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Quentin Petit

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**THÈSE DE DOCTORAT**

**DE L'UNIVERSITÉ PSL**

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**Mean field games and optimal transport in urban modelling**

Soutenue par

**Quentin Petit**

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# Contents

<b>1</b>	<b>Introduction</b>	<b>11</b>
1.1	Vers un modèle de jeux à champ moyen appliqué au marché locatif de l'immobilier professionnel	11
1.1.1	Équilibres économiques	11
1.1.2	La théorie des jeux à champ moyen	12
1.1.3	Un modèle pour le marché locatif de l'immobilier professionnel	16
1.2	Le marché de l'immobilier locatif pour les particuliers	16
1.2.1	La théorie du transport optimal	17
1.2.2	Les jeux non-atomiques	19
1.3	Organisation du mémoire	20
1.4	Contributions	21
1.4.1	Une approche jeu à champ moyen pour modéliser les interactions entre différents marchés	21
1.4.2	Un modèle pour la distribution spatiale des résidences	26
1.4.3	Perspectives et futurs développements	28
<b>I</b>	<b>A model for the interactions of firms in factor markets</b>	<b>29</b>
<b>2</b>	<b>A mean field game approach to model equilibria in perfectly competitive markets</b>	<b>31</b>
2.1	Introduction	31
2.1.1	General framework	32
2.2	An example with an almost explicit solution	34
2.2.1	Hamilton-Jacobi equation	35
2.2.2	The continuity equation	37
2.2.3	Equilibrium	37
2.2.4	Comments	38
2.3	The optimal control problem of a representative firm	39
2.3.1	Setting and main assumptions	39
2.3.2	Some properties of the Hamiltonian	41
2.3.3	A solution of the Hamilton-Jacobi equation on $[k^*, +\infty)$	44
2.3.4	A solution of the Hamilton-Jacobi equation on $(0, k^*]$	46
2.3.5	Conclusion	49
2.3.6	Extension to the case when there is depreciation of capital	50
2.4	Distribution of capital	52
2.5	Equilibrium	56
2.6	The net output deduced from a production function	60
2.7	Conclusion	64

<b>3</b>	<b>A Lagrangian approach to solve the individual optimal control problem for the firms</b>	<b>67</b>
3.1	Introduction	67
3.2	Preliminaries	68
3.3	Study of the reduced and relaxed optimal control problem	70
3.4	Conclusion	84
<b>4</b>	<b>The labour demand: two important examples</b>	<b>85</b>
4.1	Introduction	85
4.2	Standing assumptions on the optimal investment	86
4.3	The Cobb-Douglas model	86
4.4	A model with constant elasticity of substitution	90
4.5	Conclusion	91
<b>5</b>	<b>Existence and uniqueness of equilibria</b>	<b>93</b>
5.1	Introduction	93
5.2	Preliminaries	94
5.2.1	The Brouwer degree	94
5.2.2	$Z$ -matrices and $M$ -matrices and their applications in economics	95
5.2.3	A labour market model	96
5.2.4	A useful lemma	97
5.2.5	Standing assumptions on the labour supply	98
5.2.6	Standing assumptions on the labour demand	105
5.3	Main results	108
5.4	A numerical approach to solve the problem	111
5.4.1	Newton and quasi-Newton methods	111
5.4.2	Continuation method	112
5.5	Conclusion	114
<b>6</b>	<b>Alternative models for the distribution of capital</b>	<b>115</b>
6.1	Introduction	115
6.2	Standing assumptions on the drift	116
6.3	A model in which the number of firms is not known a priori	116
6.3.1	Application to the entry of entrepreneurs	117
6.4	A non local model	118
6.5	Conclusion	123
<b>7</b>	<b>Numerical simulations</b>	<b>125</b>
7.1	Introduction	125
7.2	The finite difference operators	126
7.2.1	The scheme	126
7.2.2	The algorithm	127
7.3	Numerical simulations	128
7.3.1	Solution of (7.1)-(7.5) applied to the audiovisual, publishing and distribution sector	128
7.3.2	A model with a labour market composed of two types of workers linked to the rental market for professionals	133
7.3.3	Behaviour of the distribution of capital near $k^*(w)$ .	134
7.3.4	A model in which the aggregate capital is not known a priori	136
7.3.5	A non local model for the distribution of capital	137
7.3.6	Sensitivity with respect to the parameters of the Cobb-Douglas model	139
7.4	Conclusion	148

<b>II</b>	<b>A model linking the labour market to the rental market for individuals</b>	<b>149</b>
<b>8</b>	<b>A static model for the spatial distribution of residences</b>	<b>151</b>
8.1	The model . . . . .	152
8.2	Standing assumptions and main results . . . . .	154
8.3	Existence of equilibria . . . . .	155
8.4	Uniqueness of equilibria . . . . .	161
8.5	Extensions . . . . .	164
8.5.1	A home-based telecommuting model . . . . .	164
8.5.2	The zero-noise limit case . . . . .	166
8.6	Conclusion . . . . .	170
8.A	Appendix: a random revenue model . . . . .	170
<b>9</b>	<b>Generalization for different types of workers</b>	<b>173</b>
9.1	The model . . . . .	174
9.1.1	Standing assumptions and a preliminary result . . . . .	176
9.2	Existence of equilibria . . . . .	177
9.3	Conclusion . . . . .	187
<b>10</b>	<b>Numerical simulations</b>	<b>189</b>
10.1	Introduction . . . . .	189
10.2	The finite difference operators . . . . .	190
10.2.1	The scheme . . . . .	191
10.2.2	The method . . . . .	191
10.3	Some comparative statics . . . . .	192
10.3.1	Comparative statics as the preference parameter $\theta$ varies . . . . .	192
10.3.2	Comparative statics as the capital is transferred to a given workplace . . . . .	195
10.3.3	A home-based telecommuting model . . . . .	198
10.3.4	A home based telecommuting model in a two-dimensional space . . . . .	201
10.4	Conclusion . . . . .	204





# List of symbols and notations

$\ \cdot\ $	the Euclidean norm on $\mathbb{R}^p$ .
$\ \cdot\ _\infty$	the supremum norm (on the relevant space)
$C^n$	the set of functions such that the 0th through $n$ th derivatives are continuous.
$C_c^\infty$	the set of infinitely differentiable functions with compact support.
id	the identity function.
$D_x f(x, y)$	the gradient of $f(\cdot, y)$ .
$\mathcal{M}(X)$	the space of Radon measure on a metric space $X$ .
$\mathcal{M}_+(X)$	the space of positive Radon measure on a metric space $X$ .
$\mathcal{P}(X)$	the probability space on a metric space $X$ .
$\delta_x$	the Dirac mass in $x$ .
$\delta_{i,j}$	the Kronecker delta.
$\ \cdot\ _{\mathcal{M}(X)}$	the total variation norm.
$\mu_{ac}(\cdot)$	the absolutely continuous part of the measure $\mu$ with respect to the Lebesgue measure.
$ X $	the Lebesgue measure of a Lebesgue measurable set $X$ .
$BV$	the set of the functions of bounded variation.
$BV_{loc}$	the set of functions of locally bounded variation.
$g(\cdot) = o(f(\cdot))$	$g(x)/f(x) \rightarrow 0$ as $x \rightarrow x_0$ (where $x_0$ is clear from the context).
$f(\cdot) \sim g(\cdot)$	$f(x)/g(x) \rightarrow 0$ as $x \rightarrow x_0$ (where $x_0$ is clear from the context).
$a.a$	means <i>almost all</i> with respect to the Lebesgue measure.
$a.e$	means <i>almost everywhere</i> with respect to the Lebesgue measure.



# Chapitre 1

## Introduction

Au moment où ces lignes sont écrites, la crise du COVID-19 semble en passe d'être mieux contrôlée. Cette pandémie mondiale a bouleversé nombre d'habitudes et a eu des effets sur les marchés du travail et de l'immobilier. En France, dès le début de l'épidémie, le télétravail a été encouragé dans le but de limiter les interactions physiques et de freiner la propagation du virus. Cette pratique qui tend à se répandre, diminue les coûts de transport entre l'habitat et le lieu de travail ; elle semble influencer la productivité des travailleurs et diminuer la demande des entreprises en immobilier locatif. Ces changements auront certainement un effet sur les prix sur le marché du travail et sur ceux de l'immobilier locatif pour les professionnels et les particuliers.

Peut-on expliquer les variations de ces prix, comprendre le fonctionnement de ces marchés, notamment comment l'offre et la demande globale de main d'oeuvre et de surface locative peuvent être affectées par la productivité des entreprises ?

Ce mémoire est motivé par ces questions. Il est composé de deux parties indépendantes pour la majorité de leur contenu. La première partie traitera plus particulièrement du marché de l'immobilier locatif professionnel, et le marché locatif pour les particuliers sera abordé dans la seconde partie. La différence majeure réside dans le fait que dans le premier cas, la surface est considérée comme un facteur de production, tandis que dans le deuxième cas, elle est considérée comme une source d'utilité. Dans la première partie, nous proposerons et étudierons un modèle de jeux à champ moyen prenant en compte les interactions des marchés des facteurs de production. Dans la seconde partie, nous nous intéresserons à un modèle basé sur des notions de transport optimal et de jeux à champ moyen, permettant de comprendre plus précisément la compétition des entreprises sur le marché du travail, et celle des travailleurs sur celui de l'immobilier locatif.

### 1.1 Vers un modèle de jeux à champ moyen appliqué au marché locatif de l'immobilier professionnel

Dans la première partie de ce manuscrit, nous nous intéresserons aux liens entre le marché du travail et le marché locatif de l'immobilier professionnel. De ce fait, la main d'oeuvre et l'espace de travail seront considérés comme des facteurs de production. Dans ce cadre, peut-on comprendre les interactions entre ces marchés via des arguments d'équilibre ?

#### 1.1.1 Équilibres économiques

En économie, la théorie des jeux permet de modéliser certains marchés. Un exemple typique est le modèle de duopole que Cournot a introduit dans son livre [37] en 1838. Il met en concurrence deux firmes qui cherchent à maximiser leurs profits (hypothèse de rationalité). Elles ont toutes deux un pouvoir de marché, c'est-à-dire que le prix du bien est influencé par la quantité produite selon la loi de l'offre

et de la demande. Cournot définit alors un jeu non-coopératif à deux joueurs (les deux firmes) : les stratégies des entreprises consistent à choisir leur volume de production pour maximiser leur profit. Dans ce cas particulier, Cournot montre qu'il existe un équilibre. Plus tard, dans les années 1950, Nash définit une classe d'équilibres portant son nom et contenant les équilibres décrits par Cournot. Des résultats d'existence pour les équilibres de Nash sont établis dans [86, 85, 87]. Ils permettent d'étudier des jeux plus généraux.

Certains modèles d'équilibre général permettent de comprendre le marché des facteurs de production. Le modèle de référence est celui d'Arrow-Debreu-McKenzie, ces derniers prouvant l'existence d'équilibres en 1954, voir [9, 78]. Le but de leur modèle est de décrire l'équilibre sur les marchés des biens produits et celui des facteurs de production. Il permet de prévoir la quantité de biens produits, la quantité de facteurs utilisés, et leurs prix d'équilibre. Comme expliqué en détail dans le deuxième chapitre du livre écrit par Debreu, [40], les facteurs de production peuvent recouvrir l'espace de travail et la main d'oeuvre. Dans ce modèle, il y a un nombre fini de consommateurs et de producteurs. Arrow et Debreu introduisent la notion d'*économie abstraite*, qui est une généralisation des jeux évoqués ci-dessus, puisque l'ensemble des stratégies de certains joueurs dépend des stratégies jouées par les autres. Arrow et Debreu montrent d'une part que les conditions d'équilibre sont équivalentes à celles d'un équilibre de Nash pour une économie abstraite qu'ils définissent, et prouvent d'autre part l'existence d'un équilibre de Nash pour ce jeu généralisé. Dans leur modèle les agents ne font pas d'anticipation, c'est à dire que dans leur décision, ils ne prennent pas en compte leur niveau de consommation futur. De plus, Arrow et Debreu font l'hypothèse d'un marché parfaitement compétitif, c'est à dire qu'un agent ne peut pas individuellement influencer le prix des biens et des facteurs de production. Or, puisqu'il y a un nombre fini d'agents, un seul agent devrait pouvoir influencer les prix...

En 1964, Aumann introduit dans [12] un jeu modélisant un marché parfaitement compétitif avec une infinité de joueurs. Il définit pour cela la notion d'équilibre compétitif, et en prouve l'existence dans [13]. Depuis, une abondante littérature s'est développée. En particulier, la définition d'équilibre s'est affinée et le lien avec les équilibres de Nash pour un jeu avec un nombre fini de joueurs a été étudié, voir [32] par exemple.

L'idée des anticipations rationnelles a été introduite par Muth [84], et développée par Lucas dans les années 1970. Par exemple, dans [74], Lucas et Prescott supposent que les entreprises prennent des décisions en se basant sur un critère d'utilité considérant leur niveau de consommation futur. Les firmes sont supposées homogènes, ce qui facilite l'agrégation des productions individuelles pour passer à l'échelle macroscopique. Le but du modèle est de comprendre comment le prix du bien produit par les entreprises évolue lorsque la demande est soumise à des chocs aléatoires. Si le modèle de Lucas et Prescott n'explique pas les prix des facteurs de production, il permet de déterminer la stratégie d'accumulation du capital et le prix du bien produit au cours du temps. En particulier, Lucas et Prescott montrent que le niveau de capital accumulé converge vers une certaine valeur déterminée par les paramètres du modèle.

## 1.1.2 La théorie des jeux à champ moyen

Au milieu des années 2000, Lasry et Lions [68, 69, 70] et indépendamment Caines, Malhamé et Huang [62, 61, 60] introduisent la théorie des jeux à champ moyen. Elle permet d'étudier les équilibres de Nash d'un jeu différentiel avec une infinité de joueurs. Cette théorie a d'ailleurs été utilisée dès son invention pour modéliser certains marchés [70]. Une abondante littérature s'est développée : en particulier, on peut citer [28, 72, 33, 34] pour des introductions détaillées. Dans le paragraphe suivant, nous résumerons certains aspects de cette théorie puis nous présenterons quelques jalons dans la littérature traitant des applications économiques. Nous terminerons cette section par la présentation de la refonte du modèle de Aiyagari-Bewley-Huggett dans le cadre des jeux à champ moyen proposé par Achdou, Han, Lasry, Lions et Moll [3].

### Une présentation formelle des jeux à champ moyen

Comme évoqué plus haut, la théorie des jeux à champ moyen permet d'étudier les équilibres de Nash d'un jeu différentiel avec une infinité de joueurs. Dans les cas les plus favorables, l'étude de ces équilibres conduit au système d'équations aux dérivées partielles suivant :

$$-\partial_t u - \nu \Delta u + H(x, m, Du) = F(x, m) \quad \text{dans } \mathbb{R}^d \times (0, T) \quad (1.1)$$

$$\partial_t m - \nu \Delta m - \operatorname{div}(D_p H(x, m, Du)m) = 0 \quad \text{dans } \mathbb{R}^d \times (0, T) \quad (1.2)$$

$$m(0) = m_0, \quad u(\cdot, T) = G(\cdot, m(T)). \quad (1.3)$$

Le coefficient de diffusion  $\nu$  est une constante positive ou nulle. La première équation, (1.1), est une équation de Hamilton-Jacobi-Bellman (ou HJB) ; elle traduit le fait que les agents font des anticipations rationnelles. Elle permet de définir la stratégie optimale des agents étant donnée la distribution des états dans le futur (plus exactement la distribution des états qu'anticipent les agents). Ainsi, cette équation est posée de manière rétrograde dans le temps et est associée à une condition terminale donnée par le coût terminal  $G(\cdot, \cdot)$ . La seconde équation, (1.2), est une équation de Fokker-Planck (ou FP) : elle décrit l'évolution de la distribution des états si les joueurs adoptent tous la stratégie optimale mentionnée précédemment. On voit donc que l'équilibre est atteint si la distribution des états, solution de l'équation de Fokker-Planck, coïncide avec la prévision que les joueurs font de celle-ci. L'équation de Fokker-Planck est posée dans le sens direct du temps et est associée à une condition initiale, traduisant la connaissance de la distribution initiale des états, décrite par  $m_0$ , une mesure de probabilité sur l'espace des états. Le système ci-dessus a donc une structure mathématique particulière : une des équations est posée dans le sens direct du temps, l'autre est dans le sens rétrograde, et l'équation de Fokker-Planck est une équation adjointe de l'équation de Hamilton-Jacobi-Bellman linéarisée.

Formellement, l'interprétation du système est la suivante : l'état d'un agent évolue selon l'équation différentielle stochastique contrôlée

$$dX_t = \alpha_t dt + \sqrt{2\nu} dB_t, \quad (1.4)$$

où  $(B_t)$  est un mouvement Brownien standard et le contrôle  $\alpha$  est un processus adapté à la filtration associée à  $(B_t)$ . Anticipant l'évolution de la distribution des états, i.e.  $m(\cdot, \cdot)$ , un agent représentatif cherche à minimiser le coût suivant

$$\mathbb{E} \left[ \int_0^T L(X_s, m(s), \alpha_s) ds + G(X_T, m(T)) \right],$$

où  $H(\cdot, \cdot, \cdot)$  est la conjuguée de Fenchel de  $L(\cdot, \cdot, \cdot)$  par rapport à sa troisième variable (sous des hypothèses convenables). Ainsi, la fonction valeur du problème de contrôle optimal précédemment défini est l'unique solution de (1.1) avec pour condition terminale  $u(\cdot, T) = G(\cdot, m(T))$ . Par conséquent, au moins formellement, le contrôle optimal est donné par la loi de feedback  $\alpha^*(t, x) = -D_p H(x, m, Du)$ . En supposant que tous les agents font les mêmes anticipations, leur déplacement sera donné par (1.4), avec  $\alpha_t = \alpha^*(t, X_t)$ . À l'équilibre, l'évolution de la distribution des états sera régie par l'équation de Fokker-Planck (1.2) avec pour condition initiale  $m(0) = m_0$ .

### Les jeux à champ moyen en économie

Dès son introduction, la théorie des jeux à champ moyen a permis de modéliser certains marchés [70]. Depuis, le nombre d'applications des jeux à champ moyen dans le domaine économique n'a cessé de croître.

Les articles [36, 76, 26, 25, 20, 21] étudient le modèle de formation de prix initialement proposé dans la section 3 de [70]. Ce modèle met en relation deux groupes : les acheteurs et les vendeurs d'un même bien. Les agents interagissent par rapport au prix et changent de groupe une fois la transaction réalisée. Les acheteurs (resp. vendeurs) sont prêts à acheter (resp. vendre) le bien à des prix différents. De ce fait, le modèle permet de déterminer la distribution des acheteurs (resp. vendeurs) en fonction du prix

d'achat (resp. de vente) qu'ils pratiquent. Le prix d'équilibre est déterminé lorsque la loi de l'offre et de la demande est satisfaite. Les agents effectuent la transaction dès que le prix atteint le niveau souhaité. En ce sens, ils sont rationnels, mais le modèle n'explique pas en quoi ce niveau de prix est intéressant pour eux. De plus, les agents ne font pas d'anticipation rationnelle ; c'est pourquoi la structure mathématique de ce modèle diffère de celle présentée dans le paragraphe 1.1.2 ci-dessus.

Un autre modèle de formation de prix a été étudié dans [51, 52, 49, 10]. Il a été introduit pour modéliser la formation de prix sur le marché de l'électricité. Les agents font des anticipations rationnelles et ont la capacité de stocker l'énergie achetée sur le marché. Le modèle permet donc de comprendre comment le prix évolue et comment les agents choisissent de stocker l'énergie au cours du temps. Il se résume en trois équations : une équation de Hamilton-Jacobi-Belmann qui modélise des anticipations rationnelles des agents, une équation de Fokker-Planck permettant de comprendre l'évolution de la distribution des états, et une équation traduisant l'égalité de l'offre et la demande sur le marché de l'électricité.

D'autres travaux appliqués à l'économie admettent une structure similaire. Le modèle introduit par Guéant, Lasry et Lions [57] permet de déterminer le prix d'une ressource non renouvelable. Toujours dans le domaine de l'énergie, Chan et Sicar [35], et Graber et Mouzouni [55] ont aussi étudié ce type de système. Graber et Bensoussan, puis Graber et Mouzouni [54, 53] obtiennent aussi un système du même type en reformulant les modèles de Bertrand et Cournot dans le cadre de la théorie des jeux à champ moyen. Dans [50], Gomes développe d'autres applications économiques comme des modèles de croissance. Cette problématique a intéressé plusieurs mathématiciens et économistes. Par exemple, Lucas et Moll [73] développent un modèle permettant de comprendre comment évolue la productivité des individus dans une économie lorsque ceux-ci maximisent leur capacité de production actualisée au cours de leur vie. Les individus divisent leur temps entre la production et l'apprentissage qui leur permet de gagner en productivité. Gomes, Lafleche et Nurbekyan [50], et Achdou, Han, Lasry, Lions et Moll [3] développent des modèles de croissance permettant de comprendre la répartition des richesses dans une économie. Dans [3] les auteurs mettent en œuvre un algorithme permettant de simuler numériquement l'équilibre, en s'appuyant sur des travaux réalisés par Achdou et ses co-auteurs [2, 1, 6, 5, 4]. Dans le prochain paragraphe, nous développerons le modèle introduit dans [3]. Ce modèle aboutit à un système de jeux à champ moyen à trois équations. Sa description permettra d'introduire certaines idées et résultats qui seront importants pour le modèle développé dans la première partie de ce manuscrit.

### Le modèle de Aiyagari-Bewley-Huggett vu comme un jeu à champ moyen

Nous présentons le modèle dans un cadre stationnaire. Nous supposons qu'il y a un continuum d'agents hétérogènes dans leur niveau de richesse ; la richesse ( $a_t$ ) d'un individu est solution de l'équation différentielle ordinaire

$$\frac{da_t}{dt} = y_t + ra_t - c_t, \quad (1.5)$$

où  $r$  représente le taux d'intérêt,  $y_t$  le revenu (la production) de l'agent, et  $c_t$  sa consommation. Le revenu évolue de manière stochastique au cours du temps. Nous supposons qu'il suit une loi de Poisson à deux états  $y_t \in \{y_1, y_2\}$  avec  $0 \leq y_1 < y_2$ . Le processus saute de l'état 1 à l'état 2 avec l'intensité  $\lambda_1$ , et dans le sens inverse avec l'intensité  $\lambda_2$ . L'individu maximise l'utilité

$$\mathbb{E} \int_0^{+\infty} e^{-\rho t} U(c_t) dt, \quad (1.6)$$

où  $U(\cdot)$  est la fonction d'utilité instantannée et  $\rho > 0$  un taux d'actualisation. Enfin, les individus font face à une contrainte d'endettement maximal, qui est une contrainte sur l'état :  $a_t \geq \underline{a}$ , avec  $\underline{a} \in (-\infty, 0)$ . Lorsque le taux d'intérêt  $r$  est fixé, le problème de contrôle optimal (1.6) se résume à deux équations de Hamilton-Jacobi couplées (car  $y_t$  admet deux valeurs). Puis, il est possible de déduire la distribution des capitaux en résolvant les équations de Fokker-Planck : on détermine la distribution du capital pour les agents dont le revenu est  $y_1$ , notée  $m_1$ , et celle pour les agents dont le revenu est  $y_2$ , notée  $m_2$ .

Le modèle peut être fermé de plusieurs manières, voir [3] pour plus de détails. La plus simple est de

considérer que la richesse agrégée est constante et égale à  $B \in [0, +\infty)$ . Cette relation d'équilibre s'écrit

$$\int_{\underline{a}}^{+\infty} adm_1(a) + \int_{\underline{a}}^{+\infty} adm_2(a) = B,$$

De ce fait, le système de jeu à champ moyen est :

$$\rho u_j(a) = H(u'_j(a)) + u'_j(a)(y_j + ra) + \lambda_j(u_{-j}(a) - u_j(a)) \quad (1.7)$$

$$\frac{d}{da} ((y_j + ra + H'(u'_j(a))m_j(a)) = \lambda_j m_j(a) - \lambda_{-j} m_{-j}(a) \quad (1.8)$$

$$\int_{\underline{a}}^{+\infty} adm_1(a) + \int_{\underline{a}}^{+\infty} adm_2(a) = B, \quad (1.9)$$

pour  $j = 1, 2$ , avec la convention que  $-j = 2$  si  $j = 1$  et vice versa. Dans [3], les auteurs proposent une méthode numérique pour simuler ce système. Il établissent aussi certains résultats théoriques, en particulier le fait que la distribution des états des agents dont le revenu est  $y_1$  comporte une masse de Dirac localisée au niveau de richesse correspondant à la limite d'endettement.

Soit  $s_j(\cdot)$  la stratégie optimale d'épargne des agents dans l'état  $j$  : la dynamique du capital (1.5) peut se réécrire

$$\frac{da_t}{dt}(t) = \mathbf{1}_{\{1\}}(j)s_1(a_t) + \mathbf{1}_{\{2\}}(j)s_2(a_t),$$

où  $\mathbf{1}_E(\cdot)$  représente la fonction indicatrice d'un ensemble  $E$ . Si pour un niveau de richesse  $a^*$ ,  $s_j(a^*) = 0$  et si  $a^*$  est atteint en temps fini, alors la distribution des richesses  $m_j$  admet une masse de Dirac en  $a^*$ . Si au contraire,  $s_j(a^*) = 0$  et si ce niveau de richesse n'est pas atteint en temps fini, alors la distribution n'admet pas de masse de Dirac en  $a^*$ . En formalisant ces considérations heuristiques, les auteurs de [3] donnent des estimations sur les stratégies optimales d'épargne.

**Proposition 1.1** *Si  $r < \rho$  et si le coefficient d'aversion absolu au risque  $-U''(c)/U'(c)$  reste fini au voisinage de  $\underline{a}$ , alors*

- $s_1(a) < 0$  pour tout  $a > \underline{a}$
- il existe une constante  $C > 0$  telle que  $s_1(a) \sim C\sqrt{a - \underline{a}}$  quand  $a$  tend vers  $\underline{a}$ .

**Proposition 1.2** *Si  $r < \rho$  et si l'aversion relative au risque  $-cU''(c)/U'(c)$  est majorée uniformément par rapport à  $c$ , alors*

- il existe  $a_{\max} < +\infty$  tel que  $s_2(a) < 0$  pour  $a > a_{\max}$  et  $s_2(a) > 0$  pour  $a < a_{\max}$
- il existe une constante  $C' > 0$  telles que  $s_2(a) \sim C'(a_{\max} - a)$  quand  $a$  tend vers  $a_{\max}$ .

Le comportement en racine carrée de  $s_1(\cdot)$  au voisinage de  $\underline{a}$  permet de déduire que la richesse des agents dans l'état 1 stationne à  $\underline{a}$  en temps fini. Ce résultat provient du fait que la fonction  $(\underline{a}, +\infty) \ni a \mapsto 1/s_1(a)$  est intégrable au voisinage de  $\underline{a}$ . En revanche, l'estimation sur la stratégie d'épargne des individus dans l'état 2 montre que  $(\underline{a}, +\infty) \ni a \mapsto 1/s_2(a)$  n'est pas intégrable au voisinage de  $a_{\max}$ . Ainsi, les agents dans l'état 2 mettent un temps infini à atteindre  $a_{\max}$  et  $m_2$  n'admet pas de masse de Dirac en ce point. Les estimations ci-dessus sont des étapes permettant d'obtenir des résultats d'existence et d'unicité pour les équilibres, voir [3]. L'existence est basée sur le théorème des valeurs intermédiaires. En effet, les auteurs de [3] vérifient que la fonction

$$S : (-\infty, \rho) \ni r \mapsto \int_{\underline{a}}^{+\infty} adm_1(a) + \int_{\underline{a}}^{+\infty} adm_2(a)$$

est continue et que

$$\lim_{r \rightarrow -\infty} S(r) = \underline{a} \quad \text{et} \quad \lim_{r \rightarrow \rho^-} S(r) = +\infty.$$



L'unicité est basée sur le fait que  $S(\cdot)$  est strictement croissante, ce qui assure qu'il y a au plus une solution à l'équation  $S(r) = B$ .

### 1.1.3 Un modèle pour le marché locatif de l'immobilier professionnel

Dans la première partie de ce manuscrit, nous proposons un modèle du même type : les agents sont des entreprises cherchant à maximiser un critère d'utilité comme dans (1.6). Nous prouverons l'existence d'équilibres (et dans des cas particuliers leur unicité). En particulier, nous justifierons la continuité de l'application qui joue le rôle de  $S(\cdot)$  ci-dessus. Il faut cependant noter plusieurs différences avec le modèle de [3]. La première est que les solutions du problème de contrôle optimal interviendront dans l'équation analogue à (1.9). Ainsi, l'argument de monotonie permettant d'établir l'unicité ne fonctionnera plus. Deuxièmement, puisque nous voulons modéliser les interactions entre différents marchés, l'équation analogue à (1.9) sera multivariée. Alors, nous ne pourrons pas utiliser la stratégie basée sur le théorème des valeurs intermédiaires. Enfin, nous étudierons le cas où la fonction d'utilité instantanée  $U(\cdot)$  peut exploser en  $\underline{a}$ , par exemple avec un comportement logarithmique.

## 1.2 Le marché de l'immobilier locatif pour les particuliers

Dans la deuxième partie de ce mémoire, nous nous intéressons aux liens entre les marchés du travail et de l'immobilier locatif pour les particuliers. On cherchera à comprendre la répartition de l'habitat locatif en fonction de la position des pôles d'activité et d'emploi, en tenant compte des coûts de déplacement habitat-lieu de travail. La surface habitée louée sera considérée comme une source d'utilité pour les individus. La question posée sera : peut-on déterminer les salaires et les loyers, ainsi que la distribution des résidences des individus ? D'autre part, est-il possible de comprendre comment certaines modifications de l'économie peuvent influencer ces prix et cette distribution ?

Comprendre la structure spatiale du territoire a intéressé depuis longtemps les économistes. Le premier modèle documenté est proposé par von Thünen en 1826 dans [101], et est consacré au fermage en agriculture. Il permet de déterminer les loyers que les agriculteurs versent aux propriétaires terriens, ainsi que d'expliquer le zonage géographique associé à la production d'un produit agricole donné. Dans ce modèle, von Thünen fait l'hypothèse que les agriculteurs sont rationnels et choisissent le produit qu'ils cultiveront parmi plusieurs possibles, en maximisant leur profit avant la soustraction des loyers. Les produits cultivables se différencient par les coûts de transport qu'ils occasionnent et par leur prix de vente qui sont fixes et exogènes au modèle. Le choix rationnel des agriculteurs est à l'origine du zonage géographique. A l'équilibre, il est supposé que le profit des agriculteurs est constant et nul. Il est alors possible de déduire les loyers. Ce modèle a permis de comprendre plus précisément le fonctionnement du marché du fermage à une époque où la société était majoritairement rurale. De ce fait, la terre y est considérée comme un facteur de production et non comme une source d'utilité. Le modèle de von Thünen ne s'applique pas à la distribution des résidences des agriculteurs.

Burgess [22] et Hoyt [59] ont par la suite proposé des modèles toujours basés sur l'idée de zonage mais appliqués aux villes modernes. En 1945, Harris et Ullman introduisent dans [58] le modèle des noyaux multiples, après avoir remarqué que les modèles précédents faisaient l'hypothèse d'une ville monocentrique, c'est à dire, doté d'un unique quartier d'affaires situé au centre de la ville. Harris et Ullman suppriment cette hypothèse et déduisent un zonage plus complexe et plus proche de la réalité.

Dans les années 1960, Alonso, Mills and Muth introduisent dans [8, 80, 99] un modèle permettant de déterminer la distribution des résidences et les loyers. Ils supposent que la ville est monocentrique : les agents travaillent tous dans le quartier d'affaires situé au centre de la ville. Les travailleurs ont un coût de transport dépendant de la distance de leur résidence au quartier d'affaires. Alonso, Mills et Muth montrent que les loyers sont hauts et que la densité de population est forte près du quartier d'affaires. A l'inverse, loin du quartier d'affaires, les loyers sont bas et la densité faible. Ce modèle permet donc de déterminer les loyers ainsi que la distribution des résidences. Il est découplé du marché du travail, puisque

le salaire est une constante exogène.

Rosen et Roback introduisent dans les années 1980 [95, 94] un modèle permettant de lier le marché du travail à celui de l'immobilier. Ils distinguent deux types d'agents : les firmes et les travailleurs. Ils introduisent aussi une variable  $s$  mesurant l'activité économique et culturelle à un endroit donné. À l'équilibre, le loyer et le salaire à un endroit donné sont déterminés pour que l'utilité des travailleurs soit égale à un niveau exogène donné, et pour que le coût marginal de production des firmes soit égal à un prix exogène fixé. L'hypothèse que les agents travaillent à l'endroit où ils vivent permet d'éviter de tenir compte des coûts de transport. De ce fait, les hétérogénéités spatiales du modèle ne viennent que du niveau d'activité économique et culturel qui est exogène. En d'autre terme, la variable spatiale n'intervient que via le paramètre  $s$ .

Après cet article, plusieurs travaux dont ceux de Fujita et Ogawa [45], Krugman [66], Fujita et Krugman [43], Fujita, Krugman et Venables [44], et Lucas et Rossi-Hansberg [75], ont porté sur la formation des villes, sans supposer que la localisation des firmes ou des habitations soit connue a priori. Dans ces modèles, la formation des agglomérations résulte de plusieurs effets antagonistes. Par exemple dans [45] et [75], la concentration des firmes au même endroit améliore leur productivité. De la même manière, les coûts de transport favorisent la concentration des habitats près des zones de production. En revanche, la compétition sur le marché de l'immobilier, la loi de l'offre et de la demande sur le marché locatif des particuliers, vont à l'encontre de cette concentration. Ces modèles ne supposent plus l'existence d'un quartier d'affaires situé au centre de la ville. Cependant, ils nécessitent des hypothèses simplificatrices ; en particulier, la zone géographique considérée est souvent supposée unidimensionnelle, c'est à dire une ligne ou un segment.

Le modèle que nous proposerons dans la deuxième partie de ce mémoire se base sur des notions de transport optimal et de jeux non-atomiques. Dans les deux paragraphes qui suivent, nous présentons une introduction rapide et formelle de ces théories.

### 1.2.1 La théorie du transport optimal

Le premier problème de transport optimal a été formulé par Monge en 1781 dans un mémoire soumis à l'Académie des sciences sur la théorie des déblais et remblais. Il étudie comment déplacer un tas de sable d'un lieu à un autre de manière optimale. Il a fallu attendre les années 1940 pour que Kantorovitch relaxe le problème et puisse le résoudre. Depuis, le transport optimal permet de répondre à des problèmes économiques, voir [47, 97] pour une introduction détaillée.

Concernant l'économie urbaine, plusieurs mathématiciens ont construit des modèles se basant sur la théorie du transport optimal. Par exemple, les travaux de Carlier et Ekeland [30, 31] permettent de généraliser les idées de Lucas et Rossi-Hansberg [75]. En particulier, ces auteurs peuvent s'affranchir de l'hypothèse, centrale dans [75], d'une ville circulaire, et parviennent à montrer l'existence d'équilibres en utilisant des techniques de transport optimal. Buttazzo et Santambrogio proposent dans [23] un modèle permettant à un planificateur d'optimiser l'espace urbain. Même si ce modèle n'explique pas la structure d'une ville déjà existante, il peut permettre de planifier la construction d'une ville future, d'un campus universitaire, ou d'un village touristique par exemple. Plus récemment, Barilla, Carlier et Lasry [16] présentent un modèle liant le transport optimal et les jeux à champ moyen, qui vise à décrire l'évolution d'une ville au cours du temps. C'est un jeu à champ moyen à deux populations : les firmes et les travailleurs, avec un couplage par un problème de transport optimal.

#### Présentation formelle de quelques notions de transport optimal

Soient  $X$  et  $Y$  deux espaces métriques compacts,  $\mu \in \mathcal{P}(X)$  et  $\nu \in \mathcal{P}(Y)$  deux mesures de probabilité, et  $c : X \times Y \rightarrow [0, +\infty]$  une fonction de coût borélienne. Le problème de transport optimal qui a été initialement formulé par Monge dans [81], se présente de la manière suivante : on cherche une application

$T : X \rightarrow Y$  qui réalise l'infimum

$$\inf \left\{ \int_X c(x, T(x)) d\mu(x) : T_{\#}\mu = \nu \right\}, \quad (1.10)$$

où  $T_{\#}\mu$  est la mesure image de  $\mu$  par l'application  $T : X \rightarrow Y$ , c'est à dire que pour tout ensemble borélien  $A \subset Y$ ,  $T_{\#}\mu(A) = \mu(T^{-1}(A))$ . Si le problème de Monge admet une solution, alors elle permet de transporter  $\mu$  sur  $\nu$  de manière optimale notamment lorsque les coûts de transport sont donnés par  $c(\cdot, \cdot)$ .

Le problème de Monge peut être mal posé si l'ensemble des applications  $T : X \rightarrow Y$  telles que  $T_{\#}\mu = \nu$  est vide. Ce n'est que dans les années 1940 que Kantorovitch propose et résout le problème relaxé suivant :

$$\inf \left\{ \int_X c(x, y) d\gamma(x, y) : \gamma \in \Pi(\mu, \nu) \right\}, \quad (1.11)$$

où  $\Pi(\mu, \nu)$  est l'ensemble des plans de transport, i.e.

$$\Pi(\mu, \nu) = \{ \gamma \in \mathcal{P}(X \times Y) : (\pi_x)_{\#}\gamma = \mu \text{ et } (\pi_y)_{\#}\gamma = \nu \},$$

où  $\pi_x(\cdot)$  et  $\pi_y(\cdot)$  sont les projections de  $X \times Y$  sur  $X$  et  $Y$  respectivement. C'est une relaxation de (1.10) dans le sens suivant : si (1.10) admet une solution  $T(\cdot)$ , alors toute particule en  $x$  est transportée en  $T(x)$ . En revanche, si (1.11) admet une solution  $\gamma$ , alors  $\gamma(x, y)$  représente la masse de particules en  $x$  transportée en  $y$ . Nous allons maintenant voir des applications économiques du transport optimal.

Supposons que  $X$  est la fermeture d'un domaine borné de  $\mathbb{R}^2$ , qui représente une ville, et  $Y = \{y_1, \dots, y_N\}$  est un ensemble discret contenant les positions de toutes les fontaines de la ville. Soient  $\mu \in \mathcal{P}(X)$  une mesure de probabilité sur  $X$  représentant la distribution des résidences des habitants, et  $\nu \in \mathcal{P}(Y)$  qui représente la capacité des fontaines. Ceci veut dire que si on identifie  $\nu$  à un vecteur de

$$\left\{ \nu \in [0, +\infty)^N : \sum_{i=1}^N \nu_i = 1 \right\},$$

la  $i^{eme}$  fontaine peut être utilisée par au maximum  $\nu_i$  habitants. En introduisant les coûts de transport  $c(\cdot, \cdot)$  des habitants pour atteindre les fontaines, le problème de transport optimal (1.11) permet de déterminer les usagers de chaque fontaine. C'est un problème de transport optimal semi-discret.

Intéressons nous au problème dual de (1.11) qui s'écrit formellement

$$\sup_{\phi(\cdot) \in C(X), \psi(\cdot) \in C(Y)} \left\{ \int_X \phi(x) d\mu(x) + \int_Y \psi(y) d\nu(y) : c(x, y) \geq \phi(x) + \psi(y) \forall (x, y) \in X \times Y \right\}. \quad (1.12)$$

Puisque  $Y$  est un ensemble discret, nous pouvons identifier  $C(Y)$  à  $\mathbb{R}^N$ . En introduisant pour tout vecteur  $\psi \in \mathbb{R}^N$  la transformation

$$\psi^c(x) = \min_{i=1}^N \{c(x, y_i) - \psi_i\}, \quad \forall x \in X,$$

nous pouvons montrer sous certaines conditions que le problème (1.12) admet un minimiseur de la forme  $(\psi^c(\cdot), \psi)$ . De ce fait, on peut étudier le problème réduit

$$\sup_{\psi \in \mathbb{R}^d} \left\{ \int_X \psi^c(x) d\mu(x) + \sum_{i=1}^N \psi_i \nu_i \right\}. \quad (1.13)$$

Il est alors intéressant de voir  $\psi$  comme une collection de prix, et de remarquer que les conditions d'optimalité du problème (1.13) correspondent à la loi de l'offre et de la demande sur le marché de l'eau.

En effet, si  $\mu$  est une mesure de probabilité admettant une densité régulière, les conditions d'optimalité de (1.13) s'écrivent

$$-\int_X \frac{\partial \psi^c}{\partial \psi_i}(x) \mu(x) dx = \nu_i, \quad \forall i \in \{1, \dots, N\}. \quad (1.14)$$

Dans (1.14), le terme de droite traduit l'offre en eau, tandis que le terme de gauche correspond à la demande. En effet, un habitant vivant en  $x$  et choisissant la fontaine  $i$  aura les coûts de transport  $c(x, y_i)$  et devra payer  $\psi_i$  pour la quantité d'eau qui lui sera attribuée, soit un coût total de  $c(x, y_i) - \psi_i$ . Étant rationnels, les agents choisissent de minimiser leurs coûts. En définissant les cellules de Laguerre généralisées

$$V_i(\psi) = \left\{ x \in X : c(x, y_i) - \psi_i = \min_{j=1}^N (c(x, y_j) - \psi_j) \right\} \quad \forall i \in \{1, \dots, N\},$$

qui correspondent pour chaque  $i$  à l'ensemble des positions tel que la  $i^{\text{eme}}$  fontaine minimise le coût global, nous obtenons formellement

$$\frac{\partial \psi^c}{\partial \psi_i}(x) = \mathbf{1}_{V_i(\psi)}(x), \quad \forall i \in \{1, \dots, N\},$$

pour presque tout  $x$  par rapport à la mesure de Lebesgue. Ainsi, (1.14) peut se réécrire comme

$$\int_{V_i(\psi)} \mu(x) dx = \nu_i, \quad \forall i \in \{1, \dots, N\}$$

et  $\psi$  est la collection de prix permettant de définir les zones délimitant l'usage des fontaines de telle sorte que l'offre et la demande coïncident. Pour faciliter l'analyse mathématique, il est possible d'introduire un coût régularisé dans le but de manipuler un minimum régularisé plutôt qu'un minimum dans (1.13) ; ceci permet en particulier de différencier  $\psi^c(\cdot)$  par rapport à  $\psi$  facilement. Pour plus de détails nous faisons références aux ouvrages de Santambrogio [97], Peyré [90] et Galichon [47].

Dans la deuxième partie de ce manuscrit nous obtiendrons plusieurs conditions d'équilibre dont l'une traduit l'équilibre sur le marché du travail. L'offre et la demande doivent coïncider et nous montrerons que ces équations sont les conditions d'optimalité d'un problème d'optimisation du même type que (1.13).

## 1.2.2 Les jeux non-atomiques

Les jeux statiques non-atomiques ont été introduits par Aumann en 1964 dans [12]. Ils permettent d'étudier les équilibres de Nash d'un jeu statique ayant une infinité de joueurs. Nous introduisons dans ce paragraphe la définition d'un équilibre dans un tel jeu.

Supposons qu'il y ait une infinité de joueurs indistinguables, rationnels et qui n'ont individuellement aucun impact sur le système global. Soient  $X$  la fermeture d'un domaine borné de  $\mathbb{R}^d$  où  $d \geq 1$  est un entier naturel fixé représentant l'ensemble des stratégies des joueurs, et  $F : X \times \mathcal{P}(X) \rightarrow [-\infty, +\infty]$  une fonction d'utilité. Chaque joueur choisit une stratégie dans  $X$  dans le but de résoudre

$$\sup_{x \in X} F(x, \mu),$$

où  $\mu \in \mathcal{P}(X)$  représente la distribution des stratégies jouées par les autres joueurs. Un équilibre de Nash pour ce jeu sera décrit par une mesure  $\mu \in \mathcal{P}(X)$  satisfaisant l'équation

$$\int_X F(x, \mu) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X F(x, \nu) d\nu(x). \quad (1.15)$$

En effet, en considérant que  $F(\cdot, \mu)$  est continue alors (1.15) est équivalent à

$$\text{supp } \mu \subset \text{argmax } F(\cdot, \mu),$$

où  $\text{supp } \mu$  correspond au support de  $\mu$  qui est défini comme l'intersection de tous les fermés  $\mathcal{F} \subset X$  tel que  $\mu(X) = \mu(\mathcal{F})$ , autrement dit, pour tout  $x$  dans le support de  $\mu$ ,

$$x \in \operatorname{argmax}_{y \in X} F(y, \mu).$$

Cette définition généralise la définition d'un équilibre de Nash dans le sens où aucun joueur n'a intérêt à changer sa stratégie lorsque l'état du système est donné par  $\mu$  ; à l'équilibre, la stratégie de chaque joueur est la meilleure réponse lorsque le système est dans l'état  $\mu$ .

Dans la deuxième partie de ce manuscrit, nous verrons plusieurs conditions d'équilibre. Deux d'entre elles, à savoir les conditions d'équilibre sur le marché de l'immobilier et une condition de mobilité des agents, pourront s'écrire comme un équilibre de Nash pour un jeu statique non-atomique comme dans (1.15).

### 1.3 Organisation du mémoire

Ce mémoire est consacré à l'élaboration et à l'étude de modèles mathématiques liés à la théorie des jeux à champ moyen et au transport optimal. Il est divisé en deux parties indépendantes pour la majorité de leur contenu. La première partie sera consacrée à modéliser les marchés des facteurs de production des entreprises, et à étudier théoriquement l'existence d'équilibres. Le modèle sera utilisé pour comprendre les interactions entre le marché du travail et de l'immobilier locatif professionnel. Cette première partie est composée de six chapitres consacrés à la définition du modèle, son étude théorique et à des simulations numériques des équilibres. La deuxième partie porte sur un modèle couplant les marchés du travail et de l'immobilier locatif des particuliers, et dont un des paramètres de sortie est la distribution spatiale des résidences des travailleurs. Cette partie est composée de trois chapitres, consacrés respectivement à la définition du modèle, à l'étude théorique de ce dernier, et à des simulations numériques.

Dans le deuxième chapitre, nous proposons un jeu à champ moyen modélisant le marché des facteurs de production des entreprises. Nous considérons le problème de contrôle optimal auquel les firmes font face. Nous présentons ensuite l'équation de continuité de type Fokker-Planck permettant de déterminer la distribution du capital des entreprises. L'équilibre sur les différents marchés vient alors de la loi de l'offre et de la demande.

Une fois le problème mathématique introduit, nous commençons par exhiber des solutions explicites dans des cas simples, retrouvant ainsi certaines lois économiques. Dans un cadre plus général, nous proposons ensuite un premier résultat d'existence d'équilibres. La plus grande difficulté consiste à résoudre le problème de contrôle optimal qui se pose aux firmes. Nous utilisons dans ce chapitre le principe de programmation dynamique, qui consiste à étudier une équation de type Hamilton-Jacobi pour laquelle le Hamiltonien présente une singularité. Nous prouvons l'existence et l'unicité de solutions de cette équation, retrouvant ainsi la règle d'or d'accumulation du capital : nous montrons qu'il existe un unique niveau de capital permettant de maximiser la consommation des entreprises. Puis, sous des hypothèses raisonnables sur l'offre de travail, nous prouvons l'existence d'équilibres en utilisant le théorème du point fixe de Brouwer.

Dans le troisième chapitre, nous revisitons le problème de contrôle optimal mentionné ci-dessus en utilisant cette fois une approche lagrangienne. Nous suivons la stratégie développée par F. Santambrogio dans [98] pour un problème assez voisin, qui consiste à relaxer le problème pour établir l'existence d'un contrôle optimal. En montrant que ce dernier est régulier, nous prouvons qu'il est solution du problème de départ (non relaxé).

Le quatrième chapitre est consacré à des exemples importants : nous supposons que la production des entreprises est donnée par des fonctions de production du type Cobb-Douglas ou CES (*Constant Elasticity of Substitution*). Nous déduisons en particulier certaines propriétés vérifiées par la fonction modélisant la demande de travail.

Dans le Chapitre 5, nous proposons des hypothèses plus générales sur la fonction modélisant l'offre de travail pour lesquelles l'existence d'équilibres reste vraie. Plusieurs exemples permettent de justifier ces hypothèses. Le résultat d'existence est obtenu grâce à une méthode de continuation (théorème du degré).

Le sixième chapitre est consacré à des extensions du modèle. On considère deux modifications concernant l'entrée des entreprises dans l'économie. La première fait dépendre la création d'entreprise de la fonction valeur associée au problème de contrôle optimal mentionné ci-dessus : ainsi, une entreprise est créée si l'utilité qu'elle procure est suffisante. Le deuxième changement fait dépendre l'entrée des entreprises dans l'économie de la distribution des capitaux. Dans ce cadre, on suppose que lorsqu'une entreprise meurt, elle donne naissance à un certain nombre d'entreprises plus petites (deux dans l'exemple considéré). Nous prouvons l'existence et l'unicité des solutions des équations de continuité associées. Ceci permet ensuite d'obtenir l'existence d'équilibres pour ces extensions, de la même manière qu'au deuxième chapitre.

Le septième chapitre est consacré à des simulations numériques. Nous décrivons la discrétisation et l'algorithme que nous utilisons pour approcher numériquement les équilibres du modèle. Puis, nous présentons plusieurs simulations permettant de comprendre comment certains paramètres de l'économie influencent l'équilibre calculé.

Le modèle décrit au huitième chapitre (le premier chapitre de la deuxième partie) couple les marchés du travail et de l'immobilier locatif des particuliers. Il permet de prévoir la distribution spatiale des habitants d'une ville, les salaires d'équilibre sur le marché du travail ainsi que les loyers d'équilibre sur le marché de l'immobilier. L'existence d'équilibres est prouvée. On donne aussi un résultat d'unicité pour certaines valeurs des paramètres du modèle. On propose ensuite une extension permettant de modéliser simplement l'introduction du télétravail dans l'économie. Des simulations numériques seront d'ailleurs présentées dans le dernier chapitre.

Le neuvième chapitre est consacré à une extension du modèle précédent au cas où il y a plusieurs types de travailleurs. Certains arguments de la preuve d'existence du chapitre huit ne peuvent pas être adaptés, et la preuve de l'existence d'équilibres nécessite des idées nouvelles.

Le dernier chapitre contient des résultats numériques sur la sensibilité de l'équilibre calculé aux variations de certains des paramètres du modèle.

## 1.4 Contributions

### 1.4.1 Une approche jeu à champ moyen pour modéliser les interactions entre différents marchés

Nous commençons par présenter une nouvelle approche de type jeu à champ moyen pour modéliser d'une manière générale un ou plusieurs marchés de facteurs de production. Pour simplifier la présentation, nous nous concentrons sur le marché du travail, avec éventuellement plusieurs types de travailleurs. Dans un cas simple, nous montrons alors qu'il est possible de trouver des équilibres admettant une forme explicite. Dans un cadre plus général, nous utilisons le principe de programmation dynamique pour aborder le problème de contrôle optimal intervenant dans la décision des entreprises : nous étudions donc l'équation de Hamilton-Jacobi associée. Nous présentons ensuite une approche lagrangienne, qui reprend celle développée par Santambrogio dans [98]. Puis, nous prouvons plusieurs résultats d'existence d'équilibres dans ce cadre général. Enfin, nous présentons quelques extensions possibles du modèle et effectuons des simulations numériques.

**Le modèle.** Nous supposons qu'il y a un continuum de firmes hétérogènes dans leur capital. Nous faisons l'hypothèse que le marché de travail est parfaitement compétitif, c'est à dire qu'individuellement, les entreprises n'ont pas d'impact sur les salaires. La dynamique du capital d'une firme est donnée par

$$\frac{dk}{dt}(t) = F(k(t), \ell(t)) - w \cdot \ell(t) - \delta k(t) - c(t), \quad (1.16)$$

où  $F : [0, +\infty)^{d+1} \rightarrow [0, +\infty)$  est la fonction de production,  $d \in \mathbb{N}^*$  est le nombre de types de travailleurs,  $\delta \geq 0$  le taux de dépréciation du capital,  $\ell(t) \in \mathbb{R}^d$  et  $c(t) \geq 0$  sont respectivement le niveau d'emploi et

la consommation à l'instant  $t$ . La stratégie d'une entreprise est déterminée en résolvant le problème de contrôle optimal suivant :

$$u(\kappa) = \sup_{c(t), \ell(t)} \int_0^{+\infty} U(c(t))e^{-\rho t} dt$$

sous les contraintes

$$\begin{cases} c(\cdot), \ell(\cdot) \in L_{\text{loc}}^1(0, +\infty), k(\cdot) \in W_{\text{loc}}^{1,1}(0, +\infty); \\ c(t) \geq 0, \ell(t) \geq 0, \text{ pour } t > 0; \\ k(\cdot) \text{ est une solution positive de (1.16) avec pour condition initiale } k(0) = \kappa. \end{cases} \quad (1.17)$$

La fonction  $u(\cdot)$  est la fonction valeur du problème. Au moins formellement, elle est la solution de l'équation de Hamilton-Jacobi :

$$\rho u(k) = H(k, u'(k)), \quad \forall k \in (0, +\infty), \quad (1.18)$$

où le Hamiltonien est donné par

$$H(k, q) = \sup_{c \geq 0, \ell \in [0, +\infty)^d} \{U(c) + q(F(k, \ell) - w \cdot \ell - \delta k - c)\}. \quad (1.19)$$

Au moins formellement, la stratégie d'accumulation du capital d'une entreprise dont le niveau de capital est  $k \in (0, +\infty)$  s'exprime alors par  $D_q H(k, u'(k))$ , où  $D_q H(\cdot, \cdot)$  est la dérivée de  $H(\cdot, \cdot)$  par rapport à sa deuxième variable.

Dans notre modèle, l'évolution de la distribution du capital est régie par l'équation de transport :

$$\frac{d}{dk} \left( D_q H(\cdot, u'(\cdot)) m(\cdot) \right) (k) = \eta(k) - \nu m(k), \quad \forall k \in (0, +\infty), \quad (1.20)$$

où  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  modélise l'entrée de nouvelles firmes dans le système et  $\nu > 0$  le taux d'extinction des entreprises. L'inconnue est la distribution du capital  $m$ ; nous normalisons le capital agrégé à 1, ce qui fait que  $m$  est une mesure de probabilité sur  $[0, +\infty)$ .

L'équilibre est atteint lorsque l'offre et la demande de travail coïncident. Nous modélisons l'offre par une fonction exogène  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$ . Au moins formellement, la demande individuelle d'une firme dont le niveau de capital est  $k \in (0, +\infty)$  est donnée par  $-D_w D_q H(k, u'(k))$ , où  $D_w$  est la Jacobienne associée à la différentiation par rapport à  $w$ . Ainsi, la demande agrégée est modélisée par

$$- \int_0^{+\infty} D_w D_q H(k, u'(k)) dm(k).$$

Dans ce contexte, la loi de l'offre et de la demande s'écrit

$$S(w) + \int_0^{+\infty} D_w D_q H(k, u'(k)) dm(k) = 0.$$

Finalement, l'équilibre sur le marché du travail est décrit par un triplet  $(u, m, w)$  solution du système suivant :

$$\rho u(k) = H\left(k, \frac{\partial u}{\partial k}(k)\right), \quad (1.21)$$

$$\frac{d}{dk} \left( D_q H\left(\cdot, \frac{\partial u}{\partial k}(\cdot)\right) m(\cdot) \right) (k) = \eta(k) - \nu m(k), \quad (1.22)$$

$$1 = \int_0^{+\infty} dm(k), \quad (1.23)$$

$$S(w) = - \int_0^{+\infty} D_w D_q H(k, u'(k)) dm(k). \quad (1.24)$$

C'est un jeu à champ moyen de contrôle puisque l'interaction entre les firmes s'exprime au travers du niveau d'emploi. Ce système à trois équations ressemble par exemple au modèle de croissance étudié par Achdou, Han, Lasry, Lions et Moll dans [3]. Cependant, l'équation d'équilibre (1.24) est nouvelle. En effet, la dynamique (1.16) n'est pas linéaire par rapport au contrôle  $\ell(t)$  et le feedback pour le niveau d'emploi optimal s'écrit  $-D_w D_q H(\cdot, u'(\cdot))$ . Cette particularité empêche notamment d'appliquer l'argument de monotonie décrit par exemple dans [51] pour montrer l'unicité d'un équilibre.

**Résolution dans le cas homogène de degré 1 et pour  $d = 1$ .** Lorsque la fonction d'utilité  $U(\cdot)$  est un logarithme et que la fonction de production est donnée par

$$F(k, \ell) = Ak^\alpha \ell^{1-\alpha}, \quad \forall (k, \ell) \in [0, +\infty)^2,$$

il est possible de montrer que (1.21)-(1.24) admet une unique solution ayant une forme presque explicite qui dépendra du choix de  $\eta(\cdot)$  et de  $S(\cdot)$ . Dans ce cadre nous obtenons :

**Théorème 1.1** *Sous les hypothèses supplémentaires que  $\int_0^\infty \kappa \eta(\kappa) d\kappa < +\infty$ , et que  $S(\cdot)$  est continue, croissante, non identiquement nulle et telle que  $S(0) = 0$ , il existe une unique solution à (1.21)-(1.24).*

Les formules explicites nous permettent de retrouver certaines lois économiques. La première est la loi de Pareto introduite dans [82]. Une étonnante régularité a été remarquée dans la distribution des richesses, de la taille des entreprises, ou celle des villes : les queues de ces distributions sont bien approchées par l'inverse d'une fonction puissance, voir [14] pour le cas des firmes. Ce sont des distributions à queues lourdes puisque la queue de distribution ne peut pas être bornée par une fonction exponentielle strictement décroissante.

La deuxième loi retrouvée est la loi de Gibrat, voir [48]. Elle affirme que le taux de croissance d'une firme est indépendant de sa taille. C'est ce que nous retrouvons en montrant que la stratégie d'accumulation du capital d'une firme  $\Sigma(\cdot)$  est linéaire par rapport à son niveau de capital.

**Problème de contrôle optimal dans le cas général : approche programmation dynamique de Bellman.** Dans le cadre général (les fonctions  $U(\cdot)$  et  $F(\cdot)$  ne sont plus explicitées), plusieurs difficultés surgissent. Premièrement, comme le montrent les solutions explicites, si la production est à rendement d'échelle constant, alors la distribution du capital des entreprises est à queue lourde. Ceci fait craindre un défaut de compacité : par exemple, l'ensemble  $\mathcal{P}([0, +\infty))$  muni de la topologie étroite n'est pas compact. Nous faisons donc l'hypothèse que la production des entreprises est à rendement d'échelle strictement décroissant. Dans ce cadre, et en fixant une collection de salaires  $w \in (0, +\infty)^d$ ,  $d \geq 1$ , nous étudions l'équation de Hamilton-Jacobi (1.21). Les résultats classiques de la théorie des solutions de viscosité ne s'appliquent pas directement. En effet, le Hamiltonien admet une singularité en  $k = 0$  lorsque

$$\sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} \rightarrow 0,$$

quand  $k$  tend vers 0, et que  $U(c)$  tend vers  $-\infty$  quand  $c$  tend vers 0. Ces comportements sont pourtant classiques, puisqu'on les rencontre lorsque  $F(\cdot)$  est une fonction de Cobb-Douglas et que  $U(\cdot)$  est un logarithme. Ainsi, pour prouver l'existence d'une solution classique de l'équation de Hamilton-Jacobi, nous exploitons des propriétés de monotonie du Hamiltonien par rapport à sa deuxième variable. Ces propriétés nous permettent d'inverser localement le Hamiltonien et de résoudre (1.21) comme une équation différentielle ordinaire par une méthode de tir. Nous aboutissons à l'existence d'une solution classique ; nous pouvons ensuite appliquer un principe de comparaison des solutions de viscosité pour des problèmes avec contraintes d'état et obtenons le résultat suivant :

**Theorem 1.1** *Sous des hypothèses sur  $F(\cdot)$  et  $U(\cdot)$  qui seront précisées dans le Chapitre 2, il existe une unique solution  $u(\cdot)$  de classe  $C^1(0, +\infty)$  de (1.21). De plus, il existe une valeur  $k^* \in (0, +\infty)$  telle que :*

$$D_q H(k, u'(k)) > 0, \quad \text{pour } 0 < k < k^*, \quad (1.25)$$

$$D_q H(k, u'(k)) < 0, \quad \text{pour } k > k^*, \quad (1.26)$$

$$D_q H(k, u'(k)) = 0, \quad \text{pour } k = k^*. \quad (1.27)$$



Le niveau de capital optimal  $k^*$  est déterminé par la règle d'or d'accumulation du capital. Cette règle a été introduite par Allais dans [7] et démontrée pour la première fois par Phelps dans [91]. Elle donne la stratégie d'accumulation du capital qui permet de maximiser à très long terme la consommation ; autrement dit, cette règle permet de déterminer le niveau de capital optimal pour maximiser la consommation des entreprises.

Pour des raisons économiques, il est important de connaître le comportement des agents dont le capital est proche de  $k^*$ . Dans le modèle de Aiyagari-Bewley-Huggett étudié par Achdou, Han, Lasry, Lions et Moll dans [3], les auteurs présentent des estimations que nous avons rappelées dans les Propositions 1.1 et 1.2. Dans notre cadre, nous montrons :

**Lemme 1.1** *Sous des hypothèses sur  $F(\cdot)$  et  $U(\cdot)$  que nous précisons dans le Chapitre 2, il existe  $\epsilon > 0$  et  $M > 0$  tels que*

$$0 \leq D_q H(k, u'(k)) \leq M(k^* - k), \quad \text{si } k \in [k^* - \epsilon, k^*], \quad (1.28)$$

$$M(k^* - k) \leq D_q H(k, u'(k)) \leq 0, \quad \text{si } k \in [k^*, k^* + \epsilon]. \quad (1.29)$$

Ce résultat assure que la distribution  $m$  n'admet pas de masse de Dirac en  $k^*$ .

**Problème de contrôle optimal dans le cas général : approche lagrangienne.** Dans le Chapitre 3, nous appliquons une méthode lagrangienne au problème de contrôle optimal auquel les entreprises font face. La stratégie adoptée suit celle proposée par Santambrogio [98]. Elle consiste à relaxer le problème (1.17) pour établir l'existence de solutions. Une fois l'existence obtenue, nous montrons que le maximiseur est régulier et est solution du problème initial.

En définissant

$$f(k) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} - \delta k, \quad \forall k \in [0, +\infty),$$

le problème relaxé que nous étudions est le suivant :

$$\sup_{(k, c) \in \mathcal{A}(\kappa)} \int_0^{+\infty} U(c^{ac}(t)) e^{-\rho t} dt, \quad (1.30)$$

où

$$\mathcal{A}(\kappa) = \{(k(\cdot), c) : k(\cdot) \in BV_{\text{loc}}(\mathbb{R}_+), c \in \mathcal{M}_+(\mathbb{R}_+), k(0) = \kappa, k(\cdot) \geq 0, c \geq 0, k' + c \leq f(k(\cdot))\}.$$

L'enjeu est de montrer des bornes a priori sur  $\mathcal{A}(\kappa)$  pour obtenir des propriétés de compacité. La différence majeure entre nos estimations et celles de [98] repose sur le fait que nous ne supposons pas que la fonction  $f(\cdot)$  admet des valeurs négatives pour  $k$  assez grand. Nous traitons en particulier le cas où le taux de dépréciation du capital est nul.

**Lemme 1.2 (Estimations a priori de  $\mathcal{A}(\kappa)$ )** *Sous des hypothèses sur  $f(\cdot)$  qui seront précisées dans le Chapitre 3, pour tout  $C_1 > 0$  il existe  $C_2, C_3$  et  $C_4$  tels que pour tout  $\kappa > 0$ , pour toute paire  $(k, c) \in \mathcal{A}(\kappa)$  et pour tout  $t \in [0, +\infty)$ ,*

$$\begin{cases} \sup_{[0, t]} |k| \leq \left(\kappa + \frac{C_2}{C_1}\right) e^{C_1 t} - \frac{C_2}{C_1}, \\ \|k'\|_{\mathcal{M}([0, t])} \leq (C_3 \kappa + C_4) e^{C_1 t}, \\ \|c\|_{\mathcal{M}_+([0, t])} \leq (C_3 \kappa + C_4) e^{C_1 t}. \end{cases} \quad (1.31)$$

En montrant que l'application  $\mathcal{A}(\kappa) \ni (k, c) \mapsto \int_0^{+\infty} U(c^{ac}(t)) e^{-\rho t} dt$  est semi-continue supérieurement pour une topologie bien choisie, nous parvenons à établir l'existence d'un maximiseur. En appliquant les résultats de [98], nous déduisons que ce maximiseur est unique et régulier. Nous obtenons alors un résultat analogue au Théorème 1.1 :

**Proposition 1.3** *Il existe une unique constante  $k^* \in (0, +\infty)$  telle que si  $(k(\cdot), c)$  est un maximiseur du problème (1.30), alors  $k(\cdot) \in C^2(\mathbb{R}_+)$  et  $c(\cdot) \in C^1(\mathbb{R}_+)$ . De plus,*

- 1) si  $0 < \kappa < k^*$ , alors pour tout  $t \geq 0$ ,  $f(k(t)) - c(t) \geq 0$
- 2) si  $\kappa > k^*$ , alors pour tout  $t \geq 0$ ,  $f(k(t)) - c(t) \leq 0$
- 3) si  $\kappa = k^*$ , alors pour tout  $t \geq 0$ ,  $f(k(t)) - c(t) = 0$ .

Il faut cependant remarquer que, par rapport à l'approche précédente, il a été nécessaire de faire plusieurs hypothèses supplémentaires sur  $f(\cdot)$  et  $U(\cdot)$ . En particulier, nous supposons que  $f(0) = 0$  et que  $U(\cdot)$  est positive. L'hypothèse supplémentaire sur  $f(\cdot)$  permet d'assurer que le niveau de capital d'une firme reste positif, tandis que celle sur  $U(\cdot)$  permet de montrer la semi-continuité du critère à optimiser.

**Quelques résultats d'existence d'équilibres.** Dans ce cadre général, nous avons étudié deux approches pour montrer l'existence d'équilibres. En faisant d'abord des hypothèses plus restrictives sur  $S(\cdot)$ , nous pouvons utiliser le théorème du point-fixe de Brouwer. Puis, dans le Chapitre 4, nous présentons des exemples fondamentaux, en particulier les cas où la fonction de production est donnée par une fonction de type Cobb-Douglas et par une fonction de type CES. Ceci nous permet, dans le Chapitre 5 de faire des hypothèses plus générales sur  $S(\cdot)$  pour montrer l'existence d'équilibres par une méthode de continuation : nous utilisons le degré de Brouwer pour conclure.

**Extensions.** Nous avons souhaité que le modèle décrit ci-dessus puisse être adapté en fonction de l'objet d'étude. Nous considérons deux extensions, qui concernent l'équation de continuité permettant de déterminer la distribution d'équilibre du capital. Ces deux exemples montrent qu'il est possible d'enrichir le modèle (1.21)-(1.24).

Dans le problème initial, la distribution  $m$  était l'unique solution de

$$\begin{aligned} \frac{d}{dk} (D_q H(k, u'(k))m(k)) &= \eta(k) - \nu m(k), \\ m &\in \mathcal{P}((0, +\infty)), \end{aligned}$$

où  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  est une fonction positive et continue telle que  $\int_0^{+\infty} \eta(k) dk = \nu$ . Dans le Chapitre 6, nous modifions ce terme pour le faire dépendre soit de la fonction valeur du problème de contrôle optimal  $u(\cdot)$ , soit de la distribution du capital  $m$ . Ainsi, dans un premier temps, nous considérons une équation de continuité du type

$$\begin{aligned} \frac{d}{dk} (D_q H(\cdot, u'(\cdot))m(\cdot))(k) &= \eta(k, u(k)) - \nu m(k), \quad \forall k \in (0, +\infty), \\ \int_0^{+\infty} dm(k) &= \frac{1}{\nu} \int_0^{+\infty} \eta(k) dk. \end{aligned}$$

Puis, dans un second temps, nous nous intéressons à un processus particulier de renouvellement des entreprises : nous supposons que lorsqu'une entreprise avec un capital  $k > 0$  cesse d'exister, elle donne naissance à deux entreprises plus petites avec un capital  $k/2$  chacune. Ainsi, le terme source va dépendre de la distribution des entreprises et l'équation de continuité devient non locale :

$$\frac{d}{dk} (D_q H(\cdot, u'(\cdot))m) = \nu ((\text{id}/2)_\# m - m), \quad \text{on } (0, +\infty), \quad (1.32)$$

$$m \in \mathcal{P}(0, +\infty). \quad (1.33)$$

Ici encore, nous montrons l'existence et l'unicité de solutions.

**Simulations numériques.** Dans le Chapitre 7, nous discrétisons le problème et proposons un algorithme dans le but d’approcher les solutions de (1.21)-(1.24). Nous nous appuyons sur les travaux de Y. Achdou et ses co-auteurs [2, 1, 6, 3, 5, 4]. Nous présentons des simulations numériques avec des paramètres choisis pour concorder avec la littérature économique, et pour approcher certaines données provenant du CONSEIL SUPERIEUR DE L’AUDIOVISUEL et de l’INSEE [39, 63]. Plusieurs autres simulations sont menées à bien pour montrer que l’algorithme fonctionne lorsque  $d = 3$  et lorsque l’équation de continuité est remplacée par les extensions discutées plus haut. Nous montrons numériquement que la distribution  $m$  au voisinage du point  $k^*$  peut avoir plusieurs comportements : elle peut exploser ou converger vers une valeur réelle. Enfin, nous étudions la sensibilité de l’équilibre calculé aux variations de certains paramètres du modèle.

## 1.4.2 Un modèle pour la distribution spatiale des résidences

Nous proposons un nouveau modèle permettant de comprendre plus en détail les interactions entre les marchés du travail et de l’immobilier locatif des particuliers. Nous prouvons d’abord l’existence et l’unicité d’équilibres dans un cas régulier. Plusieurs extensions sont ensuite étudiées. D’abord, nous étendons ce modèle pour prendre en compte le télétravail. Ensuite nous étendons le résultat d’existence au cas non régulier. Enfin nous étudions le modèle lorsqu’il y a plusieurs types de travailleurs. Le dernier chapitre décrit diverses simulations numériques.

**Le modèle.** Soit  $X$  la fermeture d’un domaine borné de  $\mathbb{R}^d$  représentant l’ensemble des lieux possibles de résidences. Nous supposons qu’il y a  $N \in \mathbb{N}^*$  lieux de travail. Pour tout  $i \in \{1, \dots, N\}$ , les données relatives au lieu de travail  $i$  sont

1. Les coûts de transport  $c_i : X \rightarrow [0, +\infty)$ .
2. La demande de travail  $L_i : [0, +\infty) \rightarrow [0, +\infty)$ .

Ici,  $c_i(x)$  représente le coût de transport pour atteindre le  $i^{\text{eme}}$  lieu de travail en venant du point  $x \in X$ . La quantité  $L_i(w_i)$  modélise la demande de travail au lieu  $i$  lorsque le salaire est  $w_i$ .

Nous supposons que les agents (les travailleurs) sont rationnels. Pour  $R > 0$  le revenu d’un agent et  $Q$  le loyer par unité de surface, l’utilité du travailleur est donnée par :

$$\mathcal{U}_\theta(R, Q) = \sup \{ C^\theta S^{1-\theta} : C + QS \leq R, C \geq 0, S \geq 0 \}, \quad (1.34)$$

où  $C$  et  $S$  sont des variables représentant respectivement sa consommation et la surface qu’il loue. Nous désignons par  $C_\theta(R)$  et  $S_\theta(R, Q)$  les maximiseurs de (1.34).

En outre, nous supposons que le revenu d’un agent est donné par :

$$R_\sigma(x, w) = \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right), \quad (1.35)$$

où le paramètre  $w_0 > 0$  représente l’allocation ou le revenu qu’un agent perçoit en restant chez lui (il ne travaille donc pas dans les entreprises indexées par  $i$  et son coût de transport est  $c_0(\cdot) = 0$ ), et  $\sigma$  est un paramètre strictement positif. En fait, la formule (1.35) est une régularisation de  $\max_{0 \leq i \leq N} w_i - c_i(x)$ , voir l’appendice 8.A pour plus de détails. En particulier, lorsque  $w \in (0, +\infty)^d$  est fixé, la probabilité qu’un travailleur à la position  $x \in X$  choisisse de travailler dans le lieu  $i$  est donnée par la distribution de Gibbs :

$$\frac{\partial R_\sigma}{\partial w_i}(x, w_i) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}}, \quad \forall x \in X. \quad (1.36)$$

Nous pouvons alors définir l’offre de travail et la demande de surface :

**Définition 1.1** *Pour toute distribution de résidences  $\mu \in \mathcal{P}(X)$  et pour toute collection de salaires  $w \in (0, +\infty)^N$ , l'offre de travail au lieu  $i$  est donnée par*

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x). \quad (1.37)$$

**Définition 1.2** *Pour toute distribution de résidences  $\mu \in \mathcal{P}(X)$ , toute collection de salaires  $w \in (0, +\infty)^N$  et toute fonction de loyer unitaire :  $Q : X \rightarrow [0, +\infty)$ , la demande de surface locative est donnée par la mesure positive*

$$S_\theta(R_\sigma(\cdot, w), Q(\cdot))\mu.$$

Finalement, l'équilibre est décrit par un triplet  $(w, Q(\cdot), \mu)$  solution du système suivant :

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x) = L_i(w_i), \quad \forall i \in \{1, \dots, N\}, \quad (1.38)$$

$$S_\theta(R_\sigma(x, w), Q(x))\mu(x) = 1, \quad \forall x \in X, \quad (1.39)$$

$$\int_X \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) d\nu(x) < +\infty, \quad (1.40)$$

où (1.38) traduit l'équilibre sur le marché du travail, (1.39) traduit l'équilibre sur le marché de l'immobilier où l'offre est normalisée à un. Nous supposons que cette dernière équation est satisfaite sur l'ensemble  $X$  ce qui assurera qu'à l'équilibre  $\text{supp } \mu = X$ . De plus, (1.40) assure qu'aucun agent n'a intérêt à déménager.

**Existence et unicité d'équilibres.** Nous démontrons l'existence de solutions de (1.38)-(1.40), et l'unicité dans des cas plus restrictifs :

**Théorème 1.2** *Sous des hypothèses sur les fonctions  $L_i(\cdot)$  et  $c_i(\cdot)$  qui seront précisées dans le Chapitre 8, il existe au moins une solution de (1.38)-(1.40).*

*De plus, si les fonctions  $L_i(\cdot)$  sont régulières, alors il existe  $\theta_0 > 0$  (admettant une forme quasiment explicite), tel que pour tout  $\theta \in [0, \theta_0]$ , le système (1.38)-(1.40) admet une unique solution.*

Le résultat d'existence est démontré en utilisant une stratégie de point fixe, qui peut être mise en œuvre grâce à une reformulation des conditions d'équilibres décrites par (1.38)-(1.40). En particulier, (1.38) est reformulée comme un problème d'optimisation du même type que (1.13), ce qui permet de montrer des bornes a priori sur  $w$ . D'autre part, pour  $w \in (0, +\infty)^N$  fixé, si  $(Q(\cdot), \mu)$  est solution de (1.39)-(1.40), alors  $\mu$  et  $Q(\cdot)$  admettent des formes explicites. Ces deux dernières conditions peuvent aussi être interprétées comme décrivant l'équilibre dans un jeu non-atomique. La preuve de l'unicité utilise quant à elle le théorème des fonctions implicites et repose largement sur la formule explicite pour  $\mu$  déduite de (1.39)-(1.40).

**Extensions.** Plusieurs extensions sont étudiées. La première est liée au télétravail. Dans ce cadre, la fonction de production des entreprises admet deux paramètres : le nombre de travailleurs en présentiel et le nombre de télétravailleurs. Ce nouveau problème admet des similitudes avec une autre extension que nous développons dans le Chapitre 9, où nous étudions l'extension du modèle au cas où il y a plusieurs types de travailleurs. Cette extension vise notamment à répondre à la question : est-ce que seule la différence salariale explique les phénomènes de ségrégation ? Enfin, nous étudions le cas où  $\sigma = 0$  dans (1.35) : il s'agit du cas limite non régulier. Pour chacune de ces extensions, le résultat d'existence est étendu. Nous ne parvenons à généraliser le résultat d'unicité que pour le modèle de télétravail.

**Simulations numériques.** Dans le dernier chapitre de cette thèse, nous proposons un schéma numérique pour approcher les solutions de (1.38)-(1.40). Nous nous ramenons à la résolution d'une équation en exploitant le fait que les conditions (1.39)-(1.40) permettent d'explicitier la distribution des résidences. Puis nous présentons trois simulations. La première permet de comprendre l'influence sur l'équilibre du paramètre modélisant la préférence des agents  $\theta$ . Dans la deuxième simulation, nous transférons progressivement le capital depuis deux lieux de travail vers un autre. Ce test permet de comparer différents scénarios et de déterminer quel est celui qui maximise l'utilité des agents. La dernière simulation est consacrée à l'introduction du télétravail.

### 1.4.3 Perspectives et futurs développements

Les modèles développés et étudiés dans ce mémoire permettent d'appréhender les liens entre le marché du travail et celui du marché locatif de l'immobilier. Nous avons notamment pu voir comment les prix d'équilibre sur ces marchés varient lorsqu'un des paramètres de l'économie est modifié. Le modèle proposé dans la deuxième partie est lié à l'organisation de l'espace urbain. Par exemple, il est utile pour tenter de comprendre les conséquences de changements majeurs comme le développement du télétravail.

Bien-sûr, ces modèles sont sources de questionnements.

Pour le modèle de jeu à champ moyen développé dans la première partie, nous avons observé numériquement l'existence d'un seul équilibre. En revanche, très peu de résultats d'unicité ont pu être établis. Il serait intéressant d'essayer d'obtenir des résultats d'unicité en utilisant de nouveaux arguments. En outre, l'étude de ce modèle est restreinte au cadre stationnaire. Il faudrait donc étudier la version évolutive (horizon fini) du modèle. On pourrait notamment s'intéresser à la question suivante : sous quelles conditions et à quelle vitesse le système converge vers son état stationnaire ? D'autres extensions importantes peuvent être étudiées comme la prise en compte des externalités, ou l'introduction d'un bruit idiosyncratique ou commun dans la dynamique du capital des entreprises.

Concernant le modèle développé dans la deuxième partie, une amélioration naturelle serait d'étendre les résultats au cas où les lieux de travail sont distribués dans tout le domaine et non pas localisés en un nombre fini de points. Un autre axe de recherche concernerait la mise au point d'une méthode numérique permettant de simuler les équilibres du modèle avec plusieurs types de travailleurs. De telles méthodes permettraient de mieux comprendre les phénomènes de ségrégation possiblement dus à des hétérogénéités salariales.

Enfin, il serait intéressant de coupler les modèles proposés dans les deux parties. En effet, dans le cadre du modèle spatial, supposons que pour chaque lieu d'activité, il existe un grand nombre d'entreprises pouvant être décrite par le modèle du jeu à champ moyen de la première partie. Si de plus, le terme modélisant la création des entreprises fait intervenir la fonction valeur issue du problème de contrôle optimal dans le jeu à champ moyen, voir le Chapitre 6, alors le nombre de firme en un lieu donné devient une inconnue du problème. En couplant les deux modèles, on pourrait donc étudier la concurrence entre ces centres d'activité, en enrichissant progressivement le modèle ; on pourrait par exemple supposer que les firmes sont distribuées sur tout le territoire urbain, prendre en compte des externalités, modéliser la compétition des firmes et des travailleurs sur le marché locatif de l'immobilier...

## Part I

# A model for the interactions of firms in factor markets



## Chapter 2

# A mean field game approach to model equilibria in perfectly competitive markets

### 2.1 Introduction

Apart from capital, firms need several inputs to produce goods such as raw materials, workforce, space, energy... In this chapter, we focus on the modelling of markets for the factors of production. Therefore, the model developed here permits to understand how the modifications of a market, such as an increase of the supply, impact the prices on the others. The general assumptions are:

1. The economy is reduced to one sector of activity with a (very) large number of firms.
2. These firms are indistinguishable in the sense that two different firms with the same quantities of inputs will produce the same quantities of goods.
3. Each firm has a negligible impact on markets.
4. Firms are rational and aim at maximizing a utility function.
5. The equilibrium is reached when supply matches demand.

In this context, the use of the *Mean Field Games* theory is natural. It was recently introduced by P.-L. Lions and J.-M. Lasry in a series of papers [68, 69, 70], and independently by M. Huang, P.-E. Caines and R.-P. Malhamé in [62, 61, 60]. This theory aims at studying Nash equilibria for a game with a (very) large number of players. This theory was early used to model price formation in [70] and [57]. Such models are still being developed in a number of areas such as energy [51], exhaustive resources [57], growth theory [3], trading [67, 29]...

The approach used has several advantages. It can model a large range of markets: if we consider that labour is an input, then it models the labour market, similarly, if the workspace is considered as an input, then it models the rental market for professionals. Note that both markets can be taken into account. In such case, the model gives a way to understand interactions between these two markets. In addition, the state variable of firms belongs to  $(0, +\infty)$ , therefore, every equation which appears in the development is posed on  $(0, +\infty)$ , so the dimension of the problem, even if we take into account several markets, remains small.

The chapter is structured as follows: we first develop a toy model where we assume constant returns to scale, the computations can be made almost explicitly. We obtain the existence and uniqueness of



an equilibrium. We also find that the right tail of the distribution of capital of firms decays as a power function: this is known in economics as the Pareto's law [82, 46], while the growth rate does not depend on the capital of the firm, which is known as the Gibrat's law [48, 96]. Secondly, we make the assumption of decreasing returns to scale, where the computations can no longer be explicit. However, in a rather general case, we are able to solve the Hamilton-Jacobi (or HJ) equation associated with the optimal control problem that firms face and we notice that the unique solution is concave. This solution gives us the optimal investment policy of a firm depending on its capital, and we establish that:

1. there exists a capital level  $\kappa^* \in (0, +\infty)$  such that all capital of firms converges towards  $\kappa^*$ : this is known as the golden rule of accumulation of capital [7, Chapter 7],
2. this target capital  $\kappa^*$  is reached in infinite time, i.e. the capital distribution of firms does not admit a Dirac mass in  $\kappa^*$ .

Thirdly, the regularity of the solution of the HJ equation allow us to deduce the capital distribution of firms by solving a transport equation: the continuity equation. We finally establish in the last part, the existence of equilibria i.e. when the market clearing conditions are fulfilled, by using Brouwer fixed-point theorem.

### 2.1.1 General framework

The aim of this paragraph is to set a model describing the interactions on the labour market of a continuum of identical firms which are heterogeneous in their capital (the state variable). An output of the model is the level of wages at the equilibrium. Note that for the purpose of keeping the notations as simple as possible we only consider workers as inputs. Nonetheless, rather than considering another type of workers, it is possible to consider the working place as an input and link the rental market for professionals. The main assumptions are as follows:

- A change in strategy of a single firm does not impact the equilibrium.
- The labour market can admit several types of workers, say  $d$  types.
- The production of a given firm is a function of its capital and the level of employment of each type, namely  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$ . A classical example is the Cobb-Douglas production function  $F(k, \ell) = Ak^\alpha \ell^\beta$  where  $\beta \in (0, 1)^d$ ,  $\sum_{i=1}^d \beta_i < 1$ ,  $\ell^\beta = \prod_{i=1}^d \ell_i^{\beta_i}$ , and  $\alpha \in (0, 1 - \sum_{i=1}^d \beta_i)$ . The quantities  $\beta$  and  $\alpha$  respectively stand for the output elasticities of labour and capital,  $A > 0$  is the total-factor productivity.
- The benefits of production are cut by the wages and the depreciation of capital. Therefore the total benefits are  $F(k, \ell) - w \cdot \ell - \delta k$ , where  $w \in (0, +\infty)^d$  contains the unitary wages for each type of workers and  $\delta \geq 0$  is the rate of capital depreciation.

The firms face the problem of how to split their benefits into consumption or into investment in order to produce growth. The dynamics of capital is then given by

$$\frac{dk}{dt}(t) = F(k(t), \ell(t)) - w \cdot \ell(t) - \delta k(t) - c(t), \quad (2.1)$$

where  $c(t)$  stands for the consumption of capital at time  $t$ ; the consumption  $c(t)$  is one control variable of each firm, the other being the vector  $\ell(t) \in \mathbb{R}^d$  already introduced. The strategy of a given firm is determined by solving an optimal control problem to maximize the payoff

$$\int_0^{+\infty} U(c(t))e^{-\rho t} dt, \quad (2.2)$$

where  $U : [0, +\infty) \rightarrow \{-\infty\} \times \mathbb{R}$  is a utility function and  $\rho$  is a positive discount factor. Firms aim at finding the optimal controls  $c(t) \in [0, +\infty)$  and  $\ell(t) \in [0, +\infty)^d$  which maximize (2.2), under the constraint that their capital stay non negative (borrowing constraint). The value of the optimal control problem is

$$u(\kappa) = \sup_{c(t), \ell(t)} \int_0^{+\infty} U(c(t))e^{-\rho t} dt$$

subject to

$$\left\{ \begin{array}{l} c(\cdot), \ell(\cdot) \in L^1_{\text{loc}}(0, +\infty), k(\cdot) \in W^{1,1}_{\text{loc}}(0, +\infty); \\ c(t) \geq 0, \quad \ell(t) \geq 0, \quad \text{for all } t > 0; \\ k(\cdot) \text{ is a non negative solution of (2.1) with initial condition } k(0) = \kappa. \end{array} \right. \quad (2.3)$$

It is expected that the function  $u(\cdot)$  is a solution of the following Hamilton-Jacobi equation

$$\rho u(k) = H(k, u'(k)), \quad \forall k \in (0, +\infty), \quad (2.4)$$

where the Hamiltonian  $H : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is given by

$$H(k, q) = \sup_{c \geq 0, \ell \in [0, +\infty)^d} \{U(c) + q(F(k, \ell) - w \cdot \ell - \delta k - c)\}. \quad (2.5)$$

The Hamiltonian can be written in equivalent manner as follows:

$$H(k, q) = \sup_{c \geq 0} \{U(c) - cq\} + f(k)q, \quad (2.6)$$

where  $f : [0, +\infty) \rightarrow \mathbb{R}$  is the net output function:

$$f(k) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} - \delta k. \quad (2.7)$$

If the value function  $u(\cdot)$  and the Hamiltonian  $H(\cdot, \cdot)$  are regular enough, it is classical in optimal control theory to express the optimal investment of an agent with capital  $k \in (0, +\infty)$  by  $D_q H(k, u'(k))$ , where  $D_q H(\cdot, \cdot)$  stands for the derivative of  $H(\cdot, \cdot)$  with respect to its second variable. We complete (2.4) with the state constraint boundary condition:

$$D_q H(0, u'(0)) \geq 0.$$

If in the model, there is neither creation nor destruction of firms, then the probability density function  $m(\cdot)$  of the distribution of capital is found by solving the following continuity equation, whose drift is the optimal investment function  $D_q H(\cdot, u'(\cdot))$ :

$$\frac{d}{dk} \left( D_q H(\cdot, u'(\cdot)) m(\cdot) \right) (k) = 0, \quad k > 0, \quad (2.8)$$

which has to be understood in the sense of distributions and supplemented with the equation:

$$\int_0^{+\infty} dm(k) = 1.$$

In a more general framework, it is possible to consider exogenous phenomena leading to the creation and destruction of firms: for example, one may assume that there is a destruction rate  $\nu \geq 0$  and that the creation of firms is modelled by a source term of the form  $\eta = \eta(k)$  in the continuity equation. The existence of a stationary equilibrium then requires that creations compensate destructions, i.e. that

$$\int_0^{+\infty} \eta(k) dk - \nu = 0.$$

The continuity equation then becomes

$$\frac{d}{dk} \left( D_q H(\cdot, u'(\cdot)) m(\cdot) \right) (k) = \eta(k) - \nu m(k), \quad k > 0, \quad (2.9)$$

supplemented with  $\int_0^{+\infty} dm(k) = 1$ .

Whereas all the above considerations have been made given  $w$ , the vector of wages, the present work consists in finding an equilibrium that satisfies the labour market clearing conditions. For a firm whose capital is  $k$ , the optimal numbers of employees for each type is denoted by  $\ell^*(k, w) \in \mathbb{R}^d$ . The labour supply for each type is an exogenous function of the wages. It is denoted by  $S(w)$  where  $S(\cdot)$  is a bounded and monotonous function from  $\mathbb{R}_+^d$  to  $\mathbb{R}_+^d$ . The labour market clearing condition takes the form of a system of  $d$  equations:

$$S(w) = \int_0^{+\infty} \ell^*(k, w) dm(k). \quad (2.10)$$

Assuming enough regularity on  $F(\cdot, \cdot)$ ,  $\ell^*(\cdot, \cdot)$  is given by

$$\ell^*(k, w) = -D_w f(k), \quad (2.11)$$

where  $D_w f(\cdot)$  stands for the gradient of  $f(\cdot)$  with respect to the parameter  $w$ . To summarize, the equilibrium on the labour market is described by the following system of equations:

$$\rho u(k) = H(k, u'(k)), \quad (2.12)$$

$$\frac{d}{dk} (D_q H(\cdot, u'(\cdot)) m(\cdot)) (k) = \eta(k) - \nu m(k), \quad (2.13)$$

$$S(w) = - \int_0^{+\infty} D_w f(k) dm(k), \quad (2.14)$$

completed with the following conditions:

$$D_q H(0, u'(0)) \geq 0, \quad (2.15)$$

$$1 = \int_0^{+\infty} dm(k). \quad (2.16)$$

The unknowns of this problem are the value function  $u(\cdot)$ , the measure of probability  $m$  and the vector of wages  $w$ . For keeping the notations reasonably light, it has been chosen not to explicitly write  $w$  as an argument of  $H(\cdot, \cdot)$ ,  $f(\cdot)$ ,  $u(\cdot)$  and  $m$ .

## 2.2 An example with an almost explicit solution

This paragraph is devoted to an example in which (2.12)-(2.16) may be solved almost explicitly. Here  $d = 1$  and the production is modelled by a Cobb-Douglas function

$$F(k, \ell) = Ak^\alpha \ell^\beta, \quad \text{with } \alpha + \beta = 1, \quad (2.17)$$

the utility is logarithmic:

$$U(c) = \ln(c). \quad (2.18)$$

Then, the Hamiltonian has the following form:

$$H(k, q) = -(\ln(q) + 1) + \left( Cw^{-\frac{\beta}{1-\beta}} - \delta \right) kq, \quad (2.19)$$

where

$$C = (A)^{\frac{1}{1-\beta}} \left( (\beta)^{\frac{\beta}{1-\beta}} - (\beta)^{\frac{1}{1-\beta}} \right) = (A\beta)^{\frac{1}{1-\beta}} \left( \frac{1}{\beta} - 1 \right). \quad (2.20)$$

In the two next paragraphs, we explicitly solve the HJ equation (2.12) completed with (2.15) and the continuity equation (2.13) supplemented with (2.16). This leads to an explicit formula for the labour demand. Then system (2.12)-(2.16) reduces to a single equation whose unknown is the unitary salary  $w \in (0, +\infty)$ . The last paragraph is devoted to solving the latter equation.

### 2.2.1 Hamilton-Jacobi equation

Here, the parameter  $w \in (0, +\infty)$  is fixed. The purpose is to find an explicit solution of (2.12) when the Hamiltonian is given by (2.19).

**Proposition 2.1** *Let  $H(\cdot, \cdot)$  be given by (2.19), with  $C$  given by (2.20). A solution of (2.12) is the function  $v : (0, +\infty) \rightarrow \mathbb{R}$ :*

$$v(k) = \frac{1}{\rho} (\ln(k) + \xi), \quad \forall k > 0, \quad (2.21)$$

where

$$\xi = \ln(\rho) - 1 + \frac{1}{\rho} \left( Cw^{-\frac{\beta}{1-\beta}} - \delta \right). \quad (2.22)$$

**Proof** It is easy to see that  $v'(k) = 1/(\rho k)$ . Then, straightforward calculus yields

$$\begin{aligned} H(k, v'(k)) &= \ln(\rho k) - 1 + \frac{1}{\rho} \left( Cw^{-\frac{\beta}{1-\beta}} - \delta \right) \\ &= \ln(k) + \xi \\ &= \rho v(k), \end{aligned}$$

which achieves the proof. □

Since the solution  $v(\cdot)$  is smooth on  $(0, +\infty)$ , it is possible to prove a verification theorem:

**Theorem 2.1 (Verification theorem)** *Let  $H(\cdot, \cdot)$  be given by (2.19), with  $C$  given by (2.20). The function  $v(\cdot)$  defined in (2.21) coincides with the value function  $u(\cdot)$  defined in (2.3).*

**Proof** The first step consists in proving that  $v(\cdot) \leq u(\cdot)$ .

For every  $k \in (0, +\infty)$ , it is straightforward to check that

$$\operatorname{argmax}_{c \geq 0} (U(c) - v'(k)c) = \{c^*(k)\} \quad \text{and} \quad \operatorname{argmax}_{\ell \geq 0} (F(k, \ell) - w\ell) = \{\ell^*(k)\},$$

where

$$c^*(k) = \rho k \quad \text{and} \quad \ell^*(k) = \left( \frac{A\beta}{w} \right)^{\frac{1}{1-\beta}} k, \quad (2.23)$$

by writing the necessary optimality condition.

Fix  $k \in (0, +\infty)$  and consider  $\tilde{k} : [0, +\infty) \rightarrow \mathbb{R}$ , the unique solution of

$$\begin{aligned} \frac{d\tilde{k}}{dt}(t) &= F(\tilde{k}(t), \ell^*(\tilde{k}(t))) - w\ell^*(\tilde{k}(t)) - \delta\tilde{k}(t) - c^*(\tilde{k}(t)) \\ &= \left( Cw^{-\frac{\beta}{1-\beta}} - \delta - \rho \right) \tilde{k}(t), \end{aligned} \quad (2.24)$$

with the initial condition  $\tilde{k}(0) = k$ .

For simplicity, set

$$c(t) = c^*(\tilde{k}(t)), \quad \ell(t) = \ell^*(\tilde{k}(t)).$$

On the other hand, since  $v(\cdot)$  is a solution of (2.12), we see that

$$0 = U(c(t)) + v'(\tilde{k}(t)) \left\{ F(\tilde{k}(t), \ell(t)) - w\ell(t) - \delta\tilde{k}(t) - c(t) \right\} - \rho v(\tilde{k}(t)),$$

for every  $t \in [0, +\infty)$ . We obtain, by multiplying this equation by  $e^{-\rho t}$ , integrating on  $[0, +\infty)$  and using (2.24):

$$0 = \int_0^{+\infty} U(c^*(\tilde{k}(t)))e^{-\rho t} dt - v(\tilde{k}(0)).$$

Therefore

$$v(k) = \int_0^{+\infty} U(c(t))e^{-\rho t} dt,$$

which implies  $v(\cdot) \leq u(\cdot)$ .

The second step consists in obtaining the converse inequality, i.e.  $u(\cdot) \leq v(\cdot)$ . Consider an admissible trajectory  $c(\cdot)$ ,  $\ell(\cdot)$ , and  $\tilde{k}(\cdot)$ , i.e.

1.  $\tilde{k}(t) \geq 0$ ,  $c(t) \geq 0$ ,  $\ell(t) \geq 0$ , for a.a.  $t > 0$
2.  $\frac{d\tilde{k}}{dt}(t) = F(\tilde{k}(t), \ell(t)) - w\ell(t) - \delta\tilde{k}(t) - c(t)$ , for a.a.  $t > 0$
3.  $\tilde{k}(0) = k$ .

We observe that for almost every  $t \geq 0$ ,

$$\begin{aligned} & \sup_{\bar{c} \geq 0, \bar{\ell} \geq 0} \left\{ U(\bar{c}) + v'(\tilde{k}(t)) \left( F(\tilde{k}(t), \bar{\ell}) - w\bar{\ell} - \delta\tilde{k}(t) - \bar{c} \right) \right\} \\ & \geq U(c(t)) + v'(\tilde{k}(t)) \left( F(\tilde{k}(t), \ell(t)) - w\ell(t) - \delta\tilde{k}(t) - c(t) \right). \end{aligned}$$

The left hand side coincides with  $H(\tilde{k}(t), v'(\tilde{k}(t)))$ . Therefore, multiplying both sides by  $e^{-\rho t}$ , integrating and using (2.24) yields

$$v(k) \geq \int_0^{+\infty} U(c(t))e^{-\rho t} dt.$$

Taking the supremum over the admissible trajectories yields  $v(\cdot) \geq u(\cdot)$ . □

**Corollary 2.1** *The optimal consumption and hiring policy are characterized by (2.23).*

Observe that the optimal consumption is a linear function of the capital: this is a consequence of the special choice of a logarithmic utility function. On the other hand, the optimal level of employment is a decreasing function of the unitary salary  $w$ , and also grows linearly with the capital, due to the homogeneity of the Cobb-Douglas production function.

**Corollary 2.2** *The optimal investment policy of a firm whose capital is  $k$  is given by*

$$\Sigma^*(k) = F(k, \ell^*(k)) - w\ell^*(k) - \delta k - c^*(k) = \left( Cw^{-\frac{\beta}{1-\beta}} - \delta - \rho \right) k. \quad (2.25)$$

Note that by extending  $\Sigma^*(\cdot)$  by continuity in 0, we can see that (2.15) holds.

### 2.2.2 The continuity equation

The knowledge of the optimal investment policy of the firms allows to obtain the distribution of capital, by solving the continuity equation (2.13). For brevity, set

$$b = Cw^{-\frac{\beta}{1-\beta}} - \delta - \rho, \quad (2.26)$$

where the constant  $C$  is given by (2.20). Note that  $\Sigma^*(k) = bk$ , i.e. the optimal investment is proportional to the capital, because  $b$  does not depend on  $k$ .

We make the assumption that the rate  $\nu$  is positive. This is necessary for the existence of solutions to (2.13) indeed, if  $b > 0$  and  $\nu = 0$ , then there is no solution of (2.13) with total mass one.

**Proposition 2.2** *Assume that  $\nu > 0$  and that the function  $\eta : (0, +\infty) \rightarrow [0, +\infty)$  is such that  $k \mapsto k^\gamma \eta(k)$  is integrable for all  $\gamma \in \mathbb{R}$ . The density of  $m$ , (still denoted by  $m(\cdot)$  with a slight abuse of notation) is the unique solution of (2.13) given by*

$$m(k) = \begin{cases} \frac{\eta(k)}{\nu}, & \text{if } b = 0, \\ \frac{k^{-(1+\frac{\nu}{b})}}{b} \int_0^k \kappa^{\frac{\nu}{b}} \eta(\kappa) d\kappa, & \text{if } b > 0, \\ -\frac{k^{-(1+\frac{\nu}{b})}}{b} \int_k^{+\infty} \kappa^{\frac{\nu}{b}} \eta(\kappa) d\kappa, & \text{if } b < 0, \end{cases} \quad (2.27)$$

for  $k > 0$ .

**Remark 2.1** *If  $b > 0$  and  $\eta(\cdot)$  has a compact support, then*

$$m(k) \sim k^{-(1+\frac{\nu}{b})}, \quad \text{as } k \rightarrow \infty. \quad (2.28)$$

### 2.2.3 Equilibrium

The last step consists in finding equilibria, i.e. the unitary wages  $w$  for which the labour market clearing condition is fulfilled. The first thing to do is to compute the labour demand:

**Lemma 2.1** *Assume that  $\int_0^{+\infty} \kappa \eta(\kappa) d\kappa < +\infty$ . Under the assumptions made in Propositions 2.1 and 2.2,  $(u(\cdot), m, w)$  is a solution of (2.12)-(2.16) if  $u(\cdot)$  is given by (2.21),  $m$  is given by (2.27) and  $w$  satisfies*

$$b(w) < \nu, \quad (2.29)$$

and

$$S(w) = \frac{1}{\nu - b(w)} \left( \frac{A\beta}{w} \right)^{\frac{1}{1-\beta}} \int_0^{+\infty} \kappa \eta(\kappa) d\kappa, \quad (2.30)$$

where  $b(w)$  is given by (2.26).

**Proof** The optimal individual labour demand  $\ell^*(k, w)$  is given by (2.23). The labour demand is found as the integral  $\int_0^\infty \ell^*(\kappa, w) dm(\kappa)$  where  $m$  is given by (2.27). Then, one can see that the latter integral  $\int_0^{+\infty} \ell^*(\kappa, w) dm(\kappa)$  is finite if and only if (2.29) holds, and in this case, its value is given by (2.30).  $\square$

Lemma 2.1 allows to solve (2.12)-(2.16) under natural assumptions on the data.

**Theorem 2.2** *Assume that  $\int_0^\infty \kappa \eta(\kappa) d\kappa < +\infty$ . Under the assumptions made in Propositions 2.1 and 2.2, if  $S(\cdot)$  is a continuous and non decreasing function, non identically 0 and such that  $S(0) = 0$ , then there exists a unique solution of (2.12)-(2.16).*

**Proof** We observe that there exists  $\underline{w} > 0$  such that  $\nu - b(\underline{w}) = 0$  and

$$\begin{aligned}\nu - b(w) &> 0, & \forall w > \underline{w}, \\ \nu - b(w) &< 0, & \forall w < \underline{w}.\end{aligned}$$

From (2.26),

$$(\nu - b(w))w^{\frac{1}{1-\beta}} = \left( (\nu + \delta + \rho)w^{\frac{1}{1-\beta}} - \frac{C}{w} \right).$$

Thus,  $w \mapsto S(w)(\nu - b(w))w^{\frac{1}{1-\beta}}$  is a continuous and non decreasing function satisfying

$$\begin{aligned}S(\underline{w})(\nu - b(\underline{w}))\underline{w}^{\frac{1}{1-\beta}} &= 0, \\ \lim_{w \rightarrow +\infty} S(w) \left( (\nu + \delta + \rho)w^{\frac{1}{1-\beta}} - \frac{C}{w} \right) &= +\infty.\end{aligned}$$

It is clear that  $\int_0^{+\infty} \kappa \eta(\kappa) d\kappa > 0$  since  $\eta(\cdot)$  is a non negative function such that  $\int_0^{+\infty} \eta(\kappa) d\kappa = \nu > 0$ . Hence,  $(A\beta)^{\frac{1}{1-\beta}} \int_0^{+\infty} \kappa \eta(\kappa) d\kappa > 0$ . Therefore, there exists a solution in  $(\underline{w}, +\infty)$  of

$$S(w)(\nu - b(w))w^{\frac{1}{1-\beta}} = (A\beta)^{\frac{1}{1-\beta}} \int_0^{+\infty} \kappa \eta(\kappa) d\kappa.$$

In addition, we observe that the function  $w \mapsto S(w)(\nu - b(w))w^{\frac{1}{1-\beta}}$  is increasing in the interval

$$\{w > 0 : S(w) > 0\},$$

therefore the solution is unique. □

## 2.2.4 Comments

It has been proved that solving (2.12)-(2.16) amounts to solving an equation for  $w$ , which has a unique solution.

Moreover, the model has interesting features:

- first, the optimal investment of a firm is proportional to its capital: this is known as Gibrat's law in economics [48, 96]. Gibrat's law is usually stated in a stochastic setting in which the dynamics of capital is of the form :

$$dk_t = k_t (\gamma dt + \sigma dB_t),$$

with  $\gamma, \sigma \in \mathbb{R}$  and  $(B_t)_{t \geq 0}$  a standard Brownian motion. But it is possible to modify the dynamics (2.1) as follows

$$dk_t = (F(k_t, l_t) - wl_t - \delta k_t - c_t) dt + \sigma F(k_t, l_t) dB_t,$$

in order to take into account for instance the idiosyncratic sales risk, and obtain the same results as above.

- Second, if  $\eta(\cdot)$  has a compact support, then the right tail of the distribution  $m$  decays as a power of the capital: this is known as Pareto's law in economics [82, 46].
- The example is therefore a situation when Gibrat's and Pareto's law are compatible with each other.

Let us now consider the more general case where we no longer assume an explicit form for the utility or the production function.

## 2.3 The optimal control problem of a representative firm

We study the optimal control problem defined in (2.3) via the associated Hamilton-Jacobi equation. In the paragraph above, we showed that when the production function has constant returns to scale, then the distribution may decay like a power function at infinity. This may cause a default of compactness: for instance,  $\mathcal{P}([0, +\infty))$  is not compact for the narrow topology. Therefore, we make the assumption of decreasing returns to scale. These assumptions will be given in detail in the next paragraph. Then, we prove the existence and uniqueness of a solution of the Hamilton-Jacobi equation, with  $w \in (0, +\infty)^d$  fixed. The dependency of the value function with respect to  $w$  will be studied in Section 2.5.

### 2.3.1 Setting and main assumptions

We have already seen that the firms are heterogeneous in the capital variable  $k \in [0, +\infty)$  (the state variable). Given a discount factor  $\rho > 0$  and a collection of wages  $w \in (0, +\infty)^d$ , they solve an infinite horizon optimal control problem in which the control variable is their consumption, a non negative quantity noted  $c$  in what follows.

We expect that the value function  $u(\cdot)$  of the optimal control problem satisfies the following:

- $u(\cdot)$  is an increasing function of  $k$ ,
- $u(\cdot)$  is a solution of the following Hamilton-Jacobi equation

$$-\rho u(k) + H(k, u'(k)) = 0, \quad \text{for } k > 0, \quad (2.31)$$

with suitable assumptions on the Hamiltonian  $H : [0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}$ ,  $(k, q) \mapsto H(k, q)$  that will be made soon, see Assumption 2.6,

- there exists a critical value  $\kappa^* > 0$  such that the optimal investment of an agent with capital  $k$ , given by the expression  $D_q H(k, u'(k))$ , satisfies:

$$D_q H(k, u'(k)) > 0, \quad \text{for } 0 < k < \kappa^*, \quad (2.32)$$

$$D_q H(k, u'(k)) < 0, \quad \text{for } \kappa^* < k < +\infty, \quad (2.33)$$

where  $D_q H(\cdot, \cdot)$  stands for the partial derivative of  $H(\cdot, \cdot)$  with respect to  $q$ .

Equations (2.32) and (2.33) tell us that if the firm has a capital  $k < \kappa^*$  (resp.  $k > \kappa^*$ ), then it will accumulate (resp. decrease) capital. The existence and uniqueness of the critical value  $\kappa^*$  is connected with the golden rule of accumulation of capital which was first presented by Maurice Allais [7, Chapter 7]: the capital of a firm that solves the above mentioned optimal control will converge toward  $\kappa^*$ .

Note that in the previous example, either  $\kappa^* = 0$  or  $\kappa^* = +\infty$ . In what follows,  $\kappa^*$  will belong to  $(0, +\infty)$ .

In (2.31), the Hamiltonian  $H : [0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}$  is of the form:

$$H(k, q) = \sup_{c \geq 0} \{U(c) - cq\} + f(k)q, \quad (2.34)$$

and we make appropriate assumptions on the utility function  $U : [0, +\infty) \rightarrow \{-\infty\} \cup \mathbb{R}$  and on the net output  $f : [0, +\infty) \rightarrow [0, +\infty)$ .

#### Standing assumptions

**Assumption 2.1 (Assumptions on  $U(\cdot)$ )** *The utility function  $U : [0, +\infty) \rightarrow \{-\infty\} \cup \mathbb{R}$  has the following properties:*

- i)  $U(\cdot)$  is  $C^2$  on  $(0, +\infty)$ .



ii)  $U(\cdot)$  is increasing and strictly concave on  $(0, +\infty)$ .

iii)  $\lim_{c \rightarrow 0^+} U'(c) = +\infty$  and  $\lim_{c \rightarrow +\infty} U'(c) = 0$ .

**Assumption 2.2 (Assumptions on  $f(\cdot)$ )** The net output  $f : [0, +\infty) \rightarrow \mathbb{R}$  has the following properties:

i)  $f(\cdot)$  is continuous on  $[0, +\infty)$ .

ii)  $f(\cdot)$  is locally of class  $C^{1,1}$  on  $(0, +\infty)$ .

iii)  $f(\cdot)$  is strictly concave,  $\lim_{k \rightarrow 0^+} f'(k) = +\infty$  and  $\lim_{k \rightarrow +\infty} f'(k) = 0$ .

iv)  $f(\cdot) \geq 0$ .

**Remark 2.2** Assumption 2.2 implies that  $f(k)$  may vanish only at  $k = 0$  and has a non negative limit as  $k$  tends to  $+\infty$ .

**Example of utility function.** A common example is the constant relative risk aversion (CRRA) utility:

$$U(c) = \ln(c) \quad \text{or} \quad U(c) = \frac{1}{b}c^b, \quad \text{with } b \in (0, 1).$$

**Examples of net output functions.** In the applications that we are concerned with, the net output  $f(\cdot)$  has the form:

$$f(k) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} - \delta k. \quad (2.35)$$

Here, the production  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  has two arguments, the capital  $k$  and the vector  $\ell$  which contains the numbers of employees in the different types of workers. The vector  $w \in (0, +\infty)^d$  contains the wages corresponding to the different types. With  $\delta \geq 0$ , the term  $-\delta k$  models the depreciation of capital.

Taking  $d = 1$ , the two classical forms of the production function:

- 1)  $F(k, l) = k^\alpha l^\beta$ , where  $\alpha, \beta > 0$  and  $\alpha + \beta < 1$ . With this class of Cobb-Douglas production functions, the net output is given by

$$f(k) = w^{-\frac{\alpha\beta}{1-\beta}} \left( \beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \right) k^{\frac{\alpha}{1-\beta}} - \delta k.$$

Note that  $f(0) = 0$  in this example.

- 2)  $F(k, l) = (k^\alpha + l^\beta)^\gamma$ , where  $\alpha, \beta > 0$ ,  $\alpha\gamma < 1$  and  $\beta\gamma < 1$ . For example, if  $\gamma = 1$ , then the net output takes the form

$$f(k) = k^\alpha + w^{-\frac{\beta}{1-\beta}} \left( \beta^{\frac{\beta}{1-\beta}} - \beta^{\frac{1}{1-\beta}} \right) - \delta k$$

Assumption 2.2 is fulfilled when the depreciation rate  $\delta = 0$  in (2.35), which prevents  $f(\cdot)$  from being negative. We will discuss about extension in section 2.3.6 where the case  $\delta > 0$  is taken into account.

**Remark 2.3** From Assumption 2.2, there exists a unique value  $k^*$  such that

$$f'(k^*) = \rho. \quad (2.36)$$

We will state that  $\kappa^* = k^*$ .

We are ready to state the main result:

### Existence and uniqueness of the Hamilton-Jacobi equation

**Theorem 2.3** *Under Assumptions 2.1 and 2.2, there exists a unique classical solution  $u(\cdot) \in C^1(0, +\infty)$  of (2.31)-(2.33), where the critical value  $\kappa^* = k^*$ .*

The difficulty to build a solution of the HJ equation lies in the fact that the level-sets of the function  $q \mapsto H(k, q)$  are not reduced to one element (see Figure 2.1): at first, this function is decreasing, and then increasing. Therefore, the strategy to show the existence of solutions is based on the resolution of two ordinary differential equations (or ODEs) where for one, we inverted the increasing part of  $H(k, \cdot)$ , and for the second, we inverted the decreasing part.

### 2.3.2 Some properties of the Hamiltonian

**Lemma 2.2** *Under Assumption 2.1, for any  $k > 0$ , the map  $(0, +\infty) \ni q \mapsto H(k, q)$  is strictly convex and of class  $C^2$ .*

**Proof** We fix  $k > 0$  and for brevity, we set  $h(q) = H(k, q)$ . From Assumption 2.1, there exists a unique value  $c^*(q) > 0$  achieving the supremum in (2.34). It is characterized by the equation

$$U'(c^*(q)) = q. \quad (2.37)$$

Moreover, again from Assumption 2.1, the map  $c^*(\cdot)$  is  $C^1$  and its derivative is  $1/U''(c^*(\cdot))$ . From the envelope theorem, we see that

$$h'(q) = -c^*(q) + f(k). \quad (2.38)$$

Hence,  $h(\cdot)$  is a  $C^2$  function defined on  $(0, +\infty)$  and

$$h''(q) = -\frac{1}{U''(c^*(q))} > 0.$$

This implies the strict convexity of  $h(\cdot)$ . □

**Remark 2.4** *Note that  $c^*(q)$  does not depend on  $k$ .*

**Lemma 2.3** *Under Assumptions 2.1 and 2.2, for any  $k > 0$ ,*

$$\min_{q>0} H(k, q) = U(f(k)), \quad (2.39)$$

$$\operatorname{argmin}_{q>0} H(k, q) = \{U'(f(k))\}. \quad (2.40)$$

**Proof** We keep the notation  $h(\cdot)$  introduced in the proof of Lemma 2.2. We have seen that the unique value  $c^*(q)$  achieving the supremum in (2.34) is characterized by (2.37), and that  $h'(q) = D_q H(k, q)$  is given by (2.38). From Assumption 2.2,  $f(k) > 0$ , so  $U'(f(k))$  is well defined, and it is easily checked that  $h'(U'(f(k))) = 0$ . We deduce (2.39) and (2.40) from the strict convexity of  $h(\cdot)$ , see Lemma 2.2. □

**Remark 2.5** *Note that from Assumption 2.1, if  $f(0) = 0$ , then  $\lim_{k \rightarrow 0} U'(f(k)) = +\infty$ . On the contrary, from Remark 2.2, if  $f(0) > 0$ , then  $U'(f(\cdot))$  remains bounded on bounded subsets of  $[0, +\infty)$ .*

**Lemma 2.4** *Under Assumption 2.1,*

$$\lim_{q \rightarrow 0^+} H(k, q) = \lim_{c \rightarrow +\infty} U(c) - cU'(c) = \lim_{c \rightarrow +\infty} U(c) \in \mathbb{R} \cup \{+\infty\}, \quad (2.41)$$

$$\lim_{q \rightarrow 0^+} D_q H(k, q) = -\infty. \quad (2.42)$$

**Proof** From (2.37) and Assumption 2.1, we see that  $\lim_{q \rightarrow 0} c^*(q) = +\infty$ . Therefore, from (2.38), with the notation used in the proof of Lemma 2.2,  $\lim_{q \rightarrow 0} D_q H(k, q) = \lim_{q \rightarrow 0} h'(q) = -\infty$ .

We know that  $U(\cdot)$  is increasing: we set  $\ell_1 = \lim_{c \rightarrow +\infty} U(c) = \sup_{c > 0} U(c)$ ,  $\ell_1 \in \mathbb{R} \times \{+\infty\}$ . On the other hand, the function  $c \mapsto U(c) - cU'(c)$  is increasing on  $\mathbb{R}_+$ , because its derivative is  $c \mapsto -cU''(c)$ : we set  $\ell_2 = \lim_{c \rightarrow +\infty} U(c) - cU'(c)$ ,  $\ell_2 \in \mathbb{R} \times \{+\infty\}$

We see that  $H(k, q) \sim U(c^*(q)) - c^*(q)U'(c^*(q))$  as  $q \rightarrow 0$ . Therefore,  $\lim_{q \rightarrow 0} H(k, q) = \ell_2$ .

We need to compare  $\ell_1$  and  $\ell_2$ . It is obvious that  $\ell_2 \leq \ell_1$ . We wish to prove that  $\ell_2 = \ell_1$ . We argue by contradiction and assume that  $\ell_2 < \ell_1$ . We make out two cases:

1.  $\ell_1 \in \mathbb{R}$  and  $\ell_2 < \ell_1$ : we see that  $cU'(c)$  tends to  $\ell_1 - \ell_2 > 0$  as  $c$  tends to  $+\infty$ . This implies that  $U(c)$  blows up like a logarithm of  $c$  as  $c$  tends to  $+\infty$ , in contradiction with the fact that  $\ell_1 < +\infty$ . Therefore, if  $\ell_1$  is finite, then  $\ell_1 = \ell_2$ .
2.  $\ell_1 = +\infty$  and  $\ell_2 \in \mathbb{R}$ . Since  $c \mapsto U(c) - cU'(c)$  is increasing, we infer that  $U'(c) \geq (U(c) - \ell_2)/c$  for all  $c > 0$ . This implies that there exists a constant  $u_0$  such that  $U(c) \geq c - u_0$  for  $c$  large enough. Thus,  $\liminf_{c \rightarrow \infty} U'(c) \geq 1$  in contradiction with Assumption 2.1.

The proof is completed. □

Lemmas 2.2 and 2.3 above allow us to define the increasing and decreasing parts of the Hamiltonian, see Figure 2.1:

**Definition 2.1** We define the sets

$$\begin{aligned} \Theta^\uparrow &= \{(k, q) \text{ such that } k > 0, q \geq U'(f(k))\}, \\ \Theta^\downarrow &= \{(k, q) \text{ such that } k > 0, q \leq U'(f(k))\}. \end{aligned}$$

Let

- $H^\uparrow(\cdot, \cdot)$  be the restriction of  $H(\cdot, \cdot)$  to  $\Theta^\uparrow$ . The function  $q \mapsto H^\uparrow(k, q)$  is increasing on  $[U'(f(k)), \infty)$ .
- $H^\downarrow(\cdot, \cdot)$  be the restriction of  $H(\cdot, \cdot)$  to  $\Theta^\downarrow$ . The function  $q \mapsto H^\downarrow(k, q)$  is decreasing on  $(0, U'(f(k))]$ .

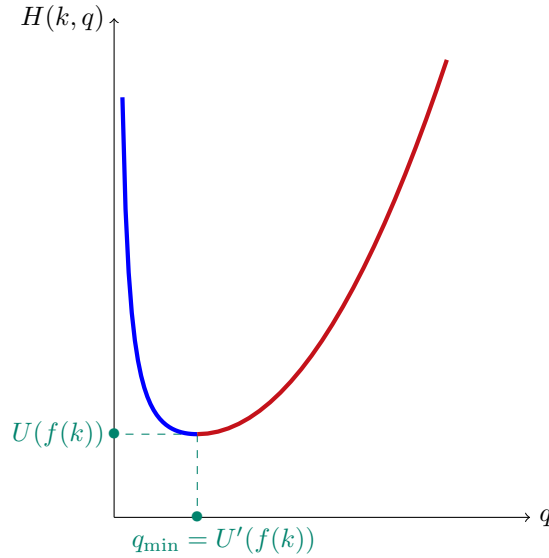


Figure 2.1: The bold line (blue and red) is the graph of the function  $H(k, \cdot)$ . The blue line is the graph of  $H^\downarrow(k, \cdot)$ . The red line is the graph of  $H^\uparrow(k, \cdot)$ . In the present figure,  $\lim_{q \rightarrow 0_+} H(k, q) = +\infty$ , but it is also possible that  $\lim_{q \rightarrow 0_+} H(k, q) \in \mathbb{R}$ .

**Lemma 2.5** Under Assumption 2.1 and 2.2,  $H^\downarrow(\cdot, \cdot)$  (respectively  $H^\uparrow(\cdot, \cdot)$ ) is of class  $C^1$  on  $\Theta^\downarrow$  (respectively  $\Theta^\uparrow$ ).

**Proof** We have already seen in the proof of Lemma 2.2 that  $q \mapsto H(k, q)$  is of class  $C^2$ . Moreover, from Assumption 2.2,  $k \mapsto f(k)q$  is of class  $C^1$ , so  $k \mapsto H(k, q)$  is also of class  $C^1$ . Hence  $(k, q) \mapsto H^\downarrow(k, q)$  is of class  $C^1$  on  $\Theta^\downarrow$ , and so is  $(k, q) \mapsto H^\uparrow(k, q)$  on  $\Theta^\uparrow$ .  $\square$

**Remark 2.6 (General orientation)** The target capital must be  $k^*$ . The optimal investment of a firm with the level of capital  $\kappa^*$  must be zero, i.e.  $D_q H(\kappa^*, u'(\kappa^*)) = 0$ . Heuristically, if  $u(\cdot)$  is a classical solution of (2.31)-(2.33), take formally the derivative of (2.31): we obtain that

$$(f'(k) - \rho)u'(k) = -D_q H(k, u'(k))u''(k), \quad \text{for } k > 0.$$

If  $u'(k) > 0$  for  $k > 0$ . Assuming also that  $u''(\cdot)$  is locally bounded, the equation  $D_q H(\kappa^*, u'(\kappa^*)) = 0$  then implies that  $f'(\kappa^*) = \rho$ , and  $\kappa^* = k^*$ . This is why we are going to look for a solution of (2.31)-(2.33) with  $\kappa^* = k^*$ .

**Remark 2.7** Heuristically, going back to (2.31)-(2.33) with  $\kappa^* = k^*$ , we see that

$$H(k, u'(k)) = \begin{cases} H^\uparrow(k, u'(k)), & \text{if } k < k^*, \\ H^\downarrow(k, u'(k)), & \text{if } k > k^*. \end{cases}$$

If  $u(\cdot)$  is a classical solution, then it must satisfy  $D_q H(k^*, u'(k^*)) = 0$ , which yields that  $u'(k^*) = U'(f(k^*))$  and from (2.31), this implies that  $u(k^*) = U(f(k^*))/\rho$ , see Lemma 2.3 and Figure 2.1. This is why we expect that the solution of (2.31)-(2.33) will be such that  $\kappa^* = k^*$ ,  $u(k^*) = U(f(k^*))/\rho$  and  $u'(k^*) = U'(f(k^*))$ .

In order to construct the restrictions of the solution to (2.31)-(2.33) to  $(0, k^*)$  and  $(k^*, +\infty)$ , we are going to use arguments from the theory of ordinary differential equations, and for that, we need to consider the inverse maps of  $q \mapsto H^\uparrow(k, q)$  and  $q \mapsto H^\downarrow(k, q)$ :

**Definition 2.2** We define the sets

$$\Omega^\uparrow = \{(k, v) : k \in (0, k^*], \rho v \in (U(f(k)), +\infty)\}, \quad (2.43)$$

$$\Omega^\downarrow = \left\{ (k, v) : k \in [k^*, +\infty), \rho v \in \left( U(f(k)), \lim_{q \rightarrow 0^+} H(k, q) \right) \right\}. \quad (2.44)$$

Set

$$\mathcal{F}^\uparrow(k, v) = (H^\uparrow(k, \cdot))^{-1}(\rho v), \quad \forall (k, v) \in \Omega^\uparrow, \quad (2.45)$$

$$\mathcal{F}^\downarrow(k, v) = (H^\downarrow(k, \cdot))^{-1}(\rho v), \quad \forall (k, v) \in \Omega^\downarrow. \quad (2.46)$$

**Lemma 2.6** Under Assumption 2.1 and 2.2,  $\mathcal{F}^\downarrow(\cdot, \cdot)$  (respectively  $\mathcal{F}^\uparrow(\cdot, \cdot)$ ) is of class  $C^1$  on  $\Omega^\downarrow$  (respectively  $\Omega^\uparrow$ ).

**Proof** Let us write the proof for  $\mathcal{F}^\downarrow(\cdot, \cdot)$  only, since the arguments are similar for  $\mathcal{F}^\uparrow(\cdot, \cdot)$ . Instead of trying to adapt a general result, we choose to write the proof completely.

Take first a sequence  $(k_n, v_n)_{n \in \mathbb{N}}$  that converges to  $(k, v) \in \Omega^\downarrow$  as  $n \rightarrow \infty$ . The real number  $q_n = \mathcal{F}^\downarrow(k_n, v_n) \in (0, U'(f(k_n)))$  is such that  $\rho v_n = H(k_n, q_n)$ . From Remark 2.2,  $f(k_n)$  stays bounded away from 0, and  $U'(f(k_n))$  tends to  $U'(f(k))$ . This implies that  $q_n$  is bounded from above. Up to the extraction of a subsequence, we may assume that  $q_n \rightarrow q$  as  $n \rightarrow \infty$ . Note that it is not possible that  $q = 0$ , because it would imply that  $\rho v = \lim_{q \rightarrow 0} H(k, q)$  (which does not depend on  $k$  from the structure of  $H(\cdot, \cdot)$ ), in contradiction with the fact that  $(k, v) \in \Omega^\downarrow$ . Passing to the limit, we see that  $\rho v = H(k, q)$ .

Since  $q \in (0, U'(f(k))]$ , this proves that  $q = \mathcal{F}^\downarrow(k, v)$ . We have proven that  $\mathcal{F}^\downarrow(\cdot, \cdot)$  is continuous on  $\Omega^\downarrow$ . It is also easy to check that  $\mathcal{F}^\downarrow(\cdot, \cdot)$  can be extended by continuity to  $\overline{\Omega^\downarrow}$ .

Let us prove that  $\mathcal{F}^\downarrow(\cdot, \cdot)$  is continuously differentiable on  $\Omega^\downarrow$ . For that, take  $(k, v)$  in  $\Omega^\downarrow$  and set  $q = \mathcal{F}^\downarrow(k, v)$ . Since  $H(k, \cdot)$  is decreasing in  $(0, U'(f(k)))$ , there exists a neighbourhood  $\mathcal{V}$  of  $\rho v$  in  $(U(f(k)), \lim_{q \rightarrow 0} H(k, q))$  and a neighbourhood  $\mathcal{Q}$  of  $q$  in  $[0, U'(f(k))]$  such that  $H(k, \cdot)$  defines a  $C^1$  diffeomorphism from  $\mathcal{Q}$  onto  $\mathcal{V}$ . Hence  $v \mapsto \mathcal{F}(k, v)$  is  $C^1$  and

$$\frac{\partial \mathcal{F}^\downarrow}{\partial v}(k, v) = \frac{\rho}{D_q H^\downarrow(k, \mathcal{F}^\downarrow(k, v))}.$$

This proves that  $(k, v) \mapsto \frac{\partial \mathcal{F}^\downarrow}{\partial v}(k, v)$  is continuous on  $\Omega^\downarrow$ .

Then, taking  $(k_0, v_0)$  in  $\Omega^\downarrow$  and setting  $q = \mathcal{F}^\downarrow(k_0, v_0)$ , we deduce from the implicit function theorem that there exists a neighbourhood  $\mathcal{C}$  of  $k_0$  in  $[k^*, +\infty)$ , a neighbourhood  $\mathcal{O}$  of  $(k_0, v_0)$  in  $\Omega^\downarrow$  and a map  $\varphi(v_0; \cdot) : \mathcal{C} \rightarrow \mathbb{R}$  of class  $C^1$  such that

$$(k, q) \in \mathcal{O} \text{ and } H^\downarrow(k, q) = \rho v_0 \quad \Leftrightarrow \quad k \in \mathcal{C} \text{ and } q = \varphi(v_0; k).$$

Hence, for all  $k \in \mathcal{C}$ ,  $\mathcal{F}^\downarrow(k, v_0) = \varphi(v_0; k)$ , and

$$\frac{\partial \mathcal{F}^\downarrow}{\partial k}(k_0, v_0) = - \frac{\frac{\partial H^\downarrow}{\partial k}(k_0, \mathcal{F}^\downarrow(k_0, v_0))}{D_q H^\downarrow(k_0, \mathcal{F}^\downarrow(k_0, v_0))}.$$

Therefore  $\frac{\partial \mathcal{F}^\downarrow}{\partial k}(\cdot, \cdot)$  is continuous on  $\Omega^\downarrow$ , which achieves the proof.  $\square$

### 2.3.3 A solution of the Hamilton-Jacobi equation on $[k^*, +\infty)$

We aim at showing the existence of a solution of (2.31), (2.33) on  $[k^*, +\infty)$  by considering first the unique maximal solution  $\phi_\lambda(\cdot)$  of the Cauchy problem

$$\phi'_\lambda(k) = \mathcal{F}^\downarrow(k, \phi_\lambda(k)), \quad \text{for } k \geq k^*, \quad (2.47)$$

$$(k, \phi_\lambda(k)) \in \Omega^\downarrow, \quad (2.48)$$

$$\phi_\lambda(k^*) = \lambda, \quad (2.49)$$

for  $\lambda$  such that  $(k^*, \lambda) \in \Omega^\downarrow$ , see (2.43). Note that  $\mathcal{F}^\downarrow(\cdot, \cdot)$  is regular enough on  $\Omega^\downarrow$  so that the Cauchy-Lipschitz theorem may be applied. Then, after having proved the existence and uniqueness of  $\phi_\lambda(\cdot)$ , we will let  $\lambda$  tend to  $U(f(k^*))/\rho$  and show that the sequence  $\phi_\lambda(\cdot)$  converges to a solution of (2.31), (2.33). One reason for not applying directly the standard existence results for ODEs to the equation with initial condition  $\lambda = U(f(k^*))/\rho$  is that  $\mathcal{F}^\downarrow(\cdot, \cdot)$  is not regular at the boundary of  $\Omega^\downarrow$ . In particular,  $v \mapsto \mathcal{F}^\downarrow(k^*, v)$  is not Lipschitz continuous in the neighbourhood of  $(k^*, U(f(k^*))/\rho)$ . Moreover, the point  $(k^*, U(f(k^*))/\rho)$  belongs to the boundary of  $\Omega^\downarrow$ ; this forbids the direct use of Cauchy-Peano-Arzelà theorem for obtaining the existence of a solution.

**Proposition 2.3** *Under Assumptions 2.1 and 2.2, for every  $\lambda$  such that  $(k^*, \lambda) \in \Omega^\downarrow$ , there exists a unique global solution  $\phi_\lambda(\cdot)$  of (2.47)-(2.49) in  $[k^*, +\infty)$ , and  $\phi_\lambda(\cdot)$  is increasing and strictly concave.*

**Proof** Setting  $\Theta(k) = (k, \phi_\lambda(k))$ , it is convenient to rewrite (2.47)-(2.49) in the equivalent form: find  $k \mapsto \Theta(k) \in \Omega^\downarrow$  such that

$$\Theta'(k) = (1, \mathcal{F}^\downarrow(\Theta(k))), \quad k \geq k^*, \quad (2.50)$$

$$\Theta(k^*) = (k^*, \lambda). \quad (2.51)$$

We may apply Cauchy-Lipschitz theorem; indeed, from Lemma, the map  $\Theta \mapsto (1, \mathcal{F}^\downarrow(\Theta))$  is  $C^1$  on  $\Omega^\downarrow$ . Therefore, there exists a unique maximal solution  $\Theta$  of (2.50)-(2.51) in  $[k^*, \bar{k}]$ . We observe that for  $k \in [k^*, \bar{k})$ ,  $\phi'_\lambda(k) = \mathcal{F}^\downarrow(k, \phi_\lambda(k)) > 0$ , so  $\lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k)$  exists. Moreover, by taking the derivative,

$$\phi''_\lambda(k) = \frac{\rho - f'(k)}{D_q H(k, \phi'_\lambda(k))} \phi'_\lambda(k) < 0.$$

Therefore  $\phi_\lambda(\cdot)$  is strictly concave in  $[k^*, \bar{k})$ .

If  $\bar{k} \neq \infty$ , then from Cauchy-Lipschitz theorem,  $\rho \lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k)$  must be equal either to  $U(f(\bar{k}))$  or to  $\lim_{q \rightarrow 0} H(k, q) = \lim_{c \rightarrow +\infty} U(c)$  (which does not depend on  $k$ ).

1. We wish to rule out the case when  $\rho \lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k) = \lim_{q \rightarrow 0} (\sup_{c \geq 0} \{U(c) - qc\})$ ; we make out two subcases:
  - (a) If  $\lim_{c \rightarrow +\infty} U(c) = +\infty$ , then  $\lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k) = +\infty$ , which yields that  $\lim_{k \rightarrow \bar{k}^-} \mathcal{F}^\downarrow(k, \phi_\lambda(k)) = 0$ . Using the ODE, this implies that  $\lim_{k \rightarrow \bar{k}^-} \phi'_\lambda(k) = 0$ , which contradicts the fact that  $\lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k) = +\infty$ . Therefore, if  $\lim_{c \rightarrow +\infty} U(c) = +\infty$  then it is not possible that  $\lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k) = +\infty$ .
  - (b) If  $\lim_{c \rightarrow +\infty} U(c) = \ell \in \mathbb{R}$ , then it is possible to extend continuously  $\phi_\lambda(\cdot)$  to  $\bar{k}$  by setting  $\phi_\lambda(\bar{k}) = \ell/\rho$ . On the other hand, the limit of the optimal control  $c^*(q)$  as  $q$  tends to 0 must be  $+\infty$ , from (2.37) and Assumption 2.1. From this, we deduce that  $\mathcal{F}^\downarrow(k, \ell/\rho) = 0$  for all  $k$  and that  $\frac{\partial \mathcal{F}^\downarrow}{\partial v}(k, \ell/\rho) = 0$  for all  $k$ . This makes it impossible for the state  $\ell/\rho$  to be reached in finite time by a solution of (2.47)-(2.48). Therefore, it is not possible that  $\phi_\lambda(\bar{k}) = \ell/\rho$ .
2. We wish to rule out the case when  $\lim_{k \rightarrow \bar{k}^-} \phi_\lambda(k) = U(f(\bar{k}))/\rho$ . In this case, it is possible to extend continuously  $\phi_\lambda(\cdot)$  to  $\bar{k}$  by setting  $\phi_\lambda(\bar{k}) = U(f(\bar{k}))/\rho$ , and the differential equations holds in  $[k^*, \bar{k}]$ . We see that

$$\frac{d}{dk} \left( \frac{U(f(k))}{\rho} \right) - \mathcal{F}^\downarrow \left( k, \frac{U(f(k))}{\rho} \right) = U'(f(k)) \frac{f'(k) - \rho}{\rho} < 0, \quad \text{for } k > k^*, \quad (2.52)$$

from the definition of  $k^*$  and Assumption 2.2. Hence  $k \mapsto U(f(k))/\rho$  is a subsolution of the ODE satisfied by  $\phi_\lambda(\cdot)$ . This implies that if  $\phi_\lambda(\bar{k}) = U(f(\bar{k}))/\rho$ , then  $U(f(k))/\rho > \phi_\lambda(k)$  for  $k < \bar{k}$ , which contradicts the fact that  $(k, \phi_\lambda(k)) \in \Omega^\downarrow$  for  $k < \bar{k}$ .

We have proven that  $\bar{k} = +\infty$ . The unique maximal solution of (2.50)-(2.51) is a global solution.  $\square$

Letting  $\lambda$  tend to  $U(f(k^*))/\rho$ , we shall prove the following result:

**Proposition 2.4** *Under Assumption 2.1 and 2.2, there exists a unique  $u^\downarrow : [k^*, +\infty) \rightarrow \mathbb{R}$  such that*

$$\frac{du^\downarrow}{dk}(k) = \mathcal{F}^\downarrow(k, u^\downarrow(k)), \quad \text{for } k \geq k^*, \quad (2.53)$$

$$(k, u^\downarrow(k)) \in \Omega^\downarrow, \quad \text{for } k > k^*, \quad (2.54)$$

$$u^\downarrow(k^*) = \frac{1}{\rho} U(f(k^*)). \quad (2.55)$$

**Proof** Consider a decreasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$ , such for all  $n \in \mathbb{N}$ ,  $(k^*, \lambda_n) \in \Omega^\downarrow$  and  $\lim_{n \rightarrow \infty} \lambda_n = U(f(k^*))/\rho$ .

A direct consequence of Cauchy-Lipschitz theorem is that  $\phi_{\lambda_n}(k) > \phi_{\lambda_{n+1}}(k)$  for all  $k \geq k^*$ .

On the other hand, we know that  $\phi_{\lambda_n}(k) \geq U(f(k))/\rho$ . This implies that there exists a function  $\phi : [k^*, +\infty) \rightarrow \mathbb{R}$  such that  $\phi_{\lambda_n}(\cdot)$  converge to  $\phi(\cdot)$  pointwise as  $n$  tends to  $+\infty$ . Since  $(\phi_{\lambda_n}(\cdot))_{n \in \mathbb{N}}$  is a sequence of concave functions locally uniformly bounded, we see from [27, Theorem 3.3.3] that the

convergence is uniform on every compact set, so the limit  $\phi(\cdot)$  is continuous. Since  $\mathcal{F}^\downarrow(\cdot, \cdot)$  is continuous on the closure of  $\Omega^\downarrow$ , we see that for all  $k \geq k^*$ ,

$$\lim_{n \rightarrow \infty} \int_{k^*}^k \mathcal{F}^\downarrow(\kappa, \phi_{\lambda_n}(\kappa)) d\kappa = \int_{k^*}^k \mathcal{F}^\downarrow(\kappa, \phi(\kappa)) d\kappa.$$

Passing to the limit in the integral form of (2.47), we see that for all  $k \geq k^*$ ,

$$\phi(k) = \frac{1}{\rho} U(f(k^*)) + \int_{k^*}^k \mathcal{F}^\downarrow(\kappa, \phi(\kappa)) d\kappa.$$

This implies that  $\phi(\cdot) \in C^1([k^*, +\infty))$  and that  $\phi(\cdot)$  satisfies (2.53) and (2.55). Hence  $\phi(\cdot)$  is an increasing function.

On the other hand, (2.52) implies that  $\phi(k) > U(f(k))/\rho$  for all  $k > k^*$ . This shows that  $\phi(\cdot)$  satisfies (2.54).

Arguing as in the proof of Proposition 2.3, we see that  $\phi(\cdot)$  is  $C^2$  on  $(k^*, +\infty)$  and strictly concave. We have proved the existence of a solution of (2.53)-(2.55).

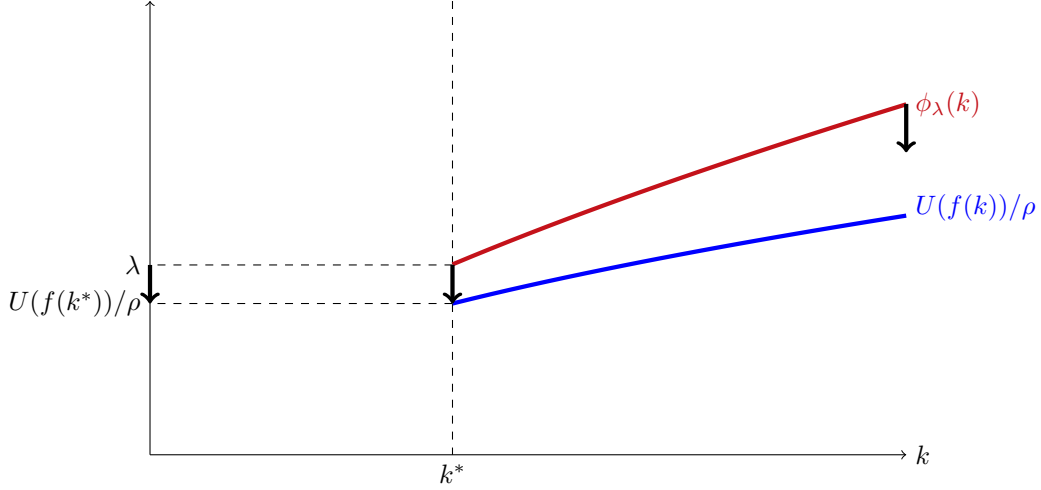


Figure 2.2: The principle of the proof: to let  $\lambda$  tend to  $U(f(k^*))/\rho$

Assume that there are two solutions  $\phi_1(\cdot)$  and  $\phi_2(\cdot)$  of (2.53)-(2.55). If there exists  $k_0 > k^*$  such that  $\phi_1(k_0) = \phi_2(k_0)$ , then  $\phi_1(\cdot)$  and  $\phi_2(\cdot)$  coincide from Cauchy-Lipschitz theorem. Let us assume that  $\phi_1(k) < \phi_2(k)$  for  $k > k^*$ . Then for every  $k > k^*$ ,

$$0 > \phi_1(k) - \phi_2(k) = \int_{k^*}^k \mathcal{F}^\downarrow(\kappa, \phi_1(\kappa)) - \mathcal{F}^\downarrow(\kappa, \phi_2(\kappa)) d\kappa \geq 0,$$

where the last inequality comes from the fact that  $\mathcal{F}^\downarrow(k, \cdot)$  is non increasing. This shows that the considered situation is not possible. This achieves the proof of uniqueness.  $\square$

### 2.3.4 A solution of the Hamilton-Jacobi equation on $(0, k^*]$

To prove the existence of a solution of (2.31), (2.32) on  $(0, k^*]$ , we start by studying the Cauchy problem

$$\psi'_{\epsilon, \lambda}(k) = \mathcal{F}^\uparrow(k, \psi_{\epsilon, \lambda}(k)), \quad (2.56)$$

$$(k, \psi_{\epsilon, \lambda}(k)) \in \Omega^\uparrow, \quad (2.57)$$

$$\psi_{\epsilon, \lambda}(\epsilon) = \lambda, \quad (2.58)$$

for  $(\epsilon, \lambda) \in \Omega^\uparrow$ , see (2.46) (thus  $0 < \epsilon < k^*$ ). As above,  $\mathcal{F}^\uparrow(\cdot, \cdot)$  is regular enough so that the Cauchy-Lipschitz theorem may be applied. Then, after having proved the existence and uniqueness of  $\psi_{\epsilon, \lambda}(\cdot)$ , we will show that there exists  $\lambda$  such that the solution is global, i.e. defined on  $(0, k^*]$  and such that  $\psi_{\epsilon, \lambda}(k^*) = U(f(k^*))/\rho$ . Here also,  $\mathcal{F}^\uparrow(k, \cdot)$  is not Lipschitz continuous in the neighbourhood of  $(k^*, U(f(k^*))/\rho)$  and  $(k^*, U(f(k^*))/\rho)$  belongs to the boundary of  $\Omega^\uparrow$ , which prevents us from applying directly the standard existence results for ODEs to the problem with terminal condition  $\psi(k^*) = U(f(k^*))/\rho$ .

**Lemma 2.7** *Under Assumptions 2.1 and 2.2, for every  $(\epsilon, \lambda) \in \Omega^\uparrow$  with  $\epsilon < k^*$ , there exists a unique maximal solution of the Cauchy problem (2.56)-(2.58) of the form  $\left( (0, \bar{k}(\epsilon, \lambda)), \psi_{\epsilon, \lambda}(\cdot) \right)$  where  $\epsilon < \bar{k}(\epsilon, \lambda) \leq k^*$ . The function  $\psi_{\epsilon, \lambda}(\cdot)$  is strictly concave and increasing on  $(0, \bar{k}(\epsilon, \lambda))$ .*

**Proof** The existence and uniqueness of a maximal solution follows from the Cauchy-Lipschitz theorem. The strict monotonicity and concavity of  $\psi_{\epsilon, \lambda}(\cdot)$  are obtained as in Proposition 2.3. Assume that the maximal solution was not defined in an interval of the form  $(0, \bar{k}(\epsilon, \lambda))$ . This would imply that there exists  $\underline{k} \in (0, \lambda)$  such that either  $\lim_{k \rightarrow \underline{k}} \psi_{\epsilon, \lambda}(k) = -\infty$  or  $\psi_{\epsilon, \lambda}(\underline{k}) = U(f(\underline{k}))/\rho$ . Let us rule out these two situations:

- If  $\lim_{k \rightarrow \underline{k}} \psi_{\epsilon, \lambda}(k) = -\infty$ , then  $\lim_{k \rightarrow \underline{k}} \psi'_{\epsilon, \lambda}(k) = +\infty$  which is only possible if  $\lim_{k \rightarrow \underline{k}} \psi_{\epsilon, \lambda}(k) = U(f(\underline{k}))/\rho$ .
- If  $\psi_{\epsilon, \lambda}(\underline{k}) = U(f(\underline{k}))/\rho$ , then proceeding as in the end of the proof of Proposition 2.3, we see that this would imply that  $\psi_{\epsilon, \lambda}(k) \leq U(f(k))/\rho$  for all  $k \in [\underline{k}, \epsilon]$  while we know that  $\psi_{\epsilon, \lambda}(\epsilon) = \lambda > U(f(\epsilon))/\rho$ .

Therefore the maximal solution is defined in an interval of the form  $(0, \bar{k}(\epsilon, \lambda))$ . □

**Remark 2.8** *Note that if  $f(0) = 0$ , then  $\psi'_{\epsilon, \lambda}(k)$  may blow up when  $k \rightarrow 0^+$ : indeed, from (2.38)*

$$0 < D_q H(k, \psi'_{\epsilon, \lambda}(k)) = f(k) - c^*(\psi'_{\epsilon, \lambda}(k)),$$

hence  $c^*(\psi'_{\epsilon, \lambda}(k)) < f(k)$ . Therefore,  $U'(c^*(\psi'_{\epsilon, \lambda}(k))) > U'(f(k))$ . Thus, from (2.37)

$$\psi'_{\epsilon, \lambda}(k) = U'(c^*(\psi'_{\epsilon, \lambda}(k))) > U'(f(k))$$

tends to  $+\infty$  as  $k$  tends to 0.

We then define the set  $\Lambda_\epsilon$  as follows:

$$\Lambda_\epsilon = \{ \lambda > U(f(\epsilon))/\rho \text{ such that } \bar{k}(\epsilon, \lambda) = k^* \}. \quad (2.59)$$

**Lemma 2.8** *Under Assumptions 2.1 and 2.2, for every  $\epsilon \in (0, k^*)$ , the set  $\Lambda_\epsilon$  is not empty.*

**Proof** Take  $\lambda > U(f(k^*))/\rho$ . The maximal solution  $\psi_{\epsilon, \lambda}(\cdot)$  of the Cauchy problem (2.56)-(2.58) is defined in  $(0, \bar{k}(\epsilon, \lambda))$ , with  $\bar{k}(\epsilon, \lambda) > \epsilon$ .

Assume by contradiction that  $\bar{k}(\epsilon, \lambda) < k^*$ .

Observe first that  $\psi_{\epsilon, \lambda}(\cdot)$  cannot blow up when  $k \rightarrow \bar{k}(\epsilon, \lambda)$ . Indeed  $v \mapsto \mathcal{F}^\uparrow(k, \rho v)$  is Lipschitz continuous on  $[\max_{k \in [\epsilon, k^*]} U(f(k)) + 1, +\infty)$  with a Lipschitz constant that does not depend on  $k \in [\epsilon, k^*]$ . This property prevents  $\psi_{\epsilon, \lambda}(\cdot)$  from blowing up in finite time.

Therefore, the function  $\psi_{\epsilon, \lambda}(\cdot)$  can be extended to  $\bar{k}(\epsilon, \lambda)$  by continuity, and

$$\psi_{\epsilon, \lambda}(\bar{k}(\epsilon, \lambda)) = U(f(\bar{k}(\epsilon, \lambda)))/\rho, \quad (2.60)$$



otherwise it would not be the maximal solution. On the other hand, we know that  $f(\cdot)$  is increasing on  $(0, k^*]$ , so  $U(f(k^*)) > U(f(k))$  for all  $k < k^*$ . From the monotonicity of  $\psi_{\epsilon, \lambda}(\cdot)$ , we obtain that

$$\psi_{\epsilon, \lambda}(\bar{k}(\epsilon, \lambda)) \geq \psi_{\epsilon, \lambda}(\epsilon) = \lambda > U(f(k^*))/\rho > U(f(\bar{k}(\epsilon, \lambda)))/\rho,$$

which contradicts (2.60).

We have thus proved that if  $\lambda > U(f(k^*))/\rho$ , then the maximal solution is defined on  $(0, k^*]$ . Therefore  $\Lambda_\epsilon$  is not empty.  $\square$

**Proposition 2.5** *For all  $\epsilon < k^*$ , there exists  $\lambda$  such that  $(\epsilon, \lambda) \in \Omega^\dagger$  and a global solution  $\psi_{\epsilon, \lambda}(\cdot)$  of the Cauchy problem (2.56)-(2.58) (thus defined on  $(0, k^*]$ ) such that  $\psi_{\epsilon, \lambda}(k^*) = U(f(k^*))/\rho$ .*

**Proof** Consider a decreasing sequence  $(\lambda_n)_{n \in \mathbb{N}}$  in  $\Lambda_\epsilon$  such that  $\lim_{n \rightarrow \infty} \lambda_n = \underline{\lambda}_\epsilon = \inf_{\lambda \in \Lambda_\epsilon} \lambda$ . It is clear that  $(\psi_{\epsilon, \lambda_n}(\cdot))_{n \in \mathbb{N}}$  is a decreasing sequence of functions defined on  $(0, k^*]$ . Moreover, since

$$(k, \psi_{\epsilon, \lambda_n}(k)) \in \Omega^\dagger$$

for  $k \in (0, k^*)$ , we see that  $\psi_{\epsilon, \lambda_n}(\cdot)$  is bounded from below by  $U(f(\cdot))/\rho$ . Hence, there exists a function  $\psi_\epsilon(\cdot)$  defined on  $(0, k^*]$  such that  $\lim_{n \rightarrow +\infty} \psi_{\epsilon, \lambda_n}(k) = \psi_\epsilon(k)$  for all  $k \in (0, k^*]$ , see Figure 2.3.

Since  $(\psi_{\epsilon, \lambda_n}(\cdot))_{n \in \mathbb{N}}$  is a sequence of concave functions locally uniformly bounded, we see from [27, Theorem 3.3.3] that the convergence is uniform on every compact set, so  $\psi_\epsilon(\cdot)$  is continuous on  $(0, k^*]$ . Extending  $\mathcal{F}^\dagger(\cdot, \cdot)$  by continuity on the set  $\{(k, U(f(k))/\rho) : k \in (0, k^*)\}$ , we may pass to the limit in the integral form:

$$\psi_{\epsilon, \lambda_n}(k) = \lambda_n + \int_\epsilon^k \mathcal{F}^\dagger(\kappa, \psi_{\epsilon, \lambda_n}(\kappa)) d\kappa,$$

which yields that

$$\psi_\epsilon(k) = \underline{\lambda}_\epsilon + \int_\epsilon^k \mathcal{F}^\dagger(\kappa, \psi_\epsilon(\kappa)) d\kappa.$$

We see that  $\psi_\epsilon(\cdot)$  is a solution of (2.56) on  $(0, k^*)$ , which implies that  $\psi_\epsilon(\cdot)$  is  $C^1$  and increasing on  $(0, k^*)$ . We are left with proving that  $\psi_\epsilon(k^*) = U(f(k^*))/\rho$ . We already know that  $\psi_\epsilon(k^*) \geq U(f(k^*))/\rho$ . Assume by contradiction that  $\psi_\epsilon(k^*) > U(f(k^*))/\rho$ . Then, set

$$b = \frac{\psi_\epsilon(k^*) + U(f(k^*))/\rho}{2},$$

and consider the Cauchy problem on  $(0, k^*]$

$$\begin{aligned} \xi'(k) &= \mathcal{F}^\dagger(k, \xi(k)), \\ (k, \xi(k)) &\in \Omega^\dagger, \\ \xi(k^*) &= b. \end{aligned}$$

It can be proved by contradiction (with the same kind of argument as in the end of the proof of Proposition 2.3) that the maximal solution of this problem is in fact global, therefore defined in  $(0, k^*]$ . Cauchy-Lipschitz theorem implies that  $\xi(k) < \psi_\epsilon(k)$  for all  $k \in (0, k^*]$ . Therefore, we find that  $\xi(\epsilon) \in \Lambda_\epsilon$  and  $\xi(\epsilon) < \psi_\epsilon(\epsilon) = \underline{\lambda}_\epsilon$ , which contradicts the definition of  $\underline{\lambda}_\epsilon$ .

Therefore,  $\psi_\epsilon(k^*) = U(f(k^*))/\rho$ . With the same arguments as in the proof of Proposition 2.3, we may also prove that  $\psi_\epsilon(k) > U(f(k))/\rho$  for all  $k \in (0, k^*)$ . Hence  $\psi_\epsilon(\cdot) = \psi_{\epsilon, \underline{\lambda}_\epsilon}(\cdot)$ . This achieves the proof.

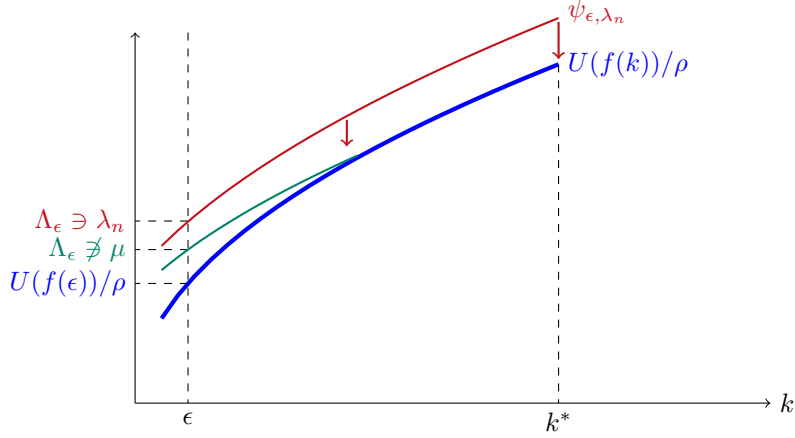


Figure 2.3: The principle of the proof: the solution corresponds to  $\inf \Lambda_\epsilon$

□

**Corollary 2.3** Under Assumption 2.1 and 2.2, there exists a unique  $u^\uparrow : (0, k^*] \rightarrow \mathbb{R}$  such that

$$\frac{du^\uparrow}{dk}(k) = \mathcal{F}^\uparrow(k, u^\uparrow(k)), \quad \text{for } 0 < k \leq k^*, \quad (2.61)$$

$$(k, u^\uparrow(k)) \in \Omega^\uparrow, \quad \text{for } 0 < k < k^*, \quad (2.62)$$

$$u^\uparrow(k^*) = \frac{1}{\rho} U(f(k^*)). \quad (2.63)$$

**Proof** Existence is a consequence of Proposition 2.5. Uniqueness is proved exactly with the same arguments as in the proof of Proposition 2.4. □

**Remark 2.9** From Remark 2.8, it is possible that  $\frac{du^\uparrow}{dk}(k)$  tends to  $+\infty$  when  $k \rightarrow 0$ . It is also possible that  $u^\uparrow(k)$  tends to  $-\infty$  as  $k \rightarrow 0$ .

### 2.3.5 Conclusion

Let us define

$$u(k) = \begin{cases} u^\uparrow(k), & \text{if } k \in (0, k^*], \\ u^\downarrow(k), & \text{if } k \in [k^*, +\infty), \end{cases} \quad (2.64)$$

where  $u^\downarrow(\cdot)$  and  $u^\uparrow(\cdot)$  are the functions uniquely defined in Propositions 2.3 and 2.4. Note that  $u(k^*)$  is well defined since

$$u^\uparrow(k^*) = \frac{1}{\rho} U(f(k^*)) = u^\downarrow(k^*).$$

Similarly,

$$\frac{du^\uparrow}{dk}(k^*) = U'(f(k^*)) = \frac{du^\downarrow}{dk}(k^*).$$

Therefore  $u(\cdot)$  is of class  $C^1$  in  $(0, +\infty)$ , and  $C^2$  in  $(0, k^*) \cup (k^*, +\infty)$ . From Proposition 2.3 and 2.4,  $u(\cdot)$  is a classical solution of (2.31) in  $(0, k^*) \cup (k^*, +\infty)$ , thus in  $(0, +\infty)$  since  $u(\cdot)$  is  $C^1$ . Moreover,

$$D_q H(k, u'(k)) \begin{cases} > 0, & \text{if } k \in (0, k^*), \\ < 0, & \text{if } k \in (k^*, +\infty), \\ = 0 & \text{if } k = k^*. \end{cases}$$

We have proved the existence of a classical solution of (2.31)-(2.33).

Regarding uniqueness, for any positive numbers  $a, b$  such that  $0 < a < k^* < b$ , it is well known that there is at most one constrained viscosity solution of the following Hamilton-Jacobi in  $[a, b]$  (see [15, Theorem 5.8 p.278]). Therefore there exists a unique solution in  $C^1([a, b])$  of

$$-\rho v(k) + H(k, v'(k)) = 0, \quad \text{for } a < k < b, \quad (2.65)$$

$$D_q H(a, v'(a)) \geq 0, \quad (2.66)$$

$$D_q H(b, v'(b)) \leq 0. \quad (2.67)$$

On the other hand, if  $u(\cdot)$  is a solution of (2.31)-(2.33), then it is obviously a constrained viscosity solution of (2.65)-(2.67) in  $[a, b]$ . Therefore, the restriction of a solution of (2.31)-(2.33) must be unique. Since  $a$  and  $b$  are arbitrary, the solution  $u(\cdot)$  given in (2.64) is unique.

**Remark 2.10** *Note that we can extend  $D_q H(\cdot, u'(\cdot))$  by continuity in  $k = 0$ , in that case (2.15) holds. Moreover, using the same arguments as in Theorem 2.1, we can show that every classical solution in  $C^1(0, +\infty)$  of the Hamilton-Jacobi equation (2.12) supplemented with (2.15) is the value function of the optimal control problem (2.2). In particular, the solution given in (2.64) is the value function of (2.2).*

### 2.3.6 Extension to the case when there is depreciation of capital

We wish to extend Theorem 2.3 to the case when Assumption 2.2 is no longer valid, i.e. when  $f(\cdot)$  may take negative values. Indeed, going back to the examples in paragraph 2.3.1, with a rate of depreciation of capital  $\delta$  positive, the net output function is no longer positive on  $(0, +\infty)$ .

The following set of assumptions does not contain the non negativity of  $f(\cdot)$ :

**Assumption 2.3** *The net output function  $f : [0, +\infty) \rightarrow \mathbb{R}$  has the following properties:*

- i)  $f(\cdot)$  is continuous on  $[0, +\infty)$ .
- ii)  $f(\cdot)$  is locally of class  $C^{1,1}$  on  $(0, +\infty)$ .
- iii)  $f(0) \geq 0$ ,  $f(\cdot)$  is strictly concave and  $\lim_{k \rightarrow 0^+} f'(k) = +\infty$ .
- iv)  $f(k)$  is negative for  $k$  large enough.

**Lemma 2.9** *Under Assumption 2.3, there exists a unique value  $k_0 \in (0, +\infty)$  such that*

$$f(k_0) = 0,$$

*and  $f(\cdot)$  is positive on  $(0, k_0)$  and negative on  $(k_0, +\infty)$ .*

*Moreover, for  $k^*$  defined in (2.36) (whose existence and uniqueness is still guaranteed by Assumption 2.3),*

$$f'(k_0) < 0, \quad \text{and} \quad k^* < k_0.$$

**Proof** From Assumption 2.3,  $\lim_{k \rightarrow 0^+} f'(k) = +\infty$  and  $f(0) \geq 0$ ; this implies that  $f(k) > 0$  for  $k > 0$  small enough. On the other hand,  $f(k) < 0$  for  $k$  large enough. The continuity of  $f(\cdot)$  then yields the existence of  $k_0 \in (0, +\infty)$  such that  $f(k_0) = 0$ .

Observe that  $f'(k_0) < 0$ . Indeed, since  $f(\cdot)$  is continuous, there exists  $\hat{k} \in (0, k_0)$  such that

$$f(\hat{k}) = \max_{k \in [0, k_0]} f(k).$$

From the strict concavity of  $f(\cdot)$  we deduce that

$$f'(k_0) < f'(\hat{k}) = 0. \quad (2.68)$$

This yields the uniqueness of  $k_0$ . Indeed, assume that for some  $k_1 > k_0$ ,  $f(k_1) = 0$ . Then, from Rolle theorem,  $f'(\cdot)$  vanishes in the interval  $(k_0, k_1)$ , in contradiction with (2.68) and the strict concavity of  $f(\cdot)$ . There is no loss of generality in taking  $k_1 > k_0$ , because the roles of  $k_0$  and  $k_1$  can be exchanged. Note that (2.68) also implies that  $f'(k)$  has a negative limit as  $k \rightarrow +\infty$ .

Finally, by definition of  $k^*$ ,  $f(k^*) = \rho > 0 > f'(k_0)$ ; this implies that  $k^* < k_0$  from the strict concavity of  $f(\cdot)$ .  $\square$

For solving (2.31)-(2.33), Lemma 2.9 suggests to use the same construction as in paragraphs 2.3.3 and 2.3.4 in the interval  $(0, k_0)$  which contains  $k^*$  and where  $f(\cdot)$  is positive, and to apply new arguments in  $[k_0, +\infty)$ .

**Theorem 2.4** *All the conclusions of Theorem 2.3 hold under Assumptions 2.1 and 2.3.*

**Proof** In  $(0, k_0)$ , it is possible to repeat the construction made in paragraphs 2.3.3 and 2.3.4: there exists a unique classical solution  $u_1(\cdot)$  of the following problem:

$$-\rho u_1(k) + H(k, u_1'(k)) = 0, \quad \text{for } 0 < k < k_0, \quad (2.69)$$

$$D_q H(k, u_1'(k)) > 0, \quad \text{for } 0 < k < k^*, \quad (2.70)$$

$$D_q H(k, u_1'(k)) < 0, \quad \text{for } k^* < k < k_0, \quad (2.71)$$

where  $H(\cdot, \cdot)$  is defined in (2.34). The function  $u_1(\cdot)$  is of class  $C^1$  in  $(0, k_0)$ , strictly concave and increasing in  $(0, k_0)$ .

As already seen in the proof of Lemma 2.9, there exists  $\hat{k} \in (k^*, k_0)$  such that  $f(\hat{k}) = \max_{k \in [0, k_0]} f(k)$ . Since  $u_1(\cdot)$  is increasing,  $\lim_{k \rightarrow k_0} u_1(k_0) \geq u_1(\hat{k})$ . On the other hand,  $\rho u_1(\hat{k}) > U(f(\hat{k}))$  (see paragraph 2.3.3). Since  $U(\cdot)$  is increasing,  $U(f(\hat{k})) > \lim_{k \rightarrow k_0} U(f(k)) = \lim_{c \rightarrow 0} U(c)$  (which may be  $-\infty$ ). Therefore,

$$\rho \lim_{k \rightarrow k_0} u_1(k_0) > \lim_{c \rightarrow 0} U(c).$$

With the same kind of arguments as in the proof of Proposition 2.3, it can also be proved that  $\rho u_1(k_0) < \lim_{c \rightarrow +\infty} U(c)$ . This implies that  $u_1(\cdot)$  can be extended by continuity to  $(0, k_0]$  and that

$$\lim_{c \rightarrow 0} U(c) < \rho u_1(k_0) < \lim_{c \rightarrow \infty} U(c).$$

The function  $u_1'(\cdot)$  can then be extended by continuity to  $k = k_0$  and (2.69) holds up to  $k = k_0$ .

There remains to deal with  $(k_0, +\infty)$ . Observe that, for a given  $k \geq k_0$ ,  $H(k, \cdot)$  is decreasing from (2.38), and that

1.  $\lim_{q \rightarrow 0} H(k, q) = \lim_{c \rightarrow +\infty} U(c)$  for all  $k \geq k_0$ .
2. for all  $k \geq k_0$ ,  $\lim_{q \rightarrow +\infty} c^*(q) = 0$  and  $U(c) - cq + f(k)q \leq U(c)$ . Hence,

$$\lim_{q \rightarrow +\infty} H(k, q) \leq \lim_{c \rightarrow 0} U(c), \quad \forall k \geq k_0.$$

Hence,

$$\left( \lim_{c \rightarrow 0} U(c), \lim_{c \rightarrow +\infty} U(c) \right) \subset \{H(k, q) : 0 < q < +\infty\}, \quad \forall k \geq k_0,$$

and for each  $z \in (\lim_{c \rightarrow 0} U(c), \lim_{c \rightarrow +\infty} U(c))$ , there is a unique number  $\mathcal{F}(k, z)$  such that  $H(k, \mathcal{F}(k, z)) = z$ .

Let  $\varepsilon > 0$  be small enough so that  $\rho(u_1(k_0) - \varepsilon) > \lim_{c \rightarrow 0} U(c)$  and set

$$\Omega = \left\{ (k, v) : k_0 \leq k \text{ and } \rho(u_1(k_0) - \varepsilon) < \rho v < \lim_{c \rightarrow +\infty} U(c) \right\}. \quad (2.72)$$

Note that  $(k_0, u_1(k_0)) \in \Omega$ . Similar arguments as in the proof of Lemma 2.6 show that  $\mathcal{F}(\cdot, \cdot)$  is of class  $C^1$  on  $\Omega$ . Furthermore, it can be seen that  $\mathcal{F}(k, \cdot)$  is Lipschitz continuous on  $[u_1(k_0) - \varepsilon, \lim_{c \rightarrow \infty} U(c)/\rho]$  with a constant which does not depend on  $k \in [k_0, +\infty)$ .

Consider the Cauchy problem

$$u_2'(k) = \mathcal{F}(k, u_2(k)), \quad \text{for } k \geq k_0, \quad (2.73)$$

$$(k, u_2(k)) \in \Omega, \quad (2.74)$$

$$u_2(k_0) = u_1(k_0). \quad (2.75)$$

From Cauchy-Lipchitz theorem, there is a unique maximal solution of (2.73)-(2.75). The same arguments as in the proof of Proposition 2.3 yield that the solution is indeed global, i.e. defined on  $[k_0, +\infty)$ , and increasing and strictly concave.

Set

$$u(k) = \begin{cases} u_1(k), & \text{if } k \in (0, k_0], \\ u_2(k), & \text{if } k \in [k_0, +\infty). \end{cases}$$

The function  $u(\cdot)$  is clearly  $C^1$  from what precedes, and for any  $k \in (0, +\infty)$ ,  $\rho u(k) = H(k, u'(k))$ . Note that  $u(\cdot)$  is also  $C^2$  in  $(0, k^*) \cup (k^*, +\infty)$ . Hence  $u(\cdot)$  is a classical solution of (2.31)-(2.33). Uniqueness is obtained as above.  $\square$

## 2.4 Distribution of capital

Given the wages, the optimal control problem addressed by the firms has been studied previously. The optimal strategy of the firms and the related dynamics of their capital have been obtained by solving the Hamilton-Jacobi equation

$$\rho u(k) = H(k, u'(k)), \quad (2.76)$$

where  $u(\cdot)$  is the value function. Under Assumptions 2.1 and 2.2, or 2.1 and 2.3, it has been proved that the optimal investment strategy of the firms have the effect of driving their capital to a value  $k^*$ , which is characterized by

$$f'(k^*) = \rho,$$

The purpose of the present discussion is to study the distribution of capital corresponding to the optimal investment strategy of the firms. It is characterized by the following continuity equation:

$$\frac{d}{dk} \left( D_q H(\cdot, u'(\cdot)) m(\cdot) \right) (k) = \eta(k) - \nu m(k), \quad k > 0, \quad (2.77)$$

where  $\nu \geq 0$  is the rate of death of firms and the source term  $\eta(k)$  captures the exogenous creation of firms. Here the source term does not depend on the value function of the optimal control problem  $u(\cdot)$ , neither on the capital distribution  $m$ . We will see some extensions in Chapter 6 which study these cases. The existence of a stationary equilibrium then requires that creations compensate destructions, i.e. that

$$\int_0^{+\infty} \eta(k) dk - \nu = 0.$$

It will be proved that (2.77) admits a unique solution in the space of probability measures on  $\mathbb{R}_+^*$ . Two cases will be addressed:

1.  $\nu = 0$  and  $\eta(\cdot) \equiv 0$ .
2.  $\nu > 0$  and  $\eta(\cdot)$  is a continuous function with compact support.

In the first case, it will be shown that the distribution of capital is a Dirac mass at  $k = k^*$ .

In the second case, it is natural to split the distribution of capital into the sum a regular part and a singular part consisting in a Dirac mass at  $k = k^*$ . The distribution will exhibit a Dirac mass at  $k = k^*$  if and only if the latter capital can be driven to  $k^*$  in finite time. The following lemma rules out this situation under suitable assumptions.

**Lemma 2.10** *Under Assumptions 2.1 and 2.2, or 2.1 and 2.3, there exists  $\epsilon > 0$  and  $M > 0$  such that*

$$0 \leq D_q H(k, u'(k)) \leq M(k^* - k), \quad \text{if } k \in [k^* - \epsilon, k^*], \quad (2.78)$$

$$M(k^* - k) \leq D_q H(k, u'(k)) \leq 0, \quad \text{if } k \in [k^*, k^* + \epsilon]. \quad (2.79)$$

**Proof** Since the value function is  $C^1$  on  $(0, +\infty)$ , and  $C^2$  on  $(0, k^*) \cup (k^*, +\infty)$ , it is possible to take the derivative of (2.76) and get

$$\rho u'(k) - D_k H(k, u'(k)) = D_q H(k, u'(k)) u''(k), \quad \forall k \neq k^*. \quad (2.80)$$

Recall that the optimal consumption is named  $c^*(k)$ . Using that

$$D_k H(k, u'(k)) = f'(k) U'(c^*(k)), \quad U'(c^*(k)) = u'(k) \quad \text{and} \quad D_q H(k, u'(k)) = f(k) - c^*(k),$$

then (2.80) can be written as follows:

$$U'(c^*(k))(\rho - f'(k)) = (f(k) - c^*(k)) U''(c^*(k)) (c^*)'(k). \quad (2.81)$$

Proof of (2.78). The inequality in the left side of (2.78) comes from the already proved fact that  $f(k) - c^*(k) > 0$  for  $k < k^*$ . There remains to prove the other inequality for  $k$  sufficiently close to  $k^*$ .

The first step consists in proving that there exist  $\epsilon > 0$  and  $C_2 > 0$  such that for every  $k \in [k^* - \epsilon, 0]$ ,

$$c^*(k^*) - c^*(k) = f(k^*) - c^*(k) \leq C_2(k^* - k).$$

For  $\epsilon$  small enough, dividing (2.81) by  $U''(c^*(k))$  and integrating between  $k \geq k^* - \epsilon$  and  $k^*$  yields

$$\begin{aligned} \int_k^{k^*} \frac{U'(c^*(\kappa))}{U''(c^*(\kappa))} (\rho - f'(\kappa)) d\kappa + \int_k^{k^*} (f(k^*) - f(\kappa)) (c^*)'(\kappa) d\kappa &= \frac{1}{2} (f(k^*) - c^*(k))^2 \\ &= \frac{1}{2} (c(k^*) - c^*(k))^2. \end{aligned} \quad (2.82)$$

Since  $f(\cdot) \in W_{\text{loc}}^{2,\infty}$ , there exists  $C_1 > 0$  such that

$$\rho - f'(\kappa) = f'(k^*) - f'(\kappa) = \int_\kappa^{k^*} f''(z) dz \geq -C_1(k^* - \kappa).$$

The first integral in (2.82) is bounded above as follows:

$$\begin{aligned} \int_k^{k^*} \frac{U'(c^*(\kappa))}{U''(c^*(\kappa))} (\rho - f'(\kappa)) d\kappa &= \int_k^{k^*} \frac{U'(c^*(\kappa))}{U''(c^*(\kappa))} \int_\kappa^{k^*} f''(z) dz d\kappa \\ &\leq -C_1 \int_k^{k^*} \frac{U'(c^*(\kappa))}{U''(c^*(\kappa))} (k^* - \kappa) d\kappa \end{aligned} \quad (2.83)$$

Since  $U'(c^*(k))/U''(c^*(k))$  admits a negative limit as  $k \rightarrow k^*$ , there exists  $C_2 > 0$  such that

$$2 \int_k^{k^*} \frac{U'(c^*(\kappa))}{U''(c^*(\kappa))} (\rho - f'(\kappa)) d\kappa \leq C_2(k^* - k)^2.$$

Next, integrating by part the second integral in (2.82) yields

$$\begin{aligned} \int_k^{k^*} (f(k^*) - f(\kappa))(c^*)'(\kappa)d\kappa &= \int_k^{k^*} f'(\kappa)(c^*(\kappa) - c^*(k))d\kappa \\ &= (c^*(k^*) - c^*(k)) \int_k^{k^*} f'(\kappa) \frac{c^*(\kappa) - c^*(k)}{c^*(k^*) - c^*(k)} d\kappa. \end{aligned} \quad (2.84)$$

Setting  $J(k) = \int_k^{k^*} f'(\kappa) \frac{c^*(\kappa) - c^*(k)}{c^*(k^*) - c^*(k)} d\kappa$ , the monotonicity of  $c^*(\cdot)$  and of  $f(\cdot)$  on  $(0, k^*]$  imply that

$$0 \leq J(k) \leq f(k^*) - f(k).$$

Hence, for  $\epsilon > 0$  small enough, we can find  $M_1 > 0$  such that

$$0 \leq J(k) \leq M_1(k^* - k), \quad \text{for all } k \in [k^* - \epsilon, k^*]. \quad (2.85)$$

From (2.82), (2.83) and (2.84), we deduce

$$c^*(k^*) - c^*(k) \leq (C_2(k^* - k)^2 + 2(c^*(k^*) - c^*(k))J(k))^{\frac{1}{2}}. \quad (2.86)$$

Taking the square, elementary algebra yields that for  $\epsilon > 0$  small enough, for all  $k \in [k^* - \epsilon, k^*]$ ,

$$\begin{aligned} 0 \leq c^*(k^*) - c^*(k) &\leq J(k) + \left( J^2(k) + C_2(k^* - k)^2 \right)^{\frac{1}{2}} \\ &\leq \left( M_1 + \left( M_1^2 + C_2 \right)^{\frac{1}{2}} \right) (k^* - k), \end{aligned} \quad (2.87)$$

where the last inequality is a consequence of (2.85).

Finally, observe that  $f(k) - c^*(k^*) = f(k) - f(k^*) = -\rho(k^* - k) + o(k^* - k)$ . Therefore, from (2.87), there exists a constant  $M > 0$  and  $\epsilon > 0$  such that for all  $k \in [k^* - \epsilon, k^*]$ ,

$$0 \leq D_q H(k, u'(k)) = f(k) - c(k) \leq M(k^* - k),$$

which achieves the proof of (2.78).

Proof of (2.79). The proof of (2.79) is done exactly in the same way. □

**Remark 2.11** *Let us remark that under the additional assumption that  $f(\cdot)$  is locally uniformly concave, i.e. for every compact set  $K \subset (0, +\infty)$ , there exists  $\theta > 0$  such that*

$$f''(k) \leq -\theta, \quad \forall k \in K,$$

*it can be checked using the same argument in the previous proof that there exists  $\epsilon > 0$  and  $M_1 > 0$  such that for every  $k \in [k^* - \epsilon, k^* + \epsilon]$ ,*

$$|D_q H(k, u'(k))| \geq M_1 |k^* - k|. \quad (2.88)$$

*Consider  $k \neq k^*$  such that  $|k - k^*| \leq \epsilon$ ; by differentiating equation (2.76) at point  $k$ , we obtain*

$$\rho u'(k) = D_k H(k, u'(k)) + D_q H(k, u'(k))u''(k) \Leftrightarrow u''(k) = \frac{u'(k)(\rho - f'(k))}{D_q H(k, u'(k))}.$$

*Using the estimate (2.88) and the regularity of  $f(\cdot)$ , we deduce that there exists a constant  $M_2 > 0$  independent of  $k$  taken in  $[k^* - \epsilon, k^* + \epsilon]$  and such that*

$$|u''(k)| \leq M_2 u'(k).$$

*This shows that  $u''(\cdot)$  belongs to  $L^\infty([k^* - \epsilon, k^* + \epsilon])$ . Finally,  $u(\cdot) \in W_{loc}^{2,\infty}(0, +\infty)$ .*

**Proposition 2.6** *Under Assumptions 2.1 and 2.2, or 2.1 and 2.3,*

1. *If  $\nu = 0$  and  $\eta \equiv 0$ , then the unique solution of (2.77) is the Dirac mass at  $k = k^*$ , i.e.*

$$m = \delta_{k^*}.$$

2. *If  $\nu > 0$  and  $\eta(\cdot)$  is a continuous and non negative function with compact support, then the unique solution of (2.77) is given by*

$$m(k) = \begin{cases} \frac{1}{b(k)} \int_0^k \eta(\kappa) \exp\left(-\int_\kappa^k \frac{\nu}{b(z)} dz\right) d\kappa, & \text{if } k \in [0, k^*), \\ -\frac{1}{b(k)} \int_k^{+\infty} \eta(\kappa) \exp\left(\int_k^{\kappa} \frac{\nu}{b(z)} dz\right) d\kappa, & \text{if } k \in (k^*, +\infty), \end{cases} \quad (2.89)$$

where, for brevity,

$$b(k) = D_q H(k, u'(k)).$$

### Proof

1. If  $\nu = 0$  and  $\eta \equiv 0$ , for any test function  $\varphi(\cdot) \in C_c^\infty(0, +\infty)$ ,

$$\int_0^{+\infty} \varphi(k) D_q H(k, u'(\cdot)) dm(k) = 0.$$

Hence, the support of  $m$  is  $\{k^*\}$ , since  $k^*$  is the only point in  $(0, +\infty)$  where  $D_q H(\cdot, u'(\cdot))$  vanishes. Therefore,  $m = \delta_{k^*}$ .

2. Using the same arguments than in Lemma 6.2 and Remark 6.2 in Chapter 6, it can be proved that if  $m \in \mathcal{P}((0, +\infty))$  is a solution of (2.77) in the distributional sense on  $(0, +\infty)$ , then  $m \in \mathcal{P}((0, +\infty)) \cap C^1((0, k^*) \cup (k^*, +\infty))$ . Then, the density  $m(\cdot)$  is found by integrating (2.77) in the intervals  $(0, k^*)$  and  $(k^*, +\infty)$ , and choosing the integration constants such that  $m(\cdot) \geq 0$  and  $\int_0^{k^*} m(k) dk + \int_{k^*}^{+\infty} m(k) dk < +\infty$ . This leads to (2.89). To show that  $m$  has no singular part at  $k = k^*$ , it is enough to prove that  $\int_0^{k^*} m(k) dk + \int_{k^*}^{+\infty} m(k) dk = 1$ . Set  $I_1 = \int_0^{k^*} m(k) dk$  and  $I_2 = \int_{k^*}^{+\infty} m(k) dk$ .

Focusing on  $I_1$ ,

$$\begin{aligned} I_1 &= \int_0^{k^*} \frac{1}{b(k)} \int_0^k \eta(\kappa) \exp\left(-\int_\kappa^k \frac{\nu}{b(z)} dz\right) d\kappa dk, \\ &= \int_0^{k^*} \eta(\kappa) \int_\kappa^{k^*} \frac{1}{b(k)} \exp\left(-\int_\kappa^k \frac{\nu}{b(z)} dz\right) dk d\kappa, \\ &= \frac{1}{\nu} \int_0^{k^*} \eta(\kappa) d\kappa. \end{aligned} \quad (2.90)$$

The second line in (2.90) is obtained using the non negativity of the integrand and Tonelli's theorem. The third line in (2.90) comes from the fact that

$$\int_k^{k^*} \frac{\nu}{b(z)} dz = +\infty,$$

which is a consequence of Lemma 2.10. It can be proved in the same way that

$$I_2 = \frac{1}{\nu} \int_{k^*}^{+\infty} \eta(\kappa) d\kappa.$$

Hence  $I_1 + I_2 = 1$ . Therefore,  $m$  given by (2.89) defines a density of probability on  $\mathbb{R}_+$ . Since (2.89) is implied by (2.77),  $m$  given by (2.89) is the unique solution of (2.77).

□



## 2.5 Equilibrium

This paragraph is devoted to the existence of equilibria. In what follows, the dependency of the Hamiltonian  $H(\cdot, \cdot, \cdot)$ , the value function  $u(\cdot, \cdot)$  and the distribution of capital  $m$  with respect to the vector of wages  $w$  is made explicit. Assumption 2.2 or 2.3 will be made on the net output  $f(\cdot, \cdot)$ ; further assumptions on  $f(\cdot, \cdot)$  are needed for the existence of an equilibrium:

**Assumption 2.4 (Further assumptions on the net output  $f(\cdot, \cdot)$ )** *The net output  $f : [0, +\infty) \times [0, +\infty)^d \rightarrow \mathbb{R}$  has the following properties*

- i)  $f(\cdot, \cdot)$  is of class  $C^1$  on  $(0, +\infty) \times (0, +\infty)^d$ .
- ii)  $f(\cdot, \cdot)$  is monotone with respect to  $w$ , i.e. for every  $w, \tilde{w} \in [0, +\infty)^d$ ,

$$w \leq \tilde{w} \quad \Rightarrow \quad f(\cdot, w) \geq f(\cdot, \tilde{w}),$$

where the partial order  $\leq$  on  $[0, +\infty)^d$  is defined by

$$w \leq \tilde{w} \quad \Leftrightarrow \quad \forall i \in \{1, \dots, d\}, \quad w_i \leq \tilde{w}_i.$$

**Assumption 2.5 (Assumptions on the labour supply)** *The labour supply is a function of the unitary wages, named  $S : [0, +\infty)^d \rightarrow [0, 1]^d$ . It is assumed that*

- i)  $S(\cdot)$  is continuous.
- ii)  $S(\cdot)$  is onto.
- iii) Set  $W = \overline{S^{-1}((0, 1)^d)}$ : the restriction of  $S(\cdot)$  to  $W$  is injective.
- iv)  $W$  is a compact and convex subset of  $(0, +\infty)^d$ .

In Chapter 5, we will develop assumptions on the supply function in order to have more cases covered. However, a simpler existence result can be presented here with Assumption 2.5 which takes into account the following example:

**Example of supply function satisfying Assumption 2.5.** Consider a family of positive numbers  $(\underline{w}_i, \overline{w}_i)$ ,  $i \in \{1, \dots, d\}$ , such that  $0 < \underline{w}_i < \overline{w}_i < +\infty$ . Let  $(S_i(\cdot))_{i=1, \dots, d}$  be a set of continuous and non decreasing functions from  $[0, +\infty)$  to  $[0, 1]$  such that, for any  $i \in \{1, \dots, d\}$ ,

- $S_i(\underline{w}_i) = 0$
- $S_i(\overline{w}_i) = 1$
- $S_i(\cdot)$  is increasing on  $[\underline{w}_i, \overline{w}_i]$ .

The map  $S : [0, +\infty)^d \rightarrow [0, 1]^d$ , defined by  $S(w) = (S_1(w_1), \dots, S_d(w_d))$ , satisfies Assumption 2.5.

**Lemma 2.11** *Under Assumptions 2.1, 2.2 or 2.3, and 2.4, the value function  $u(\cdot, \cdot)$  is monotone with respect to  $w$ , i.e. for every  $w, \tilde{w} \in [0, +\infty)^d$ ,*

$$w \leq \tilde{w} \quad \Rightarrow \quad u(\cdot, w) \geq u(\cdot, \tilde{w}).$$

**Proof** It is clear that the Hamiltonian  $H(\cdot, \cdot, \cdot)$  has the same monotonicity as  $f(\cdot, \cdot)$  with respect to  $w$ . Hence, if  $w \leq \tilde{w}$ , then

$$0 = -\rho u(k, w) + H(k, w, D_k u(k, w)) \geq -\rho u(k, \tilde{w}) + H(k, \tilde{w}, D_k u(k, \tilde{w})), \quad \forall k \in (0, +\infty).$$

By comparison, this yields that  $u(\cdot, w) \geq u(\cdot, \tilde{w})$ .  $\square$

**Lemma 2.12** *Under Assumptions 2.2 or 2.3 and 2.4, the map  $(0, +\infty)^d \ni w \mapsto k^*(w)$ , where  $k^*(w)$  is defined in (2.36), takes its values in  $(0, +\infty)$  and is continuous.*

**Proof** By definition,  $k^*(\cdot)$  takes its values in  $(0, +\infty)$ . Consider a sequence  $(w_n)_{n \in \mathbb{N}}$  in  $(0, +\infty)^d$  such that  $w_n$  tends to  $w$ , an element of  $(0, +\infty)^d$ , as  $n \rightarrow +\infty$ .

- Assume by contradiction that after the extraction of subsequence,  $k^*(w_n)$  tends to  $+\infty$  as  $n \rightarrow +\infty$ . From the strict concavity and the  $C^1$  regularity of  $f(\cdot, \cdot)$ , this implies that for any  $k \in (0, +\infty)$ ,  $\frac{\partial f}{\partial k}(k, w) > \rho$ ; this contradicts either Assumption 2.2 or Assumption 2.3, namely, either that  $\lim_{k \rightarrow +\infty} \frac{\partial f}{\partial k}(k, w) = 0$  or that  $f(k, w) \leq 0$  for  $k$  large enough. Hence, the sequence  $(k^*(w_n))_{n \in \mathbb{N}}$  is bounded above.
- Assume by contradiction that  $k^*(w_n)$  tends to 0 as  $n \rightarrow +\infty$ : then from the  $C^1$  regularity of  $f(\cdot, \cdot)$ , this implies that for any  $k \in (0, +\infty)$ ,  $\frac{\partial f}{\partial k}(k, w) < \rho$ ; this contradicts the assumption that  $\lim_{k \rightarrow 0} \frac{\partial f}{\partial k}(k, w) = +\infty$ , see Assumptions 2.2 or 2.3. Hence, the sequence  $(k^*(w_n))_{n \in \mathbb{N}}$  is bounded below by a positive constant which may depend on  $w$ .

This implies that, possibly after the extraction of subsequence,  $k^*(w_n)$  tends to a positive limit  $\tilde{k}$ , and again from the  $C^1$  regularity of  $f(\cdot, \cdot)$ ,  $\frac{\partial f}{\partial k}(\tilde{k}, w) = \rho$ . Therefore  $\tilde{k}$  must coincide with  $k^*(w)$ , and the uniqueness of the cluster point implies that the whole sequence  $k^*(w_n)$  tends to  $k^*(w)$ . This achieves the proof.  $\square$

**Lemma 2.13 (Stability)** *Let  $(w_n)_{n \in \mathbb{N}}$  be a sequence in  $(0, +\infty)^d$  converging to  $w \in (0, +\infty)^d$  as  $n$  tends to  $+\infty$ . Under Assumptions 2.1, 2.2 or 2.3, and 2.4,*

$$u(\cdot, w_n) \rightarrow u(\cdot, w)$$

*in  $C^1(K)$  for every compact  $K$  of  $(0, +\infty)$ .*

**Proof** Since  $w_n \rightarrow w \in (0, +\infty)^d$  as  $n \rightarrow +\infty$ , there exists an integer  $N \in \mathbb{N}$  and two vectors  $\underline{w}, \bar{w} \in (0, +\infty)^d$  such that for every  $n \geq N$ ,

$$\underline{w} \leq w_n \leq \bar{w}.$$

Without loss of generality, it may be assumed that  $N = 0$ . From Lemma 2.11, the following inequalities hold:

$$u(\cdot, \bar{w}) \leq u(\cdot, w_n) \leq u(\cdot, \underline{w}), \quad \forall n \in \mathbb{N}.$$

Since  $(u(\cdot, w_n))_{n \in \mathbb{N}}$  is a sequence of concave functions uniformly bounded on every compact subset of  $(0, +\infty)$ , there exists a continuous and concave function  $v : (0, +\infty) \rightarrow \mathbb{R}$  such that for any compact subset  $K$  of  $(0, +\infty)$  and up to the extraction of a subsequence,

1.  $u(\cdot, w_n) \rightarrow v(\cdot)$  uniformly on  $K$ .
2.  $D_k u(\cdot, w_n) \rightarrow v'(\cdot)$  almost everywhere in  $K$ .

From Lemma 2.12, there exist  $\underline{k} > 0$  and  $\bar{k} > \underline{k}$  such that

$$\underline{k} < \min_{w \leq w \leq \bar{w}} k^*(w) \leq \max_{w \leq w \leq \bar{w}} k^*(w) < \bar{k}.$$

From the continuity of  $H(\cdot, \cdot, \cdot)$  on  $(0, +\infty) \times (0, +\infty)^d \times (0, +\infty)^d$  and the uniform convergence of  $(u(\cdot, w_n))_{n \in \mathbb{N}}$  towards  $v(\cdot)$  on  $[\underline{k}, \bar{k}]$ , stability results on viscosity solutions can be used and yield that  $v(\cdot)$  is a viscosity solution of

$$\rho v(\cdot) = H(v(\cdot), w, v'(\cdot))$$

on  $(\underline{k}, \bar{k})$ , with state constrained boundary conditions at  $k = \underline{k}$  and  $k = \bar{k}$ . From the uniqueness of such a solution,  $v(\cdot) = u(\cdot, w)$ .

In fact, the convergence holds locally in  $C^1$ . Consider  $[a, b]$  a compact interval contained in  $(0, +\infty)$ :

- $u(\cdot, w_n)$  tends to  $u(\cdot, w)$  uniformly in  $[a, b]$
- there exists a measurable subset  $E$  of  $[a, b]$ , such that the Lebesgue measure of  $[a, b] \setminus E$  is zero and that  $D_k u(\cdot, w_n)$  tends to  $D_k u(\cdot, w)$  pointwise in  $E$ .

Note that it is always possible to modify a little  $a$  and  $b$  in such a way that  $a \in E$  and  $b \in E$ . Dini's theorem yields that the latter convergence is in fact uniform in  $[a, b]$ : for completeness, the proof is given in what follows.

The function  $D_k u(\cdot, w)$  is continuous, thus uniformly continuous on  $[a, b]$ ; hence, given  $\epsilon > 0$ , it is possible to choose  $\delta > 0$  small enough such that

$$|k - k'| \leq \delta \quad \Rightarrow \quad |D_k u(k, w) - D_k u(k', w)| < \frac{\epsilon}{2}, \quad \forall k, k' \in [a, b].$$

For such a choice of  $\delta > 0$ , it is possible to define a finite subdivision  $(\sigma_i)_{i \in \{0, \dots, I\}}$  of  $[a, b]$  such that

- for every  $i \in \{0, \dots, I\}$ ,  $\sigma_i \in E$ .
- for any  $i \in \{0, \dots, I-1\}$ ,  $0 < \sigma_{i+1} - \sigma_i < \delta$ .

On the other hand, for any  $k \in [a, b]$ , there exists  $i_0 \in \{0, \dots, I-1\}$  such that  $\sigma_{i_0} \leq k \leq \sigma_{i_0+1}$ . Then the concavity of  $u(\cdot, \cdot)$  with respect to  $k$  yields

$$\begin{aligned} D_k u(k, w_n) - D_k u(k, w) &\leq D_k u(\sigma_{i_0}, w_n) - D_k u(\sigma_{i_0+1}, w) \\ &= D_k u(\sigma_{i_0}, w_n) - D_k u(\sigma_{i_0}, w) + D_k u(\sigma_{i_0}, w) - D_k u(\sigma_{i_0+1}, w). \end{aligned}$$

Taking  $N \in \mathbb{N}$  large enough such that for every  $n \geq N$ ,

$$\max_{0 \leq i \leq I} |D_k u(\sigma_i, w_n) - D_k u(\sigma_i, w)| < \frac{\epsilon}{2}$$

yields that

$$D_k u(k, w_n) - D_k u(k, w) < \epsilon, \quad \forall n \geq N.$$

A similar argument can be used to bound  $D_k u(k, w_n) - D_k u(k, w)$  from below. Finally, for any  $\epsilon > 0$  there exists  $N > 0$  such that

$$\sup_{k \in [a, b]} |D_k u(k, w_n) - D_k u(k, w)| < \epsilon, \quad \forall n \geq N.$$

This achieves the proof. □

**Theorem 2.5 (Existence)** *Under Assumptions 2.1, 2.2 or 2.3, 2.4, 2.5, and if  $\nu > 0$  and  $\eta(\cdot)$  is a continuous and non negative function with compact support in  $(0, +\infty)$ , there exists an equilibrium, i.e.  $(u(\cdot, w), m(\cdot, w), w)$  such that*

1.  $u(\cdot, w)$  is a classical solution of (2.31)-(2.33), which is  $C^1$  in  $(0, +\infty)$ .

2.  $m(\cdot, w)$  is a distributional solution of (2.77).

3. the market clearing conditions hold, i.e.

$$S(w) = - \int_0^{+\infty} D_w f(k, w) dm(k, w). \quad (2.91)$$

**Proof** Consider the map  $\Phi : W \rightarrow W$  ( $W$  has been introduced in Assumption 2.5), defined as follows: for any  $w \in W$ ,

1.  $u(\cdot, w)$  is the unique solution of (2.31)-(2.33)

2.  $m(\cdot, w)$  is the unique solution of the continuity equation (2.77)

3. Set

$$\Psi(w) = \max \left( \mathbf{1}, - \int_0^{+\infty} D_w f(k, w) m(k, w) dk \right) \in [0, 1]^d, \quad (2.92)$$

where  $\mathbf{1} \in \mathbb{R}^d$  is the vector whose components are all equal to 1, and the maximum is taken componentwise.

4.

$$\Phi(w) = \mathcal{S}^{-1}(\Psi(w)), \quad (2.93)$$

where  $\mathcal{S}(\cdot)$  is the restrictions of  $S(\cdot)$  to the set  $W$ , which is a one-to-one map from  $W$  onto  $[0, 1]^d$ .

In order to apply Brouwer fixed-point theorem to  $\Phi(\cdot)$ , one must first check that  $\Phi(\cdot)$  is continuous. Let  $(w_n)_{n \in \mathbb{N}}$  be a sequence of elements of  $W$  converging to  $w$  as  $n \rightarrow +\infty$ . From Lemma 2.13, for any compact subset  $K$  of  $(0, +\infty)$ ,  $u(\cdot, w_n) \rightarrow u(\cdot, w)$  in  $C^1(K)$ .

As a consequence of Lemma 2.12,  $\bar{k}_1 = \max_{w \in W} k^*(w)$  is a real number.

Let  $\bar{k}_2$  be the supremum of  $k \in \text{supp}(\eta(\cdot))$ . Proposition 2.6 implies that for all  $n \in \mathbb{N}$ ,  $m(\cdot, w_n)$  is supported in  $[0, \bar{k}]$ , where  $\bar{k} = \max(\bar{k}_1, \bar{k}_2)$ .

This yields that there exists a probability measure  $\mu$  and a subsequence still named  $(w_n)_{n \in \mathbb{N}}$  such that  $(m(\cdot, w_n))_{n \in \mathbb{N}}$  converges weakly  $*$  to  $\mu$ .

The next step consists in proving that  $\mu = m(\cdot, w)$ , i.e that  $\mu$  is the solution of the continuity equation (2.77) associated with  $w$ , in the distributional sense. Consider a test function  $\varphi(\cdot) \in C_c^\infty(0, +\infty)$ . By definition of  $m(\cdot, w_n)$ ,

$$- \int_0^{+\infty} \varphi'(k) b(k, w_n) m(k, w_n) dk = \int_0^{+\infty} \varphi(k) \eta(k) dk - \nu \int_0^{+\infty} \varphi(k) m(k, w_n) dk.$$

The right-hand side converges to

$$\int_0^{+\infty} \varphi(k) \eta(k) dk - \nu \int_0^{+\infty} \varphi(k) \mu(k) dk.$$

On the other hand, note that the  $C^1$  convergence of  $u(\cdot, w_n)$  to  $u(\cdot, w)$  on every compact subset of  $(0, +\infty)$  implies the uniform convergence of  $D_q H(\cdot, u'(\cdot, w_n))$  to  $D_q H(\cdot, u'(\cdot, w))$  on  $[0, \bar{k}] \cap \text{supp} \varphi'(\cdot)$ . Hence,

$$- \int_0^{+\infty} \varphi'(k) D_q H(k, u'(k, w_n)) m(k, w_n) dk \rightarrow - \int_0^{+\infty} \varphi'(k) D_q H(k, u'(k, w)) \mu(k) dk.$$

Therefore  $\mu$  is a solution of (2.77) which implies that  $\mu = m(\cdot, w)$ . The uniqueness of the cluster point yields that the whole sequence  $m(\cdot, w_n)$  weakly  $*$  converges to  $m(\cdot, w)$  as  $n \rightarrow \infty$ .

The regularity of  $f(\cdot, \cdot)$  and the fact that  $m(\cdot, w_n)$  is supported in a compact subset of  $(0, +\infty)$  independent of  $n$  imply that

$$-\int_0^{+\infty} D_w f(k, w_n) m(k, w_n) dk \rightarrow -\int_0^{+\infty} D_w f(k, w) m(k, w) dk.$$

Therefore  $\Psi(\cdot)$  is a continuous map and so is  $\Phi(\cdot)$ .

From Brouwer theorem,  $\Phi(\cdot)$  has a fixed point, and the thesis follows.  $\square$

## 2.6 The net output deduced from a production function

Recall that the Hamiltonian is given by

$$H(k, w, q) = \sup_{c \geq 0, \ell \in [0, +\infty)^d} \{U(c) + q(F(k, \ell) - w \cdot \ell - \delta k - c)\}.$$

It is clear that when  $q \geq 0$ ,

$$H(k, w, q) = \sup_{c \geq 0} \{U(c) - qc\} + f(k, w)q, \quad (2.94)$$

where the net output is given by

$$f(k, w) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} - \delta k. \quad (2.95)$$

An interesting question is to determine what kind of assumptions on the production function imply that Assumptions 2.2 or 2.3 and 2.4 are satisfied by the net output. Consider the following set of assumptions on  $F(\cdot, \cdot)$ :

**Assumption 2.6 (Assumptions on  $F(\cdot, \cdot)$ )** *The production function  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  has the following properties:*

- i)  $F(\cdot, \cdot)$  is continuous on  $[0, +\infty) \times [0, +\infty)^d$  and non negative.
- ii)  $F(\cdot, \cdot)$  is strictly concave and of class  $C^1$  on  $(0, +\infty) \times (0, +\infty)^d$ , and for any  $k \in (0, +\infty)$ , it is possible to extend  $\ell \mapsto D_k F(k, \ell)$  by continuity at  $\ell = 0$ .
- iii) For any  $L > 0$  and  $K > 1$ , there exists  $C > 0$  such that for any  $\ell \in [0, L]^d$  and any  $k, k' \in [1/K, K]$ ,

$$F(\lambda k + (1 - \lambda)k', \ell) - \lambda F(k, \ell) - (1 - \lambda)F(k', \ell) \geq \frac{C}{2} \lambda(1 - \lambda)(k - k')^2.$$

- iv) For any  $\ell \in [0, +\infty)^d$ , the function  $[0, +\infty) \ni k \mapsto F(k, \ell)$  is non decreasing.
- v) For any  $w \in (0, +\infty)^d$ , the function  $[0, +\infty)^d \ni \ell \mapsto F(0, \ell) - w \cdot \ell$  admits a unique maximum point.
- vi) Either a) or b) below holds:
  - a) For any  $w \in (0, +\infty)^d$ ,  $D_k F(k, \ell^*(0, w)) \rightarrow +\infty$  as  $k \rightarrow 0$
  - b) There exists  $a \in (0, 1)$  such that for any  $(k, \ell) \in [0, +\infty) \times [0, +\infty)^d$  and any  $0 \leq \lambda \leq 1$ ,  $F(0, \ell) = 0$  and  $F(\lambda(k, \ell)) \geq \lambda^a F(k, \ell)$ .
- vii) There exists  $b \in (0, 1)$  such that for any  $(k, \ell) \in [0, +\infty) \times [0, +\infty)^d$  and any  $\lambda \geq 1$ ,  $F(\lambda(k, \ell)) \leq \lambda^b F(k, \ell)$ .

**Examples of production functions.** Here are two common examples of production functions which satisfy Assumption 2.6:

- The Cobb-Douglas production function  $F(k, \ell) = Ak^\alpha \ell^\beta$  where  $\beta \in (0, 1)^d$ ,  $\sum_{i=1}^d \beta_i < 1$ ,  $\ell^\beta = \prod_{i=1}^d \ell_i^{\beta_i}$ , and  $\alpha \in (0, 1 - \sum_{i=1}^d \beta_i)$ . The quantities  $\beta$  and  $\alpha$  respectively stand for the output elasticities of labour and capital,  $A > 0$  is the global factor of productivity.
- The Constant Elasticity Substitution (CES) production function  $F(k, \ell) = (Ak^\alpha + B \cdot \ell^\beta)^\gamma$  where  $\beta \in (0, 1]^d$ ,  $\ell^\beta = (\ell_1^{\beta_1}, \dots, \ell_d^{\beta_d})$ ,  $\alpha \in (0, 1]$ , and  $\gamma \in (0, 1]$ . The number  $A > 0$  is the factor of productivity of the capital whereas  $B \in (0, +\infty)^d$  is the vector of factors of productivity of the labour.

Note that the Cobb-Douglas type function satisfies Assumption 2.6-vi)-b), whereas the CES type function satisfies Assumption 2.6-vi)-a).

**Lemma 2.14** *Under Assumption 2.6, for any  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$  there exists a unique maximizer  $\ell^*(k, w)$  of  $[0, +\infty)^d \ni \ell \mapsto F(k, \ell) - w \cdot \ell$*

**Proof** Fix  $(k, w) \in [0, +\infty) \times (0, +\infty)^d$ ; we make out two cases:

1. If  $k = 0$ , the result is a direct consequence of Assumption 2.6-v).
2. If  $k > 0$ , note that the map  $g : [0, +\infty)^d \ni \ell \mapsto F(k, \ell) - w \cdot \ell$  is continuous and strictly concave. Since  $w \in (0, +\infty)^d$  and Assumption 2.6-vii) holds,  $g(\ell) \rightarrow -\infty$  when  $\|\ell\| \rightarrow +\infty$ ; then, from the continuity of  $g(\cdot)$ , there exists a maximizer which is unique by the strict concavity of  $g(\cdot)$ .

□

**Remark 2.12** *Let us notice two features of the production functions given in the examples :*

- *In the case of a Cobb-Douglas type production function, for any  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$ ,  $\ell^*(k, w) \in (0, +\infty)^d$ .*
- *In the case of a CES type production function,  $F(\cdot, \cdot)$  is strictly concave on the closed set  $[0, +\infty) \times [0, +\infty)^d$ .*

**Proposition 2.7** *Under Assumption 2.6, if  $F(\cdot, \cdot)$  also has one of the following properties,*

1. *for any  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$ ,  $\ell^*(k, w) \in (0, +\infty)^d$ , or,*
2.  *$F(\cdot, \cdot)$  is strictly concave on  $[0, +\infty) \times [0, +\infty)^d$ ,*

*then the net output function  $f(\cdot, \cdot)$  defined for all  $(k, w) \in [0, +\infty) \times (0, +\infty)^d$  by (2.95) has the following properties:*

- i)  *$f(\cdot, \cdot)$  is continuous on  $[0, +\infty) \times (0, +\infty)^d$*
- ii)  *$f(\cdot, \cdot)$  is  $C^1$  on  $(0, +\infty) \times (0, +\infty)^d$*
- iii) *For any  $w \in (0, +\infty)^d$ , the map  $f(\cdot, w)$  is strictly concave, locally  $C^{1,1}$  on  $(0, +\infty)$ , and*

$$\lim_{k \rightarrow 0^+} \frac{\partial f}{\partial k}(k, w) = +\infty \text{ and } \lim_{k \rightarrow +\infty} \frac{\partial f}{\partial k}(k, w) = -\delta.$$

**Proof** We focus on the case when  $\delta = 0$ . Let us start with point *i*). We claim that  $\ell^*(\cdot, \cdot)$  is continuous on  $[0, +\infty) \times (0, +\infty)^d$ . Indeed, let  $(k_n, w_n)_{n \in \mathbb{N}}$  be a sequence in  $[0, +\infty) \times (0, +\infty)^d$  converging to  $(k, w) \in [0, +\infty) \times (0, +\infty)^d$ . The sequence  $(\ell^*(k_n, w_n))_{n \in \mathbb{N}}$  is bounded. Indeed, if it were not the case, then possibly after the extraction of a subsequence,  $\|\ell^*(k_n, w_n)\| \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Let  $K > 0$  be an upper bound for  $(k_n)_{n \in \mathbb{N}}$ . Assumption 2.6-i) and 2.6-iv) imply that

$$0 \leq F(k_n, \ell^*(k_n, w_n)) - w_n \cdot \ell^*(k_n, w_n) \leq F(K, \ell^*(k_n, w_n)) - w_n \cdot \ell^*(k_n, w_n).$$

Hence, as in the proof of Lemma 2.14 (the second case),

$$F(K, \ell^*(k_n, w_n)) - w_n \cdot \ell^*(k_n, w_n) \rightarrow -\infty$$

as  $n$  tends to  $+\infty$ . This is the desired contradiction. At this point, we know that  $(\ell^*(k_n, w_n))_{n \in \mathbb{N}}$  is a bounded sequence. Therefore, there exists  $\ell \in [0, +\infty)^d$  such that, then possibly after the extraction of a subsequence,

$$\ell^*(k_n, w_n) \rightarrow \ell.$$

For any  $l \in [0, +\infty)^d$ , the definition of  $(\ell^*(k_n, w_n))_{n \in \mathbb{N}}$  implies that

$$F(k_n, \ell^*(k_n, w_n)) - w_n \cdot \ell^*(k_n, w_n) \geq F(k_n, l) - w_n \cdot l, \quad \forall n \in \mathbb{N}.$$

Passing to the limit in both sides, the continuity of the map  $[0, +\infty) \times [0, +\infty)^d \times (0, +\infty)^d \ni (k, l, w) \mapsto F(k, l) - w \cdot l$  implies that

$$F(k, \ell) - w \cdot \ell \geq F(k, l) - w \cdot l.$$

Since, it is true for any  $l \in [0, +\infty)^d$ ,

$$F(k, \ell) - w \cdot \ell = \max_{l \in [0, +\infty)^d} \{F(k, l) - w \cdot l\}.$$

The uniqueness of the maximizer yields  $\ell = \ell^*(k, w)$ . Since the accumulation point is unique, the whole sequence  $(\ell^*(k_n, w_n))_{n \in \mathbb{N}}$  converges towards  $\ell^*(k, w)$ . The continuity of  $\ell^*(\cdot, \cdot)$  on  $[0, +\infty) \times (0, +\infty)^d$  then implies the continuity of  $f(\cdot, \cdot)$  on the same set.

Let us turn to point *ii*). The envelope theorem implies that  $f(\cdot, \cdot)$  is differentiable at  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$  and

$$(D_k f, D_w f)(k, w) = (D_k F(k, \ell^*(k, w)), -\ell^*(k, w)).$$

Since

$$(0, +\infty) \times (0, +\infty)^d \ni (k, w) \mapsto (D_k F(k, \ell^*(k, w)), -\ell^*(k, w))$$

is a continuous map,  $f(\cdot, \cdot)$  is of class  $C^1$  on  $(0, +\infty) \times (0, +\infty)^d$ .

For point *iii*), take  $w \in (0, +\infty)$ . We start by proving that  $f(\cdot, w)$  is strictly concave. Let  $k, k'$  be two distinct numbers in  $(0, +\infty)$  and  $\lambda$  belong to  $(0, 1)$ . To keep the notation simple, set

$$k_\lambda = \lambda k + (1 - \lambda)k' \text{ and } \ell_\lambda = \lambda \ell^*(k, w) + (1 - \lambda)\ell^*(k', w).$$

Then,

$$f(k_\lambda, w) \geq F(k_\lambda, \ell_\lambda) - w \cdot \ell_\lambda, \tag{2.96}$$

$$> \lambda (F(k, \ell^*(k, w)) - w \cdot \ell^*(k, w)) + (1 - \lambda) (F(k', \ell^*(k', w)) - w \cdot \ell^*(k', w)), \tag{2.97}$$

$$= \lambda f(k, w) + (1 - \lambda)f(k', w), \tag{2.98}$$

where (2.96) comes from the definition of  $f(\cdot, \cdot)$  and the particular choice of  $\ell_\lambda$ , (2.97) is obtained from the strict concavity of  $F(\cdot, \cdot)$  and one of the additional assumptions made at the beginning of Proposition 2.7, and (2.98) comes from the definition of  $f(\cdot, \cdot)$ . This yields that  $f(\cdot, w)$  is strictly concave on  $(0, +\infty)$ . Then, let us prove that  $f(\cdot, w)$  is locally of class  $C^{1,1}$ . Since  $f(\cdot, w)$  is concave, it is enough to establish that

it is locally semiconvex with a linear modulus (see [27, Corollary 3.3.8]). Take  $K > 1$  and  $k, k' \in [1/K, K]$ . We wish to show that there exists  $C > 0$  which only depends of  $K$  such that

$$\lambda f(k, w) + (1 - \lambda)f(k', w) - f(\lambda k + (1 - \lambda)k', w) \geq -\frac{C}{2}\lambda(1 - \lambda)|k - k'|^2.$$

Set  $\ell = \ell^*(\lambda k + (1 - \lambda)k', w)$ . Note that the map

$$g : [0, 1] \ni \theta \mapsto \ell^*(\theta k + (1 - \theta)k', w)$$

is continuous and takes its value in  $[0, +\infty)^d$ . Therefore, there exists a constant  $L > 0$ , such that  $g([0, 1]) \subset [0, L]^d$ . In particular, for any choice of  $\lambda \in (0, 1)$ ,  $\ell \in [0, L]^d$ . From the definition of  $f(\cdot, \cdot)$ , we obtain the following inequalities:

$$f(k, w) \geq F(k, \ell) - w \cdot \ell \quad \text{and} \quad f(k', w) \geq F(k', \ell) - w \cdot \ell.$$

Therefore,

$$\begin{aligned} & \lambda f(k, w) + (1 - \lambda)f(k', w) - f(\lambda k + (1 - \lambda)k', w) \\ & \geq \lambda F(k, \ell) + (1 - \lambda)F(k', \ell) - F(\lambda k + (1 - \lambda)k', \ell) \\ & \geq -\frac{C}{2}\lambda(1 - \lambda)|k - k'|^2, \end{aligned} \tag{2.99}$$

where the constant  $C > 0$  is independent of  $\lambda$  and depends only on  $K$ . The first line of (2.99) comes from the definition of  $f(\cdot, \cdot)$  and the inequalities above. The second line is implied by Assumption 2.6-iii). We have shown that  $f(\cdot, w)$  is locally semiconvex with a linear modulus on  $(0, +\infty)$ . This yields that it is locally of class  $C^{1,1}$  on  $(0, +\infty)$ .

There remains to show that

$$\lim_{k \rightarrow 0^+} D_k f(k, w) = +\infty \quad \text{and} \quad \lim_{k \rightarrow +\infty} D_k f(k, w) = -\delta = 0.$$

Concerning the limit when  $k \rightarrow 0^+$ , we make out two cases:

1. Assumption 2.6-vi)-a) holds; for any  $k \in (0, +\infty)$ ,

$$\frac{1}{k}(f(k, w) - f(0, w)) \geq \frac{1}{k}(F(k, \ell^*(0, w)) - F(0, \ell^*(0, w))).$$

The concavity of  $F(\cdot, \cdot)$  then leads to:

$$\frac{1}{k}(f(k, w) - f(0, w)) \geq D_k F(k, \ell^*(0, w)),$$

This inequality is true for any  $k \in (0, +\infty)$  and  $\lim_{k \rightarrow 0^+} D_k F(k, \ell^*(0, w)) = +\infty$  by assumption. Therefore, from the concavity of  $f(\cdot, w)$ ,  $\lim_{k \rightarrow 0^+} D_k f(k, w) = +\infty$ .

2. Assumption 2.6-vi)-b) holds. For any  $k \in (0, +\infty)$  and  $\lambda \in (0, 1)$ ,

$$\begin{aligned} f(\lambda k, w) &= \sup_{\ell \in [0, +\infty)^d} \{F(\lambda k, \lambda \ell) - w \cdot \lambda \ell\} \\ &\geq \lambda^a \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - \lambda^{1-a} w \cdot \ell\} \\ &\geq \lambda^a f(k, w). \end{aligned}$$

In particular, for any  $k \in (0, 1)$ ,

$$f(k, w) \geq k^a f(1, w).$$



Then, since  $f(0, w) = 0$ ,

$$\frac{1}{k}(f(k, w) - f(0, w)) \geq \frac{f(1, w)}{k^{1-a}}.$$

Note also that  $f(1, w) > 0$  since  $f(0, w) = 0$ ,  $f(\cdot, w)$  is non decreasing on  $[0, +\infty)$  and strictly concave in  $(0, +\infty)$ . Therefore,  $\lim_{k \rightarrow 0^+} D_k f(k, w) = +\infty$ .

In order to show that  $\lim_{k \rightarrow +\infty} D_k f(k, w) = 0$ , let us notice that the map  $(0, +\infty) \ni k \mapsto D_k f(k, w)$  is bounded by 0 from below since  $(0, +\infty) \ni k \mapsto \sup_{\ell \in (0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\}$  is increasing; it is decreasing since  $f(\cdot, w)$  is strictly concave. Then, there exists  $\xi \in [0, +\infty)$  such that

$$D_k f(k, w) \rightarrow \xi,$$

when  $k \rightarrow +\infty$ . With the same kind of arguments as those used immediately above, Assumption 1 – *vii*) implies that for any  $k \geq 1$ ,

$$f(k, w) \leq k^b f(1, w). \quad (2.100)$$

If  $\xi > 0$ , then there exists  $M \in \mathbb{R}$  such that for any  $k \in [0, +\infty)$ ,

$$f(k, w) \geq \xi k + M.$$

This contradicts (2.100) since  $b \in (0, 1)$ . Therefore  $\xi = 0$ .

In the case when  $\delta > 0$ , the map

$$0, +\infty) \times (0, +\infty)^d \ni (k, w) \mapsto \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\}$$

has already been studied and the additional term  $-\delta k$  does not change the qualitative properties of the map.  $\square$

**Remark 2.13** *If moreover*

- $F(\cdot, \cdot)$  is locally uniformly concave on  $(0, +\infty) \times (0, +\infty)^d$  when the additional assumption 1 of Proposition 2.7 holds, or,
- $F(\cdot, \cdot)$  is locally uniformly concave on  $(0, +\infty) \times [0, +\infty)^d$  when the additional assumption 2 of Proposition 2.7 holds,

then by continuity of the map  $l^*(\cdot, w)$ , one can add the term

$$\lambda(1 - \lambda)M \left( |k - k'|^2 + |l^*(k, w) - l^*(k', w)|^2 \right)$$

on the right-hand side of the inequality (2.97), where the positive constant  $M$  depends only on a ball included in  $(0, +\infty)$  and which contains  $k$  and  $k'$ . Fix a such ball  $B$ , this leads to

$$f(\lambda k + (1 - \lambda)k') \geq \lambda f(k, w) + (1 - \lambda)f(k', w) + \lambda(1 - \lambda)M |k - k'|^2. \quad (2.101)$$

It is clear that for every elements  $k$  and  $k'$  in  $B$ , (2.101) holds. Therefore,  $f(\cdot, w)$  is locally uniformly concave on  $(0, +\infty)$ .

## 2.7 Conclusion

We have introduced a mean field game in order to address the interactions on the labour market. More generally, the model can deal with different factors of production, and can be used to link the rental market for professionals with the labour market. For instance, in Chapter 7, we show numerically how the parameters of the model impact the equilibrium.

We have presented the model in a stationary setting for two reasons. First, this gives information on the long term behaviour of firms. It is expected that if the associated dynamical problem is well defined, the solution converges towards the equilibria characterised in this chapter. Second, the mathematical analysis is easier to handle. Indeed, we have used the theory of ordinary differential equation to study the HJ equation associated to the optimal control problem faced by firms. In a dynamical setting this strategy no longer works.

We have showed that the model admits equilibria. In one particular case, we have given an almost explicit formula for the solution of (2.12)-(2.16) and we have found that Pareto's law and Gibrat's law are recovered.

In a more general setting, the assumption that the production has decreasing returns to scale has proved necessary in order to obtain compactness properties. The more important difficulty is related to the optimal control problem. We have solved it by proving that there exists a unique classical solution of the related Hamilton-Jacobi equation. The decreasing returns to scale has permitted to find the golden rule of capital accumulation; in other words, we have proved that there is a unique level of capital that maximises the utility of the firms in infinite horizon. We have showed that this level of capital is never reached. Therefore, the capital distribution does not admit a Dirac mass at this point.

We have proven a first existence result by using the Brouwer fixed-point theorem. In Chapter 4, we present two important examples which allow us, in Chapter 5, to introduce more general assumptions on the supply function and extend the existence result.



## Chapter 3

# A Lagrangian approach to solve the individual optimal control problem for the firms

### 3.1 Introduction

In Chapter 2, we introduced a mean-field game model for factor markets, which applies in particular to the labour market with possibly several types of workers, say  $d$  types. In this model, there is a continuum of identical firms which are heterogeneous in their capital (the state variable). The main characteristics of firms are as follows:

- The production of a given firm is a function of its capital and the level of employment of each type, namely  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$ . A classical example is the Cobb-Douglas production function:  $F(k, \ell) = Ak^\alpha \ell^\beta$  where  $\beta \in (0, 1)^d$ ,  $\sum_{i=1}^d \beta_i < 1$ ,  $\ell^\beta = \prod_{i=1}^d \ell_i^{\beta_i}$ , and  $\alpha \in (0, 1 - \sum_{i=1}^d \beta_i)$ . The quantities  $\beta$  and  $\alpha$  respectively stand for the output elasticities of labour and capital,  $A > 0$  is the global factor of productivity.
- The benefits of production are cut by the wages and the depreciation of the capital. Therefore the total benefits are  $F(k, \ell) - w \cdot \ell - \delta k$ , where  $w \in (0, +\infty)^d$  contains the unitary wages for each type of employees and  $\delta \geq 0$  is the rate of capital depreciation.

The firms face the problem of how to split their benefits into consumption or into investment in order to produce growth. The dynamics of capital is then given by

$$k'(t) = F(k(t), \ell(t)) - w \cdot \ell(t) - \delta k(t) - c(t), \quad (3.1)$$

where  $c(t)$  stands for the consumption of capital at time  $t$ ; the consumption  $c(t)$  is a control variable of each firm, the other being the vector  $\ell(t) \in \mathbb{R}^d$  already introduced. The strategy of a given firm is determined by solving an optimal control problem in order to maximize the payoff

$$\int_0^{+\infty} U(c(t)) e^{-\rho t} dt, \quad (3.2)$$

where  $U : [0, +\infty) \rightarrow \{-\infty\} \times \mathbb{R}$  is a utility function and  $\rho$  is a positive discount factor. Firms aim at finding the optimal controls  $c(t) \in [0, +\infty)$  and  $\ell(t) \in [0, +\infty)^d$  which maximize (3.2), under the constraint that their capital stays non negative (borrowing constraint). The value of the optimal control

problem is

$$\begin{aligned}
u(\kappa) &= \sup_{c(t), \ell(t)} \int_0^{+\infty} U(c(t))e^{-\rho t} dt \\
&\text{subject to} \\
&\left\{ \begin{array}{l} c(\cdot), \ell(\cdot) \in L^1_{\text{loc}}(0, +\infty), k(\cdot) \in W^{1,1}_{\text{loc}}(0, +\infty); \\ c(t) \geq 0, \quad \ell(t) \in [0, +\infty)^d, \quad \text{for almost every } t > 0; \\ k(t) \text{ is a non negative solution of (3.1) with initial condition } k(0) = \kappa. \end{array} \right.
\end{aligned} \tag{3.3}$$

In Chapter 2, given the vector of unitary wages  $w \in (0, +\infty)^d$ , we solved problem (3.3) by showing that the value function is a classical solution of a Hamilton-Jacobi (HJ) equation and deducing the existence and uniqueness of a closed-loop optimal control. The purpose of the present chapter is

1. to prove with different arguments other than those developed in Chapter 2 that under rather general assumptions on the production function, the control  $\ell(\cdot)$  admits an explicit closed-loop form and can be eliminated
2. to obtain the existence and uniqueness of an open-loop optimal consumption strategy, avoiding the analysis of the HJ equation.

We find essentially the same properties obtained in Chapter 2 but with a different method. The present approach is similar to the one proposed in [98], i.e. it consists in introducing a relaxed form of the initial problem, then obtaining compactness properties that lead to the existence of an optimizer. Once the existence of a minimizer of the relaxed problem is obtained, the strategy consists in proving some regularity properties of the latter. This finally leads to the existence of a solution of the original (non-relaxed) problem.

## 3.2 Preliminaries

In this paragraph, we define a relaxation of the problem defined by (3.3). Then, we show that under Assumption 3.1 below (which are assumptions on the production function), we can eliminate the control  $\ell$  by showing that it admits an explicit closed-loop form.

### Relaxation of the optimal control problem defined by (3.3)

We relax the problem in order to obtain some compactness properties of the set of admissible trajectories and controls; these properties will yield the existence of an optimizer.

**Definition 3.1** *Let  $w \in (0, +\infty)^d$  be the vector of wages and  $\delta \geq 0$  the depreciation rate of capital. For all  $\kappa \geq 0$ , we define the set of admissible trajectories and controls*

$$\mathcal{A}_0(\kappa) = \left\{ \begin{array}{l} (k(\cdot), c, \ell) : k(\cdot) \in BV_{\text{loc}}(\mathbb{R}_+), c, \ell \in \mathcal{M}_+(\mathbb{R}_+), k(0) = \kappa, k(\cdot) \geq 0, \\ c \geq 0, \ell \geq 0, k' + c \leq F(k(\cdot), \ell^{ac}(\cdot)) - w \cdot \ell - \delta k. \end{array} \right\},$$

where  $\ell^{ac}(\cdot)$  is the regular part of  $\ell$  with respect to the Lebesgue measure.

**Definition 3.2** *Let  $\rho > 0$  be the discount factor and  $U : \mathbb{R}_+ \rightarrow \mathbb{R}$  be the utility function satisfying Assumption 3.3 below. We define the payoff of firms by*

$$J(c) = \int_0^{+\infty} U(c^{ac}(t))e^{-\rho t} dt,$$

for all consumption strategy  $c \in \mathcal{M}_+(\mathbb{R}_+)$ .

The value function of the relaxed problem associated to (3.3) is defined as follows:

**Definition 3.3** For all  $\kappa \geq 0$ , we define the value function

$$u_0(\kappa) = \sup \{J(c) : (k(\cdot), c, \ell) \in \mathcal{A}_0(\kappa)\}. \quad (3.4)$$

**Reduction of the relaxed optimal control problem defined by (3.4)**

As shown in Chapter 2, under Assumption 3.1-3.2 below, the control  $\ell$  is deduced from the level of capital, i.e.  $\ell$  admits a close-loop form. The purpose of this paragraph is to show the equivalence between two optimization problems and thus to reduce the initial one.

**Assumption 3.1 (Assumptions on  $F(\cdot, \cdot)$ )** The production function  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  has the following properties:

- i)  $F(\cdot, \cdot)$  is continuous on  $[0, +\infty) \times [0, +\infty)^d$  and non negative.
- ii)  $F(\cdot, \cdot)$  is strictly concave and of class  $C^1$  on  $(0, +\infty) \times (0, +\infty)^d$ , and for any  $k \in (0, +\infty)$ , it is possible to extend  $\ell \mapsto D_k F(k, \ell)$  by continuity at  $\ell = 0$ .
- iii) For any  $L > 0$  and  $K > 1$ , there exists  $C > 0$  such that for any  $\ell \in [0, L]^d$  and any  $k, k' \in [1/K, K]$ ,

$$F(\lambda k + (1 - \lambda)k', \ell) - \lambda F(k, \ell) - (1 - \lambda)F(k', \ell) \geq \frac{C}{2} \lambda(1 - \lambda)(k - k')^2.$$

- iv) For any  $\ell \in [0, +\infty)^d$ , the function  $[0, +\infty) \ni k \mapsto F(k, \ell)$  is non decreasing.
- v) There exists  $a \in (0, 1)$  such that for any  $(k, \ell) \in [0, +\infty) \times [0, +\infty)^d$  and any  $0 \leq \lambda \leq 1$ ,  $F(0, \ell) = 0$  and  $F(\lambda(k, \ell)) \geq \lambda^a F(k, \ell)$ .
- vi) There exists  $b \in (0, 1)$  such that for any  $(k, \ell) \in [0, +\infty) \times [0, +\infty)^d$  and any  $\lambda \geq 1$ ,  $F(\lambda(k, \ell)) \leq \lambda^b F(k, \ell)$ .

As shown in Chapter 2, Assumption 3.1 ensures the existence and uniqueness of

$$\ell^*(k, w) = \operatorname{argmax}_{\ell \in [0, +\infty)^d} (F(k, \ell) - w \cdot \ell).$$

This leads us to make the further assumption:

**Assumption 3.2** For any  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$ ,  $\ell^*(k, w) \in (0, +\infty)^d$ .

**Definition 3.4** Let us introduce the net output:

$$f(k) = \sup_{\ell \in [0, +\infty)^d} (F(k, \ell) - w \cdot \ell) - \delta k. \quad (3.5)$$

**Definition 3.5** For all  $\kappa \geq 0$ , we define the set

$$\mathcal{A}(\kappa) = \{(k(\cdot), c) : k(\cdot) \in BV_{\text{loc}}(\mathbb{R}_+), c \in \mathcal{M}_+(\mathbb{R}_+), k(0) = \kappa, k(\cdot) \geq 0, c \geq 0, k' + c \leq f(k(\cdot))\}.$$

We now establish the equivalence of two optimization problems, namely

1. maximize the payoff on  $\mathcal{A}(\kappa)$
2. maximize the payoff on  $\mathcal{A}_0(\kappa)$ .

This will allow us to focus on the reduced problem where the control  $\ell(\cdot)$  is in closed-loop form, i.e. we will work with the set of admissible trajectories and controls  $\mathcal{A}(\kappa)$ .

**Proposition 3.1** If Assumptions 3.1 and 3.2 hold, then

$$u_0(\kappa) = \sup_{(k(\cdot), c) \in \mathcal{A}(\kappa)} J(c). \quad (3.6)$$

**Proof** We start by showing that  $u_0(\kappa) \leq \sup_{(k(\cdot), c) \in \mathcal{A}(\kappa)} J(c)$ . For this aim, let us consider  $(k(\cdot), c, \ell) \in \mathcal{A}_0(\kappa)$ . We claim that  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ . Indeed, the definition of  $f(\cdot)$  implies that  $F(k(\cdot), \ell^{ac}(\cdot)) - w \cdot \ell - \delta k \leq f(k)$ , therefore

$$k' + c \leq F(k(\cdot), \ell^{ac}(\cdot)) - w \cdot \ell - \delta k(\cdot) \leq f(k(\cdot)).$$

Thus,  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ . This yields that

$$u_0(\kappa) \leq \sup_{(k(\cdot), c) \in \mathcal{A}(\kappa)} J(c).$$

Conversely, let us prove that  $u_0(\kappa) \geq \sup_{(k(\cdot), c) \in \mathcal{A}(\kappa)} J(c)$ . For that, consider  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ ; Assumption 3.1 yields that for almost every  $t \geq 0$ ,

$$f(k(t)) = F(k(t), \ell^*(k(t), w)) - w \cdot \ell^*(k(t), w) - \delta k(t),$$

where  $\ell^*(\cdot, \cdot)$  is defined as in Lemma 2.14. The continuity of  $\ell^*(\cdot, \cdot)$ , obtained in the proof of Proposition 2.7, implies that the function  $t \in [0, +\infty) \mapsto \ell^*(k(t), w)$  is measurable. This ensures that

$$(k(\cdot), c, \ell^*(k(\cdot), w)) \in \mathcal{A}_0(\kappa),$$

therefore  $u_0(\kappa) \geq \sup_{(k(\cdot), c) \in \mathcal{A}(\kappa)} J(c)$ .  $\square$

When the optimization is done on the set  $\mathcal{A}_0(\kappa)$ , the corresponding non relaxed problem is defined by (3.3). On the other hand, when the optimization is done on the set  $\mathcal{A}(\kappa)$ , the corresponding non relaxed problem is as follows:

$$\begin{aligned} & \sup_{c(\cdot)} \int_0^{+\infty} U(c(t)) e^{-\rho t} dt \\ & \text{subject to} \\ & \begin{cases} (c(\cdot), k(\cdot)) : & c(\cdot) \in L^1_{\text{loc}}(0, +\infty), k(\cdot) \in W^{1,1}_{\text{loc}}(0, +\infty), \\ & c(t) \geq 0; \\ & k(\cdot) \text{ is a non negative solution on } (0, +\infty) \text{ of } k'(\cdot) = f(k(\cdot)) - c(\cdot), \\ & \text{with initial condition } k(0) = \kappa. \end{cases} \end{aligned} \quad (3.7)$$

We will see that problems (3.3) and (3.7) are equivalent and admit a unique maximizer.

### 3.3 Study of the reduced and relaxed optimal control problem

We saw that under Assumption 3.1-3.2, the control  $\ell(\cdot)$  admits a closed-loop form. This allows us to work with the net output rather than with the production function and the extra control  $\ell(\cdot)$ . Hence, we will study the optimal control problem in which the pair  $(k(\cdot), c)$  belongs to  $\mathcal{A}(\kappa)$  instead of the initial problem in which the triplet  $(k(\cdot), c, \ell)$  belongs to  $\mathcal{A}_0(\kappa)$ . Let us make some assumptions on the utility function  $U(\cdot)$  and the net output  $f(\cdot)$ .

#### Standing assumptions

**Assumption 3.3 (Assumptions on  $U(\cdot)$ )** *The utility function  $U : [0, +\infty) \rightarrow \{-\infty\} \cup \mathbb{R}$  has the following properties:*

- i)  $U(\cdot)$  is  $C^2$  on  $(0, +\infty)$ .
- ii)  $U(\cdot)$  is increasing and strictly concave on  $(0, +\infty)$ .
- iii)  $\lim_{c \rightarrow 0^+} U'(c) = +\infty$  and  $\lim_{c \rightarrow +\infty} U'(c) = 0$ .

iv)  $U(\cdot)$  is non negative.

In Chapter 2, we proved that the following assumptions on the net output is a consequence of Assumptions 3.1 and 3.2 when  $f(\cdot)$  is defined by (3.5).

**Assumption 3.4 (Assumptions on  $f(\cdot)$ )** *The net output  $f : [0, +\infty) \rightarrow [0, +\infty)$  has the following properties:*

i)  $f(\cdot)$  is continuous on  $[0, +\infty)$ .

ii)  $f(\cdot)$  is locally of class  $C^{1,1}$  on  $(0, +\infty)$ .

iii)  $f(\cdot)$  is strictly concave,  $\lim_{k \rightarrow 0^+} f'(k) = +\infty$  and  $-\infty < \lim_{k \rightarrow +\infty} f'(k) \leq 0$ .

iv)  $f(0) = 0$ .

These assumptions are more restrictive than the ones used in Chapter 2. Indeed, we demand that the utility function stays non negative which permits to establish the semi-continuity of the criterion to optimize. Moreover, we impose that  $f(0) = 0$  which ensures that the level of capital stays positive. In particular, these additional constraints remove the cases where the utility function is a logarithm or the production function is a Constant Elasticity of Substitution (or CES) type function.

### Properties of the optimal trajectory and the optimal consumption strategy

The purpose of this paragraph is

1. to prove that any  $(k(\cdot), c) \in \mathcal{A}(\kappa)$  admits a priori bounds, which will yield key compactness properties;
2. to deduce the existence of an optimizer;
3. to prove the uniqueness of the optimizer and obtain regularity properties.

Let us fix for a while the initial level of capital  $\kappa > 0$  and the vector of unitary wages  $w \in (0, +\infty)^d$ .

**Lemma 3.1 (A priori bounds for  $\mathcal{A}(\kappa)$ )** *Under Assumptions 3.4, for every  $C_1 > 0$  there exists  $C_2, C_3, C_4 > 0$  such that for any pair  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ , and for all  $t \in [0, +\infty)$ ,*

$$\begin{cases} \sup_{[0,t]} |k(\cdot)| \leq \left( \kappa + \frac{C_2}{C_1} \right) e^{C_1 t} - \frac{C_2}{C_1}, \\ \|k'\|_{\mathcal{M}([0,t])} \leq (C_3 \kappa + C_4) e^{C_1 t}, \\ \|c\|_{\mathcal{M}_+([0,t])} \leq (C_3 \kappa + C_4) e^{C_1 t}. \end{cases} \quad (3.8)$$

**Remark 3.1** *The a priori bounds in Lemma 3.1 lead to some compactness properties of the set  $\mathcal{A}(\kappa)$ .*

*Indeed, let  $(k_n(\cdot), c_n)_{n \in \mathbb{N}}$  be a sequence of elements of  $\mathcal{A}(\kappa)$ . Therefore, fixing  $T > 0$  and considering, for all  $n \in \mathbb{N}$ , the restriction of the capital trajectory  $k_n(\cdot)$  and the consumption strategy  $c_n$  to the time interval  $[0, T]$ , we define uniformly bounded sequences respectively on the set of the functions of bounded variation on  $[0, T]$  (i.e. on  $BV([0, T])$ ) and on the set of non negative measures on  $[0, T]$  (i.e. on  $\mathcal{M}_+([0, T])$ ). Therefore, the sequence  $k_n(\cdot)$  is compact in  $L^1(0, T)$  and the sequence  $c_n$  is compact for the weak topology of  $\mathcal{M}_+([0, T])$ .*



**Proof** Let us consider  $\delta > 1$  and  $C_1 > 0$ , and set  $T = \frac{1}{C_1} \left(1 - \frac{1}{\delta}\right)$ .

**Step 1:** We claim that there exists  $C_2 > 0$  such that

$$\begin{cases} \sup_{[0,T]} |k(\cdot)| \leq \delta (\kappa + C_2 T), \\ \|k'\|_{\mathcal{M}([0,T])} \leq (2\delta - 1)\kappa + 2\delta C_2 T, \\ \|c\|_{\mathcal{M}_+([0,T])} \leq \delta (\kappa + C_2 T). \end{cases} \quad (3.9)$$

We start by proving the bound on  $\sup_{[0,T]} |k(\cdot)|$ ; let us fix  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ . By definition of  $\mathcal{A}(\kappa)$ ,

$$k' + c \leq f(k(\cdot)) \leq C_1 k(\cdot) + C_2, \quad (3.10)$$

where the last inequality is true for some constant  $C_2 > 0$  which depends on  $C_1$ , since  $f(\cdot)$  is a concave function and  $\lim_{k \rightarrow +\infty} f'(k) \leq 0$ . For every  $s \in [0, T]$ , by integrating inequality (3.10) between 0 and  $s$  and adding  $\kappa$  on each part of the inequality, we obtain

$$k(s) + c([0, s]) \leq \kappa + C_1 \|k(\cdot)\|_{L^1(0,s)} + C_2 s. \quad (3.11)$$

Recall that  $k(\cdot)$  belongs to  $BV_{\text{loc}}([0, +\infty)) \subset L^1_{\text{loc}}(0, +\infty)$ ; hence  $\|k\|_{L^1(0,s)} < +\infty$ . Furthermore,  $c$  is a non negative measure and  $k(\cdot)$  is non negative; this and (3.11) yield that

$$\sup_{[0,s]} |k(\cdot)| \leq \kappa + C_1 \|k(\cdot)\|_{L^1(0,s)} + C_2 s < +\infty. \quad (3.12)$$

We also remark that

$$\|k(\cdot)\|_{L^1(0,T)} \leq T \sup_{[0,T]} |k(\cdot)|. \quad (3.13)$$

Combining inequality (3.12) with  $s = T$ , and (3.13), we deduce

$$(1 - C_1 T) \sup_{[0,T]} |k(\cdot)| \leq \kappa + C_2 T.$$

The definition of  $T$  leads to the identity:  $1 - C_1 T = 1/\delta$ . Therefore,

$$\sup_{[0,T]} |k(\cdot)| \leq \delta (\kappa + C_2 T). \quad (3.14)$$

We have obtained the first inequality in (3.9). For the second one, let us split  $k'$  as follows:  $k' = k'_+ - k'_-$  where  $k'_+$  (resp.  $k'_-$ ) is the non negative (resp. non positive part) of  $k'$ . Inequality (3.10) implies that  $k'_+$  is absolutely continuous with respect to the Lebesgue measure on  $[0, T]$ . With a slight abuse, let us use the same notation, i.e.  $k'_+(\cdot)$ , for the density with respect to the Lebesgue measure on  $[0, T]$ . From inequality (3.10) and (3.14), we deduce

$$\|k'_+(\cdot)\|_{L^\infty(0,T)} \leq C_1 \delta (\kappa + C_2 T) + C_2.$$

Using the identity  $\delta C_1 T = \delta - 1$ , we deduce that

$$\|k'_+(\cdot)\|_{L^\infty(0,T)} \leq \delta (C_1 \kappa + C_2). \quad (3.15)$$

Turning to  $k'_-$ , we observe that  $k(\cdot)$  is non negative. Therefore,

$$k'_-([0, T]) \leq \kappa + \int_0^T k'_+(t) dt.$$

Combining this and (3.15) yields

$$k'_-([0, T]) \leq \kappa + T \delta (C_1 \kappa + C_2).$$

Since  $k'_-$  is a positive measure and  $1 + \delta C_1 T = \delta$ ,

$$\|k'_-\|_{\mathcal{M}_+([0,T])} \leq \delta (\kappa + C_2 T). \quad (3.16)$$

Combining inequality (3.15) and (3.16), we finally obtain

$$\begin{aligned} \|k'\|_{\mathcal{M}([0,T])} &= \|k'_+\|_{\mathcal{M}_+([0,T])} + \|k'_-\|_{\mathcal{M}_+([0,T])} \\ &\leq T\delta (C_1\kappa + C_2) + \delta (\kappa + C_2 T) \\ &\leq (2\delta - 1)\kappa + 2\delta C_2 T. \end{aligned}$$

This is the second inequality in (3.9). The bound on  $c$  is deduced from (3.11) with  $s = T$ ,

$$\|c\|_{\mathcal{M}_+([0,T])} \leq \kappa + C_1 \|k(\cdot)\|_{L^1(0,T)} + C_2 T.$$

This yields

$$\|c\|_{\mathcal{M}_+([0,T])} \leq \delta (\kappa + C_2 T),$$

i.e. the third estimate in (3.9).

**Step 2:** From the previous step, we deduce by induction that for all  $n \in \mathbb{N}$ ,

$$\sup_{[0,(n+1)T]} |k(\cdot)| \leq \delta \left( \sup_{[0,nT]} |k(\cdot)| + C_2 T \right),$$

then that for all  $n > 0$ ,

$$\begin{aligned} \sup_{[0,nT]} |k(\cdot)| &\leq \left( \kappa + C_2 T \sum_{k=1}^n \delta^{k-n} \right) \delta^n \\ &= \kappa \delta^n + C_2 T \delta \frac{\delta^n - 1}{\delta - 1} \\ &= \kappa \delta^n + \frac{C_2 T \delta}{C_1 T \delta} (\delta^n - 1) \\ &= \left( \kappa + \frac{C_2}{C_1} \right) \delta^n - \frac{C_2}{C_1}. \end{aligned}$$

Let us fix  $t \in (0, +\infty)$ . Consider  $n > 0$  and define  $\delta$  such that  $t = nT$ , i.e.

$$\delta = \frac{n}{n - C_1 t}.$$

By letting  $n$  tend to  $+\infty$ , we observe that  $\delta^n \sim e^{C_1 t}$ ; therefore, passing to the limit, we obtain the following bound:

$$\sup_{[0,t]} |k(\cdot)| \leq \left( \kappa + \frac{C_2}{C_1} \right) e^{C_1 t} - \frac{C_2}{C_1}, \quad (3.17)$$

i.e. the first line in (3.8).

**Step 3:** Let us use the first two steps in order to deduce a bound for  $\|k'\|_{\mathcal{M}([0,t])}$  when  $t \geq 0$ . Let us fix  $C_1 \in (0, 1)$  for a while, and choose  $\delta = \frac{1}{1-C_1} > 1$  so that  $T = 1$ , (recall that  $T = (1 - 1/\delta)/C_1$ ). In this case, the second line in (3.9) becomes

$$\|k'\|_{\mathcal{M}_+([0,1])} \leq (2\delta - 1)\kappa + 2\delta C_2.$$

By induction, for every  $i \in \mathbb{N}$ ,

$$\|k'\|_{\mathcal{M}_+([i,i+1])} \leq (2\delta - 1) \sup_{[0,i]} |k(\cdot)| + 2\delta C_2.$$

Hence, for every  $n \in \mathbb{N}$ ,

$$\|k'\|_{\mathcal{M}_+([0, n+1])} \leq \sum_{i=0}^n \left( (2\delta - 1) \sup_{[0, i]} |k(\cdot)| + 2\delta C_2 \right).$$

Using estimate (3.17), we obtain

$$\begin{aligned} \|k'\|_{\mathcal{M}_+([0, n+1])} &\leq (2\delta - 1) \left( \kappa + \frac{C_2}{C_1} \right) \sum_{i=0}^n e^{C_1 i} + 2\delta C_2 (n+1) \\ &= (2\delta - 1) \left( \kappa + \frac{C_2}{C_1} \right) \frac{e^{C_1(n+1)} - 1}{e^{C_1} - 1} + 2\delta C_2 (n+1). \end{aligned}$$

Therefore, there exists  $C_3 > 0$  and  $C_4 > 0$  such that, for all  $n \in \mathbb{N}$  and  $t \in [n, n+1]$ ,

$$\|k'\|_{\mathcal{M}_+([0, t])} \leq \|k'\|_{\mathcal{M}_+([0, n+1])} \leq (C_3 \kappa + C_4) e^{C_1 n} \leq (C_3 \kappa + C_4) e^{C_1 t}.$$

The same argument leads to an estimate of the same form for  $\|c\|_{\mathcal{M}_+([0, t])}$ .

In Step 3, we have imposed  $C_1 \in (0, 1)$  so far. Yet, the same kind of argument works for  $C_1$  arbitrarily chosen in  $(0, +\infty)$  provided a different value of  $T$  is chosen (different than  $T = 1$ ), in such a way that there exists  $\delta > 1$  with  $T = \frac{1}{C_1} \left(1 - \frac{1}{\delta}\right)$ . □

**Remark 3.2** Note that the constants  $C_i$ ,  $i = 1, \dots, 4$  introduced in Lemma 3.1 do not depend of the initial capital  $\kappa > 0$ .

**Lemma 3.2** Under Assumption 3.3 and 3.4,  $J(\cdot)$  is upper semi-continuous on  $\mathcal{A}(\kappa)$ .

**Proof** We follow the lines of the proof of [98, Lemma 3.2], where the authors have obtained a similar result. We first build a map  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , such that

- i)  $\varphi(\tau) \rightarrow 0$  as  $\tau \rightarrow +\infty$
- ii) for every pair  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ ,

$$J(c) \leq \int_0^\tau U(c^{ac}(t)) e^{-\rho t} dt + \varphi(\tau). \quad (3.18)$$

Second, we show that the function

$$\mathcal{A}(\kappa) \ni (k(\cdot), c) \mapsto \int_0^\tau U(c^{ac}(t)) e^{-\rho t} dt$$

is upper semi-continuous for every  $\tau \geq 0$ . These two points will lead to the upper semi-continuity of  $J(\cdot)$  on  $\mathcal{A}(\kappa)$ .

Let us fix  $\tau \geq 0$  and focus on the quantity  $\int_\tau^{+\infty} U(c^{ac}(t)) e^{-\rho t} dt$ . Set  $\lambda(\tau) = \frac{e^{-\rho\tau}}{\rho}$ . Since  $U(\cdot)$  is a concave function, we may use Jensen's inequality:

$$\int_\tau^{+\infty} U(c^{ac}(t)) e^{-\rho t} dt \leq \lambda(\tau) U \left( \int_\tau^{+\infty} c^{ac}(t) e^{-\rho t} \frac{dt}{\lambda(\tau)} \right). \quad (3.19)$$

Then from (3.8), we see that, for every  $t \geq \tau$ ,

$$\int_\tau^t c^{ac}(s) ds \leq (C_3 \kappa + C_4) e^{C_1 t},$$

where  $0 < C_1 < \rho$ ,  $C_3$  and  $C_4$  are suitably chosen positive constants. Therefore, integration by part makes sense for the right hand side of (3.19); it yields

$$\int_{\tau}^{+\infty} c^{ac}(t)e^{-\rho t} dt = \rho \int_{\tau}^{+\infty} \left( \int_{\tau}^t c^{ac}(s) ds \right) e^{-\rho t} dt.$$

From the monotony of  $U(\cdot)$  and the estimates on  $c$ , we see that

$$\int_{\tau}^{+\infty} U(c^{ac}(t))e^{-\rho t} dt \leq \lambda(\tau)U \left( \rho \int_{\tau}^{+\infty} (C_3\kappa + C_4) e^{(C_1-\rho)t} \frac{dt}{\lambda(\tau)} \right) = \frac{e^{-\rho\tau}}{\rho} U \left( \rho^2 \left( \frac{C_3\kappa + C_4}{\rho - C_1} \right) e^{C_1\tau} \right).$$

Let us define for every  $\tau \geq 0$ ,

$$\varphi(\tau) = \frac{e^{-\rho\tau}}{\rho} U \left( \rho^2 \left( \frac{C_3\kappa + C_4}{\rho - C_1} \right) e^{C_1\tau} \right). \quad (3.20)$$

Since  $U(\cdot)$  is concave, it is bounded from above by an affine function. We also know that  $U(\cdot)$  is non negative. On the other hand, since  $C_1 < \rho$ , there exists  $A_1, A_2 > 0$  which depend continuously on  $\rho, \kappa$  and  $C_1$ , such that

$$0 \leq \varphi(\tau) \leq (A_1 e^{C_1\tau} + A_2) e^{-\rho\tau},$$

so  $\varphi(\tau) \rightarrow 0$  as  $\tau \rightarrow +\infty$ . Moreover, by construction, (3.18) holds for the map defined in (3.20).

As in the proof of the second point of [98, Lemma 3.2], using [11, Theorem 13.3.1] and the concavity of  $U(\cdot)$ , we obtain that for every  $\tau > 0$ , the map

$$\mathcal{M}_+([0, \tau]) \ni c \mapsto \int_0^{\tau} U(c^{ac}(t))e^{-\rho t} dt$$

is upper semi-continuous for the weak  $*$  topology of measures.

Let us consider a sequence  $(k_n(\cdot), c_n)_{n \in \mathbb{N}}$  of  $\mathcal{A}(\kappa)$  such that  $c_n \xrightarrow{*} c$  in  $\mathcal{M}_{\text{loc}}(\mathbb{R}_+)$ . We know that

$$J(c_n) \leq \int_0^{\tau} U(c_n^{ac}(t))e^{-\rho t} dt + \varphi(\tau), \quad \forall n \in \mathbb{N}. \quad (3.21)$$

Furthermore, the non negativity of  $U(\cdot)$  yields

$$\int_0^{\tau} U(c^{ac}(t))e^{-\rho t} dt \leq J(c). \quad (3.22)$$

Taking the limit supremum on both sides of (3.21), we obtain

$$\limsup_n J(c_n) \leq \limsup_n \int_0^{\tau} U(c_n^{ac}(t))e^{-\rho t} dt + \varphi(\tau) \leq \int_0^{\tau} U(c^{ac}(t))e^{-\rho t} dt + \varphi(\tau) \leq J(c) + \varphi(\tau),$$

where we used first the upper semi-continuity of  $c \mapsto \int_0^{\tau} U(c^{ac}(t))e^{-\rho t} dt$ , then (3.22). We conclude by letting  $\tau$  tend to  $+\infty$ .  $\square$

**Remark 3.3** *The proof above also supplies a uniform bound for  $J(\cdot)$  on  $\mathcal{A}(\kappa)$ , namely that for every  $(k(\cdot), c) \in \mathcal{A}(\kappa)$ ,*

$$J(c) \leq \varphi(0) = \frac{1}{\rho} U \left( \rho^2 \left( \frac{C_3\kappa + C_4}{\rho - C_1} \right) \right).$$

**Remark 3.4** *More generally, the arguments in the proof of Lemma 3.2 show that if there exist two constants  $A > 0$ ,  $B \in (0, \rho)$  and a sequence of non negative measures  $(c_n)_{n \in \mathbb{N}}$  which satisfies*

$$i) \text{ for every } n \in \mathbb{N} \text{ and } t \geq 0, \|c_n\|_{\mathcal{M}_+([0, t])} \leq A e^{Bt},$$

ii)  $c_n \xrightarrow{*} c$  in  $\mathcal{M}_{\text{loc}}(\mathbb{R}_+)$ ,

then  $\limsup_{n \rightarrow +\infty} J(c_n) \leq J(c)$ .

From the upper semi-continuity of  $J(\cdot)$  and the compactness properties of the set  $\mathcal{A}(\kappa)$ , we obtain the existence of an optimizer. On the other hand, following the lines of [98], more properties of regularity of the optimizers are deduced from the regularity of  $f(\cdot)$ ; the strict concavity of  $f(\cdot)$  yields the uniqueness of the optimizer. The proofs of the three following statements can be found in [98].

**Proposition 3.2** *Under Assumptions 3.3 and 3.4, for every  $\kappa > 0$ , the problem  $\sup_{\mathcal{A}(\kappa)} J(\cdot)$  admits at least one maximizer.*

**Lemma 3.3** *Under Assumptions 3.3 and 3.4, any optimizer  $(k(\cdot), c)$  of the problem  $\sup_{\mathcal{A}(\kappa)} J(\cdot)$  is such that  $c \in L^1_{\text{loc}}(\mathbb{R}_+)$ .*

**Corollary 3.1** *Under Assumptions 3.3 and 3.4, for every optimizer  $(k(\cdot), c(\cdot))$  of the problem  $\sup_{\mathcal{A}(\kappa)} J(\cdot)$ ,  $k(\cdot)$  belongs to  $W^{1,1}_{\text{loc}}(\mathbb{R}_+)$  and the constraint  $k'(\cdot) + c(\cdot) = f(k(\cdot))$  is saturated for almost all  $t \geq 0$ . Moreover, the optimal pair  $(k(\cdot), c(\cdot))$  is unique.*

We have thus obtained the existence and uniqueness of an optimizer. Moreover, its regularity allows us to conclude that the initial (non-relaxed) problem defined by (3.7) has a unique optimizer. Under Assumptions 3.1 and 3.3, when  $f(\cdot)$  has the form (3.5), this and Proposition 3.1 yield that the problem defined by (3.3) has a unique optimal control. We now state some qualitative properties of the optimizer  $(k(\cdot), c(\cdot))$  of problem (3.6):

**Proposition 3.3** *Under Assumptions 3.3 and 3.4, let  $k^*$  be the unique positive number such that*

$$f'(k^*) = \rho, \quad (3.23)$$

(whose existence and uniqueness is guaranteed by Assumption 3.4). If  $(k(\cdot), c(\cdot))$  is an optimizer of problem (3.6), then  $k(\cdot) \in C^2(\mathbb{R}_+)$  and  $c(\cdot) \in C^1(\mathbb{R}_+)$ . Moreover,

- 1) if  $0 < \kappa < k^*$ , then for all  $t \geq 0$ ,  $f(k(t)) - c(t) \geq 0$
- 2) if  $\kappa > k^*$ , then for all  $t \geq 0$ ,  $f(k(t)) - c(t) \leq 0$
- 3) if  $\kappa = k^*$ , then for all  $t \geq 0$ ,  $f(k(t)) - c(t) = 0$ .

## Proof

**Step 1: The Euler-Lagrange condition.** The first step consists in obtaining the Euler-Lagrange condition. Let  $(k(\cdot), c(\cdot))$  be an optimizer. Let us fix  $T \geq 0$  and  $h > 0$ . Let us prove by contradiction that the restriction of  $k(\cdot)$  to the time interval  $[T, T+h]$  belongs to

$$\mathcal{S} = \operatorname{argmax} \left\{ \int_T^{T+h} U(f(y(t)) - y'(t)) e^{-\rho t} dt : \begin{array}{l} y(\cdot) \in W^{1,1}([T, T+h]), \\ y(T) = k(T), y(T+h) = k(T+h), \\ y(\cdot) \geq 0, \\ f(y(\cdot)) - y'(\cdot) \geq 0 \end{array} \right\}. \quad (3.24)$$

By assumption,

$$k(\cdot) \in \operatorname{argmax} \left\{ \int_0^{+\infty} U(f(y(t)) - y'(t)) e^{-\rho t} dt : y(\cdot) \in W^{1,1}_{\text{loc}}(0, +\infty), y(0) = \kappa, y(\cdot) \geq 0, f(y(\cdot)) - y'(\cdot) \geq 0 \right\}.$$

Let us assume that the restriction of  $k(\cdot)$  to the set  $[T, T+h]$  does not belong to  $\mathcal{S}$ . Then, there exists  $y(\cdot) \in W^{1,1}([T, T+h])$  with  $y(T) = k(T)$ ,  $y(T+h) = k(T+h)$ ,  $y(\cdot) \geq 0$  and  $f(y(\cdot)) - y'(\cdot) \geq 0$  such that

$$\int_T^{T+h} U(f(k(t)) - k'(t))e^{-\rho t} dt < \int_T^{T+h} U(f(y(t)) - y'(t))e^{-\rho t} dt.$$

It is possible to build the following admissible competitor:

$$k_2(t) = \begin{cases} y(t), & \text{if } t \in [T, T+h], \\ k(t), & \text{otherwise.} \end{cases}$$

Indeed,  $0 \leq k_2(\cdot) \in W_{\text{loc}}^{1,1}(0, +\infty)$ ,  $k_2(0) = \kappa$  and  $c_2(\cdot) = f(k_2(\cdot)) - k_2'(\cdot) \geq 0$ . We see from the monotonicity of  $U(\cdot)$  that

$$\int_0^{+\infty} U(c(t))e^{-\rho t} dt < \int_0^{+\infty} U(c_2(t))e^{-\rho t} dt,$$

which is the desired contradiction.

Therefore, the restriction of  $k(\cdot)$  to the set  $[T, T+h]$  belongs to  $\mathcal{S}$ . Since  $T \geq 0$  and  $h > 0$  were taken arbitrarily, the latter property holds for any  $T \geq 0$  and  $h > 0$ .

In order to obtain the Euler-Lagrange equation on the whole interval  $(0, +\infty)$ , we finally need to prove that the inequality constraints are not active. From [98, Proposition 3.6], we know that for any  $\tau > 0$ , there exists  $\underline{c} > 0$  such that the optimal consumption strategy  $c(\cdot) = f(k(\cdot)) - k'(\cdot)$  is bounded from below by  $\underline{c}$  on the time interval  $[0, \tau]$ . Moreover, as in the proof of [98, Corollary 3.9], we see that  $k(\cdot)$  is positive. Therefore, the inequality constraints are not active and the following Euler-Lagrange equation holds:

$$e^{-\rho t} U'(f(k(t)) - k'(t)) f'(k(t)) + \frac{d}{dt} \left\{ e^{-\rho t} U'(f(k(t)) - k'(t)) \right\} = 0, \text{ for almost all } t \in (0, +\infty).$$

Hence the map  $\xi : [0, +\infty) \ni t \mapsto e^{-\rho t} U'(f(k(t)) - k'(t))$  is such that

$$\xi'(t) = -f'(k(t))\xi(t), \quad \text{almost everywhere in } (0, +\infty),$$

thus

$$\xi(t) = \lambda \exp\left(-\int_0^t f'(k(s)) ds\right), \quad \forall t \in [0, +\infty).$$

From the assumptions on  $U(\cdot)$ ,  $\xi(\cdot)$  is positive on  $(0, +\infty)$ , thus  $\lambda > 0$ . Therefore, the optimal consumption strategy  $c(\cdot) = f(k(\cdot)) - k'(\cdot)$  satisfies

$$U'(c(t)) = \lambda \exp\left(\int_0^t (\rho - f'(k(s))) ds\right), \quad \forall t \in [0, +\infty). \quad (3.25)$$

From the continuity of  $k(\cdot)$ , we deduce that  $c(\cdot)$  is of class  $C^1$  on  $[0, +\infty)$ . Then, since  $k'(\cdot) = f(k(\cdot)) - c(\cdot)$ ,  $k(\cdot)$  is of class  $C^2$  on the time interval  $\{t > 0 : k(t) > 0\} = [0, +\infty)$ .

Let us now study the properties of  $k(t)$  for  $t \geq 0$ .

**Step 2:** we assume that  $0 < k(t) < k^*$  and we aim at proving by contradiction that  $k'(t) \geq 0$ . Set  $T^* = \inf\{s > t : k(s) \notin (0, k^*)\}$ . Recall that (3.25) holds for  $s \in [t, T^*)$ , with  $\lambda > 0$ . Since

$$\rho - f' \circ k(\cdot) < 0$$

on  $(t, T^*)$  and  $U(\cdot)$  is strictly concave, we obtain by taking the derivative of (3.25) that  $c(\cdot)$  is increasing on  $[t, T^*)$ .

Let us assume that  $k'(t) < 0$ . Note first that  $f'(\cdot) > \rho$  on  $(0, k^*)$ , thus  $f(\cdot)$  is increasing on  $(0, k^*)$ . This implies that  $k(s) < k(t)$  for all  $s \in (t, T^*)$ ; otherwise, there would exist  $s_0 \in (t, T^*)$  such that  $k(s_0) \leq k(t)$  and  $k'(s_0) \geq 0$ , which would lead to the following contradiction:

$$0 \leq k'(s_0) = f(k(s_0)) - c(s_0) < f(k(t)) - c(t) = k'(t) < 0$$

Hence,  $k(s) < k(t)$  and  $k'(s) \leq k'(t) < 0$  for all  $s \in (t, T^*)$ . This implies that the inequality  $k(s) \leq k(t) + k'(t)(s - t)$  holds for every  $s \in (t, T^*)$ . Therefore,  $T^*$  must be finite and  $k(T^*) = 0$  and  $k'(T^*) < 0$  from the regularity of  $k(\cdot)$ . This is the desired contradiction, since we know that  $k(\cdot)$  only takes non negative values.

**Step 3:** we assume that  $k(t) > k^*$  and we aim at proving that  $k'(t) \leq 0$ .

As in Step 2, set  $T^* = \inf \{s > t : k(s) \leq k^*\}$ . The same arguments as in the beginning of step 2 show that  $c(\cdot)$  is decreasing on  $[t, T^*)$ . We make out two cases:

1. Let us first consider the case when  $f(\cdot)$  is decreasing in a neighbourhood of  $k(t)$ , i.e. that  $f(\cdot)$  is decreasing on the interval  $[k(t) - \epsilon, +\infty)$  for some  $\epsilon \in (0, k(t) - k^*)$  (using the concavity of  $f(\cdot)$ ). This may occur for instance if  $f(\cdot)$  has the form (3.5) and if  $\delta$ , the rate of capital depreciation, is positive.

We first claim that if  $k'(t) \geq 0$  then for all  $s \in [t, T^*)$ ,  $k'(s) \geq 0$ ; indeed, if it was not true, there would exist  $\bar{s} \in (t, T^*)$  such that  $k'(\bar{s}) < 0$ . Set  $\hat{s} = \sup \{s \in [t, \bar{s}) : k'(s) \geq 0\}$ : we see that  $t \leq \hat{s} < \bar{s}$  and  $k'(\hat{s}) = 0$ . We know that  $c(\hat{s}) > c(\bar{s})$ . On the other hand,  $k(\hat{s}) \geq k(\bar{s})$  thus  $f(k(\hat{s})) \leq f(k(\bar{s}))$ . This implies that

$$0 > k'(\bar{s}) = f(k(\bar{s})) - c(\bar{s}) > f(k(\hat{s})) - c(\hat{s}) = k'(\hat{s}) = 0,$$

i.e. a contradiction.

Therefore, if  $k'(t) \geq 0$  and  $T^*$  is finite then  $k(T^*) \geq k(t) > k^*$  in contradiction with the definition of  $T^*$ . Hence  $k'(t) \geq 0$  implies that  $T^* = +\infty$ .

Finally, let us show that  $k'(t) \geq 0$  leads to a contradiction with the optimality of  $k$ : for  $s > t$ , we see that

$$c(s) = f(k(s)) - k'(s) \leq f(k(s)) < f(k(t) - \epsilon),$$

because  $k'(\cdot) \geq 0$  in  $[t, T^*) = [t, +\infty)$  and  $f(\cdot)$  is decreasing in  $[k(t) - \epsilon, +\infty)$ .

We then see that the competitor  $(k_2(\cdot), c_2(\cdot))$  defined by

$$(k_2(s), c_2(s)) = \begin{cases} (k(s), c(s)), & \text{if } s < t, \\ (k(t) - \epsilon, f(k(t) - \epsilon)), & \text{otherwise,} \end{cases}$$

belongs to  $\mathcal{A}(\kappa)$  and is such that  $J(c_2) > J(c)$ , in contradiction with the fact that  $(k(\cdot), c(\cdot))$  is an optimizer. Therefore,  $k'(t)$  must be negative.

2. Let us consider the case when  $f(\cdot)$  is non decreasing in a neighbourhood of  $k(t)$ ; we make out two sub-cases:

- (a) there exists  $\bar{k} > k^*$  such that  $f'(\bar{k}) = 0$  (we then know that  $\bar{k}$  is unique from the strict concavity of  $f(\cdot)$ ). Then, if  $k'(t) > 0$ , set

$$I = \{s : t \leq s < T^* \text{ and } k(t) \leq k(s) < \bar{k}\}$$

and  $T = \sup I$ . Since for all  $s \in I$ ,  $k'(s) = f(k(s)) - c(s) \geq f(k(t)) - c(t) = k'(t) > 0$ , we see that either  $T = T^*$  or  $k(T) = \bar{k}$ . But  $T = T^*$  is not possible, because  $k(T) \geq k(t) > k^*$ . Hence  $k(T) = \bar{k}$ . Since  $k'(T) > 0$ , there exists  $\epsilon > 0$  such that  $k(T + \epsilon) > \bar{k}$  and  $k'(T + \epsilon) > 0$ , i.e.  $f(\cdot)$  is decreasing in a neighbourhood of  $k(T + \epsilon)$  and  $k'(T + \epsilon) > 0$ : we are back to case 1 and we know that this situation cannot happen.

- (b) The function  $f'(\cdot)$  is positive on  $(0, \infty)$ . Then, if  $k'(t) > 0$ , set  $I = \{s : t \leq s < T^* \text{ and } k(t) \leq k(s)\}$  and  $T = \sup I$ . Then for all  $s \in I$ ,  $k'(s) = f(k(s)) - c(s) \geq f(k(t)) - c(t) = k'(t) > 0$ , which shows that  $T = T^* = +\infty$  and that  $k(\cdot)$  is increasing in  $[t, +\infty)$ . It is easy to show that  $\lim_{s \rightarrow +\infty} k(s) = +\infty$  and that  $\lim_{s \rightarrow +\infty} c(s) = 0$ . The competitor  $(\tilde{k}(\cdot), \tilde{c}(\cdot))$  obtained by modifying  $(k(\cdot), c(\cdot))$  only for  $s > t$  by setting  $\tilde{k}(s) = k(t)$  and  $\tilde{c}(s) = f(k(t))$  is such that  $J(\tilde{c}(\cdot)) > J(c(\cdot))$ , which contradicts the optimality of  $(k(\cdot), c(\cdot))$ .

We have proved that  $k'(t) \leq 0$ .

**Step 4:** we assume that  $k(t) = k^*$  and we aim at proving that  $k(s) = k^*$  for all  $s \geq t$ .

Indeed, if there exists  $s > t$  such that  $k(s) > k^*$ , then, from the regularity of  $k(\cdot)$ , there must exist  $\tilde{s} \in (t, s)$  such that  $k(\tilde{s}) > k^*$  and  $k'(\tilde{s}) > 0$ , which is not possible from Step 3. Similarly, by using Step 2, we see that there does not exist  $s > t$  such that  $k(s) < k^*$ .

### Conclusion.

- From Step 4,  $\kappa = k^*$  implies that  $k'(t) = 0$  for all  $t \in [0, +\infty)$ .
- If  $0 < \kappa < k^*$ , let us prove that  $k'(t) \geq 0$  for all  $t \in [0, +\infty)$ . We proceed by contradiction. We know from Step 3 that  $k'(t) < 0$  implies that  $k(t) \geq k^*$ , hence  $t > 0$ . By continuity, there exists  $\tilde{t} \in (0, t]$  such that  $k(\tilde{t}) = k^*$ , and from Step 4,  $k(t) = k^*$ ; then, since  $k'(t) < 0$  and thanks to the regularity of  $k(\cdot)$ , there exists  $\epsilon > 0$  such that  $k(t + \epsilon) < k(t) = k^*$  and  $k'(t + \epsilon) < 0$ , which is impossible from step 3. Therefore,  $k'(t) \geq 0$  for all  $t \in [0, +\infty)$ .
- The case when  $\kappa > k^*$  is addressed in the same way, using Steps 2 and 4.

□

An important argument in Chapter 2 is the continuous dependency of the optimal control upon the initial capital  $\kappa > 0$  and the vector of unitary wages  $w \in (0, +\infty)^d$ . In what follows, we focus on the dependency of the optimal control problem upon on  $(\kappa, w)$ , namely of the set of admissible trajectories and controls  $\mathcal{A} = \mathcal{A}(\kappa, w)$ , the value function  $u = u(\kappa, w)$ , the net output  $f = f(k, w)$ , the capital  $k = k(t, \kappa, w)$ , the optimal consumption  $c = c(t, \kappa, w)$  and the capital target  $k^* = k^*(w)$ .

In order to obtain results about the stability of the optimal trajectory, we need a further assumption on the net output:

### Assumption 3.5 (A further assumption on the net output $f(\cdot, \cdot)$ )

The net output  $f : [0, +\infty) \times [0, +\infty)^d \rightarrow \mathbb{R}$  is of class  $C^1$  on  $(0, +\infty) \times (0, +\infty)^d$ .

The next lemma is a general stability result:

**Lemma 3.4** Let  $(\kappa_n, w_n)$  be a sequence of  $(0, +\infty) \times (0, +\infty)^d$  converging to  $(\kappa, w) \in (0, +\infty) \times (0, +\infty)^d$ . Under Assumption 3.3, 3.4 and 3.5, if  $k(\cdot, \kappa_n, w_n)$  converges towards  $h(\cdot) \in BV_{\text{loc}}(0, +\infty)$  in the following sense :

- $k(\cdot, \kappa_n, w_n) \rightarrow h(\cdot)$  almost everywhere,
- $\frac{\partial k}{\partial t}(\cdot, \kappa_n, w_n) \rightarrow h'$  in  $\mathcal{M}_{\text{loc}}([0, +\infty))$ ,

then  $h(\cdot) = k(\cdot, \kappa, w)$ .



**Proof** We argue by contradiction and assume that  $h(\cdot)$  differs from  $k(\cdot, \kappa, w)$ . For brevity, let us use the notations:  $k_n(\cdot) = k(\cdot, \kappa_n, w_n)$  and

- $c_h = f(h(\cdot), w) - h'$ ,
- $c_n : [0, +\infty) \ni t \mapsto f(k_n(t), w_n) - \frac{\partial k_n}{\partial t}(t, \kappa, w)$ ,
- $c : [0, +\infty) \ni t \mapsto f(k(t, \kappa, w), w) - \frac{\partial k}{\partial t}(t, \kappa, w)$ .

The regularity of  $f(\cdot, \cdot)$  and the convergence of the sequence  $(k_n(\cdot))_{n \in \mathbb{N}}$  yield that  $(h(\cdot), c_h) \in \mathcal{A}(\kappa, w)$ . From the uniqueness of the optimizer, we infer that

$$J(c_h) < J(c(\cdot)) = u(\kappa, w).$$

On the other hand, we know that

$$\limsup_{n \rightarrow +\infty} J(c_n(\cdot)) \leq J(c_h),$$

see Remark 3.4.

Let us set  $c_\varepsilon(\cdot) = (1 - \varepsilon)c(\cdot)$ . From the monotone convergence theorem,

$$\int_0^\tau U(c_\varepsilon(t))e^{-\rho t} dt \rightarrow J(c) \quad \text{as } \varepsilon \rightarrow 0 \text{ and } \tau \rightarrow +\infty.$$

Since  $J(c_h) < J(c(\cdot))$ , there exist  $\varepsilon > 0$  small enough,  $\tau > 0$  large enough and  $N_1 \in \mathbb{N}$  such that, for every  $n \geq N_1$ ,

$$J(c_n(\cdot)) < \int_0^\tau U(c_\varepsilon(t))e^{-\rho t} dt. \quad (3.26)$$

Let us fix such  $\varepsilon > 0$ ,  $\tau > 0$  and  $N_1 \in \mathbb{N}$ . For every  $n \in \mathbb{N}$ , we set

$$\begin{aligned} \tilde{k}_n(t) &= \begin{cases} k(t, \kappa, w) - \kappa + \kappa_n, & \text{if } t \in [0, \tau], \\ k(\tau, \kappa, w) - \kappa + \kappa_n, & \text{otherwise.} \end{cases} \\ \tilde{c}_n(t) &= \begin{cases} c_\varepsilon(t), & \text{if } t \in [0, \tau], \\ f(k(\tau, \kappa, w) - \kappa + \kappa_n, w_n), & \text{otherwise.} \end{cases} \end{aligned}$$

The continuity and the positivity of the optimal capital trajectory and the uniform convergence of  $\tilde{k}_n(\cdot)$  towards  $k(\cdot)$  ensure that for  $n$  large enough,  $\tilde{k}_n(\cdot) \geq 0$  on  $[0, \tau]$ . From [98, Proposition 3.6], we know that there exists a constant  $\underline{c} > 0$  such that  $c(t) \geq \underline{c}$  for  $t \in [0, \tau]$ . Therefore, for  $t \in [0, \tau]$ ,

$$\tilde{k}'_n(t) + \tilde{c}(t) = k'(t, \kappa, w) + (1 - \varepsilon)c(t), \quad (3.27)$$

$$\leq f(k(t, \kappa, w), w) - \varepsilon \underline{c}, \quad (3.28)$$

$$\leq f(\tilde{k}_n(t), w) + \left| f(k(t, \kappa, w), w) - f(\tilde{k}_n(t), w) \right| - \varepsilon \underline{c}, \quad (3.29)$$

hence  $\tilde{k}'_n(t) + \tilde{c}(t) \leq f(\tilde{k}_n(t), w)$  for  $n$  large enough.

If  $t > \tau$ , then

$$\tilde{k}'_n(t) = f(\tilde{k}_n(t), w) - \tilde{c}_n(t) = 0.$$

On the other hand,  $\tilde{c}_n(\cdot) \geq 0$  and  $\tilde{k}_n(0) = \kappa_n$ ; therefore, there exists  $N_2 \in \mathbb{N}$  such that, for every  $n \geq N_2$ ,  $(\tilde{k}_n(\cdot), \tilde{c}_n(\cdot)) \in \mathcal{A}(\kappa_n, w_n)$ .

We then see that, for every  $n \geq \max(N_1, N_2)$ ,

$$u(\kappa_n, w_n) = J(c_n) < \int_0^\tau U(c_\varepsilon(t))e^{-\rho t} dt \leq J(\tilde{c}_n) \leq u(\kappa_n, w_n),$$

from (3.26), the non negativity of  $U(\cdot)$  and the definition of  $u(\cdot, \cdot)$ . This is a contradiction; hence  $h(\cdot) = k(\cdot, \kappa, w)$ .  $\square$

Before stating the stability of the optimal trajectories with respect to  $\kappa$  and  $w$ , we need to obtain estimates  $k(\cdot, \kappa, w)$  which are locally uniform in  $(\kappa, w)$ . We first discuss the equiboundedness and equicontinuity of the optimal trajectories when  $(\kappa, w)$  varies in compact subsets of  $(0, +\infty) \times (0, +\infty)^d$ , leading to compactness properties in spaces of continuous functions on bounded intervals. Then we will see that  $k(\cdot, \kappa, w)$  is either semi-concave or semi-convex with respect to  $t$ , and that the constant in the semi-concavity or semi-convexity estimates is locally uniform with respect to  $(\kappa, w)$ . This will lead to compactness properties for  $\frac{\partial k}{\partial t}(\cdot, \cdot, \cdot)$ .

**Lemma 3.5** *Let us fix  $0 < \underline{w} < \bar{w} < +\infty$  and  $0 < \underline{\kappa} < \bar{\kappa} < +\infty$ . Under Assumptions 3.3, 3.4 and 3.5, there exists  $M > 0$  such that for every  $(\kappa, w) \in [\underline{\kappa}, \bar{\kappa}] \times [\underline{w}, \bar{w}]^d$ ,*

$$\|k(\cdot, \kappa, w)\|_{W^{1,\infty}(0, +\infty)} \leq M.$$

**Proof** Let  $w \in [\underline{w}, \bar{w}]^d$  be a vector of unitary wages and  $\kappa \in [\underline{\kappa}, \bar{\kappa}]$  be an initial capital. To keep the notations simple, we omit the dependency of  $k(\cdot, \cdot, \cdot)$  and  $c(\cdot, \cdot, \cdot)$  upon  $(\kappa, w)$ , i.e.  $(k(\cdot), c(\cdot))$  stands for the optimizer associated to  $(\kappa, w)$ .

We deduce from Proposition 3.3 that

$$0 < \underline{K} \leq k(t) \leq \bar{K} < +\infty, \quad \text{for any } t \in \mathbb{R}_+, \quad (3.30)$$

where

$$\underline{K} = \min \left( \underline{\kappa}, \inf_{\omega \in [\underline{w}, \bar{w}]^d} k^*(\omega) \right), \quad (3.31)$$

$$\bar{K} = \max \left( \bar{\kappa}, \sup_{\omega \in [\underline{w}, \bar{w}]^d} k^*(\omega) \right). \quad (3.32)$$

The infimum in (3.31) and the supremum (3.32) are actually achieved in  $[\underline{w}, \bar{w}]^d$ , thanks to the continuity of  $k^*(\cdot)$ , see Chapter 2.

We are left with finding a uniform bound on  $|k'(\cdot)|$ . Recall that for any  $t \geq 0$ ,  $k'(t) = f(k(t), w) - c(t)$ .

Since  $c(\cdot)$  is non negative, (3.30) implies that

$$k'(t) \leq f(k(t), w) \leq \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega) < +\infty.$$

To obtain a lower bound on  $k'(\cdot)$ , let us first bound  $c(\cdot)$  from above. We make out two cases:

1. If  $\kappa \leq k^*(w)$ , then, from Proposition 3.3,  $c(t) \leq f(k(t), w)$  for all  $t \geq 0$ . In this case

$$\|c(\cdot)\|_{L^\infty(0, +\infty)} \leq \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega).$$

2. Let us now discuss the case when  $\kappa \geq k^*(w)$ . The Legendre transform of  $U(\cdot)$ ,

$$U^*(y) = \inf_{c \geq 0} \{y \cdot c - U(c)\}, \quad \forall y > 0,$$

is concave and such that for all  $y > 0$ ,  $(U^*)'(y) = (U')^{-1}(y)$ . Then (3.25) implies that

$$k'(t) = f(k(t), w) - (U^*)' \left( \lambda \exp \left( \int_0^t \rho - \frac{\partial f}{\partial k}(k(s), w) ds \right) \right).$$

Let  $A$  be a non negative number such that

$$A \geq \sup_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} -\frac{\partial f}{\partial k}(\chi, \omega).$$

From the concavity of  $U^*(\cdot)$ , we infer that

$$k'(t) \leq f(k(t), w) - (U^*)' \left( \lambda e^{t(\rho+A)} \right).$$

We now need to bound  $\lambda$  from below. For a positive number  $T$ , set  $\eta = \kappa/T > 0$ . From Assumption 3.3, we see that

$$\lambda \leq U' \left( \eta + \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega) \right) e^{-T(\rho+A)} \quad (3.33)$$

is equivalent to

$$-(U^*)' \left( \lambda e^{T(\rho+A)} \right) + \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega) \leq -\eta.$$

Therefore, (3.33) implies that for any  $t \in [0, T]$ ,

$$\begin{aligned} k'(t) &= f(k(t), w) - (U^*)' \left( \lambda \exp \left( \int_0^t \rho - \frac{\partial f}{\partial k}(k(s), w) ds \right) \right) \\ &\leq \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega) - (U^*)' \left( \lambda e^{T(\rho+A)} \right) \\ &\leq -\eta, \end{aligned} \quad (3.34)$$

and

$$k(t) \leq \kappa - \eta t. \quad (3.35)$$

From the choice of  $\eta$ , we see that that  $k(T) = 0$  which is impossible since  $k(\cdot)$  does not vanish (see [98, Corollary 3.9]). This shows that (3.33) is impossible. Therefore, for all  $T > 0$ ,

$$\lambda > U' \left( \frac{\kappa}{T} + \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega) \right) e^{-T(\rho+A)}. \quad (3.36)$$

Let us set

$$\underline{\lambda} = \max_{T>0} U' \left( \frac{\bar{\kappa}}{T} + \max_{(\chi, \omega) \in [\underline{K}, \bar{K}] \times [\underline{w}, \bar{w}]^d} f(\chi, \omega) \right) e^{-T(\rho+A)} > 0.$$

Then (3.36) implies that  $\lambda > \underline{\lambda}$ . From (3.25) and since  $\kappa \geq k^*(w)$  implies that  $\frac{\partial f}{\partial k}(k(t), w) \leq \rho$  for all  $t \geq 0$ , we see that

$$\begin{aligned} 0 \leq c(t) &\leq (U')^{-1} \left( \underline{\lambda} \exp \left( \int_0^t (\rho - f'(k(s), w)) ds \right) \right) \\ &\leq (U')^{-1}(\underline{\lambda}). \end{aligned}$$

where  $\underline{\lambda}$  only depends of  $\underline{\kappa}$ ,  $\bar{\kappa}$ ,  $\underline{w}$  and  $\bar{w}$ .

Since  $k'(\cdot) = f(k(\cdot), w) - c(\cdot)$ , we have proved that there exists a positive constant  $M = M(\underline{\kappa}, \bar{\kappa}, \underline{w}, \bar{w})$  such that  $\|k(\cdot)\|_{W^{1,\infty}(0,+\infty)} \leq M$ .  $\square$

The next lemma is devoted to semi-concavity or semi-convexity estimates on the optimal trajectory; its proof uses the monotony of the optimal consumption strategy.

**Lemma 3.6** *Let us fix  $0 < \underline{w} < \bar{w} < +\infty$  and  $0 < \underline{\kappa} < \bar{\kappa} < +\infty$ . Under Assumptions 3.3, 3.4 and 3.5, there exists  $C > 0$  such that for every  $(\kappa, w) \in [\underline{\kappa}, \bar{\kappa}] \times [\underline{w}, \bar{w}]^d$ ,*

- *if  $\kappa \leq k^*(w)$ , then  $k(\cdot, \kappa, w)$  is semi-concave with a constant  $C$ ,*
- *if  $\kappa \geq k^*(w)$ , then  $k(\cdot, \kappa, w)$  is semi-convex with a constant  $C$ .*

**Proof** Let  $w \in [\underline{w}, \bar{w}]^d$  be a vector of unitary wages and  $\kappa \in [\underline{\kappa}, \bar{\kappa}]$  be an initial capital. As in the proof of Lemma 3.5, we omit the dependency of  $k(\cdot, \cdot, \cdot)$  and  $c(\cdot, \cdot, \cdot)$  upon  $(\kappa, w)$ .

Let us assume that  $\kappa \geq k^*(w)$ . From Proposition 3.3, we know that the related optimal consumption strategy  $c(\cdot)$  is non increasing. Since  $c(\cdot)$  is  $C^1$  in  $[0, +\infty)$ , this implies that  $c'(\cdot) \leq 0$ , i.e.

$$\frac{\partial f}{\partial k}(k(\cdot), w)k'(\cdot) - k''(\cdot) \leq 0.$$

Moreover, still from Proposition 3.3,  $\frac{\partial f}{\partial k}(k(t), w) \leq \rho$  and  $k'(t) \leq 0$  for all  $t \in [0, +\infty)$ . Hence, with  $M$  as in Lemma 3.5,

$$-M\rho \leq k''(\cdot).$$

Therefore,  $k(\cdot)$  is semi-convex with a constant independent of  $(\kappa, w) \in [\underline{\kappa}, \bar{\kappa}] \times [\underline{w}, \bar{w}]^d$ . Similarly, if  $\kappa \leq k^*(w)$ , then Lemma 3.5 yields

$$M \max_{(\chi, \omega) \in [\underline{\kappa}, \bar{\kappa}] \times [\underline{w}, \bar{w}]^d} \frac{\partial f}{\partial k}(\chi, \omega) \geq k''(\cdot).$$

Therefore,  $k(\cdot)$  is semi-concave with a constant independent of  $(\kappa, w) \in [\underline{\kappa}, \bar{\kappa}] \times [\underline{w}, \bar{w}]^d$ . □

**Proposition 3.4** *Let  $(\kappa_n, w_n)_{n \in \mathbb{N}}$  be a sequence of  $(0, +\infty) \times (0, +\infty)^d$  which tends to  $(\kappa, w) \in (0, +\infty) \times (0, +\infty)^d$ . Under Assumptions 3.3, 3.4 and 3.5, for any compact  $I$  contained in  $(0, +\infty)$ ,  $k(\cdot, \kappa_n, w_n) \rightarrow k(\cdot, \kappa, w)$  converges in  $C^1(I)$  and  $c(\cdot, \kappa_n, w_n) \rightarrow c(\cdot, \kappa, w)$  in  $C^0(I)$ .*

**Proof**

1. Assume first that  $\kappa < k^*(w)$ . From the continuity of  $k^*(\cdot)$  and the convergence of  $(\kappa_n, w_n)_{n \in \mathbb{N}}$ , we may assume that for every  $n \in \mathbb{N}$ ,  $\kappa_n < k^*(w_n)$ . From Lemmas 3.5 and 3.6, using a diagonal extraction argument, we see that there exist a semi-concave function  $h(\cdot) \in W^{1,\infty}(0, \infty)$  and a subsequence still indexed by  $n$  such that, for each compact  $I$ ,  $(k(\cdot, \kappa_n, w_n))_{n \in \mathbb{N}}$  converges to  $h(\cdot)$  in  $C^0(I)$  and  $(\frac{\partial k}{\partial t}(\cdot, \kappa_n, w_n))_{n \in \mathbb{N}}$  converges almost everywhere to  $h'(\cdot)$ . From Lemma 3.4, we know that  $h(\cdot) = k(\cdot, \kappa, w)$ . The uniqueness of the cluster point implies that the whole sequence converges to  $k(\cdot, \kappa, w)$ . There remains to prove the locally uniform convergence of  $\frac{\partial k}{\partial t}(\cdot, \kappa_n, w_n)$  to  $\frac{\partial k}{\partial t}(\cdot, \kappa, w)$ . For that, we consider the sequence  $\varphi_n : [0, +\infty) \ni t \mapsto \frac{\partial k}{\partial t}(t, \kappa_n, w_n) - Ct$  (for the constant  $C$  appearing in the uniform semi-concavity estimate): the functions  $\varphi_n(\cdot)$  are decreasing, and  $(\varphi_n(\cdot))_{n \in \mathbb{N}}$  converges almost everywhere to  $t \mapsto \frac{\partial k}{\partial t}(t, \kappa, w) - Ct$  which is continuous and decreasing. We can then apply the slight modification of the second theorem of Dini which is proved in Lemma 2.13: the sequence  $(\varphi_n(\cdot))_{n \in \mathbb{N}}$  converges locally uniformly to  $t \mapsto \frac{\partial k}{\partial t}(t, \kappa, w) - Ct$ . This yields that  $\frac{\partial k}{\partial t}(\cdot, \kappa_n, w_n)$  converges to  $\frac{\partial k}{\partial t}(\cdot, \kappa, w)$  locally uniformly in  $[0, +\infty)$ .
2. If  $\kappa > k^*(w)$ , then we argue in the same way, using the semi-convexity estimates this time.
3. If  $\kappa = k^*(w)$ , then let us introduce  $\mathcal{I}$  the set of indexes  $n$  such that  $\kappa_n \geq k^*(w_n)$ . If  $n \in \mathcal{I}$ , then  $k(\cdot, \kappa_n, w_n)$  is non increasing on  $[0, +\infty)$ . In the opposite case,  $k(\cdot, \kappa_n, w_n)$  is increasing. We make out two cases:
  - (a) if  $\mathcal{I}$  or  $\mathbb{N} \setminus \mathcal{I}$  is finite, then we may argue as in one among the first two cases.

- (b) If both  $\mathcal{I}$  or  $\mathbb{N} \setminus \mathcal{I}$  are infinite, then we consider separately the subsequence corresponding to the indexes in  $\mathcal{I}$  for which we use the arguments of case 2, and the subsequence corresponding to the indexes in  $\mathbb{N} \setminus \mathcal{I}$  for which we use the arguments of case 1.

We conclude the proof by noticing that  $c(\cdot, \kappa_n, w_n) = f(k(\cdot, \kappa_n, w_n)) - \frac{\partial k}{\partial t}(\cdot, \kappa_n, w_n)$ ; therefore, the sequence  $(c(\cdot, \kappa_n, w_n))_{n \in \mathbb{N}}$  converges locally uniformly to  $c(\cdot, \kappa, w)$ .  $\square$

**Remark 3.5** *A weaker stability result than the one stated in Proposition 3.4 can be obtained by using Lemmas 3.1 and 3.4 but not Lemmas 3.5 and 3.6.*

## 3.4 Conclusion

A key idea of the present chapter comes from Santambrogio [98]: we have introduced a relaxed version of the optimal control problem (3.3), which is equivalent to the relaxation of problem (3.7), because there is an explicit closed loop form for one of the two controls. Given an initial capital  $\kappa \in (0, +\infty)$  and a vector of unitary wages  $w \in (0, +\infty)^d$ , we have obtained the existence and uniqueness of an optimizer  $(k(\cdot), c(\cdot))$ . The capital  $k(\cdot, \kappa, w)$  is  $C^2$  while the optimal consumption strategy  $c(\cdot, \kappa, w)$  is  $C^1$ , and  $k'(\cdot) + c(\cdot) = f(k(\cdot), w)$ . Thanks to these regularity results, we have deduced that problems (3.7) and (3.3) are equivalent and have a unique optimizer. Finally, we have obtained the stability of the optimal capital trajectory  $k(\cdot, \kappa, w)$  and the optimal consumption strategy  $c(\cdot, \kappa, w)$  with respect to variations of  $(\kappa, w)$ .

Note that the results obtained in the present chapter are similar to those contained in Chapter 2, but require more restrictive assumptions. Yet, this approach gives a way to handle dynamical problems, in which  $w$  is no longer a vector in  $(0, +\infty)^d$  but a function from  $[0, +\infty)$  to some compact cube contained in  $(0, +\infty)^d$ . In such a case, if a lower bound on  $k(\cdot)$  can be proved, it would lead to a modified version of Lemma 3.5.

## Chapter 4

# The labour demand: two important examples

### 4.1 Introduction

In this chapter, we consider the model introduced in Chapter 2. The purpose is to provide two main examples of the labour demand which is of the form

$$L(w) = - \int_0^{+\infty} D_w f(k, w) dm(k, w),$$

with the net output  $f : [0, +\infty) \times (0, +\infty)^d \rightarrow \mathbb{R}$  defined by equation (2.7), i.e.

$$f(k) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} - \delta k, \quad \forall k \geq 0,$$

where  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  is the production function,  $w \in (0, +\infty)^d$  the vector of unitary wages, and  $\delta > 0$  the depreciation rate of capital. Here,  $(u(\cdot), m)$  is the unique solution of the system (2.12)-(2.13) completed with (2.15)-(2.16), i.e.

$$\rho u(k) = H(k, u'(k)), \quad (4.1)$$

$$\frac{d}{dk} (D_q H(\cdot, u'(\cdot)) m(\cdot))(k) = \eta(k) - \nu m(k), \quad (4.2)$$

completed with the following conditions:

$$D_q H(0, u'(0)) \geq 0, \quad (4.3)$$

$$1 = \int_0^{+\infty} dm(k), \quad (4.4)$$

where the Hamiltonian

$$H(k, q) = \sup\{U(c) - c\} + f(k)q, \quad \forall (k, q) \in [0, +\infty) \times \mathbb{R},$$

with  $U : [0, +\infty) \rightarrow \{-\infty\} \cup \mathbb{R}$  the utility function.

We make explicit the production function of firms. First, we recall the properties satisfied by the optimal investment established in Chapter 2, then we assume that the production function is a Cobb-Douglas function, we give the formulas of the net output and the target capital, and deduce the form of the labour demand. Finally, we assume that the production function is a Constant Elasticity of Substitution (or CES) function and explore the form of the net output, the target capital and the labour demand in this case.

## 4.2 Standing assumptions on the optimal investment

Some properties of the optimal investment,  $b : (0, +\infty) \times (0, +\infty)^d \rightarrow \mathbb{R}$ , have been given in Chapter 2. In particular, we have proved that under rather general assumptions, the golden rule of accumulation of capital holds, i.e. given a vector of wages  $w \in (0, +\infty)^d$ , the capital of firms converges towards a target capital  $k^*(w)$  solution of equation (2.36) which is

$$\frac{\partial f}{\partial k}(k, w) = \rho. \quad (4.5)$$

**Assumption 4.1** *The optimal investment of the firms  $b : (0, +\infty) \times (0, +\infty)^d \rightarrow \mathbb{R}$  has the following properties: for any  $w \in (0, +\infty)^d$ ,*

- i)  $b(\cdot, w)$  is continuous on  $(0, +\infty)$  and of class  $C^1$  on  $(0, k^*(w)) \cup (k^*(w), +\infty)$ .
- ii)  $b(\cdot, w)$  is positive on  $(0, k^*(w))$  and negative on  $(k^*(w), +\infty)$ .
- iii) there exist  $\varepsilon > 0$  and  $M > 0$  such that

$$0 \leq b(k, w) \leq M(k^*(w) - k), \quad \text{if } k \in [k^*(w) - \varepsilon, k^*(w)], \quad (4.6)$$

$$M(k^*(w) - k) \leq b(k, w) \leq 0, \quad \text{if } k \in [k^*(w), k^*(w) + \varepsilon]. \quad (4.7)$$

Thus, by ease of notation, instead of working with the solution  $(u(\cdot), m)$  of the system (4.1)-(4.4), we will consider that  $m$  is solution of

$$\frac{d}{dk} (b(\cdot, w)m(\cdot))(k) = \eta(k) - \nu m(k), \quad k > 0, \quad (4.8)$$

$$m \in \mathcal{P}((0, +\infty)), \quad (4.9)$$

where the function  $b(\cdot, \cdot)$  satisfies Assumption 4.1. The function  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  is a non negative and continuous function such that  $\int_0^{+\infty} \eta(k) dk = \nu$ , which models the entries of firms in the economy.

## 4.3 The Cobb-Douglas model

In this paragraph, we consider that the production is described by the Cobb-Douglas function,  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$ , by

$$F(k, \ell) = Ak^\alpha \ell^\beta,$$

where  $A > 0$ ,  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)^d$ , with  $\ell^\beta = \prod_{i=1}^d \ell_i^{\beta_i}$  and  $\alpha + \sum_{i=1}^d \beta_i < 1$ .

**Lemma 4.1** *For every  $(k, w) \in [0, +\infty) \times (0, +\infty)^d$ ,*

- the net output is given by

$$f(k, w) = \left( Ak^\alpha \prod_{i=1}^d \left( \frac{\beta_i}{w_i} \right)^{\beta_i} \right)^{\frac{1}{1-|\beta|}} (1 - |\beta|) - \delta k. \quad (4.10)$$

- the first derivative of  $f(\cdot, \cdot)$  with respect to  $k$  is given by

$$\frac{\partial f}{\partial k}(k, w) = \alpha \left( A \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} k^{-\frac{1-\alpha-|\beta|}{1-|\beta|}} - \delta.$$

- the first derivative of  $f(\cdot, \cdot)$  with respect to  $w_i$  is

$$\frac{\partial f}{\partial w_i}(k, w) = - \left( Ak^\alpha \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} \frac{\beta_i}{w_i}. \quad (4.11)$$

- the target capital is

$$k^*(w) = \left( \frac{\alpha}{\alpha + \rho} \right)^{\frac{1-|\beta|}{1-(\alpha+|\beta|)}} \left( A \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-(\alpha+|\beta|)}}. \quad (4.12)$$

**Proof** For any  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$ , we recall that the net output is defined as follows

$$f(k, w) = \sup_{\ell \in [0, +\infty)^d} \{F(k, \ell) - w \cdot \ell\} - \delta k. \quad (4.13)$$

Let us take  $\ell(k, w)$  a maximizer in (4.13). The conditions of optimality are

$$\beta_i Ak^\alpha \ell(k, w)^\beta = w_i \ell_i(k, w), \quad \forall i \in \{1, \dots, d\}.$$

Setting  $\lambda(k, w) = Ak^\alpha \ell(k, w)^\beta$ , we obtain that for every  $i \in \{1, \dots, d\}$ ,

$$\ell_i(k, w) = \lambda(k, w) \frac{\beta_i}{w_i}. \quad (4.14)$$

The definition of  $\lambda(k, w)$  leads to the following equation:

$$\lambda(k, w) = Ak^\alpha \ell(k, w)^\beta = Ak^\alpha \prod_{i=1}^d \left( \lambda(k, w) \frac{\beta_i}{w_i} \right)^{\beta_i}.$$

Hence,

$$\lambda(k, w) = \left( Ak^\alpha \prod_{i=1}^d \left( \frac{\beta_i}{w_i} \right)^{\beta_i} \right)^{\frac{1}{1-|\beta|}}.$$

From (4.14), we deduce that

$$f(k, w) = \lambda(k, w) - \sum_{i=1}^d w_i \lambda(k, w) \frac{\beta_i}{w_i} - \delta k = \lambda(k, w)(1 - |\beta|) - \delta k.$$

This is equation (4.10). From (4.10), it is possible to deduce the first derivatives of  $f(\cdot, \cdot)$  and compute the target capital  $k^*(w)$  as the unique solution of the equation (4.5).  $\square$

**Lemma 4.2** For every  $i \in \{1, \dots, d\}$  and  $w \in (0, +\infty)^d$ , the labour demand is given by

$$L_i(w) = \tilde{L}(w) \frac{\beta_i}{w_i}, \quad (4.15)$$

where

$$\tilde{L}(w) = \left( A \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} \frac{1}{\nu} \int_0^{+\infty} \eta(\kappa) \left( \kappa^{\frac{\alpha}{1-|\beta|}} + \int_{\kappa}^{k^*(w)} \frac{\alpha}{1-|\beta|} k^{-\frac{1-(\alpha+|\beta|)}{1-|\beta|}} e^{\int_{\kappa}^k \frac{\nu}{b(z, w)} dz} \right) d\kappa.$$



**Proof** For any  $w \in (0, +\infty)^d$ , the labour demand for the  $i^{\text{th}}$  type of workers is given by

$$L_i(w) = - \int_0^{+\infty} \frac{\partial f}{\partial w_i}(k, w) m(k, w) dk,$$

where, as in Proposition 2.6,  $m$  is given by

$$m(k, w) = \begin{cases} \frac{1}{b(k, w)} \int_0^k \eta(\kappa) \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) d\kappa, & \text{if } k \in [0, k^*(w)], \\ -\frac{1}{b(k, w)} \int_k^{+\infty} \eta(\kappa) \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) d\kappa, & \text{if } k \in (k^*(w), +\infty), \end{cases}$$

Let us fix  $a \in (0, +\infty)$  and compute

$$I_1 = \int_0^{k^*(w)} \frac{k^a}{b(k, w)} \int_0^k \eta(\kappa) \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) d\kappa dk.$$

We see that,

$$I_1 = \int_0^{k^*(w)} \eta(\kappa) \int_\kappa^{k^*(w)} \frac{k^a}{b(k, w)} \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) dk d\kappa, \quad (4.16)$$

$$= \int_0^{k^*(w)} \eta(\kappa) \left( \left[ -\frac{k^a}{\nu} \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) \right]_\kappa^{k^*(w)} + \int_\kappa^{k^*(w)} \frac{a}{\nu} k^{a-1} \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) dk \right) d\kappa, \quad (4.17)$$

$$= \frac{1}{\nu} \int_0^{k^*(w)} \eta(\kappa) \left( \kappa^a + \int_\kappa^{k^*(w)} a k^{a-1} \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) dk \right) d\kappa, \quad (4.18)$$

where we applied Tonelli's theorem to deduce (4.16), (4.17) comes from an integration by parts, and (4.18) uses point *iii*) in Assumption 4.1 which yields

$$\left[ -\frac{k^a}{\nu} e^{-\int_\kappa^k \frac{\nu}{b(z, w)} dz} \right]_\kappa^{k^*(w)} = \frac{\kappa^a}{\nu},$$

since  $k \mapsto -\int_\kappa^k \frac{\nu}{b(z, w)} dz$  blows up like a logarithm of  $(k^*(w) - k)$  as  $k$  tends to  $k^*(w)$ .

We compute in the same way the quantity

$$I_2 = \int_{k^*(w)}^{+\infty} \frac{k^a}{b(z, w)} \int_k^{+\infty} \eta(\kappa) \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) d\kappa dk$$

and obtain

$$I_2 = \frac{1}{\nu} \int_{k^*(w)}^{+\infty} \eta(\kappa) \left( \kappa^a + \int_\kappa^{k^*(w)} a k^{a-1} \exp\left(-\int_\kappa^k \frac{\nu}{b(z, w)} dz\right) dk \right) d\kappa.$$

From equation (4.11), and the expressions of  $I_1$  and  $I_2$  with  $a = \frac{\alpha}{1-|\beta|}$ , we deduce the result.  $\square$

The following technical lemma which will be used in Chapter 5 to justify the assumptions made on the labour demand.

**Lemma 4.3** *Let  $(w^n)_{n \in \mathbb{N}}$  be a sequence of vectors in  $(0, +\infty)^d$ ; if  $\tilde{L}(w^n) \rightarrow 0$  when  $n$  goes towards  $+\infty$ , then there exists  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} w_i^n = +\infty$ .*

**Proof** For any  $w \in (0, +\infty)^d$ , let us introduce

$$I(w) = \int_0^{+\infty} \eta(\kappa) \left( \kappa^{\frac{\alpha}{1-|\beta|}} + \int_{\kappa}^{k^*(w)} \frac{\alpha}{1-|\beta|} k^{-\frac{1-(\alpha+|\beta|)}{1-|\beta|}} e^{\int_{\kappa}^k \frac{\nu}{b(z,w)} dz} dk \right) d\kappa.$$

This permits us to write

$$\tilde{L}(w) = \left( A \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} \frac{I(w)}{\nu}. \quad (4.19)$$

Let us introduce the positive function  $J : (0, +\infty) \rightarrow (0, +\infty)$  defined for every  $z \in (0, +\infty)$  by

$$J(z) = \int_0^z \eta(\kappa) \kappa^{\frac{\alpha}{1-|\beta|}} d\kappa + z^{\frac{\alpha}{1-|\beta|}} \int_z^{+\infty} \eta(\kappa) d\kappa.$$

We claim that for every  $w \in (0, +\infty)^d$ ,

$$J(k^*(w)) \leq I(w). \quad (4.20)$$

Indeed, let us fix  $w \in (0, +\infty)^d$  and split  $I(w)$  into  $I_1(w) + I_2(w)$ :

$$I_1(w) = \int_0^{k^*(w)} \eta(\kappa) \left( \kappa^{\frac{\alpha}{1-|\beta|}} + \int_{\kappa}^{k^*(w)} \frac{\alpha}{1-|\beta|} k^{-\frac{1-(\alpha+|\beta|)}{1-|\beta|}} e^{\int_{\kappa}^k \frac{\nu}{b(z,w)} dz} dk \right) d\kappa,$$

and

$$I_2(w) = \int_{k^*(w)}^{+\infty} \eta(\kappa) \left( \kappa^{\frac{\alpha}{1-|\beta|}} + \int_{\kappa}^{k^*(w)} \frac{\alpha}{1-|\beta|} k^{-\frac{1-(\alpha+|\beta|)}{1-|\beta|}} e^{\int_{\kappa}^k \frac{\nu}{b(z,w)} dz} dk \right) d\kappa.$$

It can be observed that

$$I_1(w) \geq \int_0^{k^*(w)} \eta(\kappa) \kappa^{\frac{\alpha}{1-|\beta|}} d\kappa.$$

On the other hand, since  $b(z, w) < 0$  for  $z \geq k^*(w)$ ,

$$\begin{aligned} I_2(w) &\geq \int_{k^*(w)}^{+\infty} \eta(\kappa) \left( \kappa^{\frac{\alpha}{1-|\beta|}} - \int_{k^*(w)}^{\kappa} \frac{\alpha}{1-|\beta|} k^{-\frac{1-(\alpha+|\beta|)}{1-|\beta|}} dk \right) d\kappa \\ &= k^*(w)^{\frac{\alpha}{1-|\beta|}} \int_{k^*(w)}^{+\infty} \eta(\kappa) d\kappa. \end{aligned}$$

Combining the estimates on  $I_1(\cdot)$  and  $I_2(\cdot)$  yields (4.20). Moreover,  $J(\cdot)$  satisfies:

- $J(\cdot)$  is positive on  $(0, +\infty)$ .
- $\liminf_{z \rightarrow +\infty} J(z) \geq \int_0^{+\infty} \kappa^{\frac{\alpha}{1-|\beta|}} \eta(\kappa) d\kappa > 0$ .
- $J(z) \rightarrow 0$  when  $z \rightarrow 0^+$

Therefore, from (4.19), (4.20), the expression of  $k^*(w)$  given by (4.12) and the properties of  $J(\cdot)$ , it is possible to deduce that there exists a continuous function  $\Lambda : (0, +\infty) \rightarrow (0, +\infty)$  satisfying

- $\Lambda(\cdot)$  is positive on  $(0, +\infty)$ ,
- $\Lambda(z) \rightarrow +\infty$  when  $z \rightarrow +\infty$ ,
- $\Lambda(z) \rightarrow 0$  when  $z \rightarrow 0^+$ ,

such that for every  $w \in (0, +\infty)^d$ ,

$$0 \leq \Lambda \left( \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right) \leq \tilde{L}(w).$$

Thus, let  $(w^n)_{n \in \mathbb{N}}$  be a sequence of elements of  $(0, +\infty)^d$ ; if  $\tilde{L}(w^n) \rightarrow 0$  when  $n \rightarrow +\infty$ , then the latter inequalities ensure that  $\prod_{j=1}^d \left( \frac{\beta_j}{w_j^n} \right)^{\beta_j} \rightarrow 0$ . This yields the existence of  $i \in \{1, \dots, d\}$  such that  $\limsup w_i^n = +\infty$ .  $\square$

## 4.4 A model with constant elasticity of substitution

In this paragraph, we consider that the production function is a CES function, i.e. the production function  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  is

$$F(k, \ell) = (k^\alpha + \ell^\beta)^\gamma,$$

where  $\alpha \in (0, 1)$ ,  $\beta \in (0, 1)^d$  and  $\gamma \in (0, 1)$ , with  $\ell^\beta = \sum_{i=1}^d \ell_i^{\beta_i}$ .

**Lemma 4.4** *For every  $(k, w) \in [0, +\infty) \times (0, +\infty)^d$ , let us introduce  $\lambda(k, w)$  the unique solution of*

$$\lambda \left( k^\alpha + \sum_{j=1}^d \left( \frac{\lambda \beta_j}{w_j} \right)^{\frac{\beta_j}{1-\beta_j}} \right)^{1-\gamma} = \gamma, \quad (4.21)$$

then

- the net output is

$$f(k, w) = \left( k^\alpha + \sum_{j=1}^d \left( \frac{\lambda(k, w) \beta_j}{w_j} \right)^{\frac{\beta_j}{1-\beta_j}} \right)^\gamma - \sum_{j=1}^d w_j \left( \frac{\lambda(k, w) \beta_j}{w_j} \right)^{\frac{1}{1-\beta_j}} - \delta k.$$

- the first derivative of  $f(\cdot, \cdot)$  with respect to  $k$  is

$$\frac{\partial f}{\partial k}(k, w) = \alpha \lambda(k, w) k^{\alpha-1} - \delta.$$

- the first derivative of  $f(\cdot, \cdot)$  with respect to  $w_i$  is

$$\frac{\partial f}{\partial w_i}(k, w) = - \left( \frac{\lambda(k, w) \beta_i}{w_i} \right)^{\frac{1}{1-\beta_i}}. \quad (4.22)$$

- the target capital  $k^*(w)$  satisfies

$$\alpha \lambda(k, w) k^{\alpha-1} = \delta + \rho. \quad (4.23)$$

**Proof** For any  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$ , we recall that the net output is defined by (4.10). We consider  $\ell(k, w)$  a maximizer in (4.10). The conditions of optimality are

$$\gamma (k^\alpha + \ell(k, w)^\beta)^{\gamma-1} \beta_i \ell_i(k, w)^{\beta_i-1} = w_i, \quad \forall i \in \{1, \dots, d\}.$$

Setting  $\lambda(k, w) = \gamma (k^\alpha + \ell(k, w)^\beta)^{\gamma-1}$ , we can write  $\ell_i(k, w) = \left( \frac{\lambda(k, w)\beta_i}{w_i} \right)^{\frac{1}{1-\beta_i}}$ . This leads to the following equation on  $\lambda(k, w)$ :

$$\begin{aligned} \lambda(k, w) &= \gamma \left( k^\alpha + \sum_{i=1}^d \left( \frac{\lambda(k, w)\beta_i}{w_i} \right)^{\frac{\beta_i}{1-\beta_i}} \right)^{\gamma-1} \\ \Leftrightarrow \lambda(k, w) \left( k^\alpha + \sum_{i=1}^d \left( \frac{\lambda(k, w)\beta_i}{w_i} \right)^{\frac{\beta_i}{1-\beta_i}} \right)^{1-\gamma} &= \gamma. \end{aligned}$$

This is equation (4.21). Finally, we obtain the first derivatives of  $f(\cdot, \cdot)$  by using the envelope theorem and we deduce that the target capital  $k^*(w)$  satisfies equation (4.23).  $\square$

**Lemma 4.5** *The labour demand is given for every  $i \in \{1, \dots, d\}$  and  $w \in (0, +\infty)^d$  by*

$$L_i(w) = \int_0^{+\infty} \left( \frac{\lambda(k, w)\beta_i}{w_i} \right)^{\frac{1}{1-\beta_i}} m(k, w) dk. \quad (4.24)$$

**Proof** This is direct from (4.22).  $\square$

## 4.5 Conclusion

In the present chapter, we have supplied two important examples for the labour demand function. We have considered successively the Cobb-Douglas and CES production functions. In both cases, we have characterized key quantities such as the net output or the target capital. Building on this, in the next chapter, we propose another set of assumptions for the labour supply, and prove the existence of equilibria when the labour demand is given by the two abovementioned examples. The interest of this new set of assumptions is to allow supply functions with more complex dependencies between the different types of workers.



## Chapter 5

# Existence and uniqueness of equilibria

### 5.1 Introduction

In this chapter, we aim at establishing an existence result, and to discuss the uniqueness of solutions of a non-linear system of equations. This system appears in the modelling of factor markets (see Chapter 2 for more details). To fix ideas and notations, we assume that the only inputs are workers (with possibly several types), therefore, the market modelled is the labour market. Thus, the system that we aim to solve has the form

$$S(w) = L(w),$$

where  $w \in (0, +\infty)^d$  is considered as a vector of wages, with  $d$  the number of different types of workers, and  $S(w)$  and  $L(w)$  are respectively the supply and the demand for labour, when the wages are given by the vector  $w$ .

The strategy for proving existence of solutions, is based on the Brouwer degree, i.e. for a well-chosen element  $w_0 \in (0, +\infty)^d$ , we use the following deformation:

$$h(t, \cdot) = S(\cdot) - (1 - t)S(w_0) + tL(\cdot),$$

for every  $t \in [0, 1]$ . We are able to state a priori bounds for solutions of the equation  $h(t, \cdot) = 0$  which are independent of  $t$ , then the properties of the Brouwer degree lead to the existence of solutions. These a priori bounds require assumptions on the supply and the demand for labour. The assumptions on  $L(\cdot)$ , the labour demand, are driven by the examples developed in Chapter 4. The assumptions made on  $S(\cdot)$ , are as general as possible. In addition, it should be noted that when  $(S - L)(\cdot)$  is regular enough, the Brouwer degree is given by a formula from which the uniqueness of solutions can be deduced.

The chapter is structured as follows. First, we state preliminary results which are necessary for the next sections. Namely, we introduce the Brouwer degree and give some of its properties, and the notions of  $Z$ -matrices and  $M$ -matrices which are important tools in economics. We then describe the model that we choose for the labour market. Lastly, we state a technical lemma that serves to prove several results. In a second part, we state the assumptions made on the supply and demand for labour. We discuss these assumptions in some particular cases. The third section is devoted to the proof of the existence of solutions under the latter assumptions. Under additional assumptions, uniqueness can be proved as well. The Brouwer degree is a non constructive method. Then, in the last section, we see that under more restrictive assumptions, existence (and uniqueness of solutions) can be proved by a continuation method. Since it is a constructive method, it gives us a way to solve this problem numerically.

## 5.2 Preliminaries

First, we introduce the Brouwer degree. It is the main ingredient on which the proofs of existence and uniqueness of solutions are based. Second, we recall the notions of  $Z$ -matrices and  $M$ -matrices, which has important applications in economics. Then, we present the labour market model used and the assumptions on the labour demand. Finally, we state a technical lemma which will serve the subsequent development.

### 5.2.1 The Brouwer degree

The Brouwer degree permits to ensure that the problem

$$f(x) = y,$$

admits solutions when  $f : \bar{\Omega} \rightarrow \mathbb{R}^d$  is a continuous map,  $\Omega$  a bounded and open set of  $\mathbb{R}^d$ , and  $y \notin f(\partial\Omega)$ . Formally, let us assume that  $f(\cdot)$  is regular and that the system  $f(x) = y$  admits a finite number (possibly zero) of solutions. Therefore, the Brouwer degree will count in a certain way the solutions of  $f(x) = y$ . Similarly to the determinant for linear maps, if the degree is different from zero, then the non-linear system admits at least one solution. Here is its definition:

**Definition 5.1** *The Brouwer degree, denoted by  $\deg$ , is a map from*

$$\mathcal{M} = \{(\mathcal{O}, f(\cdot), y) : \mathcal{O} \text{ is open and bounded, } f : \bar{\mathcal{O}} \rightarrow \mathbb{R}^d \text{ is continuous, and } \mathbb{R}^d \ni y \notin f(\partial\mathcal{O})\},$$

into  $\mathbb{Z}$  such that

1.  $\deg(\text{id}, \mathcal{O}, y) = 1$  for  $y \in \mathcal{O}$ .
2.  $\deg(f(\cdot), \mathcal{O}, y) = \deg(f(\cdot), \mathcal{O}_1, y) + \deg(f(\cdot), \mathcal{O}_2, y)$  whenever  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are disjoint open subsets of  $\mathcal{O}$  such that  $y \notin f(\bar{\mathcal{O}} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2))$
3.  $\deg(h(t, \cdot), \mathcal{O}, y(t))$  is independent of  $t \in [0, 1]$  whenever  $h : [0, 1] \times \bar{\mathcal{O}} \rightarrow \mathbb{R}^d$  is continuous,  $y : [0, 1] \rightarrow \mathbb{R}^d$  is continuous, and  $y(t) \notin h(t, \partial\mathcal{O})$  for every  $t \in [0, 1]$ .

The third property gives a way to compute the degree of complicated maps as soon as they can be continuously deformed into a nicer one, for which we know the value of the degree. Here are important properties of the degree:

**Theorem 5.1** *There exists a unique map satisfying Definition 5.1. Moreover, it satisfies the following properties:*

1.  $\deg(f(\cdot), \mathcal{O}, y) \neq 0$  implies  $f^{-1}(y) \neq \emptyset$ .
2.  $\deg(g(\cdot), \mathcal{O}, y) = \deg(f(\cdot), \mathcal{O}, y)$  whenever  $g|_{\partial\mathcal{O}} = f|_{\partial\mathcal{O}}$ .
3. If  $\mathcal{O}$  is an open and bounded set,  $f \in C^1(\bar{\mathcal{O}})$  and  $y \in \mathbb{R}^d \setminus f(\partial\mathcal{O} \cup S_f)$  where  $S_f$  is defined as follows:

$$S_f = \{x \in \mathcal{O} : \det Df(x) = 0\},$$

then,

$$\deg(f(\cdot), \mathcal{O}, y) = \sum_{x \in f^{-1}(y)} \text{sign det } Df(x).$$

This theorem states that the degree is uniquely determined and is characterized by the boundary values of the continuous function considered. Moreover, in a regular case, the degree has an explicit formula.

### 5.2.2 $Z$ -matrices and $M$ -matrices and their applications in economics

**Definition 5.2** Let  $A$  be a  $d \times d$  real matrix. The matrix  $A$  is said to be a  $Z$ -matrix if every off-diagonal entry is non positive, i.e.

$$A_{ij} \leq 0, \quad \forall i \neq j.$$

**Definition 5.3** Let  $A$  be a  $d \times d$  real matrix. The matrix  $A$  is said to be a  $M$ -matrix if it is a  $Z$ -matrix and if it has the form  $A = sI - B$ , where  $B = (B_{ij})$  with  $B_{ij} \geq 0$ , for every  $1 \leq i, j \leq d$ ,  $s$  is at least as large as  $\rho(B)$  the maximum of the moduli of the eigenvalues of  $B$ , and  $I$  is the identity matrix. Moreover, if  $s > \rho(B)$ , then  $A$  is non-singular.

There exist several characterizations of a non-singular  $M$ -matrix (R. J. Plemmons listed 40 characterizations in [92]). Let us present, only three of them:

**Proposition 5.1** Let  $A$  be a  $Z$ -matrix. Then  $A$  is a non-singular  $M$ -matrix if and only if one of the following properties holds:

- there exists  $x \geq 0$  such that  $Ax > 0$ .
- all the principal minors of  $A$  are positive.
- for every  $x \in \mathbb{R}^d$ ,  $Ax \geq 0 \Rightarrow x \geq 0$ .

The concepts of  $Z$  and  $M$ -matrices are important in economics. Concerning the notion of  $M$ -matrices, W. Leontief developed in [71] an economic model, in order to understand the relationship between different industries which depend on each others. Let us present here the model, since it provides interesting ideas for the next paragraphs.

We assume that there are  $d$  industries, which are interdependent in the sense that each industry needs some of the goods produced by the others in order to produce its own good. Let us denote by  $y \in [0, +\infty)^d$  the vector of the quantities of the different goods produced by the industries. The production  $y$  must match the demand which is the sum of

1. the goods needed in order to produce  $y$ .
2. an exterior demand, denoted by  $x \in [0, +\infty)^d$ .

Let us introduce  $B$  a  $d \times d$  real matrix whose coefficients are non negative. We assume that for every  $y \in [0, +\infty)^d$ ,  $By$  represents the quantities of goods which permit to produce  $y$ , i.e for every couple of indices  $(i, j)$ ,  $B_{ij}$  corresponds to the quantity of the  $i^{\text{th}}$  good necessary in order to produce one unit of the  $j^{\text{th}}$  good. Therefore, the market clearing conditions are given by the following equation:

$$y = By + x \Leftrightarrow (I - B)y = x.$$

We observe that  $I - B$  is a  $Z$ -matrix. To ensure that there exists a general equilibrium, i.e. a solution  $y \in [0, +\infty)^d$  of the latter equation, economists often assume that the Hawkins-Simon condition holds, i.e. they require that all the principal minors of  $I - B$  are positive. According to Proposition 5.1, it is equivalent that  $I - B$  is a non-singular  $M$ -matrix. The Hawkins-Simon condition makes sense in this setting since it guarantees that producing one unit of every good requires (directly and indirectly, see [64] for more details) less than one unit of the same good. So, the notions of  $Z$ -matrices and  $M$ -matrices appear naturally in this context. We observe however that this model is linear and cannot represent the reality in general. Nevertheless, it is perfectly sensible to use a linear model in order to study small perturbations around an equilibrium.

Concerning the labour market, let us assume that the labour supply is modelled by a regular function  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$ . The substitution effects between the types of workers imply that the off-diagonal entries of  $DS(w)$  are non positive. Therefore, it is a  $Z$ -matrix.



### 5.2.3 A labour market model

We use the model developed in Chapter 2 in order to model the labour market. We recall in this paragraph the main assumptions and notations.

There is an infinite number of companies which have individually no impact on the labour market. Companies are assumed to be similar in the sense that they share the same production function  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  and therefore, the same net output:

$$f(k, w) = \sup_{\ell \in [0, +\infty)^d} (F(k, \ell) - w \cdot \ell) - \delta k, \quad (5.1)$$

where

- $\delta \geq 0$  is the depreciation rate of the capital.
- the control  $\ell \in [0, +\infty)^d$  corresponds to the level of employment for each type of workers.

Given an initial level of capital  $\kappa \in (0, +\infty)$  and a vector of wages  $w \in (0, +\infty)^d$ , each firm solves an individual optimal control problem, namely

$$u(\kappa, w) = \sup_{c(t), \ell(t)} \int_0^{+\infty} U(c(t)) e^{-\rho t} dt$$

subject to

$$\left\{ \begin{array}{l} c(\cdot), \ell(\cdot) \in L_{\text{loc}}^1(0, +\infty), k(\cdot) \in W_{\text{loc}}^{1,1}(0, +\infty); \\ (c(\cdot), \ell(\cdot), k(\cdot)) : c(t) \geq 0, \ell(t) \in [0, +\infty)^d, \text{ for almost every } t > 0; \\ k(t) \text{ is a non negative solution of } k'(t) = f(k(t), w) - c(t) \text{ with initial condition } k(0) = \kappa; \end{array} \right.$$

where  $U : \mathbb{R} \rightarrow \mathbb{R}$  is the utility function, and  $\rho > 0$  the discount factor. It has been proved that under the assumptions made in Chapter 2,  $u(\cdot, w)$  is the unique classical solution of the Hamilton-Jacobi equation:

$$\rho v(k) = H(k, v'(k)), \quad \forall k \in (0, +\infty),$$

where the Hamiltonian  $H : (0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is defined for every  $(k, q) \in (0, +\infty) \times \mathbb{R} \cup \{+\infty\}$  by

$$H(k, q) = \sup \{ U(c) + q(F(k, \ell) - w \cdot \ell - \delta k - c) : c \geq 0, \ell \in [0, +\infty)^d \}$$

Therefore, the optimal investment can be computed and are given for every  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$  by

$$b(k, w) = D_q H \left( k, \frac{\partial u}{\partial k}(k, w) \right).$$

More properties on the function  $b(\cdot, \cdot)$  can be found in Chapter 2. It has also been proved that the capital of firms converges towards a target value  $k^*(w)$  given by the following equation:

$$\frac{\partial f}{\partial k}(k, w) = \rho.$$

The distribution of the capital  $m$  is assumed to be the unique solution of the following problem:

$$\frac{d}{dk} (b(\cdot, w)m(\cdot, w))(k) = \eta(k) - \nu m(k, w), \quad (5.2)$$

$$(k, w) \in (0, +\infty) \times (0, +\infty)^d \quad (5.3)$$

where  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  is a non negative continuous function with compact support on  $(0, +\infty)$  such that

$$\int_0^{+\infty} \eta(k) dk = \nu$$

which models the entries of companies in the economy.

Since for every  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$  the optimal level of employment for each type is given by  $-D_w f(k, w)$ . Then, for every  $w \in (0, +\infty)^d$ , the labour demand for each type is given by the vector

$$L(w) = - \int_0^{+\infty} D_w f(k, w) m(k, w) dk.$$

### 5.2.4 A useful lemma

Here is a lemma which will allow us to wisely extract a subsequence from a sequence appearing in the proof of several results.

**Lemma 5.1** *Let  $\omega : [0, +\infty) \rightarrow (0, +\infty)^d$  be a continuous trajectory such that for every  $i \in \{1, \dots, d\}$ ,*

1.  $\lim_{t \rightarrow +\infty} \omega_i(t) = +\infty$ .
2. *the function  $\omega_i(\cdot)$  is non decreasing.*

*Let  $(w^n)_{n \in \mathbb{N}}$  be a sequence in  $(0, +\infty)^d$  such that the set of indices*

$$\mathcal{I}_\infty = \left\{ i \in \{1, \dots, d\} : \limsup_{n \rightarrow +\infty} w_i^n = +\infty \right\},$$

*is non empty. Then, there exists an index  $i_\infty \in \mathcal{I}_\infty$  such that, possibly after extracting a subsequence of  $(w^n)_{n \in \mathbb{N}}$ , there exists a sequence of positive real numbers  $(t^n)_{n \in \mathbb{N}}$  such that*

1. *for every  $n \in \mathbb{N}$ ,  $w_{i_\infty}^n = \omega_{i_\infty}(t^n)$ .*
2. *for every  $n \in \mathbb{N}$  and for every  $j \in \{1, \dots, d\}$ ,  $w_j^n \leq \omega_j(t^n)$ .*

**Proof** From the assumptions, the set of indexes  $\mathcal{I}_\infty$  is non empty. We denote by  $I_0$  its cardinal. Let us fix  $\tau \in [0, +\infty)$  such that

$$\sup_{k \in \mathbb{N}} w_i^k \leq \omega_i(\tau), \quad \forall i \notin \mathcal{I}_\infty. \quad (5.4)$$

Let us take an arbitrary element  $i_0$  of  $\mathcal{I}_\infty$ . By extracting a subsequence of  $(w^n)_{n \in \mathbb{N}}$ , if necessary, we can assume that

$$\omega_{i_0}(\tau) \leq w_{i_0}^n, \quad \forall n \in \mathbb{N}.$$

The continuity of  $\omega(\cdot)$  and the fact that  $\lim_{t \rightarrow +\infty} \omega_{i_0}(t) = +\infty$  yield the existence of  $t_0^n \geq \tau$  such that

$$w_{i_0}^n = \omega_{i_0}(t_0^n), \quad \forall n \in \mathbb{N}.$$

Moreover, the monotonicity of  $\omega_i(\cdot)$  yields by (5.4)

$$w_i^n \leq \omega_i(t_0^n), \quad \forall i \notin \mathcal{I}_\infty, \forall n \in \mathbb{N}.$$

Let us consider

$$N_0 = \{n \in \mathbb{N} : \forall j \in \{1, \dots, d\}, w_j^n \leq \omega_j(t_0^n)\}.$$

If  $I_0 = 1$ , then  $N_0 = \mathbb{N}$  and the result follows. If not, we make out two cases:

1.  $N_0$  is infinitely countable. In this case, let us consider an increasing function  $\phi : \mathbb{N} \rightarrow N_0$  and consider the subsequence  $(w^{\phi(n)})_{n \in \mathbb{N}}$  which, by construction, satisfies:

- (a) for every  $n \in \mathbb{N}$ ,  $w_{i_0}^{\phi(n)} = \omega_{i_0}(t_0^{\phi(n)})$ .
- (b) for every  $n \in \mathbb{N}$  and for every  $j \in \{1, \dots, d\}$ ,  $w_j^{\phi(n)} \leq \omega_j(t_0^{\phi(n)})$ .

Therefore  $(w^{\phi(n)})_{n \in \mathbb{N}}$  is the desired subsequence.

2.  $N_0$  is finite. Then, we consider the set of indexes  $\mathcal{I}_\infty^1 = \mathcal{I}_\infty \setminus \{i_0\}$ . Let us choose  $i_1 \in \mathcal{I}_\infty^1$  such that it is possible to extract a subsequence of  $(w^n)_{n \in \mathbb{N}}$  such that:

$$\omega_{i_1}(t_0^n) < w_{i_1}^n, \quad \forall n \in \mathbb{N},$$

we call it in the same way with a slight abuse of notation. The continuity of  $\omega(\cdot)$  and the fact that  $\lim_{t \rightarrow +\infty} \omega_{i_1}(t) = +\infty$  yield the existence of  $t_1^n > t_0^n$  such that

$$w_{i_1}^n = \omega_{i_1}(t_1^n) \quad \forall n \in \mathbb{N}.$$

Moreover, the monotonicity of  $\omega(\cdot)$  ensures that

$$w_i^n \leq \omega_i(t_0^n) \leq \omega_i(t_1^n), \quad \forall i \notin \mathcal{I}_\infty^1, \forall n \in \mathbb{N}.$$

Let us consider

$$N_1 = \{n \in \mathbb{N} : \forall j \in \{1, \dots, d\}, w_j^n \leq \omega_j(t_1^n)\}.$$

We notice that we are in the exact same situation than in the beginning of this proof, except that the cardinal of  $\mathcal{I}_\infty^1$  is  $I_1 = I_0 - 1$ . We continue the argument by doing the same disjunction of cases.

By induction, the desired subsequence is built after at most  $I_0 - 1$  steps. □

## 5.2.5 Standing assumptions on the labour supply

### Presentation

In this paragraph we make the main assumptions on the labour supply. We start by introducing a notation: given a function  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$ , let us introduce for every  $i, j \in \{1, \dots, d\}$  and for every  $z \in [0, +\infty)^{d-1}$  the function  $S_i^{j,z} : [0, +\infty) \rightarrow [0, +\infty)$  defined for every  $v \in [0, +\infty)$  by

$$S_i^{j,z}(v) = S_i(z_1, \dots, z_{j-1}, v, z_j, \dots, z_{d-1}).$$

With a slight abuse of notation, if  $z \in [0, +\infty)^d$ , then we denote in the same way  $S_i^{j,z} : [0, +\infty) \rightarrow [0, +\infty)$  the function defined for every  $v \in [0, +\infty)$  by

$$S_i^{j,z}(v) = S_i(z_1, \dots, z_{j-1}, v, z_{j+1}, \dots, z_d).$$

The assumptions on the labour supply are:

**Assumption 5.1 (Assumption on the labour supply)** *We assume that the labour supply  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$  satisfies:*

- i)  $S(\cdot)$  is continuous and bounded.
- ii) There exist  $w_0 \in (0, +\infty)^d$  and  $\varepsilon_0 \in (0, 1)$  such that
 
$$\deg(S(\cdot), (\varepsilon, 1/\varepsilon)^d, S(w_0)) \neq 0, \quad \forall \varepsilon \in (0, \varepsilon_0).$$
- iii) For every  $t \in [0, 1)$ , there exists  $\varepsilon_t > 0$  such that for every  $w \in [0, +\infty)^d$ ,
 
$$S(w) \geq (1-t)S(w_0) \Rightarrow w \in [\varepsilon_t, +\infty)^d.$$

- iv) There exists a continuous trajectory  $\omega : [0, +\infty) \rightarrow [0, +\infty)^d$  such that for every  $i \in \{1, \dots, d\}$ ,

- (a) the function  $\omega_i(\cdot)$  is non decreasing.
- (b)  $\lim_{t \rightarrow +\infty} \omega_i(t) = +\infty$ .
- (c)  $\liminf_{t \rightarrow +\infty} \inf_{z \in \prod_{j \neq i} (0, \omega_j(t))} S_i^{i,z}(\omega_i(t)) > S_i(w_0)$ .

To better understand this set of hypotheses we will present in the following paragraph more explicit assumptions that imply Assumption 5.1 and which are satisfied in concrete examples.

### Discussion

In practice, Assumption 5.1 is not easy to verify. Let us therefore derive these assumptions from another set of hypotheses.

**Proposition 5.2** *Let  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$  be a function satisfying the following properties:*

1.  $S(\cdot)$  is continuous and bounded.
2. There exists a continuous trajectory  $\omega : [0, +\infty) \rightarrow [0, +\infty)^d$  such that for every  $i \in \{1, \dots, d\}$ ,
  - (a) the map  $\omega_i(\cdot)$  is non decreasing.
  - (b)  $\lim_{t \rightarrow +\infty} \omega_i(t) = +\infty$ .
  - (c)  $\liminf_{t \rightarrow +\infty} S_i(\omega(t)) > 0$ .
  - (d) there exists  $t_0 \in [0, +\infty)$  such that  $S(\omega(t_0)) \in \prod_{i=1}^d (0, \liminf_{t \rightarrow +\infty} S_i(\omega(t)))$ .
3. For every  $w \in [0, +\infty)^d$  and for every  $i \in \{1, \dots, d\}$ ,

$$w_i = 0 \Rightarrow S_i(w) = 0.$$

4. For every  $z \in [0, +\infty)^{d-1}$  and for every  $i \in \{1, \dots, d\}$ , if  $j \neq i$ , then the function  $S_i^{j,z}(\cdot)$  is non increasing.

Then  $S(\cdot)$  satisfies Assumption 5.1.

Let us comment the four assumptions in Proposition 5.2:

1. The first point lies on the regularity of the labour supply but most importantly, it requires that the total number of workers is finite.
2. The second point concerns the existence of a continuous trajectory such that all its coordinates grow towards  $+\infty$ , and along which the labour supply never vanishes. Moreover, it contains the existence of a point  $w_0 = \omega(t_0)$  which plays the same role as  $w_0$  defined in the point *ii*) in Assumption 5.1.
3. The third point says that the labour supply of type  $i$  is 0 if  $w_i = 0$ . In particular, this ensures the same kind of a priori bounds assumed in point *iii*) in Assumption 5.1.
4. From the fourth point, we can recover point *iv*) in Assumption 5.1. It concerns a particular monotony structure of the map  $S(\cdot)$ . Indeed, for a fixed index  $i \in \{1, \dots, d\}$ , if the wages of the other types of workers increase, then the type  $i$  becomes less attractive, therefore the labour supply of the type  $i$  decreases.

**Remark 5.1** *Assume that  $S(\cdot)$  is differentiable at  $w \in (0, +\infty)^d$ . According to the fourth point in Proposition 5.2,  $DS(w)$  is a  $Z$ -matrix.*

**Proof of Proposition 5.2** Let us check that Assumption 5.1 holds:

- i*) This point holds trivially.
- ii*) Let us set  $w_0 = \omega(t_0)$ . We check that there exists  $\varepsilon_0 \in (0, 1)$  such that for every  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\deg(S(\cdot), (\varepsilon, 1/\varepsilon)^d, S(w_0)) \neq 0.$$

For this aim, let us consider  $h : [0, 1] \times [0, +\infty)^d \rightarrow \mathbb{R}$  defined for every  $w \in (0, +\infty)^d$  by

$$h(t, w) = (1 - t)(w - w_0) + t(S(w) - S(w_0)).$$

Let us assume by contradiction that for every  $\varepsilon > 0$ , there exists  $(t, w) \in [0, 1] \times (0, +\infty)^d$  and  $i \in \{1, \dots, d\}$ , such that  $h(t, w) = 0$  and  $w_i < \varepsilon$ . This yields the existence of a sequence  $(t^n, w^n)_{n \in \mathbb{N}}$  such that  $h(t^n, w^n) = 0$  and  $\lim_{n \rightarrow +\infty} w_i^n = 0$ . Since  $w_i^n \rightarrow 0$  when  $n \rightarrow +\infty$ , there exists an integer  $N \in \mathbb{N}$  such that for every  $n \geq N$ ,  $w_i^n < (w_0)_i$  and

$$0 > S_i^{i,0}(w_i^n) - S_i(w_0) \geq S_i(w^n) - S_i(w_0),$$

where the last inequality comes from point 4, we deduce that  $h_i(t^n, w^n) < 0$ . Thus, for every  $n \geq N$ ,  $0 > h_i(t^n, w^n) = 0$ . So, for every  $n \geq N$ ,  $0 > h_i(t^n, w^n) = 0$ . This is impossible; hence, there exists  $\varepsilon_0 > 0$  such that for every  $(t, w) \in [0, 1] \times [0, +\infty)^d$ , if  $h(t, w) = 0$ , then  $w \in [\varepsilon_0, +\infty)^d$ . This is the desired bound from below since we have just showed that for every  $(t, w) \in [0, 1] \times (0, +\infty)^d$ , if  $h(t, w) = 0$  then  $w \in [\varepsilon_0, +\infty)^d$ . Let us now prove a bound from above for every  $w \in (0, +\infty)^d$  satisfying  $h(t, w) = 0$ . Let  $(t^n, w^n)_{n \in \mathbb{N}}$  be a sequence such that  $h(t^n, w^n) = 0$ . By contradiction, we assume that the following set of indexes

$$\mathcal{I}_\infty = \{i \in \{1, \dots, d\} : \limsup_{n \rightarrow +\infty} w_i^n = +\infty\},$$

is non empty. Lemma 5.1 yields the existence of an index  $i_\infty \in \mathcal{I}_\infty$  such that, passing through a subsequence, there exists a sequence of positive real numbers  $(t^n)_{n \in \mathbb{N}}$  such that

1. for every  $n \in \mathbb{N}$ ,  $w_{i_\infty}^n = \omega_{i_\infty}(t^n)$ .
2. for every  $n \in \mathbb{N}$ , and for every  $j \in \{1, \dots, d\}$ ,  $w_j^n \leq \omega_j(t^n)$ .

Hence we have

$$\limsup_{n \rightarrow +\infty} S_{i_\infty}(w^n) \geq \limsup_{n \rightarrow +\infty} \inf_{z \in \prod_{j \neq i_\infty} (0, \omega_j(t^n))} S_{i_\infty}^{i_\infty, z}(w_{i_\infty}^n) \geq \liminf_{n \rightarrow +\infty} S_{i_\infty}(\omega(t^n)) > S_{i_\infty}(w_0).$$

This yields the existence of an index  $N \in \mathbb{N}$  such that  $h(t^N, w^N)_{i_\infty} > 0$ . Thus  $0 > h_i(t^N, w^N) = 0$ , which is impossible. Hence, by taking  $\varepsilon_0 \in (0, 1)$  smaller than before if necessary, we have just established that for every  $(t, w) \in [0, 1] \times [0, +\infty)^d$ , if  $h(t, w) = 0$ , then  $w \in [\varepsilon_0, 1/\varepsilon_0]^d$ . Thus, for every  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\begin{aligned} \deg(S(\cdot) - S(w_0), (\varepsilon, 1/\varepsilon)^d, 0) &= \deg(h(1, \cdot), (\varepsilon, 1/\varepsilon)^d, 0) \\ &= \deg(h(0, \cdot), (\varepsilon, 1/\varepsilon)^d, 0) \\ &= \deg(\text{id} - w_0, (\varepsilon, 1/\varepsilon)^d, 0) = 1 \neq 0. \end{aligned}$$

iii) Let us consider  $w_0 = \omega(t_0)$ , where  $t_0$  satisfies 2.(d). Let us fix  $t \in [0, 1)$  and consider the set

$$\mathcal{W}_t = \{w \in [0, +\infty)^d : S(w) \geq (1-t)S(w_0)\}.$$

Let  $(w^n)_{n \in \mathbb{N}}$  be a sequence of elements of  $\mathcal{W}_t$ . By contradiction, we assume that there exists  $i_0 \in \{1, \dots, d\}$  such that  $\liminf_{n \rightarrow +\infty} w_{i_0}^n = 0$ . We remark that point 4 yields

$$S_{i_0}(w^n) \leq S_{i_0}^{i_0,0}(w_{i_0}^n).$$

Therefore

$$0 \leq \liminf_{n \rightarrow +\infty} S_{i_0}(w^n) \leq \liminf_{n \rightarrow +\infty} S_{i_0}^{i_0,0}(w_{i_0}^n) = 0,$$

which yields the desired contradiction, since  $(1-t)S_{i_0}(w_0) > 0$ . Thus, there exists  $\varepsilon_t > 0$  such that  $\mathcal{W}_t \subset (\varepsilon_t, +\infty)$ .

*iv)* From point 4 for every  $i \in \{1, \dots, d\}$  and for every  $t \in [0, +\infty)$ ,

$$\inf_{z \in \prod_{j \neq i} (0, \omega_j(t))} S_i^{i,z}(\omega_i(t)) = S_i(\omega(t)).$$

Therefore, the choice of  $w_0$  ensures

$$S_i(w_0) < \liminf_{t \rightarrow +\infty} S_i(\omega(t)) = \liminf_{t \rightarrow +\infty} \inf_{z \in \prod_{j \neq i} (0, \omega_j(t))} S_i^{i,z}(\omega_i(t)).$$

This is precisely the point *iv)* in Assumption 5.1. □

The following proposition links the notion of  $M$ -matrix and the existence of a trajectory satisfying point 2 in Proposition 5.2.

**Proposition 5.3** *Let  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$  be a function satisfying the following properties:*

1.  $S(\cdot)$  is continuous and bounded on  $[0, +\infty)^d$ .
2.  $S(\cdot)$  is of class  $C^1$  on  $(0, +\infty)^d$  and there exists  $\varepsilon \in (0, 1)$  such that for every  $w \in (0, +\infty)^d$ ,  $DS(w)$  is a  $M$ -matrix and there exists  $z \in [\varepsilon, 1/\varepsilon]^d$ ,

$$DS(w)z \geq 0.$$

3. For every  $w \in [0, +\infty)^d$  and for every  $i \in \{1, \dots, d\}$ ,

$$w_i = 0 \Rightarrow S_i(w) = 0.$$

4. For every  $z \in [0, +\infty)^{d-1}$  and for every  $i \in \{1, \dots, d\}$ , if  $j \neq i$ , then the function  $S_i^{j,z}(\cdot)$  is non increasing.

Then  $S(\cdot)$  satisfies the assumptions of Proposition 5.2.

**Proof** We just need to check point 2 in Proposition 5.2. We will build in three steps a trajectory  $\omega(\cdot)$  satisfying the requirements.

**Step 1.** Let us consider  $\tilde{w} \in (0, +\infty)^d$ . Since  $DS(\tilde{w})$  is a  $M$ -matrix, from Proposition 5.1 there exists  $\tilde{z} \in [0, +\infty)^d$  such that

$$DS(\tilde{w})\tilde{z} > 0.$$

Let us fix  $t_0 > 0$  small enough to ensure that for every  $i \in \{1, \dots, d\}$  the function

$$[0, 2t_0] \ni t \mapsto S_i(\tilde{w} + t\tilde{z})$$

is increasing. Setting

$$\omega^1(t) = \tilde{w} + t\tilde{z}, \quad \forall t \in [0, 2t_0],$$

we remark that

$$S(\omega^1(t_0)) > 0. \tag{5.5}$$

**Step 2.** We denote by  $\mathcal{B}((0, +\infty)^d)$  the Borel  $\sigma$ -algebra on  $(0, +\infty)^d$ . Let

$$\Phi : ((0, +\infty)^d, \mathcal{B}((0, +\infty)^d)) \rightrightarrows ((0, +\infty)^d, \mathcal{B}((0, +\infty)^d))$$

be a multivalued map defined as follows

$$\Phi(w) = \{z \in [\varepsilon, 1/\varepsilon]^d : DS(w)z \geq 0\}, \quad \forall w \in (0, +\infty)^d.$$

$\Phi(\cdot)$  has non empty, compact and convex values. Then, to apply the Kuratowski and Ryll-Nardzewski measurable selection theorem we need to show that  $\Phi(\cdot)$  is  $\mathcal{B}((0, +\infty)^d)$ -weakly measurable. Let us fix  $\mathcal{F}$  a closed set of  $(0, +\infty)^d$ , we aim to prove that

$$A = \{w \in (0, +\infty)^d : \Phi(w) \cap \mathcal{F} \neq \emptyset\}$$

is closed. We observe that

$$A = \{w \in (0, +\infty)^d : \exists x \in [\varepsilon, 1/\varepsilon]^d \cap \mathcal{F}, DS(w)x \geq 0\}. \quad (5.6)$$

Let us consider a sequence  $(w_n)_{n \in \mathbb{N}}$  in  $A$  which converges towards  $w^* \in (0, +\infty)^d$ . From (5.6) it is possible to define a sequence  $(x_n)_{n \in \mathbb{N}}$  such that for every  $n \in \mathbb{N}$

- $x_n \in [\varepsilon, 1/\varepsilon]^d \cap \mathcal{F}$ .
- $DS(w_n)x_n \geq 0$ .

Since  $[\varepsilon, 1/\varepsilon]^d \cap \mathcal{F}$  is a compact set there exists  $x^* \in [\varepsilon, 1/\varepsilon]^d \cap \mathcal{F}$  such that, passing through a subsequence,  $x_n \rightarrow x^*$  when  $n \rightarrow +\infty$ . The continuity of the function  $(0, +\infty)^d \times \mathbb{R} \ni (w, x) \mapsto DS(w)x$  yields that

$$DS(w^*)x^* \geq 0,$$

and then  $w^* \in A$ . Therefore,  $A$  is a closed set and belongs to  $\mathcal{B}((0, +\infty)^d)$ . Thus, the multivalued map  $\Phi(\cdot)$  is  $\mathcal{B}((0, +\infty)^d)$ -weakly measurable and by applying the Kuratowski and Ryll-Nardzewski measurable selection theorem we conclude the existence of

$$\phi : ((0, +\infty)^d, \mathcal{B}((0, +\infty)^d)) \rightarrow ((0, +\infty)^d, \mathcal{B}((0, +\infty)^d))$$

a measurable function such that for every  $w \in (0, +\infty)^d$ ,  $\phi(w) \in \Phi(w)$ , namely:

- $\phi(w) \in [\varepsilon, 1/\varepsilon]^d$ .
- $DS(w)\phi(w) \geq 0$ .

**Step 3.** Let us consider the integral equation

$$\omega^2(t) = \omega^1(2t_0) + \int_{2t_0}^t \phi(\omega^2(s)) ds. \quad (5.7)$$

$$(t, \omega^2(t)) \in [2t_0, +\infty) \times (0, +\infty)^d. \quad (5.8)$$

We denote by  $\omega^2(\cdot)$  a solution of (5.7)-(5.8). We claim that the trajectory

$$\omega(t) = \begin{cases} \omega^1(t) & \text{if } t \in [0, 2t_0], \\ \omega^2(t) & \text{if } t \in (2t_0, +\infty), \end{cases}$$

satisfies the requirements. Indeed, for every  $i \in \{1, \dots, d\}$ ,  $\omega_i(\cdot)$  is

- continuous and increasing on  $[0, +\infty)$ .

- for every  $t \geq 2t_0$ ,  $\omega_i(t) \geq \omega_i(2t_0) + \varepsilon(t - 2t_0)$ , then  $\omega_i(t) \rightarrow +\infty$  when  $t$  goes towards  $+\infty$ .
- $\lim_{t \rightarrow +\infty} S_i(\omega(t)) \geq S_i(2t_0) > S_i(t_0) > 0$ .

□

**Remark 5.2** For simplicity, in point 2 in Proposition 5.3 the constant  $\varepsilon \in (0, 1)$  does not depend on  $w \in (0, +\infty)^d$ . However, it is possible to make it depend on  $w$ . For instance, we can assume that there exists two non decreasing Lipschitz functions  $\psi^i : (0, +\infty) \rightarrow (0, +\infty)$  ( $i = 1, 2$ ) satisfying  $\psi^1(\cdot) \leq \psi^2(\cdot)$  and that for every  $w_1 \in (0, +\infty)$ , the unique solution of

$$\begin{aligned}\omega'(t) &= \psi^1(\omega(t)) \\ \omega(t) &= w_1 \\ (t, \omega(t)) &\in [0, +\infty) \times (0, +\infty)\end{aligned}$$

blows up towards  $+\infty$  when  $t$  goes towards  $+\infty$ ; such that for every  $w \in (0, +\infty)^d$ ,  $DS(w)$  is a  $M$ -matrix and there exists  $z \in \prod_{i=1}^d [\psi^1(w_i), \psi^2(w_i)]$ ,

$$DS(w)z \geq 0.$$

We conclude in the same way the existence of trajectory satisfying the point 2 in Proposition 5.3.

In the subsequent paragraph, we present some examples satisfying the assumptions in Proposition 5.2, and therefore Assumption 5.1.

**Example 5.1** Let us consider that there are  $d$  types of workers. We make the assumption that the labour supply of one type is not influenced by the wages offered to the others. We observe that in the long run, this assumption may not hold because of the occupational mobility of workers (see [24] and [65] for more details). However, we have seen in Chapter 2, that the model is not restricted to the labour market and can model the market of factors of production. Therefore, if  $S(\cdot)$  models, for instance, the supply of labour, materials and workspace this independence assumption makes sense.

In that case,  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$  is given for every  $w \in (0, +\infty)^d$  by

$$S(w) = (S_i(w_i))_{i=1, \dots, d}$$

with  $S_i : [0, +\infty) \rightarrow [0, +\infty)$ . If for every  $i \in \{1, \dots, d\}$ ,

1.  $S_i(\cdot)$  is continuous and bounded.
2.  $\liminf_{t \rightarrow +\infty} S_i(t) > 0$ .
3.  $S_i(0) = 0$ .

Then, the assumptions in Proposition 5.2 hold. Indeed, point 1, 3 and 4 in Proposition 5.2 hold trivially. On the other hand, point 2 holds because the trajectory  $\omega : [0, +\infty) \ni t \mapsto t(1, \dots, 1)$  satisfies all the requirements.

**Example 5.2** This example is inspired by [100] where the labour supply is treated as a discrete choice. We assume that individuals can either take a job among  $d$  possibilities, or not work. Given  $w_0 > 0$  the income of the unemployed, for any vector of wages  $w \in [0, +\infty)^d$ , the utility for an individual who chooses the type  $i$  ( $i = 0, \dots, d$ ) is given by the random variable

$$U_i(w_i) = \ln(w_i) + \epsilon_i,$$

where by convention  $\ln(0) = -\infty$ . Moreover, as in [100], we assume that  $\epsilon_i$  is an idiosyncratic noise which follows the standard Gumbel distribution with the following cumulative distribution function

$$\mathbb{P}(\epsilon_i \leq \varepsilon) = e^{-e^{-\varepsilon}}, \quad \forall \varepsilon \in \mathbb{R}.$$



We suppose that the random variables  $(\epsilon_j)_{j=1,\dots,d}$  are independent and that individuals choose  $i$  for which  $U_i(w_i)$  is largest. Therefore, the probability for  $i$  to be chosen is given by

$$\begin{aligned} \mathbb{P}(U_i(w_i) > U_j(w_j), \forall j \neq i) &= \begin{cases} \mathbb{P}(\ln(w_i) - \ln(w_j) > \epsilon_j - \epsilon_i, \forall j \neq i) & \text{if } w_i > 0, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} w_i / \sum_{k=0}^d w_k & \text{if } w_i > 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If the total number of individuals is normalized to one, then for every  $i \in \{1, \dots, d\}$  the labour supply of type  $i$  is given by

$$S_i(w) = \begin{cases} w_i / \sum_{k=0}^d w_k & \text{if } w_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The supply function  $S(\cdot)$  satisfies the assumptions of Proposition 5.2. We just need to check point 2 in Proposition 5.2.

Let us fix  $\tilde{w} \in (0, +\infty)^d$ , we consider the trajectory  $\omega : [0, +\infty) \rightarrow [0, +\infty)^d$  defined for every  $i \in \{1, \dots, d\}$  by the differential equation:

$$\omega'_i(t) = \omega_i(t), \quad (5.9)$$

$$\omega_i(0) = \tilde{w}_i, \quad (5.10)$$

$$(t, \omega(t)) \in [0, +\infty) \times (0, +\infty)^d. \quad (5.11)$$

For every  $i \in \{1, \dots, d\}$  and  $t \geq 0$ ,

$$\omega_i(t) = \tilde{w}_i e^t.$$

We claim that the trajectory  $\omega(\cdot)$  satisfies the assumptions of Proposition 5.2. Indeed, for every  $i \in \{1, \dots, d\}$ ,

- $\omega_i(\cdot)$  is defined, continuous and increasing on  $[0, +\infty)$ .
- $\omega_i(t) \rightarrow +\infty$  when  $t$  goes towards  $+\infty$ .
- for every  $j \in \{1, \dots, d\}$  and  $w \in (0, +\infty)^d$ ,

$$\frac{\partial S_i}{\partial w_j}(w) = \frac{w_i}{\sum_{k=0}^d w_k} \left( \frac{\delta_{i,j}}{w_i} - \frac{1}{\sum_{k=0}^d w_k} \right).$$

So for every  $t \in [0, +\infty)$ ,

$$\begin{aligned} \frac{d}{dt} S_i(\omega(t)) &= DS_i(\omega(t)) \omega'(t) \\ &= \sum_{j=1}^d \frac{\omega_i(t)}{\sum_{k=0}^d \omega_k(t)} \left( \frac{\delta_{i,j}}{\omega_i(t)} - \frac{1}{\sum_{k=0}^d \omega_k(t)} \right) \omega'_j(t) \\ &= \frac{\omega_i(t)}{\sum_{k=0}^d \omega_k(t)} \left( 1 - \frac{\sum_{j=1}^d \omega_j(t)}{\sum_{k=0}^d \omega_k(t)} \right) > 0. \end{aligned}$$

Therefore,

$$\lim_{t \rightarrow +\infty} S_i(\omega(t)) > S_i(\tilde{w}).$$

- $S_i(\omega(0)) > 0$ .

**Remark 5.3**

1. In Example 5.2, Proposition 5.1 yields that for every  $w \in (0, +\infty)^d$ ,  $DS(w)$  is a  $M$ -matrix.
2. Similarly, in Example 5.2 we could use Proposition 5.3 and Remark 5.2 with  $\psi^1(\cdot) = \psi^2(\cdot) = \text{id}_{(0, +\infty)^d}$  to conclude that  $S(\cdot)$  satisfies the assumptions of Proposition 5.2.
3. Example 5.2 can easily be generalized to the case where the utility for an individual who chooses the type  $i$  ( $i = 0, \dots, d$ ) is given by the random variable

$$U_i(w_i) = V(w_i) + \epsilon_i.$$

where the deterministic function  $V_i : [0, +\infty) \rightarrow \{-\infty\} \cup \mathbb{R}$  is assumed to

- be non decreasing, concave and of class  $C^2$  on  $(0, +\infty)$ .
- satisfy  $V(0) = -\infty$  and  $V_i(t) \rightarrow -\infty$  when  $t$  goes towards  $0^+$ .

In this case, given a vector of wages  $w \in (0, +\infty)^d$ , the labour supply for the type  $i$  will be given by

$$S_i(w) = \begin{cases} e^{V_i(w_i)} / \sum_{k=0}^d e^{V_k(w_k)}, & \text{if } w_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

**5.2.6 Standing assumptions on the labour demand****Presentation**

In this paragraph we make the main assumptions on the labour supply. We start by introducing a notation: given a function  $L : (0, +\infty)^d \rightarrow (0, +\infty)^d$ , let us introduce for every  $i, j \in \{1, \dots, d\}$  and for every  $z \in [0, +\infty)^{d-1}$  the function  $L_i^{j,z} : (0, +\infty) \rightarrow (0, +\infty)$  defined for every  $v \in (0, +\infty)$  by

$$L_i^{j,z}(v) = L_i(z_1, \dots, z_{j-1}, v, z_j, \dots, z_{d-1}).$$

With a slight abuse of notation, if  $z \in (0, +\infty)^d$ , then we denote in the same way  $L_i^{j,z} : (0, +\infty) \rightarrow (0, +\infty)$  the function defined for every  $v \in (0, +\infty)$  by

$$L_i^{j,z}(v) = L_i(z_1, \dots, z_{j-1}, v, z_{j+1}, \dots, z_d).$$

Regarding the labour demand, we make the following set of assumptions:

**Assumption 5.2 (Assumptions on the labour demand)** We assume that the labour demand  $L : (0, +\infty)^d \rightarrow (0, +\infty)^d$  satisfies:

- i)  $L(\cdot)$  is continuous.
- ii) If  $(w^n)_{n \in \mathbb{N}}$  is a sequence of elements of  $(0, +\infty)^d$  such that there exists an index  $i_0 \in \{1, \dots, d\}$  and  $w_{i_0}^n \rightarrow 0$  when  $n \rightarrow +\infty$ . Then,
  - either, there exists  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} L_i(w^n) = +\infty$ .
  - or, if the sequence  $(L(w^n))_{n \in \mathbb{N}}$  is bounded, then the following set

$$\mathcal{I}_\infty = \{i \in \{1, \dots, d\} : \limsup_{n \rightarrow +\infty} w_i^n = +\infty\}$$

is non empty, and, for every  $i \in \mathcal{I}_\infty$ , passing through a subsequence,  $w_i^n \rightarrow +\infty$  and  $L_i(w^n) \rightarrow 0$  when  $n$  tends to  $+\infty$ .

iii) For every continuous trajectory  $\omega : [0, +\infty) \rightarrow (0, +\infty)^d$  such that for every  $i \in \{1, \dots, d\}$ ,  $\omega_i(\cdot)$  is non decreasing and  $\omega_i(t) \rightarrow +\infty$  when  $t \rightarrow +\infty$ , then

$$\liminf_{t \rightarrow +\infty} \sup_{z \in \prod_{j \neq i} (\varepsilon, \omega_j(t))} L_i^{i,z}(\omega_i(t)) = 0, \quad \forall \varepsilon > 0, \forall i \in \{1, \dots, d\}.$$

**Remark 5.4** Point iii) is slightly stronger than necessary since to prove the main results, it is sufficient that this condition holds only for the continuous trajectory defined in Assumption 5.1.

### Discussion

Assumption 5.2 is motivated by the model introduced in Chapter 2 and recalled in section 5.2.3 of this chapter. In particular, we derive assumptions from two fundamental examples: when the production function  $F(\cdot, \cdot)$  has the form of a Cobb-Douglas or a Constant Elasticity of Substitution (or CES) function. The formulas presented in this paragraph are justified in Chapter 4.

**Example 5.3 (Cobb-Douglas)** If the production function is a Cobb-Douglas function, i.e.  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  is defined for every  $(k, \ell) \in [0, +\infty) \times [0, +\infty)^d$  by

$$F(k, \ell) = Ak^\alpha \ell^\beta,$$

where  $A > 0$ ,  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)^d$ , with  $\ell^\beta = \prod_{i=1}^d \ell_i^{\beta_i}$  and  $\alpha + \sum_{i=1}^d \beta_i < 1$ . Then, for every  $w \in (0, +\infty)^d$ , the labour demand is given for every  $i \in \{1, \dots, d\}$  by

$$L_i(w) = \left( A \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-|\beta|}} \frac{1}{\nu} \int_0^{+\infty} \eta(\kappa) \left( \kappa^{\frac{\alpha}{1-|\beta|}} + \int_{\kappa}^{k^*(w)} \frac{\alpha}{1-|\beta|} k^{-\frac{1-(\alpha+|\beta|)}{1-|\beta|}} e^{\int_{\kappa}^k \frac{\nu}{b(z,w)} dz} \right) d\kappa \frac{\beta_i}{w_i}, \quad (5.12)$$

with the target capital  $k^*(w) = \left( \frac{\alpha}{\alpha+\rho} \right)^{\frac{1-|\beta|}{1-(\alpha+|\beta|)}} \left( A \prod_{j=1}^d \left( \frac{\beta_j}{w_j} \right)^{\beta_j} \right)^{\frac{1}{1-(\alpha+|\beta|)}}.$

**Example 5.4 (CES)** If the production function is a CES function, i.e.  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  is defined for every  $(k, \ell) \in [0, +\infty) \times [0, +\infty)^d$  by

$$F(k, \ell) = (k^\alpha + \ell^\beta)^\gamma,$$

where  $\alpha \in (0, 1)$ ,  $\beta \in (0, 1)^d$  and  $\gamma \in (0, 1)$ , with  $\ell^\beta = \sum_{i=1}^d \ell_i^{\beta_i}$ . Then, for every  $w \in (0, +\infty)^d$ , the labour demand is given, for every  $i \in \{1, \dots, d\}$ , by

$$L_i(w) = \int_0^{+\infty} \left( \frac{\lambda(k, w) \beta_i}{w_i} \right)^{\frac{1}{1-\beta_i}} m(k, w) dk, \quad (5.13)$$

where  $m$  is solution of (5.2)-(5.3) and  $\lambda(k, w)$  is the unique solution of the following equation:

$$\lambda \left( k^\alpha + \sum_{j=1}^d \left( \frac{\lambda \beta_j}{w_j} \right)^{\frac{\beta_j}{1-\beta_j}} \right)^{1-\gamma} = \gamma. \quad (5.14)$$

The purpose of the following paragraph is to verify that the two latter examples of the labour demand satisfy Assumption 5.2. We will need three lemmas.

**Lemma 5.2** Let us assume that for every  $i \in \{1, \dots, d\}$ ,  $L_i : (0, +\infty)^d \rightarrow (0, +\infty)^d$  is given by (5.12). Let  $(w^n)_{n \in \mathbb{N}}$  be a sequence of elements of  $(0, +\infty)^d$ . If there exists an index  $i_0 \in \{1, \dots, d\}$  such that  $w_{i_0}^n \rightarrow 0$  when  $n \rightarrow +\infty$ . Then,

- either, there exists  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} L_i(w^n) = +\infty$ .
- or, if the sequence  $(L(w^n))_{n \in \mathbb{N}}$  is bounded, then the following set

$$\mathcal{I}_\infty = \{i \in \{1, \dots, d\} : \limsup_{n \rightarrow +\infty} w_i^n = +\infty\}$$

is non empty, and, for every  $i \in \mathcal{I}_\infty$ , passing through a subsequence,  $w_i^n \rightarrow +\infty$  and  $L_i(w^n) \rightarrow 0$  when  $n$  tends to  $+\infty$ .

**Proof** We observed in Lemma 4.2 that there exists a continuous function  $\tilde{L} : (0, +\infty)^d \rightarrow (0, +\infty)^d$  such that for every  $i \in \{1, \dots, d\}$ ,  $L_i(w) = \tilde{L}(w)\beta_i/w_i$ . We make out two cases:

1. If  $\liminf_{n \rightarrow +\infty} \tilde{L}(w^n) = 0$ , possibly after the extraction of a subsequence, we may suppose that  $\tilde{L}(w^n) \rightarrow 0$  when  $n \rightarrow +\infty$ . From Lemma 4.3, there exists  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} w_i^n = +\infty$ . Passing through a subsequence we can deduce that when  $n$  goes towards  $+\infty$ ,

- $w_i^n \rightarrow +\infty$ .
- $L(w^n) = \tilde{L}(w^n)\beta_i/w_i^n \rightarrow 0$ .

2. There exists  $\varrho > 0$  such that for every  $n \in \mathbb{N}$ ,  $\tilde{L}(w^n) > \varrho$ . Thus, it is clear that

$$\liminf_{n \rightarrow +\infty} L_{i_0}(w^n) \geq \lim_{n \rightarrow +\infty} \varrho \beta_{i_0}/w_{i_0}^n = +\infty.$$

□

**Lemma 5.3** *Let us assume that for every  $i \in \{1, \dots, d\}$ ,  $L_i : (0, +\infty)^d \rightarrow (0, +\infty)^d$  is given by (5.13). Let  $(w^n)_{n \in \mathbb{N}}$  be a sequence of elements of  $(0, +\infty)^d$ . If there exists an index  $i_0 \in \{1, \dots, d\}$  such that  $w_{i_0}^n \rightarrow 0$  when  $n \rightarrow +\infty$ . Then, there exists  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} L_i(w^n) = 0$ .*

**Proof** First, we notice that since  $w_{i_0}^n \rightarrow 0$  and  $\lambda(k, w^n)$  is the solution of (5.14), then  $\lambda(k, w^n) \rightarrow 0$  when  $n \rightarrow +\infty$ . Therefore, the target capital  $k^*(w^n) \rightarrow 0$  when  $n \rightarrow +\infty$ . Indeed, Lemma 4.4 yields that  $k^*(w)$  is the unique solution of the equation

$$\lambda(k, w)k^{\alpha-1} = \delta + \rho.$$

This means that for  $n$  sufficiently large,  $\text{supp } m(\cdot, w^n) \subset [0, \text{supp } \eta(\cdot)]$ . Moreover, Lemma 4.4 also justifies that for every  $(k, w) \in (0, +\infty) \times (0, +\infty)^d$ , and every  $i \in \{1, \dots, d\}$ ,

$$\frac{\partial f}{\partial w_i}(k, w) = - \left( \frac{\lambda(k, w)\beta_i}{w_i} \right)^{\frac{1}{1-\beta_i}}.$$

In addition, we notice that  $\lambda(\cdot, w)$  is non increasing. Let us fix  $\bar{k} \geq \text{supp } \eta(\cdot)$ . There exists an integer  $N$ , such that for every  $n \geq N$ , and for every  $i \in \{1, \dots, d\}$ ,

$$L_i(w^n) = - \int_0^{+\infty} \frac{\partial f}{\partial w_i}(k, w^n) m(k, w^n) dk \geq \left( \frac{\lambda(\bar{k}, w^n)\beta_i}{w_i^n} \right)^{\frac{1}{1-\beta_i}}.$$

It remains to show that there exists an index  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} \lambda(\bar{k}, w^n)/w_i^n = +\infty$ . Equation (5.14) gives:

$$\lambda(\bar{k}, w^n) = \frac{\gamma}{\left( \bar{k}^\alpha + \sum_{j=1}^d \left( \frac{\lambda(\bar{k}, w^n)\beta_j}{w_j^n} \right)^{\frac{\beta_j}{1-\beta_j}} \right)^{1-\gamma}}.$$

Since the left hand side tends to zero,  $\sum_{j=1}^d \left( \frac{\lambda(\bar{k}, w^n)\beta_j}{w_j^n} \right)^{\frac{\beta_j}{1-\beta_j}} \rightarrow +\infty$  as  $n \rightarrow +\infty$ . Therefore, there exists  $i \in \{1, \dots, d\}$  such that  $\limsup_{n \rightarrow +\infty} \lambda(\bar{k}, w^n)\beta_i/w_i^n = +\infty$ , so  $\limsup_{n \rightarrow +\infty} L_i(w^n) \geq +\infty$ . □

**Lemma 5.4** *Let  $\omega : [0, +\infty) \rightarrow (0, +\infty)^d$  be a continuous trajectory such that for every  $i \in \{1, \dots, d\}$ ,  $\omega_i(t) \rightarrow +\infty$  when  $t \rightarrow +\infty$ . Let us assume that, for every  $i \in \{1, \dots, d\}$   $L_i : (0, +\infty)^d \rightarrow (0, +\infty)^d$  is given by (5.12), or by (5.13). Then,*

$$\liminf_{t \rightarrow +\infty} \sup_{z \in \prod_{j \neq i} (\varepsilon, \omega_j(t))} L_i^{i,z}(\omega_i(t)) = 0, \quad \forall \varepsilon > 0.$$

**Proof** In the case when the  $L_i(\cdot)$  are given by (5.12), the desired result comes from (5.12). In the case when they are given by (5.13), let us remark that the monotonicity of  $[0, +\infty) \ni k \mapsto \lambda(k, w)$  yields

$$\lambda(k, w) \leq \lambda(0, w), \quad \forall k \in (0, +\infty), \forall w \in (0, +\infty)^d.$$

Therefore, for every  $i \in \{1, \dots, d\}$  and every  $w \in (0, +\infty)^d$ ,

$$L_i(w) \leq \left( \frac{\lambda(0, w)\beta_i}{w_i} \right)^{\frac{1}{1-\beta_i}}.$$

On the other hand, using equation (5.14), we get

$$\lambda(0, w) \left( \frac{\lambda(0, w)\beta_i}{w_i} \right)^{\frac{\beta_i(1-\gamma)}{1-\beta_i}} \leq \gamma.$$

Hence,

$$\left( \frac{\lambda(0, w)\beta_i}{w_i} \right)^{1 + \frac{\beta_i(1-\gamma)}{1-\beta_i}} \leq \frac{\gamma\beta_i}{w_i}.$$

Therefore, for every  $\varepsilon > 0$ ,

$$\begin{aligned} 0 &\leq \liminf_{t \rightarrow +\infty} \sup_{z \in \prod_{j \neq i} (\varepsilon, \omega_j(t))} L_i^{i,z}(\omega_i(t)) \\ &\leq \liminf_{t \rightarrow +\infty} \sup_{z \in \prod_{j \neq i} (\varepsilon, \omega_j(t))} \left\{ \left( \frac{\lambda(0, (z_1, \dots, z_{i-1}, \omega_i(t), z_i, \dots, z_d))\beta_i}{\omega_i(t)} \right)^{\frac{1}{1-\beta_i}} \right\} \\ &\leq \liminf_{t \rightarrow +\infty} \left( \frac{\gamma\beta_i}{\omega_i(t)} \right)^{\frac{1}{1-\beta_i}\gamma} = 0. \end{aligned}$$

□

From the three lemmas above, we deduce that the demand coming from a Cobb-Douglas or CES type production function satisfies Assumption 5.2.

## 5.3 Main results

We saw in the previous paragraphs that relevant models satisfy Assumptions 5.1 and 5.2. Under these assumptions, we investigate the existence and possibly the uniqueness of equilibria, i.e. an element  $w \in (0, +\infty)^d$  satisfying  $S(w) = L(w)$ .

**Theorem 5.2** *If  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$  and  $L : (0, +\infty)^d \rightarrow (0, +\infty)^d$  satisfy Assumption 5.1 and 5.2, then there exists  $w \in (0, +\infty)^d$  satisfying  $S(w) = L(w)$ .*

**Proof** Let us introduce the homotopy  $h : [0, 1] \times (0, +\infty)^d \rightarrow \mathbb{R}$  defined for every  $(t, w) \in [0, 1] \times (0, +\infty)^d$  by

$$h(t, w) = S(w) - (1-t)S(w_0) - tL(w),$$

where  $w_0$  is given by Assumption 5.1. Let us find  $\varepsilon \in (0, 1)$  such that  $\deg(S(\cdot), (\varepsilon, 1/\varepsilon)^d, S(w_0)) \neq 0$  and that if  $w \in \partial(\varepsilon, 1/\varepsilon)^d$ , then for every  $t \in [0, 1]$ ,  $h(t, w) \neq 0$ .

**Step 1: a bound from below.**

For any  $t \in [0, 1]$ , assume that  $w \in (0, +\infty)^d$  satisfies  $h(t, w) = 0$ . Since  $L(\cdot)$  takes its values in  $(0, +\infty)^d$ ,

$$S(w) \geq (1 - t)S(w_0).$$

The third point in Assumption 5.1 yields the existence of  $\varepsilon_t^0 > 0$  such that  $w \in (\varepsilon_t^0, +\infty)^d$ . Note that we can choose  $\varepsilon_t^0$  in such way that the map  $[0, 1] \ni t \mapsto \varepsilon_t^0$  is non increasing and  $\varepsilon_1^0 = 0$ .

In the same way, for any  $t \in (0, 1]$ , since  $S_i(\cdot)$  is non negative for every  $i \in \{1, \dots, d\}$ ,

$$L_i(w) \leq \|S_i\|_\infty / t,$$

and

$$L_i(w) \geq (S_i(w) - S_i(w_0))/t.$$

Since  $L_i(\cdot)$  is positive, the latter inequality implies that

$$L_i(w) \geq S_i(w) - S_i(w_0).$$

To summarize,

$$S_i(w) - S_i(w_0) \leq L_i(w) \leq \|S_i\|_\infty / t, \quad \forall i \in \{1, \dots, d\}. \quad (5.15)$$

We claim that there exists  $\varepsilon_t^1 > 0$  such that for every  $w \in (0, +\infty)^d$ , if (5.15) holds, then  $w \in (\varepsilon_t^1, +\infty)^d$ . By contradiction, let us assume that there exists a sequence  $(w^n)_{n \in \mathbb{N}}$  of elements of  $(0, +\infty)^d$  satisfying (5.15) and an index  $i_0 \in \{1, \dots, d\}$  such that  $w_{i_0}^n \rightarrow 0$  when  $n \rightarrow +\infty$ . From the point *ii*) in Assumption 5.2 we make out two cases:

- there exists  $i \in \{1, \dots, d\}$ , such that  $\limsup_{n \rightarrow +\infty} L_i(w^n) = +\infty$ , which enters in contradiction with (5.15).
- the following set

$$\mathcal{I}_\infty = \{i \in \{1, \dots, d\} : \limsup_{n \rightarrow +\infty} w_i^n = +\infty\}$$

is non empty, and, for every  $i \in \mathcal{I}_\infty$ , passing through a subsequence,  $w_i^n \rightarrow +\infty$  and  $L_i(w^n) \rightarrow 0$  when  $n$  goes towards  $+\infty$ .

Then, Lemma 5.1 yields the existence of an index  $i_\infty \in \mathcal{I}_\infty$  such that, possibly after the extraction of a subsequence, there exists a sequence of positive real numbers  $(t_n)_{n \in \mathbb{N}}$  such that

1. for every  $n \in \mathbb{N}$ ,  $w_{i_\infty}^n = \omega_{i_\infty}(t_n)$ .
2. for every  $n \in \mathbb{N}$ , and for every  $j \in \{1, \dots, d\}$ ,  $w_j^n \leq \omega_j(t_n)$ .

From (5.15) we deduce that for every  $n \in \mathbb{N}$ ,

$$S_{i_\infty}(w^n) - S_{i_\infty}(w_0) \leq L_{i_\infty}(w^n),$$

so

$$\inf_{z \in \prod_{j \neq i_\infty} (0, \omega_j(t_n))} S_{i_\infty}^{i_\infty, z}(\omega_{i_\infty}(t_n)) - S_{i_\infty}(w_0) \leq L_{i_\infty}(w^n).$$

Therefore,

$$0 < \liminf_{n \rightarrow +\infty} \inf_{z \in \prod_{j \neq i_\infty} (0, \omega_j(t_n))} S_{i_\infty}^{i_\infty, z}(\omega_{i_\infty}(t_n)) - S_{i_\infty}(w_0) \leq \liminf_{n \rightarrow +\infty} L_{i_\infty}(w^n) = 0,$$

which is impossible.

This yields the existence of  $\varepsilon_t^1 > 0$  such that for every  $w \in (0, +\infty)^d$  if (5.15) holds, then  $w \in (\varepsilon_t^1, +\infty)^d$ . Note that we can choose  $\varepsilon_t^1$  in such a way that the map  $[0, 1] \ni t \mapsto \varepsilon_t^1$  is non decreasing and  $\varepsilon_0^1 = 0$ .

Finally, we have proved that if  $w \in (0, +\infty)^d$  satisfies  $h(t, w) = 0$ , then

$$w \in \cup_{s \in [0, 1]} ((\varepsilon_s^0, +\infty)^d \cap (\varepsilon_s^1, +\infty)^d) = \cup_{s \in [0, 1]} (\max(\varepsilon_s^0, \varepsilon_s^1), +\infty)^d = (\underline{\varepsilon}, +\infty)^d,$$

where  $\underline{\varepsilon} = \inf_{s \in [0, 1]} \max(\varepsilon_s^0, \varepsilon_s^1)$ . Let us show that  $\underline{\varepsilon}$  is positive. We argue by contradiction and assume that  $\underline{\varepsilon} = 0$ . So we take a sequence  $(t_n)_{n \in \mathbb{N}}$  of elements of  $[0, 1]$  such that  $\max(\varepsilon_{t_n}^0, \varepsilon_{t_n}^1) \rightarrow 0$  when  $n \rightarrow +\infty$ . By extraction, we can assume that the sequence  $(t_n)_{n \in \mathbb{N}}$  is monotone and converges towards  $t_\infty \in [0, 1]$ . Without loss of generality, let us assume that the sequence is non decreasing. Therefore,

- either for every  $n \in \mathbb{N}$ ,  $t_n = t_\infty = 0$  which is impossible since  $\max(\varepsilon_0^0, \varepsilon_0^1) = \varepsilon_0^0 > 0$ ;
- or,  $t_\infty > 0$  and  $\liminf_{n \rightarrow +\infty} \max(\varepsilon_{t_n}^0, \varepsilon_{t_n}^1) \geq \liminf_{n \rightarrow +\infty} \varepsilon_{t_n}^1 \geq \varepsilon_{t_\infty/2}^1 > 0$ , which is impossible.

Thus, this yields that  $\underline{\varepsilon} > 0$ .

### Step 2: a bound from above.

Let us assume that there exists a sequence  $(t_n, w^n)_{n \in \mathbb{N}}$  of  $[0, 1] \times (0, +\infty)^d$  such that  $h(t_n, w^n) = 0$ . Moreover, let us assume that the set

$$\mathcal{I}_\infty = \{i \in \{1, \dots, d\} : \limsup_{n \rightarrow +\infty} w_i^n = +\infty\},$$

is non empty. Lemma 5.1 gives the existence of an index  $i_\infty \in \mathcal{I}_\infty$  such that possibly after the extraction of a subsequence, there exists a sequence of positive real numbers  $(\tau_n)_{n \in \mathbb{N}}$  such that

1. for every  $n \in \mathbb{N}$ ,  $w_{i_\infty}^n = \omega_{i_\infty}(\tau_n)$ .
2. for every  $n \in \mathbb{N}$ , for every  $j \in \{1, \dots, d\}$ ,  $w_j^n \leq \omega_j(\tau_n)$ .

Then, from (5.15),

$$S_{i_\infty}(w^n) - S_{i_\infty}(w_0) \leq L_{i_\infty}(w^n) \leq \sup_{z \in \prod_{j \neq i_\infty} (\underline{\varepsilon}, \omega_j(\tau_n))} L_{i_\infty}^{i_\infty, z}(w_{i_\infty}^n),$$

which, by taking the lim inf, leads to the following contradiction:

$$0 < \liminf_{n \rightarrow +\infty} S_{i_\infty}(w^n) - S_{i_\infty}(w_0) \leq \liminf_{n \rightarrow +\infty} \sup_{z \in \prod_{j \neq i_\infty} (\underline{\varepsilon}, \omega_j(\tau_n))} L_{i_\infty}^{i_\infty, z}(w_{i_\infty}^n) = 0.$$

Therefore, possibly after taking a smaller value for  $\underline{\varepsilon}$ , this implies that for every  $n \in \mathbb{N}$ ,  $w^n \in (\underline{\varepsilon}, 1/\underline{\varepsilon})^d$ . Thus, for every  $(t, w) \in [0, 1] \times (0, +\infty)^d$ :

$$h(t, w) = 0 \Rightarrow w \in (\underline{\varepsilon}, 1/\underline{\varepsilon})^d.$$

### Step 3: Conclusion.

We end the proof by taking  $\varepsilon \in (0, \min(\underline{\varepsilon}, \varepsilon_0))$ . On the first hand,  $\deg(S(\cdot), (\varepsilon, 1/\varepsilon)^d, S(w_0)) \neq 0$ , and on the other hand, step 1 and 2 ensure that if  $w \in \partial(\varepsilon, 1/\varepsilon)^d$ , then for every  $t \in [0, 1]$ ,  $h(t, w) \neq 0$ , hence

$$\begin{aligned} \deg((S - L)(\cdot), (\varepsilon, 1/\varepsilon)^d, 0) &= \deg(h(1, \cdot), (\varepsilon, 1/\varepsilon)^d, 0) \\ &= \deg(h(0, \cdot), (\varepsilon, 1/\varepsilon)^d, 0) \\ &= \deg(S(\cdot) - S(w_0), (\varepsilon, 1/\varepsilon)^d, 0) \neq 0. \end{aligned}$$

Therefore, there exists at least one solution of the system in  $(\varepsilon, 1/\varepsilon)^d \subset (0, +\infty)^d$ . □

**Theorem 5.3** *Let  $S : [0, +\infty)^d \rightarrow [0, +\infty)^d$  and  $L : (0, +\infty)^d \rightarrow (0, +\infty)^d$  satisfy Assumption 5.1 and 5.2. Moreover, we assume that for every  $\varepsilon \in (0, \varepsilon_0)$ ,*

$$\deg(S(\cdot), (\varepsilon, 1/\varepsilon)^d, S(w_0)) = 1,$$

*and that the functions  $S(\cdot)$  and  $L(\cdot)$  are of class  $C^1$  on  $(0, +\infty)^d$  and satisfy*

$$\det(D(S - L)(w)) > 0, \quad \forall w \in (0, +\infty)^d.$$

*Then, there exists a unique element  $w \in (0, +\infty)^d$  satisfying  $S(w) = L(w)$ .*

**Proof** The existence of a solution comes from Theorem 5.2. Their uniqueness must be checked. From the a priori bounds established in the proof of Theorem 5.2 and the additional assumptions made in the statement of Theorem 5.3, it is possible to find  $\varepsilon \in (0, 1)$  such that

1. for every  $(t, w) \in [0, 1] \times (0, +\infty)^d$ , if  $S(w) - (1 - t)S(w_0) - tL(w) = 0$ , then  $w \in (\varepsilon, 1/\varepsilon)^d$ .
2.  $\deg(S(\cdot), (\varepsilon, 1/\varepsilon)^d, S(w_0)) = 1$ .

In addition, since  $(S - L)(\cdot)$  is of class  $C^1$ ,  $0 \notin (S - L)(\partial(\varepsilon, 1/\varepsilon)^d)$  and for every  $w \in (S - L)^{-1}(\{0\})$ ,  $\det D(S - L)(w) \neq 0$ , then Theorem 5.1 yields

$$\deg((S - L)(\cdot), (\varepsilon, 1/\varepsilon)^d, 0) = \sum_{w \in (S - L)^{-1}(\{0\})} \text{sign}(\det(D(S - L)(w))).$$

Therefore,

$$\sum_{w \in (S - L)^{-1}(\{0\})} \text{sign}(\det(D(S - L)(w))) = 1,$$

and we deduce from the assumption on the sign of the determinant of  $D(S - L)(\cdot)$  that there is at most one term in the sum, hence the desired uniqueness property.  $\square$

## 5.4 A numerical approach to solve the problem

The Brouwer degree is a powerful method to establish the existence of solutions of a non-linear system of equations. However, it is a non constructive method. Therefore, in order to compute numerically the solutions of this system, we need to resort to other methods.

### 5.4.1 Newton and quasi-Newton methods

The Newton's method is an effective way to solve a system of non linear equations. Let us assume that we aim to solve  $F(x) = 0$ , where  $F : \mathcal{O} \rightarrow \mathbb{R}^d$  is differentiable and  $\mathcal{O}$  is an open subset of  $\mathbb{R}^d$ . Then, we study the sequence defined for every  $n \in \mathbb{N}^*$  by

$$y_{n+1} = y_n - DF(y_n)^{-1}F(y_n), \tag{5.16}$$

with  $y_0$  an arbitrary element of  $\mathcal{O}$ . The main advantage of this method lies in its convergence, which is known to be locally quadratic in lots of cases (see [19] for more details). However, the main issue resides in the difficulty to implement or to numerically compute the inverse of the derivative of  $F(\cdot)$ . In our case,  $F(\cdot) = (S - L)(\cdot)$ , and the expression of  $L(\cdot)$  does not allow easy computations (see equations (5.12) and (5.13)).

To avoid this issue, we will use quasi-Newton methods. They are based on an approximation of  $DF(y_n)^{-1}$  in equation (5.16) by a matrix  $H_n$  easier to compute. They usually satisfy the following quasi-Newton relation:

$$y_n - y_{n-1} = H_n(F(y_n) - F(y_{n-1})), \quad \forall n \in \mathbb{N}^*, \tag{5.17}$$



which is the generalization, in several dimensions, of the coefficient used in the secant method in one dimension. Two examples of these methods are the David-Fletcher-Powell method (see [38] and [42] for more details), and the Broyden's method (see [18] for more details) which are based on different approximations of  $DF(y_n)^{-1}$ , although both satisfy the quasi-Newton relation (5.17). These methods allow the local convergence of the approximation sequence when  $F(\cdot)$  is locally regular in a neighbourhood of the solution, however at a lower speed than the Newton's method in general (see [19] and [41] for more details).

To summarize, the Newton and quasi-Newton methods are effective to solve a system of non-linear equations. In general, the Newton's method solves the problem quicker than quasi-Newton methods since there is no loss of information due to an approximation of the jacobian. However, the exact calculation of this quantity can be numerically costly and difficult to implement. Therefore, we would prefer a quasi-Newton method where the local convergence can also be ensured.

### 5.4.2 Continuation method

We have seen that the Newton and quasi-Newton methods converge towards the solution of the problem, if the starting point of the sequence of approximation is sufficiently close to the solution. Nevertheless, without a priori information on the function  $(S - L)(\cdot)$ , it can be difficult to initialize well the sequence. Thus, the continuation method is a way to handle this problem. Indeed, the strategy will be to fix a certain element  $w_0 \in (0, +\infty)^d$  and notice that the equation  $S(w) - S(w_0) = 0$  admits  $w_0$  as solution. Therefore, we will follow a continuous trajectory  $(w_t)_{t \in [0,1]}$ , such that for each  $t \in [0, 1]$ ,  $w_t$  is solution of

$$S(w) - (1 - t)S(w_0) - tL(w) = 0.$$

We observe that the element  $w_1$  is a solution of the problem we aim to solve. Let us state a theoretical result which permits to prove the existence and uniqueness of solutions using a continuation method. It will give us an algorithm to numerically solve the problem.

**Proposition 5.4** *If assumptions in Theorem 5.2 hold, and if moreover,*

- $S^{-1}(\{S(w_0)\}) = \{w_0\}$ ,
- $S(\cdot)$  and  $L(\cdot)$  are continuously differentiable and for every  $(t, w) \in [0, 1] \times (0, +\infty)^d$ ,  $D(S - tL)(w)$  is an isomorphism of  $\mathbb{R}^d$ ,

*then, there exists a unique  $w \in (0, +\infty)^d$  such that  $S(w) = L(w)$ .*

**Proof** Let us define  $h : [0, 1] \times (0, +\infty)^d \rightarrow \mathbb{R}$  such that

$$h(t, w) = S(w) - (1 - t)S(w_0) - tL(w), \quad \forall (t, w) \in [0, 1] \times (0, +\infty)^d.$$

Step 1 and 2 in the proof of Theorem 5.2 state that there exists  $\varepsilon \in (0, 1)$  such that for every  $(t, w) \in [0, 1] \times (0, +\infty)^d$  if  $h(t, w) = 0$ , then  $w \in (\varepsilon, 1/\varepsilon)^d$ . Let us introduce

$$T = \{t \in [0, 1] : \text{There exists a unique } w \in (0, +\infty)^d \text{ satisfying } h(t, w) = 0\}.$$

From the assumptions,  $T$  is non empty since  $0 \in T$ .

**$T$  is closed.** Indeed, let  $(t_n)_{n \in \mathbb{N}}$  be a sequence of  $T$  such that  $t_n \rightarrow t \in [0, 1]$  when  $n \rightarrow +\infty$ . Then, for every  $n \in \mathbb{N}$  let us denote by  $w_n$  the unique solution in  $(0, +\infty)^d$  of  $h(t_n, \cdot) = 0$ , we observe that  $w_n \in (\varepsilon, 1/\varepsilon)^d$ . Thus, there exists an element  $w \in [\varepsilon, 1/\varepsilon]^d$  such that, passing through a subsequence,  $w_n \rightarrow w$  when  $n \rightarrow +\infty$ . The continuity of  $h(\cdot, \cdot)$  yields that  $h(t, w) = 0$ , and so  $w \in (\varepsilon, 1/\varepsilon)^d$ .

Let us check that there does not exist a distinct element  $\hat{w} \in (0, +\infty)^d$  of  $w$  such that  $h(t, \hat{w}) = 0$ . We prove this claim by contradiction. Since  $D(S - tL)(w)$  and  $D(S - tL)(\hat{w})$  are isomorphisms, we can apply the implicit function theorem in order to deduce that

- there exist  $\mathcal{V}$  an open neighbourhood of  $w$  and  $\mathcal{V}'$  an open neighbourhood of  $\hat{w}$ .
- there exist  $\mathcal{T}$  and  $\mathcal{T}'$  two open neighbourhoods of  $t$  in  $[0, 1]$ .
- there exist  $W : \mathcal{T} \rightarrow \mathcal{V}$  and  $\hat{W} : \mathcal{T}' \rightarrow \mathcal{V}'$  two  $C^1$  trajectories such that:
  - for every  $s \in \mathcal{T}$ ,  $h(s, W(s)) = 0$  and for every  $s \in \mathcal{T}'$ ,  $h(s, \hat{W}(s)) = 0$ .
  - if  $(s, z) \in \mathcal{T} \times \mathcal{V}$  satisfies  $h(s, z) = 0$  then  $z = W(s)$ .
  - if  $(s, z) \in \mathcal{T}' \times \mathcal{V}'$  satisfies  $h(s, z) = 0$  then  $z = \hat{W}(s)$ .

Therefore, let us fix  $\rho = \|w - \hat{w}\|/3$ . From the continuity of the trajectories  $W$  and  $\hat{W}$ , there exists  $\delta > 0$  satisfying  $((t - \delta, t + \delta) \cap [0, 1]) \subset \mathcal{T} \cap \mathcal{T}'$ , such that

$$W(s) \in B(w, \rho) \text{ and } \hat{W}(s) \in B(\hat{w}, \rho), \quad \forall s \in (t - \delta, t + \delta) \cap [0, 1].$$

Thus, for every  $s \in (t - \delta, t + \delta) \cap [0, 1]$ , the equation  $h(s, \cdot) = 0$  admits at least two distinct solutions:  $W(s)$  and  $\hat{W}(s)$ . It is in contradiction with the fact that  $t \in \bar{T}$ . Finally, we have established that  $t \in T$ .

**T is open.** Let us fix  $t \in T$ . Let  $w \in (\varepsilon, 1/\varepsilon)^d$  be the unique element satisfying  $h(t, w) = 0$ . By applying the implicit function theorem, there exists  $\delta > 0$  such that for every  $s \in (t - \delta, t + \delta) \cap [0, 1]$ ,  $h(s, \cdot) = 0$  admits at least a solution.

By contradiction, let us assume that there exists a sequence  $t_n \rightarrow t$  such that the equation  $h(t_n, \cdot) = 0$  admits at least two distinct solutions. We denote by  $w_n^1$  and  $w_n^2$  these solutions. Since these sequences are compact there exist  $w^1$  and  $w^2$  such that, passing through a subsequence,  $w_n^1 \rightarrow w^1$  and  $w_n^2 \rightarrow w^2$  when  $n \rightarrow +\infty$ . The continuity of  $h(\cdot, \cdot)$  yields

$$h(t, w^1) = h(t, w^2) = 0.$$

The uniqueness of the solution ensures that  $w^1 = w^2 = w$ . This enters in contradiction with the implicit function theorem which guarantees the local uniqueness of the solution. Thus, there exists  $\delta > 0$  such that  $(t - \delta, t + \delta) \cap [0, 1] \subset T$ .

**Conclusion.** We have shown that  $T$  is non empty, closed and open for the topology induced on  $[0, 1]$ . Thus,  $T = [0, 1]$ . In particular,  $S(\cdot) = L(\cdot)$  admits a unique solution.  $\square$

This proof allows us to build a continuous trajectory  $\omega : [0, 1] \rightarrow (0, +\infty)^d$  such that for every  $t \in [0, 1]$ ,  $h(t, \omega(t)) = 0$ . The continuity of the trajectory suggests the use of a predictor-corrector algorithm in order to reach the solution of the initial problem. Namely, let us introduce

$$t_0 = 0 < t_1 < \dots < t_N = 1$$

a discretization of the interval  $[0, 1]$ . We fix  $i \in \{0, \dots, N - 1\}$ , and assume that the solution of the equation  $h(t_i, w) = 0$ , given by  $\omega(t_i)$ , is known. Since  $\omega(\cdot)$  is a continuous trajectory, the prediction we make to guess the solution of the equation

$$h(t_{i+1}, w) = 0$$

is  $\omega(t_i)$ . Then, we will use the Newton's method or a quasi-Newton method with the predictor  $\omega(t_i)$  as initial point for the approximation sequence to compute  $\omega(t_{i+1})$ . By induction, we build a sequence  $(w_i)$  such that for every  $i \in \{0, \dots, N\}$ ,  $w_i = \omega(t_i)$ . In particular,  $w_N$  is the element we are looking for, since it satisfies  $S(w_N) = L(w_N)$ . The advantage of this method lies in the fact that  $w_0$  is easier to guess than the solution of  $S(\cdot) = L(\cdot)$  since  $S(\cdot)$  is an input of the model.

## 5.5 Conclusion

In this chapter, we have used the properties of the demand function established in Chapter 4 and proposed another set of assumptions on the supply function; they allow us to consider new models with more complex labour supply functions. We have given examples to support the assumptions made and we have proven the existence of equilibria using the Brouwer degree.

## Chapter 6

# Alternative models for the distribution of capital

### 6.1 Introduction

The purpose of this chapter is to introduce variants in the model of Chapter 2. We recall that the model leads to the following MFG system:

$$\rho u(k) = H(k, u'(k)), \quad (6.1)$$

$$\frac{d}{dk} (D_q H(\cdot, u'(\cdot)) m(\cdot))(k) = \eta(k) - \nu m(k), \quad (6.2)$$

$$S(w) = - \int_0^{+\infty} D_w f(k) dm(k), \quad (6.3)$$

completed with the following conditions:

$$D_q H(0, u'(0)) \geq 0, \quad (6.4)$$

$$1 = \int_0^{+\infty} dm(k). \quad (6.5)$$

where  $w$  is the collection of wages,  $u(\cdot)$  is the value function of the individual optimal control problem faced by firms,  $m$  the capital distribution,  $f(\cdot)$  the net output and  $S(\cdot)$  models the labour supply. The Hamiltonian is given by

$$H(k, q) = \sup\{U(c) - c\} + f(k)q, \quad \forall (k, q) \in [0, +\infty) \times \mathbb{R},$$

with  $U : [0, +\infty) \rightarrow \{-\infty\} \cup \mathbb{R}$  the utility function. This model captures the labour market when there are several types of workers. More generally, it can be used to model factor markets.

This chapter proposes two different modifications to the transport equation (6.2) completed with (6.5). Firstly, we make the assumption that the number of incoming firms is no longer fixed. Instead, we assume that  $\eta(\cdot)$  depends on  $u(\cdot)$ . Thus, the total number of firms also depends on  $u(\cdot)$ . Secondly, we present a special process for renewing firms. We assume that when a firm with capital  $k \in (0, +\infty)$  ceases to exist, it gives birth to two smaller firms with capital  $k/2$  each. In this context,  $\eta(\cdot)$  depends on the capital distribution  $m$ , and the transport equation is no longer local. In both cases, we aim at proving the existence and uniqueness of solutions. Given a modelling problem, these results give the possibility to switch the transport equation in the MFG system (6.1)-(6.5), and to use the best adapted one.

It is possible to make  $\eta(\cdot)$  depend on  $u(\cdot)$  without changing the existence and uniqueness results presented in Proposition 2.6 in Chapter 2. In particular, the semi-explicit formula still hold. Then, we present a

model for the entry of entrepreneurs which justifies the assumptions made on the function  $\eta(\cdot)$ . The non local transport equation is more challenging. Indeed, we need to establish the existence and uniqueness of solutions with two different strategies. Concerning existence, we show by a fixed-point strategy that there exist *solutions*, in a sense we make precise below. To establish uniqueness, we need more regularity properties. Thus, we use the notion of *integral solutions* defined below. We prove that if there are two distinct integral solutions,  $m_1$  and  $m_2$ , then the difference of their respective cumulative distribution function:  $F_1(\cdot)$  and  $F_2(\cdot)$ , admits a maximizer  $\hat{k} > 0$ . Using the regularity of  $m_1$  and  $m_2$ , and the fact that they are solutions of the transport equation, we prove that for every  $n \in \mathbb{N}^*$ ,  $2^n \hat{k}$  is a maximizer of  $(F_1 - F_2)(\cdot)$ . On the other hand, we establish that there exists  $\bar{k}$  such that for every  $k \geq \bar{k}$ ,  $(F_1 - F_2)(k) = 0$ . Therefore, the maximum of  $(F_1 - F_2)(\cdot)$  equals zero. Exchanging the roles of  $m_1$  and  $m_2$ , we conclude that  $F_1(\cdot) = F_2(\cdot)$ , and that the non local transport equation admits a unique integral solution. By proving that a solution of the non local transport equation is an integral solution, we conclude the existence and uniqueness of solutions.

The chapter is structured as follows. Firstly, we recall the properties established in Chapter 2 on the optimal investment policy. Secondly, we introduce the first modifications on the transport equation where the mass of incoming firms is no longer fixed. Then, we study a special process for renewing firms where the source term depends on the distribution of capital  $m$ , and the transport equation is no longer local.

## 6.2 Standing assumptions on the drift

Since, in this chapter, we focus on the transport equation, the value of the collection of wages  $w \in (0, +\infty)^d$  is fixed, as is the optimal investment policy  $b(\cdot) = D_q H(\cdot, u'(\cdot))$ . We present here the properties on  $b(\cdot)$  that were proved in Chapter 2 and that we assume throughout the chapter.

**Assumption 6.1 (Assumption on  $b(\cdot)$ )** *The optimal investment policy of firms  $b : (0, +\infty) \rightarrow \mathbb{R}$  satisfies:*

- i)  $b(\cdot)$  is continuous on  $(0, +\infty)$  and of class  $C^1$  on  $(0, k^*) \cup (k^*, +\infty)$ .
- ii) There exists a unique level of capital  $k^* \in (0, +\infty)$  such that  $b(k^*) = 0$ .
- iii)  $b(\cdot)$  is positive on  $(0, k^*)$  and negative on  $(k^*, +\infty)$ .
- iv) There exist  $\varepsilon > 0$  and  $M > 0$  such that

$$0 \leq b(k) \leq M(k^* - k), \quad \text{if } k \in [k^* - \varepsilon, k^*], \quad (6.6)$$

$$M(k^* - k) \leq b(k) \leq 0, \quad \text{if } k \in [k^*, k^* + \varepsilon]. \quad (6.7)$$

In the following, the drift  $b(\cdot)$  will always satisfies Assumption 6.1.

## 6.3 A model in which the number of firms is not known a priori

In this section, we assume that  $\eta(\cdot)$  depends on the value function. Here are the assumptions we make on the source term  $\eta(\cdot)$ :

**Assumption 6.2** *We assume that  $\eta : [0, +\infty) \times \mathbb{R} \rightarrow [0, +\infty)$  as the following form*

$$\frac{\eta_0(k)}{C} \leq \eta(k, u) \leq C\eta_0(k), \quad \forall (k, u) \in [0, +\infty) \times \mathbb{R}.$$

where

- $\eta_0 : [0, +\infty) \rightarrow [0, +\infty)$  is a density of probability belonging in  $L^\infty(0, +\infty)$  with compact support in  $(0, +\infty)$ .
- $C > 1$ .

We observe that under Assumption 6.1, the problem

$$\frac{d}{dk} (b(\cdot)m)(k) = \eta(k, u(k)) - \nu m(k) \quad \text{on } (0, +\infty), \quad (6.8)$$

$$m((0, +\infty)) = \frac{1}{\nu} \int_0^{+\infty} \eta(k, u(k)) dk, \quad (6.9)$$

$$m \in \mathcal{M}_+((0, +\infty)). \quad (6.10)$$

where equation (6.8) holds in the distributional sense, is well defined. Indeed, it can be checked, using the same argument as in the proof of Proposition 2.6 in Chapter 2, that the unique solution of the latter problem is given by

$$m(k) = \begin{cases} \frac{1}{b(k)} \int_0^k \eta(\kappa, u(\kappa)) \exp\left(-\int_\kappa^k \frac{\nu}{b(z)} dz\right) d\kappa, & \text{if } k \in [0, k^*), \\ -\frac{1}{b(k)} \int_k^{+\infty} \eta(\kappa, u(\kappa)) \exp\left(\int_k^\kappa \frac{\nu}{b(z)} dz\right) d\kappa, & \text{if } k \in (k^*, +\infty), \end{cases} \quad (6.11)$$

where  $k^*$  is the target capital. Therefore, under Assumption 6.2 the important examples of demand functions saw in Chapter 4 can be extended to the case where  $\eta(\cdot)$  depends locally on the value function of the optimal control problem of firms.

**Proposition 6.1** *Under Assumption 6.2 every result established in Chapter 4 holds.*

Moreover, we saw in Lemma 2.13 in Chapter 2 that when  $w_n \rightarrow w$ , the value function of the optimal control problem associated to  $w_n$ , denoted by  $u_n(\cdot)$ , converges uniformly towards  $u(\cdot)$ , the value function of the optimal control problem associated to  $w$ , on every compact set of  $(0, +\infty)$ . Then, the same arguments developed in the proof of Theorem 2.5 hold to show that  $m_n$ , the solution of (6.8)-(6.10) associated to  $w_n$ , converges for the weak \* topology of measures towards  $m$ , the solution of (6.8)-(6.10) associated to  $w$ . Thus, it is possible to extend the existence result of Chapter 5 to this case.

In the next paragraph we show that Assumption 6.2 may model entrepreneurship.

### 6.3.1 Application to the entry of entrepreneurs

Economists established that there are four main factors of production which permit to produce, see Marshall [77, Book IV] for more details. First, there is land which is associated to every natural resources such as oil or gold. Second, there is labour which is the work done by people. Third, there is capital which corresponds to man-made objects like machines or equipments which are used in production. Finally, there is entrepreneurship: the entrepreneur will combine the other three factors of production to produce.

Until now the first three factors of production are well-modelled. Indeed, the state variable is the capital owned by firms, moreover, if the control variables in the individual optimal control problem faced by firms are the level of land and labour, then the modelling takes into account these three factors of production.

Moreover, the model can handle the case where the capital is not entirely held by firms. In this case, a third control is introduced and corresponds to the borrowed capital by firms. Thus, the model will give the price of land, the wages for labour and an interest rate for the borrowed capital. Nonetheless, in this modelling entrepreneurship misses.

Thus, let us assume that for each period of time, there are new entrepreneurs. We make the assumption that they are heterogeneous in their productivity which is measured by a coefficient  $a \in \mathbb{R}$ . An entrepreneur with a level of capital  $k \in (0, +\infty)$  and a coefficient of productivity  $a \in \mathbb{R}$  has two choices:

1. either, to launch their firm in the economy and obtain an utility  $u(k)$ .
2. or, to invest somewhere else where he can get an utility  $u_0(k, a)$ .

We assume that these entrepreneurs are distributed randomly on the space  $(0, +\infty) \times \mathbb{R}$  according to a couple  $(K, A)$  of random variables. We assume that  $K$  and  $A$  are independent and that the law of probability of  $K$  (resp.  $A$ ) admits a density  $g_K : (0, +\infty) \rightarrow [0, +\infty)$  (resp.  $g_A : \mathbb{R} \rightarrow [0, +\infty)$ ). Therefore, the probability for an entrepreneur to launch his firm in this economy is

$$\mathbb{P}(u_0(K, A) < u(K)) = \int_{(0, +\infty) \times \mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u(k)\}} g_K(k) g_A(a) dk da.$$

Thus, if the total number of new entrepreneurs is  $\nu > 0$ , then the number of firms created with capital  $k$  is given by the quantity:

$$\eta(k, u(k)) = \nu \int_{\mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u(k)\}} g_K(k) g_A(a) da. \quad (6.12)$$

**Lemma 6.1** *If there exists  $\underline{a} \in \mathbb{R}$  such that*

1. *there exists a continuous function  $\psi : (\underline{a}, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$  such that*

$$u_0(k, a) = \begin{cases} -\infty, & \text{if } a \leq \underline{a}, \\ \psi(k, a), & \text{otherwise.} \end{cases}$$

2.  $\int_{(-\infty, \underline{a}]} g_A(a) da > 0$ .
3.  $g_K(\cdot)$  *is continuous and has a compact support.*

*Then the function  $\eta : (0, +\infty) \times \mathbb{R} \rightarrow [0, +\infty)$  defined in (6.12) satisfies Assumption 6.2.*

**Proof** For any  $(k, u) \in (0, +\infty) \times \mathbb{R}$ ,

$$\begin{aligned} \eta(k, u) &= \nu \int_{\mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u\}} g_K(k) g_A(a) da \\ &= g_K(k) \nu \int_{\mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u\}} g_A(a) da \end{aligned}$$

Setting for every  $(k, u) \in (0, +\infty) \times \mathbb{R}$ ,

$$\eta_0(k) = g_K(k) \quad \text{and} \quad \eta_1(k, u) = \nu \int_{\mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u\}} g_A(a) da,$$

we observe that there exists  $C > 1$  such that  $1/C \leq \eta_1(\cdot, \cdot) \leq C$ . Therefore  $\eta(\cdot, \cdot)$  satisfies Assumption 6.2.  $\square$

## 6.4 A non local model

The second change concerns a new process for renewing firms. We seek to prove the existence and uniqueness of a solution of the following problem:

$$\frac{d}{dk} (bm) = \nu ((\text{id}/2)_{\#} m - m), \quad \text{on } (0, +\infty), \quad (6.13)$$

$$m \in \mathcal{P}(0, +\infty), \quad (6.14)$$

where  $\text{id}$  stands for the identity function on  $[0, +\infty)$  and  $(\text{id}/2)_\#m$  is the push-forward measure of  $m$ , i.e

$$\int_{(0, +\infty)} g(k)(\text{id}/2)_\#m(k) = \int_{(0, +\infty)} g\left(\frac{k}{2}\right) dm(k), \quad \forall g \in C_b(0, +\infty).$$

The map  $b : (0, +\infty) \rightarrow \mathbb{R}$  satisfies Assumptions 6.1 above and is understood as the optimal investment. The right-hand side of (6.13) corresponds to a renewal of firms at a renewal rate  $\nu > 0$ : formally, we assume that for a level of capital  $k \in (0, +\infty)$  the firms die at a rate  $\nu$  and when a firm ceases to exist, it gives birth to two smaller firms of capital  $k/2$  each. Hence, the creation of firms is  $2\nu m(2k)$ . In other words we assume that at each period of time a firm has the probability  $\nu$  to disappear which leads to the creation of 2 firms with half its capital.

**Definition 6.1** *We say that  $m$  is a solution of (6.13)-(6.14) if (6.13) holds in the sense of distribution and (6.14) is satisfied, i.e :*

$$-\int_0^{+\infty} \varphi'(k)b(k)dm(k) = \nu \int_0^{+\infty} \left( \varphi\left(\frac{k}{2}\right) - \varphi(k) \right) dm(k), \quad \forall \varphi \in C_c^\infty(0, +\infty),$$

and  $m \in \mathcal{P}(0, +\infty)$ .

**Remark 6.1** *It is important to note that we are looking for solutions in  $\mathcal{P}(0, +\infty)$  and not in  $\mathcal{P}([0, +\infty))$ . Indeed, since the state  $k = 0$  is unstable because  $b(\cdot)$  is positive on  $(0, k^*)$ , we are not interested in the trivial solution  $\delta_0$ .*

**Definition 6.2** *We say that  $m$  is an integral solution of (6.13)-(6.14) if*

- $(bm)(\cdot)$  is continuous on  $(0, +\infty)$ .
- $m$  is in  $L^1(0, +\infty) \cap \mathcal{P}(0, +\infty)$ .
- For every  $k, k' \in (0, +\infty)$ ,

$$(bm)(k') - (bm)(k) = \nu \int_k^{k'} 2m(2k) - m(k)dk. \quad (6.15)$$

We wish to show that there exists a unique solution of (6.13)-(6.14). For this aim, we will need four lemmas. The first two will lead to uniqueness of solutions: we prove that if a solution exists, then it admits interesting regularity properties, in particular that it is an integral solution. The last two lemmas are focused on a map defined on the space of probability measures on  $[0, k^*]$  which will permit to conclude the existence of solutions by using the Schauder fixed-point theorem.

**Lemma 6.2** *Under Assumption 6.1, if  $m$  is a solution of (6.13)-(6.14), then  $m$  is an integral solution of (6.13)-(6.14).*

**Proof** Let us remark that  $(bm)(\cdot) \in BV_{loc}(0, +\infty)$ . Let  $K$  be a compact subset of  $(0, +\infty)$ . Since  $m$  is a solution of (6.13) in the distributional sense on  $(0, +\infty)$ , the weak derivative of  $(bm)(\cdot)$  on  $K$  is  $\nu((\text{id}/2)_\#m - m)$  which is a finite Radon measure. Since  $(bm)(\cdot) \in BV_{loc}(0, +\infty)$ ,  $1/b(\cdot) \in C^1((0, k^*) \cup (k^*, +\infty))$ , and

$$m = (bm)(\cdot) \times \frac{1}{b(\cdot)},$$

in the distributional sense on  $(0, k^*) \cup (k^*, +\infty)$ , then  $m \in BV_{loc}((0, k^*) \cup (k^*, +\infty))$ . Outside the set  $\{k^*\}$ , the measure  $m$  can be locally identify to a  $BV$  function; then using the Lebesgue decomposition of  $m = m_{ac}(\cdot) + m_s$ , the support of its singular part  $m_s$  is included in  $\{k^*\}$ , while its regular part  $m_{ac}(\cdot)$  is in  $L^1(0, +\infty)$ . Because  $m$  is a non negative measure,  $m_s$  must be non negative as well. Therefore, it must be a Dirac mass in  $k^*$ , i.e. it has the form  $\lambda \delta_{k^*}$  with  $\lambda \geq 0$ .

We claim that  $\lambda = 0$ . We already know that  $\lambda \geq 0$ ; let us show the converse inequality. For this aim, consider a sequence  $(\varphi_\varepsilon(\cdot))_{\varepsilon > 0}$  such that for every  $\varepsilon > 0$ ,



- $\varphi_\varepsilon(\cdot) \in C_c^\infty(0, +\infty)$ .
- $\text{supp } \varphi_\varepsilon(\cdot) \subset [k^* - \varepsilon, k^* + \varepsilon]$ .
- $\varphi_\varepsilon(k^*) = 1$ .
- $\varphi_\varepsilon(\cdot)$  is non decreasing on  $[0, k^*]$ , and non increasing on  $[k^*, +\infty)$ .
- $\|\varphi'_\varepsilon(\cdot)\|_\infty \leq 2/\varepsilon$ .

Since  $m$  is solution of (6.13)-(6.14), we see that

$$-\int_0^{+\infty} \varphi'_\varepsilon(k)b(k)dm(k) = \nu \int_0^{+\infty} \left( \varphi_\varepsilon\left(\frac{k}{2}\right) - \varphi_\varepsilon(k) \right) dm(k), \quad \forall \varepsilon > 0,$$

which can be rewritten for  $\varepsilon \in (0, k^*/2)$  as

$$-\int_{k^*-\varepsilon}^{k^*+\varepsilon} \varphi'_\varepsilon(k)b(k)m_{ac}(k)dk = \nu \int_{2(k^*-\varepsilon)}^{2(k^*+\varepsilon)} \varphi_\varepsilon\left(\frac{k}{2}\right) m_{ac}(k)dk - \nu \int_{k^*-\varepsilon}^{k^*+\varepsilon} \varphi_\varepsilon(k)m_{ac}(k)dk - \lambda\nu, \quad (6.16)$$

since  $b(k^*) = 0$  and  $m = m_{ac} + \lambda\delta_{k^*}$ . The construction of  $\varphi_\varepsilon(\cdot)$  and Assumption 6.1 ensure that:

$$\sup_{[k^*-\varepsilon, k^*+\varepsilon]} |\varphi'_\varepsilon(k)b(k)| \leq 2M.$$

Using this estimate in (6.16), the fact  $\varphi_\varepsilon(\cdot)$  is bounded above by 1 and  $\nu \int_{k^*-\varepsilon}^{k^*+\varepsilon} \varphi_\varepsilon(k)m_{ac}(k)dk \geq 0$ , the following inequality holds:

$$\lambda \leq \frac{2M}{\nu} \int_{k^*-\varepsilon}^{k^*+\varepsilon} m_{ac}(k)dk + \int_{2(k^*-\varepsilon)}^{2(k^*+\varepsilon)} m_{ac}(k)dk.$$

The Lebesgue convergence theorem yields  $\lambda \leq 0$  by letting  $\varepsilon$  vanishes, and so  $\lambda = 0$ .

Therefore  $m(\cdot) \in L^1(0, +\infty)$ , and using equation (6.13), we deduce  $(bm)(\cdot)$  is  $W_{loc}^{1,1}(0, +\infty)$  so (6.15) holds. Finally,  $m$  is an integral solution of (6.13)-(6.14).  $\square$

**Remark 6.2** *It is easy to remark that more regularity holds for  $m(\cdot)$  and  $(bm)(\cdot)$ . Indeed, using that almost everywhere  $m(\cdot) = (bm)(\cdot) \times 1/b(\cdot)$ , we can identify  $m(\cdot)$  to a continuous function on  $(0, k^*) \cup (k^*, +\infty)$ . Which leads, thanks to equation (6.13), to  $(bm)(\cdot)$  is of class  $C^1$  on  $\mathbb{R} \setminus \{k^*/2, k^*\}$  and so does  $m(\cdot)$ .*

**Lemma 6.3** *Under Assumption 6.1, if  $m(\cdot)$  is a solution of (6.13)-(6.14), then  $m(\cdot)$  vanishes on  $(k^*, +\infty)$  and  $\lim_{k \rightarrow 0^+} (bm)(k) = \lim_{k \rightarrow +\infty} (bm)(k) = 0$ .*

**Proof** From Lemma 6.2,  $m(\cdot)$  is an integral solution of (6.13). Let us take two constants  $0 < \underline{k} < \bar{k} < +\infty$ , then

$$(bm)(\bar{k}) - (bm)(\underline{k}) = \nu \int_{\underline{k}}^{\bar{k}} 2m(2\kappa) - m(\kappa) d\kappa. \quad (6.17)$$

This equation shows that

1.  $\lim_{k \rightarrow +\infty} (bm)(k)$  and  $\lim_{k \rightarrow 0^+} (bm)(k)$  exist.
2.  $\lim_{k \rightarrow +\infty} (bm)(k) = \lim_{k \rightarrow 0^+} (bm)(k)$ .

Assumption 6.1-iii) leads to

$$0 \geq \lim_{k \rightarrow +\infty} (bm)(k) = \lim_{k \rightarrow 0^+} (bm)(k) \geq 0.$$

Therefore

$$\lim_{k \rightarrow +\infty} (bm)(k) = \lim_{k \rightarrow 0^+} (bm)(k) = 0.$$

Taking equation (6.17) with  $\bar{k} = +\infty$  and  $\underline{k} = k \in (k^*, +\infty)$ ,

$$-(bm)(k) = \nu \int_k^{+\infty} 2m(2\kappa) - m(\kappa) d\kappa = -\nu \int_k^{2k} m(\kappa) d\kappa.$$

Since Assumption 6.1-iii) holds and  $m(\cdot)$  is non negative, then the last equation yields that

$$m(k) \leq 0,$$

where  $m(k)$  does make sense, as shown in Remark 6.2. Therefore,  $m(\cdot) \equiv 0$  on  $(k^*, +\infty)$ .  $\square$

**Lemma 6.4** *Let  $\mu$  be a probability measure with support in  $[0, k^*]$ , then  $\Phi[\mu](\cdot)$  defined for all  $k \in [0, +\infty)$ , by*

$$\Phi[\mu](k) = \begin{cases} \frac{\nu}{b(k)} \int_0^{\min(2k, k^*)} e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} d\mu(\kappa), & \text{if } k \in (0, k^*), \\ 0, & \text{otherwise.} \end{cases} \quad (6.18)$$

is a solution in the sense of distributions of

$$\frac{d}{dk} (bm) = \nu ((\text{id}/2)_\# \mu - m), \quad \text{on } (0, +\infty), \quad (6.19)$$

$$m \in \mathcal{P}(0, +\infty). \quad (6.20)$$

**Proof** Let  $\varphi(\cdot) \in C_c^\infty(0, +\infty)$  be a test function.

$$-\int_0^{+\infty} \varphi'(k) b(k) \Phi[\mu](k) dk = -\nu \int_0^{k^*} \int_0^{\min(2k, k^*)} \varphi'(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} d\mu(\kappa) dk \quad (6.21)$$

$$= -\nu \int_0^{k^*} \int_{\kappa/2}^{k^*} \varphi'(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} dk d\mu(\kappa) \quad (6.22)$$

$$= -\nu \int_0^{k^*} \left\{ \left[ \varphi(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} \right]_{\kappa/2}^{k^*} + \frac{\nu}{b(k)} \int_{\kappa/2}^{k^*} \varphi(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} dk \right\} d\mu(\kappa) \quad (6.23)$$

$$= \nu \int_0^{k^*} \varphi\left(\frac{\kappa}{2}\right) d\mu(\kappa) - \nu \int_0^{k^*} \varphi(k) \Phi[\mu](k) dk \quad (6.24)$$

$$= \nu \left( \int_0^{+\infty} \varphi \circ f(k) d\mu(\kappa) - \int_0^{+\infty} \varphi(k) \Phi[\mu](k) dk \right). \quad (6.25)$$

Here, (6.21) comes from the definition of  $\Phi[\mu](\cdot)$ , (6.22) is obtained by applying Fubini theorem, (6.23) comes from an integration by part, and (6.24) uses Tonelli theorem and Assumption 6.1-iv) to deduce

$$\left[ \varphi(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} \right]_{\kappa/2}^{k^*} = -\varphi\left(\frac{\kappa}{2}\right),$$

since  $k \mapsto -\int_{\kappa/2}^k \frac{\nu}{b(z)} dz$  blows up like a logarithm of  $(k^* - k)$  as  $k$  tends to  $k^*$ . Therefore,  $\Phi[\mu](\cdot)$  is a solution of (6.19) in the distributional sense. Let us check that it defines a density of probability. It is

easy to check that  $\Phi[\mu](\cdot)$  is measurable and non-negative. With essentially the same computations as above,

$$\int_0^{+\infty} \Phi[\mu](k) dk = \int_0^{k^*} \int_0^{\min(2k, k^*)} \frac{\nu}{b(k)} e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} d\mu(\kappa) dk \quad (6.26)$$

$$= \int_0^{k^*} \int_{\kappa/2}^{k^*} \frac{\nu}{b(k)} e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} dk d\mu(\kappa) \quad (6.27)$$

$$= \int_0^{k^*} \left[ -e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} \right]_{\kappa/2}^{k^*} d\mu(\kappa) \quad (6.28)$$

$$= \int_0^{k^*} d\mu(\kappa) \quad (6.29)$$

$$= 1. \quad (6.30)$$

Here, (6.26) comes from the definition of  $\Phi[\mu](\cdot)$ , (6.27) is obtained by applying Tonelli theorem, (6.28) comes from an integration, (6.29) uses once again Assumption 6.1-iv) to deduce

$$\left[ -e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} \right]_{\kappa/2}^{k^*} = 1,$$

since  $k \mapsto -\int_{\kappa/2}^k \frac{\nu}{b(z)} dz$  blows up like a logarithm of  $(k^* - k)$  as  $k$  tends to  $k^*$ ; and (6.30) holds because  $\mu \in \mathcal{P}([0, k^*])$ . Then  $\Phi[\mu](\cdot)$  defines a density of probability.  $\square$

**Remark 6.3** For every  $\mu \in \mathcal{P}([0, k^*])$ , the proof of the Lemma 6.4 shows that  $\Phi[\mu](\cdot) \in L^1(0, +\infty)$ .

**Lemma 6.5** The map  $\Phi : \mathcal{P}([0, k^*]) \rightarrow \mathcal{P}([0, k^*])$  is continuous for the weak \* topology of measures.

**Proof** Let us take  $(\mu_n)_{n \in \mathbb{N}}$  a sequence of measures in  $\mathcal{P}([0, k^*])$  converging for the weak \* topology of measures towards  $\mu$ . Let us show that  $\Phi[\mu_n] \rightarrow \Phi[\mu]$  for the same topology. We first remark that  $\mathcal{P}([0, k^*])$  is compact for this topology, therefore there exists  $m \in \mathcal{P}([0, k^*])$  such that (up to a subsequence)  $\Phi[\mu_n] \rightarrow m$ . Second, we remark that  $\Phi[\mu_n] \rightarrow \Phi[\mu]$  in the distributional sense. We fix  $\varphi(\cdot) \in C_c^\infty([0, k^*])$  a test function:

$$\int_0^{k^*} \varphi(k) \Phi[\mu_n](k) dk = \int_0^{k^*} \varphi\left(\frac{\kappa}{2}\right) d\mu_n(\kappa) + \frac{1}{\nu} \int_0^{k^*} \varphi'(k) b(k) \Phi[\mu_n](k) dk, \quad (6.31)$$

$$= \int_0^{k^*} \varphi\left(\frac{\kappa}{2}\right) d\mu_n(\kappa) + \int_0^{k^*} \int_0^{\min(2k, k^*)} \varphi'(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} d\mu_n(\kappa) dk, \quad (6.32)$$

$$= \int_0^{k^*} \varphi\left(\frac{\kappa}{2}\right) d\mu_n(\kappa) + \int_0^{k^*} \int_{\kappa/2}^{k^*} \varphi'(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} dk d\mu_n(\kappa). \quad (6.33)$$

Here, (6.31) holds since  $\Phi[\mu]$  is a distributional solution of (6.19), (6.32) is obtained by definition of  $\Phi[\cdot]$  and (6.33) comes from Fubini theorem. The regularity of  $\varphi(\cdot)$  and the convergence of the sequence  $(\mu_n)_{n \in \mathbb{N}}$  ensure that

$$\begin{aligned} \int_0^{k^*} \varphi\left(\frac{\kappa}{2}\right) d\mu_n(\kappa) &\rightarrow \int_0^{k^*} \varphi\left(\frac{\kappa}{2}\right) d\mu(\kappa), \\ \int_0^{k^*} \int_{\kappa/2}^{k^*} \varphi'(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} dk d\mu_n(\kappa) &\rightarrow \int_0^{k^*} \int_{\kappa/2}^{k^*} \varphi'(k) e^{-\int_{\kappa/2}^k \frac{\nu}{b(z)} dz} dk d\mu(\kappa), \end{aligned}$$

which leads to

$$\int_0^{k^*} \varphi(k) d\Phi[\mu_n](k) \rightarrow \int_0^{k^*} \varphi(k) d\Phi[\mu](k).$$

The uniqueness of the limit in the distributional sense shows that  $m = \Phi[\mu]$  and then the whole sequence converges towards  $\Phi[\mu]$  for the weak \* topology of measures.  $\square$

We are now ready to state the existence and uniqueness of the solution of (6.13)-(6.14):

**Theorem 6.1** *Under Assumption 6.1 there exists a unique solution of the problem (6.13)-(6.14).*

**Proof**

**Uniqueness** Let  $m_1$  and  $m_2$  be two distinct solutions of (6.13)-(6.14). Let us take the repartition function

$$F_i(k) = \begin{cases} m_i((0, k]), & \text{if } k > 0, \\ 0, & \text{otherwise.} \end{cases}$$

for  $i = 1, 2$ . From Lemma 6.2 and 6.3, and Remark 6.2,

1.  $F_i(\cdot)$  is continuous on  $\mathbb{R}$  and of class  $C^1$  on  $(0, k^*) \cup (k^*, +\infty)$ ,
2. For every  $k > k^*$ ,  $F_1(k) = F_2(k)$ .

Let us assume by contradiction that  $\max_{[0, +\infty)}(F_1 - F_2)(\cdot) > 0$  and let us take  $\hat{k} \in (0, k^*)$  a maximal point of  $(F_1 - F_2)(\cdot)$  to deduce that

$$0 = (F_1 - F_2)'(\hat{k}) = m_1(\hat{k}) - m_2(\hat{k}).$$

Integrating both the equations satisfied by  $m_i$  ( $i = 1, 2$ ) between 0 and  $\hat{k}$  and taking the difference, lead to

$$b(\hat{k})(m_1(\hat{k}) - m_2(\hat{k})) = (F_1 - F_2)(2\hat{k}) - (F_1 - F_2)(\hat{k}) \Leftrightarrow (F_1 - F_2)(2\hat{k}) = (F_1 - F_2)(\hat{k}).$$

We deduce that  $2\hat{k}$  is a maximal point as well. Since  $(F_1 - F_2)(k^*) = 0$  then  $2\hat{k} \neq k^*$ , therefore, by induction for every  $n \in \mathbb{N}$ ,  $2^n \hat{k}$  is a maximal point. Let us fix  $n \in \mathbb{N}$  large enough to ensure  $2^n \hat{k} > k^*$ ; since  $(F_1 - F_2)(2^n \hat{k}) = 0$ , it shows that  $\max_{[0, +\infty)}(F_1 - F_2)(\cdot) = 0$  which contradicts  $\max_{[0, +\infty)}(F_1 - F_2)(\cdot) > 0$ . Hence  $\max_{[0, +\infty)}(F_1 - F_2)(\cdot) \leq 0$ .

Exchanging the roles of  $m_1$  and  $m_2$  we deduce that  $F_1 = F_2$  and therefore  $m_1 = m_2$ .

**Existence** Let us apply Schauder fixed-point theorem to the map  $\Phi : \mathcal{P}([0, k^*]) \rightarrow \mathcal{P}([0, k^*])$ . Since  $\mathcal{P}([0, k^*])$  is a convex and compact set for the weak \* topology of measures and  $\Phi[\cdot]$  is continuous by Lemma 6.5, Schauder fixed-point theorem yields the existence of  $m \in \mathcal{P}([0, k^*])$  solution of (6.13). We finish by observing that  $m = \Phi[m]$  is also in  $L^1(0, +\infty)$  as noted in Remark 6.3, therefore it is an element of  $\mathcal{P}(0, +\infty)$ .  $\square$

## 6.5 Conclusion

In this chapter, we have presented two extensions of the model where the transport equation has been modified. In both cases, the changes concern the term which models the entries of firms in the economy. First, we assumed that it depends on the value function of the optimal control problem faced by firms. We have seen that it allows to model the entries of firms as the result of entrepreneurial decisions. In this framework, the total mass of firms is no longer fixed. The existence and uniqueness of solutions of the modified problem (6.8)-(6.10) hold and the examples given in Chapter 4 remain valid. Then, we have changed the modelling of the entries of firms of the economy by assuming that when a firm ceases to exist it is acquired by two new firms which split in half its capital. We have obtained the problem (6.13)-(6.14) with a non local transport equation where we have shown existence and uniqueness of solutions. This chapter has extended the possibilities of modelling proposed in Chapter 2 by replacing the transport problem studied in section 2.4 by those introduced in this chapter.



# Chapter 7

## Numerical simulations

### 7.1 Introduction

In this chapter we aim at solving numerically the MFG model introduced in Chapter 2. We are also interested in the extensions presented in Chapter 6. Since the algorithms are basically the same, we present the method only for the equations introduced in Chapter 2. Namely, we aim at solving the following MFG system

$$\rho u(k) = H(k, u'(k)), \quad (7.1)$$

$$\frac{d}{dk} (D_q H(\cdot, u'(\cdot)) m(\cdot))(k) = \eta(k) - \nu m(k), \quad (7.2)$$

$$S(w) = - \int_0^{+\infty} D_w f(k) dm(k), \quad (7.3)$$

completed with the following conditions:

$$D_q H(0, u'(0)) \geq 0, \quad (7.4)$$

$$1 = \int_0^{+\infty} dm(k). \quad (7.5)$$

where the Hamiltonian  $H : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined by (2.5), i.e.

$$H(k, q) = \sup_{c \geq 0} \{U(c) - cq\} + f(k)q, \quad \forall (k, q) \in (0, +\infty)^2. \quad (7.6)$$

We assume that the net output  $f : [0, +\infty) \times (0, +\infty)^d \rightarrow \mathbb{R}$  and the labour supply  $S : (0, +\infty)^d \rightarrow (0, +\infty)^d$  are given and essentially satisfy the assumptions stated in Chapter 2 and Chapter 5 respectively. The system (7.1)-(7.5) characterizes an equilibrium where equation (7.1) completed with (7.4) gives the strategy of the firms, (7.2) completed with (7.5) gives the capital distribution, and (7.3) corresponds to the market clearing conditions on the labour market.

For a given vector of wages  $w \in (0, +\infty)^d$ , the approximation used is inspired from the one proposed by Achdou and Capuzzo-Dolcetta in [2]. Once the two equations are solved we can approximate the integral in (7.2) and compute  $S(w) + \int_0^{+\infty} D_w f(k) dm(k)$ . We then aim at finding a zero of the function

$$(0, +\infty)^d \ni w \mapsto S(w) + \int_0^{+\infty} D_w f(k) dm(k)$$

in order to exhibit a solution of the problem.

Then, we use this method to make simulations. We fix some parameters from the economic literature

[89], while the others are fixed in order to have solutions which well represent the data we get from the CONSEIL SUPERIEUR DE L'AUDIOVISUEL and from INSEE [39, 63]. From these simulations, we numerically show that the capital distribution may or may not explode near the target capital  $k^*$ . Then, we keep essentially the same parameters and we run some simulations when the transport equation in the system (7.1)-(7.5) is given by the extensions discussed in Chapter 6. Finally, we do comparative statics to examine the impact that the variations of the model's parameters have on the equilibrium.

## 7.2 The finite difference operators

### 7.2.1 The scheme

In this paragraph, we present the approximation of (7.1)-(7.5). Let us fix  $w \in (0, +\infty)^d$ . We saw in Chapter 2 that if the source term  $\eta(\cdot)$  has compact support, then the density of probability  $m(\cdot)$  also has a compact support. Under this assumption, let us introduce  $K = \max \text{supp } m(\cdot)$  which is in practice a well known real number since  $K = \max(\text{supp } \eta(\cdot), k^*(w))$  (see Theorem 2.3 and Proposition 2.6), where  $k^*(w)$  is the unique solution of

$$D_k f(k, w) = \rho. \quad (7.7)$$

For  $\epsilon > 0$ , we set  $\Gamma = (0, K + \epsilon]$  and let  $\Gamma_h$  be a uniform grid on  $\Gamma$  with mesh step  $h$  (assuming that  $1/h$  is an integer  $N_h$ ). Let  $k_i$  denotes a generic point in  $\Gamma_h$ . The values of  $u(\cdot)$  and  $m(\cdot)$  at  $k_i$  will respectively be approximated by  $U_i$  and  $M_i$ . We introduce the finite difference operator:

$$(D^+U)_i = \frac{U_{i+1} - U_i}{h}, \quad (i = 1, \dots, N_h - 1).$$

Given a level of capital  $k \in (0, +\infty)$ , we can split the Hamiltonian into its non decreasing part and its non increasing part with respect to  $q$ :  $H(k, \cdot) = H^\downarrow(k, \cdot) + H^\uparrow(k, \cdot) - \min_{q \in (0, +\infty)} H(k, q)$ , see Definition 2.1 in Chapter 2 for more details. If there is no increasing part with respect to  $q$ , then  $H(k, \cdot) = H^\downarrow(k, \cdot)$ . The approximation of (7.1) is therefore

$$\rho U_i = \tilde{G}_i(k, U) \quad \forall i = 1, \dots, N_h, \quad (7.8)$$

with

$$\tilde{G}_i(k, U) = H^\downarrow(k_i, (D^+U)_{i-1}) + H^\uparrow(k_i, (D^+U)_i) - a(k_i),$$

where

$$a(k_i) = \begin{cases} 0, & \text{if } H^\uparrow(k_i, \cdot) \equiv 0, \\ \min_{q \in (0, +\infty)} H(k_i, q), & \text{otherwise,} \end{cases}$$

and where  $(D^+U)_0$  and  $(D^+U)_{N_h}$  are arbitrarily chosen to ensure that  $H^\downarrow(k_1, (D^+U)_0) = 0$  and  $H^\uparrow(k_{N_h}, (D^+U)_{N_h}) = 0$ . Indeed, for every  $k \in (0, k^*(w))$ ,  $D_q H(k, u'(k))$  must be positive. Therefore, in a neighbourhood of 0, the contribution of the Hamiltonian comes from its increasing part. Similarly, near  $K + \epsilon$  the contribution of the Hamiltonian comes from its decreasing part.

In order to introduce the approximation used for the continuity equation (7.2), let us differentiate  $\tilde{G}_i(\cdot, \cdot)$  with respect to  $U$ :

$$\begin{aligned} D_U \tilde{G}_i(k, U) V &= -D_q H^\downarrow(k_i, (D^+U)_{i-1}) V_{i-1}/h \\ &\quad + \{D_q H^\downarrow(k_i, (D^+U)_{i-1}) - D_q H^\uparrow(k_i, (D^+U)_i)\} V_i/h \\ &\quad + D_q H^\uparrow(k_i, (D^+U)_i) V_{i+1}/h \end{aligned} \quad (7.9)$$

for  $i = 2, \dots, N_h - 1$ . In the case  $i = 1$  and  $i = N_h$  we obtain:

$$D_U \tilde{G}_0(k, U) V = -D_q H^\uparrow(k_1, (D^+U)_1) V_1/h + D_q H^\uparrow(k_1, (D^+U)_1) V_2/h,$$

and

$$D_U \tilde{G}_{N_h}(k_i, U)V = D_q H^\downarrow(k_{N_h}, (D^+U)_{N_h-1})V_{N_h-1}/h + D_q H^\downarrow(k_{N_h}, (D^+U)_{N_h-1})V_{N_h}/h.$$

We can summarize the  $N_h$  previous lines as  $D_U \tilde{G}(k, U)V$ . Let us consider  $\Sigma$  the vector in  $\mathbb{R}^{N_h}$  such that for every  $i = 1, \dots, N_h$ ,  $\Sigma_i = \eta(k_i)$ . The approximation of the continuity equation (7.2) is given by

$$(I_d \nu + D_q \tilde{H}(k, U)^T)M = \Sigma. \quad (7.10)$$

Given  $M \in \mathbb{R}^{N_h}$  the solution of (7.10), we approximate the integral in (7.3) as follows:

$$-\int_0^{+\infty} D_w f(k) dm(k) \simeq -h \sum_{i=1}^{N_h} D_w f(k_i) M_i. \quad (7.11)$$

## 7.2.2 The algorithm

### Solution of the HJ equation

Given a vector of wages  $w \in (0, +\infty)^d$ , the problem defined in (7.8) is non linear. Therefore, the Newton's method is used to compute the unique solution  $U$ .

Let us introduce  $G : \mathbb{R}^{N_h} \rightarrow \mathbb{R}^{N_h}$  given for every  $V \in \mathbb{R}^{N_h}$  by

$$G_i(V) = \rho V_i - \tilde{G}_i(k, V), \quad \forall i = 1, \dots, N_h,$$

where  $a(\cdot)$  has been previously defined. Given an arbitrary initial guess  $U^0 \in \mathbb{R}^{N_h}$ , for every  $n$ , we compute  $U^{n+1}$  from  $U^n$  by the Newton's iteration:

$$DG(U^n)(U^{n+1} - U^n) = -G(U^n).$$

This sequence converges towards the zero of  $G(\cdot)$ .

### Solution of the problem

Once the solution of (7.8) is computed, we solve the system of linear equations (7.10) and obtain  $M$ . This allows us to determinate the residual

$$\Lambda(w) = S(w) + h \sum_{i=1}^{N_h} D_w f(k_i) M_i.$$

When  $d = 1$ , we use the secant method in order to solve the equation  $\Lambda(w) = 0$ . When  $d \geq 2$ , we use the so called "good" Broyden's iterations. For completeness we recall that iterations of the secant method are of the form

$$w^{n+1} = w^n - \frac{w^n - w^{n-1}}{\Lambda(w^n) - \Lambda(w^{n-1})} \Lambda(w^n).$$

The "good" Broyden's method consists in the iterations:

$$B^n s^n = -\Lambda(w^n),$$

where  $s^n = w^n - w^{n-1}$  and

$$B^n = B^{n-1} + \frac{r^n (s^n)^T}{(s^n)^T s^n}$$

with  $r_n = \Lambda(w^n) - \Lambda(w^{n-1})$ . Note that both methods need two initial guesses. For the secant method, we simply choose two points  $w^0, w^1$  in  $(0, +\infty)$ . For the "good" Broyden's method, we choose one point  $w^0$  in  $(0, +\infty)^d$  and we specify  $B^0 = I_d$ .



## 7.3 Numerical simulations

The numerical simulations reported below deal with the sector of the audiovisual production and distribution. The data come from the CONSEIL SUPERIEUR DE L'AUDIOVISUEL, see [39], and INSEE the French institute for statistics, see [63]. We also use data coming from the economic literature, see [89]. The data from INSEE are summarized in Table 7.1 below.

Annual production / Nb of firms ( in $10^4$ Euro)	66.4
Total payroll / Nb of firms ( in $10^4$ Euro)	47.1
Nb of Employees / Nb of firms	5.81
Annual production / Nb of Employees ( in $10^4$ Euro)	11.4
Total payroll / Nb of Employees ( in $10^4$ Euro)	8.11

Table 7.1: Data from INSEE.

### 7.3.1 Solution of (7.1)-(7.5) applied to the audiovisual, publishing and distribution sector

We make two tests in this paragraph. In Test 1, the only factor of production of firms is the workforce. We are able to find a numerical solution in this case. Then, we run the Test 2 where we link the labour market with rental market for professionals, and see how the equilibrium is impacted. We also compare the speed of convergence of the algorithms used for these simulations.

#### Choice of models and parameters

We assume that the production function  $F : [0, +\infty) \times [0, +\infty)^d \rightarrow [0, +\infty)$  is a Cobb-Douglas function:

$$F(k, \ell) = Ak^\alpha \ell^\beta, \quad \forall (k, \ell) \in [0, +\infty) \times [0, +\infty)^d.$$

Then, if labour is a control, the elasticity of the production with respect to the variations of the total payroll is given by the following quantity

$$\frac{\text{Total payroll}}{\text{Annual production}} \simeq \frac{8.11}{11.4} = 0.710.$$

From [89], we set the elasticity with respect to capital to  $\alpha = 0.21$ . We also fix the depreciation rate  $\delta = 0.07$ , i.e. within a year, firms lose 7% percent of their capital.

The report [39] states that, in 2018 around 26% of the firms in the audiovisual sector were less than three year old. Assuming that the death of firms follows an exponential law, it gives a rate of death of 0.10 in this sector of activity. We retain  $\nu = 0.1$  for the whole sector of "Audiovisual, publishing and distribution".

We set the discount factor to  $\rho = 0.1$ .

We assume that the labour supply is given by a logistic function, i.e.  $S_{labour} : [0, +\infty) \rightarrow [0, +\infty)$  is given for every  $w \in [0, +\infty)$  by

$$S_{labour}(w) = \frac{K}{(1 + e^{-r(w-\mu)})},$$

with  $K = 6.5$ ,  $r = 2 \times 10^{-4}$ , and  $\mu = 7 \times 10^4$  (see Figure 7.1) and that the instantaneous utility function is a logarithm, i.e.

$$U(c) = \ln(c), \quad \forall c \in (0, +\infty).$$

Moreover, we model the entries of firms by a Gaussian function, times the rate  $\nu$ , centred in  $30 \times 10^4$ , with a standard deviation of  $9 \times 10^4$  (see Figure 7.1).

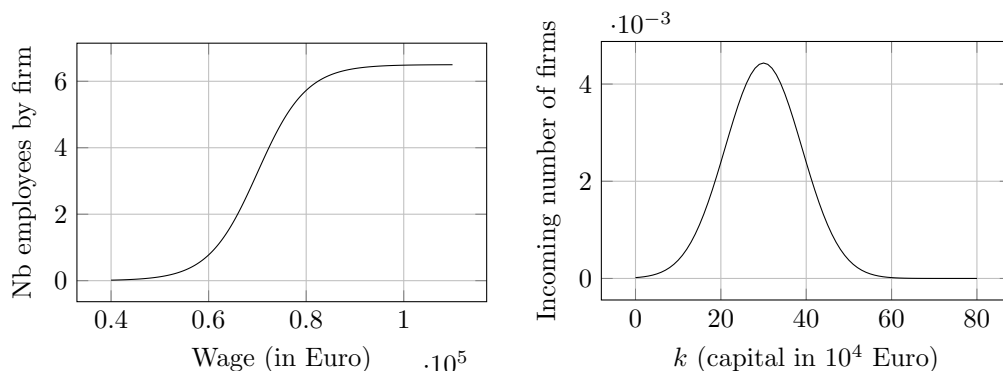


Figure 7.1: Labour supply and the source.

In Test 2, we also need to model the rental market for professionals. We assume that the workspace supply is given by a logistic function i.e.  $S_{workspace} : [0, +\infty) \rightarrow [0, +\infty)$

$$S_{workspace}(p) = \frac{K_2}{1 + e^{-r_2(p - \mu_2)}},$$

with  $K_2 = 100$ ,  $r_2 = 2 \times 10^{-2}$ , and  $\mu_2 = 300 \times 10^4$ . We also fix the output elasticity with respect to workspace in the production function at 0.05.

We choose the global productivity factor in order to obtain equilibrium wages close to  $81.1 \times 10^4$  Euro. Therefore, we choose the global productivity factor to be  $1.16 \times 10^4$  in Test 1 and  $0.93 \times 10^4$  in Test 2.

Parameter	Test 1	Test 2
$d$	1	2
$\alpha$	0.21	0.21
$\beta$	0.71	(0.71, 0.05)
$\delta$	0.07	0.07
$\nu$	0.1	0.1
$\rho$	0.1	0.1
$A$	$1.16 \cdot 10^4$	$0.93 \cdot 10^4$
$S_{labour}(w)$	$\frac{6.5}{1 + \exp(2 \cdot 10^{-4}(w - 7 \cdot 10^4))}$	$\frac{6.5}{1 + \exp(2 \cdot 10^{-4}(w - 7 \cdot 10^4))}$
$S_{workspace}(p)$	$\frac{100}{1 + \exp(2 \cdot 10^{-2}(p - 300))}$	$\frac{100}{1 + \exp(2 \cdot 10^{-2}(p - 300))}$
$\eta(k)$	$\frac{\nu}{\sqrt{2\pi} \cdot 9 \cdot 10^4} e^{-\frac{(k - 3 \cdot 10^5)^2}{2(9 \cdot 10^4)^2}}$	$\frac{\nu}{\sqrt{2\pi} \cdot 9 \cdot 10^4} e^{-\frac{(k - 3 \cdot 10^5)^2}{2(9 \cdot 10^4)^2}}$

Table 7.2: Summary of parameters used in Test 1 and Test 2.

### Numerical results.

The table below summarizes the results of the simulations:

	Test 1	Test 2	Data of INSEE
Annual production / Nb of firms ( in $10^4$ Euro)	67.3	67.4	66.4
Total payroll / Nb of firms ( in $10^4$ Euro)	47.8	47.9	47.1
Nb of Employees / Nb of firms	5.88	5.89	5.81
Annual production / Nb of Employees ( in $10^4$ Euro)	11.4	11.5	11.4
Total payroll / Nb of Employees ( in $10^4$ Euro)	8.13	8.13	8.11

Table 7.3: Comparative table for Test 1 and Test 2.

The equilibrium annual wages for both tests are  $w \simeq 8.13 \times 10^4$  Euro. It shows that the employment rate in this sector is

$$\frac{S(w)}{K} \times 100 = 90.6\%.$$

The important outputs of the model are the value function, the distribution the capital of the firms, their optimal consumption, their individual demand on the labour market, and for Test 2 also on the rental market for professionals, and their level of investment. The later figures present these outputs.

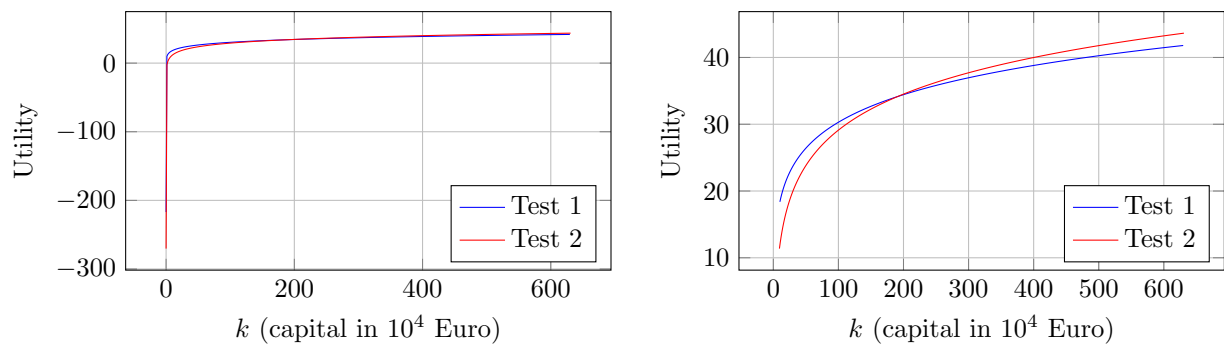


Figure 7.2: Value functions. This is the value functions corresponding to Test 1 and 2. On the left the curves are displayed on the interval  $[10^{-13}, 630]$ , and on the right on  $[10, 630]$ . As proved in Chapter 2, the value function is strictly increasing and concave. It blows up at  $k = 0$ : this comes from the choice of the logarithm as a utility function and the fact that the net output vanishes at  $k = 0$ .

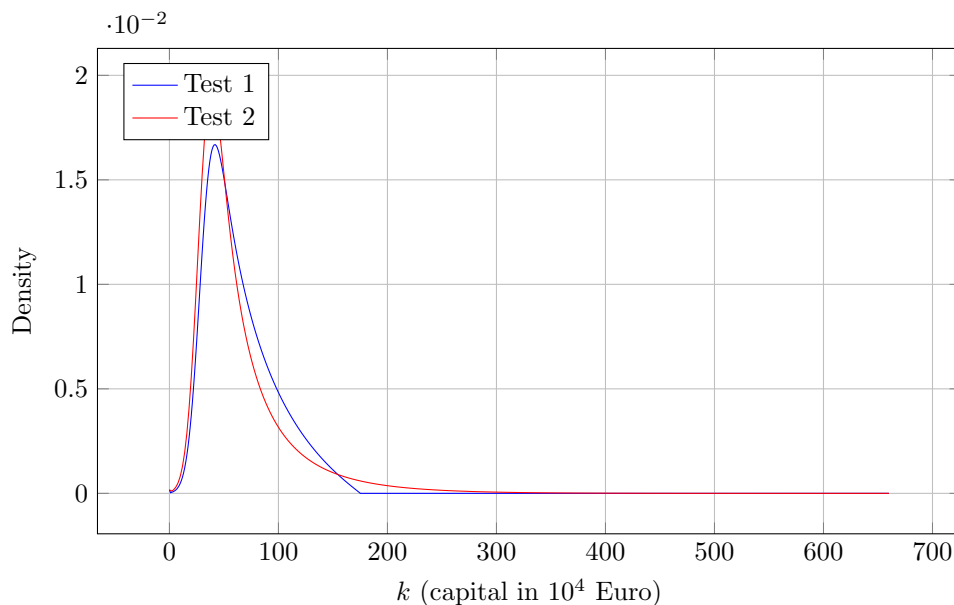


Figure 7.3: Distributions of capital. The source term  $\eta(\cdot)$  (see Figure 7.1) explains the peak on the left of the curves. Since the investment is positive on  $(0, k^*(w))$  with  $k^*(w) \simeq 175 \times 10^4$  Euro for Test 1, and  $k^*(w) \simeq 635 \times 10^4$  Euro for Test 2 (see Figure 7.6), the distribution is shifted to the right. Note that in both cases, the density vanishes at  $k = k^*(w)$ .

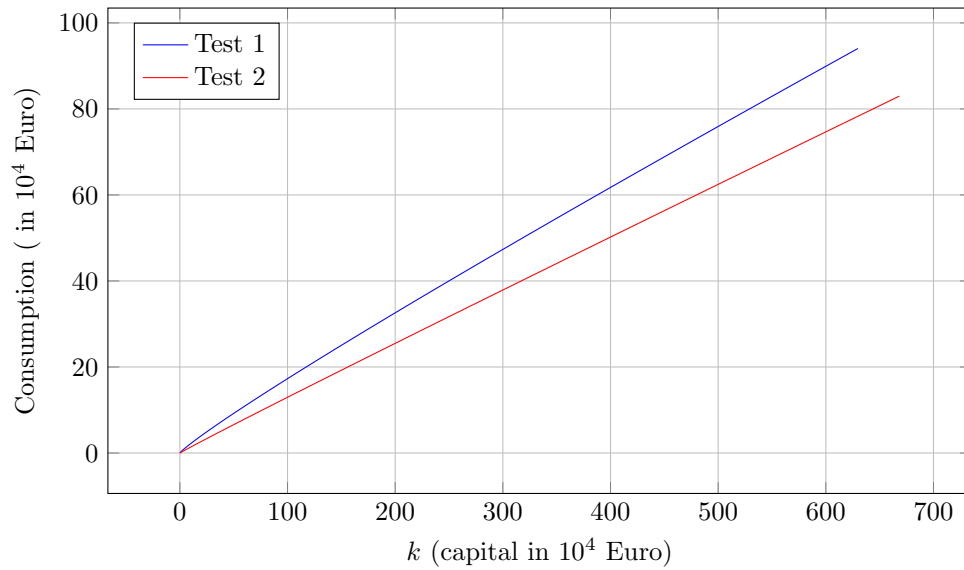


Figure 7.4: Optimal consumption. The optimal consumption is increasing with respect to the capital. This was proved in Chapter 2 and is linked with the concavity of the value function. Indeed, since for every  $k \in (0, +\infty)$ ,  $U'(c(k)) = u'(k)$  where  $c(k)$  is the optimal consumption of the firms with capital  $k > 0$ , then the strict concavity of  $U(\cdot)$  and  $u(\cdot)$  implies that  $c(\cdot)$  is increasing.

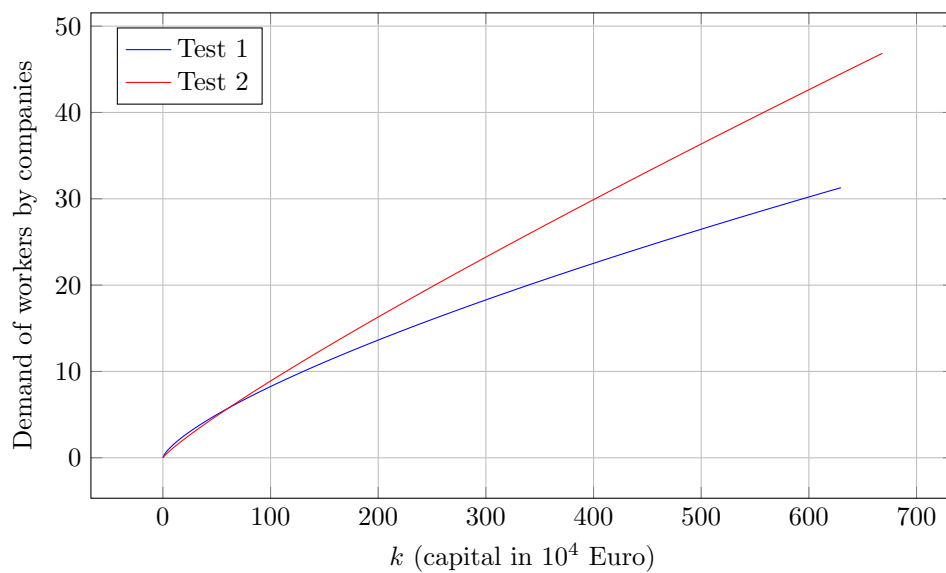


Figure 7.5: Individual labour demand. The individual labour demand is increasing with respect to the capital.

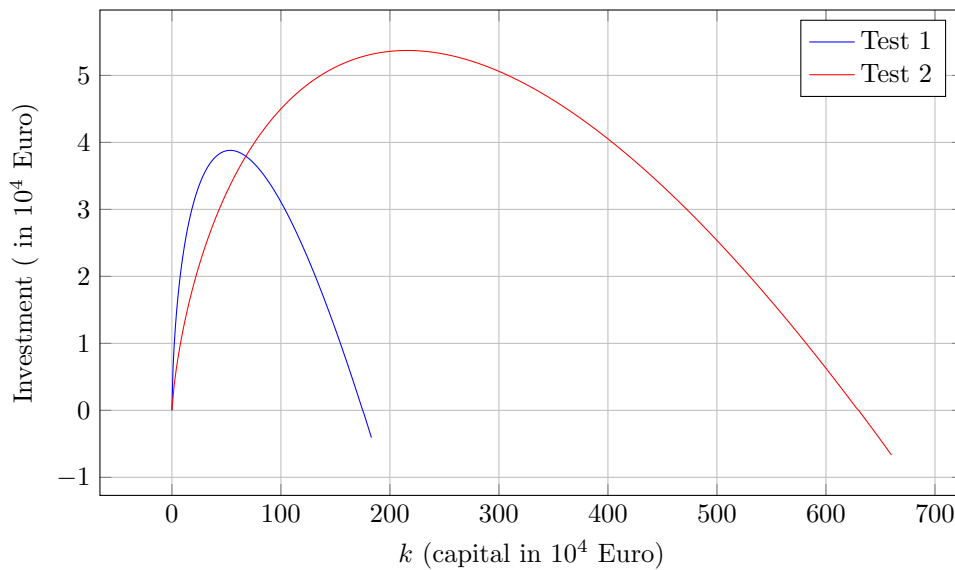


Figure 7.6: Optimal investment. The curves are smooth on the interval  $(0, k^*)$ . They also admit a derivative at  $k^*(w)$ , at least numerically. Since the value of  $k^*(w)$  varies much from Test 1 to Test 2, the investment policy differ very much in both tests. However, in average a company in Test 1 accumulates  $3.47 \times 10^4$  Euro in capital and  $3.39 \times 10^4$  Euro in Test 2. The relative variation is of 2.31%.

Concerning the rental market for professionals (Test 2), we see that at equilibrium, the rental price per square meter for a year is 391 Euro. We also see that 86.1% of the available workspace is used. Moreover, we observe that the workspace divided by the number of employees is constant with respect to the capital of firms. This is a consequence of the choice of the Cobb-Douglas production function and the optimality conditions which occur in the determination of the net output  $f(\cdot)$ . This ratio is equal to 14.6 in this simulation.

From the outputs, it is possible to extract useful data such as the distribution of firms with respect to the number of employees (see the Figure 7.7 below).

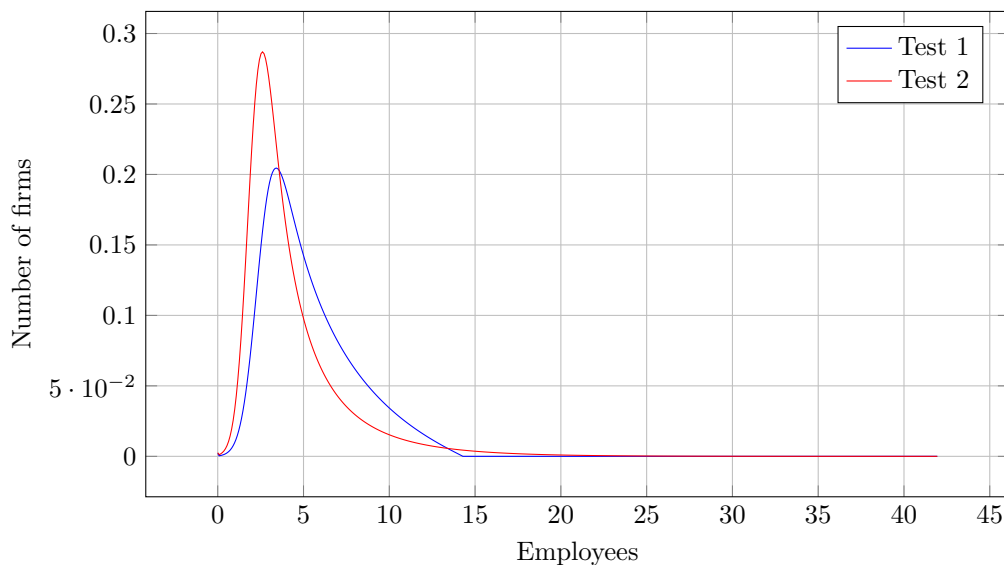


Figure 7.7: Distribution of firms with respect to the number of employees. We observe that the sizes of

the firms in term of number of employees are similar in both tests.

### Convergence

Concerning the convergence, we plot the relative residual  $\hat{r}_n$ , where  $n$  refers to the  $n^{\text{th}}$  iteration, defined by

$$\hat{r}_n = \frac{\Lambda(w^n)}{\Lambda(w^0)}.$$

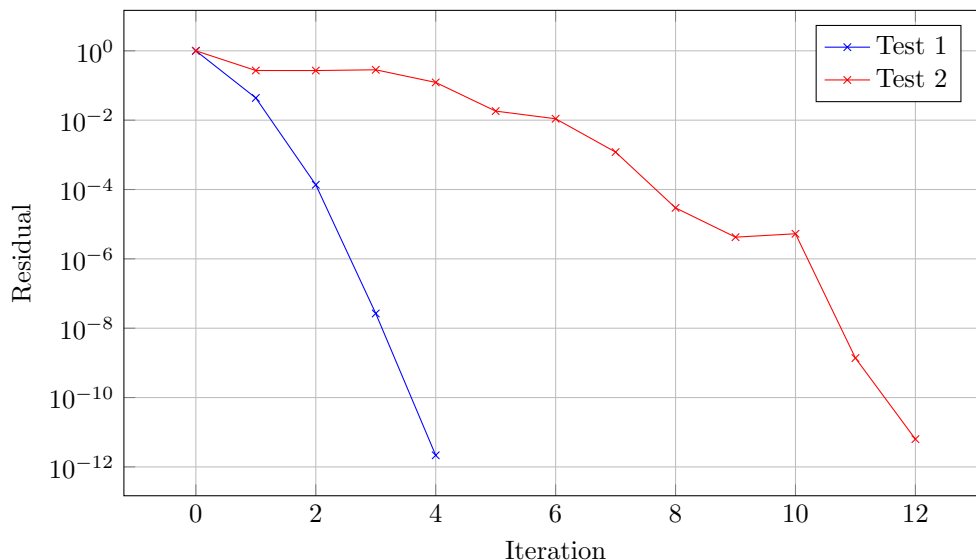


Figure 7.8: Convergence. To find the equilibrium, we used the secant method in Test 1 and the Broyden's method in Test 2. Note that the convergence is super-linear in both cases and slower in Test 2.

### 7.3.2 A model with a labour market composed of two types of workers linked to the rental market for professionals

We consider that the labour is composed of two types of workers, for a schematic example, the white and blue-collars. We assume that the first type of workers is more productive than the second. We essentially take the same parameters as for Test 1 and 2 (see Table 7.2) with changes in the labour supply and the elasticity of labour. Namely, concerning the labour supply we take Example 5.2 and Remark 5.2 in Chapter 5 where we define the utility of an agent without noise by

$$\int_0^{+\infty} \ln(w_i) e^{-\rho t} dt - \ln(1 + c_i) = \frac{\ln(w_i)}{\delta} - \ln(1 + c_i),$$

where  $i \in \{1, 2\}$ ,  $\delta > 0$  is the discount factor and  $c_i$  corresponds to the cost of education. We choose to fix

$$c_1 = 15000, \text{ and } c_2 = 0.$$

Therefore, the labour supply of type  $i$  is given by

$$S_i(w) = K \frac{\frac{w_i^{\frac{1}{\delta}}}{1+c_i}}{w_0^{\frac{1}{\delta}} + \frac{w_1^{\frac{1}{\delta}}}{1+c_1} + \frac{w_2^{\frac{1}{\delta}}}{1+c_2}},$$

where  $w_0 = 3 \times 10^4$  Euro is the minimum wages an agent get if he chooses to work from home as an independent. The table below summarizes the parameters used for Test 3.

Parameter	Test 3
$d$	3
$\alpha$	0.21
$\beta$	(0.41, 0.3, 0.05)
$\delta$	0.07
$\nu$	0.1
$\rho$	0.1
$A$	$10.73 \cdot 10^4$
$K$	5.81
$c_1$	0
$c_2$	$1.5 \cdot 10^4$
$w_0$	$3 \cdot 10^4$
$S_1(w)$	$K \frac{w_1^{1/\delta} / (1+c_1)}{w_0^{1/\delta} + w_1^{1/\delta} / (1+c_1) + w_2^{1/\delta} / (1+c_2)}$
$S_2(w)$	$K \frac{w_2^{1/\delta} / (1+c_2)}{w_0^{1/\delta} + w_1^{1/\delta} / (1+c_1) + w_2^{1/\delta} / (1+c_2)}$
$S_{workspace}(p)$	$\frac{100}{1 + \exp(2 \cdot 10^{-2}(p-300))}$
$\eta(k)$	$\frac{\nu}{\sqrt{2\pi} \cdot 9 \cdot 10^4} e^{-\frac{(k-3 \cdot 10^9)^2}{2(9 \cdot 10^4)^2}}$

Table 7.4: Summary of parameters used Test 3.

**Numerical results** The table below summarizes the results of the simulation:

	Test 3	Data of INSEE
Annual production / Nb of firms ( in $10^4$ Euro)	66.3	66.4
Total payroll / Nb of firms ( in $10^4$ Euro)	47.1	47.1
Nb of Employees / Nb of firms	5.81	5.81
Annual production / Nb of Employees ( in $10^4$ Euro)	11.4	11.4
Total payroll / Nb of Employees ( in $10^4$ Euro)	8.10	8.11

Table 7.5: Comparative table for Test 3.

The equilibrium wages are respectively 101713 Euro for the white-collars and 63395 Euro for the blue-collars. In average, there is 2.67 white-collars and 3.14 blue-collars by company. In this simulation, every worker is hired by a firm since the minimum wages  $w_0$  is too low to be attractive. Thus, to have in average 5.81 workers by company, we imposed  $K = 5.81$ . The rental price on the rental market for professionals is 388 Euro.

Note that the important outputs of the model for Test 2 and Test 3 are similar, we do not plot them for this test. In particular, the target capital  $k^*(w)$  for both tests are close: it respectively equals to  $635 \times 10^4$  Euro and  $610 \times 10^4$  Euro.

### 7.3.3 Behaviour of the distribution of capital near $k^*(w)$ .

We saw in Test 1 and 2 that the distribution of capital vanishes at point  $k^*(w)$ . The purpose of this test is to highlight the fact that it is not true in general.

Let  $w$  be the wages of the employees at equilibrium. We already know from Lemma 2.10 of Chapter 2 that the distribution of capital does not exhibit a Dirac mass at  $k^*(w)$ . Indeed, the investment decreases at least as fast as an affine function which vanishes at  $k^*(w)$ . This yields that the capital of firms can not reach the level  $k^*(w)$  in finite time. Formally, two behaviours may be expected:

1. The distribution  $m(\cdot)$  has a finite limit when  $k$  tends to  $k^*(w)$ .
2. The distribution  $m(\cdot)$  blows up as  $k \rightarrow k^*(w)$ .

We take the same parameters as in Test 1 expect the rate  $\nu$  whose value varies between 0.04 and 0.14. Figure 7.9 shows the distribution  $m(\cdot)$  for different values of  $\nu$ . We remark that the density of capital near  $k^*(w)$  increases as  $\nu$  decreases, and blows up at  $k^*(w)$  for  $\nu$  small. In other words, the less the rate  $\nu$ , the more firms have their capital close to  $k^*(w)$ .

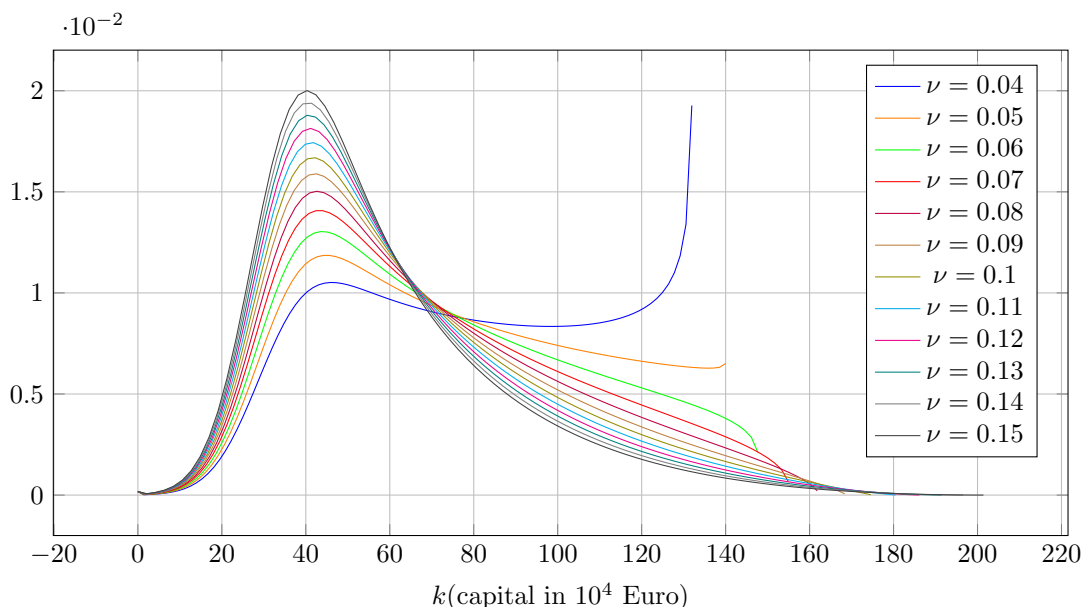


Figure 7.9: Distribution of capital

Let us state a proposition which explains this behaviour in a case where the investment is regular:

**Proposition 7.1** *Under Assumption 2.1 and 2.2, or 2.1 and 2.3, let us fix  $w \in (0, +\infty)^d$ . Let  $u(\cdot)$  be the unique classical solution of (7.1), see Chapter 2. Let us assume furthermore that the investment  $b(\cdot) = D_q H(\cdot, u'(\cdot))$  is continuously differentiable on  $(0, k^*]$  where  $k^*$  is defined by equation (7.7) with  $b'(k^*) < 0$ . Moreover, we assume that  $\eta : [0, +\infty) \rightarrow [0, +\infty)$  is continuous with compact support and such that  $\max \text{supp } \eta(\cdot) < k^*$ . In this case:*

1. *If  $b'(k^*) + \nu < 0$ , then  $m(\cdot)$  blows up at  $k^*$ .*
2. *If  $b'(k^*) + \nu = 0$ , then  $m(\cdot)$  admits a finite limit at  $k^*$ .*
3. *If  $b'(k^*) + \nu > 0$ , then  $m(\cdot)$  vanishes at  $k^*$ .*

**Proof** Let us fix  $\varepsilon > 0$  such that  $\eta \equiv 0$  on  $(k^* - \varepsilon, k^*)$ . Since  $b(\cdot)$  is regular, on  $(k^* - \varepsilon, k^*)$  equation (7.2) becomes:

$$b(k)m'(k) = 0 - (b'(k) + \nu)m(k). \quad (7.12)$$

The solutions of (7.12) have the form

$$(k^* - \varepsilon, k^*) \ni k \mapsto \lambda \exp \left( -(\ln(b(k)) - \ln(b(k^* - \varepsilon))) - \int_{k^* - \varepsilon}^k \frac{\nu}{b(\kappa)} d\kappa \right), \quad (7.13)$$



with  $\lambda \in \mathbb{R}$ . We know that  $\lambda > 0$  since  $m(\cdot)$  is positive on  $(k^* - \varepsilon, k^*)$ . Indeed, let us observe that the restriction of  $m(\cdot)$  on  $[k_0, k^*)$ , where  $k_0 \in (0, k^*)$  is chosen such that  $m(k_0) > 0$ , is a super-solution of

$$\mu'(k) = -\frac{\nu + b'(k)}{b(k)}\mu(k), \quad (7.14)$$

$$(k, \mu(k)) \in (k_0, k^*) \times \mathbb{R}, \quad (7.15)$$

$$\mu(k_0) = 0. \quad (7.16)$$

On the other hand, the unique solution of (7.14)-(7.16) is given by the zero function. A strong comparison principle (see for example [79]) applied on every interval of the form  $[k_0, k^* - \varepsilon]$  yields that  $m(\cdot) > 0$  on  $(k_0, k^*)$ . Since  $b(k) \sim b'(k^*)(k - k^*)$  near  $k^*$ , then  $m(\cdot)$  behaves like a power function in a neighbourhood of  $k^*$  with exponent  $-(1 + \nu/b'(k^*))$ , i.e there exists a constant  $M > 0$  such that

$$m(k) \sim M(k^* - k)^{-\left(1 + \frac{\nu}{b'(k^*)}\right)},$$

when  $k \rightarrow (k^*)^-$ . □

### 7.3.4 A model in which the aggregate capital is not known a priori

In this test we have the source term  $\eta(\cdot)$  depend on the value function. As in Chapter 6, let us assume that for each period of time, there are new entrepreneurs. We make the assumption that they are heterogeneous in their productivity which is measured by a coefficient  $a \in \mathbb{R}$ . An entrepreneur with a level of capital  $k \in (0, +\infty)$  and a coefficient of productivity  $a \in \mathbb{R}$  has two choices:

1. either to start a new company and obtain a utility  $u(k)$ ,
2. or to invest somewhere else and get a utility  $u_0(k, a)$ .

We assume that these entrepreneurs are distributed randomly on the space  $(0, +\infty) \times \mathbb{R}$  according to a couple  $(K, A)$  of random variables. We assume that  $K$  and  $A$  are independent and that the law of  $K$  (resp.  $A$ ) admits a density  $g_K : (0, +\infty) \rightarrow [0, +\infty)$  (resp.  $g_A : \mathbb{R} \rightarrow [0, +\infty)$ ). Therefore, the probability for an entrepreneur to start his company is

$$\mathbb{P}(u_0(K, A) < u(K)) = \int_{(0, +\infty) \times \mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u(k)\}} g_K(k) g_A(a) dk da.$$

Thus, if the total quantity of new entrepreneurs is  $\nu > 0$ , then the quantity of firms created with capital  $k$  is given by

$$\eta(k, u(k)) = \nu \int_{\mathbb{R}} \mathbf{1}_{\{u_0(k, a) < u(k)\}} g_K(k) g_A(a) da. \quad (7.17)$$

In this simulation, we assume that for any  $(k, a) \in (0, +\infty)^2$ ,

$$g_K(k) = C_1 \mathbf{1}_{(0, +\infty)}(k) \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(k-\mu_1)^2}{2\sigma_1^2}}, \quad \text{and} \quad g_A(a) \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(a-\mu_2)^2}{2\sigma_2^2}},$$

where  $C_1 > 0$  is a positive constant (hence  $\int_{\mathbb{R}^2} g_1(k) dk = 1$ ). In practice,  $C_1 \simeq 1$  when  $\sigma_1$  is small enough. We also assume that

$$u_0(k, a) = (C_2 + a) \ln(k) + C_3,$$

with  $C_2 > 0$  and  $C_3$  two constants. The parameters of the model are contained in Table 7.6.

Parameter	Test 5
$d$	2
$\alpha$	0.21
$\beta$	(0.71,0.05)
$\delta$	0.07
$\nu$	0.1
$\rho$	0.1
$A$	$0.93 \cdot 10^4$
$S_{labour}(w)$	$\frac{6.5}{1+\exp(2.10^{-4}(w-7.10^4))}$
$S_{workspace}(p)$	$\frac{100}{1+\exp(2.10^{-2}(p-300))}$
$\eta(k, u(k))$	$\nu g_K(k) \int_{\mathbb{R}} \mathbf{1}_{\{u_0(k,a) < u(k,w)\}} g_A(a) da$
$C_1$	1
$C_2$	15
$C_3$	-31
$\sigma_1$	9
$\mu_1$	30
$\sigma_2$	4
$\mu_2$	15

Table 7.6: Summary: the parameters used for Test 5.

The results are very similar to those in Test 2. The results are summarized in Table 7.7.

	Test 5	Data of INSEE
Annual production / Nb of firms ( in $10^4$ Euro)	66.1	66.4
Total payroll / Nb of firms ( in $10^4$ Euro)	48.0	47.1
Nb of Employees / Nb of firms	5.77	5.81
Annual production / Nb of Employees ( in $10^4$ Euro)	11.2	11.4
Total payroll / Nb of Employees ( in $10^4$ Euro)	8.14	8.11

Table 7.7: Comparative table for Test 5.

The target capital  $k^*(w)$  is found to be close to  $654 \times 10^4$  Euro in this test. The total mass of firms is 1.02. This comes with the fact that 51% of the entrepreneurs start a company.

### 7.3.5 A non local model for the distribution of capital

Let us consider another extension. We now assume that  $\eta(\cdot)$  depend on the distribution of capital  $m(\cdot)$ . As in Chapter 6, we change (7.2) and (7.5) into

$$\frac{d}{dk} \left( D_q H \left( k, \frac{\partial u}{\partial k}, w \right) m \right) = \nu ((id/2)_{\#} m - m), \quad \text{on } (0, +\infty), \quad (7.18)$$

$$m \in \mathcal{P}(0, +\infty), \quad (7.19)$$

It has been proved in Chapter 6 that for a fixed  $w \in (0, +\infty)^d$  this problem admits a unique solution. The numerical results are similar to those obtained in Test 1. They are summarized in Table 7.8.

	Test 6	Data of INSEE
Annual production / Nb of firms ( in $10^4$ Euro)	67.8	66.4
Total payroll / Nb of firms ( in $10^4$ Euro)	48.1	47.1
Nb of Employees / Nb of firms	5.90	5.81
Annual production / Nb of Employees ( in $10^4$ Euro)	11.5	11.4
Total payroll / Nb of Employees ( in $10^4$ Euro)	8.15	8.11

Table 7.8: Comparative table for Test 6.

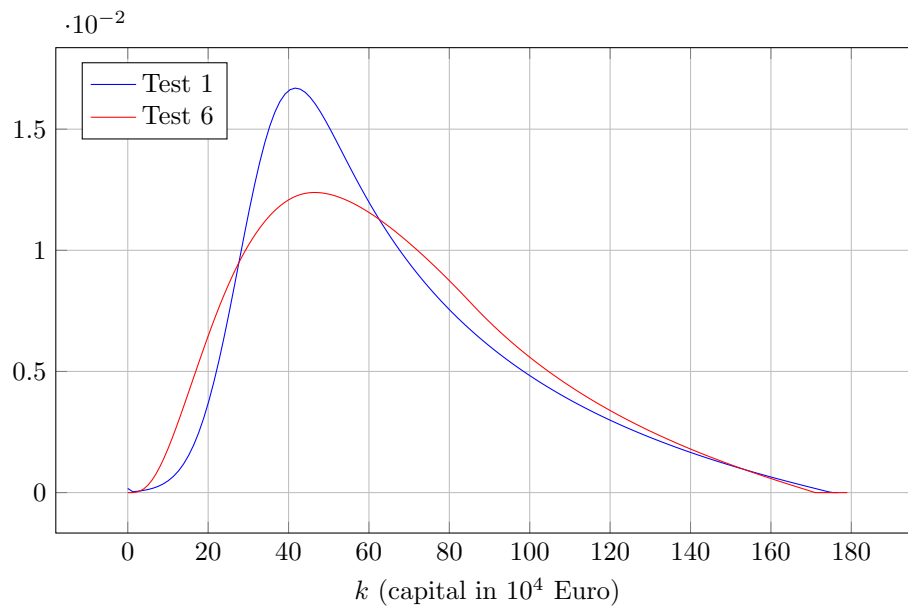


Figure 7.10: Distribution of capital. We observe that the distributions are similar even if the distribution of incoming firms differ from Test 1 and Test 6 as we can see in the figure below.

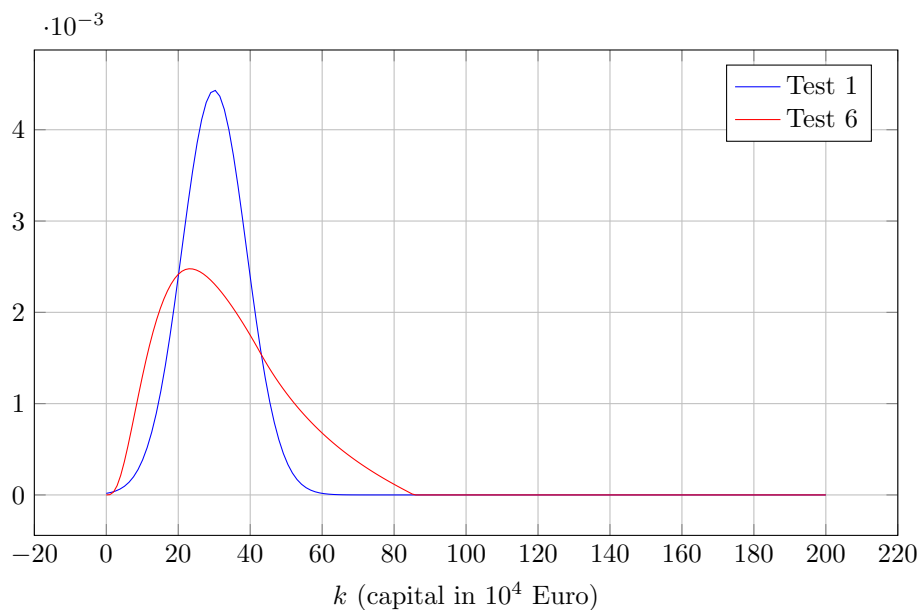


Figure 7.11: Distribution of incoming firms. The distributions of the entries of firms in Test 1 is more concentrated than the one for Test 6. Indeed, in Test 6, the support of the distribution  $m(\cdot)$  is  $[0, k^*(w)]$ , therefore the incoming firms may have capital in the range  $[0, k^*(w)/2]$ .

### Discussion on the numerical results

We observe that the data in the Table 7.1 are well approximated by the simulations. The model admits enough parameters to approach these values. To improve this work, it would be interesting to find more precise data which are more difficult to fit. For instance, the distribution of firms with respect to their size seems relevant. The parameters of the model impact directly the Hamiltonian and indirectly the value function of the optimal control problem. Therefore, the solution of the transport equation computed at equilibrium will be more difficult to fit to the data. Thus, there will be the necessity to find the right form of the production function and the utility function, and then improve the model.

### 7.3.6 Sensitivity with respect to the parameters of the Cobb-Douglas model

We are interested in the impact of the parameters of the model on the computed equilibrium. We present sensitivity when only the labour market is considered ( $d = 1$ ), then when it is coupled with the rental market for professionals ( $d = 2$ ).

#### ► Sensitivity tests for the labour market.

We consider the labour market alone, so  $d = 1$ . For each sensitivity test, we fix every parameter according to the column associated to Test 1 in Table 7.2, except the one for which we study the impact on the equilibrium. We will look at several quantities and each figure in this paragraph will be presented in the same way. A figure will consist of four sub-figures:

1. The first one (on the top left) concerns aggregate quantities:
  - the mean production (—), which is  $\int_0^{+\infty} F(k, -f'(k)) m(k) dk$ .
  - the mean payroll (—), which is  $-w \int_0^{+\infty} \frac{\partial f}{\partial w}(k) m(k) dk$ .
  - the mean consumption (—), which is  $\int_0^{+\infty} (U^*)'(u'(k)) m(k) dk$ .
  - the mean depreciation of capital (—), which is  $\delta \int_0^{+\infty} km(k) dk$ .
  - the mean investment (—), which is  $\int_0^{+\infty} (f(k) - (U^*)'(u'(k))) m(k) dk$ .
2. In the top-right sub-figure, we display the mean capital of a company (—), which is  $\int_0^{+\infty} km(k) dk$ .
3. In the bottom-left sub-figure, we display the wages at equilibrium (—).
4. In the bottom-right sub-figure, we display the rate of employment (—).

**Remark 7.1** *The production is equal to the sum of the payroll, the consumption the depreciation of capital and the investment. Therefore, we can recover the mean production by adding all the curves of the first graph unless the one corresponding to the mean production.*

We model the production of firms by means of a Cobb-Douglas function:

$$F(k, \ell) = Ak^\alpha \ell^\beta, \quad \forall (k, \ell) \in [0, +\infty)^2,$$

where  $A > 0$ ,  $\alpha \in (0, 1)$  and  $\beta \in (0, 1)$  such that  $\alpha + \beta < 1$ . Let us also recall from Chapter 4 that given the wages  $w$ , the optimal labour demand for a company with capital  $k$  is

$$\left( \frac{A\beta k^\alpha}{w} \right)^{\frac{1}{1-\beta}}, \quad (7.20)$$

and the net output is

$$f(k) = (1 - \beta) \left( Ak^\alpha \left( \frac{\beta}{w} \right)^\beta \right)^{\frac{1}{1-\beta}} - \delta k.$$

Note that the labour demand in (7.20) is an increasing function in  $\beta$ . Moreover, since the utility function  $U(c) = \ln(c)$ , it is possible to compute

- the Hamiltonian  $H(k, q) = \sup_{c \geq 0} \{U(c) + q(f(k) - c)\} = -\ln(q) - 1 + qf(k)$ .
- the derivative of the Hamiltonian with respect to  $q$ ,  $D_q H(k, q) = -1/q + f(k)$ .

These formulas are important to understand how the optimal consumption  $c(k)$  and the investment  $f(k) - c(k)$  vary. Indeed, by considering  $u(\cdot)$  the unique classical solution of the HJ equation (7.1), we obtain  $f(k) - c(k) = D_q H(k, u'(k))$ , so

$$c(k) = \frac{1}{u'(k)}.$$

In the following lemma, we compare the decisions of the firms when their productivity grows. For the majority of the tests, the distribution of capital will be supported on  $(0, k^*(w)]$ . Therefore, we focus on the decisions of those firms with a capital below the target capital.

**Lemma 7.1** *Let  $f_i : (0, +\infty) \rightarrow [0, +\infty)$  ( $i = 1, 2$ ) be two net output functions satisfying Assumption 2.2 or 2.3, and  $u_i(\cdot)$  be the unique classical solution of*

$$\rho u_i(k) = H_i(k, u'_i(k)), \quad \forall k \in (0, +\infty), \quad (7.21)$$

where the Hamiltonian  $H_i(k, q) = -\ln(q) - 1 + qf_i(k)$ . If, moreover,  $f_1(\cdot) \leq f_2(\cdot)$  on  $(0, +\infty)$ , then

$$u_1(\cdot) \leq u_2(\cdot), \quad \text{on } (0, +\infty),$$

and

$$u'_1(\cdot) \leq u'_2(\cdot) \quad \text{on } (0, \min(k_1^*, k_2^*)].$$

We recall that  $k_i^*$  is the target capital defined as the unique solution of

$$f'_i(k_i^*) = \rho.$$

**Proof** We start by proving that  $u_1(\cdot) \leq u_2(\cdot)$  on  $(0, +\infty)$ . Since  $H_1(\cdot, \cdot) \leq H_2(\cdot, \cdot)$  on  $(0, +\infty)^2$ , and that  $u_i(\cdot)$  ( $i = 1, 2$ ) is a classical solution of (7.21), we can apply the comparison principle in [15, Theorem 5.8] to deduce that for any real number  $a$  and  $b$  satisfying  $0 < a < b < +\infty$ ,  $u_1(\cdot) \leq u_2(\cdot)$  on  $[a, b]$ . This yields that  $u_1(\cdot) \leq u_2(\cdot)$  on  $(0, +\infty)$ .

Let us now establish that  $u'_1(\cdot) \leq u'_2(\cdot)$  on  $(0, \min(k_1^*, k_2^*)]$ . We recall from Section 2.3 in Chapter 2, that  $u'_1(\cdot) \leq 1/f_1(\cdot) \leq 1/f_2(\cdot)$  and  $u'_2(\cdot) \leq 1/f_2(\cdot)$  on  $(0, \min(k_1^*, k_2^*)]$ . Moreover, by definition,

$$\rho u_i(k) = H_i(k, u'_i(k)), \quad \forall k \in (0, +\infty) \quad (i = 1, 2).$$

Therefore,

$$\rho(u_1(k) - u_2(k)) = H_1(k, u'_1(k)) - H_2(k, u'_2(k)), \quad \forall k \in (0, +\infty).$$

The latter equation with the fact that  $u_1(\cdot) \leq u_2(\cdot)$  yields

$$-\ln(u'_2(k)) + u'_2(k)f_2(k) \leq -\ln(u'_1(k)) + u'_1(k)f_1(k), \quad \forall k \in (0, +\infty).$$

The positivity of  $u'_1(\cdot)$  and the fact that  $f_1(\cdot) \leq f_2(\cdot)$  yield

$$-\ln(u'_2(k)) + u'_2(k)f_2(k) \leq -\ln(u'_1(k)) + u'_1(k)f_2(k), \quad \forall k \in (0, +\infty).$$

Note that  $(0, 1/f_2(k)] \ni q \mapsto -\ln(q) + qf_2(k)$  is decreasing. Thus, for every  $k \in (0, \min(k_1^*, k_2^*)]$ ,  $u'_1(k) \leq u'_2(k)$ .  $\square$

**Remark 7.2** Let us assume that the firms become more productive (i.e.  $f(\cdot)$  increases). The latter lemma suggests that they individually tend to increase their investment and decrease their consumption. Indeed, the optimal consumption of a company with capital  $k$  is given by  $1/u'(k)$ . Thus, the latter lemma yields that the gain of production implies an increase of  $u'(\cdot)$  on an interval of the form  $(0, a]$ , with  $a > 0$ . Therefore, the consumption of firms with the level of capital  $k \in (0, a]$  will decrease.

Assuming that most of the mass of the distribution of capital belongs to the interval  $(0, a]$  before and after the gain of productivity, we already can see two conflicting effects:

1. The firms with a level of capital  $k \in (0, a]$  will decrease their consumption and will increase their investment. This effect can lead to think that the mean consumption, i.e.

$$\int_0^{+\infty} \frac{1}{u'(k)} dm(k) \quad (7.22)$$

will decrease and the mean capital, i.e.

$$\int_0^{+\infty} k dm(k)$$

will increase since the distribution of capital will be shifted on the right.

2. However, if the distribution of capital is shifted on the right, then the mean consumption given by (7.22) should increase since the individual consumption given by  $1/u'(\cdot)$  is an increasing function of the capital.

**Sensitivity with respect to  $\alpha$  the output elasticity with respect to capital.** We fix every parameter of the model but the output elasticity of capital  $\alpha$ , whose value will vary between 0.15 to 0.25.

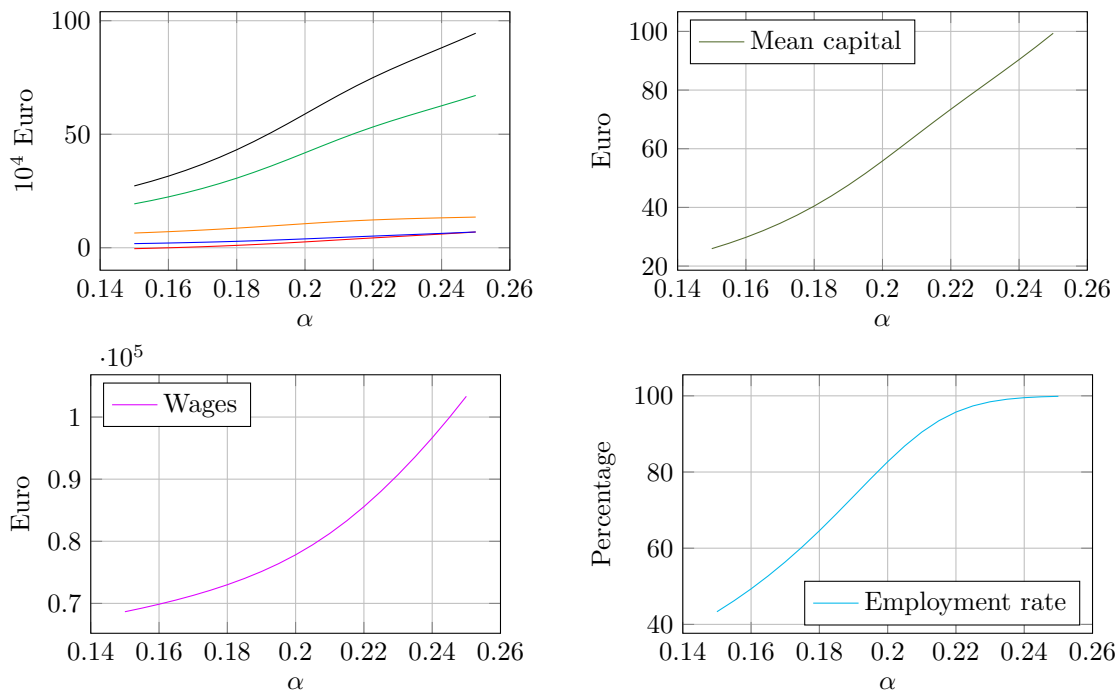


Figure 7.12: Sensitivity with respect to  $\alpha$ . If  $\alpha$  increases, then the capital becomes more productive. As a result the production and the net output  $f(\cdot)$  increase. The hints given by lemma 7.1 yield that

the mean investment (—), and the mean capital (—) increase. Note that the mean consumption (—) is increasing. So the prevailing effect is the increase in mean capital rather than the local decrease of consumption. The fact that  $\alpha$  and the mean capital increase imply an increase in the labour demand. Thus, the wages increase (—), and the employment rate increases (—).

**Sensitivity with respect to  $\beta$  the output elasticity with respect to labour.** This test is exactly the same as before, we only consider the output elasticity with respect to labour rather than capital. Thus,  $\beta$  will vary between 0.65 to 0.75.

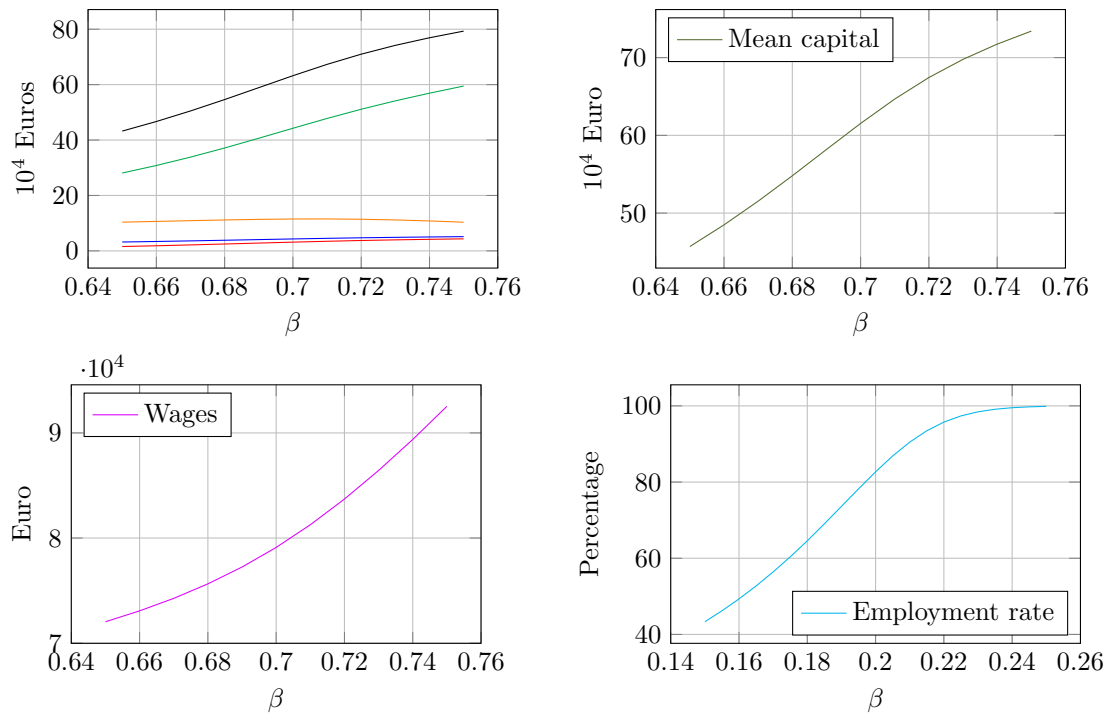


Figure 7.13: Sensitivity with respect to  $\beta$ . If  $\beta$  increases, then the labour becomes more productive. As a result the production and the net output  $f(\cdot)$  increase. The hints given by lemma 7.1 yield that the mean investment (—), and the mean capital (—) increase. Note that the mean consumption (—) is increasing until the value  $\beta = 0.72$ , then it decreases. Thus, at first, the prevailing effect is the increase in mean capital rather than the local decrease of consumption, and when  $\beta \geq 0.72$  it is the opposite. The fact that  $\beta$  and the mean capital increase implies an increase in the labour demand. Thus, the wages increase (—), and the employment rate increases (—).

**Sensitivity with respect to the labour supply.** The labour supply is modelled by a function  $S_{labour}(\cdot)$  defined in Table 7.2. Let us introduce a real number  $\lambda$  which ranges 0.8 to 1.2. We run the test when the labour supply is modelled by  $S_{\lambda}(\cdot) = \lambda S_{labour}(\cdot)$ .

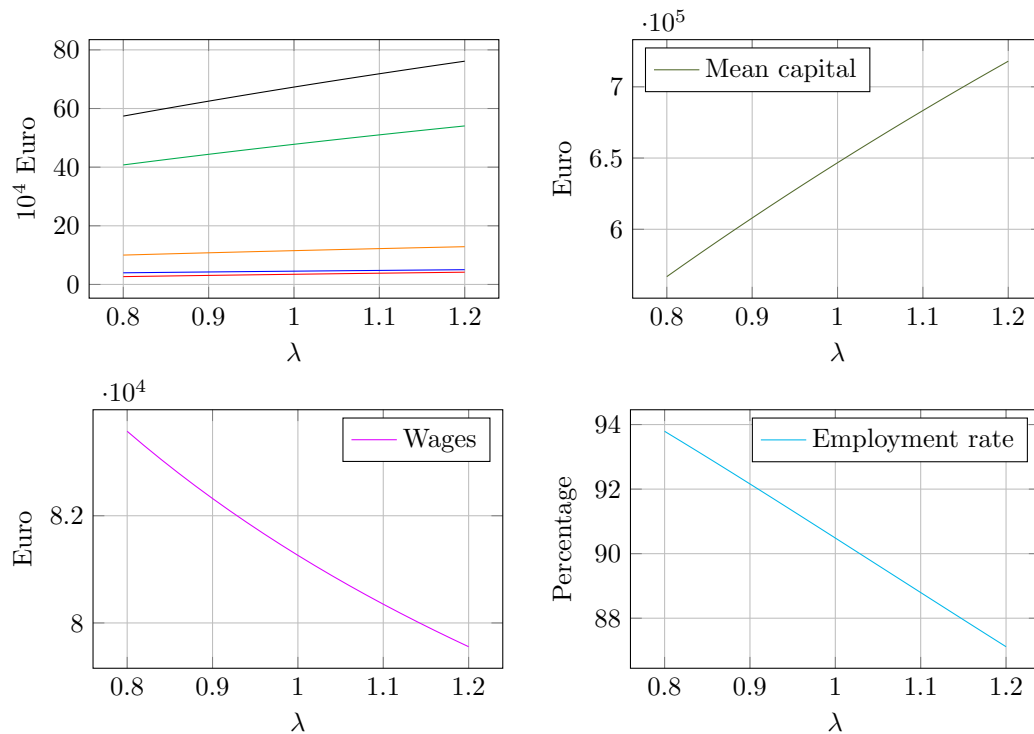


Figure 7.14: Sensitivity with respect to the labour supply. If  $\lambda$  increases, then the labour supply increases. This implies a decrease of the wages (—), and of the employment rate (—). Since the wages drop each firm becomes more productive. So, as before, the net output increases. This yields an increase of investment (—), and then the mean capital by company increases (—). Note that the mean consumption (—) is increasing. So the prevailing effect is the increase in mean capital rather than the local decrease of consumption. Moreover, we observe that the total payroll (—) increases due to the increase in the total amount of workers. Indeed, for  $\lambda = 1$  the employment rate is  $\sim 90.5\%$ . An increase of 20% in the labour supply ( $\lambda = 1.2$ ) implies only a decrease of  $\sim 3.5\%$  of employment. Hence the total number of workers increases.



**Sensitivity with respect to  $\nu$  the death rate of company.** In this test  $\nu$ , the death rate of company, will vary between 0.05 to 0.2.

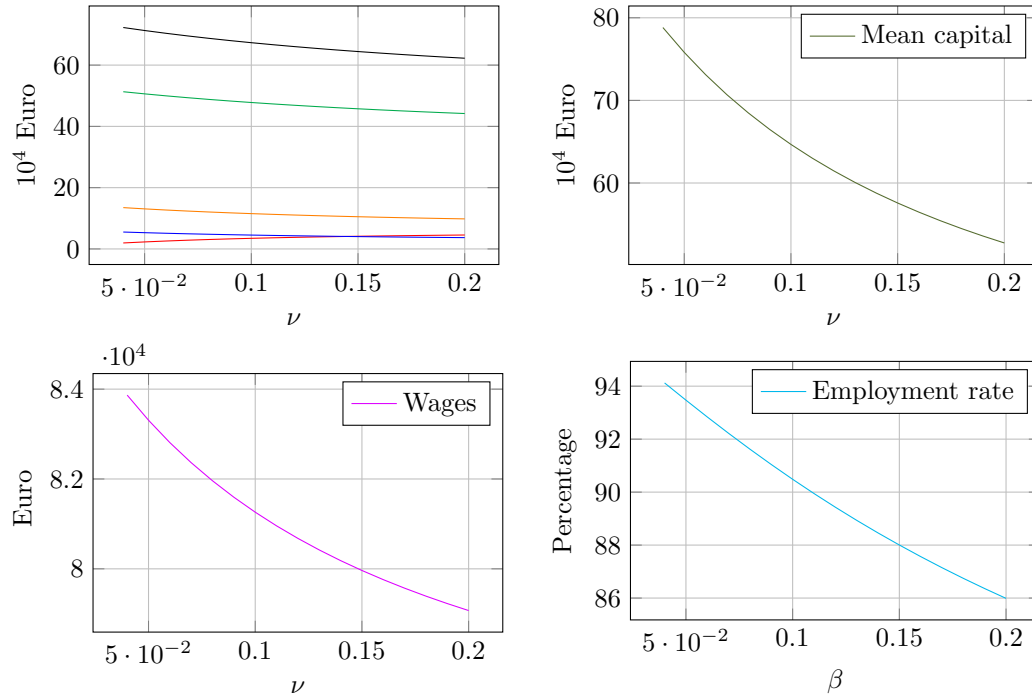


Figure 7.15: Sensitivity with respect to  $\nu$ . If  $\nu$  increases, then the probability for a company to die increases. This implies that at the equilibrium the distribution of capital is shifted to the left as we saw it in Figure 7.9. Therefore the mean capital (—) decreases. This implies that firms are less productive in average and so the total demand for labour decreases. Thus, the wages decrease (—), which implies an decrease of the employment rate (—). On the other hand, since consumption is an increasing function of capital, the mean consumption decreases (—). We observe that the investment grows (—), so the consumption decreases faster than the net output of a company in average.

**Sensitivity with respect to  $\rho$  the discount factor.** We focus now on  $\rho$  the discount factor. It appears in the definition of the value function  $u(\cdot)$  in (2.3) that when  $\rho$  increases, then the present becomes more important and the future less important. The parameter  $\rho$  will vary between 0.05 to 0.2

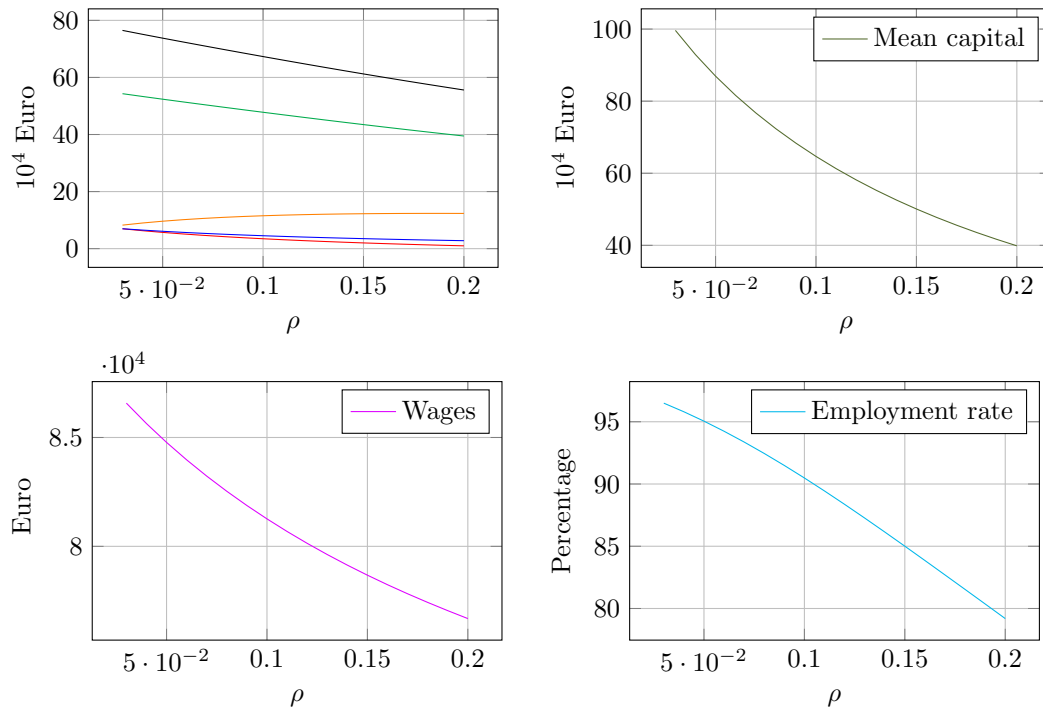


Figure 7.16: Sensitivity with respect to  $\rho$ . If  $\rho$  increases, then the decision making process of the firms gives more importance to the near future, and this implies an increase in consumption (—) and a decrease of the investment (—) in average. Therefore, the total amount of capital decreases (—) and the firms are in average less productive. Thus, the total demand for labour decreases which implies that the total payroll (—), the wages (—), and the employment rate(—) decrease.

**Sensitivity with respect to  $A$  the global factor of productivity.** Let us study the sensitivity of the equilibrium with respect to the parameter  $A$ , the global factor of productivity. Let us introduce a real number  $\lambda_A$  which will vary between 0.8 to 1.2. We run the test when the global factor of productivity is  $\lambda_A A$ .

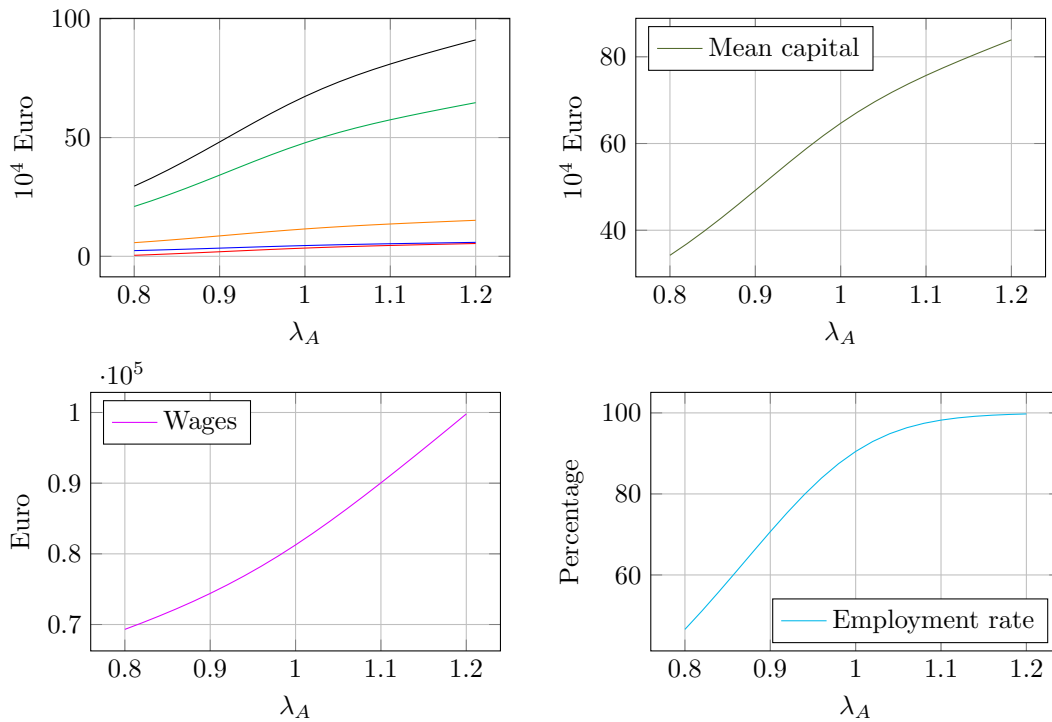


Figure 7.17: Sensitivity with respect to  $A$ . If  $A$  increases, then firms becomes more productive, and the monotonicity of the curves can be explained in the same way as in the sensitivity test with respect to  $\alpha$ .

**Conclusion.** This paragraph permits to understand how the equilibrium is impacted by the parameters of the model. Lemma 7.1 gives a theoretical tool to understand the behaviour of the key quantities of the economy. However, as noticed in Remark 7.2 there may be conflicting effects. In this case, the numerical computations permit to show which one has more impact than the others.

Let us summarize the main observations:

- when the production increases, then
  1. the mean investment increases.
  2. the mean capital increases.
  3. the local consumption decreases.
- when the mean capital increases, then
  1. the mean consumption increases.
  2. the labour demand increases

Then, as stated in Remark 7.2 the monotonicity of the consumption is unclear. Nevertheless, we observe that when the production increases, it is more likely that the mean consumption grows as well. It means that the increase of consumption due to the increase of the mean capital compensate the decrease of the local consumption. However, it is important to moderate this statement since the sensitivity test with respect to  $\beta$  (see Figure 7.3.6) provides a counterexample.

### ► Sensitivity tests for the combined labour market and the rental market for professionals

The latter paragraph was aimed at understanding the impact of the parameters of the model on the equilibrium. We now want to understand how the labour market and the rental market for professionals are linked. Note that the labour supply and the workspace supply are independent, i.e. the supply in one market does not depend on the prices on the other market. The curves presented below deal with the sensitivities of the wages and the rental prices. The parameters that we do not modify are the same as those used in Test 2 presented in Table 7.2.

**Sensitivity with respect to the labour supply.** As before, we model the labour supply with a function  $S_{labour}(\cdot)$  defined in Table 7.2. Let us introduce a real number  $\lambda$  which will vary between 0.01 to 1.2. We run the test when the labour supply is modelled by  $S_{\lambda}(\cdot) = \lambda S_{labour}(\cdot)$ .

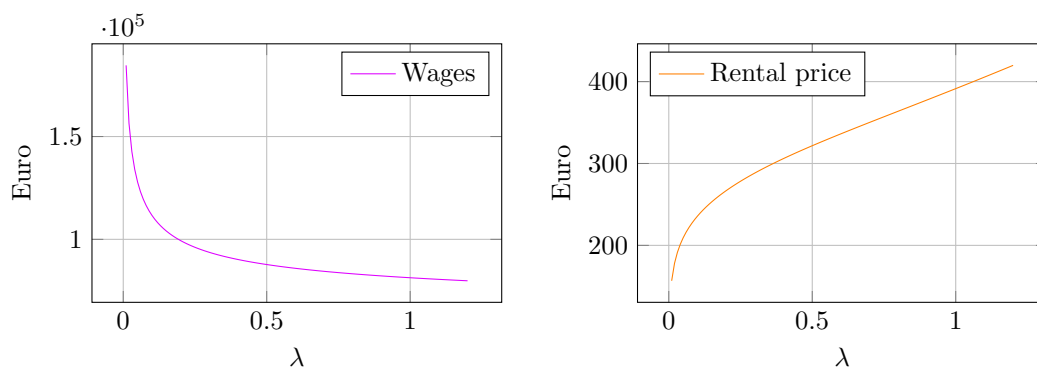


Figure 7.18: Sensitivity with respect to  $\lambda$ . If  $\lambda$  increases, then the labour supply increases. Thus, the market clearing conditions imply a decrease of the wage. Therefore, firms are more productive which yields an increase of the demand for workspace, which increases the rental price. The wages explode when  $\lambda \rightarrow 0^+$ , while the rental price should vanish. However, we observe that this behaviour is slow since for the value  $\lambda = 0.01$  the wages are 184618 Euro and the rental price is 157 Euro.

**Sensitivity with respect to the workspace supply.** We model the workspace supply with a function  $S_{workspace}(\cdot)$  defined in Table 7.2. Let us introduce a real number  $\lambda$  which will vary between 0.01 to 1.2. We run the test when the workspace supply is modelled by  $S_{\lambda}(\cdot) = \lambda S_{workspace}(\cdot)$ .

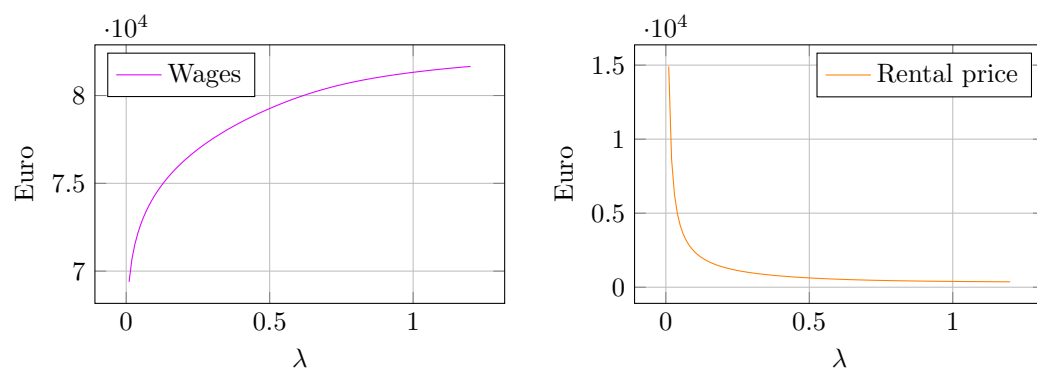


Figure 7.19: Sensitivity with respect to  $\lambda$ . If  $\lambda$  increases, then the workspace supply increases. Thus, the market clearing conditions imply a decrease of the rental price. Therefore, firms are more productive which yields an increase of the demand for labour, which in turn makes the wages grow. Similarly, in

the latter simulation, the rental price explodes when  $\lambda \rightarrow 0^+$ , while the wages should vanish. However, we observe that this behaviour is slow since for the value  $\lambda = 0.01$  the rental price is 14913 Euro and the wages are 69390 Euro.

## 7.4 Conclusion

We have proposed a numerical method and carried out several simulations. In the first set of simulations, the parameters have been chosen in accordance with the economic literature and in order to fit the data supplied by the CONSEIL SUPERIEUR DE L'AUDIOVISUEL and INSEE [39, 63]. Then, we have increased the value of  $d$ , connecting several markets, and showed that the numerical method succeeds in computing an equilibrium. We have found a good agreement between the simulations and the data. However, more data would be needed in order to improve the model, such as the distribution of the size of the firms.

We have observed numerically that at  $k^*$ , the capital distribution may either admit a limit or blow up. In Proposition 7.1, we have characterized the different regimes.

Then, we have showed that the numerical method can be adapted to the extensions of the model proposed in Chapter 6.

We have done comparative statics in order to understand the impact of some parameters on the equilibrium. We have observed that when the production increases, the aggregate investment and capital increase. Moreover, when the aggregate capital increases, the mean consumption and the labour demand increase as well. Finally, we have presented comparative statics in the case when the rental market for the professionals and the labour market are linked. When the supply of one factor (labour or space) tends to zero, the price of the factor blows up while the price of the other factor decays slowly.

## Part II

**A model linking the labour market  
to the rental market for individuals**



## Chapter 8

# A static model for the spatial distribution of residences

The purpose of this chapter is to develop a model aimed at understanding the spatial repartition of residences for the workers in a given region. Indeed, given the position of the workplaces and the demand function associated to each workplace, the model is designed to compute the equilibrium in the labour market as well as the one in the rental market for individuals. In other words, the outputs of the model are

- the collection of wages such that the demand for labour is equal to the supply in each workplace.
- for each position, the number of workers who have a job in each workplace.
- the residential rental price at each position such that the demand for space is equal to the supply.

This model is based on the following general assumptions, which are close to the ones we usually make in a mean field game setting (see for instance the introduction of [56]):

1. There is a continuum of agents or workers which are rational and indistinguishable.
2. The agents can freely choose where they live (mobility assumption).
3. The agents have individually no impact on the global system (mean field assumption).
4. There is a finite number of workplaces.
5. The equilibrium is reached when the following conditions are simultaneously satisfied:
  - the market clearing conditions on the labour and rental market for individuals hold.
  - no agent has any interest in moving.

Assumption 1 and 2 specify the behaviour of workers or agents, while the mean field assumption simplified the interactions between agents as it is done in the mean field game theory introduced by Lasry and Lions [68, 69, 70], and independently by Caines, Huang and Malhamé [62, 61, 60]. Assumption 4 simplified a lot the labour market since the market clearing conditions will be interpreted as a convex optimization problem in finite dimension. As in Part 1, the rental market is linked to the labour market. However, in Part 1, indoor surface was considered as a factor of production. In this model, it contributes to the well-being of people.

The model possess several good features. First, from a theoretical point of view, we are able to show the existence of equilibria, and under more restrictive assumptions its uniqueness. Indeed, under the assumptions introduced in Section 8.2, we prove that given a distribution of residences, the equilibrium



in the labour market can be interpreted as a convex optimization problem. This allows us to establish a priori bound on the collection of wages which satisfies the market clearing conditions on the labour market. On the other hand, given a collection of wages, we find that the distribution which satisfies the equilibrium in the rental market for individuals and the mobility assumption has an explicit formula, as well as the rental price. This allows us to use a fixed-point strategy to establish the existence of equilibria. Concerning the uniqueness, the explicit forms with the market clearing conditions on the labour market allow us to characterize an equilibrium as a solution of a non-linear system posed in finite dimension (the dimension is exactly the number of workplaces). Under more restrictive assumptions, this system admits at most one solution. Second, the model is fairly simple. Indeed, the distribution of residences and the rental price admit explicit forms. Moreover, since an equilibrium can be characterized as a solution of a non-linear system posed in finite dimension the numerical method to approach it will be simple. Third, it models interesting situations. For instance, to model the structure of a city, the circular assumption is not needed. This is an important assumption made in several papers such as [22, 88, 75] to facilitate the analysis of the model which is not necessary here. On the other hand, it is possible to introduce several workplaces. For instance, in [75] the business has different locations and is not only concentrated in the Central Business district (or CBD). Fourth, the model facilitates extensions. For instance, it is possible to study the impact of the home-based telecommuting. If we assume that firms can hire workers who work on-site and others at home, then we can see how the distribution of residences is impacted.

The chapter is structured as follows. In Section 8.3 we present the model specifying the behaviour of agents and introducing the mathematical definition of an equilibrium. Then, in Section 8.2 we present the standing assumptions used throughout the chapter. The sections 8.3 and 8.4 deal with the existence and uniqueness of equilibria respectively. We conclude the chapter by presenting some extensions of the model in Section 8.5.

**Notations.** Throughout the chapter, for every  $D \in \mathbb{N}$  and vector  $z \in \mathbb{R}^D$  we denote by  $z_i$  the  $i^{\text{th}}$  coordinate of  $z$  and

$$\|z\|_{\infty} = \max_{i=1}^D |z_i|.$$

The measure of a Lebesgue-measurable set  $E$  is denoted by  $|E|$ . We also consider that for every set  $X$  which is the closure of a bounded domain of  $\mathbb{R}^d$ , that  $\mathcal{P}_{ac}(X)$  is the set of probability measures on  $X$  admitting a density with respect to the Lebesgue measure. The notation *a.e.* will stand for *almost everywhere*. Note that *almost everywhere* will always implicitly mean *almost everywhere with respect to the Lebesgue measure*.

## 8.1 The model

Let us introduce  $X$  the closure of a bounded domain of  $\mathbb{R}^d$ :  $X$  is the set of all possible places of residence. We assume that there are  $N \in \mathbb{N}^*$  workplaces. For any  $i \in \{1, \dots, N\}$ , two functions are associated to the  $i^{\text{th}}$  workplace:

1. The transportation costs  $c_i : X \rightarrow [0, +\infty)$ .
2. The labour demand  $L_i : [0, +\infty) \rightarrow [0, +\infty)$ .

Here,  $c_i(x)$  corresponds to the transport cost to reach the  $i^{\text{th}}$  workplace coming from  $x \in X$ , and  $L_i(w_i)$  corresponds to the labour demand when the wages in the  $i^{\text{th}}$  workplace is  $w_i \in (0, +\infty)$ .

### Utility

Let us consider a generic agent. Let  $R$  and  $Q$  be two non-negative real numbers corresponding respectively to the revenue of the agent and the rental price by surface unit.

Let  $\theta$  be a number in  $[0, 1]$ . We assume that the utility of a given agent is

$$\mathcal{U}_\theta(R, Q) = \sup \{ C^\theta S^{1-\theta} : C + QS \leq R, C \geq 0, S \geq 0 \}, \quad (8.1)$$

where  $C$  and  $S$  are variables which respectively stand for the level of consumption and the surface of the residence.

**Lemma 8.1** *For any  $(R, Q) \in (0, +\infty)^2$ , the optimal consumption and demand of surface are respectively given by*

$$C_\theta(R) = \theta R, \quad \text{and} \quad S_\theta(R, Q) = (1 - \theta) \frac{R}{Q}. \quad (8.2)$$

**Corollary 8.1** *For any  $(R, Q) \in (0, +\infty)^2$ , the utility of an agent is given by*

$$\mathcal{U}_\theta(R, Q) = \theta^\theta (1 - \theta)^{1-\theta} \frac{R}{Q^{1-\theta}} \quad (8.3)$$

## Revenue

We assume that given a collection of wages  $w \in (0, +\infty)^N$ , agents at the position  $x \in X$  can

- either choose to work at the  $i^{\text{th}}$  workplace and receive the income  $w_i - c_i(x)$ ,
- or choose to stay at home and receive  $w_0 > 0$ .

From Corollary 8.1, the utility defined in (8.1) is an increasing function of the revenue. Therefore, given a collection of wages  $w \in (0, +\infty)^N$ , agents living in  $x \in X$  will choose the workplace  $i$  that maximizes the difference  $w_i - c_i(x)$ . Thus, the revenue of an agent is defined by

$$R(x, w) = \max_{i=0}^N (w_i - c_i(x)), \quad (8.4)$$

with  $c_0(\cdot)$  constant equals to 0. The maximum in (8.4) is a source of irregularities and mathematical difficulties. We will treat this case in section 8.5. Therefore, let us consider a regularization:

$$R_\sigma(x, w) = \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right), \quad (8.5)$$

where  $\sigma > 0$  measures the noise on the effective wages. Indeed, we assume that choosing the workplace indexed by  $i$  an agent will earn

$$w_i + a_i,$$

where  $a_i$  is a random variable following the Gumbel law with parameters  $-\sigma\gamma$  and  $\sigma$ , more details are provided in Appendix 8.A. Moreover, Proposition 8.2 in Appendix 8.A states that  $R_\sigma(\cdot, \cdot)$  is an approximation of  $R(\cdot, \cdot)$ . In this setting, the probability for an agent in the position  $x \in X$  to choose the workplace indexed by  $i$  is given by the Gibbs distribution:

$$\frac{\partial R_\sigma}{\partial w_i}(x, w) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}}, \quad (8.6)$$

as we can see it in Appendix 8.A.

Before giving the definition of equilibria, we define the labour supply and the demand of surface.

**Definition 8.1** *For any distribution of residences  $\mu \in \mathcal{P}(X)$  and any collection of wages  $w \in (0, +\infty)^N$ , the labour supply for the  $i^{\text{th}}$  workplace is given by*

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x). \quad (8.7)$$

**Definition 8.2** For any distribution of residences  $\mu \in \mathcal{P}(X)$ , any collection of wages  $w \in (0, +\infty)^N$  and rental price function:  $Q : X \rightarrow [0, +\infty)$ , the demand of surface is given by the non negative measure

$$S_\theta(R_\sigma(\cdot, w), Q(\cdot))\mu.$$

We are now ready to define the notion of equilibrium:

**Definition 8.3** We say that the triplet

$$(w, Q(\cdot), \mu) \in (0, +\infty)^N \times C(X) \times (C(X) \cap \mathcal{P}(X))$$

is an equilibrium if

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x) = L_i(w_i), \quad \forall i \in \{1, \dots, N\}, \quad (8.8)$$

$$S_\theta(R_\sigma(x, w), Q(x))\mu(x) = 1, \quad \forall x \in X, \quad (8.9)$$

$$\int_X \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) d\nu(x) < +\infty. \quad (8.10)$$

In Definition 8.3, (8.8) reflects the equilibrium in the labour market and (8.9) the one on the rental market for individuals where the total supply is normalized to one for convenience. Instead, it is possible to use a heterogeneous supply function  $\varphi : X \rightarrow [0, +\infty)$  continuous and positive. Note that (8.9) must hold on the subset of  $X$  such that the revenue of agents is positive, and the minimum income  $w_0 > 0$  ensures that the revenue is positive on the whole set  $X$ . Moreover, (8.10) comes from the assumption on the mobility of the agents. Indeed, when  $X \ni x \mapsto \mathcal{U}_\theta(R_\sigma(x, w), Q(x))$  is continuous, equation (8.10) is equivalent to

$$\text{supp } \mu \subset \text{argmax}_{x \in X} \mathcal{U}_\theta(R_\sigma(x, w), Q(x)),$$

which ensures that no agent has any interest in moving.

## 8.2 Standing assumptions and main results

We collect here the standings assumptions:

**Assumption 8.1** For every  $i \in \{1, \dots, N\}$ , the labour demand  $L_i : (0, +\infty) \rightarrow (0, +\infty)$  of the  $i^{\text{th}}$  workplace satisfies:

- i)  $L_i(\cdot)$  is continuous.
- ii)  $L_i(\cdot)$  is decreasing.
- iii)  $\lim_{t \rightarrow 0^+} L_i(t) = +\infty$  and  $\lim_{t \rightarrow +\infty} L_i(t) = 0$ .

**Assumption 8.2** For every  $i \in \{1, \dots, N\}$ , the transport cost associated to the  $i^{\text{th}}$  workplace  $c_i : X \rightarrow [0, +\infty)$  is continuous.

Let us summarize our main results as follows.

**Theorem 8.1** (Theorem 8.2, Theorem 8.3) Under Assumption 8.1 and 8.2, there exists an equilibrium  $(w, Q(\cdot), \mu)$  in the sense of Definition 8.3.

Moreover, if for every  $i \in \{1, \dots, N\}$ ,  $L_i(\cdot)$  is of class  $C^1$  on  $(0, +\infty)$ , then for every  $\theta \in [0, \theta_0]$  the equilibrium is unique, where

$$\theta_0 = \frac{\alpha_0(Y)}{1 + \alpha_0(Y)},$$

and

$$\alpha_0(Y) = \min \left\{ \frac{w_0}{2N} \inf_{w \in Y} \min_{i=1}^N (-L'_i(w_i)), 1 \right\},$$

with  $Y$  the compact and convex set introduced in Lemma 8.2 below.

### 8.3 Existence of equilibria

The goal of this paragraph is to establish the existence of equilibria via a fixed-point strategy. We show that it is possible to build a map  $\Upsilon(\cdot)$  such that the fixed-points of  $\Upsilon(\cdot)$  are exactly the equilibria of Definition 8.3. The construction of the function  $\Upsilon(\cdot)$  will be, first, based on the fact that the market clearing conditions of the labour market, namely (8.8), are equivalent to an optimization problem (see Lemma 8.2 below), and that the market clearing conditions on the rental market for individuals and the mobility condition, namely (8.9)-(8.10), give an explicit formula of the distribution of the residences  $\mu$  (see the proof of Lemma 8.4 for more details).

The lemma below characterizes the unique collection of wages for which the equilibrium in the labour market holds.

**Lemma 8.2** *Under Assumption 8.1 and 8.2, fix any distribution of residences  $\mu \in \mathcal{P}(X)$ , (8.8) holds for  $w \in (0, +\infty)^N$  if and only if  $w$  is the unique minimizer of*

$$\min_{z \in (0, +\infty)^N} \left\{ \phi_\mu(z) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds \right\}, \quad (8.11)$$

where  $\varepsilon > 0$  is fixed and

$$\phi_\mu(z) = \int_X R_\sigma(x, z) d\mu(x), \quad \forall z \in [0, +\infty)^N.$$

Moreover, the minimizer  $w$  of (8.11) belongs to a compact and convex set  $Y$  of  $(0, +\infty)^N$  which is independent of  $\mu$ .

**Remark 8.1** *In the optimization problem (8.11),  $\varepsilon > 0$  can be taken arbitrarily since the optimality conditions do not depend on this parameter (see the second step of the proof below). Nevertheless, it is important to take it positive since the functions  $L_i(\cdot)$  may not be integrable in 0.*

**Proof** Let us first note that the map  $\Lambda_\mu : (0, +\infty)^N \rightarrow \mathbb{R}$  defined by

$$\Lambda_\mu(z) = \left( \phi_\mu(z) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds \right), \quad \forall z \in (0, +\infty)^N,$$

is strictly convex since  $\phi_\mu(\cdot)$  is convex and  $(0, +\infty)^N \ni z \mapsto \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds$  is strictly concave.

**Step 1: existence of a minimizer.** This step will be split into two:

- first, we establish a priori bounds to deduce that there exists a compact and convex set  $Y \subset (0, +\infty)^N$  such that

$$\min_{z \in Y} \Lambda_\mu(z) = \min_{z \in (0, +\infty)^N} \Lambda_\mu(z);$$

- second, we use the continuity of  $\Lambda_\mu(\cdot)$  to deduce the existence of solutions, and its strict convexity to establish uniqueness.

**First a priori bound.** Let us fix  $z$  and  $\hat{z}$  two elements of  $(0, +\infty)^N$  such that

$$\Lambda_\mu(z) \leq \Lambda_\mu(\hat{z}).$$

We note that

$$\Lambda_\mu(z) = \phi_\mu(z) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds \geq \|z\|_\infty - \|c(\cdot)\|_{L^\infty} - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds,$$

since from (8.5) and (8.4),  $R_\sigma(x, z) \geq R(x, z) \geq \|z\|_\infty - \|c(\cdot)\|_{L^\infty}$ . Therefore

$$\Lambda_\mu(\hat{z}) \geq \|z\|_\infty - \|c(\cdot)\|_{L^\infty} - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds,$$

and

$$\|z\|_\infty \leq \|c(\cdot)\|_{L^\infty} + \phi_\mu(\hat{z}) + \sum_{k=1}^N \int_{\hat{z}_k}^{z_k} L_k(s) ds. \quad (8.12)$$

Using the fact that

- $R_\sigma(x, \hat{z}) \leq R(x, \hat{z}) + \sigma \ln(N)$ ,
- $R(x, \hat{z}) \leq \|\hat{z}\|_\infty$ ,

we deduce that

$$\phi_\mu(\hat{z}) \leq \|\hat{z}\|_\infty + \sigma \ln(N).$$

Then, using the monotonicity of  $L_i(\cdot)$ , (8.12) yields

$$\|z\|_\infty \leq \|c(\cdot)\|_{L^\infty} + \|\hat{z}\|_\infty + \sigma \ln(N) + \sum_{k=1}^N L_k(\hat{z}_k)(z_k - \hat{z}_k)$$

and due to the positivity of the functions  $L_i(\cdot)$ ,

$$\|z\|_\infty \leq \|c(\cdot)\|_{L^\infty} + \|\hat{z}\|_\infty + \sigma \ln(N) + \sum_{k=1}^N L_k(\hat{z}_k) \|z\|_\infty.$$

Thus, if  $z$  and  $\hat{z}$  belong to  $(0, +\infty)^N$  and satisfy

- $\Lambda_\mu(z) \leq \Lambda_\mu(\hat{z})$ ,
- $\sum_{k=1}^N L_k(\hat{z}_k) < 1$ ,

we deduce

$$\|z\|_\infty \leq \frac{\|c(\cdot)\|_{L^\infty} + \|\hat{z}\|_\infty + \sigma \ln(N)}{1 - \sum_{k=1}^N L_k(\hat{z}_k)} \quad (8.13)$$

**Second a priori bound.** We are now interested in a bound from below. Take  $z \in (0, +\infty)^N$ , we show that if there exists  $i \in \{1, \dots, N\}$  such that  $z_i < L_i^{-1}(1)$ , then it is possible to build a competitor  $\tilde{z}$  strictly better than  $z$ , i.e. such that

$$\Lambda_\mu(\tilde{z}) < \Lambda_\mu(z). \quad (8.14)$$

Indeed, if  $z_i < L_i^{-1}(1)$ , then  $L(z_i) > 1$  by Assumption 8.1-ii) and

$$\frac{\partial \Lambda_\mu}{\partial z_i}(z) = \int_X \frac{\partial R_\sigma}{\partial z_i}(x, z) d\mu(x) - L_i(z_i) < 0,$$

since  $R_\sigma(x, \cdot)$  is non-expansive. Then, the competitor

$$\tilde{w} = (z_1, \dots, z_{i-1}, z_i + t, z_{i+1}, \dots, z_N),$$

with  $t > 0$  small enough satisfies (8.14).

Let us introduce the compact and convex set

$$Y = \{z \in (0, +\infty)^N : \|z\|_\infty \leq M \text{ and } z_i \geq (L_i)^{-1}(1)\},$$

where

$$M = \min \left\{ \frac{\|c(\cdot)\|_{L^\infty} + \|\hat{z}\|_\infty + \sigma \ln(N)}{1 - \sum_{k=1}^N L_i(\hat{z}_i)} : \hat{z} \in (0, +\infty)^N \text{ and } \sum_{k=1}^N L_i(\hat{z}_i) < 1 \right\}.$$

We established that

$$\min_{z \in Y} \Lambda_\mu(z) = \min_{z \in (0, +\infty)^N} \Lambda_\mu(z).$$

**Conclusion.** Let us take a minimizing sequence  $(w^n)_{n \in \mathbb{N}}$  of the problem

$$\min_{w \in Y} \Lambda_\mu(w).$$

The compactness of  $Y$  and the continuity of  $\Lambda_\mu(\cdot)$  yield the existence of a minimizer  $w \in (0, +\infty)^N$ . The uniqueness is ensured by the strict convexity of  $\Lambda_\mu(\cdot)$ . This provides the existence and uniqueness of a solution of (8.11).

**Step 2: Characterization of the minimizer.** Since  $\Lambda_\mu(\cdot)$  is strictly convex and smooth,  $w \in (0, +\infty)^N$  is a minimizer of  $\Lambda_\mu(\cdot)$  if and only if for every  $i \in \{1, \dots, N\}$ ,

$$\frac{\partial \Lambda_\mu}{\partial w_i}(w) = 0 \Leftrightarrow L_i(w_i) = \int_X \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}} d\mu(x).$$

□

The following lemma addresses the stability of solutions of the problem (8.11) with respect to the variations of  $\mu$ .

**Lemma 8.3** *Let us suppose that Assumption 8.1 and 8.2 hold. Let  $(\mu_n)_{n \geq 0}$  be a sequence in  $\mathcal{P}(X)$  and  $(w^n)$  be the sequence of associated minimizers in (8.11). If  $\mu_n \rightarrow \mu$  for the weak \* topology of measures then  $(w^n)$  converges to  $w$ , the minimizer in (8.11) associated to  $\mu$ .*

**Proof** The proof of this result is split into two steps. The first step establishes that the map

$$\mathcal{P}(X) \times Y \ni (\mu, w) \mapsto \Lambda_\mu(w)$$

is uniformly continuous. Thus, an estimation is proved which implies that if  $\mu_n$  converges to  $\mu$  in  $\mathcal{P}(X)$  for the weak \* topology of measures then

$$\min_Y \Lambda_{\mu_n}(\cdot) \rightarrow \min_Y \Lambda_\mu(\cdot),$$

where  $Y$  is the compact and convex set introduced in Lemma 8.2.

The second step is dedicated to use the estimation to deduce that it induces the stability of the minimizers.

**Step 1.** Let  $Y$  be the convex and compact set previously introduced in Lemma 8.2. We show that the map

$$\mathcal{P}(X) \times Y \ni (\mu, w) \mapsto \Lambda_\mu(w) = \int_X R_\sigma(x, w) d\mu(x) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds$$

is continuous for the product topology when  $\mathcal{P}(X)$  is endowed with the weak \* topology and  $(0, +\infty)^N$  the standard topology. Let us suppose that  $(\mu_n, w_n)$  converges to  $(\mu, w)$  in  $\mathcal{P}(X) \times (0, +\infty)^N$  for the product topology. Then

$$R_\sigma(\cdot, w_n) \rightarrow R_\sigma(\cdot, w)$$

uniformly on  $X$  when  $n$  tends to  $+\infty$ . This with the fact that  $\mu_n \rightarrow \mu$  for the weak  $*$  topology of measures imply that

$$\int_X R_\sigma(x, w_n) d\mu_n(x) \rightarrow \int_X R_\sigma(x, w) d\mu(x),$$

hence the continuity of  $(\mu, w) \mapsto \Lambda_\mu(w)$ .

Note that  $\mathcal{P}(X) \times Y$  endowed with the product topology is a metrizable space when considering the metric  $d(\cdot, \cdot)$  defined by

$$d((\mu_1, w_1), (\mu_2, w_2)) = d_1(\mu_1, \mu_2) + \|w_1 - w_2\|, \quad \forall (\mu_1, \mu_2, w_1, w_2),$$

where  $d_1(\cdot, \cdot)$  is the Kantorovich-Rubinstein distance, i.e.

$$d_1(\mu_1, \mu_2) = \sup \left\{ \int_X f(x) d(\mu_1 - \mu_2)(x) : f : X \rightarrow \mathbb{R} \text{ is } 1\text{-Lipschitz} \right\}.$$

Since  $(\mathcal{P}(X) \times Y, d(\cdot, \cdot))$  is a compact and metric space, the Heine-Cantor theorem yields that the map  $(\mu, w) \mapsto \Lambda_\mu(w)$  is uniformly continuous on  $(\mathcal{P}(X) \times Y, d(\cdot, \cdot))$ .

By introducing  $\omega(\cdot)$  a modulus of continuity, we obtain

$$|\Lambda_{\mu_1}(w_1) - \Lambda_{\mu_2}(w_2)| \leq \omega(d((\mu_1, w_1), (\mu_2, w_2))), \quad \forall (\mu_1, \mu_2, w_1, w_2).$$

**Step 2.** From the latter paragraph, we deduce that for every  $(\mu_1, \mu_2) \in \mathcal{P}(X)^2$  and every  $w \in Y$ ,

$$|\Lambda_{\mu_1}(w) - \Lambda_{\mu_2}(w)| \leq \omega(d_1(\mu_1, \mu_2)). \quad (8.15)$$

Let  $\mu_n$  be a sequence converging to  $\mu$  in  $\mathcal{P}(X)$  for the weak  $*$  topology of measures. The estimation (8.15) ensures that

$$\min_Y \Lambda_{\mu_n}(\cdot) \rightarrow \min_Y \Lambda_\mu(\cdot)$$

when  $n$  tends to  $+\infty$ .

Let us now prove that the sequence of minimizers is stable. Let  $w^n$  be the unique minimizer of  $\Lambda_{\mu_n}(\cdot)$ . From Lemma 8.2 it belongs to  $Y$ . The compactness of  $Y$  yields that there exists  $\tilde{w} \in Y$  such that, up to the extraction of a subsequence,  $w^n \rightarrow \tilde{w}$ . Therefore

$$\begin{aligned} \left| \min_Y \Lambda_{\mu_n}(\cdot) - \Lambda_\mu(\tilde{w}) \right| &= |\Lambda_{\mu_n}(w^n) - \Lambda_\mu(\tilde{w})| \\ &\leq |\Lambda_{\mu_n}(w^n) - \Lambda_\mu(w^n)| + |\Lambda_\mu(w^n) - \Lambda_\mu(\tilde{w})| \\ &\leq \omega(d_1(\mu_n, \mu)) + |\Lambda_\mu(w^n) - \Lambda_\mu(\tilde{w})|. \end{aligned}$$

This ensures that  $\min_Y \Lambda_{\mu_n}(\cdot)$  converges to  $\Lambda_\mu(\tilde{w})$  when  $n$  tends to  $+\infty$ . The uniqueness of the limit ensures that

$$\Lambda_\mu(\tilde{w}) = \min_Y \Lambda_\mu(\cdot).$$

In addition, from Lemma 8.2, there is a unique minimizer of  $\Lambda_\mu(\cdot)$ , namely  $w$ . Hence,  $\tilde{w} = w$ . The uniqueness of the cluster point ensures that the whole sequence  $(w^n)_{n \in \mathbb{N}}$  converges towards  $w$ .  $\square$

We now want to construct a function  $\Upsilon(\cdot)$  such that its fixed-points are exactly the equilibria in the sense of Definition 8.3.

**Lemma 8.4** *Let  $Y$  be the compact and convex set introduced in Lemma 8.2. Under Assumption 8.1 and 8.2, consider the function  $\Upsilon : Y \rightarrow Y$  defined as follows:*

1. to any  $w \in Y$ , we associate the probability  $\mu(w)$  on  $X$  with density

$$X \ni x \mapsto \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy}.$$

with respect to the Lebesgue measure.

2. Then, we define  $\Upsilon(w)$  as the unique solution of (8.11) associated to  $\mu(w)$ , i.e.  $\Upsilon(w)$  is the unique minimizer of

$$\min_{z \in (0, +\infty)^N} \left\{ \phi_\mu(z) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds \right\}.$$

The fixed-points of  $\Upsilon(\cdot)$  are exactly the equilibria in the sense of Definition 8.3.

**Proof** We split the proof into two steps. Firstly, let  $(w, Q(\cdot), \mu)$  be an equilibrium. We claim that

$$\mu(x) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy}, \quad \forall x \in X. \quad (8.16)$$

Then, we establish that the fixed-points of  $\Upsilon(\cdot)$  are exactly the equilibria.

**Step 1.** Let us assume that  $(w, Q(\cdot), \mu)$  is an equilibrium.

Equation (8.9) ensures that  $Q(\cdot)$  takes positive values since the function  $S_\theta(\cdot, \cdot)$  is defined on  $(0, +\infty)^2$  (see Lemma 8.1). The same lemma yields that  $S_\theta((0, +\infty)^2) = (0, +\infty)$ . Therefore, (8.9) implies that  $\mu$  is positive. From Corollary 8.1, the utility of agents is given by:

$$\mathcal{U}_\theta(R_\sigma(x, w), Q(x)) = \theta^\theta \frac{R_\sigma(x, w)^\theta}{\mu(x)^{1-\theta}}, \quad \forall x \in X. \quad (8.17)$$

Moreover, (8.10) ensures that there exists a real number  $\lambda$  such that

$$\begin{cases} \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) \leq \lambda, & \forall x \in X, \\ \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) = \lambda, & \forall x \in \text{supp } \mu. \end{cases} \quad (8.18)$$

Thus, from (8.17) and (8.18) we deduce that

$$\theta^\theta \frac{R_\sigma(x, w)^\theta}{\mu(x)^{1-\theta}} = \lambda, \quad \forall x \in X.$$

Hence

$$\mu(x) = \left( \frac{\theta^\theta}{\lambda} \right)^{\frac{1}{1-\theta}} R_\sigma(x, w)^{\frac{\theta}{1-\theta}}, \quad \forall x \in X.$$

Since  $\mu$  is a probability measure on  $X$ ,

$$\mu(x) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy}, \quad \forall x \in X.$$

**Step 2.** A fixed-point of  $\Upsilon(\cdot)$  characterizes an equilibrium. First, note that  $\Upsilon(\cdot)$  is well-defined since, from Lemma 8.2, the solution of (8.11) is unique and belongs to  $Y$ . Second, if  $w = \Upsilon(w)$  then let us consider  $\mu$  the probability measure defined by (8.16), and the rental price  $Q : X \rightarrow [0, +\infty)$  given by

$$Q(x) = (1 - \theta) R_\sigma(x, w) \mu(x), \quad \forall x \in X.$$

The triplet

$$(w, Q(\cdot), \mu) \in Y \times C(X) \times (\mathcal{P}(X) \cap C(X))$$

is an equilibrium since (8.8) holds because  $w = \Upsilon(w)$  is the unique solution of (8.11) associated to  $\mu$ . Moreover, the definition of  $Q(\cdot)$  yields (8.9). Finally, the mobility condition (8.10) holds since the function

$$X \ni x \mapsto \mathcal{U}_\theta(R_\sigma(x, w), Q(x))$$



is constant and equal to  $\theta^\theta \left( \int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy \right)^{1-\theta}$ . Conversely, if  $(w, Q(\cdot), \mu)$  is an equilibrium, from Step 1  $\mu$  is given by (8.16), and  $w$  is the solution of (8.11) associated to  $\mu$ . Therefore  $w = \Upsilon(w)$ .  $\square$

**Remark 8.2** *The present model can be rewritten by using the notion of static mean field games. Indeed, equation (8.9) links the rental price  $Q(\cdot)$  with the distribution  $\mu$ . We saw in the latter proof that the utility of agents can be written at the equilibrium as*

$$U(x, w, \mu) = \begin{cases} \theta^\theta \frac{R_\sigma(x, w)^\theta}{\mu(x)^{1-\theta}}, & \text{if } \mu \in \mathcal{P}_{ac}(X), \\ -\infty, & \text{otherwise,} \end{cases}$$

with the convention  $1/0 = +\infty$ . Thus, equation (8.10) can be seen as the mean field equation for potential static games:

$$\int_X U(x, w, \mu) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X U(x, w, \mu) d\nu(x) < +\infty. \quad (8.19)$$

Thus, the following definition is equivalent to Definition 8.3.

**Definition 8.4** *The pair  $(w, \mu) \in (0, +\infty)^N \times (C(X) \times \mathcal{P}(X))$  is an equilibrium if*

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x) = L_i(w_i), \quad \forall i \in \{1, \dots, N\}, \quad (8.20)$$

$$\int_X U(x, w, \mu) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X U(x, w, \mu) d\nu(x) < +\infty. \quad (8.21)$$

**Remark 8.3** *The rental housing market is taken into account in Definition 8.4 via the utility function  $U(\cdot)$ . The rental price  $Q : X \rightarrow [0, +\infty)$  can be recovered thanks to (8.9)*

We are now ready to prove

**Theorem 8.2** *Under Assumption 8.1 and 8.2, there exists an equilibrium  $(w, Q(\cdot), \mu)$  in the sense of Definition 8.3.*

**Proof** Let  $Y$  be the compact and convex set introduced in Lemma 8.2. We aim at proving, by using Brouwer fixed-point theorem, the existence of a fixed-point of the function  $\Upsilon : Y \rightarrow Y$  defined by the following construction:

1. To any  $w \in Y$ , we associate the probability  $\mu(w)$  on  $X$  with density

$$X \ni x \mapsto \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy},$$

with respect to the Lebesgue measure.

2. Then, we define  $\Upsilon(w)$  as the unique solution of (8.11) associated to  $\mu(w)$ , i.e.  $\Upsilon(w)$  is the unique minimizer of

$$\min_{z \in (0, +\infty)^N} \left\{ \phi_\mu(z) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds \right\}.$$

Since  $Y$  is compact and convex, we only need to check the continuity of  $\Upsilon(\cdot)$ . For any  $w \in Y$ , if a sequence  $(w^n)_{n \in \mathbb{N}}$  in  $Y$  converges to  $w$ , then  $\mu(w^n) \rightarrow \mu(w)$  uniformly on  $X$ , which implies the weak \* convergence of measures. Applying Lemma 8.3, we see that  $\Upsilon(w^n) \rightarrow \Upsilon(w)$  when  $n \rightarrow +\infty$ .

Brouwer fixed-point theorem yields the existence of a fixed-point of  $\Upsilon(\cdot)$  and then the existence of an equilibrium.  $\square$

**Remark 8.4** Let us introduce  $\alpha = \theta/(1 - \theta)$ . The latter proof yields the existence of solutions of

$$\Psi(w, \alpha) = L(w), \quad (8.22)$$

where

- $L(w) = (L_1(w_1), \dots, L_N(w_N))$ ,
- $\Psi(w, \alpha) = \int_X D_w R_\sigma(x, w) \tilde{\mu}(x, w, \alpha) dx$ ,
- $\tilde{\mu}(x, w, \alpha) = \frac{R_\sigma(x, w)^\alpha}{\int_X R_\sigma(y, w)^\alpha dy}$ .

Indeed, with the explicit formula of  $\mu$  (see (8.16)), it is easy to see that the triplet  $(w, Q(\cdot), \mu)$  is an equilibrium if and only if

- $w$  is solution of (8.22),
- $Q(\cdot) = (1 - \theta)R_\sigma(\cdot, w)\tilde{\mu}(\cdot, w, \alpha)$ ,
- $\mu = \tilde{\mu}(\cdot, w, \alpha)$ .

**Remark 8.5** If  $\theta = 0$ , then for every collection of wages, the measure  $\mu$  defined in (8.16) is a uniformly distributed measure on  $X$  with mass 1. Moreover, in this case, there is a unique equilibrium namely the triplet  $(w, Q(\cdot), \mu)$  where

1.  $d\mu(x) = 1/|X| dx$ ,
2.  $w$  is the unique solution of (8.11) associated to  $\mu$ ,
3. for every  $x \in X$ ,

$$Q(x) = \frac{R_\sigma(x, w)}{|X|}.$$

## 8.4 Uniqueness of equilibria

Remark 8.5 ensures the uniqueness of equilibria when  $\theta = 0$ . To establish a uniqueness result, we use Remark 8.4 and the implicit function theorem in order to extend the uniqueness property to  $\theta \in [0, \theta_0]$  for some  $\theta_0 > 0$ .

**Theorem 8.3** Let Assumption 8.1, 8.2 hold, and assume that for every  $i \in \{1, \dots, N\}$ ,  $L_i(\cdot)$  is of class  $C^1$  on  $(0, +\infty)$ . Then, for every  $\theta \in [0, \theta_0]$  the equilibrium is unique, where

$$\theta_0 = \frac{\alpha_0(Y)}{1 + \alpha_0(Y)},$$

and

$$\alpha_0(Y) = \min \left\{ \frac{w_0}{2N} \inf_{w \in Y} \min_{i=1}^N (-L'_i(w_i)), 1 \right\},$$

with  $Y$  the convex compact set introduced in Lemma 8.2.

**Proof** We introduce the variable  $\alpha = \theta/(1 - \theta)$ . Let us study the uniqueness for the following problem:

$$\Psi(w, \alpha) = L(w), \quad (8.23)$$

where

- $L(w) = (L_1(w_1), \dots, L_N(w_N))$ ,
- $\Psi(w, \alpha) = \int_X D_w R_\sigma(x, w) \tilde{\mu}(x, w, \alpha) dx$ ,
- $\tilde{\mu}(x, w, \alpha) = \frac{R_\sigma(x, w)^\alpha}{\int_X R_\sigma(y, w)^\alpha dy}$ .

Recall that  $Y$  is the convex compact set given in Lemma 8.2. If  $w \in (0, +\infty)^N$  is a solution of (8.23), then  $w$  solves (8.11) associated to  $\tilde{\mu}(\cdot, w, \alpha)$ . Therefore, from Lemma 8.2,  $w \in Y$ . On the other hand, Remark 8.5 yields the existence and uniqueness of a solution of (8.23) for  $\alpha = 0$ . We extend this result by applying the implicit function theorem.

**Step 1: a sufficient condition on  $\alpha$  for the invertibility of  $D_w(\Psi(\cdot, \alpha) - L(\cdot))$  on  $Y$ .**

For any  $w \in (0, +\infty)^N$ ,  $\alpha \in [0, +\infty)$  and  $i, j \in \{1, \dots, N\}$ ,

$$\frac{\partial \Psi_i}{\partial w_j}(w, \alpha) = \int_X \frac{\partial^2 R_\sigma}{\partial w_j \partial w_i}(x, w) \tilde{\mu}(x, w, \alpha) dx + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha) dx.$$

We see that for any  $x \in X$ ,

- $\frac{\partial R_\sigma}{\partial w_i}(x, w) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}}$ .
- $\frac{\partial^2 R_\sigma}{\partial w_i \partial w_j}(x, w) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sigma \sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}} \left( \delta_{i,j} - \frac{e^{\frac{w_j - c_j(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}} \right)$ .
- $\frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha) = \frac{\alpha R_\sigma^\alpha(x, w)}{\int_X R_\sigma(y, w)^\alpha dy} \left( \frac{\partial R_\sigma}{\partial w_j}(x, w)}{R_\sigma(x, w)} - \frac{\int_X R_\sigma(y, w)^{\alpha-1} \frac{\partial R_\sigma}{\partial w_j}(y, w) dy}{\int_X R_\sigma(y, w)^\alpha dy} \right)$ .

Since the matrix

$$\left( \frac{\partial^2 R_\sigma}{\partial w_i \partial w_j}(x, w) \right)_{i,j=1,\dots,N}$$

is strictly diagonal dominant with positive diagonal entries, the matrix

$$\left( \int_X \frac{\partial^2 R_\sigma}{\partial w_i \partial w_j}(x, w) \tilde{\mu}(x, w, \alpha) dx \right)_{i,j=1,\dots,N}$$

has the same properties. Then, a sufficient condition for the matrix  $D_w(\Psi(w, \alpha) - L(w))$  to be invertible is that the matrix

$$\left( -L'_i(w_i) \delta_{i,j} + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha) dx \right)_{i,j=1,\dots,N}$$

is diagonal dominant with positive entries. This is equivalent to

$$-L'_i(w_i) + \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \frac{\partial \tilde{\mu}}{\partial w_i}(x, w, \alpha) dx \geq \sum_{j \neq i} \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha) dx, \quad \forall i \in \{1, \dots, N\}. \quad (8.24)$$

We observe that for any  $i, j \in \{1, \dots, N\}$ ,

$$\left| \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \frac{\partial \tilde{\mu}}{\partial w_j}(x, w) dx \right| \leq \int_X \left| \frac{\partial R_\sigma}{\partial w_i}(x, w) \right| \left| \frac{\partial \tilde{\mu}}{\partial w_j}(x, w) \right| dx \quad (8.25)$$

$$\leq \int_X \frac{\alpha R_\sigma(x, w)^\alpha}{\int_X R_\sigma(y, w)^\alpha dy} \left( \frac{\frac{\partial R_\sigma}{\partial w_j}(x, w)}{R_\sigma(x, w)} + \frac{\int_X R_\sigma(x, w)^{\alpha-1} \frac{\partial R_\sigma}{\partial w_j}(y, w) dy}{\int_X R_\sigma(y, w)^\alpha dy} \right) dx \quad (8.26)$$

$$\leq 2\alpha \frac{\int R_\sigma^{\alpha-1} \frac{\partial R_\sigma}{\partial w_j} dx}{\int R_\sigma^\alpha dx}, \quad (8.27)$$

where (8.26) comes from the fact that  $\left| \frac{\partial R_\sigma}{\partial w_i}(x, w) \right| \leq 1$  and the expression of  $\frac{\partial \tilde{\mu}}{\partial w_j}(x, w, \alpha)$ , while (8.27) is straightforward. In addition, if  $\alpha \leq 1$ , then

$$\frac{\int R_\sigma^{\alpha-1}(x, w) \frac{\partial R_\sigma}{\partial w_j}(x, w) dx}{\int R_\sigma^\alpha(x, w) dx} \leq \frac{|X| \times w_0^{\alpha-1} \times 1}{|X| \times w_0^\alpha} = \frac{1}{w_0}.$$

Therefore (8.27) yields that

$$\left| \int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) \frac{\partial \tilde{\mu}}{\partial w_j}(x, w) dx \right| \leq \frac{2\alpha}{w_0}. \quad (8.28)$$

Let us introduce the positive constant

$$\alpha_0(Y) = \min \left\{ \frac{w_0}{2N} \min_{w \in Y} \min_{i=1}^N (-L'_i(w_i)), 1 \right\}.$$

We notice that

$$\alpha \leq \alpha_0(Y)$$

is equivalent to

$$\frac{2\alpha N}{w_0} \leq -L'_i(w_i) \text{ and } \alpha \leq 1, \quad \forall i \in \{1, \dots, N\}, \forall w \in Y.$$

Therefore, if  $\alpha \leq \alpha_0(Y)$ , then (8.28) implies (8.24) and that  $D_w(\Psi(w, \alpha) - L(w))$  is invertible for every  $w \in Y$ .

**Step 2: uniqueness of equilibria for every  $\alpha \leq \alpha_0(Y)$ .**

The proof of this result is the same as in the proof of Proposition 5.4 in Chapter 5. We choose to write it here for completeness.

Let us define  $h : [0, 1] \times (0, +\infty)^d \rightarrow \mathbb{R}$  such that

$$h(\alpha, w) = \Psi(w, \alpha) - L(w), \quad \forall (\alpha, w) \in [0, \alpha_0] \times (0, +\infty)^d.$$

Lemma 8.2 states that for every  $(\alpha, w) \in [0, \alpha_0] \times (0, +\infty)^d$  if  $h(\alpha, w) = 0$ , then  $w \in Y$ . Let us introduce

$$A = \{ \alpha \in [0, \alpha_0] : \text{There exists a unique } w \in (0, +\infty)^d \text{ satisfying } h(\alpha, w) = 0 \}.$$

The set  $A$  is non empty since  $0 \in A$ .

**$A$  is closed.** Indeed, let  $(\alpha_n)_{n \in \mathbb{N}}$  be a sequence of  $A$  such that  $\alpha_n \rightarrow \alpha \in [0, \alpha_0]$  when  $n \rightarrow +\infty$ . Then, for every  $n \in \mathbb{N}$  let us denote by  $w_n$  the unique solution in  $(0, +\infty)^d$  of  $h(\alpha_n, \cdot) = 0$ , we observe that  $w_n \in Y$ . Thus, there exists an element  $w \in Y$  such that, passing through a subsequence,  $w_n \rightarrow w$  when  $n \rightarrow +\infty$ . The continuity of  $h(\cdot, \cdot)$  yields that  $h(\alpha, w) = 0$ , and so  $w \in Y$ .

Let us check that there does not exist a distinct element  $\hat{w} \in (0, +\infty)^d$  of  $w$  such that  $h(\alpha, \hat{w}) = 0$ . We prove this claim by contradiction. Since  $D_w h(\alpha, w)$  and  $D_w h(\alpha, \hat{w})$  are isomorphisms, we can apply the implicit function theorem in order to deduce that

- there exist  $\mathcal{V}$  an open neighbourhood of  $w$  and  $\mathcal{V}'$  an open neighbourhood of  $\hat{w}$ .
- there exist  $\mathcal{A}$  and  $\mathcal{A}'$  two open neighbourhoods of  $\alpha$  in  $[0, \alpha_0]$ .
- there exist  $W : \mathcal{A} \rightarrow \mathcal{V}$  and  $\hat{W} : \mathcal{A}' \rightarrow \mathcal{V}'$  two  $C^1$  trajectories such that:
  - for every  $s \in \mathcal{A}$ ,  $h(s, W(s)) = 0$  and for every  $s \in \mathcal{A}'$ ,  $h(s, \hat{W}(s)) = 0$ .
  - if  $(s, z) \in \mathcal{A} \times \mathcal{V}$  satisfies  $h(s, z) = 0$  then  $z = W(s)$ .
  - if  $(s, z) \in \mathcal{A}' \times \mathcal{V}'$  satisfies  $h(s, z) = 0$  then  $z = \hat{W}(s)$ .

Therefore, let us fix  $\rho = \|w - \hat{w}\|/3$ . From the continuity of the trajectories  $W(\cdot)$  and  $\hat{W}(\cdot)$ , there exists  $\delta > 0$  satisfying  $((\alpha - \delta, \alpha + \delta) \cap [0, 1]) \subset \mathcal{A} \cap \mathcal{A}'$ , such that

$$W(s) \in B(w, \rho) \text{ and } \hat{W}(s) \in B(\hat{w}, \rho), \quad \forall s \in (\alpha - \delta, \alpha + \delta) \cap [0, \alpha_0].$$

Thus, for every  $s \in (\alpha - \delta, \alpha + \delta) \cap [0, \alpha_0]$ , the equation  $h(s, \cdot) = 0$  admits at least two distinct solutions:  $W(s)$  and  $\hat{W}(s)$ . It is in contradiction with the fact that  $\alpha$  belongs to the closure of  $A$ . Finally, we have established that  $\alpha \in A$ .

**$A$  is open.** Let us fix  $\alpha \in A$ . Let  $w \in Y$  be the unique element satisfying  $h(\alpha, w) = 0$ . By applying the implicit function theorem, there exists  $\delta > 0$  such that for every  $s \in (\alpha - \delta, \alpha + \delta) \cap [0, \alpha_0]$ ,  $h(s, \cdot) = 0$  admits at least a solution.

By contradiction, let us assume that there exists a sequence  $\alpha_n \rightarrow \alpha$  such that the equation  $h(\alpha_n, \cdot) = 0$  admits at least two distinct solutions. We denote by  $w_n^1$  and  $w_n^2$  these solutions. Since these sequences are compact there exist  $w^1$  and  $w^2$  such that, passing through a subsequence,  $w_n^1 \rightarrow w^1$  and  $w_n^2 \rightarrow w^2$  when  $n \rightarrow +\infty$ . The continuity of  $h(\cdot, \cdot)$  yields

$$h(\alpha, w^1) = h(\alpha, w^2) = 0.$$

The uniqueness of the solution ensures that  $w^1 = w^2 = w$ . This enters in contradiction with the implicit function theorem which guarantees the local uniqueness of the solution. Thus, there exists  $\delta > 0$  such that  $(\alpha - \delta, \alpha + \delta) \cap [0, \alpha_0] \subset A$ .

**Conclusion.** We have shown that  $A$  is non empty, closed and open for the topology induced on  $[0, \alpha_0]$ . Thus,  $A = [0, \alpha_0]$ . In particular,  $\Psi(\cdot, \alpha) = L(\cdot)$  admits a unique solution for every  $\alpha \in [0, \alpha_0]$ . □

## 8.5 Extensions

### 8.5.1 A home-based telecommuting model

We aim at applying the model to home-based telecommuting. We use the same notation as before; let  $M_{N \times 2}(0, +\infty)$  denote the space of real matrices with positive entries, with  $N$  rows and 2 columns. We assume that for every index  $i$ , the labour demand at  $y_i$  is modelled by the function  $L_i : (0, +\infty)^2 \rightarrow (0, +\infty)^2$ , in which the first argument is the demand of commuters and the second argument is the

demand of telecommuters. In this setting, the  $i^{\text{th}}$  workplace proposes two wages:  $W_1^i$  for the commuters and  $W_2^i$  for the telecommuters. Therefore, given  $W \in M_{N \times 2}(0, +\infty)$ , we define the revenue of an agent at the position  $x \in X$  by

$$R_\sigma(x, W) = \sigma \ln \left( e^{\frac{w_0}{\sigma}} + \sum_{i=1}^N e^{\frac{W_1^i - c_i(x)}{\sigma}} + \sum_{i=1}^N e^{\frac{W_2^i}{\sigma}} \right).$$

For every  $i \in \{1, \dots, N\}$ , we suppose that the output of  $y_i$  is given by the production function

$$F^i : [0, +\infty)^2 \rightarrow [0, +\infty).$$

Then, for a given collection of wages  $W^i \in (0, +\infty)^2$ , the  $i^{\text{th}}$  company aims at solving:

$$(F^i)^*(W^i) = \sup \{ F^i(\ell) - W^i \cdot \ell : \ell \in [0, +\infty)^2 \}. \quad (8.29)$$

Here,  $(F^i)^*(\cdot)$  is the payoff of the  $i^{\text{th}}$  company, and  $\ell \in [0, +\infty)^2$  is the level of employment of each type of workers. At least formally, the envelope theorem yields the labour demand:

$$L_i(W^i) = -D(F^i)^*(W^i). \quad (8.30)$$

We thus see that under the assumption made below and when the envelope theorem can be applied, the functions  $L_i(\cdot)$  admit a potential and are given by (8.30). In the latter analysis, the only change implied by this extension concerns the labour market. Using the arguments contained in the proof of Lemma 9.2, we extend Lemma 8.2 to the present case under the following set of assumptions:

**Assumption 8.3** For every  $i \in \{1, \dots, N\}$ ,  $F^i : [0, +\infty)^K \rightarrow [0, +\infty)$ , the production function of the  $i^{\text{th}}$  workplace, satisfies:

1.  $F^i(\cdot)$  is increasing with respect to each coordinate on  $(0, +\infty)^2$
2. for every  $\bar{\ell} \in (0, +\infty)$ ,

$$\lim_{\ell \rightarrow +\infty} F^i(\ell, \bar{\ell}) = +\infty \text{ and } \lim_{\ell \rightarrow +\infty} F^i(\bar{\ell}, \ell) = +\infty.$$

3.  $F^i(\cdot)$  is strictly concave on  $(0, +\infty)^2$
4.  $F^i(\cdot)$  is continuous on  $[0, +\infty)^2$  and of class  $C^1$  on  $(0, +\infty)^2$
5.  $F(\ell)/\|\ell\| \rightarrow 0$  when  $\|\ell\|$  tends to  $+\infty$ .
6. If  $(\ell^n)_{n \in \mathbb{N}}$  is a bounded sequence of  $(0, +\infty)^2$  such that  $\min_{k=1,2} \ell_k^n \rightarrow 0$  when  $n$  converges to  $+\infty$ , then

$$\max_{k=1,2} \frac{\partial F^i}{\partial \ell_k}(\ell^n) \rightarrow +\infty$$

when  $n$  tends to  $+\infty$ .

The following lemma holds:

**Lemma 8.5** Under Assumption 8.3 and 8.2, fix any distribution of residences  $\mu \in \mathcal{P}(X)$ , the market clearing conditions on the labour market, i.e

$$\int_X D_W R_\sigma(x, W) d\mu(x) = L_i(W^i), \quad \forall i \in \{1, \dots, N\}, \quad (8.31)$$

holds for  $W \in M_{N \times 2}(0, +\infty)$  if and only if  $W$  is the unique minimizer of

$$\min_{Z \in M_{N \times 2}(0, +\infty)} \left\{ \phi_\mu(Z) + \sum_{i=1}^N (F^*)(Z^i) \right\}, \quad (8.32)$$

and

$$\phi_\mu(Z) = \int_X R_\sigma(x, Z) d\mu(x), \quad \forall Z \in M_{N \times 2}(0, +\infty).$$

Moreover, the minimizer  $W$  of (8.32) belongs to a compact and convex set  $Y$  of  $M_{N \times 2}(0, +\infty)$  which is independent of  $\mu$ .

The definition of equilibria is extended by replacing equation (8.8) in Definition 8.3 by (8.31). Using the same analysis as in Sections 8.3 and 8.4, we conclude that the following result holds:

**Theorem 8.4** *Under Assumption 8.3 and 8.2, there exists an equilibrium  $(W, Q(\cdot), \mu)$ .*

*Moreover, if for every  $i \in \{1, \dots, N\}$ ,  $L_i(\cdot, \cdot)$  is of class  $C^1$  on  $(0, +\infty)^2$ , then there exists  $\theta_0 > 0$  such that for every  $\theta \in [0, \theta_0]$  the equilibrium is unique.*

### 8.5.2 The zero-noise limit case

Let us tackle the case when the revenue of the agents is given by (8.4) instead of (8.5). We first extend the definition of an equilibrium to this case and address existence results.

For every  $w \in (0, +\infty)^N$ , let us introduce the following sets:

$$V_i(w) = \{x \in X : R(x, w) = w_i - c_i(x)\},$$

and

$$V_i^s(w) = \{x \in X : \forall j \neq i, R(x, w) > w_j - c_j(x)\}.$$

For  $i \in \{1, \dots, N\}$  fixed,  $V_i^s(w)$  (resp.  $V_i(w)$ ) is the set of positions in  $X$  where the revenue is only reached (resp. reached) when individuals work at the  $i^{\text{th}}$  workplace. Note that  $V_i^s(w) \subset V_i(w)$ . Therefore, the extension of equation (8.8) to this setting is:

$$L_i(w_i) \in [\mu(V_i^s(w)), \mu(V_i(w))], \quad \forall i \in \{1, \dots, N\},$$

supplemented with

$$1 - \sum_{i=1}^N L_i(w_i) \in [\mu(V_0^s(w)), \mu(V_0(w))].$$

**Definition 8.5** *We say that the triplet*

$$(w, Q(\cdot), \mu) \in (0, +\infty)^N \times C(X) \times (\mathcal{P}(X) \cap C(X))$$

*is an equilibrium if*

$$L_i(w_i) \in [\mu(V_i^s(w)), \mu(V_i(w))], \quad \forall i \in \{1, \dots, N\}, \quad (8.33)$$

$$1 - \sum_{i=1}^N L_i(w_i) \in [\mu(V_0^s(w)), \mu(V_0(w))], \quad (8.34)$$

$$S(R(x, w), Q(x))\mu(x) = 1, \quad \forall x \in X, \quad (8.35)$$

$$\int_X \mathcal{U}_\theta(R(x, w), Q(x)) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X \mathcal{U}_\theta(R(x, w), Q(x)) d\nu(x) < +\infty. \quad (8.36)$$

**Remark 8.6** Note that in the latter definition we imposed  $\mu$  to be continuous, then (8.33)-(8.34) is equivalent to

$$L_i(w_i) = \mu(V_i(w)), \quad \forall i \in \{1, \dots, N\},$$

and

$$1 - \sum_{i=1}^N L_i(w_i) = \mu(V_0(w)).$$

The purpose of keeping this condition in this form is to show that for every  $\mu \in \mathcal{P}(X)$ , (8.33)-(8.34) is equivalent to an optimization problem. The main interest of working with  $\mathcal{P}(X)$  lies in the fact that the  $\mathcal{P}(X)$  endowed with the weak \* topology of measures is compact.

### Existence of equilibria

We show the existence of equilibria in the zero-noise limit case by letting  $\sigma$  goes to 0. Indeed, we prove that each sequence of associated equilibria is compact and each accumulation point is a solution in the sense of Definition 8.5.

Note that it is possible to apply the same fixed-point strategy developed in the proof of Theorem 8.2 to show the existence of equilibria. Nevertheless, the approach we use has the advantage to link the equilibria with  $\sigma > 0$  and the ones of the limit case.

In the same manner as in the regular case, we relate (8.33)-(8.34) to an optimization problem. The lemma below extends Lemma 8.2.

**Lemma 8.6** Under Assumptions 8.1 and 8.2, for every distribution of residences  $\mu \in \mathcal{P}(X)$ , (8.33)-(8.34) hold for  $w \in (0, +\infty)^N$  if and only if  $w$  is the unique minimizer of

$$\min_{z \in (0, +\infty)^N} \left\{ \phi_\mu(z) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds \right\}, \quad (8.37)$$

where  $\varepsilon > 0$  is fixed and

$$\phi_\mu(w) = \int_X R(x, w) d\mu(x), \quad \forall w \in [0, +\infty)^N.$$

Moreover, the minimizer  $w$  of (8.37) belongs to a compact  $Y$  which is independent of  $\mu$ .

**Proof** Note that the map  $\Lambda_\mu : (0, +\infty)^N \rightarrow \mathbb{R}$  defined for every  $w \in (0, +\infty)^N$  by

$$\Lambda_\mu(w) = \left( \phi_\mu(w) - \sum_{i=1}^N \int_\varepsilon^{w_i} L_i(s) ds \right)$$

is strictly convex. Indeed,  $\phi_\mu(\cdot)$  is convex and  $(0, +\infty)^N \ni w \mapsto \sum_{i=1}^N \int_\varepsilon^{w_i} L_i(s) ds$  is strictly concave.

**Step 1: existence of a minimizer.** Essentially the same arguments as in the proof of Lemma 8.2 yield the existence of a unique solution of (8.37). The only change concerns the second a priori bound. It is no longer possible to differentiate  $\Lambda_\mu(\cdot)$  since  $R(x, \cdot)$  is only a Lipschitz function. However, if there exists  $i \in \{1, \dots, N\}$  such that  $w_i < L_i^{-1}(1)$  then using the fact that  $R(x, \cdot)$  is 1-Lipschitz and  $L_i(\cdot)$  is continuous we have that the competitor

$$\tilde{w} = (w_1, \dots, w_{i-1}, w_i + t, w_{i+1}, \dots, w_N),$$

is strictly better than  $w$  (i.e.  $\Lambda_\mu(\tilde{w}) < \Lambda_\mu(w)$ ), for  $t > 0$  small enough.

Therefore, as before, the bounds and the continuity of  $\Lambda_\mu(\cdot)$  yield the existence of a solution. The strict convexity of  $\Lambda_\mu(\cdot)$  implies the uniqueness of the minimizer.



**Step 2: Characterization of the minimizer.** Since  $\Lambda_\mu(\cdot)$  is convex,  $w \in (0, +\infty)^N$  is a minimizer of  $\Lambda_\mu(\cdot)$  if and only if  $0 \in \partial\Lambda_\mu(w)$ , i.e.

$$(L_1(w_1), \dots, L_N(w_N)) \in \prod_{i=1}^N [\mu(V_i^s(w)), \mu(V_i(w))].$$

Indeed, for every  $w \in (0, +\infty)^N$ ,  $\partial\Lambda_\mu(w) = \partial\phi_\mu(w) - (L_1(w), \dots, L_N(w))^T$ . Thus, let us fix  $w \in (0, +\infty)^N$ , we focus on the sub-differential  $\partial\phi_\mu(w)$ . We introduce for every  $x \in X$ , the set of indexes  $I(x, w)$  such that for every  $i \in I(x, w)$  an agent at the position  $x$  and working at the  $i^{\text{th}}$  workplace receives the revenue  $R(x, w)$ , i.e

$$I(x, w) = \{i \in \{1, \dots, N\} : R(x, w) = w_i - c_i(x)\}.$$

Since  $R(x, \cdot)$  is convex,  $R(\cdot, w)$  is  $\mu$ -integrable because  $R(\cdot, w)$  belongs to  $L^\infty$  then [17, Proposition 2.2] ensures that

$$\partial\phi_\mu(w) = \left\{ \int_X z(x) d\mu(x) : z(\cdot) \text{ is measurable, and } z(x) \in \partial_w R(x, w) \text{ a.e.} \right\}. \quad (8.38)$$

For ease of notations, let us introduce

$$\psi_i(w, t) = (w_1, \dots, w_{i-1}, w_i + t, w_{i+1}, \dots, w_N).$$

Moreover, since  $R(\cdot, x)$  is the maximum of linear maps,

$$\partial_w R(x, w) = \text{co} \left( \cup_{i \in I(x, w)} \{\psi_i(0_{\mathbb{R}^N}, 1)\} \right). \quad (8.39)$$

Equation (8.38) and (8.39) yield that

$$\partial\phi_\mu(w) = \prod_{i=1}^N [\mu(V_i^s(w)), \mu(V_i(w))]. \quad (8.40)$$

Therefore,

$$\begin{aligned} 0 \in \partial\Lambda_\mu(w) &\Leftrightarrow 0 \in \prod_{i=1}^N [\mu(V_i^s(w)) - L_i(w_i), \mu(V_i(w)) - L_i(w_i)] \\ &\Leftrightarrow (L_1(w_1), \dots, L_N(w_N)) \in \prod_{i=1}^N [\mu(V_i^s(w)), \mu(V_i(w))]. \end{aligned}$$

There remains to prove that if  $w \in (0, +\infty)^N$  is a minimizer of  $\Lambda_\mu(\cdot)$ , then

$$1 - \sum_{i=1}^N L_i(w_i) \in [\mu(V_0^s(w)), \mu(V_0(w))].$$

To this aim, we notice that the function

$$\tilde{\Lambda} : (-\epsilon, \epsilon) \ni t \mapsto \int_X R(x, w + t(1, \dots, 1)) d\mu(x) - \sum_{i=1}^N \int_\epsilon^{w_i+t} L_i(s) ds$$

is convex. Let us consider

$$H = \{(t, x, w) : w \in (0, +\infty)^N, (w_1 + t, \dots, w_N + t) \in (0, +\infty)^N \text{ and } x \in X\},$$

and  $\psi : H \rightarrow \mathbb{R}$  given by

$$\psi(t, x, w) = R(x, w + t(1, \dots, 1)), \quad \forall (t, x, w) \in H.$$

We have that

$$\partial_t \psi(0, x, w) = \begin{cases} 1, & \text{if } x \in X \setminus V_0(w), \\ 0, & \text{if } x \in V_0^s(w), \\ [0, 1], & \text{if } x \in V_0(w) \setminus V_0^s(w). \end{cases} \quad (8.41)$$

If  $w$  is a minimizer of  $\Lambda_\mu(\cdot)$  then  $t = 0$  is a minimizer of  $\tilde{\Lambda}(\cdot)$ . Hence,  $0 \in \partial \tilde{\Lambda}(0)$ , which is equivalent to (8.34).  $\square$

**Theorem 8.5** *If Assumptions 8.1 and 8.2 hold, then there exists at least an equilibrium.*

*Moreover, if  $(\sigma_n)_{n \in \mathbb{N}}$  is a sequence of positive real numbers tending to 0, and if  $(w_n, Q_n(\cdot), \mu_n)$  is a sequence of solutions of (8.8)-(8.10) with  $\sigma = \sigma_n$ , then the sequence is relatively compact and each cluster point is an equilibrium in the sense of Definition 8.5.*

**Proof** Let  $\theta$  be an element of  $[0, 1]$ . Let  $(w, Q(\cdot), \mu)$  be an equilibrium. The arguments developed in the first step of the proof of Lemma 8.4 hold. Therefore, if  $(w, Q(\cdot), \mu)$  is an equilibrium then

$$\mu(x) = \frac{R(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R(y, w)^{\frac{\theta}{1-\theta}} dy}, \quad \forall x \in X.$$

#### Existence of equilibria.

Let  $\bar{\sigma}$  be a positive real number. From Theorem 8.2 for every  $\sigma \in (0, \bar{\sigma}]$ , there exists an equilibrium  $(w_\sigma, Q_\sigma(\cdot), \mu_\sigma(\cdot))$ .

Moreover, Lemma 8.2 yields the existence of a compact set  $Y \subset (0, +\infty)^N$  such that  $w_\sigma \in Y$  for every  $\sigma \in (0, \bar{\sigma}]$ . Let  $\sigma_n$  be a sequence in  $(0, \bar{\sigma}]$  converging to 0. The compactness of  $Y$  yields the existence of  $w \in Y$  such that, up to the extraction of a subsequence,  $w_{\sigma_n} \rightarrow w$  when  $n$  tends to  $+\infty$ . We deduce that

$$R_{\sigma_n}(\cdot, w_{\sigma_n}) \rightarrow R(\cdot, w)$$

uniformly on  $X$ . Let us introduce for every  $x \in X$ ,

$$\mu(x) = \frac{R(x, w)^\alpha}{\int_X R(y, w)^\alpha dy} \text{ and } Q(x) = (1 - \theta)R(x, w)\mu(x).$$

Let us show that the triplet  $(\mu, Q(\cdot), w)$  is an equilibrium.

We only need to check that  $w$  is the unique minimizer of  $\Lambda_\mu(\cdot)$ .

We proved in the proof of Lemma 8.4 that for every  $n$ ,

$$\mu_{\sigma_n}(x) = \frac{R_{\sigma_n}(x, w_{\sigma_n})^\alpha}{\int_X R_{\sigma_n}(y, w_{\sigma_n})^\alpha dy}, \quad \forall x \in X.$$

Then  $\mu_{\sigma_n}$  converges to  $\mu$  uniformly on  $X$ . The same arguments developed in the proof of Lemma 8.3 yield that the minimizers of

$$(0, +\infty)^N \ni z \mapsto \int_X R_{\sigma_n}(x, z) d\mu_{\sigma_n}(x) - \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds$$

converge to the minimizer of  $\Lambda_\mu(\cdot)$  when  $n$  tends to  $+\infty$ . Therefore,

$$w \in \operatorname{argmin}_{(0, +\infty)^N} \Lambda_\mu(\cdot)$$

and  $(\mu, Q(\cdot), w)$  is an equilibrium.  $\square$

## 8.6 Conclusion

We have introduced a model that links the labour market and the rental market for individuals. Given the locations of the workplaces, the model allows one to determinate the distribution of housing for the workers, the rental price and the wages. First, we have proved an existence and a uniqueness result in a regular case. Existence is based on two reformulations of the equilibrium conditions. The market clearing conditions on the labour market are reformulated as a convex optimisation problem which yields a priori bounds on the collection of wages. Then, the market clearing conditions on the rental market for individuals and the mobility condition yield an explicit formula for the distribution of residences. These two reformulations have been sufficient to obtain the existence of equilibria, with a fixed-point strategy. By exploiting the fact that for  $\theta = 0$ , the unique distribution satisfying the market clearing conditions on the rental market for individuals and the mobility condition is the uniform probability measure on  $X$ , we have proven the uniqueness of equilibria. Using the implicit function theorem, we have extended the uniqueness property for  $\theta > 0$ .

Then, we have extended the model to telecommuting and to the limit case when some regularity properties are lost. In the home-based telecommuting model, there is one type of workers. Every worker can choose to work from home or on-site. However, there are some jobs for which telecommuting is not possible. In this case, we need to consider several types of workers. This is the extension addressed in the next chapter.

## 8.A Appendix: a random revenue model

Let us consider  $X$  the closure of a bounded domain of  $\mathbb{R}^d$ , for every  $i \in \{1, \dots, N\}$   $c_i(\cdot)$  be a continuous function with non-negative values, and  $c_0(\cdot)$  be the zero function.

In this paragraph, we wish to prove that  $R_\sigma(\cdot, \cdot)$ , defined by

$$R_\sigma(x, w) = \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right) \quad \forall (x, w) \in X \times (0, +\infty)^K$$

can be viewed as an expected revenue and is an approximation of  $R(\cdot, \cdot)$  defined by

$$R(x, w) = \max_{i=0}^N w_i - c_i(x), \quad \forall (x, w) \in X \times (0, +\infty)^K.$$

We assume that the agents are heterogeneous in their ability to obtain a well-paid job. An agent at the position  $x \in X$  and choosing  $i$  receives the revenue

$$w_i + a_i - c_i(x),$$

where  $a_i$  is an idiosyncratic noise following the Gumbel law with parameters  $-\sigma\gamma$  and  $\sigma$ ,  $\sigma > 0$  is fixed and

$$\gamma = - \int_0^{+\infty} \ln(\xi) e^{-\xi} d\xi. \quad (8.42)$$

is the Euler–Mascheroni constant. The cumulative distribution of  $a_i$  is given by

$$F(y) = \exp \left( - \exp \left( - \frac{y}{\sigma} - \gamma \right) \right), \quad \forall y \in \mathbb{R},$$

and its mean value is 0. We suppose that the random variables  $(a_j)_{j=0, \dots, N}$  are independent and that agents choose the index  $i$  for which  $w_i + a_i - c_i(x)$  is maximal. Therefore, the probability for  $i$  to be chosen is given by

$$\begin{aligned} \mathbb{P}(w_i + a_i - c_i(x) \geq w_j + a_j - c_j(x), \forall j \neq i) &= \mathbb{P}(w_i - c_i(x) - (w_j - c_j(x)) \geq a_j - a_i, \forall j \neq i) \\ &= \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}}. \end{aligned}$$

Therefore, given  $w \in (0, +\infty)^N$  a collection of wages, we define the expected revenue of an agent at the position  $x \in X$  by

$$R_\sigma(x, w) = \mathbb{E} \left( \max_{i=0}^N (w_i + a_i - c_i(x)) \right).$$

**Proposition 8.1** For every  $(x, w) \in X \times (0, +\infty)^N$ ,

$$R_\sigma(x, w) = \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right). \quad (8.43)$$

**Proof** The independence and the law of the random variables  $(a_i)_{i=0, \dots, N}$  yield that the cumulative distribution of the random variable  $\max_{i=0}^N (w_i + a_i - c_i(x))$  is given for every  $y \in \mathbb{R}$  by

$$\begin{aligned} G(y) &= \mathbb{P}(\max_{i=0}^N (w_i + a_i - c_i(x)) \leq y) \\ &= \prod_{i=0}^N \mathbb{P}(a_i \leq y - w_i + c_i(x)) \\ &= \prod_{i=0}^N F(y - w_i + c_i(x)) \\ &= \prod_{i=0}^N \exp \left( - \exp \left( - \frac{y - w_i + c_i(x)}{\sigma} - \gamma \right) \right) \\ &= \exp \left( - \sum_{i=0}^N \exp \left( - \frac{y - w_i + c_i(x)}{\sigma} - \gamma \right) \right) \\ &= \exp \left( - \exp \left( - \frac{y}{\sigma} \right) \sum_{i=0}^N \exp \left( \frac{w_i - c_i(x)}{\sigma} - \gamma \right) \right). \end{aligned}$$

We deduce that the density of probability of  $\max_{i=0}^N (w_i + a_i - c_i(x))$  is given for every  $y \in \mathbb{R}$  by

$$g(y) = \exp \left( - \exp \left( - \frac{y}{\sigma} \right) \sum_{i=0}^N \exp \left( \frac{w_i - c_i(x)}{\sigma} - \gamma \right) \right) \exp \left( - \frac{y}{\sigma} \right) \sum_{i=0}^N \exp \left( \frac{w_i - c_i(x)}{\sigma} - \gamma \right) \frac{1}{\sigma}, \quad \forall y \in \mathbb{R}.$$

Let  $\varphi : [0, +\infty) \rightarrow \mathbb{R}$  be defined by

$$\varphi(z) = -\sigma \ln(z) - \sigma\gamma + \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right).$$

We see that

$$R_\sigma(x, w) = \mathbb{E} \left( \max_{i=0}^N (w_i + a_i - c_i(x)) \right) \quad (8.44)$$

$$= \int_{\mathbb{R}} yg(y)dy \quad (8.45)$$

$$= \int_0^{+\infty} \varphi(z)g \circ \varphi(z) |\varphi'(z)| dz \quad (8.46)$$

$$= \int_0^{+\infty} \left( -\sigma \ln(z) - \sigma\gamma + \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right) \right) e^{-z} dz \quad (8.47)$$

$$= \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right) \int_0^{+\infty} e^{-z} dz - \sigma\gamma \int_0^{+\infty} e^{-z} dz + \sigma\gamma \quad (8.48)$$

$$= \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_i - c_i(x)}{\sigma}} \right), \quad (8.49)$$

where (8.46) comes from the change of variable formula, (8.47) comes from direct computations, (8.48) uses (8.42), while (8.49) comes from the fact that

$$\int_0^{+\infty} e^{-z} dz = 1.$$

□

**Remark 8.7** For every  $i \in \{1, \dots, N\}$ ,  $(x, w) \in X \times (0, +\infty)^N$  and  $\sigma > 0$ ,

$$\frac{\partial R_\sigma}{\partial w_i}(x, w) = \frac{e^{\frac{w_i - c_i(x)}{\sigma}}}{\sum_{k=0}^N e^{\frac{w_k - c_k(x)}{\sigma}}} = \mathbb{P}(w_i + a_i - c_i(x) > w_j + a_j - c_j(x), \forall j \neq i).$$

Thus  $\frac{\partial R_\sigma}{\partial w_i}(x, w)$  is the probability for an agent at the position  $x$  to choose a job at the workplace with label  $i$ .

**Proposition 8.2** For every  $(x, w) \in X \times (0, +\infty)^N$  and every  $\sigma > 0$ ,

$$R(x, w) \leq R_\sigma(x, w) \leq \sigma \ln(N) + R(x, w). \quad (8.50)$$

**Remark 8.8** The latter proposition ensures the uniform convergence of  $R_\sigma(\cdot, \cdot)$  towards  $R(\cdot, \cdot)$  when  $\sigma \rightarrow 0$ .

## Chapter 9

# Generalization for different types of workers

Here, our purpose is to extend the model introduced in Chapter 8 to the case when there are several types of workers. Except for that aspect, the general assumptions are the same as in Chapter 8. In the model proposed below, the total number of workers of a given type may differ from one type to another. Naturally, depending on their type, the workers have different productivities, and, from the law of supply and demand, earn different wages. Regarding the rental market for individuals, this implies that at the same location, households with different income will pay the same rent. Thus, several types of tenants compete in the rental market: is this heterogeneity in income sufficient to cause segregation ?

Our goal is to extend the system (8.8)-(8.10) to the situation described above and to prove the existence of solutions. As in Chapter 8, we prove that for a fixed distribution of living places, the equilibrium on the labour market can be seen as a convex optimization problem. This allow us to establish a priori bounds on the collection of wages which satisfy the market clearing conditions on the labour market. On the other hand, given a collection of wages, the equilibrium on the rental market for individuals and the mobility condition no longer yield an explicit formula for the distribution of residences. Instead, we interpret these conditions, first as the equilibrium conditions in a finite number of coupled static mean field games, second as a Nash equilibrium in a game with a finite number of players. Finally, we introduce an optimization problem whose optimizers are Nash equilibria of the game. Without an explicit formula, we are no longer able to prove that the distribution of living places of a given type of workers is continuous. Therefore, to prove the existence of an equilibrium, we have to drop the continuity condition in the definition of the latter. The above-mentioned two reformulations with a weaker definition of an equilibrium are sufficient to prove existence via a fixed-point strategy. By contrast, we are not able to extend the uniqueness result contained in Chapter 8, since this was based on the explicit formula that is no longer available.

The chapter is structured as follows. In Section 9.1 we present the model, specify the behaviour of agents, give the mathematical definition of an equilibrium, and list the standing assumptions used throughout the chapter. Section 9.2 deals with the existence of equilibria.

**Notations.** Throughout the chapter, we denote by  $M_{N \times K}(0, +\infty)$  the space of real matrices with positive entries, with  $N$  rows and  $K$  columns. If  $Z \in M_{N \times K}(\mathbb{R})$ , then we denote by  $Z_k^i$  the entry of  $Z$  at the  $i^{th}$  row and  $k^{th}$  column. We also denote by  $Z^i$  and  $Z_k$  the  $i^{th}$  row and the  $k^{th}$  column respectively. The notation  $\|\cdot\|_\infty$  is used for the infinity norm, i.e for every  $Z \in M_{N \times K}(0, +\infty)$ ,

$$\|Z\| = \max_{i=1}^N \max_{k=1}^K |Z_k^i|.$$

The notation *a.e.* will stand for *almost everywhere*. Note that *almost everywhere* will always implicitly mean *almost everywhere with respect to Lebesgue measure*.

## 9.1 The model

Let  $X$  be the closure of a bounded domain of  $\mathbb{R}^d$ :  $X$  is the set of all possible places of residence. We assume that there are  $N \in \mathbb{N}^*$  workplaces and  $K \in \mathbb{N}^*$  types of workers. As in Chapter 8, for any  $i \in \{1, \dots, N\}$ , two functions are associated to the  $i^{\text{th}}$  workplace:

1. The transportation costs  $c_i : X \rightarrow [0, +\infty)$ .
2. The demand of labour  $L_i : [0, +\infty)^K \rightarrow [0, +\infty)^K$ .

Here,  $c_i(x)$  corresponds to the transport cost to reach the  $i^{\text{th}}$  workplace coming from  $x \in X$ , and  $L_i(W^i)$  corresponds to the labour demand when the collection of wages in the  $i^{\text{th}}$  workplace is  $W^i \in (0, +\infty)^K$ . Note that the function  $L_i(\cdot)$  takes its values in  $\mathbb{R}^K$ , and the  $k^{\text{th}}$  component of  $L_i(W^i)$  is the demand of workers of type  $k$ .

In Chapter 8, we used the fact that the demand of labour was continuous and defined on  $(0, +\infty)$  to deduce that it admitted a potential. This allowed us to see the market clearing conditions on the labour market (8.8) as an optimization problem, see Lemma 8.2. Here, since the functions  $L_i(\cdot)$  are multivariate, they may not admit any potential. We are thus led to assume that each workplace is the location of a firm which aims at maximizing its profits. For every  $i \in \{1, \dots, N\}$ , we suppose that the output of the firm occupying the workplace  $i$  is given by the production function

$$F^i : [0, +\infty)^K \rightarrow [0, +\infty).$$

Then, for a given collection of wages  $W^i \in (0, +\infty)^K$ , the  $i^{\text{th}}$  firm aims at solving:

$$(F^i)^*(W^i) = \sup \{F^i(\ell) - W^i \cdot \ell : \ell \in [0, +\infty)^K\}. \quad (9.1)$$

Here,  $(F^i)^*(\cdot)$  is the profit of the  $i^{\text{th}}$  firm, and  $\ell \in [0, +\infty)^K$  is the level of employment for each type. At least formally, the envelope theorem yields the demand of labour of this firm:

$$L_i(W^i) = -D(F^i)^*(W^i). \quad (9.2)$$

We thus see that under the assumption made and when the envelope theorem can be applied, the functions  $L_i(\cdot)$  admit a potential and are given by (9.2).

### The behaviour of the agents

The agents behave as in Chapter 8. Therefore, Lemma 8.1 and Corollary 8.1 hold. In particular, the demand of surface by an agent with revenue  $R$  when the rental price by surface unit is  $Q$  is given by

$$S_\theta(R, Q) = (1 - \theta) \frac{R}{Q}, \quad (9.3)$$

and her utility is

$$\mathcal{U}_\theta(R, Q) = \theta^\theta (1 - \theta)^{1-\theta} \frac{R}{Q^{1-\theta}}. \quad (9.4)$$

Moreover, we assume that the revenue of an agent of type  $k \in \{1, \dots, K\}$  who lives at the position  $x \in X$  is given by:

$$R_\sigma(x, W_k) = \sigma \ln \left( \sum_{i=0}^N e^{\frac{w_k^i - c_i(x)}{\sigma}} \right), \quad (9.5)$$

when the collection of wages proposed is  $W_k \in (0, +\infty)^N$ . Here,  $\sigma$  is a positive constant,  $W_k^0 > 0$  is the income of an agent when he chooses to stay at home, and  $c_0(\cdot) \equiv 0$ . More details on  $R_\sigma(\cdot, \cdot)$  are provided in Appendix 8.A. In this setting, the probability for an agent of type  $k$  living at the position  $x \in X$  to choose to work in the  $i$ th workplace is given by the Gibbs distribution:

$$\frac{\partial R_\sigma}{\partial W_k^i}(x, W_k) = \frac{e^{\frac{W_k^i - c_i(x)}{\sigma}}}{\sum_{j=0}^N e^{\frac{W_k^j - c_j(x)}{\sigma}}}. \quad (9.6)$$

Before defining the equilibria, let us define the labour supply and the demand of surface. Let  $(m_k)_{k=1, \dots, K}$  be a sequence of positive real numbers. Let  $\Theta_k$  be the set of admissible distributions of residences for the agents of type  $k$ :

$$\Theta_k = \{\nu \in \mathcal{M}_+(X) : \nu(X) = m_k\},$$

and let  $\Theta$  be the set of admissible distributions of residences:

$$\Theta = \{\mu \in \mathcal{M}_+(X)^K : \mu_k \in \Theta_k\}.$$

We say that the distribution of residences is given by  $\mu \in \Theta$  if for  $1 \leq k \leq K$ , the distribution of residences for the workers of type  $k$  is given by  $\mu_k$ . Let  $\Theta_{ac}$  be the subset of  $\Theta$  made of the distributions  $\mu$  such that every coordinate  $\mu_k$  is absolutely continuous with respect to Lebesgue measure.

**Definition 9.1** For any distribution of residences  $\mu \in \Theta$  and any collection of wages  $W \in M_{N \times K}(0, +\infty)$ , the labour supply of agents of type  $k$  at the  $i$ th workplace is given by

$$\int_X \frac{\partial R_\sigma}{\partial W_k^i}(x, W_k) d\mu_k(x). \quad (9.7)$$

**Definition 9.2** For any distribution of residences  $\mu \in \Theta$ , any collection of wages  $W \in M_{N \times K}(0, +\infty)$ , and a rental price function  $Q : X \rightarrow [0, +\infty)$ , the demand of surface is given by the non negative measure

$$\sum_{k=1}^K S_\theta(R_\sigma(\cdot, W_k), Q(\cdot)) \mu_k.$$

These definitions are the direct extensions of Definition 8.1 and 8.2.

We are now ready to define the notion of equilibrium:

**Definition 9.3** The triplet  $(W, Q(\cdot), \mu) \in M_{N \times K}(0, +\infty) \times L^1(X) \times \Theta_{ac}$  is an equilibrium if

$$\int_X \frac{\partial R_\sigma}{\partial W_k^i}(x, W_k) d\mu_k(x) = -\frac{\partial(F^i)^*}{\partial W_k^i}(W^i), \quad \forall (i, k) \in \{1, \dots, N\} \times \{1, \dots, K\}, \quad (9.8)$$

$$\sum_{k=1}^K S_\theta(R_\sigma(\cdot, W_k), Q(\cdot)) \mu_k = 1, \quad \text{for a.e. } x \in X, \quad (9.9)$$

$$\int_X \mathcal{U}_\theta(R_\sigma(x, W_k), Q(x)) d\mu_k(x) = \sup_{\nu \in \Theta_k} \int_X \mathcal{U}_\theta(R_\sigma(x, W_k), Q(x)) d\nu(x), \quad \forall k \in \{1, \dots, K\}. \quad (9.10)$$

The system (9.8)-(9.10) is the counterpart of (8.8)-(8.10) in Chapter 8. Note that in Definition 9.3, we do not require that  $\mu$  be continuous. This is the main difference with the definition of equilibria in Definition 8.3. This weaker definition will make it possible to prove the existence of equilibria in the present setting. Indeed, given  $W \in M_{N \times K}(0, +\infty)$ , a solution of (9.9)-(9.10) will not be expected to be continuous.



### 9.1.1 Standing assumptions and a preliminary result

Let us list the standing assumptions:

**Assumption 9.1** For every  $i \in \{1, \dots, N\}$ ,  $F^i : [0, +\infty)^K \rightarrow [0, +\infty)$ , the production function of the  $i^{\text{th}}$  workplace, satisfies:

1.  $F^i(\cdot)$  is increasing with respect to each coordinate on  $(0, +\infty)^K$
2. for every  $k_0 \in \{1, \dots, K\}$  and  $\bar{\ell} \in (0, +\infty)^{K-1}$ ,

$$\lim_{\ell_{k_0} \rightarrow +\infty} F^i(\bar{\ell}_1, \dots, \bar{\ell}_{k_0-1}, \ell_{k_0}, \bar{\ell}_{k_0}, \dots, \bar{\ell}_{K-1}) = +\infty$$

3.  $F^i(\cdot)$  is strictly concave on  $(0, +\infty)^K$
4.  $F^i(\cdot)$  is continuous on  $[0, +\infty)^K$  and of class  $C^1$  on  $(0, +\infty)^K$
5.  $F(\ell)/\|\ell\| \rightarrow 0$  when  $\|\ell\|$  tends to  $+\infty$ .
6. If  $(\ell^n)_{n \in \mathbb{N}}$  is a bounded sequence of  $(0, +\infty)^K$  such that  $\min_{k=1}^K \ell_k^n \rightarrow 0$  when  $n$  converges to  $+\infty$ , then

$$\max_{k=1}^K \frac{\partial F^i}{\partial \ell_k}(\ell^n) \rightarrow +\infty$$

when  $n$  tends to  $+\infty$ .

**Assumption 9.2** For every  $i \in \{1, \dots, N\}$ ,  $c_i : X \rightarrow [0, +\infty)$ , the transport cost associated to the  $i^{\text{th}}$  workplace, is continuous.

Before ending this section, we show that Assumption 9.1 yields that for every  $i$  the problem (9.1) admits a unique solution in  $(0, +\infty)^K$ . In particular,  $DF^i(\cdot)$  induces a homeomorphism on  $(0, +\infty)^K$ .

**Lemma 9.1** Under Assumption 9.1, fix any  $w \in (0, +\infty)^K$ . Then, for every  $i \in \{1, \dots, N\}$  the problem

$$\sup_{\ell \in [0, +\infty)^K} \{F^i(\ell) - w \cdot \ell\} \quad (9.11)$$

admits a unique solution in  $(0, +\infty)^K$ . Moreover,  $DF^i(\cdot)$  is a homeomorphism on  $(0, +\infty)^K$ .

**Proof** Fix any  $w \in (0, +\infty)^K$ . It is clear that Assumption 9.1-5 implies that there exists a positive constant  $L$  such that

$$\sup_{\ell \in [0, +\infty)^K} \{F^i(\ell) - w \cdot \ell\} = \sup_{\ell \in [0, L]^K} \{F^i(\ell) - w \cdot \ell\}.$$

Moreover, if for some  $\ell \in [0, +\infty)^K$  there exists  $k \in \{1, \dots, K\}$  such that  $\frac{\partial F^i}{\partial \ell_k}(\ell) > w_k$ , then it is possible to build a competitor  $\tilde{\ell}$  strictly better than  $\ell$ , i.e.

$$F^i(\tilde{\ell}) - w \cdot \tilde{\ell} > F^i(\ell) - w \cdot \ell.$$

This comes from the fact that for a positive constant  $T$  small enough, the function

$$[0, T] \ni t \mapsto F^i(\ell_1, \dots, \ell_{k-1}, \ell_k + t, \ell_{k+1}, \dots, \ell_K) - w \cdot \ell - w_k t$$

is increasing.

Therefore, we can restrict our attention to  $\ell$  such that  $\frac{\partial F^i}{\partial \ell_k}(\ell) \leq w_k$  for every  $k \in \{1, \dots, K\}$ . Let us introduce the set

$$\Gamma = \left\{ \ell \in (0, +\infty)^K : \ell_k \leq L \text{ and } \frac{\partial F^i}{\partial \ell_k}(\ell) \leq w_k, \forall k \in \{1, \dots, K\} \right\}.$$

From Assumption 9.1-6, there exists  $\varepsilon > 0$  such that

$$\ell \in \Gamma \Rightarrow \ell_k \geq \varepsilon, \quad \forall k \in \{1, \dots, K\}.$$

Finally,  $\Gamma$  is a compact subset of  $(0, +\infty)^K$  such that

$$\sup_{\ell \in [0, +\infty)^N} \{F^i(\ell) - w \cdot \ell\} = \sup_{\ell \in \Gamma} \{F^i(\ell) - w \cdot \ell\}.$$

Therefore, there exist solutions of the problem. The strict concavity of  $F^i(\cdot)$  yields their uniqueness.

Since (9.11) admits a unique solution in  $(0, +\infty)^K$  and that  $DF^i(\cdot)$  is continuous, then  $DF^i(\cdot)$  is a homeomorphism on  $(0, +\infty)^K$ .  $\square$

## 9.2 Existence of equilibria

Below, we prove the existence of equilibria via a fixed-point strategy. As in Chapter 8, we will build a map  $\Psi(\cdot)$  whose fixed points are equilibria in the sense of Definition 9.3. Its construction will be based on the same ideas as in Chapter 8. Nevertheless, several differences can be emphasized. First, the reformulation of (9.9)-(9.10) is more challenging than in Chapter 8. Indeed, without an explicit formula for the distribution of residences, we introduce an optimization problem on  $\Theta$  whose maximizers are associated to solutions of (9.9)-(9.10) (see Corollary 9.1 and Lemma 9.5 below). Second, the uniqueness of the maximizers of the latter optimization problem does not hold. Therefore, to prove the existence of fixed-points, we will use Kakutani fixed-point theorem in the space  $\Theta$ .

Lemma 9.2 below is a straightforward extension of Lemma 8.2. It characterizes the unique collection of wages for which the equilibrium on the labour market holds.

**Lemma 9.2** *Under Assumption 9.1 and 9.2, fix any distribution  $\mu \in \Theta$ . Then, (9.8) holds for  $W \in M_{N \times K}(0, +\infty)$  if and only if  $W$  is the unique minimizer of*

$$\min_{Z \in M_{N \times K}(0, +\infty)} \left\{ \sum_{k=1}^K \int_X R_\sigma(x, Z_k) d\mu_k(x) + \sum_{i=1}^N (F^i)^*(Z^i) \right\}. \quad (9.12)$$

Moreover, the minimizer  $W$  of (9.12) belongs to a convex compact set  $Y$  which is independent of  $\mu$ .

**Proof** Let us first introduce the strictly convex function  $\Lambda_\mu : M_{N \times K}(0, +\infty) \rightarrow \mathbb{R}$  defined by

$$\Lambda_\mu(Z) = \sum_{k=1}^K \int_X R_\sigma(x, Z_k) d\mu_k(x) + \sum_{i=1}^N (F^i)^*(Z^i).$$

As in the proof of Lemma 8.2 in Chapter 8, we proceed in two steps. The first step is dedicated to the existence of minimizers. The arguments will be in the same spirit as those used in the first step in the proof of Lemma 8.2. However, since we manipulate the functions  $F^i(\cdot)$  instead of the function  $L^i(\cdot)$ , we have decided to write the full arguments. The second step deals with a characterization of the minimizer of (9.12).

**Step 1: Existence of a minimizer** As in the proof of Lemma 8.2, this step will be split into two:

- first, we establish a priori bounds and deduce that there exists a compact and convex set  $Y \subset M_{N \times K}(0, +\infty)$  such that

$$\min_{z \in Y} \Lambda_\mu(Z) = \min_{Z \in M_{N \times K}(0, +\infty)} \Lambda_\mu(Z);$$

- second, we use the direct method of the calculus of variations and deduce the existence of solutions. We use the strict convexity of  $\Lambda_\mu(\cdot)$  to obtain their uniqueness.

**A first a priori bound.** Let us fix  $Z$  and  $\hat{Z}$  two elements of  $M_{N \times K}(0, +\infty)$  such that

$$\Lambda_\mu(Z) \leq \Lambda_\mu(\hat{Z}).$$

Note that

$$\Lambda_\mu(Z) = \sum_{k=1}^K \int_X R_\sigma(x, Z_k) d\mu_k(x) + \sum_{i=1}^N (F^i)^*(Z^i) \geq \sum_{k=1}^K (\|Z_k\|_\infty - \|c\|_{L^\infty}) m_k + \sum_{i=1}^N (F^i)^*(Z^i),$$

since  $R_\sigma(x, Z_k) \geq R_\sigma(x, Z_k) \geq \|Z_k\|_\infty - \|c\|_{L^\infty}$ . Therefore

$$\Lambda_\mu(\hat{Z}) \geq \sum_{k=1}^K (\|Z_k\|_\infty - \|c\|_{L^\infty}) m_k + \sum_{i=1}^N (F^i)^*(Z^i),$$

and

$$\sum_{k=1}^K \|Z_k\|_\infty m_k \leq \sum_{k=1}^K \int_X R_\sigma(x, \hat{Z}_k) d\mu_k(x) + \|c\|_{L^\infty} \sum_{k=1}^K m_k + \sum_{i=1}^N \left( (F^i)^*(\hat{Z}^i) - (F^i)^*(Z^i) \right). \quad (9.13)$$

Using the fact that

$$R_\sigma(x, \hat{Z}_k) \leq \left\| \hat{Z}_k \right\|_\infty + \sigma \ln(N),$$

we deduce

$$\sum_{k=1}^K \int_X R_\sigma(x, \hat{Z}_k) d\mu_k(x) \leq \sum_{k=1}^K \left( \left\| \hat{Z}_k \right\|_\infty + \sigma \ln(N) \right) m_k. \quad (9.14)$$

Then, using the convexity of  $(F^i)^*(\cdot)$ , (9.13) and (9.14) yield

$$\sum_{k=1}^K \|Z_k\|_\infty m_k \leq \sum_{k=1}^K \left( \left\| \hat{Z}_k \right\|_\infty + \sigma \ln(N) + \|c\|_{L^\infty} \right) m_k - \sum_{i=1}^N D(F^i)^*(\hat{Z}^i)(Z^i - \hat{Z}^i).$$

The monotonicity of the functions  $(F^i)^*(\cdot)$  implies that  $-\frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i)$  is positive for every  $k \in \{1, \dots, K\}$ . Therefore, the latter inequality yields

$$\sum_{k=1}^K \|Z_k\|_\infty m_k \leq \sum_{k=1}^K \left( \left\| \hat{Z}_k \right\|_\infty + \sigma \ln(N) + \|c\|_{L^\infty} \right) m_k - \sum_{i=1}^N \sum_{k=1}^K D(F^i)^*(\hat{Z}^i) \|Z^i\|_\infty.$$

Since,  $\|Z\|_\infty \min_{k=1}^K m_k \leq \sum_{k=1}^K \|Z_k\|_\infty m_k$  and  $\|Z^i\|_\infty \leq \|Z\|_\infty$ , we obtain

$$\|Z\|_\infty \left( \min_{k=1}^K m_k + \sum_{i=1}^N \sum_{k=1}^K D(F^i)^*(\hat{Z}^i) \right) \leq \sum_{k=1}^K \left( \left\| \hat{Z}_k \right\|_\infty + \sigma \ln(N) + \|c\|_{L^\infty} \right) m_k.$$

Thus, if  $Z$  and  $\hat{Z}$  belong to  $M_{N \times K}(0, +\infty)$  and satisfy

- $\Lambda_\mu(Z) \leq \Lambda_\mu(\hat{Z})$ ,
- $-\sum_{i=1}^N \sum_{k=1}^K \frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i) < \min_{k=1}^K m_k$ ,

then

$$\|Z\|_\infty \leq \frac{\sum_{k=1}^K \left( \|\hat{Z}_k\|_\infty + \sigma \ln(N) + \|c\|_{L^\infty} \right) m_k}{\min_{k=1}^K m_k + \sum_{i=1}^N \sum_{k=1}^K \frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i)}. \quad (9.15)$$

Note that there exists  $\hat{Z} \in M_{N \times K}(0, +\infty)$  such that

$$-\sum_{i=1}^N \sum_{k=1}^K \frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i) < \min_{k=1}^K m_k.$$

Indeed, let us take  $L \in M_{N \times K}(0, +\infty)$  such that all the entries of  $L$  are less than  $\min_{k=1}^K m_k / (NK)$ , and set for every  $i \in \{1, \dots, N\}$ ,  $\hat{Z}^i = D(F^i)(L^i)$ . From the envelope theorem,  $L^i = -D(F^i)^*(\hat{Z}^i)$ , hence

$$\hat{Z}^i = D(F^i)(L^i) = D(F^i)(-D(F^i)^*(\hat{Z}^i)).$$

We see that

$$-\sum_{i=1}^N \sum_{k=1}^K \frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i) = \sum_{i=1}^N \sum_{k=1}^K L_k^i < \min_{k=1}^K m_k.$$

**A second a priori bound.** We are now interested in finding a bound from below. Take  $Z \in M_{N \times K}(0, +\infty)$ ; let us show that if there exists  $(i, k) \in \{1, \dots, N\} \times \{1, \dots, K\}$  such that

$$0 > m_k + \frac{\partial(F^i)^*}{\partial W_k^i}(Z^i), \quad (9.16)$$

then it is possible to build a competitor  $\tilde{Z}$  strictly better than  $Z$ , i.e. such that

$$\Lambda_\mu(\tilde{Z}) < \Lambda_\mu(Z). \quad (9.17)$$

Indeed, if (9.16) holds, then

$$\frac{\partial \Lambda_\mu}{\partial W_k^i}(Z) = \int_X \frac{\partial R_\sigma}{\partial W_k^i}(x, Z_k) d\mu_k(x) + \frac{\partial(F^i)^*}{\partial W_k^i}(Z^i) < 0,$$

since  $R_\sigma(x, \cdot)$  is non-expansive. Therefore, the competitor

$$\tilde{Z} = Z + t\psi(i, k),$$

with  $\psi(i, k) \in M_{N \times K}(\mathbb{R})$  such that

$$\psi(i, k)_\kappa^j = \delta_{(i,k)}(j, \kappa), \quad \forall (j, \kappa) \in \{1, \dots, N\} \times \{1, \dots, K\}$$

with  $t > 0$  small enough, satisfies (9.17).

We have just proven that if (9.16) holds, then  $Z$  is not a minimizer of  $\Lambda_\mu$ . Therefore, we can restrict our attention to  $Z$  such that  $-D(F^i)^*(Z) \leq m$ , where the inequality is understood componenwise. Similarly, from (9.15), we may consider only those  $Z$  such that  $\|Z\|_\infty \leq M$ , where

$$M = \inf \left\{ \frac{\sum_{k=1}^K \left( \|\hat{Z}_k\|_\infty + \sigma \ln(N) + \|c\|_{L^\infty} \right) m_k}{\min_{k=1}^K m_k - \sum_{i=1}^N \sum_{k=1}^K \frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i)} : \hat{Z} \in M_{N \times K}((0, +\infty)), -\sum_{i=1}^N \frac{\partial(F^i)^*}{\partial W_k^i}(\hat{Z}^i) < \min_{k=1}^K m_k \right\}.$$

Note that  $M$  is well defined as the infimum of a non-empty subset of  $\mathbb{R}_+$ .

For  $i \in \{1, \dots, N\}$ , consider the set

$$Y_i = \{z \in (0, +\infty)^K : \|z\|_\infty \leq M \text{ and } -D(F^i)^*(z) \leq m\}.$$

We aim at proving that there exists  $\varepsilon > 0$  such that if  $Z$  belongs to  $\tilde{Y} = \prod_{i=1}^N Y_i$ , then

$$Z_k^i \geq \varepsilon, \quad \forall (i, k) \in \{1, \dots, N\} \times \{1, \dots, K\}.$$

Fix any  $i \in \{1, \dots, N\}$ . Observe that for any  $z \in Y_i$ ,

$$(F^i)^*(z) = F^i(-D(F^i)^*(z)) + z \cdot D(F^i)^*(z) \leq F^i(m), \quad (9.18)$$

since  $F^i(\cdot)$  is non decreasing with respect to each of its coordinate and  $z \cdot D(F^i)^*(z)$  is non positive. Let  $\bar{\ell}$  be an arbitrary vector in  $(0, +\infty)^{K-1}$ ; we observe that

$$(F^i)^*(z) = \max_{\ell \in [0, +\infty)^K} \{F^i(\ell) - z \cdot \ell\} \quad (9.19)$$

$$\geq \max_{\ell \in [0, +\infty)^K} \left\{ F^i(\ell) - z_1 \ell_1 - M \sum_{k=2}^K \ell_k \right\} \quad (9.20)$$

$$\geq F^i(1/z_1, \bar{\ell}_1, \dots, \bar{\ell}_{K-1}) - 1 - M \sum_{k=2}^K \bar{\ell}_{k-1}, \quad (9.21)$$

where (9.19) comes from the definition of  $(F^i)^*(\cdot)$ , (9.20) holds by monotonicity, and (9.21) is deduced from the fact that  $(1/z_1, \bar{\ell}_1, \dots, \bar{\ell}_{K-1})$  is a vector in  $[0, +\infty)^K$ . From (9.18) and (9.21), we deduce that

$$F^i(1/z_1, \bar{\ell}_1, \dots, \bar{\ell}_{K-1}) \leq F^i(m) + 1 + M \sum_{k=2}^K \bar{\ell}_{k-1}.$$

Since  $F^i(1/z_1, \bar{\ell}_1, \dots, \bar{\ell}_{K-1}) \rightarrow +\infty$  when  $z_1$  tends to  $0^+$ , the latter inequality yields a bound from below for the first coordinate of  $z$ , which only depends on  $\bar{\ell}$ ,  $M$  and  $m$ . The same argument holds for all coordinates. Thus, we deduce that there exists  $\varepsilon > 0$  such that if  $z \in Y_i$  then  $z \geq \varepsilon$ .

Since this is true for every  $i$ , we can extend this property to the set  $\tilde{Y}$ , i.e. there exists  $\varepsilon > 0$  such that

$$Z \in \tilde{Y} \Rightarrow Z_k^i \geq \varepsilon, \quad \forall (i, k) \in \{1, \dots, N\} \times \{1, \dots, K\}.$$

The set

$$Y = \{Z \in M_{N \times K}(0, +\infty) : \varepsilon \leq Z_k^i \leq M, \forall (i, k) \in \{1, \dots, N\} \times \{1, \dots, K\}\}$$

is a compact and convex subset of  $M_{N \times K}((0, +\infty))$  which contains  $\tilde{Y}$ . Therefore,

$$\inf_{Z \in \tilde{Y}} \Lambda_\mu(Z) = \inf_{Z \in M_{N \times K}((0, +\infty))} \Lambda_\mu(Z).$$

**Conclusion.** Let us take a minimizing sequence  $(W^n)_{n \in \mathbb{N}}$  of the problem

$$\min_{Z \in Y} \Lambda_\mu(Z).$$

The compactness of  $Y$  and the continuity of  $\Lambda_\mu(\cdot)$  yield the existence of a minimizer  $W \in M_{N \times K}(0, +\infty)$ . The uniqueness is ensured by the strict convexity of  $\Lambda_\mu(\cdot)$ . We have obtained the existence and uniqueness of a solution of (9.12).

**Step 2: Characterization of the minimizer.** Since  $\Lambda_\mu(\cdot)$  is strictly convex and smooth,  $W \in M_{N \times K}(0, +\infty)$  is a minimizer of  $\Lambda_\mu(\cdot)$  if and only if for every  $(i, k) \in \{1, \dots, N\} \times \{1, \dots, K\}$ ,

$$\frac{\partial \Lambda_\mu}{\partial W_k^i}(W) = 0 \Leftrightarrow -\frac{\partial (F^i)^*}{\partial W_k^i}(W^i) = \int_X \frac{e^{\frac{W_k^i - c_i(x)}{\sigma}}}{\sum_{j=0}^N e^{\frac{W_k^j - c_j(x)}{\sigma}}} d\mu_k(x).$$

□

The following lemma addresses the stability of solutions of the problem (9.2) with respect to the variations of  $\mu$ .

**Lemma 9.3** *Under Assumptions 9.1 and 9.2, let  $(\mu_n)_{n \geq 0}$  be a sequence in  $\Theta$  and  $(W_n)$  be the sequence of associated minimizers in (9.12). If  $\mu_n \rightarrow \mu$  in the weak  $*$  topology, then  $(W_n)$  converges to  $W$ , the minimizer in (9.12) associated to  $\mu$ .*

**Proof** The proof follows exactly the same arguments as that of Lemma 8.3. □

Lemma 9.2 tells us that the market clearing conditions on the labour market, namely (9.8), are equivalent to a convex optimization problem. We now wish to propose an optimization problem whose maximizers are linked to the solutions of (9.9)-(9.10).

For this aim, given  $W \in M_{N \times K}(0, +\infty)$ , we are going to see in Lemma 9.4 below that a solution  $(\mu, Q(\cdot))$  of (9.9)-(9.10) can be seen as a solution of  $K$  coupled static mean field games. Then, in Lemma 9.5 below, we will see that there is a simultaneous game with  $K$  players such that each Nash equilibrium is a solution of the coupled static mean field games appearing in Lemma 9.4. Then, noticing that the payoffs of the players are the same, we will be able to propose an optimization problem whose maximizers are Nash equilibria of the simultaneous game; these maximizers are thus linked to a solution of (9.9)-(9.10).

**Lemma 9.4** *Under Assumption 9.1 and 9.2, fix any  $W \in M_{N \times K}(0, +\infty)$ . Let us introduce for every  $k \in \{1, \dots, K\}$  and  $\mu \in \Theta$ ,*

$$U(k, W, \mu; x) = \begin{cases} \theta^\theta \frac{R_\sigma(x, W_k)}{\left(\sum_{\kappa=1}^K R_\sigma(x, W_\kappa) \mu_\kappa^{ac}(x)\right)^{1-\theta}}, & \text{if } \frac{\mu_k}{m_k}(\cdot) \in \mathcal{P}_{ac}(X), \\ -\infty, & \text{otherwise,} \end{cases}, \quad \forall x \in X. \quad (9.22)$$

Equations (9.9) and (9.10) hold for a couple  $(Q(\cdot), \mu) \in L^1(X) \times \Theta_{ac}$  if and only if

1. for every  $k \in \{1, \dots, K\}$ ,  $\mu_k \in \Theta_k$  is an equilibrium for the static mean field game in which the utility of the agents of type  $k$  is  $U(k, W, \mu; \cdot)$

2.

$$Q(x) = (1 - \theta) \sum_{k=1}^K R_\sigma(x, W_k) \mu_k^{ac}(x), \quad \text{for a.e. } x \in X. \quad (9.23)$$

The two points above are equivalent to

1. for all  $k \in \{1, \dots, K\}$ , there holds

$$\int_X U(k, W, \mu; x) d\mu_k(x) = \sup_{\nu \in \Theta_k} \int_X U(k, W, \mu; x) d\nu(x) < +\infty \quad (9.24)$$

2. Equation (9.23) holds.

**Remark 9.1** In (9.22), there may exist  $x \in X$  such that  $\sum_{\kappa=1}^K R_{\sigma}(x, W_{\kappa})\mu_{\kappa}^{ac}(x)$  may take the value 0. In this case, we use the convention  $1/0 = +\infty$ .

Note that, for the solutions of the system of static mean field games, all the  $\mu_k$  are absolutely continuous with respect to Lebesgue measure. Moreover, (9.24) ensures  $\text{supp}\left(\sum_{k=1}^K \mu_k\right) = X$ , since otherwise

$$\sup_{\nu \in \Theta_k} \int_X U(k, W, \mu; x) d\nu(x) = +\infty.$$

**Proof** Let us fix  $W \in M_{N \times K}(0, +\infty)$ . Let  $(Q(\cdot), \mu) \in L^1(X) \times \Theta_{ac}$  be a solution of (9.9)-(9.10). Equation (9.9) gives

$$Q(\cdot) = (1 - \theta) \sum_{k=1}^K R_{\sigma}(\cdot, W_k) \mu_k, \text{ for a.e. } x \in X,$$

i.e. (9.23). Then, combining (9.23) with (9.4) yields that for almost every  $x \in X$ , the utility of an agent of type  $k$  living at  $x$  is given by

$$\mathcal{U}_{\theta}(R_{\sigma}(x, W_k), Q(x)) = \theta^{\theta} \frac{R_{\sigma}(x, W_k)}{\left(\sum_{\kappa=1}^K R_{\sigma}(x, W_{\kappa}) \mu_{\kappa}(x)\right)^{1-\theta}}.$$

Thus, since  $\mu \in \Theta_{ac}$ , (9.10) implies (9.24) for every  $k \in \{1, \dots, K\}$ .

Conversely, if for every  $k \in \{1, \dots, K\}$ ,  $\mu_k \in \Theta_k$  is the equilibrium of the static mean field game with utility defined by (9.22), i.e if (9.24) holds, then from Remark 9.1, all the  $\mu_k$  are absolutely continuous with respect to the Lebesgue measure. Thus, equation (9.23) yields (9.9).

On the other hand, the identity

$$U(k, W, \mu; x) = \mathcal{U}(R_{\sigma}(x, W_k), Q(x)), \text{ for a.e } x \in X$$

holds for every  $k \in \{1, \dots, K\}$ . Thus, the mean field equation (9.24) yields (9.10).  $\square$

Noting that the mean field games introduced in the latter lemma admit a potential, it is possible to introduce a simultaneous game with  $K$  players such that a Nash equilibrium is a solution of (9.9)-(9.10).

**Lemma 9.5** Under Assumption 9.1 and 9.2, fix any  $W \in M_{N \times K}(0, +\infty)$ . Let us introduce the payoff  $V(W; \cdot) : \Theta \rightarrow [0, +\infty)$

$$V(W; \mu) = \int_X \left( \sum_{\kappa=1}^K R_{\sigma}(x, W_{\kappa}) \mu_{\kappa}^{ac}(x) \right)^{\theta} dx, \quad \forall \mu \in \Theta. \quad (9.25)$$

If  $\mu \in \Theta$  is a Nash equilibrium for the game

$$(P, (\Theta_k)_{k \in \{1, \dots, K\}}, (V_k^W(\cdot))_{k \in \{1, \dots, K\}}),$$

where

- $P = \{1, \dots, K\}$  is the set of players
- for each  $k \in \{1, \dots, K\}$ ,  $\Theta_k$  is the set of strategies of player  $k$ , and  $\Theta = \prod_{k=1}^K \Theta_k$  is the set of strategy profiles
- for each  $k \in \{1, \dots, K\}$ ,  $V_k^W : \Theta \rightarrow \mathbb{R}$  is the payoff function defined by

$$V_k^W(\mu) = V(W; \mu),$$

and  $Q(\cdot)$  is given by

$$Q(x) = (1 - \theta) \sum_{k=1}^K R_{\sigma}(x, W_k) \mu_k^{ac}(x), \quad \text{for a.e. } x \in X, \quad (9.26)$$

then  $(Q(\cdot), \mu) \in L^1(X) \times \Theta_{ac}$  and satisfies (9.9) and (9.10).

**Proof** Assume that  $\mu$  corresponds to a Nash equilibrium of the game

$$(P, (\Theta_k)_{k \in \{1, \dots, K\}}, (V_k^W(\cdot))_{k \in \{1, \dots, K\}}),$$

and that  $Q(\cdot)$  is given by (9.26). Consider the function

$$J : (\mathbb{R}_+)^K \ni \mu \mapsto \left( \sum_{\kappa=1}^K R_{\sigma}(x, W_{\kappa}) \mu_{\kappa} \right)^{\theta}.$$

We claim that if

$$\mu_k \in \operatorname{argmax} \left\{ \int_X J(\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \nu^{ac}(x), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x)) dx : \nu \in \Theta_k \right\}, \quad (9.27)$$

then

$$\mu_k \in \operatorname{argmax} \left\{ \int_X U(k, W, \mu; x) d\nu(x) : \nu \in \Theta_k \right\}. \quad (9.28)$$

with a finite maximal value. Indeed, fix  $k \in \{1, \dots, K\}$ ; if  $\mu_k$  is solution of (9.27), then  $\mu_k$  is absolutely continuous with respect to the Lebesgue measure. If it was not the case, then  $\mu_k$  would be decomposed as  $\mu_k = \mu_k^{ac} + \mu_k^s$ , with  $\mu_k^s(X) > 0$ , where  $\mu_k^s$  refers to the singular part of  $\mu_k$  with respect to the Lebesgue measure. The competitor

$$\tilde{\nu}(x) = \mu_k^{ac}(x) + \frac{\mu_k^s(X)}{|X|}$$

would then satisfy

$$J(\mu_1^{ac}(x), \dots, \mu_K^{ac}(x)) < J(\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \tilde{\nu}^{ac}(x), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x)), \quad \text{for a.e. } x \in X,$$

thus

$$\int_X J(\mu_1^{ac}(x), \dots, \mu_K^{ac}(x)) dx < \int_X J(\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \tilde{\nu}^{ac}(x), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x)) dx,$$

which is impossible. We have proved that  $\mu_k \in \Theta_k \cap L^1(X)$ .

We now claim that there exists  $\rho > 0$  such that  $\sum_{\kappa=1}^K \mu_{\kappa} \geq \rho$  a.e. Indeed, for some positive constant  $\rho$  that we will choose later, consider the set

$$E_{\rho} = \left\{ x \in X : \sum_{\kappa=1}^K \mu_{\kappa}^{ac}(x) \leq \rho \right\},$$

and assume that  $E_{\rho}$  has positive Lebesgue measure.

For  $0 < t < 1$ , let us consider the competitor

$$\tilde{\nu}_t^{\rho}(x) = \mu_k^{ac}(x)(1 - t|E_{\rho}|) + t m_k \mathbf{1}_{E_{\rho}}(x),$$

which belongs to  $\Theta_k \cap L^1(X)$ . Let us define the function

$$I_{\rho}(t) = \int_X J(\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \tilde{\nu}_t^{\rho}(x), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x)) dx.$$

Observe that

$$I_{\rho}(0) = \int_X J(\mu_1^{ac}(x), \dots, \mu_K^{ac}(x)) dx$$

and that  $I_{\rho}(\cdot)$  is continuous in 0. We claim that for  $\rho$  small enough, there exists  $t > 0$  such that

$$I_{\rho}(t) > I_{\rho}(0), \quad (9.29)$$



in contradiction with the optimality of  $\mu_k$ . We are going to obtain (9.29), by proving that for  $\rho$  small enough,  $I_\rho(\cdot)$  is increasing in an interval of the form  $[0, T)$  for some positive constant  $T$ . Note first that for any  $t \in [0, 1/|E_\rho|]$ ,

$$\begin{aligned} I_\rho(t) &= \int_E J((\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \tilde{\nu}_t^\rho(x), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x))) dx \\ &\quad + \int_{E_\rho^c} J(x)(\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \tilde{\nu}_t^{ac}(x), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x)) dx \\ &= \int_{E_\rho} \left( \sum_{\kappa=1}^K R_\sigma(x, W_\kappa) \mu_\kappa^{ac}(x) + R_\sigma(x, W_k) (m_k - \mu_k^{ac}(x)t |E_\rho|) \right)^\theta dx \\ &\quad + \int_{E_\rho^c} \left( \sum_{\kappa=1}^K R_\sigma(x, W_\kappa) \mu_\kappa^{ac}(x) - R_\sigma(x, W_k) \mu_k^{ac}(x) |E_\rho| t \right)^\theta dx. \end{aligned}$$

Using Lebesgue dominated convergence theorem, we differentiate  $I_\rho(\cdot)$  in  $t > 0$  and obtain

$$\begin{aligned} I'_\rho(t) &= \theta \int_{E_\rho} \left( \sum_{\kappa=1}^K R_\sigma(x, W_\kappa) \mu_\kappa^{ac}(x) + R_\sigma(x, W_k) (m_k - \mu_k^{ac}(x) |E_\rho| t) \right)^{\theta-1} R_\sigma(x, W_k) (m_k - \mu_k^{ac}(x) |E_\rho|) dx \\ &\quad - \theta \int_{E_\rho^c} \left( \sum_{\kappa=1}^K R_\sigma(x, W_\kappa) \mu_\kappa^{ac}(x) - R_\sigma(x, W_k) \mu_k^{ac}(x) |E_\rho| t \right)^{\theta-1} R_\sigma(x, W_k) \mu_k^{ac}(x) |E_\rho| dx. \end{aligned}$$

Recall that  $R_\sigma(\cdot, W_k) \geq W_k^0$  (see (9.5)) and that  $\sum_{\kappa=1}^K \mu_\kappa^{ac}(x) \leq \rho$  a.e. on  $E_\rho$ . From these facts, the positivity of  $R_\sigma(\cdot, W_k) m_k$  and the straightforward inequality

$$\sum_{\kappa=1}^K R_\sigma(x, W_\kappa) \mu_\kappa^{ac}(x) - R_\sigma(x, W_k) \mu_k^{ac}(x) |E_\rho| t \geq R_\sigma(x, W_k) (1 - |E_\rho| t) \mu_k^{ac}(x),$$

we deduce that

$$\begin{aligned} I'(t) &\geq \theta \int_{E_\rho} \frac{W_k^0 m_k}{(\rho \max_{\kappa=1}^K \|R_\sigma(\cdot, W_\kappa)\|_\infty + R_\sigma(x, W_k) (m_k - \mu_k^{ac}(x) |E_\rho| t))^{1-\theta}} dx \\ &\quad - \theta \int_X R_\sigma(x, W_k)^\theta \mu_k^{ac}(x)^\theta |E_\rho| (1 - |E_\rho| t)^{\theta-1} dx. \end{aligned}$$

Using Lebesgue dominated convergence theorem, we see that the right-hand side converges to

$$\theta \int_{E_\rho} \frac{W_k^0 m_k}{\max_{\kappa=1}^K \|R_\sigma(\cdot, W_\kappa)\|_\infty^{1-\theta}} dx \rho^{\theta-1} - \theta \int_X R_\sigma(x, W_k)^\theta \mu_k^{ac}(x)^\theta |E_\rho| dx$$

as  $t$  tends to 0. Let us now fix  $\rho > 0$  small enough such that this quantity is positive.

For such a choice of  $\rho$ , we have therefore proved that there exists  $T > 0$  such that  $I_\rho(\cdot)$  is increasing on  $[0, T)$ , hence there exists  $t > 0$  such that (9.29) holds. This is impossible, hence we see that for  $\rho$  small enough,  $\sum_{\kappa=1}^K \mu_\kappa \geq \rho$  a.e. in  $X$ . The claim is proved.

Thus, if  $\mu_k$  satisfies (9.27), then

$$\mu_k \in \operatorname{argmax} \left\{ \int_X J((\mu_1^{ac}, \dots, \mu_{k-1}^{ac}, \nu^{ac}, \mu_{k+1}^{ac}, \dots, \mu_K^{ac})(x)) dx : \nu \in \Theta_k \cap L^1(X) \right\}, \quad (9.30)$$

and  $\sum_{\kappa=1}^K \mu_\kappa^{ac}(x) \geq \rho$  for almost every  $x \in X$ , where the positive constant  $\rho$  is chosen above.

Take now  $\nu \in \Theta_k \cap L^1(X)$ . For  $t > 0$ ,

$$\int_X J(\mu_1^{ac}(x), \dots, \mu_{k-1}^{ac}(x), \mu_k^{ac}(x) + t(\nu(x) - \mu_k^{ac}(x)), \mu_{k+1}^{ac}(x), \dots, \mu_K^{ac}(x)) dx - \int_X J(\mu_1^{ac}(x), \dots, \mu_K^{ac}(x)) dx \leq 0.$$

We divide the latter inequality by  $t$  and let  $t$  tend to 0. The Lebesgue dominated convergence theorem yields

$$\int_X J_k((\mu_1^{ac}, \dots, \mu_K^{ac})(x)) (\nu(x) - \mu_k^{ac}(x)) dx \leq 0,$$

where  $J_k(\cdot)$  stands for the derivative of  $J(\cdot)$  with respect to its  $k^{\text{th}}$  variable. This yields (9.28).

Finally, we proved that (9.28) is true for every  $k \in \{1, \dots, K\}$ . Since,  $Q(\cdot)$  is given by (9.26), Lemma 9.4 yields that  $(Q(\cdot), \mu) \in L^1(X) \times \Theta_{ac}$  and (9.9) and (9.10) hold.  $\square$

Since the payoffs of the players in the game introduced in Lemma 9.5 are all the same, we obtain the following corollary.

**Corollary 9.1** *Under Assumption 9.1 and 9.2, fix any  $W \in M_{N \times K}(0, +\infty)$ . If*

$$\mu \in \operatorname{argmax}_{\lambda \in \Theta} V(W; \lambda), \quad (9.31)$$

*then  $\mu$  is a Nash equilibrium for the game introduced in Lemma 9.5.*

**Lemma 9.6** *Under Assumption 9.1 and 9.2, fix any  $W \in M_{N \times K}(0, +\infty)$ . There exists a solution of (9.31).*

**Proof** The set  $\Theta$  is compact in the weak  $*$  topology. Using [11, Theorem 13.3.1] the function

$$\Theta \ni \lambda \mapsto V(W; \lambda)$$

is upper-semi continuous in the weak  $*$  topology. Then, the existence of a maximizer is obtained by the direct method of the calculus of variations.  $\square$

We are going to construct a multi-valued map  $\Psi(\cdot)$  whose fixed-points are equilibria in the sense of Definition 9.3.

**Lemma 9.7** *Under Assumption 9.1 and 9.2, consider the multi-valued map  $\Psi : \Theta \rightrightarrows \Theta$  defined as follows:*

1. *to any  $\mu \in \Theta$ , we associate the unique minimizer  $W(\mu)$  of (9.12), i.e.  $W(\mu)$  is the unique solution of*

$$\min_{Z \in M_{N \times K}(0, +\infty)} \left\{ \sum_{k=1}^K \int_X R_\sigma(x, W_k) d\mu_k(x) + \sum_{i=1}^N (F^i)^*(W^i) \right\}$$

2. *Then,*

$$\Psi(\mu) = \operatorname{argmax}_{\lambda \in \Theta} V(W(\mu); \lambda),$$

*where  $V(\cdot; \cdot)$  has been introduced in Lemma 9.5.*

*The fixed-points of  $\Psi(\cdot)$  are equilibria in the sense of Definition 9.3.*

**Proof** We claim that if  $\mu \in \Psi(\mu)$ ,  $W = W(\mu)$  and

$$Q(x) = (1 - \theta) \sum_{k=1}^K R_\sigma(x, W_k(\mu)) \mu_k(x), \quad \text{for a.e } x \in X, \quad (9.32)$$

then  $(W, Q(\cdot), \mu)$  is an equilibrium in the sense of Definition 9.3. Indeed, from Lemma 9.2,  $W = W(\mu)$  is equivalent to equation (9.8) associated to  $\mu$ . Moreover, Lemma 9.4, 9.5, and Corollary 9.1 yield that if

$$\mu \in \operatorname{argmax}_{\lambda \in \Theta} V(W; \lambda)$$

and  $Q(\cdot)$  is defined by (9.32) then  $(Q(\cdot), \mu) \in L^1(X) \times \Theta_{ac}$  and (9.9) and (9.10) associated to  $W$  hold.  $\square$

We are now ready to prove our main result.

**Theorem 9.1** *Under Assumption 9.1 and 9.2, there exists an equilibrium  $(W, Q(\cdot), \mu)$  in the sense of Definition 9.3.*

**Proof** We aim at proving the existence of a fixed-point of  $\Psi(\cdot)$ . Recall that  $\Theta$  is convex and compact in the weak  $*$  topology. From Lemma 9.6, the concavity and the upper semi continuity of  $V(W; \cdot)$  for all  $W \in M_{N \times K}(0, +\infty)$ , we see that  $\Psi(\cdot)$  maps elements of  $\Theta$  to non-empty, compact and convex subsets of  $\Theta$ .

In order to apply Kakutani fixed-point theorem, there remains to check that the graph of  $\Psi(\cdot)$  is closed. Let  $\mu_n \rightarrow \mu$  weakly  $*$ . Consider  $\nu_n \in \Lambda(\mu_n)$ . Let us prove that if  $\nu_n$  converges to  $\nu$  weakly  $*$ , then  $\nu \in \Psi(\mu)$ , i.e.

$$\nu \in \operatorname{argmax}_{\lambda \in \Theta} V(W(\mu); \lambda). \quad (9.33)$$

Lemma 9.3 yields  $W(\mu_n) \rightarrow W(\mu)$  when  $n$  tends to  $+\infty$ . Moreover, from the definition of  $\nu_n$ ,

$$V(W(\mu_n); \nu_n) = \max_{\lambda \in \Theta} V(W(\mu_n); \lambda), \quad \forall \lambda \in \Theta.$$

On the other hand, note that for any ,

$$|V(W(\mu_n); \lambda) - V(W(\mu); \lambda)| = \left| \int_X \left( \sum_{k=1}^K R_\sigma(x, W_k(\mu_n)) \lambda_k^{ac}(x) \right)^\theta dx - \int_X \left( \sum_{k=1}^K R_\sigma(x, W_k(\mu)) \lambda_k^{ac}(x) \right)^\theta dx \right| \quad (9.34)$$

$$\leq \int_X \left| \sum_{k=1}^K (R_\sigma(x, W_k(\mu_n)) - R_\sigma(x, W_k(\mu)) \lambda_k^{ac}(x)) \right|^\theta dx \quad (9.35)$$

$$\leq \left( \int_X \left| \sum_{k=1}^K (R_\sigma(x, W_k(\mu_n)) - R_\sigma(x, W_k(\mu)) \lambda_k^{ac}(x)) \right| \frac{dx}{|X|} \right)^\theta |X| \quad (9.36)$$

$$\leq \left( \max_{k=1}^K \|R_\sigma(\cdot, W_k(\mu_n)) - R_\sigma(\cdot, W_k(\mu))\|_\infty \sum_{k=1}^K m_k \right)^\theta |X|^{1-\theta}, \quad (9.37)$$

where (9.35) comes from the properties of the function  $\mathbb{R}_+ \ni t \mapsto t^\theta$ , (9.36) is obtained with Jensen's inequality, and (9.37) uses the facts that for any  $k \in \{1, \dots, K\}$ ,

$$|R_\sigma(x, W_k(\mu_n)) - R_\sigma(x, W_k(\mu))| \leq \|R_\sigma(\cdot, W(\mu_n)_k) - R_\sigma(\cdot, W(\mu)_k)\|_\infty$$

and

$$\int_X \lambda_k^{ac}(x) dx \leq m_k.$$

From Lemma 9.3,

$$\lim_{n \rightarrow +\infty} \|R_\sigma(\cdot, W_k(\mu_n)) - R_\sigma(\cdot, W_k(\mu))\|_\infty = 0, \quad \forall k \in \{1, \dots, K\},$$

and (9.37) ensures that

$$\lim_{n \rightarrow +\infty} \max_{\lambda \in \Theta} V(W(\mu_n); \lambda) = \max_{\lambda \in \Theta} V(W(\mu); \lambda).$$

Moreover,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} V(W(\mu_n); \nu_n) - V(W(\mu); \nu) &= \limsup_{n \rightarrow +\infty} (V(W(\mu_n); \nu_n) - V(W(\mu); \nu_n) \\ &\quad + V(W(\mu); \nu_n) - V(W(\mu); \nu)) \\ &\leq 0, \end{aligned}$$

since  $|V(W(\mu_n); \nu_n) - V(W(\mu); \nu_n)|$  vanishes and  $V(W(\mu); \cdot)$  is upper-semi-continuous in the weak \* topology. Finally, since  $V(W(\mu_n); \nu_n) = \max_{\lambda \in \Theta} V(W(\mu_n); \lambda)$  for any  $n \in \mathbb{N}$ , we deduce that  $\max_{\lambda \in \Theta} V(W(\mu); \lambda) \leq V(W(\mu); \nu)$  and therefore

$$\max_{\lambda \in \Theta} V(W(\mu); \lambda) = V(W(\mu); \nu),$$

i.e.  $\nu \in \Psi(\mu)$ . We conclude that the graph of  $\Psi(\cdot)$  is closed.

From Kakutani fixed-point theorem,  $\Psi(\cdot)$  admits a fixed-point  $\mu$ , (i.e.  $\mu \in \Lambda(\mu)$ ). □

## 9.3 Conclusion

In this chapter, we have extended the model presented in Chapter 8 to the case when there are several types of agents. The analysis has been performed for a smooth revenue function, see (9.5). The existence of equilibria in the sense of Definition 9.3 has been proved.

Concerning the existence of equilibria, the global strategy in both Chapters 8 and 9 is the same:

1. The market clearing conditions on the labour market are reformulated as an optimization problem.
2. We reformulate the market clearing conditions on the rental market for individuals and the mobility condition.
3. From these two reformulations, we build a function whose fixed-point are equilibria, and we prove that it, indeed, admits fixed-points.

However, several differences can be noted:

1. In Chapter 8, from the market clearing conditions on the rental market for individuals and the mobility condition, it was possible to deduce an explicit formula for the distribution of residences. In the present case, we have not been able to do so. Instead, we have reformulated these conditions as a convex optimization problem with possibly several maximizers. Therefore, to obtain the existence of fixed-points, we have been led to use Kakutani fixed-point theorem instead of Brouwer fixed-point theorem as in Chapter 8.
2. Besides, since the uniqueness result in Theorem 8.3 was strongly based on the explicit formula for the distribution of residences, we have not been able to extend this result to the present case.



# Chapter 10

## Numerical simulations

### 10.1 Introduction

We aim at making numerical simulations with the model introduced in Chapter 8. We carry out simulations with three different workplaces. The first three are done on a straight line while the last one presents an extension to a two-dimensional space.

In the first simulation, we focus our attention on the impact of the parameter  $\theta$  which models the preferences of the agents. When  $\theta$  is small, the utility of agents is mainly determined by the surface of their apartments. On the contrary, when  $\theta$  is close to 1, the utility is mainly determined by the consumption level.

In the second simulation, we transfer progressively the capital from two workplaces to the third one. We aim at understanding how these variations in the spatial distribution of the capital impact the equilibrium.

In the third one, we consider the case when the workers have the possibility to telecommute, and we let the productivity of telecommuters vary.

Finally, we recast the latter home-based telecommuting model to a two-dimensional space.

The first and second simulations use the same numerical method in order to approximate an equilibrium. In the third and fourth cases, slight modifications are needed for taking into account the two different types of workers: the commuters and the telecommuters and for handling the two-dimensional space case. However, the methods are similar, and we restrict ourselves to the discussion of the numerical method used for system (8.8)-(8.10); more precisely, we aim at approximating a triplet  $(w, Q(\cdot), \mu) \in (0, +\infty)^N \times C(X) \times (C(X) \cap \mathcal{P}(X))$  such that

$$\int_X \frac{\partial R_\sigma}{\partial w_i}(x, w) d\mu(x) = L_i(w_i), \quad \forall i \in \{1, \dots, N\}, \quad (10.1)$$

$$S_\theta(R_\sigma(x, w), Q(x))\mu(x) = 1, \quad \forall x \in X, \quad (10.2)$$

$$\int_X \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) d\mu(x) = \sup_{\nu \in \mathcal{P}(X)} \int_X \mathcal{U}_\theta(R_\sigma(x, w), Q(x)) d\nu(x) < +\infty, \quad (10.3)$$

where

- $X \subset \mathbb{R}$  is the closure of a bounded domain.
- the revenue of the agents  $R_\sigma : X \times (0, +\infty)^N \rightarrow \mathbb{R}$  is defined by

$$R_\sigma(x, w) = \sigma \ln \left( \sum_{i=0}^N e^{\frac{w^i - c_i(x)}{\sigma}} \right), \quad \forall (x, w) \in X \times (0, +\infty)^K.$$

- the optimal surface  $S_\theta : (0, +\infty)^2 \rightarrow (0, +\infty)$  is defined by

$$S_\theta(R, Q) = (1 - \theta) \frac{R}{Q}, \quad \forall (R, Q) \in (0, +\infty)^2.$$

- the utility of the agents  $\mathcal{U}_\theta : (0, +\infty)^2 \rightarrow (0, +\infty)$  is defined by

$$\mathcal{U}_\theta(R, Q) = \theta^\theta (1 - \theta)^{1-\theta} \frac{R}{Q^{1-\theta}}, \quad \forall (R, Q) \in (0, +\infty)^2.$$

In this setting, it has been proved (see the first step of the proof of Lemma 8.4) that if  $(w, Q(\cdot), \mu)$  is an equilibrium, then  $\mu$  is given by the explicit formula:

$$\mu(x) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy}, \quad \forall x \in X. \quad (10.4)$$

This allows us to find a solution  $(w, Q(\cdot), \mu)$  of (10.1)-(10.3) by solving

$$\Psi(w, \theta) = L(w), \quad (10.5)$$

where

- $L(w) = (L_1(w_1), \dots, L_N(w_N))$
- $\Psi(w, \theta) = \int_X D_w R_\sigma(x, w) \tilde{\mu}(x, w, \theta) dx$
- $\tilde{\mu}(x, w, \theta) = \frac{R_\sigma(x, w)^{\frac{\theta}{1-\theta}}}{\int_X R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy}$ .

Indeed, if  $w$  is a solution of (10.5) and if we set

$$Q(x) = (1 - \theta) R_\sigma(x, w) \mu(x), \quad \text{and} \quad \mu(x) = \tilde{\mu}(x, w, \theta) \quad \forall x \in X,$$

then  $(w, Q(\cdot), \mu)$  is an equilibrium. Therefore, we may focus on solving (10.5).

We first briefly discuss the numerical scheme used to approximate (10.5), then we present the three above-mentioned simulations.

## 10.2 The finite difference operators

It has been proved in Lemma 8.2 in Chapter 8 that given  $\mu \in \mathcal{P}(X)$ , (10.1) admits a variational structure. Therefore, a way to solve (10.1)-(10.3) is to find  $w \in (0, +\infty)^N$  such that

$$\psi(w) - w = 0, \quad (10.6)$$

where  $\psi(w)$  is the unique solution of

$$\min_{z \in (0, +\infty)^N} \int_X R_\sigma(x, z) \mu[z](x) dx + \sum_{i=1}^N \int_\varepsilon^{z_i} L_i(s) ds, \quad (10.7)$$

when  $\mu[w](\cdot)$  given by (10.4).

The problem of this approach lies in the fact that the solution of (10.7) needs to be approximated each time  $\psi(\cdot)$  is evaluated. Therefore, we prefer to approximately solve (10.5) since  $\Psi(\cdot, \theta)$  is easier to compute.

### 10.2.1 The scheme

We aim at approximating (10.5). Since the problem is reduced to a system of non-linear equations where the functions  $L_i(\cdot)$  are given, we only need to approximate  $\Psi(\cdot, \theta)$ . For convenience, we focus on the case when  $X = [0, 1]$ . Note that the method can be extended to the case when  $X$  is the closure of domain of  $\mathbb{R}^d$ , with  $d > 1$ . Fix  $w \in (0, +\infty)^N$ . Let  $X_h$  be a uniform grid on  $X$  with step  $h = 1/N_h$ ,  $N_h \in \mathbb{N}$ . Let  $x_k$  denotes a generic point in  $X_h$ . The values of  $R_\sigma(\cdot, w)$ ,  $\frac{\partial R_\sigma}{\partial w^i}(\cdot, w)$  and  $\tilde{\mu}(\cdot, w, \theta)$  at  $x_k$  will be respectively approximated by  $R_k$ ,  $DR_k^i$  and  $\mu_k$ . Since  $R_\sigma(\cdot, w)$  and  $D_w R_\sigma(\cdot, w)$  have explicit forms,  $R_k$  is simply given by  $R_\sigma(x_k, w)$ , and  $DR_k$  by  $D_w R_\sigma(x_k, w)$ . To approximate the integrals, we use the trapezoidal rule, i.e.

$$\int_{[0,1]} R_\sigma(y, w)^{\frac{\theta}{1-\theta}} dy \simeq h \left( \frac{R_0^{\frac{\theta}{1-\theta}} + R_{N_h}^{\frac{\theta}{1-\theta}}}{2} + \sum_{k=1}^{N_h-1} R_k^{\frac{\theta}{1-\theta}} \right).$$

Therefore, we can define  $\mu_k$  as follows

$$\mu_k = \frac{R_k^{\frac{\theta}{1-\theta}}}{h \left( \frac{R_0^{\frac{\theta}{1-\theta}} + R_{N_h}^{\frac{\theta}{1-\theta}}}{2} + \sum_{k=1}^{N_h-1} R_k^{\frac{\theta}{1-\theta}} \right)}.$$

This yields an approximation for  $\Psi(w, \theta)$  :

$$\Psi(w, \theta) = \int_{[0,1]} D_w R_\sigma(x, w) \tilde{\mu}(x, w, \theta) dx \simeq h \left( \frac{DR_0 \mu_0 + DR_{N_h} \mu_{N_h}}{2} + \sum_{k=1}^{N_h-1} DR_k \mu_k \right).$$

When the dimension of  $X$  is greater than one, the only change in the discretization introduced concerns the approximation of the integrals above.

### 10.2.2 The method

To solve (10.5), which consists in finding a root of a vector function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , we use the method `scipy.optimize.root` contained in the library `scipy` of python. It is based on the Powell hybrid method [93] which consists in finding a minimizer of

$$G(\cdot) = \sum_{i=1}^N f_i(\cdot)^2$$

by constructing a sequence  $x_{n+1} = x_n + \delta_n$ , where  $\delta_n$  combines the Gauss-Newton and the gradient descent step. A finite difference approximation of the gradient of  $f(\cdot)$  is used in `scipy.optimize.root` if  $Df(\cdot)$  is not specified. As explained in [83],

”It is a modification of the Powell hybrid method. (...) The choice of the corrections guarantees (under reasonable conditions) global convergence for starting points far from the solution and a fast rate of convergence.”

This last property is crucial since for the ”good” and ”bad” Broyden methods, which are quasi-Newton methods and have local convergence properties, the sequence of iterates does not converge. As pointed out in Chapter 5, another way to solve this problem, which in this case involves more calculations, is to use a continuation method.



### 10.3 Some comparative statics

We present four simulations aimed at understanding the impact of some of the parameters on the equilibrium. The first three are carried out on a one-dimensional space while the fourth is achieved on a two-dimensional space.

The standard framework which will be slightly modified for the last simulation is as follows: we set  $X = [-10, 10]$ , and we assume that there are three workplaces ( $N = 3$ ) located at three different points in  $X$ . Let  $y_i \in X$  be the location of the  $i$ th workplace. We assume that

$$y_1 = -7, \quad y_2 = 0, \quad y_3 = 3,$$

and that each workplace corresponds to a firm that seeks to maximise its profits; with a slight abuse of language, let  $y_i$  be the name of the firm located at  $y_i$ . The company  $y_i$  solves the following optimization problem

$$\sup_{\ell \in [0, +\infty)^K} \{F_i(\ell) - w_i \ell\},$$

where  $K \in \{1, 2\}$  is the number of types of workers,  $F_i : [0, +\infty)^K \rightarrow \mathbb{R}$  is the production function and  $w_i \in \mathbb{R}^K$  is the collection of the wages of the different types of workers.

We also assume that the transport cost to reach  $y_i$  from  $x \in X$  is given by

$$c_i(x) = \frac{|x - y_i|}{2}, \quad \forall i \in \{1, 2, 3\}.$$

Note that we could use any other continuous function on  $X$  to model the transport costs without changing the scheme or the method.

#### 10.3.1 Comparative statics as the preference parameter $\theta$ varies

##### Definition of the model and the parameters

We assume that the production of the firm  $y_i$  is given by

$$F_i(\ell) = A^\alpha \ell^{1-\alpha}, \quad \forall \ell \in [0, +\infty).$$

where  $\ell$  is the number of workers hired by the firm. Therefore, the demand of labour at  $y_i$  is

$$L_i(w_i) = A \left( \frac{1-\alpha}{w_i} \right)^{\frac{1}{\alpha}}, \quad \forall w_i > 0.$$

The parameters used in Test 1 are listed in Table 10.1 below.

Parameter	Value
$\alpha$	0.3
$A$	$10^4$
$w_0$	12
$\sigma$	0.1

Table 10.1: The parameters used in Test 1

##### Numerical results

In the following three figures, we compare the results obtained for different values of  $\theta$ . In Figure 10.1, we display the distribution of the houses of the people working at the different workplaces. Recall that these distributions are given by

$$X \ni x \mapsto \frac{\partial R_\sigma}{\partial w_i}(x, w) \mu(x), \quad \forall i \in \{0, 1, 2, 3\}.$$

The lines (—), (—) and (—) are associated to the residences of the agents working at  $y_1$ ,  $y_2$  and  $y_3$  respectively. The curve (—) corresponds to the residences of the independent workers.

In Figure 10.2, we display the wages and the number of workers in each workplace (we use the same color code as in Figure 10.1). Finally, the rental price as a function of the spatial variable  $x$  is plotted in Figure 10.3.

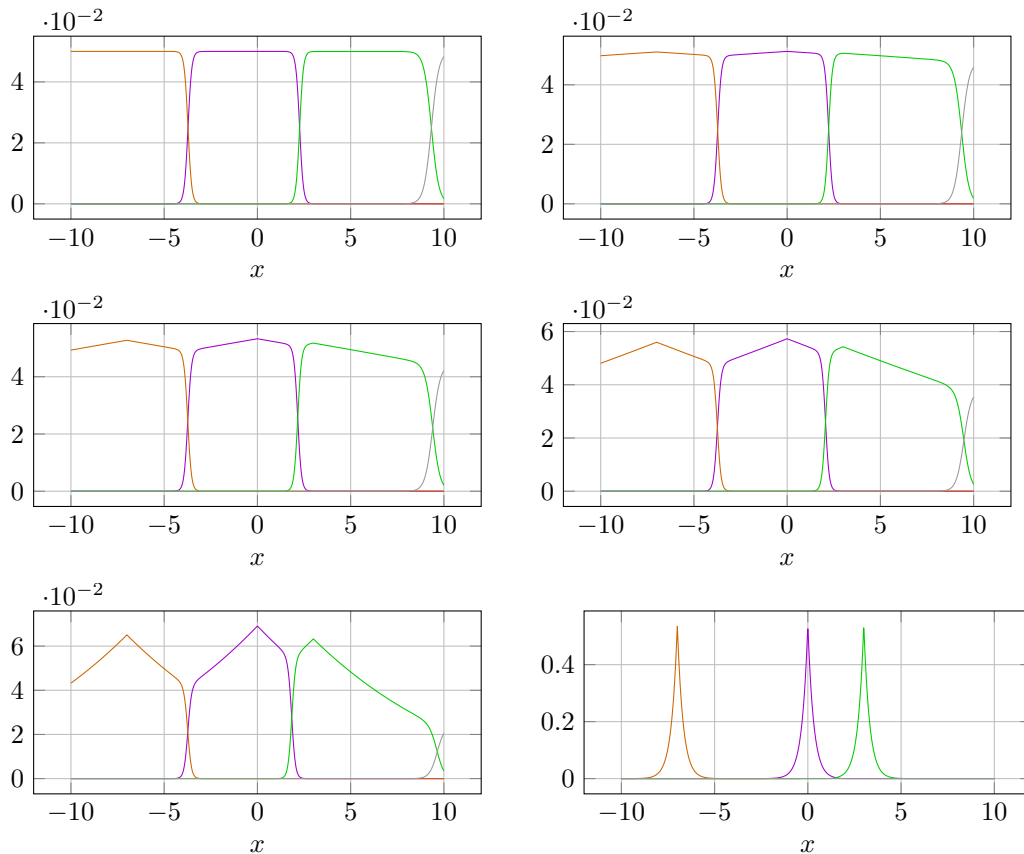


Figure 10.1: Distributions of the houses of the people working at the different workplaces and of the independent workers, with  $\theta = 0$  (top-left),  $\theta = 0.2$  (top-right),  $\theta = 0.4$  (middle-left),  $\theta = 0.6$  (middle-right),  $\theta = 0.8$  (bottom-left),  $\theta = 0.99$  (bottom-right).

Let us focus on the case when  $\theta = 0$ . The agents have only one source of utility, the surface that they rent. Therefore, as it clearly appears on Figure 10.1, they distribute themselves uniformly on  $X$  (the supply of space is constant). This gives to  $y_3$  a positional advantage; indeed the basin of attraction of  $y_3$  is larger than those of  $y_1$  and  $y_2$ . Therefore, the supply of labour at  $y_3$  is larger than at  $y_1$  or  $y_2$ . We see that different locations lead to differences in labour supply.

Due to the advantage that  $y_3$  has, it attracts more workers and may pay less than  $y_1$  or  $y_2$ . Similarly,  $y_1$  has a positional advantage with respect to  $y_2$ . This explains why  $w_3 < w_1 < w_2$ , even though  $y_3$  attracts more workers than  $y_1$ , which attracts more workers than  $y_2$ , see Figure 10.2.

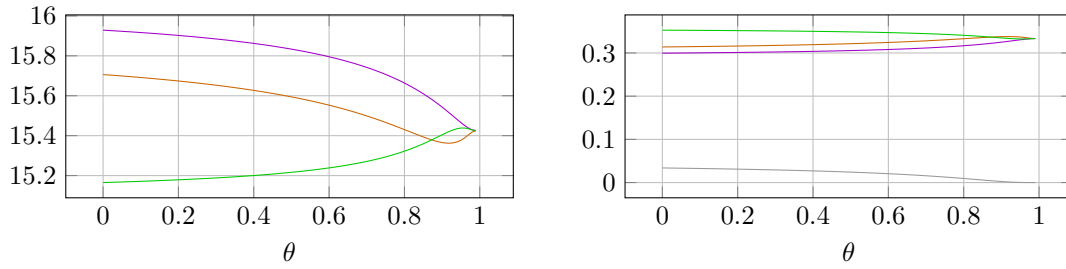


Figure 10.2: Wages versus  $\theta$  (on the left) and the number of workers versus  $\theta$  (on the right).

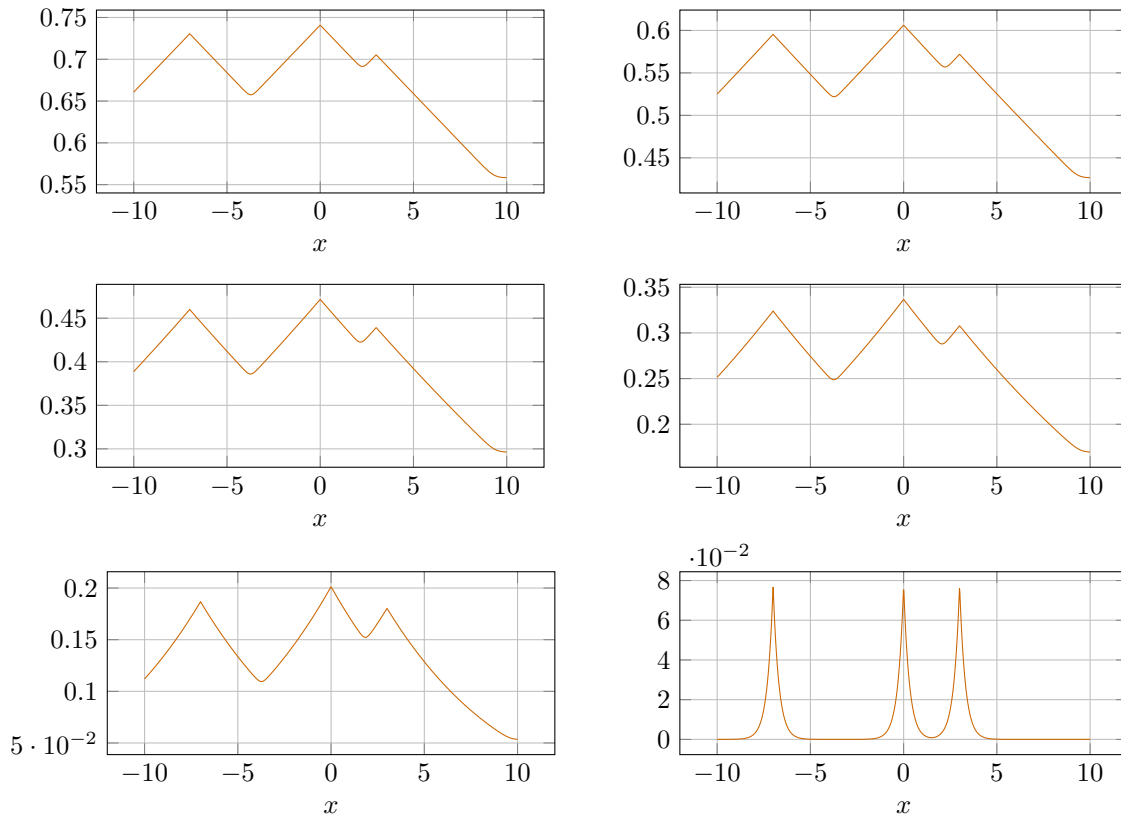


Figure 10.3: Rental price versus  $x$ : comparison for  $\theta = 0$  (top-left),  $\theta = 0.2$  (top-right),  $\theta = 0.4$  (middle-left),  $\theta = 0.6$  (middle-right),  $\theta = 0.8$  (bottom-left),  $\theta = 1$  (bottom-right).

As  $\theta$  increases, the relative importance of the surface of the house in the utility function of the agents decreases. As a consequence, the demand for surface decreases and so does the rental price, see Figure 10.3. On the other hand, the relative importance of consumption in the utility function of the agents increases. Therefore, there tends to be a concentration of housing close to the workplaces, because the agents choose to reduce their transport costs in order to increase their consumption, see Figure 10.1. In Figure 10.2, we observe that, when  $\theta$  varies from 0 to 0.8, the wages in  $y_1$  and  $y_2$  decrease, while the wages in  $y_3$  increase. We may give two reasons for that. First, the concentration of houses near the workplaces tends to reduce the competition on the labour market between  $y_1$  and  $y_2$ . Second, the number of workers living in the interval  $[5, 10]$  decreases, so the positional advantage of  $y_3$  decreases.

For  $\theta > 0.8$ , the concentration phenomenon progressively isolates  $y_1$  from  $y_2$  and  $y_3$ . Therefore,  $y_1$  enjoys a positional advantage similar to the one that  $y_3$  had when  $\theta$  was small. This allows  $y_1$  to decrease

the level of wage. On the other hand, this extra competition pushes  $y_3$  to increase its wage.

Finally, when  $\theta$  is close to 1, the size of the basins of attraction of the different workplaces becomes small, so that they are isolated of each other. When  $\theta = 0.99$ , the wages and the number of agents in each workplace is the almost the same.

## 10.3.2 Comparative statics as the capital is transferred to a given workplace

### Description of the model and the parameters

We assume that the production of the firm  $y_i$  is given by

$$F_i(\ell) = (A_i)^\alpha \ell^{1-\alpha}, \quad \forall \ell \in [0, +\infty),$$

where  $A_i$  is the capital of  $y_i$ . For the firm  $y_i$ , the demand of labour is thus given by

$$L_i(w_i) = A_i \left( \frac{1-\alpha}{w_i} \right)^{\frac{1}{\alpha}}, \quad \forall w_i > 0.$$

The goal is to compare situations in which the capital is progressively transferred from the firms  $y_1$  and  $y_3$  to the firm  $y_2$ . Consider a parameter  $t \in [0, 1]$  and set

$$A_1 = 1.5A(1-t), \quad A_2 = 3At, \quad A_3 = 1.5A(1-t),$$

so that the global capital is always  $3A$ .

The parameters used in Test 2 are listed in Table 10.2

Parameter	Value
$\alpha$	0.3
$A$	$10^4$
$w_0$	12
$\sigma$	0.1
$\theta$	0.7

Table 10.2: The parameters used in Test 2.

### Numerical results

In the following three figures, we compare the obtained results for different values of  $t$ . In Figure 10.4, we display the distribution of the houses of the people working at the different workplaces. The colors are the same as in Figure 10.1. In Figure 10.6, we plot the the wages and the number of workers in each workplace versus  $t$  (with the same color code). In Figure 10.5, we plot the rental price versus  $t$ . In Figure 10.7, we plot the utility of the agents versus  $t$ .

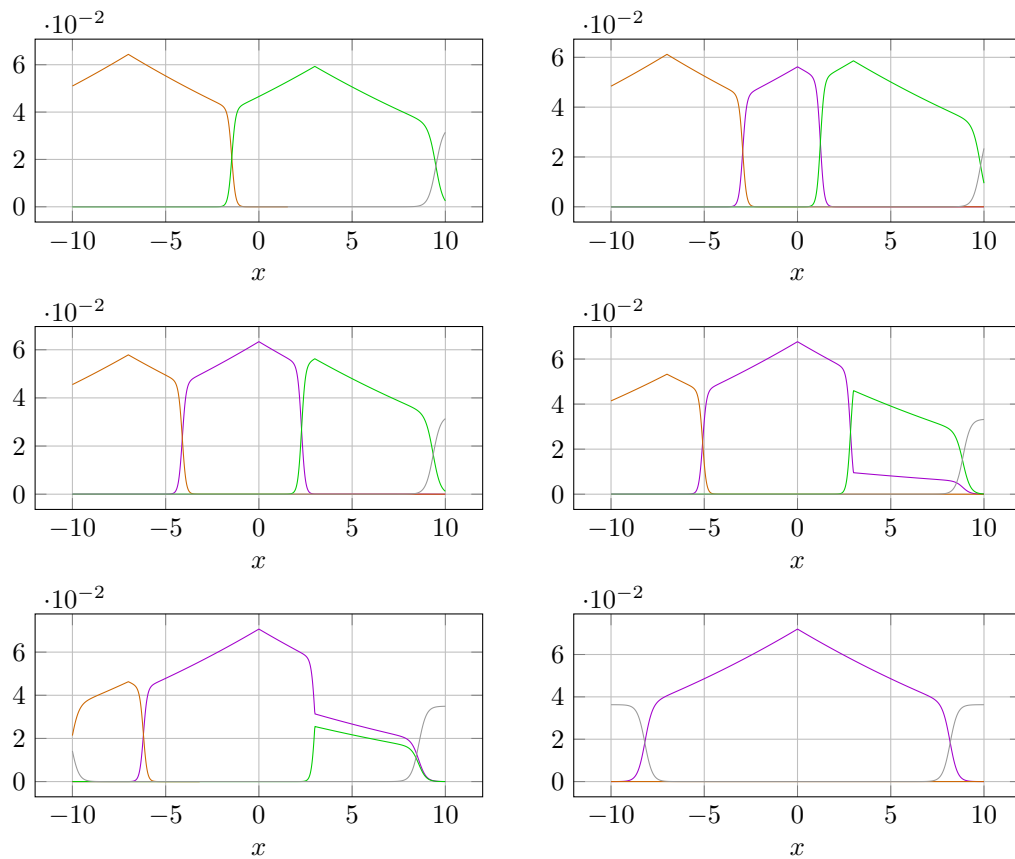


Figure 10.4: The distributions of the houses of the people working at the different workplaces and of the independent workers, for  $t = 0$  (top-left),  $t = 0.2$  (top-right),  $t = 0.4$  (middle-left),  $t = 0.6$  (middle-right),  $t = 0.8$  (bottom-left),  $t = 1$  (bottom-right).

In Figure 10.4, we see that the basin of attraction of  $y_2$  is empty for  $t = 0$ , then grows with  $t$ , and occupies most of the domain for  $t = 1$ . The basins of attraction of  $y_1$  and  $y_3$  become smaller and smaller as  $t$  grows, and vanish for  $t$  large enough. For  $t = 1$ , all the agents work in  $y_2$ , except those who live far from  $y_2$  and remain independent of any firm. The behaviour of the rental price is very much connected to the changes in the distribution of the houses of the people working at the different workplaces, see Figure 10.5. The wages and the number of workers in  $y_2$  increase, while these two quantities decrease in  $y_1$  and  $y_3$ , see Figure 10.6.

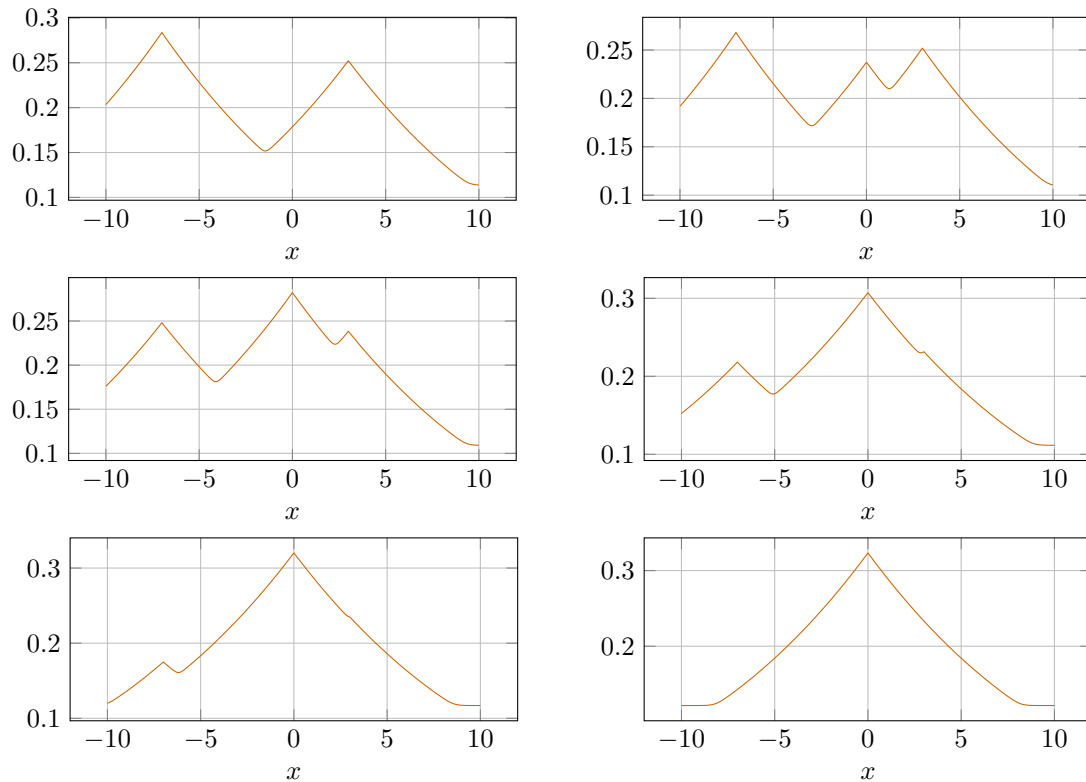


Figure 10.5: Rental price versus  $x$ : comparison for  $t = 0$  (top-left),  $t = 0.2$  (top-right),  $t = 0.4$  (middle-left),  $t = 0.6$  (middle-right),  $t = 0.8$  (bottom-left),  $t = 1$  (bottom-right).

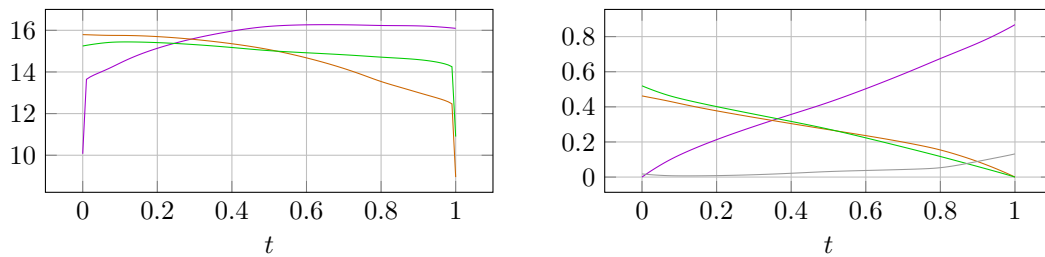


Figure 10.6: Wages versus  $t$  (on the left) and the number of workers versus  $t$  (on the right).

In Figure 10.7, we see that the utility at equilibrium (the same for all agents) is maximal for  $t^* \simeq 0.33$ , which corresponds to the case when the capital is uniformly distributed between  $y_1$ ,  $y_2$  and  $y_3$ . Note that we optimized with respect to only one parameter. Therefore, it is possible to find a more complex allocation of capital that leads to a better level of utility at equilibrium.

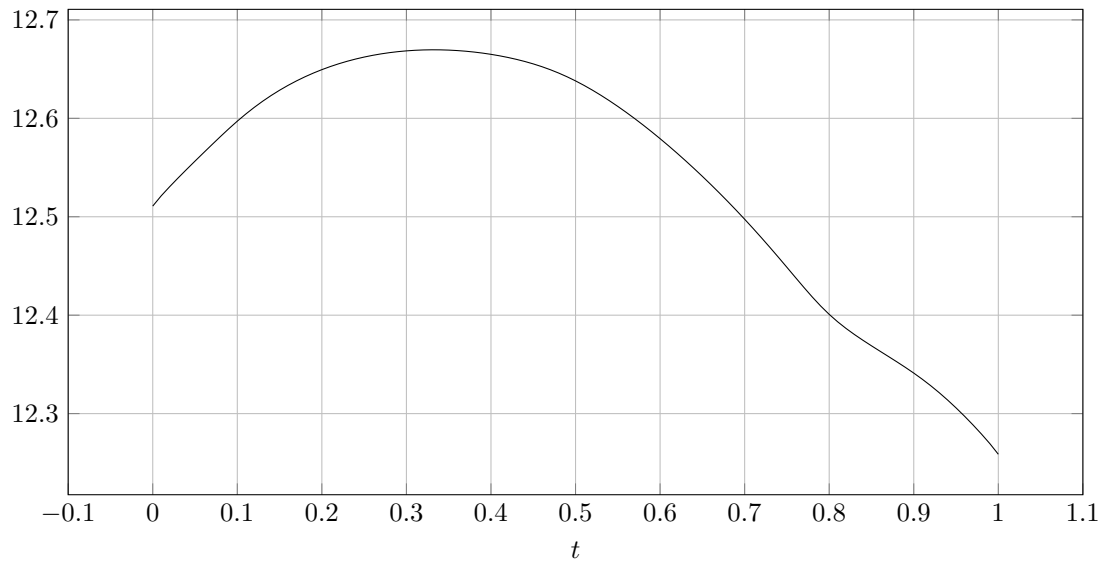


Figure 10.7: The utility of the agents at equilibrium (all the agents have the same utility at equilibrium) versus  $t$

### 10.3.3 A home-based telecommuting model

#### Description of the model and the parameters

We assume that the production of  $y_i$  is given by

$$F_i(\ell) = A^\alpha (\ell_1^\gamma + B\ell_2^\gamma)^{\frac{1-\alpha}{\gamma}}, \quad \forall \ell \in [0, +\infty)^2.$$

where  $\ell_1$  is the number of commuters and  $\ell_2$  is the number of telecommuters. The parameter  $B$  is related to the productivity of the telecommuters. In the simulation, we are going to let  $B$  vary. For each firm, there are two wages, the first one for the commuters and the second one for the telecommuters. Therefore, there are six equations in (10.5). Our simulation is essentially the same as before, except that everything is done as if there were six workplaces ( $N = 6$ ).

The parameters used in Test 3 are listed in Table 10.3 below.

Parameter	Value
$\alpha$	0.3
$\gamma$	0.9
$A$	$10^4$
$w_0$	12
$\sigma$	0.1
$\theta$	0.7

Table 10.3: The parameters used in Test 3.

#### Numerical results

In the three figures below, we compare the results obtained for different values of  $B$ . In Figure 10.8, we display the distribution of the houses of the people working at the different workplaces. The lines (—), (—) and (—) are associated to the residences of the commuters working at  $y_1$ ,  $y_2$  and  $y_3$  respectively. The curves (—), (—) and (—) are respectively associated to the residences of the telecommuters

working for  $y_1$ ,  $y_2$  and  $y_3$ . The curve (—) corresponds to the residences distribution of independent workers. In Figure 10.9, we plot the wages and the number of workers in each workplace versus  $B$  (we use the same color code). In Figure 10.10, we plot the rental price versus  $x$ .

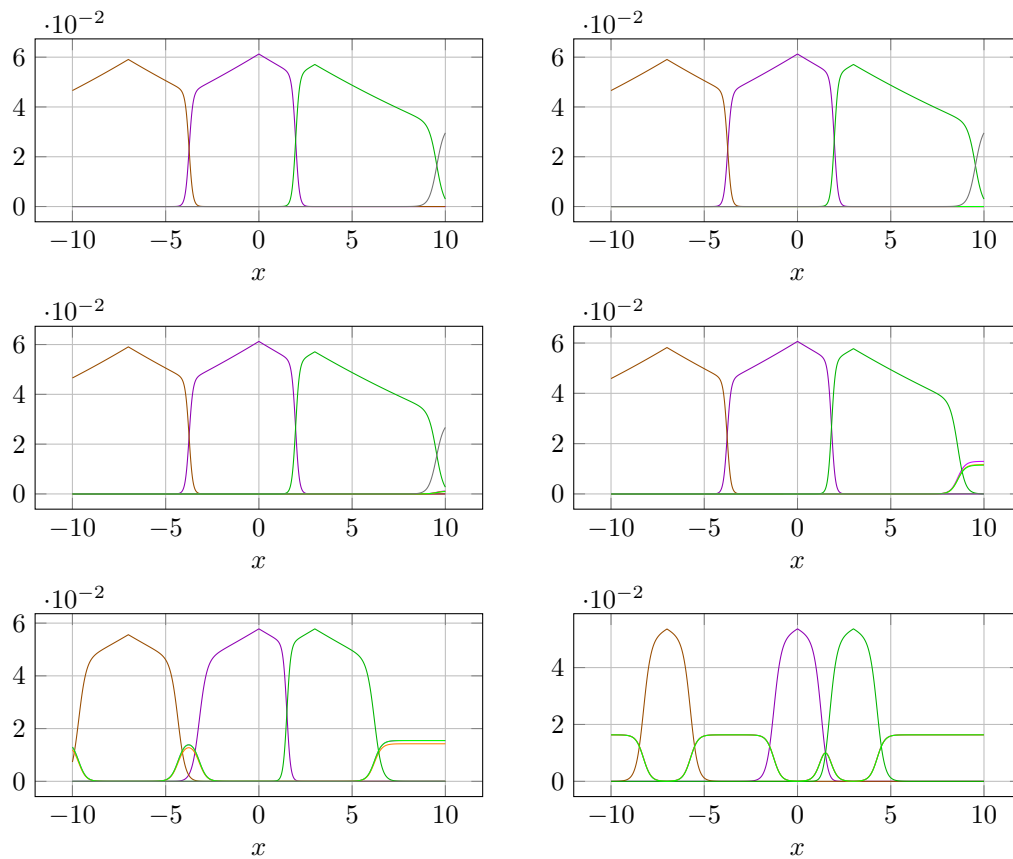


Figure 10.8: The distributions of the houses of the people working for the different workplaces and of the independent workers, for  $B = 0$  (top-left),  $B = 0.2$  (top-right),  $B = 0.4$  (middle-left),  $B = 0.6$  (middle-right),  $B = 0.8$  (bottom-left),  $B = 1$  (bottom-right).

On Figure 10.9, we observe a phenomenon similar to what happened in the first simulation related to the sensitivity with respect to  $\theta$ . Indeed, for  $B = 0$ , the wages depend on the workplace. Then, as the parameter  $B$  increases, people choose to telecommute when their transport costs are high, see Figure 10.8. As a result,  $y_3$  loses some of its positional advantage, since  $y_1$  and  $y_2$  may hire telecommuters to the right of  $y_3$  because the latter do not incur transportation costs. Progressively, as in Test 1,  $y_3$  loses its positional advantage whereas  $y_1$  becomes more attractive because its basin of attraction becomes isolated from those of the other two workplaces. As in Test 1, the housing of commuters tends to concentrate in smaller and smaller areas, so that when  $B$  is large, no firm has a geographical advantage on the others. Therefore, when  $B = 1$ , the wages and the number of workers in each workplace are the same, see Figure 10.9. Note that for small values of  $\sigma$ , the wages of the telecommuters are the same in each workplace, because the firms compete for hiring the telecommuters and because the latter do not have transportation costs. This is what we observe here for  $\sigma = 0.1$  where the differences when  $B \geq 0.2$  are of the order of  $10^{-4}$ . The differences observed for  $B < 0.2$  are due to numerical approximations.



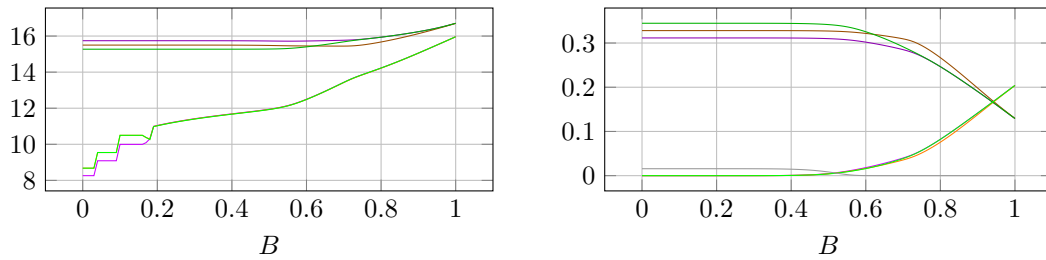


Figure 10.9: Wages versus  $B$  (on the left) and the number of workers versus  $B$  (on the right).

We also observe on Figure 10.9 that commuters are more paid than telecommuters, even if  $B = 1$ . This comes from the fact that commuters have transport costs. Besides, the commuters live in areas where the rental price is higher. Moreover, we observe that when  $B$  increases, the wages of both commuters and telecommuters increase. There are two reasons which explain this phenomenon. First, the demand for telecommuters increases with their productivity, and so does their wage. Second, the form of the production function and the fact that  $\gamma \in (0, 1)$  imply that hiring both types of workers is more productive than hiring just one type.

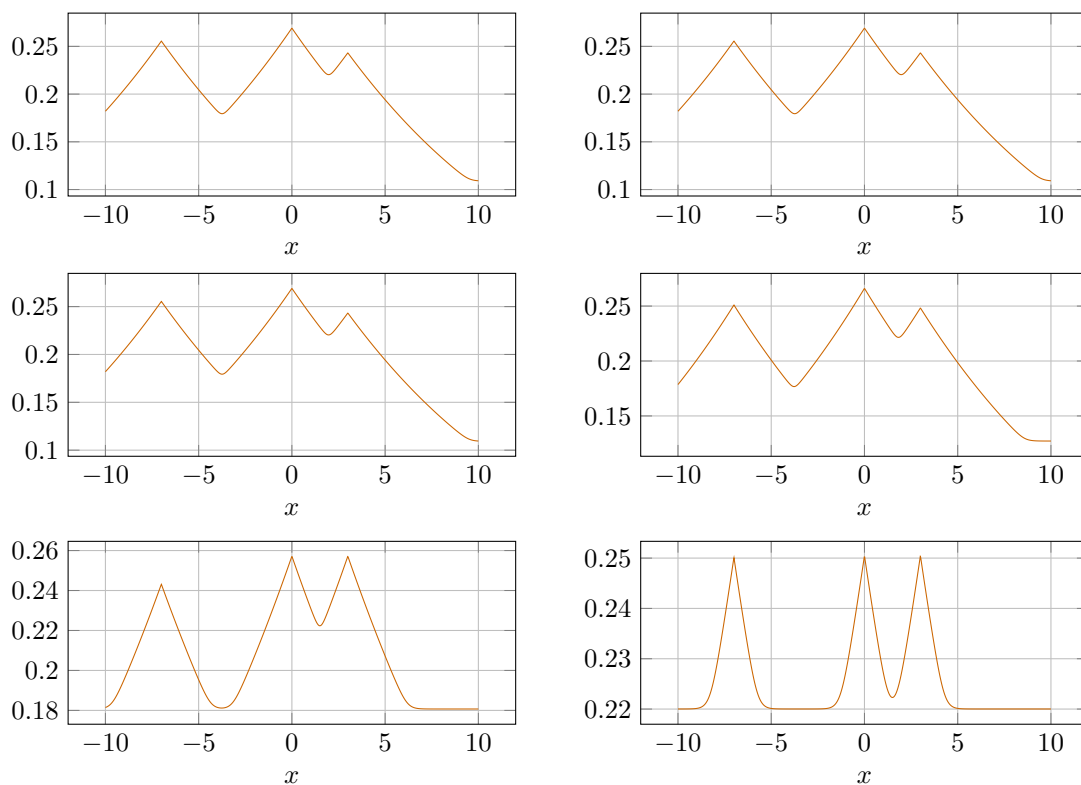


Figure 10.10: Rental price versus  $x$ , for  $B = 0$  (top-left),  $B = 0.2$  (top-right),  $B = 0.4$  (middle-left),  $B = 0.6$  (middle-right),  $B = 0.8$  (bottom-left),  $B = 1$  (bottom-right).

On Figure 10.10, we see that the rental price increases firstly in the area with the highest transport costs. Then, due to the fact that the geographical position of workplaces is no longer important, the rental price in  $y_1$ ,  $y_2$  and  $y_3$  is the same. Moreover, note that the rental price in the area occupied by telecommuters is constant.

### 10.3.4 A home based telecommuting model in a two-dimensional space

We aim at extending the latter simulation to a two-dimensional space. The results are similar with those obtained in Test 3. Moreover, we observe that the computation time increases significantly due to the calculation of the integrals presented in section 10.2.1 where we use the trapezoidal rule adapted to a two-dimensional space.

#### Description of the model and the parameters

We assume that  $X = [-10, 10]^2$ , that the workplaces are located in

$$y_1 = (-7, 7), \quad y_2 = (0, 0), \quad y_3 = (3, -3)$$

and that the production of  $y_i$  is given by

$$F_i(\ell) = A^\alpha (\ell_1^\gamma + B\ell_2^\gamma)^{\frac{1-\alpha}{\gamma}}, \quad \forall \ell \in [0, +\infty)^2,$$

where  $\ell_1$  is the number of commuters and  $\ell_2$  is the number of telecommuters. In this setting, the transport cost to reach the  $i$ th workplace for the commuters living in  $x \in X$  is given by

$$c_i(x) = \frac{\|x - y_i\|_2}{2},$$

and 0 for the telecommuters.

As before, we are going to let  $B$  vary from 0 to 1. The parameters used in Test 4 are the same as in Test 3 and are listed in Table 10.4 below.

Parameter	Value
$\alpha$	0.3
$\gamma$	0.9
$A$	$10^4$
$w_0$	12
$\sigma$	0.1
$\theta$	0.7

Table 10.4: The parameters used in Test 4.

#### Numerical results

In the figures below, we display the distribution of the houses of the commuters working at the different workplaces for different values of  $B$ . The color (—), (—) and (—) are associated to the residences of the commuters working at  $y_1$ ,  $y_2$  and  $y_3$  respectively. The color (—) is associated to the residences of the telecommuters, and (—) corresponds to the residences distribution of independent workers.

The analysis is the same as in the latter simulation. When  $B$  increases, telecommuting develops in areas with high transport costs, see Figure 10.11, 10.12, 10.13 and 10.14. Figure 10.15 highlights the same phenomenon of concentration identified in Test 1 and 3, namely the houses of commuters are located in smaller and smaller areas.

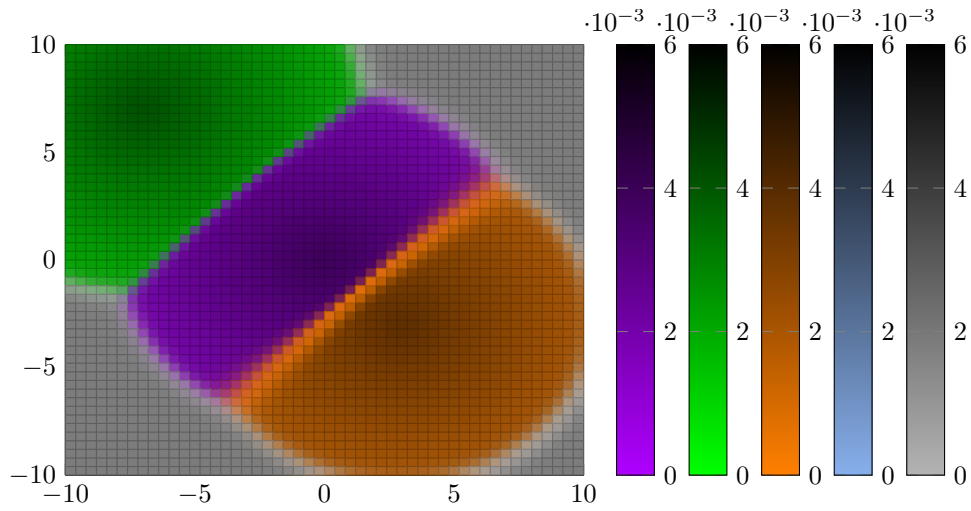


Figure 10.11: The distributions of the houses of commuters working for the different workplaces, of telecommuters and of the independent workers for  $B = 0$ .

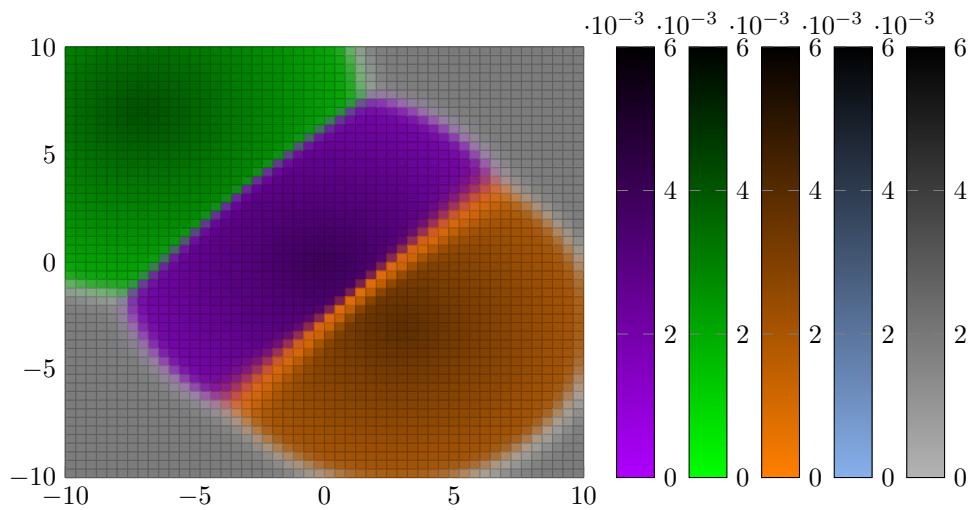


Figure 10.12: The distributions of the houses of commuters working for the different workplaces, of telecommuters and of the independent workers for  $B = 0.33$ .

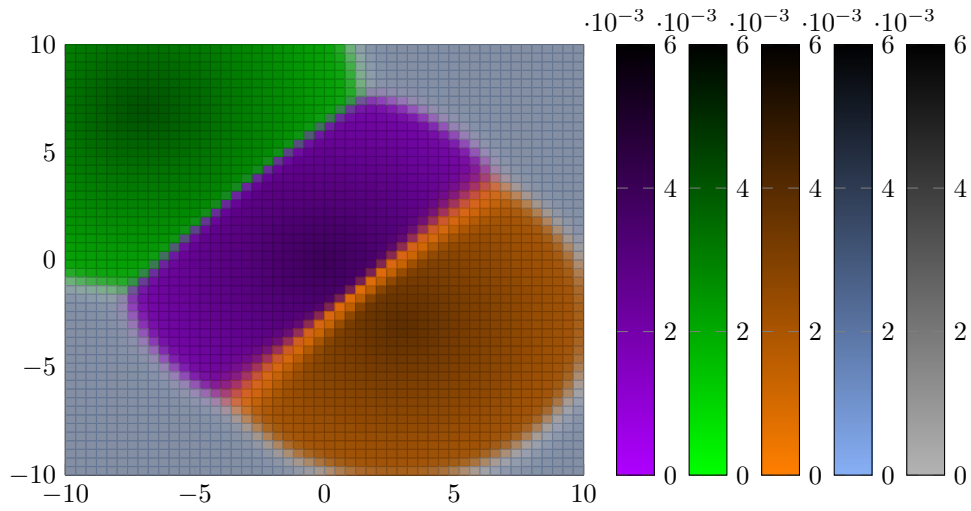


Figure 10.13: The distributions of the houses of commuters working for the different workplaces, of telecommuters and of the independent workers for  $B = 0.5$ .

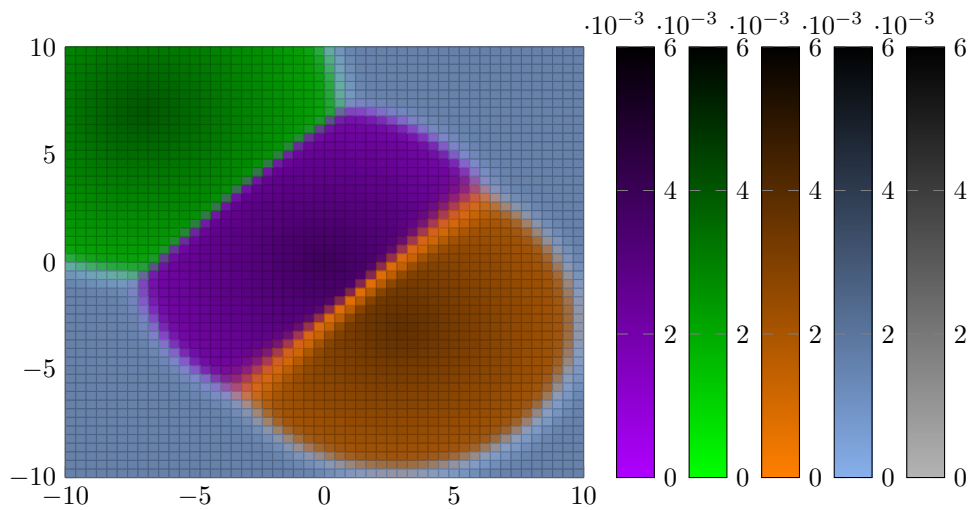


Figure 10.14: The distributions of the houses of commuters working for the different workplaces, of telecommuters and of the independent workers for  $B = 0.66$ .

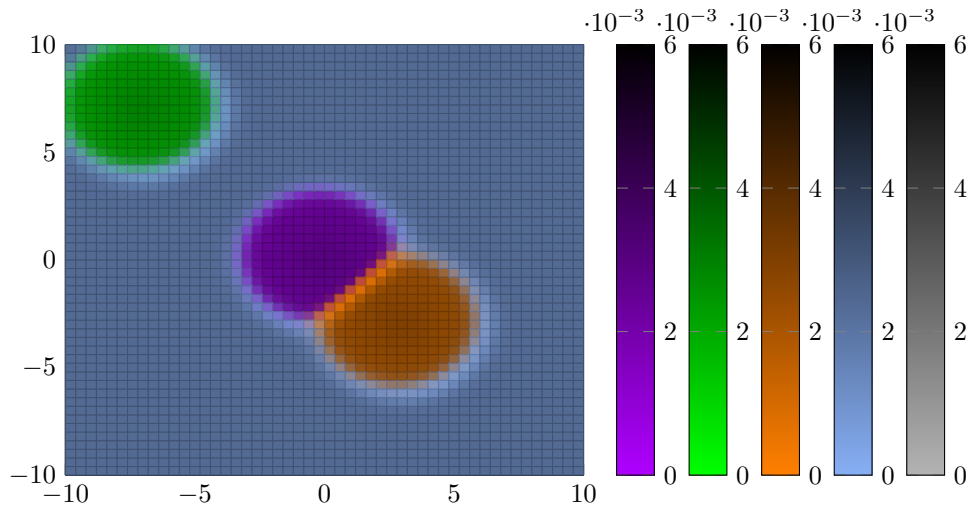


Figure 10.15: The distributions of the houses of commuters working for the different workplaces, of telecommuters and of the independent workers for  $B = 1$ .

## 10.4 Conclusion

We have carried out three simulations in one dimension and one in two dimensions. In the first test, we have highlighted the fact that the location of the firms plays an important role for fixing the wages at the equilibrium. Indeed, the labour supply differs according to the location of the companies. The second test has indicated that this model may be useful to design a urban area by allowing several scenarios. An example is the implantation of a shopping mall. In this case, the simulations can help to determine the size and the location of the shopping center in order to maximize the utility of the agents.

The third simulation has been devoted to telecommuting. We have seen that the number of telecommuters increases as their productivity grows, especially in the regions where the transport costs are high. When  $\sigma$  is small, the competition on the labour market implies that the wages of the telecommuters in each location are almost the same. The last test has recast the home-based telecommuting model in a two-dimensional space and has shown that the numerical method does not need to change for computing an equilibrium. We have noted that the computation time has increased due to the calculation of integrals that are on a two-dimensional space.

Future developments will consist in developing new simulations to study the impact of other parameters such as  $\sigma$  or the transport costs.

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## RÉSUMÉ

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Le marché du travail est étroitement lié aux marchés de l'immobilier locatif pour les professionnels et pour les particuliers. L'objet de cette thèse est l'étude des interactions de ces marchés.

Dans un premier temps, nous développons et étudions un modèle de jeux à champ moyen permettant de lier le marché du travail avec celui de l'immobilier locatif pour les professionnels. Dans un cadre spécifique où la production des firmes est supposée à rendement d'échelle constant, nous montrons que les équilibres admettent une forme explicite qui nous permet d'établir leur existence et leur unicité. Plusieurs lois économiques, dont celle de Pareto et celle de Gibrat, se vérifient. Puis, dans un cadre plus général où nous supposons que la production est à rendement d'échelle strictement décroissant, nous établissons plusieurs résultats d'existence, et retrouvons la règle d'or d'accumulation du capital. Enfin, nous présentons une méthode numérique pour approcher les équilibres. Nous détaillons plusieurs simulations et étudions l'influence de certains paramètres sur l'équilibre calculé en faisant de la statique comparative.

Dans un deuxième temps, nous nous intéressons à un modèle liant le marché du travail avec celui de l'immobilier locatif pour les particuliers. Il admet une composante spatiale et permet de déterminer la distribution des résidences, les salaires et les loyers. Ces trois résultats du modèle vérifient trois conditions d'équilibres : celle du marché du travail, celle du marché de l'immobilier, et une condition de mobilité. La condition sur le marché du travail est liée à un problème de transport optimal, tandis que les deux autres sont liées à un jeu statique non-atomique. Les résultats d'existence et d'unicité d'équilibres que nous établissons exploitent le fait, qu'à l'équilibre, la distribution des résidences admet une forme explicite. Puis plusieurs extensions sont considérées comme l'adaptation du modèle au télétravail. Nous terminons par la présentation d'une méthode numérique développée dans le but d'approcher les équilibres et l'étude, en faisant de la statique comparative, de l'influence de certains paramètres du modèle sur l'équilibre calculé.

## MOTS CLÉS

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Modélisation, Problème de contrôle optimal, Jeux à champ moyen, Transport optimal, Jeux non-atomiques, Simulations numériques.

## ABSTRACT

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The labour market is closely linked to the rental markets for professionals and for individuals. The purpose of this thesis is to study the interactions of these markets.

In a first step, we develop and study a mean field game model that links the labour market with the rental market for professionals. In a specific framework where firms' production is assumed to have constant returns to scale, we show that the equilibria admit an explicit form that allows us to establish their existence and uniqueness. Several economic laws, including Pareto's law and Gibrat's law, hold true. Then, in a more general framework where we assume that the production has decreasing returns to scale, we establish several existence results and find the golden rule of capital accumulation. Finally, we present a numerical method to approximate the equilibria. We detail several simulations and study the influence of some parameters on the computed equilibrium by doing comparative statics.

In a second step, we focus on a model linking the labour market with the rental market for individuals. It admits a spatial component and allows us to determine the distribution of residences, wages and rents. These three outputs verify three equilibrium conditions: the labour market condition, the housing market condition, and a mobility condition. The labour market condition is related to an optimal transport problem, while the other two are related to a static non-atomic game. The existence and uniqueness results we establish, exploit the fact that, at equilibrium, the distribution of residences admits an explicit form. Then, several extensions are considered such as the adaptation of the model to telecommuting. We conclude with the presentation of a numerical method developed to approach equilibria and the study, by doing comparative statics, of the influence of some parameters of the model on the calculated equilibrium.

## KEYWORDS

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Modelling, Optimal control problem, Mean field games, Optimal transport, Non-atomic games, Numerical simulations.