\beta-1/2}^0(\mathbb{R}_+) := \{u \text{ such that } r^{\beta-1/2}u \in L^2(\mathbb{R}_+)\}$  and the space  $L^2(\ell_{-\beta})$ .

- If  $u \in V_{\beta_1-1/2}^0(\mathbb{R}_+) \cap V_{\beta_2-1/2}^0(\mathbb{R}_+)$  with  $\beta_1 < \beta_2$ , then  $\lambda \mapsto \hat{u}(\lambda)$  is well-defined and holomorphic in the strip  $\Re(\lambda) \in (-\beta_2; -\beta_1)$ .
- The inverse Mellin transformation of  $\hat{u}(\lambda) \in L^2(\ell_{-\beta})$  is given by

$$u(r) = \frac{1}{2i\pi} \int_{\ell_{-\beta}} r^\lambda \hat{u}(\lambda) d\lambda \in V_{\beta-1/2}^0(\mathbb{R}_+).$$

Now, for  $u \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\})$ , we denote by  $(\lambda, \omega) \mapsto \hat{u}(\lambda, \omega)$  the partial Mellin transform of  $u$  (with respect to  $r = |x|$ ) such that for all  $\lambda \in \mathbb{C}$  and  $\omega \in \mathbb{S}^2$

$$\hat{u}(\lambda, \omega) = \int_0^{+\infty} r^{-\lambda-1} u(r\omega) dr.$$

Using the properties above, one can easily see that for all  $u \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\})$  and all  $v \in L^2(\mathbb{S}^2)$  we have

$$\begin{aligned} \langle \hat{u}(\lambda, \omega), v(\omega) \rangle_{L^2(\mathbb{S}^2)} &= \int_{\mathbb{S}^2} \hat{u}(\lambda, \omega) \overline{v(\omega)} d\omega = \int_0^{+\infty} \int_{\mathbb{S}^2} r^{-\lambda-1} u(r\omega) \overline{v(\omega)} dr d\omega = \int_{\mathbb{R}^3} r^{-\lambda-3} u \overline{v} dx \\ &= \langle u, r^{-\bar{\lambda}-3} v \rangle_{L^2(\mathbb{R}^3)}. \end{aligned}$$

The previous identity will be used to define the Mellin transform of elements of the space  $(V_\beta^1(\mathbb{R}))^*$ . In the rest of this paragraph, we focus our attention on the study of the action of the (partial) Mellin transform on the weighted Sobolev spaces  $V_\beta^0(\mathbb{R}^3)$ ,  $V_\beta^1(\mathbb{R}^3)$  and  $(V_\beta^1(\mathbb{R}^3))^*$  for an arbitrary  $\beta \in \mathbb{R}$ . Let us start with the case of the space  $V_\beta^0(\mathbb{R}^3)$ . This is the subject of the next

**Lemma 2.4.2.** *A function  $x \mapsto u(r\omega)$  belongs to the space  $V_\beta^0(\mathbb{R}^3)$  if and only if  $(\lambda, \omega) \mapsto \hat{u}(\lambda, \omega)$  (its partial Mellin transform with respect to  $r$ ) belongs to the space  $L^2(\ell_{-\beta+1/2}, L^2(\mathbb{S}^2))$ . Additionally, the norm*

$$\|u\| = \left( \frac{1}{2i\pi} \int_{\ell_{-\beta+1/2}} \|\hat{u}(\lambda, \omega)\|_{L^2(\mathbb{S}^2)}^2 d\lambda \right)^{1/2}$$

is equivalent to  $\|\cdot\|_{V_\beta^0(\mathbb{R}^3)}$ .

**Proof.** By definition of  $V_\beta^0(\mathbb{R}^3)$ , we can say that  $u \in V_\beta^0(\mathbb{R}^3)$  if and only if  $r \mapsto u(r\omega) \in V_{\beta+1}^0(\mathbb{R}_+)$  for almost all  $\omega \in \mathbb{S}^2$  and  $\omega \mapsto u(r\omega) \in L^2(\mathbb{S}^2)$  for almost all  $r \in \mathbb{R}_+$ . Then The result follows by applying the second item of the previous lemma.  $\blacksquare$

With the same idea as in the previous proof, we obtain the

**Lemma 2.4.3.** *A function  $r\omega \mapsto u(r\omega)$  belongs to the space  $V_\beta^1(\mathbb{R}^3)$  if and only if the functions  $\lambda \mapsto \hat{u}(\lambda, \omega)$  and  $\lambda \mapsto \lambda \hat{u}(\lambda, \omega)$  belong respectively to  $L^2(\ell_{-\beta-1/2}, H^1(\mathbb{S}^2))$  and  $L^2(\ell_{-\beta-1/2}, L^2(\mathbb{S}^2))$ . Furthermore, the norm*

$$\|u\| = \left( \frac{1}{2i\pi} \int_{\ell_{-\beta-1/2}} \|\hat{u}(\lambda, \omega)\|_{H^1(\mathbb{S}^2)}^2 + |\lambda|^2 \|\hat{u}(\lambda, \omega)\|_{L^2(\mathbb{S}^2)}^2 d\lambda \right)^{1/2}$$

is equivalent to  $\|\cdot\|_{V_\beta^1(\mathbb{R}^3)}$ .

For all  $\lambda \in \mathbb{C}$ , we introduce the norm  $\|\cdot\|_{H^1(\mathbb{S}^2, |\lambda|)}$  such that

$$\|u\|_{H^1(\mathbb{S}^2, |\lambda|)} =: (\|u\|_{H^1(\mathbb{S}^2)}^2 + |\lambda|^2 \|u\|_{L^2(\mathbb{S}^2)}^2)^{1/2} \text{ for all } u \in H^1(\mathbb{S}^2).$$

Clearly, for a fixed  $\lambda \in \mathbb{C}$ , it is equivalent to the classical  $H^1(\mathbb{S}^2)$  norm. But when  $|\lambda|$  tends to  $+\infty$  this is no-longer the case. In  $(H^1(\mathbb{S}^2))^*$ , we introduce the norm  $\|\cdot\|_{(H^1(\mathbb{S}^2, |\lambda|))^*}$  which is defined as follows

$$\|f\|_{(H^1(\mathbb{S}^2, |\lambda|))^*} = \sup_{u \in H^1(\mathbb{S}^2) \setminus \{0\}} \frac{|\langle f, u \rangle|}{\|u\|_{H^1(\mathbb{S}^2, |\lambda|)}} \text{ for all } f \in (H^1(\mathbb{S}^2))^*.$$

The last part of this section is dedicated to the study of the Mellin transform of elements of the space  $(V_\beta^1(\mathbb{R}^3))^*$  with  $\beta \in \mathbb{R}$ . For simplicity, we limit ourselves to the case of distributions with compact support in  $\mathbb{R}^3 \setminus \{O\}$ . Consider  $r\omega \mapsto f(r\omega) \in (V_\beta^1(\mathbb{R}^3))^*$  with compact support in  $\mathbb{R}^3 \setminus \{O\}$ . Its Mellin transform  $\hat{f}(\lambda, \cdot)$  belongs to  $(H^1(\mathbb{S}^2))^*$  and is defined by the relation

$$\langle \hat{f}(\lambda, \cdot), v \rangle_{(H^1(\mathbb{S}^2))^*, H^1(\mathbb{S}^2)} = \langle f(r, \omega), r^{-\bar{\lambda}-3} v(\omega) \rangle_{(V_\beta^1(\mathbb{R}^3))^*, V_\beta^1(\mathbb{R}^3)} \text{ for all } v \in H^1(\mathbb{S}^2). \quad (2.11)$$

Clearly, the last duality product is well-defined because  $f$  has a compact support in  $\mathbb{R}^3 \setminus \{O\}$ . This means that  $\hat{f}(\lambda, \cdot)$  is well-defined for all  $\lambda \in \mathbb{C}$ .

**Lemma 2.4.4.** *Let  $f(r, \omega) \in (V_\beta^1(\mathbb{R}^3))^*$  with compact support in  $\mathbb{R}^3 \setminus \{O\}$ . Define  $g(r, \omega) = r^2 f(r, \omega)$ . We have the equality*

$$\|f\|_{(V_\beta^1(\mathbb{R}^3))^*} = \frac{1}{2i\pi} \int_{\ell_{\beta-1/2}} \|\hat{g}(\lambda, \omega)\|_{(H^1(\mathbb{S}^2, |\lambda|))^*} d\lambda.$$

**Proof.** The proof closely follows the one of Lemma 5.3 of [25] (that deals with the 2D case). Given that  $V_{\beta}^1(\mathbb{R}^3)$  is a Hilbert space, there exists a unique  $F \in V_{\beta}^1(\mathbb{R}^3)$  such that

$$\langle f, v \rangle_{(V_{\beta}^1(\mathbb{R}^3))^*, V_{\beta}^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |x|^{2(\beta-1)} F \bar{v} + |x|^{2\beta} \nabla F \cdot \nabla \bar{v} dx \quad \text{for all } v \in V_{\beta}^1(\mathbb{R}^3).$$

Furthermore, we have  $\|f\|_{(V_{\beta}^1(\mathbb{R}^3))^*} = \|F\|_{V_{\beta}^1(\mathbb{R}^3)}$ . Since  $f$  has a compact support  $\mathbb{R}^3 \setminus \{O\}$ , it follows that  $g$  is also compactly supported in  $\mathbb{R}^3 \setminus \{O\}$ . This implies that  $\hat{g}(\lambda, \cdot)$  is well-defined for all  $\lambda \in \mathbb{C}$ . Moreover, one can easily check that for all  $\lambda \in \mathbb{C}$ , we have

$$\langle \hat{g}(\lambda, \omega), h(\omega) \rangle_{H^1(\mathbb{S}^2)^*, H^1(\mathbb{S}^2)} = \langle f, r^{-\bar{\lambda}-1} h(\omega) \rangle_{(V_{\beta}^1(\mathbb{R}^3))^*, V_{\beta}^1(\mathbb{R}^3)} \quad \text{for all } h \in H^1(\mathbb{S}^2).$$

In particular for all  $\lambda_{\gamma} = \beta - 1/2 + i\gamma$  with  $\gamma \in \mathbb{R}$ , there holds

$$\begin{aligned} \langle f, r^{-\bar{\lambda}_{\gamma}-1} h(\omega) \rangle_{(V_{\beta}^1(\mathbb{R}^3))^*, V_{\beta}^1(\mathbb{R}^3)} &= \int_{\mathbb{S}^2} \int_0^{\infty} r^{2\beta-\lambda_{\gamma}-1} F(r, \omega) \bar{h}(\omega) dr d\omega \\ &\quad + \int_{\mathbb{S}^2} \int_0^{\infty} r^{2\beta-\lambda_{\gamma}-1} \nabla_S F(r, \omega) \cdot \nabla_S \bar{h}(\omega) dr d\omega \\ &\quad - (\lambda_{\gamma} + 1) \int_{\mathbb{S}^2} \int_0^{\infty} r^{2\beta-\lambda_{\gamma}} \partial_r F \bar{h}(\omega) dr d\omega. \end{aligned}$$

On the other hand, since  $F \in V_{\beta}^1(\mathbb{R}^3)$ , the function  $\lambda \mapsto \hat{F}(\lambda, \omega)$  is well-defined for all  $\lambda \in \ell_{-\beta-1/2}$ . As a result, we obtain

$$\begin{aligned} \langle f, r^{-\lambda_{\gamma}-1} h(\omega) \rangle_{(V_{\beta}^1(\mathbb{R}^3))^*, V_{\beta}^1(\mathbb{R}^3)} &= \langle \hat{F}(-\beta - 1/2 + i\gamma), h(\omega) \rangle_{H^1(\mathbb{S}^2)} \\ &\quad - (\lambda_{\gamma} + 1)(\lambda_{\gamma} - 2\beta) \langle \hat{F}(-\beta - 1/2 + i\gamma), h(\omega) \rangle_{L^2(\mathbb{S}^2)} \\ &= \langle \hat{F}(-\beta - 1/2 + i\gamma), h(\omega) \rangle_{H^1(\mathbb{S}^2)} \\ &\quad + |\lambda_{\gamma}|^2 \langle \hat{F}(-\beta - 1/2 + i\gamma), h(\omega) \rangle_{L^2(\mathbb{S}^2)}. \end{aligned}$$

Above, we have used the fact that  $(\lambda_{\gamma} + 1)(2\beta - \lambda_{\gamma}) = |\lambda_{\gamma}|^2 = |-\beta - 1/2 + i\gamma|^2$ . Consequently, one obtains that for all  $\gamma \in \mathbb{R}$ , we have

$$\|\hat{g}(\beta - 1/2 + i\gamma, \cdot)\|_{H^1(\mathbb{S}^2)^*} = \|\hat{F}(-\beta - 1/2 + i\gamma, \cdot)\|_{H^1(\mathbb{S}^2, |\ell_{-\beta-1/2+i\gamma})^*}.$$

By integrating the previous estimate with respect to  $\gamma \in \mathbb{R}$  and by using the fact that  $\|f\|_{(V_{\beta}^1(\mathbb{R}^3))^*} = \|F\|_{V_{\beta}^1(\mathbb{R}^3)}$ , we obtain the wanted result.  $\blacksquare$

### 2.4.3 Definition of the problem

Before defining the problem that we want to study, let us start by observing that for all  $u \in V_{\beta}^1(\mathbb{R}^3)$  with  $\beta \in \mathbb{R}$  and all  $\varphi \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\})$  we have

$$- \int_{\mathbb{R}^3} \operatorname{div}(\sigma \nabla u) \bar{\varphi} dx = \int_{\mathbb{R}^3} \sigma \nabla u \cdot \nabla \bar{\varphi} dx.$$

Thanks to the Cauchy-Schwarz inequality, we obtain the estimate

$$\left| \int_{\mathbb{R}^3} \operatorname{div}(\sigma \nabla u) \bar{\varphi} dx \right| \leq C \|u\|_{V_{\beta}^1(\mathbb{R}^3)} \|\varphi\|_{V_{-\beta}^1(\mathbb{R}^3)}$$

with  $C$  independent of  $u$  and of  $\varphi$ . This means that for all  $u \in V_{\beta}^1(\mathbb{R}^3)$ , we have  $\operatorname{div}(\sigma \nabla u) \in (V_{-\beta}^1(\mathbb{R}^3))^*$ . The main goal of this paragraph is to study the well-posedness of the problem

$$\text{Find } u \in V_{\beta}^1(\mathbb{R}^3) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f \in (V_{-\beta}^1(\mathbb{R}^3))^*. \quad (2.12)$$

Clearly, the well-posedness (in the Fredholm sense) of the previous problem is equivalent to study the Fredholmness of the operator  $W_\sigma^\beta : V_\beta^1(\mathbb{R}^3) \rightarrow (V_{-\beta}^1(\mathbb{R}^3))^*$  such that

$$\langle W_\sigma^\beta u, v \rangle = \int_{\mathbb{R}^3} \sigma \nabla u \cdot \nabla \bar{v} \quad \forall (u, v) \in V_\beta^1(\mathbb{R}^3) \times V_{-\beta}^1(\mathbb{R}^3).$$

Even in the classical configuration, i.e. when the function  $\sigma$  has a constant sign, the study of the Fredholmness of the operator  $W_\sigma^\beta$  is not an easy problem. The main difficulty comes from the fact that we are dealing with a non-symmetric problem (except when  $\beta = 0$ ) that is set in an unbounded domain. The classical tool to deal with such difficulty is to use the Mellin transform. The goal of the next paragraph is to investigate how this idea can be extended to the study of the Fredholmness of the operator  $W_\sigma^\beta$  when the sign of  $\sigma$  is not constant.

#### 2.4.4 Mellin symbol of the problem

As mentioned above, to study the properties of the operator  $W_\sigma^\beta$ , we are going to use the Mellin transform. For this, we need to define the so-called Mellin symbol of the problem. For all  $\lambda \in \mathbb{C}$ , introduce the operator  $\mathcal{L}_\sigma(\lambda) : H^1(\mathbb{S}^2) \rightarrow (H^1(\mathbb{S}^2))^*$  such that

$$\langle \mathcal{L}_\sigma(\lambda)\Phi, \Phi' \rangle =: \int_{\mathbb{S}^2} \sigma \nabla_S \Phi \cdot \nabla_S \bar{\Phi}' d\omega - \lambda(\lambda + 1) \int_{\mathbb{S}^2} \sigma \Phi \bar{\Phi}' d\omega \quad \forall \Phi, \Phi' \in H^1(\mathbb{S}^2).$$

The link between the operators  $W_\sigma^\beta$  and  $\mathcal{L}_\sigma(\lambda)$  is clarified in the next

**Lemma 2.4.5.** *Let  $u \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\})$ . Then for all  $\lambda \in \mathbb{C}$ , we have*

$$\widehat{r^2 W_\sigma^\beta u}(\lambda, \cdot) = \mathcal{L}_\sigma(\lambda) \hat{u}(\lambda, \cdot).$$

**Proof.** It is not difficult to check that the transformation  $u \mapsto r^2 u$  is continuous from  $W_\beta^1(\mathbb{R}^3)$  to  $W_{\beta-2}^1(\mathbb{R}^3)$ . This implies that for all  $u \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\})$  we have  $r^2 W_\sigma^\beta(u) \in W_{-\beta-2}^1(\mathbb{R}^3)^*$ . Furthermore, one can see that  $r^2 W_\sigma^\beta(u)$  is compactly supported in  $\mathbb{R}^3 \setminus \{O\}$ . As a result, for all  $v \in H^1(\mathbb{S}^2)$  we have

$$\langle \widehat{r^2 W_\sigma^\beta u}(\lambda, \cdot), v \rangle_{H^1(\mathbb{S}^2)^*, H^1(\mathbb{S}^2)} = \langle W_\sigma^\beta u, r^{-\bar{\lambda}-1} v \rangle_{(V_{-\beta}^1(\mathbb{R}^3))^*, V_\beta^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \sigma(\omega) \nabla u \cdot \nabla r^{-\lambda-1} \bar{v} dx.$$

On the other hand, there holds

$$\begin{aligned} \langle W_\sigma^\beta u, r^{-\lambda-1} v \rangle_{(V_{-\beta}^1(\mathbb{R}^3))^*, V_\beta^1(\mathbb{R}^3)} &= \int_{\mathbb{S}^2} \sigma(\omega) \left( \int_0^{+\infty} r^{-\lambda-1} \nabla_S u(r\omega) \cdot \nabla_S \bar{v}(\omega) dr \right) d\omega \\ &\quad - (\lambda + 1) \int_{\mathbb{S}^2} \sigma(\omega) \left( \int_0^{+\infty} r^{-\lambda-1} (r \partial_r u(r\omega)) \bar{v}(\omega) dr \right) d\omega \\ &= \int_{\mathbb{S}^2} \sigma(\omega) (\nabla_S \hat{u}(\lambda, \omega) \cdot \nabla_S \bar{v}(\omega) - \lambda(\lambda + 1) \hat{u}(\lambda, \omega) \bar{v}(\omega)) d\omega. \end{aligned}$$

Consequently, by using (2.11), we infer that for all  $\lambda \in \mathbb{C}$  and  $v \in H^1(\mathbb{S}^2)$  we have

$$\langle \widehat{r^2 W_\sigma^\beta u}(\lambda, \cdot), v \rangle_{H^1(\mathbb{S}^2)^*, H^1(\mathbb{S}^2)} = \langle \mathcal{L}_\sigma(\lambda) \hat{u}(\lambda, \cdot), v(\cdot) \rangle_{H^1(\mathbb{S}^2)^*, H^1(\mathbb{S}^2)}$$

This means that  $\widehat{r^2 W_\sigma^\beta u}(\lambda, \cdot) = \mathcal{L}_\sigma(\lambda) \hat{u}(\lambda, \cdot)$ . The lemma is then proved.  $\blacksquare$

### 2.4.5 Solvability of the problem

In this paragraph, we come back to the study of the Fredholmness of the operators  $W_\sigma^\beta$  with  $\beta \in \mathbb{R}$ . By combining the results of Lemma 2.4.5 and Lemma 2.4.4, we can say that the features of the operator  $W_\sigma^\beta$  is directly related to the behavior of  $\mathcal{L}_\sigma(\lambda)$  along the energy line  $\ell_{-\beta-1/2}$ . Intuitively, one expects that if  $\mathcal{L}_\sigma(\lambda)$  is invertible along the line  $\ell_{-\beta-1/2}$  the operator  $W_\sigma^\beta$  must also be invertible. To prepare the ground, let us recall some classical definitions and notations, that we borrow from [101], concerning the spectral properties of  $\mathcal{L}_\sigma$ .

**Spectrum of  $\mathcal{L}_\sigma$ :** a complex number  $\lambda$  is said to be a non-regular point of  $\mathcal{L}_\sigma$  if and only if  $\mathcal{L}_\sigma(\lambda)$  is not invertible. Otherwise we say that  $\lambda$  is regular. The set of non-regular points is called the spectrum of  $\mathcal{L}_\sigma$  and is denoted by  $\Lambda(\mathcal{L}_\sigma)$ . Clearly, the set  $\Lambda(\mathcal{L}_\sigma)$  is closed in  $\mathbb{C}$ .

**Eigenvalues and eigenfunctions:** a complex number  $\lambda_0 \in \Lambda(\mathcal{L}_\sigma)$  is said to be an eigenvalue of  $\mathcal{L}_\sigma$  if and only if  $\mathcal{L}_\sigma(\lambda_0)$  is not injective. All the elements of  $\text{Ker}(\mathcal{L}_\sigma(\lambda_0)) \setminus \{0\}$  are called eigenfunctions of  $\mathcal{L}_\sigma$  associated to  $\lambda_0$ . The number  $\iota_g(\lambda_0) := \dim(\text{ker}(\lambda_0)) \in \mathbb{N}^* \cup \{\infty\}$  is known as the geometric multiplicity of  $\lambda_0$ .

**Jordan chain:** Let  $(\lambda, \varphi_0)$  be a pair of eigenvalue and eigenfunction. If there is some ordered family  $\varphi_1, \dots, \varphi_n$  (with  $n \in \mathbb{N}^* \cup \{\infty\}$ ) such that the system of equations

$$\begin{cases} \frac{d\mathcal{L}_\sigma}{d\lambda}(\lambda_0)\varphi + \mathcal{L}_\sigma(\lambda_0)\varphi_1 = 0 \\ \frac{1}{2} \frac{d^2\mathcal{L}_\sigma}{d\lambda^2}(\lambda_0)\varphi_{k-2} + \frac{d\mathcal{L}_\sigma}{d\lambda}(\lambda_0)\varphi_{k-1} + \mathcal{L}_\sigma(\lambda_0)\varphi_k = 0, & k = 2, \dots, n \end{cases}$$

is satisfied, we say that  $\varphi_0, \dots, \varphi_n$  is a Jordan chain of  $\mathcal{L}_\sigma$  associated to  $\lambda_0$  of length  $n + 1$ . The functions  $\varphi_1, \dots, \varphi_n$  are called generalized eigenfunctions associated to  $\lambda$ . The maximal length of the Jordan chain associated to  $(\lambda, \varphi_0)$  is called the rank of  $\varphi_0$  and is denoted by  $\text{rank}(\varphi_0)$ .

**Index, partial/algebraic multiplicity:** Let  $\lambda$  be an eigenvalue of  $\mathcal{L}_\sigma$  and let  $(\varphi_j)_{j=1, \dots, \iota_g(\lambda)}$  be a basis of  $\text{Ker}(\mathcal{L}_\sigma(\lambda))$ . For each  $j = 1, \dots, \iota_g(\lambda)$  we denote by  $(\varphi_{j,k})_{k=0, \dots, \text{rank}(\varphi_j)-1}$  (with  $\varphi_{j,0} = \varphi_j$ ) a Jordan chain associated to  $(\lambda, \varphi_j)$ . The numbers  $\kappa_j = \text{rank}(\varphi_j)$  are called the partial multiplicities of  $\lambda$ . The largest one is called the index of  $\lambda$  and is denoted by  $\iota(\lambda)$ . The sum of these partial multiplicities is called the algebraic multiplicity of  $\lambda$  and is denoted by  $\iota_a(\lambda)$ .

The set  $\{\varphi_{j,k}\}_{j=1, \dots, \iota_g(\lambda), k=0, \dots, \kappa_j(\lambda)-1}$ , is called a canonical system of Jordan chains associated to  $\lambda$ .

**Remark 2.4.1.** Note that in the definition of Jordan chains, the generalized eigenfunctions are not necessarily linearly independent. In particular, some of them may be zero.

Since the symbol  $\mathcal{L}_\sigma$  is associated with a second order PDE, we have the following

**Lemma 2.4.6.** Let  $(\lambda_0, \varphi_0) \in \mathbb{C} \times H^1(\mathbb{S}^2) \setminus \{0\}$  be a eigenpair of  $\mathcal{L}_\sigma$ . If there is no  $\varphi_1 \in H^1(\mathbb{S}^2) \setminus \{0\}$  such that

$$\frac{d\mathcal{L}_\sigma}{d\lambda}(\lambda_0)\varphi + \mathcal{L}_\sigma(\lambda_0)\varphi_1 = 0. \tag{2.13}$$

then, the  $\text{rank}(\varphi_0) = 1$ .

**Proof.** In the particular case when  $\lambda_0 = -1/2$ , one can see that (2.13) holds when  $\varphi_1 = \varphi_0$ . The result is then proved for this particular case. Now, let us suppose that  $\lambda_0 \neq -1/2$  and assume that  $2 \leq \text{rank}(\varphi_0)$ . As a consequence, by using the assumption made on  $\varphi_0$ , we deduce that  $(2\lambda_0 + 1)\sigma\varphi_0 = 0$ . Therefore, we have  $\varphi_0 = 0$  which contradicts the fact that  $\varphi_0$  is an eigenfunction of  $\mathcal{L}_\sigma(\lambda_0)$ . ■

Because of the change of sign of  $\sigma$ , the study of the spectral properties of the family of operators  $(\mathcal{L}_\sigma(\lambda))_{\lambda \in \mathbb{C}}$  does not fit into the general theory presented in [101]. A detailed study of the spectral properties of  $(\mathcal{L}_\sigma(\lambda))_{\lambda \in \mathbb{C}}$  is given in the next chapter. By adapting the results of §3.2.3, one can prove (see Theorem 3.2.1) the

**Proposition 2.4.1.** *Assume that  $\kappa_\sigma \neq -1$ . The spectrum of  $\mathcal{L}_\sigma$  is composed by isolated eigenvalues with finite algebraic multiplicities. Furthermore, there exist two positive constants  $r_0$  and  $\gamma_0$  such that*

$$D_{\gamma_0}^{r_0} := \{z \in \mathbb{C} \text{ such that } r_0 < |z - 1/2| \text{ and } |\Re(z + 1/2)| < \gamma_0 |\Im(z + 1/2)|\} \subset \mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma).$$

Besides, for all  $\beta \in \mathbb{R}$  such that  $\ell_{-\beta-1/2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ , there exists  $0 < C^\beta$  such that

$$\|u\|_{\mathbb{H}^1(\mathbb{S}^2, |\lambda|)} \leq C^\beta \|\mathcal{L}_\sigma(\lambda)u\|_{\mathbb{H}^1(\mathbb{S}^2, |\lambda|)^*} \quad u \in \mathbb{H}^1(\mathbb{S}^2, |\lambda|).$$

**Remark 2.4.2.** *It is worth to mention that the discreteness of  $\Lambda(\mathcal{L}_\sigma)$  combined with the fact that  $D_{\gamma_0}^{r_0} \subset \mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma)$  allow us to say that for all  $\beta_1, \beta_2 \in \mathbb{R}$  such that  $\beta_1 < \beta_2$ , the set  $\Lambda(\mathcal{L}_\sigma) \cap \{\lambda \in \mathbb{C} \mid \Re(\lambda) \in (\beta_1, \beta_2)\}$  is finite.*

Now, we have all the tools to prove the following

**Theorem 2.4.1.** *Assume that  $\kappa_\sigma \neq -1$ . If  $\beta \in \mathbb{R}$  is such that  $\ell_{-\beta-1/2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ , then  $W_\sigma^\beta$  is an isomorphism. Moreover, we have the estimate*

$$\|u\|_{V_\beta^1(\mathbb{R}^3)} \leq C^\beta \|W_\sigma^\beta u\|_{(V_{-\beta}^1(\mathbb{R}^3))^*}, \quad u \in V_\beta^1(\mathbb{R}^3)$$

where  $C^\beta$  is a constant that depends only in  $\beta$ .

**Proof.** Let  $u \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\})$ . According to Lemma 2.4.5 we know that for all  $\lambda \in \mathbb{C}$  we have

$$\widehat{r^2 W_\sigma^\beta u}(\lambda, \cdot) = \mathcal{L}_\sigma(\lambda) \hat{u}(\lambda, \cdot).$$

Since by assumption the operator  $\mathcal{L}_\sigma(\lambda)$  is invertible for all  $\lambda \in \ell_{-\beta-1/2}$ , by using the results of Proposition 2.4.1, we deduce that there is some constant  $0 < C^\beta$  that depends only on  $\beta$  such that

$$\|\hat{u}(\lambda, \cdot)\|_{\mathbb{H}^1(\mathbb{S}^2, |\lambda|)} \leq C^\beta \|\widehat{r^2 W_\sigma^\beta u}(\lambda, \cdot)\|_{\mathbb{H}^1(\mathbb{S}^2, |\lambda|)}, \quad \lambda \in \ell_{-\beta-1/2}.$$

By integrating the previous estimate with respect to  $\lambda$  along the line  $\ell_{-\beta-1/2}$  and by using the result of Proposition 2.4.4, we arrive to the estimate

$$\|u\|_{V_\beta^1(\mathbb{R}^3)} \leq C^\beta \|W_\sigma^\beta u\|_{(V_{-\beta}^1(\mathbb{R}^3))^*}, \quad u \in \mathcal{D}(\mathbb{R}^3 \setminus \{O\}).$$

The density of  $\mathcal{D}(\mathbb{R}^3 \setminus \{O\})$  in  $V_\beta^1(\mathbb{R}^3)$  allows us to deduce the estimate

$$\|u\|_{V_\beta^1(\mathbb{R}^3)} \leq C^\beta \|W_\sigma^\beta u\|_{(V_{-\beta}^1(\mathbb{R}^3))^*}, \quad u \in V_\beta^1(\mathbb{R}^3).$$

This shows that the operator  $W_\sigma^\beta$  is injective and that its range is closed. By observing that for all  $\beta \in \mathbb{R}$  we have  $\mathcal{L}_\sigma(-\beta - 1/2) = \mathcal{L}_\sigma(\beta - 1/2)$  we infer that the operator  $W_\sigma^{-\beta}$  is also injective and its range is closed. By noticing that  $(W_\sigma^\beta)^* = W_\sigma^{-\beta}$  we then deduce that  $W_\sigma^\beta$  is bijective. Finally, the open map theorem allows us to say that  $W_\sigma^\beta$  is an isomorphism.  $\blacksquare$

The proof of the previous theorem combined with the expression of the inverse Mellin transform leads us to the

**Corollary 2.4.1.** *Assume that  $\kappa_\sigma \neq -1$  and that  $\ell_{-\beta-1/2}$  is free of eigenvalues of  $\mathcal{L}_\sigma$  then the solution of (2.12) can be expressed in the following way*

$$u(r\omega) = \frac{1}{2i\pi} \int_{\ell_{-\beta-1/2}} r^\lambda \mathcal{L}_\sigma(\lambda)^{-1} (\widehat{r^2 f}(\lambda, \omega)) d\lambda. \quad (2.14)$$

### 2.4.6 Asymptotic of the solution

One of the most important results of the Kondratiev theory for linear strongly elliptic PDE [100] is the fact that it allows us to derive an asymptotic expansion of the solution at infinity and near the origin. Interestingly, these results can be extended to a more general class of PDE. All we need is to have a precise information about the Mellin symbol generated by the problem under study. Before explaining how to obtain this expansion for the solutions of (2.12), we need to define the so-called singularities of the problem.

**Definition 2.4.1.** Let  $\chi \in \mathcal{D}(\mathbb{R}^3)$  be a cut-off function that depends only on  $r = |x|$  and that is equal to 1 near the origin. We say that  $\mathfrak{s}$  is a singularity of (2.12) if and only if it has the form

$$\mathfrak{s}(r\omega) = \chi(r)r^\lambda \sum_{s=0}^n \frac{1}{s!} \log(r)^s \varphi_{n-s}(\omega)$$

where  $(\varphi_s)_{s=0, \dots, n}$  is a Jordan chain of  $\mathcal{L}_\sigma$  corresponding to  $\lambda$ .

Now, we explain how these singularities are related to Problem (2.12).

**Lemma 2.4.7.** We have the equivalence:  $(\varphi_s)_{s=0, \dots, n}$  is a Jordan chain of  $\mathcal{L}_\sigma$  corresponding to  $\lambda$  if and only if the functions

$$\mathfrak{s}_k(r\omega) = r^\lambda \sum_{s=0}^k \frac{1}{s!} \log(r)^s \varphi_{k-s}(\omega)$$

satisfy  $\operatorname{div}(\sigma \nabla \mathfrak{s}_k) = 0$  for  $k = 1, \dots, n$ .

**Proof.** The proof of this result is given in [101, Theorem 1.1.5]. We limit ourselves to the proof of the result for  $\mathfrak{s}_0$ . Starting from the identity

$$\operatorname{div}(\sigma \nabla(r^\lambda \varphi_0(\omega))) = r^{\lambda-2} (\operatorname{div}_S(\sigma \nabla_S \varphi_0) + \lambda(\lambda+1)\sigma \varphi_0) \text{ in } \mathbb{R}^3 \setminus \{O\},$$

we infer that

$$\operatorname{div}(\sigma \nabla(r^\lambda \varphi_0(\omega))) = 0 \text{ in } \mathbb{R}^3 \setminus \{O\} \iff \operatorname{div}_S(\sigma \nabla_S \varphi_0) + \lambda(\lambda+1)\sigma \varphi_0 = 0 \text{ in } \mathbb{S}^2.$$

The result follows from the identity

$$\langle \operatorname{div}_S(\sigma \nabla_S \varphi_0) + \lambda(\lambda+1)\sigma \varphi_0, \varphi' \rangle = \langle \mathcal{L}_\sigma(\lambda) \varphi_0, \varphi' \rangle, \quad \varphi' \in H^1(\mathbb{S}^2).$$

■

From the previous lemma, we deduce that all the singularities of Problem (2.2) satisfy the equation  $\operatorname{div}(\sigma \nabla_S \cdot) = 0$  near the origin.

### Asymptotic of the solution

Now, we explain how one can find an asymptotic expansion of the solution to (2.12). The starting point is to apply the Residue theorem to the formula (2.14) and to take profit from the fact (thanks to Proposition 2.4.1) that if  $\kappa_\sigma \neq -1$ , then near any  $\lambda_0 \in \Lambda(\mathcal{L}_\sigma)$  the operator  $\mathcal{L}(\lambda)^{-1}$  has the representation

$$\mathcal{L}_\sigma(\lambda)^{-1} = \sum_{j=1}^{\iota(\lambda_0)} \frac{1}{(\lambda - \lambda_0)^j} A_j + \sum_{j=0}^{+\infty} (\lambda - \lambda_0)^j B_j.$$

Here all the  $A_j$  are finite-dimensional operators, the  $B_j$  are continuous operators and this result is proved in [101, Theorem 1.1.2]. By adapting the proof of [102, Theorem 6.1.5], we can prove the



**Proposition 2.4.2.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\beta_1 < \beta_2 \in \mathbb{R}$  such that the lines  $\ell_{-\beta_1-1/2}$  and  $\ell_{-\beta_2-1/2}$  are free of eigenvalues of  $\mathcal{L}_\sigma$ . Denote by  $\lambda_1, \dots, \lambda_N$  (with  $N \in \mathbb{N}^*$ ) the set of eigenvalues of  $\mathcal{L}_\sigma$  that are located in the strip  $-\beta_2 - 1/2 < \Re(\lambda) < -\beta_2 - 1/2$ . For each  $j = 1, \dots, N$  we denote by*

$$\{\varphi_{k,s}^j\}_{k=1, \dots, \iota_g(\lambda_j), s=0, \dots, \kappa_k(\lambda_j)-1}$$

*a canonical system of Jordan chains associated to  $\lambda_j$ . The number  $\kappa_j(\lambda_j)$  stands for the  $j$ -th partial multiplicity of  $\lambda_j$ . Let  $f_0 \in (V_{-\beta_1}^1(\mathbb{R}^3))^* \cap (V_{-\beta_2}^1(\mathbb{R}^3))^*$  and denote by  $u_1 \in V_{\beta_1}^1(\mathbb{R}^3)$  (resp.  $u_2 \in V_{\beta_2}^1(\mathbb{R}^3)$ ) the solution of (2.12) with  $f = f_0$ . The function  $u_1 - u_2$  admits the decomposition*

$$u_1(x) - u_2(x) = \sum_{j=1}^N \sum_{k=1}^{\iota_g(\lambda_j)} \sum_{s=0}^{\kappa_k(\lambda_j)-1} c_{j,k,s} r^{\lambda_j} \frac{1}{s!} \log(r)^s \varphi_{j,k-s}^i(\omega) \quad \text{for almost all } x \in \mathbb{R}^3$$

*in which all the  $c_{j,k,s}$  are complex numbers.*

**Remark 2.4.3.** *Clearly, the coefficients  $c_{j,k,s}$  depend on the choice of the canonical system of Jordan chains associated to each  $(\lambda_j)_{j=1, \dots, N}$ . An explicit formula for the coefficient  $c_{i,j,k}$  can be found in [101]. The idea is based on the use of a well-chosen canonical system of Jordan chains of  $\mathcal{L}_\sigma$  that are associated to  $(-\bar{\lambda}_j - 1)_{j=1, \dots, N}$  (see [108, §5.4.1]) for which the so-called biorthogonality condition (see [101, Theorem 5.1.1]) is satisfied.*

## 2.5 Application: study of the problem in the unit ball

The results of this section are not essential to understand those of the next one. Therefore, this section can be skipped in a first reading. In this paragraph, we are going to study the Fredholmness of the problem

$$\text{Find } u \in H_0^1(B(O, 1)) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f \in (H_0^1(B(O, 1)))^* \quad (2.15)$$

where  $\sigma = \sigma_2$  in  $B(O, 1) \cap \mathcal{K}$  and  $\sigma = \sigma_1$  in  $B(O, 1) \setminus \overline{\mathcal{K}}$ . In order to simplify notations, we shall denote by  $B$  the open unit ball of  $\mathbb{R}^3$ . Moreover, we denote by  $B_1, B_2$  the domains  $B_2 := B \cap \mathcal{K}$  and  $B_1 := B \setminus \overline{B_2}$ . For all  $m \in \mathbb{N}$  and  $\beta \in \mathbb{R}$ , we define the space  $\dot{V}_\beta^m(\mathbb{R}^3)$  as the closure of  $\mathcal{D}(B \setminus \{O\})$  for the norm

$$\|\varphi\|_{\dot{V}_\beta^m(B)} = \left( \sum_{|\alpha| \leq m} \|r^{|\alpha|-m+\beta} \partial_x^\alpha \varphi\|_{L^2(B)}^2 \right)^{1/2}.$$

Note that for all  $m \in \mathbb{N}^*$  and  $\beta \in \mathbb{R}$  we have  $V_\beta^m(B) \subset V_{\beta-1}^{m-1}(B)$ . Besides, one can see that for all  $m \in \mathbb{N}$  and  $\beta_1, \beta_2 \in \mathbb{R}$  such that  $\beta_1 < \beta_2$  we have the embedding  $V_{\beta_1}^m(B) \subset V_{\beta_2}^m(B)$ . In addition to that, by using the [101, Theorem 7.1.1] we can prove that  $H_0^1(B) = V_0^1(B)$ . We also have the following

**Lemma 2.5.1.** *Let  $\Phi \in H^1(\mathbb{S}^2)$  then the function  $x \rightarrow r^\lambda \Phi(\omega)$  (where  $(r, \omega)$  are the classical spherical coordinates) belongs to the space  $V_\beta^1(B)$  if and only if  $-1/2 - \beta/2 < \Re(\lambda)$*

**Proof.** Easily one can show that  $x \rightarrow r^\lambda \Phi(\omega) \in V_\beta^1(B)$  if and only if  $x \rightarrow r^\lambda \Phi(\omega) \in V_{\beta-1}^0(B)$ . This means that  $x \rightarrow r^\lambda \Phi(\omega) \in V_\beta^1(B)$  if and only if  $-1 < 2(\Re(\lambda) + \beta - 1) + 2$ . This ends the proof.  $\blacksquare$

Instead of studying the well-posedness of the (2.15) we are going to study the solvability of the family of problems:

$$\text{Find } u \in \dot{V}_\beta^1(B) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f \in (\dot{V}_{-\beta}^1(B))^*.$$



Note that when  $\beta = 0$ , the previous problem is nothing else than the problem (2.15). Without any difficulty, one can check that the study of the Fredholmness of the previous problem is equivalent to the study of the Fredholmness of the operator  $L_\sigma^\beta : \mathring{V}_\beta^1(B) \rightarrow (\mathring{V}_{-\beta}^1(B))^*$  such that

$$\langle L_\sigma^\beta u, v \rangle := \int_B \sigma \nabla u \cdot \nabla v \text{ for all } (u, v) \in \mathring{V}_\beta^1(B) \times \mathring{V}_{-\beta}^1(B).$$

**Theorem 2.5.1.** *Assume that  $\kappa_\sigma \neq -1$ . If  $\beta \in \mathbb{R}$  is such that  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-\beta-1/2} = \emptyset$  then the operator  $L_\sigma^\beta$  is a Fredholm operator.*

**Proof.** Let  $\chi \in \mathcal{D}(\mathbb{R}^3, [0; 1])$  that depends only on  $r = |x|$  such that  $\chi(x) = 1$  for all  $|x| \in [0; 1/2]$  and  $\chi(x) = 0$  for all  $|x| \in [3/4, +\infty)$ . To simplify notations, we introduce  $D := \{x \in \mathbb{R}^3 \mid |x| \in [1/2, 1)\}$ . By observing that the function  $1 - \chi$  is supported in  $D$ , we obtain the following estimate

$$\|u\|_{\mathring{V}_\beta^1(B)} \leq C(\|\chi u\|_{\mathring{V}_\beta^1(B)} + \|(1 - \chi)u\|_{H_0^1(D)}), \quad u \in V_\beta^1(B(0, 1))$$

in which  $C$  is independent of  $u$ . Now by extending the function  $\chi u$  by 0 in  $\mathbb{R}^3 \setminus B$  we can say that  $\chi u$  is then an element of the space  $V_\beta^1(\mathbb{R}^3)$ . The assumptions made on  $\kappa_\sigma$  and  $\beta$  allow us to use the results of Proposition 2.4.1. In particular, we have the estimate

$$\|\chi u\|_{\mathring{V}_\beta^1(B)} \leq C\|\chi u\|_{V_\beta^1(\mathbb{R}^3)} \leq C\|\operatorname{div}(\sigma \nabla \chi u)\|_{(V_{-\beta}^1(\mathbb{R}^3))^*}$$

with  $C$  independent of  $u$ . Using the fact that the function  $\chi$  depends only on  $r = |x|$ , we can say that for all  $u \in V_\beta^1(B)$  the function  $\operatorname{div}(\sigma u \nabla \chi)$  belongs to  $L^2(B)$  which is supported in  $D$ . Furthermore, we have the estimate

$$\|\operatorname{div}(\sigma u \nabla \chi)\|_{V_{-\beta}^1(\mathbb{R}^3)} \leq C\|u\|_{L^2(D)}$$

in which  $C$  is independent of  $u$ . By Combining this result with the identity: for all  $v \in V_{-\beta}^1(\mathbb{R}^3)$  we have

$$\langle \operatorname{div}(\sigma \nabla \chi u), v \rangle = -(\sigma \nabla u, \nabla \chi v)_{L^2(\mathbb{R}^3)} - (u, \operatorname{div}(\sigma v \nabla \chi))_{L^2(\mathbb{R}^3)} + (\operatorname{div}(\sigma u \nabla \chi), v)_{L^2(\mathbb{R}^3)} \quad (2.16)$$

we obtain the estimate

$$\|\operatorname{div}(\sigma \nabla \chi u)\|_{(V_{-\beta}^1(\mathbb{R}^3))^*} \leq C(\|L_\sigma^\beta(u)\|_{(\mathring{V}_{-\beta}^1(B))^*} + \|u\|_{L^2(D)}).$$

By adapting the results of §2.3, we also have the estimate

$$\|(1 - \chi)u\|_{H_0^1(D)} \leq C(\|\operatorname{div}(\sigma \nabla (1 - \chi)u)\|_{(H_0^1(D))^*} + \|u\|_{L^2(D)}).$$

Using the same idea as in (2.16), we get the estimate

$$\|(1 - \chi)u\|_{H_0^1(D)} \leq C(\|L_\sigma^\beta(u)\|_{(\mathring{V}_{-\beta}^1(B))^*} + \|u\|_{L^2(B_1)}).$$

As a consequence, we obtain the following estimate

$$\|u\|_{\mathring{V}_\beta^1(B)} \leq C(\|L_\sigma^\beta(u)\|_{(\mathring{V}_\beta^1(B(O, 1)))^*} + \|u\|_{L^2(B_1)}), \quad u \in \mathring{V}_\beta^1(B(O, 1)). \quad (2.17)$$

By observing that the map  $u \rightarrow u|_D$  from  $\mathring{V}_\beta^1(B)$  to  $L^2(D)$  is compact and by using the results of Proposition 2.8.1, we deduce that  $L_\sigma^\beta$  has a closed range and that its kernel is finite dimensional. Using the fact that  $\mathcal{L}_\sigma(-\beta - 1/2) = \mathcal{L}_\sigma(\beta - 1/2)$ , we infer that  $L_\sigma^{-\beta}$  has also a closed range and a finite dimensional kernel. By noticing that  $L_\sigma^{-\beta} = (L_\sigma^\beta)^*$ , we conclude that  $\operatorname{coker}(L_\sigma^\beta)$  is finite dimensional, this implies that  $L_\sigma^\beta$  is of Fredholm type.  $\blacksquare$

**Proposition 2.5.1.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $0 < \beta$  such that  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-\beta-1/2} = \emptyset$  then  $L_\sigma^{-\beta}$  is injective and  $L_\sigma^\beta$  is surjective.*

**Proof.** According to the previous proposition the operators  $L_\sigma^{\pm\beta}$  are of Fredholm type. Since  $L_\sigma^\beta$  is the adjoint of  $L_\sigma^{-\beta}$  it suffices to show that  $L_\sigma^{-\beta}$  is injective. For this, let us suppose that there exists some  $u \in \dot{V}_{-\beta}^1(B(O, 1))$  such that  $\operatorname{div}(\sigma \nabla u) = 0$ . The goal is then to prove that  $u = 0$ . For this we are going to use the Kelvin transform and the fact that in our case the operators  $W_\sigma^{\pm\beta}$  are isomorphism. We define the function  $\tilde{u}$  such that

$$\tilde{u}(r, \omega) = \begin{cases} u(r, \omega) & \text{if } r < 1 \\ -u(1/r, \omega)/r & \text{if } 1 < r. \end{cases}$$

Denote by  $B^c = \mathbb{R}^3 \setminus \overline{B(O, 1)}$ . According to Lemma 2.8.3, we can say that  $\tilde{u}|_{B^c} \in V_\beta^1(B^c)$ . Since  $\tilde{u}$  is continuous across the unit sphere  $\mathbb{S}^2$  and since  $V_{-\beta}^1(B(O, 1)) \subset V_\beta^1(B(O, 1))$  we conclude that  $\tilde{u} \in V_\beta^1(\mathbb{R}^3)$ . To proceed, we denote by  $B_1^{-1}$  (resp.  $B_2^{-1}$ ) the image of  $B_1$  (resp.  $B_2$ ) by the map  $(r, \omega) \rightarrow (1/r, \omega)$ . Note that we have  $B_1 \cup B_1^{-1} = \mathcal{K}$  and  $B_2 \cup B_2^{-1} = \mathbb{R}^3 \setminus \mathcal{K}$ . The next step is to extend the function  $\sigma$  to  $\mathbb{R}^3$  as follows

$$\sigma = \begin{cases} \sigma_1 & \text{in } \mathbb{R}^3 \setminus \mathcal{K} \\ \sigma_2 & \mathcal{K}. \end{cases}$$

To end the proof, we are going to show that the function  $\tilde{u}$  satisfies the equation  $\operatorname{div}(\sigma \nabla \tilde{u}) = 0$  in  $\mathbb{R}^3 \setminus \{O\}$ . Since  $\tilde{u} \in V_\beta^1(\mathbb{R}^3)$  this will implies that  $W_\sigma^\beta(\tilde{u}) = 0$  and then by applying Theorem 2.4.1 we will be able to deduce that  $\tilde{u} = 0$  and thus  $u = 0$ . Starting from the fact that the function  $u$  is harmonic in  $B_1$  and  $B_2$  and by using Lemma 2.8.1 we deduce that  $\tilde{u}$  is harmonic in  $B_1^{-1} \cup B_2^{-1}$ . It remains to prove  $\sigma \partial_n \tilde{u}$  is continuous across the unit sphere and across the interface between  $B_1^{-1}$  and  $B_2^{-1}$ . For the case of the unit sphere, the continuity of  $\sigma \partial_n \tilde{u}$  follows from the fact that  $\sigma$  is continuous. It remains to explain why  $\sigma \partial_n \tilde{u}$  is continuous across  $\overline{B_1} \cap \overline{B_2}$ . This comes from the fact that  $\sigma \partial_n \tilde{u}$  is continuous across  $\overline{B_1^{-1}} \cap \overline{B_2^{-1}}$  and from the fact that the Kelvin transform acts only in the radial direction. The Lemma is then proved.  $\blacksquare$

In the particular case  $\beta = 0$ , the results of the previous proposition can be refined.

**Theorem 2.5.2.** *Assume that  $\kappa_\sigma \neq -1$ . We have the assertions*

- if  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} = \emptyset$  then  $L_\sigma^0$  is an isomorphism.
- If  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} \neq \emptyset$  then the operator  $L_\sigma^0$  is not of Fredholm type.

**Proof.** The proof of first statement is easy. Since  $L_\sigma^\beta = L_\sigma^{-\beta}$ , the previous proposition allows us to say that  $L_\sigma^0$  is bijective. Since  $L_\sigma^0$  is continuous the result is then a direct consequence of the open map theorem. The proof of the second statement follows the lines of the proof of Proposition 2.6.1.  $\blacksquare$

The previous theorem gives us a simple way to characterize the set of contrast  $\kappa_\sigma$  for which the near problem (2.15) is ill-posed in the Fredholm sense. Since the existence of eigenvalue on the energy line  $\ell_{-1/2}$  is equivalent to say that the problem (2.12) has singularities that coincide near the origin with

$$\mathfrak{s}_{\eta, k}(r\omega) = r^{-1/2+i\eta} \sum_{s=0}^k \log(r)^s / s! \varphi_{k-s}(\omega)$$

where  $\eta \in \mathbb{R}$  and  $(\varphi_0, \dots, \varphi_{s-1})$  is a Jordan chain associated to  $-1/2 + i\eta$ . The previous theorem tell us us that that the existence of such singularities is the main cause of the absence of Fredholmness of the problem (2.15). In accordance with the vocabulary used in the 2D configuration

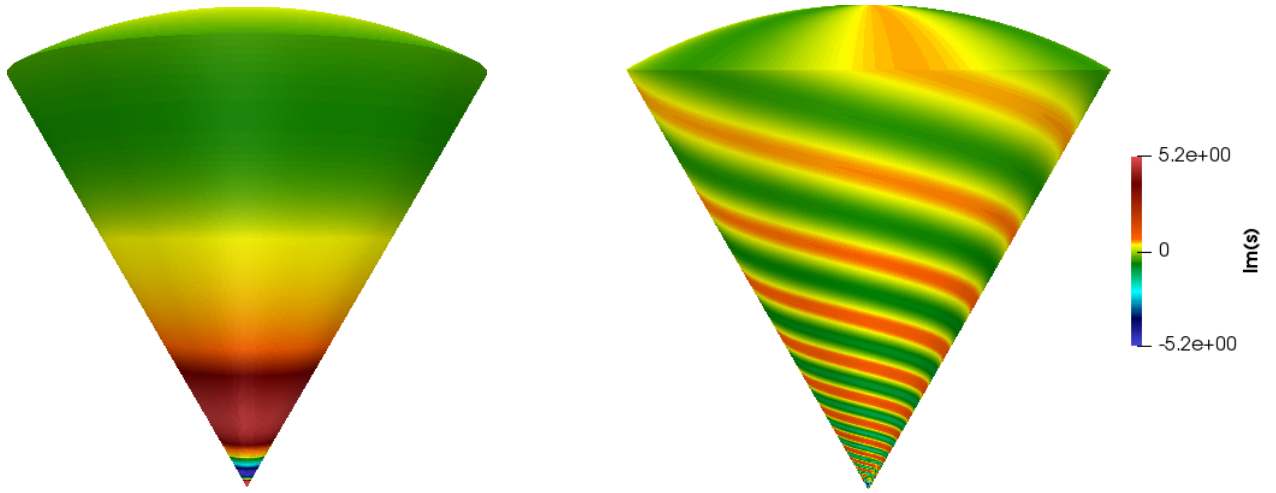


Figure 2.3: Examples of propagating singularities (imaginary parts).

[25, 50], these singularities will be called **propagating (or black hole) singularities**. A more visual description of the behavior of these singularities is given in Figure 2.3.

From a physical point of view these singularities can be interpreted as waves that propagate toward or outward the conical tip, see the next section for more details. From a mathematical point of view the existence of these singularities implies that the space  $H_0^1(B)$  is no long the appropriate framework in which one has to set the problem. Instead, one has introduce a wider framework that contains these singular functions (or some of them) in order to restore Fredholmness. Since for all  $0 < \beta$  the functions  $\mathfrak{s}_{\eta,s}$  belongs to the space  $\mathring{V}_\beta^1(B)$ , a natural choice is to work in the space  $\mathring{V}_\beta^1(B)$ . Unfortunately the next proposition shows that this is not possible.

**Proposition 2.5.2.** *Assume that  $\kappa \neq -1$  and  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} \neq \emptyset$  then for all  $0 < \beta$  the operator  $L_\sigma^\beta$  (resp.  $L_\sigma^{-\beta}$ ) is not injective (resp. not surjective).*

**Proof.** Since the  $L_\sigma^{-\beta}$  is the adjoint of  $L_\sigma^\beta$  it suffices to prove that  $L_\sigma^\beta$  is not injective. For this we shall distinguish two situations: the first one when we can find  $\eta \in \mathbb{R}^*$  such that  $\lambda_\eta := -1/2 + i\eta \in \Lambda(\mathcal{L}_\sigma)$ , the second one is when  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} = \{-1/2\}$ .

**The first case:** We suppose that there exists  $\eta \in \mathbb{R}^*$  such that  $\lambda_\eta = -1/2 + i\eta \in \Lambda(\mathcal{L}_\sigma)$ . Denote by  $\varphi_\eta$  a real valued an eigenfunction of  $\mathcal{L}_\sigma$  associated to  $\lambda_\eta$  (this is possible because  $\mathcal{L}_\sigma(\lambda)$  is symmetric when  $\Re(\lambda) = -1/2$ ). Since  $(\overline{\lambda_\eta}, \varphi_\eta)$  is a pair of eigenvalue and eigenfunction of  $\mathcal{L}_\sigma$ . We then introduce the function  $\Phi_\eta(r\omega) = r^{\lambda_\eta} \varphi_\eta(\omega) - r^{\overline{\lambda_\eta}} \varphi_\eta(\omega)$ . Clearly, the function  $\Phi_\eta$  belongs to  $\mathring{V}_\beta^1(B) \setminus \{0\}$  and satisfies  $\text{div}(\sigma \nabla \Phi_\eta) = 0$ . This ends the proof for this case.

**The second case:** Here, we suppose that  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} = \{-1/2\}$ . Denote by  $\varphi_0$  an eigenfunction of  $\mathcal{L}_\sigma$  associated to  $-1/2$ . Without any difficulty we can check that  $(\varphi_0, \varphi_0)$  is a Jordan chain associated  $-1/2$ . This means that the functions  $\Phi_0(r\omega) := r^{-1/2} \varphi_0(\omega)$  and  $\Phi_1(r\omega) := r^{-1/2}(1 + \log(r))\varphi_0(\omega)$  satisfy the equation  $\text{div}(\sigma \nabla \Phi_i) = 0$  for  $i = 0, 1$ . This implies that the function  $x \mapsto \log(r)\phi(\omega) \in \mathring{V}_\beta^1(B) \setminus \{0\}$  belongs to the kernel of  $L^\beta$ . The result is then proved.  $\blacksquare$

What we learn from the proof of the previous proposition is the fact that in order to construct a new functional framework in which the problem is again well-posed one need to incorporate some of the propagating singularities and not all of them. Using the waveguides terminology, one has to work with just the outgoing ones in order to construct a functional framework that leads to physical solution of the problem. This will be clarified in the next section.

## 2.6 Study of the initial problem

In this section, we return our attention to the analysis of the initial problem (2.2). The goal is to explain how to combine the analysis of the far problem and the near one in order to get a clear information about the well-posedness of (2.2). This section is divided into three parts. In the first one, we explain how to use the existence of propagating singularities of the problem (see Definition 2.6.1) in order to characterize the critical interval  $I_\Sigma$ . After that, by making use of the Mandelstam principle [112, 103] we explain how to construct several (an infinite number) of functional frameworks in which the Fredholmness of the problem is recovered for contrasts inside the critical interval. The last part is devoted to explain how to use the limiting absorption principle in order to select, among these mathematical frameworks the one that leads to the physical solution of the problem.

### 2.6.1 Characterization of the critical interval

Let us start by defining the propagating singularities of the problem (2.2). Once for all, in all this section, we denote by  $\chi$  a cutoff function that is equal to 1 near the origin and that is supported in  $B(O, 1)$ . It is important to mention that all the results obtained below are independent of the choice of the function  $\chi$ .

**Definition 2.6.1.** *Assume that the function  $\sigma$  is such that  $\kappa_\sigma \neq -1$  and  $\ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma) \neq \emptyset$ . Let  $\lambda_\eta = -1/2 + i\eta \in \Lambda(\mathcal{L}_\sigma)$  with  $\eta \in \mathbb{R}$ . We say that a function  $\mathfrak{s}$  is a propagating singularity of the problem (2.2) (or equivalently of  $A_\sigma$ ) if and only if it has the form*

$$\mathfrak{s}(r\omega) = \chi(r)r^{-1/2+i\eta} \sum_{s=0}^k \frac{\log(r)^s}{s!} \varphi_{k-s}(\omega)$$

where  $k \in \mathbb{N}$  is such that  $(\varphi_0, \dots, \varphi_k)$  is a Jordan chain of  $\mathcal{L}_\sigma$  associated to  $\lambda_\eta$ .

It is worthy to note that any propagating singularity of the problem (2.2) belongs to  $L^2(\Omega) \setminus H^1(\Omega)$  and satisfies the equation  $\operatorname{div}(\sigma \nabla \cdot) = 0$  near the origin (see Lemma 2.4.7). Furthermore, it will be interesting to mention that any propagating singularity  $\mathfrak{s}$  is such that  $\operatorname{div}(\sigma \nabla \mathfrak{s})$  is compactly supported in  $\Omega$  and belongs to the space  $L^2(\Omega)$  (this a consequence, in particular, of the fact that the cutoff function  $\chi$  depends only in  $r = |x|$  while  $\sigma|_{B(O,1)}$  does not depend on it).

**Proposition 2.6.1.** *Assume that the function  $\sigma$  is such that  $A_\sigma$  has a propagating singularity. Then  $A_\sigma$  is not of Fredholm type.*

**Proof.** We proceed by contradiction. Let us suppose that the operator  $A_\sigma$  is of Fredholm type. Given that the embedding  $H_0^1(\Omega) \subset L^2(\Omega)$  is compact, one can then use the Theorem 2.8.1 to deduce that there exists  $0 < C$  such that

$$\|u\|_{H_0^1(\Omega)} \leq C(\|A_\sigma u\|_{(H_0^1(\Omega))^*} + \|u\|_{L^2(\Omega)}) \text{ for all } u \in H_0^1(\Omega).$$

Our goal is then to contradict this estimate. To do so, we shall explain how to construct a sequence  $(u_n)_{n \in \mathbb{N}}$  of elements of  $H_0^1(\Omega)$  such that

$$\lim_{n \rightarrow +\infty} \|u_n\|_{H_0^1(\Omega)} = +\infty \text{ and } \|A_\sigma u_n\|_{(H_0^1(\Omega))^*} + \|u_n\|_{L^2(\Omega)} \text{ remains bounded as } n \rightarrow \infty.$$

Since by assumption we know that  $A_\sigma$  has a least one propagating singularity, we can say that there exists  $\eta \in \mathbb{R}$  and  $\Phi \in H^1(\mathbb{S}^2) \setminus \{0\}$  such that the function  $\mathfrak{s}(r\omega) = r^{-1/2+i\eta} \Phi(\omega)$  satisfies the equation  $\operatorname{div}(\sigma \nabla \mathfrak{s}) = 0$  in  $\mathbb{R}^3$  (see Lemma 2.4.7). For all  $n \in \mathbb{N}$ , we denote by  $s_n, u_n$  the functions  $s_n(r\omega) := r^{1/n} \mathfrak{s}(r\omega)$  and  $u_n(r\omega) := \chi(r)r^{1/n} \mathfrak{s}(r\omega)$  (recall that  $\chi$  is supported in  $B(0, 1)$  and equal to 1 near the origin). Without any difficulty, one can see that we have  $\lim_{n \rightarrow +\infty} \|u_n\|_{H_0^1(\Omega)} = +\infty$ .

To end the proof it remains to study the behavior of  $\|A_\sigma u_n\|_{(\mathbb{H}_0^1(\Omega))^*} + \|u_n\|_{L^2(\Omega)}$  as  $n \rightarrow +\infty$ . By observing that  $\mathfrak{s}$  belongs to  $L^2$  near the origin and by using the dominated convergence theorem, we infer that  $\|u_n\|_{L^2(\Omega)}$  converges as  $n$  tends to  $+\infty$  to  $\|\chi \mathfrak{s}\|_{L^2(\Omega)}$ . As a consequence, we just need to study the behavior of  $\|A_\sigma u_n\|_{(\mathbb{H}_0^1(\Omega))^*}$  as  $n \rightarrow +\infty$ . Given that  $\mathcal{D}(\Omega \setminus \{O\})$  is dense in  $\mathbb{H}_0^1(\Omega)$  (see [101, Theorem 7.1.1]), we deduce that

$$\|A_\sigma(u_n)\|_{\mathbb{H}_0^1(\Omega)^*} = \sup_{v \in \mathcal{D}(\Omega \setminus \{O\}) \setminus \{0\}} \frac{(\sigma \nabla u_n, \nabla v)_{L^2(\Omega)}}{\|v\|_{\mathbb{H}_0^1(\Omega)}}.$$

Interestingly, it can be shown (following the results of [101, Theorem 7.1.1]) that we have the estimate

$$\|r^{-1}v\|_{L^2(\Omega)} \leq C\|v\|_{\mathbb{H}_0^1(\Omega)} \quad (2.18)$$

with  $C$  that does not depend in  $v$ . In the other hand, one can check that for all  $v \in \mathcal{D}(\Omega \setminus \{O\})$  we have

$$(\sigma \nabla u_n, \nabla v)_{L^2(\Omega)} = (\sigma s_n \nabla \chi, \nabla v)_{L^2(\Omega)} + (s_n, \operatorname{div}(\sigma v \nabla \chi))_{L^2(\Omega)} - (\operatorname{div}(\sigma \nabla s_n), \chi v)_{L^2(\operatorname{supp}(\chi))}.$$

Note that above, we have used the fact that  $\operatorname{div}(\sigma \nabla s_n) \in L^2(\Omega)$  and also the fact that for all function  $v \in \mathbb{H}_0^1(\Omega)$  we have  $\operatorname{div}(\sigma v \nabla \chi) \in L^2(\Omega)$  which is true because  $\chi$  depends only on  $r$  and then its normal derivative vanishes at  $\Sigma$ . The next step is to observe that we have the following estimate

$$|(\sigma s_n \nabla \chi, \nabla v)_{L^2(\Omega)} + (s_n, \operatorname{div}(\sigma v \nabla \chi))_{L^2(\Omega)}| \leq C\|s_n\|_{L^2(\Omega)}\|v\|_{\mathbb{H}_0^1(\Omega)}$$

in which  $C$  is independent of  $v \in \mathbb{H}_0^1(\Omega)$  and of  $n \in \mathbb{N}$ . As a result, to finish the proof it remains to study the term  $(\operatorname{div}(\sigma \nabla s_n), \chi v)_{L^2(\operatorname{supp}(\chi))}$ . By observing that

$$(\operatorname{div}(\sigma \nabla s_n), \chi v)_{L^2(\operatorname{supp}(\chi))} = (r \operatorname{div}(\sigma \nabla s_n), r^{-1} \chi v)_{L^2(\operatorname{supp}(\chi))}$$

and by means of (2.18) For this, we are going to show that  $\|r \operatorname{div}(\sigma \nabla s_n)\|_{L^2(\operatorname{supp}(\chi))}$  tends to 0 as  $n$  tends to  $\infty$ . A direct calculus (using the relation  $\operatorname{div}(\sigma \nabla \mathfrak{s}) = 0$  in  $B(O, 1)$ ) yields

$$r \operatorname{div}(\sigma \nabla s_n) = \sigma r^{1/n-3/2+i\eta} (2(-1/2 + i\eta) + 1 - 1/n) \Phi(\theta, \varphi) / n \text{ in } B(O, 1).$$

Introduce some  $0 < r_0 < 1$  such that  $\operatorname{supp}(\chi) \subset \{x \in \mathbb{R}^3 \mid |x| < r_0\}$ . By remarking that  $(2(-1/2 + i\eta) + 1 - 1/n) \Phi(\theta, \varphi)$  is uniformly bounded in  $L^2(\mathbb{S}^2)$  with respect to  $n$  and by means of the identity

$$\int_0^{r_0} |r^{1/n-3/2+i\eta}/n|^2 r^2 dr = \frac{1}{n^2} \int_0^{r_0} r^{2/n-1} dr = \frac{1}{2n} (r_0)^{2/n} \leq C/n$$

with  $C$  independent of  $n$ , we obtain the wanted result. ■

This leads us to the

**Theorem 2.6.1.** *Assume that  $\kappa_\sigma \neq -1$ . Then the following statements are equivalent:*

1. *The operator  $A_\sigma$  is a Fredholm operator of index zero.*
2. *The function  $\sigma$  is such that  $A_\sigma$  does not have any propagating singularity.*

**Proof.** Regarding what we have proved in the previous proposition, it is enough to show the direct implication ('2' implies '1'). Since by assumption  $A_\sigma$  does not have any propagating singularity and  $\kappa_\sigma \neq -1$ , we infer, thanks to Theorem 2.4.1, that the operator  $W_\sigma^0 : V_0^1(\mathbb{R}^3) \rightarrow (V_0^1(\mathbb{R}^3))^*$  is an isomorphism. Given that  $V_0^1(\mathbb{R}^3) = H^1(\mathbb{R}^3)$  (see §2.4.1), we then obtain the estimate

$$\|u\|_{\mathbb{H}_0^1(\mathbb{R}^3)} \leq C_1 \|\operatorname{div}(\sigma \nabla u)\|_{H^1(\mathbb{R}^3)^*} \text{ for all } u \in V_0^1(\mathbb{R}^3) = H^1(\mathbb{R}^3)$$

with  $C$  independent of  $u$ . Now, take  $\chi_0 \in \mathcal{D}(\mathbb{R}_+, [0; 1])$  a cutoff function that is supported in  $[0; 1]$  and equal to 1 near 0. Thanks to the previous estimate we get

$$\|\chi_0(r)u\|_{\mathbf{H}_0^1(\Omega)} \leq C \|\operatorname{div}(\sigma \nabla(\chi_0(r)u))\|_{\mathbf{H}_0^1(\mathbb{R}^3)^*} \text{ for all } u \in \mathbf{H}_0^1(\Omega).$$

Using the fact that  $\chi_0$  depends only on  $r$  and working as in the proof of Theorem 2.5.1, we arrive to the estimate

$$\|\chi_0(r)u\|_{\mathbf{H}_0^1(\Omega)} \leq C(\|\operatorname{div}(\sigma \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*} + \|u\|_{L^2(\Omega)}) \text{ for all } u \in \mathbf{H}_0^1(\Omega)$$

with  $C$  that does not depend on  $u$ . On the other hand, by working as in the proof of Proposition 2.3.1, we deduce that there exists  $0 < C$  such that for all  $u \in \mathbf{H}_0^1(\Omega)$ , we have

$$\|(1 - \chi(r))u\|_{\mathbf{H}_0^1(\Omega)}^2 \leq C(\|\operatorname{div}(\sigma \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*} + \|u\|_{L^2(\Omega)})\|u\|_{\mathbf{H}_0^1(\Omega)} \text{ for all } u \in \mathbf{H}_0^1(\Omega).$$

By combining the last two estimates, we conclude that there is  $0 < C$  we have

$$\|u\|_{\mathbf{H}_0^1(\Omega)} \leq C(\|\operatorname{div}(\sigma \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*} + \|u\|_{L^2(\Omega)}) \text{ for all } u \in \mathbf{H}_0^1(\Omega).$$

This is enough to deduce, thanks to Proposition 2.8.2, that  $A_\sigma$  is a Fredholm operator of index zero.  $\blacksquare$

**Remark 2.6.1.** *In the next section, we will show that when propagating singularities exist (and  $\kappa_\sigma \neq -1$ ) the kernel of  $A_\sigma$  is finite dimensional. Since  $A_\sigma$  is symmetric, the dimension of  $\operatorname{coker}(A_\sigma)$  is then finite. As a consequence, we then deduce that when  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$ , the absence of Fredholmness of  $A_\sigma$  is caused by the fact that its range is not closed in  $(\mathbf{H}_0^1(\Omega))^*$ .*

As a consequence of the previous theorem, we conclude that the set  $I_\Sigma \setminus \{-1\}$  coincides with the set of contrasts  $\kappa_\sigma$  for which  $A_\sigma$  has at least one propagating singularity. In other words,  $I_\Sigma \setminus \{-1\}$  is equal to the set of contrast  $\kappa_\sigma$  for which the the symbol  $\mathcal{L}_\sigma$  has at least one eigenvalue in the energy line  $\Re e(\lambda) = -1/2$ . With this in mind and by using the results of §3.4.2 we arrive to the following

**Theorem 2.6.2.** *In the case of circular conical tips  $g(\theta) = \alpha \in (0, \pi/2]$ , the critical interval  $I_\Sigma$  (that will be also denoted by  $I_\alpha$ ) is given by*

$$I_\Sigma = I_\alpha = \left[-1, -\frac{{}_2F_1(1/2, 1/2, 1, \cos^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \sin^2(\alpha/2))}{{}_2F_1(1/2, 1/2, 1, \sin^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \cos^2(\alpha/2))}\right]$$

in which  ${}_2F_1$  stands for the Gauss hypergeometric function (see Appendix §3.6.2).

When  $\alpha = \pi/2$  (the locally symmetric case), one can easily see that  $I_\Sigma = \{-1\}$  (this is coherent with the results of [50, Theorem 1.2.1]). For the case  $\alpha \in (\pi/2; \pi)$  the expression of  $I_\Sigma$  is given by  $I_\Sigma = 1/I_{\pi-\alpha}$ . Compared to the 2D case, the result of the previous theorem is a little bit surprising because  $I_\Sigma$  is from one side of the value  $\kappa_\sigma = -1$ . For the case of a general smooth conical tip,  $I_\Sigma$  cannot be calculated by hand and numerical tools must be developed to do so.

## 2.6.2 On the use of the Mandelstam principle to recover Fredholmness of the problem

Along this section, we suppose that the function  $\sigma$  is such that  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$ . This means that the operator  $A_\sigma$  is not of Fredholm type. Our goal is to explain how to use the Mandelstam principle in order to construct a functional framework in which the scalar problem is again well-posed in the Fredholm sense. Before getting into details let us start with some preliminary results.



### Preliminaries

We start by defining the weighted Sobolev spaces that are associated to the domain  $\Omega \setminus \{O\}$ . For all  $\beta \in \mathbb{R}$ , we define the spaces  $V_\beta^0(\Omega)$  and  $\mathring{V}_\beta^1(\Omega)$  such that

$$V_\beta^0(\Omega) = \{u \mid r^\beta u \in L^2(\Omega)\} \text{ and } \mathring{V}_\beta^1(\Omega) = \{u \mid r^{\beta-1}u \in L^2(\Omega) \text{ and } r^\beta \nabla u \in L^2(\Omega)\}.$$

Classically (see [102]), the space  $\mathcal{D}(\Omega \setminus \{O\})$  is dense in  $V_\beta^0(\Omega)$  and  $\mathring{V}_\beta^1(\Omega)$  for all  $\beta \in \mathbb{R}$ . From their definitions, one see that  $\mathring{V}_\beta^1(\Omega) \subset V_{\beta-1}^0(\Omega)$  for all  $\beta \in \mathbb{R}$ . It will be also interesting to mention that for all  $u \in \mathring{V}_\beta^1(\Omega)$  we have  $u|_\omega \in H^1(\omega)$  for all open subset  $\omega \subset \Omega \setminus \{O\}$ . When  $\beta = 0$ , we have  $V_0^0(\Omega) = L^2(\Omega)$  and  $\mathring{V}_0^1(\Omega) = H_0^1(\Omega)$  (see [101, Theorem 7.1.1]). Now, we introduce for all  $\beta \in \mathbb{R}$ , the operator such that  $A_\sigma^\beta : \mathring{V}_\beta^1(\Omega) \rightarrow (\mathring{V}_{-\beta}^1(\Omega))^*$  such that

$$\langle A_\sigma^\beta u, v \rangle =: \int_\Omega \sigma \nabla u \cdot \nabla \bar{v} \quad \text{for all } u \in \mathring{V}_\beta^1(\Omega) \text{ and } v \in \mathring{V}_{-\beta}^1(\Omega).$$

By means of localization techniques (using radial cutoff functions) and using the results of Theorem 2.3.1 and Theorem 2.4.1, one obtains the

**Proposition 2.6.2.** *Assume that  $\kappa_\sigma \neq -1$ , then for all  $\beta \in \mathbb{R}$  such that  $\ell_{-\beta+1/2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$  the operator  $A_\sigma^{\pm\beta}$  is of Fredholm type.*

To proceed, we denote by  $N_t$  the number of eigenvalues of  $\mathcal{L}_\sigma$  that are located on the energy line  $\ell_{-1/2}$  and denote by  $\lambda_1, \dots, \lambda_{N_t}$  the elements of the set  $\Lambda_{-1/2} := \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$ . For each  $j = 1, \dots, N_t$  we denote by  $\{\varphi_{k,s}^j\}_{k=1, \dots, \iota_g(\lambda_j), s=0, \dots, \kappa_k(\lambda_j)-1}$  a canonical system of Jordan chains associated to  $\lambda_j$ . Each  $\lambda_j$  ( $j = 1, \dots, N_t$ ) generates  $\iota_a(\lambda_j)$  propagating singularities that are defined as follows: for all  $j = 1, \dots, N_t, k = 1, \dots, \iota_g(\lambda_j), s = 0, \dots, \kappa_k(\lambda_j) - 1$ , we have

$$\mathfrak{s}_{j,k,s} = \chi(r) r^{\lambda_j} \sum_{p=0}^s \frac{\log(r)^p}{p!} \varphi_{k,s-p}^j(\omega) \quad (2.19)$$

As consequence, we have defined  $T_\sigma$  propagating singularities with

$$T_\sigma = \sum_{\lambda \in \Lambda_{-1/2}} \iota_a(\lambda).$$

As by assumption the function  $\sigma$  is such that  $\kappa_\sigma \neq -1$ , the set  $\Lambda(\mathcal{L}_\sigma)$  is discrete without any finite accumulation point (see Proposition 2.4.1). Besides, since  $\mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma) \subset D_{r_0}^{\gamma_0}$  (see Proposition 2.4.1), we can define the positive number

$$\beta_0 := \min\{1/2 + \Re e(\lambda) \mid \lambda \in \Lambda(\mathcal{L}_\sigma) \text{ and } -1/2 < \Re e(\lambda)\}.$$

Since the set  $\Lambda(\mathcal{L}_\sigma)$  is symmetric with respect to  $(-1/2, 0)$  we can say that

$$\{\lambda \in \mathbb{C} \mid \Re e(\lambda) \in (-\beta_0 - 1/2; \beta_0 - 1/2)\} \cap \Lambda(\mathcal{L}_\sigma) = \Lambda_{-1/2}.$$

By adapting the results of [102, Chapter 6 ], we obtain the next

**Proposition 2.6.3.** *Assume that  $\kappa_\sigma \neq -1$  and let  $\beta \in (0; \beta_0)$ . Then we have the following assertions:*

1. *If there exists  $u \in \mathring{V}_\beta^1(\Omega)$  such that  $\text{div}(\sigma \nabla u) \in (\mathring{V}_\beta^1(\Omega))^* \subset (\mathring{V}_{-\beta}^1(\Omega))^*$  then  $u$  decomposes as*

$$u = \tilde{u} + \sum_{j=1}^{N_t} \sum_{k=1}^{\iota_g(\lambda_j)} \sum_{s=0}^{\kappa_k(\lambda_j)-1} c_{j,k,s} \mathfrak{s}_{j,k,s} \text{ with } \tilde{u} \in \mathring{V}_{-\beta}^1(\Omega) \text{ and } c_{j,k,s} \in \mathbb{C}$$

*in which the functions  $\mathfrak{s}_{i,j,k}$  are defined in (2.19).*



$$2. \text{index}(A_\sigma^\beta) - \text{index}(A_\sigma^{-\beta}) = T_\sigma.$$

$$3. \ker(A_\sigma^{\pm\beta}) \text{ is independent of } \beta \in (0; \beta_0).$$

**Remark 2.6.2.** Since  $H_0^1(\Omega) \subset \mathring{V}_\beta^1(\Omega)$  for all  $\beta \in (0; \beta_0)$ , the Proposition 2.6.2 allows us to say that for  $\kappa_\sigma \neq -1$  the kernel of  $A_\sigma$  is finite dimensional. Furthermore, thanks to the previous proposition we obtain the following property: if  $u \in H_0^1(\Omega)$  is such that  $\text{div}(\sigma \nabla u) \in (\mathring{V}_\beta^1(\Omega))^*$  for some  $\beta \in (0; \beta_0)$  then  $u \in \mathring{V}_{-\beta}^1(\Omega)$ .

As a consequence of the previous proposition we obtain the

**Lemma 2.6.1.** Assume that  $\kappa_\sigma \neq -1$ . The number  $T_\sigma$  is even. Furthermore, for all  $\beta \in (0; \beta_0)$  we have

$$-\text{index}(A^{-\beta}) = \text{index}(A^\beta) = T_\sigma/2.$$

**Proof.** Since  $A_\sigma^\beta$  is the adjoint of  $A_\sigma^{-\beta}$ , we obtain  $\text{index}(A^\beta) = -\text{index}(A^{-\beta})$ . Combining this with the fact that  $\text{index}(A^\beta) - \text{index}(A^{-\beta}) = T_\sigma$  we get  $T_\sigma = 2 \text{index}(A^\beta)$ . ■

We also have the

**Lemma 2.6.2.** Let  $\lambda_0 \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$  then  $\overline{\lambda_0} \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$ . Furthermore if  $(\varphi_0, \dots, \varphi_s)$  is a Jordan chain of  $\mathcal{L}_\sigma$  associated to  $\lambda$  then  $(\overline{\varphi_0}, \dots, \overline{\varphi_s})$  is a Jordan chain of  $\mathcal{L}_\sigma$  associated to  $\overline{\lambda_0}$ . Furthermore, if  $-1/2 \in \Lambda(\mathcal{L}_\sigma)$  then  $\iota_a(-1/2)$  is even.

**Proof.** By remarking that for all  $q \in \mathbb{N}$  and all  $\varphi, v \in H^1(\mathbb{S}^2)$  we have

$$\overline{\left\langle \frac{d\mathcal{L}_\sigma}{d\lambda^q}(\lambda_0)\varphi, \overline{v} \right\rangle} = \left\langle \frac{d\mathcal{L}_\sigma}{d\lambda^q}(\overline{\lambda_0})\overline{\varphi}, v \right\rangle$$

we obtain the first part of the statement. The second part follows from the fact that  $T_\sigma$  is even. ■

From a physical point of view, the fact that the number of propagating singularities is even can be explained by the fact that we have two kind of propagating singularities: those which propagate toward the conical tip and those which propagate outward conical tip. For each  $\lambda_j, j = 1, \dots, N_t$ , we define the space of propagating singularities of singular exponent  $\lambda_j$  that is defined by

$$S(\lambda_j) := \text{span}\{\mathfrak{s}_{j,k,s}, k = 1, \dots, \iota_g(\lambda_j), s = 0, \dots, \kappa_k(\lambda_j) - 1\}$$

in which  $\mathfrak{s}_{j,k,s}$  are defined in (2.19). Next, we denote by  $\mathcal{S}$  the space of propagating singularities of the operator  $A_\sigma$  :

$$\mathcal{S} := \bigoplus_{\lambda \in \Lambda_{-1/2}} S(\lambda) = \text{span}\{\mathfrak{s}_{j,k}, j = 1, \dots, N_t, k = 1, \dots, \iota_g(\lambda_j), s = 0, \dots, \kappa_k(\lambda_j) - 1\}$$

again  $\mathfrak{s}_{i,j,k}$  are defined in (2.19). Clearly  $T_\sigma = \dim(\mathcal{S})$ . To simplify notations, we denote by  $N_\sigma = T_\sigma/2 \in \mathbb{N}$  and we enumerate the singularities  $\mathfrak{s}_{j,k,s}$  in the following way  $\mathfrak{s}_1, \dots, \mathfrak{s}_{2N_\sigma}$ . This means that  $\mathcal{S} = \text{span}\{\mathfrak{s}_j | j = 1, \dots, 2N_\sigma\}$ . In the next paragraph, we are going to explain how to use the so-called Mandelstam energy radiation principle [112, 103] in order to decompose the space  $\mathcal{S}$  into the sum of two sub-spaces  $\mathcal{S}^+$  (a space of outgoing propagating singularities) and  $\mathcal{S}^-$  (a space of incoming propagating singularities). The reason why we have chosen to work with Mandelstam principle (i.e the direction of propagation of a propagating singularity is determined by the sign of its energy flux near the origin) instead of the classical Sommerfeld radiation principle (i.e. the direction of propagation of a propagating singularity that is associated to  $\lambda_\eta = -1/2 + i\eta \in \Lambda(\mathcal{L}_\sigma)$  with  $\eta \in \mathbb{R}$  is determined by the sign of  $\eta$ ) is the fact that Mandelstam principle allows us to incorporate the case when propagating singularities have a logarithmic growth near the origin (see [114, §5.3] for more details).

### The classification of propagating singularities by the Mandelstam principle

The Mandelstam principle relates the direction of propagation of a propagating singularity  $\mathfrak{s} \in \mathcal{S}$  to the sign of the imaginary part of its energy flux

$$Q(\mathfrak{s}) := \lim_{\varepsilon \rightarrow 0} \int_{|x|=\varepsilon} \sigma(\bar{\mathfrak{s}}\partial_r \mathfrak{s} - \mathfrak{s}\partial_r \bar{\mathfrak{s}}) d\omega.$$

Let us explain why the function  $Q(\mathfrak{s})$  is well-defined for all  $\mathfrak{s} \in \mathcal{S}$ . To do so, we start by observing that for all  $\mathfrak{s} \in \mathcal{S}$  the functions  $\operatorname{div}(\sigma\nabla\mathfrak{s})$ ,  $\operatorname{div}(\sigma\nabla\bar{\mathfrak{s}})$  belong to  $L^2(\Omega)$  and are compactly supported in  $\Omega \setminus \{O\}$ . This implies that  $\operatorname{div}(\sigma\nabla\mathfrak{s})\bar{\mathfrak{s}} - \operatorname{div}(\sigma\nabla\bar{\mathfrak{s}})\mathfrak{s} \in L^1(\Omega)$ . Applying the dominated convergence theorem and integrating by parts we can write that

$$\int_{\Omega} \operatorname{div}(\sigma\nabla\bar{\mathfrak{s}})\mathfrak{s} - \operatorname{div}(\sigma\nabla\mathfrak{s})\bar{\mathfrak{s}} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B(O,\varepsilon)} \operatorname{div}(\sigma\nabla\bar{\mathfrak{s}})\mathfrak{s} - \operatorname{div}(\sigma\nabla\mathfrak{s})\bar{\mathfrak{s}} = \lim_{\varepsilon \rightarrow 0} \int_{|x|=\varepsilon} \sigma(\bar{\mathfrak{s}}\partial_r \mathfrak{s} - \mathfrak{s}\partial_r \bar{\mathfrak{s}}) ds.$$

This shows that  $Q(\mathfrak{s})$  is well-defined for all  $s \in \mathcal{S}$ . Furthermore, one can see that  $Q(\mathfrak{s})$  is purely imaginary for all  $\mathfrak{s} \in \mathcal{S}$ . Observe that the value of  $Q(\mathfrak{s})$  is independent of the choice of the cutoff function  $\chi$  in (2.19). Now, we present the definition of outgoing and incoming (with respect to the Mandelstam principle) propagating singularities.

**Definition 2.6.2.** *A propagating singularity  $\mathfrak{s} \in \mathcal{S} \setminus \{0\}$  is said to be incoming (resp. outgoing) if  $0 < \Im m(Q(\mathfrak{s}))$  (resp.  $\Im m(Q(\mathfrak{s})) < 0$ ). If  $Q(\mathfrak{s}) = 0$ , we say that  $\mathfrak{s}$  is unclassified.*

In the following, we will prove that one can find a basis of the space  $\mathcal{S}$  that contains  $N_\sigma$  outgoing propagating singularities and  $N_\sigma$  incoming ones. For this we start by introducing  $q : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$  the symplectic, i.e. sesquilinear and anti-Hermitian, form associated to the quadratic form  $Q$ . For all  $u, v \in \mathcal{S}$ , we set

$$q(u, v) = \int_{\Omega} \operatorname{div}(\sigma\nabla\bar{v})u - \int_{\Omega} \operatorname{div}(\sigma\nabla u)\bar{v}.$$

It will be interesting to note that, by means of the dominated convergence theorem, we have

$$q(u, v) = \lim_{\varepsilon \rightarrow 0} \int_{|x|=\varepsilon} \sigma(\partial_r u \bar{v} - u \partial_r \bar{v}) ds \text{ for all } u, v \in \mathcal{S}.$$

Observe that for all  $\mathfrak{s} \in \mathcal{S}$  we have  $Q(\mathfrak{s}) = q(\mathfrak{s}, \mathfrak{s})$ . It will be also interesting to mention that for all  $\mathfrak{s} \in \mathcal{S}$  we have

$$q(\bar{\mathfrak{s}}, \bar{\mathfrak{s}}) = -q(\mathfrak{s}, \mathfrak{s}).$$

Let us recall the definition of a non-degenerate symplectic form.

**Definition 2.6.3.** *Let  $h : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$  be a symplectic form. We say that  $h$  is non-degenerate if the matrix  $(h(\mathfrak{s}_j, \mathfrak{s}_k))_{j,k=1,\dots,2N_\sigma}$  is nonsingular or equivalently if the following statement is satisfied:*

$$x \in \mathcal{S} \text{ such that } h(x, y) = 0 \text{ for all } y \in \mathcal{S} \implies x = 0.$$

**Proposition 2.6.4.** *The symplectic form  $q$  is non-degenerate.*

Before starting the proof of the previous proposition, let us, first, prove the

**Lemma 2.6.3.** *We have the following assertions:*

1. Let  $\lambda, \lambda' \in \Lambda_{-1/2}$  such that  $\lambda \neq \lambda'$  then for all  $(u, v) \in S(\lambda) \times S(\lambda')$  we have  $q(u, v) = 0$ .
2. Let  $\lambda \in \Lambda_{-1/2}$ . For all  $u \in S(\lambda) \setminus \{0\}$  there exists  $u' \in S(\lambda)$  such that  $q(u, u') = 1$ .

**Proof.** For each  $j = 1, \dots, N_t$  we denote by  $\{\varphi_{k,s}^j\}_{k=1, \dots, \iota_g(\lambda_j), s=0, \dots, \kappa_k(\lambda_j)-1}$  a canonical system of Jordan chains associated to  $\lambda_j \in \Lambda_{-1/2}$  and we define the functions  $\mathfrak{s}_{j,k,s}$  as in (2.19). As explained in [70, §3] or [108, §5.4.1], for each  $j = 1, \dots, N_t$ , we can find  $\{\psi_{k,s}^j\}_{k=1, \dots, \iota_g(\lambda_j), s=0, \dots, \kappa_k(\lambda_j)-1}$  a canonical system of Jordan chains of  $\mathcal{L}_\sigma$  (this is because the operator  $\operatorname{div}(\sigma \nabla \cdot)$  is formally self-adjoint) associated to  $-\lambda_j - 1 = \lambda_j$  (because  $\lambda_j \in \ell_{-1/2}$ ) such that the functions

$$\mathfrak{s}'_{j,k,s} := -r^{\lambda_j} \sum_{p=0}^s \frac{\log(r)^p}{p!} \psi_{k,s-p}^j(\omega), k = 1, \dots, \iota_g(\lambda_j), s = 0, \dots, \kappa_k(\lambda_j) - 1,$$

satisfy the relations

$$-\int_{\mathbb{R}^3} \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,s}) \overline{\mathfrak{s}'_{j',k',s'}} = \delta_{j,j'} \delta_{k,k'} \delta_{s,\kappa_k(\lambda_{j'})-s'-1} \quad (2.20)$$

where  $\delta_{\cdot, \cdot}$  stands for the Kronecker symbol. Given that the functions  $\operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,s})$  are supported in  $B(O, 1)$  and since  $\operatorname{div}(\sigma \nabla \mathfrak{s}'_{j,k,s}) = 0$  in  $\mathbb{R}^3$  (see Lemma 2.4.7), we obtain

$$\begin{aligned} \delta_{j,j'} \delta_{k,k'} \delta_{s,\kappa_k(\lambda_{j'})-s'-1} &= - \int_{\Omega} \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,s}) \overline{\mathfrak{s}'_{j',k',s'}} \\ &= - \int_{\Omega} \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,s}) \overline{\mathfrak{s}'_{j',k',s'}} + \int_{\Omega} \operatorname{div}(\sigma \nabla \overline{\mathfrak{s}'_{j',k',s'}}) \mathfrak{s}_{j,k,s} \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|x|=\varepsilon} \sigma(\overline{\mathfrak{s}'_{j',k',s'}} \partial_r \mathfrak{s}_{j,k,s} - \mathfrak{s}_{j,k,s} \partial_r \overline{\mathfrak{s}'_{j',k',s'}}) = q(\mathfrak{s}_{j,k,s}, \chi(r) \overline{\mathfrak{s}'_{j',k',s'}}). \end{aligned}$$

The first item is then proved by observing that the functions  $(\chi(r) \overline{\mathfrak{s}'_{j,k,s}})_{j,k,s}$  form a basis of  $S(\lambda_j)$ . The second item is direct consequence of the previous biorthonormality relation.  $\blacksquare$

**Remark 2.6.3.** Another interesting way to prove the previous result is to take profit of the fact that the operator  $\operatorname{div}(\sigma \nabla \cdot)$  is formally self adjoint and to use the results of [114, Chapter 5].

Now, we can give a proof to the Proposition 2.6.4.

*Proof of Proposition 2.6.4.* Assume that there exists  $u \in \mathcal{S} \setminus \{0\}$  such that  $q(u, v) = 0$  for all  $v \in \mathcal{S}$ . Since  $\mathcal{S} = \bigoplus_{\lambda \in \Lambda_{-1/2}} S(\lambda)$ , the function  $u$  decomposes as  $u = u_1 + \dots + u_{N_t}$  where  $u_i \in S(\lambda_i)$  for  $i = 1, \dots, N_t$ . Since  $u \neq 0$ , there exists  $i^* \in \{1, \dots, N_t\}$  such that  $u_{i^*} \neq 0$ . According to the previous lemma, we can find  $u'_{i^*} \in S(\lambda_{i^*})$  such that  $q(u_{i^*}, u'_{i^*}) = 1$  and  $q(u_j, u'_{i^*}) = 0$  for all  $j \neq i^*$ . This means that  $q(u, u'_{i^*}) = 1$ , which leads to a contradiction.  $\blacksquare$

The fact that  $q$  is non-degenerate implies that its rank is equal to  $2N_\sigma$ . Now, we can show the

**Theorem 2.6.3** (The Mandelstam principle). *There exists  $\mathfrak{s}_1^\pm, \dots, \mathfrak{s}_{N_\sigma}^\pm$  a basis of the space  $\mathcal{S}$  such that*

$$q(\mathfrak{s}_j^\pm, \mathfrak{s}_k^\pm) = \pm i \delta_{j,k}, q(\mathfrak{s}_j^\mp, \mathfrak{s}_k^\pm) = 0 \text{ and } \mathfrak{s}_j^\mp = \overline{\mathfrak{s}_j^\pm} \text{ for all } j, k = 1, \dots, N_\sigma \quad (2.21)$$

where  $\delta_{j,k}$  is the Kronecker symbol.

**Proof.** The starting point is to observe that the sesquilinear form  $(u, v) \mapsto -iq(u, v)$  is hermitian. Since  $q$  is non-degenerate,  $-iq$  is also non-degenerate. By applying the Sylvester's law of inertia, we deduce that there exists  $(K^+, K^-) \in \mathbb{N}^*$  ( $K^+, K^- \in \mathbb{N}^*$  (the pair  $(K^+, K^-)$  is called the signature of  $-iq$ ) such that  $K^+ + K^- = 2N_\sigma$  and a basis  $(\mathfrak{s}_1^\pm, \dots, \mathfrak{s}_{K^\pm}^\pm)$  for the space  $\mathcal{S}$  such that

$$-iq(\mathfrak{s}_j^\pm, \mathfrak{s}_k^\pm) = \pm \delta_{j,k} \text{ and } q(\mathfrak{s}_j^\mp, \mathfrak{s}_k^\pm) = 0 \text{ for all } j = 1, \dots, K^+, k = 1, \dots, K^-.$$

Furthermore, the numbers  $K^+$  and  $K^-$  are defined as follows:

$$K^\pm = \max\{\dim(\mathcal{A} \text{ subspace of } \mathcal{S} \text{ such that } \pm 0 \leq -iq(x, x) \text{ for all } x \in \mathcal{A})\}$$

Since the space  $\mathcal{S}$  is stable by complex conjugation and  $-iq(\mathfrak{s}, \mathfrak{s}) = iq(\overline{\mathfrak{s}}, \overline{\mathfrak{s}})$  for all  $\mathfrak{s} \in \mathcal{S}$ , we deduce that  $K^+ = K^- = N_\sigma$  and then the theorem is proved.  $\blacksquare$

**Remark 2.6.4.** *It is important to mention that the basis  $(\mathfrak{s}_j^\pm)_{j=1,\dots,N_\sigma}$  is not unique in the previous theorem. Indeed, one can easily see that for all  $a, b \in \mathbb{R}$  such that  $a^2 - b^2 = 1$  the set of functions  $(\mathfrak{w}_j^\pm)_{j=1,\dots,N_\sigma}$  that are defined by the relation*

$$\mathfrak{w}_j^+ = a\mathfrak{s}_j^+ + b\mathfrak{s}_j^- \quad \text{and} \quad \mathfrak{w}_j^- = b\mathfrak{s}_j^+ + a\mathfrak{s}_j^-$$

*form a basis of the space  $\mathcal{S}$ . Moreover, they satisfy the same orthogonality relations as  $(\mathfrak{s}_j^\pm)_{j=1,\dots,N_\sigma}$ .*

### Definition of the problem and its well-posedness

In this paragraph, we denote by  $(\mathfrak{s}_j^\pm)_{j=1,\dots,N_\sigma}$  a basis of the space  $\mathcal{S}$  such that (2.21) is satisfied. We define the space  $\mathcal{S}^+$  (resp.  $\mathcal{S}^-$ ) the space of outgoing (resp. incoming) propagating singularities such that

$$\mathcal{S}^\pm = \text{span}\{\mathfrak{s}_j^\pm, j = 1, \dots, N_\sigma\}.$$

Thanks to the previous theorem we can write that  $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ : any propagating singularity is the sum of an outgoing and an incoming one. Note that since  $(\mathfrak{s}_j^\pm)_{j=1,\dots,N_\sigma}$  satisfies (2.21), we obtain  $\mathcal{S}^+ = \overline{\mathcal{S}^-}$ . Following the Mandelstam principle (the physical solution must be outgoing), we define for all  $\beta \in (0; \beta_0)$ , the space  $V_\beta^{\text{out}}(\Omega) := \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}^+$ . Endowed with the norm

$$\|\tilde{u} + \sum_{j=1}^{N_\sigma} c_j \mathfrak{s}_j^+\| = (\|\tilde{u}\|_{\mathring{V}_{-\beta}^1(\Omega)}^2 + \sum_{j=1}^{N_\sigma} |c_j|^2)^{1/2} \quad \text{for all } \tilde{u} \in \mathring{V}_{-\beta}^1(\Omega) \text{ and } c_j \in \mathbb{C},$$

the space  $V_\beta^{\text{out}}(\Omega)$  is a Hilbert space. Then, we introduce the operator  $A_\beta^{\text{out}} : V_\beta^{\text{out}}(\Omega) \rightarrow (\mathring{V}_\beta^1(\Omega))^*$  such that for all  $u = \tilde{u} + s^+$  with  $(\tilde{u}, s^+) \in \mathring{V}_{-\beta}^1(\Omega) \times \mathcal{S}^+$  and  $v \in \mathring{V}_\beta^1(\Omega)$  we have

$$\langle A_\beta^{\text{out}} u, v \rangle := \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla \bar{v} - \int_\Omega \text{div}(\sigma \nabla s^+) \bar{v}.$$

Note that  $\langle A_\beta^{\text{out}} u, v \rangle$  is well-defined for all  $u \in V_\beta^{\text{out}}(\Omega)$  and  $v \in \mathring{V}_\beta^1(\Omega)$  because the function  $\text{div}(\sigma \nabla s^+)$  belongs to  $L^2(\Omega)$  and is compactly supported in  $\Omega \setminus \{O\}$ . Before getting into details, let us explain why  $A_\beta^{\text{out}}$  is continuous for all  $\beta \in (0; \beta_0)$ . This is a consequence of the following

**Lemma 2.6.4.** *There exists a positive constant  $C$  such that*

$$\left| \int_\Omega \text{div}(\sigma \nabla s) \bar{v} \right| \leq C \left( \sum_{j=1}^{N_\sigma} |c_j| \right) \|v\|_{\mathring{V}_\beta^1(\Omega)} \quad \text{for all } s = \sum_{j=1}^{N_\sigma} c_j \mathfrak{s}_j^+ \in \mathcal{S} \text{ and } v \in \mathring{V}_\beta^1(\Omega).$$

**Proof.** It suffices to prove the result with  $s = \mathfrak{s}_j^+$  for all  $j = 1, \dots, N_\sigma$ . For this we start by recalling that for all  $j = 1, \dots, N_\sigma$  the function  $\mathfrak{s}_j^+$  has the form  $\mathfrak{s}_j^+ = \chi(r) \tilde{\mathfrak{s}}_j^+$  where  $\text{div}(\sigma \nabla \tilde{\mathfrak{s}}_j^+) = 0$  in  $B(O, 1)$  (we remind the reader that the function  $\chi$  is also supported in  $B(O, 1)$ ). With this in mind, we can write that for all  $\varphi \in \mathcal{D}(\Omega \setminus \{O\})$  we have

$$\begin{aligned} \int_\Omega \text{div}(\sigma \nabla \mathfrak{s}_i^+) \bar{\varphi} &= - \int_\Omega \sigma \nabla(\chi \tilde{\mathfrak{s}}_i^+) \cdot \nabla \bar{\varphi} = - \int_\Omega \sigma \chi \nabla \tilde{\mathfrak{s}}_i^+ \cdot \nabla \bar{\varphi} - \int_\Omega \sigma \tilde{\mathfrak{s}}_i^+ \nabla \chi \cdot \nabla \bar{\varphi} \\ &= \int_\Omega \sigma \bar{\varphi} \nabla \tilde{\mathfrak{s}}_i^+ \cdot \nabla \chi - \int_\Omega \sigma \tilde{\mathfrak{s}}_i^+ \nabla \chi \cdot \nabla \bar{\varphi}. \end{aligned}$$

As  $\chi = 1$  near the origin, the support of  $\nabla \chi$  is then detached from the origin. This leads us to the estimate

$$\left| \int_\Omega \text{div}(\sigma \nabla \mathfrak{s}_i^+) \bar{\varphi} \right| \leq C \|\varphi\|_{\mathring{V}_\beta^1(\Omega)}$$

with  $0 < C$  independent of  $\varphi$ . The wanted result follows from the density of  $\mathcal{D}(\Omega \setminus \{O\})$  into the space  $\mathring{V}_\beta^1(\Omega)$ . ■

**Remark 2.6.5.** By observing that for all  $v \in \mathcal{D}(\Omega \setminus \{O\})$  we have  $\langle A_\beta^{\text{out}} u, v \rangle = -\langle \text{div}(\sigma \nabla u), v \rangle$  and by using the continuity of  $A_\beta^{\text{out}}$ , we can that we have the equivalence

$$u \in \mathring{V}_\beta^{\text{out}}(\Omega) \text{ such that } \text{div}(\sigma \nabla u) = 0 \quad \text{if and only if} \quad A^{\text{out}} u = 0.$$

To proceed, we define the sesquilinear form  $q^{\text{out}} : \mathring{V}_\beta^{\text{out}}(\Omega) \times \mathring{V}_\beta^{\text{out}}(\Omega) \rightarrow \mathbb{C}$  such that for all  $u, v \in \mathring{V}_\beta^{\text{out}}(\Omega)$ , we have

$$q^{\text{out}}(u, v) = \langle A_\beta^{\text{out}} u, v \rangle - \langle u, A_\beta^{\text{out}} v \rangle = \langle A_\beta^{\text{out}} u, v \rangle - \overline{\langle A_\beta^{\text{out}} v, u \rangle}.$$

It will be interesting to note that the value of  $q(u, v)$  for  $u, v \in \mathring{V}_\beta^{\text{out}}(\Omega)$  depends only in the singular part of  $u$  and  $v$ . Indeed, for all  $u = \tilde{u} + s_u, v = \tilde{v} + s_v \in \mathring{V}_\beta^{\text{out}}(\Omega)$ , with  $\tilde{u}, \tilde{v} \in \mathring{V}_{-\beta}^1(\Omega)$  and  $s_u, s_v \in \mathcal{S}^+$ , we have

$$\begin{aligned} q^{\text{out}}(u, v) &= \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla \bar{\tilde{v}} + \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla \bar{s}_v - \int_\Omega \text{div}(\sigma \nabla s_u) \bar{\tilde{v}} - \int_\Omega \text{div}(\sigma \nabla s_u) \bar{s}_v \\ &\quad - \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla \bar{\tilde{v}} - \int_\Omega \sigma \nabla \bar{\tilde{v}} \cdot \nabla s_u + \int_\Omega \text{div}(\sigma \nabla \bar{s}_v) \tilde{u} + \int_\Omega \text{div}(\sigma \nabla \bar{s}_v) s_u \\ &= \int_\Omega \text{div}(\sigma \nabla \bar{s}_v) s_u - \int_\Omega \text{div}(\sigma \nabla s_u) \bar{s}_v = q^{\text{out}}(s_u, s_v) = q(s_u, s_v). \end{aligned} \quad (2.22)$$

**Remark 2.6.6.** For all  $0 < \beta$ , we define the space  $\mathring{V}_\beta^{\text{in+out}} := \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}$ . We also introduce the operator  $A^{\text{in+out}} : \mathring{V}_\beta^{\text{in+out}} \rightarrow (\mathring{V}_\beta^1(\Omega))^*$  such that for all  $u = \tilde{u} + s_u \in \mathring{V}_\beta^{\text{in+out}}$  (with  $\tilde{u} \in \mathring{V}_{-\beta}^1(\Omega)$ ) and  $s_u \in \mathcal{S}$  and  $v \in \mathring{V}_\beta^1(\Omega)$  we have

$$\langle A^{\text{in+out}} u, v \rangle := \int_\Omega \sigma \nabla \tilde{u} \cdot \nabla \bar{v} - \int_\Omega \text{div}(\sigma \nabla s) \bar{v}.$$

Observe that for all  $u \in \mathring{V}_\beta^{\text{out}}(\Omega)$  we have  $A^{\text{in+out}} u = A^{\text{out}} u$ . Working as in the case of the operator  $A_\beta^{\text{out}}$ , we can show that  $A^{\text{in+out}}$  is continuous. We also define the sesquilinear form  $q^{\text{in+out}} : \mathring{V}_\beta^{\text{in+out}} \rightarrow \mathbb{C}$  such that for all  $u, v \in \mathring{V}_\beta^{\text{in+out}}$  we set

$$q^{\text{in+out}}(u, v) = \langle A_\beta^{\text{in-out}} u, v \rangle - \langle u, A_\beta^{\text{in-out}} v \rangle = \langle A_\beta^{\text{in-out}} u, v \rangle - \overline{\langle A_\beta^{\text{in-out}} v, u \rangle}.$$

By arguing as in the case of the sesquilinear form  $q^{\text{out}}$ , we can show that for all  $u, v \in \mathring{V}_{-\beta}^1(\Omega)$  and  $s_u, s_v \in \mathcal{S}$  we have

$$q^{\text{in+out}}(u + s_u, v + s_v) = q(s_u, s_v). \quad (2.23)$$

To simplify a little bit the analysis below, we will make the

**Assumption 2.6.1.** Assume that  $\kappa_\sigma \neq -1$  and that there exists  $\beta^* \in (0; \beta_0)$  such that  $A_\sigma^{-\beta^*}$  is injective.

Using the last item of Proposition 2.6.3, we obtain the

**Lemma 2.6.5.** Assume that Assumption 2.6.1 holds then for all  $\beta \in (0; \beta_0)$  the operator  $A_\sigma^{-\beta}$  is injective. In particular, we have the estimate: there exists  $0 < C_\beta$  such that

$$\|u\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C_\beta \|A_\sigma^{-\beta} u\|_{(\mathring{V}_\beta^1(\Omega))^*}, \quad u \in \mathring{V}_{-\beta}^1(\Omega)$$

Using the terminology of the waveguides theory, the previous assumption is equivalent to say that we suppose that trapped modes do not exist. When it is not satisfied, a modified version of our results can be obtained (see Remark 2.6.7). The remaining part of this paragraph is devoted to prove that under Assumption 2.6.1, the operator  $A_\beta^{\text{out}}$  is an isomorphism for all  $\beta \in (0; \beta_0)$ . The injectivity of  $A_\beta^{\text{out}}$  is the subject of the next

**Proposition 2.6.5.** *Assume that Assumption 2.6.1 holds. Then for all  $\beta \in (0; \beta_0)$  the operator  $A_\beta^{\text{out}}$  is injective.*

**Proof.** Let  $u = \tilde{u} + s^+ \in V_\beta^{\text{out}}(\Omega)$  with  $(\tilde{u}, s^+) \in \mathring{V}_{-\beta}^1(\Omega) \times \mathcal{S}^+$  such that  $A_\beta^{\text{out}}u = 0$ . Since by assumption  $A_\sigma^{-\beta}$  is injective it suffices to show that  $s^+ = 0$ . Given that  $\sigma$  is real valued, it follows that  $\text{div}(\sigma \nabla \tilde{u}) = 0$ . This leads us to write that  $q^{\text{out}}(u, u) = 0$ . Using (2.22), we deduce that  $q(s^+, s^+) = 0$ . The definition of the space  $\mathcal{S}^+$  suggests that we can decompose the function  $s^+$  as follows

$$s^+ = \sum_{j=1}^{N_\sigma} c_j^+ \mathfrak{s}_j^+.$$

By observing that  $q(s^+, s^+) = i(\sum_{j=1}^{N_\sigma} |c_j^+|^2)$ , we infer that  $s^+ = 0$ . The result is then proved.  $\blacksquare$

Now, we turn our attention to the study of the surjectivity of  $A_\beta^{\text{out}}(\Omega)$ . Before that, we will prove the following useful result.

**Proposition 2.6.6.** *Assume that Assumption 2.6.1 holds and let  $\beta \in (0; \beta_0)$ . Then for all  $j = 1, \dots, N_\sigma$  there exists a unique pair of functions  $(s_j^\pm, \tilde{u}_j^\pm) \in \mathcal{S}^\pm \times \mathring{V}_{-\beta}^1(\Omega)$  such that*

$$\mathfrak{d}_j^\pm = \mathfrak{s}_j^\pm + s_j^\mp - \tilde{u}_j^\pm \in \ker(A_\sigma^\beta).$$

Classically, for all  $j = 1, \dots, N_\sigma$ , the function  $\mathfrak{d}_j^\pm$  defined above is known as the dual singularity associated to  $\mathfrak{s}_j^\pm$ .

**Proof.** The proof follows the lines of the proof of [114, Proposition 5.3.3]. The fact that  $A_\sigma^{-\beta}$  is injective implies that  $A_\sigma^\beta$  is surjective and that  $N_\sigma = \text{index}(A_\sigma^\beta) = \dim(\text{Ker}(A_\sigma^\beta))$ . Denote by  $u_1, \dots, u_{N_\sigma}$  a basis of  $\text{Ker}(A_\sigma^\beta)$ . Thanks to Proposition 2.6.3, for each  $j = 1, \dots, N_\sigma$  the function  $u_j$  decomposes as

$$u_j = \sum_{k=1}^{N_\sigma} c_{j,k} \mathfrak{s}_k^+ + \sum_{k=1}^{N_\sigma} d_{j,k} \mathfrak{s}_k^- + \tilde{u}_j$$

where  $\tilde{u}_j \in \mathring{V}_{-\beta}^1(\Omega)$  and all the  $c_{j,k}, d_{j,k} \in \mathbb{C}$ . Denote by  $C, D \in \mathbb{M}_{N_\sigma}(\mathbb{C})$  the matrices

$$C = (c_{j,k})_{j,k=1,\dots,N_\sigma} \text{ and } D = (d_{j,k})_{j,k=1,\dots,N_\sigma}.$$

To end the proof we are going to show the matrices  $C$  and  $D$  are nonsingular (with this in mind one can then find linear combinations of the functions  $u_j$  that lead to the wanted results). We start with the case of the matrix  $C$ . Suppose that  $C$  is not injective. Then there exists a function  $u \in \text{Ker}(A_\sigma^\beta) \setminus \{0\}$  that decomposes as

$$u = \sum_{j=0}^{N_\sigma} \gamma_j \mathfrak{s}_j^- + \tilde{u} \text{ with } \tilde{u} \in \mathring{V}_{-\beta}^1(\Omega) \text{ and } \gamma_j \in \mathbb{C}.$$

By working as in the proof of Proposition 2.6.5, we infer that  $u = 0$ . This leads to a contradiction. Thus the matrix  $C$  is nonsingular. With the same arguments, we show that  $D$  is nonsingular.  $\blacksquare$

Now, we can prove the surjectivity of  $A_\beta^{\text{out}}$ .

**Proposition 2.6.7.** *Assume that Assumption 2.6.1 holds. For all  $\beta \in (0; \beta_0)$  the operator  $A_\beta^{\text{out}}$  is surjective.*

**Proof.** Since  $A_\sigma^\beta$  is the adjoint of  $A_\sigma^{-\beta}$  and the latter is assumed to be injective, we infer that  $A_\sigma^\beta$  is surjective. Take  $f \in (\mathring{V}_\beta^1(\Omega))^*$ . Since  $0 < \beta$ , we have the embedding  $\mathring{V}_{-\beta}^1(\Omega) \subset \mathring{V}_\beta^1(\Omega)$  and then, by duality, infer that  $(\mathring{V}_\beta^1(\Omega))^* \subset (\mathring{V}_{-\beta}^1(\Omega))^*$ . This allows us to say that there exists  $u \in \mathring{V}_\beta^1(\Omega)$  such that  $A_\sigma^\beta u^\beta = f$ . Since  $f \in (\mathring{V}_\beta^1(\Omega))^*$  we know, thanks to Proposition 2.6.3, that the function  $u^\beta$  decomposes as

$$u^\beta = u^{-\beta} + \sum_{j=1}^{N_\sigma} c_j^+ \mathfrak{s}_j^+ + c_j^- \mathfrak{s}_j^-$$

with  $u^{-\beta} \in \mathring{V}_{-\beta}^1(\Omega)$  and all the  $c_j^\pm \in \mathbb{C}$ . Thanks to Proposition 2.6.3 we know that there exist  $u_1^\beta, \dots, u_{N_\sigma}^\beta \in \text{Ker}(A_\sigma^\beta)$  such that for all  $j = 1, \dots, N_\sigma$  we have

$$u_j^\beta = u_j^{-\beta} + \mathfrak{s}_j^- + \sum_{k=1}^{N_\sigma} \gamma_k \mathfrak{s}_k^+ \text{ with } u_j^{-\beta} \in \mathring{V}_{-\beta}^1(\Omega) \text{ and } \gamma_k \in \mathbb{C}.$$

By observing that the function  $u = u^\beta - \sum_{j=1}^{N_\sigma} c_j^- u_j^\beta$  belongs to the space  $\mathring{V}_\beta^{\text{out}}(\Omega)$  and satisfies the equation  $A_\beta^{\text{out}} u = f$ , we obtain the wanted result.  $\blacksquare$

Since the operator  $A_\beta^{\text{out}}$  is continuous for all  $\beta \in (0; \beta_0)$ , the open map theorem, combined with the results of the previous propositions, leads us to the

**Theorem 2.6.4.** *Assume that Assumption 2.6.1 holds. Then for all  $\beta \in (0; \beta_0)$  the operator  $A_\beta^{\text{out}}$  is an isomorphism. Moreover, there exists a constant  $0 < C$  such that for all  $u = \tilde{u} + c_1 \mathfrak{s}_1^+ + \dots + c_{N_\sigma} \mathfrak{s}_{N_\sigma}^+$  with  $\tilde{u} \in \mathring{V}_{-\beta}^1(\Omega)$  and all  $c_j \in \mathbb{C}$  we have the estimate*

$$\|\tilde{u}\|_{\mathring{V}_{-\beta}^1(\Omega)} + \sum_{j=1}^{N_\sigma} |c_j| \leq C^\beta \|A_\beta^{\text{out}}(u)\|_{(\mathring{V}_\beta^1(\Omega))^*}.$$

For all  $\beta \in (0; \beta_0)$ , the expression of the singular coefficients of the solution (i.e. the coefficients in front of the singularities  $\mathfrak{s}_j^+$  in the decomposition of the solution  $u$ ) to the well-posed problem:

$$\text{Find } u \in \mathring{V}_\beta^{\text{out}}(\Omega) \text{ such that } A_\beta^{\text{out}} u = f \in (\mathring{V}_\beta^1(\Omega))^* \quad (2.24)$$

can be determined thanks to the following

**Lemma 2.6.6.** *Let  $0 < \beta$  and let  $u = \tilde{u} + \sum_{j=1}^{N_\sigma} c_j^+ \mathfrak{s}_j^+ \in \mathring{V}_\beta^{\text{out}}(\Omega)$  (with  $\tilde{u} \in \mathring{V}_{-\beta}^1(\Omega)$ ). Then for all  $j = 1, \dots, N_\sigma$ , we have*

$$c_j^+ = \langle A^{\text{out}} u, \mathfrak{d}_j^+ \rangle / i$$

where the function  $\mathfrak{d}_j^+$  are defined in Proposition 2.6.6.

**Proof.** For all  $j = 1, \dots, N_\sigma$ , the function  $\mathfrak{d}_j^+$  belongs to the space  $\mathring{V}_\beta^{\text{in+out}}(\Omega)$  (see Remark 2.6.6). Furthermore since  $\text{div}(\sigma \nabla \mathfrak{d}_j^+) = 0$  in  $\Omega$  and thanks to the continuity of  $A^{\text{in+out}}$  (see Remark 2.6.6) we can say that  $\langle A^{\text{in+out}} \mathfrak{d}_j^+, u \rangle = 0$ . On the other hand, since  $u \in \mathring{V}_\beta^{\text{out}}(\Omega) \subset \mathring{V}_\beta^{\text{in+out}}(\Omega)$  we have  $\langle A^{\text{out}} u, \mathfrak{d}_j^+ \rangle = \langle A^{\text{in+out}} u, \mathfrak{d}_j^+ \rangle$ . As a result we can write

$$\langle A^{\text{out}} u, \mathfrak{d}_j^+ \rangle = \langle A^{\text{in+out}} u, \mathfrak{d}_j^+ \rangle - \langle A^{\text{in+out}} \mathfrak{d}_j^+, u \rangle = q^{\text{in+out}}(u, \mathfrak{d}_j^+).$$

Given that the function  $\mathfrak{d}_j^+$  decomposes as  $\mathfrak{d}_j^+ = \mathfrak{s}_j^+ + s^- + \tilde{u}_j$  with  $s^- \in \mathcal{S}^-$  and  $\tilde{u}_j \in \mathring{V}_{-\beta}^1(\Omega)$  we deduce (thanks to (2.23)) that

$$\langle A^{\text{out}} u, \mathfrak{d}_j^+ \rangle = q \left( \sum_{k=1}^{N_\sigma} c_k^+ \mathfrak{s}_k^+ + s^- \right) = i c_j^+.$$

$\blacksquare$



This leads us to the following

**Corollary 2.6.1.** *Assume that Assumption 2.6.1 holds and let  $\beta \in (0; \min(\beta_0, 1))$ . Then for all  $f \in L^2(\Omega)$  the problem (2.24) is well-posed. Moreover, its solution decomposes as  $u = \tilde{u} + c_1^+ \mathfrak{s}_1^+ + \dots + c_{N_\sigma}^+ \mathfrak{s}_{N_\sigma}^+$  with  $\tilde{u} \in \mathring{V}_{-\beta}^1(\Omega)$  where the coefficients  $(c_j^+)_{j=1, \dots, N_\sigma}$  are given by*

$$c_j = \frac{1}{i} \int_{\Omega} f \overline{\mathfrak{d}_j^+}.$$

**Proof.** It suffices to mention that for all  $\beta \in (0; 1)$  we have the embedding  $L^2(\Omega) \subset (\mathring{V}_{-\beta}^1(\Omega))^*$  and then to apply Theorem 2.6.4 and Lemma 2.6.6. ■

We finish this paragraph with some remarks.

**Remark 2.6.7.**

- *It is worth to note that since  $(\mathring{V}_{-\beta}^1(\Omega))^* \subset (H_0^1(\Omega))^*$  for all  $0 < \beta$ , the framework that we have proposed above does not allow us to define a solution to (2.2) for all given source term  $f \in (H_0^1(\Omega))^*$ .*
- *In the case when the Assumption 2.6.1 is not satisfied, i.e. when the operator  $A_\sigma^{-\beta}$  is not injective, the operator  $A_\beta^{\text{out}}$  is no longer isomorphism. However, one can show that for  $\beta \in (0; \beta_0)$  the operator  $A_\beta^{\text{out}}$  is Fredholm of index zero. Let us explain, briefly, how to show this result. Starting from Proposition 2.15, using the results of §2.3 we can show that for all  $\beta \in (0; \beta_0)$  the operator  $A_\beta^{\text{out}}$  has closed range and a finite dimensional kernel. Furthermore, by working as in the proof of Proposition 2.6.5 we can easily prove that for all  $\beta \in (0; \beta_0)$ , we have  $\ker(A_\beta^{\text{out}}) = \ker(A_\sigma^{-\beta})$ . The last step is to show that  $A_\beta^{\text{out}}$  is of index zero. To do that, one can follow the lines of the proof of [25, Proposition 4.4].*

### 2.6.3 Selection of the physical solution by means of the limiting absorption principle

In the previous section, we have explained how it is possible, even in the case of propagating singularities with logarithmic growth near the origin, to define a radiation condition that allows us to construct a functional framework in which the scalar problem is well-posed in the Fredholm sense. However, as explained in Remark 2.6.4, it is possible to construct an infinite number of functional frameworks that are coherent with the Mandelstam radiation principle and in which the problem is also well-posed. This means that almost all the functional frameworks that can be constructed using the Mandelstam radiation principle do not lead to the physical solution of the problem. Obviously, the main difficulty is to define a space of outgoing propagating singularities that has a physical meaning. To do that, we are going to use the limiting absorption principle. The idea is to say that the physical solution of the problem 2.2 must be defined as the limit when  $\delta \rightarrow 0^+$  (in some space to be defined) of the  $(u_\delta)_\delta$  where  $u_\delta$  solves the well-posed problem

$$\text{Find } u^\delta \in H_0^1(\Omega) \text{ such that } -\operatorname{div}((\sigma + i\delta)\nabla u^\delta) = f \in (H_0^1(\Omega))^*. \quad (2.25)$$

The well-posedness of the previous problem for all  $\delta \in (0; +\infty)$  is guaranteed by the Lax-Milgram lemma. Introduce the operator  $A_{\sigma+i\delta} : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  such that

$$\langle A_{\sigma+i\delta} u, v \rangle := \int_{\Omega} (\sigma + i\delta) \nabla u \cdot \nabla \bar{v} \quad \text{for all } u, v \in H_0^1(\Omega).$$

The case of non-critical coefficients is treated in the following

**Lemma 2.6.7.** *Assume that the function  $\sigma$  is such that  $\kappa_\sigma \notin I_\Sigma$  and suppose that the source term  $f$  is such that the problem (2.2) is well-posed in the Hadamard sense. Then the sequence  $(u_\delta)_\delta$  converges as  $\delta \rightarrow 0^+$ , in  $H_0^1(\Omega)$  to  $u$  the solution of (2.2).*

**Proof.** By definition of  $I_\Sigma$ , we know that since  $\kappa_\sigma \notin I_\Sigma$ , the operator  $A_\sigma$  is a Fredholm operator of index 0. Furthermore, since by assumption the problem (2.2) has a unique solution, we infer that  $A_\sigma$  is injective and thus it is an isomorphism. As a result, we have the estimate

$$\|u\|_{\mathbf{H}_0^1(\Omega)} \leq C \|\operatorname{div}(\sigma \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*}, \quad u \in \mathbf{H}_0^1(\Omega)$$

in which  $C$  is a constant that does not depend on  $u$ . Combining this with the obvious estimate

$$\|\operatorname{div}(\sigma \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*} \leq \|\operatorname{div}((\sigma + i\delta) \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*} + |\delta| \|u\|_{\mathbf{H}_0^1(\Omega)} \text{ for all } u \in \mathbf{H}_0^1(\Omega)$$

we arrive to

$$\|u\|_{\mathbf{H}_0^1(\Omega)} \leq C \|\operatorname{div}((\sigma + i\delta) \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*} + C|\delta| \|u\|_{\mathbf{H}_0^1(\Omega)}.$$

By taking  $\delta$  such that  $0 < \delta < \delta_0 := 1/2C$ , we obtain the following estimate

$$\|u\|_{\mathbf{H}_0^1(\Omega)} \leq 2C \|\operatorname{div}((\sigma + i\delta) \nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*}, \quad u \in \mathbf{H}_0^1(\Omega).$$

Applying the previous estimate to the function  $u - u_\delta$  where  $u$  and  $u_\delta$  are respectively the solutions to (2.2) and (2.25), we conclude that for all  $\delta \in (0; \delta_0)$  we have

$$\|u - u_\delta\|_{\mathbf{H}_0^1(\Omega)} \leq 2C|\delta| \|\operatorname{div}(\nabla u)\|_{(\mathbf{H}_0^1(\Omega))^*}.$$

The lemma is then proved. ■

Now, we turn our attention to the study of the case where the function  $\sigma$  is such that  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$ .

### Definition of the space of physical outgoing propagating singularities

The starting point is to introduce the Mellin symbol of the problem (2.25). For all  $\delta \in \mathbb{R}_+^*$  and all  $\lambda \in \mathbb{C}$  we introduce the operator  $\mathcal{L}_{\sigma+i\delta}(\lambda) : \mathbf{H}^1(\mathbb{S}^2) \rightarrow (\mathbf{H}^1(\mathbb{S}^2))^*$  such that for all  $\Phi, \Phi' \in \mathbf{H}^1(\mathbb{S}^2)$  we have

$$\langle \mathcal{L}_{\sigma+i\delta}(\lambda) \Phi, \Phi' \rangle =: \int_{\mathbb{S}^2} (\sigma + i\delta) \nabla_S \Phi \cdot \nabla_S \bar{\Phi}' d\omega - \lambda(\lambda + 1) \int_{\mathbb{S}^2} (\sigma + i\delta) \Phi \bar{\Phi}' d\omega.$$

We denote by  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  the spectrum of the family of operators  $(\mathcal{L}_{\sigma+i\delta}(\lambda))_{\lambda \in \mathbb{C}}$ . In 3.3, we will present a study of the spectral properties of  $\mathcal{L}_{\sigma+i\delta}$ . In particular, we will prove the following

**Lemma 2.6.8.** *Assume that the function  $\sigma$  is such that  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$  and let  $\beta \in (0; \beta_0)$ . Then there exists  $0 < \delta_\beta$  such that for all  $0 < \delta < \delta_\beta$  the operator  $\mathcal{L}_{\sigma+i\delta}$  has  $N_t$  eigenvalues in the strip  $\{\lambda \in \mathbb{C} \mid -\beta < 1/2 + \Re(\lambda) < \beta\}$  of total algebraic multiplicity (i.e. the sum of all the algebraic multiplicity of these eigenvalues) equal to  $2N_\sigma$ . Furthermore, we have*

$$\lim_{\delta \rightarrow 0^+} \Lambda(\mathcal{L}_{\sigma+i\delta}) \cap \{\lambda \in \mathbb{C} \mid -\beta < 1/2 + \Re(\lambda) < \beta\} = \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} = \Lambda_{-1/2}. \quad (2.26)$$

In the rest of this section, we are going to work under the following

**Assumption 2.6.2.** *We suppose that the function  $\sigma$  is such that  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$  and such that*

- *All the eigenvalues of  $\mathcal{L}_\sigma$  that are located on the energy line  $\Re(\lambda) = -1/2$  are semi-simple<sup>7</sup>. We denote them by  $\lambda_1, \dots, \lambda_{N_t}$ .*
- *There exists  $0 < \delta_0$  and  $0 < r_0$  such for all  $0 < \delta < \delta_0$  and all  $j = 1, \dots, N_t$ , we have  $B(\lambda_j, r_0) \cap \Lambda(\mathcal{L}_{\sigma+i\delta}) = \{\lambda_{j,\delta}\}$ .*
- *All the  $\lambda_{j,\delta}$  ( $j = 1, \dots, N_t$ ) are semi-simple.*

<sup>7</sup>We say that an eigenvalue of  $\mathcal{L}_\sigma$  is semi-simple if  $\iota_a(\lambda) = \iota_g(\lambda)$ .

**Remark 2.6.8.** Clearly, when all the eigenvalues of  $\mathcal{L}_\sigma$  that are located on  $\Re(\lambda) = -1/2$  are simple<sup>8</sup> the previous assumption is satisfied thanks to Theorem 3.3.1.

**Lemma 2.6.9.** Assume that Assumption 2.6.2 holds. Then  $N_t$  is even. Moreover, for all  $\beta \in (0; \beta_0)$  there exists  $0 < \delta_\beta$  such that for all  $\delta \in (0; \delta_\beta)$ , we have

- the strip  $\{\lambda \in \mathbb{C} \mid 1/2 + \Re(\lambda) \in (0; \beta)\}$  contains  $N_t/2$  eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  of total algebraic geometric equal to  $N_\sigma$ .
- For all  $j = 1, \dots, N_t$  we have  $\iota_g(\lambda_\delta^j) = \iota_g(\lambda_j)$ .

**Proof.** We start by proving that, under Assumption 2.6.2,  $-1/2 \notin \Lambda(\mathcal{L}_\sigma)$ . For this it suffices to see that if  $\varphi \in \text{Ker}(\mathcal{L}_\sigma(-1/2)) \setminus \{0\}$  then  $(\varphi, \varphi)$  is a Jordan chain of  $\mathcal{L}_\sigma$  associated to  $-1/2$ . This means that  $-1/2$  can not be a semi-simple eigenvalue of  $\mathcal{L}_\sigma$ . Given that the spectrum of  $\mathcal{L}_\sigma$  is symmetric with respect to  $(-1/2, 0)$ , we infer that  $N_t$  is even. This implies that the strip  $\{\lambda \in \mathbb{C} \mid 1/2 + \Re(\lambda) \in (0; \beta)\}$  contains  $N_t/2$  eigenvalue(s) of  $\mathcal{L}_\sigma$ .

According to Proposition 3.3.1, we know that  $\Lambda(\mathcal{L}_{\sigma+i\delta}) \cap \ell_{-1/2} = \emptyset$  for all  $0 < \delta$ . As a result, by using the fact that  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  is also symmetric with respect to  $(-1/2, 0)$  and by means of (2.26), we deduce that, under Assumption 2.6.2, there exists  $0 < \delta_\beta$  such that for all  $\delta \in (0; \delta_\beta)$  the strip  $\{\lambda \in \mathbb{C} \mid 1/2 + \Re(\lambda) \in (0; \beta)\}$  contains  $N_t/2$  eigenvalue(s) of  $\mathcal{L}_{\sigma+i\delta}$ . Now, let us explain why  $\iota_g(\lambda_\delta^j)$  coincides with  $\iota_g(\lambda_j)$ . This is a consequence of Proposition 3.3.3 in which we prove that the sum of all the algebraic multiplicities of the eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  that are near  $\lambda_j$  must be equal to the algebraic multiplicity of  $\lambda_j$ . ■

Now, let us assume that Assumption 2.6.2 holds. For all  $\beta \in (0; \beta_0)$  and all  $0 < \delta$  sufficiently small, we denote by  $(\lambda_{j,\delta}^+)_{j=1, \dots, N_t/2}$  the set of eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  that are located in the strip  $\{\lambda \in \mathbb{C} \mid 1/2 + \Re(\lambda) \in (0; \beta)\}$ . For each  $j = 1, \dots, N_t/2$ , we denote by  $(\varphi_{k,\delta}^j)_{k=1, \dots, \iota_g(\lambda_{j,\delta}^+)}$  an orthonormal (with respect to the inner product of  $H^1(\mathbb{S}^2)$ ) basis of  $\text{Ker} \mathcal{L}_{\sigma+i\delta}(\lambda_{j,\delta}^+)$ . Next, we introduce the functions

$$\mathfrak{s}_{j,k,\delta}^+(r\omega) = \chi(r)r^{\lambda_{j,\delta}^+} \varphi_{k,\delta}^j(\omega), j = 1, \dots, N_t/2, k = 1, \dots, \iota_g(\lambda_{j,\delta}^+).$$

Then, we define the space  $\mathcal{S}_\delta^+ = \text{span}\{\mathfrak{s}_{j,k,\delta}^+, j = 1, \dots, N_t/2, k = 1, \dots, \iota_g(\lambda_{j,\delta}^+)\}$ . It is obvious that if Assumption 2.6.2 is valid, then for all  $0 < \delta$  small enough the space  $\mathcal{S}_\delta^+$  is of dimension  $N_\sigma$ . For this reason, we can introduce  $(\mathfrak{s}_{j,\delta}^+)_{j=1, \dots, N_\sigma}$  a basis of the space  $\mathcal{S}_\delta^+ := \text{span}\{\mathfrak{s}_{j,\delta}^+, j = 1, \dots, N_\sigma\}$ . It will be interesting to note that for all  $j = 1, \dots, N_t/2$  and all  $k = 1, \dots, \iota_g(\lambda_{j,\delta}^+)$  the function  $\mathfrak{s}_{j,k,\delta}^+$  belongs to the space  $H_0^1(\Omega)$ . Moreover, one can easily see that the functions  $\text{div}((\sigma + i\delta)\nabla \mathfrak{s}_{j,k,\delta}^+)$  vanishes near the origin and then they belong to the space  $L^2(\Omega) \cap (\mathring{V}_\beta^1(\Omega))^*$  for all  $\beta \in \mathbb{R}$ . The behavior of these functions as  $\delta \rightarrow 0^+$  is the subject of the next

**Lemma 2.6.10.** Assume that Assumption 2.6.2 is valid and let  $\beta \in (0; \beta_0)$ . Then for  $j = 1, \dots, N_t/2$  and  $k = 1, \dots, \iota_g(\lambda_{j,\delta}^+)$ , the sequence of functions  $(\mathfrak{s}_{j,k,\delta}^+)_\delta$  converges, up to a sub-sequence, as  $\delta \rightarrow 0^+$ , in  $\mathring{V}_\beta^1(\Omega)$  to the function

$$\mathfrak{s}_{j,k,0}^+(r\omega) = \chi(r)r^{\lambda_j^+} \varphi_k^j(\omega)$$

where  $\Lambda_{-1/2} \ni \lambda_j^+ = \lim_{\delta \rightarrow 0^+} \lambda_{j,\delta}^+$  is such that  $\iota_g(\lambda_j^+) = \iota_g(\lambda_{j,\delta}^+)$  and  $\varphi_k^j \in \text{ker}(\mathcal{L}_\sigma(\lambda_j^+))$ . Furthermore, we have

$$\text{span}\{\varphi_k^j, k = 1, \dots, \iota_g(\lambda_j^+)\} = \text{ker}(\mathcal{L}_\sigma(\lambda_j^+)).$$

In addition to that the sequence of functions  $\text{div}((\sigma + i\delta)\nabla \mathfrak{s}_{j,k,\delta}^+)$  converges, up to a sub-sequence, as  $\delta \rightarrow 0^+$  in  $(\mathring{V}_\beta^1(\Omega))^*$  to  $\text{div}(\sigma \nabla \mathfrak{s}_{j,k,0}^+)$ .

<sup>8</sup>i.e. if  $\iota_a(\lambda) = \iota_g(\lambda) = 1$ .

**Proof.** The fact that  $(\lambda_{j,\delta}^+)_{\delta}$  converges to some  $\lambda_j^+ \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$  is guaranteed by Lemma 2.6.8. The fact that  $\iota_g(\lambda_j^+) = \iota_g(\lambda_{j,\delta}^+)$  follows from the application of Lemma 2.6.9. The convergence, up to a sub-sequence, of  $(\varphi_{k,\delta}^j)_{\delta}$  as  $\delta \rightarrow 0^+$  in  $H^1(\mathbb{S}^2)$  to an element of  $\varphi_k^j \in \ker(\mathcal{L}_\sigma(\lambda_j^+))$  is guaranteed by Proposition 3.3.4. Thanks to the fact that  $(\varphi_{k,\delta}^j)_{k=1,\dots,\iota(\lambda_{j,\delta}^+)}$  are orthogonal allows to say that  $(\varphi_k^j)_{k=1,\dots,\iota(\lambda_j^+)}$  are linearly independent. This means that  $(\varphi_k^j)_{k=1,\dots,\iota(\lambda_j^+)}$  is a basis of  $\ker(\mathcal{L}_\sigma(\lambda_j^+))$ .

The convergence of  $(\mathfrak{s}_{j,k,\delta}^+)_{\delta}$  to  $\mathfrak{s}_{j,k,0}^+$  in  $\mathring{V}_\beta^1(\Omega)$  follows from the application of the dominated convergence theorem. By observing that for all  $v \in \mathring{V}_\beta^1(\Omega)$  we have

$$\left| \int_{\Omega} (\operatorname{div}((\sigma + i\delta)\nabla \mathfrak{s}_{j,k,\delta}^+) - \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,0}^+)) \bar{v} \right| = \left| \int_{\Omega \setminus \{r \mid \chi(r)=1\}} (\operatorname{div}((\sigma + i\delta)\nabla \mathfrak{s}_{j,k,\delta}^+) - \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,0}^+)) \bar{v} \right|$$

We infer that we have the estimate

$$\|\operatorname{div}((\sigma + i\delta)\nabla \mathfrak{s}_{j,k,\delta}^+) - \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,0}^+)\|_{(\mathring{V}_\beta^1(\Omega))^*} \leq C \|\operatorname{div}((\sigma + i\delta)\nabla \mathfrak{s}_{j,k,\delta}^+) - \operatorname{div}(\sigma \nabla \mathfrak{s}_{j,k,0}^+)\|_{L^2(\Omega)}$$

with  $C$  independent of  $\delta$ . The result follows, again, from application of dominated convergence theorem.  $\blacksquare$

In the sequel, when Assumption 2.6.2 is satisfied, we denote by  $\Lambda_{-1/2}^+ := \{\lambda_j^+, j = 1, \dots, N_t/2\}$ . Furthermore, we define the space  $\mathcal{S}_0^+ := \operatorname{span}\{\mathfrak{s}_{j,k,0}^+, j = 1, \dots, N_\sigma\}$ . Thanks to the result of the previous lemma, we can say that  $\dim(\mathcal{S}_0^+) = N_\sigma$ . To simplify notations, for all  $\lambda \in \Lambda_{-1/2}$ , we denote by  $S(\lambda)$  the space

$$S(\lambda) = \chi(r)r^\lambda \ker(\mathcal{L}_\sigma(\lambda)) = \{\mathfrak{s}(r\omega) = \chi(r)r^\lambda \varphi(\omega) \quad \text{with} \quad \varphi \in \operatorname{Ker}(\mathcal{L}_\sigma(\lambda))\}.$$

Without any difficulty, one can see that  $\mathcal{S}_0^+ = \bigoplus_{\lambda \in \Lambda_{-1/2}^+} S(\lambda)$ . In the remaining part of this paragraph,

we are going to explain how to find a simple characterization of the space  $\mathcal{S}_0^+$  (or equivalently the set  $\Lambda_{-1/2}^+$ ). The starting point is the next

**Lemma 2.6.11.** *Assume that Assumption 2.6.2 is valid then the space  $\mathcal{S}_0^+$  is of dimension  $N_\sigma$ . Furthermore, we have*

$$0 \leq \Im m q(u, u) \text{ for all } u \in \mathcal{S}_0^+.$$

**Proof.** The fact that the dimension of  $\mathcal{S}_0^+$  is equal to  $N_\sigma$  follows from its definition and thanks to the previous lemma. Furthermore, we know that for all  $u \in \mathcal{S}_0^+$  there exists a sequence  $(u_\delta)_\delta$  of elements of  $\mathcal{S}_\delta^+$  such that  $(u_\delta)_\delta$  and  $(\operatorname{div}((\sigma + i\delta)\nabla u_\delta))_\delta$  converges, as  $\delta \rightarrow 0^+$ , respectively in  $\mathring{V}_\beta^1(\Omega)$  and in  $(\mathring{V}_\beta^1(\Omega))^*$  to  $u$  and  $\operatorname{div}(\sigma \nabla u)$ . As a result, we deduce that

$$\lim_{\delta \rightarrow 0^+} \int_{\Omega} \operatorname{div}((\sigma + i\delta)\nabla u_\delta) \bar{u}_\delta - \overline{\operatorname{div}((\sigma + i\delta)\nabla u_\delta)} u_\delta = q(u, u).$$

Since  $\mathcal{S}_\delta^+ \subset H_0^1(\Omega)$ , one obtains (thanks to an integration by parts) that  $q(u, u) = \lim_{\delta \rightarrow 0^+} 2i\delta \int_{\Omega} |\nabla u_\delta|^2$ .

This ends the proof.  $\blacksquare$

Thanks to the previous lemma, we can then introduce  $(\mathfrak{s}_{j,0}^+)_{j=1,\dots,N_\sigma}$  a basis of the space  $\mathcal{S}_0^+ : \mathcal{S}_0^+ = \operatorname{span}\{\mathfrak{s}_{j,0}^+, j = 1, \dots, N_\sigma\}$ . The second key result to find a characterization of the space  $\mathcal{S}_0^+$  is the following

**Proposition 2.6.8.** *Assume that Assumption 2.6.2 holds, then for all  $\lambda \in \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  we have two possible situations: either  $0 \leq \Im m(q(u, u))$  for all  $u \in S(\lambda)$  or  $\Im m(q(u, u)) \leq 0$  for all  $u \in S(\lambda)$ .*

**Proof.** By combining the two previous lemmas, we deduce that for all  $\lambda \in \Lambda_{-1/2}^+$ , we have

$$0 \leq \Im m(q(u, u)), \quad u \in S(\lambda).$$

This shows the result for all  $\lambda \in \Lambda_{-1/2}^+$ . Given that  $\Lambda(\mathcal{L}_\sigma)$  is symmetric with respect to  $(-1/2, 0)$  and since  $-\lambda - 1 = \bar{\lambda}$  for all  $\lambda \in \Lambda_{-1/2}$ , we infer that  $\Lambda_{-1/2} = \Lambda_{-1/2}^+ \cup \overline{\Lambda_{-1/2}^+}$ . According to Lemma 2.6.2 we know that  $\text{Ker}(\mathcal{L}_\sigma(\bar{\lambda})) = \overline{\text{Ker}(\mathcal{L}_\sigma(\lambda))}$ . Given that  $q(\bar{u}, \bar{u}) = -q(u, u)$ , for all  $u \in \mathcal{S}$ . Consequently, for all  $\lambda \in \Lambda_{-1/2}^+$  we have  $\Im m(q(u, u)) \leq 0$  for all  $u \in S(\lambda)$ . The lemma is then proved.  $\blacksquare$

Without any difficulty, one can check that for all  $\lambda = -1/2 + i\eta \in \Lambda_{-1/2}$  (i.e.  $\eta \in \mathbb{R}$ ) and  $\varphi \in \text{Ker} \mathcal{L}_\sigma(\lambda)$ , the function  $\mathfrak{s}(r\omega) := \chi(r)r^\lambda \varphi(\omega) \in S(\lambda)$  satisfies the relation:

$$q(\mathfrak{s}, \mathfrak{s}) = 2i\eta \int_{\mathbb{S}^2} \sigma |\varphi|^2 d\omega. \quad (2.27)$$

With this in mind, we can show the following result that gives us a very simple characterization of set  $\Lambda_{-1/2}^+$  and the space  $\mathcal{S}_0^+$ .

**Proposition 2.6.9.** *Assume that Assumption 2.6.2 holds. Let  $\lambda = -1/2 + i\eta \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$  and let  $\varphi$  be an arbitrary eigenfunction of  $\mathcal{L}_\sigma$  associated to  $\lambda$ . Then, we have the equivalence*

$$\lambda \in \Lambda_{-1/2}^+ \text{ if and only if } 0 < \eta \int_{\mathbb{S}^2} \sigma |\varphi|^2.$$

**Proof.** We already know thanks to Lemma 2.6.11 we have  $0 \leq -iq(u, u)$  for all  $u \in \mathcal{S}_0^+$ . This means that  $-iq$  is positive hermitian form on  $\mathcal{S}_0^+ \times \mathcal{S}_0^+$ . By making use of the Cauchy-Schwarz (applied to  $-iq$ ) and using the fact that  $q$  is non-degenerate, we infer that  $0 < -iq(u, u)$  for all  $u \in \mathcal{S}_0^+ \setminus \{0\}$ . This proves the direct implication. The reverse implication follows from the Proposition 2.6.8 and the relation (2.27).  $\blacksquare$

**Lemma 2.6.12.** *There exists  $(\mathfrak{s}_j^+)_{j=1, \dots, N_\sigma}$  a basis of  $\mathcal{S}_0^+$  such that*

$$q(\mathfrak{s}_j^+, \mathfrak{s}_k^+) = i\delta_{j,k}, \text{ for all } j, k = 1, \dots, N_\sigma.$$

**Proof.** We denote by  $q_0$  the symplectic form that is the restriction of the symplectic form  $q$  to the space  $\mathcal{S}_0^+$ . Thanks to Lemma 2.6.11, we know that  $-iq_0$  is hermitian and positive, i.e.  $0 \leq -iq_0(u, u)$  for all  $u \in \mathcal{S}_0^+$ . Given that

$$\mathcal{S}_0^+ = \bigoplus_{\lambda \in \Lambda_{-1/2}^+} S(\lambda).$$

and thanks to the second item of Lemma 2.6.3, we deduce that  $q_0$  is non-degenerate. The wanted result follows then from the application of Sylvester's law of inertia.  $\blacksquare$

For all  $\beta \in (0; \beta_0)$  we introduce the space  $\mathring{V}_{0,\beta}^{\text{out}} := \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}_0^+$  and the operator  $A_{0,\beta}^{\text{out}} : \mathring{V}_{0,\beta}^{\text{out}} \rightarrow (\mathring{V}_{-\beta}^1(\Omega))^*$  such that for all  $u = \tilde{u} + s^+$  with  $\tilde{u} \in \mathring{V}_{-\beta}^1(\Omega)$  and  $s^+ \in \mathcal{S}_0^+$  we have

$$\langle A_{0,\beta}^{\text{out}} u, v \rangle := \int_{\Omega} \sigma \nabla \tilde{u} \cdot \nabla \bar{v} - \int_{\Omega} \text{div}(\sigma \nabla s^+) \bar{v} \quad , \quad v \in \mathring{V}_{-\beta}^1(\Omega).$$

Using the results of the previous section and with the help of Lemma 2.6.12, we obtain the

**Proposition 2.6.10.** *Assume that the Assumptions 2.6.1-2.6.2 are satisfied. Then the operator  $A_{0,\beta}^{\text{out}}$  is an isomorphism.*

### Final proof of the limiting absorption principle

The main result of this section is given by the following

**Theorem 2.6.5.** *Assume that Assumptions 2.6.1-2.6.2 hold and let  $f \in (\mathring{V}_\beta^1(\Omega))^*$  with  $\beta \in (0; \beta_0)$ . Then the sequence  $(u_\delta)_\delta$  ( $u_\delta$  is the solution of (2.25)) converges in  $\mathring{V}_\beta^1(\Omega)$  to  $u \in \mathring{V}_{0,\beta}^{\text{out}}(\Omega) = \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}_0^+$  the unique solution to the well-posed problem  $A_{0,\beta}^{\text{out}}u = f$ .*

The proof of the previous theorem is based on a succession of lemmas. The first one is the

**Lemma 2.6.13.** *Assume that Assumption 2.6.2 holds and that  $f \in (\mathring{V}_\beta^1(\Omega))^*$  then there exists  $\delta_\beta$  such that for all  $0 < \delta < \delta_\beta$  the function  $u_\delta$  (the solution to (2.25)) decomposes as*

$$u_\delta = \sum_{j=1}^{N_\sigma} c_\delta^j \mathfrak{s}_{\delta,j}^+ + \tilde{u}_\delta \quad (2.28)$$

where  $\tilde{u}_\delta \in \mathring{V}_{-\beta}^1(\Omega)$  and  $c_\delta^j \in \mathbb{C}$ .

**Proof.** Thanks to Lemma 2.6.9, we know that there exists  $\delta_\beta$  such that for all  $\delta \in (0; \delta_\beta)$ , we have  $\{\lambda \in \mathbb{C} \mid \Re e(\lambda) \in (-1/2, -1/2 + \beta)\} \cap \Lambda(\mathcal{L}_{\sigma+i\delta}) = \{\lambda_{j,\delta}^+, j = 1, \dots, N_t/2\}$ . Since by Assumption the eigenvalues  $\lambda_{j,\delta}^+$  are semi-simple for  $\delta \in (0; \delta_0)$  ( $\delta_0$  is defined in the statement of Assumption 2.6.2), the result follows then by replacing  $\delta_\beta$  by  $\min(\delta_\beta, \delta_0)$  and by adapting the classical results of [102, Chapter 6]. ■

**Lemma 2.6.14.** *Assume that Assumption 2.6.1 holds. Then for all  $\beta \in (0; \beta_0)$  there exists  $0 < \delta_\beta$  such that for all  $\delta \in (0; \delta_\beta)$ , we have the estimate*

$$\|u\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C_\beta \|\text{div}((\sigma + i\delta)\nabla u)\|_{(\mathring{V}_\beta^1(\Omega))^*} \text{ for all } u \in \mathring{V}_{-\beta}^1(\Omega)$$

in which the constant  $C_\beta$  is independent of  $u$  and of  $\delta$ .

**Proof.** Thanks to the Assumption 2.6.1, we know that for all  $\beta \in (0; \beta_0)$  we have the estimate (see Lemma 2.6.5)

$$\|u\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C_\beta \|\text{div}(\sigma\nabla u)\|_{(\mathring{V}_\beta^1(\Omega))^*} \text{ for all } u \in \mathring{V}_{-\beta}^1(\Omega)$$

where  $0 < C_\beta$  does not depend on  $u$ . By combining the estimate

$$\|\text{div}(\nabla u)\|_{(\mathring{V}_\beta^1(\Omega))^*} \leq C'_\beta \|u\|_{\mathring{V}_{-\beta}^1(\Omega)} \text{ for all } u \in \mathring{V}_{-\beta}^1(\Omega)$$

(in which  $C'_\beta$  is independent of  $u$ ) with the fact that for all  $\delta \in \mathbb{R}$  and all  $u \in \mathring{V}_{-\beta}^1(\Omega)$  we have  $\text{div}(\sigma\nabla u) = \text{div}((\sigma + i\delta)\nabla u) - i\delta\text{div}(\nabla u)$ , we obtain the estimate

$$\|u\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C_\beta \|\text{div}((\sigma + i\delta)\nabla u)\|_{(\mathring{V}_\beta^1(\Omega))^*} + C'_\beta |\delta| \|u\|_{\mathring{V}_{-\beta}^1(\Omega)} \text{ for all } u \in \mathring{V}_{-\beta}^1(\Omega).$$

Taking  $\delta$  small enough (e.g.  $|\delta| < (2C'_\beta)^{-1}$ ), we get the estimate

$$\|\tilde{u}_\delta\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq 2C_\beta \|\text{div}((\sigma + i\delta)\nabla u)\|_{(\mathring{V}_\beta^1(\Omega))^*}$$

which ends the proof. ■

**Lemma 2.6.15.** *Assume that Assumption 2.6.1 holds and let  $\beta \in (0; \beta_0)$ . Let  $(u_\delta)_\delta$  be a sequence of elements of  $\mathring{V}_{-\beta}^1(\Omega)$  such that  $(f_\delta := \text{div}((\sigma + i\delta)\nabla u_\delta))_\delta$  converges, as  $\delta \rightarrow 0^+$ , in  $(\mathring{V}_\beta^1(\Omega))^*$  then  $(u_\delta)_\delta$  converges in  $\mathring{V}_{-\beta}^1(\Omega)$  as  $\delta \rightarrow 0^+$ .*



**Proof.** Thanks to the previous lemma, we infer that  $(u_\delta)_\delta$  is bounded in  $\mathring{V}_{-\beta}^1(\Omega)$ . In order to prove our claim we are going to show that  $(u_\delta)_\delta$  is a Cauchy sequence. Let  $\delta, \delta' \in \mathbb{R}_+^*$ . Starting from the identity

$$f_\delta - f_{\delta'} = \operatorname{div}((\sigma + i\delta)\nabla(u_\delta - u_{\delta'})) + i(\delta - \delta')\operatorname{div}\nabla u_{\delta'}$$

and by using the estimate  $\|\operatorname{div}(\nabla u)\|_{(\mathring{V}_{-\beta}^1(\Omega))^*} \leq C\|u\|_{\mathring{V}_{-\beta}^1(\Omega)}$  for all  $u \in \mathring{V}_{-\beta}^1(\Omega)$  (with  $C$  independent of  $u$ ) we obtain (thanks to Lemma 2.6.14) the estimate

$$\|u_\delta - u_{\delta'}\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C(\|f_\delta - f_{\delta'}\|_{(\mathring{V}_{-\beta}^1(\Omega))^*} + |\delta - \delta'|)$$

with  $C$  that does not depend on  $\delta$ . Since by assumption  $(f_\delta)_\delta$  converges in  $(\mathring{V}_{-\beta}^1(\Omega))^*$  its then a Cauchy sequence and then the result is proved.  $\blacksquare$

As a consequence, we can now show the following result.

**Lemma 2.6.16.** *Assume that Assumptions 2.6.1-2.6.2 hold and let  $\beta \in (0; \beta_0)$ . Then the sequences  $(c_\delta^j)_\delta$  in (2.28) are bounded as  $\delta$  tends to 0.*

**Proof.** For all  $\delta$  small enough, we denote by  $R_\delta = \max_j |c_\delta^j|$ . To prove our claim it suffices to show that  $(R_\delta)_\delta$  is bounded as  $\delta$  vanishes. If this not the case, one can say that there exists a sub-sequence of  $(R_\delta)_\delta$ , that will be indexed by  $\delta$  for the reader convenience, such that  $|R_\delta| \rightarrow +\infty$  as  $\delta \rightarrow 0$ . To simplify notations, we introduce for all  $j = 1, \dots, N_\sigma$  the sequences  $(\hat{c}_\delta^j := c_\delta^j/R_\delta)_\delta$ . Note that from the definition of  $R_\delta$ , we infer that

$$\max_j |\hat{c}_\delta^j| = 1. \tag{2.29}$$

This implies that  $(\hat{c}_\delta^1, \dots, \hat{c}_\delta^{N_\sigma})_\delta$  is bounded in  $\mathbb{C}^{N_\sigma}$ . As a result, we deduce that up to a sub-sequence, still indexed by  $\delta$ , the sequence  $(\tilde{c}_\delta^1, \dots, \tilde{c}_\delta^{N_\sigma})_\delta$  converges to some  $(\hat{c}^1, \dots, \hat{c}^{N_\sigma})$  in  $\mathbb{C}^{N_\sigma}$ . Note that thanks to (2.29), we deduce  $\max_j |\hat{c}_j| = 1$ . By observing that  $\hat{c}_1^\delta \mathfrak{s}_{\delta,1}^+ + \dots + \hat{c}_{N_\sigma}^\delta \mathfrak{s}_{\delta,N_\sigma}^+$  converges, as  $\delta \rightarrow 0^+$  to  $\hat{c}_1 \mathfrak{s}_{0,1}^+ + \dots + \hat{c}_{N_\sigma} \mathfrak{s}_{0,N_\sigma}^+$ , by using the fact that  $-\operatorname{div}((\sigma + i\delta)\nabla u^\delta/R_\delta) = f/R_\delta \rightarrow 0$  in  $(\mathring{V}_{-\beta}^1(\Omega))^*$  and the result of Lemma 2.6.5, we deduce that  $(\operatorname{div}((\sigma + i\delta)\nabla \tilde{u}_\delta/R_\delta))_\delta$  converges in  $(\mathring{V}_{-\beta}^1(\Omega))^*$  as  $\delta \rightarrow 0^+$ .

Since  $\tilde{u}_\delta/R_\delta \in \mathring{V}_{-\beta}^1(\Omega)$  for all  $\delta \in (0; \delta_\beta)$  and by applying Lemma 2.6.15 we conclude that  $\tilde{u}_\delta/R_\delta$  converges in  $\mathring{V}_{-\beta}^1(\Omega)$ , as  $\delta \rightarrow 0$ , to some  $\tilde{u}_0 \in \mathring{V}_{-\beta}^1(\Omega)$ . Consequently, the function  $u = \tilde{u}_0 + \hat{c}^1 \mathfrak{s}_{0,1}^+ + \dots + \hat{c}^{N_\sigma} \mathfrak{s}_{0,N_\sigma}^+ \in \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}_0^+$  and satisfies the equation

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ in } (\mathring{V}_{-\beta}^1(\Omega))^*.$$

Applying the Proposition 2.6.10, we find that  $\hat{c}^j = 0$  for all  $j = 1, \dots, N_\sigma$  which contradicts the fact that  $\max_j |\hat{c}_j| = 1$ . The Lemma is then proved.  $\blacksquare$

*Proof of Theorem 2.6.5.* We know that for  $0 < \delta$  small enough, the function  $u_\delta$  decomposes as

$$u_\delta = \tilde{u}_\delta + \sum_{j=1}^{N_\sigma} c_\delta^j \mathfrak{s}_{\delta,j}^+ \text{ with } c_\delta^j \in \mathbb{C} \text{ and } \tilde{u}_\delta \in \mathring{V}_{-\beta}^1(\Omega).$$

The previous lemma ensures that  $(c_\delta^1, \dots, c_\delta^{N_\sigma})$  is bounded in  $\mathbb{C}^{N_\sigma}$ . This means that up to a sub-sequence (that will be indexed by  $\delta$ ),  $(c_\delta^1, \dots, c_\delta^{N_\sigma})$  converges as  $\delta \rightarrow 0$  in  $\mathbb{C}^{N_\sigma}$  to some  $(c_1, \dots, c_{N_\sigma}) \in \mathbb{C}^{N_\sigma}$ . Starting from the fact that

$$\operatorname{div}((\sigma + i\delta)\nabla \tilde{u}_\delta) = -f - \operatorname{div}((\sigma + i\delta)\nabla(\sum_{j=1}^{N_\sigma} c_\delta^j \mathfrak{s}_{\delta,j}^+))$$



and by using Lemma 2.6.10 and Lemma 2.6.15, we deduce that  $(u_\delta)_\delta$  converges in  $\mathring{V}_\beta^1(\Omega)$  to some  $u \in \mathring{V}_{0,\beta}^{\text{out}}(\Omega) = \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}_0^+$  that satisfies the equation

$$-\text{div}(\sigma \nabla u) = f \text{ in } (\mathring{V}_\beta^1(\Omega))^*.$$

Thanks to Proposition 2.6.10, we know that the latter problem has a unique solution. This implies that  $(u_\delta)_\delta$  converges in  $\mathring{V}_\beta^1(\Omega)$ , as  $\delta \rightarrow 0$ , to the unique solution of  $A_{0,\beta}^{\text{out}} u = f$ . Since this limit is independent of the chosen sub-sequence, we obtain the wanted result.  $\blacksquare$

### On the relaxation of Assumption 2.6.2

The results obtained in the previous section are also valid if one replaces Assumption 2.6.2 by the following

**Assumption 2.6.3.** *We suppose that the function  $\sigma$  is such that  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$  and such that*

- *All the eigenvalues of  $\mathcal{L}_\sigma$  that are located on the energy line  $\Re(\lambda) = -1/2$  are semi-simple<sup>9</sup>. We denote them by  $\lambda_1, \dots, \lambda_{N_t}$ .*
- *There exists  $0 < \delta_0$  and  $0 < r_0$  such for all  $0 < \delta < \delta_0$  and all  $j = 1, \dots, N_t$ , the set  $B(\lambda_j, r_0) \cap \Lambda(\mathcal{L}_{\sigma+i\delta})$  is either a subset of  $\{\lambda \in \mathbb{C} \mid -1/2 < \Re(\lambda)\}$  or a subset  $\{\lambda \in \mathbb{C} \mid \Re(\lambda) < -1/2\}$ . We use the notation  $B(\lambda_j, r_0) \cap \Lambda(\mathcal{L}_{\sigma+i\delta}) = \{\lambda_{j,k,\delta}, k = 1, \dots, N_j\}$  with  $N_j \in \mathbb{N}$ .*
- *All the  $\lambda_{j,k,\delta}$  ( $j = 1, \dots, N_t, k = 1, \dots, N_j$ ) are semi-simple.*

The only point that needs to be clarified is the proof of the fact that, under the previous assumption, the dimension of the space  $\mathcal{S}_0^+$  is equal to  $N_t/2$ . To do this, we have to modify a little the proof of Proposition 2.6.10. Instead of performing a Gram-Schmidt process on  $\mathcal{S}_\delta^+$  with respect to their angular component in  $H^1(\mathbb{S}^2)$  (which was the case in the proof of Proposition 2.6.10), one must perform a Gram-Schmidt process on  $\mathcal{S}_\delta^+$  with respect to  $\mathring{V}_\beta^1(\Omega)$  with  $0 < \beta$  (which is a Hilbert space).

Unfortunately, we are not able to find a weaker assumption under which we can explain how to choose, among the functional frameworks constructed by Mandelstam's radiation principle, the one that is consistent with the limiting absorption principle.

The difficulty comes from the fact that, in general, any assumption made on the nature of the eigenvalues of  $\mathcal{L}_\sigma$  which belong to  $\ell_{-1/2}$  does not imply, a priori, any information on the nature of the eigenvalue of  $\mathcal{L}_{\sigma+i\delta}$  which are near  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$ . Note that this difficulty occurs even in the case of finite dimensional problems. To be convinced of this, consider for all  $0 < \delta$  the matrix

$$A_\delta := \begin{pmatrix} 1 + \delta & \delta \\ 0 & 1 + \delta \end{pmatrix}.$$

We can clearly see that  $A_\delta$  tends as  $\delta \rightarrow 0^+$  to the identity matrix  $I_2$ . Moreover, the spectrum of  $A_\delta$  is equal to  $\{1 + \delta\}$  which converges as expected to  $\{1\}$  which is the spectrum of  $I_2$ . However, when we come to the question of the convergence of the eigenfunctions, the situation is totally different: while 1 is a semi-simple eigenvalue of  $I_2$ , for all  $0 < \delta$  the matrix  $A_\delta$  has an generalized eigenfunction associated to  $1 + \delta$ .

### Application to the case of circular conical tips

In §3.4.1 we shall prove that, when  $\kappa_\sigma \neq -1$ , the set  $\Lambda(\mathcal{L}_\sigma)$  can be characterized by means of dispersion relations. Moreover, we will explain that  $\Lambda_{-1/2}$  coincides with

$$\{-1/2 \pm i\tau \text{ s.t. } \exists m \in \mathbb{N} \text{ s.t. } a_m(\tau) = \kappa_\sigma\}$$

where  $a_m : \mathbb{R}_+ \rightarrow \mathbb{R}$  are continuous functions. The curves of the functions  $a_m$  for  $m = 0, \dots, 3$  are displayed in Figure 2.4.

<sup>9</sup>We say that an eigenvalue of  $\mathcal{L}_\sigma$  is semi-simple if  $\iota_a(\lambda) = \iota_g(\lambda)$ .

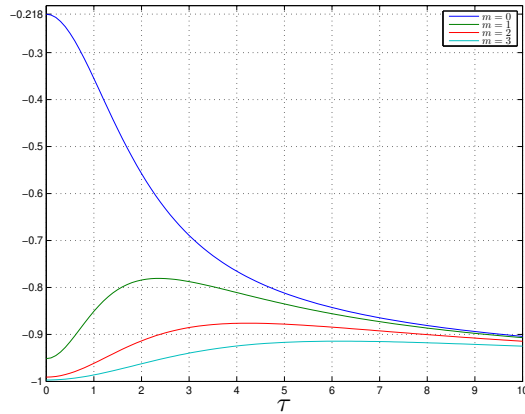


Figure 2.4: Curves of the functions  $\tau \mapsto a_m(\tau)$  for  $m = 0, 1, 2, 3$  and  $\alpha = \pi/4$ .

Clearly the curves of the functions seems to be disjoint. However, we did not succeed in proving this observation. In §3.4.3, we will show that except for the particular values of  $\lambda = -1 + i\tau^*$  where  $a'_m(\tau^*) = 0$  for some  $m \in \mathbb{N}$ , the assumption 2.6.2 is valid. Furthermore we are going to show that for these particular values of  $\lambda$ , propagating singularities with logarithmic growth exist.

**On the existence of inverse modes and the numerical approximation of the problem**

In this paragraph, we will discuss in very brief way the question of the numerical approximation of the scalar problem. Clearly, one has to distinguish two situations: the case  $\kappa_\sigma \notin I_\Sigma$  and the case  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$ . In the first case the approximation of the solution can be done thanks to the numerical method that we are going to present in Chapter 4. In the case  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$ , propagating singularities exist. To the best of our knowledge the only existing method to deal with the problem in 2D has been proposed in [45] and is based on the use of PMLs near the origin. The adaption of this method to the 3D configuration is not done yet. This adaptation does not seem to be an easy task because of the possible existence of inverse modes in the expression of the physical solution of the problem (i.e the solution obtained by the limiting absorption principle contains propagating singularities which are associated with singular exponents with opposite signs). This is exactly the case illustrated by Figure 2.5: we observe that in this situation Assumption 2.6.2 is valid and that the space of the physical propagating singularities contains propagating singularities with singular exponents that have opposite sign.

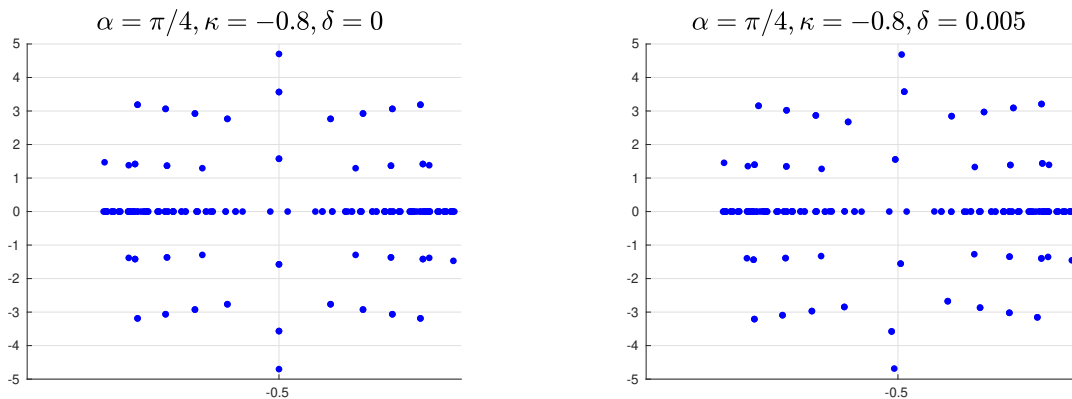


Figure 2.5: The spectrum of  $\mathcal{L}_{\sigma+i\delta}$  for  $\delta = 0, \delta = 0.005$  for the case of a circular conical tip ( $\alpha = \pi/4$ ) and  $\kappa_\sigma = -0.8$ .

## 2.7 Concluding remarks and open questions

In this chapter, we presented a detailed study of the scalar problem (2.2). In particular, we explained how to characterize the critical interval  $I_\Sigma$  by means of the existence of propagating singularities. When  $\kappa_\sigma \in I_\Sigma \setminus \{-1\}$ , a general approach based on the use of Mandelstam’s radiation principle has been proposed in order to construct functional frameworks in which Fredholmness of the problem is recovered (even in the presence of propagating singularities with logarithmic growth near the origin which has not been treated in [25]). The selection of the physical framework has been done, under Assumption 2.6.2 (or Assumption 2.6.3), by means of the limiting absorption principle. It seems (thanks to numerical calculations) that Assumption 2.6.2 is satisfied for the case of circular conical tips, except for a discrete set of contrasts for which there are propagating singularities with logarithmic growth near the origin. Of course, all the results we obtained above hold if we replace the homogeneous Dirichlet boundary conditions by any other elliptic boundary condition. In addition to that, we expect that our results remain true when the conical tip touches the domain boundary (see Figure 2.6). Let us conclude this chapter by mentioning two of the most important questions that can be studied in future works:

1. How to select the physical framework when Assumption 2.6.3 is not satisfied? In the literature, it seems that the most important reference, which can help us to deal with this question, is the book [138].
2. How to adapt the use of PMLs near the origin in order to construct a numerical approximation of the solution to the scalar problem with propagating singularities? How to deal with the possible existence of inverse modes? An interesting work that can help us in this direction is done in [13].

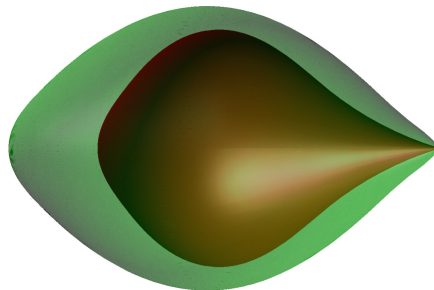


Figure 2.6: An example of a geometry where the conical tip touches the boundary of the domain.

## 2.8 Appendix

### 2.8.1 The Kelvin transform

The Kelvin transform is a classical geometrical mapping that permits us to transform problems set in unbounded domains into other ones set in bounded domains and vice versa. As we shall see below (Lemma 2.8.1), the Kelvin transform preserves harmonic functions. This property makes it very adapted to the study of “Laplacian-based” problems. It is also interesting to note that the Kelvin transform can be used for numerical purposes as an alternative approach to solve scattering problems (see [69, 111] and the references therein). Along this paragraph, we denote by  $B$  the unit ball of  $\mathbb{R}^3$ . The Kelvin transform of a function  $u$  defined in  $B \setminus \{O\}$  is the function  $\tilde{u}$  defined in  $B^c := \mathbb{R}^3 \setminus \bar{B}$  by the relation:

$$\tilde{u}(r\omega) = u(\omega/r)/r$$

in which  $(r, \omega)$  are the classical spherical coordinates. The first interesting property is the fact that the Kelvin transformation of a harmonic function is also harmonic.

**Lemma 2.8.1.** *Let  $u \in \mathcal{C}^2(\mathbb{B} \setminus \{O\})$  be such that  $\Delta u = 0$ . Then we have  $\Delta \tilde{u} = 0$  in  $\mathbb{B}^c$ .*

**Proof.** A direct calculus yields  $\Delta \tilde{u}(r\omega) = \frac{1}{r^5} \Delta u(\omega/r)$  for all  $r\omega \in \mathbb{B}^c$ . ■

Now, we turn our attention to the study of the action of the Kelvin transform on weighted Sobolev spaces. We limit ourselves to the spaces  $V_\beta^0(\mathbb{B})$  and  $V_\beta^1(\mathbb{B})$  for arbitrary  $\beta \in \mathbb{R}$ . The case of the spaces  $V_\beta^0(\mathbb{B})$  is the subject of the following

**Lemma 2.8.2.** *If  $u \in V_\beta^0(\mathbb{B})$  then  $\tilde{u} \in V_{-\beta-2}^0(\mathbb{B}^c)$ .*

**Proof.** By definition of  $V_\beta^0(\mathbb{B})$ , we have  $\int_0^1 \int_{\mathbb{S}^2} r^{2\beta} u^2(r\omega) r^2 dr d\omega < \infty$ . By performing the change of variables  $r \mapsto 1/r$ , we get

$$\int_1^\infty \int_{\mathbb{S}^2} r^{-2\beta} u^2(\omega/r) r^{-4} dr d\omega = \int_1^\infty \int_{\mathbb{S}^2} r^{-2\beta-4} \tilde{u}^2(r\omega) r^2 dr d\omega < \infty.$$

As a result  $r^{-\beta-2} \tilde{u}$  belongs to  $L^2(\mathbb{B}^c)$  and then the lemma is proved. ■

The case of the spaces  $V_\beta^1(\mathbb{B})$  is treated in the following

**Lemma 2.8.3.** *If  $u \in V_\beta^1(\mathbb{B})$  then  $\tilde{u} \in V_{-\beta}^1(\mathbb{B}^c)$ .*

**Proof.** Since  $u \in V_\beta^1(\mathbb{B})$ , we deduce that  $u \in V_{\beta-1}^0(\mathbb{B})$  and then by using the result of the previous lemma we can say that  $\tilde{u} \in V_{-\beta-1}^0(\mathbb{B}^c)$ . To make things as clear as possible, instead of working with the variable  $r$  for the function  $\tilde{u}$ , we use the variable  $t = 1/r$ . With this in mind, we have the relation  $t\tilde{u}(t\omega) = u(r\omega)$  for all  $r \in (0; 1)$ . To end the proof, we need to show that  $t\omega \mapsto \partial_t \tilde{u}(t\omega)$  and  $t\omega \mapsto |\nabla_S \tilde{u}(t\omega)/t|$  belong to  $V_{-\beta}^0(\mathbb{B}^c)$ . It is important to note that using the variable  $t$  instead of  $r$ , the space  $V_{-\beta}^0(\mathbb{B}^c)$  is defined as follows

$$V_{-\beta}^0(\mathbb{B}^c) = \{f : \mathbb{B}^c \rightarrow \mathbb{C} \text{ such that } \int_1^\infty \int_{\mathbb{S}^2} t^{-2\beta} f(t\omega)^2 t^2 dt d\omega < \infty\}.$$

For the case of the function  $t\omega \mapsto |\nabla_S \tilde{u}(t, \cdot)/t|$ , this follows from the equality (that is obtained thanks to the change of variable  $r \mapsto 1/r$ )

$$\int_0^1 \int_{\mathbb{S}^2} r^{2\beta} |\nabla_S u(r\omega)/r|^2 r^2 dr d\omega = \int_1^\infty \int_{\mathbb{S}^2} t^{-2\beta} |\nabla_S \tilde{u}(t\omega)/t|^2 t^2 dt d\omega.$$

The case of the function  $\partial_t \tilde{u}$  is a little bit more involved. The starting point is to observe that we have

$$t\tilde{u}(t\omega) = u(r\omega) \implies \partial_r u(r\omega) = -t^2 \tilde{u}(t\omega) - t^3 \partial_t \tilde{u}(t\omega).$$

Thus we can write that  $t^3 \partial_t \tilde{u}(t\omega) = \partial_r u(r\omega) + u(r\omega)/r$ . Using the fact that  $r\omega \mapsto \partial_r u(r\omega) + u(r\omega)/r$  belongs to the space  $V_\beta^0(\mathbb{B})$ , we then deduce that

$$\int_1^\infty \int_{\mathbb{S}^2} t^{-2\beta} (\partial_t \tilde{u}(t, \omega))^2 t^2 dt d\omega = \int_0^1 \int_{\mathbb{S}^2} r^{2\beta} (\partial_r u(r\omega) + u(r\omega)/r)^2 r^2 dr d\omega < \infty.$$

This ends the proof. ■

### 2.8.2 The Peetre's Lemma

In this paragraph, we present some variants of the classical Peetre's lemma. These results are a very powerful tools that allow us to prove that a given operator is of Fredholm type. The classical Peetre's lemma is given by the following

**Theorem 2.8.1.** [101, Lemma 3.4.1] *Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  and  $(Z, \|\cdot\|_Z)$  be three Banach spaces such that  $X$  is compactly embedded in  $Z$ . Let  $A : X \rightarrow Y$  be a continuous linear operator. Then the following assertions are equivalent*

1.  $A$  has a closed range and its kernel is finite dimensional.
2. The estimate

$$\|u\|_X \leq C(\|A(u)\|_Y + \|u\|_Z), \quad u \in X$$

holds with  $C$  independent of  $u$ .

In some configurations, we may need to use the following alternative version of the Peetre's Lemma.

**Proposition 2.8.1.** [124] *Let  $(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  and  $(Z, \|\cdot\|_Z)$  be three Banach spaces and let  $K : X \rightarrow Z$  be a compact operator. If there exists  $0 < C$  such that we have the estimate*

$$\|u\|_X \leq C(\|A(u)\|_Y + \|K(u)\|_Z), \quad u \in X$$

then  $A$  has a closed range and its kernel is finite dimensional.

For any Banach space  $X$ , we denote by  $X^*$  its topological anti-dual. An operator  $A : X \rightarrow X^*$  is said to be symmetric if and only if  $\langle Au, v \rangle = \langle Av, u \rangle$  for all  $u, v \in X$ . A direct application of the Theorem 2.8.1 yields

**Proposition 2.8.2.** *Let  $(X, \|\cdot\|_X)$  and  $(Z, \|\cdot\|_Z)$  be two Banach spaces such that  $X$  is compactly embedded in  $Z$ . Let  $A : X \rightarrow X^*$  be a continuous linear symmetric operator. Then the following assertions are equivalent*

1.  $A$  is a Fredholm operator of index zero.
2. The estimate

$$\|u\|_X \leq C(\|A(u)\|_{X^*} + \|u\|_Z), \quad u \in X$$

holds with  $C$  independent of  $u$ .

# Chapter 3

## The study of the Mellin symbol of the problem

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### 3.1 Introduction

This chapter is devoted to the study of the "spectral" properties of the Mellin symbol generated by the scalar problem (2.2) that we have studied in the previous chapter. More precisely, we are interested in the study of spectral properties of the family of operators  $(\mathcal{L}_\sigma(\lambda))_{\lambda \in \mathbb{C}}$  that is defined as follows: for all  $\lambda \in \mathbb{C}$ , we introduce  $\mathcal{L}_\sigma(\lambda) : H^1(\mathbb{S}^2) \rightarrow (H^1(\mathbb{S}^2))^*$  such that for all  $\psi, \psi' \in H^1(\mathbb{S}^2)$  we have

$$\langle \mathcal{L}_\sigma(\lambda)\psi, \psi' \rangle := \int_{\mathbb{S}^2} \sigma \nabla_S \psi \cdot \nabla_S \overline{\psi'} d\omega - \lambda(\lambda + 1) \int_{\mathbb{S}^2} \sigma \psi \overline{\psi'} d\omega.$$

Above  $d\omega = \sin(\theta)d\theta d\varphi$  where  $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$  are the classical (angular) spherical coordinates. Recall that  $\sigma$  is a piecewise constant function such that  $\sigma = \sigma_1 \in \mathbb{R}_+^*$  in  $S_1$  and  $\sigma = \sigma_2 \in \mathbb{R}_-^*$  in which  $S_1$  and  $S_2$  are two subdomains of  $\mathbb{S}^2$  that are defined as follows:

$$S_1 = \{(\theta, \varphi) \in [0, \pi] \times [0, 2\pi] \text{ such that } g(\varphi) < \theta\} \text{ and } S_2 = \{(\theta, \varphi) \in [0, \pi] \times [0, 2\pi] \text{ such that } \theta < g(\varphi)\}$$

where  $g : [0, 2\pi] \rightarrow [0, \pi]$  is a periodic function of class  $\mathcal{C}^2$  (see Figure 3.1). As in the previous chapter, we denote  $\kappa_\sigma := \sigma_2/\sigma_1$ . Observe that the particular case where  $g$  coincides with a constant function corresponds to the case of circular conical tips.

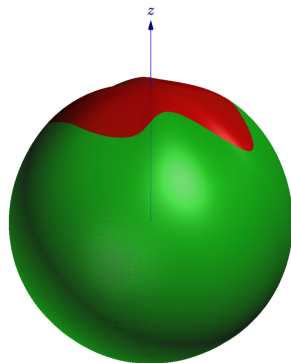


Figure 3.1: An example of the geometry considered: the red (resp. green ) part is filled with a negative (resp. positive) material.

Classically, we say that  $\lambda \in \mathbb{C}$  is a regular point of  $\mathcal{L}_\sigma$  if and only if the operator  $\mathcal{L}_\sigma(\lambda)$  is invertible otherwise we say that  $\lambda$  is an eigenvalue of  $\mathcal{L}_\sigma$ . The set of eigenvalues of  $\mathcal{L}_\sigma$  is called the spectrum of  $\mathcal{L}_\sigma$  and is denoted by  $\Lambda(\mathcal{L}_\sigma)$ . As we have seen in §2.4, having an accurate information about the location of the spectrum  $\Lambda(\mathcal{L}_\sigma)$  in the complex plane is important for the study of the well-posedness of the problems:

$$\text{Find } u \in W_\beta^1(\mathbb{R}^3) \text{ such that } -\operatorname{div}(\sigma \nabla u) = f \in (W_{-\beta}^1(\mathbb{R}^3))^*$$

for  $\beta \in \mathbb{R}$ . More precisely, the formula (2.14) tells us that the solvability of the previous problem is directly related to the invertibility of  $\mathcal{L}_\sigma(\lambda)$  along the energy line  $\Re e(\lambda) = -\beta - 1/2$  and on the behavior of  $\mathcal{L}_\sigma^{-1}(\lambda)$  on this line. In addition to that, we have also seen that to obtain an asymptotic expansion of its solution, near the origin, one needs to have a precise information about the associated eigenfunctions/generalized eigenfunctions and the algebraic multiplicities of its eigenvalues (see §2.4.4 for the definition of these objects).

Because of the sign-change in the density function  $\sigma$ , the study of the spectral properties of  $\mathcal{L}_\sigma$  does not fit into the general theory presented in [101] that concerns the study of the spectral properties of the Mellin symbols generated by strongly elliptic operators. Our goal is to show that, even in our situation, some of the well-known results of the classical theory of Fredholm operator pencils can be recovered. Note that, to the best of our knowledge, the results that we shall present below are new. In some way, these results can be seen as an extension of the ones presented in [25] for the case of two dimensional transmission problem with sign-changing coefficients.

The results of this chapter are organized as follows. In §3.2, we address the question of the discreteness of the spectrum of  $\mathcal{L}_\sigma$  and the behavior of its resolvent (i.e.  $\lambda \mapsto \mathcal{L}_\sigma(\lambda)^{-1}$ ) for large values of  $|\lambda|$ . Next, in §3.3, we turn our attention to the study of the behavior of the spectrum and the associated eigenvectors when one replaces  $\sigma$  by  $\sigma + i\delta$  where  $\delta$  is a small parameter. In the last section (§3.4), thanks to some explicit computations, we explain how the general results, obtained in the previous two sections, can be made more precise in the particular case of circular conical tips.



## 3.2 Fredholmness of the symbol and discreteness of the spectrum

In the classical configuration (when  $\sigma$  has a constant sign), one can prove (for instance see the proof of [101, Lemma 3.6.3]), by means of the analytic Fredholm theorem (see [101, Corollary 1.1.1]), that the spectrum of  $\mathcal{L}_\sigma(\lambda)$  is discrete and consists of isolated eigenvalues with finite algebraic multiplicities. In our configuration, because of the sign-change in  $\sigma$ , the operator  $\mathcal{L}_\sigma(\lambda)$  is not necessarily of Fredholm type. This means that  $\Lambda(\mathcal{L}_\sigma)$  may contain some eigenvalues of infinite algebraic multiplicity or even worse than that,  $\Lambda(\mathcal{L}_\sigma)$  can be not discrete (or possibly equal to the complex plane).

### 3.2.1 Fredholmness of the symbol

As in the classical configuration the first step is to endow the space  $H^1(\mathbb{S}^2)$  with the norm  $\|u\|_{H^1(\mathbb{S}^2, |\lambda|)}^2$  such that for all  $u \in H^1(\mathbb{S}^2)$  we have

$$\|u\|_{H^1(\mathbb{S}^2, |\lambda|)}^2 = \|u\|_{H^1(\mathbb{S}^2)}^2 + |\lambda|^2 \|u\|_{L^2(\mathbb{S}^2)}^2.$$

Obviously, one can say that for all  $\lambda \in \mathbb{C}$ , the norm  $\|u\|_{H^1(\mathbb{S}^2, |\lambda|)}$  is equivalent to the classical one (which, by the way, coincides with  $\|u\|_{H^1(\mathbb{S}^2, 1)}$ ). However, when  $|\lambda|$  goes to  $+\infty$  these two norms have two different behaviours. Note that the introduction of this norm is motivated by the expression of the inverse Mellin transform (2.4.3). We also endow the space  $(H^1(\mathbb{S}^2), |\lambda|)^*$  with the norm  $\|\cdot\|_{(H^1(\mathbb{S}^2, |\lambda|))^*}$  such that for all  $f \in (H^1(\mathbb{S}^2), |\lambda|)^*$  we set

$$\|f\|_{(H^1(\mathbb{S}^2, |\lambda|))^*} = \sup_{v \in H^1(\mathbb{S}^2) \setminus \{0\}} \frac{|\langle f, v \rangle|}{\|v\|_{H^1(\mathbb{S}^2, |\lambda|)}}.$$

As mentioned above, because of the sign-change in  $\sigma$ , the Fredholmness of  $\mathcal{L}_\sigma(\lambda)$  may be lost. In this paragraph, we shall explain how to use the  $\Gamma$ -coercivity approach in order to prove, under some condition on the contrast  $\kappa_\sigma$ , the Fredholmness of  $\mathcal{L}_\sigma(\lambda)$ . We have the

**Lemma 3.2.1.** *Assume that  $\kappa_\sigma \neq -1$ , then there exists  $t_0 \in \mathbb{R}^+$  such that for all  $t \in \mathbb{R}$  such that  $t_0 < |t|$  the operator  $\mathcal{L}_\sigma(-1/2 + it)$  is an isomorphism. More precisely, there exists  $0 < C$  such that for all  $t_0 < |t|$  and  $\lambda = -1/2 + it$  we have the estimate*

$$\|u\|_{H^1(\mathbb{S}^2, |\lambda|)} \leq C \|\mathcal{L}_\sigma(\lambda)(u)\|_{(H^1(\mathbb{S}^2, |\lambda|))^*} \text{ for all } u \in H^1(\mathbb{S}^2).$$

**Remark 3.2.1.** *The proof of the previous result is a little bit technical. For pedagogical purposes, we will limit ourselves here to the study of the particular case of a circular conical tip (i.e.  $\theta = g(\varphi) = \alpha$ ) and the study of the general case ( $g \in \mathcal{C}^2[0; 2\pi]$ ) will be left as an appendix (see Appendix 3.6.1).*

**Proof in the particular case**  $g(\varphi) = \alpha \in (0; \pi)$ . The main idea is to use the  $\Gamma$ -coercivity approach. By dividing  $\mathcal{L}_\sigma$  by  $\sigma_1$  we come back to the study of the particular case where  $\sigma = 1$  in  $S_1$  and  $\sigma = \kappa_\sigma$  in  $S_2$ . To prove our claim, one has just to study the case  $-1 < \kappa_\sigma < 0$ , the other case (when  $\kappa_\sigma < -1$ ) can be studied in the same way by exchanging the roles of  $S_1$  and  $S_2$ . For this reason, we are going to suppose that  $\kappa_\sigma > -1$ . Then, we define the operator  $T : H^1(\mathbb{S}^2) \rightarrow H^1(\mathbb{S}^2)$  such that

$$T(u)(\theta, \varphi) = \begin{cases} u_1(\theta, \varphi) & \text{in } S_1 \\ -u_2(\theta, \varphi) + 2\chi(\theta)u_1(2\alpha - \theta, \varphi) & \text{in } S_2 \end{cases}$$

where the functions  $u_1$  and  $u_2$  are such that  $u_1 = u|_{S_1}$  and  $u_2 = u|_{S_2}$  and in which  $\chi : [0, \pi] \rightarrow [0, 1]$  is a cutoff function that is equal to one for  $\theta \in (\alpha - \gamma; \alpha + \gamma)$  and vanishes for  $\theta \in (0; \alpha - 2\gamma) \cup$

$(\alpha + 2\gamma; \pi)$ . The parameter  $\gamma$  must be chosen such that  $2\gamma < \min(\alpha, \pi - \alpha)$ . We also need to define the positive numbers  $M_\gamma, L_\gamma$  and  $N_\gamma$  such that

$$M_\gamma = \sup_{\theta \in [\alpha - 2\gamma; \alpha + 2\gamma]} \frac{\sin(2\alpha - \theta)}{\sin(\theta)}, \quad L_\gamma = \sup_{\theta \in [\alpha - 2\gamma; \alpha + 2\gamma]} \frac{\sin(\theta)}{\sin(2\alpha - \theta)} \quad \text{and} \quad N_\gamma = \sup_{\theta \in [\alpha - 2\gamma; \alpha + 2\gamma]} \chi'(\theta).$$

It will be useful to note that, at least for  $\gamma$  small enough, the functions  $\gamma \mapsto M_\gamma$  and  $\gamma \mapsto L_\gamma$  are continuous. Since  $M_0 = L_0 = 1$  one deduces that there exists some  $\gamma^*$  small enough such that for all  $\gamma \in (0; \gamma^*)$  one has  $\max(M_\gamma, L_\gamma) < 1/|\kappa_\sigma|$  (this is true because we have supposed that  $-1 < \kappa_\sigma < 0$ ). Now, let us come back to the study of the operator  $\mathbb{T}$ . First of all, by observing that for all  $u \in \mathbf{H}^1(\mathbb{S})$ , the function  $\mathbb{T}(u)$  is continuous across the interface  $\{\theta = \alpha\}$ , we infer that  $\mathbb{T}(u) \in \mathbf{H}^1(\mathbb{S})$ . This means that the operator  $\mathbb{T}$  is well-defined. Since for all  $u \in \mathbf{H}^1(\mathbb{S})$  we have  $\mathbb{T} \circ \mathbb{T}(u) = u$ , we deduce that  $\mathbb{T}$  is a bijective operator. The continuity of  $\mathbb{T}$  follows from the following estimates: for all  $u \in \mathbf{H}^1(\mathbb{S}^2)$  we have

$$\begin{aligned} \int_{\mathbb{S}^2} |\mathbb{T}(u)|^2 d\omega &\leq \int_{S_1} |u_1|^2 d\omega + 2 \int_{S_2} |u_2|^2 d\omega + 8 \int_{S_2} |\chi(\theta)u_1(2\alpha - \theta, \varphi)|^2 d\omega \\ &\leq 2 \int_{\mathbb{S}^2} |u|^2 d\omega + 8M_\gamma \int_{S_1} |u_1|^2 d\omega \leq (2 + 8M_\gamma) \int_{\mathbb{S}^2} |u|^2 d\omega \\ \int_{\mathbb{S}^2} |\nabla_S \mathbb{T}(u)|^2 d\omega &\leq 2 \int_{\mathbb{S}^2} |\nabla_S u|^2 d\omega + 8 \int_{S_2} |\nabla_S(\chi(\theta)u_1(2\alpha - \theta, \varphi))|^2 d\omega \\ &\leq 2 \int_{\mathbb{S}^2} |\nabla_S u|^2 d\omega + 8 \int_{S_2} \left| \frac{\chi(\theta)\partial_\varphi u_1(2\alpha - \theta, \varphi)}{\sin(\theta)} \right|^2 d\omega \\ &\quad + 8 \int_{S_2} |\partial_\theta(\chi(\theta)u_1(2\alpha - \theta, \varphi))|^2 d\omega \\ &\leq 2 \int_{\mathbb{S}^2} |\nabla_S u|^2 d\omega + 8L_\gamma \int_{S_1} |\nabla_S u_1|^2 d\omega + 8N_\gamma M_\gamma \int_{S_1} |u_1|^2 d\omega \\ &\quad + 8M_\gamma \int_{S_1} |\nabla_S u_1|^2 d\omega. \end{aligned}$$

The next step is to compute  $\langle \mathcal{L}(-1/2 + it)u, \mathbb{T}(u) \rangle$  for an arbitrary  $u \in \mathbf{H}^1(\mathbb{S})$  and an arbitrary  $t \in \mathbb{R}$ . To simplify notations, we shall denote by  $\tilde{u}_1$  the function  $(\theta, \varphi) \mapsto \tilde{u}_1(\theta, \varphi) = \chi(\theta)u_1(2\alpha - \theta, \varphi)$  and by  $\gamma_t$  the real positive number  $\rho_t = 1/4 + t^2$ . For all  $t \in \mathbb{R}$  and all  $u \in \mathbf{H}^1(\mathbb{S}^2)$ , we have

$$\begin{aligned} \langle \mathcal{L}(-1/2 + it)u, \mathbb{T}(u) \rangle &= \int_{\mathbb{S}^2} \sigma \nabla_S u \cdot \nabla_S \overline{\mathbb{T}(u)} d\omega + \rho_t \int_{\mathbb{S}^2} \sigma u \overline{\mathbb{T}(u)} d\omega \\ &= (|\sigma| \nabla_S u, \nabla_S u)_{\mathbb{S}^2} + \rho_t (|\sigma| u, u)_{\mathbb{S}^2} + 2\kappa_\sigma (\nabla u_2, \nabla(\tilde{u}_1))_{S_2} + 2\kappa_\sigma \rho_t (u_2, \tilde{u}_1)_{S_2}. \end{aligned}$$

Now, by means of the Young's inequality and the definition of  $M_\gamma$  one finds for all  $0 < a$  that

$$2|(u_2, \tilde{u}_1)_{S_2}| = \left| \int_{S_2} u_2(\theta, \varphi) \chi(\theta) \overline{u_1(2\alpha - \theta, \varphi)} d\omega \right| \leq a|(u_2, u_2)_{S_2}| + a^{-1}M_\gamma |(u_1, u_1)_{S_1}|.$$

For the term  $(\nabla_S u_2, \nabla_S \tilde{u}_1)_{S_2}$ , we decompose it into the sum of  $(\chi(\theta)\nabla_S u_2, \nabla_S(u_1(2\alpha - \theta, \varphi)))_{S_2}$  and of  $(\nabla_S u_2, u_1(2\alpha - \theta, \varphi)\nabla_S \chi(\theta))_{S_2}$ . Applying the Young's inequality, one obtains that for all  $0 < b, c$

$$\begin{aligned} 2|(\chi(\theta)\nabla_S u_1, \nabla_S \tilde{u}_1)_{S_2}| &= 2 \left| \int_{S_2} \chi(\theta) (-\partial_\theta u_2(\theta, \varphi) \partial_\theta \overline{u_1(2\alpha - \theta, \varphi)} d\omega + \frac{\partial_\varphi u_2(\theta, \varphi)}{\sin(\theta)} \frac{\partial_\varphi \overline{u_1(2\alpha - \theta, \varphi)}}{\sin(\theta)} d\omega \right| \\ &\leq b(\nabla_S u_2, \nabla_S u_2)_{S_2} + \frac{\max(M_\gamma, L_\gamma)}{b} (\nabla_S u_1, \nabla_S u_1)_{S_1} \\ 2|(\nabla_S u_2, u_1(2\alpha - \theta, \varphi)\nabla_S \chi(\theta))_{S_2}| &= \left| \int_{S_2} (\partial_\theta u_2(\theta, \varphi) \partial_\theta \overline{\chi(\theta)u_1(2\alpha - \theta, \varphi)} d\omega \right| \\ &\leq c(\nabla_S u_2, \nabla_S u_2)_{S_2} + \frac{N_\gamma M_\gamma}{c} (u_1, u_1)_{S_1}. \end{aligned}$$

With these estimates, one concludes that for all  $t \in \mathbb{R}$  and  $u \in H^1(\mathbb{S}^2)$  we have for all  $0 < a, b, c$

$$\begin{aligned} |\langle \mathcal{L}(-1/2 + it)u, T(u) \rangle| &\geq \left(1 - \frac{|\kappa_\sigma| \max(M_\gamma, \mathbb{L}_\gamma)}{b}\right) (\nabla_S u_1, \nabla_S u_1)_{S_1} + |\kappa_\sigma| (1 - b - c) (\nabla_S u_2, \nabla_S u_2)_{S_2} \\ &\quad + \left(\rho_t \left(1 - \frac{|\kappa_\sigma| M_\gamma}{a}\right) - \frac{|\kappa_\sigma| N_\gamma M_\gamma}{c}\right) (u_1, u_1)_{S_1} + (\rho_t |\kappa_\sigma| (1 - a)) (u_2, u_2)_{S_2}. \end{aligned}$$

Given that for all  $\gamma \in [0, \gamma^*)$ , we have  $\max(M_\gamma, L_\gamma) < 1/|\kappa_\sigma|$ , we then deduce that for all  $\gamma \in (0; \gamma^*)$ , one can find  $a, b \in (0; 1)$  such that

$$0 < 1 - \frac{|\kappa_\sigma| \max(M_\gamma, \mathbb{L}_\gamma)}{b} \quad \text{and} \quad 0 < 1 - \frac{|\kappa_\sigma| M_\gamma}{a}.$$

By taking  $c \in (0; 1 - b)$  and  $t$  large enough so that  $0 < \left(\rho_t \left(1 - \frac{|\kappa_\sigma| M_\gamma}{a}\right) - \frac{|\kappa_\sigma| N_\gamma M_\gamma}{c}\right)$ , one deduces that there exists some  $0 < t_0$  such that for all  $t \in \mathbb{R}$  satisfying  $t_0 < |t|$ , we have the estimate

$$C_0 \|u\|_{H^1(\mathbb{S}^2, |\lambda|)}^2 \leq |\langle \mathcal{L}(-1/2 + it)u, T(u) \rangle| \quad (3.1)$$

with  $C_0$  independent of  $t$ . Note that to obtain the previous estimate, we have used the fact that for all  $\lambda = -1/2 + it$  with  $t \in \mathbb{R}$  we have  $|\lambda|^2 = \rho_t$ . Since  $T : H^1(\mathbb{S}) \rightarrow H^1(\mathbb{S})$  is continuous (here  $H^1(\mathbb{S})$  is endowed with its natural norm), the operator  $T : H^1(\mathbb{S}, |\lambda|) \rightarrow H^1(\mathbb{S})$  is continuous and uniformly bounded. This, simply, means that for all  $\lambda \in \mathbb{C}$  and all  $u \in H^1(\mathbb{S}^2)$  we have the estimate,

$$\|T(u)\|_{H^1(\mathbb{S}^2)} \leq C \|u\|_{H^1(\mathbb{S}^2, |\lambda|)} \quad \text{for all } u \in H^1(\mathbb{S}^2)$$

with  $C$  independent of  $\lambda$  and of  $u$ . Inserting this into (3.1), one deduces there is some  $0 < C$  independent of  $u \in H^1(\mathbb{S}^2)$  and  $\lambda = -1/2 + it$  (with  $t_0 < |t|$ ) such that

$$\|u\|_{H^1(\mathbb{S}^2, |\lambda|)} \leq C \|\mathcal{L}_\sigma(\lambda)(u)\|_{H^1(\mathbb{S}^2, |\lambda|)^*}.$$

This furnishes the wanted estimate and shows that for all  $\lambda = -1/2 + it$  (with  $t_0 < |t|$ ) the operator  $\mathcal{L}_\sigma(\lambda)$  is injective and its range is closed. By observing that for all  $t \in \mathbb{R}$  the operator  $\mathcal{L}_\sigma(-1/2 + it)$  is self-adjoint (because it is bounded and symmetric), we deduce that  $\mathcal{L}_\sigma(-1/2 + it)$  is an isomorphism for all  $t \in \mathbb{R}$  satisfying  $t_0 < |t|$ .  $\blacksquare$

### 3.2.2 Discreteness of the spectrum

Given that the embedding of  $H^1(\mathbb{S}^2)$  into  $L^2(\mathbb{S}^2)$  is compact (see [91, Proposition 2.4]), one can easily see that for all  $\lambda, \lambda' \in \mathbb{C}$  the operator  $\mathcal{L}_\sigma(\lambda) - \mathcal{L}_\sigma(\lambda')$  is compact. Taking  $\lambda' = -1/2 + it_0$  with  $t_0$  as in Lemma 3.2.1, we can say that if  $\kappa_\sigma \neq -1$  the operator  $\mathcal{L}_\sigma(\lambda)$  is a Fredholm operator (of index zero) for all  $\lambda \in \mathbb{C}$ . Furthermore, by applying the analytic Fredholm theorem (see [101, Corollary 1.1.1]), one obtains the following

**Lemma 3.2.2.** *Assume that  $\kappa_\sigma \neq -1$ . The spectrum of  $\mathcal{L}_\sigma$  is composed by isolated eigenvalues with finite algebraic multiplicities. Furthermore,  $\mathcal{L}_\sigma(\lambda)^{-1}$  is analytic in  $\mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma)$ .*

### 3.2.3 Localization of the spectrum and boundedness of the resolvent

In this paragraph, we intend to explain how to obtain a more precise information about the location of the spectrum of  $\mathcal{L}_\sigma$  in the complex plane. In addition to that, we are going to address the question of the behaviour of  $\|\mathcal{L}_\sigma^{-1}(\lambda)\|$  when  $|\lambda|$  is large (this result is important to show that the solution constructed by means of the inverse Mellin transform is uniformly bounded with respect to the source term). Before getting into details, one can easily see that  $\Lambda(\mathcal{L}_\sigma)$  is symmetric with respect to the point  $(-1/2, 0)$  (i.e. if  $\lambda \in \Lambda(\mathcal{L}_\sigma)$  then  $-\lambda - 1$  also belongs to it). Furthermore, since  $\sigma$  is real-valued one can also observe that  $\Lambda(\mathcal{L}_\sigma)$  is symmetric with respect to the lines  $\Im m(\lambda) = 0$  (i.e. if  $\lambda \in \Lambda(\mathcal{L}_\sigma)$  then  $\bar{\lambda} \in \Lambda(\mathcal{L}_\sigma)$ ).

**Theorem 3.2.1.** *Assume that  $\kappa_\sigma \neq -1$ . Then all the eigenvalues of  $\mathcal{L}_\sigma(\lambda)$ , with the possible exception of finitely many, are located outside of some double sector centered at  $(-1/2, 0)$  (i.e.  $\{z \in \mathbb{C} \mid |\Re(z + 1/2)| \leq \delta |\Im(z + 1/2)|\}$  with  $0 < \delta$ ) of the complex plane (see Figure 3.2). More precisely, there exist  $0 < \gamma_0, r_0$  such that*

$$D_{\gamma_0}^{r_0} := \{z \in \mathbb{C} \text{ such that } r_0 < |z + 1/2| \text{ and } |\Re(z + 1/2)| < \gamma_0 |\Im(z + 1/2)|\} \subset \mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma).$$

Furthermore, there exists some positive constant  $C$  independent of  $\lambda \in D_{\gamma_0}^{r_0}$  such that

$$\|u\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda|)} \leq C \|\mathcal{L}_\sigma(\lambda)u\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda|)^*} \text{ for all } u \in \mathbf{H}^1(\mathbb{S}^2).$$

**Proof.** Let us start with the case  $\lambda = -1/2 + it$  with  $t \in \mathbb{R}$ . We have shown in Lemma 3.2.1 that there is some  $0 < t_0$  such that for all  $|t| < t_0$  we have the estimate

$$C_0 \|u\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda|)} \leq \|\mathcal{L}_\sigma(\lambda)(u)\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda|)^*} \quad (3.2)$$

with  $C_0$  independent of  $t$ . In the rest of the proof we are going to suppose that  $1/2 < t_0$ . Now, for all  $\theta \in [-\pi/2; \pi/2]$  and all  $t \in \mathbb{R}$  such that  $t_0 < |t|$  we denote by  $\lambda_\theta^t$  the complex number  $\lambda_\theta^t = -1/2 + ite^{i\theta}$ . One can easily check that for all  $u, v \in \mathbf{H}^1(\mathbb{S}^2)$  we have

$$\langle (\mathcal{L}_\sigma(\lambda_\theta^t) - \mathcal{L}_\sigma(\lambda_0^t))u, v \rangle = (\lambda_\theta^t(\lambda_\theta^t + 1) - \lambda_0^t(\lambda_0^t + 1)) \int_{\mathbb{S}^2} \sigma u \bar{v} d\omega = t^2(1 - e^{2i\theta}) \int_{\mathbb{S}^2} \sigma u \bar{v} d\omega.$$

Given that the function  $x \mapsto x/(x - 1/2)$  is decreasing in  $[t_0, +\infty)$ , we deduce that for all  $t_0 < |t|$  and  $\theta \in [-\pi/2, \pi/2]$  we have

$$\frac{|t|}{|\lambda_\theta^t|} \leq \frac{|t|}{|t| - 1/2} \leq \frac{t_0}{t_0 - 1/2}.$$

As a result, for all  $t_0 < |t|$  and all  $\theta \in [-\pi/2, \pi/2]$  we obtain the estimate

$$\|\mathcal{L}_\sigma(\lambda_\theta^t) - \mathcal{L}_\sigma(\lambda_0^t)\|_{(\mathbf{H}^1(\mathbb{S}^2, |\lambda_\theta^t|))^*} \leq C_1 |1 - e^{2i\theta}|$$

with  $C_1$  independent of  $t$  and of  $\theta$ . Starting from the fact that the operator satisfies the estimate (3.2), one obtains the following estimate: for all  $u \in \mathbf{H}^1(\mathbb{S}^2)$  we have

$$(C_0 - C_1 |1 - e^{2i\theta}|) \|u\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda_\theta^t|)} = (C_0 - 2C_1 |\sin(\theta)|) \|u\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda_\theta^t|)} \leq \|\mathcal{L}_\sigma(\lambda_\theta^t)(u)\|_{\mathbf{H}^1(\mathbb{S}^2, |\lambda_\theta^t|)^*}. \quad (3.3)$$

Given that  $\theta \mapsto \sin(\theta)$  is continuous, we infer that there exists  $\theta^* \in (0; \pi/2)$  such that for all  $\theta \in (-\theta^*; \theta^*)$ , we have  $0 < C_0 - C_1 |1 - e^{2i\theta}|$ . Consequently, we deduce that for all  $\theta \in (-\theta^*; \theta^*)$  and all  $t_0 < |t|$  we have  $\lambda_\theta^t \notin \Lambda(\mathcal{L}_\sigma)$ . Since  $|t| = |\lambda_\theta^t + 1/2|$  and  $\theta = \arg(\lambda_\theta^t + 1/2) - \pi/2$  we infer that the region

$$\{z \in \mathbb{C} \text{ such that } t_0 < |z + 1/2| \text{ and } |\Re(z + 1/2)| \leq \tan(\theta^*) |\Im(z + 1/2)|\}$$

is free of eigenvalues of  $\mathcal{L}_\sigma$ . To end the proof, it remains to see that inside the ball  $B =: \{z \in \mathbb{C} \text{ such that } |z + 1/2| < t_0\}$  there is a finite number of eigenvalues of  $\mathcal{L}_\sigma$ . This is a direct consequence of the fact that  $\Lambda(\mathcal{L}_\sigma)$  consists of isolated eigenvalues. The theorem is then proved by taking  $r_0 = t_0$ ,  $\gamma_0 = \tan(\theta^*)$  and  $C = 2/C_0$ . ■

**Remark 3.2.2.** *One of the consequences of the previous theorem is the fact that, when  $\kappa_\sigma \neq -1$ , for all  $\beta_1 < \beta_2$  the strip  $\Re(\lambda) \in (\beta_1, \beta_2)$  contains a finite number of eigenvalues of  $\mathcal{L}_\sigma$ .*

It is worth to note that in the statement of the previous theorem, the parameters  $\gamma_0$  and  $r_0$  depend on the contrast  $\kappa_\sigma$ . Let  $\beta \in \mathbb{R}$ , we define  $\ell_\beta =: \{\lambda \in \mathbb{C} \text{ such that } \Re(\lambda) = \beta\}$ . Using the same idea as in the proof of the previous theorem, one shows the

**Corollary 3.2.1.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\beta \in \mathbb{R}$  such that  $\ell_\beta \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$  then there exists some constant  $C_\beta$  (independent of  $\lambda$ ) such that the estimate*

$$\|u\|_{H^1(\mathbb{S}^2, |\lambda|)} \leq C_\beta \|\mathcal{L}_\sigma(\lambda)u\|_{(H^1(\mathbb{S}^2, |\lambda|))^*}$$

holds for all  $u \in H^1(\mathbb{S}^2)$  and all  $\lambda \in \ell_\beta$ .

**Proof.** The Theorem 3.2.1, shows that when  $\lambda \in \ell_\beta \cap (D_{\gamma_0}^{r_0})$  (see the statement of Theorem 3.2.1 for definition of  $D_{\gamma_0}^{r_0}$ ), the estimate

$$\|u\|_{H^1(\mathbb{S}^2, \lambda)} \leq C \|\mathcal{L}_\sigma(\lambda)u\|_{H^1(\mathbb{S}^2, \lambda)^*} \text{ for all } u \in H^1(\mathbb{S}^2)$$

holds with some  $C$  independent of  $\lambda$ . By combining the fact that  $D_{\gamma_0}^{r_0} \subset \mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma)$ , the fact that  $\Lambda(\mathcal{L}_\sigma)$  is composed by isolated points and by using the assumption  $\ell_\beta \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ , one can say that for  $\eta$  small enough the strip  $C_\eta =: \{\lambda \in \mathbb{C}; \Re(\lambda) \in [\beta - \eta, \beta + \eta]\}$  is free of eigenvalues of  $\mathcal{L}_\sigma$ . The wanted estimate follows, then, by combining the fact that  $\mathcal{L}_\sigma^{-1}(\lambda)$  is analytic in  $C_\eta$  and the compactness of  $C_\eta \cap (\mathbb{C} \setminus D_{\gamma_0}^{r_0})$ . ■

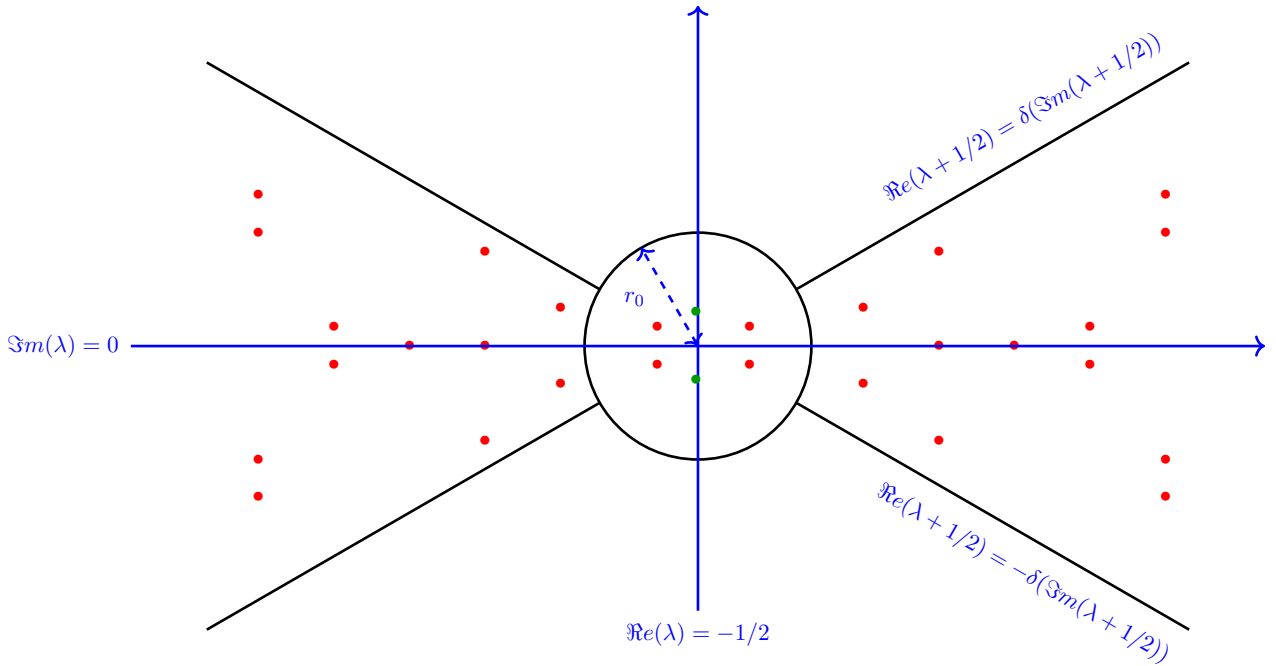


Figure 3.2: A possible distribution of the spectrum of  $\mathcal{L}_\sigma$  (the red points) for  $\kappa_\sigma \neq -1$ . The green points are associated to propagating singularities (see Definition 2.6.1).

The previous corollary tells us that when  $\mathcal{L}_\sigma(\lambda)$  is invertible along the energy line  $\lambda \in \ell_\beta$ , then, seen as an operator from  $(H^1(\mathbb{S}^2, |\lambda|))^*$  to  $(H^1(\mathbb{S}^2, |\lambda|))$ , the operator  $\mathcal{L}_\sigma(\lambda)^{-1}$  is uniformly bounded with respect to  $\lambda \in \ell_\beta$ . However, when it is considered as an operator from  $(H^1(\mathbb{S}^2))^*$  to  $H^1(\mathbb{S}^2)$ , the result of previous corollary implies that the norm of  $\mathcal{L}_\sigma(\lambda)^{-1}$  does not grow faster than  $|\lambda|^2$  (when  $\lambda \in \ell_\beta$ ).

Now, let us consider two real constants  $\beta_1 < \beta_2$  such that  $\ell_{\beta_1} \cap \Lambda(\mathcal{L}_\sigma) = \ell_{\beta_2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ . For all  $r \in \mathbb{R}_+^*$ , we introduce the closed set

$$D(r, \beta_1, \beta_2) := \{\lambda \in \mathbb{C} \mid \beta_1 \leq \Re(\lambda) \leq \beta_2\} \setminus \left( \bigcup_{\lambda \in \Lambda(\mathcal{L}_\sigma)} B(\lambda, r) \right). \quad (3.4)$$

**Corollary 3.2.2.** *Suppose that  $\kappa_\sigma \neq -1$  and let  $\beta_1 < \beta_2 \in \mathbb{R}$  such that  $\ell_{\beta_1} \cap \Lambda(\mathcal{L}_\sigma) = \ell_{\beta_2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ . Then, for all  $0 < r$  there exists  $0 < C$  such that*

$$\|u\|_{\mathbb{H}^1(\mathbb{S}^2, |\lambda|)} \leq C \|\mathcal{L}_\sigma(\lambda)u\|_{(\mathbb{H}^1(\mathbb{S}^2, |\lambda|))^*} \text{ for all } \lambda \in D(r, \beta_1, \beta_2) \text{ and for all } u \in \mathbb{H}^1(\mathbb{S}^2).$$

**Proof.** Thanks to Theorem 3.2.1, we obtain the wanted estimate for all  $\lambda \in D(r, \beta_1, \beta_2) \cap D_{\gamma_0}^{r_0}$ . To obtain the wanted estimate for  $\lambda \in D(r, \beta_1, \beta_2) \cap (\mathbb{C} \setminus D_{\gamma_0}^{r_0})$  is enough to see that the latter is a compact subset of  $\mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma)$ . ■

### 3.2.4 Algebraic multiplicities of eigenvalues in the energy line $\Re e(\lambda) = -1/2$

In this paragraph, we are going to prove some useful results concerning the algebraic multiplicities of eigenvalues of  $\mathcal{L}_\sigma$  that are located in the energy line  $\ell_{-1/2}$ . The starting point of our discussion is the following:

**Proposition 3.2.1.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\lambda_0 \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$ . Let  $(\varphi_1, \dots, \varphi_{\iota_g(\lambda_0)})$  be a basis of  $\text{Ker}(\mathcal{L}_\sigma(\lambda_0))$ . Then, we have the equivalence*

$$\iota_g(\lambda) < \iota_a(\lambda) \text{ iff } \exists k \in \{1, \dots, \iota_g(\lambda_0)\} \text{ s.t. } (2\lambda_0 + 1) \int_{\mathbb{S}^2} \sigma \varphi_k \overline{\varphi_j} = 0 \text{ for } j = 1, \dots, \iota_g(\lambda_0)$$

where we refer to §2.4.4 for the definitions.

**Proof.** Since  $\kappa_\sigma \neq -1$ , we know that  $\mathcal{L}_\sigma(\lambda_0)$  is a Fredholm operator of index 0. The fact that  $\lambda_0 \in \ell_{-1/2}$  implies that  $\mathcal{L}_\sigma(\lambda_0)$  is self-adjoint. By definition of  $\iota_g(\lambda_0)$  and  $\iota_a(\lambda_0)$ , we know that  $\iota_g(\lambda_0) < \iota_a(\lambda_0)$  if and only if there exists  $k \in \{1, \dots, \iota_g(\lambda_0)\}$  for which the function  $\varphi_k$  has at least a generalized eigenfunction. This is equivalent to say that the problem

$$\text{Find } u \in \mathbb{H}^1(\mathbb{S}^2) \text{ such that } \mathcal{L}_\sigma(\lambda_0)u = -\frac{d\mathcal{L}_\sigma}{d\lambda}(\lambda_0)\varphi_k$$

has a solution. By the Fredholm alternative, we know that the previous equation has a solution if and only if

$$\left\langle \frac{d\mathcal{L}_\sigma}{d\lambda}(\lambda_0)\varphi_k, \varphi_j \right\rangle = 0 \text{ for } j = 1, \dots, \iota_g(\lambda_0).$$

The result is then proved by observing that  $\left\langle \frac{d\mathcal{L}_\sigma}{d\lambda}(\lambda_0)\varphi_k, \varphi_j \right\rangle = (2\lambda_0 + 1) \int_{\mathbb{S}^2} \sigma \varphi_k \overline{\varphi_j}$ . ■

A direct consequence of the previous proposition is the following

**Lemma 3.2.3.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\lambda \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$  such that  $\iota_g(\lambda) = 1$ . Let  $\varphi \in \text{Ker}(\mathcal{L}_\sigma(\lambda)) \setminus \{0\}$ . Then*

$$\iota_a(\lambda) = 1 \text{ if and only if } (2\lambda + 1) \int_{\mathbb{S}^2} \sigma |\varphi|^2 \neq 0.$$

We also obtain the following

**Lemma 3.2.4.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\lambda \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$  such that  $\iota_g(\lambda) = 2$ . Let  $\varphi \in \text{Ker}(\mathcal{L}_\sigma(\lambda))$  such that  $(\varphi, \overline{\varphi})$  is a basis of  $\text{Ker}(\mathcal{L}_\sigma(\lambda))$ . Then*

$$(2\lambda + 1) \int_{\mathbb{S}^2} \sigma |\varphi|^2 \neq 0 \implies \iota_a(\lambda) = 2.$$

### 3.3 Stability of $\Lambda(\mathcal{L}_\sigma)$ with respect to perturbations of $\sigma$

In the present section, we will be concerned with the study of the spectrum of the operator  $\mathcal{L}_{\sigma+i\delta}(\lambda)$ , where  $\delta$  is a real parameter. Our main goal, is to study the convergence of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  and the associated eigenfunctions when  $\delta$  tends to 0. Note that the study of such convergence is essential when one wants to define the physical solution of the original scalar problem by means of the limiting absorption principle (see §2.6.3). Since for all  $\lambda \in \mathbb{C}$  the operator  $\mathcal{L}_{\sigma+i\delta}(\lambda)$  can be seen as a small perturbation (of course when  $\delta$  is small) of the operator  $\mathcal{L}_\sigma(\lambda)$ , one may expect that when  $\delta$  goes to 0 the set  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  will converge to  $\Lambda(\mathcal{L}_\sigma)$  (here the convergence must be understood with respect to the Hausdorff distance (see Definition 3.3.1)).

#### 3.3.1 Properties of the spectrum of the perturbed problem

Before getting into details, let us start by proving that for all  $0 \neq \delta$ , the set  $\Lambda(\mathcal{L}_\sigma)$  is discrete. This is the object of the following

**Proposition 3.3.1.** *Let  $0 < \delta$ . The spectrum of  $\mathcal{L}_{\sigma+i\delta}$  is discrete and composed by isolated eigenvalues. Furthermore, we have  $\ell_{-1/2} \cap \Lambda(\mathcal{L}_{\sigma+i\delta}) = \emptyset$ .*

**Proof.** By observing that for all  $t \in \mathbb{R}$ ,  $\delta \in \mathbb{R}^*$  and all  $u \in H^1(\mathbb{S}^2)$ , we have

$$|\Im(\langle \mathcal{L}_{\sigma+i\delta}(-1/2 + it)u, u \rangle)| = |\delta| \|u\|_{H^1(\mathbb{S}^2, 1/4+t^2)}^2,$$

we deduce that for all  $t \in \mathbb{R}$  and  $\delta \in \mathbb{R}^*$  the operator  $\mathcal{L}_{\sigma+i\delta}^*(-1/2 + it)$  is injective and that its range is closed. By observing that  $\mathcal{L}_{\sigma+i\delta}^*(-1/2 + it) = \mathcal{L}_{\sigma-i\delta}(-1/2 + it)$ , we deduce that for all  $\delta \in \mathbb{R}^*$  and  $t \in \mathbb{R}$  the operator  $\mathcal{L}_{\sigma+i\delta}^*(-1/2 + it)$  is an isomorphism. The rest of the proof is a direct application of the analytic Fredholm theorem. ■

Note that in the previous proposition, there is no assumption about the value of the contrast  $\kappa_\sigma$ . We have the analogue of Theorem 3.2.1.

**Lemma 3.3.1.** *Assume that  $\kappa_\sigma \neq -1$ . There exist two positive constants  $r_0, \gamma_0$  independent of  $\delta$  and  $0 < \delta_0$  such that for all  $\delta$  satisfying  $|\delta| < \delta_0$  we have*

$$D_{\gamma_0}^{r_0} := \{z \in \mathbb{C} \text{ such that } r_0 < |z - 1/2| \text{ and } |\Re(z + 1/2)| < \gamma_0 |\Im(z + 1/2)|\} \subset \mathbb{C} \setminus \Lambda(\mathcal{L}_{\sigma+i\delta}).$$

Moreover, there is some  $0 < C$  independent of  $\delta$  such that the estimate

$$\|u\|_{H^1(\mathbb{S}^2, |\lambda|)} \leq C \|\mathcal{L}_{\sigma+i\delta}(\lambda)u\|_{H^1(\mathbb{S}^2, |\lambda|)^*} \text{ for all } u \in H^1(\mathbb{S}^2)$$

holds for all  $\delta$  such that  $|\delta| < \delta_0$  and all  $\lambda \in D_{\gamma_0}^{r_0}$ .

**Proof.** From the results of Theorem 3.2.1, we already know that there exist two positive constants  $r_0, \gamma_0$  such that  $D_{\gamma_0}^{r_0} \subset \mathbb{C} \setminus \Lambda(\mathcal{L}_\sigma)$ . Furthermore, we know that when  $\lambda \in D_{\gamma_0}^{r_0}$ , the operator  $\mathcal{L}_\sigma(\lambda)^{-1} : ((H^1(\mathbb{S}^2))^*, |\lambda|) \rightarrow (H^1(\mathbb{S}^2), |\lambda|)$  is uniformly bounded with respect to  $\lambda$ . As a result, to prove our claim, we need to find a uniform estimate of  $\|\mathcal{L}_\sigma(\lambda) - \mathcal{L}_{\sigma+i\delta}(\lambda)\|_{(H^1(\mathbb{S}^2, |\lambda|) \rightarrow (H^1(\mathbb{S}^2, |\lambda|))^*)}$  for  $\lambda \in D_{\gamma_0}^{r_0}$ . To do so, we start from the fact that for all  $u \in H^1(\mathbb{S}^2)$ ,  $\delta \in \mathbb{R}^*$  and  $\lambda \in D_{\gamma_0}^{r_0}$  we have

$$\|\mathcal{L}_\sigma(\lambda)u - \mathcal{L}_{\sigma+i\delta}(\lambda)u\|_{(H^1(\mathbb{S}^2, |\lambda|))^*} \leq |\delta| (\|u\|_{H^1(\mathbb{S}^2)} + |\lambda(\lambda + 1)| \|u\|_{L^2(\mathbb{S}^2)}).$$

Next, given that  $0 \notin \overline{D_{\gamma_0}^{r_0}}$ , we infer that there exists  $0 < \rho$  such that  $\rho < |\lambda|$  for all  $\lambda \in D_{\gamma_0}^{r_0}$ . As a result, we conclude that there exists a constant  $0 < C_0$  independent of  $\lambda$  such that

$$|\lambda(\lambda + 1)| \leq C_0 |\lambda|^2 \text{ for all } \lambda \in D_{\gamma_0}^{r_0}.$$

As a consequence, we deduce that for all  $\lambda \in D_{\gamma_0}^{r_0}$  we have

$$\|\mathcal{L}_\sigma(\lambda) - \mathcal{L}_{\sigma+i\delta}(\lambda)\|_{(H^1(\mathbb{S}^2, |\lambda|) \rightarrow (H^1(\mathbb{S}^2, |\lambda|))^*)} \leq C |\delta|$$

where  $C$  is independent of  $\lambda \in D_{\gamma_0}^{r_0}$  and of  $\delta \in \mathbb{R}^*$ . This leads to the wanted result. ■



### 3.3.2 Convergence of the spectrum

In this paragraph, we are going to address the question of the convergence of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$ . Before getting into the details, for all  $\beta_1 < \beta_2$  we denote by  $B(\beta_1, \beta_2)$  the strip

$$B(\beta_1, \beta_2) := \{\lambda \in \mathbb{C} \mid \beta_1 < \Re(\lambda) < \beta_2\}.$$

The main result of this part is given by the

**Theorem 3.3.1.** *Assume that  $\kappa_\sigma \neq -1$  and let  $\beta_1 < \beta_2 \in \mathbb{R}$  such that  $\ell_{\beta_1} \cap \Lambda(\mathcal{L}_\sigma) = \ell_{\beta_2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ . We have  $\lim_{\delta \rightarrow 0} (\Lambda(\mathcal{L}_{\sigma+i\delta}) \cap B(\beta_1, \beta_2)) = \Lambda(\mathcal{L}_\sigma) \cap B(\beta_1, \beta_2)$ .*

In the statement of the previous result, the convergence must be understood in the sense of convergence with respect to the Hausdorff distance. To be more precise, we adopt the following

**Definition 3.3.1.** *Let  $E$  be a closed subset of the complex plane. Let  $(E^\delta)_{\delta \in \mathbb{R}}$  be a family of closed subsets of the complex plane. We say that  $(E^\delta)_{\delta \in \mathbb{R}}$  converges to  $E$  (or briefly  $\lim_{\delta \rightarrow 0} E^\delta = E$ ) if and only if*

$$\lim_{\delta \rightarrow 0} \max(\sup_{x \in E^\delta} \inf_{y \in E} |x - y|, \sup_{x \in E} \inf_{y \in E^\delta} |x - y|) = 0.$$

The proof of the Theorem, will be done thanks to the two following propositions. By working as in the proof of Lemma 3.3.1 and by using the results of Corollary 3.2.2, one can easily prove the

**Proposition 3.3.2.** *Suppose that  $\kappa_\sigma \neq -1$  and let  $\beta_1, \beta_2 \in \mathbb{R}$  satisfying  $\beta_1 < \beta_2$  and such that  $\ell_{\beta_1} \cap \Lambda(\mathcal{L}_\sigma) = \ell_{\beta_2} \cap \Lambda(\mathcal{L}_\sigma) = \emptyset$ . Then for all  $0 < r$ , there exists  $0 < \delta_r$  such that for all  $\delta$  satisfying  $|\delta| < \delta_r$ , we have  $\Lambda(\mathcal{L}_{\sigma+i\delta}) \cap D(r, \beta_1, \beta_2) = \emptyset$  (see (3.4)). Moreover, for all  $\delta$  satisfying  $|\delta| < \delta_r$ , we have the estimate*

$$\| \mathcal{L}_{\sigma+i\delta}(\lambda)^{-1} \|_{((H^1(\mathbb{S}^2, |\lambda|))^* \rightarrow H^1(\mathbb{S}^2, |\lambda|))} \leq C_r \text{ for all } \lambda \in D(r, \beta_1, \beta_2)$$

in which  $C_r$  is a constant that does not depend on  $\lambda$ .

Note that the previous result does not apply when  $\beta_1 = -\infty$  or when  $\beta_2 = +\infty$ . This is due to the possible existence of accumulation points at infinity.

**Remark 3.3.1.** *It is important to mention that near an eigenvalue of  $\mathcal{L}_\sigma$  one can, eventually, find several eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  for  $\delta$  small enough. This will be indeed illustrated in the next paragraph.*

The second result that we need, is given by

**Proposition 3.3.3.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\lambda_0 \in \Lambda(\mathcal{L}_\sigma)$  and denote by  $\iota_a(\lambda_0)$  its algebraic multiplicity. Let  $0 < r$  such that  $\Lambda(\mathcal{L}_\sigma) \cap B(\lambda_0, r) = \{\lambda_0\}$ . There exists  $0 < \delta_0$  such that for all  $\delta$  satisfying  $|\delta| < \delta_0$ , we have*

$$\Lambda(\mathcal{L}_{\sigma+i\delta}) \cap B(\lambda_0, r) \neq \emptyset \text{ and } \varkappa(\mathcal{L}_{\sigma+i\delta}, B(\lambda_0, r)) = \iota_a(\lambda_0)$$

where  $\varkappa(\mathcal{L}_{\sigma+i\delta}, B(\lambda_0, r))$  is the sum of the algebraic multiplicities of the eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  that are located in  $B(\lambda_0, r)$ .

The previous result is a direct consequence of [101, Corollary 1.1.2]. Let us just mention the idea of the proof. Its is based on three important points. The first one is that  $\lambda \mapsto \mathcal{L}_\sigma(\lambda)$  and  $\lambda \mapsto \mathcal{L}_{\sigma+i\delta}(\lambda)$  are two meromorphic functions. The second one is the fact that  $\iota_a(\lambda_0)$  can be expressed as follows [101, Theorem 1.1.3]:

$$\iota_a(\lambda_0) = \frac{1}{2i\pi} \int_{\partial B(\lambda_0, r)} \frac{d\mathcal{L}_\sigma(\lambda)}{d\lambda} \mathcal{L}_\sigma^{-1}(\lambda) d\lambda.$$

The last one is the generalization of Rouché's theorem [101, Theorem 1.1.4]. Now, we have all the tools to prove the Theorem 3.3.1.

*Proof of Theorem 3.3.1.* The Proposition 3.3.3 tells us that every  $\lambda \in \Lambda(\mathcal{L}_\sigma) \cap B(\beta_1, \beta_2)$  is a limit, when  $\delta$  tends to 0, of eigenvalues of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$ . By means of Proposition 3.3.2, we ensure that  $\lim_{\delta \rightarrow 0} (\Lambda(\mathcal{L}_{\sigma+i\delta} \cap B(\beta_1, \beta_2)))$  is a subset of  $\Lambda(\mathcal{L}_\sigma) \cap B(\beta_1, \beta_2)$ . Since the set  $\Lambda(\mathcal{L}_\sigma) \cap B(\beta_1, \beta_2)$  is finite, we obtain the wanted result. ■

We have the following

**Corollary 3.3.1.** *Assume that  $\kappa_\sigma \neq -1$  and let  $\lambda \in \Lambda(\mathcal{L}_\sigma)$ . If  $\lambda$  is simple then there exists  $0 < r_0, \delta_0$  such that for all  $0 < |\delta| < \delta_0$ , the ball  $B(\lambda, r_0)$  contains one eigenvalue of  $\mathcal{L}_{\sigma+i\delta}$ .*

**Proof.** The result follows from the fact that the sum of the algebraic multiplicities of the eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  that are near  $\lambda$  is equal to 1. ■

### 3.3.3 Numerical illustration

To illustrate the results obtained above concerning the convergence of the spectrum of  $\mathcal{L}_{\sigma+i\delta}$  to the one of  $\mathcal{L}_\sigma$ , we shall use the numerical approximation of the spectrum of  $\mathcal{L}_{\sigma+i\delta}$  and  $\mathcal{L}_\sigma$ , by the FEM. Instead of approximating the problem directly in  $\mathbb{S}^2$ , we shall start by write a an equivalent formulation of the problem that will be posed in  $B = (0; 2\pi) \times (0; \pi)$  (this will allow us to avoid the discretization of the unit sphere which is not an easy task in general). To do so, we use the classical angular spherical coordinates  $(\varphi, \theta) \in B$  to parameterize  $\mathbb{S}^2$ . With this in mind, we can say that when  $\kappa_\sigma \neq -1$ ,  $\lambda \in \Lambda(\mathcal{L}_\sigma)$  if and only if there exists  $u \in H_{\#}^1(B)$  such that for all  $v \in H_{\#}^1(B)$  we have

$$\int_B \left( \frac{\sigma(\theta)}{\sin(\theta)} \partial_\varphi u \partial_\varphi \bar{v} + \sigma(\theta) \sin(\theta) \partial_\theta u \partial_\theta \bar{v} \right) d\theta d\varphi = \lambda_h(\lambda_h + 1) \int_B \sigma(\theta) \sin(\theta) u \bar{v} d\theta d\varphi$$

in which

$$H_{\#}^1(B) := \{(\varphi, \theta) \mapsto u(\varphi, \theta) \mid \sqrt{\sin(\theta)}u, \sqrt{\sin(\theta)}\partial_\theta u, \partial_\varphi u / \sqrt{\sin(\theta)} \in L^2(B) \text{ and } u(0, \theta) = u(2\pi, \theta)\}.$$

Naturally, this leads us to the following discrete problem: Find  $(u_h, \lambda_h) \in V_{h,\#}(B) \setminus \{0\} \times \mathbb{C}$  such that for all  $v_h \in V_{h,\#}(B)$

$$\int_B \left( \frac{\sigma(y)}{\sin(y)} \partial_x u_h \partial_x \bar{v}_h + \sigma(y) \sin(y) \partial_y u_h \partial_y \bar{v}_h \right) dx dy = \lambda_h(\lambda_h + 1) \int_B \sigma(y) \sin(y) u_h \bar{v}_h dx dy$$

where the space  $V_{h,\#}^k(B) := \{u \in P^k(B) \mid \text{such that } u(0, y) = u(2\pi, y)\}$ , where  $P^k(B)$  stands for the space of polynomials (of 2 variables) of degree at most equal to  $k$ . In order to take into account the periodicity condition with respect to  $x$ , the mesh of  $B$  must be, then, periodic with respect to  $x$ . Moreover, because of the sign-change in  $\sigma$  and following the results of [46], we need to use a mesh that is periodic in the  $x$  direction and that is symmetric near the interface  $\Sigma := \{y = \pi/4\}$  (we say that the mesh is T-conforming). See Figure 3.3, for an example of T-conforming mesh that is periodic in the  $x$  direction. In our work, we used the library `FreeFem++` for the construction of the matrices associated to the discrete formulation and we used the `eig` function of `MATLAB` in order to approximate the eigenvalues. To approximate the eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$ , we used the same strategy as in the case of  $\mathcal{L}_\sigma$  (one, simply, needs to replace  $\sigma$  by  $\sigma + i\delta$  in the formulation above).

To proceed, we will work with two different values of the contrast  $\kappa_\sigma$  :  $\kappa_\sigma = -0.7807, -0.8$ . For these particular choices of  $\kappa_\sigma$ , we can guaranty that  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} \neq \emptyset$ . For this reason, we shall focus our attention on the behavior of the eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  that are near  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  as  $\delta \rightarrow 0$ . The numerical results for the case  $\kappa = -0.8$  are displayed in Figure 3.4 and those associated with the case  $\kappa_\sigma = -0.7907$  are presented in Figure 3.5.

What we can learn from these results are the following facts:

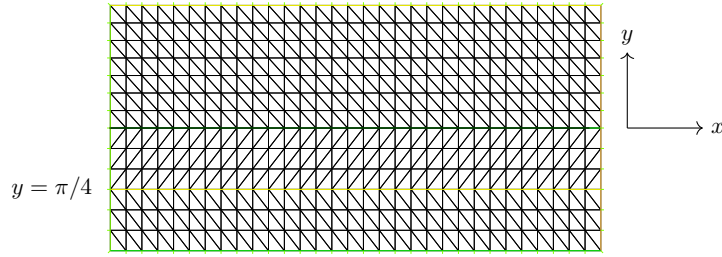
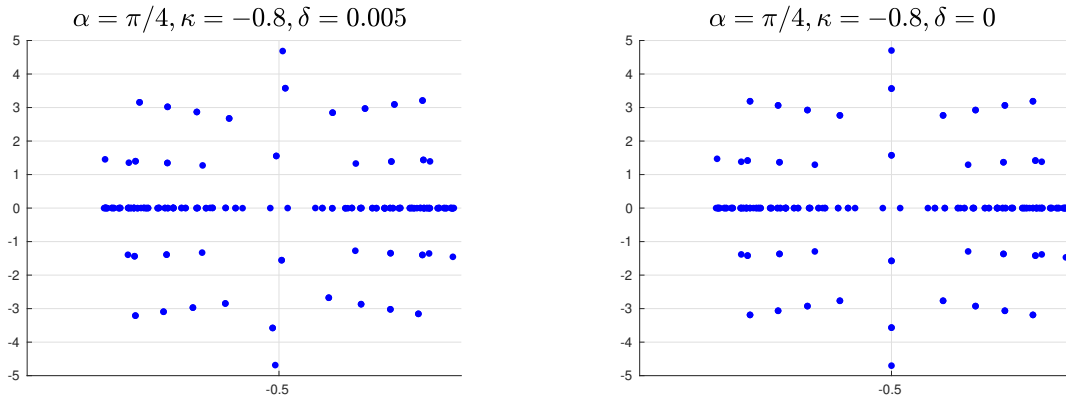
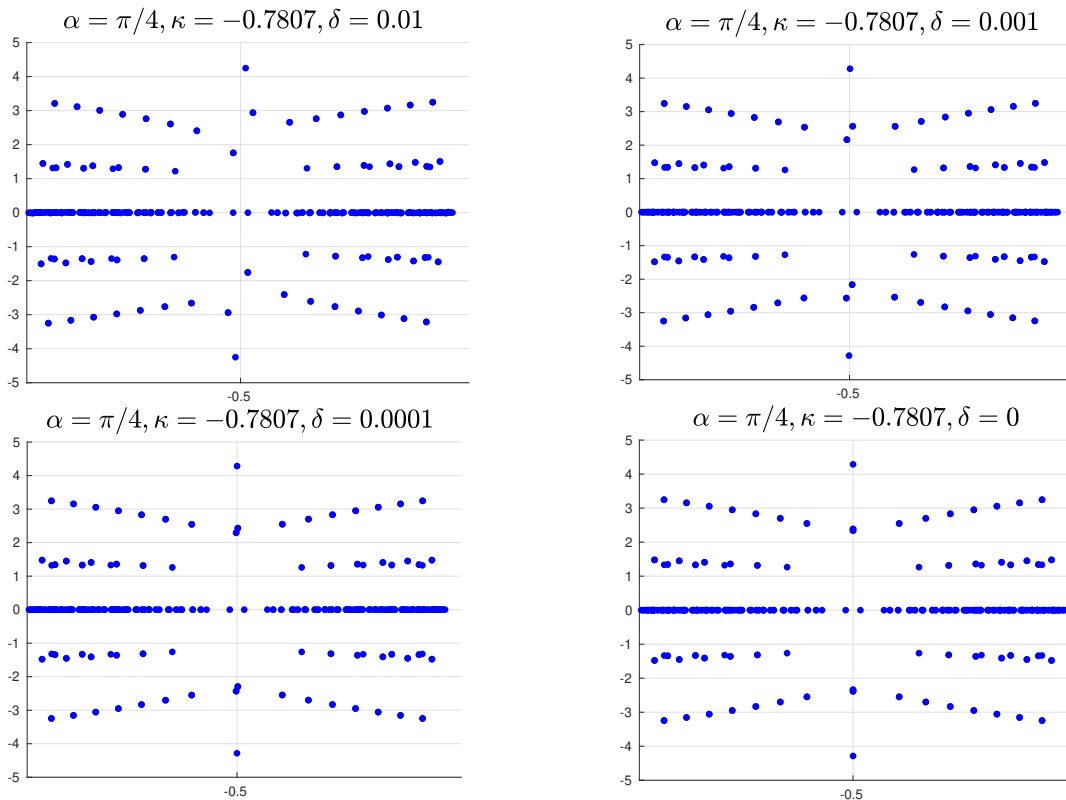


Figure 3.3: An example of a periodic T-conforming mesh.

Figure 3.4: Behavior of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  for  $\kappa_\sigma = -0.8$ .Figure 3.5: Behavior of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  for the case  $\kappa_\sigma = -0.7807$ .

- In both cases, we observe that the convergence of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  to  $\Lambda(\mathcal{L}_\sigma)$  occurs.
- It seems that in the case  $\kappa_\sigma = -0.7807$ , the assumption 2.6.2 is not valid. Indeed, we observe that there exists an eigenvalue  $\lambda \in \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  which corresponds to the coalescence of

two eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$ , one of which comes from the left (i.e. that  $\Re e(\lambda) < -1/2$ ) and the other from the right (i.e.  $-1/2 < \Re e(\lambda)$ ).

- We also observe that the rate of convergence depends on the nature of the eigenvalue: we notice that for the eigenvalue which is the limit of two coalescent eigenvalues, the convergence is slower than for the case of the other eigenvalues that belong to  $\ell_{-1/2}$ .

**Remark 3.3.2.** *For the case of general smooth conical point ( $g \in \mathcal{C}^2([0; 2\pi])$ ), the same numerical approach can be used but this time the construction of  $T$ -conforming mesh seems to be a little bit complicated (see [45, §2.B]). We will leave this question for a future work.*

### 3.3.4 Convergence of the eigenfunctions

Up to now, we have proved that when  $\kappa_\sigma \neq -1$ , the spectrum  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  converges, when  $\delta$  tends to 0, to  $\Lambda(\mathcal{L}_\sigma)$ . Unfortunately, this result alone is not sufficient to derive the theory we need to define the physical solution of the scalar problem by means of the limiting absorption principle (see §2.6.3). To complete it, one has to study the behaviour of the associated eigenfunctions and the generalized eigenfunctions of  $\mathcal{L}_{\sigma+i\delta}$  as  $\delta \rightarrow 0$ .

As we have seen in the previous subsection that, when  $\kappa_\sigma \neq -1$ , for any  $\lambda \in \Lambda(\mathcal{L}_\sigma)$  and  $0 < \delta$  small enough  $\mathcal{L}_{\sigma+i\delta}$  has one or several eigenvalues near  $\lambda$ . The only information that we can guarantee about the nature of these eigenvalues is that the sum of their algebraic multiplicities is equal to the algebraic multiplicity of  $\lambda$ . This means that even if  $\lambda$  is a semisimple eigenvalue of  $\mathcal{L}_\sigma$  there is no guarantee about the fact that all the eigenvalues of  $\mathcal{L}_{\sigma+i\delta}$  are semisimple. To be convinced, let us consider the following example that comes from the finite dimensional setting. For all  $\delta \in \mathbb{R}^*$ , we define the matrix

$$A^\delta = \begin{pmatrix} 1 + \delta & \delta \\ 0 & 1 + \delta \end{pmatrix}$$

Clearly,  $A_\delta$  is a small analytic perturbation of the identity matrix  $I_2$ . The spectrum of  $A_\delta$  coincides with  $\{1 + \delta\}$  which converges when  $\delta \rightarrow 0$ , as expected, to  $\{1\}$  which is equal to the spectrum of  $I_2$ . We can also see that the algebraic multiplicity of  $1 + \delta$  (as an eigenvalue of  $A_\delta$ ) is equal to 2. While 1 is a semisimple eigenvalue of  $I_2$ ,  $1 + \delta$  is geometrically simple (there is a Jordan chain of length 2 composed of an eigenfunction and a generalized eigenfunction). This example shows that, in general, we are not able to guarantee that the eigenvalues of the perturbed problem and those of the unperturbed problem have the same nature. This explains, in a way, why we have made the Assumption 2.6.2 when we used the limiting absorption principle to define a physical solution to the scalar problem. The main result of this section is given by

**Proposition 3.3.4.** *Assume that  $\kappa_\sigma \neq -1$ . Let  $\lambda \in \Lambda(\mathcal{L}_\sigma)$  and let  $(\lambda_\delta)_\delta$  be a sequence<sup>1</sup> of elements of  $\Lambda(\mathcal{L}_{\sigma+i\delta})$  that converges to  $\lambda$  when  $\delta$  tends to 0. Consider a sequence  $(\varphi_\delta)_\delta$  of elements of  $\ker(\mathcal{L}_{\sigma+i\delta}(\lambda_\delta))$  such that  $\|\varphi_\delta\|_{H^1(\mathbb{S}^2)} = 1$  for  $\delta$  small enough. Then,  $(\varphi_\delta)_\delta$  converges (up to a sub-sequence), in  $H^1(\mathbb{S}^2)$ , to some  $\varphi_0 \in H^1(\mathbb{S}^2)$  that belongs to  $\ker(\mathcal{L}_\sigma(\lambda))$ .*

**Proof.** Since  $(\varphi_\delta)_\delta$  is bounded in  $H^1(\mathbb{S}^2)$ , one can extract a sub-sequence from it that converges (when  $\delta$  goes to 0) weakly in  $H^1(\mathbb{S}^2)$  and strongly in  $L^2(\mathbb{S}^2)$  to some  $\varphi_0 \in H^1(\mathbb{S}^2)$ . To simplify, this sub-sequence is still denoted by  $(\varphi_\delta)_\delta$ .

Since  $(\lambda_\delta)$  converges to  $\lambda$ , as  $\delta \rightarrow 0$ , one deduces that  $\varphi_0$  belongs to  $\ker(\mathcal{L}_\sigma(\lambda))$ . It remains, then, to explain why the convergence of  $(\varphi_\delta)_\delta$  to  $\varphi_0$  occurs in the strong sense. For this, we start by

<sup>1</sup>Here and in what follows, a sequence indexed with a non integer parameter refers to an indexed family of elements.

observing that since  $\mathcal{L}_{\sigma+i\delta}(\lambda_\delta)\varphi_\delta = 0$ , we have for all  $t \in \mathbb{R}$ ,

$$\begin{aligned} \langle \mathcal{L}_\sigma(t)\varphi_\delta, \varphi' \rangle &= -i\delta \int_{\mathbb{S}^2} \nabla_S \varphi_\delta \nabla_S \overline{\varphi'} + i\delta \lambda_\delta (\lambda_\delta + 1) \int_{\mathbb{S}^2} \varphi_\delta \overline{\varphi'} \\ &\quad + (t^2 + 1/4 + \lambda_\delta (\lambda_\delta + 1)) \int_{\mathbb{S}^2} \sigma \varphi_\delta \overline{\varphi'}. \end{aligned}$$

As  $(\varphi_\delta)_\delta$  is bounded in  $H^1(\mathbb{S}^2)$  and converges in  $L^2(\mathbb{S}^2)$  to  $\varphi_0$ , the previous identity shows that  $(\mathcal{L}_\sigma(t)\varphi_\delta)_\delta$  converges, as  $\delta \rightarrow 0$ , in  $((H^1(\mathbb{S}^2))^*)$ . Owing to Lemma 3.1, we know that for  $t$  large enough, the operator  $\mathcal{L}_\sigma(t)$  becomes isomorphism. This implies that  $(\varphi_\delta)_\delta$  converges, as  $\delta \rightarrow 0$ , strongly in  $H^1(\mathbb{S}^2)$  to some  $\varphi_1 \in H^1(\mathbb{S}^2)$ . By uniqueness of the limit in  $L^2(\mathbb{S}^2)$ , we deduce that  $\varphi_0 = \varphi_1$ . ■

### 3.4 The particular case of circular conical tips

In the previous paragraphs, the main spectral properties of the family  $(\mathcal{L}_\sigma(\lambda))_{\lambda \in \mathbb{C}}$  have been investigated in the case of a general smooth conical tip  $g \in \mathcal{C}_{per}^2([0; 2\pi])$ . The main goal of this paragraph is to study the particular case of circular conical points, in other words, when the function  $g$  coincides with a constant  $\alpha \in (0; \pi)$ . In this particular case, some of the results that we have established before can be improved. In addition to that some new results can be obtained.

The main idea is to take advantage from the fact that circular conical tips are rotationally symmetric. With this in mind, any function of the space  $H^1(\mathbb{S}^2)$  can be decomposed, by means of the Fourier decomposition, into a sum of separated variable functions. This will help us in getting a deeper information about the spectrum. More precisely, the spectrum of  $\mathcal{L}_\sigma$  can be characterized by means of a dispersion relation. Furthermore, since the eigenfunctions are also known explicitly, some results concerning the existence of generalized eigenvectors for the particular case of eigenvalues that are on the line  $\Re(\lambda) = -1/2$  can be obtained. This will help us studying the validity of Assumption 2.6.2 in this particular configuration.

#### 3.4.1 Dispersion relation

The goal of this part is to determine a dispersion relation that allows us to characterize the spectrum  $\mathcal{L}_\sigma$  when the contrast  $\kappa_\sigma \neq -1$ . By this, we mean finding a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  such that  $\lambda \in \Lambda(\mathcal{L}_\sigma) \iff f(\lambda) = 0$ . According to Lemma 3.2.2, we already know that when  $\kappa_\sigma \neq -1$  the spectrum of  $\mathcal{L}_\sigma$  is composed by discrete eigenvalues. Consequently, it suffices to find the set of  $\lambda \in \mathbb{C}$  for which the equation  $\mathcal{L}_\sigma(\lambda)u = 0$  has a non trivial solution in  $H^1(\mathbb{S}^2)$ . As in this particular geometry, the function  $\sigma$  is independent of the variable  $\varphi \in [0, 2\pi]$ , we then obtain the equivalence:  $\lambda \in \Lambda(\mathcal{L}_\sigma)$  if and only if

$$\exists u \in H^1(\mathbb{S}^2) \setminus \{0\} \text{ s.t. } -\frac{1}{\sin(\theta)} \partial_\theta (\sigma(\theta) \sin(\theta) \partial_\theta u) - \frac{\sigma(\theta)}{\sin(\theta)^2} \partial_\varphi^2 u = \lambda(\lambda + 1) \sigma(\theta) u \quad (3.5)$$

in which the last equation is written in the distributional sense. The key idea (which is also used in [92, 104]) is to use the fact that every function  $u \in H^1(\mathbb{S}^2)$  can be decomposed as

$$u(\theta, \varphi) = \sum_{m \in \mathbb{Z}} u_m(\theta) e^{im\varphi} \text{ where } u_m(\theta) = \frac{1}{2\pi} \int_0^{2\pi} u(\theta, \varphi) e^{-im\varphi} d\varphi \text{ for all } m \in \mathbb{Z}.$$

Note that the previous decomposition is nothing but the classical Fourier decomposition with respect to  $\varphi \in [0, 2\pi]$ . It is interesting to observe that, in the decomposition above, since  $u \in H^1(\mathbb{S}^2)$  one can show that for all  $m \in \mathbb{Z}$ , the function  $u_m(\theta)$  is such that  $\sqrt{\sin(\theta)} u_m, u_m / \sqrt{\sin(\theta)}$

and  $\sqrt{\sin(\theta)} d_\theta u_m$  belong to  $L^2(0; \pi)$ . Now, if  $u \in H^1(\mathbb{S}^2)$  is a solution of (3.5), one deduces that for all  $m \in \mathbb{Z}$  the function  $u_m$  is such that

$$-\frac{1}{\sin(\theta)} \frac{d}{d\theta} (\sigma(\theta) \sin(\theta) \frac{d}{d\theta}) u_m + \frac{\sigma(\theta)}{\sin(\theta)^2} m^2 u_m = \lambda(\lambda + 1) \sigma(\theta) u_m \quad \text{for all } m \in \mathbb{Z}.$$

To proceed, let us denote, respectively, by  $u_{1m}, u_{2m}$  the restriction of  $u_m$  to  $(0; \alpha)$  and to  $(\alpha; \pi)$ . With this in mind, we arrive to the conclusion that for all  $m \in \mathbb{Z}$  the function  $u_m$  satisfies following transmission problem:

$$\begin{cases} -\frac{1}{\sin(\theta)} \frac{d}{d\theta} (\sin(\theta) \frac{du_{1m}}{d\theta}) + \frac{1}{\sin(\theta)^2} m^2 u_{1m} = \lambda(\lambda + 1) u_{1m} & \text{in } (0; \alpha) \\ -\frac{1}{\sin(\theta)} \frac{d}{d\theta} (\sin(\theta) \frac{du_{2m}}{d\theta}) - \frac{1}{\sin(\theta)^2} m^2 u_{2m} = \lambda(\lambda + 1) u_{2m} & \text{in } (\alpha; \pi) \\ u_{1m}(\alpha) = u_{2m}(\alpha), \quad \sin(\alpha) \frac{du_{1m}}{d\theta}(\alpha) = \kappa_\sigma \sin(\alpha) \frac{du_{2m}}{d\theta}(\alpha). \end{cases}$$

At this stage, and in order to write simpler equations that we can solve by means of classical special functions, we need to perform the change of variable  $\theta \rightarrow \gamma = \cos(\theta)$ . After this change of variable, the new function, which is still denoted by  $u_m(\gamma)$ , is a solution to the following transmission problem: For all  $m \in \mathbb{Z}$  we have

$$\begin{cases} \frac{d}{d\gamma} ((1 - \gamma^2) \frac{d}{d\gamma} u_{1m}(\gamma)) + \lambda(\lambda + 1) u_{1m}(\gamma) - \frac{m^2}{(\gamma^2 - 1)} u_{1m}(\gamma) = 0 & \text{in } (\cos(\alpha); 1] \\ \frac{d}{d\gamma} ((1 - \gamma^2) \frac{d}{d\gamma} u_{2m}(\gamma)) + \lambda(\lambda + 1) u_{2m}(\gamma) - \frac{m^2}{(\gamma^2 - 1)} u_{2m}(\gamma) = 0 & \text{in } [-1; \cos(\alpha)) \\ u_{1m}(\cos(\alpha)) = u_{2m}(\cos(\alpha)), \quad \sin(\alpha)^2 \frac{d}{d\gamma} u_{1m}(\cos(\alpha)) = \kappa_\sigma \sin(\alpha)^2 \frac{d}{d\gamma} u_{2m}(\cos(\alpha)). \end{cases} \quad (3.6)$$

Starting from the fact that the function  $(\theta, \varphi) \mapsto u_m(\theta) e^{-im\varphi} \in H^1(\mathbb{S}^2)$  and using the change of variable  $\theta \rightarrow \gamma$ , one can show that the functions  $\gamma \mapsto u_m(\gamma), u_m(\gamma)/\sqrt{1 - \gamma^2}, \sqrt{1 - \gamma^2} d_\gamma u_m(\gamma)$  belong to the space  $L^2(-1; 1)$ . This implies, in particular thanks to the fact that  $\gamma \mapsto u_m(\gamma)/\sqrt{1 - \gamma^2} \in L^2(-1; 1)$ , that  $u_m(\gamma) \rightarrow 0$  as  $\gamma \rightarrow \pm 1$ . As a result, we are only interested in the solutions to (3.6) that vanish near  $\gamma = \pm 1$ .

The equation (3.6) tells us that in each of intervals  $(\cos(\alpha); 1]$  and  $[-1; \cos(\alpha))$  the function  $u_m(\gamma)$  is a solution to the associated Legendre equation in which  $\lambda \in \mathbb{C}$  plays the role of the degree and  $m \in \mathbb{Z}$  is the order.

The literature about the associated Legendre's equations is very rich especially when  $\lambda \in \mathbb{N}$ , in that case the solutions are the associated Legendre polynomials (for instance see [3]). In addition to that, the approximation of these functions is available in almost all scientific computing software.

On the other hand, when  $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ , many results are also available (see [3]) but when it comes to the approximation of the associated Legendre functions, almost all open source software do not provide it.

In order to make this chapter self-contained, we shall present, in §3.6.2, a brief overview about the basic properties of these functions. Furthermore, we will explain how to write a C++ program that can be used to approximate these functions for the general case  $\lambda \in \mathbb{C}$  and  $m \in \mathbb{Z}$ .

To, proceed, for all  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C} \setminus \llbracket -m, -1 \rrbracket \cup \llbracket 0, m - 1 \rrbracket^2$ , we denote by  $P_\lambda^m(x)$  (with  $x \in (-1; 1)$ ) the associated Legendre function of first kind of order  $m$  and of degree  $\lambda$  and by  $(P_\lambda^m)'(x)$  its derivative with respect to  $x$ . Besides, for all  $m \in \mathbb{N}$ , and  $\lambda \in \mathbb{C} \setminus \llbracket -m, -1 \rrbracket \cup \llbracket 0, m - 1 \rrbracket$ , we introduce the functions  $f^m(\lambda, \kappa_\sigma)$  such that

$$f^m(\lambda, \kappa_\sigma) = \kappa_\sigma P_\lambda^m(-\cos(\alpha)) (P_\lambda^m)'(\cos(\alpha)) + P_\lambda^m(\cos(\alpha)) (P_\lambda^m)'(-\cos(\alpha)).$$

<sup>2</sup>For all  $a, b \in \mathbb{Z}$ , we denote by  $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ .



**Proposition 3.4.1.** *Assume that  $\kappa_\sigma \neq -1$ . We have the equivalence*

$$\lambda \in \Lambda(\mathcal{L}_\sigma) \iff \exists m \in \mathbb{N} \text{ such that } \lambda \in \mathbb{C} \setminus \llbracket -m, -1 \rrbracket \cup \llbracket 0, m-1 \rrbracket \text{ and } f^m(\lambda, \kappa_\sigma) = 0. \quad (3.7)$$

**Proof.** Thanks to the modal decomposition (3.6), one can say that the problem  $\mathcal{L}_\sigma(\lambda)u = 0$  has a non trivial solution in  $H^1(\mathbb{S}^2)$  if and only there exists at least one  $m \in \mathbb{Z}$  for which the equation (3.6) has a non zero solution. Given that the function  $u_m$  must vanish at  $\gamma = \pm 1$ , and by using the results of §3.6.2, we infer that this is possible if and only if  $\lambda \in \mathbb{C} \setminus \llbracket -|m|, -1 \rrbracket \cup \llbracket 0, |m| - 1 \rrbracket$ . In that case, we have

$$u_m(\gamma) = \begin{cases} A P_\lambda^m(\gamma) & \text{in } (\cos(\alpha); 1] \\ B P_\lambda^m(-\gamma) & \text{in } [-1; \cos(\alpha)). \end{cases}$$

The continuity and the transmission conditions satisfied by  $u_m$  at  $\gamma = \cos(\alpha)$  lead us to the following system of equations:

$$\begin{cases} A P_\lambda^m(\cos(\alpha)) = B P_\lambda^m(-\cos(\alpha)) \\ A \sin^2(\alpha) (P_\lambda^m)'(\cos(\alpha)) = -\kappa_\sigma \sin^2(\alpha) (P_\lambda^m)'(-\cos(\alpha)). \end{cases}$$

The previous linear system of equations has a non trivial solution if and only if

$$\kappa_\sigma P_\lambda^m(\cos(\alpha)) (P_\lambda^m)'(-\cos(\alpha)) = -P_\lambda^m(-\cos(\alpha)) (P_\lambda^m)'(\cos(\alpha)).$$

Given that the functions  $P_\lambda^m$  and  $P_\lambda^{-m}$  are collinear (see (3.16)), we obtain the wanted result. ■

The proof of the previous proposition allows us to find the expression of the eigenfunctions associated to  $\lambda \in \Lambda(\mathcal{L}_\sigma)$ . Indeed, if we denote by

$$\mathcal{A}(\lambda) := \{m \in \mathbb{N} \mid \exists \lambda \in \mathbb{C} \setminus \llbracket -m, -1 \rrbracket \cup \llbracket 0, m-1 \rrbracket \text{ and } f^m(\lambda, \kappa_\sigma) = 0\}, \quad (3.8)$$

we can easily prove that

$$\text{Ker}(\mathcal{L}_\sigma(\lambda)) = \text{span}\{u^m, u^{-m} \mid m \in \mathcal{A}(\lambda)\}, \quad (3.9)$$

where the functions  $u^{\pm m}$  are defined as follows

$$u_1^{\pm m}(\theta, \varphi) = \begin{cases} P_\lambda^m(-\cos(\alpha)) P_\lambda^m(\cos(\theta)) e^{\pm im\varphi} & \text{if } \theta \in (0; \alpha) \\ P_\lambda^m(\cos(\alpha)) P_\lambda^m(-\cos(\theta)) e^{\pm im\varphi} & \text{if } \theta \in (\alpha; \pi). \end{cases}$$

### 3.4.2 Expression of the critical interval

In this paragraph, we shall explain how to find an explicit expression of the critical interval  $I_\Sigma$  in the particular case of circular conical tips. Recall that for a general interface  $\Sigma$ ,  $I_\Sigma$  is defined as the set of contrasts  $\kappa_\sigma$  for which the problem (2.2) is ill-posed in the Fredholm sense. In Theorem 2.6.1, we have proved that for the case of an interface with smooth conical tip, the critical interval  $I_\Sigma$  can be defined as the set of contrasts for which the problem (2.2) has propagating singularities (see Definition 2.6.1) or equivalently the set of  $\kappa_\sigma$  for which  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} \neq \emptyset$ .

To simplify notations, we shall denote by  $I_\alpha$  the critical interval in the case of an interface that has a circular conical tip of opening angle  $\alpha$  (i.e.  $g(\varphi) = \alpha$ ).

As we have seen in §2.2, the determination of  $I_\alpha$  is directly related to the determination of the essential spectrum of the Neumann-Poincaré operator on the infinite cone

$$W_\alpha := \{\mathbf{x} = r(\sin(\alpha) \cos(\varphi), \sin(\alpha) \sin(\varphi), \cos(\alpha)); r \in \mathbb{R}_+, \varphi \in (0; 2\pi)\}.$$

This latter question was investigated in details in [92, 104]. But it seems that the results obtained there are not sufficient to obtain a simple expression of the critical interval.

During an exchange with Karl-Mikael Perfekt, he told us that the missing argument is to show that when  $\alpha \in (0; \pi/2)$  (resp.  $\alpha \in (\pi/2; \pi)$ ), the spectrum of the Neumann-Poincaré operator is positive (resp. negative). In this paragraph, we are going to explain how to combine the results of [92, 104] and the T-coercivity approach in order to obtain an explicit expression of  $I_\alpha$  and by the way, we also answer the question about the sign of the essential spectrum of the Neumann-Poincaré operator that was left unanswered in [92, 104].



### Expression of the critical interval

According to Proposition 3.4.1, one can say that  $\lambda = -1/2 + it$  (with  $t \in \mathbb{R}$ ) belongs to  $\Lambda(\mathcal{L})$  if and only if there exists some  $m \in \mathbb{N}$  such that

$$\kappa_\sigma P_{-1/2+it}^m(-\cos(\alpha))(P_{-1/2+it}^m)'(\cos(\alpha)) + P_{-1/2+it}^m(\cos(\alpha))(P_{-1/2+it}^m)'(-\cos(\alpha)) = 0.$$

In the literature, the functions  $P_{-1/2+it}^m$  (with  $m \in \mathbb{N}$  and  $t \in \mathbb{R}$ ) are known as the conical functions or the Mehler functions of the first kind. They play an important role in the area of mathematical physics (see [87, 80, 122, 123] for more details). The study of these functions was the subject of the book [146]. Some basic properties of these functions are, briefly, recalled in §3.6.2. Let us introduce, for all  $m \in \mathbb{N}$ , the function  $a_m : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$a_m(t) = \frac{P_{-1/2+it}^m(\cos(\alpha))(P_{-1/2+it}^m)'(-\cos(\alpha))}{P_{-1/2+it}^m(-\cos(\alpha))(P_{-1/2+it}^m)'(\cos(\alpha))} \quad \text{for all } t \in \mathbb{R}.$$

As it is proved in §3.6.2, for all  $m \in \mathbb{Z}$  and all  $t \in \mathbb{R}$  the function  $P_{-1/2+it}^m$  as well as its derivative are real valued. The functions  $a_m$  are then real valued. Given that these functions are continuous, we then denote, for all  $m \in \mathbb{N}$ , by  $I_m$  the interval

$$I_m = \{-a_m(t) : 0 \leq t \leq +\infty\}.$$

Given that  $I_\alpha$  coincides with the set of contrasts  $\kappa_\sigma$  for which  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2} \neq \emptyset$ , we then obtain, thanks to Proposition 3.4.1, the following

**Proposition 3.4.2** (First definition of the critical interval). *Assume that  $\alpha \in (0; \pi)$ . We have*

$$I_\alpha = \bigcup_{m \in \mathbb{N}} I_m.$$

Observe that when  $\alpha = \pi/2$ , the intervals  $I_m$  are, all, reduced to the singleton  $\{-1\}$ . In Figure 3.6, we represent the functions  $\tau \rightarrow -a_m(\tau)$  for  $m = 0, 1, 2, 3$  and  $\tau \in \mathbb{R}_+$  (the approximation of the conical functions is achieved by using hypergeometric function of Matlab and by using the results of §3.6.2).

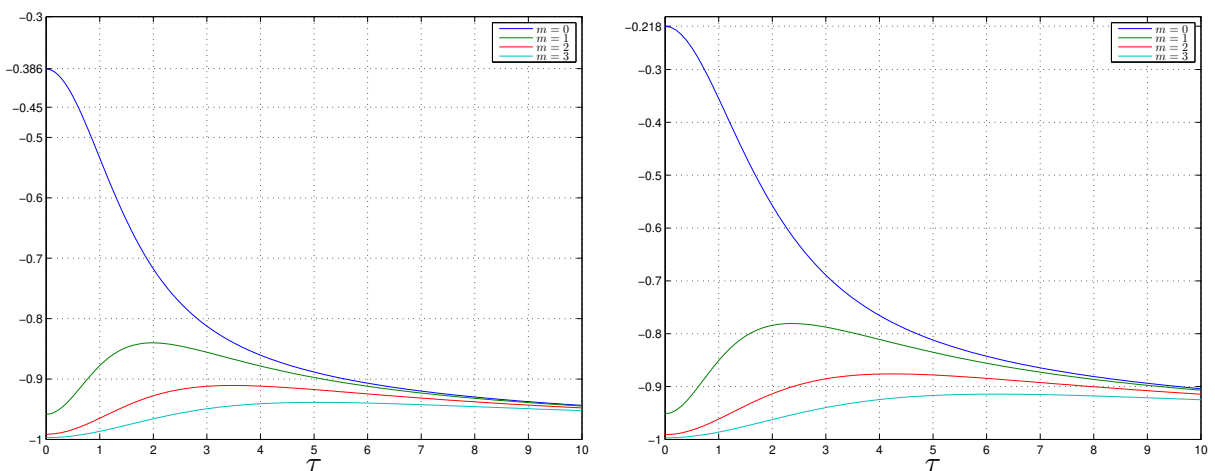


Figure 3.6: The graphs of the functions  $\tau \mapsto a_m(\tau)$  for the cases  $\alpha = \pi/3$  (left) and  $\alpha = \pi/4$  (right).

What we can take away from this figure is the fact that, when  $\tau$  tends to  $+\infty$  all the functions  $-a_m$  tend to a fixed valued independent of  $m$ . Moreover, we also observe that the range of the functions  $-a_m$  is contained in the one of  $-a_0$ . These observations, will be confirmed theoretically, in the next two Propositions.

**Remark 3.4.1.** One can also observe that the curves of the functions  $\tau \mapsto -a_m(\tau)$  are all disjoint. Unfortunately we do not succeed in proving this observation. Note that this is equivalent to say that when  $\kappa_\sigma \in I_\alpha \setminus \{-1\}$  then for all  $\lambda \in \ell_{-1/2}$ , we have  $\dim(\ker(\mathcal{L}_\sigma(\lambda))) \leq 2$ . More precisely, this is equivalent to say that for all  $\lambda = -1/2 + it \in \ell_{-1/2}$  we have

$$\dim(\ker(\mathcal{L}_\sigma(\lambda))) = \begin{cases} 1 & \text{if } \kappa_\sigma = -a_0(|t|) \\ 2 & \text{if } \exists m \in \mathbb{N}^* \text{ such that } \kappa_\sigma = -a_m(|t|). \end{cases}$$

In particular, when  $\alpha = \pi/4$  and  $\kappa_\sigma = -0.8$ , one can see from Figure 3.6 that there exist 3 eigenvalues of  $\mathcal{L}_\sigma$  that belong to  $\ell_{-1/2}^+ := \ell_{-1/2} \cap \{\lambda \in \mathbb{C} \mid 0 < \Im m(\lambda)\}$ . These eigenvalues are approximately equal to  $\lambda_1 = -1/2 + 1.6i$ ,  $\lambda_2 = -1/2 + 3.6i$  and  $\lambda_3 = -1/2 + 4.7i$ . While  $\lambda_3$  is geometrically simple,  $\lambda_1$  and  $\lambda_2$  have geometric multiplicity equal to 2. The corresponding eigenfunctions for  $\lambda_1, \lambda_2$  and  $\lambda_3$  are known explicitly. In Figures 3.7-3.8, we display an eigenfunction associated to  $\lambda_3$  and another one associated to  $\lambda_1$ .

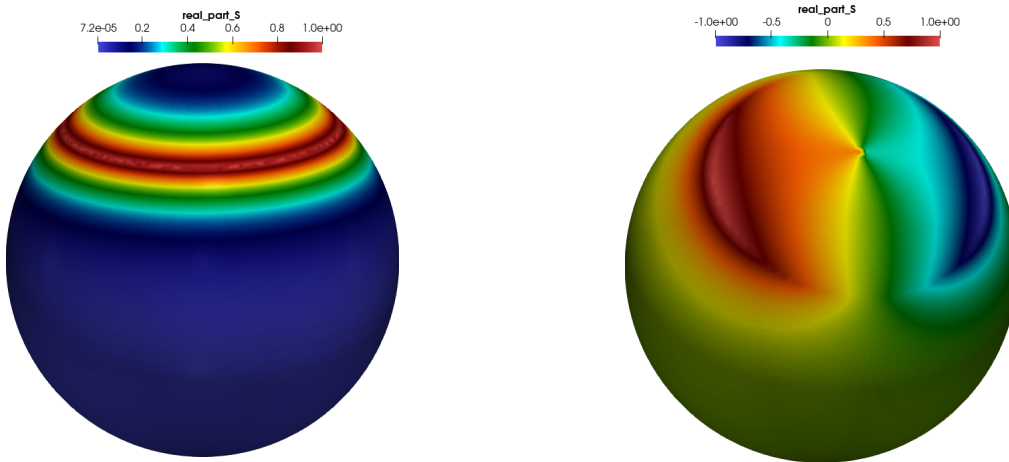


Figure 3.7: An eigenfunction associated to  $\lambda_3$ . Figure 3.8: An eigenfunction associated to  $\lambda_1$ .

To proceed, we have the

**Proposition 3.4.3.** Let  $\alpha \in (0; \pi)$ . Then for all  $m \in \mathbb{N}$ , we have  $\lim_{t \rightarrow +\infty} a_m(t) = 1$ .

It is worth mentioning that this result has been already proved in a very brief way in [122]. Again, in order to make our work self-contained, we will propose a more detailed proof.

**Proof.** The idea is to use an asymptotic expansions of  $P_{-1/2+it}^m(\cos(\theta))$  when  $t$  tends to  $+\infty$  for a given  $\theta \in (0; \pi)$  and a fixed  $m \in \mathbb{N}$ . According to [146], one has for all  $\theta \in (0; \pi)$  and  $m \in \mathbb{N}$  the expansion

$$P_{-1/2+it}^m(\cos(\theta)) = \frac{t^{m-1/2} e^{t\theta}}{\sqrt{2\pi} \sin(\theta)} \left( 1 - \frac{m^2 - 1/4}{2t} \cot(\theta) + O(1/t^2) \right).$$

Using the recurrence relation (see §3.6.2)

$$(P_{-1/2+it}^m)'(\cos(\theta)) = \frac{m \cos(\theta)}{\sin(\theta)^2} P_{-1/2+it}^m(\cos(\theta)) - \frac{1}{\sin(\theta)} P_{-1/2+it}^{m+1}(\cos(\theta)),$$

we infer that

$$(P_{-1/2+it}^m)'(\cos(\theta)) / P_{-1/2+it}^m(\cos(\theta)) = \frac{m \cos(\theta)}{\sin(\theta)^2} - \frac{t}{\sin(\theta)} + O(1) \underset{t \rightarrow +\infty}{\sim} -\frac{t}{\sin(\theta)}.$$

Exchanging  $\theta$  with  $\pi - \theta$ , we find that

$$\frac{(\mathbf{P}_{-1/2+it}^m)'(-\cos(\theta))}{\mathbf{P}_{-1/2+it}^m(-\cos(\theta))} \underset{t \rightarrow +\infty}{\sim} -\frac{t}{\sin(\theta)}.$$

The lemma is then proved by taking  $\theta = \alpha$  in the previous two equivalences and then considering their quotient. ■

The last proposition tells us that for all  $\alpha \in (0; \pi)$ , the particular value  $\kappa_\sigma = -1$  belongs to the critical interval  $I_\alpha$ . Indeed, we have shown that  $-1$  is an accumulation point of  $I_\alpha$  but since the latter is a closed subset of  $\mathbb{R}_-$ , it follows that  $-1 \in I_\alpha$ . In the rest of this paragraph, we are going to present two key results that will allow us to obtain an explicit expression of  $I_\alpha$ . The first result is given in the following

**Proposition 3.4.4.** *Assume that  $\alpha \in (0; \pi/2)$ . The critical interval  $I_\alpha$  is a subset of  $[-1; 0]$ .*

**Proof.** It was proved in [50, Theorem 1.6.5], by means of the T-coercivity approach, that when the contrast  $\kappa_\sigma$  does not belong to  $[-1, -\frac{1 - \cos(\alpha)}{1 + \cos(\alpha)}]$  the problem 2.15 is well-posed. This implies

$$\text{that } I_\alpha \subset [-1, -\frac{1 - \cos(\alpha)}{1 + \cos(\alpha)}] \subset [-1, 0]. \quad \blacksquare$$

The second result that we need in order to obtain an explicit expression of  $I_\alpha$  is more involved to be proved. In fact, as we shall see, it is an adaptation of some of the results obtained by Johan Helsing and Karl-Mikael Perfekt (see [92]) in the context of the study of the essential spectrum of the Neumann-Poincaré operator in the case of rotationally symmetric conical points.

**Proposition 3.4.5.** *Assume that  $\alpha \in (0; \pi/2)$ . Then for all  $m \in \mathbb{N}$  and all  $t \in \mathbb{R}$ , we have*

$$|a_m(t)| \leq |a_0(0)|.$$

**Proof.** Since the function  $x \rightarrow (x - 1)/(x + 1)$  is increasing in  $(-1; 0)$ , and thanks to Proposition 3.4.4, it suffices to show that

$$(a_m(t) - 1)/(a_m(t) + 1) \leq (a_0(0) - 1)/(a_0(0) + 1).$$

On the other hand using the same notations of [92], we can write that

$$(a_m(t) - 1)/(a_m(t) + 1) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} s^{1/2+it} e^{-im\varphi} K(s) ds d\varphi$$

in which  $K(s)$  is a real valued positive function that is associated to the modal kernel of the Neumann-Poincaré operator (i.e. associated to the space of functions that have the form  $u(\theta, \varphi) = u_m(\theta)e^{-im\varphi} \in H^1(\mathbb{S}^2)$ ) on the infinite cone  $W_\alpha = \{r(\sin(\alpha) \cos(\varphi), \sin(\alpha) \sin(\varphi), \cos(\alpha)); r \in \mathbb{R}_+, \varphi \in (0; 2\pi)\}$  (see [92, 104] for more details). It is important to note that the positivity of the function  $K(s)$  is a consequence of the convexity of the interior of the cone  $W_\alpha$ . As a result one can easily see that

$$\begin{aligned} 0 \leq \frac{a_m(t) - 1}{a_m(t) + 1} &\leq \left| \frac{1}{2\pi} \int_0^{+\infty} \int_0^\pi s^{1/2+it} e^{-im\varphi} K(s) ds d\varphi \right| \\ &\leq \frac{1}{2\pi} \int_0^{+\infty} \int_0^\pi s^{1/2} K(s) ds d\varphi = \frac{a_0(0) - 1}{a_0(0) + 1}. \end{aligned}$$

which ends the proof. ■

We now have all the needed tools to state the final expression of the critical interval.

**Proposition 3.4.6.** *[Final expression of the critical interval] Let  $\alpha \in (0; \pi/2)$ , we have  $I_\alpha = I_0 = [-1, -a_0(0)]$ . The value of  $a_0(0)$  is given by*

$$a_0(0) = \frac{{}_2F_1(1/2, 1/2, 1, \cos^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \sin^2(\alpha/2))}{{}_2F_1(1/2, 1/2, 1, \sin^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \cos^2(\alpha/2))}$$

in which  ${}_2F_1$  stands for the Gauss hypergeometric function (see §3.6.2).

**Proof.** The proposition 3.4.4 tells us that for all  $m \in \mathbb{N}$ , the interval  $I_m$  is a connected subset of  $[-1, 0]$ . Thanks to Proposition 3.4.3, we can say that for all  $m \in \mathbb{N}$  the interval  $\overline{I_m}$  has the form  $[-1, -\gamma_m]$  with  $\gamma_m \in [0, 1]$ . This implies that  $I_\alpha$  has the form  $I_\alpha = [-1, -\gamma]$  where the value of  $\gamma$  is given by  $\gamma = \sup_{m \in \mathbb{N}} \gamma_m$ . To finish the proof, one has to use the Proposition 3.4.5, to deduce  $\gamma = \gamma_0 = a_0(0)$ . Thanks to the results of §3.6.2, in particular the relation (3.19), we find that

$$a_0(0) = \frac{P_{-1/2}^0(\cos(\alpha))(P_{-1/2}^0)'(-\cos(\alpha))}{P_{-1/2}^0(-\cos(\alpha))(P_{-1/2}^0)'(\cos(\alpha))} = \frac{{}_2F_1(1/2, 1/2, 1, \cos^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \sin^2(\alpha/2))}{{}_2F_1(1/2, 1/2, 1, \sin^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \cos^2(\alpha/2))}.$$

■

**Remark 3.4.2.** *When the opening angle  $\alpha$  belongs to  $(\pi/2; \pi)$ , the critical interval can be determined by exchanging the roles of  $\Omega_1$  and  $\Omega_2$ . More precisely, one has  $I_\alpha = 1/I_{\pi-\alpha}$ . When  $\alpha = \pi/2$ , one has  $I_{\pi/2} = \{-1\}$ .*

Using the results of §2.2, in particular Lemma 2.2.4, we obtain the

**Lemma 3.4.1.** *Let  $\alpha \in (0; \pi/2)$  and denote by  $W_\alpha$  the cone*

$$W_\alpha = \{r(\sin(\alpha) \cos(\varphi), \sin(\alpha) \sin(\varphi), \cos(\alpha)); r \in \mathbb{R}_+, \varphi \in (0; 2\pi)\}$$

. *Then the essential spectrum of Neumann-Poincaré operator  $\sigma_{ess}(K_{W_\alpha}, H^{1/2}(W_\alpha))$  is positive and given by*

$$\sigma_{ess}(K_{W_\alpha}, H^{1/2}(W_\alpha)) = [0, \frac{a_0(0) - 1}{2(a_0(0) + 1)}].$$

The previous results require some comments.

1. Unlike the 2D configuration, for a given contrast  $\kappa_\sigma \in I_\alpha \setminus \{-1\}$ , more than two propagating singularities can exist. In fact, as can be seen in Figure 3.6 and as justified by Proposition 3.4.3, when the contrast  $\kappa_\sigma$  approaches  $-1$ , the number of propagation singularities that appear tends to infinity.
2. It can be proved that the critical interval widens as the opening angle  $\alpha \in (0, \pi/2)$  gets smaller. This can be observed from Figure 3.9. In particular, we show that  $I_\alpha$  tends to  $[-1, 0]$  as  $\alpha \rightarrow 0$ .
3. As mentioned above, the critical interval is located on one side of the  $-1$ . value. This is a bit surprising compared to the 2D case. The reader may wonder if this is the case for all smooth conical tips. To the best of our knowledge, this question remains open.

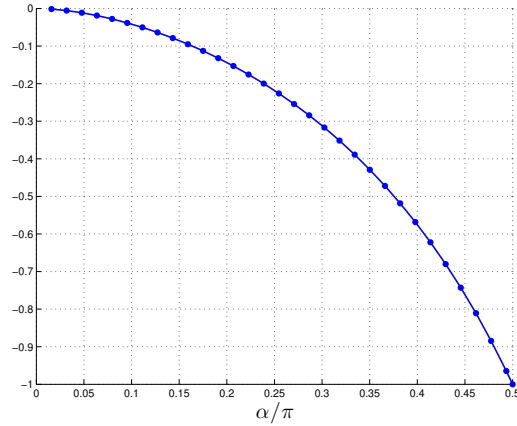


Figure 3.9: The behaviour of  $-a_0(0)$  as a function of  $\alpha \in (0; \pi/2)$ .

### 3.4.3 On the validity of Assumption 2.6.2 for circular conical tips

We are interested here in the question of the validity of Assumption 2.6.2 in the case of circular conical tips. Recall that the Assumption 2.6.2 is valid if and only if:

1. All the elements of  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  are semi-simple.
2. Near any element of  $\Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$ , there exists a unique eigenvalue of  $\mathcal{L}_{\sigma+i\delta}$  for  $\delta$  small enough.

#### Validity of the first condition

**Proposition 3.4.7.** *Assume that  $\kappa_\sigma \in I_\alpha \setminus \{-1\}$  and let  $\lambda_0 = -1/2 + it_0 \in \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$ . Suppose that there exists a unique  $m_0 \in \mathbb{N}$  such that  $a_{m_0}(|t_0|) = -\kappa_\sigma$ . Then, we have the equivalence*

$$\lambda_0 \text{ is a semi-simple eigenvalue of } \mathcal{L}_\sigma \text{ iff } \exists \varphi \in \text{Ker}(\mathcal{L}_\sigma(\lambda_0)) \text{ s.t. } t_0 \int_{\mathbb{S}^2} \sigma |\varphi|^2 \neq 0.$$

**Proof.** As explained in Remark 3.4.1, we know, under the assumption made on  $\lambda_0$ , that  $\iota_g(\lambda_0) \leq 2$ . If  $\iota_g(\lambda_0) = 1$ , the wanted result is a direct application of Lemma 3.2.3. So, let us suppose that  $\iota_g(\lambda_0) = 2$  (this implies that  $m_0 \in \mathbb{N}^*$ ). Thanks to the assumption made on  $\lambda_0$  we infer that  $\mathcal{A}(\lambda_0) = \{\pm m_0\}$  (the definition of  $\mathcal{A}(\lambda_0)$  is given in (3.8)). Moreover, we also know that  $\ker(\mathcal{L}_\sigma(\lambda)) = \text{span}(u_1^{m_0}, u_2^{m_0})$  where  $u_1^{m_0}, u_2^{m_0}$  have the form  $u_1^{m_0}(\theta, \varphi) = f(\theta)e^{im_0\varphi}$ ,  $u_2^{m_0} = f(\theta)e^{-im_0\varphi}$  in which  $f$  is a real valued function. By observing that

$$\int_{\mathbb{S}^2} \sigma u_1^{m_0} \overline{u_2^{m_0}} d\omega = 0 \text{ and } \int_{\mathbb{S}^2} \sigma |u_1^{m_0}|^2 d\omega = \int_{\mathbb{S}^2} \sigma |u_2^{m_0}|^2 d\omega = \int_{\mathbb{S}^2} \sigma |f|^2 d\omega, \quad (3.10)$$

we infer that for all  $\alpha, \beta \in \mathbb{C}$  the function  $\varphi = \alpha u_1^{m_0} + \beta u_2^{m_0}$  satisfies

$$\int_{\mathbb{S}^2} \sigma |\varphi|^2 d\omega = (|\alpha|^2 + |\beta|^2) \int_{\mathbb{S}^2} \sigma |f|^2 d\omega.$$

This means that  $\exists \varphi \in \text{Ker}(\mathcal{L}_\sigma(\lambda_0))$  s.t.  $t_0 \int_{\mathbb{S}^2} \sigma |\varphi|^2 \neq 0 \iff t_0 \int_{\mathbb{S}^2} \sigma |f|^2 d\omega \neq 0$ . By using Proposition 3.2.1 and owing to (3.10), we obtain the wanted result.  $\blacksquare$

As mentioned before, the figure 3.6 shows that the hypothesis of the previous proposition is valid, but unfortunately we are not able to prove it theoretically. Moreover, the previous result gives us, then, a very simple way to check if a  $\lambda \in \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  is a semi-simple or not. Now, let us explain how to use the previous proposition in order to find the set  $J$  of contrasts  $\kappa_\sigma$  for which

there exists  $\lambda \in \ell_{-1/2} \cap \Lambda(\mathcal{L}_\sigma)$  which is not semi-simple. Given that  $I_\alpha = \bigcup_{m \in \mathbb{N}} I_m$ , it is enough to find for each  $m \in \mathbb{N}$  the  $J_m := J \cap I_m$ . Since the interval  $I_m$  corresponds to the range of the function  $a_m$  and given that for all  $0 < t$  the function

$$u_t^m(\theta, \varphi) = \begin{cases} P_{-1/2+it}^m(-\cos(\alpha)) P_{-1/2+it}^m(\cos(\theta)) e^{im\varphi} & \text{if } \theta \in (0; \alpha) \\ P_{-1/2+it}^m(\cos(\alpha)) P_{-1/2+it}^m(-\cos(\theta)) e^{im\varphi} & \text{if } \theta \in (\alpha; \pi). \end{cases} \quad (3.11)$$

belongs to  $\text{Ker}(\mathcal{L}_{\sigma_t}(-1/2 + it))$  where  $\sigma_t(\theta) = 1$  for  $\theta < \alpha$  and  $\sigma_t(\theta) = -a_m(t)$  for  $\alpha < \theta < \pi$ , we can then write that

$$J_m = \{-a_m(t) \mid t \in \mathbb{R}_+, t \int_{\mathbb{S}^2} \sigma_t |u_t^m|^2 d\omega = 0\} = \{-a_m(0)\} \cup \{-a_m(t) \mid t \in \mathbb{R}_+, \int_{\mathbb{S}^2} \sigma_t |u_t^m|^2 d\omega = 0\}.$$

For all  $m \in \mathbb{N}$ , we define the function  $b_m : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $b_m(t) = \text{sign}(\int_{\mathbb{S}^2} \sigma_t |u_t^m|^2 d\omega)$  for all  $t \in \mathbb{R}_+$ , where the function  $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$  is such that  $\text{sign}(\pm x) = \pm 1$  if  $0 < \pm x$  and  $\text{sign}(0) = 0$ . In Figure 3.10, we display the curves of the functions  $a_m, b_m$  for  $m = 0, \dots, 3$  for  $\alpha = \pi/4$ .

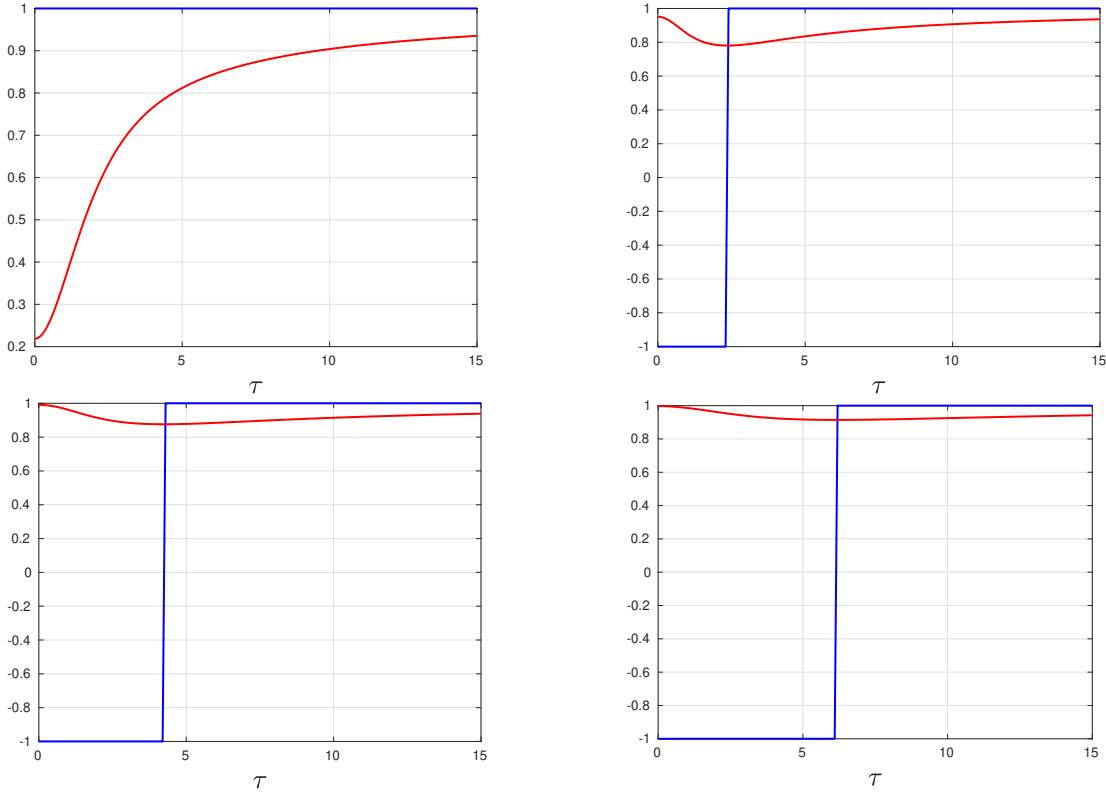


Figure 3.10: Curves of  $a_m$  (in red) and  $b_m$  (in blue) for  $m = 0$  (top left),  $m = 1$  (top right),  $m = 2$  (bottom left) and  $m = 3$  (bottom right).

The approximation of the integral in the expression of the functions  $b_m$  has been done using the `integral` function of `MATLAB`. What we can take from these results is the following fact: it seems that for all  $m \in \mathbb{N}$ , the set  $J_m$  corresponds to the set  $\{-a_m(0)\} \cup \{-a_m(t) \mid 0 < t \text{ and } a_m'(t) = 0\}$ . We also notice that for the case  $m = 0$ , generalized eigenfunction exists only when  $\kappa_\sigma = -a_0(0)$  (which corresponds to one of bounds of  $I_0$ ). For the case  $m \in \mathbb{N}$  the situation seems to be different: it seems that for all  $1 < m$  eigenvalues (that belong to  $\ell_{-1/2}$ ) with generalized eigenfunctions exist for two particular values of  $\kappa_\sigma$ . The first one is when  $\kappa_\sigma$  coincides with the opposite of the minimum of  $a_m$ , this value corresponds to one of the bounds on  $I_m$  (the other bound is  $-1$ ). The second one corresponds to the case  $\kappa = -a_m(0) \in \overset{\circ}{I}_m$ .

Given that the function  $a_m$  is in fact a function of  $t^2$ , we infer that for all  $m \in \mathbb{N}$  we have  $a'_m(0) = 0$ . Thus, can then formulate the following conjecture:

$$J_m = \{-a_m(t) \mid 0 \leq t \text{ and } a'_m(t) = 0\} \text{ for all } m \in \mathbb{N}.$$

Unfortunately, we are not able to prove the previous equality (which seems to be true). However, we succeed in proving the following

**Proposition 3.4.8.** *Assume that  $\kappa_\sigma \in I_\alpha \setminus \{-1\}$  and let  $\lambda_0 = -1/2 + it_0 \in \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  such that there exists a unique  $m_0 \in \mathbb{N}$  such that  $a_{m_0}(|t_0|) = -\kappa_\sigma$ . Then, we have the implications:*

$$a'_{m_0}(t_0) = 0 \implies \lambda_0 \text{ has generalized eigenfunctions} \implies \iota_g(\lambda_0) < \iota_a(\lambda_0).$$

**Proof.** We know that for all  $0 \leq t$  the function

$$u_t^m(\theta, \varphi) = \begin{cases} P_{-1/2+it}^m(-\cos(\alpha)) P_{-1/2+it}^m(\cos(\theta)) e^{im\varphi} & \text{if } \theta \in (0; \alpha) \\ P_{-1/2+it}^m(\cos(\alpha)) P_{-1/2+it}^m(-\cos(\theta)) e^{im\varphi} & \text{if } \theta \in (\alpha; \pi). \end{cases} \quad (3.12)$$

belongs to  $\text{Ker}(\mathcal{L}_{\sigma_t}(-1/2+it))$  where  $\sigma_t(\theta) = 1$  for  $\theta \in (0; \alpha)$  and  $\sigma_t(\theta) = -a_{m_0}(t)$  for  $\theta \in (\alpha; \pi)$ . This means that for all  $v \in \mathbb{H}^1(\mathbb{S}^2)$  we have

$$\int_{\mathbb{S}^2} \sigma(t) \nabla_S u_t^m \cdot \nabla_S \bar{v} d\omega + (t^2 + 1/4) \int_{\mathbb{S}^2} \sigma(t) u_t^m \bar{v} d\omega = 0$$

Taking the derivative with respect to  $t$  of the previous relation at  $t_0$ , using the fact that the derivative of  $\sigma_t$  at  $t_0$  vanishes and since  $\frac{du_t^m}{dt}|_{t=t_0} \in \mathbb{H}^1(\mathbb{S}^2)$  (see Remark 3.6.5), we infer that  $(u_m^{t_0}, i \frac{du_m^t}{dt}|_{t=t_0})$  is Jordan chain of  $\mathcal{L}_\sigma$  associated to  $\lambda_0$ . ■

The previous proposition shows therefore that for all  $m \in \mathbb{N}$  we have

$$\{-a_m(t) \mid 0 \leq t \text{ and } a'_m(t) = 0\} \subset J_m.$$

### Validity of the second condition

**Proposition 3.4.9.** *Assume that  $\kappa_\sigma \in I_\alpha \setminus \{-1\}$  and let  $\lambda_0 = -1/2 + it_0 \in \Lambda(\mathcal{L}_\sigma) \cap \ell_{-1/2}$  such that there exists a unique  $m_0 \in \mathbb{N}$  such that  $a_{m_0}(|t_0|) = -\kappa_\sigma$ . Suppose that  $\lambda_0$  is semi-simple. Then, there exist  $0 < r_0, \delta_0$  such that for all  $0 < |\delta| < \delta_0$ , the ball  $B(\lambda_0, r_0)$  contains one eigenvalue of  $\mathcal{L}_{\sigma+i\delta}$  that is semi-simple.*

**Proof.** In the case where  $\lambda_0$  is a simple eigenvalue of  $\mathcal{L}_\sigma$ , the result follows from Corollary 3.3.1. It remains, then (thanks to the assumption made on  $\lambda$ ), to study the case when  $\iota_a(\lambda) = \iota_g(\lambda) = 2$  (in this case we necessarily have  $m_0 \in \mathbb{N}^*$ ). Let  $\kappa_{\sigma+i\delta} = (\sigma_2 + i\delta)/(\sigma_1 + i\delta)$ . Using the Fourier decomposition and working exactly as in the beginning of §3.4.1, one obtains the same dispersion relation as in (3.7) where  $\kappa_\sigma$  is replaced by  $\kappa_{\sigma+i\delta}$ . Given the fact that  $\lambda \mapsto P_\lambda^m$  is an analytic function, one deduces that  $\lambda \mapsto f^m(\lambda, \kappa_{\sigma+i\delta})$  is analytic near  $\lambda_0$  (see Remarque 3.6.3). Given that  $f^{m_0}(\lambda_0, \kappa_\sigma) = 0$  and by using the Rouché's theorem, we can say that there exists  $0 < r_0, \delta_0$  such that  $\lambda \mapsto f^{m_0}(\lambda, \kappa_\sigma)$  has a solution  $\lambda_\delta$  that belongs to  $B(\lambda_0, r)$  for all  $0 < |\delta| < \delta_0$ . Since the associated eigenfunctions to  $\lambda_\delta$  have the form  $(\theta, \varphi) \mapsto f(\theta) e^{\pm im_0 \varphi}$ , they are then independent and thus  $\iota_g(\lambda_\delta) = 2$  for all  $0 < |\delta| < \delta_0$ . Given that  $\iota_a(\lambda_0)$  must be greater or equal to  $\iota_a(\lambda_\delta)$ , we obtain that  $\iota_a(\lambda_\delta) = 2$ . This ends the proof. ■

### Final conclusion

Thanks to the results proved in the previous two paragraphs and thanks to the numerical results presented above, we can say that for the case of circular conical tips, Assumption 2.6.2 seems to be true expect for a discrete set of contrasts for which the energy line  $\ell_{-1/2}$  contains eigenvalues of  $\mathcal{L}_\sigma$  with generalized eigenvectors.



### 3.5 Concluding remarks

In this chapter, we presented a detailed study of the Mellin symbol generated by the scalar problem (2.2). We proved that all the classical results that we need to apply the Kondratiev theory are valid. Moreover, we studied the effect of the introduction of a small dissipation on the spectrum and on the behavior of the eigenfunctions of the perturbed problem. In the particular case of circular conical tips, the spectrum of  $\mathcal{L}_\sigma$  has been characterized by means of a dispersion relation. Furthermore, in this particular configuration, we investigated the validity of Assumption 2.6.2. We will conclude this chapter by mentioning that the most important question left unanswered in this work is about the validity of the Assumptions 2.6.2-2.6.3 in the case of general smooth conical tip. For this, one needs to have a better understanding of the behavior of the spectrum of  $\mathcal{L}_{\sigma+i\delta}$  and the associated eigenfunctions/ generalized eigenfunctions as  $\delta \rightarrow 0$ . Again an interesting reference that can help us dealing with this question is the book [138].

## 3.6 Appendix

### 3.6.1 The T-coercivity approach for the anisotropic scalar problem

The main goal of this section is to present a detailed proof of the Lemma 3.1 when the function  $g$  belongs to the space  $\mathcal{C}_{per}^2[0, 2\pi]$  (an example of such configuration is given in Figure 3.11).

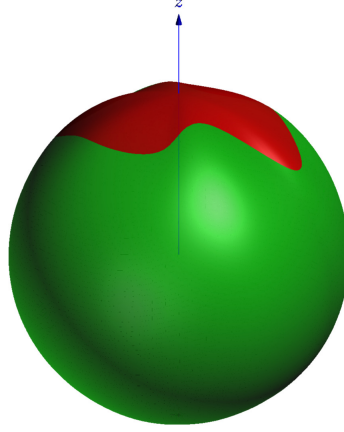


Figure 3.11: The red (resp. green) part is filled with a negative (resp. positive) materiel.

As in the case of circular conical tip ( $g(\varphi) = \alpha \in (0; \pi)$ ), the proof will be based on the use of the T-coercivity method, but this time we need to combine it with some localization techniques. This will make the proof a little bit technical. In order to make its presentation as simple as possible, we are going to start with the study of a related problem that will help us simplifying the final proof which will be presented in §3.6.1.

#### A simplified version of the problem

For all  $0 < L$ , we define the domains  $\Omega_1^L, \Omega_2^L \subset \mathbb{R}^2$  such that

$$\Omega_1^L = \{(x, y) \mid x \in (0; 2\pi) \text{ and } -L < y < 0\} \text{ and } \Omega_2^L = \{(x, y) \mid x \in (0; 2\pi) \text{ and } 0 < y < L\}.$$

Denote by  $\Sigma = \{(x, y) \mid y = 0\}$ , the interface between  $\Omega_1^L$  and  $\Omega_2^L$  and by  $\Omega^L$  the union of  $\Omega_1^L, \Omega_2^L$  and  $\Sigma$ . We introduce continuous real valued matrix (resp. scalar) function  $\mathbb{A} : \Omega^L \rightarrow \mathbb{M}_{2,2}(\mathbb{R})$  (resp.  $\mu : \Omega^L \rightarrow \mathbb{R}$ ). We suppose that  $\mathbb{A}(x, y)$  (resp.  $\mu(x, y)$ ) is symmetric and positive definite (resp. positive) for all  $(x, y) \in \Omega^L$ . This allows us to endow the spaces  $L^2(\Omega^L)$  and  $H^1(\Omega^L)$  with the norms:

$$\|u\|_{L^2(\Omega^L)} := \left( \int_{\Omega^L} |u|^2 \mu \, dx dy \right)^{1/2} \text{ and } \|u\|_{H^1(\Omega^L)} := \left( \int_{\Omega^L} \mathbb{A} \nabla u \cdot \nabla \bar{u} \, \mu \, dx dy + \int_{\Omega^L} |u|^2 \mu \, dx dy \right)^{1/2}.$$

Clearly, endowed with these norms the spaces  $L^2(\Omega^L)$  and  $H^1(\Omega^L)$  are of Hilbert type. We finish this series of notations by introducing a piecewise constant density function  $\sigma$  that is equal to  $0 < \sigma_1$  in  $\Omega_1^L$  and is equal to  $\sigma_2 < 0$  in  $\Omega_2^L$ . The contrast  $\kappa_\sigma$  is defined by  $\kappa_\sigma = \sigma_2/\sigma_1$ . Now, we have all the tools to define the linear operator  $A_t^\sigma : H^1(\Omega^L) \rightarrow (H^1(\Omega^L))^*$  that is defined as follows: for all  $u, v \in H^1(\Omega^L)$  we set

$$\langle A_t^\sigma u, v \rangle := \int_{\Omega^L} \sigma \mathbb{A}(x, y) \nabla u \cdot \nabla \bar{v} \, \mu(x, y) \, dx dy + t^2 \int_{\Omega^L} \sigma u \bar{v} \, \mu(x, y) \, dx dy.$$

Because of the sing-change in  $\sigma$ , the operator  $A_t^\sigma$  may be not of Fredholm type. On the other hand, because of the fact that the matrix valued function  $\mathbb{A}$  does not coincide with the identity

matrix, the theory presented in [50] can not be used. Our goal is then to find some conditions on the contrast  $\kappa_\sigma$ , on the function  $\mathbb{A}$  and on  $\mu$  under of which  $A_t^\sigma$  is an isomorphism for  $|t|$  large enough. For the case of the functions  $\mathbb{A}$  and  $\mu$ , we need to make the following

**Assumption 3.6.1.** *The exists  $0 < \eta_0 < L$  such that the function  $\mathbb{A}^{-1}$  and  $1/\mu$  are continuous in  $\Omega^{\eta_0}$ .*

The main result of this paragraph is given by the following

**Theorem 3.6.1.** *Let  $0 < L$ . Assume that the function  $\mathbb{A}$  and  $\mu$  are such that Assumption 3.6.1 is satisfied with  $0 < \eta_0 < L$ . Then for all  $\kappa_\sigma \neq -1$ , there exists  $0 < t_0$  such that for all  $t_0 < |t|$  the operator  $A_t^\sigma : H^1(\Omega^L) \rightarrow (H^1(\Omega^L))^*$  is an isomorphism. Moreover, we have the estimate*

$$(\|u\|_{H^1(\Omega^L)}^2 + t^2\|u\|_{L^2(\Omega^L)}^2)^{1/2} \leq C\|A_t^\sigma(u)\|_{(H^1(\Omega^L))^*} \text{ for all } u \in H^1(\Omega^L)$$

with  $C$  independent of  $u$  and  $t$ .

The proof of the previous theorem will be done thanks to a succession of two lemmas. Before getting into details, let us start by recalling some useful results about the classical Euclidean norm. For all  $R \in \mathbb{M}_{2,2}(\mathbb{R})$ , the Euclidean norm of  $R$  will be denoted by  $\varrho(R)$  and is defined by  $\varrho(R) = \sqrt{\lambda_{\max}(R^t R)}$ , in which  $\lambda_{\max}(R^t R)$  refers to the largest eigenvalue of the matrix  $R^t R$ . Note that when  $R$  is positive and symmetric, one has  $\varrho(R) = \lambda_{\max}(R)$ . Furthermore, we have the following interesting property: for all non-singular symmetric matrices  $R_1, R_2 \in \mathbb{M}_{2,2}(\mathbb{R})$ , we have the identity

$$\varrho(R_1 R_2) = \varrho(R_2 R_1). \quad (3.13)$$

This is true because the matrices  $R_1 R_2^2 R_1$  and  $R_2 R_1^2 R_2$  are similar. The first result that we need is the following

**Lemma 3.6.1.** *Let  $0 < \eta$  and let  $u_1 \in H^1(\Omega_1^\eta)$ . We define the function  $u_2(x, y) =: u_1(x, -y) \in H^1(\Omega_2^\eta)$ . The following estimate*

$$\int_{\Omega_2^\eta} \mathbb{A}(x, y) \nabla u_2 \cdot \nabla \bar{u}_2 \mu(x, y) dx dy \leq C^\eta \int_{\Omega_1^\eta} \mathbb{A}(x, y) \nabla u_1 \cdot \nabla \bar{u}_1 \mu(x, y) dx dy$$

holds with  $C^\eta = \sup_{\Omega_1^\eta} \varrho(\mathbb{A}(x, -y) \mathbb{A}^{-1}(x, y) \frac{\mu(x, -y)}{\mu(x, y)})$ .

**Proof.** The first step is to perform the change of variable  $(x, y) \rightarrow (x, -y)$ . This transformation maps  $\Omega_2^\eta$  and  $\Omega_1^\eta$ . Furthermore, one can easily check that the jacobian matrix of this transformation coincides with the identity matrix. This leads us to write that

$$I_0 := \int_{\Omega_2^\eta} \mathbb{A}(x, y) \nabla u_2 \cdot \nabla u_2 \mu(x, y) dx dy = \int_{\Omega_1^\eta} \mathbb{A}(x, -y) \nabla u_1 \cdot \nabla u_1 \mu(x, -y) dx dy$$

As a result we have

$$\begin{aligned} I_0 &= \int_{\Omega_1^\eta} \sqrt{\mathbb{A}^{-1}}(x, y) \mathbb{A}(x, -y) \sqrt{\mathbb{A}^{-1}}(x, y) \sqrt{\mathbb{A}}(x, y) \nabla u_1 \cdot \sqrt{\mathbb{A}}(x, y) \nabla u_1 \mu(x, -y) dx dy \\ &\leq \sup_{\Omega_1^\eta} \varrho(\sqrt{\mathbb{A}^{-1}}(x, y) \mathbb{A}(x, -y) \sqrt{\mathbb{A}^{-1}}(x, y) \frac{\mu(x, -y)}{\mu(x, y)}) \int_{\Omega_1^\eta} \mathbb{A}(x, y) \nabla u_1 \cdot \nabla u_1 \mu(x, y) dx dy \\ (\text{by means of (3.13)}) &= \sup_{\Omega_1^\eta} \varrho(\mathbb{A}(x, -y) \mathbb{A}^{-1}(x, y) \frac{\mu(x, -y)}{\mu(x, y)}) \int_{\Omega_1^\eta} \mathbb{A}(x, y) \nabla u_1 \cdot \nabla u_1 \mu(x, y) dx dy. \end{aligned}$$

Note that, above the matrix  $\sqrt{\mathbb{A}}(x, y)$  is defined as the unique positive definite matrix  $R(x, y) \in \mathbb{M}_{2,2}(\mathbb{R})$  satisfying the equation  $R^2(x, y) = \mathbb{A}(x, y)$ .  $\blacksquare$

In order to study the behaviour of  $C^\eta$  when  $\eta$  tends to 0, we are going to use the following

**Lemma 3.6.2.** *Let  $0 < \eta_0$  and let  $f$  be a continuous function in  $\overline{\Omega^{\eta_0}}$  then the function  $F(\eta) =: \sup_{x \in \Omega^\eta} f(x)$  is non-decreasing continuous in  $[0; \eta_0]$ .*

**Proof.** The fact that  $F$  is non-decreasing is obvious. Let  $\eta \in [0; \eta_0]$  and  $(\eta_n)_{n \in \mathbb{N}}$  be a sequence of  $[0; \eta_0]$  that converges, as  $n$  tends to  $+\infty$  to  $\eta$ . We will show that  $F(\eta_n)_{n \in \mathbb{N}}$  converges to  $F(\eta)$ . To simplify notations, let us denote by  $\Omega^n$  the domain  $\Omega^{\eta_n}$ . The starting point is to observe that  $d_H(\overline{\Omega^n}, \overline{\Omega^\eta})$  the Hausdorff distance between  $\overline{\Omega^{\eta_n}}$  and  $\overline{\Omega^\eta}$  tends to 0 as  $n$  tends to  $\infty$ . Since the function  $f$  is continuous in  $\overline{\Omega^{\eta_0}}$  it is then uniformly continuous. This means that for all  $0 < \varepsilon$  there is  $0 < \delta_\varepsilon$  such that for all  $x_1, x_2 \in \Omega^{\eta_0}$  satisfying  $\|x_1 - x_2\| < \delta_\varepsilon$  we have  $|f(x_1) - f(x_2)| < \varepsilon$ . Now take, some  $x \in \Omega^\eta$  and  $0 < \varepsilon$ . Since  $d_H(\overline{\Omega^{\eta_n}}, \overline{\Omega^\eta})$  tends to 0 as  $n$  tends to  $\infty$ , there exists  $n_0^\varepsilon \in \mathbb{N}$  large enough such that for all  $n_0^\varepsilon < n$  the domain  $\overline{\Omega^n}$  contains an element  $y_n$  such that  $\|x - y_n\| < \delta_\varepsilon$ . This implies that for all  $n_0 < n$ , we have

$$f(x) \leq f(y_n) + \varepsilon \implies f(x) \leq F(\eta_n) + \varepsilon \text{ for all } n_0^\varepsilon < n.$$

By letting  $\varepsilon$  tend to 0 and  $n_0^\varepsilon$  to  $\infty$ , we deduce that  $f(x) \leq \liminf_{n \rightarrow \infty} F(\eta_n)$  for all  $x \in \Omega^\eta$ . Thus we obtain the inequality  $F(\eta) \leq \liminf_{n \rightarrow \infty} F(\eta_n)$ . By exchanging the roles of  $\Omega^\eta$  and  $\Omega^n$  in the previous reasoning, we get  $\limsup_{n \rightarrow \infty} F(\eta_n) \leq F(\eta)$ . The lemma is then proved.  $\blacksquare$

Since, under Assumption 3.6.1, the function  $(x, y) \rightarrow \rho(\mathbb{A}(x, y)\mathbb{A}^{-1}(x, -y)\mu(x, -y)/\mu(x, y))$  is continuous in  $\overline{\Omega^\eta}$  for  $\eta$  small enough, the previous Lemma allows us to deduce that  $C^\eta$  tends to 1 when  $\eta$  tends to 0. At this stage, we have all the needed tools to present a clear proof of Theorem 3.6.1.

*Proof of Theorem 3.6.1.* Let  $\chi$  be a cutoff function defined in  $\mathbb{R}$  such that  $\chi(r) = 1$  for  $|r| \leq 1/2$  and  $\chi(r) = 0$  for  $1 \leq |r|$ . For all  $\eta \in \mathbb{R}$ , we introduce the function  $\chi^\eta$  such that  $\chi^\eta(r) = \chi(r/\eta)$ . Note that for all  $\eta \in \mathbb{R}^*$  the function  $\chi^\eta$  is supported in  $[-|\eta|, |\eta|]$ . From now on, we are going to assume that  $0 < \eta < L/4$ , other assumptions on  $\eta$  will be made later. As mentioned previously, the main idea is to use the T-coercivity method. For this, let us start by introducing the map  $T : H^1(\Omega^L) \rightarrow H^1(\Omega^L)$  such that for all  $u \in H^1(\Omega^L)$ , we set

$$T(u)(x, y) = \begin{cases} u_1(x, y) & \text{in } \Omega_1^L \\ -u_2(x, y) + 2\chi^\eta(y)u_1(x, -y) & \text{in } \Omega_2^L. \end{cases}$$

One, can easily check that for all  $u \in H^1(\Omega^L)$ , the function  $T(u)$  belongs also to the space  $H^1(\Omega^L)$ . Moreover, we can also check that  $T(T(u)) = u$  for all  $u \in H^1(\Omega^L)$  and then  $T$  is bijective. Using the result of Lemma 3.6.1, one can prove that  $T(u)$  is continuous and satisfies the estimate

$$\|T(u)\|_{H^1(\Omega^L)} \leq C_1^\eta \|u\|_{H^1(\Omega^L)} \text{ for all } u \in H^1(\Omega^L)$$

with  $C_1^\eta$  independent of  $u \in H^1(\Omega^L)$ . Now, let us fix some  $u \in H^1(\Omega^L) \setminus \{0\}$ , and let us compute  $\langle A_t^\sigma u, T(u) \rangle$ . Before getting into that, let us denote by  $\tilde{u}_1 \in H^1(\Omega_1^L)$  and by  $\tilde{\chi}^\eta$  the functions defined by  $\tilde{u}_1(x, y) = u_1(x, -y)$  and  $\tilde{\chi}^\eta(x, y) = \chi^\eta(-y)$  for all  $(x, y) \in \Omega_2^L$ . Using these notations, one finds that

$$\begin{aligned} \langle A_t^\sigma u, T(u) \rangle &= \int_{\Omega^L} |\sigma| \mathbb{A}(x, y) |\nabla u \cdot \nabla \bar{u}| \mu(x, y) dx dy + t^2 \int_{\Omega^L} |\sigma| |u|^2 \mu(x, y) dx dy \\ &\quad + 2\sigma_2 \int_{\Omega_2^L} \mathbb{A}(x, y) \nabla u_2 \cdot \nabla (\tilde{u}_1 \tilde{\chi}^\eta) \mu(x, y) dx dy + 2\sigma_2 t^2 \int_{\Omega_2^L} u_2 \tilde{\chi}^\eta \bar{u} \mu(x, y) dx dy. \end{aligned}$$

The next step is to find some upper bounds of the magnitudes of the third and of the last terms in the previous identity. For that we are going to use the Young's inequality. For the case of the fourth term, one finds that for all  $0 < a$  we have

$$|2 \int_{\Omega_2^L} u_2 \tilde{\chi}^\eta \bar{u} \mu(x, y) dx dy| \leq a \int_{\Omega_2^L} |u_2|^2 \mu(x, y) dx dy + s_1^\eta a^{-1} \int_{\Omega_1^L} |u_1|^2 \mu(x, y) dx dy$$

in which  $s_1^\eta = \sup_{(x, y) \in \overline{\Omega^\eta}} \frac{\mu(x, -y)}{\mu(x, y)}$ . The third term can be decomposed as

$$\int_{\Omega_2^L} \mathbb{A} \nabla u_2 \cdot \nabla (\chi^\eta \bar{u}) \mu dx dy = \int_{\Omega_2^L} \mathbb{A} \nabla u_2 \cdot \nabla (\chi^\eta) \bar{u} \mu dx dy + \int_{\Omega_2^L} \chi^\eta \mathbb{A} \nabla u_2 \cdot \nabla \bar{u} \mu dx dy$$

The Young's inequality tells us that for all  $0 < b$  we have

$$|2 \int_{\Omega_2^L} \mathbb{A} \nabla u_2 \cdot \nabla (\chi^\eta) \bar{u} \mu dx dy| \leq b \int_{\Omega_2^L} \mathbb{A} \nabla u_2 \cdot \nabla \bar{u}_2 \mu dx dy + s_2^\eta b^{-1} \int_{\Omega_1^L} |u_1|^2 \mu dx dy$$

where  $s_2^\eta = s_1^\eta \sup_{(x, y) \in \overline{\Omega^\eta}} |\mathbb{A} \nabla \chi^\eta \cdot \nabla \chi^\eta|$ . Furthermore, one obtains that for all  $0 < c$ ,

$$|2 \int_{\Omega_2} \chi^\eta \mathbb{A} \nabla u_2 \cdot \nabla \bar{u} \mu dx dy| \leq c \int_{\Omega_2} \mathbb{A} \nabla u_2 \cdot \nabla \bar{u}_2 \mu dx dy + s_3^\eta c^{-1} \int_{\Omega_1} \mathbb{A} \nabla u_1 \cdot \nabla \bar{u}_1 \mu dx dy$$

in which  $s_3^\eta = C_\eta$  where  $C_\eta$  is given in the statement of Lemma 3.6.1. Inserting all these estimates in the expression of  $\langle A_t^\sigma u, T(u) \rangle$ , we get

$$\begin{aligned} |\langle A_t^\sigma u, T(u) \rangle| &\geq |\sigma_1| \left( (1 - |\kappa_\sigma| s_3^\eta c^{-1}) \int_{\Omega_1} \mathbb{A} \nabla u_1 \cdot \nabla \bar{u}_1 \mu dx dy + |\kappa_\sigma| (1 - b - c) \int_{\Omega_2} \mathbb{A} \nabla u_2 \cdot \nabla \bar{u}_2 \mu dx dy \right. \\ &\quad \left. + (t^2 (1 - s_1^\eta |\kappa_\sigma| a^{-1}) - |\kappa_\sigma| s_2^\eta b^{-1}) \int_{\Omega_1} |u_1|^2 \mu dx dy + |\kappa_\sigma| t^2 (1 - a) \int_{\Omega_2} |u_2|^2 \mu dx dy \right). \end{aligned}$$

According to Lemma 3.6.2, we know that, at least for  $\eta$  small enough, under Assumption 3.6.1 the functions  $\eta \rightarrow s_1^\eta, s_2^\eta, s_3^\eta$  are non-decreasing continuous in  $\overline{\Omega^\eta}$ . In the other hand, one can easily check that  $s_3^0 = s_1^0 = 1$ . This means that if the contrast  $\kappa_\sigma$  is such that  $1 < |\kappa_\sigma|$ , there exists some  $0 < \eta^*$  such that

$$\max(1/s_3^\eta, 1/s_1^\eta) < |\kappa_\sigma| \text{ for all } \eta \in (0; \eta^*).$$

By taking  $\eta = \eta^*/2$ ,  $a, c \in (0; 1)$  such that the coefficients  $(1 - |\kappa_\sigma| s_3^\eta c^{-1})$  and  $(1 - s_1^\eta |\kappa_\sigma| a^{-1})$  become positive and then by taking  $b \in (0; 1 - c)$  and  $t$  large enough we deduce that there exists some  $0 < t_0$  such that for all  $t_0 < |t|$  we have the estimate

$$\|u\|_{\mathbf{H}^1(\Omega^L)}^2 + t^2 \|u\|_{\mathbf{L}^2(\Omega^L)}^2 \leq C \langle A_t^\sigma u, T(u) \rangle \text{ for all } u \in \mathbf{H}^1(\Omega^L)$$

with  $C$  independent of  $u$ . By recalling the continuity of  $T$ , we deduce the wanted result for all  $\kappa_\sigma$  satisfying  $1 < |\kappa_\sigma|$ . The case  $|\kappa_\sigma| < 1$  can be treated, similarly, by exchanging the roles of  $\Omega_1^L$  and  $\Omega_2^L$ . ■

### Remark 3.6.1.

- Using the the fact that the embedding of  $\mathbf{H}^1(\Omega_L)$  into  $\mathbf{L}^2(\Omega_L)$  is compact, one deduces that when Assumption 3.6.1 is satisfied and when  $\kappa_\sigma \neq -1$  the operator  $A_0^\sigma$  is a Fredholm operator of index 0.

- Since the proof of the Theorem 3.6.1 is based on the use of local reflection with respect to the interface  $\Sigma$ , one can easily see that if we replace the domain  $\Omega_L$  by  $\Omega_{L_1} \cup \Omega_{L_2}$  with  $L_1 \neq L_2$  then its the statement remains true.
- In the particular case when the domain  $\Omega^L$  coincides with the domain  $(0; 2\pi) \times (0; \pi)$  and the interface  $\Sigma$  is defined by the equation  $y = \alpha \in (0; \pi)$  and when the matrix function  $\mathbb{A}$  and the scalar function  $\mu$  are defined by  $\mathbb{A}(x, y) = \begin{pmatrix} 1/\sin(y) & 0 \\ 0 & 1 \end{pmatrix}$  and  $\mu(x, y) = \sin(y)$  (in this case the Assumption 3.6.1 is valid thanks to the smoothness of the functions  $(x, y) \mapsto \sin(y)$  and  $(x, y) \mapsto 1/\sin(y)$  near  $\alpha \in (0; \pi)$ ) we obtain a proof of Lemma 3.2.1 in the particular case of circular conical tips (here  $\varphi$  is replaced by  $x$  and the variable  $\theta$  is replaced by  $y$ ).

### Proof of Lemma 3.2.1

Here, we go back to the proof of Lemma 3.2.1 when the function  $g$  belongs to the space  $\mathcal{C}^2([0, 2\pi])$ . The starting point is to observe that when we use the classical spherical coordinates  $(\theta, \varphi) \in (0; \pi) \times (0; 2\pi)$  to parameterize the unit sphere  $\mathbb{S}^2$ , the expression of  $\langle \mathcal{L}_\sigma(\lambda)u, v \rangle$  writes

$$\langle \mathcal{L}_\sigma(\lambda)u, v \rangle = \int_0^{2\pi} \int_0^\pi \sigma(\varphi, \theta) \mathbb{A}(\varphi, \theta) \nabla u \cdot \nabla \bar{v} \mu(\varphi, \theta) d\varphi d\theta + \lambda(\lambda+1) \int_0^{2\pi} \int_0^\pi \sigma(\varphi, \theta) u \bar{v} \mu(\varphi, \theta) d\varphi d\theta$$

in which  $\mathbb{A}(\varphi, \theta) = \begin{pmatrix} 1/\sin(\theta) & \\ & 1 \end{pmatrix}$ ,  $\mu(\varphi, \theta) = \sin(\theta)$ ,  $\nabla = \begin{pmatrix} \partial_\varphi \\ \partial_\theta \end{pmatrix}$  and  $\sigma(\varphi, \theta) = \begin{cases} 0 < \sigma_1 & \text{if } \theta < g(\varphi) \\ \sigma_2 < 0 & \text{if } g(\varphi) < \theta \end{cases}$ .

Using these notations, the norms  $\|\cdot\|_{\mathbb{H}^1(\mathbb{S}^2)}$  and  $\|\cdot\|_{\mathbb{L}^2(\mathbb{S}^2)}$  are given by

$$\|u\|_{\mathbb{L}^2(\mathbb{S}^2)} = \left( \int_0^{2\pi} \int_0^\pi |u(\varphi, \theta)|^2 \mu(\theta) d\varphi d\theta \right)^{1/2},$$

and

$$\|u\|_{\mathbb{H}^1(\mathbb{S}^2)} = \left( \|u\|_{\mathbb{L}^2(\mathbb{S}^2)}^2 + \int_0^{2\pi} \int_0^\pi \mathbb{A}(\varphi, \theta) \nabla u \cdot \nabla \bar{u} \mu(\varphi, \theta) d\varphi d\theta \right)^{1/2}.$$

To simplify notations, we shall denote by  $\Omega_1 = \{(\varphi, \theta) \in (0; 2\pi) \times (0; \pi) \mid \theta < g(\varphi)\}$ ,  $\Omega_2 = \{(\varphi, \theta) \in (0; 2\pi) \times (0; \pi) \mid g(\varphi) < \theta\}$  and by  $\Sigma = \{(\varphi, \theta) \mid \theta = g(\varphi)\}$ . For the reader convenience, we will denote by  $\Omega$  the union of  $\Omega_1$ ,  $\Omega_2$  and  $\Sigma$ . It is worthy to note that in the topological sense  $\Omega$  coincides with  $\mathbb{S}^2$ . Before presenting the final proof, let us recall the definition of the so-called  $\varepsilon$ -neighborhood to a curve.

**Definition 3.6.1.** Let  $O$  be a curve in  $\mathbb{R}^2$  that can be parameterized by a function  $f : [a, b] \rightarrow \mathbb{R}^2$  of class  $\mathcal{C}^1$ . For all  $0 < \varepsilon$ , we define  $O^\varepsilon = \{x \in \mathbb{R}^2 \text{ such that } d(x, O) < \varepsilon\}$ . We say that  $O^\varepsilon$  is an  $\varepsilon$ -neighborhood if the following conditions are satisfied:

1. each  $x \in O^\varepsilon$  possesses a unique closest point  $\pi_\varepsilon(x)$  in  $O$ .
2. the map  $\pi_\varepsilon : O^\varepsilon \rightarrow O$  is onto.

In this case,  $O^\varepsilon = \{(s, t) \mid (s, t) \in [a, b] \times (-\varepsilon, \varepsilon)\}$  in which  $(s, t)$  are the curvilinear coordinates associated to  $O$ .

Unfortunately for the case of  $\mathcal{C}^1$  curves, the existence of such tubular neighborhood is not guaranteed (a counterexample can be constructed by taking the inner parallel curve of the ellipse (that is not a circle) which passes through the foci<sup>3</sup>). For the case of  $\mathcal{C}^2$  curves, a proof of the existence of tubular neighborhood can be found in [57, Theorem 3.1.1].

<sup>3</sup>Details can be found [here](#).

**Final proof of Lemma 3.2.1.** Given that the function  $g$  is periodic and of class  $\mathcal{C}^2$ , this means that there exists a least  $\varphi^* \in (0; 2\pi)$  such that  $g'(\varphi^*) = 0$ . Without loss of generality, we can suppose that  $\varphi^* = 0$  (or equivalently  $\varphi^* = 2\pi$ ). Moreover, since  $g$  is of class  $C^2$ , one can find an  $L$ -tubular neighborhood to  $\Sigma$  with some  $0 < L$ . This neighborhood will be denoted by  $O^L$ . Using the fact that  $g'(0) = g'(2\pi)$ , we infer that  $O^L$  can be chosen such that  $O^L \subset \Omega$ . We denote respectively by  $\Omega_1^L$  and by  $\Omega_2^L$  the domains  $\Omega_1 \cap O^L$  and  $\Omega_2 \cap O^L$ . As in the proof of Theorem 3.6.1, for all  $0 < \eta < L/4$  we introduce a smooth cut-off function  $\chi^\eta$  that is supported in  $[-\eta, \eta]$  and equal to 1 in  $[-\eta/2, \eta/2]$ . Now, for all  $0 < \eta < L$  we introduce the map  $T^\eta : H^1(\Omega) \rightarrow H^1(\Omega)$  such that for all  $u \in H^1(\Omega)$

$$T^\eta(u)(\varphi, \theta) = \begin{cases} u_1(\varphi, \theta) & \text{in } \Omega_1 \\ -u_2(\varphi, \theta) + 2R^\eta(u_1)(\varphi, \theta) & \text{in } \Omega_2, \end{cases}$$

where  $R^\eta(u_1)$  is defined by  $R^\eta(u)(s, t) = \chi^\eta(t)u_1(s, -t)$  for all  $(s, t) \in \Omega_2^L$ , here  $(s, t)$  are the curvilinear coordinates that are associated to  $\Sigma$  and that are well-defined in  $\Omega_1^L$  (thanks to its definition) and  $R^\eta(u)(x, y) = 0$  for all  $(x, y) \in \Omega_2 \setminus \Omega_2^L$ . One can easily see that the map  $T^\eta$  is well-defined (this is true because of the continuity of  $T^\eta(u)$  at  $\Sigma$ ). Moreover, one can easily check that for all  $u \in H^1(\Omega)$  we have  $T^\eta \circ T^\eta(u) = u$  and this shows the bijectivity of  $T^\eta$ . The continuity of  $T^\eta$  can be shown in the same way as in the proof of Theorem 3.6.1. To proceed, take some  $t \in \mathbb{R}$  and observe that for all  $u \in H^1(\Omega)$  we have

$$\begin{aligned} \langle \mathcal{L}_\sigma(-1/2 + it)u, T^\eta(u) \rangle &= \int_{\Omega \setminus \Omega^L} |\sigma| \mathbb{A} \nabla u \cdot \nabla \bar{u} \mu d\varphi d\theta + (1/4 + t^2) \int_{\Omega \setminus \Omega^L} |\sigma| |u|^2 \mu d\varphi d\theta \\ &+ \int_{\Omega^L} \sigma \mathbb{A} \nabla u \cdot \nabla \overline{T^\eta(u)} \mu dx dy + (1/4 + t^2) \int_{\Omega^L} \sigma u \overline{T^\eta(u)} \mu d\varphi d\theta. \end{aligned}$$

This means that to prove our claim, we just need to study the behaviour of the second part of the previous sum (the one in which all the integrals are taken over  $\Omega^L$ ). Clearly this fits into the general problem studied in the previous paragraph, but it is worthy to note that since the operator  $R^\eta$  is written in local coordinates the Theorem 3.6.1 can not be used directly. To be able to apply it, we need to write all the integrals over  $\Omega^L$  in local coordinates  $(s, t)$ . If we denote by  $\mathbb{J}(s, t)$  the jacobian matrix of the change of variables that allows to pass from  $(\varphi, \theta)$  to  $(s, t)$  in  $\Omega^L$  and by  $|\mathbb{J}(s, t)|$  its jacobian (this transformation is well-defined thanks to the definition of  $\Omega^L$ ), one can write that for all  $u \in H^1(\Omega_L)$ , we have

$$\begin{aligned} \int_{\Omega^L} \sigma \mathbb{A} \nabla u \cdot \nabla \overline{T^\eta(u)} \mu d\varphi d\theta &= \int_{\hat{\Omega}^L} \sigma \mathbb{J}^t \mathbb{A} \mathbb{J} \nabla u \cdot \nabla \overline{T^\eta(u)} \mu |\mathbb{J}| ds dt \\ \int_{\Omega^L} \sigma u \overline{T^\eta(u)} \mu dx dy &= \int_{\hat{\Omega}^L} \sigma u \overline{T^\eta(u)} \mu |\mathbb{J}| ds dt \end{aligned}$$

where  $\hat{\Omega}^L = \{(t, s) \in (0; 2\pi) \times (-L; L)\}$ . Since  $g$  is of class  $\mathcal{C}^2$  the matrix valued function  $(s, t) \mapsto \mathbb{J}^t(s, t) \mathbb{A}(s, t) \mathbb{J}(s, t)$  as well as the scalar function  $(s, t) \mapsto \mu(s, t) |\mathbb{J}(s, t)|$  are continuous in  $\hat{\Omega}^\eta$  for all  $0 < \eta < L$ . Given that in local coordinates  $\Sigma$  is given by the equation  $t = 0$ , a direct application of Theorem 3.6.1, leads us to say that for all  $\kappa_\sigma \neq -1$  there exists  $0 < t_0$  that depends only on  $\kappa_\sigma$  and  $\eta$  such that for all  $t_0 < |t|$  and all  $u \in H^1(\Omega_L)$  we have the estimate

$$\begin{aligned} C \left( \int_{\Omega^L} \mathbb{A} \nabla u \cdot \nabla \bar{u} \mu dx dy + (1/4 + t^2) \int_{\Omega^L} |u|^2 \mu dx dy \right) &\leq \int_{\Omega^L} \sigma \mathbb{A} \nabla u \cdot \nabla \overline{T^\eta(u)} \mu dx dy \\ &+ (1/4 + t^2) \int_{\Omega^L} \sigma u \overline{T^\eta(u)} \mu dx dy. \end{aligned}$$

with  $0 < C$  independent of  $u$ . Inserting this in the expression of  $\langle \mathcal{L}_\sigma(-1/2 + it)u, T^\eta(u) \rangle$ , we arrive to the estimate

$$C \left( \int_{\Omega} \mathbb{A} \nabla u \cdot \nabla u \mu dx dy + (1/4 + t^2) \int_{\Omega} |u|^2 \mu dx dy \right) \leq |\langle \mathcal{L}_\sigma(-1/2 + it)u, T^\eta(u) \rangle|$$



for all  $u \in H^1(\Omega)$  and  $t_0 < |t|$  with  $C$  independent of  $u$ . The rest of the proof is the same as in the case of a circular conical tip.  $\blacksquare$

**Remark 3.6.2.** Clearly the result of Lemma 3.2.1 can be, easily, extended to the following cases:

- When the function  $g$  belongs to the space  $\mathcal{C}_{per}^1([0; 2\pi]) \cap P\mathcal{C}^2([0; 2\pi])$ .<sup>4</sup>
- When the interface  $\Sigma$  is connected curve of class  $\mathcal{C}^2$  (i.e. can be locally parameterized by  $\mathcal{C}^2$  functions).
- More generally, the result holds if  $\Sigma$  is connected curve of class  $\mathcal{C}^1$  that has  $\varepsilon$ -tubular neighborhood.

### 3.6.2 Associated Legendre functions

The goal of this section is to recall some of the basic properties of the associated Legendre functions. Let  $m \in \mathbb{Z}$  and  $\lambda \in \mathbb{C}$ , the associated Legendre equation of order  $m$  and of degree  $\lambda \in \mathbb{C}$  writes: Find a function  $u \neq 0$  such that

$$\frac{d}{dx}((1-x^2)\frac{du}{dx}) + \lambda(\lambda+1)u - \frac{m^2}{1-x^2}u = 0 \text{ for all } x \in (-1; 1). \quad (3.14)$$

To be more precise, we are interested in the solutions of (3.14) that are bounded near the point  $x = 1$ . As all second order differential equations, the space of solutions of associated Legendre equation is a vector space of dimension two. At  $x = \pm 1$  the equation (3.14) degenerates and its solution may be singular near these points. In general, except in particular situations that will be specified later, the space of solutions of (3.14) is generated by a pair of linearly independent functions denoted by  $P_\lambda^m$  and  $Q_\lambda^m$  and known respectively as the associated Legendre functions of first (resp. second) kind. It is important to note that, in the literature, these functions are also called Ferrers functions of first and second kind (see [3]). While the functions  $P_\lambda^m$  are known to have a regular behaviour near the points  $\pm 1$ , the functions  $Q_\lambda^m$  are singular near these points (see the discussion below for more details).

Since the equation (3.14) is unchanged when we exchange  $m$  by  $-m$  or  $\lambda$  by  $-\lambda - 1$  one expects the functions  $P_\lambda^{\pm m}$  and  $P_{-\lambda-1}^{\pm m}$  (resp.  $Q_\lambda^{\pm m}$  and  $Q_{-\lambda-1}^{\pm m}$ ) are linearly dependent. In order to make the presentation of properties of the associated Legendre function as clear as possible, we shall distinguish two situations: the first one is when  $\lambda$  belongs to  $\mathbb{Z}$  and the second one is when  $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ .

#### The case $\lambda \in \mathbb{Z}$

Before getting into details, for all  $a, b \in \mathbb{Z}$ , we denote by  $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$ . Clearly, if  $b < a$  we have  $\llbracket a, b \rrbracket = \emptyset$ . The starting point is the following

**Lemma 3.6.3** (Appendix A of [95]). Assume that  $m \in \mathbb{Z}$ .

- If  $\lambda \in \llbracket -|m|, -1 \rrbracket \cup \llbracket 0, |m| - 1 \rrbracket$ , then any solution of (3.14) is unbounded near  $x = 1$ .
- When  $\lambda \in \mathbb{Z} \setminus \llbracket -|m|, -1 \rrbracket \cup \llbracket 0, |m| - 1 \rrbracket$  the function  $P_\lambda^m$  and  $Q_\lambda^m$  are well-defined.
- For all  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{Z} \setminus \llbracket -m, -1 \rrbracket \cup \llbracket 0, m - 1 \rrbracket$ . Then, we have the relations:  $P_\lambda^m = P_{-\lambda-1}^m, Q_\lambda^m = Q_{-\lambda-1}^m$  and

$$P_\lambda^{-m} = \frac{(\lambda - m)!}{(\lambda + m)!} P_\lambda^m, Q_\lambda^{-m} = \frac{(\lambda - m)!}{(\lambda + m)!} Q_\lambda^m.$$

<sup>4</sup> $P\mathcal{C}^2([0; 2\pi])$  is the space of piecewise  $\mathcal{C}^2$  functions.

As a result, we just need to explain how to define the functions  $P_\lambda^m$  and  $Q_\lambda^m$  for  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{N}$  such that  $m \leq \lambda$ . Let us start by treating the case  $m = 0$ . In this particular case the equation (3.14) is known as the Legendre equation. The functions  $P_\lambda^0$  and  $Q_\lambda^0$  for  $\lambda \in \mathbb{N}$  are defined as follows: for all  $x \in (-1; 1)$  we have

$$P_\lambda^0(x) = \frac{1}{2^{\lambda\lambda!}} \frac{d^\lambda}{dx^\lambda} \{(1-x^2)^\lambda\}, Q_\lambda^0(x) = \frac{1}{2^{\lambda\lambda!}} \frac{d^\lambda}{dx^\lambda} \left\{ (1-x^2)^\lambda \log\left(\frac{1+x}{1-x}\right) \right\} - \frac{1}{2} \log\left(\frac{1+x}{1-x}\right) P_\lambda^0(x).$$

When  $m \in \mathbb{N}^*$  and  $\lambda \in \mathbb{N} \setminus \llbracket 0, m-1 \rrbracket$ , the functions  $P_\lambda^m$  and  $Q_\lambda^m$  can be defined thanks to the relations: for  $x \in (-1; 1)$  we have

$$P_\lambda^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\lambda^0(x) \text{ and } Q_\lambda^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} Q_\lambda^0(x).$$

More explicit formulas to calculate the functions  $P_\lambda^m$  and  $Q_\lambda^m$  will be given in next paragraph.

### The case $\lambda \in \mathbb{C} \setminus \mathbb{Z}$

In such configuration, the functions  $P_\lambda^m$  and  $Q_\lambda^m$  are well-defined for all  $m \in \mathbb{Z}$ . Let us start with the case of the second kind ones. As mentioned above, the function  $Q_\lambda^m$  is singular near  $x = 1$ . This can be seen from the following expansions and identities (see [3]):

$$\left| \begin{array}{l} Q_\lambda^m(x) \underset{x=1}{\sim} \frac{1}{2} (-1)^m (m-1)! \left(\frac{2}{1-x}\right)^{m/2}, \quad Q_\lambda^{-m}(x) = (-1)^m \frac{\Gamma(\lambda-m+1)}{\Gamma(\lambda+m+1)} Q_\lambda^m(x) \quad \text{for all } m \in \mathbb{N} \setminus \{0\} \\ Q_\lambda^0 = -\log(1-x)/2 + b_\lambda + O(x-1) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{Z}_-, \end{array} \right.$$

Above  $\Gamma(z)$  is the analytic continuation of the classical Euler gamma function which is defined for  $z \in \mathbb{C} \setminus \mathbb{Z}$  (see [3]) and  $b_\lambda$  is a constant that depends, only, on  $\lambda$ . For this reason, we shall then limit ourselves, in the rest of this appendix, to the presentation of the main properties of the function  $P_\lambda^m$ . Classically these functions are defined by using the so-called Gauss hypergeometric function  ${}_2F_1$ . For given complex numbers  $a, b, c$ , we define the Gauss hypergeometric function  ${}_2F_1$  such that

$${}_2F_1(a, b, c, z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \text{ for all } z \in \mathbb{C} \text{ such that } |z| < 1$$

where for all complex number  $y$ , the Pochhammer symbol  $(y)_n$  is defined by  $(y)_n =: y(y+1)(y+2)\dots(y+n-1)$  for  $n \in \mathbb{N}^*$  and  $(y)_0 = 1$ . One can also write for all  $y \in \mathbb{C} \setminus \mathbb{Z}_-$  that  $(y)_n = \Gamma(y+n)/\Gamma(y)$ . It can be shown that, except for this case  $c \in \mathbb{Z}_-$  (in the case  $(c)_n = 0$  for  $n$  large), the function  $z \mapsto {}_2F_1(a, b, c, z)$  is well-defined for all  $z \in \mathbb{C}$  satisfying  $|z| < 1$ .

For  $m \in \mathbb{N}$  and  $\lambda \in \mathbb{C} \setminus \mathbb{Z}$  the function  $P_\lambda^m(x)$  is defined by the following expression:

$$P_\lambda^m(x) = \frac{(\lambda+1)_m (-\lambda)_m}{2^m m!} (1-x^2)^{\frac{m}{2}} {}_2F_1(-\lambda+m, \lambda+m+1, m+1, \frac{1-x}{2})^5. \quad (3.15)$$

With this in mind, one can immediately see that for all  $\lambda \in \mathbb{C} \setminus \mathbb{Z}$  we have  $P_\lambda^m = P_{-\lambda-1}^m$ . In order to define the function  $P_\lambda^m(x)$  for  $m \in \mathbb{Z}_-$ , one has to use the relation

$$P_\lambda^{-m} = (-1)^m \frac{\Gamma(\lambda-m+1)}{\Gamma(\lambda+m+1)} P_\lambda^m = \frac{P_\lambda^m}{(\lambda+1)_m (-\lambda)_m} \text{ for all } m \in \mathbb{N}. \quad (3.16)$$

**Remark 3.6.3.** Given that for all  $m \in \mathbb{N}$  the map  $(a, b) \mapsto {}_2F_1(a, b, m)$  is analytic with respect to  $a, b \in \mathbb{C}$  (see [75, §15.2]) we then deduce that for all  $m \in \mathbb{Z}$  and  $x \in [-1, 1]$  the map  $\lambda \mapsto P_\lambda^m(x)$  is analytic in  $\mathbb{C} \setminus \mathbb{Z}$ .

<sup>5</sup>In many references the term  $(\lambda+1)_m (-\lambda)_m$  is written differently:  $(\lambda+1)_m (-\lambda)_m = (-1)^m \Gamma(\lambda+m+1)/\Gamma(\lambda-m+1)$ .

**Remark 3.6.4.** *It is important to note that the relations (3.15)-(3.16) are also valid for the case  $\lambda \in \mathbb{Z}$  provided that  $\lambda \notin \llbracket -|m|, -1 \rrbracket \cup \llbracket 0, |m| - 1 \rrbracket$ . This explains the presence of the normalization factor  $(\lambda + 1)_m (-\lambda)_m / 2^m m!$  in the expression of the function  $P_\lambda^m$ .*

Let us finish this paragraph with this useful relation that allows us to compute the derivative of the function  $P_\lambda^m(x)$  with respect to  $x$ :

$$\frac{d}{dx} P_\lambda^m(x) = \frac{mx}{1-x^2} P_\lambda^m(x) - \frac{1}{\sqrt{1-x^2}} P_\lambda^{m+1}(x)$$

provided that all the functions  $P_\lambda^m$  and  $P_\lambda^{m+1}$  are well-defined. Other useful relations are also available (see [3, 95]).

### Approximations of the first kind associated Legendre functions

Starting from the relations (3.15)-(3.16), one can write a simple code that approximates the function  $P_\lambda^m$  with  $\lambda \in \mathbb{C}$  and  $m \in \mathbb{Z}$  provided that  $\lambda \notin \llbracket -m, -1 \rrbracket \cup \llbracket 0, m - 1 \rrbracket$ . The easiest way to do that, is to use **MATLAB**, in which the Gauss hypergeometric function with complex arguments is already defined. However, for practical purposes (visualization, finite elements approximations, ...), it would be useful to write a **C++** program that computes an approximation of these functions. In this case, one has to implement an approximation of the Gauss hypergeometric function  ${}_2F_1$  which is not available in **C++**. An implementation of the algorithm that approximates the functions  $P_\lambda^m$  for  $m \in \mathbb{Z}$  and  $\lambda \in \mathbb{C} \setminus \llbracket -|m|, -1 \rrbracket \cup \llbracket 0, |m| - 1 \rrbracket$ , is as follows:

```

1 | Complex P( Number m, Complex lam, Complex x ){
2 | %Compute _2F1(-lam+abs(m), lam+abs(m)+1, abs(m)+1)
3 | Complex res1=1.; if(abs(m)>0){for (int i = 0; i < abs(m); ++i){res=res*(-lam+i)*
   |   lam+1+i);}}
4 | Real tol=1.0e-9; Complex a=-lam+abs(m), b=lam+abs(m)+1, c=abs(m)+1, term=a*b*x/c,
   |   value=1.+term;
5 | Number n=1;
6 | while (abs(term)>tol){ a=a+1, b=b+1, c=c+1, n=n+1; term= (term*a*b*x)/(c*n); value
   |   += term;} value=value*pow(1-x*x,m/2); if(m<0) res1=1/res1; return value;}

```

Note that the previous code was implemented using the **C++** library **Xlife++**.

### The case of conical (or Mehler) functions

Conical functions are a particular class of the associated Legendre functions. More precisely, they correspond to the particular case when the degree  $\lambda$  has the form  $\lambda = -1/2 + it$  with  $t \in \mathbb{R}$ . For the same reasons as above, we shall restrict ourselves to the case of first kind ones. Since for all  $t \in \mathbb{R}$  we have  $-1/2 + it \in \mathbb{C} \setminus \mathbb{Z}$ , it follows that the functions  $x \mapsto P_{-1/2+it}^m(x)$  are well-defined for all  $t \in \mathbb{R}$ . Moreover, one can easily check that for all  $t \in \mathbb{R}$  the function  $x \mapsto P_{-1/2+it}^m(x)$  admits the representation (see [146])

$$P_{-1/2+it}^m(x) = \frac{\prod_{k=1}^m (t^2 + (2k-1)^2/4)}{2^m m!} {}_2F_1(1/2 + m + it, 1/2 + m - it, m + 1, \frac{1-x}{2}). \quad (3.17)$$

This implies that these functions are real-valued and positive for all  $x \in [-1; 1]$ . In addition to that, it can be seen that for all  $t \in \mathbb{R}$ , we have

$$P_{-1/2+it}^m = P_{-1/2-it}^m. \quad (3.18)$$

Using the results of the previous paragraph, one can say that for all  $m \in \mathbb{Z}$  and  $t \in \mathbb{R}$  we have

$$\frac{d}{dx} P_{-1/2+it}^m(x) = \frac{mx}{1-x^2} P_{-1/2+it}^m(x) - \frac{1}{\sqrt{1-x^2}} P_{-1/2+it}^{m+1}(x) \text{ for all } x \in [-1; 1].$$

This implies in particular that

$$\frac{d}{dx} P_{-1/2+it}^0(x) = -\frac{1}{\sqrt{1-x^2}} P_{-1/2+it}^1(x) \text{ for all } x \in [-1; 1]. \quad (3.19)$$

**Remark 3.6.5.** *Taking the derivative of the relation (3.17) with respect to  $t$ , one can see that the function  $x \mapsto \frac{dP_{-1/2+it}^m}{dt}(x)$  is analytic.*

## Chapter 4

# An optimal control-based numerical method for scalar transmission problems with sign-changing coefficients

The results of this chapter were obtained in collaboration with Patrick Ciarlet (POEMS/ENSTA) and David Lassounon (IRMAR-INSA/Université de Rennes 1).

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### 4.1 Introduction

In the present chapter, we study the numerical approximation of the scalar problem with sign-changing coefficients. To fix ideas, consider  $\Omega$  a domain (an open connected subset) of  $\mathbb{R}^d$ ,

$d \in \{2, 3\}$ , formed by the union of two disjoint subdomains  $\Omega_1$ ,  $\Omega_2$  that are separated by an interface  $\Sigma$  (see Figure 4.1 for an example). We assume that  $\partial\Omega$ ,  $\partial\Omega_1$ ,  $\partial\Omega_2$  and  $\Sigma$  are Lipschitz regular.

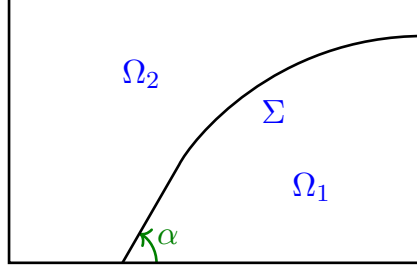


Figure 4.1: Example of geometry.

We also introduce a piecewise constant function  $\varepsilon$  such that  $\varepsilon = \varepsilon_1 > 0$  in  $\Omega_1$  and  $\varepsilon = \varepsilon_2 < 0$  in  $\Omega_2$ . The contrast  $\kappa_\varepsilon$  is defined by  $\kappa_\varepsilon := \varepsilon_2/\varepsilon_1 < 0$ . For a given source term  $f \in L^2(\Omega)$ , we consider the problem

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } -\operatorname{div}(\varepsilon \nabla u) = f \in L^2(\Omega). \quad (4.1)$$

The equivalent variational formulation to (4.1) writes

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \int_{\Omega} \varepsilon \nabla u \cdot \nabla v = \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega). \quad (4.2)$$

Because of the change of sign of  $\varepsilon$ , the well-posedness of this problem does not fit into the classical theory of elliptic PDEs and it can be ill-posed. On the other hand, one can show that for large or small contrasts, Problem 4.2 is T-coercive, i.e. there exists an operator  $T : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that  $(u, v) \mapsto \int_{\Omega} \varepsilon \nabla u \cdot \nabla(T(v))$  is coercive, and then it is well-posed. For the case of polygonal interfaces, the construction of such operator T is based on the use of local geometrical transformations (such as reflections, rotations, ...) near the interface.

The implementation of a general conforming finite element methods to discretize (4.2) leads us to consider the problem

$$\text{Find } u_h \in V_h(\Omega) \text{ such that } \int_{\Omega} \varepsilon \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h, \quad \forall v_h \in V_h, \quad (4.3)$$

where  $V_h(\Omega)$  is a well-chosen subspace  $H_0^1(\Omega)$ . Even in the case where (4.2) is T-coercive, one can not guaranty that Problem (4.3) is also T-coercive. Indeed, it may happen that for some  $v_h \in V_h(\Omega)$ , there holds  $T(v_h) \notin V_h(\Omega)$ . To overcome this difficulty, an interesting idea is to try to construct meshes such that the approximation spaces  $V_h(\Omega)$  are stable by operators T for which Problem (4.2) is T-coercive. This type of meshes are called T-conforming meshes. Such an approach has been investigated in [147, 49, 45]. It works quite well but presents two main drawbacks:

- The construction of well-suited meshes for curved interfaces, interfaces with corners or 3D interfaces is not an easy task [45].
- Sometimes the operator T for which the problem is T-coercive is constructed by abstract tools and therefore is not explicit. In these situations, one cannot find adapted meshes.

Two first alternatives have been proposed. The first one, presented in [51], consists in adding some dissipation to the problem (considering  $\varepsilon + i\delta$  instead of  $\varepsilon$  in (4.2) where  $\delta$  depends on the meshsize). The second one is developed in [119] and is based on the use of mesh refinement techniques. The essential limitation of these two approaches is that, for interfaces with general shapes, the convergence can not be assured for all contrasts for which Problem (4.2) is well-posed.

A new technique relying on the use of an optimal control reformulation has been introduced in [1]. It is proved to be convergent for general meshes (that respect the interface) as soon as the exact solution of (4.1) belongs to the space  $\text{PH}^s(\Omega) := \{u|_{\Omega_1} \in \text{H}^s(\Omega_1) \text{ and } u|_{\Omega_2} \in \text{H}^s(\Omega_2)\}$  with  $s > 3/2$ . Unfortunately, this regularity condition is not always satisfied, especially when  $\Sigma$  has corners in 2D or conical points in 3D.

In this chapter, we present a new strategy which relies on the use of a different optimal control reformulation and which converges without any restriction neither on the mesh (the interface simply needs to coincide with edges of the mesh) nor on the regularity of the exact solution. This method is inspired by the smooth extension method that was used (without proof of convergence) in [73] to approximate the solution of some classical scalar transmission problems.

The chapter is organized as follows. In Section 4.2, we start by giving a detailed description of the problem. Then we explain how to derive an equivalent optimal control reformulation. Section 4.4 is dedicated to the study of some basic properties of the optimization problem and its regularization. The proposed numerical method and the proof of its convergence are given in Section 4.5. Our results are illustrated by some numerical experiments in Section 4.6. Finally we give a few words of conclusion and discuss some possible extensions.

## 4.2 Main assumption on $\varepsilon$ and reformulation of the problem

Introduce the bounded operator  $A_\varepsilon : \text{H}_0^1(\Omega) \rightarrow (\text{H}_0^1(\Omega))^*$  such that

$$\langle A_\varepsilon u, v \rangle = \int_{\Omega} \varepsilon \nabla u \cdot \nabla v, \quad \forall u, v \in \text{H}_0^1(\Omega).$$

Obviously if  $A_\varepsilon$  is an isomorphism then Problem (4.1) is well-posed in the Hadamard sense. In this chapter, we shall work under the following

**Assumption 4.2.1.** *Assume that  $\varepsilon$  is such that  $A_\varepsilon$  is an isomorphism and that the source term  $f$  in (4.1) belongs to  $\text{L}^2(\Omega)$ .*

As we have seen in the first chapter of this thesis, the previous assumption is satisfied when the contrast  $\kappa_\varepsilon$  does not belong to the critical interval that will be denoted by  $I_\Sigma$ . The expression of this interval in general is not known analytically, except for particular geometries like symmetric domains, simple 2D interface with corners, simple 3D interfaces with circular conical tips..., but can be approximated numerically.

**Remark 4.2.1.** *In Problem (4.1), we consider homogeneous Dirichlet boundary conditions. Let us mention that the results below extend quite straightforwardly to other situations, for example with Neumann or Robin-Fourier boundary conditions which can be homogeneous or not.*

To introduce the method, we start by writing an equivalent version of (4.1) in which the unknown  $u \in \text{H}_0^1(\Omega)$  is splitted into two partial unknowns defined in  $\Omega_1$  and  $\Omega_2$ . To do so, we observe that since  $f \in \text{L}^2(\Omega)$ , the solution  $u$  of (4.1) is such that the vector field  $\varepsilon \nabla u$  belongs to the space  $\text{H}(\text{div}, \Omega) = \{\mathbf{u} \in (\text{L}^2(\Omega))^d \text{ such that } \text{div}(\mathbf{u}) \in \text{L}^2(\Omega)\}$ . Consequently, the pair of functions  $(u|_{\Omega_1}, u|_{\Omega_2})$  satisfies the problem

$$\text{Find } (u_1, u_2) \in \text{V}_1(\Omega_1) \times \text{V}_2(\Omega_2) \text{ such that } \begin{cases} -\varepsilon_1 \Delta u_1 = f_1 =: f|_{\Omega_1} \\ -\varepsilon_2 \Delta u_2 = f_2 =: f|_{\Omega_2} \\ \partial_n u_1 = \kappa_\varepsilon \partial_n u_2 \text{ and } u_1 = u_2 \text{ on } \Sigma \end{cases} \quad (4.4)$$

in which  $n$  stands for the unit normal vector to  $\Sigma$  oriented to the exterior of  $\Omega_1$  and

$$\text{V}_1(\Omega_1) := \{u \in \text{H}^1(\Omega_1), u = 0 \text{ on } \partial\Omega_1 \setminus \Sigma\}, \quad \text{V}_2(\Omega_2) := \{u \in \text{H}^1(\Omega_2), u = 0 \text{ on } \partial\Omega_2 \setminus \Sigma\}.$$



On the other hand, one can check that if  $(u_1, u_2)$  is a solution of (4.4), then the function  $u$  defined by  $u|_{\Omega_j} = u_j$  for  $j = 1, 2$  solves (4.1). The equations satisfied by  $u_1$  and  $u_2$  are elliptic but they are coupled by the transmission conditions on  $\Sigma$ . As a consequence, we cannot solve them independently. The purpose of the next paragraph is to explain how to proceed to write an alternative formulation (an optimization-based one), which can be solved via an iterative procedure such that at each step one has to solve a set of elliptic problems.

### 4.3 The smooth extension method for the scalar transmission problem

The smooth extension method was proposed in [79] and can be considered as a special case of the fictitious domain methods (see [11]). It has been adapted to study the classical scalar transmission problem, i.e. with constant sign coefficients, in [73]. In this section, we explain how to apply it to our problem. In order to make the presentation as simple as possible, we start with a formal presentation of the technique, and then we will make things more rigorous.

#### 4.3.1 Formal presentation of the smooth extension method

The idea behind the smooth extension method is the following: instead of looking for  $(u_1, u_2) \in V_1(\Omega) \times V_2(\Omega_2)$  solution of (4.4), we search for a pair of functions  $(\tilde{u}, u_2) \in H_0^1(\Omega) \times V_2(\Omega)$  such that  $(\tilde{u}|_{\Omega_1}, u_2)$  is the solution of (4.4). The function  $\tilde{u}$  is then a particular continuous extension of  $u_1$  to the whole domain  $\Omega$ . The difficulty is to find a good way to define the function  $\tilde{u}$ . In particular, we have to describe the equation satisfied by  $\tilde{u}$  in  $\Omega_2$ . Formally, the idea of the smooth extension method is to extend the equation satisfied by  $u_1$  to the whole domain  $\Omega$ . More precisely, the idea is to suppose that the function  $\tilde{u}$  satisfies the problem

$$-\varepsilon_1 \Delta \tilde{u} = f_1 1_{\Omega_1} + \ell 1_{\Omega_2}$$

in which  $\ell$  is a function to determine so that  $(\tilde{u}|_{\Omega_1}, u_2)$  solves (4.4). If one finds a way to compute such a  $\ell$ , since the problem that relates  $\ell$  to  $\tilde{u}$  is elliptic, the function  $\tilde{u}$  can be approximated by the classical FEM. After that, the function  $u_2$  can be then approximated by solving the problem satisfied by  $u_2$  in  $\Omega_2$  completed by  $\tilde{u}|_{\Sigma}$  (resp.  $\kappa_\varepsilon^{-1} \partial_n \tilde{u}|_{\Sigma}$ ) as a Dirichlet (resp. Neumann) boundary condition on  $\Sigma$  which is also elliptic. Note that at first sight, neither the existence nor the construction of such  $\ell$  are clear. This will be done thanks to an optimal control reformulation of (4.4). This is the main goal of the next paragraph in which we also reformulate the idea presented above in a more rigorous way.

#### 4.3.2 An optimal control reformulation of the problem

For  $\ell \in (V_2(\Omega_2))^*$ , introduce  $u^\ell$  the uniquely defined function satisfying the problem

$$\text{Find } u^\ell \in H_0^1(\Omega) \text{ such that } \varepsilon_1 \int_{\Omega} \nabla u^\ell \cdot \nabla v = \int_{\Omega_1} f_1 v + \ell(v|_{\Omega_2}), \quad \forall v \in H_0^1(\Omega).$$

Clearly the function  $u_1^\ell =: u^\ell|_{\Omega_1} \in V_1(\Omega_1)$  is such that  $-\varepsilon_1 \Delta u_1^\ell = f_1$  in  $\Omega_1$ . This is the equation satisfied by  $u_1$  in (4.4). Using this and an integration by parts, for all  $v_2 \in V_2(\Omega_2)$ , we obtain

$$\langle \varepsilon_1 \partial_n u^\ell, v_2 \rangle = -\varepsilon_1 \int_{\Omega_2} \nabla u^\ell \cdot \nabla v_2 + \ell(v_2).$$

Now, assume that one finds  $\ell^* \in (V_2(\Omega_2))^*$  for which  $u_1^{\ell^*}$  coincides with  $u_1$  (the solution of (4.4)) in  $\Omega_1$ . Then the function  $u_2$  can be deduced either by using  $u_1^{\ell^*}|_{\Sigma}$  or  $\kappa_\varepsilon^{-1} \partial_n u_1^{\ell^*}|_{\Sigma}$  as a Dirichlet or as Neumann trace of  $u_2$  on  $\Sigma$ . More precisely, if one uses  $\kappa_\varepsilon^{-1} \partial_n u_1^{\ell^*}|_{\Sigma}$  as a Neumann boundary data

for  $u_2$  on  $\Sigma$ , the problem satisfied by  $u_2$  writes: Find  $u_2 \in V_2(\Omega_2)$  such that for all  $v_2 \in V_2(\Omega_2)$ , we have

$$\varepsilon_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 = \int_{\Omega_2} f_2 v_2 - \langle \varepsilon_1 \partial_n u^{\ell^*}, v_2 \rangle = \int_{\Omega_2} f_2 v_2 - \ell^*(v_2) + \varepsilon_1 \int_{\Omega_2} \nabla u^{\ell^*} \cdot \nabla v_2.$$

Obviously, the previous problem is well-posed. This leads us to define for all  $\ell \in (V_2(\Omega_2))^*$  the (well-posed) problem: Find  $(u^\ell, u_2^\ell) \in H_0^1(\Omega) \times V_2(\Omega_2)$  such that

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla u^\ell \cdot \nabla v = \int_{\Omega_1} f_1 v + \ell(v|_{\Omega_2}) & \forall v \in H_0^1(\Omega) \\ \varepsilon_2 \int_{\Omega_2} \nabla u_2^\ell \cdot \nabla v_2 = \int_{\Omega_2} f_2 v_2 - \ell(v_2) + \varepsilon_1 \int_{\Omega_2} \nabla u^\ell \cdot \nabla v_2 & \forall v_2 \in V_2(\Omega_2). \end{cases} \quad (4.5)$$

Using the optimal control terminology, the previous equation plays the role of the state equation in which  $\ell$  is the control function and  $\ell^*$  (that we are looking for) is the optimal control. In order to write an optimal control reformulation of our problem, it remains to find an adapted objective (or cost) function. To do so, the starting point is the following

**Proposition 4.3.1.** *For  $\ell \in (V_2(\Omega_2))^*$ , the functions  $u_1^\ell$  and  $u_2^\ell$  are such that*

$$\begin{cases} -\varepsilon_1 \Delta u_1^\ell = f_1 & \text{in } \Omega_1 \\ -\varepsilon_2 \Delta u_2^\ell = f_2 & \text{in } \Omega_2 \\ \partial_n u_1^\ell = \kappa_\varepsilon \partial_n u_2^\ell & \text{on } \Sigma. \end{cases}$$

**Proof.** Take  $\varphi_1 \in \mathcal{C}_0^\infty(\Omega_1)$  and extend it by 0 to the whole  $\Omega$  to obtain the function  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ . Take  $v = \varphi$  in the problem satisfied by  $u^\ell$ . One finds that  $-\varepsilon_1 \Delta u_1^\ell = f_1$  in  $\Omega_1$ . Next, take some  $\varphi_2 \in \mathcal{C}_0^\infty(\Omega_2)$ , extend it by 0 in  $\Omega_1$  and denote by  $\varphi$  the new function. By taking  $v = \varphi$  in the problem satisfied by  $u^\ell$  and  $v_2 = \varphi_2$  in the problem satisfied by  $u_2^\ell$  one finds that

$$\varepsilon_1 \int_{\Omega_2} \nabla u^\ell \cdot \nabla \varphi_2 = \ell(\varphi_2) \quad \text{and} \quad \varepsilon_2 \int_{\Omega_2} \nabla u_2^\ell \cdot \nabla \varphi_2 = \int_{\Omega_2} f_2 \varphi_2 - \ell(\varphi_2) + \varepsilon_1 \int_{\Omega_2} \nabla u^\ell \cdot \nabla \varphi_2.$$

By considering the sum of the two formulations, we conclude that  $-\varepsilon_2 \Delta u_2^\ell = f_2$  in  $\Omega_2$ . To end the proof, it remains to show that  $\partial_n u^\ell = \kappa_\varepsilon \partial_n u_2^\ell$ . To do so, taking  $v_2 = v|_{\Omega_2}$  for an arbitrary  $v \in H_0^1(\Omega)$  in (4.5), integrating by parts in both formulations and then, using the equations satisfied by  $u_1^\ell$  and  $u_2^\ell$ , we infer that

$$-\langle \varepsilon_1 \partial_n u_1^\ell, v \rangle = -\langle \varepsilon_2 \partial_n u_2^\ell, v \rangle, \quad v \in H_0^1(\Omega).$$

This gives  $\varepsilon_1 \partial_n u_1^\ell = \varepsilon_2 \partial_n u_2^\ell$  on  $\Sigma$  and ends the proof. ■

Thus the introduction of an auxiliary control function  $\ell \in (V_2(\Omega_2))^*$  allows us to construct pseudo-solutions of the equation (4.4) for which the condition on the normal derivatives is automatically satisfied. However we do not have in general continuity of the field at the interface. Taking this into account, we get the

**Lemma 4.3.1.** *If there exists  $\ell^* \in (V_2(\Omega_2))^*$  such that the solution of (4.5) satisfies  $u|_\Sigma^{\ell^*} = u_2|_\Sigma^{\ell^*}$ , then  $(u_1^{\ell^*}, u_2^{\ell^*})$  solves (4.4).*

The existence of such  $\ell^*$  is the subject of the following

**Lemma 4.3.2.** *There exists  $\ell^* \in (V_2(\Omega_2))^*$  such that the solution of (4.5) satisfies  $u|_\Sigma^{\ell^*} = u_2|_\Sigma^{\ell^*}$ .*

**Proof.** We proceed by construction, i.e. we find  $\ell^* \in (V_2(\Omega_2))^*$  for which the condition  $u_{|\Sigma}^{\ell^*} = u_2^{\ell^*}|_{\Sigma}$  is satisfied. Since by assumption Problem (4.1) is uniquely solvable, the functions  $u_1$  and  $u_2$  are well-defined. The function  $u_2$  can be seen as a continuous extension of  $u_1$  to the domain  $\Omega_2$ . Moreover, one can check that  $u$  the solution of (4.1) satisfies the problem

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega_1} f_1 v + \varepsilon_1 \int_{\Omega_2} \nabla u_2 \cdot \nabla v - \varepsilon_1 \langle \partial_n u_1, v \rangle & \forall v \in H_0^1(\Omega) \\ \varepsilon_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 = \int_{\Omega_2} f_2 v_2 + \varepsilon_1 \langle \partial_n u_1, v_2 \rangle & \forall v \in V_2(\Omega_2). \end{cases}$$

Now, by observing that the linear form  $\ell^*$  defined by

$$\ell^*(v_2) = \varepsilon_1 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 - \varepsilon_1 \langle \partial_n u_1, v_2 \rangle \quad \forall v_2 \in V_2(\Omega)$$

is continuous, we obtain the desired result.  $\blacksquare$

**Remark 4.3.1.** As pointed out in [73] for the classical transmission problem, the optimal function  $\ell^*$  (for which  $u_{|\Sigma}^{\ell^*} = u_2^{\ell^*}|_{\Sigma}$ ) is not unique. Indeed, if one denotes by  $E(u_1) \in H_0^1(\Omega)$  any continuous extension of the function  $u_1$  to  $\Omega_2$ , one can show that  $(E(u_1), u_2)$  satisfies the problem

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla E(u_1) \cdot \nabla v = \int_{\Omega_1} f_1 v + \varepsilon_1 \int_{\Omega_2} \nabla E(u_1) \cdot \nabla v - \varepsilon_1 \langle \partial_n E(u_1), v \rangle & \forall v \in H_0^1(\Omega) \\ \varepsilon_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla v_2 = \int_{\Omega_2} f_2 v_2 + \varepsilon_1 \langle \partial_n E(u_1), v_2 \rangle & \forall v \in V_2(\Omega_2). \end{cases}$$

The linear form  $\ell^* \in (V_2(\Omega_2))^*$  defined by

$$\ell^*(v_2) = \varepsilon_1 \int_{\Omega_2} \nabla E(u_1) \cdot \nabla v_2 - \varepsilon_1 \langle \partial_n u_1, v_2 \rangle \quad \forall v_2 \in V_2(\Omega_2) \quad (4.6)$$

can be then considered as another optimal function. This implies that the set of optimal functions  $\ell^* \in (V_2(\Omega_2))^*$  is infinite. More precisely, we observe that the set of optimal functions  $\ell^*$  is in bijection with the set of continuous extensions of  $u_1 \in V_1(\Omega_1)$  to a function of  $H_0^1(\Omega)$ .

Now, we have all the tools to write an equivalent optimal control formulation to (4.4). To do that, it suffices to observe that since  $u_1^{\ell} \in H^{1/2}(\Sigma) \subset L^2(\Sigma)$  and by means of Lemma 4.3.1 we can say that  $\ell^*$  is an optimal control if and only if  $\|u_1^{\ell^*} - u_2^{\ell^*}\|_{L^2(\Sigma)}^2 = 0$ . This allows us to say that  $\ell^*$  is an optimal control if and only if it solves the problem

$$\text{Find } \ell^* \in (V_2(\Omega_2))^* \text{ solution of } \begin{cases} \min_{\ell \in (V_2(\Omega_2))^*} \frac{1}{2} \int_{\Sigma} |u_1^{\ell} - u_2^{\ell}|^2 d\sigma \\ \text{where } (u^{\ell}, u_2^{\ell}) \text{ is the solution of (4.5).} \end{cases} \quad (4.7)$$

Regarding what we have proved previously, it follows the

**Corollary 4.3.1.** Problem (4.7) has an infinite number of solutions.

Since  $V_2(\Omega_2)$  is a Hilbert space, the Riesz representation theorem guarantees that for any element  $\ell \in (V_2(\Omega_2))^*$ , there is a unique  $w_{\ell} \in V_2(\Omega_2)$  such that

$$\ell(v) = \int_{\Omega_2} \nabla w_{\ell} \cdot \nabla v \quad \forall v \in V_2(\Omega_2).$$

Then the optimal control problem (4.7) can be reformulated in the following way

$$\text{Find } w^* \in V_2(\Omega_2) \text{ solution of } \min_{w \in V_2(\Omega_2)} J(w) \quad \text{with} \quad J(w) = \frac{1}{2} \int_{\Sigma} |u_1^w - u_2^w|^2 d\sigma, \quad (4.8)$$

where  $(u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2)$  is the solution of the problem (the state equation)

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla u^w \cdot \nabla v = \int_{\Omega_1} f_1 v + \int_{\Omega_2} \nabla w \cdot \nabla v & \forall v \in H_0^1(\Omega), \\ \varepsilon_2 \int_{\Omega_2} \nabla u_2^w \cdot \nabla v_2 = \int_{\Omega_2} f_2 v_2 + \varepsilon_1 \int_{\Omega_2} \nabla u^w \cdot \nabla v_2 - \int_{\Omega_2} \nabla w \cdot \nabla v_2 & \forall v_2 \in V_2(\Omega_2). \end{cases} \quad (4.9)$$

The objective of this section is then achieved. The next step is to propose a discretization of the problem based on FEM and to study its convergence. Before that, we have to prepare the ground and present some basic properties of the above problem which will help us to prove the convergence of the proposed discretization method.

As we have seen previously, the minimization problem (4.8) has an infinite number of solutions. Therefore, a regularization method may be necessary in order to propose a convergent discretization. For that, we will work in §4.4.4 with the classical Tikhonov regularization of Problem (4.8). This will guide our intuition in the construction of a convergent numerical method.

## 4.4 Basic properties of the optimization problem and its regularization

In this section, we focus our attention on the properties of the cost function  $J$ . In addition, we compute in §4.4.3 an explicit expression for the derivative of  $J$  with respect to  $w$ . In the process, we give useful properties of the set of minimizers of  $J$ .

### 4.4.1 Properties of the objective function

The fact that we have used the  $L^2(\Sigma)$  norm instead of the  $H^{1/2}(\Sigma)$  norm in the definition of  $J$  allows us to get the following compactness result.

**Lemma 4.4.1.** *Let  $(w_n)$  be a sequence of elements of  $V_2(\Omega_2)$  that converges weakly to  $w_0 \in V_2(\Omega_2)$ . Then,  $(J(w_n))$  converges to  $J(w_0)$ .*

**Proof.** For all  $n \in \mathbb{N}$ , denote by  $(u^n, u_2^n) \in H_0^1(\Omega) \times V_2(\Omega)$  the solution of (4.9) with  $w = w_n$ . From the ellipticity of the problems involved in (4.9), it follows that  $(u^n)$  (resp.  $(u_2^n)$ ) converges weakly in  $H_0^1(\Omega)$  (resp.  $V_2(\Omega_2)$ ) to some  $u \in H^1(\Omega)$  (resp.  $u_2 \in V_2(\Omega_2)$ ) such that  $(u, u_2)$  is the solution of (4.9) with  $w = w_0$ .

The continuity of the trace operator from  $H^1(\Omega)$  to  $H^{1/2}(\Sigma)$  implies that  $u_{|\Sigma}^n - u_{|\Sigma}^n$  converges weakly to  $u_{|\Sigma} - u_{|\Sigma}$  in  $H^{1/2}(\Sigma)$ . Given that the embedding of  $H^{1/2}(\Sigma)$  into  $L^2(\Sigma)$  is compact,  $u_{|\Sigma}^n - u_{|\Sigma}^n$  converges strongly to  $u_{|\Sigma} - u_{|\Sigma}$  in  $L^2(\Sigma)$ . Thus  $(J(w_n))$  converges to  $J(w_0)$ . The result is proved. ■

A direct consequence of the previous Lemma is the following

**Lemma 4.4.2.** *The function  $J$  is continuous and convex on  $V_2(\Omega_2)$ .*

**Proof.** While the continuity is a direct consequence of the previous lemma, the convexity follows from the fact that  $J : V_2(\Omega_2) \rightarrow \mathbb{R}$  is the composition of the affine map  $j_1 : V_2(\Omega_2) \rightarrow L^2(\Sigma)$  and of the convex map  $j_2 : L^2(\Sigma) \rightarrow \mathbb{R}$  such that for all  $w \in V_2(\Omega_2)$ ,  $g \in L^2(\Sigma)$

$$\begin{cases} j_1(w) = (u^w - u_2^w)_{|\Sigma} \text{ where } (u^w, u_2^w) \in H_0^1(\Omega) \times V_2(\Omega_2) \text{ is the solution of (4.9)} \\ j_2(g) = \frac{1}{2} \int_{\Sigma} |g|^2 d\sigma. \end{cases} \quad (4.10)$$

■

#### 4.4.2 The set of minimizers of the function $J$

As explained above, the set of minimizers of the functional  $J$ , denoted by  $M_J$ , is infinite. More precisely, one can check the equivalence

$$w^* \in M_J \quad \text{if and only if} \quad J(w^*) = 0.$$

This allows us to prove the

**Lemma 4.4.3.**  $M_J$  is a closed and convex subset of  $V_2(\Omega)$ .

**Proof.** Clearly  $M_J$  coincides with the set of zeros of  $J$ . Since  $J : V_2(\Omega_2) \rightarrow \mathbb{R}$  is continuous, we infer that  $M_J$  is closed. The convexity of  $M_J$  is a direct consequence of the positivity and convexity of  $J$ . ■

As a direct result of the previous lemma, we can say that the following minimization problem:

$$\min_{w \in M_J} \|\nabla w\|_{L^2(\Omega_2)}^2 \quad (4.11)$$

has a unique solution (this is a consequence of the strict convexity of  $\|\nabla \cdot\|_{L^2(\Omega_2)}^2$  and of the fact that  $M_J$  is a closed subset of  $V_2(\Omega_2)$ ). In the following, we shall denote by  $w_J^*$  the solution of (4.11), i.e. the smallest minimizer of the function  $J$ . By definition, we know that for all  $w \in M_J$ , the function  $u^w \in H_0^1(\Omega)$  is a continuous extension of the  $u_1$  (the restriction of the solution of (4.1) to  $\Omega_1$ ). In particular, this means that for all  $w_1, w_2 \in M_J$  we have  $u_{|\Omega_1}^{w_1} = u_{|\Omega_1}^{w_2}$ . Our next goal is to find a simple characterization of  $u^{w_J^*}$ .

#### On the smallest minimizer of $J$

We already know that for any  $w \in V_2(\Omega_2)$ , the function  $u^w$  satisfies

$$\varepsilon_1 \int_{\Omega} \nabla u^w \cdot \nabla v = \int_{\Omega_1} f v + \int_{\Omega_2} \nabla w \cdot \nabla v \quad \forall v \in H_0^1(\Omega).$$

This means that that for all  $w_1, w_2 \in V_2(\Omega_2)$  we have

$$\varepsilon_1 \int_{\Omega} \nabla(u^{w_1} - u^{w_2}) \cdot \nabla v = \int_{\Omega_2} \nabla(w_1 - w_2) \cdot \nabla v \quad \forall v \in H_0^1(\Omega).$$

Using the fact that for all  $w_1, w_2 \in M_J$ , we have  $u_{|\Omega_1}^{w_1} = u_{|\Omega_1}^{w_2}$  we then obtain

$$\varepsilon_1 \int_{\Omega_2} \nabla(u_2^{w_1} - u_2^{w_2}) \cdot \nabla v_2 = \int_{\Omega_2} \nabla(w_1 - w_2) \cdot \nabla v_2 \quad \forall v_2 \in V_2(\Omega), \quad (4.12)$$

in which for  $j = 1, 2$ , we set  $u_j^w := u_{|\Omega_j}^w$ . Note that, for all  $w_1, w_2 \in M_J$  the function  $u_2^{w_1} - u_2^{w_2}$  belongs to  $H_0^1(\Omega_2)$ . To proceed, we denote by  $E_H(u_1) \in H_0^1(\Omega)$  the continuous harmonic extension of  $u_1$ . In particular, the function  $E_H(u_1)|_{\Omega_2}$  satisfies

$$\Delta E_H(u_1)|_{\Omega_2} = 0 \text{ in } \Omega_2 \quad \text{and} \quad E_H(u_1) = u_1 \text{ on } \Sigma.$$

To this particular extension of  $u_1$ , we can introduce a unique  $w_H \in M_J$  such that  $E_H(u_1) = u^{w_H}$  (see Remark 4.3.1). More precisely, the functions  $w_H$  can be defined as the unique solution of the well-posed problem: Find  $w_H \in V_2(\Omega_2)$  such that

$$\int_{\Omega_2} \nabla w_H \cdot \nabla v_2 = \varepsilon_1 \int_{\Omega_2} \nabla E_H(u_1) \cdot \nabla v_2 - \varepsilon_1 \langle \partial_n u_1, v_2 \rangle \quad \forall v_2 \in V_2(\Omega_2).$$

In particular, we have

$$\int_{\Omega_2} \nabla w_H \cdot \nabla v_2 = 0 \quad \forall v_2 \in H_0^1(\Omega_2). \quad (4.13)$$

By taking  $v_2 = w_H$ ,  $w_1 = w_h$  and  $w_2 = w_J^* \in M_J$  in (4.12) and by making use of (4.13) we obtain

$$\int_{\Omega_2} \nabla(w_H - w_J^*) \cdot w_H = 0.$$

This shows, by means of the Cauchy-Schwarz lemma, that  $\|\nabla w_H\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)}$  and then, thanks to the definition of  $w_J^*$ , we infer that  $w_J^* = w_H$ . This leads us to state the following

**Proposition 4.4.1.** *The function  $w_J^*$  coincides with  $w_H$  and  $u^{w_J^*} = E_H(u_1)$ .*

### 4.4.3 Gradient of the function $J$

As indicated in the introduction of this chapter, the main objective of this work is to propose a new numerical method for approximating the solution of (4.1). This method will be based on the numerical approximation of the solution of the optimization problem (4.8). In this section, we will explain how to obtain an explicit expression of  $J'(w)$  the gradient of  $J$  at some  $w \in V_2(\Omega)$ . The starting point is to explain why the function  $J$  is differentiable. Again, this is the consequence of the fact that  $J$  can be written as a composition of the two differentiable maps  $j_1$  (which is differentiable because it is an affine map) and  $j_2$  (which is differentiable thanks to the differentiability of the square of the  $L^2(\Sigma)$  norm) that are defined in (4.10). Since the function  $J$  is scalar valued, its differential at any  $w \in V_2(\Omega_2)$  is then a continuous linear form  $\ell_w$  on  $V_2(\Omega_2)$ . By means of the Riesz representation theorem,  $\ell_w$  can be represented by a unique element of  $V_2(\Omega_2)$ , this element will be denoted by  $J'(w)$  and is defined as follows:

$$\text{For all } h \in V_2(\Omega_2), \text{ we have } \int_{\Omega_2} \nabla J'(w) \cdot \nabla h = \lim_{t \rightarrow 0} \frac{J(w + th) - J(w)}{t} = \ell_w(h).$$

Given the fact that  $J = j_2 \circ j_1$ , the natural idea to compute  $J'(w)$ , for all  $w \in V_2(\Omega_2)$ , is to use the chain rule formula. For this, we need to start by computing the derivative of  $w \rightarrow u^w$  and of  $w \rightarrow u_2^w$  (where  $(u^w, u_2^w)$  is the solution of (4.9)) with respect to  $w \in V_2(\Omega_2)$ . The differential of these maps will be denoted by

$$\frac{du^w}{dw} \in \mathcal{L}(V_2(\Omega_2), H_0^1(\Omega)) \text{ and } \frac{du_2^w}{dw} \in \mathcal{L}(V_2(\Omega_2), V_2(\Omega)).$$

Without any difficulty, one can check that for any  $h \in V_2(\Omega_2)$  we have

$$\frac{du^w}{dw}(h) = \tilde{u}^h \text{ and } \frac{du_2^w}{dw}(h) = \tilde{u}_2^h$$

where  $(\tilde{u}^h, \tilde{u}_2^h) \in H_0^1(\Omega) \times V_2(\Omega_2)$  is the unique solution of the well-posed system of equations:

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla \tilde{u}^h \cdot \nabla v = \int_{\Omega_2} \nabla h \cdot \nabla v & \forall v \in H_0^1(\Omega), \\ \varepsilon_2 \int_{\Omega_2} \nabla \tilde{u}_2^h \cdot \nabla v_2 = \varepsilon_1 \int_{\Omega_2} \nabla \tilde{u}^h \cdot \nabla v_2 - \int_{\Omega_2} \nabla h \cdot \nabla v_2 & \forall v_2 \in V_2(\Omega_2). \end{cases} \quad (4.14)$$

Note that since  $w \rightarrow u^w$  and  $w \rightarrow u_2^w$  are affine maps, for all  $w, h \in V_2(\Omega_2)$  we have the relation

$$u^{w+h} = u^w + \frac{du^w}{dw}(h) = u^w + \tilde{u}^h \text{ and } u_2^{w+h} = u_2^w + \frac{du_2^w}{dw}(h) = u_2^w + \tilde{u}_2^h.$$

Using these notations, and the fact that  $J$  is the composition of  $j_1$  and  $j_2$ , we obtain the following

**Proposition 4.4.2.** *For all  $w, h \in V_2(\Omega_2)$ , we have*

$$\int_{\Omega_2} \nabla J'(w) \cdot \nabla h = \int_{\Sigma} (\tilde{u}^h - \tilde{u}_2^h)(u^w - u_2^w) d\sigma.$$

where  $(u^w, u_2^w)$  and  $(\tilde{u}^h, \tilde{u}_2^h)$  are the solutions to (4.9) and (4.14) respectively.

**Proof.** Let  $t \in \mathbb{R}$  be a damping parameter. Thanks to direct calculus, for all  $w, h \in V_2(\Omega_2)$  one finds that

$$J(w + th) = 1/2 \int_{\Sigma} |u^{w+th} - u_2^{w+th}|^2 d\sigma = 1/2 \int_{\Sigma} |u^w - u_2^w + t(\tilde{u}^h - \tilde{u}_2^h)|^2 d\sigma.$$

As a result, we obtain

$$J(w + th) = J(w) + t \int_{\Sigma} (\tilde{u}^h - \tilde{u}_2^h)(u^w - u_2^w) d\sigma + o(t).$$

The lemma is then proved. ■

The expression  $J'(w)$  that we have obtained above, is not explicit. A more elegant way to get a simpler expression of  $J'(w)$  was proposed in [73]. The idea is based on the use of more general theory called the adjoint approach that was introduced in [47], and that allows us to compute the gradient of objective functions that depends in non-explicit way of the main variable of the problem, but via the solution of PDE (the state equations) in which the main variable plays the role of a parameter. Here, we are going to explain how to apply this method to our case. The idea is to introduce a Lagrangian function  $\mathcal{L} : V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \rightarrow \mathbb{R}$  such that

$$\mathcal{L}(w, u, u_2, \lambda, \lambda_2) = \frac{1}{2} \int_{\Sigma} |u - u_2|^2 d\sigma + a_1(w, u, \lambda) + a_2(w, u, u_2, \lambda_2)$$

in which  $a_1(w, u, \lambda)$  and  $a_2(w, u, u_2, \lambda_2)$  are given by

$$\begin{cases} a_1(w, u, \lambda) = \varepsilon_1 \int_{\Omega} \nabla u \cdot \nabla \lambda - \int_{\Omega_1} f \lambda - \int_{\Omega_2} \nabla w \cdot \nabla \lambda \\ a_2(w, u, u_2, \lambda_2) = \varepsilon_2 \int_{\Omega_2} \nabla u_2 \cdot \nabla \lambda_2 - \int_{\Omega_2} f_2 \lambda_2 - \varepsilon_1 \int_{\Omega_2} \nabla u \cdot \nabla \lambda_2 + \int_{\Omega_2} \nabla w \cdot \nabla \lambda_2. \end{cases}$$

The functions  $\lambda \in H_0^1(\Omega)$ ,  $\lambda_2 \in V_2(\Omega_2)$  are called the adjoint variables associated to  $u, u_2$  respectively. It will be useful to observe that when  $(u, u_2)$  coincides with  $(u^w, u_2^w)$  (the solution of (4.9)), we have

$$\mathcal{L}(w, u^w, u_2^w, \lambda, \lambda_2) = J(w) \quad \forall \lambda \in H_0^1(\Omega), \lambda_2 \in V_2(\Omega_2). \quad (4.15)$$

Clearly, the function  $\mathcal{L}$  is differentiable with respect to all its variables. In what follows, for all  $(w, u, u_2, \lambda, \lambda_2) \in V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2) \times H_0^1(\Omega) \times V_2(\Omega_2)$ , the partial derivative of  $\mathcal{L}$  with respect to the variable  $w, u, u_2, \lambda, \lambda_2$  are denoted, respectively, by

$$\frac{\partial \mathcal{L}}{\partial w}, \frac{\partial \mathcal{L}}{\partial u}, \frac{\partial \mathcal{L}}{\partial u_2}, \frac{\partial \mathcal{L}}{\partial \lambda} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \lambda_2}.$$

They belong, respectively, to the spaces  $(V_2(\Omega_2))^*, (H_0^1(\Omega))^*, (V_2(\Omega_2))^*, (H_0^1(\Omega))^*$  and  $(V_2(\Omega_2))^*$ . As a result, for a fixed  $\lambda \in H_0^1(\Omega)$  and  $\lambda_2 \in V_2(\Omega_2)$ , by taking the derivative of the relation (4.15) with respect to  $w$ , we can say, by applying the chain rule formula, that for all  $h \in V_2(\Omega_2)$  we have,

$$\begin{aligned} \langle J'(w), h \rangle &= \left\langle \frac{\partial \mathcal{L}}{\partial w}(w, u^w, u_2^w, \lambda, \lambda_2), h \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial u}(w, u^w, u_2^w, \lambda, \lambda_2), \frac{du^w}{dw}(h) \right\rangle \\ &\quad + \left\langle \frac{\partial \mathcal{L}}{\partial u_2}(w, u^w, u_2^w, \lambda, \lambda_2), \frac{du_2^w}{dw}(h) \right\rangle \end{aligned}$$



or equivalently,

$$\langle J'(w), h \rangle = \left\langle \frac{\partial \mathcal{L}}{\partial w}(w, u^w, u_2^w, \lambda, \lambda_2), h \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial u}(w, u^w, u_2^w, \lambda, \lambda_2), \tilde{u}^h \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial u_2}(w, u^w, u_2^w, \lambda, \lambda_2), \tilde{u}_2^h \right\rangle$$

where  $(\tilde{u}^h, \tilde{u}_2^h)$  is defined by (4.14).

Now, suppose that we find  $(\lambda^w, \lambda_2^w)$  for which the equations

$$\frac{\partial \mathcal{L}}{\partial u}(w, u^w, u_2^w, \lambda_1^w, \lambda_2^w) = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial u_2}(w, u^w, u_2^w, \lambda_1^w, \lambda_2^w) = 0$$

are satisfied for all  $w \in V_2(\Omega_2)$ , this will implies that

$$J'(w) = \frac{\partial \mathcal{L}}{\partial w}(w, u^w, u_2^w, \lambda_1^w, \lambda_2^w) \quad \forall w \in V_2(\Omega_2).$$

To investigate the existence of such functions, we need to write down, for an arbitrary  $(\lambda, \lambda_2) \in H_0^1(\Omega) \times V_2(\Omega_2)$ , the expression of

$$\frac{\partial \mathcal{L}}{\partial u}(w, u^w, u_2^w, \lambda, \lambda_2) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial u_2}(w, u^w, u_2^w, \lambda, \lambda_2).$$

By a direct calculus, we find, for all  $w, \lambda_2 \in V_2(\Omega_2)$ ,  $\lambda \in H_0^1(\Omega)$ ,

$$\begin{aligned} \left\langle \frac{\partial \mathcal{L}}{\partial u}(w, u^w, u_2^w, \lambda, \lambda_2), v \right\rangle &= \varepsilon_1 \int_{\Omega} \nabla \lambda^w \cdot \nabla v - \varepsilon_1 \int_{\Omega_2} \nabla \lambda_2^w \cdot \nabla v + \int_{\Sigma} (u^w - u_2^w) v \quad \forall v \in H_0^1(\Omega) \\ \left\langle \frac{\partial \mathcal{L}}{\partial u_2}(w, u^w, u_2^w, \lambda, \lambda_2), v_2 \right\rangle &= \varepsilon_2 \int_{\Omega_2} \nabla \lambda_2^w \cdot \nabla v_2 - \int_{\Sigma} (u^w - u_2^w) v_2 \quad \forall v_2 \in V_2(\Omega_2). \end{aligned}$$

As a consequence, the functions  $\lambda^w$  and  $\lambda_2^w$  that we are looking for, must satisfy the following system of equations:

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla \lambda^w \cdot \nabla v = \varepsilon_1 \int_{\Omega_2} \nabla \lambda_2^w \cdot \nabla v - \int_{\Sigma} (u^w - u_2^w) v & \forall v \in H_0^1(\Omega) \\ \varepsilon_2 \int_{\Omega_2} \nabla \lambda_2^w \cdot \nabla v_2 = \int_{\Sigma} (u^w - u_2^w) v_2 & \forall v_2 \in V_2(\Omega_2). \end{cases} \quad (4.16)$$

Clearly the previous system of equations is well-posed. Therefore the function  $\lambda^w, \lambda_2^w$  are well-defined. Note that, in the literature (see [73, 47]), the previous equations are known as the adjoint system. To summarize, we have the

**Lemma 4.4.4.** *For all  $w \in V_2(\Omega_2)$ , there holds  $J'(w) = \lambda_2^w - \lambda_1^w|_{\Omega_2}$ , where  $\lambda^w, \lambda_2^w$  are given by Equation (4.16).*

**Proof.** Take  $w \in V_2(\Omega_2)$ . From the definition of  $\lambda^w$  and  $\lambda_2^w$ , we deduce that for all  $h \in V_2(\Omega_2)$ , we have

$$\langle \nabla J'(w), h \rangle = \left\langle \frac{\partial \mathcal{L}}{\partial w}(w, u^w, u_2^w, \lambda_1^w, \lambda_2^w), h \right\rangle.$$

Now, let us compute explicitly the value of  $\left\langle \frac{\partial \mathcal{L}}{\partial w}(w, u, u_2, \lambda, \lambda_2), h \right\rangle$  for any  $u, u_2, \lambda, \lambda_2$ . Easily, one finds that

$$\left\langle \frac{\partial \mathcal{L}}{\partial w}(w, u, u_2, \lambda, \lambda_2), h \right\rangle = \int_{\Omega_2} \nabla h \cdot \nabla (\lambda_2 - \lambda|_{\Omega_2}).$$

This shows that  $J'(w) = \lambda_1^w|_{\Omega_2} - \lambda_2^w$  and then the result is proved. ■

We have the following optimality result

**Corollary 4.4.1.** *We have the equivalence*

$$[w^* \in V_2(\Omega_2) \text{ is such that } J'(w^*) = 0] \iff w^* \in M_J.$$

**Proof.** Let us start with the proof of the direct implication. Suppose that there exists some  $w^* \in V_2(\Omega_2)$  such that  $\lambda^{w^*}|_{\Omega_2} = \lambda_2^{w^*}$ . By taking the sum of the variational formulations of (4.16), we deduce that

$$\int_{\Omega} \varepsilon \nabla \lambda^{w^*} \cdot \nabla v = 0 \quad \forall v \in H_0^1(\Omega).$$

This means  $A_\varepsilon(\lambda^{w^*}) = 0$  and then, thanks to Assumption 4.2.1,  $\lambda^{w^*} = 0$ . This implies that  $\lambda_2^{w^*} = 0$  and then by using the second equation of (4.16), that  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ . This shows that  $w^*$  is a minimizer of  $J$ . The reverse implication is a consequence of the fact that if  $w_2^* \in M_J$  we have  $J(w^*) = 0$  and then  $u^{w^*} = u_2^{w^*}$  on  $\Sigma$ . This implies that  $\lambda_2^{w^*} = 0$  and that  $\lambda^{w^*} = 0$ . ■

We end this paragraph with the following result that can be useful to prove the convergence of the classical gradient descent algorithm.

**Corollary 4.4.2.** *The function  $J' : V_2(\Omega_2) \rightarrow V_2(\Omega_2)$  is Lipschitz continuous.*

**Proof.** Starting from (4.9), we deduce that  $w \mapsto u^w$ ,  $w \mapsto u_2^w$  are Lipschitz continuous. Inserting this into (4.16), we obtain the result. ■

#### 4.4.4 Tikhonov regularization of the problem

Tikhonov regularization, which was originally introduced in [137], is a classical method to regularize a convex optimization problem. Classically, this method is used in the context of regularization of ill-posed inverse problems (see [76] and the references therein). In this paragraph, we study the convergence of such regularization when it is applied to our problem. For  $\delta > 0$ , we introduce the functional  $J^\delta : V_2(\Omega_2) \rightarrow \mathbb{R}$  defined by

$$J^\delta(w) = J(w) + \delta \|\nabla w\|_{L^2(\Omega_2)}^2 \quad \forall w \in V_2(\Omega_2).$$

Since  $J$  is convex and  $\delta > 0$ , the functional  $J^\delta$  is strictly convex and coercive. Therefore the minimization problem

$$\min_{w \in V_2(\Omega_2)} J^\delta(w)$$

has a unique solution that we denote by  $w_\delta^*$ . Our goal is to study the behaviour of  $w_\delta^*$  as  $\delta$  tends to zero. One may expect that  $w_\delta^*$  converge to one of the solutions (4.8). If this is the case and because the problem (4.8) has an infinite number of solutions, it will be interesting to characterize the particular solution to which  $w_\delta^*$  converges. Our findings are given in the following

**Proposition 4.4.3.** *The sequence  $(w_\delta^*)$  converges when  $\delta \rightarrow 0$  to  $w_J^*$  the smallest minimizer of  $J$ .*

**Proof.** From the definition of  $w_\delta^*$ , we can write that

$$\delta \|\nabla w_\delta^*\|_{L^2(\Omega_2)} \leq J^\delta(w_\delta^*) \leq J^\delta(w_J^*) = J(w_J^*) + \delta \|\nabla w_J^*\|_{L^2(\Omega_2)} = \delta \|\nabla w_J^*\|_{L^2(\Omega_2)}.$$

This means that for all  $0 < \delta$ , there holds  $\|\nabla w_\delta^*\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)}$ . As a result  $(w_\delta^*)$  is bounded in  $V_2(\Omega_2)$ . This implies that, up to a sub-sequence,  $(w_\delta^*)$  converges, as  $\delta$  tends to 0, weakly in  $V_2(\Omega_2)$  to some  $w_0 \in V_2(\Omega_2)$ . For the reader convenience, this sequence is also denoted by  $(w_\delta)$ . Now, let us prove that  $w_0$  is a minimizer of  $J$ . To do that, we start by observing that for all  $\delta > 0$ , we have

$$0 \leq J(w_\delta^*) \leq J^\delta(w_\delta^*) \leq J^\delta(w_J^*) = \delta \|\nabla w_J^*\|_{L^2(\Omega_2)}^2.$$

This shows that  $(J(w_\delta^*))$  converges to zero as  $\delta$  tends to zero. On the other hand, by using the result of Lemma 4.4.1, we know that  $(J(w_\delta^*))$  converges to  $J(w_0)$ . Consequently,  $J(w_0) = 0$  and then  $w_0$  is a minimizer of  $J$ .

The next step is to show that the convergence of  $(w_\delta^*)$  to  $w_0$  occurs in the strong sense and that  $w_0 = w_J^*$ . To do so, we observe that

$$\left\{ \begin{array}{l} \|\nabla w_\delta^*\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)} \implies \limsup_{\delta \rightarrow 0} \|\nabla w_\delta^*\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)} \\ w_\delta^* \rightharpoonup w_0 \text{ in } V_2(\Omega_2) \implies \|\nabla w_0\|_{L^2(\Omega_2)} \leq \liminf_{\delta \rightarrow 0} \|\nabla w_\delta^*\|_{L^2(\Omega_2)} \end{array} \right. \quad (1)$$

This implies that  $\|\nabla w_0\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)}$ . Thanks to the definition of  $w_J^*$ , we deduce that  $w_0 = w_J^*$ .

With this in mind and with the help of the previous inequality, we conclude that

$$\lim_{\delta \rightarrow 0} \|\nabla w_\delta^*\|_{L^2(\Omega_2)} = \|\nabla w_J^*\|_{L^2(\Omega_2)}.$$

Since  $V_2(\Omega_2)$  is a Hilbert space, it follows (see [40, Proposition III.30]) that  $w_\delta \rightarrow w_J^*$  in  $V_2(\Omega_2)$ . By noticing that  $w_J^*$  is independent of the considered sub-sequence, the result is then proved. ■

In conclusion, we can say that the Tikhonov regularization allows us to obtain a stabilized version of the optimization problem (4.8). This will be used in order to introduce a stabilization of the finite element discretization of the problem (4.8), but in that case the stabilization parameter  $\delta$  must be chosen as a function of the meshsize. This will be detailed in §4.5.3. Note that the same idea was employed in [2].

## 4.5 Numerical discretization of the problem

In this part, we are concerned with the numerical approximation of (4.8) by means of the Finite Elements Method. To do so, we start by presenting some details and notations about the sequence of meshes that will be used.

### 4.5.1 Mesh assumptions

**Assumption 4.5.1.** *Let  $T$  be a regular (see [55]) mesh of  $\overline{\Omega}$  composed by triangles (resp. tetrahedrons) when  $d = 2$  (resp.  $d = 3$ ). We suppose that*

- *each element of  $T$  belongs either to  $\overline{\Omega_1}$  or to  $\overline{\Omega_2}$ .*
- *$T$  does not have any hanging node on  $\Sigma$ : each vertex  $v$  of  $T$  that belongs to  $\Sigma$ , is a common vertex between  $T_1, T_2 \in T$  such that  $\overline{T_1} \subset \overline{\Omega_1}$  and  $\overline{T_2} \subset \overline{\Omega_2}$ . See Figure 4.2.*

Let  $(\mathcal{T}_h)_{h>0}$  be a family of meshes of  $\overline{\Omega}$  such that  $\mathcal{T}_h$  satisfy Assumption 4.5.1 for all  $h > 0$ . The subscript  $h$  stands for the meshsize. For all  $k \in \mathbb{N}^*$ , we set

$$V_h^k(\Omega) := \{u \in H_0^1(\Omega) \mid u|_T \in P^k(T) \text{ for all } T \in \mathcal{T}_h\}.$$

Here  $P^k(T)$  stands for the space of polynomials (of  $d$  variables) of degree at most equal to  $k$ . In the same way, we define the spaces  $V_{i,h}^k(\Omega_i)$ ,  $i = 1, 2$ , such that

$$V_{i,h}^k(\Omega_i) := \{u \in H^1(\Omega_i) \mid u|_T \in P^k(T) \text{ for all } T \in \mathcal{T}_h \text{ and } u = 0 \text{ on } \partial\Omega_i \setminus \Sigma\}.$$

**Remark 4.5.1.** *Since for all  $h > 0$  the mesh  $\mathcal{T}_h$  is conforming to  $\Sigma$ , the space  $V_{i,h}^k(\Omega_i)$  coincides with  $\{u|_{\Omega_i}, u \in V_h^k(\Omega)\}$ ,  $i \in \{1, 2\}$ .*

<sup>1</sup>This is a consequence of the fact that the norm of a Banach space is weakly lower semicontinuous.

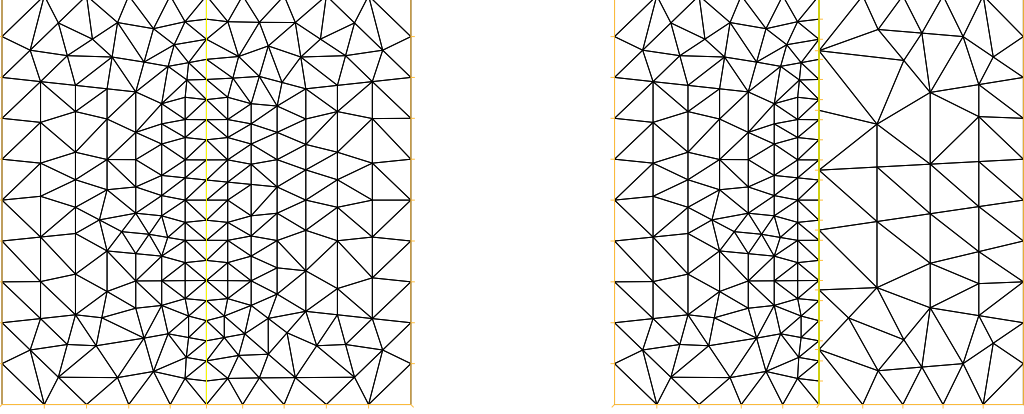


Figure 4.2: Examples of meshes without (left) and with (right) hanging nodes.

### 4.5.2 Discretization strategy

For  $h > 0$  and  $w \in V_2(\Omega)$ , define the functions  $u_h^w \in V_h^k(\Omega)$  and  $u_{2,h}^w \in V_{2,h}^k(\Omega_2)$  as the solutions to the following well-posed problems:

$$\begin{cases} \varepsilon_1 \int_{\Omega} \nabla u_h^w \cdot \nabla v_h = \int_{\Omega_1} f v_h + \int_{\Omega_2} \nabla w \cdot \nabla v_h, & \forall v_h \in V_h^k(\Omega) \\ \varepsilon_2 \int_{\Omega_2} \nabla u_{2,h}^w \cdot \nabla v_{2,h} = \int_{\Omega_2} f_2 v_{2,h} + \varepsilon_1 \int_{\Omega_2} \nabla u_1^w \cdot \nabla v_{2,h} - \int_{\Omega_2} \nabla w \cdot \nabla v_{2,h}, & \forall v_{2,h} \in V_{2,h}^k(\Omega_2). \end{cases} \quad (4.17)$$

Then introduce the projection operator  $\pi_h^k : V_2(\Omega_2) \rightarrow V_{2,h}^k(\Omega_2)$  such that for all  $w \in V_2(\Omega_2)$ ,  $\pi_h^k(w)$  is defined as the unique element of  $V_{2,h}^k(\Omega_2)$  that satisfies the problem

$$\int_{\Omega_2} \nabla \pi_h^k(w) \cdot \nabla v_{2,h} = \int_{\Omega_2} \nabla w \cdot \nabla v_{2,h} \quad \forall v_{2,h} \in V_{2,h}^k(\Omega_2).$$

Note that we have the estimate

$$\|\nabla \pi_h^k(w)\|_{L^2(\Omega_2)} \leq \|\nabla w\|_{L^2(\Omega_2)}. \quad (4.18)$$

From the definition of  $\pi_h(w)$ , one can easily see that for all  $w \in V_2(\Omega_2)$  we have the identities

$$u_h^{\pi_h^k(w)} = u_h^w \quad \text{and} \quad u_{2,h}^{\pi_h^k(w)} = u_{2,h}^w. \quad (4.19)$$

Now, let us turn our attention to the discretization of the optimization problem (4.8). The natural way to do that is to replace it by the problem

$$\inf_{w_h \in V_{2,h}^k(\Omega_2)} J_0^h(w_h) := \frac{1}{2} \int_{\Sigma} |u^{w_h} - u_2^{w_h}|^2 d\sigma. \quad (4.20)$$

One can proceed as in the proof of Lemma 4.4.2 to show that the objective function  $J_0^h : V_{h,k} \rightarrow \mathbb{R}$  (defined in (4.20)) is convex and continuous. Unfortunately this result is not sufficient to justify that the problem (4.20) is well-posed for  $h > 0$  small enough.

The difficulty comes from the fact that, even under Assumption 4.2.1, we do not have the discrete version of Lemma 4.3.2 since we can not guaranty that the problem

$$\text{Find } u_h \in V_h^k(\Omega) \text{ such that } \int_{\Omega} \sigma \nabla u_h \cdot \nabla v_h = \int_{\Omega} f v_h \quad \forall v_h \in V_h^k(\Omega)$$

is well-posed even for  $h$  small enough. To cope with this difficulty, an idea is to use the Tikhonov regularization approach (see §4.4.4) but with a regularization parameter that depends on  $h$ . This idea was originally proposed in [86] for the case of elliptic equations and then, was used by Assyr Abdulle et al. in [2] for the case of problems with sign-changing coefficients. Here, we explain how to adapt it to our case. The idea is to replace the objective function  $J_0^h$  in (4.20) by the function  $J^h : V_{2,h}^k(\Omega_2) \rightarrow \mathbb{R}_+$  such that for all  $w_h \in V_{2,h}^k(\Omega_2)$ , we have

$$J^h(w_h) := \frac{1}{2} \int_{\Sigma} |u^{w_h} - u_2^{w_h}|^2 d\sigma + \lambda(h) \|\nabla w_h\|_{L^2(\Omega_2)}^2$$

, where  $\lambda(h)$  is a positive function of  $h$  that tends to zero as  $h$  goes to 0. Since  $\lambda(h) > 0$  for all  $h > 0$ , the function  $J^h$  is strictly convex and coercive. This guarantees that the optimization problem

$$\min_{w_h \in V_{2,h}^k(\Omega_2)} J^h(w_h) \tag{4.21}$$

has a unique solution that we denote by  $w_{k,h}^*$ . All the difficulty now is to choose the function  $\lambda(h)$  in order to be able to ensure the convergence of  $(w_{k,h}^*)$  to a solution of (4.8) as  $h$  tends to zero. This is the main goal of the next paragraph.

### 4.5.3 Convergence of the method

The starting point of our discussion is the following

**Lemma 4.5.1.** *We have the estimate*

$$J_h(w_{k,h}^*) \leq \frac{1}{2} \int_{\Sigma} |u_h^{w_J^*} - u_{2,h}^{w_J^*}|^2 d\sigma + \lambda(h) \|\nabla w_J^*\|_{L^2(\Omega_2)}^2 \tag{4.22}$$

where  $w_J^*$  is defined in (4.11).

**Proof.** Starting from the fact that  $\pi_h^k(w_J^*) \in V_{2,h}^k(\Omega_2)$  and using that  $w_{k,h}^*$  is the unique solution of the optimization problem (4.21), we conclude that  $J_h(w_{k,h}^*) \leq J_h(\pi_h^k(w_J^*))$ . On the other hand, the identity (4.19) allows us to write

$$J_h(\pi_h^k(w_J^*)) = \frac{1}{2} \int_{\Sigma} |u_h^{w_J^*} - u_{2,h}^{w_J^*}|^2 d\sigma + \lambda(h) \|\nabla \pi_h(w_J^*)\|_{L^2(\Omega_2)}^2.$$

The Lemma is then proved by recalling the estimate (4.18). ■

In order to simplify notations, for  $h > 0$  and  $w \in V_2(\Omega_2)$ , we denote by  $A_h(w)$  the real number

$$A_h(w) = \frac{1}{2} \|u_h^w - u_{2,h}^w\|_{L^2(\Sigma)}^2.$$

From (4.19), we know that for all  $w \in V_2(\Omega_2)$ , we have  $A_h(w) = J_0^h(\pi_h^k(w))$ . The main result of this paragraph is the following theorem.

**Theorem 4.5.1.** *Assume that the function  $\lambda(h)$  can be chosen such that the sequences  $(\lambda(h))$  and  $(A^h(w_J^*)/\lambda(h))$  converge to zero as  $h$  tends to zero. Then,*

- *the sequence  $(w_{k,h}^*)$  converges, as  $h \rightarrow 0$ , in  $V_2(\Omega_2)$  to  $w_J^*$ .*
- *The sequences  $(u_h^{w_{k,h}^*})$  and  $(u_{2,h}^{w_{k,h}^*})$  converge respectively in  $H_0^1(\Omega)$  and  $V_2(\Omega_2)$  to  $E_H(u_1)$  and  $u_2$  where  $(u_1, u_2)$  is the solution of (4.4) and  $E_H(u_1)$  is the harmonic extension of  $u_1$ .*

**Proof.** The strategy of proof is similar to the one of Proposition 4.4.3. To simplify notations, we denote by  $u^{k,h} \in V_h^k(\Omega)$  and  $u_2^{k,h} \in V_{2,h}^k(\Omega_2)$  the functions

$$u^{k,h} = u_h^{w_{k,h}^*} \quad \text{and} \quad u_2^{k,h} = u_{2,h}^{w_{k,h}^*}.$$

In order to make the proof as clear as possible, we divide it into four steps.

**Step 1: weak convergence of  $(w_{k,h}^*)$ ,  $(u^{k,h})$  and  $(u_2^{k,h})$ .** Starting from the estimate

$$\|\nabla w_{k,h}^*\|_{L^2(\Omega_2)}^2 \leq J_h(w_{k,h}^*)/\lambda(h) \leq A_h(w_J^*)/\lambda(h) + \|\nabla w_J^*\|_{L^2(\Omega_2)}^2$$

and using the fact that  $A_h(w_J^*)/\lambda(h)$  tends to 0 as  $h$  vanishes, we infer that  $(w_{k,h}^*)$  is bounded in  $V_2(\Omega_2)$ . This implies that, up to a sub-sequence,  $(w_{k,h}^*)$  converges weakly to some  $w_0 \in V_2(\Omega)$ . For the reader convenience, this sub-sequence is still denoted by  $(w_{k,h}^*)$ .

Since the problem (4.17) is elliptic, we know that the sequence  $(u^{k,h})$  (resp.  $(u_2^{k,h})$ ) converges weakly in  $H_0^1(\Omega)$  (resp. in  $V_2(\Omega_2)$ ) to some  $u \in H_0^1(\Omega)$  (resp.  $u_2 \in V_2(\Omega_2)$ ). Using the density of  $V_h^k(\Omega)$  (resp.  $V_{2,h}^k(\Omega_2)$ ) in  $H_0^1(\Omega)$  (resp.  $V_2(\Omega_2)$ ), we infer that  $u = u^{w_0}$  and  $u_2 = u_2^{w_0}$  (these functions are defined in (4.9) by replacing  $w$  by  $w_0$ ).

**Step 2:  $w_0$  is a minimizer of  $J$ .** The compactness of the embedding  $H^{1/2}(\Sigma) \subset L^2(\Gamma)$  and the continuity of trace operator, ensures that

$$u^{k,h}|_{\Sigma} - u_2^{k,h}|_{\Sigma} \rightarrow u^{w_0} - u_2^{w_0}$$

in  $L^2(\Sigma)$  as  $h \rightarrow 0$ . By noticing that

$$\frac{1}{2} \int_{\Sigma} |u^{k,h} - u_2^{k,h}|^2 d\sigma = J_0^h(w_{k,h}^*) \leq J_h(w_{k,h}^*) \leq \lambda(h)(A_h(w_J^*)/\lambda(h) + \|w_J^*\|_{L^2(\Omega_2)}^2)$$

and using that  $\lambda(h), A_h(w_J^*)/\lambda(h) \rightarrow 0$  as  $h$  goes to zero, we deduce that  $u^{w_0} - u_2^{w_0} = 0$ . This shows that  $w_0$  is a minimizer of  $J$ .

**Step 3: strong convergence of  $(w_{k,h}^*)$  to  $w_J^*$ .** Thanks to the fact that  $A_h(w_J^*)/\lambda(h) \rightarrow 0$  as  $h \rightarrow 0$  and by means of the estimate

$$\|\nabla w_{k,h}^*\|_{L^2(\Omega_2)}^2 \leq J_h(w_{k,h}^*)/\lambda(h) \leq A_h(w_J^*)/\lambda(h) + \|\nabla w_J^*\|_{L^2(\Omega_2)}^2,$$

We can write

$$\limsup_{h \rightarrow 0} \|\nabla w_{k,h}^*\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)}.$$

On the other hand, since  $(w_{k,h}^*)$  converges weakly to  $w_0$  as  $h \rightarrow 0$ , we infer that

$$\|\nabla w_0\|_{L^2(\Omega_2)} \leq \liminf_{h \rightarrow 0} \|\nabla w_{k,h}^*\|_{L^2(\Omega_2)}.$$

This implies that  $\|\nabla w_0\|_{L^2(\Omega_2)} \leq \|\nabla w_J^*\|_{L^2(\Omega_2)}$ . Since  $w_0$  is a minimizer of  $J$ , we conclude that  $w_0 = w_J^*$ . Furthermore, we also deduce that

$$\lim_{h \rightarrow 0} \|\nabla w_{k,h}^*\|_{L^2(\Omega_2)} = \|\nabla w_0\|_{L^2(\Omega_2)}.$$

As a result, by applying [40, Proposition III.30], we infer that  $(w_{k,h}^*)$  converges, strongly, in  $V_2(\Omega_2)$  to  $w_0 = w_J^*$ .

**Step 4: strong convergence of  $(u^{k,h})$  and  $(u_2^{k,h})$ .** The ellipticity of Problem (4.17), combined with the strong convergence of  $(w_{k,h}^*)$  to  $w_J^*$ , imply the convergence of  $(u^{k,h})$  in  $H_0^1(\Omega)$  to  $u^{w_J^*}$  and of  $(u_2^{k,h})$  in  $V_2(\Omega_2)$  to  $u_2^{w_J^*}$ .

The Lemma is then proved by using that  $u^{w_J^*} = E_H(u_1)$  (see Proposition 4.4.1) and by observing that these limits are independent of the chosen sub-sequences. ■

The rest of this paragraph is devoted to explain why it is possible to choose the function  $\lambda(h)$  in such a way that  $\lambda(h)$  and  $A_h(w_j^*)$  converge to 0 as  $h$  tends to 0. To do so, one needs to study the behaviour of  $A_h(w_j^*)$  as  $h$  tends to 0. For all  $s \geq 1$ , we denote by  $\text{PH}^s(\Omega)$  the space

$$\text{PH}^s(\Omega) := \{u \in \text{H}_0^1(\Omega) \text{ such that } u|_{\Omega_i} \in \text{H}^s(\Omega_i) \text{ for } i = 1, 2\}.$$

Before studying the behavior of  $A_h(w_j^*)$ , let us start with the following

**Proposition 4.5.1.** *Assume that the solution  $u$  of (4.1) belongs to  $\text{PH}^{1+s}(\Omega)$  with  $s > 0$ . Then there exists  $s \geq s' > 0$  that depends only on the geometry of  $\Omega_2$  and  $\sigma \in (0, 1]$  that depends only on the geometry of  $\Omega$  such that*

$$\begin{aligned} \|u^{w_j^*} - u_h^{w_j^*}\|_{\text{H}_0^1(\Omega)} &\leq Ch^{p'} \|u\|_{\text{PH}^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{V}_2(\Omega_2)} \leq Ch^{p'} \|u_2\|_{\text{H}^{1+p'}(\Omega_2)}, \\ \|u^{w_j^*} - u_h^{w_j^*}\|_{\text{L}^2(\Omega)} &\leq Ch^{p'+\sigma} \|u\|_{\text{H}^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{L}^2(\Omega_2)} \leq Ch^{p'+\sigma} \|u_2\|_{\text{H}^{1+p'}(\Omega_2)} \end{aligned}$$

with  $C$  independent of  $h$  and  $p' = \min(s', k)$ .

**Proof.** Along this proof,  $C$  denotes a positive constant whose value can change from line to line but does not depend on  $h$ . Given that  $u^{w_j^*} = E_H(u_1)$  (see Proposition 4.4.1) and since  $u_1 \in \text{H}^{1+s}(\Omega_1)$  then, by means of classical regularity results, we can say that there exists  $0 < s' \leq s$  such that  $u^{w_j^*} \in \text{PH}^{1+s'}(\Omega)$ . Given that  $u_2^{w_j^*} = u_2 \in \text{H}^s(\Omega_2) \subset \text{H}^{s'}(\Omega_2)$  and since the problem (4.17) is elliptic, we obtain the estimates (see [55])

$$\|u^{w_j^*} - u_h^{w_j^*}\|_{\text{H}_0^1(\Omega)} \leq Ch^{p'} \|u\|_{\text{PH}^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{V}_2(\Omega_2)} \leq Ch^{p'} \|u_2\|_{\text{H}^{1+p'}(\Omega_2)}.$$

By applying the classical Aubin–Nitsche Lemma (see [55, Theorem 3.2.4]), we infer that there exists  $0 < \sigma \leq 1$  such that

$$\|u^{w_j^*} - u_h^{w_j^*}\|_{\text{L}^2(\Omega)} \leq Ch^{p'+\sigma} \|u\|_{\text{H}^{1+p'}(\Omega)} \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{L}^2(\Omega_2)} \leq Ch^{p'+\sigma} \|u_2\|_{\text{H}^{1+p'}(\Omega_2)}.$$

■

**Remark 4.5.2.** *It is worth to note that the value of  $s'$  depends only on the regularity of the harmonic extension of the function  $u_1$ . In particular, if  $\Omega_2$  is smooth or convex then we have  $s' = s$ .*

Now we have all the tools to study the behavior  $A_h(w_j^*)$  as  $h$  vanishes.

**Proposition 4.5.2.** *Assume that  $u$  the solution of (4.1) belongs to  $\text{PH}^{1+s}(\Omega)$  with  $0 < s$ . There exists  $0 < s' \leq s$  that depends only on the geometry of  $\Omega_2$  and  $\sigma \in (0, 1]$  that depends only on the geometry of  $\Omega$  such that*

$$A_h(w_j^*) \leq Ch^{2p'+\sigma}$$

with  $C$  independent of  $h$  and  $p' = \min(s', k)$ .

**Proof.** Applying the multiplicative trace inequality (Proposition 4.8.1) and using the estimates of Proposition 4.5.1 yield the estimates

$$\|u^{w_j^*} - u_h^{w_j^*}\|_{\text{L}^2(\Sigma)}^2 \leq Ch^{2p'+\sigma} \|u\|_{\text{PH}^{1+p'}(\Omega)}^2 \quad \text{and} \quad \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{L}^2(\Sigma)}^2 \leq Ch^{2p'+\sigma} \|u_2\|_{\text{H}^{1+p'}(\Omega_2)}^2.$$

By observing that

$$\|u_h^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{L}^2(\Sigma)}^2 \leq 2(\|u^{w_j^*} - u_h^{w_j^*}\|_{\text{L}^2(\Sigma)}^2 + \|u_2^{w_j^*} - u_{2,h}^{w_j^*}\|_{\text{L}^2(\Sigma)}^2),$$

we conclude that  $A_h(w_j^*) \leq Ch^{2p'+\sigma}$ .

■

The previous proposition gives us a simple way to choose the function  $\lambda(h)$  in order to ensure that  $(\lambda(h))$  and  $(A_h(w_j^*)/\lambda(h))$  tend both to 0 as  $h$  tends to 0.

**Proposition 4.5.3.** *Any function  $\lambda(h)$  of the form  $\lambda(h) = Ch^q$  with  $C > 0$  independent of  $h$  and  $0 < q < 2p' + \sigma$  satisfies the conditions of Theorem 4.5.1.*



## 4.6 Numerical experiments

In this section we turn our attention to the validation of the numerical method that we have proposed. We limit ourselves to the case of 2D domains. The numerical results that we present below have been obtained using the library `FreeFem++`<sup>2</sup>. To solve the optimization problem (4.21), we used the `BFGS` function of `FreeFem++`.

Since the well-posedness of (4.1) depends on the shape of the interface  $\Sigma$ , we test the performance of our method in three different configurations. In the first one,  $\Sigma$  is flat, in the second one,  $\Sigma$  is circular interface and in the last one,  $\Sigma$  has a corner.

### 4.6.1 Flat interface

In this paragraph, we take

$$\Omega_1 = \{(x, y) \in (0; 1/2) \times (0; 1)\} \quad \text{and} \quad \Omega_2 = \{(x, y) \in (1/2; 1) \times (0; 1)\}$$

(a flat interface and a domain which is symmetric with respect to  $\Sigma$ ). We consider a mesh sequence of  $\bar{\Omega}$  satisfying Assumption 4.5.1 (see Figure 4.3).

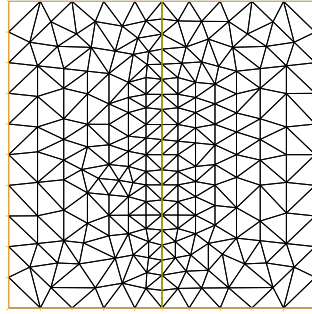


Figure 4.3: An example of mesh.

It has been shown in particular with the T-coercivity approach that  $A_\varepsilon$  is an isomorphism if and only if  $\kappa_\varepsilon \neq -1$ . In the rest of this paragraph we suppose that  $\kappa_\varepsilon \neq -1$ . To test the performance of our method, we work with the same example considered in [2, 51]. Define the function  $u_{\kappa_\varepsilon}$  such that

$$u_\kappa(x, y) = \begin{cases} (x^2 + bx) \sin(\pi y) & \text{if } x < 1/2 \\ a(x-1) \sin(\pi y) & \text{if } 1/2 < x \end{cases}, \quad \text{where } a = \frac{1}{2(\kappa_\varepsilon + 1)} \quad \text{and} \quad b = -\frac{\kappa_\varepsilon + 2}{2(\kappa_\varepsilon + 1)}.$$

and consider it as an exact solution of (4.1). This is possible because  $\text{div}(\varepsilon \nabla u_\kappa) \in L^2(\Omega)$ . The source term  $f$  is computed accordingly. Since  $u_\kappa \in \text{PH}^2(\Omega)$  and since  $\Omega_2$  is convex, we can take  $s = s' = 1$  in Propositions 4.5.1 and 4.5.2.

Furthermore, given that  $\Omega$  is convex, we have  $\sigma = 1$ . As a result, if we use the Lagrange  $P1$  finite elements, i.e.  $p = 1$ , a direct application of Proposition 4.5.1 guarantees that by choosing  $\lambda(h) = Ch^q$  with  $0 < q < 3$ , the method is convergent. In our experiment, we take  $\lambda(h) = 0.002h^2$ . We work with two values of contrasts  $\kappa_\varepsilon = -2$  and  $\kappa_\varepsilon = -1.001$ . The behavior of the  $L^2$  and  $H_0^1$  errors with respect to the exact solution in these two configurations are given in Figure 4.4.

We observe that in both situations, the method is of order 2 in the  $L^2$  norm. We also remark that the order of convergence in the  $H_0^1$  norm is greater than 1. In the particular case  $\kappa_\varepsilon = -1.001$ , we note a super-convergence in the  $H_0^1$  norm.

<sup>2</sup>See <https://freefem.org/>.

**Remark 4.6.1.** The constant  $C$  in  $\lambda(h) = Ch^q$  must be adjusted by the user according to the contrast  $\kappa_\varepsilon$  in order to obtain a fast convergence of the method. Clearly this depends on  $\|\nabla w_j^*\|_{L^2(\Omega_2)}$ . When the solution is such that its normal derivative jump across the interface is large (his the case when  $\kappa_\varepsilon$  approaches  $-1$ ), one expects that  $\|\nabla w_j^*\|_{L^2(\Omega_2)}$  must be large and then  $C$  must be chosen small. It is also important to note that that when  $h$  is small enough the choice of  $C$  does not affect the convergence of the method.

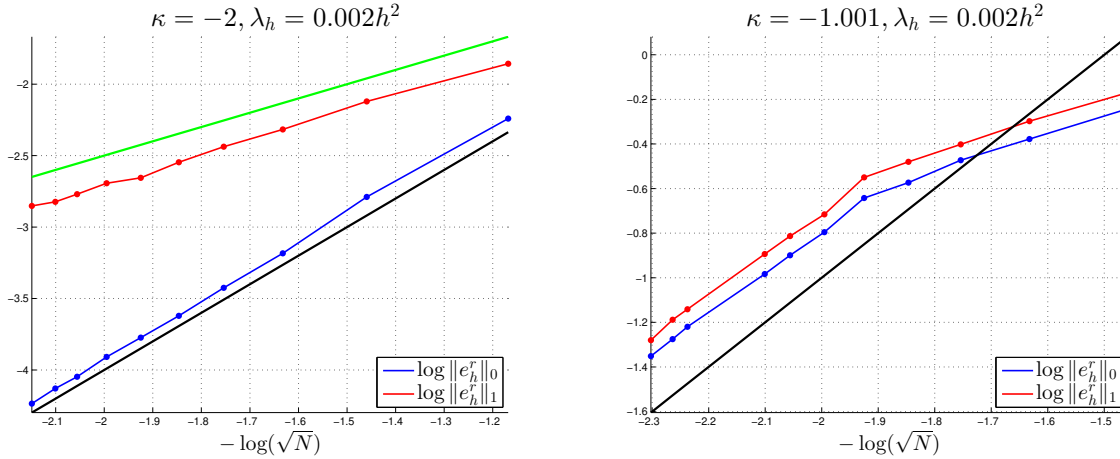


Figure 4.4: Behavior of the relative  $L^2$  and  $H_0^1$  errors with respect to the meshsize  $h \sim \sqrt{N}$ . Here  $N$  is the total number of nodes of the mesh.

### 4.6.2 The case of a circular interface

In this paragraph, we consider the case where the domains  $\Omega_1$  and  $\Omega_2$  are such that  $\Omega_1 = \{x \in \mathbb{R}^2 \mid |x| < 1\}$  and  $\Omega_2 = \{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}$ . In Proposition 4.8.2, we prove that  $A_\varepsilon$  is an isomorphism  $\kappa_\varepsilon \notin \{-1\} \cup \mathcal{S}$  with  $\mathcal{S} := \{-(1 - (1/2)^{2n})/(1 + (1/2)^{2n}) \mid n \in \mathbb{N}^*\}$ . For this we shall limit ourselves to the case where  $\kappa_\varepsilon = -2$ . Given that both  $\Omega_2$  and  $\Omega$  are smooth, we infer that  $\sigma = 1$  and  $s' = s$ . Again, we are going to work with the Lagrange  $P1$  finite elements (i.e.  $p = 1$ ). By taking  $f$  as the source term associated to the function

$$u_{\kappa_\varepsilon}(x, y) = \begin{cases} r^2 + b & \text{if } r < 1 \\ a(r - 2)^2 & \text{if } 1 < r < 2. \end{cases}, \text{ with } r = \sqrt{x^2 + y^2}, a = -1/\kappa_\varepsilon \text{ and } b = a - 1$$

and by taking  $\lambda_h = 0.002h^2$ . We obtain the results displayed in Figure 4.5. We observe that the method converges with optimal rate.

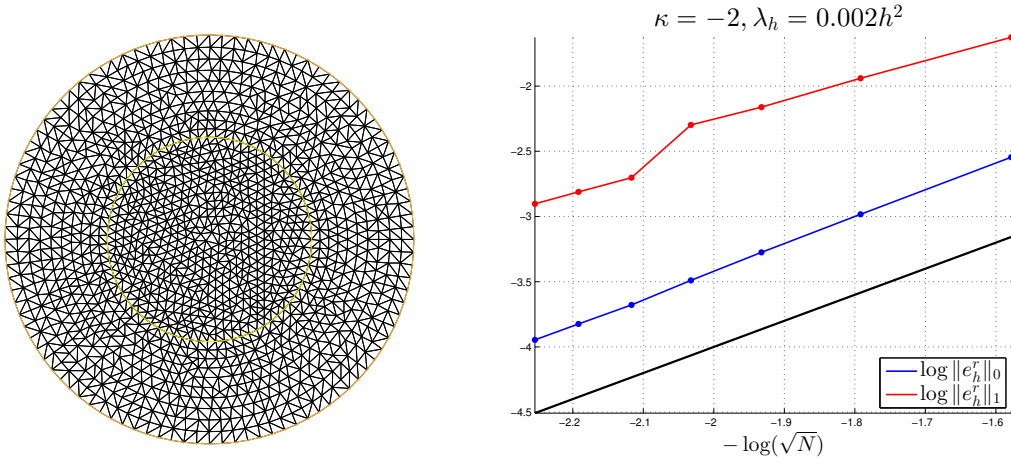


Figure 4.5: A member of the mesh sequence (left). The behavior of the  $L^2$  and  $H_0^1$  errors with respect to the meshsize  $h \sim \sqrt{N}$  where  $N$  is the number of nodes of the mesh(right).

### 4.6.3 The case of an interface with corner

Now, we consider the configuration where the interface  $\Sigma$  has a corner. More precisely, we assume that  $\Omega := \{x \in \mathbb{R}^2 \mid |x| < 1 \text{ and } \arg(x) \in (0; \pi/2)\}$  and  $\Omega_1 := \{x \in \Omega \mid \arg(x) \in (0; \pi/4)\}$  (see Figure 4.6). In such configuration, it can be proved (see [74]) that  $A_\varepsilon$  is an isomorphism if and only if  $\kappa_\varepsilon \in \mathbb{R}_-^* \setminus [-3, -1]$ . Furthermore, in contrary to the two previous cases, in this configuration the solution of (4.1) can be very singular near the origin. Indeed, it was proved in [49, Chapter 2] that the regularity of the solution of (4.1) depends in  $\kappa_\varepsilon$  and can be very low as  $\kappa_\varepsilon$  approaches  $[-3, -1]$ . To be more complete, one can show that the optimal regularity of the solution of (4.1) is  $\text{PH}^{1+\Re(\lambda_0)}(\Omega) \cap H_0^1(\Omega)$  where  $\lambda_0$  is the solution of

$$\kappa_\varepsilon = -\tan(3\lambda\pi/4)/\tan(\lambda\pi/4) \quad (4.23)$$

that has the smallest positive real part. Note that one can show (see [49, Chapter 3]) that all the solutions to (4.23) are real-valued. In the particular case where  $\kappa_\varepsilon = -5$ , one finds that  $\lambda_0 \approx 0.458$ . As mentioned previously this regularity result is optimal. Indeed, one can check that the function

$$u_{\lambda_0}(r, \theta) := (1-r)r^{\lambda_0} \begin{cases} \sin(\lambda\theta)/\sin(\lambda\pi/4) & \theta \in (0; \pi/4), \\ \sin(\lambda(\pi-\theta))/\sin(3\lambda\pi/4) & \theta \in (\pi/4; \pi) \end{cases}$$

satisfies  $\text{div}(\varepsilon\nabla u_\lambda) \in L^2(\Omega)$ . Observe that  $u_{\lambda_0} \notin \text{PH}^{\lambda_0+\gamma}$  for all  $0 < \gamma$ . This means that  $u_{\lambda_0} \notin \text{PH}^{3/2}$ . Now, given that  $\Omega$  and  $\Omega_2$  are both convex, owing to Proposition 4.5.1, we can say that by choosing  $\lambda_h = Ch^q$  with  $q < 3\lambda_0$ , the convergence of the method can be guaranteed. The behaviors of the relative  $L^2$  error and of the semi- $H_0^1$  one for the case  $\lambda_h = h^{1.3}$  are given in Figure 4.6. The expected rate of convergence is equal to  $\lambda_0 \approx 0.458$  for the case of the semi- $H_0^1$  error and is equal to  $2\lambda_0 \approx 0.916$  for the case of the  $L^2$  one. In contrary to the previous two cases, the rates of convergence of these errors are not optimal but close from the expected ones.

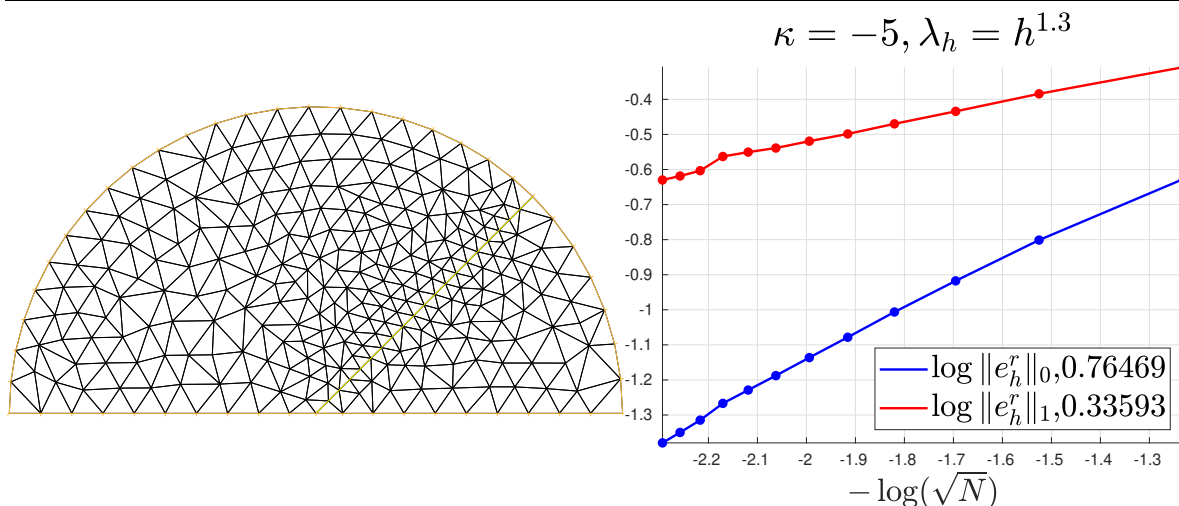


Figure 4.6: On the left: a member of the mesh sequence. On the right: the behavior of the  $L^2$  and  $H_0^1$  errors with respect to the meshsize  $h \sim \sqrt{N}$  where  $N$  is the number of nodes of the mesh.

### 4.7 Concluding remarks

In this chapter, we have presented a new numerical method for approximating the solution of the scalar transmission problem. We proved that the method converges without any restriction on the mesh sequence used or on the regularity of the solution. This result has been illustrated by numerical experiments. We mention some issues/question that can be studied in future work:

- It will be interesting to study how the function  $\lambda_h$  should be chosen in order to accelerate the convergence of the method.
- How to extend this extend to the case when the density function is critical? Is it possible to extend this method to the case of Maxwell's equations?

## 4.8 Appendix

**Proposition 4.8.1.** [39, Theorem 1.6.6] *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^d$  ( $d = 2, 3$ ). Then the estimate*

$$\|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|u\|_{H^1(\Omega)}^{1/2} \quad \forall u \in H^1(\Omega)$$

*holds with  $0 < C$  independent of  $u$ .*

**Proposition 4.8.2.** *Let  $\Omega_1 = \{x \in \mathbb{R}^2 \mid |x| < 1\}$  and  $\Omega_2 = \{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}$ . Assume that  $\kappa_\varepsilon \notin \{-1\} \cup \mathcal{S}$  with*

$$\mathcal{S} := \left\{ -\frac{1 - (1/2)^{2n}}{1 + (1/2)^{2n}} \mid n \in \mathbb{N}^* \right\}.$$

*Then the operator  $A_\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism.*

**Remark 4.8.1.** *Note that in accordance with the results concerning the Neumann-Poincaré operator, we observe that  $-1$  is an accumulation point of  $\mathcal{S}$ .*

**Proof.** [50, Theorem 1.3.3] guarantees that  $A_\varepsilon$  is Fredholm of index 0 when  $\kappa_\varepsilon \neq -1$ . Therefore it suffices to study its kernel. Let  $u \in H_0^1(\Omega)$  be such that  $A_\varepsilon u = 0$ . Then  $u_1 := u|_{\Omega_1}$  and  $u_2 = u|_{\Omega_2}$  satisfy

$$\begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_1(1, \theta) = u_2(1, \theta) & \text{and } \partial_r u_1(1, \theta) = \kappa_\varepsilon \partial_r u_2(1, \theta) \quad \forall \theta \in [0; 2\pi]. \end{cases}$$

Since the problem is invariant with respect to  $\theta$ , by Fourier decomposition for  $u_1, u_2$  we have the representations:

$$u_1(r, \theta) = \sum_{n \in \mathbb{N}} a_n r^n e^{in\theta} \quad \text{and} \quad u_2(r, \theta) = b_0 \ln(r/2) + \sum_{n \in \mathbb{Z}^*} b_n ((r/2)^n - (r/2)^{-n}) e^{in\theta},$$

where  $a_n, b_n \in \mathbb{C}$ . Using the transmission conditions, we get

$$\begin{cases} a_0 = b_0 \ln(1/2), & 0 = b_0 \kappa_\varepsilon \\ a_n = b_n ((1/2)^n - (1/2)^{-n}), & a_n = b_n ((1/2)^n + (1/2)^{-n}) \kappa_\varepsilon, \quad n \in \mathbb{N}^* \\ 0 = b_n ((1/2)^n - (1/2)^{-n}), & 0 = b_n ((1/2)^n + (1/2)^{-n}) \kappa_\varepsilon, \quad -n \in \mathbb{N}^*. \end{cases}$$

Therefore we deduce that  $A_\varepsilon$  is injective when  $\kappa_\varepsilon \notin \mathcal{S}$ . ■

# Chapter 5

## Maxwell's equations with hypersingularities at a conical plasmonic tip: the case of one critical coefficient

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## 5.1 Introduction

The present chapter inaugurates the third part of this thesis, which aims to present a detailed study of 3D (time harmonic) Maxwell's equations in presence of a negative material. Our goal is explain how the study of the scalar problems associated to the dielectric permittivity  $\varepsilon$  and the one associated to the the magnetic permeability  $\mu$  can be used to study the 3D Maxwell's equations. Unlike the study of scalar problems with changing coefficients which has been the subject of many contributions, the case of the 3D Maxwell equations has been treated in only two papers [24, 118]. While the work done in [118] deals with the case where the interface between the positive and negative material is smooth (class  $\mathcal{C}^1$ ), the results obtained in [24] are valid in the general case (i.e., when the interface separating the two materials is Lipschitz-regular).

In the present work, we consider the configuration where the interface that separates the positive and the negative material has a conical point (more details will be given later). Therefore, the only work that can help us in our study is the one presented in [24]. What we can retain from this work is the following fact: if the contrasts in  $\varepsilon$  and  $\mu$  do not take critical values, the Maxwell equations are well-posed (in the Fredholm sense) in the classical  $L^2$ -framework. The main tool used to establish this result was the T-coercivity technique. When one of the functions  $\varepsilon$  or  $\mu$  is critical or when both of them are critical, the study of the Maxwell's problem has not been done yet.

In this chapter, we will consider the case where the function  $\varepsilon$  is critical (i.e. propagating singularities exist for the scalar problem associated with  $\varepsilon$ ) and where the function  $\mu$  does not take critical values. More precisely, our goal is to explain why, in this configuration, the classical framework is no longer appropriate to study Maxwell's equations and, more importantly, to explain how to combine Mellin's analysis in Kondratiev spaces with the  $\mathbb{T}$  coercivity technique to derive an appropriate functional framework for Maxwell's equations in such configuration. We emphasize that due to the non standard singularities we have to deal with, the results we obtain are quite different from the ones existing for classical Maxwell's equations with positive materials in non smooth domains [15, 60, 16, 66, 62]. The case where both functions  $\varepsilon$  and  $\mu$  take critical values will be studied in the next chapter.

The outline is as follows. In the remaining part of the introduction, we present some general notation. In Section 5.2, we describe the assumptions made on the dielectric constants  $\varepsilon$ ,  $\mu$ . Then we propose a new functional framework for the problem for the electric field and show its well-posedness in Section 5.3. Section 5.4 is dedicated to the analysis of the problem for the magnetic field. We emphasize that due to the assumptions made on  $\varepsilon$ ,  $\mu$  (the contrast in  $\varepsilon$  is critical but the one in  $\mu$  is not), the studies in sections 5.3 and 5.4 are quite different. We give a few words of conclusion in Section 5.5 before presenting technical results needed in the analysis in two sections of appendix. The main outcomes of this work are Theorem 5.3.1 (well-posedness for the electric problem) and Theorem 5.4.1 (well-posedness for the magnetic problem).

All the study will take place in some domain  $\Omega$  of  $\mathbb{R}^3$ . More precisely,  $\Omega$  is an open, connected and bounded subset of  $\mathbb{R}^3$  with a Lipschitz-continuous boundary  $\partial\Omega$ . Once for all, we make the following assumption:

**Assumption 1.** *The domain  $\Omega$  is simply connected and  $\partial\Omega$  is connected.*

When this assumption is not satisfied, the analysis below must be adapted (see the discussion in the conclusion). For some  $\omega \neq 0$  ( $\omega \in \mathbb{R}$ ), the time-harmonic Maxwell's equations are given by

$$\mathbf{curl} \mathbf{E} - i\omega \mu \mathbf{H} = 0 \quad \text{and} \quad \mathbf{curl} \mathbf{H} + i\omega \varepsilon \mathbf{E} = \mathbf{J} \quad \text{in } \Omega. \quad (5.1)$$

$\mathbf{E}$  and  $\mathbf{H}$  above are respectively the electric and magnetic components of the electromagnetic



field. The source term  $\mathbf{J}$  is the current density. We suppose that the medium  $\Omega$  is surrounded by a perfect conductor and we impose the boundary conditions

$$\mathbf{E} \times \boldsymbol{\nu} = 0 \quad \text{and} \quad \mu \mathbf{H} \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega, \quad (5.2)$$

where  $\boldsymbol{\nu}$  denotes the unit outward normal vector field to  $\partial\Omega$ . Note that non homogeneous boundary conditions can be considered as well and that the results we obtain below also allow one to deal with the case of impedance boundary conditions (see Remark 5.3.3). The dielectric permittivity  $\varepsilon$  and the magnetic permeability  $\mu$  are real valued functions which belong to  $L^\infty(\Omega)$ , with  $\varepsilon^{-1}, \mu^{-1} \in L^\infty(\Omega)$  (without assumption of sign). Let us introduce some usual spaces in the study of Maxwell's equations:

$$\begin{aligned} \mathbf{L}^2(\Omega) &:= (\mathbf{L}^2(\Omega))^3 \\ \mathbf{H}_0^1(\Omega) &:= \{\varphi \in \mathbf{H}^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\} \\ \mathbf{H}_\#^1(\Omega) &:= \{\varphi \in \mathbf{H}^1(\Omega) \mid \int_\Omega \varphi \, dx = 0\} \\ \mathbf{H}(\mathbf{curl}) &:= \{\mathbf{H} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \, \mathbf{H} \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_N(\mathbf{curl}) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}) \mid \mathbf{E} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\} \end{aligned}$$

and for  $\xi \in L^\infty(\Omega)$ :

$$\begin{aligned} \mathbf{X}_T(\xi) &:= \{\mathbf{H} \in \mathbf{H}(\mathbf{curl}) \mid \operatorname{div}(\xi \mathbf{H}) = 0, \xi \mathbf{H} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\} \\ \mathbf{X}_N(\xi) &:= \{\mathbf{E} \in \mathbf{H}_N(\mathbf{curl}) \mid \operatorname{div}(\xi \mathbf{E}) = 0\}. \end{aligned}$$

We denote indistinctly by  $(\cdot, \cdot)_\Omega$  the classical inner products of  $L^2(\Omega)$  and  $\mathbf{L}^2(\Omega)$ . Moreover,  $\|\cdot\|_\Omega$  stands for the corresponding norms. We endow the spaces  $\mathbf{H}(\mathbf{curl})$ ,  $\mathbf{H}_N(\mathbf{curl})$ ,  $\mathbf{X}_T(\xi)$ ,  $\mathbf{X}_N(\xi)$  with the norm

$$\|\cdot\|_{\mathbf{H}(\mathbf{curl})} := (\|\cdot\|_\Omega^2 + \|\mathbf{curl} \cdot\|_\Omega^2)^{1/2}.$$

Let us recall a well-known property for the particular spaces  $\mathbf{X}_T(1)$  and  $\mathbf{X}_N(1)$  (cf. [139, 8]).

**Proposition 5.1.1.** *Under Assumption 1, the embeddings of  $\mathbf{X}_T(1)$  in  $\mathbf{L}^2(\Omega)$  and of  $\mathbf{X}_N(1)$  in  $L^2(\Omega)$  are compact. And there is a constant  $C > 0$  such that*

$$\|\mathbf{u}\|_\Omega \leq C \|\mathbf{curl} \, \mathbf{u}\|_\Omega, \quad \forall \mathbf{u} \in \mathbf{X}_T(1) \cup \mathbf{X}_N(1).$$

Therefore, in  $\mathbf{X}_T(1)$  and in  $\mathbf{X}_N(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$ .

## 5.2 Assumptions for the dielectric constants $\varepsilon$ , $\mu$

In this document, for a Banach space  $X$ ,  $X^*$  stands for the topological antidual space of  $X$  (the set of continuous anti-linear forms on  $X$ ).

In the analysis of the Maxwell's system (7.5)-(7.6), the properties of two scalar operators associated respectively with  $\varepsilon$  and  $\mu$  play a key role. Define  $A_\varepsilon : \mathbf{H}_0^1(\Omega) \rightarrow (\mathbf{H}_0^1(\Omega))^*$  such that

$$\langle A_\varepsilon \varphi, \varphi' \rangle = \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} \, dx, \quad \forall \varphi, \varphi' \in \mathbf{H}_0^1(\Omega) \quad (5.3)$$

and  $A_\mu : \mathbf{H}_\#^1(\Omega) \rightarrow (\mathbf{H}_\#^1(\Omega))^*$  such that

$$\langle A_\mu \varphi, \varphi' \rangle = \int_\Omega \mu \nabla \varphi \cdot \nabla \overline{\varphi'} \, dx, \quad \forall \varphi, \varphi' \in \mathbf{H}_\#^1(\Omega).$$

**Assumption 2.** *We assume that  $\mu$  is such that  $A_\mu : \mathbf{H}_\#^1(\Omega) \rightarrow (\mathbf{H}_\#^1(\Omega))^*$  is an isomorphism.*

Assumption 2 is satisfied in particular if  $\mu$  has a constant sign (by Lax-Milgram theorem). We underline however that we allow  $\mu$  to change sign (see in particular [68, 27, 20, 24] for examples of sign-changing  $\mu$  such that Assumption 2 is verified). The assumption on  $\varepsilon$ , that will be responsible for the presence of (hyper)singularities, requires to consider a more specific configuration as explained below.

### 5.2.1 Conical tip and scalar (hyper)singularities

We assume that  $\Omega$  contains an inclusion of a particular material (metal at optical frequency, metamaterial, ...) located in some domain  $\mathcal{M}$  such that  $\overline{\mathcal{M}} \subset \Omega$  ( $\mathcal{M}$  like metal or metamaterial). We assume that  $\partial\mathcal{M}$  is of class  $\mathcal{C}^2$  except at the origin  $O$  where  $\mathcal{M}$  coincides locally with a conical tip. More precisely, there are  $\rho > 0$  and some smooth domain  $\varpi$  of the unit sphere  $\mathbb{S}^2 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$  such that  $B(O, \rho) \subset \Omega$  and

$$\mathcal{M} \cap B(O, \rho) = \mathcal{K} \cap B(O, \rho) \quad \text{with } \mathcal{K} := \{r\boldsymbol{\theta} \mid r > 0, \boldsymbol{\theta} \in \varpi\}.$$

Here  $B(O, \rho)$  stands for the open ball centered at  $O$  and of radius  $\rho$ . We assume that  $\varepsilon$  takes the constant value  $\varepsilon_- < 0$  (resp.  $\varepsilon_+ > 0$ ) in  $\mathcal{M} \cap B(O, \rho)$  (resp.  $(\Omega \setminus \overline{\mathcal{M}}) \cap B(O, \rho)$ ). And we assume that the contrast  $\kappa_\varepsilon := \varepsilon_-/\varepsilon_+ < 0$  and  $\varpi$  (which characterizes the geometry of the conical tip) are such that there exist singularities of the form

$$\mathfrak{s}(x) = r^{-1/2+i\eta}\Phi(x/|x|) \quad (5.4)$$

satisfying  $\operatorname{div}(\varepsilon\nabla\mathfrak{s}) = 0$  in  $\mathcal{K}$  with  $\eta \in \mathbb{R}, \eta \neq 0$ . Here  $r := |x|$  while  $\Phi$  is a function which is smooth in  $\varpi$  and in  $\mathbb{S}^2 \setminus \overline{\varpi}$ . We emphasize that since the interface between the metamaterial and the exterior material is not smooth, singularities always exist at the conical tip. However, here we make a particular assumption on the singular exponent which has to be of the form  $-1/2+i\eta$  with  $\eta \in \mathbb{R}, \eta \neq 0$ . Such singularities play a particular role for the operator  $A_\varepsilon$  introduced in (5.3) because they are “just” outside  $H^1$ . More precisely, we have  $\mathfrak{s} \notin H^1(\Omega)$  but  $r^\gamma\mathfrak{s} \in H^1(\Omega)$  for all  $\gamma > 0$ . With them, we can construct a sequence of functions  $u_n \in H_0^1(\Omega)$  such that

$$\forall n \in \mathbb{N}, \quad \|u_n\|_{H^1(\Omega)} = 1 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|\operatorname{div}(\varepsilon\nabla u_n)\|_{(H_0^1(\Omega))^*} + \|u_n\|_\Omega = 0.$$

Then this allows one to prove that the range of  $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  is not closed (see [28, 20, 30] in 2D). Of course, for any given geometry, such singularities do not exist when  $\kappa_\varepsilon > 0$  because we know that in this case  $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  is an isomorphism. On the other hand, when

$$\varpi = \{(\cos\theta \cos\phi, \sin\theta \cos\phi, \sin\phi) \mid -\pi \leq \theta \leq \pi, -\pi/2 \leq \phi < -\pi/2 + \alpha\} \text{ for some } \alpha \in (0; \pi) \quad (5.5)$$

(the circular conical tip, see Figure 5.1), thanks to Theorem 3.4.6, we know that such  $\mathfrak{s}$  exists for  $\kappa_\varepsilon \in (-1; -I_\alpha)$  (resp.  $\kappa_\varepsilon \in (-I_\alpha; -1)$ ) when  $\alpha < \pi/2$  (resp.  $\alpha > \pi/2$ ). Here  $I_\alpha$  is the constant defined by

$$I_\alpha := \frac{{}_2F_1(1/2, 1/2, 1, \cos^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \sin^2(\alpha/2))}{{}_2F_1(1/2, 1/2, 1, \sin^2(\alpha/2)) {}_2F_1(3/2, 3/2, 2, \cos^2(\alpha/2))} > 0, \quad (5.6)$$

where  ${}_2F_1$  stands for the Gauss’s hypergeometric function. Note that we have  $I_\alpha = 1/I_{\pi-\alpha}$  and  $I_\alpha \in (0; 1)$  for  $\alpha \in (0; \pi/2)$ . Additionally, there holds for example  $I_{\pi/4} \approx 0.218$  as well as  $\lim_{\alpha \rightarrow \pi/2} I_\alpha = 1$ ,  $\lim_{\alpha \rightarrow 0^+} I_\alpha = 0^+$ ,  $\lim_{\alpha \rightarrow \pi^-} I_\alpha = +\infty$ .

For a general smooth domain  $\varpi \subset \mathbb{S}^2$  and a given contrast  $\kappa_\varepsilon$ , in order to know if such  $\mathfrak{s}$  exists, one has to solve the spectral problem

$$\left| \begin{array}{l} \text{Find } (\Phi, \lambda) \in H^1(\mathbb{S}^2) \setminus \{0\} \times \mathbb{C} \text{ such that} \\ \int_{\mathbb{S}^2} \varepsilon \nabla_S \Phi \cdot \nabla_S \overline{\Phi'} ds = \lambda(\lambda + 1) \int_{\mathbb{S}^2} \varepsilon \Phi \overline{\Phi'} ds, \quad \forall \Phi' \in H^1(\mathbb{S}^2), \end{array} \right. \quad (5.7)$$

and see if among the eigenvalues some of them are of the form  $\lambda = -1/2 + i\eta$  with  $\eta \in \mathbb{R}, \eta \neq 0$ . Above,  $\nabla_S$  stands for the surface gradient. With a slight abuse, when  $\varepsilon$  is involved into integrals over  $\mathbb{S}^2$ , we write  $\varepsilon$  instead of  $\varepsilon(\rho \cdot)$ . Note that since  $\varepsilon$  is real-valued, if  $\lambda = -1/2 + i\eta$  is an eigenvalue, we have  $\lambda(\lambda + 1) = -\eta^2 - 1/4$ , so that  $\lambda = -1/2 - i\eta$  is also an eigenvalue for the

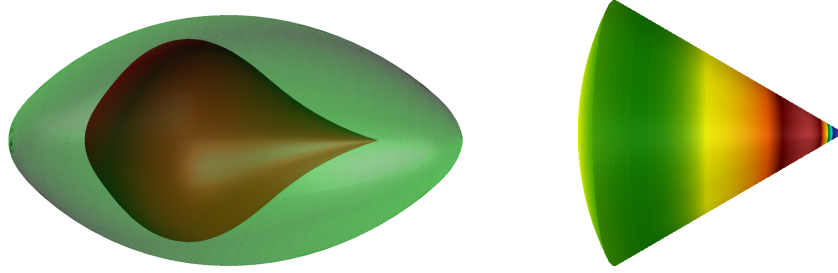


Figure 5.1: Left: the domain  $\Omega$  with the inclusion  $\mathcal{M}$  exhibiting a conical tip. Right:  $\mathfrak{s}|_{\mathcal{M}}$  for the circular conical tip with  $\alpha = \pi/6$  (the critical interval is then approximately given by  $[-1; -0.1032]$ ) and  $\kappa_\varepsilon = -0.36$ . In this situation, we have  $\eta \approx 2$ .

same eigenfunction. And since  $\lambda(\lambda + 1) \in \mathbb{R}$ , we can find a corresponding eigenfunction which is real-valued. Let us mention that this problem of existence of singularities of the form (5.4) is directly related to the problem of existence of essential spectrum for the so-called Neumann-Poincaré operator [98, 127, 36, 93]. A noteworthy difference with the 2D case of a corner in the interface is that several singularities of the form (5.4) with different values of  $|\eta|$  can exist in 3D [96] (this depends on  $\varepsilon$  and on  $\varpi$ ).

For pedagogical purposes, we shall suppose that the function  $\varepsilon$  is such that the problem (5.7) has exactly two eigenvalues that belong to  $\{\lambda \in \mathbb{C} \mid \Re(\lambda) = -1/2\} \setminus \{-1/2\}$  that will be denoted by  $\lambda_\eta^\pm := -1/2 \pm i\eta$  with  $\eta \in \mathbb{R}_+^*$ . Furthermore, we are going to suppose that  $\lambda_\eta^\pm$  are simple (of algebraic multiplicity (see §2.4.5) equal to one) eigenvalues of (5.7). In this case, using the results of §2.6.2, one can show that the operator  $A_\varepsilon$  has exactly two propagating singularities that have the form  $\mathfrak{s}^\pm(x) = r^{\lambda_\eta^\pm} \Phi(x/|x|)$  in which  $\Phi$  is real-valued eigenfunction of (5.7) associated to  $\lambda_\eta$  such that  $\|\Phi\|_{H^1(\mathbb{S}^2)} = 1$  and satisfying  $\eta \int_{\mathbb{S}^2} \sigma |\Phi|^2 \neq 0$ . Exchanging  $\eta$  by  $-\eta$  if necessary, we can set  $\eta$  so that

$$\eta \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds > 0. \quad (5.8)$$

Note that the previous condition is equivalent to suppose that  $\mathfrak{s}^+$  is outgoing (with respect to the Mandelstam radiation principle (see §2.6.2)). For the circular conical tip introduced in (5.5), say for  $\alpha < \pi/2$ , we find that the above assumptions are satisfied for contrasts  $\kappa_\varepsilon \in (-I_\alpha^\dagger; -I_\alpha)$  with a certain  $I_\alpha^\dagger \in (I_\alpha; 1)$ . For  $\kappa_\varepsilon \in (-1; -I_\alpha^\dagger)$ , the number of hypersingularities is larger than two (counting  $\pm$ ).

**Remark 5.2.1.** *In the case where several propagating singularities exist (even with logarithmic growth near the origin), the analysis below can be adapted. If the reader is interested in the treatment of this configuration, we refer him to the next chapter.*

To fix notations, we set

$$s^\pm(x) = \chi(r)r^{-1/2 \pm i\eta} \Phi(x/|x|) \quad (5.9)$$

In this definition the smooth cut-off function  $\chi$  is equal to one in a neighbourhood of 0 and is supported in  $[-\rho; \rho]$ . In particular, we emphasize that  $s^\pm$  vanish in a neighbourhood of  $\partial\Omega$ .

In order to recover Fredholm property for the scalar problem involving  $\varepsilon$ , an important idea is to add one (and only one) of the singularities (5.9) to the functional framework. From a mathematical point of view, working with the complex conjugation, it is obvious to see that adding  $s^+$  or  $s^-$  does not change the results. However physically one framework is more relevant than the other. More precisely, we will explain in §5.3.7 with the limiting absorption principle why selecting  $s^+$ , with  $\eta$  such that (5.8) holds, together with a certain convention for the time-harmonic dependence, is more natural.

### 5.2.2 Kondratiev functional framework

In this paragraph, adapting what is done in [30] for the 2D case, we describe in more details how to get a Fredholm operator for the scalar operator associated with  $\varepsilon$ . For  $\beta \in \mathbb{R}$  and  $m \in \mathbb{N}$ , let us introduce the weighted Sobolev (Kondratiev) space  $V_\beta^m(\Omega)$  (see [100]) defined as the closure of  $\mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$  for the norm

$$\|\varphi\|_{V_\beta^m(\Omega)} = \left( \sum_{|\alpha| \leq m} \|r^{|\alpha| - m + \beta} \partial_x^\alpha \varphi\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

Here  $\mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$  denotes the space of infinitely differentiable functions which are supported in  $\bar{\Omega} \setminus \{O\}$ . We also denote  $\mathring{V}_\beta^1(\Omega)$  the closure of  $\mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$  for the norm  $\|\cdot\|_{V_\beta^1(\Omega)}$ . We have the characterisation

$$\mathring{V}_\beta^1(\Omega) = \{\varphi \in V_\beta^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\}.$$

Note that using Hardy's inequality

$$\int_0^1 \frac{|u(r)|^2}{r^2} r^2 dr \leq 4 \int_0^1 |u'(r)|^2 r^2 dr, \quad \forall u \in \mathcal{C}_0^1[0; 1],$$

one can show the estimate  $\|r^{-1}\varphi\|_\Omega \leq C \|\nabla\varphi\|_\Omega$  for all  $\varphi \in \mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$ . This proves that  $\mathring{V}_0^1(\Omega) = H_0^1(\Omega)$ . Now set  $\beta > 0$ . Observe that we have

$$\mathring{V}_{-\beta}^1(\Omega) \subset H_0^1(\Omega) \subset \mathring{V}_\beta^1(\Omega) \quad \text{so that} \quad (\mathring{V}_\beta^1(\Omega))^* \subset (H_0^1(\Omega))^* \subset (\mathring{V}_{-\beta}^1(\Omega))^*.$$

Define the operators  $A_\varepsilon^{\pm\beta} : \mathring{V}_{\pm\beta}^1(\Omega) \rightarrow (\mathring{V}_{\mp\beta}^1(\Omega))^*$  such that

$$\langle A_\varepsilon^{\pm\beta} \varphi, \varphi' \rangle = \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \bar{\varphi}' dx, \quad \forall \varphi \in \mathring{V}_{\pm\beta}^1(\Omega), \varphi' \in \mathring{V}_{\mp\beta}^1(\Omega). \quad (5.10)$$

Working as in [30] for the 2D case of the corner, one can show that there is  $\beta_0 > 0$  (depending only on  $\kappa_\varepsilon$  and  $\varpi$ ) such that for all  $\beta \in (0; \beta_0)$ ,  $A_\varepsilon^\beta$  is Fredholm of index +1 while  $A_\varepsilon^{-\beta}$  is Fredholm of index -1. Note that we have  $\beta_0 = \min\{\Re \lambda + 1/2 \mid \lambda \text{ eigenvalue of (5.7) such that } \Re \lambda > -1/2\}$ . We remind the reader that for a bounded linear operator between two Banach spaces  $T : X \rightarrow Y$  whose range is closed, its index is defined as  $\text{ind } T := \dim \ker T - \dim \text{coker } T$ , with  $\dim \text{coker } T = \dim(Y/\text{range}(T))$ . On the other hand, application of Kondratiev calculus based in particular on the residue theorem (see [30, Theorem 5.2], [102, Theorem 5.4.2]) guarantees that if  $\varphi \in \mathring{V}_\beta^1(\Omega)$  is such that  $A_\varepsilon^{+\beta} \varphi \in (\mathring{V}_\beta^1(\Omega))^*$  (the important point here being that  $(\mathring{V}_\beta^1(\Omega))^* \subset (\mathring{V}_{-\beta}^1(\Omega))^*$ ), then there holds the following representation

$$\varphi = c_- s^- + c_+ s^+ + \tilde{\varphi} \quad \text{with } c_\pm \in \mathbb{C} \text{ and } \tilde{\varphi} \in \mathring{V}_{-\beta}^1(\Omega). \quad (5.11)$$

Note that  $s^\pm$ , with  $s^\pm$  defined by (5.9), belongs to  $\mathring{V}_\beta^1(\Omega)$ , but not to  $H_0^1(\Omega)$ , and a fortiori not to  $\mathring{V}_{-\beta}^1(\Omega)$ . Then introduce the space  $\mathring{V}^{\text{out}} := \text{span}(s^+) \oplus \mathring{V}_{-\beta}^1(\Omega)$ , endowed with the norm

$$\|\varphi\|_{V^{\text{out}}} = (|c|^2 + \|\tilde{\varphi}\|_{V_{-\beta}^1(\Omega)}^2)^{1/2}, \quad \forall \varphi = c s^+ + \tilde{\varphi} \in \mathring{V}^{\text{out}}, \quad (5.12)$$

which is a Banach space. Introduce also the operator  $A_\varepsilon^{\text{out}}$  such that for all  $\varphi = c s^+ + \tilde{\varphi} \in \mathring{V}^{\text{out}}$  and  $\varphi' \in \mathcal{C}_0^\infty(\bar{\Omega} \setminus \{O\})$ ,

$$\langle A_\varepsilon^{\text{out}} \varphi, \varphi' \rangle = \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \bar{\varphi}' dx = -c \int_\Omega \text{div}(\varepsilon \nabla s^+) \bar{\varphi}' dx + \int_\Omega \varepsilon \nabla \tilde{\varphi} \cdot \nabla \bar{\varphi}' dx.$$

Note that due to the features of the cut-off function  $\chi$ , we have  $\text{div}(\varepsilon \nabla s^+) \in L^2(\Omega)$ . And since  $\text{div}(\varepsilon \nabla s^+) = 0$  in a neighbourhood of  $O$ , we observe that there is a constant  $C > 0$  such that

$|\langle A_\varepsilon^{\text{out}} \varphi, \varphi' \rangle| \leq C \|\varphi\|_{V^{\text{out}}} \|\varphi'\|_{\dot{V}_\beta^1(\Omega)}$ . The density of  $\mathcal{C}_0^\infty(\Omega \setminus \{O\})$  in  $\dot{V}_\beta^1(\Omega)$  then allows us to extend  $A_\varepsilon^{\text{out}}$  as a continuous operator from  $\dot{V}^{\text{out}}$  to  $(\dot{V}_\beta^1(\Omega))^*$ . And we have

$$\langle A_\varepsilon^{\text{out}} \varphi, \varphi' \rangle = -c \int_\Omega \operatorname{div}(\varepsilon \nabla s^+) \overline{\varphi'} dx + \int_\Omega \varepsilon \nabla \tilde{\varphi} \cdot \nabla \overline{\varphi'} dx, \quad \forall \varphi = c s^+ + \tilde{\varphi}, \varphi' \in \dot{V}_\beta^1(\Omega).$$

Working as in [30] (see Proposition 4.4.) for the 2D case of the corner, one can prove that  $A_\varepsilon^{\text{out}} : \dot{V}^{\text{out}} \rightarrow (\dot{V}_\beta^1(\Omega))^*$  is Fredholm of index zero and that  $\ker A_\varepsilon^{\text{out}} = \ker A_\varepsilon^{-\beta}$ . In order to simplify the analysis below, we shall make the following assumption.

**Assumption 3.** *We assume that  $\varepsilon$  satisfies the conditions of §5.2.1 so that in particular the range of  $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  is not closed. Moreover we assume that for  $\beta \in (0; \beta_0)$ ,  $A_\varepsilon^{-\beta}$  is injective, which guarantees that  $A_\varepsilon^{\text{out}} : \dot{V}^{\text{out}} \rightarrow (\dot{V}_\beta^1(\Omega))^*$  is an isomorphism.*

The second part of this hypothesis boils down to supposing that there are no non zero regular solutions of the homogeneous problem  $\operatorname{div}(\varepsilon \nabla \varphi) = 0$  in  $\Omega$ ,  $\varphi = 0$  on  $\partial\Omega$ . Note that due to the change of sign of  $\varepsilon$ , such solutions may exist in very specific configurations, but they form at most a finite dimensional set [105, 29] which can be included in the analysis.

In what follows, we shall also need to work with the usual Laplace operator in weighted Sobolev spaces. For  $\gamma \in \mathbb{R}$ , define  $A^\gamma : \dot{V}_\gamma^1(\Omega) \rightarrow (\dot{V}_{-\gamma}^1(\Omega))^*$  such that

$$\langle A^\gamma \varphi, \varphi' \rangle = \int_\Omega \nabla \varphi \cdot \nabla \overline{\varphi'} dx, \quad \forall \varphi \in \dot{V}_\gamma^1(\Omega), \varphi' \in \dot{V}_{-\gamma}^1(\Omega)$$

(observe that there is no  $\varepsilon$  here). Combining the theory presented in [102] (see also the founding article [100] as well as the monographs [107, 113]) together with the result of [101, Corollary 2.2.1], we get the following proposition.

**Proposition 5.2.1.** *For all  $\gamma \in (-1/2; 1/2)$ , the operator  $A^\gamma : \dot{V}_\gamma^1(\Omega) \rightarrow (\dot{V}_{-\gamma}^1(\Omega))^*$  is an isomorphism.*

Note in particular that for  $\gamma = 0$ , this proposition simply says that  $\Delta : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  is an isomorphism. In order to have a result of isomorphism both for  $A_\varepsilon^{\text{out}}$  and  $A^\beta$ , we shall often make the assumption that the weight  $\beta$  is such that

$$0 < \beta < \min(1/2, \beta_0) \tag{5.13}$$

where  $\beta_0$  is defined after (5.10).

To measure electromagnetic fields in weighted Sobolev norms, in the following we shall work in the spaces

$$\begin{aligned} \mathbf{V}_\beta^0(\Omega) &:= (\mathbf{V}_\beta^0(\Omega))^3 \\ \dot{\mathbf{V}}_\beta^1(\Omega) &:= (\dot{\mathbf{V}}_\beta^1(\Omega))^3. \end{aligned}$$

Note that we have  $\mathbf{V}_{-\beta}^0(\Omega) \subset \mathbf{L}^2(\Omega) \subset \mathbf{V}_\beta^0(\Omega)$ .

### 5.3 Analysis of the problem for the electric component

In this section, we consider the problem for the electric field associated with (7.5)-(7.6). Since the scalar problem involving  $\varepsilon$  is well-posed in a non standard framework involving the propagating singularity  $s^+$  (see (5.12)), we shall add its gradient in the space for the electric field. Then we define a variational problem in this unusual space, and prove its well-posedness. In §5.3.5 we explain why the formulation in the classical framework fails to provide the solution of Maxwell problem. Finally we justify the choice of the new framework by a limiting absorption principle.

### 5.3.1 A well-chosen space for the electric field

Define the space of electric fields with the divergence free condition

$$\mathbf{X}_N^{\text{out}}(\varepsilon) := \{\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}}, c \in \mathbb{C}, \tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega), \operatorname{div}(\varepsilon \mathbf{u}) = 0 \text{ in } \Omega \setminus \{O\}, \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}. \quad (5.14)$$

In this definition, for  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}}$ , the condition  $\operatorname{div}(\varepsilon \mathbf{u}) = 0$  in  $\Omega \setminus \{O\}$  means that there holds

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega \setminus \{O\}), \quad (5.15)$$

which after integration by parts and by density of  $\mathcal{C}_0^\infty(\Omega \setminus \{O\})$  in  $H_0^1(\Omega)$  is equivalent to

$$-c \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \varphi \, dx + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega). \quad (5.16)$$

Note that we have  $\mathbf{X}_N(\varepsilon) \subset \mathbf{X}_N^{\text{out}}(\varepsilon)$  and that  $\dim(\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)) = 1$  (see Lemma 5.6.4 in Appendix). For  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}}$  with  $c \in \mathbb{C}$  and  $\tilde{\mathbf{u}} \in \mathbf{L}^2(\Omega)$ , we set

$$\|\mathbf{u}\|_{\mathbf{X}_N^{\text{out}}(\varepsilon)} = (|c|^2 + \|\tilde{\mathbf{u}}\|_{\Omega}^2 + \|\mathbf{curl} \mathbf{u}\|_{\Omega}^2)^{1/2}.$$

Endowed with this norm,  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  is a Banach space.

**Lemma 5.3.1.** *Pick some  $\beta$  satisfying (5.13). Under Assumptions 1 and 3, for any  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ , we have  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and there is a constant  $C > 0$  independent of  $\mathbf{u}$  such that*

$$|c| + \|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl} \mathbf{u}\|_{\Omega}. \quad (5.17)$$

As a consequence, the norm  $\|\cdot\|_{\mathbf{X}_N^{\text{out}}(\varepsilon)}$  is equivalent to the norm  $\|\mathbf{curl} \cdot\|_{\Omega}$  in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  and  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  endowed with the inner product  $(\mathbf{curl} \cdot, \mathbf{curl} \cdot)_{\Omega}$  is a Hilbert space.

**Proof.** Let  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}}$  be an element of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . The field  $\tilde{\mathbf{u}}$  is in  $\mathbf{L}^2(\Omega)$  and therefore decomposes as

$$\tilde{\mathbf{u}} = \nabla \varphi + \mathbf{curl} \boldsymbol{\psi} \quad (5.18)$$

with  $\varphi \in H_0^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  (item *iv*) of Proposition 5.6.1). Moreover, since  $\mathbf{u} \times \boldsymbol{\nu} = 0$  on  $\partial\Omega$  and since both  $s^+$  and  $\varphi$  vanish on  $\partial\Omega$ , we know that  $\mathbf{curl} \boldsymbol{\psi} \times \boldsymbol{\nu} = 0$  on  $\partial\Omega$ . Then noting that  $-\Delta \boldsymbol{\psi} = \mathbf{curl} \tilde{\mathbf{u}} = \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega)$ , we deduce from Proposition 5.6.2 that  $\mathbf{curl} \boldsymbol{\psi} \in \mathbf{V}_{-\beta}^0(\Omega)$  with the estimate

$$\|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl} \mathbf{u}\|_{\Omega}. \quad (5.19)$$

Using (5.15), the condition  $\operatorname{div}(\varepsilon \mathbf{u}) = 0$  in  $\Omega \setminus \{O\}$  implies

$$\int_{\Omega} \varepsilon \nabla(c s^+ + \varphi) \cdot \nabla \varphi' \, dx = - \int_{\Omega} \varepsilon \mathbf{curl} \boldsymbol{\psi} \cdot \nabla \varphi' \, dx, \quad \forall \varphi' \in \mathring{V}_{-\beta}^1(\Omega),$$

which means exactly that  $A_{\varepsilon}^{\beta}(c s^+ + \varphi) = -\operatorname{div}(\varepsilon \mathbf{curl} \boldsymbol{\psi}) \in (\mathring{V}_{-\beta}^1(\Omega))^*$ . Since additionally  $-\operatorname{div}(\varepsilon \mathbf{curl} \boldsymbol{\psi}) \in (\mathring{V}_{\beta}^1(\Omega))^*$ , from (5.11) we know that there are some complex constants  $c_{\pm}$  and some  $\tilde{\varphi} \in \mathring{V}_{-\beta}^1(\Omega)$  such that

$$c s^+ + \varphi = c_- s^- + c_+ s^+ + \tilde{\varphi}.$$

This implies  $c_- = 0$ ,  $c_+ = c$  (because  $\varphi \in H_0^1(\Omega)$ ) and so  $\varphi = \tilde{\varphi}$  is an element of  $\mathring{V}_{-\beta}^1(\Omega)$ . This shows that  $c s^+ + \varphi \in \mathring{V}^{\text{out}}$  and that  $A_{\varepsilon}^{\text{out}}(c s^+ + \varphi) = -\operatorname{div}(\varepsilon \mathbf{curl} \boldsymbol{\psi})$ . Since  $A_{\varepsilon}^{\text{out}} : \mathring{V}^{\text{out}} \rightarrow (\mathring{V}_{\beta}^1(\Omega))^*$  is an isomorphism, we have the estimate

$$|c| + \|\varphi\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C \|\operatorname{div}(\varepsilon \mathbf{curl} \boldsymbol{\psi})\|_{(\mathring{V}_{\beta}^1(\Omega))^*} \leq C \|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (5.20)$$

Finally gathering (5.18)–(5.20), we obtain that  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and that the estimate (5.17) is valid. Noting that  $\|\tilde{\mathbf{u}}\|_{\Omega} \leq C \|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}$ , this implies that the norms  $\|\cdot\|_{\mathbf{X}_N^{\text{out}}(\varepsilon)}$  and  $\|\mathbf{curl} \cdot\|_{\Omega}$  are equivalent in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ .  $\blacksquare$



Thanks to the previous lemma and by density of  $\mathcal{C}_0^\infty(\Omega \setminus \{O\})$  in  $\mathring{V}_\beta^1(\Omega)$ , the condition (5.16) for  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  is equivalent to

$$-c \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \varphi \, dx + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in \mathring{V}_\beta^1(\Omega) \quad (5.21)$$

where all the terms are well-defined as soon as  $\beta$  satisfies (5.13).

### 5.3.2 Definition of the problem for the electric field

Our objective is to define the problem for the electric field as a variational formulation set in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . For some  $\gamma > 0$ , let  $\mathbf{J}$  be an element of  $\mathbf{V}_{-\gamma}^0(\Omega)$  such that  $\operatorname{div} \mathbf{J} = 0$  in  $\Omega$ . Consider the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \text{ such that} \\ \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \bar{\mathbf{v}} \, dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}} \, dx, \quad \forall \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon), \end{array} \right. \quad (5.22)$$

where the term

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx \quad (5.23)$$

has to be carefully defined. The difficulty comes from the fact that  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  is not a subspace of  $\mathbf{L}^2(\Omega)$  so that this quantity cannot be considered as a classical integral.

Let  $\mathbf{u} = c_u \nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ . First, for  $\tilde{\mathbf{v}} \in \mathbf{V}_{-\beta}^0(\Omega)$  with  $\beta > 0$ , it is natural to set

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} \, dx := \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} \, dx. \quad (5.24)$$

To complete the definition, we have to give a sense to (5.23) when  $\mathbf{v} = \nabla s^+$ . Proceeding as for the derivation of (5.21), we start from the identity

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} \, dx = -c_u \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \bar{\varphi} \, dx + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi} \, dx, \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega \setminus \{O\}).$$

By density of  $\mathcal{C}_0^\infty(\Omega \setminus \{O\})$  in  $\mathring{V}_\beta^1(\Omega)$ , this leads to set

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} \, dx := -c_u \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \bar{\varphi} \, dx + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi} \, dx, \quad \forall \varphi \in \mathring{V}_\beta^1(\Omega). \quad (5.25)$$

With this definition, condition (5.21) can be written as

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in \mathring{V}_\beta^1(\Omega).$$

In particular, since  $s^+ \in \mathring{V}_\beta^1(\Omega)$ , for all  $\mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{s}^+ \, dx = 0 \quad \text{and so} \quad \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{s}^+ \, dx = c_u \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \bar{s}^+ \, dx. \quad (5.26)$$

Finally for all  $\mathbf{u} = c_u \nabla s^+ + \tilde{\mathbf{u}}$  and  $\mathbf{v} = c_v \nabla s^+ + \tilde{\mathbf{v}}$  in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , using (5.24) and (5.26), we find

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} \, dx = c_u \int_{\Omega} \varepsilon \nabla s^+ \cdot \bar{\tilde{\mathbf{v}}} \, dx + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} \, dx.$$

But since  $\mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ , we deduce from the second identity of (5.26) that

$$\int_{\Omega} \varepsilon \nabla s^+ \cdot \bar{\tilde{\mathbf{v}}} \, dx = \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla \bar{s}^+) \bar{s}^+ \, dx. \quad (5.27)$$

Summing up, we get

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx = c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla \bar{s}^+) \bar{s}^+ \, dx + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} \, dx, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon). \quad (5.28)$$



**Remark 5.3.1.** *Even if we use an integral symbol to keep the usual aspects of formulas and facilitate the reading, it is important to consider this new quantity as a sesquilinear form*

$$(\mathbf{u}, \mathbf{v}) \mapsto \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx$$

on  $\mathbf{X}_N^{\text{out}}(\varepsilon) \times \mathbf{X}_N^{\text{out}}(\varepsilon)$ . In particular, we point out that this sesquilinear form is not hermitian on  $\mathbf{X}_N^{\text{out}}(\varepsilon) \times \mathbf{X}_N^{\text{out}}(\varepsilon)$ . Indeed, we have

$$\overline{\int_{\Omega} \varepsilon \mathbf{v} \cdot \bar{\mathbf{u}} \, dx} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} \, dx + c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \bar{s}^+ \, dx$$

so that

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx - \overline{\int_{\Omega} \varepsilon \mathbf{v} \cdot \bar{\mathbf{u}} \, dx} = 2ic_u \bar{c}_v \Im m \left( \int_{\Omega} \operatorname{div}(\varepsilon \nabla \bar{s}^+) s^+ \, dx \right). \quad (5.29)$$

But Lemma 5.3.3 and assumption (5.8) show that

$$\Im m \left( \int_{\Omega} \operatorname{div}(\varepsilon \nabla \bar{s}^+) s^+ \, dx \right) \neq 0.$$

In the sequel, we denote by  $a_N(\cdot, \cdot)$  (resp.  $\ell_N(\cdot)$ ) the sesquilinear form (resp. the antilinear form) appearing in the left-hand side (resp. right-hand side) of (5.22).

### 5.3.3 Equivalent formulation

Before proving well-posedness in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , we have to make sure that a solution of (5.22) satisfies the initial problem (7.5)–(7.6). Proceeding as in the case of positive coefficients, this leads us to introduce the following space

$$\mathbf{H}_N^{\text{out}}(\mathbf{curl}) := \operatorname{span}(\nabla s^+) \oplus \mathbf{H}_N(\mathbf{curl}) \supset \mathbf{X}_N^{\text{out}}(\varepsilon)$$

(without the divergence free condition) and to consider the problem

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl}) \text{ such that} \\ a_N(\mathbf{u}, \mathbf{v}) = \ell_N(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl}), \end{cases} \quad (5.30)$$

where the definition of

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx$$

has to be extended to the space  $\mathbf{H}_N^{\text{out}}(\mathbf{curl})$ . Working exactly as in the beginning of the proof of Lemma 5.3.1, one can show that any  $\mathbf{u} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl})$  admits the decomposition

$$\mathbf{u} = c_u \nabla s^+ + \nabla \varphi_u + \mathbf{curl} \psi_u, \quad (5.31)$$

with  $c_u \in \mathbb{C}$ ,  $\varphi_u \in H_0^1(\Omega)$  and  $\psi_u \in \mathbf{X}_T(1)$ , such that  $\mathbf{curl} \psi_u \in \mathbf{V}_{-\beta}^0(\Omega)$ , for  $\beta$  satisfying (5.13). Then, for all  $\mathbf{u} = c_u \nabla s^+ + \nabla \varphi_u + \mathbf{curl} \psi_u$  and  $\mathbf{v} = c_v \nabla s^+ + \nabla \varphi_v + \mathbf{curl} \psi_v$  in  $\mathbf{H}_N^{\text{out}}(\mathbf{curl})$ , a natural extension of the previous definitions leads to set

$$\begin{aligned} \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx &:= \int_{\Omega} \varepsilon (\nabla \varphi_u + \mathbf{curl} \psi_u) \cdot (\nabla \bar{\varphi}_v + \mathbf{curl} \bar{\psi}_v) \, dx \\ &+ \int_{\Omega} c_u \varepsilon \nabla s^+ \cdot \overline{\mathbf{curl} \psi_v} + \bar{c}_v \varepsilon \mathbf{curl} \psi_u \cdot \nabla \bar{s}^+ \, dx \\ &- \int_{\Omega} c_u \bar{c}_v \operatorname{div}(\varepsilon \nabla s^+) \bar{s}^+ + c_u \operatorname{div}(\varepsilon \nabla s^+) \bar{\varphi}_v + \bar{c}_v \varphi_u \operatorname{div}(\varepsilon \nabla \bar{s}^+) \, dx. \end{aligned} \quad (5.32)$$

Note that (5.32) is indeed an extension of (5.28). To show it, first observe that for  $\mathbf{u} = c_u \nabla s^+ + \nabla \varphi_u + \mathbf{curl} \psi_u$ ,  $\mathbf{v} = c_v \nabla s^+ + \nabla \varphi_v + \mathbf{curl} \psi_v$  in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , the proof of Lemma 5.3.1 guarantees that  $\varphi_u, \varphi_v \in \dot{V}_{-\beta}^1(\Omega)$  with  $\beta$  satisfying (5.13). This allows us to integrate by parts in the last two terms of (5.32) to get

$$\begin{aligned} \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx &:= \int_{\Omega} \varepsilon (\nabla \varphi_u + \mathbf{curl} \psi_u) \cdot (\nabla \bar{\varphi}_v + \mathbf{curl} \bar{\psi}_v) \, dx \\ &\quad + \int_{\Omega} c_u \varepsilon \nabla s^+ \cdot (\nabla \bar{\varphi}_v + \mathbf{curl} \bar{\psi}_v) + \bar{c}_v \varepsilon (\nabla \varphi_u + \mathbf{curl} \psi_u) \cdot \nabla \bar{s}^+ \, dx \\ &\quad - c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \bar{s}^+ \, dx. \end{aligned} \quad (5.33)$$

Using (5.26), (5.27), the second line above can be written as

$$\begin{aligned} &\int_{\Omega} c_u \varepsilon \nabla s^+ \cdot (\nabla \bar{\varphi}_v + \mathbf{curl} \bar{\psi}_v) + \bar{c}_v \varepsilon (\nabla \varphi_u + \mathbf{curl} \psi_u) \cdot \nabla \bar{s}^+ \, dx \\ &= c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla \bar{s}^+) \bar{s}^+ \, dx + c_u \bar{c}_v \int_{\Omega} \operatorname{div}(\varepsilon \nabla s^+) \bar{s}^+ \, dx. \end{aligned} \quad (5.34)$$

Inserting (5.34) in (5.33) yields exactly (5.28).

**Lemma 5.3.2.** *Under Assumptions 1 and 3, the field  $\mathbf{u}$  is a solution of (5.22) if and only if it solves the problem (5.30). As a consequence, if  $\mathbf{u}$  satisfies (5.22), then  $(\mathbf{E}, \mathbf{H}) := (\mathbf{u}, (i\omega\mu)^{-1} \mathbf{curl} \mathbf{u})$  is a solution of (7.5)-(7.6).*

**Proof.** If  $\mathbf{u} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl})$  satisfies (5.30), then taking  $\mathbf{v} = \nabla \varphi$  with  $\varphi \in \mathcal{C}_0^\infty(\Omega \setminus \{O\})$  in (5.30), and using that  $\operatorname{div} \mathbf{J} = 0$  in  $\Omega$ , we get (5.15), which implies that  $\mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ . This shows that  $\mathbf{u}$  solves (5.22).

Now assume that  $\mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \subset \mathbf{H}_N^{\text{out}}(\mathbf{curl})$  is a solution of (5.22). Let  $\mathbf{v}$  be an element of  $\mathbf{H}_N^{\text{out}}(\mathbf{curl})$ . As in (5.31), we have the decomposition

$$\mathbf{v} = c_v \nabla s^+ + \nabla \varphi_v + \mathbf{curl} \psi_v, \quad (5.35)$$

with  $c_v \in \mathbb{C}$ ,  $\varphi_v \in H_0^1(\Omega)$  and  $\psi_v \in \mathbf{X}_T(1)$  such that  $\mathbf{curl} \psi_v \in \mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta$  satisfying (5.13). By Assumption 3, there is  $\zeta \in \dot{V}^{\text{out}}$  such that

$$A_\varepsilon^{\text{out}} \zeta = -\operatorname{div}(\varepsilon \mathbf{curl} \psi_v) \in (\dot{V}_\beta^1(\Omega))^*. \quad (5.36)$$

The function  $\zeta$  decomposes as  $\zeta = \alpha s^+ + \tilde{\zeta}$  with  $\tilde{\zeta} \in \dot{V}_{-\beta}^1(\Omega)$ . Finally, set

$$\hat{\mathbf{v}} = \mathbf{curl} \psi_v - \nabla \zeta = \mathbf{v} - \nabla(c_v s^+ + \varphi_v + \zeta).$$

The function  $\hat{\mathbf{v}}$  is in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , it satisfies  $\mathbf{curl} \hat{\mathbf{v}} = \mathbf{curl} \mathbf{v}$  and from (5.26), we deduce that

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\hat{\mathbf{v}}} \, dx = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx.$$

Using also that  $\mathbf{J} \in \mathbf{V}_{-\gamma}^0(\Omega)$  for some  $\gamma > 0$  and is such that  $\operatorname{div} \mathbf{J} = 0$  in  $\Omega$ , so that

$$\int_{\Omega} \mathbf{J} \cdot \bar{\hat{\mathbf{v}}} \, dx = \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}} \, dx,$$

this shows that  $a_N(\mathbf{u}, \mathbf{v}) = a_N(\mathbf{u}, \hat{\mathbf{v}}) = \ell_N(\hat{\mathbf{v}}) = \ell_N(\mathbf{v})$  and proves that  $\mathbf{u}$  is a solution of (5.30).

Now if  $\mathbf{u}$  satisfies (5.22), and so (5.30), since  $\nabla s^+ \in \mathbf{L}^1(\Omega) := \mathbf{L}^1(\Omega)^3$ , we have  $\mathbf{u} \in \mathbf{L}^1(\Omega)$ .

Therefore there holds  $\mathbf{curl} \mathbf{u} \in \mathcal{D}'(\Omega)^3$  where  $\mathcal{D}'(\Omega)$  denotes the set of distributions on  $\Omega$ . And actually one can verify that  $\mathbf{curl} \mathbf{u} \in \mathbf{L}^2(\Omega)$  because  $\mathbf{curl} \nabla s^+ = 0$  in  $\Omega$ . Thus we can set  $(\mathbf{E}, \mathbf{H}) := (\mathbf{u}, (i\omega\mu)^{-1} \mathbf{curl} \mathbf{u}) \in \mathbf{L}^1(\Omega) \times \mathbf{L}^2(\Omega)$ . Clearly we have  $\mathbf{curl} \mathbf{E} = i\omega\mu \mathbf{H}$  in  $\Omega$ . By taking  $\mathbf{v} \in \mathcal{C}_0^\infty(\Omega)^3 \subset \mathbf{H}_N^{\text{out}}(\mathbf{curl})$  in (5.30) and by observing that in this case

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx,$$

we obtain  $\mathbf{curl} \mathbf{H} + i\omega\varepsilon \mathbf{E} = \mathbf{J}$  in  $\mathcal{D}'(\Omega)^3$ . Moreover, because  $\mathbf{E} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl})$ , we have  $\mathbf{E} \times \nu = 0$  on  $\partial\Omega$ . Finally, using the relation  $\mathbf{curl} \mathbf{E} = i\omega\mu \mathbf{H}$  in  $\Omega$ , for  $\varphi \in \mathcal{C}_0^\infty(\bar{\Omega} \setminus \bar{\mathcal{M}})$ , we find

$$\langle \mu \mathbf{H} \cdot \nu, \varphi \rangle_{\partial\Omega} = \int_{\Omega} \mu \mathbf{H} \cdot \nabla \bar{\varphi} \, dx = (i\omega)^{-1} \int_{\Omega} \mathbf{curl} \mathbf{E} \cdot \nabla \bar{\varphi} \, dx = 0.$$

From the density of traces of elements of  $\mathcal{C}_0^\infty(\bar{\Omega} \setminus \bar{\mathcal{M}})$  into  $\mathbf{H}^{1/2}(\partial\Omega)$ , we infer that  $\mu \mathbf{H} \cdot \nu = 0$  on  $\partial\Omega$ .  $\blacksquare$

In the following, we shall work with the formulation (5.22) set in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . The reason being that, as usual in the analysis of Maxwell's equations, the divergence free condition will yield a compactness property allowing us to deal with the term involving the frequency  $\omega$ .

### 5.3.4 Main analysis for the electric field

Define the continuous operators  $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  and  $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  such that for all  $\mathbf{u}, \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ ,

$$\langle \mathbb{A}_N^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx, \quad \langle \mathbb{K}_N^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} \, dx.$$

With this notation, we have  $\langle (\mathbb{A}_N^{\text{out}} + \mathbb{K}_N^{\text{out}}) \mathbf{u}, \mathbf{v} \rangle = a_N(\mathbf{u}, \mathbf{v})$ .

**Proposition 5.3.1.** *Under Assumptions 1–3, the operator  $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is an isomorphism.*

**Proof.** Let us construct a continuous operator  $\mathbb{T} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow \mathbf{X}_N^{\text{out}}(\varepsilon)$  such that for all  $\mathbf{u}, \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ ,

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} (\overline{\mathbb{T}\mathbf{v}}) \, dx = \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx.$$

To proceed, we adapt the method presented in [24]. Assume that  $\mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  is given. We construct  $\mathbb{T}\mathbf{v}$  in three steps.

1) Since  $\mathbf{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)$  and  $A_\mu : \mathbf{H}_\#^1(\Omega) \rightarrow (\mathbf{H}_\#^1(\Omega))^*$  is an isomorphism, there is a unique  $\zeta \in \mathbf{H}_\#^1(\Omega)$  such that

$$\int_{\Omega} \mu \nabla \zeta \cdot \nabla \zeta' \, dx = \int_{\Omega} \mu \mathbf{curl} \mathbf{v} \cdot \nabla \zeta' \, dx, \quad \forall \zeta' \in \mathbf{H}_\#^1(\Omega).$$

Then the field  $\mu(\mathbf{curl} \mathbf{v} - \nabla \zeta) \in \mathbf{L}^2(\Omega)$  is divergence free in  $\Omega$  and satisfies  $\mu(\mathbf{curl} \mathbf{v} - \nabla \zeta) \cdot \nu = 0$  on  $\partial\Omega$ .

2) From item *ii*) of Proposition 5.6.1, we infer that there is  $\psi \in \mathbf{X}_N(1)$  such that

$$\mu(\mathbf{curl} \mathbf{v} - \nabla \zeta) = \mathbf{curl} \psi.$$

Thanks to Lemma 5.6.2, we deduce that  $\psi \in \mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in (0; 1/2)$  and a fortiori for  $\beta$  satisfying (5.13).

3) Suppose now that  $\beta$  satisfies (5.13). Then we know from the previous step that  $\operatorname{div}(\varepsilon\psi) \in (\mathring{V}_\beta^1(\Omega))^*$ . On the other hand, by Assumption 3,  $A_\varepsilon^{\text{out}} : \mathring{V}^{\text{out}} \rightarrow (\mathring{V}_\beta^1(\Omega))^*$  is an isomorphism. Consequently we can introduce  $\varphi \in \mathring{V}^{\text{out}}$  such that  $A_\varepsilon^{\text{out}}\varphi = -\operatorname{div}(\varepsilon\psi)$ .

Finally, we set  $\mathbb{T}\mathbf{v} = \psi - \nabla\varphi$ . Clearly  $\mathbb{T}\mathbf{v}$  is an element of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . Moreover, for all  $\mathbf{u}, \mathbf{v}$  in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , we have

$$\begin{aligned} \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\mathbb{T}\mathbf{v}} \, dx &= \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\psi} \, dx \\ &= \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\psi} \, dx - \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \nabla \overline{\zeta} \, dx \\ &= \int_{\Omega} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\psi} \, dx. \end{aligned}$$

From Lemma 5.3.1 and the Lax-Milgram theorem, we deduce that  $\mathbb{T}^* \mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is an isomorphism. And by symmetry, permuting the roles of  $\mathbf{u}$  and  $\mathbf{v}$ , it is obvious that  $\mathbb{T}^* \mathbb{A}_N^{\text{out}} = \mathbb{A}_N^{\text{out}} \mathbb{T}$ , which allows us to conclude that  $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is an isomorphism.  $\blacksquare$

**Proposition 5.3.2.** *Under Assumptions 1 and 3, if  $(\mathbf{u}_n = c_n \nabla s^+ + \tilde{\mathbf{u}}_n)$  is a sequence which is bounded in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , then we can extract a subsequence such that  $(c_n)$  and  $(\tilde{\mathbf{u}}_n)$  converge respectively in  $\mathbb{C}$  and in  $\mathbf{V}_{-\beta}^0(\Omega)$  for  $\beta$  satisfying (5.13). As a consequence, the operator  $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is compact.*

**Proof.** Let  $(\mathbf{u}_n)$  be a bounded sequence of elements of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . From the proof of Lemma 5.3.1, we know that for  $n \in \mathbb{N}$ , we have

$$\mathbf{u}_n = c_n \nabla s^+ + \nabla \varphi_n + \operatorname{curl} \psi_n \tag{5.37}$$

where the sequences  $(c_n)$ ,  $(\varphi_n)$ ,  $(\psi_n)$  and  $(\operatorname{curl} \psi_n)$  are bounded respectively in  $\mathbb{C}$ ,  $\mathring{V}_{-\beta}^1(\Omega)$ ,  $\mathbf{X}_T(1)$  and  $\mathbf{V}_{-\beta}^0(\Omega)$ . Observing that  $\operatorname{curl} \mathbf{u}_n = \operatorname{curl} \operatorname{curl} \psi_n = -\Delta \psi_n$  is bounded in  $\mathbf{L}^2(\Omega)$ , we deduce from Proposition 5.6.3 that there exists a subsequence such that  $(\operatorname{curl} \psi_n)$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . Moreover, by (5.20), we have

$$|c_n - c_m| + \|\varphi_n - \varphi_m\|_{\mathring{V}_{-\beta}^1(\Omega)} \leq C \|\operatorname{curl}(\psi_n - \psi_m)\|_{\mathbf{V}_{-\beta}^0(\Omega)},$$

which implies that  $(c_n)$  and  $(\varphi_n)$  converge respectively in  $\mathbb{C}$  and in  $\mathring{V}_{-\beta}^1(\Omega)$ . From (5.37), we see that this is enough to conclude about the first part of the proposition.

Finally, observing that

$$\|\mathbb{K}_N^{\text{out}} \mathbf{u}\|_{(\mathbf{X}_N^{\text{out}}(\varepsilon))^*} \leq C (\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} + |c_u|),$$

we deduce that  $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is a compact operator.  $\blacksquare$

We can now state the main theorem of the analysis of the problem for the electric field.

**Theorem 5.3.1.** *Under Assumptions 1–3, for all  $\omega \in \mathbb{R}$  the operator  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is Fredholm of index zero.*

**Proof.** Since  $\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is compact (Proposition 5.3.2) and  $\mathbb{A}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is an isomorphism (Proposition 5.3.1),  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is Fredholm of index zero.  $\blacksquare$

The previous theorem guarantees that the problem (5.22) is well-posed if and only if uniqueness holds, that is if and only if the only solution for  $\mathbf{J} = 0$  is  $\mathbf{u} = 0$ . Since uniqueness holds for  $\omega = 0$ , one can prove with the analytic Fredholm theorem that (5.22) is well-posed except for at most a

countable set of values of  $\omega$  with no accumulation points (note that Theorem 5.3.1 remains true for  $\omega \in \mathbb{C}$ ).

Note that in practice  $\varepsilon$  is itself a function of  $\omega$ . For instance, if the inclusion  $\mathcal{M}$  is metallic, it is commonly admitted that the Drude's law gives a good model for  $\varepsilon$ . But taking into account the dependence of  $\varepsilon$  with respect to  $\omega$  when studying uniqueness of problem (5.22) leads to a non-linear eigenvalue problem, where the functional space  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  itself depends on  $\omega$ . This study is beyond the scope of the present paper (see [90] for such questions in the case of the 2D scalar problem).

Nonetheless, there is a result that we can prove concerning the cases of non-uniqueness for problem (5.22).

**Proposition 5.3.3.** *If  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  is a solution of (5.22) for  $\mathbf{J} = 0$ , then  $c = 0$  and  $\mathbf{u} \in \mathbf{X}_N(\varepsilon)$ .*

**Proof.** When  $\omega = 0$ , the result is a direct consequence of Theorem 5.3.1 (because zero is the only solution of (5.22) for  $\mathbf{J} = 0$ ). From now on, we assume that  $\omega \in \mathbb{R} \setminus \{0\}$ . Suppose that  $\mathbf{u} = c\nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  is such that

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} dx = 0, \quad \forall \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon).$$

Taking the imaginary part of the previous identity for  $\mathbf{v} = \mathbf{u}$ , we get

$$\Im m \left( \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{u}} dx \right) = 0.$$

On the other hand, by (5.28), we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{u}} dx = \int_{\Omega} \varepsilon |\tilde{\mathbf{u}}|^2 dx + |c|^2 \int_{\Omega} \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+ dx,$$

so that

$$|c|^2 \Im m \left( \int_{\Omega} \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+ dx \right) = 0.$$

The result of the proposition is then a consequence of Lemma 5.3.3 where it is proved that

$$\Im m \left( \int_{\Omega} \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+ dx \right) = \eta \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds,$$

and of the assumption (5.8). ■

**Remark 5.3.2.** *As a consequence, from Lemma 5.3.1, we infer that elements of the kernel of  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}}$  are in  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta$  satisfying (5.13).*

**Remark 5.3.3.** *Using the result of Theorem 5.3.1, we could have studied a problem similar to (7.5)–(7.6) with an impedance boundary condition replacing the perfect conductor condition. In this case, using the unique continuation principle, we would have been able to prove uniqueness of the solution, and so well-posedness of the problem, for all  $\omega > 0$ . Theorem 5.3.1 can also be employed to consider the scattering of an incident wave by a bounded inclusion (with the same features as  $\mathcal{M}$ ) in freespace. In the latter situation, working as in [17, Lemma 2.1 and Proposition 2.1], in particular using the Rellich lemma, one could also establish existence and uniqueness of the solution (in a framework like  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ ) for all  $\omega > 0$ .*

### 5.3.5 Problem in the classical framework

In the previous paragraph, we have shown that the Maxwell's problem (5.22) for the electric field set in the non standard space  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , and so in  $\mathbf{H}_N^{\text{out}}(\mathbf{curl})$  according to Lemma 5.3.2, is well-posed. In order to understand what would fail with a naive approach, here we wish to analyse the properties of the problem for the electric field set in the classical space  $\mathbf{X}_N(\varepsilon)$  (which does not contain  $\nabla s^+$ ). Since this space is a closed subspace of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , it inherits the main properties of the problem in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  proved in the previous section. More precisely, we deduce from Lemma 5.3.1 and Proposition 5.3.2 the following result.

**Proposition 5.3.4.** *Under Assumptions 1 and 3, the embedding of  $\mathbf{X}_N(\varepsilon)$  in  $\mathbf{L}^2(\Omega)$  is compact, and  $\|\mathbf{curl} \cdot\|_{\Omega}$  is a norm in  $\mathbf{X}_N(\varepsilon)$  which is equivalent to the norm  $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$ .*

Note that we recover classical properties similar to what is known for positive  $\varepsilon$ , or more generally [24] for  $\varepsilon$  such that the operator  $A_{\varepsilon} : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  defined by (5.3) is an isomorphism (which allows for sign-changing  $\varepsilon$ ). But we want to underline the fact that under Assumption 3, these classical results could not be proved by using classical arguments. They require the introduction of the bigger space  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , with the singular function  $\nabla s^+$ .

Let us now consider the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_N(\varepsilon) \text{ such that} \\ \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} dx = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}} dx, \quad \forall \mathbf{v} \in \mathbf{X}_N(\varepsilon). \end{array} \right. \quad (5.38)$$

An important remark is that one cannot prove that problem (5.38) is equivalent to a similar problem set in  $\mathbf{H}_N(\mathbf{curl})$  (the analogue of Lemma 5.3.2). Again, the difficulty comes from the fact that  $A_{\varepsilon}$  is not an isomorphism, and trouble would appear when solving (5.36). Therefore, a solution of (5.38) is not in general a distributional solution of the equation

$$\mathbf{curl} \left( \mu^{-1} \mathbf{curl} \mathbf{u} \right) - \omega^2 \varepsilon \mathbf{u} = i\omega \mathbf{J}.$$

To go further in the analysis of (5.38), we recall that  $\mathbf{X}_N(\varepsilon)$  is a subspace of codimension one of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  (Lemma 5.6.4 in Appendix). Let  $\mathbf{v}_0$  be an element of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  which does not belong to  $\mathbf{X}_N(\varepsilon)$ . Then we denote by  $\ell_0$  the continuous linear form on  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  such that:

$$\forall \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon) \quad \mathbf{v} - \ell_0(\mathbf{v})\mathbf{v}_0 \in \mathbf{X}_N(\varepsilon). \quad (5.39)$$

Let us now define the operators  $\mathbb{A}_N : \mathbf{X}_N(\varepsilon) \rightarrow (\mathbf{X}_N(\varepsilon))^*$  and  $\mathbb{K}_N : \mathbf{X}_N(\varepsilon) \rightarrow (\mathbf{X}_N(\varepsilon))^*$  by

$$\langle \mathbb{A}_N \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx, \quad \langle \mathbb{K}_N \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} dx.$$

**Proposition 5.3.5.** *Under Assumptions 1–3, the operator  $\mathbb{A}_N : \mathbf{X}_N(\varepsilon) \rightarrow (\mathbf{X}_N(\varepsilon))^*$  is Fredholm of index zero.*

**Proof.** Let  $\mathbf{u} \in \mathbf{X}_N(\varepsilon)$ . By Proposition 5.3.1, for the operator  $\mathbb{T}$  introduced in the corresponding proof, one has:

$$\|\mathbf{u}\|_{\mathbf{X}_N(\varepsilon)}^2 = \|\mathbf{curl} \mathbf{u}\|_{\Omega}^2 = \langle \mathbb{A}_N^{\text{out}} \mathbf{u}, \mathbb{T} \mathbf{u} \rangle.$$

Then, using (5.39), we get:

$$\|\mathbf{u}\|_{\mathbf{X}_N(\varepsilon)}^2 = \langle \mathbb{A}_N \mathbf{u}, \mathbb{T} \mathbf{u} - \ell_0(\mathbb{T} \mathbf{u})\mathbf{v}_0 \rangle + \langle \mathbb{A}_N^{\text{out}} \mathbf{u}, \ell_0(\mathbb{T} \mathbf{u})\mathbf{v}_0 \rangle,$$

which implies that

$$\|\mathbf{u}\|_{\mathbf{X}_N(\varepsilon)} \leq C \left( \|\mathbb{A}_N \mathbf{u}\|_{(\mathbf{X}_N(\varepsilon))^*} + |\ell_0(\mathbb{T} \mathbf{u})| \right).$$

The result of the proposition then follows from a classical adaptation of Peetre's lemma (see for example [144, Theorem 12.12]) together with the fact that  $\mathbb{A}_N$  is bounded and hermitian.  $\blacksquare$

Combining the two previous propositions, we obtain the

**Theorem 5.3.2.** *Under Assumptions 1–3, for all  $\omega \in \mathbb{R}$ , the operator  $\mathbb{A}_N - \omega^2 \mathbb{K}_N : \mathbf{X}_N(\varepsilon) \rightarrow (\mathbf{X}_N(\varepsilon))^*$  is Fredholm of index zero.*

But as mentioned above, even if uniqueness holds and if Problem (5.38) is well-posed, it does not provide a solution of Maxwell's equations. Note that the phenomenon observed in this paragraph is very similar to what happens for Maxwell's equations with positive coefficients in presence of singularities when one looks at a formulation set in  $H^1(\Omega)^3$  (see e.g. [61, 88, 64]).

### 5.3.6 Expression of the singular coefficient

Under Assumptions 1–3, Theorem 5.3.1 guarantees that for all  $\omega \in \mathbb{R}$  the operator  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is Fredholm of index zero. Assuming that it is injective, the problem (5.22) admits a unique solution  $\mathbf{u} = c_u \nabla s^+ + \tilde{\mathbf{u}}$ . The goal of this paragraph is to derive a formula allowing one to compute  $c_u$  without knowing  $\mathbf{u}$ . Such kind of results are classical for scalar operators (see e.g. [85], [102, Theorem 6.4.4], [71, 72, 10, 89, 145, 121]). They are used in particular for numerical purposes. But curiously they do not seem to exist for Maxwell's equations in 3D, not even for classical situations with positive materials in non smooth domains. We emphasize that the analysis we develop can be adapted to the latter case.

In order to establish the desired expression, for  $\omega \in \mathbb{R}$ , we first introduce the field  $\mathbf{w}_N \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  such that

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{v} \cdot \mathbf{curl} \overline{\mathbf{w}_N} dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{v} \cdot \overline{\mathbf{w}_N} dx = \int_{\Omega} \varepsilon \tilde{\mathbf{v}} \cdot \nabla \overline{s^+} dx, \quad \forall \mathbf{v} \in \mathbf{X}_N^{\text{out}}(\varepsilon). \quad (5.40)$$

Note that Problem (5.40) is well-posed when  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}}$  is an isomorphism. Indeed, using (5.29), one can check that it involves the operator  $(\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}})^*$ , that is the adjoint of  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}}$ . Moreover  $\mathbf{v} \mapsto \int_{\Omega} \varepsilon \tilde{\mathbf{v}} \cdot \nabla \overline{s^+} dx$  is a linear form over  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ .

**Theorem 5.3.3.** *Assume that  $\omega \in \mathbb{R}$ , Assumptions 1–3 are valid and  $\mathbb{A}_N^{\text{out}} - \omega^2 \mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is injective. Then the solution  $\mathbf{u} = c_u \nabla s^+ + \tilde{\mathbf{u}}$  of the electric problem (5.22) is such that*

$$c_u = i\omega \int_{\Omega} \mathbf{J} \cdot \overline{\mathbf{w}_N} dx \Big/ \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{s^+} dx. \quad (5.41)$$

Here  $\mathbf{w}_N$  is the function which solves (5.40).

**Remark 5.3.4.** *Note that in practice  $\mathbf{w}_N$  can be computed once for all because it does not depend on  $\mathbf{J}$ . Then the value of  $c_u$  can be determined very simply via Formula (5.41).*

**Proof.** By definition of  $\mathbf{u}$ , we have

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \overline{\mathbf{w}_N} dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \overline{\mathbf{w}_N} dx = i\omega \int_{\Omega} \mathbf{J} \cdot \overline{\mathbf{w}_N} dx.$$

On the other hand, from (5.40), there holds

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \overline{\mathbf{w}_N} dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \overline{\mathbf{w}_N} dx = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \overline{s^+} dx.$$

From these two relations as well as (5.26), we get

$$i\omega \int_{\Omega} \mathbf{J} \cdot \overline{\mathbf{w}_N} dx = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \overline{s^+} dx = c_u \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{s^+} dx.$$

But Lemma 5.3.3 below guarantees that  $\Im \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{s^+} dx \neq 0$ . Therefore we find the desired formula.  $\blacksquare$



**Lemma 5.3.3.** *With the notations of (5.4), we have*

$$\Im m \left( \int_{\Omega} \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+ dx \right) = \eta \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds.$$

**Proof.** Set  $\Omega_{\tau} := \Omega \setminus \overline{B(O, \tau)}$ . Noticing that  $\operatorname{div}(\varepsilon \nabla \overline{s^+})$  vanishes in a neighbourhood of the origin, we can write

$$\begin{aligned} \int_{\Omega} \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+ dx &= \lim_{\tau \rightarrow 0} \int_{\Omega_{\tau}} \operatorname{div}(\varepsilon \nabla \overline{s^+}) s^+ dx \\ &= \lim_{\tau \rightarrow 0} \left( - \int_{\Omega_{\tau}} \varepsilon |\nabla s^+|^2 dx - \int_{\partial B(O, \tau)} \varepsilon \frac{\overline{\partial s^+}}{\partial r} s^+ ds \right). \end{aligned}$$

Taking the imaginary part and observing that

$$\int_{\partial B(O, \tau)} \varepsilon \frac{\overline{\partial s^+}}{\partial r} s^+ ds = - \left( \frac{1}{2} + i\eta \right) \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds,$$

the result follows. ■

### 5.3.7 Limiting absorption principle

In §5.3.4, we have proved well-posedness of the problem for the electric field in the space  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . But up to now, we have not explained why we select this framework. In particular, as mentioned in §5.2.1, well-posedness also holds in  $\mathbf{X}_N^{\text{in}}(\varepsilon)$  where  $\mathbf{X}_N^{\text{in}}(\varepsilon)$  is defined as  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  with  $s^+$  replaced by  $s^-$  (see (5.9) for the definitions of  $s^{\pm}$ ). In general, the solution in  $\mathbf{X}_N^{\text{in}}(\varepsilon)$  differs from the one in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . Therefore one can build infinitely many solutions of Maxwell's problem as linear interpolations of these two solutions. Then the question is: which solution is physically relevant? Classically, the answer can be obtained thanks to the limiting absorption principle. The idea is the following. In practice, the dielectric permittivity takes complex values, the imaginary part being related to the dissipative phenomena in the materials. Set

$$\varepsilon^{\delta} := \varepsilon + i\delta$$

where  $\varepsilon$  is defined as previously (see (5.2)) and  $\delta > 0$  (the sign of  $\delta$  depends on the convention for the time-harmonic dependence (in  $e^{-i\omega t}$  here)). Due to the imaginary part of  $\varepsilon^{\delta}$  which is uniformly positive, one recovers some coercivity properties which allow one to prove well-posedness of the corresponding problem for the electric field in the classical framework. The physically relevant solution for the problem with the real-valued  $\varepsilon$  then should be the limit of the sequence of solutions for the problems involving  $\varepsilon^{\delta}$  when  $\delta$  tends to zero. The goal of the present paragraph is to explain how to show that this limit is the solution of the problem set in  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ .

#### Limiting absorption principle for the scalar case

Our proof relies on a similar result for the 3D scalar problem which is the analogue of what has been done in 2D in [24, Theorem 4.3]. Consider the problem

$$\text{Find } \varphi^{\delta} \in H_0^1(\Omega) \text{ such that } -\operatorname{div}(\varepsilon^{\delta} \nabla \varphi^{\delta}) = f, \tag{5.42}$$

where  $f \in (H_0^1(\Omega))^*$ . Since  $\delta > 0$ , by the Lax-Milgram lemma, this problem is well-posed for all  $f \in (H_0^1(\Omega))^*$  and in particular for all  $f \in (\dot{V}_{\beta}^1(\Omega))^*$ ,  $\beta > 0$ . Our objective is to prove that  $(\varphi^{\delta})$  converges when  $\delta$  tends to zero to the unique solution of the problem

$$\text{Find } \varphi \in \dot{V}^{\text{out}} \text{ such that } A_{\varepsilon}^{\text{out}} \varphi = f. \tag{5.43}$$

We expect a convergence in a space  $\mathring{V}_{\beta}^1(\Omega)$  with  $0 < \beta < \beta_0$ . We first need a decomposition of  $\varphi^\delta$  as a sum of a singular part and a regular part. Since problem (5.42) is strongly elliptic, one can directly apply the theory presented in [102]. In particular, to characterize the singular part, one is led to consider the spectral problem

$$\left| \begin{array}{l} \text{Find } (\Phi^\delta, \lambda^\delta) \in H^1(\mathbb{S}^2) \setminus \{0\} \times \mathbb{C} \text{ such that} \\ \int_{\mathbb{S}^2} \varepsilon^\delta \nabla_S \Phi^\delta \cdot \nabla_S \bar{\Psi} ds = \lambda^\delta (\lambda^\delta + 1) \int_{\mathbb{S}^2} \varepsilon^\delta \Phi^\delta \bar{\Psi} ds, \quad \forall \Psi \in H^1(\mathbb{S}^2). \end{array} \right. \quad (5.44)$$

By assumption (see Section 5.2),  $\lambda_{\pm} := -1/2 \pm i\eta$  (where  $\eta$  is fixed in (5.8)) are eigenvalues of (5.7) of algebraic multiplicity equal to one. Using Rouché theorem, one can show that for  $\delta > 0$  small enough, there are exactly two eigenvalues  $\lambda_{\pm}^\delta$  of (5.44) such that we have  $|\lambda_{\pm} - \lambda_{\pm}^\delta| \leq C \delta$ , where  $C$  is independent of  $\delta$ . Moreover  $\lambda_{\pm}^\delta$  are of algebraic multiplicity equal to one. By observing that  $\lambda^\delta$  is an eigenvalue of (5.7) if and only  $-\lambda^\delta - 1$  is an eigenvalue of (5.7), we deduce that for  $\delta$  small enough, there exists one and only one eigenvalue of (5.7), that we denote by  $\lambda^\delta \in \mathbb{C}$ , such that  $\Re \lambda^\delta \in (-1/2; -1/2 + \beta_0 - \sqrt{\delta})$ . Let  $s^\delta$  be the corresponding singular function defined by

$$s^\delta(r, \theta, \varphi) = r^{\lambda^\delta} \Phi^\delta(x/|x|),$$

where  $\Phi^\delta$  is the eigenfunction associated with  $\lambda^\delta$  such that  $(\Phi^\delta, \Phi)_{H^1(\mathbb{S}^2)} = 1$ . Here  $\Phi$  is the function introduced in §5.2.1 and we will prove in Lemma 5.3.6 that we can indeed impose the condition  $(\Phi^\delta, \Phi)_{H^1(\mathbb{S}^2)} = 1$  for  $\delta$  small enough. Observe that  $s^\delta$  satisfies  $\text{div}(\varepsilon^\delta \nabla s^\delta) = 0$  in  $\mathcal{K}$ . As in (5.9) for  $s^\pm$ , we set

$$s^\delta(x) = \chi(r) r^{-1/2+i\eta^\delta} \Phi^\delta(x/|x|), \quad (5.45)$$

where  $\eta^\delta \in \mathbb{C}$  is the number such that  $\lambda^\delta = -1/2 + i\eta^\delta$ . By applying [102, Theorem 5.4.1], we get the following result.

**Lemma 5.3.4.** *Let  $0 < \beta < \beta_0$  and  $f \in (\mathring{V}_{\beta}^1(\Omega))^*$ . The solution  $\varphi^\delta$  of (5.42) decomposes as*

$$\varphi^\delta = c^\delta s^\delta + \tilde{\varphi}^\delta \quad (5.46)$$

where  $c^\delta \in \mathbb{C}$  and  $\tilde{\varphi}^\delta \in \mathring{V}_{-\beta}^1(\Omega)$ .

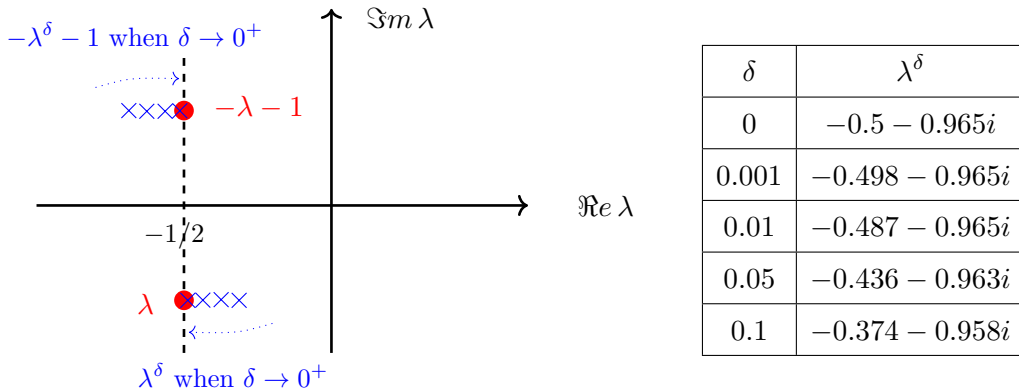


Figure 5.2: Behaviour of the eigenvalue  $\lambda^\delta$  close to the line  $\Re \lambda = -1/2$  as the dissipation  $\delta$  tends to zero. Here the values have been obtained solving the problem (5.44) with a Finite Element Method. We work in the conical tip defined via (5.5) with  $\alpha = 2\pi/3$  and  $\kappa_\varepsilon = -1.9$ . In this case, using (5.6) we find  $I_\alpha \approx 2.585$  so that the critical interval is approximately given by  $[-2.585; -1]$ .

Let us first study the limit of the singular function.

**Lemma 5.3.5.** *For all  $\beta > 0$ , when  $\delta$  tends to zero, the function  $s^\delta$  converges in  $\mathring{V}_{\beta}^1(\Omega)$  to  $s^+$  (see the definitions in (5.8) and (5.9)).*

**Proof.** The first step of the proof consists in showing that the limit of  $(\eta^\delta)$ , which we denote by  $\eta^0$ , is  $\eta$  and not  $-\eta$ . Let  $\tilde{\rho} > 0$  be such that the function  $\chi$  introduced in (5.9) satisfies  $\chi = 1$  in the ball  $B(O, \tilde{\rho})$ . From integration by parts, we get

$$0 = \int_{B(O, \tilde{\rho})} \operatorname{div}(\varepsilon^\delta \nabla s^\delta) \overline{s^\delta} dx = - \int_{B(O, \tilde{\rho})} \varepsilon^\delta |\nabla s^\delta|^2 dx + (-1/2 + i\eta^\delta) \tilde{\rho}^{-2\Im m \eta^\delta} \int_{\mathbb{S}^2} \varepsilon^\delta |\Phi^\delta|^2 ds.$$

Thus we must have

$$\Im m \left( (-1/2 + i\eta^\delta) \tilde{\rho}^{-2\Im m \eta^\delta} \int_{\mathbb{S}^2} \varepsilon^\delta |\Phi^\delta|^2 ds \right) > 0. \quad (5.47)$$

Taking the limit  $\delta \rightarrow 0^+$  in (5.47) and using Lemma 5.3.6 below which guarantees that  $(\Phi^\delta)$  converges to  $\Phi$  in  $H^1(\mathbb{S}^2)$ , we obtain the relation

$$\eta^0 \int_{\mathbb{S}^2} \varepsilon |\Phi|^2 ds > 0. \quad (5.48)$$

According to the definition (5.8) of  $\eta$ , this ensures that  $\eta^0 = \eta$  and shows that  $(\lambda^\delta)$  converges to  $\lambda$  (and not to  $-\lambda - 1$ , see an illustration with Figure 5.2). From the definitions (5.9), (5.45) of  $s$ ,  $s^\delta$ , using again that  $(\Phi^\delta)$  converges to  $\Phi$  in  $H^1(\mathbb{S}^2)$ , we infer that  $s^\delta$  converges to  $s^+$  (and not to  $s^-$ ) in  $\dot{V}_\beta^1(\Omega)$ .  $\blacksquare$

**Lemma 5.3.6.** *Let  $(\Phi^\delta)$  be a sequence of eigenfunctions associated with the eigenvalue  $\lambda^\delta$ . For  $\delta$  small enough, we can impose the condition  $(\Phi^\delta, \Phi)_{H^1(\mathbb{S}^2)} = 1$ . Then  $\Phi^\delta$  is uniquely defined and when  $\delta$  tends to zero,  $(\Phi^\delta)$  converges in  $H^1(\mathbb{S}^2)$  to the  $\Phi$  introduced in §5.2.1.*

**Proof.** Let  $(\tilde{\Phi}^\delta)$  be a sequence of eigenfunctions associated with the eigenvalue  $\lambda^\delta$  such that  $\|\tilde{\Phi}^\delta\|_{H^1(\mathbb{S}^2)} = 1$ . We can extract a subsequence, that we also denote by  $(\tilde{\Phi}^\delta)$ , which converges weakly in  $H^1(\mathbb{S}^2)$  and strongly in  $L^2(\mathbb{S}^2)$  to some  $\tilde{\Phi} \in H^1(\mathbb{S}^2)$ . For  $z \in \mathbb{C}$ , with the Riesz representation theorem, define the symbol  $\mathcal{L}^\delta(z) : H^1(\mathbb{S}^2) \rightarrow H^1(\mathbb{S}^2)$  such that

$$(\mathcal{L}^\delta(z)\Psi, \Psi')_{H^1(\mathbb{S}^2)} = \int_{\mathbb{S}^2} \varepsilon^\delta \nabla_S \Psi \cdot \nabla_S \overline{\Psi'} ds - z(z+1) \int_{\mathbb{S}^2} \varepsilon^\delta \Psi \overline{\Psi'} ds, \quad \forall \Psi, \Psi' \in H^1(\mathbb{S}^2).$$

First taking the limit  $\delta \rightarrow 0^+$  in  $(\mathcal{L}^\delta(\lambda^\delta)\tilde{\Phi}^\delta, \Psi')_{H^1(\mathbb{S}^2)} = 0$ , we get

$$(\mathcal{L}^0(\lambda)\tilde{\Phi}, \Psi')_{H^1(\mathbb{S}^2)} = 0, \quad \forall \Psi' \in H^1(\mathbb{S}^2).$$

This shows that either  $\tilde{\Phi} \equiv 0$  or  $\tilde{\Phi}$  is an eigenfunction of (5.7) associated with  $\lambda$ . On the other hand, using some T-coercivity approach on the sphere (mimic the proof [19, Theorem 6.4]), one can prove that  $\mathcal{L}^0(-1/2 + it) : H^1(\mathbb{S}^2) \rightarrow H^1(\mathbb{S}^2)$  is an isomorphism for  $t > 0$  large enough. Let us decompose  $\mathcal{L}^\delta(\lambda^\delta)$  as

$$\mathcal{L}^\delta(\lambda^\delta) = \mathcal{L}^0(-1/2 + it) + \mathcal{R}^\delta + \mathcal{K}$$

where  $\mathcal{R}^\delta, \mathcal{K} : H^1(\mathbb{S}^2) \rightarrow H^1(\mathbb{S}^2)$  are the operators such that for all  $\Psi, \Psi' \in H^1(\mathbb{S}^2)$ ,

$$\begin{aligned} (\mathcal{R}^\delta \Psi, \Psi')_{H^1(\mathbb{S}^2)} &= i\delta \left( \int_{\mathbb{S}^2} \nabla_S \Psi \cdot \nabla_S \overline{\Psi'} ds - \lambda^\delta(\lambda^\delta + 1) \int_{\mathbb{S}^2} \Psi \overline{\Psi'} ds \right) \\ &\quad - (\lambda^\delta(\lambda^\delta + 1) - \lambda(\lambda + 1)) \int_{\mathbb{S}^2} \varepsilon \Psi \overline{\Psi'} ds \\ (\mathcal{K} \Psi, \Psi')_{H^1(\mathbb{S}^2)} &= -(\lambda(\lambda + 1) - (-1/2 + it)(+1/2 + it)) \int_{\mathbb{S}^2} \varepsilon \Psi \overline{\Psi'} ds. \end{aligned}$$

Note that the norm of  $\mathcal{R}^\delta$ , as a linear operator of  $H^1(\mathbb{S}^2)$ , tends to zero when  $\delta \rightarrow 0^+$  and that  $\mathcal{K}$  is compact. Therefore, using the relations  $\mathcal{L}^\delta(\lambda^\delta)\tilde{\Phi}^\delta = \mathcal{L}^0(\lambda)\tilde{\Phi} = 0$  to get

$$\mathcal{L}^0(-1/2 + it)(\tilde{\Phi}^\delta - \tilde{\Phi}) = -\mathcal{R}^\delta \tilde{\Phi}^\delta - \mathcal{K}(\tilde{\Phi}^\delta - \tilde{\Phi}),$$

we deduce that  $(\tilde{\Phi}^\delta)$  converges strongly to  $\tilde{\Phi}$  in  $H^1(\mathbb{S}^2)$ . This implies  $\|\tilde{\Phi}\|_{H^1(\mathbb{S}^2)} = 1$  and proves that  $\tilde{\Phi}$  is an eigenfunction of (5.7) associated with  $\lambda$ . Since by assumption  $\lambda$  is a simple eigenvalue,  $\tilde{\Phi}$  is proportional to  $\Phi$ . Thus for  $\delta$  small enough, we have  $(\tilde{\Phi}^\delta, \Phi)_{H^1(\mathbb{S}^2)} \neq 0$ . Then  $(\Phi^\delta)$ , with  $\Phi^\delta = \tilde{\Phi}^\delta / (\tilde{\Phi}^\delta, \Phi)_{H^1(\mathbb{S}^2)}$ , is a sequence of eigenfunctions associated with  $\lambda^\delta$  such that  $(\Phi^\delta, \Phi)_{H^1(\mathbb{S}^2)} = 1$ . Now from the convergence of  $(\Phi^\delta)$  to  $c\Phi$  with  $|c| = 1$  and  $(\Phi^\delta, \Phi)_{H^1(\mathbb{S}^2)} = 1$ , we infer that  $(\Phi^\delta)$  converges to  $\Phi$  when  $\delta$  tends to zero. Finally, one observes that such a construction is possible for any subsequence of  $(\tilde{\Phi}^\delta)$ .  $\blacksquare$

Then proceeding exactly as in the proof of [30, Theorem 4.3], one can establish the following result.

**Lemma 5.3.7.** *Let  $0 < \beta < \beta_0$  and  $f \in (\mathring{V}_\beta^1(\Omega))^*$ . If Assumption 3 holds, then  $(\varphi^\delta = c^\delta s^\delta + \tilde{\varphi}^\delta)$  converges to  $\varphi = c s^+ + \tilde{\varphi}$  in  $\mathring{V}_\beta^1(\Omega)$  as  $\delta$  tends to zero. Moreover,  $(c^\delta, \tilde{\varphi}^\delta)$  converges to  $(c, \tilde{\varphi})$  in  $\mathbb{C} \times \mathring{V}_{-\beta}^1(\Omega)$ . In this statement,  $\varphi^\delta$  (resp.  $\varphi$ ) is the solution of (5.42) (resp. (5.43)).*

Note that the results of Lemma 5.3.7 still hold if we replace  $f$  by a family of source terms  $(f^\delta) \in (\mathring{V}_\beta^1(\Omega))^*$  that converges to  $f$  in  $(\mathring{V}_\beta^1(\Omega))^*$  when  $\delta$  tends to zero.

### Limiting absorption principle for the electric problem

The problem

$$\text{Find } \mathbf{u}^\delta \in \mathbf{X}_N(\varepsilon^\delta) \text{ such that } \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}^\delta - \omega^2 \varepsilon^\delta \mathbf{u}^\delta = i\omega \mathbf{J}, \quad (5.49)$$

with  $\mathbf{X}_N(\varepsilon^\delta) = \{\mathbf{E} \in \mathbf{H}_N(\mathbf{curl}) \mid \text{div}(\varepsilon^\delta \mathbf{E}) = 0\}$ , is well-posed for all  $\omega \in \mathbb{R}$  and all  $\delta > 0$ . This result is classical when  $\mu$  takes positive values while it can be shown by using [24] when  $\mu$  changes sign. We want to study the convergence of  $\mathbf{u}^\delta$  when  $\delta$  goes to zero. Let  $(\delta_n)$  be a sequence of positive numbers such that  $\lim_{n \rightarrow +\infty} \delta_n = 0$ . To simplify, we denote the quantities with an index  $n$  instead of  $\delta_n$  (for example we write  $\varepsilon^n$  instead of  $\varepsilon^{\delta_n}$ ).

**Lemma 5.3.8.** *Suppose that  $(\mathbf{u}^n)$  is a sequence of elements of  $\mathbf{X}_N(\varepsilon^n)$  such that  $(\mathbf{curl} \mathbf{u}^n)$  is bounded in  $\mathbf{L}^2(\Omega)$ . Then, under Assumption 3, for all  $\beta$  satisfying (5.13), for all  $n \in \mathbb{N}$ ,  $\mathbf{u}^n$  admits the decomposition  $\mathbf{u}^n = c^n \nabla s^n + \tilde{\mathbf{u}}^n$  with  $c^n \in \mathbb{C}$  and  $\tilde{\mathbf{u}}^n \in \mathbf{V}_{-\beta}^0(\Omega)$ . Moreover, there exists a subsequence such that  $(c^n)$  converges to some  $c$  in  $\mathbb{C}$  while  $(\tilde{\mathbf{u}}^n)$  converges to some  $\tilde{\mathbf{u}}$  in  $\mathbf{V}_{-\beta}^0(\Omega)$ . Finally, the field  $\mathbf{u} := c \nabla s^+ + \tilde{\mathbf{u}}$  belongs to  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ .*

**Proof.** For all  $n \in \mathbb{N}$ , we have  $\mathbf{u}^n \in \mathbf{X}_N(\varepsilon^\delta) \subset \mathbf{L}^2(\Omega)$ . Therefore, there exist  $\varphi^n \in H_0^1(\Omega)$  and  $\psi^n \in \mathbf{X}_T(1)$ , satisfying  $\mathbf{curl} \psi^n \times \nu = 0$  on  $\partial\Omega$  such that  $\mathbf{u}^n = \nabla \varphi^n + \mathbf{curl} \psi^n$ . Moreover, we have the estimate

$$\|\Delta \psi^n\|_\Omega = \|\mathbf{curl} \mathbf{u}^n\|_\Omega \leq C.$$

As a consequence, Proposition 5.6.2 guarantees that  $(\mathbf{curl} \psi^n)$  is a bounded sequence of  $\mathbf{V}_{-\beta}^0(\Omega)$ , and Proposition 5.6.3 ensures that there exists a subsequence such that  $(\mathbf{curl} \psi^n)$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . Now from the fact that  $\text{div}(\varepsilon^n \mathbf{u}^n) = 0$ , we obtain

$$\text{div}(\varepsilon^n \nabla \varphi^n) = -\text{div}(\varepsilon^n \mathbf{curl} \psi^n) \in (\mathring{V}_\beta^1(\Omega))^*.$$

By Lemmas 5.3.4 and 5.3.7, this implies that the function  $\varphi^n$  decomposes as  $\varphi^n = c^n s^n + \tilde{\varphi}^n$  with  $c^n \in \mathbb{C}$  and  $\tilde{\varphi}^n \in \mathring{V}_{-\beta}^1(\Omega)$ . Moreover,  $(c^n)$  converges to  $c$  in  $\mathbb{C}$  while  $(\tilde{\varphi}^n)$  converges to  $\tilde{\varphi}$  in  $\mathring{V}_{-\beta}^1(\Omega)$ .

Summing up, we have that  $\mathbf{u}^n = c^n \nabla s^n + \tilde{\mathbf{u}}^n$  where  $\tilde{\mathbf{u}}^n = \nabla \tilde{\varphi}^n + \mathbf{curl} \psi^n$  converges to  $\tilde{\mathbf{u}}$  in  $\mathbf{V}_{-\beta}^0(\Omega)$ . In particular, this implies that  $\mathbf{u}^n$  converges to  $\mathbf{u} = c \nabla s^+ + \tilde{\mathbf{u}}$  in  $\mathbf{V}_\gamma^0(\Omega)$  for all  $\gamma > 0$ . It remains to prove that  $\mathbf{u} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ , which amounts to showing that  $\mathbf{u}$  satisfies (5.26). To proceed, we take the limit as  $n \rightarrow +\infty$  in the identity

$$-c^n \int_\Omega \text{div}(\varepsilon^n \nabla s^n) \varphi \, dx + \int_\Omega \varepsilon^n \tilde{\mathbf{u}}^n \cdot \nabla \varphi \, dx = 0$$

which holds for all  $\varphi \in \mathring{V}_{-\beta}^1(\Omega)$  because  $\mathbf{u}^n \in \mathbf{X}_N(\varepsilon^n)$ . ■

**Theorem 5.3.4.** *Let  $\omega \in \mathbb{R}$ . Suppose that Assumptions 1, 2 and 3 hold, and that  $\mathbf{u} = 0$  is the only function of  $\mathbf{X}_N(\varepsilon)$  satisfying*

$$\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u} - \omega^2 \varepsilon \mathbf{u} = 0. \quad (5.50)$$

Then the sequence of solutions  $(\mathbf{u}^\delta = c^\delta \nabla s^\delta + \tilde{\mathbf{u}}^\delta)$  of (5.49) converges, as  $\delta$  tends to 0, to the unique solution  $\mathbf{u} = c \nabla s^+ + \tilde{\mathbf{u}} \in \mathbf{X}_N^{\text{out}}(\varepsilon)$  of (5.22) in the following sense:  $(c_\delta)$  converges to  $c$  in  $\mathbb{C}$ ,  $(\tilde{\mathbf{u}}^\delta)$  converges to  $\tilde{\mathbf{u}}$  in  $\mathbf{V}_{-\beta}^0(\Omega)$  and  $(\mathbf{curl} \mathbf{u}^\delta)$  converges to  $\mathbf{curl} \mathbf{u}$  in  $\mathbf{L}^2(\Omega)$ .

**Proof.** Let  $(\delta_n)$  be a sequence of positive numbers such that  $\lim_{n \rightarrow +\infty} \delta_n = 0$ . Denote by  $\mathbf{u}^n$  the unique function of  $\mathbf{X}_N(\varepsilon^n)$  such that

$$\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u}^n - \omega^2 \varepsilon^n \mathbf{u}^n = i\omega \mathbf{J}. \quad (5.51)$$

Note that we set again  $\varepsilon^n$  instead of  $\varepsilon^{\delta_n}$ . The proof is in two steps. First, we establish the desired property by assuming that  $(\|\mathbf{curl} \mathbf{u}^n\|_\Omega)$  is bounded. Then we show that this hypothesis is indeed satisfied.

**First step.** Assume that there is a constant  $C > 0$  such that for all  $n \in \mathbb{N}$

$$\|\mathbf{curl} \mathbf{u}^n\|_\Omega \leq C. \quad (5.52)$$

By lemma 5.3.8, we can extract a subsequence from  $(\mathbf{u}^n = c^n \nabla s^n + \tilde{\mathbf{u}}^n)$  such that  $(c^n)$  converges to  $c$  in  $\mathbb{C}$ ,  $(\tilde{\mathbf{u}}^n)$  converges to  $\tilde{\mathbf{u}}$  in  $\mathbf{V}_{-\beta}^0(\Omega)$ , with  $\mathbf{u} = \tilde{\mathbf{u}} + c \nabla s^+ \in \mathbf{X}_N^{\text{out}}(\varepsilon)$ . Besides, since for all  $n \in \mathbb{N}$ ,  $\mathbf{curl} \mathbf{u}^n \in \mathbf{L}^2(\Omega)$ , there exist  $h^n \in H_{\#}^1(\Omega)$  and  $\mathbf{w}^n \in \mathbf{X}_N(1)$ , such that

$$\mu^{-1} \mathbf{curl} \mathbf{u}^n = \nabla h^n + \mathbf{curl} \mathbf{w}^n. \quad (5.53)$$

Observing that  $(\mathbf{w}^n)$  is bounded in  $\mathbf{X}_N(1)$ , from Lemma 5.6.2, we deduce that it admits a subsequence which converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . Multiplying (5.51) taken for two indices  $n$  and  $m$  by  $\overline{(\mathbf{w}^n - \mathbf{w}^m)}$ , and integrating by parts, we obtain

$$\int_{\Omega} |\mathbf{curl} \mathbf{w}^n - \mathbf{curl} \mathbf{w}^m|^2 dx = \omega^2 \int_{\Omega} (\varepsilon^n \mathbf{u}^n - \varepsilon^m \mathbf{u}^m) \overline{(\mathbf{w}^n - \mathbf{w}^m)} dx.$$

This implies that  $(\mathbf{curl} \mathbf{w}^n)$  converges in  $\mathbf{L}^2(\Omega)$ . Then, from (5.53), we deduce that

$$\operatorname{div}(\mu \nabla h^n) = -\operatorname{div}(\mu \mathbf{curl} \mathbf{w}^n) \text{ in } \Omega.$$

By Assumption 2, the operator  $A_\mu : H_{\#}^1(\Omega) \rightarrow (H_{\#}^1(\Omega))^*$  is an isomorphism. Therefore  $(\nabla h^n)$  converges in  $\mathbf{L}^2(\Omega)$ . From (5.53), this shows that  $(\mathbf{curl} \mathbf{u}^n)$  converges to  $\mathbf{curl} \mathbf{u}$  in  $\mathbf{L}^2(\Omega)$ . Finally, we know that  $\mathbf{u}^n$  satisfies

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u}^n \cdot \mathbf{curl} \bar{v} dx - \omega^2 \int_{\Omega} \varepsilon^n \mathbf{u}^n \cdot \bar{v} dx = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{v} dx$$

for all  $v \in \mathbf{V}_{-\beta}^0(\Omega)$ . Taking the limit, we get that  $\mathbf{u}$  satisfies

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{v} dx - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{v} dx = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{v} dx \quad (5.54)$$

for all  $v \in \mathbf{V}_{-\beta}^0(\Omega)$ . Since in addition,  $\mathbf{u}$  satisfies (5.26), (5.54) also holds for  $v = \nabla s^+$  and we get that  $\mathbf{u}$  is the unique solution  $u$  of (5.22).

**Second step.** Now we prove that the assumption (5.52) is satisfied. Suppose by contradiction that there exists a subsequence of  $(\mathbf{u}^n)$  such that

$$\|\mathbf{curl} \mathbf{u}^n\|_\Omega \rightarrow +\infty$$

and consider the sequence  $(\mathbf{v}^n)$  with for all  $n \in \mathbb{N}$ ,  $\mathbf{v}^n := \mathbf{u}^n / \|\mathbf{curl} \mathbf{u}^n\|_\Omega$ . We have

$$\mathbf{v}^n \in \mathbf{X}_N(\varepsilon^n) \quad \text{and} \quad \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{v}^n - \omega^2 \varepsilon^n \mathbf{v}^n = i\omega \mathbf{J} / \|\mathbf{curl} \mathbf{u}^n\|_\Omega. \quad (5.55)$$

Following the first step of the proof, we find that we can extract a subsequence from  $(\mathbf{v}^n)$  which converges, in the sense given in the theorem, to the unique solution of the homogeneous problem (5.22) with  $\mathbf{J} = 0$ . But by Proposition 5.3.3, this solution also solves (5.50). As a consequence, it is equal to zero. In particular, it implies that  $(\mathbf{curl} \mathbf{v}^n)$  converges to zero in  $\mathbf{L}^2(\Omega)$ , which is impossible since by construction, for all  $n \in \mathbb{N}$ , we have  $\|\mathbf{curl} \mathbf{v}^n\|_\Omega = 1$ .  $\blacksquare$

## 5.4 Analysis of the problem for the magnetic component

In this section, we turn our attention to the analysis of the Maxwell's problem for the magnetic component. Importantly, in the whole section, we suppose that  $\beta$  satisfies (5.13), that is  $0 < \beta < \min(1/2, \beta_0)$ . Contrary to the analysis for the electric component, we define functional spaces which depend on  $\beta$ :

$$\mathbf{Z}_T^{\text{out}}(\mu) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \text{span}(\varepsilon \nabla s^+) \oplus \mathbf{V}_{-\beta}^0(\Omega), \text{div}(\mu \mathbf{u}) = 0 \text{ in } \Omega, \mu \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$$

and for  $\xi \in \mathbf{L}^\infty(\Omega)$ ,

$$\mathbf{Z}_T^{\pm\beta}(\xi) := \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{V}_{\pm\beta}^0(\Omega), \text{div}(\xi \mathbf{u}) = 0 \text{ in } \Omega \text{ and } \xi \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}.$$

Note that we have  $\mathbf{Z}_T^{-\beta}(\mu) \subset \mathbf{Z}_T^{\text{out}}(\mu) \subset \mathbf{Z}_T^\beta(\mu)$ . The conditions  $\text{div}(\mu \mathbf{u}) = 0$  in  $\Omega$  and  $\mu \mathbf{u} \cdot \boldsymbol{\nu} = 0$  on  $\partial\Omega$  for the elements of these spaces boil down to impose

$$\int_\Omega \mu \mathbf{u} \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in \mathbf{H}_\#^1(\Omega).$$

**Remark 5.4.1.** *Observe that the elements of  $\mathbf{Z}_T^{\text{out}}(\mu)$  are in  $\mathbf{L}^2(\Omega)$  but have a singular curl. On the other hand, the elements of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$  are singular but have a curl in  $\mathbf{L}^2(\Omega)$ . This is consistent with the fact that for the situations we are considering in this work, the electric field is singular while the magnetic field is not.*

The analysis of the problem for the magnetic component leads to considering the formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu) \text{ such that} \\ \int_\Omega \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx - \omega^2 \int_\Omega \mu \mathbf{u} \cdot \bar{\mathbf{v}} = \int_\Omega \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \bar{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{Z}_T^\beta(\mu), \end{array} \right. \quad (5.56)$$

where  $\mathbf{J} \in \mathbf{V}_{-\beta}^0(\Omega)$ . Again, the first integral in the left-hand side of (5.56) is not a classical integral. Similarly to definition (5.26), we set

$$\int_\Omega \nabla s^+ \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx := 0, \quad \forall \mathbf{v} \in \mathbf{Z}_T^\beta(\mu).$$

As a consequence, for  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$  such that  $\mathbf{curl} \mathbf{u} = c_u \varepsilon \nabla s^+ + \boldsymbol{\zeta}_u$  (we shall use this notation throughout the section) and  $\mathbf{v} \in \mathbf{Z}_T^\beta(\mu)$ , there holds

$$\int_\Omega \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx = \int_\Omega \varepsilon^{-1} \boldsymbol{\zeta}_u \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx. \quad (5.57)$$

Note that for  $\mathbf{u}, \mathbf{v}$  in  $\mathbf{Z}_T^{\text{out}}(\mu)$  such that  $\mathbf{curl} \mathbf{u} = c_u \varepsilon \nabla s^+ + \boldsymbol{\zeta}_u$ ,  $\mathbf{curl} \mathbf{v} = c_v \varepsilon \nabla s^+ + \boldsymbol{\zeta}_v$ , we have

$$\begin{aligned} \int_\Omega \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx &= \int_\Omega \varepsilon^{-1} \boldsymbol{\zeta}_u \cdot \overline{(c_v \varepsilon \nabla s^+ + \boldsymbol{\zeta}_v)} \, dx \\ &= \int_\Omega \varepsilon^{-1} \boldsymbol{\zeta}_u \cdot \bar{\boldsymbol{\zeta}}_v \, dx - \bar{c}_v \int_\Omega \text{div}(\boldsymbol{\zeta}_u) \bar{s}^+ \, dx \\ &= \int_\Omega \varepsilon^{-1} \boldsymbol{\zeta}_u \cdot \bar{\boldsymbol{\zeta}}_v \, dx + c_u \bar{c}_v \int_\Omega \text{div}(\varepsilon \nabla s^+) \bar{s}^+ \, dx. \end{aligned} \quad (5.58)$$

We denote by  $a_T(\cdot, \cdot)$  (resp.  $\ell_T(\cdot)$ ) the sesquilinear form (resp. the antilinear form) appearing in the left-hand side (resp. right-hand side) of (5.56).

**Remark 5.4.2.** Note that in (5.56), the solution and the test functions do not belong to the same space. This is different from the formulation (5.22) for the electric field but seems necessary in the analysis below to obtain a well-posed problem (in particular to prove Proposition 5.4.1). Note also that even if the functional framework depends on  $\beta$ , the solution will not if  $\mathbf{J}$  is regular enough (see the explanations in Remark 5.4.4).

### 5.4.1 Equivalent formulation

Define the spaces

$$\begin{aligned} \mathbf{H}^\beta(\mathbf{curl}) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega)\} \\ \mathbf{H}^{\text{out}}(\mathbf{curl}) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \text{span}(\varepsilon \nabla s^+) \oplus \mathbf{V}_{-\beta}^0(\Omega)\}. \end{aligned}$$

**Lemma 5.4.1.** Under Assumptions 1–2, the field  $\mathbf{u}$  is a solution of (5.56) if and only if it solves the problem

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}^{\text{out}}(\mathbf{curl}) \text{ such that} \\ a_T(\mathbf{u}, \mathbf{v}) = \ell_T(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}^\beta(\mathbf{curl}). \end{array} \right. \quad (5.59)$$

As a consequence, if  $\mathbf{u}$  satisfies (5.56), then  $(\mathbf{E}, \mathbf{H}) := (i(\omega\varepsilon)^{-1}(\mathbf{curl} \mathbf{u} - \mathbf{J}), \mathbf{u})$  is a solution of (7.5)-(7.6).

**Proof.** If  $\mathbf{u}$  satisfies (5.59), then taking  $\mathbf{v} = \nabla\varphi$  with  $\varphi \in \mathbf{H}_\#^1(\Omega)$  in (5.59), we get that  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ . This proves that  $\mathbf{u}$  solves (5.56).

Assume now that  $\mathbf{u}$  is a solution of (5.56). Let  $\mathbf{v}$  be an element of  $\mathbf{H}^\beta(\mathbf{curl})$ . Introduce  $\varphi \in \mathbf{H}_\#^1(\Omega)$  the function such that

$$\int_{\Omega} \mu \nabla\varphi \cdot \nabla\varphi' dx = \int_{\Omega} \mu \mathbf{v} \cdot \nabla\varphi' dx, \quad \forall \varphi' \in \mathbf{H}_\#^1(\Omega).$$

The field  $\hat{\mathbf{v}} := \mathbf{v} - \nabla\varphi$  belongs to  $\mathbf{Z}_T^\beta(\mu)$ . Moreover, there holds  $\mathbf{curl} \hat{\mathbf{v}} = \mathbf{curl} \mathbf{v}$  and since for  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ , we have

$$\int_{\Omega} \mu \mathbf{u} \cdot \nabla\varphi dx = 0, \quad \forall \varphi \in \mathbf{H}_\#^1(\Omega),$$

we deduce that  $a_T(\mathbf{u}, \mathbf{v}) = a_T(\mathbf{u}, \hat{\mathbf{v}}) = \ell_T(\hat{\mathbf{v}}) = \ell_T(\mathbf{v})$ .

Now if  $\mathbf{u}$  satisfies (5.56), and so (5.59), one notes that the pair  $(\mathbf{E}, \mathbf{H}) := (i(\omega\varepsilon)^{-1}(\mathbf{curl} \mathbf{u} - \mathbf{J}), \mathbf{u})$  belongs to  $\mathbf{L}^1(\Omega) \times \mathbf{L}^2(\Omega)$ . Clearly we have  $\mathbf{curl} \mathbf{H} + i\omega \varepsilon \mathbf{E} = \mathbf{J}$  in  $\Omega$ . By taking  $\mathbf{v} \in \mathcal{C}_0^\infty(\Omega)^3 \subset \mathbf{H}^\beta(\mathbf{curl})$  in (5.59) and by observing that in this case

$$\int_{\Omega} \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx = \int_{\Omega} \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx,$$

we obtain  $\mathbf{curl} \mathbf{E} = i\omega \mu \mathbf{H}$  in  $\mathcal{D}'(\Omega)^3$ . The boundary conditions (7.6) can then be deduced in a classical way. ■

### 5.4.2 Norms in $\mathbf{Z}_T^{\pm\beta}(\mu)$ and $\mathbf{Z}_T^{\text{out}}(\mu)$

We endow the space  $\mathbf{Z}_T^\beta(\mu)$  with the norm

$$\|\mathbf{u}\|_{\mathbf{Z}_T^\beta(\mu)} = (\|\mathbf{u}\|_{\Omega}^2 + \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}^2)^{1/2},$$

so that it is a Banach space.



**Lemma 5.4.2.** *Under Assumptions 1–2, there is a constant  $C > 0$  such that for all  $\mathbf{u} \in \mathbf{Z}_T^\beta(\mu)$ , we have*

$$\|\mathbf{u}\|_\Omega \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}.$$

*As a consequence, the norm  $\|\cdot\|_{\mathbf{Z}_T^\beta(\mu)}$  is equivalent to the norm  $\|\mathbf{curl} \cdot\|_{\mathbf{V}_\beta^0(\Omega)}$  in  $\mathbf{Z}_T^\beta(\mu)$ .*

**Remark 5.4.3.** *The result of Lemma 5.4.2 holds for all  $\beta$  such that  $0 \leq \beta < 1/2$  and not only for  $0 < \beta < \min(1/2, \beta_0)$ .*

**Proof.** Let  $\mathbf{u}$  be an element of  $\mathbf{Z}_T^\beta(\mu)$ . Since  $\mathbf{u}$  belongs to  $\mathbf{L}^2(\Omega)$ , according to the item *v*) of Proposition 5.6.1, there are  $\varphi \in \mathbf{H}_\#^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$  such that

$$\mathbf{u} = \nabla \varphi + \mathbf{curl} \boldsymbol{\psi}. \quad (5.60)$$

Lemma 5.6.2 guarantees that  $\boldsymbol{\psi} \in \mathbf{V}_{-\beta}^0(\Omega)$  with the estimate

$$\|\boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl} \boldsymbol{\psi}\|_\Omega. \quad (5.61)$$

Multiplying the equation  $\mathbf{curl} \mathbf{curl} \boldsymbol{\psi} = \mathbf{curl} \mathbf{u}$  in  $\Omega$  by  $\boldsymbol{\psi}$  and integrating by parts, we get

$$\|\mathbf{curl} \boldsymbol{\psi}\|_\Omega^2 \leq \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)} \|\boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (5.62)$$

Gathering (5.61) and (5.62) leads to

$$\|\mathbf{curl} \boldsymbol{\psi}\|_\Omega \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}. \quad (5.63)$$

On the other hand, using that

$$\int_\Omega \mu \mathbf{u} \cdot \nabla \varphi' dx = 0, \quad \forall \varphi' \in \mathbf{H}_\#^1(\Omega)$$

and that  $A_\mu : \mathbf{H}_\#^1(\Omega) \rightarrow (\mathbf{H}_\#^1(\Omega))^*$  is an isomorphism, we deduce that  $\|\nabla \varphi\|_\Omega \leq C \|\mathbf{curl} \boldsymbol{\psi}\|_\Omega$ . Using this estimate and (5.63) in the decomposition (5.60), we finally obtain the desired result. ■

If  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ , we have  $\mathbf{curl} \mathbf{u} = c_u \varepsilon \nabla s^+ + \boldsymbol{\zeta}_u$  with  $c_u \in \mathbb{C}$  and  $\boldsymbol{\zeta}_u \in \mathbf{V}_{-\beta}^0(\Omega)$ . We endow the space  $\mathbf{Z}_T^{\text{out}}(\mu)$  with the norm

$$\|\mathbf{u}\|_{\mathbf{Z}_T^{\text{out}}(\mu)} = (\|\mathbf{u}\|_\Omega^2 + |c_u|^2 + \|\boldsymbol{\zeta}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2)^{1/2},$$

so that it is a Banach space.

**Lemma 5.4.3.** *Under Assumptions 1–3, there is  $C > 0$  such that for all  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ , we have*

$$\|\mathbf{u}\|_\Omega + |c_u| \leq C \|\boldsymbol{\zeta}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (5.64)$$

*As a consequence, the norm  $\|\mathbf{u}\|_{\mathbf{Z}_T^{\text{out}}(\mu)}$  is equivalent to the norm  $\|\boldsymbol{\zeta}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)}$  for  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ .*

**Proof.** Let  $\mathbf{u}$  be an element of  $\mathbf{Z}_T^{\text{out}}(\mu)$ . Since  $\mathbf{Z}_T^{\text{out}}(\mu) \subset \mathbf{Z}_T^\beta(\mu)$ , Lemma 5.4.2 provides the estimate

$$\|\mathbf{u}\|_\Omega \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)} \leq C (|c_u| + \|\boldsymbol{\zeta}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)}). \quad (5.65)$$

On the other hand, taking the divergence of  $\mathbf{curl} \mathbf{u} = c_u \varepsilon \nabla s^+ + \boldsymbol{\zeta}_u$ , we obtain  $c_u \operatorname{div}(\varepsilon \nabla s^+) = -\operatorname{div} \boldsymbol{\zeta}_u$ . Using the fact that  $A_\varepsilon^{\text{out}} : \dot{\mathbf{V}}^{\text{out}} \rightarrow (\dot{\mathbf{V}}_\beta^1(\Omega))^*$  is an isomorphism, we get

$$|c_u| \leq C \|\operatorname{div} \boldsymbol{\zeta}_u\|_{(\dot{\mathbf{V}}_\beta^1(\Omega))^*} \leq C \|\boldsymbol{\zeta}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)}.$$

Using this inequality in (5.65) leads to (5.64). ■

### 5.4.3 Main analysis for the magnetic field

Define the continuous operators  $\mathbb{A}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  and  $\mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  such that for all  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ ,  $\mathbf{v} \in \mathbf{Z}_T^\beta(\mu)$ ,

$$\langle \mathbb{A}_T^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx, \quad \langle \mathbb{K}_T^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} dx. \quad (5.66)$$

With this notation, we have  $\langle (\mathbb{A}_T^{\text{out}} - \omega^2 \mathbb{K}_T^{\text{out}}) \mathbf{u}, \mathbf{v} \rangle = a_T(\mathbf{u}, \mathbf{v})$ .

**Proposition 5.4.1.** *Under Assumptions 1–3, the operator  $\mathbb{A}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is an isomorphism.*

**Proof.** We have

$$\langle \mathbb{A}_T^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \varepsilon^{-1} \zeta_{\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}} dx, \quad \forall \mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu), \forall \mathbf{v} \in \mathbf{Z}_T^\beta(\mu).$$

Let us construct a continuous operator  $\mathbb{T} : \mathbf{Z}_T^\beta(\mu) \rightarrow \mathbf{Z}_T^{\text{out}}(\mu)$  such that

$$\langle \mathbb{A}_T^{\text{out}} \mathbb{T} \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} r^{2\beta} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{Z}_T^\beta(\mu). \quad (5.67)$$

Let  $\mathbf{u}$  be an element of  $\mathbf{Z}_T^\beta(\mu)$ . Then the field  $r^{2\beta} \varepsilon \mathbf{curl} \mathbf{u}$  belongs to  $\mathbf{V}_{-\beta}^0(\Omega)$ . Since  $A_\varepsilon^{\text{out}} : \mathring{\mathbf{V}}^{\text{out}} \rightarrow (\mathring{\mathbf{V}}_\beta^1(\Omega))^*$  is an isomorphism, there is a unique  $\varphi = \alpha s^+ + \tilde{\varphi} \in \mathring{\mathbf{V}}^{\text{out}}$  such that  $A_\varepsilon^{\text{out}} \varphi = -\text{div}(r^{2\beta} \varepsilon \mathbf{curl} \mathbf{u})$ . Observing that  $\mathbf{w} := r^{2\beta} \varepsilon \mathbf{curl} \mathbf{u} - \nabla \varphi \in \mathbf{V}_\beta^0(\Omega)$  is such that  $\text{div} \mathbf{w} = 0$  in  $\Omega$ , according to the result of Proposition 5.6.4, we know that there is a unique  $\boldsymbol{\psi} \in \mathbf{Z}_T^\beta(1)$  such that

$$\mathbf{curl} \boldsymbol{\psi} = \varepsilon (r^{2\beta} \mathbf{curl} \mathbf{u} - \nabla \varphi).$$

At this stage, we emphasize that in general  $\nabla \varphi \in \mathbf{V}_\beta^0(\Omega) \setminus \mathbf{L}^2(\Omega)$ . This is the reason why we are obliged to establish Proposition 5.6.4. Since  $\boldsymbol{\psi}$  is in  $\mathbf{L}^2(\Omega)$ , when  $A_\mu : \mathbf{H}_\#^1(\Omega) \rightarrow (\mathbf{H}_\#^1(\Omega))^*$  is an isomorphism, there is a unique  $\phi \in \mathbf{H}_\#^1(\Omega)$  such that

$$\int_{\Omega} \mu \nabla \phi \cdot \nabla \bar{\phi}' dx = \int_{\Omega} \mu \boldsymbol{\psi} \cdot \nabla \bar{\phi}' dx, \quad \forall \phi' \in \mathbf{H}_\#^1(\Omega).$$

Finally, we set  $\mathbb{T} \mathbf{u} = \boldsymbol{\psi} - \nabla \phi$ . It can be easily checked that this defines a continuous operator  $\mathbb{T} : \mathbf{Z}_T^\beta(\mu) \rightarrow \mathbf{Z}_T^{\text{out}}(\mu)$ . Moreover we have

$$\mathbf{curl} \mathbb{T} \mathbf{u} = \alpha \varepsilon \nabla s^+ + \zeta_{\mathbb{T} \mathbf{u}} \quad \text{with} \quad \zeta_{\mathbb{T} \mathbf{u}} = \varepsilon (r^{2\beta} \mathbf{curl} \mathbf{u} - \nabla \tilde{\varphi}).$$

As a consequence, indeed we have identity (5.67). From Lemma 5.4.2, we deduce that  $\mathbb{A}_T^{\text{out}} \mathbb{T} : \mathbf{Z}_T^\beta(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is an isomorphism, and so that  $\mathbb{A}_T^{\text{out}}$  is onto. It remains to show that  $\mathbb{A}_T^{\text{out}}$  is injective.

If  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$  is in the kernel of  $\mathbb{A}_T^{\text{out}}$ , we have  $\langle \mathbb{A}_T^{\text{out}} \mathbf{u}, \mathbf{v} \rangle = 0$  for all  $\mathbf{v} \in \mathbf{Z}_T^\beta(\mu)$ . In particular from (5.58), we can write

$$\langle \mathbb{A}_T^{\text{out}} \mathbf{u}, \mathbf{u} \rangle = \int_{\Omega} \varepsilon^{-1} |\zeta_{\mathbf{u}}|^2 dx + |c_{\mathbf{u}}|^2 \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{s^+} dx = 0.$$

Taking the imaginary part of the above identity, we obtain  $c_{\mathbf{u}} = 0$  (see the details in the proof of Proposition 5.4.3). We deduce that  $\mathbf{u}$  belongs to  $\mathbf{Z}_T^{-\beta}(\mu)$  and from (5.58), we infer that  $\langle \mathbb{A}_T^{\text{out}} \mathbf{u}, \mathbb{T} \mathbf{u} \rangle = \overline{\langle \mathbb{A}_T^{\text{out}} \mathbb{T} \mathbf{u}, \mathbf{u} \rangle}$ . This gives

$$0 = \int_{\Omega} r^{2\beta} |\mathbf{curl} \mathbf{u}|^2 dx = 0$$

and shows that  $\mathbf{u} = 0$ . ■

**Proposition 5.4.2.** *Under Assumptions 1–3, the embedding of the space  $\mathbf{Z}_T^{\text{out}}(\mu)$  in  $\mathbf{L}^2(\Omega)$  is compact. As a consequence, the operator  $\mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  defined in (5.66) is compact.*

**Proof.** Let  $(\mathbf{u}_n)$  be a sequence of elements of  $\mathbf{Z}_T^{\text{out}}(\mu)$  which is bounded. For all  $n \in \mathbb{N}$ , we have  $\mathbf{curl} \mathbf{u}_n = c_{\mathbf{u}_n} \varepsilon \nabla s^+ + \boldsymbol{\zeta}_{\mathbf{u}_n}$ . By definition of the norm of  $\mathbf{Z}_T^{\text{out}}(\mu)$ , the sequence  $(c_{\mathbf{u}_n})$  is bounded in  $\mathbb{C}$ . Let  $\mathbf{w}$  be an element of  $\mathbf{Z}_T^{\text{out}}(\mu)$  such that  $c_{\mathbf{w}} = 1$  (if such  $\mathbf{w}$  did not exist, then we would have  $\mathbf{Z}_T^{\text{out}}(\mu) = \mathbf{Z}_T^{-\beta}(\mu) \subset \mathbf{X}_T(\mu)$  and the proof would be even simpler). The sequence  $(\mathbf{u}_n - c_{\mathbf{u}_n} \mathbf{w})$  is bounded in  $\mathbf{X}_T(\mu)$ . Since this space is compactly embedded in  $\mathbf{L}^2(\Omega)$  when  $A_\mu : \mathbf{H}_\#^1(\Omega) \rightarrow (\mathbf{H}_\#^1(\Omega))^*$  is an isomorphism (see [24, Theorem 5.3]), we infer we can extract from  $(\mathbf{u}_n - c_{\mathbf{u}_n} \mathbf{w})$  a subsequence which converges in  $\mathbf{L}^2(\Omega)$ . Since clearly we can also extract a subsequence of  $(c_{\mathbf{u}_n})$  which converges in  $\mathbb{C}$ , this shows that we can extract from  $(\mathbf{u}_n)$  a subsequence which converges in  $\mathbf{L}^2(\Omega)$ . This shows that the embedding of  $\mathbf{Z}_T^{\text{out}}(\mu)$  in  $\mathbf{L}^2(\Omega)$  is compact.

Now observing that for all  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$ , we have

$$\|\mathbb{K}_T^{\text{out}} \mathbf{u}\|_{(\mathbf{Z}_T^\beta(\mu))^*} \leq C \|\mathbf{u}\|_\Omega,$$

we deduce that  $\mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is a compact operator.  $\blacksquare$

We can now state the main theorem of the analysis of the problem for the magnetic field.

**Theorem 5.4.1.** *Under Assumptions 1–3, for all  $\omega \in \mathbb{R}$  the operator  $\mathbb{A}_T^{\text{out}} - \omega^2 \mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is Fredholm of index zero.*

**Proof.** Since  $\mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is compact (Proposition 5.4.2) and  $\mathbb{A}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is an isomorphism (Proposition 5.4.1),  $\mathbb{A}_T^{\text{out}} - \omega^2 \mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is Fredholm of index zero.  $\blacksquare$

Finally we establish a result similar to Proposition 5.3.3 by using the formulation for the magnetic field.

**Proposition 5.4.3.** *Under Assumptions 1 and 3, if  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$  is a solution of (5.56) for  $\mathbf{J} = 0$ , then  $\mathbf{u} \in \mathbf{Z}_T^{-\gamma}(\mu) \subset \mathbf{X}_T(\mu)$  for all  $\gamma$  satisfying (5.13).*

**Proof.** Assume that  $\mathbf{u} \in \mathbf{Z}_T^{\text{out}}(\mu)$  satisfies

$$\int_\Omega \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx - \omega^2 \int_\Omega \mu \mathbf{u} \cdot \bar{\mathbf{v}} = 0, \quad \forall \mathbf{v} \in \mathbf{Z}_T^\beta(\mu).$$

Taking the imaginary part of this identity for  $\mathbf{v} = \mathbf{u}$ , since  $\omega$  is real, we get

$$\Im m \left( \int_\Omega \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{u}} \, dx \right) = 0.$$

If  $\mathbf{curl} \mathbf{u} = c_{\mathbf{u}} \varepsilon \nabla s^+ + \boldsymbol{\zeta}_{\mathbf{u}}$  with  $c_{\mathbf{u}} \in \mathbb{C}$  and  $\boldsymbol{\zeta}_{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$ , according to (5.58), this can be written as

$$|c_{\mathbf{u}}|^2 \Im m \left( \int_\Omega \operatorname{div}(\varepsilon \nabla s^+) \bar{s}^+ \, dx \right) = 0.$$

Then one concludes as in the proof of Proposition 5.3.3 that  $c_{\mathbf{u}} = 0$ , so that  $\mathbf{curl} \mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega)$ . Therefore we have  $\varepsilon^{-1} \mathbf{curl} \mathbf{u} \in \mathbf{X}_N(\varepsilon) \subset \mathbf{X}_N^{\text{out}}(\varepsilon)$ . From Lemma 5.3.1, we deduce that  $\varepsilon^{-1} \mathbf{curl} \mathbf{u} \in \mathbf{V}_{-\gamma}^0(\Omega)$  for all  $\gamma$  satisfying (5.13). This shows that  $\mathbf{u} \in \mathbf{Z}_T^{-\gamma}(\mu)$  for all  $\gamma$  satisfying (5.13).  $\blacksquare$

**Remark 5.4.4.** *Assume that  $\mathbf{J} \in \mathbf{V}_{-\gamma}^0(\Omega)$  for all  $\gamma$  satisfying (5.13). Assume also that zero is the only solution of (5.56) with  $\mathbf{J} = 0$  for a certain  $\beta_0$  satisfying (5.13). Then Theorem 5.4.1 and Proposition 5.4.3 guarantee that (5.56) is well-posed for all  $\gamma$  satisfying (5.13). Moreover Proposition 5.4.3 allows one to show that all the solutions of (5.56) for  $\gamma$  satisfying (5.13) coincide.*

**Remark 5.4.5.** From Lemmas 5.3.2 and 5.4.1, we infer that if  $\mathbf{u}$  solves the electric problem (5.22), then  $(i\omega\mu)^{-1}\mathbf{curl}\mathbf{u}$  is a solution of the magnetic problem (5.56). Conversely, if  $\mathbf{u}$  solves (5.56), then  $i(\omega\varepsilon)^{-1}(\mathbf{curl}\mathbf{u} - \mathbf{J})$  is a solution of (5.22). Therefore, under Assumptions 1–3, for all  $\omega \in \mathbb{R}$ , the operator  $\mathbb{A}_N^{\text{out}} - \omega^2\mathbb{K}_N^{\text{out}} : \mathbf{X}_N^{\text{out}}(\varepsilon) \rightarrow (\mathbf{X}_N^{\text{out}}(\varepsilon))^*$  is an isomorphism if and only if  $\mathbb{A}_T^{\text{out}} - \omega^2\mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^{\text{out}}(\mu))^*$  is an isomorphism.

#### 5.4.4 Analysis in the classical framework

In the previous paragraph, we proved that the formulation (5.56) for the magnetic field with a solution in  $\mathbf{Z}_T^{\text{out}}(\mu)$  and test functions in  $\mathbf{Z}_T^{\text{out}}(\mu)$  is well-posed. Here, we study the properties of the naive problem for the magnetic field set in the classical space  $\mathbf{X}_T(\mu)$ . More precisely, we consider the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_T(\mu) \text{ such that} \\ \int_{\Omega} \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \bar{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{X}_T(\mu). \end{array} \right. \quad (5.68)$$

Working as in the proof of Lemma 5.4.1, one shows that under Assumptions 1, 2, the field  $\mathbf{u}$  is a solution of (5.68) if and only if it solves the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}(\mathbf{curl}) \text{ such that} \\ \int_{\Omega} \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \bar{\mathbf{v}}, \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}). \end{array} \right. \quad (5.69)$$

Define the continuous operators  $\mathbb{A}_T : \mathbf{X}_T(\mu) \rightarrow (\mathbf{X}_T(\mu))^*$  and  $\mathbb{K}_T : \mathbf{X}_T(\mu) \rightarrow (\mathbf{X}_T(\mu))^*$  such that for all  $\mathbf{u} \in \mathbf{X}_T(\mu)$ ,  $\mathbf{v} \in \mathbf{X}_T(\mu)$ ,

$$\langle \mathbb{A}_T \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx, \quad \langle \mathbb{K}_T \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} \, dx.$$

As for  $\mathbb{A}_N$  and  $\mathbb{K}_N$ , we emphasize that these are the classical operators which appear in the analysis of the magnetic field, for example when  $\varepsilon$  and  $\mu$  are positive in  $\Omega$ .

**Proposition 5.4.4.** Under Assumptions 1–3, for all  $\omega \in \mathbb{C}$  the operator  $\mathbb{A}_T - \omega^2\mathbb{K}_T : \mathbf{X}_T(\mu) \rightarrow (\mathbf{X}_T(\mu))^*$  is not Fredholm.

**Proof.** From [24, Theorem 5.3 and Corollary 5.4], we know that under the Assumptions 1, 2, the embedding of  $\mathbf{X}_T(\mu)$  in  $\mathbf{L}^2(\Omega)$  is compact. This allows us to prove that  $\mathbb{K}_T : \mathbf{X}_T(\mu) \rightarrow (\mathbf{X}_T(\mu))^*$  is a compact operator. Therefore, it suffices to show that  $\mathbb{A}_T : \mathbf{X}_T(\mu) \rightarrow (\mathbf{X}_T(\mu))^*$  is not Fredholm. Let us work by contradiction assuming that  $\mathbb{A}_T$  is Fredholm. Since this operator is self-adjoint (it is symmetric and bounded), necessarily it is of index zero.

★ If  $\mathbb{A}_T$  is injective, then it is an isomorphism. Let us show that in this case,  $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  is an isomorphism (which is not the case by assumption). To proceed, we construct a continuous operator  $\mathbb{T} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that

$$\langle A_\varepsilon \varphi, \mathbb{T} \varphi' \rangle = \int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla (\overline{\mathbb{T} \varphi'}) \, dx = \int_{\Omega} \nabla \varphi \cdot \nabla \overline{\varphi'} \, dx, \quad \forall \varphi, \varphi' \in H_0^1(\Omega). \quad (5.70)$$

When  $\mathbb{A}_T$  is an isomorphism, for any  $\varphi' \in H_0^1(\Omega)$ , there is a unique  $\psi \in \mathbf{X}_T(\mu)$  such that

$$\int_{\Omega} \varepsilon^{-1} \mathbf{curl} \psi \cdot \mathbf{curl} \overline{\psi'} \, dx = \int_{\Omega} \varepsilon^{-1} \nabla \varphi' \cdot \mathbf{curl} \overline{\psi'} \, dx, \quad \forall \psi' \in \mathbf{X}_T(\mu).$$

Using item *iii*) of Proposition 5.6.1, one can show that there is a unique  $\mathbb{T} \varphi' \in H_0^1(\Omega)$  such that

$$\nabla(\mathbb{T} \varphi') = \varepsilon^{-1}(\nabla \varphi' - \mathbf{curl} \psi).$$

This defines our operator  $\mathbf{T} : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and one can verify that it is continuous. Moreover, integrating by parts, we indeed get (5.70) which guarantees, according to the Lax-Milgram theorem, that  $A_\varepsilon : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  is an isomorphism.

★ If  $\mathbb{A}_T$  is not injective, it has a kernel of finite dimension  $N \geq 1$  which coincides with  $\text{span}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N)$ , where  $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N \in \mathbf{X}_T(\mu)$  are linearly independent functions such that  $(\mathbf{curl} \boldsymbol{\lambda}_i, \mathbf{curl} \boldsymbol{\lambda}_j)_\Omega = \delta_{ij}$  (the Kronecker symbol). Introduce the space

$$\tilde{\mathbf{X}}_T(\mu) := \{\mathbf{u} \in \mathbf{X}_T(\mu) \mid (\mathbf{curl} \mathbf{u}, \mathbf{curl} \boldsymbol{\lambda}_i)_\Omega = 0, i = 1, \dots, N\}.$$

as well as the operator  $\tilde{\mathbb{A}}_T : \tilde{\mathbf{X}}_T(\mu) \rightarrow \tilde{\mathbf{X}}_T(\mu)$  such that

$$\langle \tilde{\mathbb{A}}_T \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \varepsilon^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \, dx, \quad \forall \mathbf{u}, \mathbf{v} \in \tilde{\mathbf{X}}_T(\mu).$$

Then  $\tilde{\mathbb{A}}_T$  is an isomorphism. Let us construct a new operator  $\mathbf{T} : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  to have something looking like (5.70). For a given  $\varphi' \in \mathbf{H}_0^1(\Omega)$ , introduce  $\boldsymbol{\psi} \in \tilde{\mathbf{X}}_T(\mu)$  the function such that

$$\int_\Omega \varepsilon^{-1} \mathbf{curl} \boldsymbol{\psi} \cdot \mathbf{curl} \bar{\boldsymbol{\psi}}' \, dx = \int_\Omega (\varepsilon^{-1} \nabla \varphi' - \sum_{i=1}^N \alpha_i \mathbf{curl} \boldsymbol{\lambda}_i) \cdot \mathbf{curl} \bar{\boldsymbol{\psi}}' \, dx, \quad \forall \boldsymbol{\psi}' \in \tilde{\mathbf{X}}_T(\mu), \quad (5.71)$$

where for  $i = 1, \dots, N$ , we have set  $\alpha_i := \int_\Omega \varepsilon^{-1} \nabla \varphi' \cdot \mathbf{curl} \boldsymbol{\lambda}_i \, dx$ . Observing that (5.71) is also valid for  $\boldsymbol{\psi}' = \boldsymbol{\lambda}_i$ ,  $i = 1, \dots, N$ , we infer that there holds

$$\int_\Omega \varepsilon^{-1} \mathbf{curl} \boldsymbol{\psi} \cdot \mathbf{curl} \bar{\boldsymbol{\psi}}' \, dx = \int_\Omega (\varepsilon^{-1} \nabla \varphi' - \sum_{i=1}^N \alpha_i \mathbf{curl} \boldsymbol{\lambda}_i) \cdot \mathbf{curl} \bar{\boldsymbol{\psi}}' \, dx, \quad \forall \boldsymbol{\psi}' \in \mathbf{X}_T(\mu).$$

Using again item *iii*) of Proposition 5.6.1, we deduce that there is a unique  $\mathbf{T}\varphi' \in \mathbf{H}_0^1(\Omega)$  such that

$$\nabla(\mathbf{T}\varphi') = \varepsilon^{-1}(\nabla \varphi' - \mathbf{curl} \boldsymbol{\psi}) - \sum_{i=1}^N \alpha_i \mathbf{curl} \boldsymbol{\lambda}_i.$$

This defines the new continuous operator  $\mathbf{T} : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$ . Then one finds

$$\langle A_\varepsilon \varphi, \mathbf{T}\varphi' \rangle = \int_\Omega \varepsilon \nabla \varphi \cdot \nabla(\overline{\mathbf{T}\varphi'}) \, dx = \int_\Omega \nabla \varphi \cdot \nabla \overline{\varphi'} \, dx - \sum_{i=1}^N \bar{\alpha}_i \int_\Omega \varepsilon \nabla \varphi \cdot \mathbf{curl} \bar{\boldsymbol{\lambda}}_i \, dx, \quad \forall \varphi, \varphi' \in \mathbf{H}_0^1(\Omega).$$

This shows that  $\mathbf{T}$  is a left parametrix for the self adjoint operator  $A_\varepsilon$ . Therefore,  $A_\varepsilon : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  is Fredholm of index zero. Note that then, one can verify that  $\dim \ker A_\varepsilon = \dim \ker \mathbb{A}_T$ . And more precisely, we have  $\ker A_\varepsilon = \text{span}(\gamma_1, \dots, \gamma_N)$  where  $\gamma_i \in \mathbf{H}_0^1(\Omega)$  is the function such that

$$\nabla \gamma_i = \varepsilon^{-1} \mathbf{curl} \boldsymbol{\lambda}_i$$

(existence and uniqueness of  $\gamma_i$  is again a consequence of item *iii*) of Proposition 5.6.1). But by assumption,  $A_\varepsilon$  is not a Fredholm operator. This ends the proof by contradiction. ■

**Remark 5.4.6.** In the article [24], it is proved that if  $A_\varepsilon : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  is an isomorphism (resp. a Fredholm operator of index zero), then  $\mathbb{A}_T : \mathbf{X}_T(1) \rightarrow (\mathbf{X}_T(1))^*$  is an isomorphism (resp. a Fredholm operator of index zero). Here we have established the converse statement.

**Remark 5.4.7.** We emphasize that the problems (5.38) for the electric field and (5.68) for the magnetic in the usual spaces  $\mathbf{X}_N(\varepsilon)$  and  $\mathbf{X}_T(\mu)$  have different properties. Problem (5.38) is well-posed but is not equivalent to the corresponding problem in  $\mathbf{H}_N(\mathbf{curl})$ , so that its solution in general is not a distributional solution of Maxwell's equations. On the contrary, problem (5.68) is equivalent to problem (5.69) in  $\mathbf{H}(\mathbf{curl})$  but it is not well-posed.

### 5.4.5 Expression of the singular coefficient

Under Assumptions 1–3, Theorem 5.4.1 guarantees that for all  $\omega \in \mathbb{R}$  the operator  $\mathbb{A}_T^{\text{out}} - \omega^2 \mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is Fredholm of index zero. Assuming that it is injective, the problem (5.56) admits a unique solution  $\mathbf{u}$  with  $\mathbf{curl} \mathbf{u} = c_u \varepsilon \nabla s^+ + \boldsymbol{\zeta}_u$ . As in §5.3.6, the goal of this paragraph is to derive a formula for the coefficient  $c_u$  which does not require to know  $\mathbf{u}$ .

For  $\omega \in \mathbb{R}$ , introduce the field  $\mathbf{w}_T \in \mathbf{Z}_T^\beta(\mu)$  such that

$$\int_{\Omega} \varepsilon^{-1} \boldsymbol{\zeta}_v \cdot \mathbf{curl} \overline{\mathbf{w}_T} dx - \omega^2 \int_{\Omega} \mu \mathbf{v} \cdot \overline{\mathbf{w}_T} dx = \int_{\Omega} \boldsymbol{\zeta}_v \cdot \nabla \overline{s^+} dx, \quad \forall \mathbf{v} \in \mathbf{Z}_T^{\text{out}}(\mu). \quad (5.72)$$

Note that  $\mathbf{w}_T$  is well-defined because  $(\mathbb{A}_T^{\text{out}} - \omega^2 \mathbb{K}_T^{\text{out}})^* : \mathbf{Z}_T^\beta(\mu) \rightarrow (\mathbf{Z}_T^{\text{out}}(\mu))^*$  is an isomorphism.

**Theorem 5.4.2.** *Assume that  $\omega \in \mathbb{R}$ , Assumptions 1–3 are valid and  $\mathbb{A}_T^{\text{out}} - \omega^2 \mathbb{K}_T^{\text{out}} : \mathbf{Z}_T^{\text{out}}(\mu) \rightarrow (\mathbf{Z}_T^\beta(\mu))^*$  is injective. Let  $\mathbf{u}$  denote the solution of the magnetic problem (5.56). Then the coefficient  $c_u$  in the decomposition  $\mathbf{curl} \mathbf{u} = c_u \varepsilon \nabla s^+ + \boldsymbol{\zeta}_u$  is given by the formula*

$$c_u = i\omega \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \overline{\mathbf{w}_T} dx \Big/ \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{s^+} dx. \quad (5.73)$$

Here  $\mathbf{w}_T$  is the function which solves (5.72).

**Proof.** By definition of  $\mathbf{u}$ , we have

$$\int_{\Omega} \varepsilon^{-1} \boldsymbol{\zeta}_u \cdot \mathbf{curl} \overline{\mathbf{w}_T} dx - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \overline{\mathbf{w}_T} dx = i\omega \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \overline{\mathbf{w}_T} dx.$$

On the other hand, from (5.72), we can write

$$\int_{\Omega} \varepsilon^{-1} \boldsymbol{\zeta}_u \cdot \mathbf{curl} \overline{\mathbf{w}_T} dx - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \overline{\mathbf{w}_T} dx = \int_{\Omega} \boldsymbol{\zeta}_u \cdot \nabla \overline{s^+} dx.$$

From these two relations, using (5.58), we deduce that

$$i\omega \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \overline{\mathbf{w}_T} dx = \int_{\Omega} \boldsymbol{\zeta}_u \cdot \nabla \overline{s^+} dx = c_u \int_{\Omega} \text{div}(\varepsilon \nabla s^+) \overline{s^+} dx.$$

This gives (5.73). ■

## 5.5 Conclusion

In this work, we studied the Maxwell equations in presence of hypersingularities for the scalar problem involving  $\varepsilon$ . We considered both the problem for the electric field and for the magnetic field. Quite naturally, in order to obtain a framework where well-posedness holds, it is necessary to modify the spaces in different ways. More precisely, we changed the condition on the field itself for the electric problem and on the curl of the field for the magnetic problem. A noteworthy difference in the analysis of the two problems is that for the electric field, the searched solution and the test function in the corresponding sesquilinear form belong to the same space, whereas for the magnetic field we have not been able to do so. We do not know what are the numerical consequences of this difference.

Of course, we could have assumed that the scalar problem involving  $\varepsilon$  is well-posed in  $H_0^1(\Omega)$  and that hypersingularities exist for the problem in  $\mu$ . This would have been similar mathematically. Physically, however, this situation seems to be a bit less relevant because it is harder to obtain negative  $\mu$  without dissipation. More precisely, materials having an  $\varepsilon$  with a negative real part can be found easily in nature (metals for certain ranges of frequencies) and additionally they

can be very weakly dissipative (small imaginary part of  $\varepsilon$ ). On the other hand, only certain artificially designed metamaterials, made of small resonators, behave macroscopically, after an homogenization process, as homogeneous materials with a  $\mu$  having a negative real part. But for the moment, dissipation for these metamaterials still remains very important.

We assumed that the domain  $\Omega$  is simply connected and that  $\partial\Omega$  is connected. When these assumptions are not met, it is necessary to adapt the analysis (see §8.2 of [24] for the study in the case where the scalar problems are well-posed in the usual  $H^1$  framework). This has to be done. Moreover, for the conical tip, at least numerically, one finds that several singularities can exist (actually this number can be as high as we wish for a contrast close enough to  $-1$ , see the calculations in [96]). In this case, the analysis should follow the same lines but this has to be written.

On the other hand, in this work, we focused our attention on a situation where the interface between the positive and the negative material has a conical tip. It would be interesting to study a setting where there is a wedge instead. In this case, roughly speaking, one should deal with a continuum of singularities. We have to mention that the analysis of the scalar problems for a wedge of negative material in the non standard framework has not been done. Finally, considering a conical tip with both critical  $\varepsilon$  and  $\mu$  is a direction that we are investigating.

## 5.6 Appendix

### 5.6.1 Vector potentials, part 1

**Proposition 5.6.1.** *Under Assumption 1, the following assertions hold.*

i) According to [8, Theorem 3.12], if  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  satisfies  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ , then there exists a unique  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  such that  $\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}$ .

ii) According to [8, Theorem 3.17]), if  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  satisfies  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$  and  $\mathbf{u} \cdot \boldsymbol{\nu} = 0$  on  $\partial\Omega$ , then there exists a unique  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$  such that  $\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}$ .

iii) If  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  satisfies  $\mathbf{curl} \mathbf{u} = 0$  in  $\Omega$  and  $\mathbf{u} \times \boldsymbol{\nu} = 0$  on  $\partial\Omega$ , then there exists (see [110, Theorem 3.41]) a unique  $p \in H_0^1(\Omega)$  such that  $\mathbf{u} = \nabla p$ .

iv) Every  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  can be decomposed as follows ([110, Theorem 3.45])

$$\mathbf{u} = \nabla p + \mathbf{curl} \boldsymbol{\psi},$$

with  $p \in H_0^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  which are uniquely defined.

v) Every  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  can be decomposed as follows ([110, Remark 3.46])

$$\mathbf{u} = \nabla p + \mathbf{curl} \boldsymbol{\psi},$$

with  $p \in H_{\#}^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$  which are uniquely defined.

**Proposition 5.6.2.** *Under Assumption 1, if  $\boldsymbol{\psi}$  satisfies one of the following conditions*

i)  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$  and  $\Delta \boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$ ,

ii)  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$ ,  $\mathbf{curl} \boldsymbol{\psi} \times \boldsymbol{\nu} = 0$  on  $\partial\Omega$  and  $\Delta \boldsymbol{\psi} \in \mathbf{L}^2(\Omega)$ ,

then for all  $\beta < 1/2$ , we have  $\mathbf{curl} \boldsymbol{\psi} \in \mathbf{V}_{-\beta}^0(\Omega)$  and there is a constant  $C > 0$  independent of  $\boldsymbol{\psi}$  such that

$$\|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\Delta \boldsymbol{\psi}\|_{\Omega}. \quad (5.74)$$



**Proof.** It suffices to prove the result for  $\beta \in (0; 1/2)$ . Let  $\psi \in \mathbf{X}_N(1) \cup \mathbf{X}_T(1)$ . Since  $\mathbf{curl} \mathbf{curl} \psi = -\Delta \psi$ , integrating by parts we get

$$\|\mathbf{curl} \psi\|_{\Omega}^2 = - \int_{\Omega} \Delta \psi \cdot \bar{\psi} dx.$$

Note that the boundary term vanishes because either  $\psi \times \nu = 0$  or  $\mathbf{curl} \psi \times \nu = 0$  on  $\partial\Omega$ . This furnishes the estimate

$$\|\mathbf{curl} \psi\|_{\Omega} \leq C \|\Delta \psi\|_{\Omega}. \quad (5.75)$$

Now working with cut-off functions, we refine the estimate at the origin to get (5.74).

Let us consider a smooth cut-off function  $\chi$ , compactly supported in  $\Omega$ , equal to one in a neighbourhood of  $O$ . To prove the proposition, it suffices in addition to (5.75) to prove that  $\mathbf{curl}(\chi\psi) \in \mathbf{V}_{-\beta}^0(\Omega)$  together with the following estimate  $\|\mathbf{curl}(\chi\psi)\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\Delta \psi\|_{\Omega}$ .

First of all, since  $\mathbf{curl}(\chi\psi) \in \mathbf{L}^2(\Omega)$  and  $\operatorname{div}(\chi\psi) = \nabla \chi \cdot \psi \in \mathbf{L}^2(\Omega)$ , we know that  $\chi\psi_i \in \mathbf{H}_0^1(\Omega)$  for  $i = 1, 2, 3$  and we have

$$\|\mathbf{curl}(\chi\psi)\|_{\Omega}^2 + \|\operatorname{div}(\chi\psi)\|_{\Omega}^2 = \sum_{i=1}^3 \|\nabla(\chi\psi_i)\|_{\Omega}^2.$$

From the previous identity, (5.75) and Proposition 7.2.1, we deduce

$$\left( \|\psi\|_{\Omega}^2 + \sum_{i=1}^3 \|\nabla(\chi\psi_i)\|_{\Omega}^2 \right)^{1/2} \leq C \|\Delta \psi\|_{\Omega}. \quad (5.76)$$

Note that, (5.76) is also valid if we replace  $\chi$  by any other smooth function with compact support in  $\Omega$ . Now setting  $f_i = \Delta(\chi\psi_i)$  for  $i = 1, 2, 3$ , we have

$$f_i = \chi \Delta \psi_i + 2 \nabla \chi \cdot \nabla \psi_i + \psi_i \Delta \chi. \quad (5.77)$$

By writing that  $\nabla \chi \cdot \nabla \psi_i = \operatorname{div}(\psi_i \nabla \chi) - \psi_i \Delta \chi$  and replacing  $\chi$  by  $\partial_j \chi$  in (5.76) for  $j = 1, 2, 3$ , we deduce that for  $i = 1, 2, 3$ ,  $f_i$  belongs to  $\mathbf{L}^2(\Omega)$  and satisfies

$$\|f_i\|_{\Omega} \leq C \|\Delta \psi\|_{\Omega}.$$

Note that since  $\beta \in (0; 1/2)$ , we have  $\mathring{\mathbf{V}}_{\beta}^1(\Omega) \subset \mathbf{V}_{\beta-1}^0 \subset \mathbf{L}^2(\Omega)$  and so  $\mathbf{L}^2(\Omega) \subset (\mathring{\mathbf{V}}_{\beta}^1(\Omega))^*$ . Now starting from the fact that  $\chi\psi_i \in \mathbf{H}_0^1(\Omega)$  in addition to  $\Delta(\chi\psi_i) = f_i \in \mathbf{L}^2(\Omega) \subset (\mathring{\mathbf{V}}_{\beta}^1(\Omega))^*$ , by applying Proposition 6.2.1, we deduce that  $\chi\psi_i \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega)$  with the estimate

$$\|\chi\psi_i\|_{\mathring{\mathbf{V}}_{-\beta}^1(\Omega)} \leq C \|f_i\|_{(\mathring{\mathbf{V}}_{\beta}^1(\Omega))^*} \leq C \|f_i\|_{\Omega}.$$

As a consequence,  $\mathbf{curl}(\chi\psi) \in \mathbf{V}_{-\beta}^0(\Omega)$  and

$$\|\mathbf{curl}(\chi\psi)\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \sum_{i=1}^3 \|\chi\psi_i\|_{\mathring{\mathbf{V}}_{-\beta}^1(\Omega)} \leq \sum_{i=1}^3 \|f_i\|_{\Omega} \leq C \|\Delta \psi\|_{\Omega},$$

which concludes the proof. ■

**Proposition 5.6.3.** *Under Assumption 1, the following assertions hold:*

- i) if  $(\psi_n)$  is a bounded sequence of elements of  $\mathbf{X}_N(1)$  such that  $(\Delta \psi_n)$  is bounded in  $\mathbf{L}^2(\Omega)$ , then one can extract a subsequence such that  $(\mathbf{curl} \psi_n)$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in (0; 1/2)$ ;
- ii) if  $(\psi_n)$  is a bounded sequence of elements of  $\mathbf{X}_T(1)$  such that  $\mathbf{curl} \psi_n \times \nu = 0$  on  $\partial\Omega$  and such that  $(\Delta \psi_n)$  is bounded in  $\mathbf{L}^2(\Omega)$ , then one can extract a subsequence such that  $(\mathbf{curl} \psi_n)$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in (0; 1/2)$ .

**Proof.** Let us establish the first assertion, the proof of the second one being similar. Let  $(\psi_n)$  be a bounded sequence of elements of  $\mathbf{X}_N(1)$  such that  $(\Delta\psi_n)$  is bounded in  $\mathbf{L}^2(\Omega)$ . Observing that  $\mathbf{curl}\mathbf{curl}\psi_n = -\Delta\psi_n$ , we deduce that  $(\mathbf{curl}\psi_n)$  is a bounded sequence of  $\mathbf{X}_T(1)$ . Since the spaces  $\mathbf{X}_N(1)$  and  $\mathbf{X}_T(1)$  are compactly embedded in  $\mathbf{L}^2(\Omega)$  (see Proposition 7.2.1), one can extract a subsequence such that both  $(\psi_n)$  and  $(\mathbf{curl}\psi_n)$  converge in  $\mathbf{L}^2(\Omega)$ .

Then, working as in the proof of Proposition 5.6.2, we can show that for a smooth cut-off function  $\chi$  compactly supported in  $\Omega$  and equal to one in a neighbourhood of  $O$ , the sequence  $(\chi\psi_n)$  is bounded in  $\mathbf{V}_\gamma^2(\Omega) := (\mathbf{V}_\gamma^2(\Omega))^3$  for all  $\gamma > 1/2$ . To obtain this result, we use in particular the fact that if  $\mathcal{O} \subset \mathbb{R}^3$  is a smooth bounded domain such that  $O \in \mathcal{O}$ , then  $\Delta : \mathbf{V}_\gamma^2(\mathcal{O}) \cap \mathring{\mathbf{V}}_{\gamma-1}^1(\mathcal{O}) \rightarrow \mathbf{V}_\gamma^0(\mathcal{O})$  is an isomorphism for all  $\gamma \in (1/2; 3/2)$  (see [107, §1.6.2]). Finally, to conclude to the result of the proposition, we use the fact  $\mathbf{V}_\gamma^2(\mathcal{O})$  is compactly embedded in  $\mathbf{V}_{\gamma'}^1(\mathcal{O})$  as soon as  $\gamma - 1 < \gamma'$  ([102, Lemma 6.2.1]). This allows us to prove that for all  $\beta < 1/2$ , the subsequence  $(\chi\psi_n)$  converges in  $\mathbf{V}_{-\beta}^1(\Omega)$ , so that  $(\mathbf{curl}\psi_n)$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . ■

The next two lemmas are results of additional regularity for the elements of classical Maxwell's spaces that are direct consequences of Propositions 5.6.2 and 5.6.3.

**Lemma 5.6.1.** *Under Assumption 1, for all  $\beta \in (0; 1/2)$ ,  $\mathbf{X}_T(1)$  is compactly embedded in  $\mathbf{V}_{-\beta}^0(\Omega)$ . In particular, there is a constant  $C > 0$  such that*

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl}\mathbf{u}\|_\Omega, \quad \forall \mathbf{u} \in \mathbf{X}_T(1). \quad (5.78)$$

**Proof.** Let  $\mathbf{u}$  be an element of  $\mathbf{X}_T(1)$ . From the item *ii*) of Proposition 5.6.1, we know that there exists  $\psi \in \mathbf{X}_N(1)$  such that  $\mathbf{u} = \mathbf{curl}\psi$ . Using that  $-\Delta\psi = \mathbf{curl}\mathbf{u} \in \mathbf{L}^2(\Omega)$ , from Proposition 5.6.2, we get that  $\mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega)$  together with the estimate

$$\|\mathbf{curl}\psi\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl}\mathbf{u}\|_\Omega.$$

This gives (5.78). Now suppose that  $(\mathbf{u}_n)$  is a bounded sequence of elements of  $\mathbf{X}_T(1)$ . Then there exists a bounded sequence  $(\psi_n)$  of elements of  $\mathbf{X}_N(1)$  such that  $\mathbf{u}_n = \mathbf{curl}\psi_n$ . Since  $(\mathbf{curl}\mathbf{u}_n = -\Delta\psi_n)$  is bounded in  $\mathbf{L}^2(\Omega)$ , the first item of Proposition 5.6.3 implies that there is a subsequence such that  $(\mathbf{u}_n)$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . ■

**Lemma 5.6.2.** *Under Assumption 1, for all  $\beta \in (0; 1/2)$ ,  $\mathbf{X}_N(1)$  is compactly embedded in  $\mathbf{V}_{-\beta}^0(\Omega)$ . In particular, there is a constant  $C > 0$  such that*

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl}\mathbf{u}\|_\Omega, \quad \forall \mathbf{u} \in \mathbf{X}_N(1).$$

**Proof.** The proof is similar to the one of Lemma 5.6.1. ■

## 5.6.2 Vector potentials, part 2

First we establish an intermediate lemma which can be seen as a result of well-posedness for Maxwell's equations in weighted spaces with  $\varepsilon = \mu = 1$  in  $\Omega$ . Define the continuous operator  $\mathbb{B}_T : \mathbf{Z}_T^\beta(1) \rightarrow (\mathbf{Z}_T^{-\beta}(1))^*$  such that for all  $\psi \in \mathbf{Z}_T^\beta(1)$ ,  $\psi' \in \mathbf{Z}_T^{-\beta}(1)$ ,

$$\langle \mathbb{B}_T\psi, \psi' \rangle = \int_\Omega \mathbf{curl}\psi \cdot \mathbf{curl}\overline{\psi'} dx.$$

**Lemma 5.6.3.** *Under Assumption 1, for  $0 \leq \beta < 1/2$ , the operator  $\mathbb{B}_T : \mathbf{Z}_T^\beta(1) \rightarrow (\mathbf{Z}_T^{-\beta}(1))^*$  is an isomorphism.*

**Proof.** Let  $\psi$  be an element of  $\mathbf{Z}_T^\beta(1)$ . According to Proposition 6.2.1, there is a unique  $\varphi \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega)$  such that

$$\int_\Omega \nabla\varphi \cdot \nabla\overline{\varphi'} dx = \int_\Omega r^{2\beta} \mathbf{curl}\psi \cdot \nabla\overline{\varphi'} dx, \quad \forall \varphi' \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega).$$

Then denote  $\mathbb{T}\psi \in \mathbf{Z}_T^{-\beta}(1)$  the function such that

$$\mathbf{curl}(\mathbb{T}\psi) = r^{2\beta} \mathbf{curl} \psi - \nabla \varphi.$$

Observe that  $\mathbb{T}\psi$  is well-defined according to the item *i*) of Proposition 5.6.1. This defines a continuous operator  $\mathbb{T} : \mathbf{Z}_T^\beta(1) \rightarrow \mathbf{Z}_T^{-\beta}(1)$ . We have

$$\langle \mathbb{B}_T \psi, \mathbb{T}\psi \rangle = \int_{\Omega} \mathbf{curl} \psi \cdot \overline{\mathbf{curl}(\mathbb{T}\psi)} dx = \|r^{2\beta} \mathbf{curl} \psi\|_{\Omega}^2 = \|\mathbf{curl} \psi\|_{\mathbf{V}_{\beta}^0(\Omega)}^2.$$

Adapting the proof of Lemma 5.4.2, one can show that  $\|\mathbf{curl} \cdot\|_{\mathbf{V}_{\beta}^0(\Omega)}$  is a norm which is equivalent to the natural norm of  $\mathbf{Z}_T^\beta(1)$ . Therefore, from the Lax-Milgram theorem, we infer that  $\mathbb{T}^* \mathbb{B}_T$  is an isomorphism which shows that  $\mathbb{B}_T$  is injective and that its image is closed in  $(\mathbf{Z}_T^{-\beta}(1))^*$ . And from that, we deduce that  $\mathbb{B}_T$  is onto if and only if its adjoint is injective. The adjoint of  $\mathbb{B}_T$  is the operator  $\mathbb{B}_T^* : \mathbf{Z}_T^{-\beta}(1) \rightarrow (\mathbf{Z}_T^\beta(1))^*$  such that for all  $\psi \in \mathbf{Z}_T^\beta(1)$ ,  $\psi' \in \mathbf{Z}_T^\beta(1)$ ,

$$\langle \mathbb{B}_T^* \psi, \psi' \rangle = \int_{\Omega} \mathbf{curl} \psi \cdot \mathbf{curl} \overline{\psi'} dx. \quad (5.79)$$

If  $\mathbb{B}_T^* \psi = 0$ , then taking  $\psi' = \psi \in \mathbf{Z}_T^\beta(1) \subset \mathbf{Z}_T^\beta(1)$  in (5.79), we obtain  $\|\mathbf{curl} \psi\|_{\Omega} = 0$ . Since  $\mathbf{Z}_T^\beta(1) \subset \mathbf{X}_T(1)$  and  $\|\mathbf{curl} \cdot\|_{\Omega}$  is a norm in  $\mathbf{X}_T(1)$  (Proposition 7.2.1), we deduce that  $\psi = 0$ . This shows that  $\mathbb{B}_T^*$  is injective and that  $\mathbb{B}_T$  is an isomorphism.  $\blacksquare$

Now we use the above lemma to prove the following result which is essential in the analysis of the Problem (5.56) for the magnetic field. This is somehow an extension of the result of item *i*) of Proposition 5.6.1 for singular fields which are not in  $\mathbf{L}^2(\Omega)$ .

**Proposition 5.6.4.** *Under Assumption 1, for all  $0 \leq \beta < 1/2$ , if  $\mathbf{u} \in \mathbf{V}_{\beta}^0(\Omega)$  satisfies  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ , then there exists a unique  $\psi \in \mathbf{Z}_T^\beta(1)$  such that  $\mathbf{u} = \mathbf{curl} \psi$ .*

**Proof.** Let  $\mathbf{u} \in \mathbf{V}_{\beta}^0(\Omega)$  be such that  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ . According to Lemma 5.6.3, we know that there is a unique  $\psi \in \mathbf{Z}_T^\beta(1)$  such that

$$\int_{\Omega} \mathbf{curl} \psi \cdot \mathbf{curl} \overline{\psi'} dx = \int_{\Omega} \mathbf{u} \cdot \mathbf{curl} \overline{\psi'} dx, \quad \forall \psi' \in \mathbf{Z}_T^{-\beta}(1).$$

Then we have

$$\int_{\Omega} (\mathbf{u} - \mathbf{curl} \psi) \cdot \mathbf{curl} \overline{\psi'} dx = 0, \quad \forall \psi' \in \mathbf{Z}_T^{-\beta}(1). \quad (5.80)$$

Since  $\mathbf{u}$  is divergence free in  $\Omega$ , we also have

$$\int_{\Omega} (\mathbf{u} - \mathbf{curl} \psi) \cdot \nabla \overline{p'} dx = 0, \quad \forall p' \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega). \quad (5.81)$$

Now if  $\mathbf{v}$  is an element of  $\mathbf{V}_{-\beta}^0(\Omega) \subset \mathbf{L}^2(\Omega)$ , from item *iv*) of Proposition 5.6.1, we know that there holds the decomposition

$$\mathbf{v} = \nabla p' + \mathbf{curl} \psi', \quad (5.82)$$

for some  $p' \in \mathbf{H}_0^1(\Omega)$  and some  $\psi' \in \mathbf{X}_T(1)$ . Taking the divergence in (5.82), we get

$$\Delta p' = \operatorname{div} \mathbf{v} \in (\mathring{\mathbf{V}}_{\beta}^1(\Omega))^*. \quad (5.83)$$

From Proposition 6.2.1, since  $0 \leq \beta < 1/2$ , we know that (5.83) admits a solution in  $\mathring{\mathbf{V}}_{-\beta}^1(\Omega) \subset \mathbf{H}_0^1(\Omega)$ . Using uniqueness of the solution of (5.83) in  $\mathbf{H}_0^1(\Omega)$ , we obtain that  $p' \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega)$ . This implies that  $\mathbf{curl} \psi' = \mathbf{v} - \nabla p' \in \mathbf{V}_{-\beta}^0(\Omega)$  and so  $\psi' \in \mathbf{Z}_T^{-\beta}(1)$ . From (5.80) and (5.81), we infer that

$$\int_{\Omega} (\mathbf{u} - \mathbf{curl} \psi) \cdot \overline{\mathbf{v}} dx = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{-\beta}^0(\Omega).$$

This shows that  $\mathbf{u} = \mathbf{curl} \psi$ . Finally, if  $\psi_1, \psi_2$  are two elements of  $\mathbf{Z}_T^\beta(1)$  such that  $\mathbf{u} = \mathbf{curl} \psi_1 = \mathbf{curl} \psi_2$ , then  $\psi_1 - \psi_2$  belongs to  $\mathbf{X}_T(1)$  and satisfies  $\mathbf{curl}(\psi_1 - \psi_2) = 0$  in  $\Omega$ . From Proposition 7.2.1, we deduce that  $\psi_1 = \psi_2$ .  $\blacksquare$

### 5.6.3 Dimension of $\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)$

**Lemma 5.6.4.** *Under Assumptions 1–3, we have  $\dim(\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)) = 1$ .*

**Proof.** If  $\mathbf{u}_1 = c_1 \nabla s^+ + \tilde{\mathbf{u}}_1$ ,  $\mathbf{u}_2 = c_2 \nabla s^+ + \tilde{\mathbf{u}}_2$  are two elements of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ , then  $c_2 \mathbf{u}_1 - c_1 \mathbf{u}_2 \in \mathbf{X}_N(\varepsilon)$ , which shows that  $\dim(\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)) \leq 1$ .

Now let us prove that  $\dim(\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)) \geq 1$ . Introduce  $\tilde{\mathbf{s}} \in \mathring{\mathbf{V}}^{\text{out}}$  the function such that  $A_\varepsilon^{\text{out}} \tilde{\mathbf{s}} = \text{div}(\varepsilon \nabla s^-)$ . Note that since  $\text{div}(\varepsilon \nabla s^-)$  vanishes in a neighbourhood of the origin, it belongs to  $(\mathring{\mathbf{V}}_\gamma^1(\Omega))^*$  for all  $\gamma \in \mathbb{R}$ . Then set

$$\mathbf{s} = s^- + \tilde{\mathbf{s}}. \quad (5.84)$$

Observe that  $\mathbf{s} \in \mathring{\mathbf{V}}_\gamma^1(\Omega)$  for all  $\gamma > 0$  and that  $\text{div}(\varepsilon \nabla \mathbf{s}) = 0$  in  $\Omega \setminus \{O\}$  ( $\mathbf{s}$  is a non zero element of  $\ker A_\varepsilon^\gamma$  for all  $\gamma > 0$ ). Let  $\tilde{\mathbf{u}} \in (\mathcal{C}_0^\infty(\Omega \setminus \{O\}))^3$  be a field such that  $\int_\Omega \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{s}} \, dx \neq 0$ . The existence of such a  $\tilde{\mathbf{u}}$  can be established thanks to the density of  $(\mathcal{C}_0^\infty(\Omega \setminus \{O\}))^3$  in  $\mathbf{L}^2(\Omega)$ , considering for example an approximation of  $\mathbf{1}_{\bar{B}} \nabla \mathbf{s} \in \mathbf{L}^2(\Omega)$  where  $\mathbf{1}_{\bar{B}}$  is the indicator function of a ball included in  $\mathcal{M}$ . Introduce  $\zeta = c s^+ + \tilde{\zeta} \in \mathring{\mathbf{V}}^{\text{out}}$ , with  $c \in \mathbb{C}$ ,  $\tilde{\zeta} \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega)$ , the function such that  $A_\varepsilon^{\text{out}} \zeta = -\text{div}(\varepsilon \tilde{\mathbf{u}})$ . This is equivalent to have

$$-c \int_\Omega \text{div}(\varepsilon \nabla s^+) \bar{\varphi}' \, dx + \int_\Omega \varepsilon \nabla \tilde{\zeta} \cdot \nabla \bar{\varphi}' \, dx = \int_\Omega \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi}' \, dx, \quad \forall \varphi' \in \mathring{\mathbf{V}}_\beta^1(\Omega).$$

Clearly  $\nabla \zeta - \tilde{\mathbf{u}} = c \nabla s^+ + (\nabla \tilde{\zeta} - \tilde{\mathbf{u}})$  is an element of  $\mathbf{X}_N^{\text{out}}(\varepsilon)$ . Moreover taking  $\varphi' = \mathbf{s}$  above, we get

$$-c \int_\Omega \text{div}(\varepsilon \nabla s^+) \bar{\mathbf{s}} \, dx = \int_\Omega \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{s}} \, dx \neq 0.$$

This shows that  $c \neq 0$  and guarantees that  $\dim(\mathbf{X}_N^{\text{out}}(\varepsilon)/\mathbf{X}_N(\varepsilon)) \geq 1$ . ■

# Chapter 6

## Maxwell's equations with hypersingularities at a conical plasmonic tip: the case of two critical coefficients

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## 6.1 Introduction

In the previous chapter, we studied time harmonic Maxwell's problems in the presence of a conical tip of a negative material. More precisely, we studied the case where the function  $\varepsilon$  is critical (i.e. the scalar problem associated to  $\varepsilon$  is ill-posed in  $H_0^1(\Omega)$  because of the existence of propagating singularities) and where the function  $\mu$  is not critical (i.e. the scalar problem associated to  $\mu$  is well-posed in  $H_{\#}^1(\Omega)$ ). We have proved that the classical functional frameworks for the study of Maxwell's problems are no longer appropriate. More importantly, we have explained how to construct new functional frameworks in which the electric and magnetic problems are again well-posed. These functional frameworks have been constructed by making use of the weighted Sobolev spaces and cleverly taking into account the existence of propagating singularities of the scalar problem associated to  $\varepsilon$ . The justification of the adequacy with the physical reality (of the solutions obtained in these new functional frameworks) has been achieved thanks to the limiting absorption principle.

In this chapter, we are interested in studying the case where both functions  $\varepsilon$  and  $\mu$  are critical. From what has been done, in the previous chapter, we expect that, in this configuration, the classical frameworks are not suitable for the study of Maxwell problems either (this will be confirmed in §6.3). Our goal is then to explain how to construct adapted functional frameworks (that are coherent with the limiting absorption principle) that take into account both propagating singularities generated by the scalar problems associated to  $\varepsilon$  and  $\mu$ . As with the other chapters in this thesis, we will try to make this chapter self-contained (so it can be read independently of the previous one).

The plan of our work is the following. In §6.2, we start by recalling some results, which we will need, concerning scalar problems with critical coefficients. Then, in §6.3, we prove that the classical approach to study electrical and magnetic problems is no longer valid. The construction of new adapted functional frameworks for the electric problem and the magnetic problem and the study of their well-posedness in these new functional frameworks are, respectively, the object of §6.4 and §6.5. The last section is devoted to give a few words of conclusion.

## 6.2 Setting of the problem and study of the scalar problems with critical coefficients

The geometry considered is the same as in the previous chapter. Let  $\Omega$  be an open, simply connected and bounded subset of  $\mathbb{R}^3$  with Lipschitz-continuous boundary  $\partial\Omega$ . To simplify the analysis below, we shall suppose that  $\partial\Omega$  is connected. When this hypothesis is not satisfied all our results can be adapted by working as in [22, §8.2]. In  $\Omega$ , we define the piecewise constant functions  $\varepsilon$  and  $\mu$  such that

$$\varepsilon = \begin{cases} \varepsilon^+ > 0 & \text{in } \Omega \setminus \mathcal{M} \\ \varepsilon^- < 0 & \text{in } \mathcal{M} \end{cases}, \quad \mu = \begin{cases} \mu^+ > 0 & \text{in } \Omega \setminus \mathcal{M} \\ \mu^- < 0 & \text{in } \mathcal{M} \end{cases}$$

in which  $\mathcal{M}$  is a subdomain of  $\Omega$  satisfying  $\overline{\mathcal{M}} \subset \Omega$ . We suppose that  $\partial\mathcal{M}$  is of class  $\mathcal{C}^2$  except at the origin  $O = (0, 0, 0)$  where  $\mathcal{M}$  coincides, locally, with the cone  $\mathcal{K}$  such that

$$\mathcal{M} \cap \mathcal{K} = \mathcal{K} \cap B(O, \rho) = \left\{ x \in \mathbb{R}^3, |x| < \rho, \frac{x}{|x|} \in \mathcal{A} \right\}; \quad \mathcal{K} = \left\{ x \in \mathbb{R}^3, \frac{x}{|x|} \in \mathcal{A} \right\},$$

in which  $B(O, \rho)$  is the open ball of  $\mathbb{R}^3$  of center  $O$  and of radius  $\rho$  sufficiently small and  $\mathcal{A}$  is a smooth sub-domain of the unit sphere of  $\mathbb{R}^3$  (see Figure 6.1). The contrasts associated to  $\varepsilon$  and  $\mu$  are, respectively, defined by  $\kappa_\varepsilon := \varepsilon^- / \varepsilon^+$  and  $\kappa_\mu := \mu^- / \mu^+$ .

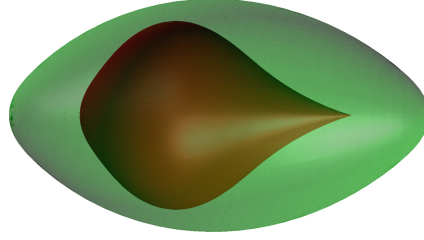


Figure 6.1: An example of the geometry considered in which  $\mathcal{M}$  is represented in red and  $\Omega \setminus \overline{\mathcal{M}}$  is in green.

In the distributional sense, the time harmonic Maxwell's problem writes: Find  $\mathbf{E}$  and  $\mathbf{H}$  such that

$$\mathbf{curl} \mathbf{E} - i\omega \mu \mathbf{H} = 0 \quad \text{and} \quad \mathbf{curl} \mathbf{H} + i\omega \varepsilon \mathbf{E} = \mathbf{J} \quad \text{in } \Omega. \quad (6.1)$$

$\mathbf{E}$  and  $\mathbf{H}$  are complex vector fields and denote, respectively, the electric and the magnetic field,  $\omega \in \mathbb{R}$  is the frequency. The vector field  $\mathbf{J}$  stands for the current density injected in the  $\Omega$  and is such that  $\text{div}(\mathbf{J}) = 0$ . In this chapter we will also suppose that  $\Omega$  is surrounded by a perfect conductor. This leads us to complete the previous system of equations with the boundary conditions:

$$\mathbf{E} \times \nu = 0 \quad \text{and} \quad \mu \mathbf{H} \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad (6.2)$$

in which  $\nu$  denotes the unit outward normal vector to  $\partial\Omega$ . In the classic configuration, when  $\varepsilon$  and  $\mu$  have constant sign, to study the time harmonic Maxwell's system one has to introduce the spaces

$$\begin{aligned} \mathbf{L}^2(\Omega) &:= (\mathbf{L}^2(\Omega))^3 \\ \mathbf{H}(\mathbf{curl}) &:= \{\mathbf{H} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{H} \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_N(\mathbf{curl}) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}) \mid \mathbf{E} \times \nu = 0 \text{ on } \partial\Omega\} \\ \mathbf{X}_T(\xi) &:= \{\mathbf{H} \in \mathbf{H}(\mathbf{curl}) \mid \text{div}(\xi \mathbf{H}) = 0, \xi \mathbf{H} \cdot \nu = 0 \text{ on } \partial\Omega\}, \text{ for } \xi \in L^\infty(\Omega) \\ \mathbf{X}_N(\xi) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}) \mid \text{div}(\xi \mathbf{E}) = 0, \mathbf{E} \times \nu = 0 \text{ on } \partial\Omega\}, \text{ for } \xi \in L^\infty(\Omega). \end{aligned}$$

We endow, the space  $\mathbf{L}^2(\Omega)$  with its natural norm  $\|\cdot\|_{\mathbf{L}^2(\Omega)}$  and the others spaces with the norm

$$\|\cdot\|_{\mathbf{H}(\mathbf{curl})} = (\|\cdot\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{curl} \cdot\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}.$$

One can check that, endowed with their natural norms all these spaces are of Hilbert type. For the particular case  $\xi = 1$ , it is well-known (see [139, 8]) that in  $\mathbf{X}_T(1)$  (resp.  $\mathbf{X}_N(1)$ ) the seminorm  $\|\mathbf{curl} \cdot\|_{\mathbf{L}^2(\Omega)}$  is a norm and it is equivalent to  $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$ . Furthermore, the embedding of  $\mathbf{X}_T(1)$  (resp.  $\mathbf{X}_N(1)$ ) in  $\mathbf{L}^2(\Omega)$  is known to be compact. It is also, well-understood thanks to results of [22], that the study of the Maxwell's system in the classical  $L^2$ -framework (see §6.3) is directly related to the study of the properties of the scalar operators  $A_\varepsilon : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  and  $A_\mu : H_{\#}^1(\Omega) \rightarrow (H_{\#}^1(\Omega))^*$  that are defined as follows:

$$\langle A_\varepsilon \varphi, \varphi' \rangle = \int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} \, dx, \quad \forall \varphi, \varphi' \in H_0^1(\Omega)$$

and

$$\langle A_\mu \varphi, \varphi' \rangle = \int_{\Omega} \mu \nabla \varphi \cdot \nabla \overline{\varphi'} \, dx, \quad \forall \varphi, \varphi' \in H_{\#}^1(\Omega).$$

Above the space  $H_{\#}^1(\Omega) := \{u \in H^1(\Omega) \mid (u, 1)_{L^2(\partial\Omega)} = 0\}$ . It is not difficult to see that the properties of  $A_\varepsilon$  and  $A_\mu$  are, respectively, related to the well-posedness of the problems:

$$\begin{aligned} \text{Find } u_d \in H_0^1(\Omega) \text{ s.t.} & \quad \text{Find } u_n \in H_{\#}^1(\Omega) \text{ s.t.} \\ \left| \begin{array}{l} -\text{div}(\varepsilon \nabla u_d) = f \in (H_0^1(\Omega))^* \\ u_d = 0 \text{ on } \partial\Omega \end{array} \right. & \quad \left| \begin{array}{l} -\text{div}(\mu \nabla u_n) = g \in (H_{\#}^1(\Omega))^* \\ \partial_\nu u_n = 0 \text{ on } \partial\Omega. \end{array} \right. \end{aligned} \quad (6.3)$$



In particular, it was proved in [22] that when the function  $\varepsilon$  and  $\mu$  are such that  $A_\varepsilon$  and  $A_\mu$  are of Fredholm type then the Maxwell's system is well-posed in the classical  $L^2$ -framework (see §6.3). In Chapter 2, we have proved that the operator  $A_\varepsilon$  (resp.  $A_\mu$ ) is a Fredholm operator if and only if  $\kappa_\varepsilon \in \mathbb{R}_-^* \setminus I_\varepsilon$  (resp.  $\kappa_\mu \in \mathbb{R}_-^* \setminus I_\mu$ ), where  $I_\varepsilon$  (resp.  $I_\mu$ ) is a closed subset of  $\mathbb{R}_-^*$  called the critical interval. As mentioned in the introduction, along this chapter we shall work under the following

**Assumption 6.2.1.** *We suppose that the function  $\varepsilon$  and  $\mu$  are such that  $\kappa_\varepsilon \in I_\varepsilon \setminus \{-1\}$ ,  $\kappa_\mu \in I_\mu \setminus \{-1\}$ .*

By definition of  $I_\varepsilon$  and  $I_\mu$ , we can say that under the previous assumption the operators  $A_\varepsilon$  and  $A_\mu$  are not of Fredholm type. Thanks to the results of Chapter 2, we know that, in our configuration, the Assumption 6.2.1 is equivalent to say that propagating singularities exist for both  $A_\varepsilon$  and  $A_\mu$ . In §6.2.2, we shall recall, briefly, how construct adapted alternative functional frameworks in which the scalar problems associated to  $A_\varepsilon$  and  $A_\mu$  are again well-posed.

To prepare the ground, we will start by recalling the definition of weighted Sobolev (Kondratiev) spaces and some useful results concerning the Laplace operator (with homogeneous Dirichlet and Neumann boundary conditions) in these spaces.

### 6.2.1 The Laplace operator in weighted Sobolev (Kondratiev) spaces

#### The weighted Sobolev spaces

For  $\beta \in \mathbb{R}$  and  $m \in \mathbb{N}$ , we introduce the weighted Sobolev (Kondratiev) space (see [100, 107, 102]) associated to the punctured domain  $\Omega \setminus \{O\}$ :  $V_\beta^m(\Omega)$  defined as the closure of  $\mathcal{D}(\bar{\Omega} \setminus \{O\})$  for the norm

$$\|\varphi\|_{V_\beta^m(\Omega)} = \left( \sum_{|\alpha| \leq m} \|r^{|\alpha|-m+\beta} \partial_x^\alpha \varphi\|_{L^2(\Omega)}^2 \right)^{1/2}$$

in which  $r = |x|$ . Here  $\mathcal{D}(\bar{\Omega} \setminus \{O\})$  denotes the space of infinitely differentiable functions which are supported in  $\bar{\Omega} \setminus \{O\}$ . For all  $m \in \mathbb{N}^*$  and  $\beta \in \mathbb{R}$  we have the inclusion

$$V_\beta^m(\Omega) \subset V_{\beta-1}^{m-1}(\Omega). \tag{6.4}$$

We also denote by  $\mathring{V}_\beta^1(\Omega)$  the closure of  $\mathcal{D}(\Omega \setminus \{O\})$  for the norm  $\|\cdot\|_{V_\beta^1(\Omega)}$ . We have the characterization

$$\mathring{V}_\beta^1(\Omega) = \{\varphi \in V_\beta^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\}.$$

It is obvious that  $V_0^1(\Omega) \subset H^1(\Omega)$ . Moreover, since  $\Omega$  is bounded, applying the results of [102, Theorem 7.1.1] yields that  $H^1(\Omega) = V_0^1(\Omega)$  and  $H_0^1(\Omega) = \mathring{V}_0^1(\Omega)$ . For  $\beta > 0$ , one has the inclusions

$$\mathring{V}_{-\beta}^1(\Omega) \subset H_0^1(\Omega) \subset \mathring{V}_\beta^1(\Omega) \quad \text{and then} \quad (\mathring{V}_\beta^1(\Omega))^* \subset (H_0^1(\Omega))^* \subset (\mathring{V}_{-\beta}^1(\Omega))^*.$$

Since for all  $0 < \beta$  we have  $V_{-\beta}^0(\Omega) \subset L^2(\Omega)$ , one deduces, thanks to (6.4), that

$$L^2(\Omega) \subset V_\beta^0(\Omega) \subset (H_0^1(\Omega))^*.$$

To obtain the previous inclusions, we have used the fact that  $(V_\beta^0(\Omega))^* = V_{-\beta}^0(\Omega)$ . For all  $\beta \in \mathbb{R}$ , we define the space  $\mathcal{V}_\beta^1(\Omega) := \{u \in V_\beta^1(\Omega) \mid (u, 1)_{L^2(\partial\Omega)} = 0\}$ . Again, by using [102, Theorem 7.1.1] we find  $H_{\#}^1(\Omega) = \mathcal{V}_0^1(\Omega)$ .

### The Laplace operator in weighted Sobolev spaces

In this paragraph, we will recall some results concerning the Laplace operator acting between weighted Sobolev spaces in the punctured domain  $\Omega \setminus \{O\}$ . These results will be very useful throughout this chapter, their proofs can be found in [102, 101, 107] and in the references therein. We will start with the homogeneous Dirichlet condition. For  $\gamma \in \mathbb{R}$ , we define the operator  $A_D^\gamma : \dot{V}_\gamma^1(\Omega) \rightarrow (\dot{V}_{-\gamma}^1(\Omega))^*$  such that

$$\langle A_D^\gamma \varphi, \varphi' \rangle = \int_{\Omega} \nabla \varphi \cdot \nabla \overline{\varphi'} dx, \quad \forall \varphi \in \dot{V}_\gamma^1(\Omega), \varphi' \in \dot{V}_{-\gamma}^1(\Omega).$$

**Proposition 6.2.1.** *For all  $\gamma \in (-1/2, 1/2)$ , the operator  $A_D^\gamma : \dot{V}_\gamma^1(\Omega) \rightarrow (\dot{V}_{-\gamma}^1(\Omega))^*$  is an isomorphism.*

Let  $0 < r_0$  and denote by  $B(O, r_0)$  the open ball of  $\mathbb{R}^3$  of center  $O$  and of radius  $r_0$ . We have the following regularity result

**Proposition 6.2.2.** [107, §1.6.2] *For all  $\gamma \in (\frac{1}{2}, \frac{3}{2})$  the operator  $\Delta : V_\gamma^2(B(O, r_0)) \cap \dot{V}_{\gamma-1}^1(B(O, r_0)) \rightarrow V_\gamma^0(B(O, r_0))$  is an isomorphism. Since for all  $\gamma \in (\frac{1}{2}, 1)$  we have  $V_\gamma^0(B(O, r_0)) \subset (H_0^1(B(O, r_0)))^*$ , the space of solutions  $V_\gamma^2(B(O, r_0)) \cap \dot{V}_{\gamma-1}^1(B(O, r_0))$  can be replaced by  $V_\gamma^2(B(O, r_0)) \cap H_0^1(B(O, r_0))$ .*

Now, we turn our attention to the case of the homogeneous Neumann boundary condition. For this, we introduce the operator  $A_N^\gamma : \mathcal{V}_\gamma^1(\Omega) \rightarrow (\mathcal{V}_{-\gamma}^1(\Omega))^*$  such that

$$\langle A_N^\gamma \varphi, \varphi' \rangle = \int_{\Omega} \nabla \varphi \cdot \nabla \overline{\varphi'} dx, \quad \text{for all } \varphi \in \mathcal{V}_\gamma^1(\Omega), \varphi' \in \mathcal{V}_{-\gamma}^1(\Omega).$$

**Proposition 6.2.3.** *For all  $\gamma \in (-1/2, 1/2)$ , the operator  $A_N^\gamma : \mathcal{V}_\gamma^1(\Omega) \rightarrow (\mathcal{V}_{-\gamma}^1(\Omega))^*$  is an isomorphism.*

Note that when  $\gamma = 0$ , we obtain the classical well-known result  $\Delta : H_{\#}^1(\Omega) \rightarrow (H_{\#}^1(\Omega))^*$  is an isomorphism.

### 6.2.2 The scalar problems with critical coefficients

Here, we recall some results, that we have proved in Chapter 2, concerning the construction of new functional frameworks for the scalar problems when the functions  $\varepsilon$  and  $\mu$  are such that Assumption 6.2.1 holds. To start, we define, for all  $\beta \in \mathbb{R}$ , the operators  $A_\varepsilon^{\pm\beta} : \dot{V}_{\pm\beta}^1(\Omega) \rightarrow (\dot{V}_{\mp\beta}^1(\Omega))^*$  such that

$$\langle A_\varepsilon^{\pm\beta} \varphi, \varphi' \rangle = \int_{\Omega} \varepsilon \nabla \varphi \cdot \nabla \overline{\varphi'} dx, \quad \text{for all } \varphi \in \dot{V}_{\pm\beta}^1(\Omega), \varphi' \in \dot{V}_{\mp\beta}^1(\Omega). \quad (6.5)$$

In the same way, for all  $\beta \in \mathbb{R}$  we introduce the operators  $A_\mu^{\pm\beta} : \mathcal{V}_{\pm\beta}^1(\Omega) \rightarrow (\mathcal{V}_{\mp\beta}^1(\Omega))^*$  such that

$$\langle A_\mu^{\pm\beta} \varphi, \varphi' \rangle = \int_{\Omega} \mu \nabla \varphi \cdot \nabla \overline{\varphi'} dx \quad \text{for all } \varphi, \varphi' \in \mathcal{V}_{\mp\beta}^1(\Omega). \quad (6.6)$$

Observe that, thanks to the fact that  $\dot{V}_0^1(\Omega) = H_0^1(\Omega)$  and  $\mathcal{V}_0^1(\Omega) = H_{\#}^1(\Omega)$ , we, then, have  $A_\varepsilon^0 = A_\varepsilon$  and  $A_\mu^0 = A_\mu$ .

**Lemma 6.2.1.** *Under Assumption 6.2.1 there exists  $0 < \beta_D$  (resp.  $0 < \beta_N$ ) such that the operator  $A_\varepsilon^{\pm\beta}$  (resp.  $A_\mu^{\pm\beta}$ ) is of Fredholm type for all  $\beta \in (0; \beta_D)$  (resp.  $\beta \in (0; \beta_N)$ ).*

To simplify the analysis below, we shall make the

**Assumption 6.2.2.** *We suppose that  $\varepsilon$  (resp.  $\mu$ ) is such that there exists  $\beta^* \in (0; \beta_D)$  (resp.  $\beta^* \in (0; \beta_N)$ ) such that  $A_\varepsilon^{-\beta^*}$  (resp.  $A_\mu^{-\beta^*}$ ) is injective.*

Using the results of Proposition 2.6.3, we obtain the

**Lemma 6.2.2.** *Assume that Assumptions 6.2.1-6.2.2 hold. Then for all  $\beta \in [0; \beta_D)$  (resp.  $\beta \in [0; \beta_N)$ ) the operator  $A_\varepsilon^{-\beta}$  (resp.  $A_\mu^{-\beta}$ ) is injective.*

Another useful result is the following

**Lemma 6.2.3.** *Assume that Assumptions 6.2.1-6.2.2 hold. If  $u \in H_0^1(\Omega)$  (resp.  $u \in H_{\#}^1(\Omega)$ ) is such that  $\operatorname{div}(\varepsilon \nabla u) \in (\dot{V}_\beta^1(\Omega))^*$  (resp.  $\operatorname{div}(\varepsilon \nabla u) \in (\mathcal{V}_\beta^1(\Omega))^*$ ) with  $\beta \in (0; \beta_D)$  (resp.  $\beta \in (0; \beta_N)$ ) then  $u \in \dot{V}_{-\beta}^1(\Omega)$  (resp.  $u \in \mathcal{V}_{-\beta}^1(\Omega)$ ).*

We denote by  $\mathcal{S}_\varepsilon$  and  $\mathcal{S}_\mu$ , respectively, the spaces of propagating singularities generated by the operators  $A_\varepsilon$  and  $A_\mu$ . Recall that these spaces have finite dimensions as soon as  $\kappa_\varepsilon \neq -1$  and  $\kappa_\mu \neq -1$ . To be more precise, the space  $\mathcal{S}_\varepsilon$  is defined as follows:

$$\mathcal{S}_\varepsilon = \operatorname{span}\left\{r\omega \mapsto \chi(r)r^{-1/2+i\eta} \sum_{p=0}^k \frac{1}{p!} \log(r)^p \varphi_{k-p} \mid \eta \in \mathbb{R}, (\varphi_p)_{p=0,\dots,k} \text{ is a Jordan chain of } \mathcal{L}_\varepsilon\right\} \quad (6.7)$$

where  $\mathcal{L}_\varepsilon$  is the Mellin symbol of  $A_\varepsilon$  and  $\chi \in \mathcal{D}(\Omega)$  is a fixed cutoff function that depends only in  $r = |x|$  and that is equal to 1 near the origin. To define the space  $\mathcal{S}_\mu$ , simply replace  $\varepsilon$  by  $\mu$ . Interestingly, we have explained in Chapter 2 that  $\mathcal{S}_\varepsilon, \mathcal{S}_\mu \subset L^2(\Omega)$  in addition to that we proved that for all  $s \in \mathcal{S}_\varepsilon$  (resp.  $s \in \mathcal{S}_\mu$ ) we have  $\operatorname{div}(\varepsilon \nabla s) \in L^2(\Omega)$  (resp.  $\operatorname{div}(\mu \nabla s) \in L^2(\Omega)$ ). This allows us to define for  $\psi = \varepsilon, \mu$ , the quadratic form  $q_\psi : \mathcal{S}_\psi \times \mathcal{S}_\psi \rightarrow \mathbb{C}$  such that

$$q_\psi(u, v) = \int_{\Omega} \operatorname{div}(\psi \nabla v) \bar{u} - \operatorname{div}(\psi \nabla u) \bar{v} \quad \text{for all } u, v \in \mathcal{S}_\psi.$$

Observe that for all  $u \in \mathcal{S}_\psi$  (with  $\psi = \varepsilon, \mu$ ) we have

$$q_\psi(u, u) = 2i \Im m \left( \int_{\Omega} \operatorname{div}(\psi \nabla u) \bar{u} \right).$$

We also have the

**Lemma 6.2.4.** *Assume that Assumptions 6.2.1-6.2.2 hold. The spaces  $\mathcal{S}_\varepsilon$  and  $\mathcal{S}_\mu$  have even dimensions denoted, respectively, by  $T_\varepsilon = 2N_\varepsilon$  and  $T_\mu = 2N_\mu$  ( $N_\varepsilon, N_\mu \in \mathbb{N}^*$ ). There exists  $(\mathfrak{s}_{\varepsilon,j}^\pm)_{j=1,\dots,N_\varepsilon}$  (resp.  $(\mathfrak{s}_{\mu,j}^\pm)_{j=1,\dots,N_\mu}$ ) a basis of  $\mathcal{S}_\varepsilon$  (resp.  $\mathcal{S}_\mu$ ) such that for  $\psi = \varepsilon, \mu$  we have*

$$q_\psi(\mathfrak{s}_{\psi,j}^\pm, \mathfrak{s}_{\psi,k}^\pm) = \pm i \delta_{j,k}, \quad q_\psi(\mathfrak{s}_{\psi,j}^\pm, \mathfrak{s}_{\psi,k}^\mp) = 0 \text{ and } \mathfrak{s}_{\psi,j}^+ = \overline{\mathfrak{s}_{\psi,j}^-} \text{ for } j, k = 1, \dots, N_\psi.$$

**Remark 6.2.1.** *As explained in §2.6.2, the choice of the bases  $(\mathfrak{s}_{\varepsilon,j}^\pm)_{j=1,\dots,N_\varepsilon}$  and  $(\mathfrak{s}_{\mu,j}^\pm)_{j=1,\dots,N_\mu}$  is not unique. One can find an infinite number of bases  $(\mathfrak{s}_{\varepsilon,j}^\pm)_{j=1,\dots,N_\varepsilon}$  and  $(\mathfrak{s}_{\mu,j}^\pm)_{j=1,\dots,N_\mu}$  satisfying all the conditions of Lemma 6.2.4. From a mathematical point of view, the choice of these bases is not important: any choice of bases will lead us to construct functional frameworks in which the scalar problems are again well posed. However, there is a particular choice of these bases which is consistent with the limiting absorption principle. We will come back to the choice of these bases in §6.4.7.*

From now on, we fix  $(\mathfrak{s}_{\varepsilon,j}^\pm)_{j=1,\dots,N_\varepsilon}$  (resp.  $(\mathfrak{s}_{\mu,j}^\pm)_{j=1,\dots,N_\mu}$ ) a basis of  $\mathcal{S}_\varepsilon$  (resp.  $\mathcal{S}_\mu$ ) satisfying the orthogonality relations in Lemma 6.2.4. Moreover, we define the spaces

$$\mathcal{S}_\varepsilon^+ := \operatorname{span}\{\mathfrak{s}_{\varepsilon,j}^+, j = 1, \dots, N_\varepsilon\}, \quad \mathcal{S}_\mu^+ := \operatorname{span}\{\mathfrak{s}_{\mu,j}^+, j = 1, \dots, N_\mu\}.$$

Easily, one can show that we have the following

**Lemma 6.2.5.** *Assume that Assumptions 6.2.1-6.2.2 hold. If  $\mathfrak{s} \in \mathcal{S}_\varepsilon^+$  (resp.  $\mathfrak{s} \in \mathcal{S}_\mu^+$ ) satisfies  $q_\varepsilon(\mathfrak{s}, \mathfrak{s}) = 0$  (resp.  $q_\mu(\mathfrak{s}, \mathfrak{s}) = 0$ ) then  $\mathfrak{s} = 0$ .*

Next, we define for all  $0 < \beta$  the spaces

$$\mathring{V}_\beta^{\text{out}}(\Omega) = \mathring{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}_\varepsilon^+, \quad \mathcal{V}_\beta^{\text{out}}(\Omega) = \mathcal{V}_{-\beta}^1(\Omega) \oplus \mathcal{S}_\mu^+.$$

For all  $0 < \beta$ , we define the operator  $A_\varepsilon^{\text{out}} : \mathring{V}_\beta^{\text{out}}(\Omega) \rightarrow (\mathring{V}_\beta^1(\Omega))^*$  such that for all  $u = \tilde{u} + \mathfrak{s}_\varepsilon^+ \in \mathring{V}_\beta^{\text{out}}(\Omega)$  (with  $\tilde{u} \in \mathring{V}_{-\beta}^1(\Omega)$  and  $\mathfrak{s}_\varepsilon^+ \in \mathcal{S}_\varepsilon^+$ ) and  $v \in \mathring{V}_\beta^1(\Omega)$  we have

$$\langle A_\varepsilon^{\text{out}} u, v \rangle := \int_\Omega \varepsilon \nabla \tilde{u} \cdot \nabla \bar{v} - \int_\Omega \text{div}(\varepsilon \nabla \mathfrak{s}_\varepsilon^+) \bar{v}.$$

In the same way, we introduce the operator  $A_\mu^{\text{out}} : \mathcal{V}_\beta^{\text{out}}(\Omega) \rightarrow (\mathcal{V}_\beta^1(\Omega))^*$  such that for all  $u = \tilde{u} + \mathfrak{s}_\mu^+ \in \mathcal{V}_\beta^{\text{out}}(\Omega)$  (with  $\tilde{u} \in \mathcal{V}_{-\beta}^1(\Omega)$  and  $\mathfrak{s}_\mu^+ \in \mathcal{S}_\mu^+$ ) and  $v \in \mathcal{V}_\beta^1(\Omega)$  we know

$$\langle A_\mu^{\text{out}} u, v \rangle := \int_\Omega \mu \nabla \tilde{u} \cdot \nabla \bar{v} - \int_\Omega \text{div}(\mu \nabla \mathfrak{s}_\mu^+) \bar{v}.$$

According to the results of §, we can prove the following

**Lemma 6.2.6.** *Assume that Assumptions 6.2.1-6.2.2 hold. Then for all  $\beta \in (0; \beta_D)$  (resp.  $\beta \in (0; \beta_N)$ ) the operator  $A_\varepsilon^{\text{out}}$  (resp.  $A_\mu^{\text{out}}$ ) is an isomorphism.*

Since in our work we are going to use at the same time the results concerning the Laplace operator with Dirichlet or Neumann boundary conditions as well as those associated with the operators  $A_\varepsilon^\beta$  and  $A_\mu^\beta$ , we are going to assume once and for all that, when the hypotheses 6.2.1-6.2.2 are satisfied, the constants  $\beta_D$  and  $\beta_N$  are such that  $\beta_N, \beta_D < 1/2$ . Moreover, we denote by  $\beta_0 := \min(\beta_D, \beta_N)$ .

### 6.3 Necessity of a new functional framework for the Maxwell's system

After eliminating  $\mathbf{H}$  and then  $\mathbf{E}$  in the problem (6.1), one concludes that the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{H}$  satisfy the problems

$$\left| \begin{array}{l} \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = i\omega \mathbf{J} \quad \text{in } \Omega \\ \mathbf{E} \times \nu = 0 \quad \text{on } \partial\Omega, \end{array} \right| \left| \begin{array}{l} \mathbf{curl} \varepsilon^{-1} \mathbf{curl} \mathbf{H} - \omega^2 \mu \mathbf{H} = \mathbf{curl} \varepsilon^{-1} \mathbf{J} \quad \text{in } \Omega \\ \mu \mathbf{H} \cdot \nu = 0, \varepsilon^{-1} (\mathbf{curl} \mathbf{H} - \mathbf{J}) \times \nu = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (6.8)$$

In the classical configuration, when  $\varepsilon$  and  $\mu$  have constant signs, the formulation associated to the electric field  $\mathbf{E}$  is set in the space  $\mathbf{H}_N(\mathbf{curl})$  and the one associated the magnetic field  $\mathbf{H}$  is set in the space  $\mathbf{H}(\mathbf{curl})$ . More precisely, when  $\mathbf{J} \in \mathbf{L}^2(\Omega)$ , the problem associated to the electric field writes

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}_N(\mathbf{curl}) \text{ such that} \\ \int_\Omega \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_\Omega \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{H}_N(\Omega). \end{array} \right. \quad (6.9)$$

Since the embedding of  $\mathbf{H}_N(\Omega)$  in  $\mathbf{L}^2(\Omega)$  is not compact (see [8]), the analysis of the previous problem cannot be treated by classical arguments. For this reason, we prefer to work with the following formulation which is posed in the space  $\mathbf{X}_N(\varepsilon) \subset \mathbf{H}_N(\Omega)$

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_N(\varepsilon) \text{ such that} \\ \int_\Omega \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_\Omega \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_\Omega \mathbf{J} \cdot \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{X}_N(\varepsilon). \end{array} \right. \quad (6.10)$$

It was proved in [22] that the previous two formulations are equivalent as soon as the operators  $A_\varepsilon$  and  $A_\mu$  are isomorphisms. Furthermore, in this situation one can show that (6.10) is well-posed

except for a discrete set of frequencies where it admits a finite dimensional kernel. When  $\varepsilon$  and  $\mu$  are critical, there is no guarantee neither on the equivalence between the formulations (6.9) and (6.10) nor on their well-posedness. To proceed, we introduce the operators  $\mathbb{A}_N : \mathbf{X}_N(\varepsilon) \rightarrow (\mathbf{X}_N(\varepsilon))^*$  and  $\mathbb{K}_N : \mathbf{X}_N(\varepsilon) \rightarrow (\mathbf{X}_N(\varepsilon))^*$  such that for all  $\mathbf{u}, \mathbf{v} \in \mathbf{X}_N(\varepsilon)$ , we have

$$\langle \mathbb{A}_N \mathbf{u}, \mathbf{v} \rangle := \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}}, \quad \langle \mathbb{K}_N \mathbf{u}, \mathbf{v} \rangle := \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}}.$$

The following lemma can be seen as an extension of the results of [54].

**Lemma 6.3.1.** *Assume that Assumptions 6.2.1-6.2.2 hold. Then the operator  $\mathbb{K}_N$  is compact.*

**Proof.** Let  $\beta \in (0; \beta_D)$ . According to Proposition 6.7.3, we know that the space  $\mathbf{X}_N(\varepsilon)$  is compactly embedded in the space  $\mathbf{V}_{-\beta}^0(\Omega) = (\mathbf{V}_{-\beta}^0(\Omega))^3$ . We finish the proof by remarking that there exists a positive constant  $C$  such that for all  $\mathbf{u} \in \mathbf{X}_N(\varepsilon)$

$$\|\mathbb{K}_N \mathbf{u}\|_{(\mathbf{X}_N(\varepsilon, \Omega))^*} \leq C \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq C \|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (6.11)$$

■

As a result, even when  $\varepsilon$  is critical, we then have the equivalence between the Fredholmness of the problem (6.10) and the Fredholmness of the operator  $\mathbb{A}_N$ . We also have the

**Proposition 6.3.1.** *Under Assumptions 6.2.1-6.2.2 the map  $\mathbf{u} \mapsto \|\mathbf{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}$  is a norm in  $\mathbf{X}_N(\varepsilon)$  that is equivalent to the  $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$  one.*

**Proof.** By the classical open map theorem, it suffices to show that  $\mathbf{u} \mapsto \|\mathbf{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}$  is a norm in  $\mathbf{X}_N(\varepsilon)$ . If  $\mathbf{u} \in \mathbf{X}_N(\varepsilon)$  such that  $\mathbf{curl} \mathbf{u} = 0$ , then by using item *iii*) Proposition 6.7.1 we infer that there exists a unique  $\varphi \in \mathbf{H}_0^1(\Omega)$  such that  $\mathbf{u} = \nabla \varphi$ . Given that  $\operatorname{div}(\varepsilon \mathbf{u}) = \operatorname{div}(\varepsilon \nabla \varphi) = 0$  and owing to Lemma 6.2.2, we obtain the wanted result. ■

The main result of this section is given by the following

**Theorem 6.3.1.** *Assume that Assumptions 6.2.1-6.2.2 hold and assume that  $\varepsilon$  and  $\mu$  are such that  $\kappa_{\varepsilon} \in I_{\varepsilon} \setminus \{-1\}$  and  $\kappa_{\mu} \in I_{\mu} \setminus \{-1\}$ . Then either the operator  $\mathbb{A}_N$  is not of Fredholm type or the problems (6.9) and (6.10) are not equivalent.*

**Remark 6.3.1.** *In the case where the operator  $\mathbb{A}_N$  is of Fredholm type, the absence of equivalence between the formulations (6.9) and (6.10) means that the solution obtained by solving (6.10) does not satisfy the equation satisfied by the electric field in the distributional sense (i.e., the first part of (6.8)).*

**Proof.** We will proceed by contradiction. Suppose that  $\mathbb{A}_N$  is a Fredholm operator and that the problems (6.9) and (6.10) are equivalent, then we will show that  $\mathbb{A}_{\mu}$  is of a Fredholm operator which is false by assumption.

The symmetric operator  $\mathbb{A}_N$  is then of Fredholm type; its index must therefore be equal to 0. Without loss of generality, we can suppose that  $\mathbb{A}_N$  is not injective. Otherwise, the following proof can be easily adapted. Since the kernel of  $\mathbb{A}_N$  is of finite dimension, say  $N \in \mathbb{N}^*$ , we can find  $N$  linearly independent elements of  $\mathbf{X}_N(\varepsilon)$  that will be denoted by  $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N$  such that  $\operatorname{Ker}(\mathbb{A}_N) = \operatorname{span}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N)$ . To proceed, with the help of Proposition 6.3.1, we introduce the closed space

$$\tilde{\mathbf{X}}_N(\varepsilon) := \{\mathbf{u} \in \mathbf{X}_N(\varepsilon) \mid (\mathbf{curl} \mathbf{u}, \mathbf{curl} \boldsymbol{\lambda}_i)_{\mathbf{L}^2(\Omega)} = 0, i = 1, \dots, N\}$$

as well as the operator  $\tilde{\mathbb{A}}_N : \tilde{\mathbf{X}}_N(\varepsilon) \rightarrow \tilde{\mathbf{X}}_N(\varepsilon)$  such that

$$\langle \tilde{\mathbb{A}}_N \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} dx, \quad \text{for all } \mathbf{u}, \mathbf{v} \in \tilde{\mathbf{X}}_N(\varepsilon).$$

Consequently, we obtain the decomposition  $\mathbf{X}_N(\varepsilon) = \tilde{\mathbf{X}}_N(\varepsilon) \oplus \text{span}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N)$ . Moreover, the operator  $\tilde{\mathbb{A}}_N$  is an isomorphism. Now, consider an element  $\varphi \in \mathbf{H}_{\#}^1(\Omega)$ . Since  $\tilde{\mathbb{A}}_N$  is an isomorphism, we define  $\boldsymbol{\psi}_\varphi \in \tilde{\mathbf{X}}_N(\varepsilon)$  the unique element of  $\tilde{\mathbf{X}}_N(\varepsilon)$  satisfying

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \boldsymbol{\psi}_\varphi \cdot \mathbf{curl} \bar{\boldsymbol{\psi}}' dx = \int_{\Omega} \mu^{-1} (\nabla \varphi - \sum_{i=1}^N \beta_i \mathbf{curl} \boldsymbol{\lambda}_i) \cdot \mathbf{curl} \bar{\boldsymbol{\psi}}' dx, \quad \forall \boldsymbol{\psi}' \in \tilde{\mathbf{X}}_N(\varepsilon) \quad (6.12)$$

in which  $\beta_i = \int_{\Omega} \mu^{-1} \nabla \varphi \cdot \mathbf{curl} \bar{\boldsymbol{\lambda}}_i dx$ . We emphasize that in (6.12) the test function  $\boldsymbol{\psi}'$  belongs to  $\tilde{\mathbf{X}}_N(\varepsilon)$ . However, thanks to the definition of  $\boldsymbol{\lambda}_i$  and  $\beta_i$ , one can check that (6.12) is also valid for all  $\boldsymbol{\psi}' \in \text{span}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N)$ . Indeed, since for all  $i \in \{1, \dots, N\}$  we have  $\boldsymbol{\lambda}_i \in \text{Ker}(\mathbb{A}_N)$  we infer that

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \boldsymbol{\psi}_\varphi \cdot \mathbf{curl} \bar{\boldsymbol{\lambda}}_i dx = 0.$$

On the other hand, thanks to the definition of  $\beta_k$  for  $k = 1, \dots, N$ , we also have

$$\int_{\Omega} \mu^{-1} (\nabla \varphi - \sum_{k=1}^N \beta_k \mathbf{curl} \boldsymbol{\lambda}_k) \cdot \mathbf{curl} \bar{\boldsymbol{\lambda}}_i dx = 0 \text{ for all } i = 1, \dots, N.$$

As a result, by linearity, we find

$$\int_{\Omega} \mu^{-1} \mathbf{curl} \boldsymbol{\psi}_\varphi \cdot \mathbf{curl} \boldsymbol{\psi}' dx = \int_{\Omega} \mu^{-1} (\nabla \varphi' - \sum_{i=1}^N \beta_i \mathbf{curl} \boldsymbol{\lambda}_i) \cdot \mathbf{curl} \boldsymbol{\psi}' dx, \quad \forall \boldsymbol{\psi}' \in \mathbf{X}_N(\varepsilon). \quad (6.13)$$

But, since by assumption the problems (6.9) and (6.10) are equivalent, the equation (6.13) is, then, valid for all  $\boldsymbol{\psi}' \in \mathbf{H}_N(\Omega)$  and then, by density of  $(\mathcal{D}(\Omega))^3$  in  $\mathbf{H}_N(\Omega)$ , we obtain

$$\mathbf{curl} (\mu^{-1} (\nabla \varphi - \sum_{i=1}^N \beta_i \mathbf{curl} \boldsymbol{\lambda}_i - \mathbf{curl} \boldsymbol{\psi}_\varphi)) = 0 \quad \text{in } \Omega.$$

From item *v*) of Proposition (6.7.1), we infer that there is a unique  $\mathbb{T}\varphi' \in \mathbf{H}_{\#}^1(\Omega)$  such that

$$\nabla(\mathbb{T}\varphi) = \mu^{-1} (\nabla \varphi' - \sum_{i=1}^N \beta_i \mathbf{curl} \boldsymbol{\lambda}_i - \mathbf{curl} \boldsymbol{\psi}_\varphi).$$

As a result, we have defined an operator  $\mathbb{T} : \mathbf{H}_{\#}^1(\Omega) \rightarrow (\mathbf{H}_{\#}^1(\Omega))^*$ . One can easily prove that  $\mathbb{T}$  is continuous. Furthermore, since for all  $\varphi \in \mathbf{H}_{\#}^1(\Omega)$ ,  $\mathbf{u} \in \mathbf{X}_N(\varepsilon)$  we have  $\int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \nabla \bar{\varphi}' = 0$ , we deduce that for all  $\varphi, \varphi' \in \mathbf{H}_{\#}^1(\Omega)$

$$\langle A_\mu \varphi, \mathbb{T}\varphi' \rangle = \int_{\Omega} \mu \nabla \varphi \cdot \nabla (\overline{\mathbb{T}\varphi'}) dx = \int_{\Omega} \nabla \varphi \cdot \nabla \bar{\varphi}' dx - \sum_{i=1}^N \int_{\Omega} \beta_i \mathbf{curl} \boldsymbol{\lambda}_i \cdot \nabla \bar{\varphi}' dx. \quad (6.14)$$

Consequently, the operator  $\mathbb{T}$  represents a left parametrix for the self adjoint operator  $A_\mu$ . As a result (see [109, Lemma 2.23]) the operator  $A_\mu : \mathbf{H}_{\#}^1(\Omega) \rightarrow (\mathbf{H}_{\#}^1(\Omega))^*$  is a Fredholm operator of index 0 which is not true by assumption ( $\mu$  is critical).  $\blacksquare$

In the classical setting, the equivalent variational formulation to the magnetic problem writes:

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}(\mathbf{curl}) \text{ such that} \\ \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{H}(\mathbf{curl}). \end{array} \right. \quad (6.15)$$

The results of [22] allow us to show that when  $\mu$  is such that  $\mathbb{A}_\mu$  is an isomorphism then the previous formulation is equivalent to the following one

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{X}_T(\mu) \text{ such that} \\ \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{X}_T(\mu). \end{array} \right. \quad (6.16)$$

If in addition to that the function  $\varepsilon$  is such that  $\mathbb{A}_\varepsilon$  is an isomorphism, it can be shown that (6.16) is well-posed except for a discrete set of frequencies at which it has a finite dimensional kernel. As in the previous paragraph, we introduce the operators  $\mathbb{A}_T, \mathbb{K}_T : \mathbf{X}_T(\mu) \rightarrow (\mathbf{X}_T(\mu))^*$  such that for all  $\mathbf{u}, \mathbf{v} \in \mathbf{X}_T(\mu)$ , we have

$$\langle \mathbb{A}_T \mathbf{u}, \mathbf{v} \rangle := \int_{\Omega} \mu^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}}, \quad \langle \mathbb{K}_T \mathbf{u}, \mathbf{v} \rangle := \int_{\Omega} \varepsilon \mathbf{u} \bar{\mathbf{v}}.$$

By working as in the case of the electric problem, one shows the

**Theorem 6.3.2.** *Assume that Assumptions 6.2.1-6.2.2 hold and assume that  $\varepsilon$  and  $\mu$  are such that  $\kappa_\varepsilon \in I_\varepsilon \setminus \{-1\}$  and  $\kappa_\mu \in I_\mu \setminus \{-1\}$ . Then the following assertions hold:*

- $\mathbf{u} \mapsto \|\mathbf{curl} \mathbf{u}\|_{\mathbf{L}^2(\Omega)}$  is a norm in  $\mathbf{X}_T(\mu)$  and is equivalent to  $\|\cdot\|_{\mathbf{H}(\mathbf{curl})}$ .
- $\mathbb{K}_T$  is compact.
- *Either the operator  $\mathbb{M}_T$  is not of Fredholm type or the problems (6.15) and (6.16) are not equivalent.*

## 6.4 The analysis the electric problem

Previously, we have shown that when  $\varepsilon$  and  $\mu$  are critical, the classical framework  $\mathbf{X}_N(\varepsilon)$  is no longer the appropriate space to solve the electric problem. In this section, we explain how to construct a new functional framework in which the problem

$$\left| \begin{array}{l} \mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = i\omega \mathbf{J} \quad \text{in } \Omega \\ \mathbf{E} \times \nu = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (6.17)$$

is again well-posed. For this, we introduce for all  $\beta \in \mathbb{R}$  the spaces

$$\begin{aligned} \mathbf{V}_\beta^0(\Omega) &:= (\mathbf{V}_\beta^0(\Omega))^3, \\ \mathbf{H}_N^\beta(\mathbf{curl}) &:= \{\mathbf{u} \in \nabla \mathcal{S}_\varepsilon^+ \oplus \mathbf{V}_{-\beta}^0(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega), \mathbf{u} \times \nu = 0 \text{ on } \partial\Omega\}, \\ \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl}) &:= \{\mathbf{u} \in \nabla \mathcal{S}_\varepsilon^+ \oplus \mathbf{V}_{-\beta}^0(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mu \nabla \mathcal{S}_\mu^+ \oplus \mathbf{V}_{-\beta}^0(\Omega), \mathbf{u} \times \nu = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Observe that the space  $\mathbf{H}_N^\beta(\mathbf{curl})$  depends on  $\varepsilon$  and that the space  $\mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})$  depends on  $\varepsilon$  and  $\mu$ . Above  $\nabla \mathcal{S}_\varepsilon^+$  and  $\nabla \mathcal{S}_\mu^+$  stand for the spaces

$$\nabla \mathcal{S}_\varepsilon^+ = \text{span}\{\nabla \mathbf{s}_{\varepsilon,j}^+, j = 1, \dots, N_\varepsilon\}, \quad \nabla \mathcal{S}_\mu^+ = \text{span}\{\nabla \mathbf{s}_{\mu,j}^+, j = 1, \dots, N_\mu\}.$$

Before getting into details, let us define the norms that we are going to use in the spaces  $\nabla \mathcal{S}_\varepsilon^+$  and  $\nabla \mathcal{S}_\mu^+$ . For  $\psi = \varepsilon, \mu$  and  $\alpha_1 \nabla \mathbf{s}_{\psi,1}^+ + \dots + \alpha_{N_\psi} \nabla \mathbf{s}_{\psi,N_\psi}^+ \in \nabla \mathcal{S}_\psi^+$  we define

$$\left\| \sum_{j=1}^{N_\psi} \alpha_j \mathbf{s}_{\psi,j}^+ \right\|_{\nabla \mathcal{S}_\psi^+} = \left( \sum_{j=1}^{N_\psi} |\alpha_j|^2 \right)^{1/2}.$$

One can check that for all  $0 < \beta$ , we have the inclusions

$$\mathbf{H}_N^{\text{out},\beta}(\mathbf{curl}) \subset \mathbf{H}_N^\beta(\mathbf{curl}) \subset \mathbf{V}_\beta^0(\Omega). \quad (6.18)$$



It is time to present the norms that we are going to use in these three spaces. We start with the case of the space  $\mathbf{V}_0^\beta(\Omega)$ . For all  $\mathbf{u} = (u_1, u_2, u_3) \in \mathbf{V}_0^\beta(\Omega)$ , we denote by

$$\|\mathbf{u}\|_{\mathbf{V}_0^\beta(\Omega)} := \left( \sum_{i=1}^3 \|u_i\|_{\mathbf{V}_0^\beta(\Omega)}^2 \right)^{1/2}.$$

For the case of the space  $\mathbf{H}_N^\beta(\mathbf{curl})$ , we proceed as follows. For all  $\mathbf{u} = \tilde{\mathbf{u}} + \nabla \mathfrak{s}_\varepsilon^+ \in \mathbf{H}_N^\beta(\mathbf{curl})$  (with  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathfrak{s}_\varepsilon^+ \in \mathcal{S}_\varepsilon^+$ ) we define

$$\|\mathbf{u}\|_{\mathbf{H}^\beta(\mathbf{curl})} := \left( \|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2 + \|\nabla \mathfrak{s}_\varepsilon^+\|_{\nabla \mathcal{S}_\varepsilon^+}^2 + \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}^2 \right)^{1/2}.$$

For all  $\mathbf{u} = \tilde{\mathbf{u}} + \nabla \mathfrak{s}_\varepsilon^+ \in \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})$  (with  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathfrak{s}_\varepsilon^+ \in \mathcal{S}_\varepsilon^+$ ) such that  $\mathbf{curl} \mathbf{u} = \boldsymbol{\psi}_\mathbf{u} + \mu \nabla \mathfrak{s}_\mu$  (with  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathfrak{s}_\mu^+ \in \mathcal{S}_\mu^+$ ) we introduce

$$\|\mathbf{u}\|_{\mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})} := \left( \|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2 + \|\nabla \mathfrak{s}_\varepsilon^+\|_{\nabla \mathcal{S}_\varepsilon^+}^2 + \|\boldsymbol{\psi}_\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2 + \|\nabla \mathfrak{s}_\mu^+\|_{\nabla \mathcal{S}_\mu^+}^2 \right)^{1/2}.$$

Given that  $\Omega$  is Lipschitz-continuous, endowed with their associated norms all the previous spaces are Hilbert spaces. In addition to that, one can show that when  $\beta$  is positive, the embeddings (6.18) are continuous.

To simplify the presentation of our results, we shall adopt the following notations: for all  $\mathbf{u} \in \mathbf{H}_N^\beta(\mathbf{curl})$ , we will write  $\mathbf{u} = \tilde{\mathbf{u}} + \nabla \mathfrak{s}_{\mathbf{u},\varepsilon}$  with  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathfrak{s}_{\mathbf{u},\varepsilon} \in \mathcal{S}_\varepsilon^+$ , for  $\mathbf{v} \in \mathbf{H}_N^{\text{out}}(\mathbf{curl})$  we will use the notation  $\mathbf{curl} \mathbf{v} = \boldsymbol{\psi}_\mathbf{v} + \mu \nabla \mathfrak{s}_{\mathbf{v},\mu}$  with  $\boldsymbol{\psi}_\mathbf{v} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathfrak{s}_{\mathbf{v},\mu} \in \mathcal{S}_\mu^+$ .

#### 6.4.1 Definition of the electric problem

In §, we will explain that the appropriate functional framework to set the electric problem is the space  $\mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})$  (some conditions on  $\beta$  that will be specified later). For this reason we are going to study the problem

$$\text{Find } \mathbf{u} \in \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl}) \text{ such that } \begin{cases} \mathbf{curl} \mu^{-1} \boldsymbol{\psi}_\mathbf{u} - \omega^2 \varepsilon \mathbf{u} = i\omega \mathbf{J} & \text{in } \Omega \setminus \{O\} \\ \mathbf{u} \times \boldsymbol{\nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.19)$$

The reason why we considered the problem in  $\Omega \setminus \{O\}$  and not in  $\Omega$  is to be able to study the problem in weighted Sobolev spaces (and we will then be able to consider very singular fields near the origin). Our goal is to write a well-posed variational formulation which is equivalent to the problem (6.19). To obtain such a variational formulation, we must choose, with care, the space of the test functions. To proceed, let us assume, for the moment, that the current density  $\mathbf{J}$  belongs to the space  $\mathbf{V}_{-\beta}^0(\Omega)$  (in the Theorem 6.4.6, we will explain how to work with more general current densities) and let us introduce the problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl}) \text{ such that} \\ \int_{\Omega} \mu^{-1} \boldsymbol{\psi}_\mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{H}_N^\beta(\mathbf{curl}). \end{array} \right. \quad (6.20)$$

in which for all  $\mathbf{u} \in \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})$  and  $\mathbf{v} \in \mathbf{H}_N^\beta(\mathbf{curl})$  we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} := \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\mathbf{v}} + \int_{\Omega} \varepsilon \nabla \mathfrak{s}_{\mathbf{u},\varepsilon} \cdot \bar{\mathbf{v}} + \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \overline{\mathfrak{s}_{\mathbf{v},\varepsilon}} - \int_{\Omega} \text{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u},\varepsilon}) \overline{\mathfrak{s}_{\mathbf{v},\varepsilon}}.$$

It will be useful to observe that for all  $\mathbf{u} \in \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})$  and  $\mathbf{v} \in \mathbf{H}_N^\beta(\mathbf{curl})$  we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} - \overline{\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}}} = \int_{\Omega} \text{div}(\varepsilon \nabla \overline{\mathfrak{s}_{\mathbf{u},\varepsilon}}) \mathfrak{s}_{\mathbf{v},\varepsilon} - \int_{\Omega} \text{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u},\varepsilon}) \overline{\mathfrak{s}_{\mathbf{v},\varepsilon}} = q_\varepsilon(\mathfrak{s}_{\mathbf{u},\varepsilon}, \mathfrak{s}_{\mathbf{v},\varepsilon}). \quad (6.21)$$

**Proposition 6.4.1.** *Assume that  $\beta \in [0; 1/2)$ . Then, the problems (6.19) and (6.20) are equivalent.*

**Proof.** Since  $\mathcal{D}(\Omega \setminus \{O\})^3 \subset \mathbf{H}_N^\beta(\Omega)$ , any solution to (6.20) is a solution to (6.19). Now, let us show the reverse statement. Starting from the fact that for all  $\beta \in [0; 1/2)$  the space  $\mathcal{D}(\Omega \setminus \{O\})^3$  is dense in the space  $\mathbf{V} = \{\mathbf{u} \in \mathbf{H}_N(\mathbf{curl}) \mid \mathfrak{s}_{\mathbf{u}, \varepsilon} = 0\}$  (see §6.7.4)), we conclude that if  $\mathbf{u}$  is a solution to (6.19) then it satisfies

$$\int_{\Omega} \mu^{-1} \psi_{\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}} \text{ for all } \mathbf{v} \in \mathbf{V}.$$

To end the proof, it remains to show that for all  $\mathbf{v} \in \nabla \mathcal{S}_\varepsilon$ , we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \mathbf{J} \cdot \bar{\mathbf{v}}.$$

To do so, let  $v \in \mathcal{S}_\varepsilon^+$  and denote by  $\mathbf{v} = \nabla v \in \nabla \mathcal{S}_\varepsilon^+$ . Given that  $\nabla \mathcal{S}_\varepsilon \subset \mathring{\mathbf{V}}_\beta^1(\Omega)$  for all  $0 < \beta$ , there exists a sequence  $(\varphi_n)_{n \in \mathbb{N}}$  of elements of  $\mathcal{D}(\Omega \setminus \{O\})$  such that  $\varphi_n \rightarrow v$  in  $\mathring{\mathbf{V}}_\beta^1(\Omega)$  as  $n \rightarrow +\infty$ . This implies that  $\nabla \varphi_n \rightarrow \nabla v$  in  $\mathbf{V}_\beta^0(\Omega)$ . Moreover, since  $\mathring{\mathbf{V}}_\beta^1(\Omega) \subset L^2(\Omega)$  for all  $\beta < 1$ , we can say that for all  $\beta \in [0; 1/2)$  we have  $\varphi_n \rightarrow v$  in  $L^2(\Omega)$  as  $n \rightarrow +\infty$ . Multiplying (6.19) by  $\nabla \varphi_n$  and integrating by parts yield

$$\int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \varphi_n - \int_{\Omega} \operatorname{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u}, \varepsilon}) \overline{\varphi_n} = \int_{\Omega} \mathbf{J} \cdot \nabla \overline{\varphi_n}.$$

By letting  $n$  tend to  $\infty$ , we deduce that

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla v = \int_{\Omega} \mathbf{J} \cdot \nabla v.$$

This leads to the wanted result. ■

### 6.4.2 Equivalent formulation for the electric field

Given that for all  $\varphi \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega)$  we have  $\mathbf{u} = \nabla \varphi \in \mathbf{H}_N^{\text{out}, \beta}(\mathbf{curl})$ , we infer that the operator associated to the sesquilinear form  $(\mathbf{u}, \mathbf{v}) \mapsto \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}}$  is not compact. As in the classical configuration, one way to deal with this absence of compactness is to impose the constraint  $\operatorname{div}(\varepsilon \cdot) = 0$  on the spaces  $\mathbf{H}_N^\beta(\mathbf{curl})$ ,  $\mathbf{H}_N^{\text{out}, \beta}(\mathbf{curl})$ . This leads us to introduce the spaces

$$\mathbf{Y}_N^\beta(\varepsilon) := \{\mathbf{u} \in \mathbf{H}_N^\beta(\mathbf{curl}) \mid \operatorname{div}(\varepsilon \mathbf{u}) = 0\}, \quad \mathbf{Y}_N^{\text{out}, \beta}(\varepsilon) := \{\mathbf{u} \in \mathbf{H}_N^{\text{out}, \beta}(\mathbf{curl}) \mid \operatorname{div}(\varepsilon \mathbf{u}) = 0\}.$$

Note that the space  $\mathbf{Y}_N^{\text{out}, \beta}(\varepsilon)$  depends also on  $\mu$ . In the sequel, we endow the space  $\mathbf{Y}_N^\beta(\varepsilon)$  and  $\mathbf{Y}_N^{\text{out}, \beta}(\varepsilon)$  respectively with the norms of the spaces  $\mathbf{H}_N^\beta(\mathbf{curl})$  and  $\mathbf{H}_N^{\text{out}, \beta}(\mathbf{curl})$ .

**Remark 6.4.1.** *Let  $\mathbf{u} = \tilde{\mathbf{u}} + \nabla \mathfrak{s}_{\mathbf{u}, \varepsilon} \in \mathbf{H}_N^\beta(\Omega)$ . At first sight the constraint  $\operatorname{div}(\varepsilon \mathbf{u})$  must be understood as follows:*

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi} - \int_{\Omega} \operatorname{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u}, \varepsilon}) \bar{\varphi} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \nabla \bar{\varphi} = 0 \text{ for all } \varphi \in \mathcal{D}(\Omega \setminus \{O\}).$$

*Given that for all  $\mathfrak{s}_{\mathbf{u}, \varepsilon} \in \mathcal{S}_\varepsilon$  the function  $\operatorname{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u}, \varepsilon})$  belongs to  $L^2(\Omega)$  and is compactly supported in  $\hat{\Omega} = \Omega \setminus \{r\omega \mid \chi(r) = 1\} \subset \Omega \setminus \{O\}$  (recall that the function  $\chi$  is a fixed cutoff function that depends only in  $r = |x|$  and that is equal to 1 near the origin, see (6.7)). With this in mind one can show that we have the estimate: there exists  $0 < C$  (independent of  $\varphi$ ) such that for all  $\varphi \in \mathcal{D}(\Omega \setminus \{O\})$  we have*

$$\left| \int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} \right| = \left| \int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{\varphi} \right| \leq C \|\varphi\|_{\mathring{\mathbf{V}}_\beta^1(\Omega)}.$$

The density of  $\mathcal{D}(\Omega \setminus \{O\})$  in  $\mathring{V}_\beta^1(\Omega)$  implies that we have the equivalence

$$\operatorname{div}(\varepsilon \mathbf{u}) = 0 \iff \int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \bar{v} = 0 \quad \text{for all } v \in \mathring{V}_\beta^1(\Omega). \quad (6.22)$$

By replacing the space  $\mathbf{H}_N^{\text{out},\beta}(\mathbf{curl})$  by  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  and the space  $\mathbf{H}_N^\beta(\mathbf{curl})$  by  $\mathbf{Y}_N^\beta(\varepsilon)$  in (6.20), one obtains the following problem

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \text{ such that} \\ \int_{\Omega} \mu^{-1} \psi_{\mathbf{u}} \cdot \mathbf{curl} \bar{v} - \omega^2 \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{v} = i\omega \int_{\Omega} \mathbf{J} \cdot \bar{v} \quad \text{for all } v \in \mathbf{Y}_N^\beta(\varepsilon). \end{array} \right. \quad (6.23)$$

Without any difficulty (using (6.22)), one can see that for all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  and  $v \in \mathbf{Y}_N^\beta(\varepsilon)$  we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{v} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{v}} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{v}} - \int_{\Omega} \operatorname{div}(\varepsilon \nabla \mathbf{s}_{\mathbf{u},\varepsilon}) \overline{\mathbf{s}_{\tilde{v},\varepsilon}}.$$

Note that to obtain the previous result, we have used the fact that for all  $v \in \mathbf{Y}_N^\beta(\varepsilon)$ , we have  $\operatorname{div}(\varepsilon v) = -\operatorname{div}(\varepsilon \nabla \mathbf{s}_{v,\varepsilon})$ .

**Theorem 6.4.1.** *Assume that  $\omega \neq 0$ .*

- *Every solution of (6.20) is a solution of (6.23).*
- *Let  $\beta \in (0; \beta_D)$ . Under Assumptions 6.2.1-6.2.2, if  $\mathbf{E}$  is a solution (6.23), then it solves (6.20). Moreover  $\{\mathbf{E}, (i\omega\mu)^{-1} \mathbf{curl} \mathbf{E}\}$  is a solution of (6.1).*

**Proof.** To prove the first part of the statement, one needs to justify that every solution  $\mathbf{u}$  of (6.20) satisfies the equation  $\operatorname{div}(\varepsilon \mathbf{u}) = 0$ . For that, it suffices to take  $v = \nabla \varphi$  in (6.20) with  $\varphi \in \mathcal{D}(\Omega \setminus \{O\})$  and then use the fact that  $\operatorname{div}(\mathbf{J}) = 0$  in  $\Omega \setminus \{O\}$ .

The proof of the second part is little bit more involved. To prove it, let  $\mathbf{u}$  be a solution of (6.23). Since  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \subset \mathbf{H}_N^{\text{out},\beta}(\Omega)$ , it suffices to show that the variational formulation (6.23) is also valid for all  $v \in \mathbf{H}_N^\beta(\Omega)$ . For this, let  $v = \tilde{v} + \nabla \mathbf{s}_{v,\varepsilon} \in \mathbf{H}_N^\beta(\Omega)$  with  $\tilde{v} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathbf{s}_{v,\varepsilon} \in \mathcal{S}_\varepsilon^+$ . By means of item *iv*) of Proposition 6.7.1, the function  $\tilde{v}$  admits the decomposition  $\tilde{v} = \nabla \varphi_v + \mathbf{curl} \zeta_v$  with  $\varphi_v \in \mathbf{H}_0^1(\Omega)$  (such that  $\nabla \varphi \in \mathbf{V}_{-\beta}^0(\Omega)$ ) and  $\zeta_v \in \mathbf{X}_T(1)$ . By remarking that  $\mathbf{curl} v = \mathbf{curl} \mathbf{curl} \zeta_v$ , we infer  $\mathbf{curl} \zeta_v \in \mathbf{X}_N(1)$  and then by Proposition 6.7.2 we deduce that  $\mathbf{curl} \zeta_v$  belongs to  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in [0; 1/2)$ . Observing that  $\operatorname{div}(\varepsilon \mathbf{curl} \zeta_v) \in (\mathring{V}_\beta^1(\Omega))^*$  for all  $\beta \in (0; \beta_D)$  allows us to define the function  $w_v \in \mathring{V}_\beta^{\text{out}}(\Omega)$  as the unique solution of the well-posed problem

$$\operatorname{div}(\varepsilon \nabla w_v) = \operatorname{div}(\varepsilon (\mathbf{curl} \zeta_v + \nabla \mathbf{s}_{v,\varepsilon})) \in (\mathring{V}_\beta^1(\Omega))^*. \quad (6.24)$$

Now, we introduce  $\hat{v}$  such that  $\hat{v} = v - \nabla \varphi_v + \nabla w_v$ . By observing that  $\operatorname{div}(\varepsilon \hat{v}) = 0$  in  $\Omega \setminus \{O\}$ , we deduce that  $\hat{v} \in \mathbf{Y}_N^\beta(\varepsilon)$ . As a result, one can take  $\hat{v}$  as a test function in (6.23). But, on the other hand, we have

$$\left\{ \begin{array}{l} \int_{\Omega} \mu^{-1} \psi_{\mathbf{u}} \cdot \mathbf{curl} \bar{v} = \int_{\Omega} \mu^{-1} \psi_{\mathbf{u}} \cdot \mathbf{curl} \bar{\hat{v}} \\ \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{v} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\hat{v}} + \int_{\Omega} \varepsilon \mathbf{u} \cdot \nabla \overline{(w_v - \varphi_v)} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\hat{v}} \\ \int_{\Omega} \mathbf{J} \cdot \bar{v} = \int_{\Omega} \mathbf{J} \cdot \bar{\hat{v}}. \end{array} \right.$$

Hence,  $\mathbf{u}$  satisfies (6.20) which ends the proof of the first part of second item. The rest of the proof can be done as in Lemma 5.3.2. ■

### 6.4.3 Equivalent norms in $\mathbf{Y}_N^\beta(\varepsilon)$ and $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$

The goal of this section is to introduce, under some condition on  $\beta$ , new "simpler" equivalent norms for the spaces  $\mathbf{Y}_N^\beta(\varepsilon)$  and  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$ . Let us start with the case of the space  $\mathbf{Y}_N^\beta(\varepsilon)$ . Following the same lines of the proof of Proposition 6.5.2, one obtains the

**Proposition 6.4.2.** *Let  $\beta \in (0; \beta_D)$  and assume that Assumptions 6.2.1-6.2.2 are satisfied. Then there exists a positive constant  $C$  such that*

$$\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} + \|\nabla \mathfrak{s}_{\mathbf{u},\varepsilon}\|_{\nabla S_\varepsilon^+} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{Y}_N^\beta(\varepsilon). \quad (6.25)$$

Consequently, the norms  $\|\cdot\|_{\mathbf{Y}_N^\beta(\varepsilon)}$  and  $\|\mathbf{curl} \cdot\|_{\mathbf{V}_\beta^0(\Omega)}$  are equivalent in  $\mathbf{Y}_N^\beta(\varepsilon)$ .

Now, we turn our attention to the case of the space  $\mathbf{Y}_N^{\text{out}}(\varepsilon)$ . We have the

**Proposition 6.4.3.** *Suppose that Assumptions 6.2.1-6.2.2 hold and let  $\beta \in (0; \beta_0)$ . Then, there exists  $0 < C$  such that*

$$\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} + \|\nabla \mathfrak{s}_{\mathbf{u},\varepsilon}\|_{\nabla S_\varepsilon^+} \leq C \|\boldsymbol{\psi}_{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \text{ with } \mathbf{curl} \mathbf{u} = \boldsymbol{\psi}_{\mathbf{u}} + \nabla \mathfrak{s}_{\mathbf{u},\mu}. \quad (6.26)$$

Consequently  $\mathbf{u} \mapsto \|\boldsymbol{\psi}_{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}$  is a norm in  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  that is equivalent to  $\|\cdot\|_{\mathbf{H}_N^\beta(\mathbf{curl})}$

**Proof.** Since for all  $0 < \beta$ , we have the inclusion  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \subset \mathbf{Y}_N^\beta(\varepsilon)$  and, by means of Proposition 6.4.2, it suffices to show that, for all  $\beta \in (0; \beta_0) \subset (0; \beta_D)$ , we have the following estimate

$$\|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)} \leq C \|\boldsymbol{\psi}_{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon).$$

By definition of the space  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$ , we know that for all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  we have  $\mathbf{curl} \mathbf{u} = \boldsymbol{\psi}_{\mathbf{u}} + \nabla \mathfrak{s}_{\mathbf{u},\mu}$ . Hence, we have the estimate

$$\|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)} \leq C (\|\nabla \mathfrak{s}_{\mathbf{u},\mu}\|_{\mathbf{V}_\beta^0(\Omega)} + \|\boldsymbol{\psi}_{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}) \leq C (\|\nabla \mathfrak{s}_{\mathbf{u},\mu}\|_{S_\mu^+} + \|\boldsymbol{\psi}_{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}) \quad (6.27)$$

with  $C$  independent of  $\mathbf{u}$ . Now, given that for all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  we have  $\text{div}(\mathbf{curl} \mathbf{u}) = 0$  we then obtain that

$$-\text{div}(\mu \nabla \mathfrak{s}_{\mathbf{u},\mu}) = \text{div} \boldsymbol{\psi}_{\mathbf{u}}.$$

Using the fact that  $\mathbf{u} \times \boldsymbol{\nu} = 0$  on  $\partial\Omega$  we deduce that  $\mathbf{curl} \mathbf{u} \cdot \boldsymbol{\nu} = 0$  on  $\partial\Omega$  and then we conclude that  $\boldsymbol{\psi}_{\mathbf{u}} \cdot \boldsymbol{\nu} = 0$  on  $\partial\Omega$ . With this in mind, we can say that  $\text{div} \boldsymbol{\psi}_{\mathbf{u}} \in (\mathcal{V}_\beta^1(\Omega))^*$ . Consequently, we can write that for all  $v \in \mathcal{V}_\beta^1(\Omega)$  we have

$$\langle A_\mu^{\text{out}} \mathfrak{s}_{\mathbf{u},\mu}, v \rangle = - \int_\Omega \text{div}(\mu \nabla \mathfrak{s}_{\mathbf{u},\mu}) \bar{v} = \int_\Omega \boldsymbol{\psi}_{\mathbf{u}} \cdot \nabla \bar{v}.$$

Given that  $A_\mu^{\text{out}}$  is an isomorphism for all  $\beta \in (0; \beta_0)$ , we infer that we have the estimate

$$\|\mathfrak{s}_{\mathbf{u},\mu}\|_{S_\mu^+} \leq \|\boldsymbol{\psi}_{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}$$

with  $C$  independent of  $\mathbf{u}$ . Inserting this into (6.27) yields the wanted result.  $\blacksquare$

### 6.4.4 Analysis of the principal part

In this section, we shall study well-posedness of the problem (6.23) when  $\omega = 0$ . For this reason, we introduce the continuous operator  $\mathbb{A}_N^\beta : \mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \mapsto (\mathbf{Y}_N^\beta(\varepsilon))^*$  such that for all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  and  $\mathbf{v} \in \mathbf{Y}_N^\beta(\varepsilon)$  we have

$$\langle \mathbb{A}_N^\beta \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mu^{-1} \boldsymbol{\psi}_{\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}}.$$

Before getting into details, observe that for all  $\mathbf{u}, \mathbf{v} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  we have

$$\langle \mathbb{A}_N^\beta \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \boldsymbol{\psi}_{\mathbf{u}} \cdot \overline{\boldsymbol{\psi}_{\mathbf{v}}} + \int_{\Omega} \operatorname{div}(\mu \nabla \mathfrak{s}_{\mathbf{u},\mu}) \overline{\mathfrak{s}_{\mathbf{v},\mu}}.$$

Note that to obtain the previous relation, we have used the fact that for all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  we have

$$\operatorname{div}(\boldsymbol{\psi}_{\mathbf{u}}) = -\operatorname{div}(\mu \nabla \mathfrak{s}_{\mathbf{u},\mu}).$$

**Theorem 6.4.2.** *Assume that Assumptions 6.2.1-6.2.2 hold. Let  $\beta \in (0; \beta_0)$ . Then, there exists a continuous operator  $\mathbf{T} : \mathbf{Y}_N^\beta(\varepsilon) \rightarrow \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  such that*

$$\langle \mathbb{A}_N^\beta \circ \mathbf{T} \mathbf{u}, \mathbf{v} \rangle := \int_{\Omega} r^{2\beta} \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \overline{\mathbf{v}} \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{Y}_N^\beta(\varepsilon).$$

As a consequence, the operator  $\mathbb{A}_N \circ \mathbf{T} : \mathbf{Y}_N^\beta(\varepsilon) \rightarrow (\mathbf{Y}_N^\beta(\varepsilon))^*$  is an isomorphism.

**Proof.** The construction of the operator  $\mathbf{T}$  will be done in three steps. Let us consider some  $\mathbf{u} \in \mathbf{Y}_N^\beta(\varepsilon)$ .

**First step:** Since  $\beta \in (0; \beta_0) \subset (0; \beta_N)$ , the operator  $\mathbb{A}_\mu^{\text{out}}$  is an isomorphism. As a result, one can introduce  $\varphi_{\mathbf{u}} = \tilde{\varphi}_{\mathbf{u}} + \mathfrak{s}_{\mathbf{u}}^\mu \in \mathcal{V}_\beta^{\text{out}}(\Omega)$  (with  $\tilde{\varphi}_{\mathbf{u}} \in \mathcal{V}_{-\beta}^1(\Omega)$  and  $\mathfrak{s}_{\mathbf{u}}^\mu \in \mathcal{S}_\mu$ ) as the unique solution of

$$\begin{cases} -\operatorname{div}(\mu \nabla \varphi_{\mathbf{u}}) = \operatorname{div}(\mu r^{2\beta} \operatorname{curl} \mathbf{u}) \in (\mathcal{V}_\beta^1(\Omega))^* & \text{in } \Omega \\ (\nabla \varphi_{\mathbf{u}} + r^{2\beta} \operatorname{curl} \mathbf{u}) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $\mathbb{A}_\mu^{\text{out}}$  is an isomorphism, the function  $\varphi_{\mathbf{u}}$  satisfies the following estimate

$$\|\tilde{\varphi}_{\mathbf{u}}\|_{\mathcal{V}_{-\beta}^1(\Omega)} + \|\nabla \mathfrak{s}_{\mathbf{u}}^\mu\|_{\nabla \mathcal{S}_\mu^+} \leq C \|r^{2\beta} \operatorname{curl} \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} = C \|\operatorname{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}. \quad (6.28)$$

**Second step:** We define the function  $\mathbf{F}_{\mathbf{u}} = \mu(\nabla \varphi_{\mathbf{u}} + r^{2\beta} \operatorname{curl} \mathbf{u}) \in \mathbf{V}_\beta^0(\Omega)$ . Easily, one can see that  $\mathbf{F}_{\mathbf{u}}$  satisfies

$$\begin{cases} \operatorname{div}(\mathbf{F}_{\mathbf{u}}) = 0 & \text{in } \Omega \\ \mathbf{F}_{\mathbf{u}} \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $\beta < 1/2$ , one can use Proposition 6.7.1 to deduce that there exists a unique  $\boldsymbol{\psi}_{\mathbf{u}} \in \mathbf{Z}_N^\beta(1)$  (see (6.42) for the definition of  $\boldsymbol{\psi}_{\mathbf{u}} \in \mathbf{Z}_N^\beta(1)$ ) such that

$$\operatorname{curl} \boldsymbol{\psi}_{\mathbf{u}} = \mathbf{F}_{\mathbf{u}} = \mu(\nabla \varphi_{\mathbf{u}} + r^{2\beta} \operatorname{curl} \mathbf{u}).$$

Furthermore, by means of Proposition 6.7.2, since  $\beta_0 < 1/2$ , the function  $\boldsymbol{\psi}_{\mathbf{u}}$  belongs to the space  $\mathbf{V}_{-\beta}^0(\Omega)$ .

**Third step:** Since by assumption  $\beta \in (0; \beta_0) \subset (0; \beta_D)$ , the operator  $\mathbb{A}_\varepsilon^{\text{out}}$  is an isomorphism. This allows us to define  $w_{\mathbf{u}} = \tilde{w}_{\mathbf{u}} + \mathfrak{s}_{\mathbf{u}}^\varepsilon \in \mathring{\mathcal{V}}_\beta^{\text{out}}(\Omega)$  (with  $\tilde{w}_{\mathbf{u}} \in \mathring{\mathcal{V}}_{-\beta}^1(\Omega)$  and  $\mathfrak{s}_{\mathbf{u}}^\varepsilon \in \mathcal{S}_\varepsilon^+$ ) as the unique solution of the problem

$$\begin{cases} \operatorname{div}(\varepsilon \nabla w_{\mathbf{u}}) = \operatorname{div}(\varepsilon \boldsymbol{\psi}_{\mathbf{u}}) \in (\mathring{\mathcal{V}}_\beta^1(\Omega))^* & \text{in } \Omega \\ w_{\mathbf{u}} = 0 & \text{on } \partial\Omega. \end{cases}$$

We set  $\mathbf{T}(\mathbf{u}) = \boldsymbol{\psi}_{\mathbf{u}} - \nabla w_{\mathbf{u}}$ . One can check that  $\mathbf{T}(\mathbf{u})$  belongs to the space  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$ . In addition to that we have

$$\begin{cases} \boldsymbol{\psi}_{\mathbf{T}(\mathbf{u})} = \mu(\nabla \tilde{\varphi}_{\mathbf{u}} + r^{2\beta} \operatorname{curl} \mathbf{u}) \\ \mathfrak{s}_{\mathbf{T}(\mathbf{u}),\mu} = \mathfrak{s}_{\mathbf{u}}^\mu. \end{cases}$$

Using (6.28), we obtain the estimate

$$\|\psi_{T(u)}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{\beta}^0(\Omega)}.$$

This shows, in particular, that the operator  $T : \mathbf{Y}_N^\beta(\varepsilon) \rightarrow \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  is continuous. Furthermore, we have

$$\langle \mathbb{A}_N^\beta \circ T \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \psi_{T\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}} = \int_{\Omega} r^{2\beta} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{Y}_N^\beta(\varepsilon).$$

Consequently, we have

$$\langle \mathbb{A}_N^\beta \circ T \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{\beta}^0(\Omega)}^2.$$

Using the Lax-Milgram lemma, we deduce that  $\mathbb{A}_N^\beta \circ T$  is an isomorphism and we finish the proof.  $\blacksquare$

Given that  $\mathbb{A}_N^\beta \circ T$  is an isomorphism, we deduce that the operator  $\mathbb{A}_N^\beta$  is onto. As a result, to show that  $\mathbb{A}_N^\beta$  is an isomorphism it suffices to prove that it is injective. This is the subject of the following

**Theorem 6.4.3.** *Assume that Assumptions 6.2.1-6.2.2 hold. Then for all  $\beta \in (0; \beta_0)$  the operator  $\mathbb{A}_N$  is an isomorphism.*

**Proof.** It is enough to prove that  $\mathbb{A}_N^\beta$  is injective. Let  $\mathbf{u}$  be an element of  $\mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  such that  $\langle \mathbb{A}_N^\beta \mathbf{u}, \mathbf{v} \rangle = 0$  for all  $\mathbf{v} \in \mathbf{Y}_N^\beta(\varepsilon)$ . In particular,  $\mathbf{u}$  satisfies

$$\int_{\Omega} \mu^{-1} \psi_{\mathbf{u}} \cdot \overline{\psi_{\mathbf{v}}} + \int_{\Omega} \operatorname{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u},\mu}) \overline{\mathfrak{s}_{\mathbf{v},\mu}} = 0 \quad \text{for all } \mathbf{v} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon). \quad (6.29)$$

By taking  $\mathbf{v} = \mathbf{u}$  in the previous equation, we obtain

$$0 = \int_{\Omega} \mu^{-1} \psi_{\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}} = \int_{\Omega} \mu^{-1} |\psi_{\mathbf{u}}|^2 - \int_{\Omega} \operatorname{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u},\mu}) \overline{\mathfrak{s}_{\mathbf{u},\mu}}.$$

Taking the imaginary part in the previous relation yields,  $\Im m(-\int_{\Omega} \operatorname{div}(\varepsilon \nabla \mathfrak{s}_{\mathbf{u},\mu}) \overline{\mathfrak{s}_{\mathbf{u},\mu}}) = 0$ . As a result, we deduce that  $q_{\varepsilon}(\mathfrak{s}_{\mathbf{u},\mu}, \mathfrak{s}_{\mathbf{u},\mu}) = 0$ . Given that  $\mathfrak{s}_{\mathbf{u},\mu} \in \mathcal{S}_{\mu}^+$  and by using Lemma 6.2.5, we infer that  $\mathfrak{s}_{\mathbf{u},\mu} = 0$ . As a result, we then have

$$\langle \mathbb{A}_N^\beta \mathbf{v}, \mathbf{u} \rangle = 0 \quad \text{for all } \mathbf{v} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon).$$

The wanted result is then proved by taking  $\mathbf{v} = T(u)$  (in the previous relation) where  $T$  is the operator defined in Theorem 6.4.2.  $\blacksquare$

### 6.4.5 Compactness result

In the classical theory of Maxwell's equation, imposing the constraint  $\operatorname{div}(\varepsilon \cdot) = 0$  leads to a compactness result. Here, we shall show that this result remains true even in our configuration. Let us introduce the operator  $\mathbb{K}_N^\beta : \mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \rightarrow (\mathbf{Y}_N^\beta(\varepsilon))^*$  such that for all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon)$  and for all  $\mathbf{v} \in \mathbf{Y}_N^\beta(\varepsilon)$  we have

$$\langle \mathbb{K}_N^\beta \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \tilde{\mathbf{v}} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}} + \int_{\Omega} \operatorname{div}(\varepsilon \overline{\mathfrak{s}_{\mathbf{v},\varepsilon}}) \mathfrak{s}_{\mathbf{u},\varepsilon}.$$

Without any difficulty, one can see that we have the estimate

$$\|\mathbb{K}_N^\beta \mathbf{u}\|_{(\mathbf{Y}_N^\beta(\varepsilon))^*} \leq C (\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0} + \|\mathfrak{s}_{\mathbf{u},\varepsilon}\|_{\mathcal{S}_{\varepsilon}^+}), \quad \text{for all } \mathbf{u} \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon, \mu). \quad (6.30)$$

with  $0 < C$  independent of  $\mathbf{u}$ .

**Theorem 6.4.4.** *For all  $\beta \in (0; \beta_D)$  and under Assumptions 6.2.1-6.2.2, the operator  $\mathbb{K}_N^\beta$  is compact.*

**Proof.** Let  $(\mathbf{u}_n)_{n \in \mathbb{N}}$  be a bounded sequence of elements of  $\mathbf{Y}_N^{\text{out}}(\varepsilon)$ . By definition of  $\mathbf{Y}_N^{\text{out}}(\varepsilon)$  one can introduce, for all  $n \in \mathbb{N}$ ,  $\tilde{\mathbf{u}}_n \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathfrak{s}_n \in \mathcal{S}_\varepsilon^+$  such that  $\mathbf{u}_n = \tilde{\mathbf{u}}_n + \mathfrak{s}_n$ . The sequences  $(\mathfrak{s}_n)_{n \in \mathbb{N}}$  and  $(\tilde{\mathbf{u}}_n)_{n \in \mathbb{N}}$  are bounded respectively in  $\mathcal{S}_\varepsilon^+$  and  $\mathbf{V}_{-\beta}^0(\Omega)$ . Since  $\mathcal{S}_\varepsilon^+$  is finite dimensional, one can extract a sub-sequence form  $(\mathfrak{s}_n)_{n \in \mathbb{N}}$  that converges in  $\mathcal{S}_\varepsilon^+$ . For simplicity, this sub-sequence will be denoted by  $(\mathfrak{s}_n)_{n \in \mathbb{N}}$ .

To prove our claim, thanks to (6.30), it is enough to show that up to a sub-sequence  $(\mathbf{u}_n)_{n \in \mathbb{N}}$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . Since  $\mathbf{V}_{-\beta}^0 \subset \mathbf{L}^2(\Omega)$ , we start by writing the Helmholtz decomposition (see Proposition 6.7.1) of  $\tilde{\mathbf{u}}_n$ :  $\tilde{\mathbf{u}}_n = \nabla \varphi_n + \mathbf{curl} \psi_n$  in which  $\psi_n \in \mathbf{X}_T(1)$  and  $\varphi_n \in \mathbf{H}_0^1(\Omega)$ . Since  $\nabla \varphi_n \times \nu = 0$  on  $\partial\Omega$  and  $\mathbf{curl} \mathbf{curl} \psi_n = \mathbf{curl} \mathbf{u}_n \in \mathbf{V}_\beta^0(\Omega)$ , we infer that  $\mathbf{curl} \psi_n \in \mathbf{Z}_N^\beta(1)$  (see (6.42)). Since by assumption  $\beta \in (0; \beta_D) \subset [0; 1/2)$  one deduces, using Proposition 6.7.5, that  $(\mathbf{curl} \psi_n)_{n \in \mathbb{N}}$  converges, up to a sub-sequence still denoted  $(\mathbf{curl} \psi_n)_{n \in \mathbb{N}}$ , in  $\mathbf{V}_{-\beta}^0(\Omega)$ . In the other hand, we know that  $\text{div}(\varepsilon \mathbf{u}_n) = 0$ . This implies that

$$-\text{div}(\varepsilon \nabla \varphi_n) = \text{div}(\varepsilon \nabla \mathfrak{s}_n) + \text{div}(\varepsilon \mathbf{curl} \psi_n) \in (\dot{\mathbf{V}}_\beta^1(\Omega))^* \quad \text{for all } \beta \in (0; \beta_D). \quad (6.31)$$

According to Lemma 6.2.3, we infer that  $\varphi_n \in \dot{\mathbf{V}}_{-\beta}^1(\Omega)$  for all  $n \in \mathbb{N}$ . Furthermore, by remarking that the right hand side of (6.31) converges in  $(\dot{\mathbf{V}}_{-\beta}^1(\Omega))^*$  and by using the fact that the operator  $\mathbb{A}_\varepsilon^{\text{out}}$  is an isomorphism for all  $\beta \in (0; \beta_D)$ , we deduce that  $(\varphi_n)_{n \in \mathbb{N}}$  converges in  $\dot{\mathbf{V}}_{-\beta}^1(\Omega)$  and then we arrive to the wanted result:  $(\tilde{\mathbf{u}}_n)_{n \in \mathbb{N}}$  converges, up to a sub-sequence, in  $\mathbf{V}_{-\beta}^0(\Omega)$ .  $\blacksquare$

#### 6.4.6 Main results about the electric problem

For all  $\omega \in \mathbb{R}$  and all  $\beta \in \mathbb{R}$ , we define the operator  $\mathcal{A}_N^\beta(\omega) := \mathbb{A}_N^\beta - \omega^2 \mathbb{K}_N^\beta$ . For all  $\mathbf{u} \in \mathbf{Y}_N^{\text{out}, \beta}(\varepsilon)$  and all  $\mathbf{v} \in \mathbf{Y}_N^\beta(\varepsilon)$  we have

$$\langle \mathcal{A}_N^\beta(\omega) \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mu^{-1} \psi_{\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_\Omega \varepsilon \tilde{\mathbf{u}} \cdot \bar{\mathbf{v}} - \omega^2 \int_\Omega \text{div}(\varepsilon \overline{\mathfrak{s}_{\mathbf{u}, \varepsilon}}) \mathfrak{s}_{\mathbf{u}, \varepsilon}.$$

In particular for all  $\mathbf{u}, \mathbf{v} \in \mathbf{Y}_N^{\text{out}, \beta}(\varepsilon)$  we have

$$\langle \mathcal{A}_N^\beta(\omega) \mathbf{u}, \mathbf{v} \rangle = \int_\Omega \mu^{-1} \psi_{\mathbf{u}} \cdot \overline{\psi_{\mathbf{v}}} + \int_\Omega \text{div}(\mu \nabla \mathfrak{s}_{\mathbf{u}, \varepsilon}) \overline{\mathfrak{s}_{\mathbf{v}, \mu}} - \omega^2 \int_\Omega \varepsilon \tilde{\mathbf{u}} \cdot \bar{\mathbf{v}} - \omega^2 \int_\Omega \text{div}(\varepsilon \overline{\mathfrak{s}_{\mathbf{v}, \varepsilon}}) \mathfrak{s}_{\mathbf{u}, \varepsilon}.$$

All this to say that for all  $\mathbf{u}, \mathbf{v} \in \mathbf{Y}_N^{\text{out}, \beta}(\varepsilon)$  we have

$$\langle \mathcal{A}_N^\beta(\omega) \mathbf{u}, \mathbf{v} \rangle - \overline{\langle \mathcal{A}_N^\beta(\omega) \mathbf{v}, \mathbf{u} \rangle} = -q_\mu(\mathfrak{s}_{\mathbf{u}, \mu}, \mathfrak{s}_{\mathbf{v}, \mu}) - \omega^2 q_\varepsilon(\mathfrak{s}_{\mathbf{u}, \varepsilon}, \mathfrak{s}_{\mathbf{v}, \varepsilon})$$

By combing Theorems 6.4.3-6.4.4 with the analytical Fredholm theorem, we obtain the

**Theorem 6.4.5.** *Assume that Assumptions 6.2.1-6.2.2 hold. For all  $\beta \in (0; \beta_0)$  the operator  $\mathcal{A}_N^\beta(\omega)$  is a Fredholm operator of index zero for all  $\omega \in \mathbb{R}$ . Furthermore, there exists  $S_N^\beta$  discrete subset of  $\mathbb{R}$  such that  $\mathcal{A}_N^\beta(\omega)$  is an isomorphism for  $\omega \in \mathbb{R} \setminus S_N^\beta$ .*

**Proposition 6.4.4.** *Assume that Assumptions 6.2.1-6.2.2 hold and let  $\omega \in \mathbb{R}^*$ . Then, if  $\mathbf{u} \in \text{Ker}(\mathcal{A}_N^\gamma(\omega))$  for some  $\gamma \in (0; \beta_0)$ . Then*

- $\mathfrak{s}_{\mathbf{u}, \varepsilon} = \mathfrak{s}_{\mathbf{u}, \mu} = 0$ .
- $\mathbf{u} \in \text{Ker}(\mathcal{A}_N^\beta(\omega))$  for all  $\beta \in (0; \beta_0)$ .



**Proof.** Starting from  $\mathcal{A}_N^\gamma(\omega)\mathbf{u} = 0$ , we deduce that  $\langle \mathcal{A}_N^\gamma(\omega)\mathbf{u}, \mathbf{u} \rangle - \overline{\langle \mathcal{A}_N^\gamma(\omega)\mathbf{v}, \mathbf{u} \rangle} = 0$ . Hence, we obtain that

$$q_\varepsilon(\mathfrak{s}_{\mathbf{u},\varepsilon}, \mathfrak{s}_{\mathbf{u},\varepsilon}) = -\omega^2 q_\mu(\mathfrak{s}_{\mathbf{u},\mu}, \mathfrak{s}_{\mathbf{u},\mu}).$$

Given that  $\mathfrak{s}_{\mathbf{u},\varepsilon} \in \mathcal{S}_\varepsilon^+$  and  $\mathfrak{s}_{\mathbf{u},\mu} \in \mathcal{S}_\mu^+$ , we infer that  $q_\varepsilon(\mathfrak{s}_{\mathbf{u},\varepsilon}, \mathfrak{s}_{\mathbf{u},\varepsilon}) = q_\mu(\mathfrak{s}_{\mathbf{u},\mu}, \mathfrak{s}_{\mathbf{u},\mu}) = 0$ . The Lemma 6.2.5 allows us to say that  $\mathfrak{s}_{\mathbf{u},\varepsilon} = \mathfrak{s}_{\mathbf{u},\mu} = 0$ . As a result the vector field  $\mathbf{u}$  belongs then to the space  $\mathbf{X}_N(\varepsilon)$  and satisfies the equation  $\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u} = \omega^2 \varepsilon \mathbf{u}$ . This shows that  $\mu^{-1} \mathbf{curl} \mathbf{u} \in \mathbf{X}_T(\mu)$ . Thanks to Propositions 6.7.3-6.7.4 and since  $\beta_0 < 1/2$ , we infer that  $\mathbf{u}, \mathbf{curl} \mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in (0; \beta_0)$ . Thus  $\mathbf{u} \in \text{Ker}(\text{Ker}(\mathcal{A}_N^\beta(\omega)))$  for all  $\beta \in (0; \beta_0)$ . ■

Because for all  $\beta \in (0; \beta_0)$  the set  $S_N^\beta$  in Theorem 6.4.5 corresponds to the set of  $\omega \in \mathbb{R}$  for which the operator  $\mathcal{A}_N^\beta(\omega)$  is not injective, we then have the

**Proposition 6.4.5.** *Assume that Assumptions 6.2.1-6.2.2 hold. Then the set  $S_N^\beta$  in Theorem 6.4.5 is independent of  $\beta \in (0; \beta_0)$ . We denote it by  $S_N$ .*

The two previous propositions allow us to deduce that the functional framework that we have proposed is independent of  $\beta$ . Indeed, we have the following

**Lemma 6.4.1.** *Suppose that Assumptions 6.2.1-6.2.2 are valid and let  $\omega \in \mathbb{R} \setminus S_N$ . Let  $0 < \beta_1 < \beta_2 < \beta_0$ . For all  $\ell \in (\mathbf{Y}_N^{\beta_1}(\varepsilon))^* \cap (\mathbf{Y}_N^{\beta_2}(\varepsilon))^*$  then the solutions to the problems:*

$$\text{Find } \mathbf{u}^\beta \in \mathbf{Y}_N^{\text{out},\beta}(\varepsilon) \text{ such that } \mathcal{A}_N^\beta(\omega)\mathbf{u}^\beta = \ell$$

with  $\beta = \beta_1$  and  $\beta = \beta_2$  coincide.

**Proof.** Since  $\beta_1 < \beta_2$  we have  $\mathbf{Y}_N^{\text{out},\beta_2}(\varepsilon) \subset \mathbf{Y}_N^{\text{out},\beta_1}(\varepsilon)$ . Therefore  $\mathbf{u}^{\beta_1} - \mathbf{u}^{\beta_2} \in \text{Ker}(\mathcal{A}_N^{\beta_1}(\omega))$ . Given that  $\omega \in \mathbb{R} \setminus S_N$ , we get  $\mathbf{u}^{\beta_1} = \mathbf{u}^{\beta_2}$ . ■

Now, we state the main result of this section.

**Theorem 6.4.6.** *Assume that Assumptions 6.2.1-6.2.2 hold. Suppose that there exists  $0 < \gamma$  such that  $\mathbf{J} \in \mathbf{V}_{-\gamma}^0(\Omega)$ . Then*

- for all  $\omega \in \mathbb{R} \setminus S_N$  the problem (6.20) (or equivalently (6.23)) is well-posed in the Hadamard sense for all  $\beta \in (0; \beta_0)$ .
- The solution to (6.20) (or equivalently to (6.23)) is independent of  $\beta \in (0; \beta_0)$ .
- When  $\omega \in S_N$  the problems (6.20) - (6.23) are well-posed in the Fredholm sense. Moreover, they have a finite dimensional kernel that is independent of  $\beta \in (0; \beta_0)$ .

### 6.4.7 The limiting absorption principle for the electric problem

As pointed out in Remark 6.2.1, one can find an infinite number of spaces  $\mathcal{S}_\varepsilon^+$  and  $\mathcal{S}_\mu^+$  that are spanned by bases for which all the conditions of Lemma 6.2.4 are satisfied. This implies that, we can construct an infinite number of functional frameworks in which the electric problem is well-posed. The goal of this paragraph is to explain explain how to chose among these functional frameworks the one that is coherent with the limiting absorption principle. For this, we start by defining for all  $0 < \delta$  the functions  $\varepsilon_\delta := \varepsilon + i\delta$  and  $\mu_\delta := \mu + i\delta$ . We introduce the operator  $A_{\varepsilon_\delta} : H_0^1(\Omega) \rightarrow (H_0^1(\Omega))^*$  and  $A_{\mu_\delta} : H_{\#}^1(\Omega) \rightarrow (H_{\#}^1(\Omega))^*$  that are, respectively, defined by replacing  $\varepsilon$  and  $\mu$  in the definition of  $A_\varepsilon$  and  $A_\mu$  by  $\varepsilon_\delta$  and  $\mu_\delta$ . One can easily check that  $A_{\varepsilon_\delta}$  and  $A_{\mu_\delta}$  are isomorphisms for all  $0 < \delta$ . From the results obtained in §2.6.2, we know that we have the following

**Proposition 6.4.6.** *Assume that the functions  $\varepsilon$  and  $\mu$  are such that Assumptions 2.6.3-6.2.1-6.2.2 are satisfied when replacing the function  $\sigma$  by  $\varepsilon$  (resp.  $\mu$ ) and let  $\beta \in (0; \beta_D)$  (resp.  $\beta \in (0; \beta_N)$ ).*

*Let  $(f_\delta)_\delta$  (resp.  $(g_\delta)_\delta$ ) be a sequence of elements of  $(\mathring{V}_\beta^1(\Omega))^*$  (resp.  $(\mathcal{V}_\beta^1(\Omega))^*$ ) and define  $u_{\varepsilon_\delta}$  (resp.  $u_{\mu_\delta}$ ) as the unique solution to  $A_{\varepsilon_\delta} u_{\varepsilon_\delta} = f_\delta$  (resp.  $A_{\mu_\delta} u_{\mu_\delta} = g_\delta$ ).*

*There exists  $0 < \delta_0$  such that for all  $\delta \in (0; \delta_0)$  there exist  $N_\varepsilon$  (resp.  $N_\mu$ ) linearly independent functions denoted by  $(\mathfrak{s}_{\varepsilon_\delta, j}^+)_{j=1, \dots, N_\varepsilon}$  (resp.  $(\mathfrak{s}_{\mu_\delta, j}^+)_{j=1, \dots, N_\mu}$ ) that belong to  $H_0^1(\Omega)$  (resp.  $H_{\#}^1(\Omega)$ ) such that*

- *the function  $u_{\varepsilon_\delta}$  (resp.  $u_{\mu_\delta}$ ) decomposes as*

$$u_{\varepsilon_\delta} = \tilde{u}_{\varepsilon_\delta} + \sum_{j=1}^{N_\varepsilon} c_{j,\delta} \mathfrak{s}_{\varepsilon_\delta, j}^+ \quad (\text{resp. } u_{\mu_\delta} = \tilde{u}_{\mu_\delta} + \sum_{j=1}^{N_\mu} c'_{j,\delta} \mathfrak{s}_{\mu_\delta, j}^+)$$

*in which  $\tilde{u}_{\varepsilon_\delta} \in \mathring{V}_{-\beta}^1(\Omega)$  (resp.  $\tilde{u}_{\mu_\delta} \in \mathcal{V}_{-\beta}^1(\Omega)$ ) and all the  $c_{j,\delta}$  (resp.  $c'_{j,\delta}$ ) are complex numbers. For all  $\delta \in (0; \delta_0)$ , denote by*

$$\mathcal{S}_{\varepsilon_\delta}^+ := \text{span}(\mathfrak{s}_{\varepsilon_\delta, j}^+, j = 1, \dots, N_\varepsilon) \quad (\text{resp. } \mathcal{S}_{\mu_\delta}^+ := \text{span}(\mathfrak{s}_{\mu_\delta, j}^+, j = 1, \dots, N_\mu)). \quad (6.32)$$

- *For each  $j = 1, \dots, N_\varepsilon$  (resp.  $j = 1, \dots, N_\mu$ ) the function  $\mathfrak{s}_{\varepsilon_\delta, j}^+$  (resp.  $\mathfrak{s}_{\mu_\delta, j}^+$ ) converges as  $\delta \rightarrow 0$  in  $\mathring{V}_\beta^1(\Omega)$  (resp.  $\mathcal{V}_\beta^1(\Omega)$ ) to  $\hat{\mathfrak{s}}_{\varepsilon, j}^+ \in \mathcal{S}_\varepsilon$  (resp.  $\hat{\mathfrak{s}}_{\mu, j}^+ \in \mathcal{S}_\mu$ ) moreover  $\{\hat{\mathfrak{s}}_{\varepsilon, j}^+, \overline{\hat{\mathfrak{s}}_{\varepsilon, j}^+}, j = 1, \dots, N_\varepsilon\}$  (resp.  $\{\hat{\mathfrak{s}}_{\mu, j}^+, \overline{\hat{\mathfrak{s}}_{\mu, j}^+}, j = 1, \dots, N_\mu\}$ ) is a basis of  $\mathcal{S}_\varepsilon$  (resp.  $\mathcal{S}_\mu$ ) satisfying the conditions of Lemma 6.2.4. Denote by*

$$\hat{\mathcal{S}}_\varepsilon^+ := \text{span}(\hat{\mathfrak{s}}_{\varepsilon, j}^+, j = 1, \dots, N_\varepsilon) \quad (\text{resp. } \hat{\mathcal{S}}_\mu^+ := \text{span}(\hat{\mathfrak{s}}_{\mu, j}^+, j = 1, \dots, N_\mu)). \quad (6.33)$$

- *If the sequence  $(f_\delta)_\delta$  (resp.  $(g_\delta)_\delta$ ) converges, as  $\delta \rightarrow 0$ , in  $(\mathring{V}_\beta^1(\Omega))^*$  (resp.  $(\mathcal{V}_\beta^1(\Omega))^*$ ) to  $f$  (resp.  $g$ ) then the sequences  $(u_{\varepsilon_\delta})_\delta$  (resp.  $(u_{\mu_\delta})_\delta$ ) converges, as  $\delta \rightarrow 0$ , in  $\mathring{V}_\beta^1(\Omega)$  (resp.  $\mathcal{V}_\beta^1(\Omega)$ ) to the unique solution of  $\hat{A}_\varepsilon^{\text{out}} u = f$  (resp.  $\hat{A}_\mu^{\text{out}} u = g$ ) where the operator  $\hat{A}_\varepsilon^{\text{out}}$  (resp.  $\hat{A}_\mu^{\text{out}}$ ) is defined in the case  $A_\varepsilon^{\text{out}}$  (resp.  $A_\mu^{\text{out}}$ ) but by replacing  $\mathcal{S}_\varepsilon^+$  by  $\hat{\mathcal{S}}_\varepsilon^+$  (resp.  $\hat{\mathcal{S}}_\mu^+$ )*

By classical arguments, we know that  $0 < \delta$  and all  $\omega \in \mathbb{R}$ , the problem

$$\text{Find } \mathbf{u}_\delta \in \mathbf{X}_N(\varepsilon^\delta) \text{ such that } \mathbf{curl}(\mu_\delta)^{-1} \mathbf{curl} \mathbf{u}_\delta - \omega^2 \varepsilon_\delta \mathbf{u}_\delta = \mathbf{J} \quad (6.34)$$

is well posed for all  $\mathbf{J} \in \mathbf{L}^2(\Omega)$  satisfying  $\text{div}(\mathbf{J}) = 0$ . Our goal is to study the behavior of  $(\mathbf{u}_\delta)$  as  $\delta \rightarrow 0$ .

By following the same steps of the proof of Lemma 5.3.8 and by using the results of Proposition 6.7.5, we can prove that we have

**Lemma 6.4.2.** *Assume that Assumptions 2.6.3-6.2.1-6.2.2 hold. Let  $\beta \in (0; \beta_D)$  (resp.  $\beta \in (0; \beta_D)$ ) and let  $(\delta_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers that converges to 0 as  $n \rightarrow +\infty$ . Denote  $\varepsilon_n = \varepsilon + i\delta_n$  (resp.  $\mu_n = \mu + i\delta_n$ ) for all  $n \in \mathbb{N}$ . Suppose that  $(\mathbf{u}_n)$  is a sequence of elements of  $\mathbf{X}_N(\varepsilon_n)$  (resp.  $\mathbf{X}_T(\mu_n)$ ) such that  $(\mathbf{curl} \mathbf{u}_n)$  is bounded in  $\mathbf{V}_\beta^0(\Omega)$ . Then,  $(\mathbf{u}_n)$  for all  $n \in \mathbb{N}$ ,  $\mathbf{u}_n$  admits the decomposition  $\mathbf{u}_n = \nabla s_n + \tilde{\mathbf{u}}_n$  with  $s_n \in \mathcal{S}_{\varepsilon_n}^+$  (resp.  $s_n \in \mathcal{S}_{\mu_n}^+$ ) (see (6.32)) and  $\tilde{\mathbf{u}}_n \in \mathbf{V}_{-\beta}^0(\Omega)$ . Moreover, there exists a sub-sequence of  $(\mathbf{u}_n)$  that converges in  $\mathbf{V}_\beta^0(\Omega)$  to an element of  $\mathbf{V}_{-\beta}^0(\Omega) \oplus \nabla \hat{\mathcal{S}}_\varepsilon^+$  (resp.  $\mathbf{V}_{-\beta}^0(\Omega) \oplus \nabla \hat{\mathcal{S}}_\mu^+$ ) (see (6.33)).*

For all  $\beta \in \mathbb{R}$ , we introduce the space  $\mathbf{W}_N^\beta(\Omega) := \{\mathbf{u} \in \mathbf{V}_\beta^0(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega)\}$ . Endowed with the norm

$$\|\mathbf{u}\|_{\mathbf{W}_\beta^0(\Omega)} = (\|\mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}^2 + \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}^2)^{1/2} \text{ for all } \mathbf{u} \in \mathbf{W}_\beta^0(\Omega),$$

the space  $\mathbf{W}_\beta^0(\Omega)$  is a Hilbert space. By replacing the spaces  $\mathcal{S}_\varepsilon^+$  and  $\mathcal{S}_\mu^+$ , respectively, by  $\hat{\mathcal{S}}_\varepsilon^+$  and  $\hat{\mathcal{S}}_\mu^+$  in the definition of  $\mathbf{H}_N^{\text{out},\beta}(\Omega)$  and  $\mathbf{Y}_N^{\text{out},\beta}(\Omega)$  we then define the spaces  $\hat{\mathbf{H}}_N^{\text{out},\beta}(\Omega)$  and  $\hat{\mathbf{Y}}_N^{\text{out},\beta}(\Omega)$ . The main result of this paragraph is given by the following

**Theorem 6.4.7.** *Assume that Assumptions 2.6.3-6.2.1-6.2.2 are satisfied and suppose that  $\omega \in \mathbb{R} \setminus S_N$  and  $\beta \in (0; \beta_0)$ . Suppose that  $\mathbf{J} \in \mathbf{V}_{-\beta}^0(\Omega)$ . Then, the sequence  $(\mathbf{u}_\delta)_\delta$  converges, as  $\delta \rightarrow 0^+$ , in  $\mathbf{W}_\beta^0(\Omega)$  to  $\mathbf{u} \in \hat{\mathbf{H}}_N^{\text{out},\beta}(\Omega)$  that is the unique solution of (6.19) (in which  $\mathbf{H}_N^{\text{out},\beta}(\Omega)$  is replaced by  $\hat{\mathbf{H}}_N^{\text{out},\beta}(\Omega)$ ).*

**Proof.** The proof closely follows the proof of Theorem 5.3.4. Let  $(\delta_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers that converges to 0 as  $n \rightarrow +\infty$ . Denote  $\varepsilon_n = \varepsilon + i\delta_n$ ,  $\mu_n = \mu + i\delta_n$  for all  $n \in \mathbb{N}$ . Denote, for all  $n \in \mathbb{N}$  by  $(\mathbf{u}_n)$  the solution to (6.34) with  $\delta = \delta_n$ . The proof will be done in two steps. First, we establish the desired result by assuming that  $(\|\mathbf{curl} \mathbf{u}_n\|_{\mathbf{V}_\beta^0(\Omega)})$  is bounded. Then we show that this hypothesis is indeed satisfied.

**First step :** Let us suppose that  $(\mathbf{curl} \mathbf{u}_n)$  is bounded in  $\mathbf{V}_\beta^0(\Omega)$ . Thus, thanks to the previous lemma, we know that up to a sequence, still indexed by  $n$ , that  $(\mathbf{u}_n)$  converges in  $\mathbf{V}_\beta^0(\Omega)$  to an element of  $\mathbf{V}_{-\beta}^0(\Omega) \oplus \nabla \hat{\mathcal{S}}_\varepsilon^+$ . This also means that  $(\mathbf{u}_n)$  is bounded in  $\mathbf{V}_\beta^0(\Omega)$ . Next, by observing that for all  $n \in \mathbb{N}$  the vector field  $\mathbf{v}_n := \mu_n^{-1} \mathbf{curl} \mathbf{u}_n$  satisfies  $\mathbf{curl} \mathbf{v}_n = \omega^2 \varepsilon_n \mathbf{u}_n + \mathbf{J}$ , we conclude that  $\mathbf{v}_n \in \mathbf{X}_T(\mu_n)$  and that  $(\mathbf{curl} \mathbf{v}_n)$  is bounded in  $\mathbf{V}_\beta^0(\Omega)$ . Then, by applying the previous lemma, we deduce that  $(\mathbf{curl} \mathbf{u}_n)$  converges in  $\mathbf{V}_\beta^0(\Omega)$  to an element of  $\mathbf{V}_{-\beta}^0(\Omega) \oplus \mu^{-1} \nabla \hat{\mathcal{S}}_\mu^+$ . Consequently,  $(\mathbf{u}_n)$  converges in  $\mathbf{W}_N^\beta(\Omega)$  to some  $\mathbf{u} \in \hat{\mathbf{Y}}_N^{\text{out},\beta}(\Omega)$  that satisfies  $\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{u} - \omega^2 \varepsilon \mathbf{u} = \mathbf{J}$ . Given that  $\omega \in \mathbb{R} \setminus S_N$ , the previous problem has a unique solution which independent of the chosen sub-sequence, we then obtain the wanted result.

**Second step:** Assume that there exists a sequence  $(\mathbf{u}_n)$  of solutions of (6.34) (associated to a some sequence  $(\delta_n)$  that tends to 0) such that  $\|\mathbf{curl} \mathbf{u}_n\|_{\mathbf{V}_\beta^0(\Omega)} \rightarrow +\infty$ . By considering the sequence  $\mathbf{u}_n / \|\mathbf{curl} \mathbf{u}_n\|_{\mathbf{V}_\beta^0(\Omega)}$  and using the result proved in the first step, we obtain a contradiction. ■

## 6.5 The analysis of the magnetic problem

This section will be dedicated to the study of the Maxwell's problem associated to the magnetic component. In section §6.3, we have explained why the problem set in the classical space  $\mathbf{X}_T(\mu)$  can not be well-posed. As a result, and as in the case of the electric component, we have to introduce a new functional framework in which the problem is well-posed in the Fredholm sense. After eliminating  $\mathbf{E}$  in (6.1), we infer that the field  $\mathbf{H}$  is a solution of the problem

$$\begin{cases} \mathbf{curl} \varepsilon^{-1} \mathbf{curl} \mathbf{H} - \omega^2 \mu \mathbf{H} = \mathbf{curl} \varepsilon^{-1} \mathbf{J} & \text{in } \Omega \\ \mu \mathbf{H} \cdot \nu = 0 & \text{on } \partial\Omega \\ \varepsilon^{-1} (\mathbf{curl} \mathbf{H} - \mathbf{J}) \times \nu = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.35)$$

One can easily show that if  $\mathbf{H}$  is a solution of (6.35), then the pair  $\{i(\omega\varepsilon)^{-1}(\mathbf{curl}\ \mathbf{H} - \mathbf{J}), \mathbf{H}\}$  is a solution of (6.1). Let us introduce, for all  $\beta \in \mathbb{R}$ , the spaces

$$\begin{cases} \mathbf{H}^\beta(\mathbf{curl}) := & \{\mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega) \oplus \mathcal{S}_\mu^+ \mid \mathbf{curl}\ \mathbf{u} \in \mathbf{V}_\beta^0(\Omega)\} \\ \mathbf{H}^{\text{out},\beta}(\mathbf{curl}) := & \{\mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega) \oplus \mathcal{S}_\mu^+ \mid \mathbf{curl}\ \mathbf{u} \in \varepsilon\nabla\mathcal{S}_\varepsilon^+ \oplus \mathbf{V}_{-\beta}^0(\Omega)\}. \end{cases}$$

We will use the same notation as in the case of the electric field:

$$\begin{cases} \mathbf{u} = \tilde{\mathbf{u}} + \nabla\mathbf{s}_{\mathbf{u},\mu} & \text{for all } \mathbf{u} \in \mathbf{H}^\beta(\Omega) \\ \mathbf{curl}\ \mathbf{v} = \boldsymbol{\psi}_v + \varepsilon\nabla\mathbf{s}_{\mathbf{v},\varepsilon} & \text{for all } \mathbf{v} \in \mathbf{H}^{\text{out},\beta}(\Omega). \end{cases} \quad (6.36)$$

We endow the spaces  $\mathbf{H}^\beta(\Omega)$  and  $\mathbf{H}^{\text{out},\beta}(\Omega)$  with the norms

$$\begin{cases} \|u\|_{\mathbf{H}^\beta(\Omega)} = (\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2 + \|\mathbf{s}_{\mathbf{u},\mu}\|_{\nabla\mathcal{S}_\mu^+}^2 + \|\mathbf{curl}\ \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)}^2)^{1/2} & \text{for all } \mathbf{u} \in \mathbf{H}^\beta(\Omega) \\ \|v\|_{\mathbf{H}^{\text{out},\beta}(\Omega)} = (\|\tilde{\mathbf{v}}\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2 + \|\mathbf{s}_{\mathbf{u},\mu}\|_{\nabla\mathcal{S}_\mu^+}^2 + \|\mathbf{s}_{\mathbf{u},\varepsilon}\|_{\nabla\mathcal{S}_\varepsilon^+}^2 + \|\boldsymbol{\psi}_v\|_{\mathbf{V}_{-\beta}^0(\Omega)}^2)^{1/2} & \text{for all } \mathbf{v} \in \mathbf{H}^{\text{out}}(\mathbf{curl}). \end{cases} \quad (6.37)$$

### 6.5.1 Definition of the magnetic problem

Regarding what we have done for the case of the electric problem and using the fact that the magnetic field  $\mathbf{H}$  and the electric field  $\mathbf{E}$  are linked by (6.1), we infer that the magnetic problem must be set in the space  $\mathbf{H}^{\text{out},\beta}(\Omega)$  (some conditions on  $\beta$  will be fixed later). Consequently, the problem (6.35) simplifies to become

$$\text{Find } \mathbf{u} \in \mathbf{H}^{\text{out},\beta}(\mathbf{curl}) \text{ such that } \begin{cases} \mathbf{curl}\ \varepsilon^{-1}\boldsymbol{\psi}_u - \omega^2\boldsymbol{\mu}\mathbf{u} = \mathbf{curl}\ \varepsilon^{-1}\mathbf{J} & \text{in } \Omega \setminus \{O\} \\ \boldsymbol{\mu}\mathbf{u} \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega \\ \varepsilon^{-1}(\mathbf{curl}\ \mathbf{u} - \mathbf{J}) \times \boldsymbol{\nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (6.38)$$

If the vector  $\mathbf{J}$  belongs to the space  $\mathbf{V}_{-\beta}^0(\Omega)$ , we introduce the variational formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{H}^{\text{out},\beta}(\mathbf{curl}) \text{ such that} \\ \int_{\Omega} \varepsilon^{-1}\boldsymbol{\psi}_u \cdot \mathbf{curl}\ \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon^{-1}\mathbf{J} \cdot \mathbf{curl}\ \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{H}^\beta(\Omega) \end{array} \right. \quad (6.39)$$

in which the term  $\int_{\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \bar{\mathbf{v}}$  is defined by

$$\int_{\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \bar{\mathbf{v}} := \int_{\Omega} \boldsymbol{\mu}\tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} + \int_{\Omega} \boldsymbol{\mu}\nabla\mathbf{s}_{\mathbf{u},\mu} \cdot \bar{\tilde{\mathbf{v}}} + \int_{\Omega} \boldsymbol{\mu}\tilde{\mathbf{u}} \cdot \nabla\overline{\mathbf{s}_{\mathbf{v},\mu}} - \int_{\Omega} \text{div}(\varepsilon\nabla\mathbf{s}_{\mathbf{u},\mu})\overline{\mathbf{s}_{\mathbf{v},\mu}}.$$

By working as in the proof of Proposition 6.4.1, we obtain the

**Proposition 6.5.1.** *If  $\beta \in [0; 1/2)$  the problems (6.38) and (6.39) are equivalent.*

### 6.5.2 Equivalent formulation for the magnetic field

The next step is to impose explicitly the constraint  $\text{div}(\boldsymbol{\mu}\cdot) = 0$  in the spaces  $\mathbf{H}^\beta(\mathbf{curl})$  and  $\mathbf{H}^{\text{out},\beta}(\mathbf{curl})$ , which leads us to define the spaces

$$\begin{cases} \mathbf{Y}_T^\beta(\boldsymbol{\mu}) := & \{\mathbf{u} \in \mathbf{H}^\beta(\mathbf{curl}) \mid \text{div}(\boldsymbol{\mu}\mathbf{u}) = 0, \boldsymbol{\mu}\mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\} \\ \mathbf{Y}_T^{\text{out},\beta}(\boldsymbol{\mu}) := & \{\mathbf{u} \in \mathbf{H}_N^{\text{out},\beta}(\mathbf{curl}) \mid \text{div}(\boldsymbol{\mu}\mathbf{u}) = 0, \boldsymbol{\mu}\mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}. \end{cases}$$

Obviously, one has the inclusion  $\mathbf{Y}_T^{\text{out},\beta}(\boldsymbol{\mu}) \subset \mathbf{Y}_T^\beta(\boldsymbol{\mu})$  for all  $0 < \beta$ . Proceeding as in Remark 6.4.1, we show that for all  $\mathbf{u} \in \mathbf{Y}_T^\beta(\boldsymbol{\mu})$  the constraint  $\text{div}(\boldsymbol{\mu}\mathbf{u}) = 0$  can also be expressed as follows:

$$\mathbf{u} \in \mathbf{Y}_T^\beta(\boldsymbol{\mu}) \text{ such that } \text{div}(\boldsymbol{\mu}\mathbf{u}) = 0 \iff \int_{\Omega} \boldsymbol{\mu}\mathbf{u} \cdot \nabla v = 0 \quad \text{for all } v \in \mathcal{V}_\beta^{\text{out}}(\Omega).$$

We introduce the following variational formulation

$$\left| \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{Y}_T^{\text{out},\beta}(\mu) \text{ such that} \\ \int_{\Omega} \varepsilon^{-1} \boldsymbol{\psi}_u \cdot \mathbf{curl} \bar{\mathbf{v}} - \omega^2 \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} = i\omega \int_{\Omega} \varepsilon^{-1} \mathbf{J} \cdot \mathbf{curl} \bar{\mathbf{v}} \quad \text{for all } \mathbf{v} \in \mathbf{Y}_T^{\beta}(\mu). \end{array} \right. \quad (6.40)$$

Note that for all  $\mathbf{u} \in \mathbf{Y}_T^{\text{out},\beta}(\mu)$  and all  $\mathbf{v} \in \mathbf{Y}_T^{\beta}(\mu)$ , we have

$$\int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \varepsilon \mathbf{u} \cdot \bar{\tilde{\mathbf{v}}} = \int_{\Omega} \varepsilon \tilde{\mathbf{u}} \cdot \bar{\tilde{\mathbf{v}}} - \int_{\Omega} \text{div}(\varepsilon \nabla \mathbf{s}_{\mathbf{u},\mu}) \overline{\mathbf{s}_{\mathbf{v},\mu}}.$$

As in the case of the electric problem, we have the

**Theorem 6.5.1.** *Assume that  $\omega \neq 0$ .*

- Every solution of (6.39) is a solution of (6.40).
- Let  $\beta \in (0; \beta_N)$ . Under Assumptions 6.2.1-6.2.2 if  $\mathbf{H}$  is a solution (6.40), then it solves (6.39). Moreover  $\{i(\omega \varepsilon)^{-1}(\mathbf{curl} \mathbf{H} - \mathbf{J}), \mathbf{H}\}$  is a solution of (6.1).

### 6.5.3 Equivalent norms in $\mathbf{Y}_T^{\beta}(\mu)$ and $\mathbf{Y}_T^{\text{out},\beta}(\mu)$

**Proposition 6.5.2.** *Let  $\beta \in (0; \beta_N)$  and assume that Assumptions 6.2.1-6.2.2 are satisfied Then there exists a positive constant  $C$  such that*

$$\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} + \|\nabla \mathbf{s}_{\mathbf{u},\mu}\|_{\mathcal{S}_{\mu}^+} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{\beta}^0(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{Y}_T^{\beta}(\mu). \quad (6.41)$$

Consequently, the norms  $\|\cdot\|_{\mathbf{Y}_T^{\beta}(\varepsilon)}$  and  $\|\mathbf{curl} \cdot\|_{\mathbf{V}_{\beta}^0(\Omega)}$  are equivalent in  $\mathbf{Y}_T^{\beta}(\mu)$ .

**Proof.** Let  $\mathbf{u} \in \mathbf{Y}_T^{\beta}(\mu)$ . By definition of  $\mathbf{Y}_T^{\beta}(\mu)$ , the function  $\mathbf{u}$  admits the decomposition  $\mathbf{u} = \tilde{\mathbf{u}} + \nabla \mathbf{s}_{\mathbf{u},\mu}$  where  $\tilde{\mathbf{u}} \in \mathbf{V}_{-\beta}^0(\Omega)$  and  $\mathbf{s}_{\mathbf{u},\mu} \in \mathcal{S}_{\mu}^+$ . Observing that  $\nabla \mathbf{s}_{\mathbf{u},\mu}$  satisfies the condition  $\mu \nabla \mathbf{s}_{\mathbf{u},\mu} \cdot \nu = 0$  on  $\partial\Omega$ , leads us to deduce that  $\mu \tilde{\mathbf{u}} \cdot \nu = 0$  on  $\partial\Omega$ . By mean of item v) of Proposition 6.7.1, one can decompose  $\tilde{\mathbf{u}}$  as  $\tilde{\mathbf{u}} = \nabla \varphi + \mathbf{curl} \boldsymbol{\psi}$  such that  $\varphi \in \mathbf{H}_{\#}^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$ . Remarking that  $\mathbf{curl}(\mathbf{curl} \boldsymbol{\psi}) = \mathbf{curl} \mathbf{u} \in \mathbf{V}_{\beta}^0(\Omega)$  and that  $\mathbf{curl} \boldsymbol{\psi} \cdot \nu = 0$  on  $\partial\Omega$  yields  $\mathbf{curl} \boldsymbol{\psi} \in \mathbf{Z}_T^{\beta}(1)$  where

$$\mathbf{Z}_N^{\beta}(1) = \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{u} \in \mathbf{V}_{\beta}^0(\Omega), \text{div} \mathbf{u} = 0, \mathbf{u} \cdot \nu = 0 \text{ on } \partial\Omega\}. \quad (6.42)$$

Since  $\beta \in [0; 1/2)$  and according to Proposition 6.7.5, we obtain  $\mathbf{curl} \boldsymbol{\psi} \in \mathbf{V}_{-\beta}^0(\Omega)$  with the estimate

$$\|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{\beta}^0}. \quad (6.43)$$

Given that  $\text{div}(\mu \mathbf{u}) = 0$ , we obtain

$$-\text{div}(\mu \nabla \varphi) = \text{div}(\mu \nabla \mathbf{s}_{\mathbf{u},\mu}) + \text{div}(\mu \mathbf{curl} \boldsymbol{\psi}) \in (\mathcal{V}_{\beta}^1(\Omega))^* \quad \text{for all } \beta \in [0; \frac{1}{2}). \quad (6.44)$$

By Lemma 6.2.3, we get  $\varphi \in \mathcal{V}_{-\beta}^1(\Omega)$  for all  $\beta \in (0; \beta_N)$  with the estimate

$$\|\nabla \varphi\|_{\mathbf{V}_{-\beta}^0(\Omega)} + \|\mathbf{s}_{\mathbf{u},\mu}\|_{\mathcal{S}_{\mu}^+} \leq C \|\mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \quad (6.45)$$

By gathering (6.43) and (6.45), we obtain the wanted result. ■

Working as in the proof of Proposition 6.4.3, we show the following result.

**Proposition 6.5.3.** *Suppose that Assumptions 6.2.1-6.2.2 hold. Then, for all  $\beta \in (0; \beta_0)$  there exists a constant  $C$  such that*

$$\|\tilde{\mathbf{u}}\|_{\mathbf{V}_{-\beta}^0(\Omega)} + \|\nabla \mathbf{s}_{\mathbf{u},\mu}\|_{\nabla \mathcal{S}_{\mu}^+} \leq C \|\boldsymbol{\psi}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{Y}_T^{\text{out},\beta}(\mu, \varepsilon). \quad (6.46)$$

Consequently  $\mathbf{u} \mapsto \|\boldsymbol{\psi}_u\|_{\mathbf{V}_{-\beta}^0(\Omega)}$  is a norm in  $\mathbf{Y}_T^{\text{out},\beta}(\mu)$  that is equivalent to  $\|\cdot\|_{\mathbf{H}^{\beta}(\mathbf{curl})}$ .

### 6.5.4 Main results about the magnetic problem

For all  $\beta \in \mathbb{R}$ , we introduce the operators  $\mathbb{A}_T^\beta, \mathbb{K}_T^\beta : \mathbf{Y}_T^{\text{out},\beta}(\mu) \mapsto (\mathbf{Y}_T^\beta(\mu))^*$  such that  $\mathbf{u} \in \mathbf{Y}_T^{\text{out},\beta}(\mu)$  and  $\mathbf{v} \in \mathbf{Y}_T^\beta(\mu)$  we have

$$\langle \mathbb{A}_T^\beta \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu^{-1} \boldsymbol{\psi}_{\mathbf{u}} \cdot \mathbf{curl} \bar{\mathbf{v}} \quad \text{and} \quad \langle \mathbb{K}_T^\beta \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}} = \int_{\Omega} \mu \mathbf{u} \cdot \bar{\mathbf{v}}.$$

By exchanging the roles of  $\varepsilon$  and  $\mu$  in study of the electric problem, one can obtain the following.

**Theorem 6.5.2.** *Assume that Assumptions 6.2.1-6.2.2 hold. For all  $\beta \in (0; \beta_0)$  the operator  $\mathbb{A}_T^\beta$  is an isomorphism and  $\mathbb{K}_T^\beta$  is compact.*

Now, define for all  $\beta \in \mathbb{R}$  the operator  $\mathcal{A}_T^\beta(\omega) := \mathbb{A}_T^\beta - \omega^2 \mathbb{K}_T^\beta$ . One can easily see that if the operator  $\mathcal{A}_T^\beta(\omega)$  is an isomorphism with  $\beta \in (0; \beta_0)$ , then the problem (6.39)-(6.39) is well-posed. By using the Fredholm analytic theorem and by working as in the proofs of Proposition 6.4.4 and Lemma 6.4.1, we can prove the following

**Theorem 6.5.3.** *Assume that Assumptions 6.2.1-6.2.2. Then, for all  $\beta \in (0; \beta_0)$  we have*

- $\mathcal{A}_T^\beta(\omega)$  is a Fredholm operator of index 0 for all  $\omega \in \mathbb{R}$ .
- There exists  $S_T$  that is independent of  $\beta \in (0; \beta_0)$  a discrete subset of  $\mathbb{R}$  such that  $\mathcal{A}_T^\beta(\omega)$  is an isomorphism for all  $\omega \in \mathbb{R} \setminus S_T$ .
- If  $\mathbf{u} \in \text{Ker}(\mathcal{A}_T^\beta(\omega))$  with  $\omega \in \mathbb{R}$  and  $\beta \in (0; \beta_0)$  then  $\mathbf{s}_{\mathbf{u},\mu} = \mathbf{s}_{\mathbf{u},\varepsilon} = 0$ .
- For all  $\omega \in \mathbb{R}$  the space  $\text{Ker}(\mathcal{A}_T^\beta(\omega))$  is independent of  $\beta \in (0; \beta_0)$ .
- Assume that  $\omega \in \mathbb{R} \setminus S_T$ . Let  $0 < \beta_1 < \beta_2 < \beta_0$  and let  $\ell \in (\mathbf{Y}_T^{\beta_1}(\varepsilon))^* \cap (\mathbf{Y}_T^{\beta_2}(\varepsilon))^*$  then the solutions to the problems

$$\mathcal{A}_T^\beta(\omega) = \ell$$

with  $\beta = \beta_1$  and  $\beta = \beta_2$  coincide.

This allows us to state the main theorem of this section

**Theorem 6.5.4.** *Assume that Assumptions 6.2.1-6.2.2 hold. Suppose that there exists  $0 < \gamma < \beta_0$  such that  $\mathbf{J} \in \mathbf{V}_{-\gamma}^0(\Omega)$ . Then*

- for all  $\omega \in \mathbb{R} \setminus S_T$  the problem (6.39)(or equivalently (6.40)) is well-posed in the Hadamard sense for all  $\beta \in (0; \beta_0)$ .
- The solution to (6.39)(or equivalently to (6.40)) is independent of  $\beta \in (0; \beta_0)$ .
- When  $\omega \in S_T$  the problems (6.39) - (6.40) are well-posed in the Fredholm sense. Moreover, they have a finite dimensional kernel that is independent of  $\beta \in (0; \beta_0)$ .

**Remark 6.5.1.** *Without any difficulty, one can check that the results of §6.4.7 hold, if one consider the magnetic problem instead of the electric one.*

## 6.6 Concluding remarks

In this chapter, we have considered the case of the time harmonic Maxwell equation when the functions  $\varepsilon$  and  $\mu$  are both critical. We have presented a general theory, which allows to construct from any functional framework for the scalar problem that respects Mandelstam's radiation principle, a functional framework in which Maxwell's problem is again well-posed. Moreover, we have established that if one uses the frameworks that respects the limiting absorption principle for scalar problems, then those provided by our theory for electric and magnetic problems are also coherent with the limiting absorption principle.



## 6.7 Appendix

### 6.7.1 Classical Helmholtz decompositions

**Proposition 6.7.1.** *If  $\Omega$  is simply connected and its boundary is connected, the following assertions hold.*

i) *According to [8, Theorem 3.12], if  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  satisfies  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$ , then there exists a unique  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  such that  $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}$ .*

ii) *According to [8, Theorem 3.17]), if  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  satisfies  $\operatorname{div} \mathbf{u} = 0$  in  $\Omega$  and  $\mathbf{u} \cdot \boldsymbol{\nu} = 0$  on  $\partial\Omega$ , then there exists a unique  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$  such that  $\mathbf{u} = \operatorname{curl} \boldsymbol{\psi}$ .*

iii) *If  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  satisfies  $\operatorname{curl} \mathbf{u} = 0$  in  $\Omega$  and  $\mathbf{u} \times \boldsymbol{\nu} = 0$  on  $\partial\Omega$ , then there exists (see [110, Theorem 3.41]) a unique  $p \in H_0^1(\Omega)$  such that  $\mathbf{u} = \nabla p$ .*

iv) *Every  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  can be decomposed as follows ([110, Theorem 3.45])  $\mathbf{u} = \nabla p + \operatorname{curl} \boldsymbol{\psi}$ , with  $p \in H_0^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  which are uniquely defined.*

v) *Every  $\mathbf{u} \in \mathbf{L}^2(\Omega)$  can be decomposed as follows ([110, Remark 3.46])  $\mathbf{u} = \nabla p + \operatorname{curl} \boldsymbol{\psi}$ , with  $p \in H_{\#}^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_N(1)$  which are uniquely defined.*

### 6.7.2 Weighted regularity of vector potentials

#### The classical case

We recall the following result, that we have proved in the annex of the previous chapter, concerning the weighted regularity of the spaces  $\mathbf{X}_T(1)$  and  $\mathbf{X}_N(1)$ .

**Proposition 6.7.2.** *For all  $\beta \in [0; 1/2)$ , the spaces  $\mathbf{X}_T(1)$  and  $\mathbf{X}_N(1)$  are compactly embedded in  $\mathbf{V}_{-\beta}^0(\Omega)$ . In particular, there exists a constant  $C$  such that*

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\operatorname{curl} \mathbf{u}\|_0, \quad \text{for all } \mathbf{u} \in \mathbf{X}_T(1) \cup \mathbf{X}_N(1). \quad (6.47)$$

Note that similar weighted regularity results can also be found in [41].

#### The case of critical contrasts

Here, we are concerned with the weighted regularity of the spaces  $\mathbf{X}_N(\varepsilon)$  and  $\mathbf{X}_T(\mu)$  when  $\varepsilon$  and  $\mu$  are critical.

**Proposition 6.7.3.** *Assume that Assumption 6.2.1-6.2.2 hold. Then for all  $\beta \in (0; \beta_D)$  the space  $\mathbf{X}_N(\varepsilon)$  is compactly embedded in  $\mathbf{V}_{-\beta}^0(\Omega)$ . We have the estimate*

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\operatorname{curl} \mathbf{u}\|_0 \quad \text{for all } \mathbf{u} \in \mathbf{X}_N(\varepsilon). \quad (6.48)$$

where  $C$  is independent of  $u$ .

**Proof.** Let  $\mathbf{u} \in \mathbf{X}_N(\varepsilon)$ . By mean of item iv) of Proposition 5.6.1, we can introduce  $\varphi \in H_0^1(\Omega)$  and  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  such that

$$\mathbf{u} = \nabla \varphi + \operatorname{curl} \boldsymbol{\psi}.$$

By observing that  $\operatorname{curl} \boldsymbol{\psi}$  belongs to  $\mathbf{X}_N(1)$  and given that  $\beta_D \leq 1/2$ , we infer by applying Proposition 6.7.2 that  $\operatorname{curl} \boldsymbol{\psi} \in \mathbf{V}_{-\beta}^0(\Omega)$ , for all  $\beta \in (0; \beta_D)$ . Furthermore, we have the estimate

$$\|\operatorname{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\operatorname{curl} \operatorname{curl} \boldsymbol{\psi}\|_0 = C \|\operatorname{curl} \mathbf{u}\|_0. \quad (6.49)$$



By observing that  $\operatorname{div}(\varepsilon \nabla \varphi) = -\operatorname{div}(\varepsilon \operatorname{curl} \psi) \in (\mathring{V}_\beta^1(\Omega))^*$  for all  $\beta \in (0; \beta_D)$ , one deduces, by means of Lemma 6.2.3, that  $\varphi \in \mathring{V}_{-\beta}^1(\Omega)$  with the estimate

$$\|\nabla \varphi\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\operatorname{curl} \psi\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\operatorname{curl} \mathbf{u}\|_0. \quad (6.50)$$

By combing (6.49) and (6.50), we obtain the wanted estimate. Now, let us prove the compactness result. Take  $(\mathbf{u}_n)_{n \in \mathbb{N}}$  a bounded sequence of  $\mathbf{X}_N(\varepsilon)$ . We introduce  $(\varphi_n)_{n \in \mathbb{N}}$  and  $(\psi_n)_{n \in \mathbb{N}}$  two sequences of  $H_0^1(\Omega)$  and  $\mathbf{X}_T(1)$ , respectively, such that  $\mathbf{u}_n = \nabla \varphi_n + \operatorname{curl} \psi_n$  for all  $n \in \mathbb{N}$ . Thanks to Proposition 6.7.2, we infer that up to a sub-sequence, still indexed by  $n$ , the sequence  $(\operatorname{curl} \psi_n)_{n \in \mathbb{N}}$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in (0; \beta_D)$ . The estimate (6.50) implies that  $(\nabla \varphi_n)_{n \in \mathbb{N}}$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . This ends the proof.  $\blacksquare$

Using the same arguments, one proves the

**Proposition 6.7.4.** *Under Assumption 6.2.1-6.2.2 the space  $\mathbf{X}_T(\mu)$  is compactly embedded in  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in (0; \beta_N)$ . In particular, there is some constant  $C$  such that*

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\operatorname{curl} \mathbf{u}\|_0 \quad \forall \mathbf{u} \in \mathbf{X}_T(\mu). \quad (6.51)$$

### 6.7.3 Vector potentials in weighted Sobolev spaces

For all  $\beta \in \mathbb{R}$ , we introduce the spaces

$$\begin{aligned} \mathbf{Z}_T^\beta(1) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega), \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\} \\ \mathbf{Z}_N^\beta(1) &:= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega), \operatorname{div} \mathbf{u} = 0, \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

We endow  $\mathbf{Z}_T^\beta(1)$  and  $\mathbf{Z}_N^\beta(1)$  with the norm  $\|\cdot\|_{\mathbf{Z}^\beta} := (\|\cdot\|_0^2 + \|\operatorname{curl} \cdot\|_{\mathbf{V}_\beta^0(\Omega)}^2)^{1/2}$ . The importance of these spaces is motivated by the next result that we have proved in the annex of the previous chapter.

**Lemma 6.7.1.** *Let  $\beta \in [0; 1/2)$ . The following assertions hold.*

- *If  $\mathbf{u} \in \mathbf{V}_\beta^0(\Omega)$  such that  $\operatorname{div} \mathbf{u} = 0$  then there exists a unique  $\psi \in \mathbf{Z}_T^\beta(1)$  such that  $\mathbf{u} = \operatorname{curl} \psi$ .*
- *If, in addition,  $\mathbf{u}$  satisfies  $\mathbf{u} \cdot \boldsymbol{\nu} = 0$  on  $\partial\Omega$ , then there exists a unique  $\psi \in \mathbf{Z}_N^\beta(1)$  such that  $\mathbf{u} = \operatorname{curl} \psi$ .*

We have the analogue of Proposition 6.7.2.

**Proposition 6.7.5.** *For all  $\beta \in [0; 1/2)$  we have*

$$\begin{aligned} \mathbf{Z}_T^\beta(1) &= \{\mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega) \mid \operatorname{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega), \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\} \\ \mathbf{Z}_N^\beta(1) &= \{\mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega) \mid \operatorname{curl} \mathbf{u} \in \mathbf{V}_\beta^0(\Omega), \operatorname{div} \mathbf{u} = 0, \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Furthermore,  $\mathbf{Z}_T^\beta(1)$  and  $\mathbf{Z}_N^\beta(1)$  are compactly embedded in  $\mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in [0; 1/2)$ . In particular, there is some constant  $C > 0$  such that

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0} \leq C \|\operatorname{curl} \mathbf{u}\|_{\mathbf{V}_\beta^0(\Omega)} \quad \text{for all } \mathbf{u} \in \mathbf{Z}_T^\beta(1) \cup \mathbf{Z}_N^\beta(1). \quad (6.52)$$

**Proof.** We will prove the result for the space  $\mathbf{Z}_N^\beta(1)$ . The case of the space  $\mathbf{Z}_T^\beta(1)$  can be treated similarly. Let  $\mathbf{u} \in \mathbf{Z}_N^\beta(1)$ . Owing to Proposition 6.7.1, we can introduce  $\boldsymbol{\psi} \in \mathbf{X}_T(1)$  such that  $\mathbf{u} = \mathbf{curl} \boldsymbol{\psi}$ . The Proposition 6.7.2, allows us say that  $\boldsymbol{\psi} \in \mathbf{V}_{-\beta}^0(\Omega)$  for all  $\beta \in [0; 1/2)$ . Observe that by applying the  $\mathbf{curl}$  operator, we infer that  $\mathbf{curl} \mathbf{curl} \boldsymbol{\psi} = \mathbf{curl} \mathbf{u}$ . Furthermore, thanks to an integration by parts, one can easily see that

$$\int_{\Omega} |\mathbf{curl} \boldsymbol{\psi}|^2 = \int_{\Omega} \mathbf{curl} \mathbf{u} \overline{\boldsymbol{\psi}}. \quad (6.53)$$

By means of the estimate (6.47), we obtain

$$\|\mathbf{u}\|_0 = \|\mathbf{curl} \boldsymbol{\psi}\|_0 \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (6.54)$$

As a consequence, to prove the wanted estimate and the regularity result, we need to refine the previous one near the origin. For this, let  $\zeta$  be a smooth cutoff function that is equal to one near the origin with support contained in  $B(O, r_0)$  ( $r_0$  sufficiently small so that  $B(O, r_0) \subset \Omega$ ). By classical results, we know that for all for  $i = 1, 2, 3$ , the component  $\zeta \boldsymbol{\psi}_i$  belongs to  $\mathbf{H}_0^1(B(O, r_0))$ . Given that  $\mathbf{curl} \mathbf{curl} \boldsymbol{\psi} = \Delta \boldsymbol{\psi} = \mathbf{curl} \mathbf{u}$ . One can show that  $\Delta(\zeta \boldsymbol{\psi}) \in \mathbf{V}_{-\beta}^0(B(O, r_0))$ . Moreover, we have the estimate

$$\|\Delta(\zeta \boldsymbol{\psi})\|_{\mathbf{V}_{-\beta}^0(B(O, r_0))} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}.$$

Since by assumption  $\beta \in [0, 1/2)$ , we deduce that  $\mathbf{V}_{-\beta}^0(\Omega) \subset \mathbf{V}_{\gamma}^0(\Omega) \subset \mathbf{V}_{1-\beta}^0(\Omega)$  for all  $\gamma \in (1/2; 1 - \beta]$ . This means that the function  $\Delta(\zeta \boldsymbol{\psi}_i)$  belongs, then, to  $\mathbf{V}_{\gamma}^0(B(O, a))$  for all  $\gamma \in (1/2; 1 - \beta] \subset (1/2, 1)$  for all  $i \in \{1, 2, 3\}$ . By applying Proposition 6.2.2, we infer that  $\zeta \boldsymbol{\psi}_i \in \mathbf{V}_{\gamma}^2(B(O, r_0))$  for all  $\gamma \in (1/2; 1 - \beta]$  with the estimate

$$\|\zeta \boldsymbol{\psi}_i\|_{\mathbf{V}_{\gamma}^2(B(O, r_0))} \leq C \|\Delta(\zeta \boldsymbol{\psi})\|_{\mathbf{V}_{\gamma}^0(B(O, r_0))} \leq C \|\Delta(\zeta \boldsymbol{\psi})\|_{\mathbf{V}_{-\beta}^0(B(O, r_0))} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (6.55)$$

Consequently, by observing that  $\zeta \mathbf{curl} \boldsymbol{\psi} = \zeta \mathbf{u} \in \mathbf{V}_{\gamma}^1(\Omega) \subset \mathbf{V}_{\gamma-1}^0(\Omega)$  for all  $\gamma \in (1/2; 1 - \beta]$  and by taking  $\gamma = 1 - \beta$ , we deduce that  $\mathbf{u} \in \mathbf{V}_{-\beta}^0(\Omega)$ . Moreover, we have the estimate

$$\|\zeta \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(B(O, r_0))} = \|\zeta \mathbf{curl} \boldsymbol{\psi}\|_{\mathbf{V}_{-\beta}^0(B(O, r_0))} \leq C \|\Delta(\zeta \boldsymbol{\psi})\|_{\mathbf{V}_{1-\beta}^0(D_{\zeta})} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}.$$

Combining this with the estimate (6.54), we find

$$\|\mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)} \leq C \|\mathbf{curl} \mathbf{u}\|_{\mathbf{V}_{-\beta}^0(\Omega)}. \quad (6.56)$$

As a result it remains to prove the compactness of the embedding  $\mathbf{Z}_N^\beta(1) \subset \mathbf{V}_{-\beta}^0(\Omega)$ . For this, let  $(\mathbf{u}_n)_{n \in \mathbb{N}}$  be a bounded sequence of  $\mathbf{Z}_N^\beta(1)$ . For all  $n \in \mathbb{N}$  we introduce  $\boldsymbol{\psi}_n \in \mathbf{X}_T(1)$  such that  $\mathbf{u}_n = \mathbf{curl} \boldsymbol{\psi}_n$ . Obviously, we can see that  $(\boldsymbol{\psi}_n)_{n \in \mathbb{N}}$  is then bounded in  $\mathbf{X}_T(1)$ . According to Proposition 6.7.2, we know, that up to a sub-sequence, still indexed by  $n$ ,  $(\boldsymbol{\psi}_n)_{n \in \mathbb{N}}$  converges in  $\mathbf{V}_{-\beta}^0(\Omega)$ . By observing that, for all  $\beta \in [0; 1/2)$  and all  $m, n \in \mathbb{N}$  we have

$$\int_{\Omega} |\mathbf{curl} \boldsymbol{\psi}_n - \mathbf{curl} \boldsymbol{\psi}_m|^2 = \int_{\Omega} \mathbf{curl} (\mathbf{u}_n - \mathbf{u}_m) (\overline{\boldsymbol{\psi}_n} - \overline{\boldsymbol{\psi}_m}) \leq \|\mathbf{curl} (\mathbf{u}_n - \mathbf{u}_m)\|_{\mathbf{V}_{-\beta}^0(\Omega)} \|\boldsymbol{\psi}_n - \boldsymbol{\psi}_m\|_{\mathbf{V}_{-\beta}^0(\Omega)}$$

we deduce that  $(\mathbf{curl} \boldsymbol{\psi}_n)_{n \in \mathbb{N}}$  converges in  $\mathbf{L}^2(\Omega)$ . Hence, up to a sub-sequence,  $(\mathbf{u}_n)_{n \in \mathbb{N}}$  converges in  $\mathbf{L}^2(\Omega)$ . This implies that the embedding of  $\mathbf{Z}_N(1) \subset \mathbf{L}^2(\Omega)$  is compact. To end the proof, it suffices to prove that  $(\zeta \mathbf{u}_n)_{n \in \mathbb{N}}$  converges, up to a sub-sequence, in  $\mathbf{V}_{-\beta}^0(B(0, r_0))$ . This consequence of the fact that  $\zeta \mathbf{u}_n \in \mathbf{V}_{\gamma-1}^1(\Omega)$  for all  $\gamma \in (1/2, 1 - \beta]$  and the compactness of  $\mathbf{V}_{\gamma}^1(B(O, r_0)) \subset \mathbf{V}_{-\beta}^0(B(O, r_0))$  for all  $\gamma \in (1/2; 1 - \beta)$  (see [102, Lemma 6.21]). ■

**Remark 6.7.1.** *The proof of the previous proposition tells us that when  $\beta \in [0; 1/2)$  then for all  $\mathbf{u} \in \mathbf{Z}_T^\beta(1) \cup \mathbf{Z}_N^\beta(1)$  we have  $\mathbf{u} \in \bigcap_{\gamma \in (-1/2; -\beta]} \mathbf{V}_{\gamma}^1(\Omega)$  near the origin.*

### 6.7.4 Density results

Let  $C$  be a sub-domain of  $\Omega$  (i.e.  $C \subset \Omega$ ), we denote by  $\mathbf{H}_N(\mathbf{curl}, C)$  the space

$$\mathbf{H}_N(\mathbf{curl}, C) := \{\mathbf{u} \in \mathbf{L}^2(C) \mid \mathbf{curl} \mathbf{u} \in \mathbf{L}^2(C); \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}.$$

Moreover, for all  $\beta \in \mathbb{R}$ , we introduce the spaces

$$\mathbf{K}_D^\beta(C) := \{\mathbf{u} \in \mathbf{V}_{-\beta}^0(C) \mid \mathbf{curl} \mathbf{u} \in \mathbf{V}_\beta^0(C); \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}, \mathbf{V}_{N,\beta}^1(C) = \{\mathbf{u} \in \mathbf{V}_\beta^1(C)^3 \mid \mathbf{u} \times \boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}.$$

**Proposition 6.7.6.** *Assume that  $\beta \in [0; 1/2)$ . Then, the space  $\mathcal{D}(\Omega \setminus \{O\})$  is dense in  $\mathbf{K}_N^\beta(\Omega)$ .*

**Proof.** Let  $0 < r_0$  such that  $B(O, r_0) \subset \Omega$ . Let  $\chi \in \mathcal{D}(\Omega)$  be a cutoff function that depends only on  $r = |x|$  that is supported  $B(O, r_0)$  and that is equal to 1 near the origin. For all  $\mathbf{u} \in \mathbf{K}_N^\beta(\Omega)$ , one can see that  $(1 - \chi)\mathbf{u} \in \mathbf{H}_N(\mathbf{curl}, \Omega \setminus \overline{B(O, r_0)})$ . Given that the space  $(\mathcal{D}(\Omega \setminus \overline{B(O, r_0)}))^3$  is dense in  $\mathbf{H}_N(\mathbf{curl}, \Omega \setminus \overline{B(O, r_0)})$  is enough to explain how to approximate  $\chi\mathbf{u}$  by functions of  $(\mathcal{D}(B(O, r_0) \setminus \{O\}))^3$  in  $\mathbf{K}_D^\beta(B(O, r_0))$ . For this, we are going to show that for all  $\beta \in [0; 1/2)$ , we have the decomposition

$$\mathbf{K}_D^\beta(B(O, r_0)) = \nabla \mathring{\mathbf{V}}_{-\beta}^1(B(O, r_0)) \oplus \mathbf{V}_{N,-\beta}^1(B(O, r_0)). \quad (6.57)$$

Obviously, one has  $\nabla \mathring{\mathbf{V}}_{-\beta}^1(B(O, r_0)) \oplus \mathring{\mathbf{V}}_\beta^1(B(O, r_0)) \subset \mathbf{K}_D^\beta(B(O, r_0))$ . The reverse inclusion is obtained as follows. Take any  $\mathbf{u} \in \mathbf{K}_D^\beta(B(O, r_0))$ . Since  $0 < \beta$ , we then have the inclusion  $\mathbf{K}_D^\beta(B(O, r_0)) \subset \mathbf{L}^2(B(O, r_0))$ . So, the function  $\mathbf{u}$  decomposes as  $\mathbf{u} = \nabla\varphi + \mathbf{curl} \boldsymbol{\psi}$  with  $\varphi \in H_0^1(B(O, r_0))$  and  $\boldsymbol{\psi} \in \mathbf{X}_N(1, B(O, r_0))$ . The vector field  $\mathbf{curl} \boldsymbol{\psi}$  belongs, then to the space  $\mathbf{Z}_N^\beta(1)$  (see §6.7.3). Thus, thanks to Remark 6.7.1, we deduce that  $\mathbf{curl} \boldsymbol{\psi}$  belongs to  $(\mathbf{V}_{-\beta}^1)^3$  near the origin and since  $B(O, r_0)$  is smooth, we infer that  $\mathbf{curl} \boldsymbol{\psi}$  belongs to the space  $(H^1)^3$  far from the origin. As a consequence we then obtain that  $\mathbf{curl} \boldsymbol{\psi} \in \mathbf{V}_{N,-\beta}^1(B(O, r_0))$ . Since  $\Delta\varphi = \text{div}(\mathbf{u}) \in (\mathring{\mathbf{V}}_\beta^1(\Omega))^*$ , Lemma 6.2.2 implies that  $\varphi \in \mathring{\mathbf{V}}_{-\beta}^1(\Omega)$ . We then deduce that (6.57) holds. Given that the space  $(\mathcal{D}(B(O, r_0) \setminus \{O\}))^3$  (resp.  $\mathcal{D}(B(O, r_0) \setminus \{O\})$ ) is dense in dense in  $\mathbf{V}_{N,-\beta}^1(B(O, r_0))$  (resp.  $\mathring{\mathbf{V}}_{-\beta}^1(B(O, r_0))$ ) and since the embeddings

$$\mathbf{V}_{N,-\beta}^1(B(O, r_0)), \nabla \mathring{\mathbf{V}}_{-\beta}^1(B(O, r_0)) \subset \mathbf{K}_D^\beta(B(O, r_0))$$

are continuous, we obtain the wanted result. ■

# Chapter 7

## Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients

The results of this chapter were obtained thanks to a collaboration with Renata Bunoiu (Université de Lorraine, CNRS, IECL), Karim Ramdani ( Université de Lorraine, CNRS, Inria, IECL) and Lucas Chesnel (Inria/Centre de mathématiques appliquées, Ecole Polytechnique).

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### 7.1 Introduction

The goal of this chapter is to study the homogenization process for time harmonic Maxwell's equations in the presence of  $\delta$ -periodically distributed inclusions of negative material embedded into a dielectric material (see Figure 7.1 for a typical configuration). The main objective is to clarify if the homogenization process is doable in this context and if so, to determine whether the corresponding homogenized material behaves like a positive or negative material as  $\delta$  tends to zero. For scalar problems, the first homogenization results have been obtained in [43] using the  $T$ -coercivity approach of [27]. More precisely, it is proved therein that for negative contrasts

close to 0 (the contrast being defined here as the ratio between the interior and exterior values, see (7.1)), the scalar problem with Dirichlet boundary conditions can be homogenized. In other words, it is proved that under this assumption on the contrast, the solution of the problem in the composite material is well-defined for  $\delta$  small enough (this is not obvious due to the loss of coercivity due to the sign-changing coefficient) and that it two-scale converges (see Definition 7.5.1 below) to the solution of a well-posed problem set in a homogeneous material. These results have been extended in [31], through the analysis of the spectrum of the Neumann-Poincaré operator. In particular, the authors show that the homogenization process is possible provided the contrast between the two media (defined using the same convention as above) belongs to  $(-\infty; -1/\alpha)$  or  $(-\alpha; 0)$ ,  $\alpha > 0$  (see Remark 7.3.1 below). The proof of this result is based on an elegant continuity argument (see [31, Corollary 5.1]). However, it does not provide a precise value for  $\alpha$ .

The chapter is organized as follows. Section 7.2 provides the mathematical setting of the problem and necessary notation. Before studying Maxwell's system, we collect in Section 7.3 some useful results concerning two associated scalar problems, a Dirichlet and a Neumann one. In particular, we prove the uniform invertibility of these operators as  $\delta$  tends to zero, for small or large values of the contrast, i.e. for contrasts in  $(-\infty; -1/m) \cup (-1/M; 0)$ , with  $0 < m < M$  (see subsections 7.3.1 to 7.3.3). A variational characterization of the bounds  $m$  and  $M$  is also obtained (see (7.45)). Next, inspired by [31], we discuss in §7.3.4 the connection with the Neumann-Poincaré operator and the optimality of the obtained conditions. In Section 7.4, we study the cell problems appearing in the homogenization of Maxwell's equations. We prove that they are well-posed under the same assumptions as the scalar problems investigated in Section 7.3. This allows us to define homogenized tensors and we show that they are positive definite under the same assumption on the contrasts, that is for contrasts in  $(-\infty; -1/m) \cup (-1/M; 0)$ . This is also an improvement of the results obtained in [43] and [31]. In Section 7.5, we finally tackle the homogenization process for Maxwell's equations with sign-changing coefficients. Combining results from [136] and [58] obtained for classical (positive) electromagnetic materials, we first derive in §7.5.1 a homogenization result under a uniform energy estimate condition. At this stage, the sign-changing of the physical parameters does not play any role. Related to this part of the work, let us mention the seminal book [14] as well as [141, 12] for the study of the time-dependent Maxwell equations. For the time harmonic case, we refer to [14, 143, 131, 77, 48, 38, 142, 6, 136, 94]. Then, in §7.5.2, we establish the needed uniform energy estimates for Maxwell's equations. This is done by using the results obtained for the scalar problems as well as the T-coercivity approach presented in [23] and a uniform compactness property. The final homogenization result for Maxwell's system with sign-changing coefficients is stated in Theorem 7.5.1. For the reader's convenience, the list of functional spaces used throughout the paper is collected in the Appendix.

## 7.2 Setting of the problem

Let  $\Omega$  be an open, connected and bounded subset of  $\mathbb{R}^3$  with a Lipschitz-continuous boundary  $\partial\Omega$ . Once and for all, we make the following assumption:

**Assumption.** *The domain  $\Omega$  is simply connected and  $\partial\Omega$  is connected.*

When this assumption is not satisfied, the analysis below must be adapted (see some preliminary ideas in [23, §8.2]). We consider a situation where  $\Omega$  is filled with a composite electromagnetic material constituted of periodically distributed inhomogeneous cells of small size  $\delta > 0$ . More precisely, let  $Y = (0; 1)^3$  denote the reference cell and assume that  $Y$  contains two materials:

- a metamaterial with negative dielectric permittivity  $\varepsilon_i < 0$  and magnetic permeability  $\mu_i < 0$  located inside a connected domain  $Y_i \subset Y$  with Lipschitz boundary  $\partial Y_i$  such that  $\overline{Y_i} \subset Y$ ;

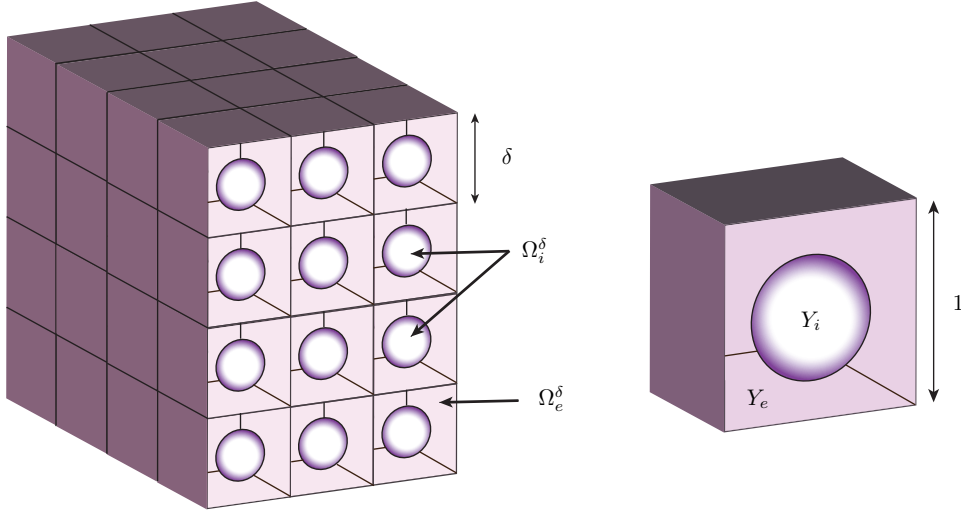


Figure 7.1: Example of a periodic material and the corresponding reference cell  $Y$ .

- a dielectric material with positive dielectric permittivity  $\varepsilon_e > 0$  and magnetic permeability  $\mu_e > 0$  filling the region  $Y_e := Y \setminus \overline{Y_i}$ .

We emphasize that the assumption  $\overline{Y_i} \subset Y$  is important. When the inclusion  $Y_i$  meets the boundary of the cell  $\partial Y$ , phenomena different from the ones described below can appear. We refer the reader to [31, Appendix A] for more details concerning the scalar problem in this case. To simplify the presentation, we assume that  $\varepsilon_i$ ,  $\varepsilon_e$ ,  $\mu_i$  and  $\mu_e$  are constant. However, we could also consider physical parameters which are elements of  $L^\infty(\Omega, \mathbb{R}^{3 \times 3})$ , the variational techniques we use below would work in a similar way. In our analysis, the following dielectric and magnetic contrasts

$$\kappa_\varepsilon := \frac{\varepsilon_i}{\varepsilon_e} < 0, \quad \kappa_\mu := \frac{\mu_i}{\mu_e} < 0 \quad (7.1)$$

will play a key role. Let us stress that the four constants  $\varepsilon_e$ ,  $\varepsilon_i$ ,  $\mu_e$ ,  $\mu_i$  are fixed once for all in the article. And when we make assumptions on the contrasts in the statements below (see in particular the final Theorem 7.5.1), they must be understood as “Assume that  $\varepsilon_e$ ,  $\varepsilon_i$ ,  $\mu_e$ ,  $\mu_i$  are such that  $\kappa_\varepsilon$ ,  $\kappa_\mu \dots$ ”. We define on the reference cell the two real-valued functions  $\varepsilon, \mu \in L^\infty(Y)$  such that

$$\varepsilon(y) = \varepsilon_e \mathbb{1}_{Y_e}(y) + \varepsilon_i \mathbb{1}_{Y_i}(y), \quad \mu(y) = \mu_e \mathbb{1}_{Y_e}(y) + \mu_i \mathbb{1}_{Y_i}(y), \quad (7.2)$$

where for a set  $S$ ,  $\mathbb{1}_S(\cdot)$  stands for the indicator function of  $S$ . For any  $\delta > 0$  and any integer vector  $k \in \mathbb{Z}^3$ , we define the shifted and scaled sets  $Y_{ik}^\delta, Y_{ek}^\delta, Y_k^\delta$  such that

$$\begin{aligned} Y_{ik}^\delta &:= \{x \in \mathbb{R}^3 \mid (x - k)/\delta \in Y_i\} \\ Y_{ek}^\delta &:= \{x \in \mathbb{R}^3 \mid (x - k)/\delta \in Y_e\} \\ Y_k^\delta &:= \{x \in \mathbb{R}^3 \mid (x - k)/\delta \in Y\}. \end{aligned} \quad (7.3)$$

We denote by  $K^\delta$  the set of  $k \in \mathbb{Z}^3$  such that  $Y_k^\delta \subset \Omega$ . We assume that the metamaterial fills the region

$$\Omega_i^\delta := \bigcup_{k \in K^\delta} Y_{ik}^\delta,$$

while the complementary set in  $\Omega$

$$\Omega_e^\delta = \Omega \setminus \overline{\Omega_i^\delta}$$

is occupied by the dielectric. We denote by  $\Omega^\delta$  the interior of  $\bigcup_{k \in K^\delta} \overline{Y_k^\delta}$  and we set  $\mathcal{U}^\delta := \Omega \setminus \overline{\Omega^\delta}$ .

We define the macroscopic dielectric permittivity  $\varepsilon^\delta$  and the magnetic permeability  $\mu^\delta$  on  $\Omega$  such

that

$$\varepsilon^\delta(x) = \varepsilon_e \mathbb{1}_{\Omega_e^\delta}(x) + \varepsilon_i \mathbb{1}_{\Omega_i^\delta}(x), \quad \mu^\delta(x) = \mu_e \mathbb{1}_{\Omega_e^\delta}(x) + \mu_i \mathbb{1}_{\Omega_i^\delta}(x). \quad (7.4)$$

For a given frequency  $\omega \neq 0$  ( $\omega \in \mathbb{R}$ ), we study time harmonic Maxwell's equations

$$\mathbf{curl} \mathbf{E}^\delta - i\omega \mu^\delta \mathbf{H}^\delta = 0 \quad \text{and} \quad \mathbf{curl} \mathbf{H}^\delta + i\omega \varepsilon^\delta \mathbf{E}^\delta = \mathbf{J} \quad \text{in } \Omega. \quad (7.5)$$

Above  $\mathbf{E}^\delta$  and  $\mathbf{H}^\delta$  are respectively the electric and magnetic components of the electromagnetic field. The source term  $\mathbf{J}$  is the current density. We suppose that the medium  $\Omega$  is surrounded by a perfect conductor and we impose the boundary conditions

$$\mathbf{E}^\delta \times n = 0 \quad \text{and} \quad \mu^\delta \mathbf{H}^\delta \cdot n = 0 \quad \text{on } \partial\Omega, \quad (7.6)$$

where  $n$  denotes the unit outward normal vector field to  $\partial\Omega$ . For an introduction to the mathematical setting of Maxwell's equations, we refer the reader to the classical monographs by Monk [110] or Nédélec [115]). We introduce some functional spaces classically used in the study of Maxwell's equations, namely

$$\begin{aligned} \mathbf{L}^2(\Omega) &:= (\mathbf{L}^2(\Omega))^3 \\ \mathbf{H}(\mathbf{curl}; \Omega) &:= \{\mathbf{H} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{H} \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_N(\mathbf{curl}; \Omega) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \mathbf{E} \times n = 0 \text{ on } \partial\Omega\} \\ \mathbf{V}_T(\xi) &:= \{\mathbf{H} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \operatorname{div}(\xi \mathbf{H}) = 0, \xi \mathbf{H} \cdot n = 0 \text{ on } \partial\Omega\}, \quad \text{for } \xi \in \mathbf{L}^\infty(\Omega) \\ \mathbf{V}_N(\xi) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \operatorname{div}(\xi \mathbf{E}) = 0, \mathbf{E} \times n = 0 \text{ on } \partial\Omega\}, \quad \text{for } \xi \in \mathbf{L}^\infty(\Omega). \end{aligned}$$

For an open set  $\mathcal{O} \subset \mathbb{R}^3$ , the inner products in  $\mathbf{L}^2(\mathcal{O})$  and  $\mathbf{L}^2(\mathcal{O})$  are denoted indistinctly by  $(\cdot, \cdot)_\mathcal{O}$  and the corresponding norm by  $\|\cdot\|_\mathcal{O}$ . To simplify, in  $\mathbf{L}^2(\Omega)$  and  $\mathbf{L}^2(\Omega)$ , we just denote  $(\cdot, \cdot)$  and  $\|\cdot\|$ . The space  $\mathbf{H}(\mathbf{curl}; \Omega)$  and its subspaces  $\mathbf{H}_N(\mathbf{curl}; \Omega)$ ,  $\mathbf{V}_N(\xi)$ ,  $\mathbf{V}_T(\xi)$  are endowed with the inner product

$$(\cdot, \cdot)_{\mathbf{curl}} := (\cdot, \cdot) + (\mathbf{curl} \cdot, \mathbf{curl} \cdot),$$

and the corresponding norm is denoted  $\|\cdot\|_{\mathbf{curl}}$ . We have the classical Green's formula for the curl operator (see for instance [110, Theorem 3.1]):

$$(\mathbf{u}, \mathbf{curl} \mathbf{v}) - (\mathbf{curl} \mathbf{u}, \mathbf{v}) = 0, \quad \forall \mathbf{u} \in \mathbf{H}_N(\mathbf{curl}; \Omega), \mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega).$$

Let us recall a well-known property for the particular spaces  $\mathbf{V}_T(1)$  and  $\mathbf{V}_N(1)$  (cf. [139, 8]).

**Proposition 7.2.1.** *The embeddings of  $\mathbf{V}_T(1)$  in  $\mathbf{L}^2(\Omega)$  and of  $\mathbf{V}_N(1)$  in  $\mathbf{L}^2(\Omega)$  are compact. Moreover, there is a constant  $C > 0$  such that*

$$\|\mathbf{u}\| \leq C \|\mathbf{curl} \mathbf{u}\|, \quad \forall \mathbf{u} \in \mathbf{V}_T(1) \cup \mathbf{V}_N(1).$$

Therefore, in  $\mathbf{V}_T(1)$  and in  $\mathbf{V}_N(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{curl}}$ .

Classically, one proves that if  $(\mathbf{E}^\delta, \mathbf{H}^\delta)$  satisfies (7.5)-(7.6), then  $\mathbf{E}^\delta$  and  $\mathbf{H}^\delta$  are respectively solutions of the problems

$$\left| \begin{array}{l} \text{Find } \mathbf{E}^\delta \in \mathbf{H}(\mathbf{curl}; \Omega) \text{ such that:} \\ \mathbf{curl}((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}^\delta) - \omega^2 \varepsilon^\delta \mathbf{E}^\delta = i\omega \mathbf{J} \quad \text{in } \Omega \\ \mathbf{E}^\delta \times n = 0 \quad \text{on } \partial\Omega, \end{array} \right. \quad (7.7)$$

$$\left| \begin{array}{l} \text{Find } \mathbf{H}^\delta \in \mathbf{H}(\mathbf{curl}; \Omega) \text{ such that:} \\ \mathbf{curl}((\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{H}^\delta) - \omega^2 \mu^\delta \mathbf{H}^\delta = \mathbf{curl}((\varepsilon^\delta)^{-1} \mathbf{J}) \quad \text{in } \Omega \\ \mu^\delta \mathbf{H}^\delta \cdot n = 0 \quad \text{on } \partial\Omega \\ (\varepsilon^\delta)^{-1} (\mathbf{curl} \mathbf{H} - \mathbf{J}) \times n = 0 \quad \text{on } \partial\Omega. \end{array} \right. \quad (7.8)$$

We emphasize that in (7.7), (7.8), the boundary conditions are the usual ones one should impose to be able to prove well-posedness of the systems. In the following, we will focus our attention on



the problem (7.7) for the electric field. The analysis for the magnetic field is quite similar. The variational formulation of (7.7) writes

$$(\mathcal{P}^\delta) \quad \left| \begin{array}{l} \text{Find } \mathbf{E}^\delta \in \mathbf{H}_N(\mathbf{curl}; \Omega) \text{ such that for all } \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega): \\ ((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}^\delta, \mathbf{curl} \mathbf{E}') - \omega^2(\varepsilon^\delta \mathbf{E}^\delta, \mathbf{E}') = i\omega(\mathbf{J}, \mathbf{E}'). \end{array} \right. \quad (7.9)$$

Before studying the behaviour of some solutions of  $(\mathcal{P}^\delta)$  as  $\delta$  tends to zero, we must clarify the properties of this problem for a fixed  $\delta > 0$ . With the Riesz representation theorem, define the linear and continuous operator  $\mathcal{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}; \Omega) \rightarrow \mathbf{H}_N(\mathbf{curl}; \Omega)$  such that for all  $\omega \in \mathbb{C}$ ,

$$(\mathcal{A}_N^\delta(\omega) \mathbf{E}, \mathbf{E}')_{\mathbf{curl}} = ((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{E}') - \omega^2(\varepsilon^\delta \mathbf{E}, \mathbf{E}'), \quad \forall \mathbf{E}, \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega). \quad (7.10)$$

The features of  $\mathcal{A}_N^\delta(\omega)$  are strongly related to the ones of two scalar operators that we define now. Set

$$\begin{aligned} \mathbf{H}_0^1(\Omega) &:= \{ \varphi \in \mathbf{H}^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega \} \\ \mathbf{H}_\#^1(\Omega) &:= \left\{ \varphi \in \mathbf{H}^1(\Omega) \mid \int_\Omega \varphi \, dx = 0 \right\}. \end{aligned}$$

In  $\mathbf{H}_0^1(\Omega)$  and in  $\mathbf{H}_\#^1(\Omega)$  (since  $\Omega$  is connected),  $\|\nabla \cdot\|$  is a norm which is equivalent to the usual norm of  $\mathbf{H}^1(\Omega)$ . We define the two linear and continuous operators  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  such that

$$\begin{aligned} (\nabla(A_\varepsilon^\delta \varphi), \nabla \varphi') &= (\varepsilon^\delta \nabla \varphi, \nabla \varphi'), & \forall \varphi, \varphi' \in \mathbf{H}_0^1(\Omega) \\ (\nabla(B_\mu^\delta \varphi), \nabla \varphi') &= (\mu^\delta \nabla \varphi, \nabla \varphi'), & \forall \varphi, \varphi' \in \mathbf{H}_\#^1(\Omega). \end{aligned}$$

With these notations, Theorem 6.1 of [23] writes as follows.

**Theorem 7.2.1.** *Assume that the scalar operators  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  are isomorphisms. Then  $\mathcal{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}; \Omega) \rightarrow \mathbf{H}_N(\mathbf{curl}; \Omega)$  is an isomorphism for all  $\omega \in \mathbb{C} \setminus \mathcal{S}$  where  $\mathcal{S}$  is a discrete set with no accumulation point.*

Note that in this statement, the set  $\mathcal{S}$  depends on the contrasts  $\kappa_\varepsilon, \kappa_\mu$  but also on the geometry and hence on  $\delta$ . In the next section, we give conditions ensuring that  $A_\varepsilon^\delta$  and  $B_\mu^\delta$  are isomorphisms.

## 7.3 Uniform invertibility of the two scalar problems

We shall say that the operators  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  are uniformly invertible as  $\delta$  tends to zero if there is  $\delta_0 > 0$  such that  $A_\varepsilon^\delta, B_\mu^\delta$  are invertible for all  $\delta \in (0; \delta_0]$  together with the estimate

$$\|(A_\varepsilon^\delta)^{-1}\| + \|(B_\mu^\delta)^{-1}\| \leq C,$$

where  $C > 0$  is a constant which is independent of  $\delta \in (0; \delta_0]$ . In this section, our goal is to find criteria on  $\kappa_\varepsilon, \kappa_\mu$  guaranteeing the uniform invertibility of  $A_\varepsilon^\delta, B_\mu^\delta$ . The uniform invertibility of  $A_\varepsilon^\delta$  has been considered in the articles [43, 31]. Below we combine the approaches presented in these two articles and we adapt the analysis in order to obtain a criterion ensuring the uniform invertibility of  $B_\mu^\delta$ .

**Remark 7.3.1.** *The result of uniform invertibility of [31, Theorem 5.2] is based on the result of Theorem 4.3 of the same article. However, its domain of validity is not completely satisfactory because the constant  $m$  defined in Theorem 4.3 is in fact equal to zero. This has been corrected by the authors and a new proof can be found in the erratum [37].*

### 7.3.1 First $\delta$ -dependent criteria

In a pedagogical aim, we first derive some criteria ensuring the invertibility of  $A_\varepsilon^\delta, B_\mu^\delta$  that are valid only for fixed  $\delta$ , and hence which are not uniform.

**Criterion of invertibility for the operator  $A_\varepsilon^\delta$**

In order to get a criterion on the contrast  $\kappa_\varepsilon$  ensuring that  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism, we start by presenting a well-chosen decomposition of the space  $H_0^1(\Omega)$  which has been introduced in [31]. We recall that  $H_0^1(\Omega)$  is endowed with the inner product  $(\nabla \cdot, \nabla \cdot)$ .

**Lemma 7.3.1.** *We have the decomposition  $H_0^1(\Omega) = \mathcal{H}_D^\delta \oplus H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$  where  $\mathcal{H}_D^\delta := \{\varphi \in H_0^1(\Omega) \mid \Delta\varphi = 0 \text{ in } \Omega_e^\delta \cup \Omega_i^\delta\}$ .*

**Remark 7.3.2.** *The index  $D$  in the notation  $\mathcal{H}_D^\delta$  stands for Dirichlet and refers to the homogeneous Dirichlet boundary condition imposed on  $\partial\Omega$  to the elements of  $\mathcal{H}_D^\delta$ . We emphasize that the functions of  $H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$  vanish on  $\partial\Omega_i^\delta$ .*

**Proof.** Let  $\varphi$  be a given element of  $H_0^1(\Omega)$ . Introduce  $\tilde{\varphi} \in H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$  the function such that  $\Delta\tilde{\varphi} = \Delta\varphi$  in  $\Omega_e^\delta \cup \Omega_i^\delta$ . Then we have  $\varphi = (\varphi - \tilde{\varphi}) + \tilde{\varphi}$  and clearly  $\varphi - \tilde{\varphi} \in \mathcal{H}_D^\delta$ . Now if  $\varphi_1$  and  $\varphi_2$  are elements of  $\mathcal{H}_D^\delta$  and  $H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , a direct integration by parts gives

$$(\nabla\varphi_1, \nabla\varphi_2) = \int_{\Omega_e^\delta \cup \Omega_i^\delta} \Delta\varphi_1 \varphi_2 \, dx + \int_{\partial\Omega_i^\delta} \frac{\partial\varphi_1}{\partial n_i} \varphi_2 \, d\sigma + \int_{\partial\Omega_e^\delta} \frac{\partial\varphi_1}{\partial n_e} \varphi_2 \, d\sigma = 0.$$

Here and below,  $n_e = -n_i$  stands for the unit normal vector to  $\partial\Omega_i^\delta$  pointing to  $\Omega_i^\delta$ . Moreover for  $x \in \partial\Omega_i^\delta$ ,  $\partial_n\varphi_i(x) = \lim_{t \rightarrow 0^+} \nabla\varphi(x - tn_i) \cdot n(x)$  and  $\partial_n\varphi_e(x) = \lim_{t \rightarrow 0^+} \nabla\varphi(x - tn_e) \cdot n(x)$ . This gives the desired result.  $\blacksquare$

In what follows, some particular elements of  $\mathcal{H}_D^\delta$  will play a key role. For  $k \in K^\delta$ , define the function  $\varphi_D^k \in \mathcal{H}_D^\delta$  such that

$$\varphi_D^k = \begin{cases} 1 & \text{in } Y_{ik}^\delta \\ 0 & \text{in } Y_{ik'}^\delta \quad \text{for } k' \neq k. \end{cases} \quad (7.11)$$

Then set

$$\hat{\mathcal{H}}_D^\delta := \{\varphi \in \mathcal{H}_D^\delta \mid (\nabla\varphi, \nabla\varphi_D^k) = 0, \quad \forall k \in K^\delta\} \quad (7.12)$$

so that we have, as in [31, Proposition 3.2],

$$H_0^1(\Omega) = \hat{\mathcal{H}}_D^\delta \oplus \text{span}_{k \in K^\delta} \{\varphi_D^k\} \oplus H_0^1(\Omega_e^\delta \cup \Omega_i^\delta). \quad (7.13)$$

Finally, we define the constants

$$m_D^\delta := \inf_{\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}, \quad M_D^\delta := \sup_{\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (7.14)$$

Before proceeding, let us discuss a few features of the constants  $m_D^\delta$ ,  $M_D^\delta$ . First, observe that the functions  $\varphi_D^k$  satisfy

$$\|\nabla\varphi_D^k\|_{\Omega_i^\delta}^2 = 0 \quad \text{and} \quad \|\nabla\varphi_D^k\|_{\Omega_e^\delta}^2 \neq 0.$$

As a consequence, the infimum of (7.14) considered over  $\hat{\mathcal{H}}_D^\delta \setminus \{0\}$  is zero. On the other hand, the next lemma guarantees that the supremum of (7.14) considered over  $\hat{\mathcal{H}}_D^\delta \setminus \{0\}$  coincides with  $M_D^\delta$ .

**Lemma 7.3.2.** *The constant  $M_D^\delta$  defined in (7.14) satisfies*

$$M_D^\delta = \sup_{\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (7.15)$$

**Proof.** Since  $\hat{\mathcal{H}}_D^\delta \subset \mathcal{H}_D^\delta$ , clearly we have

$$M_D^\delta \leq \sup_{\varphi \in \mathcal{H}_D^\delta \setminus \{0\}} \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}. \quad (7.16)$$

Now we establish the other inequality. If  $\varphi \in \mathcal{H}_D^\delta \setminus \{0\}$ , we have the decomposition  $\varphi = \hat{\varphi} + \Phi$  with  $\hat{\varphi} \in \hat{\mathcal{H}}_D^\delta$  and  $\Phi \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$ . Since  $\Phi$  is constant in each of the  $Y_{ik}^\delta$ ,  $k \in K^\delta$ , there holds

$$\|\nabla \varphi\|_{\Omega_i^\delta}^2 = \|\nabla \hat{\varphi}\|_{\Omega_i^\delta}^2. \quad (7.17)$$

As a consequence, if  $\hat{\varphi} \equiv 0$ , then  $0 = \|\nabla \varphi\|_{\Omega_i^\delta}^2 / \|\nabla \varphi\|_{\Omega_e^\delta}^2 \leq M_D^\delta$ . If  $\hat{\varphi} \not\equiv 0$ , from (7.17) and the identity  $\|\nabla \varphi\|_{\Omega}^2 = \|\nabla \hat{\varphi}\|_{\Omega}^2 + \|\nabla \Phi\|_{\Omega}^2$  (see (7.12)), we deduce that

$$\|\nabla \varphi\|_{\Omega_e^\delta}^2 = \|\nabla \hat{\varphi}\|_{\Omega_e^\delta}^2 + \|\nabla \Phi\|_{\Omega_e^\delta}^2 \geq \|\nabla \hat{\varphi}\|_{\Omega_e^\delta}^2.$$

This implies

$$M_D^\delta \geq \frac{\|\nabla \hat{\varphi}\|_{\Omega_i^\delta}^2}{\|\nabla \hat{\varphi}\|_{\Omega_e^\delta}^2} \geq \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}. \quad (7.18)$$

Taking the supremum over all  $\varphi \in \mathcal{H}_D^\delta \setminus \{0\}$  in (7.18), we deduce that (7.16) is also true with “ $\leq$ ” replaced by “ $\geq$ ”. This shows (7.15).  $\blacksquare$

Finally, we prove the following additional result.

**Lemma 7.3.3.** *The constants  $m_D^\delta$ ,  $M_D^\delta$  satisfy  $0 < m_D^\delta \leq M_D^\delta < +\infty$ .*

**Proof.** By definition of  $m_D^\delta$ ,  $M_D^\delta$ , clearly we have  $m_D^\delta \leq M_D^\delta$ . On the other hand, working by contradiction, thanks to the orthogonality conditions imposed to the elements of  $\hat{\mathcal{H}}_D^\delta$ , one can show the Poincaré-Wirtinger inequality

$$\exists C^\delta > 0 \quad \text{such that} \quad \|\varphi\|_{\Omega_i^\delta} \leq C^\delta \|\nabla \varphi\|_{\Omega_i^\delta}, \quad \forall \varphi \in \hat{\mathcal{H}}_D^\delta. \quad (7.19)$$

For  $\varphi \in \hat{\mathcal{H}}_D^\delta$ , since there holds  $\Delta \varphi = 0$  in  $\Omega_i^\delta$ , from (7.19), we obtain the estimate

$$\|\nabla \varphi\|_{\Omega_i^\delta} \leq C^\delta \|\varphi\|_{H^{1/2}(\partial\Omega_i^\delta)}.$$

Here the constant  $C^\delta$  may change from one line to another. Then the continuity of the trace from  $H^1(\Omega_e^\delta)$  into  $H^{1/2}(\partial\Omega_i^\delta)$  yields the existence of a constant  $C_1^\delta > 0$  such that

$$\|\nabla \varphi\|_{\Omega_i^\delta} \leq C_2^\delta \|\nabla \varphi\|_{\Omega_e^\delta}, \quad \forall \varphi \in \hat{\mathcal{H}}_D^\delta. \quad (7.20)$$

Similarly, using the continuity of the trace from  $H^1(\Omega_i^\delta)$  into  $H^{1/2}(\partial\Omega_i^\delta)$ , we obtain that there is  $C_2^\delta > 0$  such that

$$\|\nabla \varphi\|_{\Omega_e^\delta} \leq C_2^\delta \|\nabla \varphi\|_{\Omega_i^\delta}, \quad \forall \varphi \in \hat{\mathcal{H}}_D^\delta. \quad (7.21)$$

Estimates (7.20) and (7.21) allow one to conclude to the result of the lemma.  $\blacksquare$

After these considerations, we can now establish the following criterion concerning the invertibility of  $A_\varepsilon^\delta$ . To proceed, we work with the T-coercivity approach introduced in [27] (see also [52]). We emphasize however that we work with a different operator  $\mathbf{T}$  allowing us to obtain a sharper result.

**Proposition 7.3.1.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m_D^\delta) \cup (-1/M_D^\delta; 0)$  where  $m_D^\delta$  and  $M_D^\delta$  are defined in (7.14). Then  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism.*

**Proof.** Define the operator  $\mathbf{T}_D^+ : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  such that for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_D^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$  and  $\tilde{\varphi} \in \mathbf{H}_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , there holds

$$\mathbf{T}_D^+ \varphi = \begin{cases} \hat{\varphi}_h + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ \hat{\varphi}_h + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (7.22)$$

Note that since  $\tilde{\varphi} = 0$  on  $\partial\Omega_i^\delta$ , the operator  $\mathbf{T}_D^+$  is indeed valued in  $\mathbf{H}_0^1(\Omega)$ . Moreover we have  $\mathbf{T}_D^+ \circ \mathbf{T}_D^+ = \text{Id}$  which shows that  $\mathbf{T}_D^+$  is an isomorphism of  $\mathbf{H}_0^1(\Omega)$ . For all  $\varphi \in \mathbf{H}_0^1(\Omega)$ , we find

$$\begin{aligned} & (\nabla(A_\varepsilon^\delta(\mathbf{T}_D^+ \varphi)), \nabla \varphi) \\ &= \varepsilon_e (\nabla(\hat{\varphi}_h + \Phi_h + \tilde{\varphi}), \nabla(\hat{\varphi}_h + \Phi_h + \tilde{\varphi}))_{\Omega_e^\delta} + \varepsilon_i (\nabla(\hat{\varphi}_h - \tilde{\varphi}), \nabla(\hat{\varphi}_h + \tilde{\varphi}))_{\Omega_i^\delta}. \end{aligned} \quad (7.23)$$

Integrating by parts and using that  $\tilde{\varphi} = 0$  on  $\partial\Omega_e^\delta \cup \partial\Omega_i^\delta$ , we get

$$(\nabla \hat{\varphi}_h, \nabla \tilde{\varphi})_{\Omega_e^\delta} = (\nabla \Phi_h, \nabla \tilde{\varphi})_{\Omega_e^\delta} = (\nabla \hat{\varphi}_h, \nabla \tilde{\varphi})_{\Omega_i^\delta} = 0. \quad (7.24)$$

Besides, using again that  $\Phi_h$  is constant in each of the  $Y_{ik}^\delta$ , from the orthogonal decomposition (7.13), we infer that

$$(\nabla \hat{\varphi}_h, \nabla \Phi_h)_{\Omega_e^\delta} = (\nabla \hat{\varphi}_h, \nabla \Phi_h) = 0. \quad (7.25)$$

Inserting (7.24), (7.25) in (7.23), we obtain

$$(\nabla(A_\varepsilon^\delta(\mathbf{T}_D^+ \varphi)), \nabla \varphi) = (\varepsilon \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) + (\varepsilon_e \nabla \Phi_h, \nabla \Phi_h)_{\Omega_e^\delta} + (|\varepsilon| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (7.26)$$

For the first term of the right hand side of (7.26), we can write

$$\begin{aligned} & (\varepsilon \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) = \varepsilon_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 - |\varepsilon_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \\ & \geq (\varepsilon_e - |\varepsilon_i| M_D^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \\ & \geq \frac{1}{2} (\varepsilon_e - |\varepsilon_i| M_D^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_D^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2). \end{aligned} \quad (7.27)$$

Using this estimate in (7.26), we deduce that when  $\varepsilon_e > |\varepsilon_i| M_D^\delta \Leftrightarrow \kappa_\varepsilon = \varepsilon_i / \varepsilon_e > -1 / M_D^\delta$ , the bilinear form  $(\nabla(A_\varepsilon^\delta(\mathbf{T}_D^+ \cdot)), \nabla \cdot)$  is coercive in  $\mathbf{H}_0^1(\Omega)$  (note that Lemma 7.3.3 guarantees that  $M_D^\delta < +\infty$ ). With the Lax-Milgram theorem, we infer that when  $\kappa_\varepsilon > -1 / M_D^\delta$ , the operator  $A_\varepsilon^\delta \circ \mathbf{T}_D^+$  is an isomorphism of  $\mathbf{H}_0^1(\Omega)$  and so is  $A_\varepsilon^\delta$ .

To address the case  $\kappa_\varepsilon < -1 / m_D^\delta$ , let us work with the operator  $\mathbf{T}_D^- : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  such that

$$\mathbf{T}_D^- \varphi = \begin{cases} -\hat{\varphi}_h + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ -\hat{\varphi}_h + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (7.28)$$

We also have  $\mathbf{T}_D^- \circ \mathbf{T}_D^- = \text{Id}$  which guarantees that  $\mathbf{T}_D^-$  is an isomorphism of  $\mathbf{H}_0^1(\Omega)$ . For all  $\varphi \in \mathbf{H}_0^1(\Omega)$ , we find

$$(\nabla(A_\varepsilon^\delta(\mathbf{T}_D^- \varphi)), \nabla \varphi) = -(\varepsilon \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) + (\varepsilon_e \nabla \Phi_h, \nabla \Phi_h)_{\Omega_e^\delta} + (|\varepsilon| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (7.29)$$

This time, we can write

$$\begin{aligned} & -(\varepsilon \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) = -\varepsilon_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + |\varepsilon_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \\ & \geq (-\varepsilon_e + |\varepsilon_i| m_D^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \geq \frac{1}{2} (-\varepsilon_e + |\varepsilon_i| m_D^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_D^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2). \end{aligned} \quad (7.30)$$

As a consequence, we see from (7.29) that when  $|\varepsilon_i| m_D^\delta > \varepsilon_e \Leftrightarrow \kappa_\varepsilon = \varepsilon_i / \varepsilon_e < -1 / m_D^\delta$ , the bilinear form  $(\nabla(A_\varepsilon^\delta(\mathbf{T}_D^- \cdot)), \nabla \cdot)$  is coercive in  $\mathbf{H}_0^1(\Omega)$  (here we also use the result of Lemma 7.3.3 ensuring that  $0 < m_D^\delta \leq M_D^\delta < +\infty$ ). We can conclude as above that when  $\kappa_\varepsilon < -1 / m_D^\delta$ , the operator  $A_\varepsilon^\delta$  is an isomorphism of  $\mathbf{H}_0^1(\Omega)$ .  $\blacksquare$

**Criterion of invertibility for the operator  $B_\mu^\delta$** 

Now we show similar results for the operator  $B_\mu^\delta : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$ . First, define the space

$$H_{\diamond}^1(\Omega) := \left\{ \varphi \in H^1(\Omega) \mid \int_{\partial\Omega_i^\delta} \varphi \, d\sigma = 0 \right\}.$$

**Lemma 7.3.4.** *We have the decomposition  $H_{\diamond}^1(\Omega) = \mathcal{H}_N^\delta \oplus H_{0, \partial\Omega_i^\delta}^1(\Omega)$  where  $\mathcal{H}_N^\delta := \{\varphi \in H_{\diamond}^1(\Omega) \mid \Delta\varphi = 0 \text{ in } \Omega_e^\delta \cup \Omega_i^\delta, \partial_n\varphi = 0 \text{ on } \partial\Omega\}$  and  $H_{0, \partial\Omega_i^\delta}^1(\Omega) := \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega_i^\delta\} \subset H_{\diamond}^1(\Omega)$ .*

**Remark 7.3.3.** *This time, the index  $N$  in the notation  $\mathcal{H}_N^\delta$  stands for Neumann and refers to the homogeneous Neumann boundary condition imposed on  $\partial\Omega$  to the elements of  $\mathcal{H}_N^\delta$ .*

**Proof.** For  $\varphi$  given in  $H_{\diamond}^1(\Omega)$ , introduce  $\tilde{\varphi} \in H_{0, \partial\Omega_i^\delta}^1(\Omega)$  the function such that

$$(\nabla\tilde{\varphi}, \nabla\varphi') = (\nabla\varphi, \nabla\varphi'), \quad \forall \varphi' \in H_{0, \partial\Omega_i^\delta}^1(\Omega).$$

Note that since the Poincaré inequality holds in the space  $H_{0, \partial\Omega_i^\delta}^1(\Omega)$ , the Lax-Milgram theorem indeed guarantees that this variational problem admits a unique solution. Then we have  $\varphi = (\varphi - \tilde{\varphi}) + \tilde{\varphi}$  and one can check that  $\varphi - \tilde{\varphi}$  belongs to  $\mathcal{H}_N^\delta$ . Finally if  $\varphi_1$  and  $\varphi_2$  are elements of  $\mathcal{H}_N^\delta$  and  $H_{0, \partial\Omega_i^\delta}^1(\Omega)$ , a direct integration by parts gives  $(\nabla\varphi_1, \nabla\varphi_2) = 0$ . ■

In what follows, some particular elements of  $\mathcal{H}_N^\delta$  will play a key role. Let  $k_0$  be an arbitrary given element of  $K^\delta$  and for  $k \in K^\delta \setminus \{k_0\}$ , define the function  $\varphi_N^k \in \mathcal{H}_N^\delta$  such that

$$\varphi_N^k = \begin{cases} 1 & \text{in } Y_{ik}^\delta \\ -1 & \text{in } Y_{ik_0}^\delta \\ 0 & \text{in } Y_{ik'}^\delta \quad \text{for } k' \in K^\delta \setminus \{k_0, k\}. \end{cases}$$

Then set

$$\hat{\mathcal{H}}_N^\delta := \{\varphi \in \mathcal{H}_N^\delta \mid (\nabla\varphi, \nabla\varphi_N^k) = 0, \quad \forall k \in K^\delta \setminus \{k_0\}\}$$

so that we have

$$H_{\diamond}^1(\Omega) = \hat{\mathcal{H}}_N^\delta \oplus \text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\} \oplus H_{0, \partial\Omega_i^\delta}^1(\Omega).$$

We emphasize that the choice of  $k_0$  above does not affect this decomposition. We simply consider one particular basis for the space  $\text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\}$ . Finally, we define the constants

$$m_N^\delta := \inf_{\varphi \in \hat{\mathcal{H}}_N^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}, \quad M_N^\delta := \sup_{\varphi \in \hat{\mathcal{H}}_N^\delta \setminus \{0\}} \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (7.31)$$

Working as in the proof of Lemma 7.3.3, in particular establishing by contradiction the Poincaré-Wirtinger inequality

$$\exists C^\delta > 0 \quad \text{such that} \quad \|\varphi\|_{\Omega_i^\delta} \leq C^\delta \|\nabla\varphi\|_{\Omega_i^\delta}, \quad \forall \varphi \in \hat{\mathcal{H}}_N^\delta,$$

one can show that there holds  $0 < m_N^\delta \leq M_N^\delta < +\infty$ . As in (7.14), the functions  $\varphi_N^k$  satisfy

$$\|\nabla\varphi_N^k\|_{\Omega_i^\delta}^2 = 0 \quad \text{and} \quad \|\nabla\varphi_N^k\|_{\Omega_e^\delta}^2 \neq 0$$

so that the infimum of (7.31) considered over  $\hat{\mathcal{H}}_N^\delta \setminus \{0\}$  is zero. Working exactly as in the proof of Lemma 7.3.2, we get the following result.

**Lemma 7.3.5.** *The constant  $M_N^\delta$  defined in (7.31) satisfies*

$$M_N^\delta = \sup_{\varphi \in \mathcal{H}_N^\delta \setminus \{0\}} \frac{\|\nabla \varphi\|_{\Omega_i^\delta}^2}{\|\nabla \varphi\|_{\Omega_e^\delta}^2}. \quad (7.32)$$

Now, we give our criterion of invertibility for the operator  $B_\mu^\delta$ .

**Proposition 7.3.2.** *Assume that  $\kappa_\mu \in (-\infty; -1/m_N^\delta) \cup (-1/M_N^\delta; 0)$  where  $m_N^\delta$  and  $M_N^\delta$  are defined in (7.31). Then  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  is an isomorphism.*

**Proof.** Introduce the mappings  $\ell_\diamond : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\diamond^1(\Omega)$  and  $\ell_\# : \mathbf{H}_\diamond^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  such that

$$\ell_\diamond(\varphi) = \varphi - \frac{1}{|\partial\Omega_i^\delta|} \int_{\partial\Omega_i^\delta} \varphi \, d\sigma, \quad \ell_\#(\varphi) = \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \, dx.$$

Here and in what follows, for an open set  $\mathcal{O} \subset \mathbb{R}^3$ , we denote by  $|\mathcal{O}| = \int_{\mathcal{O}} 1 \, dx$  and  $|\partial\mathcal{O}| = \int_{\partial\mathcal{O}} 1 \, d\sigma$ .

Then define the operators  $\tilde{\mathbf{T}}_N^\pm : \mathbf{H}_\diamond^1(\Omega) \rightarrow \mathbf{H}_\diamond^1(\Omega)$  such that for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_N^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\}$  and  $\tilde{\varphi} \in \mathbf{H}_{0, \partial\Omega_i^\delta}^1(\Omega)$ , there holds

$$\tilde{\mathbf{T}}_N^\pm \varphi = \begin{cases} \pm \hat{\varphi}_h + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ \pm \hat{\varphi}_h + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (7.33)$$

Finally, we define the operators

$$\mathbf{T}_N^\pm := \ell_\# \circ \tilde{\mathbf{T}}_N^\pm \circ \ell_\diamond.$$

For  $\psi \in \mathbf{H}_\#^1(\Omega)$ , we set  $\varphi := \ell_\diamond(\psi) \in \mathbf{H}_\diamond^1(\Omega)$  and we use the notation  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_N^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta \setminus \{k_0\}} \{\varphi_N^k\}$  and  $\tilde{\varphi} \in \mathbf{H}_{0, \partial\Omega_i^\delta}^1(\Omega)$ . Observing that  $\nabla \varphi = \nabla \psi$  ( $\varphi$  and  $\psi$  differ from each other by an additive constant) and working as in (7.26), we find

$$(\nabla(B_\mu^\delta(\mathbf{T}_N^\pm \psi)), \nabla \psi) = \pm(\mu \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) + (\mu_e \nabla \Phi_h, \nabla \Phi_h)_{\Omega_e^\delta} + (|\mu| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (7.34)$$

For the first term of the right hand side of (7.34), we can write

$$\begin{aligned} (\mu \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) &= \mu_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 - |\mu_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \geq (\mu_e - |\mu_i| M_N^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \\ &\geq \frac{1}{2} (\mu_e - |\mu_i| M_N^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_N^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2) \end{aligned}$$

and

$$\begin{aligned} -(\mu \nabla \hat{\varphi}_h, \nabla \hat{\varphi}_h) &= -\mu_e \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + |\mu_i| \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2 \\ &\geq (-\mu_e + |\mu_i| m_N^\delta) \|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 \geq \frac{1}{2} (-\mu_e + |\mu_i| m_N^\delta) (\|\nabla \hat{\varphi}_h\|_{\Omega_e^\delta}^2 + (M_N^\delta)^{-1} \|\nabla \hat{\varphi}_h\|_{\Omega_i^\delta}^2). \end{aligned} \quad (7.35)$$

Using again that  $\nabla \varphi = \nabla \psi$ , we deduce from the first estimate of (7.34) that when  $\mu_e > |\mu_i| M_N^\delta \Leftrightarrow \kappa_\mu = \mu_i/\mu_e > -1/M_N^\delta$ , the bilinear form  $(\nabla(B_\mu^\delta(\mathbf{T}_N^\pm \cdot)), \nabla \cdot)$  is coercive in  $\mathbf{H}_\#^1(\Omega)$ . With the Lax-Milgram theorem, we infer that when  $\kappa_\mu > -1/M_N^\delta$ , the operator  $B_\mu^\delta \circ \mathbf{T}_N^\pm$  is an isomorphism of  $\mathbf{H}_\#^1(\Omega)$ . Since  $B_\mu^\delta$  is selfadjoint (because it is bounded and symmetric), this implies that  $B_\mu^\delta$  is an isomorphism. Working similarly with  $\mathbf{T}_N^-$ , from (7.35) one finds that when  $|\mu_i| m_N^\delta > \mu_e \Leftrightarrow \kappa_\mu = \mu_i/\mu_e < -1/m_N^\delta$ , the operator  $B_\mu^\delta$  is an isomorphism. Note that with additional few lines, one can check that we have  $\mathbf{T}_N^\pm \circ \mathbf{T}_N^\pm = \text{Id}$ .  $\blacksquare$

### 7.3.2 Comparison between the criteria of invertibility

In this section, we compare the constants involved in the criteria ensuring the invertibility of the operators  $A_\varepsilon^\delta$  (Dirichlet) and  $B_\mu^\delta$  (Neumann).

**Proposition 7.3.3.** *For all  $\delta > 0$ , the constants  $m_D^\delta$ ,  $M_D^\delta$  defined in (7.14) and the constants  $m_N^\delta$ ,  $M_N^\delta$  defined in (7.31) satisfy*

$$m_D^\delta \leq m_N^\delta \quad \text{and} \quad M_D^\delta \leq M_N^\delta. \quad (7.36)$$

**Proof.** We start by proving the second inequality of (7.36). Let  $\varphi$  be an element of  $\hat{\mathcal{H}}_D^\delta \setminus \{0\}$ . Define the function  $\zeta \in \mathcal{H}_N^\delta$  such that  $\zeta = \varphi - c$  on  $\partial\Omega_i^\delta$  where  $c = |\partial\Omega_i^\delta|^{-1} \int_{\partial\Omega_i^\delta} \varphi \, d\sigma$ . In other words,  $\zeta$  is the function such that  $\Delta\zeta = 0$  in  $\Omega_e^\delta \cup \Omega_i^\delta$ ,  $\zeta = \varphi - c$  on  $\partial\Omega_i^\delta$  and  $\partial_n\zeta = 0$  on  $\partial\Omega$ . Note that necessarily, there holds  $\zeta \neq 0$ . Then we have  $\zeta = \varphi - c$  in  $\Omega_i^\delta$  and so

$$\|\nabla\zeta\|_{\Omega_i^\delta} = \|\nabla\varphi\|_{\Omega_i^\delta}. \quad (7.37)$$

On the other hand, integrating by parts, we find

$$\begin{aligned} & (\nabla\zeta, \nabla(\zeta - \varphi))_{\Omega_e^\delta} \\ &= (\nabla\zeta, \nabla(\zeta - (\varphi - c)))_{\Omega_e^\delta} = \int_{\partial\Omega} \frac{\partial\zeta}{\partial n} (\zeta - (\varphi - c)) \, d\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\zeta_e}{\partial n_e} (\zeta - (\varphi - c)) \, d\sigma = 0. \end{aligned}$$

We deduce that

$$\|\nabla\zeta\|_{\Omega_e^\delta}^2 \leq \|\nabla\varphi\|_{\Omega_e^\delta}^2. \quad (7.38)$$

Gathering (7.37), (7.38) and using Lemma 7.3.5, we infer that

$$\frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2} \leq \frac{\|\nabla\zeta\|_{\Omega_i^\delta}^2}{\|\nabla\zeta\|_{\Omega_e^\delta}^2} \leq M_N^\delta. \quad (7.39)$$

Taking the supremum over all  $\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}$  in (7.39), we obtain that  $M_D^\delta \leq M_N^\delta$ .

Now we show the first inequality of (7.36). Let  $\varphi$  be an element of  $\hat{\mathcal{H}}_N^\delta \setminus \{0\}$ . Define the function  $\zeta \in \mathcal{H}_D^\delta$  such that  $\zeta = \varphi$  on  $\partial\Omega_i^\delta$ . In particular, we have  $\Delta\zeta = 0$  in  $\Omega_e^\delta \cup \Omega_i^\delta$  and  $\zeta = 0$  on  $\partial\Omega$ . Then decompose  $\zeta$  as  $\zeta = \hat{\zeta} + Z$  with  $\hat{\zeta} \in \hat{\mathcal{H}}_D^\delta$  and  $Z \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$ . Since  $Z$  is constant in each of the  $Y_{ik}^\delta$ ,  $k \in K^\delta$ , we have

$$\|\nabla\hat{\zeta}\|_{\Omega_i^\delta} = \|\nabla\varphi\|_{\Omega_i^\delta}. \quad (7.40)$$

On the other hand, integrating by parts, we find

$$(\nabla\varphi, \nabla(\hat{\zeta} - \varphi))_{\Omega_e^\delta} = \int_{\partial\Omega} \frac{\partial\varphi}{\partial n} (\hat{\zeta} - \varphi) \, d\sigma + \int_{\partial\Omega_i^\delta} \frac{\partial\varphi_e}{\partial n_e} (\hat{\zeta} - \varphi) \, d\sigma = - \int_{\partial\Omega_i^\delta} \frac{\partial\varphi_e}{\partial n_e} Z \, d\sigma. \quad (7.41)$$

Since the function  $\varphi$  is in  $\hat{\mathcal{H}}_N^\delta$ , for all  $k \in K^\delta \setminus \{k_0\}$ , we have  $(\nabla\varphi, \nabla\varphi_N^k) = 0$ . Integrating by parts, this implies

$$\int_{\partial Y_{ik}^\delta} \frac{\partial\varphi_e}{\partial n_e} \, d\sigma = \int_{\partial Y_{ik_0}^\delta} \frac{\partial\varphi_e}{\partial n_e} \, d\sigma.$$

But we also have  $\int_{\partial\Omega_i^\delta} \partial_{n_e}\varphi_e \, d\sigma = 0$ . As a consequence, we must have, for all  $k \in K^\delta$ ,

$$\int_{\partial Y_{ik}^\delta} \frac{\partial\varphi_e}{\partial n_e} \, d\sigma = 0.$$



Since  $Z$  is constant on each of the  $\partial Y_{ik}^\delta$ , we deduce that the terms of the equalities of (7.41) are equal to zero. Hence, there holds

$$\|\nabla\varphi\|_{\Omega_e^\delta}^2 \leq \|\nabla\hat{\zeta}\|_{\Omega_e^\delta}^2. \quad (7.42)$$

Gathering (7.40) and (7.42) leads to

$$m_D^\delta \leq \frac{\|\nabla\hat{\zeta}\|_{\Omega_i^\delta}^2}{\|\nabla\hat{\zeta}\|_{\Omega_e^\delta}^2} \leq \frac{\|\nabla\varphi\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (7.43)$$

Taking the infimum over all  $\varphi \in \hat{\mathcal{H}}_N^\delta \setminus \{0\}$  in (7.43), we obtain that  $m_D^\delta \leq m_N^\delta$ .  $\blacksquare$

### 7.3.3 Uniform criterion of invertibility

The bounds on the contrasts  $\kappa_\varepsilon, \kappa_\mu$  that we obtained in Propositions 7.3.1, 7.3.2 which ensure the invertibility of the scalar operators  $A_\varepsilon^\delta$  and  $B_\mu^\delta$ , depend on  $\delta$ . In this paragraph, we wish to get bounds which are uniform with respect to  $\delta$ .

Introduce the Hilbert spaces of functions defined in the reference cell  $Y$

$$\begin{aligned} \mathcal{H}_0 &:= \{\varphi \in H_0^1(Y) \mid \Delta\varphi = 0 \text{ in } Y_e \cup Y_i\} \\ \mathcal{H}_\diamond &:= \{\varphi \in H_\diamond^1(Y) \mid \Delta\varphi = 0 \text{ in } Y_e \cup Y_i\} \end{aligned}$$

where  $H_\diamond^1(Y) := \{\varphi \in H^1(Y) \mid \int_{\partial Y_i} \varphi \, d\sigma = 0\}$ . Define the function  $\varphi_D \in \mathcal{H}_0$  such that  $\varphi_D = 1$  in  $Y_i$  and set

$$\begin{aligned} \hat{\mathcal{H}}_0 &:= \{\varphi \in \mathcal{H}_0 \mid (\nabla\varphi, \nabla\varphi_D) = 0\} \\ \hat{\mathcal{H}}_\diamond &:= \{\varphi \in \mathcal{H}_\diamond \mid \partial_n\varphi = 0 \text{ on } \partial Y\}. \end{aligned} \quad (7.44)$$

Then we introduce the constants

$$m := \inf_{\varphi \in \hat{\mathcal{H}}_0 \setminus \{0\}} \frac{\|\nabla\varphi\|_{Y_i}^2}{\|\nabla\varphi\|_{Y_e}^2}, \quad M := \sup_{\varphi \in \hat{\mathcal{H}}_\diamond \setminus \{0\}} \frac{\|\nabla\varphi\|_{Y_i}^2}{\|\nabla\varphi\|_{Y_e}^2}. \quad (7.45)$$

We emphasize that  $m$  and  $M$  are independent of  $\delta$ .

**Lemma 7.3.6.** *The constant  $M$  defined in (7.45) satisfies*

$$M = \sup_{\varphi \in \mathcal{H}_\diamond \setminus \{0\}} \frac{\|\nabla\varphi\|_{Y_i}^2}{\|\nabla\varphi\|_{Y_e}^2} \quad (7.46)$$

(here the sup is considered over  $\mathcal{H}_\diamond \setminus \{0\}$  and not  $\hat{\mathcal{H}}_\diamond \setminus \{0\}$ ).

**Proof.** Since there holds  $\hat{\mathcal{H}}_\diamond \subset \mathcal{H}_\diamond$ , it suffices to show that

$$\sup_{\varphi \in \hat{\mathcal{H}}_\diamond \setminus \{0\}} \frac{\|\nabla\varphi\|_{Y_i}^2}{\|\nabla\varphi\|_{Y_e}^2} \leq M. \quad (7.47)$$

Let  $\varphi$  be a non zero element of  $\hat{\mathcal{H}}_\diamond$ . We have the decomposition  $\varphi = \hat{\varphi} + (\varphi - \hat{\varphi})$  where  $\hat{\varphi} \in \hat{\mathcal{H}}_\diamond$  is the function such that  $\hat{\varphi} = \varphi$  in  $Y_i$ ,  $\Delta\hat{\varphi} = 0$  in  $Y_e$ ,  $\hat{\varphi} = \varphi$  on  $\partial Y_i$  and  $\partial_n\hat{\varphi} = 0$  on  $\partial Y$ . Observing that  $\|\nabla\varphi\|_{Y_i}^2 = \|\nabla\hat{\varphi}\|_{Y_i}^2$  and that

$$\|\nabla\varphi\|_{Y_e}^2 = \|\nabla\hat{\varphi}\|_{Y_e}^2 + \|\nabla(\varphi - \hat{\varphi})\|_{Y_e}^2 \geq \|\nabla\hat{\varphi}\|_{Y_e}^2,$$

we can write

$$\|\nabla\varphi\|_{Y_i}^2 = \|\nabla\hat{\varphi}\|_{Y_i}^2 \leq M \|\nabla\hat{\varphi}\|_{Y_e}^2 \leq M \|\nabla\varphi\|_{Y_e}^2.$$

Taking the supremum over all  $\varphi \in \hat{\mathcal{H}}_\diamond \setminus \{0\}$  leads to (7.47).  $\blacksquare$

**Lemma 7.3.7.** *For all  $\delta > 0$ , we have the relations*

$$m \leq m_D^\delta \leq m_N^\delta \quad \text{and} \quad M_D^\delta \leq M_N^\delta \leq M, \quad (7.48)$$

where  $m_D^\delta, M_D^\delta$  are defined in (7.14),  $m_N^\delta, M_N^\delta$  are defined in (7.31) and  $m, M$  are defined in (7.45).

**Proof.** From Proposition 7.3.3, we know that we have  $m_D^\delta \leq m_N^\delta$  and  $M_D^\delta \leq M_N^\delta$ . Now we show that we have  $M_N^\delta \leq M$ . Let  $\varphi$  be a non zero element of  $\hat{\mathcal{H}}_N^\delta$ . For all  $k \in K^\delta$ , we define the function  $\varphi_k^\delta \in H^1(Y)$  such that  $\varphi_k^\delta(y) = \varphi(\delta(k+y))$  for  $y \in Y$  and we set  $c_k := |\partial Y|^{-1} \int_{\partial Y} \varphi_k^\delta d\sigma$ . Since  $\varphi_k^\delta - c_k \in \mathcal{H}_\circ$ , using Lemma 7.3.6, we can write

$$\begin{aligned} \|\nabla \varphi\|_{Y_{ik}^\delta}^2 &= \delta \|\nabla \varphi_k^\delta\|_{Y_i}^2 = \delta \|\nabla(\varphi_k^\delta - c_k)\|_{Y_i}^2 \\ &\leq \delta M \|\nabla(\varphi_k^\delta - c_k)\|_{Y_e}^2 \leq \delta M \|\nabla \varphi_k^\delta\|_{Y_e}^2 \leq M \|\nabla \varphi\|_{Y_{ek}^\delta}^2. \end{aligned}$$

Summing these estimates over all  $k \in K^\delta$ , we get (recall that  $\mathcal{U}^\delta = \Omega \setminus \overline{\Omega^\delta}$ )

$$\|\nabla \varphi\|_{\Omega_i^\delta}^2 \leq M \|\nabla \varphi\|_{\Omega_e^\delta \setminus \overline{\mathcal{U}^\delta}}^2 \leq M \|\nabla \varphi\|_{\Omega_e^\delta}^2. \quad (7.49)$$

Taking the supremum in (7.49) over all  $\varphi \in \hat{\mathcal{H}}_N^\delta$ , we deduce that  $M_N^\delta \leq M$ .

To establish (7.48), it remains to show that  $m \leq m_D^\delta$ . For  $\varphi$  given in  $\hat{\mathcal{H}}_D^\delta$ , introduce the function  $v \in H_0^1(\Omega)$  such that for all  $k \in K^\delta$ ,

$$\begin{cases} v = \varphi & \text{in } Y_{ik}^\delta \\ \Delta v = 0 & \text{in } Y_{ek}^\delta \\ v = 0 & \text{on } \partial Y_{ek}^\delta. \end{cases}$$

We also impose  $v = 0$  in  $\mathcal{U}^\delta = \Omega \setminus \overline{\Omega^\delta}$ . For all  $k \in K^\delta$ , define the function  $\phi_D^k \in H_0^1(\Omega)$  such that

$$\begin{cases} \phi_D^k = 1 & \text{in } Y_{ik}^\delta \\ \Delta \phi_D^k = 0 & \text{in } Y_{ek}^\delta \\ \phi_D^k = 0 & \text{in } \Omega \setminus Y_k^\delta. \end{cases}$$

Then set

$$\tilde{v} := v - \sum_{k \in K^\delta} a_k \phi_D^k \quad \text{with } a_k := (\nabla v, \nabla \phi_D^k) / \|\nabla \phi_D^k\|^2.$$

Integrating by parts, we find

$$(\nabla \varphi, \nabla(\varphi - \tilde{v}))_{\Omega_e^\delta} = \int_{\partial \Omega} \frac{\partial \varphi}{\partial n} (\varphi - \tilde{v}) d\sigma + \int_{\partial \Omega_i^\delta} \frac{\partial \varphi_e}{\partial n_e} (\varphi - \tilde{v}) d\sigma = \sum_{k \in K^\delta} a_k \int_{\partial Y_{ik}^\delta} \frac{\partial \varphi_e}{\partial n_e} \phi_D^k d\sigma. \quad (7.50)$$

Since the function  $\varphi$  is in  $\hat{\mathcal{H}}_D^\delta$ , for all  $k \in K^\delta$ , we have  $(\nabla \varphi, \nabla \phi_D^k) = 0$ . Integrating by parts, this implies

$$\int_{\partial Y_{ik}^\delta} \frac{\partial \varphi_e}{\partial n_e} d\sigma = 0.$$

Using that  $\phi_D^k$  is constant on the  $\partial Y_{ik}^\delta$ , we deduce from (7.50) that  $(\nabla \varphi, \nabla(\varphi - \tilde{v}))_{\Omega_e^\delta} = 0$ . Hence, we have

$$\|\nabla \varphi\|_{\Omega_e^\delta}^2 \leq \|\nabla \tilde{v}\|_{\Omega_e^\delta}^2 = \sum_{k \in K^\delta} \|\nabla \tilde{v}\|_{Y_{ek}^\delta}^2. \quad (7.51)$$

For  $k \in K^\delta$ , define the function  $\tilde{v}_k^\delta \in H^1(Y)$  such that  $\tilde{v}_k^\delta(y) = \tilde{v}(\delta(k+y))$  for  $y \in Y$ . Observe that we have  $\tilde{v}_k^\delta \in \hat{\mathcal{H}}_0$  so that we can write

$$\|\nabla \tilde{v}\|_{Y_{ek}^\delta}^2 = \delta \|\nabla \tilde{v}_k^\delta\|_{Y_e}^2 \leq \delta m^{-1} \|\nabla \tilde{v}_k^\delta\|_{Y_i}^2 \leq \delta m^{-1} \|\nabla \varphi_k^\delta\|_{Y_i}^2 \leq m^{-1} \|\nabla \varphi\|_{Y_{ik}^\delta}^2. \quad (7.52)$$

As a consequence, inserting (7.52) in (7.51), we obtain

$$\|\nabla \varphi\|_{\Omega_e^\delta}^2 \leq m^{-1} \|\nabla \varphi\|_{\Omega_i^\delta}^2. \quad (7.53)$$

Taking the infimum in (7.53) over all  $\varphi \in \hat{\mathcal{H}}_D^\delta$ , we deduce that  $m \leq m_D^\delta$ .  $\blacksquare$

Finally, we deduce a criterion of uniform invertibility for the operators  $A_\varepsilon^\delta$  and  $B_\mu^\delta$ .

**Theorem 7.3.1.** *Let  $m, M$  be the constants defined in (7.45).*

*When  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ ,  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is uniformly invertible as  $\delta \rightarrow 0$ .*

*When  $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ ,  $B_\mu^\delta : H_{\#}^1(\Omega) \rightarrow H_{\#}^1(\Omega)$  is uniformly invertible as  $\delta \rightarrow 0$ .*

**Proof.** Let us show the result for  $A_\varepsilon^\delta$ , the proof is completely similar for  $B_\mu^\delta$ . From the decomposition of the space  $H_0^1(\Omega)$  in (7.13), one observes that the operators  $\mathsf{T}_D^\pm = (\mathsf{T}_D^\pm)^{-1}$  defined in (7.22) and (7.28) are uniformly continuous. From the estimate (7.27) (resp. (7.30)) together with the result of Lemma 7.3.7, one infers that as  $\delta \rightarrow 0$ ,  $(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^+ \cdot)), \nabla \cdot)$  (resp.  $(\nabla(A_\varepsilon^\delta(\mathsf{T}_D^- \cdot)), \nabla \cdot)$ ) is uniformly coercive in  $H_0^1(\Omega)$  when  $\kappa_\varepsilon > -1/M$  (resp. when  $\kappa_\varepsilon < -1/m$ ). Since  $A_\varepsilon^\delta$  is also uniformly continuous, this is enough to guarantee that  $A_\varepsilon^\delta$  is uniformly invertible as  $\delta$  tends to zero.  $\blacksquare$

### 7.3.4 Optimality of the criterion and connection to the Neumann-Poincaré operator

Let us discuss the criterion we have obtained above. We focus our attention on the analysis for the operator  $A_\varepsilon^\delta$ , similar comments can be made for the operator  $B_\mu^\delta$ . We assume in this paragraph that  $\partial Y_i$ , and so  $\partial \Omega_i^\delta$ , is of class  $\mathcal{C}^2$ . Note that this assumption is important to ensure that the spectrum of Problem (7.55) below is discrete. It has been proved in [21] that in this case,  $A_\varepsilon^\delta$  is Fredholm of index zero when  $\kappa_\varepsilon \neq -1$ . Therefore when  $\kappa_\varepsilon \neq -1$ , the operator  $A_\varepsilon^\delta$  is an isomorphism if and only if it is injective. As it has been observed in different works (see in particular [31]), and as we recall below, the question of the injectivity of  $A_\varepsilon^\delta$  is directly linked to the spectrum of the so-called Neumann-Poincaré operator. The latter has been widely studied when  $\Omega$  is the whole space  $\mathbb{R}^d$ . For this problem, among the references, let us cite [128, 4, 129, 130, 99, 84, 83, 126, 33, 34, 82]. Below, we use a symmetrization argument similar to the one used in [99]. We work with Dirichlet-to-Neumann maps following the approach of [82].

#### Spectrum of the Neumann-Poincaré operator

Set  $\Sigma^\delta := \partial \Omega_i^\delta$  and introduce the two Dirichlet-to-Neumann operators  $\Lambda_e : H^{1/2}(\Sigma^\delta) \rightarrow H^{-1/2}(\Sigma^\delta)$ ,  $\Lambda_i : H^{1/2}(\Sigma^\delta) \rightarrow H^{-1/2}(\Sigma^\delta)$  such that for all  $\varphi \in H^{1/2}(\Sigma^\delta)$ , we have  $\Lambda_e \varphi = \partial_{n_e} u_e$ ,  $\Lambda_i \varphi = \partial_{n_i} u_i$  where  $u_e, u_i$  solve respectively the problems

$$\left\{ \begin{array}{l} \Delta u_e = 0 \quad \text{in } \Omega_e^\delta \\ u_e = 0 \quad \text{on } \partial \Omega \\ u_e = \varphi \quad \text{on } \Sigma^\delta \end{array} \right. \quad \left\| \quad \left\{ \begin{array}{l} \Delta u_i = 0 \quad \text{in } \Omega_i^\delta \\ u_i = \varphi \quad \text{on } \Sigma^\delta. \end{array} \right. \right. \quad (7.54)$$

Define also the lifting operator  $\mathcal{R} : H^{1/2}(\Sigma^\delta) \rightarrow H_0^1(\Omega)$  such that  $\mathcal{R}\varphi = u_e$  in  $\Omega_e^\delta$ ,  $\mathcal{R}\varphi = u_i$  in  $\Omega_i^\delta$ , where  $u_e, u_i$  are the solutions to (7.54).

If  $u$  belongs to  $\ker A_\varepsilon^\delta \setminus \{0\}$ , then  $\varphi := u|_{\Sigma^\delta} \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  satisfies  $\Lambda_e \varphi = -\kappa_\varepsilon \Lambda_i \varphi$ . By a

straightforward computation, we find that the pair  $(\alpha, \varphi)$ , with  $\alpha := (\kappa_\varepsilon + 1)/(\kappa_\varepsilon - 1) \in (-1; 1)$ , is a solution to the generalized eigenvalue problem

$$\left\{ \begin{array}{l} \text{Find } (\alpha, \varphi) \in \mathbb{R} \times (\mathbf{H}^{1/2}(\Sigma^\delta) \setminus \{0\}) \text{ such that:} \\ \Lambda_- \varphi = \alpha \Lambda_+ \varphi \end{array} \right. \quad (7.55)$$

with  $\Lambda_\pm := \Lambda_e \pm \Lambda_i$ . Reciprocally, assume that  $(\alpha, \varphi)$  is a solution to (7.55) with  $\alpha \in (-1; 1)$ . Then,  $\mathcal{R}\varphi \in \mathbf{H}_0^1(\Omega)$  is an element of  $\ker A_\varepsilon^\delta \setminus \{0\}$  for  $\kappa_\varepsilon = (\alpha + 1)/(\alpha - 1) \in (-\infty; 0)$ . This shows that it is sufficient to determine the eigenvalues of problem (7.55) to study the injectivity of  $A_\varepsilon^\delta$ . Note that the spectrum of (7.55) coincides with the spectrum of the so called Neumann-Poincaré operator studied for example in [99].

**Theorem 7.3.2.** *The spectrum of the generalized eigenvalue problem (7.55) is discrete and coincides with two sequences of real numbers*

$$-1 < \alpha_1^- \leq \alpha_2^- \leq \dots \leq 0 \quad \text{and} \quad 1 = \alpha_1^+ = \dots = \alpha_{\text{card}(K^\delta)}^+ > \alpha_{\text{card}(K^\delta)+1}^+ \geq \dots \geq 0$$

such that  $\lim_{n \rightarrow +\infty} \alpha_n^\pm = 0$ . Here  $\text{card}(K^\delta)$  is the cardinal of the set  $K^\delta$  defined after (7.3).

**Proof.** First, we show that  $\Lambda_+ : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  is an isomorphism. Consider some  $\psi \in \mathbf{H}^{-1/2}(\Sigma^\delta)$ . If  $\varphi \in \mathbf{H}^{1/2}(\Sigma^\delta)$  verifies  $\Lambda_+ \varphi = \psi$ , then  $\mathcal{R}\varphi$  is a solution to

$$\left\{ \begin{array}{l} \text{Find } u \in \mathbf{H}_0^1(\Omega) \text{ such that} \\ (\nabla u, \nabla v) = \langle \psi, v \rangle_{\Sigma^\delta}, \quad \forall v \in \mathbf{H}_0^1(\Omega). \end{array} \right. \quad (7.56)$$

Reciprocally, assume that  $u$  is a solution to (7.56). Then the function  $\varphi := u|_{\Sigma^\delta}$  satisfies  $\Lambda_+ \varphi = \psi$ . According to the Lax-Milgram theorem, Problem (7.56) admits a unique solution for all  $\psi \in \mathbf{H}^{-1/2}(\Sigma^\delta)$ . We infer that  $\Lambda_+ : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  is indeed an isomorphism.

Now, remarking that  $\Lambda_e, \Lambda_i$  have the same principal symbol and using standard arguments of pseudo-differential operators theory (work as in the proof of [83, Theorem 1]), we can show that  $\Lambda_- = \Lambda_e - \Lambda_i : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{-1/2}(\Sigma^\delta)$  is compact. We emphasize that the assumption of smoothness of  $\Sigma^\delta$  here is important.

Using the Riesz representation theorem, define the operator  $K : \mathbf{H}^{1/2}(\Sigma^\delta) \rightarrow \mathbf{H}^{1/2}(\Sigma^\delta)$  such that

$$(K\varphi, \varphi')_{\Sigma^\delta} = \langle \Lambda_- \varphi, \varphi' \rangle_{\Sigma^\delta} \quad \text{for all } \varphi, \varphi' \in \mathbf{H}^{1/2}(\Sigma^\delta). \quad (7.57)$$

Here, we use the notation  $(\cdot, \cdot)_{\Sigma^\delta} := \langle \Lambda_+ \cdot, \cdot \rangle_{\Sigma^\delta}$ . Note that according to the features of  $\Lambda_+$ , the latter form is an inner product in  $\mathbf{H}^{1/2}(\Sigma^\delta)$  equivalent to the usual one. Remark that  $(\alpha, \varphi)$  is an eigenpair for (7.55) if and only if we have  $K\varphi = \alpha\varphi$ . But due to the properties of  $\Lambda_-$ ,  $K$  is a selfadjoint and compact operator. Therefore, the spectrum of (7.55) coincides with a sequence of eigenvalues which accumulate at zero. We can use the *min-max* principle (see [140, Chapter 3]) to characterize these eigenvalues. We have

$$\alpha_1^+ = \sup_{\varphi \in \mathbf{H}^{1/2}(\Sigma^\delta) \setminus \{0\}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \quad (7.58)$$

By the *min-max* principle, we know that this sup is attained for some  $\varphi_1^+$ . By induction, for  $k \geq 2$ , we define

$$\alpha_k^+ = \sup_{\substack{\varphi \in \mathbf{H}^{1/2}(\Sigma^\delta) \setminus \{0\}, \\ \varphi \perp \{\varphi_1^+, \dots, \varphi_{k-1}^+\}}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \quad (7.59)$$

Here, if  $\varphi, \varphi'$  are two elements of  $H^{1/2}(\Sigma^\delta)$ , we write  $\varphi \perp \varphi'$  when  $(\varphi, \varphi')_{\Sigma^\delta} = \langle \Lambda_+ \varphi, \varphi' \rangle_{\Sigma^\delta} = (\nabla(\mathcal{R}\varphi), \nabla(\mathcal{R}\varphi')) = 0$ . Similarly, we define

$$\alpha_1^- = \inf_{\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}, \quad (7.60)$$

and, by induction, for  $k \geq 2$ ,

$$\alpha_k^- = \inf_{\substack{\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}, \\ \varphi \perp \{\varphi_1^-, \dots, \varphi_{k-1}^-\}}} \frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}}. \quad (7.61)$$

Observing that for all  $\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  we have

$$\frac{\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta}}{\langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta}} = \frac{1-a}{1+a}, \quad \text{with } a = \langle \Lambda_i \varphi, \varphi \rangle_{\Sigma^\delta} / \langle \Lambda_e \varphi, \varphi \rangle_{\Sigma^\delta} \geq 0,$$

we deduce that there holds  $\alpha_k^\pm \in [-1; 1]$  for all  $k \in \mathbb{N}^* := \{1, 2, \dots\}$ . Taking  $\varphi = \varphi_D^k|_{\Sigma^\delta}$  with  $\varphi_D^k$  defined in (7.11), we find  $a = 0$  and so  $\langle \Lambda_- \varphi, \varphi \rangle_{\Sigma^\delta} / \langle \Lambda_+ \varphi, \varphi \rangle_{\Sigma^\delta} = 1$ . This allows one to prove that  $\alpha_1^+ = \dots = \alpha_{\text{card}(K^\delta)}^+ = 1$ . Now, if  $\alpha_{\text{card}(K^\delta)+1}^+ = 1$ , then there is  $\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  such that  $\langle \Lambda_i \varphi, \varphi \rangle_{\Sigma^\delta} = 0$  and  $\mathcal{R}\varphi \in \hat{\mathcal{H}}_D^\delta \setminus \{0\}$ . This is impossible and therefore there holds  $\alpha_{\text{card}(K^\delta)+1}^+ < 1$ . Similarly, if  $\alpha_1^- = -1$ , then there exists  $\varphi \in H^{1/2}(\Sigma^\delta) \setminus \{0\}$  such that  $\langle \Lambda_e \varphi, \varphi \rangle_{\Sigma^\delta} = 0$ . This can not happen, which implies that  $\alpha_1^- > -1$ .  $\blacksquare$

### Optimality of the invertibility conditions

From the discussion preceding the statement of Theorem 7.3.2, we deduce the following result.

**Theorem 7.3.3.** *For  $\kappa_\varepsilon \in (-\infty; 0) \setminus \{-1\}$ , the operator  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism if and only if*

$$\kappa_\varepsilon \notin \left\{ \frac{\alpha_k^- + 1}{\alpha_k^- - 1}, k \geq 1 \right\} \cup \left\{ \frac{\alpha_k^+ + 1}{\alpha_k^+ - 1}, k \geq \text{card}(K^\delta) + 1 \right\},$$

where the  $\alpha_k^\pm$  are defined in (7.59)–(7.61).

Observing that the map  $\alpha \mapsto (\alpha + 1)/(\alpha - 1)$  is decreasing on  $(-1; 1)$ , we deduce in particular from Theorem 7.3.3 that  $A_\varepsilon^\delta$  is an isomorphism for

$$\kappa_\varepsilon \in \left( -\infty; \frac{\alpha_{\text{card}(K^\delta)+1}^+ + 1}{\alpha_{\text{card}(K^\delta)+1}^+ - 1} \right) \cup \left( \frac{\alpha_1^- + 1}{\alpha_1^- - 1}; 0 \right). \quad (7.62)$$

But one can verify that we have

$$\frac{\alpha_{\text{card}(K^\delta)+1}^+ + 1}{\alpha_{\text{card}(K^\delta)+1}^+ - 1} = -1/m_D^\delta \quad \text{and} \quad \frac{\alpha_1^- + 1}{\alpha_1^- - 1} = -1/M_D^\delta$$

where  $m_D^\delta, M_D^\delta$  are the constants defined in (7.14). As a consequence, the invertibility condition for  $A_\varepsilon^\delta$  obtained in Proposition 7.3.1 is the same as (7.62). This shows that the result of Proposition 7.3.1 is optimal in a certain sense. This is the first remark of this section.

### Comparison with existing literature

In previous articles (see in particular [27] and [43]), authors have worked with the operator  $\mathbf{T} : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that

$$\mathbf{T}\varphi = \begin{cases} \varphi & \text{in } \Omega_e^\delta \\ -\varphi + 2P\varphi & \text{in } \Omega_i^\delta \end{cases} \quad (7.63)$$

where, setting  $H_{0,\partial\Omega}^1(\Omega_e^\delta) := \{\varphi|_{\Omega_e^\delta}, \varphi \in H_0^1(\Omega)\}$ ,  $P : H_{0,\partial\Omega}^1(\Omega_e^\delta) \rightarrow H^1(\Omega_i^\delta)$  denotes the harmonic extension operator, i.e. the operator such that  $P\varphi$  solves the problem

$$\begin{cases} \Delta(P\varphi) = 0 & \text{in } \Omega_i^\delta \\ P\varphi = \varphi & \text{on } \partial\Omega_i^\delta. \end{cases} \quad (7.64)$$

We have  $\mathbf{T} \circ \mathbf{T} = \text{Id}$  which shows that  $\mathbf{T}$  is an isomorphism of  $H_0^1(\Omega)$ . On the other hand, for all  $\varphi \in H_0^1(\Omega)$ , we find

$$(\nabla(A_\varepsilon^\delta(\mathbf{T}\varphi)), \nabla\varphi) = \varepsilon_e \|\nabla\varphi\|_{\Omega_e^\delta}^2 + |\varepsilon_i| \|\nabla\varphi\|_{\Omega_i^\delta}^2 + 2\varepsilon_i (\nabla(P\varphi), \nabla\varphi)_{\Omega_i^\delta}. \quad (7.65)$$

Set

$$\tilde{M}_D^\delta := \sup_{\varphi \in H_{0,\partial\Omega}^1(\Omega_e^\delta) \setminus \{0\}} \frac{\|\nabla(P\varphi)\|_{\Omega_i^\delta}^2}{\|\nabla\varphi\|_{\Omega_e^\delta}^2}. \quad (7.66)$$

Using Young's inequality, from (7.65) we infer that for all  $\tau > 0$ , there holds

$$|(\nabla(A_\varepsilon^\delta(\mathbf{T}\varphi)), \nabla\varphi)| \geq (\varepsilon_e - \tau^{-1}|\varepsilon_i|\tilde{M}_D^\delta) \|\nabla\varphi\|_{\Omega_e^\delta}^2 + |\varepsilon_i|(1 - \tau) \|\nabla\varphi\|_{\Omega_i^\delta}^2. \quad (7.67)$$

As a consequence, we deduce that when  $\varepsilon_e > |\varepsilon_i|\tilde{M}_D^\delta \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/\tilde{M}_D^\delta$ , the operator  $A_\varepsilon^\delta$  is an isomorphism of  $H_0^1(\Omega)$ . Let us compare this operator  $\mathbf{T}$  introduced in (7.63) with the  $\mathbf{T}_D^+$  defined in (7.22). Clearly in  $\Omega_e^\delta$ , we have  $\mathbf{T}\varphi = \mathbf{T}_D^+\varphi$ . In  $\Omega_i^\delta$ , for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h \in \hat{\mathcal{H}}_D^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$  and  $\tilde{\varphi} \in H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ , we have

$$\mathbf{T}_D^+\varphi = \hat{\varphi}_h + \Phi_h - \tilde{\varphi}.$$

But one observes that

$$P\varphi = P(\hat{\varphi}_h + \Phi_h + \tilde{\varphi}) = P(\hat{\varphi}_h + \Phi_h) = \hat{\varphi}_h + \Phi_h.$$

Therefore, we have  $-\varphi + 2P\varphi = \hat{\varphi}_h + \Phi_h - \tilde{\varphi} = \mathbf{T}_D^+\varphi$  in  $\Omega_i^\delta$  which shows that the operator  $\mathbf{T}$  defined in (7.63) coincides with  $\mathbf{T}_D^+$ . Moreover, using Lemma 7.3.2, it is an exercise to prove that  $\tilde{M}_D^\delta$  is equal to the constant  $M_D^\delta$  defined in (7.14). Therefore, the simple operator  $\mathbf{T}$  in (7.63) is already very efficient. This is the second remark of this section.

### T-coercivity operator in the general case

Finally, we explain how to construct an operator of T-coercivity for contrasts  $\kappa_\varepsilon$  as in the statement of Theorem 7.3.3, in particular for contrasts in  $(-1/m_D^\delta; -1/M_D^\delta) \setminus \{-1\}$ , this case being not covered by Proposition 7.3.1. First, we reindex the eigenvalues  $\{\alpha_n^-\}_{n \geq 1}$ ,  $\{\alpha_n^+\}_{n \geq \text{card}(K^\delta)+1}$  and denote them  $\{\alpha_n\}_{n \geq 1}$ . Let  $(\varphi_n)$  be a family of eigenfunctions of the operator  $K$  introduced in (7.57) associated with the eigenvalues  $\alpha_n$ . We choose them so that the functions  $\mathcal{R}\varphi_n$ ,  $n \geq 1$ , form an orthonormal basis of  $\hat{\mathcal{H}}_D^\delta$ . Now we define the operator  $\mathbf{T}_D : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  such that for  $\varphi = \hat{\varphi}_h + \Phi_h + \tilde{\varphi}$  with  $\hat{\varphi}_h = \sum_{n \in \mathbb{N}^*} \gamma_n \mathcal{R}\varphi_n \in \hat{\mathcal{H}}_D^\delta$ ,  $\Phi_h \in \text{span}_{k \in K^\delta} \{\varphi_D^k\}$  and  $\tilde{\varphi} \in H_0^1(\Omega_e^\delta \cup \Omega_i^\delta)$ ,

there holds

$$\mathbf{T}_D\varphi = \begin{cases} \sum_{n \in \mathbb{N}^*} t_n \gamma_n \mathcal{R}\varphi_n + \Phi_h + \tilde{\varphi} & \text{in } \Omega_e^\delta \\ \sum_{n \in \mathbb{N}^*} t_n \gamma_n \mathcal{R}\varphi_n + \Phi_h - \tilde{\varphi} & \text{in } \Omega_i^\delta. \end{cases} \quad (7.68)$$

Here we take  $t_n = 1$  for  $n$  such that  $\kappa_\varepsilon > \kappa_n := (\alpha_n + 1)/(\alpha_n - 1)$  and  $t_n = -1$  otherwise. The operator  $T_D$  is valued in  $H_0^1(\Omega)$  and we have  $T_D \circ T_D = \text{Id}$  which guarantees that  $T_D$  is an isomorphism of  $H_0^1(\Omega)$ .

**Proposition 7.3.4.** *Assume that  $\kappa_\varepsilon \neq -1$  is such that for all  $n \in \mathbb{N}^*$ , we have  $\kappa_\varepsilon \neq \kappa_n$  with*

$$\kappa_n = \frac{\alpha_n + 1}{\alpha_n - 1}. \quad (7.69)$$

Let  $T_D : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  denote the isomorphism defined in (7.68). Then  $(\nabla(A_\varepsilon^\delta(T_D \cdot)), \nabla \cdot)$  is coercive in  $H_0^1(\Omega)$ . As a consequence,  $A_\varepsilon^\delta : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism.

**Proof.** For all  $\varphi \in H_0^1(\Omega)$ , we find

$$(\varepsilon \nabla \varphi, \nabla(T_D \varphi)) = \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) + (\varepsilon_e \nabla \Phi_h, \nabla \Phi_h)_{\Omega_\varepsilon^\delta} + (|\varepsilon| \nabla \tilde{\varphi}, \nabla \tilde{\varphi}). \quad (7.70)$$

But by the definition of the  $\kappa_n$ , we have, for all  $n \in \mathbb{N}^*$ ,

$$(\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_\varepsilon^\delta} = -\kappa_n (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_i^\delta}.$$

This allows us to write

$$\begin{aligned} \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) &= \varepsilon_e \sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\kappa_\varepsilon - \kappa_n) (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_i^\delta} \\ &= \varepsilon_e \sum_{n \in \mathbb{N}^*} |\gamma_n|^2 |\kappa_\varepsilon - \kappa_n| (\nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n))_{\Omega_i^\delta}. \end{aligned} \quad (7.71)$$

Observing that we have  $\|\nabla(\mathcal{R}\varphi_n)\|_{\Omega_i^\delta}^2 \geq \inf_{m \in \mathbb{N}^*} |\kappa_m|^{-1} \|\nabla(\mathcal{R}\varphi_n)\|_{\Omega_\varepsilon^\delta}^2$  (note that the sequence  $(|\kappa_m|)$  is bounded), from (7.71) we obtain

$$\sum_{n \in \mathbb{N}^*} t_n |\gamma_n|^2 (\varepsilon \nabla(\mathcal{R}\varphi_n), \nabla(\mathcal{R}\varphi_n)) \geq C \inf_{n \in \mathbb{N}^*} |\kappa_\varepsilon - \kappa_n| \sum_{n \in \mathbb{N}^*} |\gamma_n|^2 \|\nabla(\mathcal{R}\varphi_n)\|^2. \quad (7.72)$$

Using (7.72) into (7.70), we get  $(\varepsilon \nabla \varphi, \nabla(T_D \varphi)) \geq C \inf_{n \in \mathbb{N}^*} |\kappa_\varepsilon - \kappa_n| \|\nabla \varphi\|^2$  for all  $\varphi \in H_0^1(\Omega)$ . ■

**Remark 7.3.4.** *In the following, we will not work with the operator  $T_D$  defined in (7.68) to investigate what happens for contrasts in  $(-1/m; -1/M) \setminus \{-1\}$ . The reason is that the value of the  $\kappa_n$  defined in (7.69) depends on  $\delta$  and the operator  $T_D$  is useful to prove a result of uniform invertibility of  $A_\varepsilon^\delta$  only if we know that there is a segment of  $(-1/m; -1/M) \setminus \{-1\}$  of non empty interior which is uniformly free of the  $\kappa_n$  as  $\delta$  tends to zero. It is an open question to find conditions on the geometry such that this occurs.*

## 7.4 Analysis of the cell problem and properties of the homogenized tensors

In this section, we study a scalar problem set in the reference cell (supplemented with periodic boundary conditions) and the associated homogenized tensor. These quantities, which appear in the homogenization of Maxwell's equations considered in Section 7.5, are the same as the ones in [43] and [31], so that the results below complement and improve those obtained therein.



### 7.4.1 Cell problem

Denote by  $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$  the subset of functions of  $\mathcal{C}^\infty(\bar{Y})$  satisfying periodic boundary conditions on  $\partial Y$ . Let  $\mathbf{H}_{\text{per}}^1(Y)$  be the closure of  $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$  for the norm of  $\mathbf{H}^1(Y)$ . Then set

$$\mathbf{H}_{\text{per},\phi}^1(Y) := \{\varphi \in \mathbf{H}_{\text{per}}^1(Y) \mid \int_{\partial Y_i} \varphi \, d\sigma = 0\}.$$

We endow this space with the inner product  $(\nabla \cdot, \nabla \cdot)_Y$ . For  $\eta$  equal to  $\varepsilon$  or  $\mu$  as defined in (7.2), the problem we are interested in writes

$$\left| \begin{array}{l} \text{Find } \varphi \in \mathbf{H}_{\text{per},\phi}^1(Y) \text{ such that:} \\ (\eta \nabla \varphi, \nabla \varphi')_Y = \ell(\varphi'), \quad \forall \varphi' \in \mathbf{H}_{\text{per},\phi}^1(Y), \end{array} \right. \quad (7.73)$$

where  $\ell$  is a continuous linear functional on  $\mathbf{H}_{\text{per},\phi}^1(Y)$ . In order to study this problem, we introduce the closed subspace of  $\mathbf{H}_{\text{per},\phi}^1(Y)$

$$\mathbf{H}_{\text{per},0,\partial Y_i}^1(Y) := \{\varphi \in \mathbf{H}_{\text{per},\phi}^1(Y) \mid \varphi = 0 \text{ on } \partial Y_i\}.$$

Then we define the space  $\hat{\mathcal{H}}_b$  such that

$$\mathbf{H}_{\text{per},\phi}^1(Y) = \hat{\mathcal{H}}_b \oplus \mathbf{H}_{\text{per},0,\partial Y_i}^1(Y). \quad (7.74)$$

We will not look for an exact characterization of  $\hat{\mathcal{H}}_b$ . Let us simply remark that if  $\varphi \in \hat{\mathcal{H}}_b$ , then for all  $\zeta \in \mathcal{C}_0^\infty(Y_e \cup Y_i) \subset \mathbf{H}_{\text{per},0,\partial Y_i}^1(Y)$ , we have  $0 = (\nabla \varphi, \nabla \zeta)_Y$ . This implies that the elements of  $\hat{\mathcal{H}}_b$  are harmonic in  $Y_e \cup Y_i$ . Then we introduce the constants

$$m_b := \inf_{\varphi \in \hat{\mathcal{H}}_b \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}, \quad M_b := \sup_{\varphi \in \hat{\mathcal{H}}_b \setminus \{0\}} \frac{\|\nabla \varphi\|_{Y_i}^2}{\|\nabla \varphi\|_{Y_e}^2}. \quad (7.75)$$

**Theorem 7.4.1.** *Assume that  $\kappa_\varepsilon$  (resp.  $\kappa_\mu$ )  $\in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (7.45). Then the problem (7.73) with  $\eta = \varepsilon$  (resp.  $\eta = \mu$ ) admits a unique solution which depends continuously on  $\ell$ .*

**Proof.** To set ideas, we take  $\eta = \varepsilon$ , the proof is the same for  $\eta = \mu$ . With the Riesz representation theorem, define the operator  $D_\varepsilon : \mathbf{H}_{\text{per},\phi}^1(Y) \rightarrow \mathbf{H}_{\text{per},\phi}^1(Y)$  such that

$$(\nabla(D_\varepsilon \varphi), \nabla \varphi')_Y = (\varepsilon \nabla \varphi, \nabla \varphi')_Y, \quad \forall \varphi, \varphi' \in \mathbf{H}_{\text{per},\phi}^1(Y). \quad (7.76)$$

Let us show that  $D_\varepsilon$  is an isomorphism when  $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e \in (-\infty; -1/m) \cup (-1/M; 0)$ . For  $\varphi \in \mathbf{H}_{\text{per},\phi}^1(Y)$ , consider the decomposition  $\varphi = \varphi_h + \tilde{\varphi}$  with  $\varphi_h \in \hat{\mathcal{H}}_b$  and  $\tilde{\varphi} \in \mathbf{H}_{\text{per},0,\partial Y_i}^1(Y)$ . With this decomposition, we define the operators  $\mathbf{T}_b^\pm$  such that

$$\mathbf{T}_b^\pm \varphi = \begin{cases} \pm \varphi_h + \tilde{\varphi} & \text{in } Y_e \\ \pm \varphi_h - \tilde{\varphi} & \text{in } Y_i. \end{cases}$$

Working as in the proof of Proposition 7.3.1 with the operators  $\mathbf{T}^\pm$  replaced by  $\mathbf{T}_b^\pm$ , one establishes that  $D_\varepsilon$  is an isomorphism when  $\kappa_\varepsilon \in (-\infty; -1/m_b) \cup (-1/M_b; 0)$ . To obtain the desired result, it remains to show that  $m \leq m_b$  and  $M_b \leq M$ . Since  $\hat{\mathcal{H}}_b \subset \mathcal{H}_\phi$ , from Lemma 7.3.6, we clearly have  $M_b \leq M$ . Now let  $\varphi$  be an element of  $\hat{\mathcal{H}}_b \setminus \{0\}$ . Denote  $\zeta \in \mathcal{H}_0$  the function such that  $\zeta = \varphi$  on  $\partial Y_i$ . The function  $\zeta$  decomposes as  $\zeta = \hat{\zeta} + \alpha \varphi_D$  with  $\hat{\zeta} \in \hat{\mathcal{H}}_0$  and  $\alpha \in \mathbb{R}$  ( $\varphi_D$  is defined before (7.44)). Note that  $\hat{\zeta} \not\equiv 0$  otherwise we would have  $\alpha = 0$  (because  $\varphi_D = 1$  on  $\partial Y_i$  and  $\int_{\partial Y_i} \zeta \, d\sigma = 0$ ) and so  $\zeta \equiv 0$ . Observing that  $\varphi - \hat{\zeta} - \alpha$  is in  $\mathbf{H}_{\text{per},0,\partial Y_i}^1(Y)$ , due to the decomposition (7.74), we can write

$$(\nabla \varphi, \nabla(\varphi - \hat{\zeta}))_Y = (\nabla \varphi, \nabla(\varphi - \hat{\zeta} - \alpha))_Y = 0. \quad (7.77)$$

But on the other hand, since we have  $\nabla\varphi = \nabla\hat{\zeta} = \nabla\zeta$  in  $Y_i$ , so that in particular there holds

$$\|\nabla\varphi\|_{Y_i}^2 = \|\nabla\hat{\zeta}\|_{Y_i}^2, \quad (7.78)$$

we infer from (7.77) that

$$(\nabla\varphi, \nabla(\varphi - \hat{\zeta}))_{Y_e} = 0. \quad (7.79)$$

This implies

$$\|\nabla\varphi\|_{Y_e}^2 \leq \|\nabla\hat{\zeta}\|_{Y_e}^2. \quad (7.80)$$

Gathering (7.78) and (7.80), we deduce that

$$m \leq \frac{\|\nabla\hat{\zeta}\|_{Y_i}^2}{\|\nabla\hat{\zeta}\|_{Y_e}^2} \leq \frac{\|\nabla\varphi\|_{Y_i}^2}{\|\nabla\varphi\|_{Y_e}^2}. \quad (7.81)$$

Taking the infimum over all  $\varphi \in \hat{\mathcal{H}}_b \setminus \{0\}$  in (7.81), we obtain that  $m \leq m_b$ . ■

### 7.4.2 Homogenized tensors

Assume that the contrasts  $\kappa_\varepsilon$  and  $\kappa_\mu$  are located in  $(-\infty; -1/m) \cup (-1/M; 0)$ . For  $\eta = \varepsilon$  or  $\mu$  and  $j = 1, 2, 3$ , we define the function  $\chi_j^\eta \in H_{\text{per},\diamond}^1(Y)$  such that

$$(\eta\nabla\chi_j^\eta, \nabla\xi)_Y = (\eta\nabla y_j, \nabla\xi)_Y, \quad \forall \xi \in H_{\text{per},\diamond}^1(Y). \quad (7.82)$$

Note that the right hand side of (7.82) simply writes

$$(\eta\nabla y_j, \nabla\xi)_Y = \int_Y \eta \frac{\partial \xi}{\partial y_j} dy$$

and that Theorem 7.4.1 ensures that the functions  $\chi_j^\eta$  are well-defined. It is also worth noticing that by setting  $\boldsymbol{\chi}^\eta := (\chi_1^\eta, \chi_2^\eta, \chi_3^\eta)^\top$ , we have for all  $\boldsymbol{\lambda} \in \mathbb{R}^3$ :

$$(\eta\nabla(\boldsymbol{\lambda} \cdot \boldsymbol{\chi}^\eta), \nabla\xi)_Y = (\eta\nabla(\boldsymbol{\lambda} \cdot \mathbf{y}), \nabla\xi)_Y = \int_Y \eta \boldsymbol{\lambda} \cdot \nabla \xi dy, \quad \forall \xi \in H_{\text{per},\diamond}^1(Y). \quad (7.83)$$

Denoting by  $\nabla\boldsymbol{\chi}^\eta$  the jacobian matrix of  $\boldsymbol{\chi}^\eta$ :

$$\nabla\boldsymbol{\chi}^\eta = \left( \frac{\partial \chi_j^\eta}{\partial y_k} \right)_{1 \leq j, k \leq 3},$$

the homogenized tensor associated with  $\eta$  is classically defined as the  $3 \times 3$  symmetric matrix  $\mathcal{H}(\eta) = (\mathcal{H}_{jk}(\eta))_{1 \leq j, k \leq 3}$  given by (see, for instance, identity (6.35) in [59])

$$\mathcal{H}(\eta) = \frac{1}{|Y|} \int_Y \eta(y) [\text{Id} - (\nabla\boldsymbol{\chi}^\eta)^\top] dy, \quad (7.84)$$

or equivalently (see (6.37) in [59]):

$$\mathcal{H}_{jk}(\eta) = \frac{1}{|Y|} \int_Y \eta \nabla(y_j - \chi_j^\eta) \cdot \nabla(y_k - \chi_k^\eta) dy. \quad (7.85)$$

**Proposition 7.4.1.** *Assume that  $\kappa_\varepsilon$  (resp.  $\kappa_\mu$ )  $\in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (7.45). Then the matrix  $\mathcal{H}(\varepsilon)$  (resp.  $\mathcal{H}(\mu)$ ) is positive definite.*

**Proof.** The proofs for  $\mathcal{H}(\varepsilon)$  and  $\mathcal{H}(\mu)$  are the same and to set ideas, we choose to work with  $\varepsilon$ . According to formula (6.44) in [59], for all  $\xi = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3$ , we have

$$\mathcal{H}(\varepsilon) \xi \cdot \xi = \int_Y \varepsilon |\nabla \varphi_\xi|^2 \, dy$$

where the function  $\varphi_\xi$  is defined by

$$\varphi_\xi(y) = \sum_{j=1}^3 \xi_j (y_j - \chi_j^\varepsilon(y)).$$

Note that if  $\varphi_\xi$  is constant in  $Y$ , then evaluating  $\varphi_\xi$  on  $\partial Y$  and using the fact that the functions  $\chi_j^\varepsilon$  satisfy periodic boundary conditions, we find that  $\xi = 0$  and so  $\varphi_\xi \equiv 0$ . Now, we assume that  $\xi \neq 0$ . Subtracting the mean value of the test functions on  $\partial Y_i$ , we see from (7.82) that  $\chi_j^\varepsilon$  satisfy the slightly more general variational equality (the variational space is not the same as in (7.82))

$$(\varepsilon \nabla \chi_j^\varepsilon, \nabla \varphi')_Y = (\varepsilon \nabla y_j, \nabla \varphi')_Y, \quad \forall \varphi' \in H_{\text{per}}^1(Y).$$

Taking  $\varphi' \in \mathcal{C}_0^\infty(Y)$ , this implies that we have

$$\operatorname{div}(\varepsilon \nabla \varphi_\xi) = 0 \quad \text{in } Y. \tag{7.86}$$

i) Introduce the function  $\hat{\varphi}_\xi$  such that

$$\hat{\varphi}_\xi = \varphi_\xi - \frac{1}{|\partial Y_i|} \int_{\partial Y_i} \varphi_\xi \, d\sigma \in H_\diamond^1(Y).$$

From (7.86), we deduce that  $\hat{\varphi}_\xi$  is harmonic in  $Y_e \cup Y_i$ . Therefore, we have  $\hat{\varphi}_\xi \in \mathcal{H}_\diamond$  and from Lemma 7.3.6, we can write

$$\|\nabla \varphi_\xi\|_{Y_i}^2 = \|\nabla \hat{\varphi}_\xi\|_{Y_i}^2 \leq M \|\nabla \hat{\varphi}_\xi\|_{Y_e}^2 = M \|\nabla \varphi_\xi\|_{Y_e}^2.$$

This allows us to write

$$\mathcal{H}(\varepsilon) \xi \cdot \xi = \varepsilon_e \|\nabla \varphi_\xi\|_{Y_e}^2 - |\varepsilon_i| \|\nabla \varphi_\xi\|_{Y_i}^2 \geq (\varepsilon_e - |\varepsilon_i| M) \|\nabla \varphi_\xi\|_{Y_e}^2.$$

Hence, for  $\varepsilon_e > |\varepsilon_i| M \Leftrightarrow \kappa_\varepsilon = \varepsilon_i/\varepsilon_e > -1/M$ , the matrix  $\mathcal{H}(\varepsilon)$  is definite-positive. Note that we have  $\nabla \varphi_\xi \neq 0$  in  $Y_e$  otherwise we would have  $\nabla \varphi_\xi = 0$  in  $Y$  (because  $\varphi_\xi \in H^1(Y)$  is harmonic in  $Y_e$ ) which is impossible when  $\xi \neq 0$  (see the discussion above).

ii) Now, we consider the case  $\kappa_\varepsilon \in (-\infty; -1/m)$ . The proof is a bit less straightforward and we divide it into two steps. Define the quadratic form  $q_\varepsilon(\cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that

$$q_\varepsilon(\xi) = \mathcal{H}(\varepsilon) \xi \cdot \xi.$$

*Step 1.* First, we prove the following result.

**Lemma 7.4.1.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m)$ . Then the form  $q_\varepsilon$  is definite ( $q_\varepsilon(\xi) = 0 \Rightarrow \xi = 0$ ).*

**Proof.** A bit more generally (this will serve in the proof of Lemma 7.4.2 below), assume that  $\xi \in \mathbb{R}^3 \setminus \{0\}$  is such that

$$q_\varepsilon(\xi) \leq 0 \quad \Leftrightarrow \quad \int_Y \varepsilon |\nabla \varphi_\xi|^2 \, dy \leq 0.$$

Then from identity (7.86), we infer that we must have

$$\int_{\partial Y} \varepsilon_e \frac{\partial \varphi_\xi}{\partial n} \varphi_\xi \, d\sigma \leq 0. \quad (7.87)$$

Now, introduce  $\zeta \in \mathcal{H}_0$  the function such that  $\zeta = \varphi_\xi$  on  $\partial Y_i$ . The function  $\zeta$  decomposes as  $\zeta = \hat{\zeta} + \alpha \varphi_D$  with  $\hat{\zeta} \in \hat{\mathcal{H}}_0$  and  $\alpha \in \mathbb{R}$  ( $\varphi_D$  is defined before (7.44)). Observe that we have  $\hat{\zeta} \not\equiv 0$ . Indeed, otherwise  $\varphi_\xi$  would be constant in  $Y_i$ . And then (7.86) together with the unique continuation principle would imply that  $\varphi_\xi$  be constant in  $Y_e$  (because we would have that  $\Delta \varphi_\xi = 0$  in  $Y_e$ ,  $\varphi_\xi = \text{cste}$  on  $\partial Y_i$  and  $\partial_{n_e} \varphi_\xi = 0$  on  $\partial Y_i$ ) and so in  $Y$ . According to the discussion above, this is impossible when  $\xi \neq 0$ . Observing that  $\varphi_\xi - (\hat{\zeta} + \alpha) = 0$  on  $\partial Y_i$ , integrating by parts, we can write

$$\begin{aligned} (\nabla \varphi_\xi, \nabla(\varphi_\xi - \hat{\zeta}))_{Y_e} &= (\nabla \varphi_\xi, \nabla(\varphi_\xi - (\hat{\zeta} + \alpha)))_{Y_e} = \int_{\partial Y} \frac{\partial \varphi_\xi}{\partial n} \varphi_\xi \, d\sigma - \int_{\partial Y} \frac{\partial \varphi_\xi}{\partial n} \alpha \, d\sigma \\ &= \int_{\partial Y} \frac{\partial \varphi_\xi}{\partial n} \varphi_\xi \, d\sigma \leq 0. \end{aligned} \quad (7.88)$$

The last equality above has been obtained using (7.87) and identity (7.86) multiplied by  $\alpha$ . From (7.88) and the Cauchy-Schwarz inequality, we infer that

$$\|\nabla \varphi_\xi\|_{Y_e}^2 \leq \|\nabla \hat{\zeta}\|_{Y_e}^2.$$

Since on the other hand there holds  $\nabla \varphi_\xi = \nabla \hat{\zeta}$  in  $Y_i$  so that  $\|\nabla \varphi_\xi\|_{Y_i}^2 = \|\nabla \hat{\zeta}\|_{Y_i}^2$ , we deduce that

$$m \leq \frac{\|\nabla \hat{\zeta}\|_{Y_i}^2}{\|\nabla \hat{\zeta}\|_{Y_e}^2} \leq \frac{\|\nabla \varphi_\xi\|_{Y_i}^2}{\|\nabla \varphi_\xi\|_{Y_e}^2}. \quad (7.89)$$

But then, when  $\kappa_\varepsilon = \varepsilon_i/\varepsilon_e < -m^{-1} \Leftrightarrow \varepsilon_e < |\varepsilon_i|m$ , we can write

$$q_\varepsilon(\xi) = \mathcal{H}(\varepsilon) \xi \cdot \xi = \int_Y \varepsilon |\nabla \varphi_\xi|^2 \, dy = \varepsilon_e \|\nabla \varphi_\xi\|_{Y_e}^2 - |\varepsilon_i| \|\nabla \varphi_\xi\|_{Y_i}^2 \leq (\varepsilon_e - |\varepsilon_i|m) \|\nabla \varphi_\xi\|_{Y_e}^2 < 0.$$

In particular we obtain a contradiction if  $\xi \neq 0$  is such that  $q_\varepsilon(\xi) = 0$ . This proves that  $q_\varepsilon$  is definite.  $\blacksquare$

From classical results concerning quadratic forms, we deduce from Lemma 7.4.1 that for each  $\kappa_\varepsilon \in (-\infty; -1/m)$ ,  $q_\varepsilon(\cdot)$  is either positive definite or negative definite.

*Step 2.* Now consider some  $\xi \in \mathbb{R}^3 \setminus \{0\}$ . Corollary 5.6 of [31] or Lemma 7.4.2 below guarantee that  $q_\varepsilon(\xi)$  is positive for  $\kappa_\varepsilon$  tending to  $-\infty$ . Using the fact that  $\kappa_\varepsilon \mapsto q_\varepsilon(\xi)$  is continuous and that  $q_\varepsilon(\cdot)$  is always definite for  $\kappa_\varepsilon \in (-\infty; -1/m)$ , we infer that  $q_\varepsilon(\cdot)$  is positive definite for all  $\kappa_\varepsilon \in (-\infty; -1/m)$ . This achieves the proof of Proposition 7.4.1.  $\blacksquare$

Below, for the sake of completeness, we present an alternative proof to Corollary 5.6 of [31] which is a bit more direct.

**Lemma 7.4.2.** *For any given  $\xi \in \mathbb{R}^3 \setminus \{0\}$ , we have  $q_\varepsilon(\xi) > 0$  for  $\kappa_\varepsilon$  tending to  $-\infty$ .*

**Proof.** Impose that  $\kappa_\varepsilon \in (-\infty; -1/m)$  and for  $\xi \in \mathbb{R}^3 \setminus \{0\}$ , assume that we have  $q_\varepsilon(\xi) < 0$ . Define the function

$$\check{\varphi}_\xi = \varphi_\xi - \frac{1}{|\partial Y|} \int_{\partial Y} \varphi_\xi \, d\sigma.$$

From (7.86), we can write

$$\begin{aligned}
 |\varepsilon_i| \|\nabla\varphi_\xi\|_{Y_i}^2 &= |\varepsilon_i| \|\nabla\check{\varphi}_\xi\|_{Y_i}^2 = \varepsilon_e \|\nabla\check{\varphi}_\xi\|_{Y_e}^2 - \int_{\partial Y} \varepsilon_e \frac{\partial\check{\varphi}_\xi}{\partial n} \check{\varphi}_\xi \, d\sigma \\
 &\leq C \varepsilon_e \|\nabla\check{\varphi}_\xi\|_{Y_e}^2 = C \varepsilon_e \|\nabla\varphi_\xi\|_{Y_e}^2.
 \end{aligned}
 \tag{7.90}$$

The last inequality in (7.90) is a consequence of the continuity of the mappings  $\varphi \mapsto \varphi|_{\partial Y}$  and  $\varphi \mapsto \partial_n \varphi|_{\partial Y}$  from  $\{\varphi \in H^1(Y_e) \mid \Delta\varphi = 0 \text{ in } Y_e\}$  to  $H^{1/2}(\partial Y)$  and  $H^{-1/2}(\partial Y)$  respectively. Note that since the mean of  $\check{\varphi}_\xi$  over  $\partial Y$  is null, a classical Poincaré type inequality allows one to prove that the  $H^1$  norm of  $\check{\varphi}_\xi$  in  $Y_e$  is controlled by  $\|\nabla\check{\varphi}_\xi\|_{Y_e}$ . From (7.90), we get

$$\frac{\|\nabla\varphi_\xi\|_{Y_i}^2}{\|\nabla\varphi_\xi\|_{Y_e}^2} \leq \frac{C}{|\kappa_\varepsilon|}
 \tag{7.91}$$

where  $C > 0$  is independent of  $\kappa_\varepsilon$ . Taking the limit  $\kappa_\varepsilon \rightarrow -\infty$  in (7.91), we obtain a contradiction with (7.89) (here we use that  $q_\varepsilon(\xi) < 0$ ) because  $m > 0$  is independent of  $\kappa_\varepsilon$ . Therefore we must have  $q_\varepsilon(\xi) > 0$  for contrasts tending to  $-\infty$ . ■

### 7.4.3 Numerical illustrations

Proposition 7.4.1 guarantees that if  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , the matrices  $\mathcal{H}(\varepsilon), \mathcal{H}(\mu)$  are positive definite. This may seem a bit surprising and when one looks at the definition in (7.85), this is far from being obvious. The goal of this paragraph is to present some numerics to illustrate this property. To set ideas we compute  $\mathcal{H}(\varepsilon)$  and to simplify we work in 2D. In this case,  $\mathcal{H}(\varepsilon)$  is a  $2 \times 2$  symmetric matrix. We do not expect particular differences between 2D and 3D settings. Numerically, we approximate the solutions of the problems (7.82) using a P2 finite element method. To proceed, we use the library *FreeFem++*<sup>1</sup> to compute the matrix  $\mathcal{H}(\varepsilon)$  using formula (7.85). The mesh size is chosen equal to 0.02. Admittedly the numerical analysis of problems (7.82) is not standard because of the sign-changing  $\varepsilon$ . However in general, at least for contrasts  $\kappa_\varepsilon$  “not too close” to  $-1$  when  $\partial Y_i$  is smooth, we obtain a reasonable numerical solution. We refer the reader to [120, 53, 18] for more details concerning these aspects. In Figures 7.2 and 7.3 below, we display the two real eigenvalues of  $\mathcal{H}(\varepsilon)$  with respect to the contrast  $\kappa_\varepsilon \in (-10; 0)$  (we take  $\varepsilon_i = -1$  and  $\varepsilon_e$  varies) for two different geometries of  $Y_i$ . For the numerics of Figure 7.2, the inclusion  $Y_i$  is an ellipse while for Figure 7.3, it is a rectangle. We emphasize that in the latter case, problem (7.82) is not well-posed in the Fredholm sense for  $\kappa_\varepsilon \in (-3, -1/3)$  (see [26, 17]). As a consequence, for this range of contrasts, our numerical solutions have no sense. But for both settings, we observe that for contrasts large enough or small enough, the matrix  $\mathcal{H}(\varepsilon)$  is positive definite as expected. Interestingly, at least in the case of the ellipse where we know that the numerical solution is meaningful except for  $\kappa_\varepsilon \neq -1$ , we also note that  $\mathcal{H}(\varepsilon)$  is not positive definite for all contrasts. We emphasize however that we do not investigate these regimes in our analysis below.

<sup>1</sup>*FreeFem++*, <http://www.freefem.org/ff++/>.

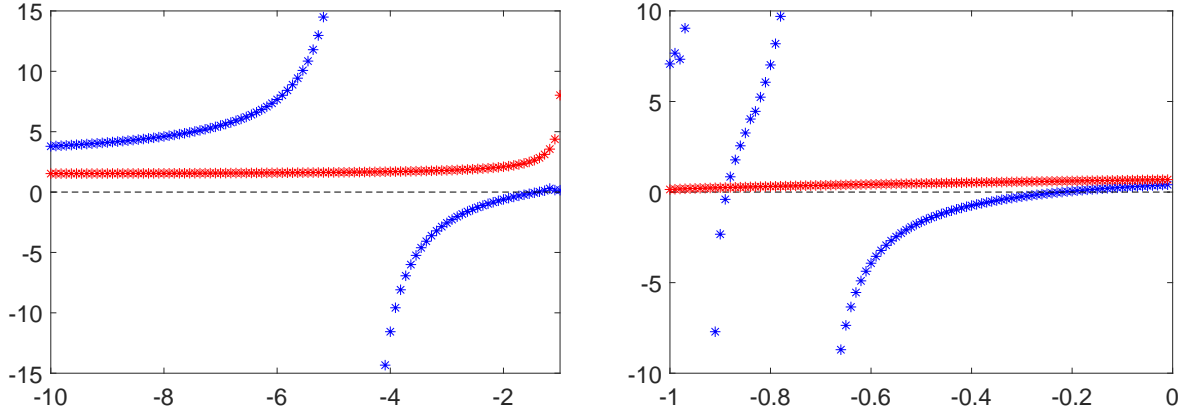


Figure 7.2: Representation of the two eigenvalues of  $\mathcal{H}(\varepsilon)$  with respect to  $\kappa_\varepsilon$  varying in  $(-10; -1)$  (left) and  $(-1; 0)$  (right). Here the inclusion  $Y_i$  coincides with the interior of the ellipse  $\{(x = 0.5 + 0.4 \cos \theta, y = 0.5 + 0.2 \cos \theta), \theta \in [0; 2\pi)\}$ .

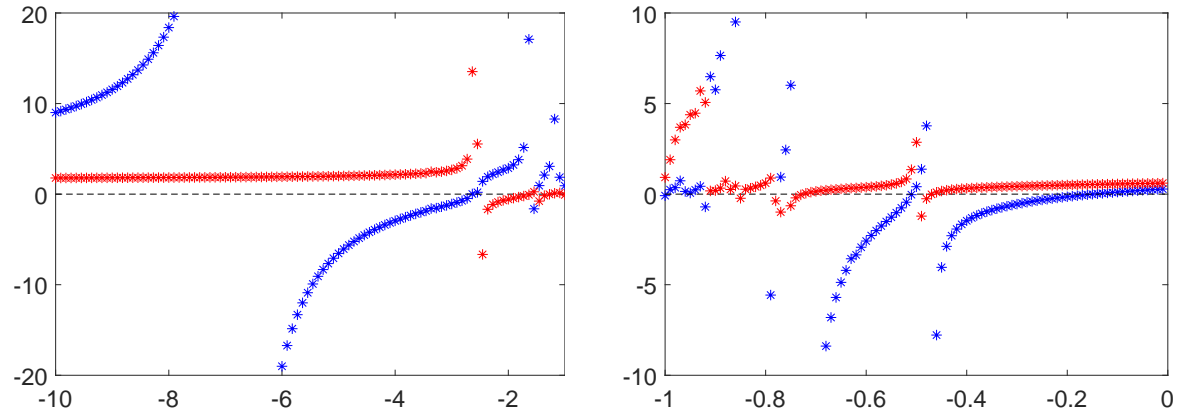


Figure 7.3: Representation of the two eigenvalues of  $\mathcal{H}(\varepsilon)$  with respect to  $\kappa_\varepsilon$  varying in  $(-10; -1)$  (left) and  $(-1; 0)$  (right). Here the inclusion  $Y_i$  coincides with the rectangle  $(0.1; 0.9) \times (0.3; 0.7)$ .

## 7.5 Homogenization of Maxwell's equations

We come back to Maxwell's problem  $(\mathcal{P}^\delta)$  for the electric field (see (7.9)). We define the bilinear form  $a_\omega^\delta(\cdot, \cdot)$  associated with (7.9) such that

$$a_\omega^\delta(\mathbf{E}, \mathbf{E}') = ((\mu^\delta)^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{E}') - \omega^2(\varepsilon^\delta \mathbf{E}, \mathbf{E}'), \quad \forall \mathbf{E}, \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega).$$

Let  $m, M$  be the constants defined in (7.45). When  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , the matrices  $\mathcal{H}(\varepsilon)$  and  $\mathcal{H}(\mu)$  are well-defined according to Theorem 7.4.1. Moreover, according to Proposition 7.4.1, these matrices are positive definite. Hence, we can introduce the homogenized problem

$$(\mathcal{P}^{\text{eff}}) \quad \left| \begin{array}{l} \text{Find } \mathbf{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}; \Omega) \text{ such that} \\ \mathbf{curl}(\mathcal{H}(\mu)^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}}) - \omega^2 \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} = i\omega \mathbf{J} \end{array} \right. \quad (7.92)$$

whose variational formulation writes

$$\left| \begin{array}{l} \text{Find } \mathbf{E}^{\text{eff}} \in \mathbf{H}_N(\mathbf{curl}; \Omega) \text{ such that for all } \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega) \\ a_\omega^{\text{eff}}(\mathbf{E}^{\text{eff}}, \mathbf{E}') = i\omega(\mathbf{J}, \mathbf{E}'). \end{array} \right. \quad (7.93)$$

Here  $a_\omega^{\text{eff}}(\cdot, \cdot)$  is the bilinear form defined on the space  $\mathbf{H}_N(\mathbf{curl}; \Omega)$  such that

$$a_\omega^{\text{eff}}(\mathbf{E}, \mathbf{E}') = (\mathcal{H}(\mu)^{-1} \mathbf{curl} \mathbf{E}, \mathbf{curl} \mathbf{E}') - \omega^2(\mathcal{H}(\varepsilon) \mathbf{E}, \mathbf{E}').$$

It is worth noticing that the above homogenized problem (which has exactly the same form as the one obtained for classical (positive) Maxwell's equations) involves the homogenized tensors of the scalar problems studied in the previous sections. This fact will be used in a crucial way in the sequel to prove our homogenization result for Maxwell's system. Classically, one can easily prove that  $(\mathcal{P}^{\text{eff}})$  admits a unique solution for all

$$\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}} \quad (7.94)$$

where  $\Lambda^{\text{eff}}$  is a discrete subset of  $[0; +\infty)$ .

The proof of a homogenization result for Maxwell's equations without sign-changing coefficients is by now quite classical (see for instance [14, 143, 136, 58]). It may be achieved by using, for instance, a notion of convergence specific to the periodic homogenization, namely the two-scale convergence, which was introduced by G. Nguetseng in [116] and further developed by G. Allaire [5]. Using this notion, a typical proof for such a homogenization result relies on three main ingredients. First, a uniform energy estimate is obtained for the sequence of solutions of  $(\mathcal{P}^\delta)$ . Next, one shows that this uniformly bounded sequence has a (two-scale) limit that solves a two-scale limit problem. Finally, this limit problem is decoupled, yielding the homogenized problem which is proved to be well-posed. Due to the sign-changing coefficients and the presence of the non sign-definite  $L^2$  term involving  $\omega^2$ , proving the first ingredient is far from being obvious. In particular, the strategy proposed for instance in [58] does not apply anymore (as the spectral decomposition available in the strongly elliptic case fails). Instead, we proceed as follows. First, we prove a homogenization result for solutions of  $(\mathcal{P}^\delta)$  under a uniform energy estimate condition. Using this result, we prove by contradiction the needed uniform energy estimate for the solutions  $(\mathcal{P}^\delta)$ . This leads to the main result of the paper (Theorem 7.5.1), namely the homogenization result for sign-changing Maxwell's equations.

### 7.5.1 Homogenization result under uniform energy estimate condition

Let  $\mathbf{J}$  be a given field of  $\mathbf{L}^2(\Omega)$ . The aim of this section is to obtain a homogenization result for a sequence of functions  $(\mathbf{E}^\delta)$  solving  $(\mathcal{P}^\delta)$  and satisfying the uniform energy estimate

$$\exists C > 0, \forall \delta \in (0; 1], \quad \|\mathbf{E}^\delta\|^2 + \|\mathbf{curl} \mathbf{E}^\delta\|^2 \leq C \|\mathbf{J}\|^2. \quad (7.95)$$

As it was already observed in [43] in the analysis of the homogenization process for the Dirichlet scalar operator  $A_\varepsilon^\delta$ , the presence of sign-changing coefficients does not affect the two-scale convergence result. However, for the sake of completeness, we give here a proof of this convergence result following [14, 143, 136] and in particular [58]. We start by recalling the definition of the two-scale convergence (see [5]). Here we set  $\mathcal{C}_{\text{per}}^\infty(\bar{Y}) := (\mathcal{C}_{\text{per}}^\infty(\bar{Y}))^3$ .

**Definition 7.5.1.** A sequence  $(\mathbf{E}^\delta)$  in  $\mathbf{L}^2(\Omega)$  two-scale converges to  $\mathbf{E}^0 \in \mathbf{L}^2(\Omega \times Y)$  if we have

$$\lim_{\delta \rightarrow 0} (\mathbf{E}^\delta, \mathbf{v}(\cdot, \cdot/\delta)) = \int_{\Omega} (\mathbf{E}^0(x, \cdot), \mathbf{v}(x, \cdot))_Y dx$$

for all  $\mathbf{v} \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(\bar{Y}))$ . Then we denote  $\mathbf{E}^\delta \xrightarrow{2s} \mathbf{E}^0$ .

The notion of two-scale convergence is interesting due to the following compactness result (see for instance [136, Proposition 2.5]). It was first obtained by N. Wellander in [141] and then by V. Tiep Chu and V.H. Hoang in [136]. Here,  $\mathbf{H}_{\text{per}}(\mathbf{curl}; Y)$  denotes the closure of  $\mathcal{C}_{\text{per}}^\infty(\bar{Y})$  for the norm  $(\|\cdot\|_Y^2 + \|\mathbf{curl} \cdot\|_Y^2)^{1/2}$ .

**Proposition 7.5.1.** Let  $(\mathbf{E}^\delta)$  be a bounded sequence in  $\mathbf{H}(\mathbf{curl}; \Omega)$ . Then, there exist a subsequence, still denoted  $(\mathbf{E}^\delta)$ , and functions  $\mathbf{E}^{\text{eff}} \in \mathbf{H}(\mathbf{curl}; \Omega)$ ,  $\Theta \in \mathbf{L}^2(\Omega; \mathbf{H}_{\text{per}}^1(Y))$ ,  $\mathbf{E}^1 \in \mathbf{L}^2(\Omega; \mathbf{H}_{\text{per}}(\mathbf{curl}, Y))$  such that the following two-scale convergence results hold as  $\delta \rightarrow 0$ :

$$\mathbf{E}^\delta \xrightarrow{2s} \mathbf{E}^{\text{eff}} + \nabla_y \Theta, \quad \mathbf{curl} \mathbf{E}^\delta \xrightarrow{2s} \mathbf{curl} \mathbf{E}^{\text{eff}} + \mathbf{curl}_y \mathbf{E}^1.$$



Moreover, we also have the following weak convergence results in  $\mathbf{L}^2(\Omega)$ :

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{in } \mathbf{L}^2(\Omega), \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{in } \mathbf{L}^2(\Omega).$$

We are now in position to prove the main result of this section, namely the convergence of a sequence of solutions of problem  $(\mathcal{P}^\delta)$  satisfying the energy estimate (7.95) to a solution of  $(\mathcal{P}^{\text{eff}})$  when  $\delta \rightarrow 0$ .

**Proposition 7.5.2.** *Assume that  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (7.45). Let  $(\mathbf{E}^\delta)$  be a sequence of solutions of  $(\mathcal{P}^\delta)$  satisfying the uniform estimate (7.95). Then as  $\delta \rightarrow 0$ , we have*

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{and} \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{in } \mathbf{L}^2(\Omega)$$

where  $\mathbf{E}^{\text{eff}}$  solves the homogenized problem  $(\mathcal{P}^{\text{eff}})$ .

**Proof.** We take in  $(\mathcal{P}^\delta)$  (see (7.9)) a test function of the form

$$\mathbf{E}'(x) = \varphi(x) + \delta \varphi^1 \left( x, \frac{x}{\delta} \right) + \delta \nabla \left( \psi \left( x, \frac{x}{\delta} \right) \right),$$

with  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ ,  $\varphi^1 \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y))$ ,  $\psi \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y))$ . By taking the limit as  $\delta \rightarrow 0$  thanks to Proposition 7.5.1, we get as in [136, Proposition 2.5] the following two-scale limit problem:

$$\begin{aligned} & \int_{\Omega \times Y} (\mu(y))^{-1} \left( \mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y) \right) \cdot \left( \mathbf{curl} \varphi(x) + \mathbf{curl}_y \varphi^1(x, y) \right) dx dy \\ & - \omega^2 \int_{\Omega \times Y} \varepsilon(y) \left( \mathbf{E}^{\text{eff}}(x) + \nabla_y \Theta(x, y) \right) \cdot \left( \varphi(x) + \nabla_y \psi(x, y) \right) dx dy \\ & = i\omega \int_{\Omega} \mathbf{J} \cdot \varphi dx + i\omega \int_{\Omega \times Y} \mathbf{J}(x) \cdot \nabla_y \psi(x, y) dx dy. \end{aligned} \quad (7.96)$$

Since  $\psi(x, \cdot)$  is  $Y$ -periodic, the second integral of the right hand side vanishes and hence, setting

$$\mathbb{R}(x, y) := (\mu(y))^{-1} \left( \mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y) \right) \quad (7.97)$$

and

$$\mathbf{S}(x, y) := \varepsilon(y) \left( \mathbf{E}^{\text{eff}} + \nabla_y \Theta \right), \quad (7.98)$$

relation (7.96) reads

$$\begin{aligned} & \int_{\Omega \times Y} \mathbb{R}(x, y) \cdot \left( \mathbf{curl} \varphi(x) + \mathbf{curl}_y \varphi^1(x, y) \right) dx dy \\ & - \omega^2 \int_{\Omega \times Y} \mathbf{S}(x, y) \cdot \left( \varphi(x) + \nabla_y \psi(x, y) \right) dx dy = i\omega \int_{\Omega} \mathbf{J} \cdot \varphi dx. \end{aligned} \quad (7.99)$$

In order to prove that  $\mathbf{E}^{\text{eff}}$  solves the homogenized problem  $(\mathcal{P}^{\text{eff}})$ , it suffices to show that the two terms of the left hand side in the above equation can also be written as follows:

$$\int_{\Omega \times Y} \mathbb{R}(x, y) \cdot \left( \mathbf{curl} \varphi(x) + \mathbf{curl}_y \varphi^1(x, y) \right) dx dy = \int_{\Omega} (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}} \cdot \mathbf{curl} \varphi dx \quad (7.100)$$

$$\int_{\Omega \times Y} \mathbf{S}(x, y) \cdot \left( \varphi(x) + \nabla_y \psi(x, y) \right) dx dy = \int_{\Omega} \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \varphi dx. \quad (7.101)$$

Indeed, once these two last relations proved, the conclusion follows immediately since problem (7.99) writes then

$$\int_{\Omega} (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}} \cdot \mathbf{curl} \boldsymbol{\varphi} \, dx - \omega^2 \int_{\Omega} \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \boldsymbol{\varphi} \, dx = i\omega \int_{\Omega} \mathbf{J} \cdot \boldsymbol{\varphi} \, dx,$$

which is exactly the weak formulation of the homogenized problem ( $\mathcal{P}^{\text{eff}}$ ).

*Step 1: proof of relation (7.100).* Taking in (7.99) test functions  $\boldsymbol{\varphi} = 0$  and  $\psi = 0$ , we obtain that

$$\int_{\Omega \times Y} \mathbb{R}(x, y) \cdot \mathbf{curl}_y \boldsymbol{\varphi}^1(x, y) \, dx \, dy = 0, \quad \forall \boldsymbol{\varphi}^1 \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y)). \quad (7.102)$$

The above relation implies the existence of a function  $\rho \in L^2(\Omega; H_{\text{per}, \diamond}^1(Y))$  such that (see for instance the proof of Proposition 1.14 of [5], and more precisely the discussion following relation (1.19) therein)

$$\mathbb{R}(x, y) = \nabla_y \rho(x, y) + \int_Y \mathbb{R}(x, \hat{y}) \, d\hat{y}. \quad (7.103)$$

Now, we follow the ideas of [14] and [58]. From the definition (7.97) of  $\mathbb{R}$  and direct calculation, one has for  $\xi \in H_{\text{per}}^1(Y)$ :

$$\int_Y \mu(y) \mathbb{R}(x, y) \cdot \nabla \xi(y) \, dy = \int_Y (\mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y)) \cdot \nabla \xi(y) \, dy = 0. \quad (7.104)$$

Combining (7.103) and (7.104) we get that

$$\int_Y \mu(y) \nabla_y \rho(x, y) \cdot \nabla \xi(y) \, dy = \int_Y \mu(y) \boldsymbol{\lambda} \cdot \nabla \xi(y) \, dy,$$

where we have set  $\boldsymbol{\lambda} = - \int_Y \mathbb{R}(x, y) \, dy \in \mathbb{R}^3$  (here,  $x$  is fixed and can be considered as a

parameter). Comparing with (7.83), we immediately obtain that  $\rho = \boldsymbol{\lambda} \cdot \boldsymbol{\chi}^\mu = \sum_{j=1}^3 \lambda_j \cdot \chi_j^\mu$ , where  $\boldsymbol{\chi}^\mu = (\chi_1^\mu, \chi_2^\mu, \chi_3^\mu)^\text{T}$  solve the cell problems (7.82) with  $\eta = \mu$ . Consequently, we have  $\nabla_y \rho = \sum_{j=1}^3 \lambda_j \cdot \nabla \chi_j^\mu = (\nabla \boldsymbol{\chi}^\mu)^\text{T} \boldsymbol{\lambda}$ , and hence

$$\mathbb{R}(x, y) = \nabla_y \rho(x, y) + \int_Y \mathbb{R}(x, \hat{y}) \, d\hat{y} = [\text{Id} - (\nabla \boldsymbol{\chi}^\mu)^\text{T}] \int_Y \mathbb{R}(x, \hat{y}) \, d\hat{y},$$

Using the above formula and expression (7.84) of  $\mathcal{H}(\mu)$ , we get that

$$\int_Y \mu(y) \mathbb{R}(x, y) \, dy = \mathcal{H}(\mu) \int_Y \mathbb{R}(x, y) \, dy.$$

But on the other hand, we also have from definition (7.97) of  $\mathbb{R}(x, y)$  that

$$\int_Y \mu(y) \mathbb{R}(x, y) \, dy = \int_Y (\mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y)) \, dy = \mathbf{curl} \mathbf{E}^{\text{eff}}(x).$$

Since  $\mathcal{H}(\mu)$  is positive definite for  $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , we obtain by combining the last two relations, that

$$\int_Y \mathbb{R}(x, y) \, dy = (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}}(x),$$

which also reads (due to the definition of  $\mathbb{R}$ )

$$\int_Y (\mu(y))^{-1} (\mathbf{curl} \mathbf{E}^{\text{eff}}(x) + \mathbf{curl}_y \mathbf{E}^1(x, y)) dy = (\mathcal{H}(\mu))^{-1} \mathbf{curl} \mathbf{E}^{\text{eff}}(x).$$

The claimed relation (7.100) simply follows by multiplying the above equation by  $\mathbf{curl} \varphi$ , integrating over  $\Omega$  and adding (7.102).

*Step 2: proof of relation (7.101).* Taking  $\varphi = \varphi^1 = 0$  in (7.99), we obtain that (since  $\omega \neq 0$ ):

$$\int_{\Omega \times Y} \mathbf{S}(x, y) \cdot \nabla_y \psi(x, y) dx dy = 0. \quad (7.105)$$

Since  $\psi$  is arbitrary in  $\mathcal{C}_0^\infty(\Omega; \mathcal{C}_{\text{per}}^\infty(Y))$ , this implies in particular that for almost every  $x \in \Omega$  and for all  $\xi \in \mathcal{C}_{\text{per}}^\infty(Y)$ :

$$\int_Y \mathbf{S}(x, y) \cdot \nabla \xi(y) dy = \int_Y \varepsilon(y) (\nabla_y \Theta(x, y) + \mathbf{E}^{\text{eff}}(x)) \cdot \nabla \xi(y) dx dy = 0. \quad (7.106)$$

Hence

$$\int_Y \varepsilon(y) \nabla_y \Theta(x, y) \cdot \nabla \xi(y) dy = \int_Y \varepsilon(y) \boldsymbol{\lambda}' \cdot \nabla \xi(y) dy,$$

where we have set  $\boldsymbol{\lambda}' = -\mathbf{E}^{\text{eff}}(x) \in \mathbb{R}^3$  (for a fixed value of  $x$ ). Comparing the above relation with (7.83) for  $\eta = \varepsilon$ , we get that  $\Theta(x, y) = \boldsymbol{\lambda}' \cdot \boldsymbol{\chi}^\varepsilon$  and hence  $\nabla_y \Theta = (\nabla \boldsymbol{\chi}^\varepsilon)^\top \boldsymbol{\lambda}' = -(\nabla \boldsymbol{\chi}^\varepsilon)^\top \mathbf{E}^{\text{eff}}(x)$ . Using expression (7.84) of the homogenized matrix, we obtain that for every  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ :

$$\begin{aligned} \int_{\Omega \times Y} \mathbf{S}(x, y) \cdot \varphi(x) dx dy &= \int_{\Omega \times Y} \varepsilon(y) (\mathbf{E}^{\text{eff}}(x) + \nabla_y \Theta(x, y)) \cdot \varphi(x) dx dy \\ &= \int_{\Omega} \mathcal{H}(\varepsilon) \mathbf{E}^{\text{eff}} \cdot \varphi dx. \end{aligned} \quad (7.107)$$

Relation (7.101) follows immediately by adding (7.105) and (7.107).  $\blacksquare$

## 7.5.2 Proof of the uniform energy estimate

This section is devoted to the proof of the uniform estimate (7.95) for solutions of  $(\mathcal{P}^\delta)$ . More precisely, we have the following proposition.

**Proposition 7.5.3.** *Assume that  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (7.45). Assume that  $\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}}$  where  $\Lambda^{\text{eff}}$  appears in (7.94). Then, there exists  $\delta_0 > 0$  such that for all  $\delta \in (0; \delta_0]$ , problem  $(\mathcal{P}^\delta)$  admits a unique solution  $\mathbf{E}^\delta$ . Moreover we have the estimate*

$$\|\mathbf{E}^\delta\| + \|\mathbf{curl} \mathbf{E}^\delta\| \leq C \|\mathbf{J}\| \quad (7.108)$$

where  $C > 0$  is independent of  $\delta \in (0; \delta_0]$ .

**Proof.** When  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , according to Theorem 7.3.1, we know that  $A_\varepsilon^\delta : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{H}_0^1(\Omega)$  and  $B_\mu^\delta : \mathbf{H}_\#^1(\Omega) \rightarrow \mathbf{H}_\#^1(\Omega)$  are isomorphisms. From the Theorem 6.1 of [23], we infer that  $\mathcal{A}_N^\delta(\omega) : \mathbf{H}_N(\mathbf{curl}; \Omega) \rightarrow \mathbf{H}_N(\mathbf{curl}; \Omega)$  is an isomorphism if it is injective. Therefore, we have to prove that  $\mathcal{A}_N^\delta(\omega)$  is injective for  $\delta$  small enough. To proceed we work by contradiction. Slightly more generally, for a given  $\mathbf{J} \in \mathbf{L}^2(\Omega)$ , assume that there is a sequence of values of  $\delta$  denoted  $(\delta_k)_{k \in \mathbb{N}}$ , with  $\delta_k \rightarrow 0$ , such that if we set  $\varepsilon_k := \varepsilon^{\delta_k}$ ,  $\mu_k := \mu^{\delta_k}$ ,  $\mathbf{E}_k := \mathbf{E}^{\delta_k} \in \mathbf{H}_N(\mathbf{curl}; \Omega)$ , we have

$$a_\omega^k(\mathbf{E}_k, \mathbf{E}') := ((\mu_k)^{-1} \mathbf{curl} \mathbf{E}_k, \mathbf{curl} \mathbf{E}') - \omega^2 (\varepsilon_k \mathbf{E}_k, \mathbf{E}') = i\omega (\mathbf{J}, \mathbf{E}'), \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega),$$

as well as

$$\|\mathbf{E}_k\|^2 + \|\mathbf{curl} \mathbf{E}_k\|^2 > k.$$

Then set  $\tilde{\mathbf{E}}_k := \mathbf{E}_k / (\|\mathbf{E}_k\|^2 + \|\mathbf{curl} \mathbf{E}_k\|^2)$  and  $\tilde{\mathbf{J}}_k := \mathbf{J} / (\|\mathbf{E}_k\|^2 + \|\mathbf{curl} \mathbf{E}_k\|^2)$ . We have

$$a_\omega^k(\tilde{\mathbf{E}}_k, \mathbf{E}') = i\omega(\tilde{\mathbf{J}}_k, \mathbf{E}'), \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega) \quad (7.109)$$

and

$$\|\tilde{\mathbf{E}}_k\|^2 + \|\mathbf{curl} \tilde{\mathbf{E}}_k\|^2 = 1, \quad \lim_{k \rightarrow +\infty} \|\tilde{\mathbf{J}}_k\| = 0.$$

Since  $(\tilde{\mathbf{E}}_k)$  is bounded in  $\mathbf{H}_N(\mathbf{curl}; \Omega)$ , we can extract a subsequence, still denoted  $(\tilde{\mathbf{E}}_k)$ , such that  $(\tilde{\mathbf{E}}_k)$  converges weakly in  $\mathbf{H}_N(\mathbf{curl}; \Omega)$  to some  $\mathbf{E}_0 \in \mathbf{H}_N(\mathbf{curl}; \Omega)$ . Thanks to Proposition 7.5.2, we can pass to the limit in (7.109) to get

$$a_\omega^{\text{eff}}(\mathbf{E}_0, \mathbf{E}') = 0, \quad \forall \mathbf{E}' \in \mathbf{H}_N(\mathbf{curl}; \Omega). \quad (7.110)$$

Since  $\omega^2 \in \mathbb{C} \setminus \Lambda^{\text{eff}}$ , this implies that  $\mathbf{E}_0 = 0$ . In order to obtain a contradiction, it remains to show that  $(\tilde{\mathbf{E}}_k)$  strongly converges to zero in  $\mathbf{H}_N(\mathbf{curl}; \Omega)$ . To proceed, we have to establish some sort of compactness result using the fact that when  $\omega \neq 0$ , we have  $\text{div}(\varepsilon_k \tilde{\mathbf{E}}_k) = 0$  in  $\Omega$  which implies that  $\tilde{\mathbf{E}}_k \in \mathbf{V}_N(\varepsilon_k)$ . For each  $k \geq 1$ , from Theorem 5.1 of [23], we know that when  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ ,  $\mathbf{V}_N(\varepsilon_k)$  is compactly embedded in  $\mathbf{L}^2(\Omega)$ . But here we need some uniform result with respect to  $k$ . To proceed, we will take in (7.109) a well-chosen test function. Let us mention that a similar difficulty appears in the justification of the approximation of Maxwell's equations with finite elements methods, the mesh size  $h$  replacing the parameter  $\delta$  (see [110, §7.3.2] and the references therein). First, introduce the unique function  $\psi_k \in \mathbf{H}_\#^1(\Omega)$  such that

$$(\mu_k \nabla \psi_k, \nabla \psi') = (\mu_k \mathbf{curl} \tilde{\mathbf{E}}_k, \nabla \psi'), \quad \forall \psi' \in \mathbf{H}_\#^1(\Omega).$$

When  $\kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$ , from Theorem 7.3.1, we know that  $\psi_k$  is well-defined. Moreover, we have  $\|\nabla \psi_k\| \leq C \|\mathbf{curl} \tilde{\mathbf{E}}_k\| \leq C$  where  $C > 0$  is independent of  $\delta$  (note that  $(\mu_k)$  is a bounded sequence of functions of  $\mathbf{L}^\infty(\Omega)$  and we have  $\|\mu_k\|_{\mathbf{L}^\infty(\Omega)} = \max(\mu_\varepsilon, |\mu_i|)$  for all  $k \in \mathbb{N}$ ). Then  $\mu_k(\mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k)$  is divergence free in  $\Omega$  and satisfies  $\mu_k(\mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k) \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . From [8, Theorem 3.17], we know that there exists a unique  $\mathbb{P}_k \tilde{\mathbf{E}}_k \in \mathbf{V}_N(1)$  such that

$$\mathbf{curl}(\mathbb{P}_k \tilde{\mathbf{E}}_k) = \mu_k(\mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k). \quad (7.111)$$

Since in  $\mathbf{V}_N(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{curl}}$  (Proposition 7.2.1), we infer that  $(\mathbb{P}_k)$  is a sequence of operators which are uniformly bounded from  $\mathbf{H}_N(\mathbf{curl}; \Omega)$  to  $\mathbf{V}_N(1)$ . Testing in (7.109) with  $\mathbf{E}' = \mathbb{P}_k \tilde{\mathbf{E}}_k$ , using (7.111) and integrating by parts, we get

$$\begin{aligned} i\omega(\tilde{\mathbf{J}}_k, \mathbb{P}_k \tilde{\mathbf{E}}_k) + \omega^2(\varepsilon_k \tilde{\mathbf{E}}_k, \mathbb{P}_k \tilde{\mathbf{E}}_k) &= ((\mu_k)^{-1} \mathbf{curl} \tilde{\mathbf{E}}_k, \mathbf{curl}(\mathbb{P}_k \tilde{\mathbf{E}}_k)) \\ &= (\mathbf{curl} \tilde{\mathbf{E}}_k, \mathbf{curl} \tilde{\mathbf{E}}_k - \nabla \psi_k) = \|\mathbf{curl} \tilde{\mathbf{E}}_k\|^2. \end{aligned} \quad (7.112)$$

Using that  $\mathbb{P}_k : \mathbf{H}_N(\mathbf{curl}; \Omega) \rightarrow \mathbf{V}_N(1)$  are uniformly bounded,  $(\tilde{\mathbf{E}}_k)$  converges weakly to zero in  $\mathbf{H}_N(\mathbf{curl}; \Omega)$  and  $\mathbf{V}_N(1)$  is compactly embedded in  $\mathbf{L}^2(\Omega)$  (Proposition 7.2.1), we deduce that we can extract a subsequence, still denoted  $(\tilde{\mathbf{E}}_k)$ , such that  $(\mathbb{P}_k \tilde{\mathbf{E}}_k)$  converges strongly to zero in  $\mathbf{L}^2(\Omega)$ . Then from (7.112), we deduce that the sequence  $(\mathbf{curl} \tilde{\mathbf{E}}_k)$  converges strongly to zero in  $\mathbf{L}^2(\Omega)$ . Using the result of Proposition 7.5.4 below which guarantees that  $\|\tilde{\mathbf{E}}_k\| \leq C \|\mathbf{curl} \tilde{\mathbf{E}}_k\|$  with some  $C > 0$  which is independent of  $k$ , we deduce that  $(\tilde{\mathbf{E}}_k)$  converges to zero in  $\mathbf{H}_N(\mathbf{curl}; \Omega)$ . This contradicts the initial assumption. As a consequence, taking first  $\mathbf{J} = 0$  above, we deduce that  $(\mathcal{P}^\delta)$  is injective and so uniquely solvable for  $\delta$  small enough. Then for a given non zero  $\mathbf{J} \in \mathbf{L}^2(\Omega)$ , the above lines imply the uniform estimate (7.108).  $\blacksquare$

**Proposition 7.5.4.** *Assume that  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (7.45). Then there is a constant  $C > 0$  independent of  $\delta$  such that*

$$\|\mathbf{E}\| \leq C \|\mathbf{curl} \mathbf{E}\|, \quad \forall \mathbf{E} \in \mathbf{V}_N(\varepsilon^\delta). \quad (7.113)$$

**Proof.** If  $\mathbf{E} \in \mathbf{V}_N(\varepsilon^\delta)$ , according to [8, Theorem 3.12], we know that there is a unique  $\mathbf{u} \in \mathbf{V}_T(1)$  such that  $\mathbf{E} = (\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{u}$ . Then integrating by parts, we find

$$((\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{u}, \mathbf{curl} \mathbf{u}') = (\mathbf{curl} \mathbf{E}, \mathbf{u}'), \quad \forall \mathbf{u}' \in \mathbf{V}_T(1). \quad (7.114)$$

Introduce the function  $\varphi \in H_0^1(\Omega)$  such that

$$(\varepsilon^\delta \nabla \varphi, \nabla \varphi') = (\varepsilon^\delta \mathbf{curl} \mathbf{u}, \nabla \varphi'), \quad \forall \varphi' \in H_0^1(\Omega).$$

Since  $\kappa_\varepsilon \in (-\infty; -1/m) \cup (-1/M; 0)$ , from Theorem 7.3.1, we know that  $\varphi$  is well-defined. Moreover, we have  $\|\nabla \varphi\| \leq C \|\mathbf{curl} \mathbf{u}\|$  where  $C > 0$  is independent of  $\delta$  (note that  $\|\varepsilon^\delta\|_{L^\infty(\Omega)} = \max(\varepsilon_e, |\varepsilon_i|)$  for all  $\delta > 0$ ). Then  $\varepsilon^\delta(\mathbf{curl} \mathbf{u} - \nabla \varphi)$  is divergence free in  $\Omega$  and again from [8, Theorem 3.12], we know that there is a unique  $\mathbf{Tu} \in \mathbf{V}_T(1)$  such that  $\mathbf{curl}(\mathbf{Tu}) = \varepsilon^\delta(\mathbf{curl} \mathbf{u} - \nabla \varphi)$ . Since in  $\mathbf{V}_T(1)$ ,  $\|\mathbf{curl} \cdot\|_\Omega$  is a norm which is equivalent to  $\|\cdot\|_{\mathbf{curl}}$  (Proposition 7.2.1), we infer that  $\mathbf{T} : \mathbf{V}_T(1) \rightarrow \mathbf{V}_T(1)$  is a uniformly bounded operator. Choosing  $\mathbf{u}' = \mathbf{Tu}$  in (7.114) and integrating by parts, we obtain

$$(\mathbf{curl} \mathbf{E}, \mathbf{Tu}) = ((\varepsilon^\delta)^{-1} \mathbf{curl} \mathbf{u}, \mathbf{curl}(\mathbf{Tu})) = \|\mathbf{curl} \mathbf{u}\|^2 - (\mathbf{curl} \mathbf{u}, \nabla \varphi) = \|\mathbf{curl} \mathbf{u}\|^2.$$

Using the Cauchy-Schwarz inequality, this gives  $\|\mathbf{curl} \mathbf{u}\| \leq C \|\mathbf{curl} \mathbf{E}\|$  where  $C > 0$  is independent of  $\delta$ . This yields the desired estimate (7.113).  $\blacksquare$

### 7.5.3 Final result

Gathering Propositions 7.5.2 and 7.5.3, we can state the final result of this article.

**Theorem 7.5.1.** *Assume that  $\kappa_\varepsilon, \kappa_\mu \in (-\infty; -1/m) \cup (-1/M; 0)$  where  $m, M$  are defined in (7.45). Assume that  $\omega \in \mathbb{C} \setminus \Lambda^{\text{eff}}$  where  $\Lambda^{\text{eff}}$  appears in (7.94). Then, there exists  $\delta_0 > 0$  such that for  $\delta \in (0, \delta_0]$ , the solution  $\mathbf{E}^\delta$  of problem  $(\mathcal{P}^\delta)$ , which is well-defined according to Proposition 7.5.3, satisfies*

$$\mathbf{E}^\delta \rightharpoonup \mathbf{E}^{\text{eff}} \quad \text{and} \quad \mathbf{curl} \mathbf{E}^\delta \rightharpoonup \mathbf{curl} \mathbf{E}^{\text{eff}} \quad \text{weakly in } \mathbf{L}^2(\Omega)$$

where  $\mathbf{E}^{\text{eff}}$  is the unique solution of problem  $(\mathcal{P}^{\text{eff}})$  given by (7.92).

Let us conclude this paper with two comments. Firstly, in this work, we only prove weak convergence results. Strong convergence results (using correctors) for Maxwell's equations with positive materials have been obtained in [133, 134]. It would be interesting to understand if we can adapt the approach proposed in these two articles to our setting. Secondly, the obtained bounds for the contrasts (involving  $m$  and  $M$ ) to ensure the homogenization process are probably not optimal. Improving them would require a sharp analysis of the asymptotic behavior of the critical contrasts given by (7.69) as  $\delta$  tends to zero (see Remark 7.3.4). Is it possible that the two scalar problems with Dirichlet and Neumann boundary conditions be uniformly well-posed as  $\delta$  tends to zero, even when some cell problems have a non zero kernel or when the homogenized tensors are not positive definite? This has still to be clarified.

## 7.6 Appendix. Table of notation for the functional spaces

For the reader's convenience, we list below the main functional spaces used throughout the paper:

$$\begin{aligned}
\mathcal{C}_0^\infty(\Omega) &:= (\mathcal{C}_0^\infty(\Omega))^3 \\
\mathcal{C}_{\text{per}}^\infty(\bar{Y}) &:= (\mathcal{C}_{\text{per}}^\infty(\bar{Y}))^3 \\
\mathbf{H}_0^1(\Omega) &:= \{\varphi \in \mathbf{H}^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\} \\
\mathbf{H}_{\#}^1(\Omega) &:= \left\{ \varphi \in \mathbf{H}^1(\Omega) \mid \int_{\Omega} \varphi \, dx = 0 \right\} \\
\mathcal{H}_D^\delta &:= \{\varphi \in \mathbf{H}_0^1(\Omega) \mid \Delta\varphi = 0 \text{ in } \Omega_e^\delta \cup \Omega_i^\delta\} \\
\hat{\mathcal{H}}_D^\delta &:= \{\varphi \in \mathcal{H}_D^\delta \mid (\nabla\varphi, \nabla\varphi_D^k) = 0, \quad \forall k \in K^\delta\} \\
\mathbf{H}_\diamond^1(\Omega) &:= \left\{ \varphi \in \mathbf{H}^1(\Omega) \mid \int_{\partial\Omega_i^\delta} \varphi \, d\sigma = 0 \right\} \\
\mathcal{H}_N^\delta &:= \{\varphi \in \mathbf{H}_\diamond^1(\Omega) \mid \Delta\varphi = 0 \text{ in } \Omega_e^\delta \cup \Omega_i^\delta, \partial_n\varphi = 0 \text{ on } \partial\Omega\} \\
\hat{\mathcal{H}}_N^\delta &:= \{\varphi \in \mathcal{H}_N^\delta \mid (\nabla\varphi, \nabla\varphi_N^k) = 0, \quad \forall k \in K^\delta \setminus \{k_0\}\} \\
\mathbf{H}_{0, \partial\Omega_i^\delta}^1(\Omega) &:= \{\varphi \in \mathbf{H}^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega_i^\delta\} \\
\mathbf{H}_{0, \partial\Omega}^1(\Omega_e^\delta) &:= \{\varphi|_{\Omega_e^\delta}, \varphi \in \mathbf{H}_0^1(\Omega)\} \\
\mathbf{H}_\diamond^1(Y) &:= \{\varphi \in \mathbf{H}^1(Y) \mid \int_{\partial Y_i} \varphi \, d\sigma = 0\} \\
\mathcal{H}_0 &:= \{\varphi \in \mathbf{H}_0^1(Y) \mid \Delta\varphi = 0 \text{ in } Y_e \cup Y_i\} \\
\mathcal{H}_\diamond &:= \{\varphi \in \mathbf{H}_\diamond^1(Y) \mid \Delta\varphi = 0 \text{ in } Y_e \cup Y_i\} \\
\hat{\mathcal{H}}_0 &:= \{\varphi \in \mathcal{H}_0 \mid (\nabla\varphi, \nabla\varphi_D) = 0\} \\
\hat{\mathcal{H}}_\diamond &:= \{\varphi \in \mathcal{H}_\diamond \mid \partial_n\varphi = 0 \text{ on } \partial Y\} \\
\mathbf{H}_{\text{per}}^1(Y) &:= \text{Closure of } \mathcal{C}_{\text{per}}^\infty(\bar{Y}) \text{ for the norm of } \mathbf{H}^1(Y) \\
\mathbf{H}_{\text{per}, \diamond}^1(Y) &:= \{\varphi \in \mathbf{H}_{\text{per}}^1(Y) \mid \int_{\partial Y_i} \varphi \, d\sigma = 0\} \\
\mathbf{H}_{\text{per}, 0, \partial Y_i}^1(Y) &:= \{\varphi \in \mathbf{H}_{\text{per}, \diamond}^1(Y) \mid \varphi = 0 \text{ on } \partial Y_i\} \\
\hat{\mathcal{H}}_b &:= \text{Orthogonal complement of } \mathbf{H}_{\text{per}, 0, \partial Y_i}^1(Y) \text{ in } \mathbf{H}_{\text{per}, \diamond}^1(Y) \\
\mathbf{H}(\mathbf{curl}; \Omega) &:= \{\mathbf{H} \in \mathbf{L}^2(\Omega) \mid \mathbf{curl} \mathbf{H} \in \mathbf{L}^2(\Omega)\} \\
\mathbf{H}_N(\mathbf{curl}; \Omega) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \mathbf{E} \times \mathbf{n} = 0 \text{ on } \partial\Omega\} \\
\mathbf{H}_{\text{per}}(\mathbf{curl}; Y) &:= \text{Closure of } \mathcal{C}_{\text{per}}^\infty(\bar{Y}) \text{ for the norm } (\|\cdot\|_Y^2 + \|\mathbf{curl} \cdot\|_Y^2)^{1/2} \\
\mathbf{L}^2(\Omega) &:= (\mathbf{L}^2(\Omega))^3 \\
\mathbf{V}_T(\xi) &:= \{\mathbf{H} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \text{div}(\xi\mathbf{H}) = 0, \xi\mathbf{H} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \\
\mathbf{V}_N(\xi) &:= \{\mathbf{E} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \text{div}(\xi\mathbf{E}) = 0, \mathbf{E} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}.
\end{aligned}$$

## Chapter 8

# Conclusions and future directions

Let us conclude this work by summarizing the contribution that has been presented in this thesis and by mentioning some future directions that we think are interesting to investigate.

### Conclusions

In the first part, we studied the scalar transmission problem between some positive and negative materials separated by an interface with a smooth conical tip. We showed that it can be studied by combining the T-coercivity approach with the Mellin analysis in weighted Sobolev spaces. We proved that the critical interval can be characterized as the set of contrasts for which propagating singularities exist. Contrary to the 2D case of interfaces with corners, the number of propagating singularities for this problem can be greater than 2 (in the particular case of the circular conical tip, this number tends to  $+\infty$  when the contrast approaches  $-1$ ). In the process, we highlighted an interesting link between the critical interval and the essential spectrum of the so-called Neumann-Poincaré operator. For contrasts inside the critical interval, the Mandelstam radiation principle has been used to construct (an infinite number of) functional frameworks in which well-posedness is restored. These frameworks are constructed by taking the sum of well-chosen weighted Sobolev spaces and particular spaces of propagating singularities (the idea is to include only the ones which have a positive energy flux). Under some assumptions on the contrast, whose validity has been investigated in details for the case of the circular conical tip, then we explained how to apply the limiting absorption principle to select among these functional frameworks that are coherent with the Mandelstam radiation principle, the one that corresponds to the physical reality.

The second part of this work has been devoted to present a new numerical strategy to approximate the solutions of the 2D/3D scalar problems with sign-changing coefficients in the classical  $H^1$  framework. The approach is based on an optimal control reformulation of the problem and is proved to be convergent without any additional assumption on the mesh near the interface as soon as the problem under study is well-posed in the classical Sobolev spaces.

Then in the third part, we considered the time harmonic Maxwell's equations with one or two critical coefficients. We explained why the classical functional framework is no longer suitable for the study of this problem. By combining new results of vector potentials in weighted Sobolev spaces, new regularity/compactness results concerning classical vector potential spaces, the Mellin analysis and the T-coercivity approach, we explained how to construct new functional frameworks for the electric and magnetic problems. These frameworks are themselves directly related to the ones obtained for the two associated scalar problems. We established that if one uses the setting that respects the limiting absorption principle for the scalar problems, then those provided by our theory for the electric and magnetic problems are also coherent with the limiting absorption



principle. Thus the important message is that the study of the Maxwell's equations can be reduced to the study of two corresponding scalar problems.

Finally, in the last part of this work, we turned our attention to the study of the homogenization process for time-harmonic Maxwell equations and related scalar problems in a 3D domain that contains a periodic distribution of inclusions made of negative material. Due to the sign-change of the electromagnetic parameters, the derivation of uniform energy estimates, which are needed to ensure that the homogenization process is possible, is not straightforward even in the case of the scalar problems. Using the T-coercivity approach, we obtained conditions on the contrasts associated to the electromagnetic parameters under which we have been able to perform the homogenization process for the scalar problems. Interestingly, we showed that the homogenized matrices associated with the limit scalar problems are either positive definite or negative definite. By combining this with a new uniform compactness result for Maxwell's spaces established thanks to the T-coercivity approach, we proved that the homogenization process applicable for the vectorial problem under the same assumptions as for the scalar equations.

## Future directions

In addition to the open questions and possible extensions that we have presented at the end of each chapter, let us mention here some more general questions that can be seen as a natural continuation of the work done in this thesis.

- **Numerical approximation of Maxwell's equations with sign-changing coefficients.**

One of the most challenging questions that has not been addressed in these pages concerns the numerical approximation of the solution of Maxwell's equations with sign-changing coefficients.

For non critical contrasts, while several convergent methods have been proposed for the scalar problem, in particular the one developed in the last chapter, the only existing work concerning Maxwell's equations is [56]. The problem considered in [56] corresponds to the particular case where  $\varepsilon$  is positive and  $\mu$  has a sign-change without being critical. The technique developed there can not be generalized to the situation where both  $\varepsilon$  and  $\mu$  are sign-changing. To overcome these limitations, an interesting idea would be to try to extend the method proposed in Chapter 4 to the case of Maxwell's equations.

When one of the electromagnetic parameters is critical, the only existing approach to approximate the solution of the scalar problem in the new framework is to use Perfectly Matched Layers (PMLs) near the origin (see [45]). The proof of convergence is a work in progress. The adaptation of this method to study Maxwell's equations is also a challenging question. Since the framework we proposed for the problem in this configuration suggests that the solution should be decomposed as the sum of a regular part and a singular part that belongs to a finite dimensional space, one might think that it is enough to enrich the classical Nedelec space with the *ad hoc* propagating singularities to obtain a convergent method. Unfortunately, this is not so simple because some terms in the new formulation are hard to compute (numerically).

- **Extension of our results to other singular geometries.** The results obtained in this work are not valid for situations involving interfaces with other types of geometric singularities such as 3D edges or 2D cusps. The determination of the critical interval in these configurations remains an open question. In this direction, natural questions arise. How to determine the critical interval for the scalar problems? How to identify, for critical contrasts, adapted functional frameworks to recover well-posedness for the scalar problems? Then does the theory presented here to construct well-chosen functional frameworks for Maxwell's equations starting from those for the scalar problems, still work?

- **Study of Maxwell's equations in waveguides.** In the literature, it seems that the study of classical (with positive coefficients) Maxwell's equations in infinite waveguides in presence of propagating modes is not yet treated. In this direction we can wonder if the work presented here can help to address this problem, the propagating modes at infinity in the waveguide playing the same role as the propagating singularities at the tip in our work.
  - **Spectral analysis of Maxwell's equations with dispersive materials.** A natural question to complete our work is to determine the configurations for which trapped modes exist. Let us recall that trapped modes are non trivial solutions of the homogeneous equations living in the classical framework. Let us also recall (see Chapter 1) that from a physical point of view, the negative materials that we have studied in this thesis are in fact dispersive materials, where  $\varepsilon$  and  $\mu$  are functions of the frequency  $\omega$ , becoming negative in some frequency ranges. Then a relevant question can be to consider  $\omega$  as a spectral parameter: for a bounded inclusion of dispersive material embedded in a bounded domain filled with a classical non-dispersive dielectric, what are the values of  $\omega$  for which trapped modes exist? Due to the dispersion, this leads to study a non-linear eigenvalue problem. Such question has been investigated in the 2D scalar case in [45]. In particular in this work, in presence of a corner, it has been shown that the critical interval of contrasts gives rise to an interval of essential spectrum. Our contributions should allow us to extend these results to the 3D scalar case. But the extension to Maxwell's equations seems much more challenging, in particular due to the fact that the spaces of divergence free fields will themselves depend on the spectral parameter  $\omega$ . Note that we would be especially interested in trapped modes corresponding to so-called embedded eigenvalues (in the essential spectrum) leading to non-uniqueness of the solution for the problems we introduced in Chapters 6 and 5.
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The results of this work have been/will be the subject of the following publications:

1. R. Bunoiu, L. Chesnel, K. Ramdani, and M. Rihani. Homogenization of Maxwell's equations and related scalar problems with sign-changing coefficients, *Annales de la Faculté des Sciences de Toulouse*, 2020. **Published.**
2. A-S. Bonnet-Ben Dhia, L. Chesnel, and M. Rihani. Maxwell's equations with hypersingularities at a conical plasmonic tip. **Accepted in Journal de mathématiques pures et appliquées.**
3. A-S. Bonnet-Ben Dhia, L. Chesnel, and M. Rihani. Maxwell's equations with hypersingularities at a conical plasmonic tip (part2), **In preparation.**
4. A-S. Bonnet-Ben Dhia, L. Chesnel, and M. Rihani. Radiation condition for a 3D interface between a dielectric and a negative material. **In preparation.**
5. P. Ciarlet Jr, D. Lassounon and M. Rihani. An optimal control-based numerical method for scalar transmission problems with sign-changing coefficients. **In preparation.**

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**Title:** Maxwell's equations in presence of negative materials.

**Key words:** Maxwell's equations, Corner singularities, Black hole waves, Radiation conditions, Mandelstam principle, Limiting absorption principle, Kondratiev theory, Fictitious domain method, Control and optimization, Homogenization.

**Abstract:** The main subject of this thesis is the study of time-harmonic electromagnetic waves in a heterogeneous medium composed of a dielectric and a negative material (i.e. with a negative dielectric permittivity  $\varepsilon$  and/or a negative magnetic permeability  $\mu$ ) which are separated by an interface with a conical tip. Because of the sign-change in  $\varepsilon$  and/or  $\mu$ , the Maxwell's equations can be ill-posed in the classical  $L^2$ -frameworks. On the other hand, we know that when the two associated scalar problems, involving respectively  $\varepsilon$  and  $\mu$ , are well-posed in  $H^1$ , the Maxwell's equations are well-posed. By combining the T-coercivity approach with the Mellin analysis in weighted Sobolev spaces, we present, in the first part of this work, a detailed study of these scalar problems. We prove that for each of them, the well-posedness in  $H^1$  is lost iff the associated contrast belong to some critical set called the critical interval. These intervals correspond to the sets of negative contrasts for which propagating singularities, also known as black hole waves, appear at the tip. Contrary to the case of a 2D corner, for a 3D tip, several black hole waves can exist. Explicit expressions of these critical intervals are obtained for the particular case of circular conical tips. For critical contrasts, using the Mandelstam radiation principle, we construct functional frameworks in which well-posedness of the scalar problems is restored. The physically relevant framework is selected by a limiting absorption principle. In the process, we present a new numerical strategy for 2D/3D scalar problems in the non-critical case. This approach, presented in the second part of this work, contrary to existing ones, does not require additional assumptions on the mesh near the interface. The third part of the thesis concerns Maxwell's equations with one or two critical coefficients. By using new results of vector potentials in weighted Sobolev spaces, we explain how to construct new functional frameworks for the electric and magnetic problems, directly related to the ones obtained for the two associated scalar problems. If one uses the setting that respects the limiting absorption principle for the scalar problems, then the settings provided for the electric and magnetic problems are also coherent with the limiting absorption principle. Finally, the last part is devoted to the homogenization process for time-harmonic Maxwell's equations and associated scalar problems in a 3D domain that contains a periodic distribution of inclusions made of negative material. Using the T-coercivity approach, we obtain conditions on the contrasts such that the homogenization results is possible for both the scalar and the vector problems. Interestingly, we show that the homogenized matrices associated with the limit problems are either positive definite or negative definite.

**Titre :** Équations de Maxwell en présence des matériaux négatifs.

**Mots clés :** Équations de Maxwell, Singularités de coin, Ondes de trou noir, Conditions de radiation, Principe de Mandelstam, Principe d'absorption limite, Théorie de Kondratiev, Méthode des domaines fictifs, Contrôle et optimisation, Homogénéisation.

**Résumé :** Le sujet principal de cette thèse est l'étude de la propagation des ondes électromagnétiques, en régime harmonique, dans un milieu hétérogène composé d'un diélectrique et d'un matériau négatif (c'est-à-dire avec une permittivité diélectrique négative  $\varepsilon$  et/ou une perméabilité magnétique négative  $\mu$ ) qui sont séparés par une interface avec une pointe conique. En raison du changement de signe de  $\varepsilon$  et/ou  $\mu$ , les équations de Maxwell peuvent être mal posées dans les cadres classiques (basés sur l'espace  $L^2$ ). D'autre part, nous savons que lorsque les deux problèmes scalaires associés, impliquant respectivement  $\varepsilon$  et  $\mu$ , sont bien posés dans  $H^1$ , les équations de Maxwell sont bien posées. En combinant la méthode de la T-coercivité avec l'analyse de Mellin dans les espaces de Sobolev à poids, nous présentons, dans la première partie de ce travail, une étude détaillée de ces problèmes scalaires. Nous prouvons que pour chacun d'entre eux, le caractère bien posé dans  $H^1$  est perdu si et seulement si le contraste associé appartient à un ensemble critique appelé intervalle critique. Ces intervalles correspondent donc à des ensembles de contrastes négatifs pour lesquels des singularités propagatives, aussi appelées ondes de trou noir, apparaissent à l'extrémité de la pointe. Contrairement au cas d'un coin 2D, pour une pointe 3D, plusieurs ondes de trou noir peuvent exister. Des expressions explicites de ces intervalles critiques sont obtenues pour le cas particulier des pointes coniques circulaires. Pour les contrastes critiques, en utilisant le principe de radiation de Mandelstam, nous construisons des cadres fonctionnels dans lesquels le caractère bien posé des problèmes scalaires est restauré. Le cadre physiquement pertinent est sélectionné par un principe d'absorption limite. En outre, nous présentons, dans la deuxième partie de ce travail, une nouvelle méthode numérique pour les problèmes scalaires dans le cas des contrastes non-critiques. Cette approche, contrairement aux techniques existantes, ne nécessite pas d'hypothèses supplémentaires sur le maillage au voisinage de l'interface. La troisième partie de la thèse concerne l'étude des équations de Maxwell avec un ou deux coefficients critiques. En utilisant de nouveaux résultats de potentiels vecteurs dans des espaces de Sobolev à poids, nous expliquons comment construire de nouveaux cadres fonctionnels pour les problèmes électrique et magnétique, qui sont directement liés à ceux obtenus pour les deux problèmes scalaires associés. Si l'on utilise le cadre qui respecte le principe d'absorption limite pour les problèmes scalaires, alors les cadres fournis pour les problèmes électrique et magnétique sont également cohérents avec le principe d'absorption limite. Enfin, la dernière partie porte sur des résultats d'homogénéisation des équations de Maxwell harmoniques et des problèmes scalaires associés dans un domaine 3D qui contient une distribution périodique d'inclusions faites de matériau négatif. En utilisant l'approche de la T-coercivité, nous obtenons des conditions sur les contrastes telles que le processus d'homogénéisation est possible pour les problèmes scalaires et vectoriels. De façon peu intuitive, nous montrons que les matrices homogénéisées associées aux problèmes limites sont soit définies positives, soit définies négatives.

