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Étude asymptotique de deux problèmes de la mécanique de fluides : formation de couches limites près des côtes rugueuses non périodiques et comportement d'ondes progressives dans le problème de Hele-Shaw

Gabriela López Ruiz

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**THÈSE DE DOCTORAT
DE SORBONNE UNIVERSITÉ**

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au **Laboratoire Jacques-Louis Lions**
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présentée par

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pour obtenir le grade de :

DOCTEUR DE SORBONNE UNIVERSITÉ

Sujet de la thèse :

**Étude asymptotique de deux problèmes de la mécanique de
fluides**

Formation de couches limite près des côtes rugueuses non périodiques et comportement
d'ondes progressives dans le problème de Hele-Shaw

soutenue le 13 décembre 2021

devant le jury composé de :

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Preface

This manuscript deals with asymptotic modeling in fluid mechanics. We are interested here in studying two singular problems: the effect of roughness on oceanic motion taking as a starting point the single-layer quasi-geostrophic model and the mathematical description of congestion phenomena in tumor growth by investigating the behavior of traveling wave solutions.

Asymptotic analysis in fluid mechanics is motivated by singular limit problems that naturally arise in physical reality, such as combustion, magneto-hydrodynamics, biology, and geophysical fluid mechanics. It provides improved insight into the mathematical structure of the problem by dealing with more straightforward but still significant models.

In the first chapter of this work, we consider the complex system that is the ocean. As a result of the multiple physical quantities determining its behavior at different time and spatial scales, computing the speed of ocean currents at any point using a universal model remains an unrealistic goal. A more reasonable strategy, both from a theoretical and numerical point of view, is to conduct an asymptotic analysis to identify approximate reduced models and their solutions. These factors motivated our study: the impact of the irregularities of the coastline on the wind-driven oceanic motion. The single-layer quasi-geostrophic system is a well-known model in meteorology used to describe the external force applied by the wind on the free surface of the ocean. From a mathematical point of view, it is a singularly perturbed quasilinear problem depending on a small parameter, which justifies the apparition of boundary layers. Since the geometry of the coasts is not meant to follow a given spatial pattern, we use functions without any particular structure to describe roughness. Consequently, nonlinear, linearized and linear PDE systems describe the behavior of the boundary layers in infinite domains, and their solutions are sought in nonlocalized Sobolev spaces. Moreover, under these hypotheses, the eastern boundary layer exhibits a singular behavior at low frequencies far from the eastern boundary. Ergodicity properties are imposed to tackle the convergence issues. In this general regime, we establish the well-posedness of the governing boundary layer equations and of the approximate solution when the small parameter goes to zero.

In the second part of this work, we address a problem inspired by biology: the asymptotics of avascular tumor growth. Establishing how cellular changes affect macroscopic distributions is especially important when examining tissue invasion, emphasizing the need for mathematical models linking multiple scales. We are interested in the singular ‘stiff pressure law’ limit of the porous medium diffusion equation with a source term towards a free boundary model of the Hele-Shaw type. We provide a rigorous mathematical description of the congestion phenomena describing the transitions between free/compressible zones and congested/incompressible zones through the behavior of their respective traveling wave solu-

tions. Furthermore, we describe the profiles in the vicinity of the transition between the free (compressible) domain and the congested (incompressible) domain. We show the nonlinear asymptotic stability of such profiles in regimes where the parameter describing the strength of the repulsive forces is very large.

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Contents

I	Introduction	1
I.1	Asymptotic models in fluid mechanics	1
I.2	A little bit of history	2
I.3	Wind-driven ocean circulation	4
I.3.1	Boundary layers and singular perturbations problems	6
I.3.2	Matched asymptotic expansion method	6
I.3.2.1	A simple example	6
I.3.3	Mathematical models of the oceanic motion	13
I.3.4	The single-layer quasi-geostrophic model	14
I.3.5	Contributions	19
I.3.5.1	Perspectives	24
I.3.5.2	Outline of the Chapter II	25
I.4	Asymptotic methods in the mathematical modeling of tissue growth	25
I.4.1	Mathematical models of tumor growth	26
I.4.2	Compressible agent-based model	26
I.4.2.1	Velocity field behavior	27
I.4.2.2	Pressure law	28
I.4.3	Free-boundary models and the incompressible limit	29
I.4.4	Traveling wave solutions of the Hele-Shaw model for tumor growth	32
I.4.5	Contributions	33
I.4.5.1	Perspectives	37
I.4.5.2	Outline of Chapter III	38
II	Boundary layer formation near nonperiodic rough coasts	39
II.1	Introduction	40
II.2	Preliminaries and main results	43
II.3	Formal asymptotic expansion and first profiles	48
II.4	Existence and uniqueness of the solutions in a rough domain: methodology	51
II.4.1	Linear case.	53
II.4.2	Nonlinear/ linearized problem.	60
II.5	Western boundary layer: the linear case	62
II.5.1	The linear problem on the half-space	63
II.5.1.1	Homogeneous linear problem	64
II.5.1.2	Non homogeneous problem	66
II.5.1.3	Bounds in Kato spaces	68

II.5.1.4	Proof of Theorem II.5.2	73
II.5.2	Differential operators at the transparent boundary	74
II.5.3	The problem in the rough channel	80
II.6	Nonlinear boundary layer formation near the western coast	86
II.6.1	Nonlinear problem in the half-space	87
II.6.2	The problem in the rough channel	88
II.6.3	Connecting the solutions at the artificial boundary	107
II.7	Linearized problem	111
II.7.1	The problem on the half space	112
II.7.2	The linearized problem on the rough domain	113
II.7.3	Joining ω_w^- and ω_w^+	116
II.8	Eastern boundary layer	118
II.8.1	The problem in the half-space	119
II.8.1.1	Behavior far from the boundary	120
II.8.2	Transparent operators	127
II.8.3	Equivalent problem and estimates on the rough channel	129
II.9	Convergence result	131
II.9.1	Construction of the approximate solution	131
II.9.2	Computing the reminder	133
II.9.2.1	Building the correctors	133
II.9.3	Energy estimates	136
Appendices		139
II.A	Study of the roots of equation of the linear western boundary layer	139
II.B	Expansion of the eigenvalues at high frequencies	142
II.C	Computations of the regularity estimates	143
II.D	Asymptotic behavior of the Green function coefficients	145
II.E	Proof of Lemma II.4.1	146
II.F	Proof of Lemmas II.5.9 and II.8.7	148
III	Travelling waves for the PME in the incompressible limit	151
III.1	Introduction	152
III.2	Traveling waves for the Hele-Shaw system and the porous media equation	157
III.2.1	TW for the limit Hele-Shaw system	157
III.2.2	Qualitative properties of travelling waves for the porous medium equation	159
III.2.3	Phase portrait of N_γ and further consequences	163
III.2.4	Quantitative bounds for the profiles N_γ	169
III.3	Stability of the profiles N_γ	177
III.3.1	Overall strategy	177
III.3.2	Control of the nonlinear terms and long-time behavior	183
III.4	Proofs of some technical results	188
III.4.1	Derivation of the equation on u	188
III.4.2	Structure of the linearized system: Lemmas III.3.1 and equality (III.3.18)	189
III.4.3	Properties of the weights \bar{w}_0 and \bar{w} : Lemma III.3.2 and estimate (III.3.19)	190
III.4.4	Proof of the weighted Poincaré inequality	194

CONTENTS

vii

Glossary

197

Bibliography

201

I – Introduction

The problems analyzed in this manuscript attest to the convenience and effectiveness of using asymptotic methods in mathematical modeling. This chapter aims to lay the foundations for later discussion of the technical results obtained during the author’s Ph.D. The first section contains a general discussion on asymptotic methods in fluid mechanics and how their evolution has been driven historically by the needs of problems belonging to mathematical physics. The remaining two sections are devoted to each one of the problems and have a similar structure. First, the evolution of research on the topic is presented, providing a glimpse of the main mathematical results and techniques. Then, a summary of the contributions of this work to the theory and its possible further development are discussed. Section I.3 presents the problem of roughness-induced effects on oceanic motion driven by the quasi-geostrophic model. Section I.4 discusses the different fluid mechanics models used in the mathematical description of tumor growth to introduce later the analysis of monotone traveling wave solutions when considering the incompressible limit.

I.1 Asymptotic models in fluid mechanics

Due to their complexity, only a few fluid flow problems may be solved (approximately) by closed-form (analytic) solutions through idealized mathematical models. Most importantly, most of them are either unsolvable by direct methods or only amenable to numerical simulation after some appropriate asymptotic modeling. Describing more physical and realistic phenomena encompasses a broad spectrum of scales and simultaneously dominant and negligible effects. Nowadays, scientific modeling confronts technological limitations beyond the capabilities of full-scale numerical simulation by the best of supercomputers known to men. For these reasons, asymptotic modeling has become an even more attractive tool to generate more straightforward (fewer variables or (and) fewer unknowns) and consistent models to develop accurate and efficient numerical methods while adding to the understanding of both the underlying physics and the mathematical structure of the investigation’s primary focus. A discussion on the role of physically motivated asymptotic analysis in the design of numerical methods for singular limit problems in fluid mechanics can be found in [1].

Perturbation (asymptotic) theory revolves around the concept of an asymptotic solution. Such a solution approaches a limit as the perturbation quantity approaches zero (or infinity). Several mathematical problems involve rigorously justifying this passage to the limit resulting in a change in the type of equations, [boundary layer](#) formation, or varying the influence of several spatial and temporal scales. In these cases, the main system of equations can be approximated by allowing the perturbation quantity to go towards its limit. Then,

an approximate solution is built in terms of an extremely small or sizeable dimensionless parameter¹.

Some relevant examples are:

- the **Reynolds number** appearing in the dimensionless representation of the Navier-Stokes equations. It leads to the inviscid and boundary-layer models when it is large, and to the Stokes and Oseen models for smaller values;
- the description of rapidly rotating flows with applications of industrial and geophysical nature, which are asymptotic models of flow at low **Rossby number**, large-scale models of flows used in meteorology and oceanography are extracted can be asymptotically derived by considering the hydrostatic balance; and,
- local low **Mach number** limits in compressible flows, where the singular limit passage links fully compressible systems with two-phase compressible/incompressible systems.

For a book on asymptotic models in fluid dynamics, see [2].

The singular parameters used in the asymptotic theory of fluid flows may be subdivided into two categories. There are the so-called dynamic parameters which can be directly built into the equations (**Reynolds number** in Navier-Stokes equations) or in the boundary conditions (inverse of high aspect ratio wings in the **no-slip condition**); and can also be found describing the coupling of two systems associated with the same physical phenomenon [2]. They determine the relative significance of physical processes competing with one another taking place in moving fluid. The remaining category corresponds to geometric parameters when used as a feature of the domain (small irregularities on a rough surface).

Asymptotic models are often classified following the nature of the problem they describe on local or global. Sometimes, understanding flow behavior in some localized regions can be pretty helpful when studying, for example, laminar separation or near interfaces with material properties. Other times, the asymptotic modeling of a flow as a whole provides the relevant information (large-scale oceanic motion).

The main reasoning behind this approach is that the approximate model is associated with an asymptotic expansion procedure, and therefore, can be improved by adding correctors, expanding in a series of successive approximations or both.

1.2 A little bit of history

Since asymptotic methods have played an essential role in the progress achieved in fluid dynamics for more than a century, this section outlines the origins of modern fluid mechanics and mathematical methods.

In 1687, Isaac Newton (1642-1727) postulated his laws of motion and the law of viscosity of Newtonian fluids - a fluid whose stress law - strain rate is linear. Eighteen-century mathematicians later developed this theory regarding an “ideal” (inviscid and incompressible) fluid. Bernoulli published in 1738 in the book *Hydrodynamica* a critical result that will be later known as Bernoulli’s Principle: the sum of pressure, kinetic energy, and potential energy per unit mass of an incompressible, non-viscous fluid in a streamlined flow remains a constant.

¹Dimensionless variables are reference variables selected by geometrical and physical arguments related to the particular system which is modeled.

Jean Le Rond D'Alembert (1749) stated the physical paradox that later will take his name: that a body immersed in a frictionless fluid has zero drag, undoubtedly a consequence of the neglected effects of viscosity. In 1755, Euler applied Newton's second law of motion to a fluid moving under an internal force known as the pressure gradient and provided the first mathematical description of the motion of this type of fluid. Later, in 1816, Pierre-Simon Laplace completed the formulation by adding the adiabatic condition.

Navier (1827) is the pioneer at deriving the equations for homogeneous incompressible viscous fluids based on considerations involving intermolecular forces' action. Later, in 1831, Poisson derived the equations for compressible fluids from a similar molecular model. Saint-Venant (1843) published a derivation of the equations applied to laminar and turbulent flows, but we owe Stokes (1845) the Navier-Stokes equations currently in use.

Experimentalists of the time showed little to no interest in the newly discovered theory of hydrodynamics. Instead, they developed empirical equations, charts, and tables to study several essential quantities. Well-known hydraulic engineers of the nineteenth century were Chézy, Poiseuille, Darcy, Bazin, Manning, and Weisbach, who performed extensive tests and original experiments on the study of flows in open channels, pipe flows, waves, and turbines.

At the end of the nineteenth century, unification between experimental hydraulics and theoretical hydrodynamics finally began. Froude (1810-1879) developed laws of model testing, Lord Rayleigh (1842-1919) proposed the technique of dimensional analysis, and Osborne Reynolds (1842-1912) published the classic pipe experiment in 1883. This study showed the importance of the dimensionless [Reynolds number](#) named after him. Meanwhile, the viscous-flow theory was available but unexploited since Navier and Stokes had successfully added the Newtonian viscous terms to the governing equations of motion. The reason behind neglect was that they were too difficult to analyze for arbitrary flows.

To this century also belongs the first use of perturbation theory in the study of physical processes. In particular, to the works of Lagrange (1811-1815) and Laplace (1799-1825) on celestial mechanics. When computing the Earth's orbit, one can disregard the existence of other planets since the Sun is the astronomical body with the highest mass in the Solar System. For a more accurate prediction, it is necessary to consider the influence of the Moon's gravitation. Perturbations to the main order approximation of the solution are obtained by taking the mass ratio of the Moon and the Earth as a small parameter. In 1843, Cauchy published his results for the Euler's [Gamma function](#) logarithm using [Bernoulli numbers](#). He called attention to the fact that the series on the right-hand side of the [Stirling formula](#) could be used to compute $\ln \Gamma(x)$ when x is large and positive. Moreover, it has been shown that the absolute error incurred by considering a finite sum of N becomes arbitrarily small with increasing x . Forty years later, Poincaré (1886) broadens the concept of development in series by introducing the *asymptotic development* to solve puzzling differential equations of celestial mechanics. Namely, he described the search for a divergent series capable of approaching functions at infinity which later became an essential theoretical analysis tool in fluid dynamics.

Asymptotic analysis and fluid mechanics become inseparable with the introduction of [boundary layer](#) theory in the seminal paper by Prandtl (1904). Although prototypes of the concept of a [boundary layer](#) associated with the [no-slip condition](#) on a solid body had existed since the derivation of the equations of motion of a viscous fluid (Stokes (1845), Rankine (1864), Froude (1872), Lorenz (1881)), it was Prandtl who noticed that while in a large [Reynolds number](#) flow past a rigid body, the Euler equations hold in the bulk of

the flow, the inviscid description was not valid near the body's surface. He introduced the idea of subdividing the entire flow field into two separate regions where different asymptotic forms of the governing equations: a small thin layer near the solid surface and the Euler equations elsewhere. This was a subject of thorough discussion in the 1950s and early 1960s involving the names of Friedrichs, Kaplun, Lagerstrom, Cole, and Van Dyke (see [3] for a survey on the first fifty years of research on [boundary layer](#) theory). Their works help to develop the singular perturbation theory as we currently know it and the method of matched asymptotic expansions. The development of the asymptotic approach in fluid mechanics is well documented; see, for example, [4]. For a recent detailed analysis of its importance in physical oceanography, the reader can turn to [5]. For an extensive discussion on the evolution of theoretical and experimental fluid mechanics, see [6].

1.3 Wind-driven ocean circulation

Ocean circulation is pivotal for climate regulation and sustaining marine life through heat, nutrients, and chemical transport [7]. A better understanding of the complex dynamical system that is the ocean would increase the precisions of climate predictions.

Wind stress is probably the most vital force acting on the upper surface of the world's oceans. In the early stages, our knowledge of its influence on general oceanic circulation was primarily observational. Its understanding entered the modeling realm with the seminal work of Ekman (1905)[8]. He showed that the frictional stress of the wind is confined to a thin (surface) layer, so the motions below can be considered frictionless. According to his theory, the velocity in the boundary layer should have a spiral structure. This layer and its flux, which currently bear Ekman's name, constitute the theoretical foundation of modern wind-driven circulation theories. It was not until 1987 [9], that in situ measurements confirmed the existence of the [Ekman spiral](#) in the upper ocean. [Ekman layers](#) are now a standard textbook topic of physical oceanography and are reasonably well-understood from the mathematical standpoint [10].

In general, oceanography and meteorology experienced an unprecedented quantitative and qualitative revolution in the late '40s and '50s with the emergence of computers and satellites. In this period occurred the second breakthrough in wind-driven circulation theory in the works of Harald Sverdrup (1947) [11]. He established the relationship between the wind stress curl and the circulation in the basin interior by assuming that the wind-driven circulation is confined to the upper layer of the ocean and the lower layer is infinitely deep. Thus, the circulation can be studied in terms of a single moving layer. Oceanographers immediately adopted this approach, and it is known today as [reduced-gravity models](#). In order to find the circulation in the basin, Sverdrup assumed and integrated the wind stress curl westward, starting with a no-zonal flux condition at the eastern boundary. No explanation was provided on dealing with the western boundary and the reasons behind the integration order. Nevertheless, the [Sverdrup balance](#) and the homogeneous model of wind-driven circulation are cornerstones of geophysical models.

The frictional or inertial western boundary layers provide a vital dynamical component that helps close the circulation in terms of the conservation of mass, energy, and potential vorticity. A dynamical explanation for [western intensification](#) phenomena was first established in 1948 when Henry Stommel studied an idealized model for the North Atlantic Ocean,

including bottom friction and the latitudinal change of the [Coriolis force](#), also known as β -effect. He used a different approach to the single-layer ocean by considering wind-driven oceanic circulation as a single moving layer: the homogeneous model. In this case, the wind-stress curl is negative, so the interior flow must move southward where planetary vorticity is lower. The interior flow must be closed by some boundary layer. Stommel proposed to close the circulation driven by evaporation at low latitudes and precipitation at high latitudes by adding the western boundary currents. This solution overcame the substantial limitations implied in Goldsbrough's model [12] concerning how the mass flux at the ocean surface induces a barotropic flow in the ocean interior through vorticity conservation needs. Accordingly, circulation in a closed basin driven by an arbitrary pattern of wind stress, or freshwater flux across the air-sea interface due to evaporation minus precipitation, can be very well described by the theory. In 1950, Munk essentially applied the same approach as Stommel to lateral friction by assuming it balanced the vorticity equation and induced an intense western flow [13].

Circa 1955, research on the Gulf Stream helped the understanding of boundary layers in physical oceanography. Here, they are worth mentioning the inertial model of Charney [14, 15] and the parallel work on the matter by Morgan [16]. Charney investigated baroclinic instability and showed the connection between the upper Ekman layer and the formation of permanent currents in the ocean interior. Morgan focused mainly on integral constraints of closed circulation in an entire basin and hinted at the connection between the interior and the inertial western boundary currents. For a detailed review on the subject, we recommend consulting [17] and the references therein.

Jules Charney is also considered the father of the quasigeostrophic (QG) theory. Although the term and some aspects of the theory were dealt with before by other authors, Charney's paper "On the scale of atmospheric motions" of 1948 [18] distinguishes for its consistent scale-analysis of the equations governing the motion of the atmosphere using all relevant dynamical and thermodynamical equations. He mathematically derived a tractable equation for unstable waves by eliminating the [inertia-gravity waves](#) solutions using the separation of time between vertical motions. This theory replaces them with an elliptic relation between a materially conserved scalar, the potential vorticity, and the geostrophic stream function. Norman Phillips in [19] believes that "earlier studies on large-scale motion in the atmosphere cannot be said to have "led" to Charney's 1948 paper", but it recognizes the influence from a methodology perspective of the works of L. Prandtl [20], and Rossby [21, 22]. For more information on the matter, the reader can refer to [19] for a detailed chronological analysis of papers preceding the one by Charney on the subject of barotropic [geostrophic approximations](#), baroclinic quasigeostrophy, and [geostrophic equilibrium](#). For more on the influence of Charney in physical oceanography, see [23].

QG theory may also be applied to shallow water equations and [stratified fluids](#). Shallow water equations are vastly used as a conceptual model for the behavior of rotating, [stratified fluids](#). They were introduced by Welander in 1957 [24] to explain the relation between wind-driven currents and surface displacements in a shallow body, extending Ekman's analysis to the case of finite depth. The development of shallow-water dynamics on account of the wind-driven circulation is examined in [25, 17]. For an extensive review of shallow-water dynamics and related models, the reader can refer to [26].

1.3.1 Boundary layers and singular perturbations problems

Our previous discussion shows oceanographers rapidly adopted the boundary layer theory developed in traditional fluid dynamics. In mathematical terms, boundary layers appear when the governing equations describing a flow field (generally a set of non-linear partial differential equations) is a singular perturbation problem depending on a small dimensionless parameter. This is often because the perturbation parameter multiplies the highest derivative of the unknown function in the main differential equation.

Boundary layer problems have attracted the attention of mathematicians for more than 60 years. Mathematical research has provided insight on many different phenomena, including, among others, the classical problem of vanishing viscosity limit in fluid mechanics and, in particular, in geophysical fluid mechanics, in relation to rotating fluids and Ekman layers. For an extensive discussion on the matter, we refer to [27] and the references therein. The reader interested in a comprehensive monograph on mathematical results in geophysical dynamics can consult [10].

1.3.2 Matched asymptotic expansion method

A widely used approach consists of the construction of an approximate solution using “matched asymptotic expansions”. This method was developed systematically between the 50s and 70s of the 20th century and used successfully in a wide variety of applications [28]. It is a powerful systematic analytical tool for asymptotically calculating solutions to singularly perturbed PDE problems. For a general presentation, we refer the reader to [29, 30, 31].

The main idea of the matched asymptotic expansions method is to introduce a local variable to describe the behavior near the wall and the corresponding local (inner) expansion when a regular (outer) expansion fails to describe the actual behavior of the fluid flow near a solid surface. Both expansions have to match to obtain a uniformly valid approximation of flow on the entire region. Thus, the name of the method. For example, when dealing with [western intensification](#), the flow can be described by the low-order dynamics in the interior ocean, i.e., essentially inviscid and linear. However, such an approximation is no longer valid near the western wall, within the boundary layer, where the fluid velocity changes rapidly and viscosity becomes essential. Consequently, other physical phenomena must be considered, such as frictional or inertial terms. In other words, the addition of correctors is necessary to match the interior solution and boundary conditions.

1.3.2.1 A simple example

We proceed to illustrate how to deal with a singularly perturbed problem through an example. For the purpose of coherence with the fluid mechanics context, we analyze the following second order linear ODE system:

$$\begin{aligned} \varepsilon \tilde{u}_\varepsilon'' + \tilde{u}_\varepsilon' &= a, \quad \text{in } (0, 1) \\ \tilde{u}_\varepsilon(0) &= 0, \quad \tilde{u}_\varepsilon(1) = 1, \end{aligned} \tag{I.3.1}$$

where $0 < \varepsilon \ll 1$ is a small parameter and a is a given positive constant of order 1 and independent of ε . This model was introduced by Friedrichs (1942) as an example of a boundary layer in a viscous fluid and has become common in perturbation theory, see for instance [32, 33].

We are interested in studying the behavior of the solution u_ε in (I.3.1) when $\varepsilon \rightarrow 0$. In this case, it is possible to find the exact solution, which is helpful for later comparisons with the approximate solution. From now on, we will denote by u_ε the solution of a perturbed problem, while u_ε refers to the corresponding limit problem at $\varepsilon = 0$. The existence and uniqueness of the solutions to (I.3.1) consist of finding the corresponding variational formulation and applying the Lax-Milgram Lemma. To facilitate the computations, we consider $u_\varepsilon(x) = \tilde{u}_\varepsilon(x) - x$ in (I.3.1) which gives the following problem on u with homogeneous boundary conditions

$$\begin{aligned} \varepsilon u_\varepsilon'' + u_\varepsilon' &= a - 1, \quad \text{in } (0, 1), \\ u_\varepsilon(0) &= u_\varepsilon(1) = 0. \end{aligned} \quad (\text{I.3.2})$$

A weak solution of (I.3.2) is a function $u_\varepsilon \in H_0^1([0, 1])$ verifying the integral condition

$$-\varepsilon \int_0^1 u_\varepsilon' v' + \int_0^1 u_\varepsilon' v = (a - 1) \int_0^1 v, \quad \forall v \in H_0^1([0, 1]). \quad (\text{I.3.3})$$

Then, the Lax-Milgram theorem is used once again to show the existence and uniqueness of a solution $u_0 \in H^1([0, 1])$ of the limit problem (I.3.6). Considering $v = u_\varepsilon$ as test function in (I.3.3) and applying Poincaré inequality, we have that

$$\begin{aligned} \varepsilon \|u_\varepsilon'\|_{L^2}^2 &= -(a - 1) \int_0^1 u_\varepsilon \leq C \|u_\varepsilon\|_{L^2} \\ &\leq C' \|u_\varepsilon'\|_{L^2} \leq \frac{C'^2}{2\varepsilon} + \frac{\varepsilon}{2} \|u_\varepsilon'\|_{L^2}^2, \end{aligned}$$

which leads to

$$\varepsilon^2 \|u_\varepsilon'\|_{L^2}^2 \leq C'^2.$$

Thus, we have found a bound for $\varepsilon u_\varepsilon'$ in $L^2([0, 1])$, independent of ε . Moreover, u_ε is bounded in $L^2([0, 1])$ independently of ε . This can easily be shown by considering a weighted test function $v = u_\varepsilon w$ in (I.3.3), where w is a decreasing function, for example, $w = 1 - x$.

$$-\varepsilon \int_0^1 (u_\varepsilon')^2 (1 - x) + \frac{1}{2} \int_0^1 (u_\varepsilon^2)' (1 - x) = \frac{a - 1}{2}. \quad (\text{I.3.4})$$

Integrating by parts the second member in right-hand side gives

$$-2\varepsilon \int_0^1 |u_\varepsilon'|^2 (1 - x) + \|u_\varepsilon\|_{L^2([0, 1])}^2 = a - 1.$$

Hence, $\|u_\varepsilon\|_{L^2([0, 1])}^2 \leq C$. We can then infer that there exists a subsequence $(\varepsilon_n)_{n \in \mathbb{N}}$ satisfying $\varepsilon_n \rightarrow 0$ and a function $u_0 \in L^2((0, 1))$ such that

$$u_{\varepsilon_n} \rightharpoonup u_0 \text{ weakly in } L^2.$$

Passing to the limit in (I.3.3), we obtain that the function $u_0 \in L^2((0, 1))$ verifies

$$-\int_0^1 u_0 v' = (a - 1) \int_0^1 v, \quad \forall v \in H_0^1([0, 1]). \quad (\text{I.3.5})$$

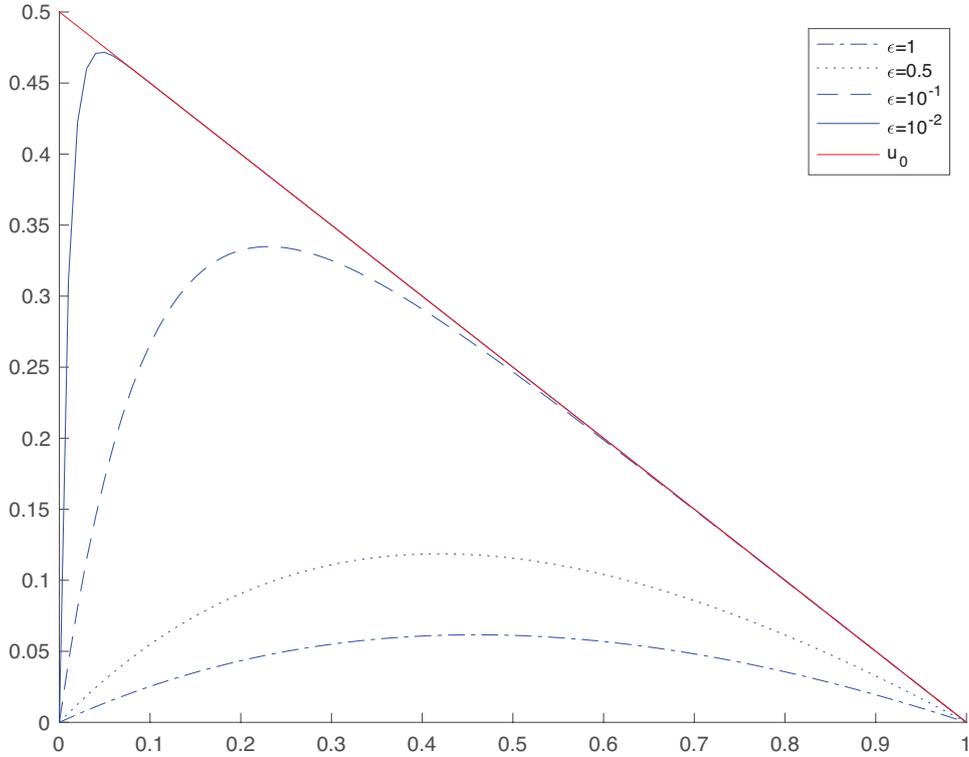


Figure I.1: Behavior of the exact solution u_ε of the problem for $a = \frac{1}{2}$ when $\varepsilon \rightarrow 0$.

The above result corresponds to the weak formulation of the problem

$$u_0' = a - 1, \quad (\text{I.3.6})$$

with no boundary conditions. It is evident that u_ε differs from u_0 in thin zones near the boundaries. This leading-order approximation is only suitable in the *outer region*, here $(0, 1)$. Note that the equation driving the first approximation of the solution differs from the one of u_ε because we have lost the second order term (I.3.1) and cannot satisfy both boundary conditions. Note this is coherent with the sharp transitions of u_ε in small area near $x = 0$, see Figure I.1.

The previous analysis gives a clear indication that relevant information is missing when considering the small parameter equal to zero and hints at the idea of rescaling the equations. The new scale allows the introduction of the local information at the boundary layer through an *inner expansion*.

If there is not an a priori knowledge of the correct scale for constructing the inner asymptotic expansion, it is usual to define the local variable X as follows

$$X = \delta(\varepsilon)x,$$

where δ is a real function of ε and $X = O(1)$. Rewriting the equation (I.3.1) in X , yields

$$\delta^2 \varepsilon \frac{d^2 U_\varepsilon}{dX^2} + \delta \frac{dU_\varepsilon}{dX} = a - 1. \quad (\text{I.3.7})$$

A quick comparison of the coefficients in (I.3.7) provides five different possibilities for the rescaling parameter.

Case (i) Taking $\delta \ll 1$, leads to the contradiction $a - 1 = o(1)$.

Case (ii) Taking $\delta = 1$, we have the regular asymptotic expansion $u'_0 = a - 1$, which it is not able to satisfy all boundary conditions as seen before.

Case (iii) When $1 \ll \delta \ll \frac{1}{\varepsilon}$, we get from equation (I.3.7) that $\frac{dU_\varepsilon}{dX} = o(1)$, which is once again impossible.

Case (iv) Now, if $\delta = \frac{1}{\varepsilon}$, then $\delta \gg 1$ and the equation behaves as follows:

$$\frac{d^2 U_\varepsilon}{dX^2} + \frac{dU_\varepsilon}{dX} = \underbrace{(a - 1)\varepsilon}_{o(1)}, \quad (\text{I.3.8})$$

which provides the correct scale.

Case (v) Lastly, if $\delta \gg \frac{1}{\varepsilon}$, i.e. $\delta\varepsilon \gg 1$, multiplying equation (I.3.7) by $\delta^{-1}\varepsilon^{-2}$ leads to

$$\frac{d^2 U_\varepsilon}{dX^2} + \underbrace{\delta^{-1}\varepsilon^{-1} \frac{dU_\varepsilon}{dX}}_{o(1)} = \underbrace{(a - 1)\delta^{-1}\varepsilon^{-2}}_{o(1)},$$

which also fails to provide a suitable scale.

In conclusion, assuming $\delta = \varepsilon^{-1}$ will transform (I.3.7) in a regular perturbation problem.

Now that we have chosen the proper scale, we can look for an approximation of u_ε in the form

$$u_\varepsilon \sim u_0(x) + u_l^{BL}(X_l) + u_r^{BL}(X_r), \quad (\text{I.3.9})$$

where $X_l = \varepsilon^{-1}x$ and $X_r = \varepsilon^{-1}(1 - x)$ are the rescaled variables; and u_r^{BL} and u_l^{BL} denote the approximate correctors whose role is to solve the discrepancy between u_ε and u_0 near $x = 0$ and $x = 1$, respectively. Here, ε is the thickness of the boundary layer. By additionally imposing a far-field or stability condition to guarantee u_ε remains bounded when $x \rightarrow +\infty$, which means both correctors converge to zero as the corresponding rescaled variable goes to infinity.

From the previous analysis, we know that the equation $u'_0 = a$ provides a first approximation of the solution, correcting the source term. Therefore, there remains to investigate the boundary layer terms. Plugging the ansatz (I.3.9) into (I.3.2) yields the following equations

$$\frac{d^2 u_l^{BL}}{dX_l} + \frac{du_l^{BL}}{dX_l} = 0, \quad (\text{I.3.10})$$

$$\frac{d^2 u_r^{BL}}{dX_r} - \frac{du_r^{BL}}{dX_r} = 0. \quad (\text{I.3.11})$$

The solutions of (I.3.10) and (I.3.11) are $u_l^{BL} = l_0 + l_1 e^{-X_l}$ and $u_r^{BL} = r_0 + r_1 e^{X_r}$, respectively, where $r_i, l_i \in \mathbb{R}$ for $i = 0, 1$. Function u_r^{BL} increases exponentially as $X_r \rightarrow +\infty$. This behavior is not compatible with the assumption that boundary layers have no influence far from the boundary (*far field condition*). Therefore, to guarantee that $\lim_{X_r \rightarrow +\infty} u_r^{BL} = 0$, one must take $r_i = 0, \forall i$, i.e., $u_r^{BL} \equiv 0$.

So far we have disregarded the boundary conditions. Note that substituting (I.3.9) in (I.3.1) leads to $u_l^{BL}(0) = -u_0(0)$, near $x = 0$; while at the other end of the interval $u_l^{BL}(1) = -u_0(1)$. If $-u_0(1) \neq 0$, the far field condition is not satisfied. Therefore, $-u_0(1)$ must forcibly be equal to zero which implies the equation driving the outer expansion term has been supplemented with the boundary condition

$$u_0(1) = 0.$$

This clearly highlights the close relationship between the nature of the boundary layer at the right and the choice of boundary conditions for the inner profile u_0 .

We conclude that:

- there is no boundary layer at $x = 1$;
- the inner profile is $u_0(x) = (a - 1)(x - 1)$; and,
- the boundary layer corrector at $x = 0$ is the solution of the system

$$\begin{aligned} \frac{d^2 u_l^{BL}}{dX_l^2} + \frac{du_l^{BL}}{dX_l} &= 0, \quad \text{in } \mathbb{R}_+ \\ u_l^{BL}(0) &= -u_0(0), \\ u_l^{BL} &\rightarrow 0, \quad \text{as } X_l \rightarrow +\infty. \end{aligned} \tag{I.3.12}$$

Therefore, $u_l^{BL}(x) = (a - 1)e^{-X_l}$.

This procedure is known as *asymptotic matching*. In the words of Van Dyke [32]:

$$\begin{aligned} & \text{Inner representation of (outer representation)} \\ & = \\ & \text{outer representation of (inner representation),} \end{aligned}$$

where the outer representation corresponds to first nonzero term in the macroscopic or global variable (here, x) and the inner ones is the equivalent in the microscopic or local variable (in our example, X_l, X_r). Therefore, the solution of (I.3.12) is an approximation of u_ε satisfying all boundary conditions

$$u_0 + u_l^{BL}, \quad u_l^{BL}(x) = -u_0(0)e^{-\frac{x}{\varepsilon}}.$$

Figure (I.2) illustrates the behavior of the new approximation for different values of ε with respect to the analytical solution u_ε and the regular (outer) approximation u_0 . Notice that for $\varepsilon = 10^{-2}$, u_ε and $u_0 + u_l^{BL}$ are indistinguishable at the scale of the graph. In mathematics, it is well-known that a graphical representation is a reference but not a proof, therefore, we need to show the convergence of our approximation to the original solution.

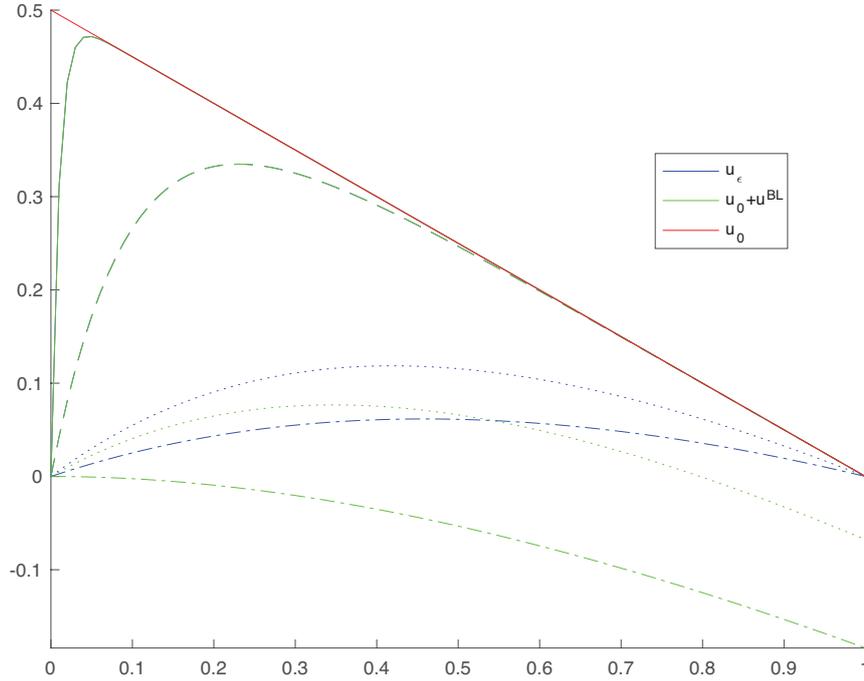


Figure I.2: Behavior of the new approximation, the analytical solution and the regular approximation u_0 for (I.3.7) for the case when $a = \frac{1}{2}$ and $\varepsilon \in \{10^{-2}, 10^{-1}, 0.5, 1\}$.

Let us define $w_\varepsilon = u_\varepsilon - (u_0 + u_t^{BL})$. This error function satisfies the following system resulting from (I.3.7), (I.3.8) and (I.3.12)

$$\begin{aligned} \varepsilon w_\varepsilon'' + w_\varepsilon' &= -\varepsilon u_0'', & 0 < x < 1, \\ w_\varepsilon(0) &= 0, & w_\varepsilon(1) = -u_t^{BL}(1) = (1-a)e^{-1/\varepsilon}. \end{aligned} \quad (\text{I.3.13})$$

Notice that $-\varepsilon u_0'' = 0$ and the value of w_ε is exponentially small as $\varepsilon \rightarrow 0$.

Therefore, we have a good approximation of the actual solution when $\varepsilon \rightarrow 0$. Let $C > 0$ be a constant whose value is independent from ε , using the estimates $\|u_t^{BL}\|_{L^2((0,1))} \leq C\sqrt{\varepsilon}$ and $\|(u_t^{BL})'\|_{L^2((0,1))} \leq C\varepsilon^{-1/2}$, we infer from (I.3.13) that

$$\begin{aligned} \|u_\varepsilon - u_0\|_{L^2((0,1))} &\leq C\sqrt{\varepsilon}, \\ \|u_\varepsilon - u_0\|_{H^1((0,1))} &\leq C\varepsilon^{-1/2}. \end{aligned}$$

This result gives a convergence rate in $L^2((0,1))$. Similar results can be obtained using L^p estimates, for $1 \leq p \leq +\infty$.

Some remarks are necessary:

- Most of the time, computing the error is not that simple. We need to compute energy estimates of w_ε in an acceptable norm and show the error becomes increasingly small as $\varepsilon \rightarrow 0$.

- $u_0 + u_l^{BL}$ is known in the literature as a *zeroth-order approximation*. Consequently, u_0 is the zeroth-order outer solution which describes the behavior of the solution outside of the boundary layer regions, and the corrector u_l^{BL} is known as zeroth-order inner solution linked to u_ε in the boundary layer.

We end our discussion by emphasizing that rarely the approximation at the main orders provides an accurate description of u_ε . One of the advantages of using asymptotic methods is that the solution can be improved by adding additional terms. Consequently, the idea discussed before can be easily generalized to higher orders by means of a power series of the small parameter ε . In practice, we look for an approximate solution of the form:

$$u_\varepsilon \sim \sum_{i=0}^{+\infty} \varepsilon^i (u_i + (u_l^{BL})_i + (u_r^{BL})_i).$$

The main steps of the analysis are the same as before. First, we substitute the asymptotic expansion in the original equation and recursively obtain a family of equations of each one of the components by balancing the powers of ε .

The main equations describing the behavior of the profiles at each order in the case of (I.3.2) is shown in Table I.1.

Order	Outer expansion	Inner expansion	
		Left boundary layer	Right boundary layer
$\mathcal{O}(\varepsilon^{-1})$		$\frac{d^2(u_l^{BL})_0}{dX_l^2} + \frac{d(u_l^{BL})_0}{dX_l} = 0$	$\frac{d^2(u_r^{BL})_0}{dX_r^2} - \frac{d(u_r^{BL})_0}{dX_r} = 0$
$\mathcal{O}(1)$	$u'_0 = a - 1$	$\frac{d^2(u_l^{BL})_1}{dX_l^2} + \frac{d(u_l^{BL})_1}{dX_l} = 0$	$\frac{d^2(u_r^{BL})_1}{dX_r^2} - \frac{d(u_r^{BL})_1}{dX_r} = 0$
$\mathcal{O}(\varepsilon)$	$u'_1 = u''_0$	$\frac{d^2(u_l^{BL})_2}{dX_l^2} + \frac{d(u_l^{BL})_2}{dX_l} = 0$	$\frac{d^2(u_r^{BL})_2}{dX_r^2} - \frac{d(u_r^{BL})_2}{dX_r} = 0$
	\vdots	\vdots	\vdots
$\mathcal{O}(\varepsilon^k)$	$u'_k = u''_{k-1}$	$\frac{d^2(u_l^{BL})_{k+1}}{dX_l^2} + \frac{d(u_l^{BL})_{k+1}}{dX_l} = 0$	$\frac{d^2(u_r^{BL})_{k+1}}{dX_r^2} - \frac{d(u_r^{BL})_{k+1}}{dX_r} = 0$
	\vdots	\vdots	\vdots

Table I.1: Equations of the inner and boundary layer profiles at order ε^k for $k \geq -1$.

A technical difficulty adds to the algebraic one: choosing the boundary conditions for each outer solution; since it is directly linked to the corrector through the matching process. The latter is far from being straightforward in most cases. There are also the issues resulting from a possible overlapping of inner and outer approximations; for details, refer to [33].

The last step corresponds to verify the error $w_\varepsilon = u_\varepsilon - \sum_{i=0}^n \varepsilon^i (u_i + u_i^{BL})$, for a certain $n \in \mathbb{N}$. The choice of n depends on the number of terms needed for the error to verify $w_\varepsilon = o(1)$ when $\varepsilon \rightarrow 0$.

In the example, $u_k'' = 0$ for all $k \in \mathbb{N}$. Therefore, the solution of the inner profiles is constant $u_k = C_k$, $C_k \in \mathbb{R}$. The equation driving the behavior of higher order correctors at $x = 1$ have an exponentially large solution. Then, similarly to the zeroth-order equation, it is necessary to choose the boundary condition for the interior profile such that $(u_r^{BL})_{k+1} \equiv 0$ for the far field conditions to be satisfied. As a consequence, the problem for higher order interior terms is as follows for $k \geq 1$

$$\begin{cases} u_k' = 0, & \text{in } (0, 1), \\ u_k(1) = 0, \end{cases} \implies u_k \equiv 0,$$

yielding in turn that $(u_t^{BL})_k \equiv 0$, $k \geq 1$.

1.3.3 Mathematical models of the oceanic motion

A classical model to describe large-scale oceanic motion is the Navier-Stokes system in a rotating frame. The model is supplemented with the first law of thermodynamics, the diffusion equation for humidity and salinity, and boundary conditions to describe the interaction of the fluid with its surroundings.

The extreme generality of the complete system of equations makes computing explicit solutions an unrealistic goal in the foreseeable future from both numerical and theoretical perspectives. Scaling and asymptotic analysis come into play to obtain approximate models that accurately predict physical phenomena while being simple enough for mathematical analysis to be possible.

The simplifying hypotheses leading to the homogeneous wind-driven circulation model establish a scale of sorts for other well-known mathematical models in geophysical dynamics.

The equations of geophysical fluid dynamics are derived from the conservation laws from continuum mechanics, namely of mass, momentum, energy, and additional elements such as salt, humidity, and chemical components. Another distinctive characteristic of this type of fluid is the influence of the Earth's rotation (the tangential speed is around 400m/s and the speed of the ocean typically around 0.1m/s).

In the ocean, it is well accepted that the differences in density are only relevant in the buoyancy term and the equation of state. Neglecting the density changes elsewhere leads to the *Boussinesq equations* (BES), which are considered as the basic equations for the ocean and removes *acoustic waves* [34]. This system is fully nonlinear and three-dimensional, making its analytical treatment extremely difficult and unsuitable for numerical simulations in its explicit form. The smallness of the aspect ratio between the vertical and horizontal scales (an average depth of 3.7 km over a typical horizontal length of 10^4 km) can be used to simplify the *vertical momentum balance*. This leads to the pressure being locally determined by the weight of fluid. This assumption on the vertical momentum equation is known as *hydrostatic approximation* and leads to the so-called *primitive equations* (PEs) introduced by Richardson in 1922. For some important mathematical results regarding the study of PEs, the reader may refer to [35, 36, 37]

From the physical point of view, *Shallow Water equations* (SW) stand as an intermediate model between the primitive and the quasi-geostrophic equations. They are essentially a vertically integrated form of the Navier–Stokes equations. Some books and survey papers the interested reader can consult are [38, 39]. Another essential feature of large-scale geophysical flows is that the Earth, in its rotation, moves much faster than the wind and the seawater.

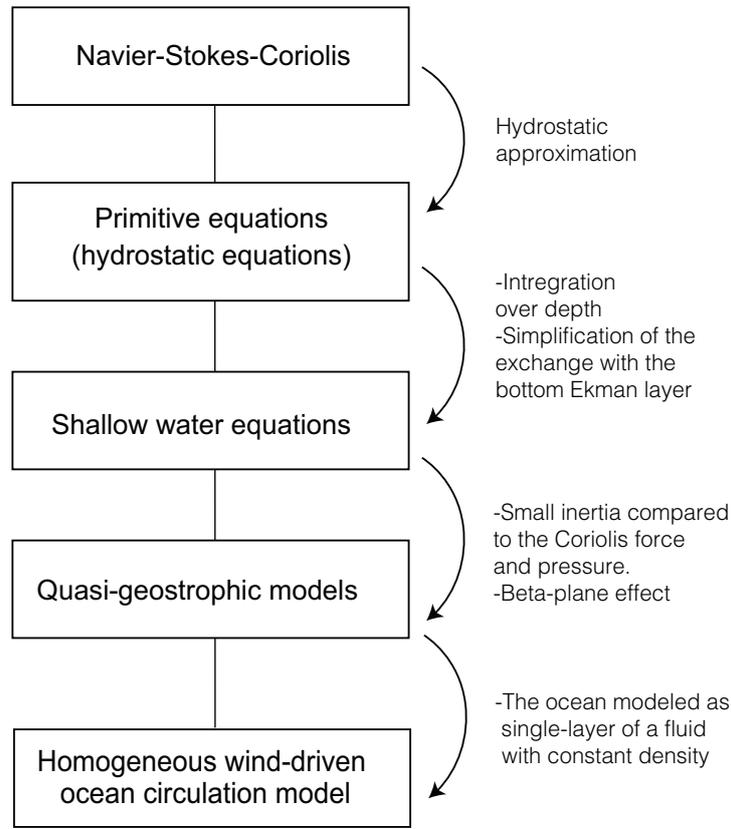


Figure I.3: Hierarchy of models in geophysical fluid dynamics leading to the homogeneous wind-driven ocean circulation model or single-layer quasi-geostrophic model.

Accounting for the fast rotation leads to the classical quasi-geostrophic equations and other balanced equations.

Although the mathematical derivation of the quasi-geostrophic (QG) equations from the free surface Navier–Stokes or primitive equations remains an open problem, many results exist with a rigid-lid hypothesis, see [10] and references therein. When a "flat" ocean is considered, the zeroth and first-order asymptotic expansions of the SW equations based on a small parameter lead to the so-called geostrophic and quasi-geostrophic equations, respectively. Formal derivations of the homogeneous wind-driven model from the Navier–Stokes equations can be found in [40] and [17].

I.3.4 The single-layer quasi-geostrophic model

This section presents the single-layer quasi-geostrophic equations and explains their most relevant physical and mathematical features. This material is well-known from geophysical fluid dynamics as applied to the ocean. Consequently, we only focus here on setting the mathematical and physical foundations leading to the discussion of our results. The reader interested in a more extensive and detailed analysis of the matter can turn to [41, 25], and for an extensive monograph on the quasi-geostrophic model, to [42].

The quasi-geostrophic model describes the almost stationary response of the ocean to the

wind forcing at the free surface and the turbulent friction resulting from Earth's rotation. The main modeling assumptions are:

- The water body is taken as a constant density fluid in the form of a single layer bounded by the seafloor and the free surface of the sea.
- The pressure is fully hydrostatic, and consequently, the horizontal pressure gradient depends only on the free-surface elevation.
- There are no dissipative mechanisms; therefore, the flow is adiabatic.
- The **Coriolis force** dominates the flow, and inertial effects can be neglected, which translates into a small **Rossby number**(Ro).
- The variations of the **Coriolis force** with respect to the latitude are small, which allows approximating its magnitude using the linear function $\beta y + f_0$. This approximation is known in the literature as the *beta-plane approximation*.
- The amplitudes of bottom topography are small.

The mathematical model describes the behavior of the stream function $\Psi = \Psi(t, x, y) \in \mathbb{R}$ associated to a two-dimensional velocity field $u = (u_1(t, x, y), u_2(t, x, y))^t$ in a simply connected domain $\Omega \subset \mathbb{R}^2$. In its dimensionless form, the system reads

$$\left(\partial_t + \nabla^\perp \Psi \cdot \nabla \right) (\Delta \Psi + \beta y - F \Psi + \eta_B) + r \Delta \Psi = \beta \operatorname{curl} \tau + \operatorname{Re}^{-1} \Delta^2 \Psi, \quad (\text{I.3.14})$$

where

- $\nabla^\perp \Psi \cdot \nabla$ is the transport operator associated to **advective time scale**;
- $\Delta \Psi$ is the vorticity, and $r \Delta \Psi$, $r > 0$, is the Ekman pumping term;
- F is the **Froude number**;
- $\beta > 0$ is the slope in the **beta-plane approximation** of the **Coriolis force**;
- η_B is smooth function associated to bottom topography;
- $\beta \operatorname{curl} \tau$ denotes the wind forcing where τ is a given stress tensor; and,
- Re is the **Reynolds number**.

When the domain is simply connected, the system is supplemented with the boundary and initial conditions

$$\Psi|_{\partial\Omega^\varepsilon} = 0, \quad \frac{\partial \Psi}{\partial n} = 0, \quad (\text{I.3.15})$$

$$\Psi|_{t=0} = \Psi_{ini}, \quad (\text{I.3.16})$$

where the first equation in (I.3.15) describes the lack of water outflow in the closed basin, while the second is known in the literature as the **no-slip condition** and characterizes how

the fluid velocity is zero in a fluid-solid interaction as a consequence of the adhesion of the particles being stronger than cohesion. Ψ_{ini} in (I.3.16) is a function for which problem (I.3.14) is “well-prepared”².

Typical values the dimensionless parameters at midlatitudes (see Table 5.3 in [26] for the case of a $1000 \times 1000\text{km}$ bassin at $\phi = 45^\circ N$) are

$$1.6 \times 10 \leq \text{Re} \leq 1.6 \times 10^2, \quad \beta \sim 10^3, \quad F \sim 1, \quad 1 \leq r \leq 10^2.$$

It is clear from the equation (I.3.14) and the values of the dimensionless parameters that the coefficient of the highest order derivative (Re^{-1}) is very small compared to others in the equation. As a consequence, thin frictional boundary layers appear near horizontal (**Ekman layers**) and vertical (**Munk layers**) boundaries.

From both physical and mathematical points of view *Ekman layers* are currently well understood, unlike vertical layers [10]. The analytical complexity of *Munk boundary layers* comes from the fact that they divide into two sublayers with different sizes, see [43, 44]. In meteorology and oceanography, vertical boundary layers appearing near continents are responsible for the *western current intensification*. This phenomenon refers to how the sea level gradients are much more robust, and the geostrophic velocities are also higher along western boundaries than those at the East. Among these subtropical western boundary currents, we have the “Kuroshio” and “East Australian” currents in the Pacific Ocean and the “Gulf Stream” and “Brazil” in the Atlantic Ocean.

B. Desjardins and E. Grenier in [45] showed the existence of global strong solutions for problem (I.3.14)–(I.3.15)–(I.3.16) using its similarities to the 2D incompressible Navier-Stokes equations and global weak solutions of the Navier-Stokes-Coriolis system by slightly modifying the classical proof by Leray [46]. They also performed a complete asymptotic analysis of (I.3.14)–(I.3.15)–(I.3.16) in a smooth bounded domain.

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \chi_w(y) \leq x \leq \chi_e(y), y_{min} \leq y \leq y_{max} \right\}.$$

Regarding the lateral boundary layers, they introduced the small scale $\varepsilon = (\text{Re}\beta)^{-1/3}$ and look for an approximate solution of (I.3.14)–(I.3.15)–(I.3.16) of the form

$$\Psi_{app}^N = \sum_{i=0}^N \Psi_{int}^i(t, x, y) + \Psi_w^i(t, y, X_w) + \Psi_e^i(t, y, X_e) + \varepsilon^N \tilde{\Psi}_N, \quad (\text{I.3.17})$$

for a given $N \in \mathbb{N}$ when $\varepsilon \rightarrow 0$. In (I.3.17), the *boundary layer variables* X_w and X_e are

$$X_w = \frac{x - \chi_w'(y)}{\varepsilon} \left(1 + \chi_w'(y)^2\right)^{-2/3}, \quad X_e = \frac{\chi_e'(y) - x}{\varepsilon} \left(1 + \chi_e'(y)^2\right)^{-2/3}.$$

Functions Ψ_{int}^i , $\Psi_w^i(t, y, X_w)$, $\Psi_e^i(t, y, X_e)$ denote the interior and boundary layer profiles, respectively. Ψ_{app}^N must be a solution of the problem

$$\begin{aligned} \left(\partial_t + \nabla^\perp \Psi_{app}^N \cdot \nabla \right) \left(\Delta \Psi_{app}^N + \varepsilon^{-3} y - \text{Fr} \Psi_{app}^N + \eta_B \right) \\ + r \Delta \Psi_{app}^N - \frac{1}{\text{Re}} \Delta^2 \Psi_{app}^N &= \varepsilon^{-3} \text{curl } \tau + r_\varepsilon^N, \quad \text{in } \Omega \\ \Psi_{app}^N |_{\partial\Omega^\varepsilon} &= \frac{\partial \Psi_{app}^N}{\partial n} |_{\partial\Omega^\varepsilon} = 0, \\ \Psi_{app}^N |_{t=0} &= \Psi_{ini}. \end{aligned} \quad (\text{I.3.18})$$

² Ψ_{ini} is chosen to be equal to the limit of the approximate solution $\Psi_{app}^\varepsilon|_{t=0}$ when $\varepsilon \rightarrow 0$ in an adequate norm.

for some $r_\varepsilon^N \rightarrow 0$ when $\varepsilon \rightarrow 0$ in a given norm.

Their main result is summarized in the following theorem:

Theorem I.3.1 (Desjardins & Grenier [34]). *Let Ω be a simply connected smooth domain given by*

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \chi_w(y) \leq x \leq \chi_e(y), y_{min} \leq y \leq y_{max} \right\}, \quad (\text{I.3.19})$$

where $\chi_w, \chi_e \in C^2([y_{min}, y_{max}])$ and r, Re and τ are known quantities. Assuming additionally that for some $\delta, \bar{\delta} > 0$, $\bar{\delta} \ll 1$, the wind forcing curl $\tau \in L^\infty((0, T), H^s(\Omega))$, for some $T > 0$ satisfies the conditions

$$\text{curl } \tau(x, y) \equiv 0 \quad \text{if } y \leq y_{min} + \delta \quad \text{or} \quad y \geq y_{max} - \delta, \quad (\text{I.3.20})$$

$$\left| \int_{\chi_w(y)}^{\chi_e(y)} \text{curl } \tau(t, x', y) dx' \right| \leq \bar{\delta}. \quad (\text{I.3.21})$$

Let Ψ^ε be a solution of system (I.3.14)–(I.3.15)–(I.3.16) and $u^\varepsilon = \nabla^\perp \Psi^\varepsilon$. Then, if s is large enough, there exists a unique approximate solution u_{app} for $N \leq 2$ rapidly decaying with respect to the reduced variables (see (I.3.17)) such that

$$u^\varepsilon - u_{app} \rightarrow 0 \quad \text{in } L^\infty((0, T), L^2(\Omega)). \quad (\text{I.3.22})$$

The hypothesis (I.3.20) aims to avoid **geostrophic degeneracy** which describes the strong singularities at the Northern and Southern coasts. This phenomenon was studied in depth by Dalibard and Saint-Raymond in [47] for a stationary linear version of (I.3.14). Assumption (I.3.21) guarantees the uniform boundedness of the coefficients stemming from the boundary layer profiles and their smallness which allows them to be absorbed in the viscosity term. Let us briefly discuss two important points regarding the zeroth-order approximation of the solution Ψ^ε since the proof of (I.3.22) follows from classical energy estimates. The first one concerns the *Sverdrup relation* and the other, the interplay between the interior profile and the eastern boundary layer. These aspects are common to all subsequent generalizations of the problem and are discussed in detail in Chapter II.

When $\varepsilon \rightarrow 0$ (which is equivalent to taking the limit $\beta \rightarrow +\infty$), a first approximation of the solution follows the Sverdrup relation

$$\partial_x \Psi^0 = \text{curl } \tau, \quad (\text{I.3.23})$$

which cannot fulfill all the boundary conditions in (I.3.15) because it is a first order partial differential equation. This confirms (I.3.14) is singularly perturbed problem when the small parameter ε goes to zero and, consequently, there is a need for correctors. Moreover, there is still a choice to be made regarding if the boundary condition for Ψ^0 is defined at the East or at the West. In its seminal work of 1947, Stommel integrated (I.3.23) from the East to the West but no explanation was provided [23]. Since then, in the literature is usual to complete problem (I.3.23) with the boundary condition

$$\Psi^0|_{\Gamma_e} = 0, \quad (\text{I.3.24})$$

where Γ_e is the eastern boundary, which for (I.3.19) is of the form

$$\Gamma_e = \{(x, y) \in \partial\Omega : x = \chi_e(y)\}. \quad (\text{I.3.25})$$

Therefore,

$$\Psi^0(t, x, y) = - \int_x^{\chi_e(y)} \operatorname{curl} \tau(t, x', y) dx'. \quad (\text{I.3.26})$$

From the mathematical standpoint, substitution of (I.3.17) leads to a family of ODE problems for the boundary layer profiles where the main fourth-order equation is supplemented with two boundary conditions and a *far field condition* (the boundary current Ψ^{bl} has no influence far its corresponding boundary, i.e., $\Psi^{bl} \rightarrow 0$ at infinity). At order ε^{-4} , we have the following equations describing the behavior of the eastern and western boundary layer

$$\partial_{X_e} \Psi_e^0 + \partial_{X_e}^4 \Psi_e^0 = 0, \quad (\text{I.3.27})$$

$$\partial_{X_w} \Psi_w - \partial_{X_w}^4 \Psi_w = 0. \quad (\text{I.3.28})$$

A quick analysis of the characteristic polynomial of the above equations shows a solution of (I.3.27) cannot simultaneously verify all boundary conditions and the assumption far from the boundary, unlike (I.3.28).

Among the physical arguments justifying the “lack” of boundary current at the East (see (I.3.25)), we can cite the nature of the **Rossby waves**, responsible for the dynamical adjustment in the ocean. They present westward group velocity, which means that at a given point, the adjustment is made on the dynamics at the eastern side; hence, to adjust the whole basin, **Rossby waves** must be at the extreme western side. Another is linked to the vorticity balance in the ocean. A boundary layer at the West re-injects the vorticity that has been previously lost due to the wind into the flow, unlike an eastern boundary layer that causes a more significant loss of vorticity [48]

A natural extension of the works by Desjardins and Grenier [45] is adding roughness to the model in an attempt to bring closer the mathematical theory to the physical reality. Indeed, it is a well-known fact that the irregularities of a given solid surface can profoundly affect the behavior of the fluid it enters in contact with. The study of roughness-induced effects in the behavior of a viscous fluid traces back to the seminal works of Achdou, Pironneau and Valentin in the 1990s [49, 50, 51] and has received much attention ever since, see [52] and the references therein.

In the context of oceanography, the burning questions are: how does the rough topography of the coasts influence the ocean currents? How can we model roughness effects, and what are the consequences of the different modeling assumptions?

D. Bresch and D. Gérard-Varet [53] provided the first answers to the discussion for the case of periodic roughness. The periodicity hypothesis is common in fluid mechanics since it reduces the complexity of the physical phenomena while remaining valid for most relevant cases, see for example [54, 55, 56, 57].

Bresch and Gérard-Varet recover the well-posedness and convergence results in [45] for (I.3.14)–(I.3.15)–(I.3.16) in the domain

$$\Omega^\varepsilon = \left\{ (x, y) \in \mathbb{R}^2 : \chi_w(y) - \varepsilon \gamma_w(\varepsilon^{-1}y) < x < \chi_e(y) + \varepsilon \gamma_e(\varepsilon^{-1}y), \quad y_{min} \leq y \leq y_{max} \right\}. \quad (\text{I.3.29})$$

In (I.3.29), regular and periodic functions γ_w, γ_e describe the roughness of the western and eastern coasts, respectively.

We can easily conclude that (I.3.29) results from adding “rough shores” to the domain

Ω in the work of Desjardins and Grenier. These portions of the domain Ω^ε are of form

$$\begin{aligned}\Omega_w^\varepsilon &= \left\{ (x, y), 0 > x - \chi_w(y) > -\varepsilon\gamma_w(\varepsilon^{-1}y) \right\}, \\ \Omega_e^\varepsilon &= \left\{ (x, y), 0 < x - \chi_e(y) < \varepsilon\gamma_e(\varepsilon^{-1}y) \right\}.\end{aligned}$$

The interface between the interior domain and the western (eastern) rough domain is defined by $\Sigma_w = \{(\chi_w(y), y), y \in (y_{min}, y_{max})\}$ ($\Sigma_e = \{(\chi_e(y), y), y \in (y_{min}, y_{max})\}$). The western (eastern) rough boundary is $\Gamma_w^\varepsilon = \{(\chi_w(y) - \varepsilon\gamma_w(\varepsilon^{-1}y), y), y \in (y_{min}, y_{max}), \}$ $\Gamma_e^\varepsilon = \{(\chi_e(y) + \varepsilon\gamma_e(\varepsilon^{-1}y), y), y \in (y_{min}, y_{max})\}$.

Choosing the same quantity ε to describe the period size of the roughness and the size of the boundary layer has no physical justification but allows for a richer mathematical analysis since the nonlinear effect also appears at the level of boundary layer. Indeed, one striking difference with respect to the work of Desjardins and Grenier is that the **Munk layers** are no longer ordinary differential equations but elliptic quasi-linear partial differential equations since a new reduced variable is added to the boundary layer profiles: $Y = \varepsilon^{-1}y$. Moreover, the dependence of the boundary layer variables X_w, X_e on the interface functions $\chi_w(y)$ and $\chi_e(y)$, respectively, prompts the introduction of more general differential operators of the form

$$\nabla_w = (\partial_{X_w}, \partial_Y - \alpha_w \partial_{X_w})^t, \quad \nabla_e = (\partial_{X_e}, -\alpha_e \partial_{X_e} - \partial_Y)^t.$$

These have similar properties as the usual ones, and consequently, the definition extends quite easily to higher orders.

As we hinted before, the equation driving the eastern boundary layer behavior inherits the issues of the linear version (the incapability of satisfying all boundary conditions and the far-field condition simultaneously). Consequently, the results of the zeroth-order interior approximation and the eastern boundary layer profile are (I.3.26) and $\Psi_e^0 \equiv 0$. A smallness assumption on the wind stress tensor here is essential to deal with the nonlinear terms associated with Ψ_w^0 in the study of the well-posedness of the western boundary layer.

I.3.5 Contributions

Although useful and well documented, the periodicity hypothesis is still way too idealized in many practical situations. As a matter of fact, the geometry of the coasts is not meant to follow a specific pattern. Consequently, we extend the analysis of the roughness-effects by relaxing the periodicity assumption. Our focus lies on the asymptotic analysis as $\varepsilon \rightarrow 0$ of problem (I.3.14)–(I.3.15)–(I.3.16) in a domain Ω^ε defined as in (I.3.29) when functions $\gamma_e, \gamma_w \in W^{2,\infty}(\mathbb{R})$ are arbitrary. Without loss of generality, we consider $\text{Re} = \mu = 1$. Assuming $F = \eta_B = 0$ results in the system

$$\begin{aligned}(\partial_t + \nabla^\perp \Psi \cdot \nabla) (\Delta \Psi + \varepsilon^{-3}y) + \Delta \Psi &= \varepsilon^{-3} \text{curl } \tau + \Delta^2 \Psi \quad \text{in } \Omega^\varepsilon, \\ \Psi|_{\partial\Omega^\varepsilon} &= \frac{\partial \Psi}{\partial n}|_{\partial\Omega^\varepsilon} = 0, \\ \Psi|_{t=0} &= \Psi_{ini}.\end{aligned}\tag{I.3.30}$$

Here, $\Omega^\varepsilon = \Omega_w^\varepsilon \cup \Sigma_w \cup \Omega \cup \Sigma_e \cup \Omega_e^\varepsilon$ and $\text{curl } \tau$ is assumed to verify the condition (I.3.20).

We use the method of matched asymptotics to look for an approximate solution of (I.3.30) depending on the natural size of the boundary layers, in this case, the small parameter ε

$$\Psi_{\text{app}}^\varepsilon(t, x, y) = \sum_{i=0}^{+\infty} \varepsilon^i \left(\Psi_{int}^i(t, x, y) + \Psi_w^i(t, y, X_w, Y) + \Psi_e^i(t, y, X_e, Y) \right), \tag{I.3.31}$$

where $Y = \varepsilon^{-1}y$, $X_w = \varepsilon^{-1}(x - \chi_w(y))$, $X_e = \varepsilon^{-1}(\chi_e(y) - x)$. Since the boundary layer terms are expected not to have an effect far from the boundaries, we assume

$$\Psi_e^i \xrightarrow{X_e \rightarrow \infty} 0, \quad \Psi_w^i \xrightarrow{X_w \rightarrow \infty} 0. \quad (\text{I.3.32})$$

We make the modeling assumption that the interior terms are zero outside Ω which induces the apparition of jumps at Σ_e and Σ_w .

Plugging (I.3.31) in (I.3.30) and then identifying the coefficient of every power of ε provides a family of problems allowing the inductive determination of the profiles.

The first interior and eastern boundary profiles are the same as those of the periodic case. Namely, Ψ_{int}^0 satisfies the Sverdrup relation (I.3.23) and $\Psi_e \equiv 0$.

In the western boundary layer domain $\omega_w = \omega_w^- \cup \sigma_w \cup \omega_w^+$, we investigate the problem

$$\begin{aligned} Q_w(\Psi_w^0, \Psi_w^0) + \partial_{X_w} \Psi_w^0 - \Delta_w^2 \Psi_w^0 &= 0 \quad \text{in } \omega_w^- \cup \omega_w^+, \\ [\Psi_w^0] |_{\sigma_w} &= \phi, \\ \left[\partial_X^k \Psi_w^0 \right] |_{\sigma_w} &= 0, \quad k = 1, 2, 3, \\ \Psi_w^0 |_{X=-\gamma_w(Y)} = \frac{\partial \Psi_w^0}{\partial n_w} |_{X=-\gamma_w(Y)} &= 0, \\ \Psi_w^0 &\longrightarrow 0 \quad \text{as } X_w \rightarrow +\infty, \end{aligned} \quad (\text{I.3.33})$$

where

$$\begin{aligned} \omega_w^+ &= \{X_w > 0, \quad Y \in \mathbb{R}\}, \quad \sigma_w = \{X_w = 0, \quad Y \in \mathbb{R}\}, \\ \omega_w^- &= \{-\gamma_w(Y) < X_w < 0, \quad Y \in \mathbb{R}\}. \end{aligned} \quad (\text{I.3.34})$$

Here, $[\cdot]_{X=X'}$ denotes the jump of a function at the interface ω_w which corresponds to the difference between its value in ω_w^+ and ω_w^- . The jump conditions in (I.3.33) were introduced to enable the match between the outer expansion (interior profile) and the inner expansion (boundary layer). Here, ϕ is used to denote the quantity

$$\phi(t, y) = -\Psi_{int}^0 |_{x=\chi_w(y)} = -\int_{\chi_w(y)}^{\chi_e(y)} \text{curl } \tau(t, x', y) dx'. \quad (\text{I.3.35})$$

The limit in (I.3.33) is the *far field condition*. The remaining conditions are associated with the boundary conditions in (I.3.30). The differential operators are defined as follows for $\alpha_w = \chi_w'(y)$

$$\begin{aligned} \Delta_w &= \partial_{X_w}^2 + (\alpha_w \partial_{X_w} - \partial_Y)^2, \\ Q_w(\Psi, \tilde{\Psi}) &= \nabla_w^\perp \cdot \left(\left(\nabla_w^\perp \Psi \cdot \nabla_w \right) \nabla_w^\perp \tilde{\Psi} \right). \end{aligned}$$

At the level of the boundary layer, t and y , and, consequently, α_w behave as parameters.

The well-posedness of (I.3.33) exhibits several difficulties. First off, ω_w is an unbounded, infinite domain that does not allow for the application of Fourier analysis due to the irregularities of γ_w . The unboundedness of the domain also prevents the use of classical methods based on energy estimates. Moreover, the quantity ϕ does not decay in the Y -direction,

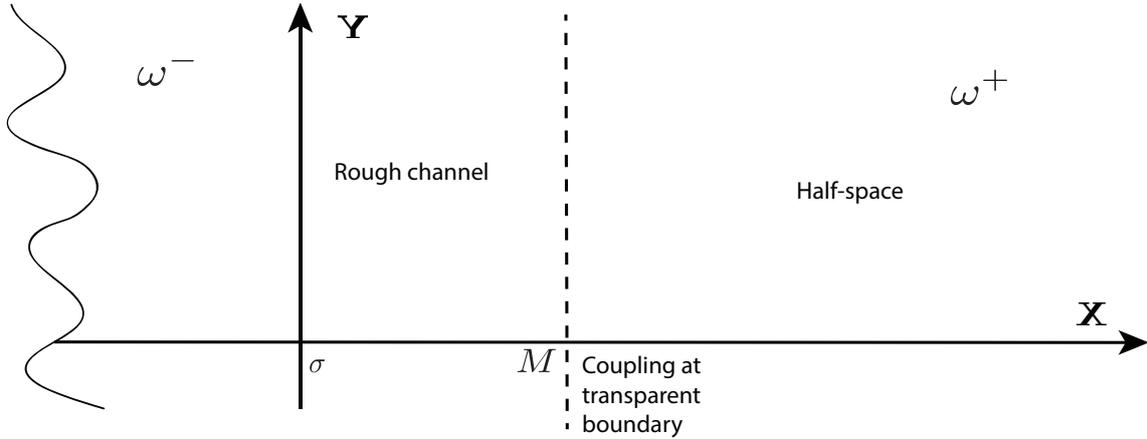


Figure I.4: Diagram of a boundary layer domain with a transparent boundary.

which raises questions concerning Ψ_w^0 's decay with respect to the tangential variable. As a consequence, we work in the framework of functional spaces of infinite energy H_{uloc}^q , $q \in \mathbb{R}$ known in the literature as uniformly local Sobolev spaces introduced by Kato in [58].

It remains to handle the issues associated with the characteristics of the domain. We apply the strategy of Masmoudi and Gérard-Varet in [59] and introduce a *transparent boundary*. This artificial interface separates the domain in two: a half-space and a bounded rough channel. A Dirichlet problem on the half-space $\{X_w > M\}$, $M > 0$, is then solved using Fourier analysis and the theory of pseudo-differential operators. In the rough channel, the proof of well-posedness relies on the method introduced by Ladyzhenskaya and Solonnikov in [60]. This method establishes an equivalence between showing the existence and uniqueness of the solution and solving the problem in a bounded truncated domain using energy estimates and then taking the limit when the truncation parameter tends to infinity. This limit provides the solution in the whole domain; hence, computing beforehand uniformly local bounds for the energy with respect to the truncation parameter is essential.

An operator of the Poincaré-Steklov type is introduced at the artificial interface to “connect” both solutions as seen in Figure I.4. These differential operators map the values of one boundary condition of the solution of an elliptic partial differential equation to another boundary condition. This last step is performed directly when $Q_w \equiv 0$; while for the nonlinear/linearized cases, the implicit function theorem is needed. The technique employed in the analysis of nonlinear boundary layers in the case of highly rotating fluids was introduced in [61]. It is important to highlight that all boundary layer problems analyzed here display the issues discussed above.

Once we have laid the foundations of the problem and its solution, we can state our first result concerning the existence and uniqueness of the solution for the western boundary layer problem under a smallness condition of ϕ .

Theorem I.3.2 (Theorem 1 in [62]). *Let γ_w be a positive $W^{2,\infty}(\mathbb{R})$ function and ω_w be defined as before. There exists a constant $\delta_0 > 0$ such that if $\|\phi\|_{W^{2,\infty}} < \delta_0$, problem (I.3.33) has a unique solution in $H_{\text{uloc}}^2(\omega_w^+ \cup \omega_w^-)$ denoted by Ψ_w . Moreover, for a certain constant*

$\delta > 0$, it satisfies the estimate

$$\|e^{\delta X_w} \Psi_w\|_{H_{\text{loc}}^2(\omega_w^+ \cup \omega_w^-)} \leq C \|\phi\|_{W^{2,\infty}}.$$

Unlike [52] and [61] where the presence of singularities at low frequencies yields a slower convergence (with polynomial weight) of the solution far from the boundary, we are able to recover the same exponentially decaying rate as $X_w \rightarrow +\infty$ for Ψ_w^0 , as in the periodic case.

To this point, we have computed the zeroth-order terms of the asymptotic expansion. The information of the first profiles is used to compute those in the following order. This process is repeated recursively. Similar to the simple model analyzed in Section 1.3.2.1, the nature of the boundary layer equation at the eastern domain prompts us to make assumptions on the interior profile and fixes the integration order at (1.3.23). The same phenomenon arises at higher orders. Assuming all the terms of the outer and inner expansions till order $i = n - 1$, $n > 1$ have been already computed, we have

$$\partial_x \Psi^n = F^n \quad \text{in } \Omega, \quad (\text{I.3.36})$$

where F_n is equal to zero for $n \leq 2$ and a smooth function depending on the interior profiles Ψ_{int}^j , $0 \leq j \leq n - 3$. Then,

$$\Psi_{int}^n(t, x, y) = C^n(t, y) - \int_x^{\chi_e(y)} F_n dx,$$

where $C^n(t, y)$ is a smooth function.

At the East, we have

$$\left\{ \begin{array}{l} -\partial_{X_e} \Psi_e^n - \Delta_e^2 \Psi_e^n = 0 \quad \text{in } \omega_e^- \cup \omega_e^+, \\ [\Psi_e^n] |_{\sigma_e} = -C^n, \\ \left[\partial_{X_e}^k \Psi_e^n \right] |_{\sigma_e} = \tilde{g}_k^n, \quad k = 1, \dots, 3, \\ \Psi_e^n |_{X_e = -\gamma_e(Y)} = \frac{\partial \Psi_e^n}{\partial n_e} |_{X_e = -\gamma_e(Y)} = 0, \\ \Psi_e^n \rightarrow 0 \quad \text{as } X_e \rightarrow +\infty, \end{array} \right. \quad (\text{I.3.37})$$

where the domain is given by

$$\begin{aligned} \omega_e^+ &= \{X_e > 0, \quad Y \in \mathbb{R}\}, & \sigma_e &= \{X_e = 0, \quad Y \in \mathbb{R}\}, \\ \omega_e^- &= \{-\gamma_e(Y) < X_e < 0, \quad Y \in \mathbb{R}\}. \end{aligned} \quad (\text{I.3.38})$$

The differential operators in (1.3.37) depend on the parameter $\alpha_e = \chi_e(y)$ and are defined as follows:

$$\nabla_e = (\partial_{X_e}, -\alpha_e \partial_{X_e} - \partial_Y), \quad \Delta_e = \partial_{X_e}^2 + (\alpha \partial_{X_e} + \partial_Y)^2.$$

Function Ψ_e^n is responsible of lifting the normal trace of Ψ_{int}^n and of the derivatives of other interior profiles at Σ_e . Namely, the jump functions are

$$\tilde{g}_k^n = - \left[\partial_x^k \Psi_{int}^{n-k} \right]_{x=\chi_e(y)}, \quad k \in \mathbb{N}, 1 \leq k \leq 3.$$

Although it is possible to have a nonzero source term depending on Ψ_e^j , $1 \leq j \leq n - 1$, it is always sufficiently small and treated as a perturbation of the equation in (1.3.37).

In addition to the aforementioned issues for the western boundary layer, the eastern boundary layer presents a singular behavior at low frequencies, which prevents its convergence far from the boundary. We distinguish three different asymptotic behaviors at low frequencies: two decaying to zero at exponential and algebraic rates in the normal direction denoted by Ψ_{exp}^n and Ψ_{alg}^n , respectively, and a third one, Ψ_{erg} , for which a probabilistic setting and ergodicity properties are needed. The far-field condition is then only guaranteed if the jump of Ψ_w^n at σ_e is chosen to be equal to the ergodic limit.

Then, $C^n(t, y)$ is uniquely determined so that $\Psi_e^n \rightarrow 0$ as X_e goes to infinity. The latter determines a fixed order in the construction of the solution: first, we solve the eastern boundary layer using a generic function \tilde{g}_0^n to describe its jump at σ_e and determine the value of the ergodic limit. Then, we fix the constant in the interior profile accordingly and then solve the boundary layer problem at the West.

This result is summarized in the following theorem:

Theorem I.3.3 (Theorem 3 in [62]). *Let $\varepsilon > 0$ and (P, Π, μ) be a probability space for which μ is a the probability measure preserved by the translation group (τ_Y) acting on P . The eastern boundary layer domain is described as follows for $m \in P$:*

$$\omega_e(m) = \left\{ (X_e, Y) \in \mathbb{R}^2 : X_e > -\gamma_e(m, Y) \right\},$$

where $\omega_e(m) = \omega_e^+(m) \cup \sigma_e \cup \omega_e^-(m)$ and $\omega_e^\pm(m) = \omega_e(m) \cap \{\pm X > 0\}$.

Let γ_e an ergodic stationary random process, K -Lipschitz almost surely, for some $K > 0$ and $\tilde{g}_k, \in L^\infty(P \times \mathbb{R}_+)$, $k = 1, 2, 3$. Then, there exist a unique measurable map C^n such that problem (I.3.37) has a unique solution $\Psi_e^n = \Psi_{\text{exp}}^n + \Psi_{\text{alg}}^n + \Psi_{\text{erg}}^n$ where

1. $\left\| \Psi_{\text{erg}}^n \right\|_{L^q(\omega_e^+)} \xrightarrow{X_e \rightarrow +\infty} 0$, locally uniformly in Y , almost surely and in $L^q(P)$ for all finite q ,

2. there exist constants $\delta, C > 0$ such that

$$\|e^{\delta X_e} \Psi_{\text{exp}}^n\|_{L^\infty(\omega_e^+)} \leq C \left(\sum_{k=1}^3 \|\tilde{g}_k\|_{L^\infty} + \|C^n\|_\infty \right), \quad (\text{I.3.39})$$

3. there exists a constant $C > 0$ such that

$$\|(1 + X_e)^{1/4} \Psi_{\text{alg}}^n\|_{L^\infty(\omega_e^+)} \leq C \left(\sum_{k=1}^3 \|\tilde{g}_k\|_{L^\infty} + \|C^n\|_\infty \right). \quad (\text{I.3.40})$$

At order n , the last step consists in solving the following system for Ψ_w^n

$$\left\{ \begin{array}{l} \partial_{X_w} \Psi_w^n + Q_w(\Psi_w^n, \Psi_w^0) + Q_w(\Psi_w^0, \Psi_w^n) - \Delta_w^2 \Psi_w^n = H_n \quad \text{in } \omega_w^- \cup \omega_w^+, \\ \left[\partial_{X_w}^k \Psi_w^n \right] |_{\sigma_w} = g_k^n, \quad k = 0, \dots, 3, \\ \Psi_w^n |_{X_w = -\gamma_w(Y)} = 0, \quad \frac{\partial \Psi_w^n}{\partial n_w} |_{X_w = -\gamma_w(Y)} = 0, \\ \Psi_w^n \rightarrow 0 \quad \text{as } X_w \rightarrow +\infty, \end{array} \right. \quad (\text{I.3.41})$$

where H_n is a small term depending on the previous western boundary layer profiles and $g_k^n \in L^\infty(\mathbb{R})$. The western boundary layer profiles lift the normal trace of interior profiles at the interface σ_w and deal additionally with the influence of the eastern boundary layers (see Theorem 1.3.3). This means that for $k = 1, 2, 3$

$$g_k^n = - \left[\partial_x^k \Psi_{int}^{n-k} \right]_{x=\chi_w(y)} - \left[\partial_{X_e}^k \Psi_e^{n-k} \right].$$

The problem when $H_n = 0$ is solved similarly to the nonlinear case using pseudo-differential analysis and the ideas introduced by Dalibard and Gérard-Varet [61].

Once we have constructed the approximate solution, our final result involves studying its convergence to that of the original problem (1.3.30). This analysis involves two main steps: finding N such that the finite sum of the first N terms of (1.3.31) satisfies (1.3.18), and then show its closeness to Ψ^ε using energy estimates on the difference $\Psi^\varepsilon - \Psi_{app}^N$.

In the periodic setting, similar to the case without roughness, the boundary layer profiles decay exponentially to zero when $\varepsilon \rightarrow 0$. Moreover, the main order approximation gives a $O(\varepsilon^{1/2})$ rate of convergence in H^2 , see [53]. In this general setting, the lack of spectral gap at the eastern boundary layer imposes using average information to guarantee the convergence of this profile far for the eastern boundary (Ergodic Theorem). Consequently, the convergence of Ψ_{erg} and, therefore, of Ψ_e , can be arbitrarily slow and impact the asymptotics of the entire approximate solution.

The convergence result is as follows:

Theorem 1.3.4. *Let Ψ^ε be the solution of problem (1.3.30) and Ψ_{app} defined as in (1.3.31). Moreover, let Ψ_{ini}^ε be such that $\Psi_{ini}^\varepsilon|_{\Omega^\varepsilon} = \partial_n \Psi_{ini}^\varepsilon = 0$ and $\|(\Psi_{ini}^\varepsilon - \Psi_{app})|_{t=0}\|_{H^1(\Omega^\varepsilon)} \rightarrow 0$. There exists C_∞ , such that if $\|\text{curl } \tau\|_{W^{2,\infty}} < C_\infty$, then*

$$\|\Psi^\varepsilon - \Psi_{app}\|_{L^2(0,T;H^2(\Omega^\varepsilon))} + \|\Psi^\varepsilon - \Psi_{app}\|_{L^\infty(0,T;H^1(\Omega^\varepsilon))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad \text{almost surely.} \quad (1.3.42)$$

The proof of 1.3.4 is classical. In this case, the difficulty comes from computing the error terms linked to the nonlinear advection component in the western boundary layer domain and building additional correctors. The reason behind this is, once again, the slow convergence of the eastern boundary layer profiles.

1.3.5.1 Perspectives

A natural question is to know whether these techniques are applicable to more complex models. It would be interesting to discard the assumption (1.3.20) on the wind stress curl with the added roughness. The linear case studied by Dalibard and Saint-Raymond [47] is already quite complex due to the North-South singularities resulting from the [geostrophic degeneracy](#). In our case, we would additionally have to deal with the nonlinear aspect of the model.

Another possibility consists of exploring the influence of roughness in the case of a domain with islands. This model characterizes for the presence of multiple boundary layers (North, South), free boundary layers in the north and the south of the islands, Stommel or Munk layer as seen in [39].

I.3.5.2 Outline of the Chapter II

The chapter is based on [62] and is organized as follows. First, we introduce the quasi-geostrophic equations and discuss their physical and mathematical relevance. In Section II.3 we formally construct an approximate solution of Ψ^ε , introduce the main modeling assumptions and establish the problems associated with the zeroth-order approximation of the solution for the interior and boundary layer profiles. Then, in Section II.4, we present the methodology behind the proof of several well-posedness results in this work for linear and nonlinear/linearized problems in an unbounded rough domain. In Section II.5 we turn to the well-posedness of the linear problem describing the behavior of the western boundary layer. These results are crucial to proving Theorem I.3.2. Later, in Section II.6, the existence and uniqueness of a solution for the nonlinear western boundary layer problem is shown in Kato spaces under a smallness condition. Before proceeding to the convergence analysis, it is necessary to show that all the approximate solution elements are well-defined. This leads in Section II.7 to the analysis of quasi-linear problems in the western boundary layer domain, which correspond to the behavior of the following order profiles at the West. Following the same direction, Section II.8 focuses on the singular behavior of the eastern boundary layer (see Theorem I.3.3). Once all the information on the different profiles has been gathered, Section II.9 is devoted to the proof of the convergence result in Theorem I.3.4.

The following section is completely independent of this one since it presents the subject of Chapter 3.

I.4 Asymptotic methods in the mathematical modeling of tissue growth

Biological systems are inherently complex due to the multiple functional networks operating at different temporal and spatial scales. From the theoretical perspective, understanding the connection between the different scales brings us closer to bridging the gap between experiments and whole-organism models [127]. Indeed, theoretical models and simulations can significantly reduce experimental efforts while providing a detailed description of the spatio-temporal evolution of a biological system and cast light on processes that are difficult to access experimentally. This type of study is usually conducted using a multiscale mathematical approach.

A usual query in medical research refers to how the behavior of the individual cells may influence tissue growth. In fact, a mass or new tissue with clear macroscopic features is still deeply influenced by the behavior of individual cells, emphasizing the multiscale nature of this biological phenomenon. At a “visible” scale, tissue growth is described either by nonlinear partial differential equations or discrete modeling approaches, where nonlinearity and the capacity of describing heterogeneous phenomena (the nonuniform distribution of the microscopic states of cells) are common features. Continuum models examine the average behavior of the cell densities or concentration, while the discrete models describe the behavior of individual cells. Furthermore, continuum models can be classified as phenomenological or mechanical models. The former disregard mechanical effects by assuming such that cells do

not move. Conversely, mechanical models assume force or momentum balance to describe the interactions between cell, matrix, and fluid components and their response to physical forces. Within the mechanical approach, one can take the solid or fluid mechanics route depending on how the cells are assumed to behave. If one considers cells conduct as a fluid, these models typically consist of reaction-diffusion-convection equations of the form

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{J} = f, \quad (\text{I.4.1})$$

where $n(t, x)$ is the cell population density, \mathbf{J} is a general flux transport and f is a source term depending on the spatio-temporal variables (x, t) and the concentration. The above equation results from applying a general conservation law stating the equivalence between the material change in a region and the flow across the boundary. A classical diffusion process is obtained when $\mathbf{J} = D\nabla n$, where D is the diffusivity coefficient. Taking D constant and $f = rn(1 - n/K)$, where r denotes the linear reproduction rate and K the carrying capacity of the environment yields the classical *Fisher–Kolmogoroff* equation. For a more information on the matter the reader can refer to [96].

I.4.1 Mathematical models of tumor growth

The classical model (I.4.1) can be adapted to tissue growth by considering that cell proliferation increases the local pressure, thus, creating a velocity field.

I.4.2 Compressible agent-based model

Consider for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$

$$\begin{aligned} \frac{\partial n}{\partial t} + \nabla \cdot (nv) &= n\Phi(p(n)), \\ n(0, x) &= n^0(x) \leq 0, \end{aligned} \quad (\text{I.4.2})$$

where $n := n(t, x) \geq 0$ denotes the cell population density, $p := p(t, x) = P(n) \geq 0$ corresponds to the pressure induced by the cell number, v is the velocity field driving cell motion and Φ is a function describing the net growth rate of cells, which is determined by contact inhibition, which means tissue growth is only limited by the availability of space. Moreover, Φ satisfies the following properties:

$$\Phi \in C^1(\mathbb{R}), \quad \Phi' < 0, \quad \Phi(p_M) = 0 \quad \text{for some } p_M > 0, \quad (\text{I.4.3})$$

where p_M is usually called **homeostatic pressure**. The left-hand side on (I.4.2) describes the cells proclivity to move towards regions where they are less compressed.

From a classical fluid mechanics standpoint, problem (I.4.2) describes a compressible fluid whose changes in pressure are bounded by differences in density through the law $P(n)$.

Choosing the appropriate framework is of paramount importance. It is determined by assumptions on the velocity field behavior and the state law $p = \Pi(n)$ in (I.4.2) and can lead to a variety of models.

I.4.2.1 Velocity field behavior

Phenomenological models are characterized by simple diffusion similar to the *Fisher–Kolmogoroff* equation. In particular, the average of the velocity over the cross-section is considered to be of the form

$$v = -D\nabla p,$$

where D refers to the diffusion coefficient. The diffusivity D can be taken as a positive constant, in which case, cell movement can be described using linear or nonlinear diffusion $D := D(n)$, which has proven to be useful when studying gliomas, and movements such as chemotaxis and haptotaxis [128].

Although these models succeed in describing the interactions between the cells, such as contact inhibition, they fail to explain the influence of the elastic membrane. In such cases, it is suitable to consider the tissue as a viscous Newtonian fluid in which the tissue grows on much longer timescales than those of elastic stress relaxation.

We now present velocity field laws that are usually considered when studying tumor growth with proliferation as a compressible fluid. It is important to notice this type of problems are supplemented with a state law $p = P(n)$. The possible choices for the pressure function will be discussed right after this paragraph.

I.4.2.1.1 Darcy’s law. If one assumes fluid-like cells, the simplest constitutive equation for the stress results from assuming that the cells move as an elastic fluid within a rigid extracellular matrix. Darcy’s law describes cells as a granular material flowing in a porous medium (the extracellular matrix). The constitutive equation is as follows:

$$v = -\frac{K}{\mu}\nabla p, \tag{I.4.4}$$

where p , μ and K are the fluid pressure, the dynamic viscosity of the fluid, and the permeability of the porous medium, respectively. This law describes a linear proportionality between the flow velocity and the pressure gradient. Henry Darcy introduced this relation in 1856 through his experiments with sand and water.

The use of Darcy’s model in the study of tumor growth traces back to the seminal work of Greenspan [138] and has been further developed by several authors. For a survey on the extensive use of this constitutive equation in the context of tissue and tumor growth, the interested reader can turn to [129, 130].

For liquids at high velocities, Darcy’s law is no longer valid. In this scenario, it is suitable to incorporate the pressure in the fluid velocity through Stokes flow, Brinkman’s or Navier–Stokes’ laws [131, 132].

I.4.2.1.2 Brinkman’s Law. Another constitutive law results from combining Darcy’s law with Stokes flow, and it is known as the Brinkman equation, see [140]. In this case, the velocity field associated with the fluid flow v is described by the equation

$$-\mu\Delta v + v = -\frac{k}{\mu}\nabla p, \tag{I.4.5}$$

where p and μ are the pressure and viscosity of the fluid, respectively; and v is the bulk velocity. Note that the term on the right-hand side coincides with Darcy’s law and is used

as a corrector. For example, when the extracellular matrix degrades and large pores appear, the viscous term applies in the large pores, while Darcy's law concerns the unscathed region, see for example [140, 131, 139] and the references therein.

I.4.2.1.3 Stokes flow. This constitutive equation is chosen when assuming the cell-matrix medium behaves as a viscous fluid, see [136], for example. Contrary to Darcy's law, velocity and pressure are no longer coupled

$$-\mu\Delta v + \nabla p - \kappa\nabla(\operatorname{div}v) = 0. \quad (\text{I.4.6})$$

Here, $\mu, \kappa > 0$ are viscosity coefficients. Several studies have concluded the Stokes flow is better suited to model tumors due to simulations being closer to experimental data, see for example [142]. Viscoelastic models for tumor growth based on Stokes' or Brinkman's differ from those using Darcy's law in that pressure is determined by tissue incompressibility instead of a state law.

I.4.2.1.4 Navier-Stokes. Navier-Stokes equation determines a more general relation between the cell density n and the pressure p since Darcy's law, Stokes' law or Brinkman's law can be formally derived from it. To (I.4.2), we add the equation

$$n(\partial_t v + v \cdot \nabla v) - \mu\Delta v - \kappa\nabla\operatorname{div}v + \nabla p = 0, \quad (\text{I.4.7})$$

where $\mu, \kappa > 0$ are viscosity coefficients.

I.4.2.2 Pressure law

Model (I.4.2) describes cellular heterogeneity by considering different pressure laws to describe the response of the tissue to external forces. Among the most widely used state laws in the mathematical modeling of tumor growth, we have power and singular pressure laws. They share a common feature: the repulsion between cells becomes "singular" at the asymptotic limit ($\gamma \rightarrow +\infty$ for the power law and $\varepsilon \rightarrow 0$, for the singular pressure).

I.4.2.2.1 Power law. Barotropic power laws are usual in fluid mechanics and have proven to provide an appropriate description for a variety of particulate and polymeric fluids[153]. One representative example is

$$p = P(n) := C(\gamma) \left(\frac{n}{n_M} \right)^\gamma, \quad (\text{I.4.8})$$

where $\gamma > 1$ provides a measure of the stiffness of the barotropic relation and accounts for the property of *finite speed of propagation*, $C(\gamma)$ is a normalization constant and n_M is generally understood as the maximum packing constraint of the cells. This law allows to recover from (I.4.2) the *porous medium equation* $\partial_t n = \Delta(n^m)$ for $m > 1$. We refer to the book [141] for more information on this equation.

I.4.2.2.2 Singular pressure. In collective motion models, it is usual to assume the non-overlap constraint on the microscopic components, particularly the cells forming the tissue. In highly dense regimes, they can alter the flow's dynamics by inducing some non-local effects. When non-overlap conditions are enforced at a population level, we encounter singular pressure laws of the type

$$P(n) = \varepsilon \frac{n}{1-n}. \quad (\text{I.4.9})$$

At the macroscopic level, the singularity in (I.4.9) acts as a barrier before the appearance of a congested zone at $n = 1$. Formally, at a microscopic level, such a limit imposes non-overlapping constraints while leading to free boundary problems.

I.4.3 Free-boundary models and the incompressible limit

It is also possible to assume the tumor as a moving domain of constant density at the macroscopic level. In this case, the evolution equations describe the changes of the boundary between the healthy tissue and the tumor. The first mathematical model in the form of a free boundary problem used to describe tumor growth appeared in the 1970s in the works of Greenspan [138, 143]. His reaction-diffusion model for the cell substrates was then improved by Byrne and Chaplain in the 1990s for radially symmetric tumors without and with necrosis [144, 145] and has attracted much attention ever since, see, for instance, the survey papers by Friedman [147] and Lowengrub et al. [146].

The last fifty years have witnessed an increasing interest on rigorously investigating the link between incompressible and compressible models within the mathematical community. The emergence of this topic in fluid mechanics was motivated by simple physical examples such as the behavior of compressible isentropic fluids when the Mach number becomes increasingly smaller [148]. The study of the *incompressible limit* is key for the derivation of macroscopic equations from microscopic dynamics (agent-based models) in the case of short-range repulsive social forces preventing contacts between individual components that are, in the case of tumor growth, the cells.

The first results in mathematical biology regarding the link between compressible and incompressible models are due to Perthame, Quirós, and Vázquez [85]. The authors considered (I.4.2) when the tissue bulk velocity abides Darcy's law (I.4.4) and the density and pressure are linked through a power law of type (I.4.8). In particular, they studied the purely mechanical model

$$\begin{cases} \partial_t n_\gamma + \nabla \cdot (n_\gamma v) &= n_\gamma \Phi(p_\gamma) & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ n_\gamma(0, \cdot) &= n_\gamma^0 \in L^1_+(\mathbb{R}^N), \\ v &= -\nabla p_\gamma, \end{cases} \quad (\text{I.4.10})$$

where $p_\gamma = P(n_\gamma)$ follows

$$P(n_\gamma) = \frac{\gamma}{\gamma-1} \left(\frac{n_\gamma}{n_M} \right)^{\gamma-1}. \quad (\text{I.4.11})$$

Here, the parameter γ refers to the strength of the repulsive forces, and the function Φ satisfies (I.4.3). Problem (I.4.10) describes a highly dense regime and displays non-local effects due to the maximal packing constraint. In the range of $\gamma > 1$, (I.4.10) degenerates at the level $p = 0$. When considering $\gamma \rightarrow +\infty$, we observe two zones: one close to the maximal

density constraint, where the pressure becomes increasingly negligible for a fixed $n < 1$ and another one, where the pressure is nonzero and $n = 1$. This produces the existence of a moving interface, separating the regions $\{n(t, \cdot) < 1\}$ and $\{n(t, \cdot) = 1\}$. This phase transition phenomenon indicates the presence of congestion. From the mathematical standpoint, the pressure p can be seen as a Lagrange multiplier associated with the incompressibility constraint in $\{n \leq 1\}$ [83]. The compressible system (I.4.10) with singular behavior is classified as a *soft congestion model*.

Using the distributional solution approach: a combination of variational formulations on the so-called *Baiocchi variable*, i.e. *weak solutions in the sense of Elliot and Janovsky* [187], and viscosity solutions; Perthame, Quirós, and Vázquez showed for initial data $n^0 \in L^1_+(\mathbb{R}^N)$ that when $\gamma \rightarrow +\infty$ in (I.4.10)–(I.4.11), the solution pair (n_γ, p_γ) converges to a solution of the Hele-Shaw model

$$\partial_t n - \nabla(n \nabla p) = n \Phi(p) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \quad (\text{I.4.12a})$$

$$0 \leq n \leq 1, \quad (1 - n)p = 0, \quad (\text{I.4.12b})$$

$$p (\Delta p + \Phi(p)) = 0. \quad (\text{I.4.12c})$$

Note that there is a transition phase with an evolving free boundary between the tumor region, where the density is saturated ($n = 1$) and p activates; and the pre-tumor zone where the density is “free”, in the sense that $n < 1$ and $p = 0$ (the support of $1 - n$). This behavior is condensed in the exclusion condition (I.4.12b). Equation (I.4.12c) is known in the literature as *complementarity condition*. Free-congested systems of type (I.4.12) are known as *hard congestion models*. For a survey on congested phenomena in fluid mechanics, see [111].

The effect of the limit $\gamma \rightarrow +\infty$ was already well known in the *porous medium equation case* (PME), i.e., equation (I.4.12) with $\Phi \equiv 0$ and can be found in [89, 90, 91, 150]. It has been proven that for initial datum larger than 1 on a nontrivial set, the solution pair (n_γ, p_γ) of (I.4.10)–(I.4.11) converges to solution of the Hele-Shaw problem (n, p) satisfying $\text{Supp } n \subseteq [0, 1]$ when $\gamma \rightarrow +\infty$. The upper part of the initial datum collapses to $n = 1$ at $t = 0^+$ as a consequence of the diffusivity $\gamma n^{\gamma-1} \rightarrow +\infty$ whenever $n > 1$. This height constraint at $\{n = 1\}$ induces the limit function n to graphically mimic the silhouette of a “mesa”: a flat-topped mountain commonly seen in the West of the USA, Australia, Israel, Germany and England. Numerical studies show the mesa formation is already apparent for relatively low values of m ($m \sim 6$) with bell-shaped initial data [188]. Consequently, the study of the incompressible limit in the PME is said to belong to the mathematical theory of the *Mesa problem*.

Mellet, Perthame and Quirós in [88] studied further the free boundary problem (I.4.12) and showed additional results on its regularity at $\partial\{n = 1\}$ using integral estimates. Moreover, they showed in a weak (distributional) sense that the velocity law normal to the free boundary in the tumor region $\partial\{n = 1\}$ is

$$V = \frac{Dp}{1 - \min(1, n^0 e^{\Phi(0)t})}, \quad (\text{I.4.13})$$

a formula that was first suggested in [85].

In 2018, Kim and Požar [133] obtained the same results of [85, 88] using pointwise arguments. In particular, the authors use a viscosity solution approach based on radially

symmetric functions with fixed boundaries to show that the density uniformly converges far from the edge of the tumor. They also proved the formula (I.4.13) for general solutions. Recently, David and Schmidtchen solved the question of the strong compactness of the pressure gradient when considering the incompressible limit in (I.4.10)–(I.4.11) and similarly to [85], they were able to obtain the *complementarity relation* for the pressure using a weak (distributional) approach, see [134].

Perthame, Quirós, and Vázquez in [85] succeeded to extend the results on the purely mechanical model to the case when the tumor growth depends additionally on the concentration of nutrients. The main system in this case is (I.4.10)–(I.4.11) with an additional equation to describe the diffusion of nutrients in the tumor:

$$\begin{aligned} \partial_t c - \Delta c &= -n\Psi(p, c), \\ c(x, t) &= c_B > 0 \quad \text{as } |x| \rightarrow +\infty, \end{aligned} \tag{I.4.14}$$

where c denotes the density of nutrients, and c_B is the far field supply of nutrients. The coupling function Ψ describes the effect of the vasculature network bringing the nutrients to the cells and their consumption. This function is considered to be smooth and satisfies

$$\partial_p \Psi \leq 0, \quad \partial_c \Psi \geq 0, \quad \Psi(p, 0) = 0.$$

The birth/death process of cells modeled by function Φ is also considered to be dependent on the amount of nutrients and therefore, the conditions

$$\partial_p \Phi < 0, \quad \partial_c \Phi \geq 0, \quad \Phi(p_M, c_B) = 0,$$

join the ones in (I.4.3). Perthame, Tang and Vauchelet proved the existence of traveling waves for the Hele-Shaw model with nutrients (I.4.12)–(I.4.14) in [121]. They showed through semi-analytical formulae that a sharp front separates the healthy part from the cancerous tissue near $n = n_M$; and, the pressure is differentiable from the edges of the tumor to the necrotic core where $p = 0$.

In [112], the convergence of a more general model to a free boundary model of Hele-Shaw type was shown following the strategy in [135] of ignoring compactness on the cell densities and prove strong compactness on the pressure gradient using an Aronson-Bénilan type estimate. It is worth noting that such a model had systems in [85] and [121] as particular cases. The same method was used in [113] in the study of (I.4.10)–(I.4.11) for the case of two types of cancerous cells.

The previous convergence results by Perthame, Quirós and Vázquez for the purely mechanical model were later extended by Hetch and Vauchelet [86] to the case where a non-overlapping constraint is imposed on the cells, leading to the singular pressure law $P(n_\varepsilon)$ in (I.4.9). Here, the incompressible limit is established when $\varepsilon \rightarrow 0$ making similar hypotheses on the growth function Φ but adding boundness to deal with the singularity in the pressure law. Regularizing effects are used to prove convergence, unlike [85] where well-known results on the PME lead to the compactness and later convergence of the solution pair $(n_\varepsilon, p_\varepsilon)$ to the solution of the free boundary model of Hele-Shaw type (I.4.12).

The question of the incompressible limit for (I.4.2) with state law (I.4.11) including viscoelastic effects has been analyzed when v follows the Brinkman equation (I.4.5) by Perthame and Vauchelet in [114], while the general case where the velocity and the pressure are related via the Navier-Stokes equation (I.4.7) was the main focus of the work by Vauchelet

and Zatorska [116]. A clear difference concerning the models based on Darcy's law is that the *complementarity condition* for the pressure depends on the viscosity. Unlike the cases of Navier-Stokes and Darcy's law, for which several mathematical results were already in existence and simplified the obtention of compactness results, Brinkman law introduces two major drawbacks: the space discontinuity of the pressure and the lack of known regularizing effects for the transport equation. Consequently, in [116] the strong compactness using the kinetic reformulation of the problem and then, carefully analyzing all possible oscillations of the pressure.

The reader interested in a more detailed discussion on the the evolution of the question of the incompressible limit in the mathematical modeling of tumor growth can refer to [134].

1.4.4 Traveling wave solutions of the Hele-Shaw model for tumor growth

In the previous section, we confirmed considering the incompressible limit in (I.4.2) is physically relevant since the study of the tumor tissue is reduced to the analysis of an incompressible elastic material in a confined environment. The limiting model is a Hele-Shaw type free boundary problem. The Hele-Shaw cell is known for being a useful tool to study pattern formation in the transition between a viscous and an effectively inviscid fluid.

Despite all the attention this question has drawn, the issue of traveling wave solutions (TWs, for short) to (I.4.12) is still not well understood. The TWs form a class of special solutions that are known to provide valuable information on general solutions of nonlinear reaction-diffusion equations and account for phase transitions between different states of physical systems, propagation of patterns, and domain invasion in population biology (see the books [122] and [123]). In the study of tumor growth, these solutions are thought to represent waves of malignant cellular invasion into healthy surrounding tissue.

To the best of our knowledge, few researchers have addressed this question. In [121], Perthame, Tang and Vauchelet investigated the existence of this type of solution for the Hele-Shaw model with nutrients (I.4.12)–(I.4.14). The existence of TWs in (I.4.10) for a fixed value of γ was not provided and the radial instabilities appearing in the numerical simulations were not explored from the analytical point of view.

Therefore, in an attempt to fill the gap in the literature, we are interested in investigating the smooth TWs when $\gamma \rightarrow \infty$ of the nonlinear parabolic equation

$$\partial_t n - \partial_x(n\partial_x p(n)) = n\Phi(p(n)) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}, \quad (\text{I.4.15})$$

where

$$p(n) = p_\gamma(n) = n^\gamma \quad \text{with } \gamma > 1, \quad \Phi(p) = 1 - p. \quad (\text{I.4.16})$$

Note that the *homeostatic pressure* p_M is unitary, which means that cells are destroyed above the *maximal packing density* $n_M = 1$. Problem (I.4.15) is endowed with the boundary conditions

$$\lim_{x \rightarrow \pm\infty} n(t, x) = n_\pm,$$

where n_\pm are constant stationary states of the equation (equilibrium solutions). It is additionally considered that $n_- = n_M = 1$, and $n_+ = 0$. In our view, this analysis takes us a step closer to a better understanding of the free boundary $\partial\{n = 1\}$ for the limit Hele-Shaw

system

$$\begin{cases} \partial_t n - \partial_x(n\partial_x p) = n\Phi(p) & \text{in } \mathbb{R}^+ \times \mathbb{R}, & \text{(I.4.17a)} \\ 0 \leq n \leq 1, \quad (1-n)p = 0, \quad p \geq 0, & & \text{(I.4.17b)} \\ p(\partial_x^2 p + \Phi(p)) = 0. & & \text{(I.4.17c)} \end{cases}$$

Equation (I.4.15) is a particular case of the quasilinear degenerate second-order parabolic type

$$\partial_t n + \beta n^\gamma \partial_x n = \partial_x(\alpha n^\gamma \partial_x n) + \zeta n(1 - n^\gamma). \quad \text{(I.4.18)}$$

Note that taking $\beta = 0$, $\alpha = \frac{\gamma}{\gamma+1}$ and $\zeta = 1$ in (I.4.18) yields (I.4.15).³ The existence and uniqueness (up to a shift) of a monotone (decreasing) TW solution of the form $n_\gamma(x, t) = N_\gamma(x - ct) = N_\gamma(\xi)$, for some sufficiently regular function N_γ (the wave profile) and constant wave speed $c \in \mathbb{R}$ when $\gamma \rightarrow \infty$; and its asymptotic behavior close to $\pm\infty$ for c larger than a threshold velocity $c_\gamma^* > 0$ result from the works of Gilding and Kersner [117].

Kamin and Rosenau [125] investigated (I.4.18) taking $\alpha = \zeta = 1$, $\beta \in \mathbb{R}$ and proved that the solutions converge to a sharp TW solution with speed c^* in a weighted L^1 -norm whenever the initial condition is non-negative, bounded, and exponentially converges towards a sharp TW. The authors relied on a method used in stability analysis of shock waves for viscous conservation laws. Although interesting, a major disadvantage of their approach is that it is specific to the critical speed c^* of the sharp fronts. In particular, the “weighted” conservation of mass property and the contraction principle cannot be adapted to the case where wavefronts propagate at speed $c > c^*$. The results in [125] were later generalized by Malaguti and Ruggerini [151] by providing a characterization of the TWs in the case where a general reaction term of *Fisher type*⁴ $f(n)$ is considered instead of $\zeta n(1 - n^\gamma)$.

To date, the only result dealing with a family of smooth fronts with speed $c > c^*$ belongs to Leyva and Plaza in [110]. Here, the authors establish the spectral stability in exponentially weighted spaces of smooth monotone traveling fronts for the equation

$$\partial_t n = \partial_x(D(n)\partial_x n) + f(n), \quad \text{(I.4.19)}$$

where the diffusion coefficient $D = D(n)$ is a nonlinear, non-negative, density-dependent function degenerating at $n = 0$, and $f(n)$ is a *Fisher type* function. They use *relative entropy* type estimates to overcome the difficulties stemming from the degeneracy and adapted to some L^2 exponential weights.

1.4.5 Contributions

This section contains the main results of Chapter 3. This work is a collaboration with Anne-Laure Dalibard and Charlotte Perrin.

We study the asymptotic behavior of TWs of the reaction-diffusion equation (I.4.15)–(I.4.16), i.e., the solutions of the form $(n_\gamma, p_\gamma) = (N_\gamma, P_\gamma)(x - ct)$, where $c > 0$ is a constant representing the traveling wave speed and N_γ, P_γ are real nonnegative functions.

³Other known mathematical equations with these characteristics are the *Fisher-KPP equation* with nonlinear diffusion ($\beta = 0$, $\alpha > 0$, $\zeta > 0$) and the degenerate *Burgers equation* ($\beta \neq 0$, $\alpha > 0$, $\zeta = 0$).

⁴The function f is said to be of Fisher type if it satisfies

$$f(0) = f(1), \quad f > 0 \text{ for } 0 < s < 1, \quad f < 0 \text{ for } s > 1.$$

Considering the incompressible limit $\gamma \rightarrow +\infty$ raises several questions. For example, do the TWs of the initial system (I.4.15)–(I.4.16) converge to those of (I.4.17)? If yes, is the convergence strong or weak? Can we provide a detailed description of the convergent subsequences? Finally -a classical query in dynamical systems- are the solutions stable under small perturbations?

We provide some answers by showing the convergence of smooth fronts of (I.4.15)–(I.4.16) towards the TWs of (I.4.17). We also conduct a nonlinear stability analysis of the solutions of (I.4.15)–(I.4.16) where the perturbations are quantified with respect to γ . The major difficulty here lies in the fact that on the one hand, we have continuous representations of phase transitions for the microscopic model, while at the other, the limiting Hele-Shaw model presents a major sharp interface at $\partial\{n = 1\}$ as a result of the discontinuous gradient.

From (I.4.15)–(I.4.16), the profile N_γ is the solution of the problem:

$$\begin{aligned} -cN'_\gamma - \gamma(N_\gamma^\gamma N'_\gamma)' &= N_\gamma(1 - N_\gamma^\gamma), \\ \lim_{\xi \rightarrow -\infty} N_\gamma(\xi) &= 1, \quad \lim_{\xi \rightarrow +\infty} N_\gamma(\xi) = 0. \end{aligned} \tag{I.4.20}$$

The existence and uniqueness (up to a shift) of a monotone (decreasing) traveling wave solution $N_\gamma(x - ct)$ of (I.4.15) for all $c \geq c_\gamma^* = \sqrt{\frac{\gamma}{\gamma+1}}$ and its asymptotic behavior close to $\pm\infty$ are a direct consequence of the works by [117]. Note that the value of P_γ is obtained by substituting N_γ in (I.4.16). For traveling wave variable $\xi \in \mathbb{R}$ defined by $\xi = x - ct$, we have

Theorem I.4.1 (Gilding & Kersner [117]). *Let $c_\gamma^* = \sqrt{\frac{\gamma}{\gamma+1}}$.*

1. *System (I.4.20) has a unique solution N_γ (up to a shift) for every $c \geq c_\gamma^*$ and no solution for $c < c_\gamma^*$.*
2. *When $c = c_\gamma^*$, N_γ is a sharp front, i.e. the support of N_γ is bounded above, and, modulo translation,*

$$N_\gamma(\xi) = \begin{cases} (1 - \exp(c\xi))^{1/\gamma} & \text{for } \xi < 0, \\ 0 & \text{for } \xi \geq 0. \end{cases}$$

3. *When $c > c_\gamma^*$, N_γ is positive, strictly monotonic and satisfies*

$$(\ln(1 - N_\gamma))'(\xi) \rightarrow \sqrt{1 + \frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma} = \frac{1}{\sqrt{1 + \frac{c^2}{4\gamma^2} + \frac{c}{2\gamma}}} \quad \text{as } \xi \rightarrow -\infty, \tag{I.4.21}$$

and

$$(\ln(N_\gamma))'(\xi) \rightarrow -\frac{1}{c}, \quad \text{as } \xi \rightarrow +\infty.$$

The TWs of the limiting system (I.4.17) satisfy the following:

Lemma I.4.1. *For $c > 1$ arbitrary, the following properties hold:*

1. Define the profile $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ by

$$P_{HS}(\xi) = \begin{cases} 0 & \text{if } \xi > 0, \\ 1 - e^\xi & \text{if } \xi < 0, \end{cases} \quad N_{HS}(\xi) = \begin{cases} \left(1 - \frac{1}{c}\right) e^{-\frac{\xi}{c}} & \text{if } \xi > 0, \\ 1 & \text{if } \xi < 0. \end{cases} \quad (\text{I.4.22})$$

Then $(N_{HS}, P_{HS})(x-ct)$ is a traveling wave moving at speed c solution of the Hele-Shaw system

$$cN' + (NP')' + N\Phi(P) = 0, \quad (\text{I.4.23})$$

$$0 \leq N \leq 1, \quad (1-N)P = 0, \quad P \geq 0, \quad (\text{I.4.24})$$

$$P(P'' + \Phi(P)) = 0. \quad (\text{I.4.25})$$

2. Let $(N, P) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ be a traveling wave profile moving at speed c of the Hele-Shaw system (I.4.17). Then there exists $\xi_0 \in \mathbb{R}$ such that $(N, P) = (N_{HS}, P_{HS})(\cdot - \xi_0)$.

The first result of this work concerns the convergence of (N_γ, P_γ) to the TWs of (I.4.15)–(I.4.16). The proof relies on the continuity of the flux $J_\gamma = cN_\gamma + N_\gamma P'_\gamma$ at the interface, similarly to [124]. The following theorem contains the convergence result in addition to the qualitative and quantitative properties of (N_γ, P_γ) that are necessary to obtain it.

Theorem I.4.2. *Let us fix the shift such that $N_\gamma(0) = \left(\frac{1}{\gamma}\right)^{1/\gamma}$. Assume $\gamma > 1$ sufficiently large, $c > 1$ be fixed, independent of γ , and let N_γ be the solution of (I.4.20) such that $P_\gamma(0) = \frac{1}{\gamma}$. Then the following properties hold true.*

- There exist $\xi_\gamma^-, \tilde{\xi}_\gamma$ with $\xi_\gamma^- = O\left(\frac{1}{\sqrt{\gamma}}\right) < 0 < \tilde{\xi}_\gamma = O\left(\frac{1}{\gamma}\right)$, such that the profile (N_γ, P_γ) satisfies
 - in the congested zone $\xi < \xi_\gamma^-$, the density N_γ converges uniformly to 1: there exists a constant $C > 0$ depending only on c such that

$$\left(\frac{C}{\sqrt{\gamma}}\right)^{\frac{1}{\gamma}} \leq N_\gamma(\xi) \leq 1 \quad \forall \xi \leq \xi_\gamma^-, \quad (\text{I.4.26})$$

and there exist constants $C' \geq C > 0$ independent of γ such that

$$1 - \left(1 - \frac{C'}{\sqrt{\gamma}}\right) e^{(1-C\gamma^{-1/2})\xi} \leq P_\gamma(\xi) \leq 1 - \left(1 - \frac{C}{\sqrt{\gamma}}\right) e^\xi \quad \forall \xi \leq \xi_\gamma^-; \quad (\text{I.4.27})$$

- in the intermediate region $\xi \in [\xi_\gamma^-, \tilde{\xi}_\gamma]$, N'_γ takes exponentially large values with respect to γ :

$$\|N'_\gamma\|_{L^\infty(\xi_\gamma^-, \tilde{\xi}_\gamma)} = O\left(\left(1 - \frac{1}{2c}\right)^{-\gamma}\right), \quad (\text{I.4.28})$$

while the pressure P_γ converges uniformly to 0 as $\gamma \rightarrow +\infty$: there exists $\delta \in (0, 1 - c^{-1})$, independent of γ such that

$$\left(1 - \frac{1}{c} - \delta\right)^\gamma \leq P_\gamma(\xi) \leq \frac{C}{\sqrt{\gamma}} \quad \forall \xi \in [\xi_\gamma^-, \tilde{\xi}_\gamma]; \quad (\text{I.4.29})$$

- in the free zone $\xi > \tilde{\xi}_\gamma$, the pressure P_γ takes exponentially small values (wrt γ): $P_\gamma(\xi) \leq \left(1 - \frac{1}{2c}\right)^\gamma$ and N_γ decreases exponentially to 0 as $\xi \rightarrow +\infty$: there exists $\delta > 0$ independent of γ , such that for γ large enough

$$\left(1 - \frac{1}{c} - \delta\right) \exp\left(-\left(\frac{1}{c} + \delta\right)\xi\right) \leq N_\gamma(\xi) \leq \left(1 - \frac{1}{c} + \delta\right) \exp\left(-\frac{1}{2c}\xi\right) \quad \forall \xi > \tilde{\xi}_\gamma; \quad (\text{I.4.30})$$

- As $\gamma \rightarrow +\infty$, there exists $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ such that $N_\gamma \rightarrow N_{HS}$ in $L^p_{loc}(\mathbb{R})$ and $P_\gamma \rightarrow P_{HS}$ in $W^{1,p}_{loc}(\mathbb{R})$ for any $p \in [1, \infty[$, and (N_{HS}, P_{HS}) is a wave-front profile of the Hele-Shaw equations (I.4.17) such that $P_{HS}(\xi) = (1 - e^\xi)\mathbf{1}_{\xi \leq 0}$, $\lim_{\xi \rightarrow 0^+} N_{HS} = 1 - \frac{1}{c}$.

Passing to the (weak) limit in equation (I.4.20) is possible thanks to the strong convergence of P_γ and the weakly-* convergence of N_γ to N_{HS} in $L^\infty(\mathbb{R})$. The limit (N_{HS}, P_{HS}) satisfies the equation

$$-cN'_{HS} - (N_{HS}P_{HS})'' = N_{HS}(1 - P_{HS}), \quad (\text{I.4.31})$$

in the sense of distributions. Making separate analysis in the free and the congested zones, we are able to show that this limit coincides with a traveling wave solution of (I.4.17). The jump of N_{HS} at $\xi = 0$ results from the continuity of the flux $J = CN + NP'$.

The second result establishes the convergence in weighted Sobolev spaces of the solution of (I.4.15)–(I.4.16) towards N_γ as $t \rightarrow +\infty$ when the initial data lies between two translations of the profile N_γ .

Theorem I.4.3. *Let*

$$W(\xi) := N_\gamma(\xi)^\gamma \exp\left(\int_{\xi_\gamma}^\xi \frac{c}{\gamma N_\gamma^\gamma}\right).$$

be a weight with double exponential growth as $\xi \rightarrow +\infty$, and a exponential decay as $\xi \rightarrow -\infty$. There exists constants $\eta_1, \eta_2 \in]0, 1[$, depending only on $c > 1$, such that the following result holds.

Let $\gamma > 1$ be fixed, sufficiently large. We make the following assumptions on the initial data n_γ^0 :

(H1) n_γ^0 lies between two shifts of N_γ , i.e. there exists $h > 0$ such that $n_\gamma^0(x) \in [N_\gamma(x + h), N_\gamma(x - h)]$ for all $x \in \mathbb{R}$;

(H2) The difference $n_\gamma^0 - N_\gamma$ is sufficiently decaying, namely

$$\int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x) dx < \infty.$$

Let n_γ be the solution of (I.4.15) associated with n_γ^0 .

Then there exists a constant $c_\gamma > 0$, $c_\gamma = O(\eta_1^\gamma)$, such that if $|h| \leq \eta_2^\gamma$, the following inequality holds:

$$\int_{\mathbb{R}} (n_\gamma(t, x) - N_\gamma(x - ct))^2 W(x - ct) dx \leq e^{-c_\gamma t} \int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x) dx \quad \forall t \geq 0.$$

Moreover, setting $u_\gamma(t, x) := (n_\gamma(t, x) - N_\gamma(x - ct))/N'_\gamma(x - ct)$, we have the additional dissipation of energy:

$$\gamma \int_0^\infty \int_{\mathbb{R}} (\partial_x u_\gamma(t, x))^2 (N_\gamma^\gamma (N'_\gamma)^2 W)(x - ct) dx dt \leq \int_{\mathbb{R}} (n_\gamma^0(x) - N_\gamma(x))^2 W(x) dx.$$

Computations lead to u satisfying an equation of the type

$$\partial_t u_\gamma + \mathcal{L}u_\gamma = \mathcal{G}[u_\gamma].$$

Here, \mathcal{L} is a linear operator resulting from the linearization of (I.4.15) around $n_\gamma = N_\gamma$,

$$\mathcal{L} := b\partial_x - a\partial_{xx},$$

where

$$a(t, x) = \bar{a}(x - ct) := \gamma N^\gamma, \quad b(t, x) = \bar{b}(x - ct) := -2\gamma \frac{(N^\gamma N')'}{N'}.$$

$\mathcal{G}[u]$ denotes the resulting quadratic operator satisfying

$$|\mathcal{G}[u]| \leq C_\gamma |u| (|u| + |\partial_x u|). \quad (\text{I.4.32})$$

The proof relies upon two main ideas using the coercivity of the linear operator \mathcal{L} and treating the nonlinear term $\mathcal{G}[u]$ as a perturbation of the linearized problem. For the latter, it is important to obtain a L^∞ control on $n_\gamma - N_\gamma$ and a L^2 weighted estimate on u_γ . The L^∞ control is a direct consequence of the parabolic nature of the equation and is obtained using a comparison principle. The L^2 weighted estimate is based on the two following inequalities:

Proposition I.4.1 (Weighted Poincaré-type inequality). *Let $v \in \mathcal{C}_c^1(\mathbb{R})$.*

Then there exists a constant \bar{C} independent of γ and a constant $C_\gamma \leq C^\gamma$ such that

$$\int_{-\infty}^{\xi_\gamma^-} v^2 \gamma N_\gamma^\gamma \bar{w}_0 d\xi + \int_{\bar{\xi}}^{+\infty} v^2 \frac{1}{\gamma N_\gamma^\gamma} \bar{w}_0 d\xi \leq \bar{C} \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a} \bar{w}_0 d\xi + C_\gamma \int_{\mathbb{R}} v^2 e^{\sqrt{\gamma} \xi} d\xi. \quad (\text{I.4.33})$$

In particular, there exists a constant c_γ , satisfying $c_\gamma \geq \eta^\gamma$ for some $\eta \in]0, 1[$ independent of γ , such that

$$c_\gamma \int_{\mathbb{R}} v^2 \bar{w}_0 d\xi \leq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a} \bar{w}_0 d\xi + \int_{\mathbb{R}} |v|^2 \exp(\sqrt{\gamma} \xi) d\xi.$$

The asymptotic behavior of N_γ and its derivatives greatly impact the size of the perturbations in the stability analysis. Indeed, the size of c_γ is linked to fact that $\|N'_\gamma\|_{L^\infty(\mathbb{R})}$ blows up exponentially as $\gamma \rightarrow \infty$. We, therefore, need $\|u\|_{L^\infty}$ to be exponentially small to treat the quadratic term as a perturbation. The remainder of the proof is classical. It relies on the coercivity of the linear operator \mathcal{L} , and $\mathcal{G}[u]$ is absorbed in the energy dissipation.

I.4.5.1 Perspectives

It would be interesting to adapt the analysis to the case where the mechanical pressure comprises the non-overlapping of cells but still generates cell displacement with a velocity field following Darcy's law. In [86] it was proven that for the singular law, taking the

incompressible limit when $\varepsilon \rightarrow 0$ leads to the Hele-Shaw model (I.4.17). For this equation, up to our knowledge, there are no known results on the existence of TW solutions for a given $\varepsilon > 0$. The general models dealing with similar reaction-diffusion equations assume the diffusion term to be $C^1([0, 1])$, where 0 and 1 are the equilibrium solutions [95]. This is not the case of $\varepsilon \frac{n_\varepsilon}{(n_\varepsilon - 1)^2}$ at $n_\varepsilon = 1$. Indeed, the lack of known results for the model with singular pressure was one of the main difficulties stated by Hecht and Vauchelet [86].

Another possible extension includes not changing the power-law but the velocity field to include viscosity effects using the Brinkman or Navier-Stokes equations as in [114] and [116], respectively.

One could also consider the same problem (I.4.15)–(I.4.16) in a higher dimension. It is a well-known fact that most realistic models of biological interest are multidimensional and involve several dependent variables. In this case, one of the main issues regards the fact that the Hele-Shaw problem does not necessarily have a global classical solution in multidimensional spaces due to cusp-like singularities⁵ appearing at the free boundary [185].

I.4.5.2 Outline of Chapter III

The chapter is organized as follows. In section III.2, we provide a description of traveling wave solution for both systems (I.4.15) and (I.4.17). Here, we also give a refined description of N_γ in the transition zone between the congested and the free region. Then in Section III.3, the nonlinear asymptotic stability of the profile N_γ is shown in an L^2 -weighted norm for small perturbation depending on γ . The key ingredients of stability analysis are the linearized system around N_γ and a weighted Poincaré inequality. The proof of this result is left to Section III.4.

⁵A cusp or spinode (in French, *point de rebroussement*) is a local singularity not resulting from self-intersection but from the point of the curve moving in a reverse direction.

II – Boundary layer formation in the quasigeostrophic model near nonperiodic rough coasts

Contents

II.1 Introduction	40
II.2 Preliminaries and main results	43
II.3 Formal asymptotic expansion and first profiles	48
II.4 Existence and uniqueness of the solutions in a rough domain: methodology	51
II.4.1 Linear case.	53
II.4.2 Nonlinear/ linearized problem.	60
II.5 Western boundary layer: the linear case	62
II.5.1 The linear problem on the half-space	63
II.5.2 Differential operators at the transparent boundary	74
II.5.3 The problem in the rough channel	80
II.6 Nonlinear boundary layer formation near the western coast	86
II.6.1 Nonlinear problem in the half-space	87
II.6.2 The problem in the rough channel	88
II.6.3 Connecting the solutions at the artificial boundary	107
II.7 Linearized problem	111
II.7.1 The problem on the half space	112
II.7.2 The linearized problem on the rough domain	113
II.7.3 Joining ω_w^- and ω_w^+	116
II.8 Eastern boundary layer	118
II.8.1 The problem in the half-space	119
II.8.2 Transparent operators	127
II.8.3 Equivalent problem and estimates on the rough channel	129
II.9 Convergence result	131
II.9.1 Construction of the approximate solution	131
II.9.2 Computing the reminder	133
II.9.3 Energy estimates	136

Appendices 139
II.A Study of the roots of equation of the linear western boundary layer 139
II.B Expansion of the eigenvalues at high frequencies 142
II.C Computations of the regularity estimates 143
II.D Asymptotic behavior of the Green function coefficients 145
II.E Proof of Lemma II.4.1 146
II.F Proof of Lemmas II.5.9 and II.8.7 148

This chapter deals with the so-called homogeneous model of wind-driven ocean circulation or the single-layer quasi-geostrophic model. Our attention is focused on performing a complete asymptotic analysis that highlights boundary layer formation along the coastal line. We assume rough coasts without any particular structure, resulting in the study of a nonlinear PDE system for the western boundary layer in an infinite domain. As a consequence, we look for the solution in nonlocalized Sobolev spaces. Under this hypothesis, the eastern boundary layer exhibits a singular behavior at low frequencies far from the rough boundary, leading to issues with convergence. The problem is tackled by imposing ergodicity properties. We establish the well-posedness of the governing boundary layer equations and the asymptotic solution. Our results generalize the ones of the paper by Bresch and Gérard-Varet [53] in the context of periodic irregularities.

II.1 Introduction

This chapter addresses roughness-induced effects on geophysical fluid motion in a context where small irregularities have very little structure. In geophysics, this phenomenon is usual when looking at the indentations on the bottom of the ocean and shores. The analysis will be conducted on the homogeneous model of wind-driven ocean circulation, also known as the 2D quasigeostrophic model. In this case, the input is the planetary wind-stress field over the ocean, while the output is the transport that takes place into the mid-depth layer and is forced by the Ekman pumping, due to wind stress, above this layer. Steady circulation is then maintained by bottom friction and lateral diffusion of relative vorticity.

The mathematical description of the model is as follows: let $\Psi = \Psi(t, \mathbf{x}) \in \mathbb{R}$ be the stream function associated to the two-dimensional velocity field $u = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}))^t$. In a simply connected domain $\Omega \subset \mathbb{R}^2$ to be described later on, the system reads

$$\begin{cases}
 (\partial_t + \nabla^\perp \Psi \cdot \nabla) (\Delta \Psi + \beta y - \text{Fr} \Psi + \eta_B) + r \Delta \Psi &= \beta \text{curl} \tau + \text{Re}^{-1} \Delta^2 \Psi, \\
 \Psi|_{\partial \Omega} &= \frac{\partial \Psi}{\partial n} = 0 \\
 \Psi|_{t=0} &= \Psi_{ini},
 \end{cases} \tag{II.1.1}$$

where

- $\nabla^\perp \Psi \cdot \nabla$ is the transport operator by the two-dimensional flow;
- $\Delta \Psi$ is the vorticity, and $r \Delta \Psi$, $r > 0$, is the Ekman pumping term due to bottom friction;

- Fr is the Froude number due to the free surface ;
- $\beta > 0$ is a parameter characterizing the beta-plane approximation which results from linearizing the Coriolis factor around a given latitude;
- η_B describes the variations of the bottom topography;
- $\beta \text{curl } \tau$ is the Ekman pumping term due to wind stress at the surface, where τ is a given stress tensor; and,
- Re denotes the Reynolds number.

Here, Ψ_{ini} is chosen such that data is “well-prepared”, i.e., we need Ψ_{ini} to converge at least in $H^1(\Omega^\varepsilon)$ to a function to be specified later on. For a formal derivation of the model (II.1.1) from the Navier–Stokes equations, we refer the reader to [40] and [17]. If the basin is closed and if we assume that there is no water outflow, then the flux corresponding to the horizontal velocity has to vanish at the boundary, from which we have the homogeneous boundary condition $\Psi|_{\partial\Omega} = 0$. Moreover, the presence of the diffusion term $\text{Re}^{-1}\Delta^2\Psi$ requires the no-slip boundary condition $\frac{\partial\Psi}{\partial n} = 0$.

Under certain hypotheses (fast rotation, thin layer domain, small vertical viscosity) and proper scaling, B. Desjardins and E. Grenier proved this model describes asymptotically a 2D fluid [45]. The authors performed a complete boundary layer analysis of the model in the domain

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 : \chi_w(y) \leq x \leq \chi_e(y), y_{min} \leq y \leq y_{max} \right\},$$

where χ_w and χ_e are smooth functions defined for $y \in [y_{min}, y_{max}]$. The forcing term τ was assumed to be identically zero when y is in a neighborhood of y_{min} and y_{max} . This assumption is crucial to avoid the strong singularities near the northern (max) and southern (min) ends of the domain, known as geostrophic degeneracy [47].

In the context of ocean currents, the Rossby parameter β and Reynolds number are very high. A first approximation of the solution confirms (II.1.1) is a singular perturbation problem and there is boundary layer formation. These problems are often tackled by a multi-scale approach. Therefore, it is natural to look for an approximate solution of (II.1.1) of the form

$$\Psi^\varepsilon \sim \Psi_{int}(t, x, y) + \Psi_{bl}(t, y, X(\varepsilon), Y(\varepsilon)),$$

where $X(\varepsilon), Y(\varepsilon)$ are the boundary layer or fast variables.

Following this reasoning, Desjardins and Grenier derived the so-called Munk layers, responsible for the western intensification of boundary currents. D. Bresch and D. Gérard-Varet [53] later generalized these results for the case of rough shores with periodic roughness. In this case, the ocean basin was described by a domain

$$\Omega^\varepsilon = \left\{ (x, y) \in \mathbb{R}^2 : \chi_w(y) - \varepsilon\gamma_w(\varepsilon^{-1}y) < x < \chi_e(y) + \varepsilon\gamma_e(\varepsilon^{-1}y), \quad y_{min} \leq y \leq y_{max} \right\}, \tag{II.1.2}$$

where ε is a small positive parameter and γ_w, γ_e are regular and periodic functions describing the roughness of the West and East coasts, respectively. Taking into account rough coast-lines leads naturally to additional mathematical difficulties. For example, the usual Munk system of ordinary differential equations is replaced by elliptic quasilinear partial differential

equations. Nevertheless, the periodicity assumption on the structural properties of γ_e, γ_w simplifies the analysis of the existence and uniqueness of the solutions.

A natural extension of this work would be dropping the periodicity assumption, since the geometry of the boundary is not meant to follow a particular spatial pattern. Our goal is to study the asymptotic behavior of problem (II.1.1) when functions γ_e, γ_w are arbitrary, therefore generalizing the results of [53].

We are able to show three main results:

1. The western boundary profiles are well-defined and decay exponentially far from the boundary (cf. Theorem II.2.1).
2. In the eastern boundary layer profiles, three components with different asymptotic behavior far from the boundary can be identified: one decaying exponentially, another converging to zero at a polynomial rate, and a third one whose convergence is guaranteed adding ergodic properties. Their well-posedness is completed by adding some constraints to the interior profile to ensure the validity of the far-field condition (see Theorem II.2.2).
3. Finally, we have $\Psi^\varepsilon - \Psi_{app}$ is $o(\varepsilon)$ in a norm that will be specified later (cf. Theorem II.2.3)

When the periodic roughness is no longer considered, the analysis, although possible, is much more involved, as shown in [63, 59, 52, 61] in other contexts. We seek the solution of the boundary layer problem in a space of infinite energy. In particular, we will be considering Kato spaces H_{uloc}^s (a definition is provided in (II.2.11)). The use of such function spaces to mathematically describe fluid systems traces back to [64, 65], in which existence is proven for weak solutions of the Navier-Stokes equations in \mathbb{R}^3 with initial data in L_{uloc}^2 . For other relevant works, see [52] and the references therein.

New difficulties arise in this context. First, due to the unboundedness of the boundary layer domains, we deal here with only locally integrable functions, leading to a completely different treatment of the energy estimates and mathematical tools, such as Poincaré inequality are needed but are no longer valid. Second, being typical for fourth-order problems, the equation lacks a maximum principle. Third, as the roughness is nonperiodic, the boundary layer system is more complex. Indeed, the absence of compactness both in the tangential and transverse variables and the presence of singularities at low frequencies for the eastern boundary layer functions make proving convergence in a deterministic setting extremely difficult. We, therefore, use the ergodic theorem to specify the behavior of the solution of the eastern boundary layer far from the boundary and, later, to find the energy estimates in the analysis of the quality of the approximation.

The rest of the paper is structured as follows. The following section contains a precise description of the domain, some simplifying assumptions and the statements of the main mathematical results. Section II.3 contains the assumptions made for constructing the profiles of the approximate solution and the detailed derivation of the functions in the main order. In Section II.4, we outline the general methodology to solve linear and nonlinear/linearized problems characterizing the boundary layer functions. The existence, uniqueness and asymptotic behavior of the first profile of the western boundary layer are discussed in Section II.5 for the linear case and Section II.6, for the nonlinear and linearized systems. Section II.8 focuses on the boundary layer analysis in the east of region. Finally, Section II.9 contains

the construction of the approximate solution and the study of its convergence to the solution of the original problem.

II.2 Preliminaries and main results

Before stating the main results, let us first state some hypotheses on the dimensionless problem (II.1.1). We will assume there is no stratification, therefore, $Fr = 0$. Our study is solely focused on the effect of rough shores on flow behavior, so the bottom topography parameter η_B is considered to be nil. Since for a basin of 1000×1000 km at a central latitude $\theta_0 = 45^\circ N$ [26], we have that $\eta_B + \mu \sim 1.0 - 100$, $\beta \sim 10^3$, and $Re \sim 1.6 - 160$, it is possible to consider $r = 1$ and $Re = 1$ to simplify the computations. Up to minor changes, equivalent results can be obtained for arbitrary values of the Reynolds number and r .

Let ε be the natural size of the boundary layers arising in this study, we consider $\beta = \varepsilon^{-3}$. This choice of scaling preserves the problem's physical accuracy. Moreover, the size of the irregularities is also assumed to be equal to ε . The last hypothesis is mainly of mathematical significance, since it allows for a richer analysis due to the interaction of the linear and non-linear terms of the equation at the main order for the boundary layer problem. Then, system (II.1.1) becomes

$$\begin{cases} (\partial_t + \nabla^\perp \Psi \cdot \nabla) (\Delta \Psi + \varepsilon^{-3} y) + \Delta \Psi = \varepsilon^{-3} \text{curl } \tau + \Delta^2 \Psi, & \text{in } \Omega^\varepsilon \\ \Psi|_{\partial\Omega^\varepsilon} = \frac{\partial \Psi}{\partial n}|_{\partial\Omega^\varepsilon} = 0, \\ \Psi|_{t=0} = \Psi_{ini}. \end{cases} \quad (\text{II.2.1})$$

Here, we adopt the notation and terminology in [53]. The domain of problem (II.2.1) is defined as follows

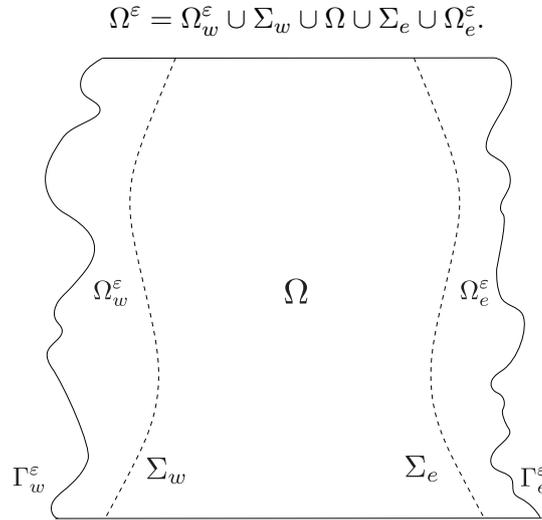


Figure II.1: Domain Ω^ε

- The “interior domain” is given by

$$\Omega = \{\chi_w(y) \leq x \leq \chi_e(y), y \in [y_{min}, y_{max}]\},$$

where χ_w and χ_e are smooth functions defined for $y \in [y_{min}, y_{max}]$.

- $\Sigma_w = \{(\chi_w(y), y), y \in (y_{min}, y_{max})\}$ and $\Sigma_e = \{(\chi_e(y), y), y \in (y_{min}, y_{max})\}$ are interfaces separating the interior domain from the “rough shores”.
- Ω_w^ε and Ω_e^ε are the rough domains. The positive smooth functions $\gamma_w = \gamma_w(Y)$ and $\gamma_e = \gamma_e(Y)$ describe the irregularities. We set

$$\Omega_w^\varepsilon = \left\{ (x, y), 0 > x - \chi_w(y) > -\varepsilon\gamma_w(\varepsilon^{-1}y) \right\}, \quad (\text{II.2.2})$$

$$\Omega_e^\varepsilon = \left\{ (x, y), 0 < x - \chi_e(y) < \varepsilon\gamma_e(\varepsilon^{-1}y) \right\}. \quad (\text{II.2.3})$$

The lateral boundaries are

$$\Gamma_w^\varepsilon = \left\{ (\chi_w(y) - \varepsilon\gamma_w(\varepsilon^{-1}y), y), y \in (y_{min}, y_{max}), \right\}$$

$$\Gamma_e^\varepsilon = \left\{ (\chi_e(y) + \varepsilon\gamma_e(\varepsilon^{-1}y), y), y \in (y_{min}, y_{max}), \right\}.$$

Let us introduce the notation n_w and n_e for the exterior unit normal vectors to the roughness curves γ_w and γ_e .

Let $T > 0$, we assume that $\tau(t, x, y) \in L^\infty((0; T); H^s)$, for s large enough. We are actually studying well-prepared data, as seen in [45]. In order to avoid steep singularities due to advection of vorticity when approaching the northern and southern extremal points, we assume additionally the irrotational part of the wind vanishes in their vicinity, see [47]. More precisely, we suppose that there exists $\lambda > 0$ such that

$$\text{curl } \tau = 0 \quad \text{for } y \in [y_{max} - \lambda, y_{max}] \cup [y_{min}, y_{min} + \lambda]. \quad (\text{II.2.4})$$

As pointed out in [45], problem (II.2.1) has a unique smooth solution Ψ^ε for all $\varepsilon > 0$.

The approximate solution is sought in form of series in powers of the small parameter ε with coefficients depending on the global variables t, x, y , and the microscopic variables $Y = Y(y, \varepsilon)$, $X = X(x, y, \varepsilon)$

$$\Psi_{\text{app}}^\varepsilon(t, x, y) \sim \sum_{k=0}^{\infty} \varepsilon^k \left(\Psi_{\text{int}}^k(t, x, y) + \Psi_w^k(t, y, X_w, Y) + \Psi_e^k(t, y, X_e, Y) \right), \quad (\text{II.2.5})$$

where $\Psi_{\text{int}}^k(t, x, y)$ correspond to the interior terms, while Ψ_w^k and Ψ_e^k refer to the corrector terms in the western and eastern boundary layer, respectively. Such a series is substituted in the original problem and a system of equations is obtained for each one of the profiles by equating to zero all coefficients associated to powers of ε . Here, X and Y are the fast or microscopic variables which depend on the small parameter. They are defined as follows:

$$Y = \varepsilon^{-1}y, \quad X_w = \varepsilon^{-1}(x - \chi_w(y)), \quad X_e = \varepsilon^{-1}(\chi_e(y) - x),$$

where ω_w and ω_e are respectively the western and eastern boundary layer domains. The former is of the form $\omega_w = \omega_w^+ \cup \sigma_w \cup \omega_w^-$, where

$$\begin{aligned} \omega_w^+ &= \{X_w > 0, \quad Y \in \mathbb{R}\}, & \sigma_w &= \{X_w = 0, \quad Y \in \mathbb{R}\} \\ \omega_w^- &= \{-\gamma_w(Y) < X_w < 0, \quad Y \in \mathbb{R}\}. \end{aligned} \quad (\text{II.2.6})$$

The domain ω_e can be defined in a similar manner.

In a first approximation of the solution, we are confronted with coastal asymmetry: it is impossible to obtain a solution in the eastern boundary layer domain satisfying all boundary conditions and decaying at infinity due to the lack of enough roots with positive real part. A usual choice under these circumstances is to consider that Ψ_{int} is tangent to the boundary Σ_e , which results in

$$\Psi_{int}^0(t, x, y) = - \int_x^{\chi_e(y)} \text{curl} \tau(t, x', y) dx', \quad (\text{II.2.7})$$

and $\Psi_e^0 \equiv 0$. Then, the key element in the construction of the approximate solution will be to determine Ψ_w^0 which formally solves the problem

$$\begin{aligned} Q_w(\Psi_w^0, \Psi_w^0) + \partial_{X_w} \Psi_w^0 - \Delta_w^2 \Psi_w^0 &= 0, \quad \text{in } \omega_w^- \cup \omega_w^+ \\ \left[\Psi_w^0 \right] |_{\sigma_w} &= \phi, \\ \left[\partial_X^k \Psi_w^0 \right] |_{\sigma_w} &= 0, \quad k = 1, 2, 3, \\ \Psi_w^0 |_{X=-\gamma_w(Y)} &= \frac{\partial \Psi_w^0}{\partial n_w} |_{X=-\gamma_w(Y)} = 0. \end{aligned} \quad (\text{II.2.8})$$

Here, $[\cdot] |_{X=X'}$ denotes the jump operator of the function f at $X = X'$ and is defined as $[f] |_{X=X'} := f(X'^+, \cdot) - f(X'^-, \cdot)$. The jump of the function at the western boundary of the interior domain is given by

$$\phi(t, y) = \int_{\chi_w(y)}^{\chi_e(y)} \text{curl} \tau(t, x', y) dx'. \quad (\text{II.2.9})$$

Moreover, for $\alpha_w = \chi'_w(y)$, the differential operators are given by

$$\nabla_w = (\partial_{X_w}, \partial_Y - \alpha_w \partial_{X_w})^t, \quad \nabla_w^\perp(y) = (\alpha_w \partial_{X_w} - \partial_Y, \partial_{X_w})^t,$$

consequently, Δ_w and Q_w are defined as follows

$$\begin{aligned} \Delta_w &= \partial_{X_w}^2 + (\alpha \partial_{X_w} - \partial_Y)^2, \\ Q_w(\Psi, \tilde{\Psi}) &= \nabla_w^\perp \cdot \left(\left(\nabla_w^\perp \Psi \cdot \nabla_w \right) \nabla_w^\perp \tilde{\Psi} \right). \end{aligned}$$

Note Δ_w and Δ_w^2 are elliptic operators with respect to the variables Y et X_w . At the level of the boundary layer, t and y behave as parameters.

Our first result is the existence and uniqueness of the solution for the boundary layer system (II.2.8). As usual in the steady Navier–Stokes equations theory, the well-posedness is obtained under a smallness hypothesis. The problem is defined in an unbounded set; therefore, we seek the solution in spaces of uniformly locally integrable functions, also known in the literature as Kato spaces [58]. They include a richer spectrum of functions, allowing for some singular behavior or non-decaying functions. Let us briefly recall the definition:

Let $\theta \in C_0^\infty(\mathbb{R}^d)$ be such that $\text{Supp} \theta \subset [-1, 1]^d$, $\theta \equiv 1$ on $[-1/4, 1/4]^d$, and

$$\sum_{k \in \mathbb{Z}^d} \tau_k \theta(x) = 1, \quad \text{for all } x \in \mathbb{R}^d, \quad (\text{II.2.10})$$

where τ_k denotes the translation operator defined by $\tau_k f(x) = f(x - k)$. Then, for $s \geq 0$, $p \in [1, +\infty)$,

$$\begin{aligned} L_{\text{uloc}}^p(\mathbb{R}^d) &= \left\{ u \in L_{\text{loc}}^p(\mathbb{R}^d) : \sup_{k \in \mathbb{Z}^d} \|(\tau_k \theta)u\|_{L^p(\mathbb{R}^d)} \right\} \\ H_{\text{uloc}}^s(\mathbb{R}^d) &= \left\{ u \in H_{\text{loc}}^s(\mathbb{R}^d) : \sup_{k \in \mathbb{Z}^d} \|(\tau_k \theta)u\|_{H^s(\mathbb{R}^d)} \right\} \end{aligned} \quad (\text{II.2.11})$$

We show the following:

Theorem II.2.1. *Let γ_w be a positive $W^{2,\infty}(\mathbb{R})$ function and ω_w be defined as before. There exists a constant $\delta_0 > 0$ such that if $\|\phi\|_{W^{2,\infty}} < \delta_0$, problem (II.2.8) has a unique solution in $H_{\text{uloc}}^2(\omega_w)$ denoted by Ψ_w . Moreover, for a certain constant $\delta > 0$, it satisfies the estimate*

$$\|e^{\delta X_w} \Psi_w\|_{H_{\text{uloc}}^2(\omega_w)} \leq C \|\phi\|_{W^{2,\infty}}. \quad (\text{II.2.12})$$

This theorem generalizes the result of [53] for to the case of nonperiodic roughness. A remarkable feature of this result is that exponential decay to zero persists, despite the arbitrary roughness, and without any additional assumption on the function describing the irregular boundary.

Following the ideas in Masmoudi and Gérard-Varet [59], we look for the solution of (II.2.8) by introducing a transparent boundary which divides the domain in two: a half-space and a bounded rough channel. Then, the problem is solved in each of the subdomains, and a pseudo-differential operator of Poincaré-Steklov is used to relate the behavior of the solutions at both sides of the interface. When (II.2.8) is considered linear, this last step can be done directly; otherwise, applying the implicit function theorem is needed to join the solutions at the artificial boundary.

Once we have shown the above result on the western boundary layer, we construct the approximate solution and analyze its closeness to the original problem. The error is computed by calculating the following profiles in a systemic scheme (see Sections II.3 and II.9) and is pretty straightforward.

At order ε^n the interior profile satisfies $\Psi_{\text{int}}^n = C^n(t, y) - \int_x^{\chi_e(y)} F_n dx$, where F_n depends on Ψ_{int}^j , $j \leq n - 1$. The value of $C^n(t, y)$ will be specified later. Then, the n -th eastern boundary layer profile meets the conditions

$$\begin{aligned} -\partial_{X_e} \Psi_e^n - \Delta_e^2 \Psi_e^n &= 0, \quad \text{in } \omega_e^- \cup \omega_e^+ \\ [\Psi_e^n] |_{\sigma_e} &= -C^n(t, y), \\ \left[\partial_{X_e}^k \Psi_e^n \right] |_{\sigma_e} &= g_k, \quad k = 1, \dots, 3, \\ \Psi_e^n |_{X_e = -\gamma_e(Y)} &= \frac{\partial \Psi_e^n}{\partial n_e} |_{X_e = -\gamma_e(Y)} = 0, \end{aligned} \quad (\text{II.2.13})$$

where $g_k \in L^\infty$, the domain is given by

$$\begin{aligned} \omega_e^+ &= \{X_e > 0, \quad Y \in \mathbb{R}\}, \quad \sigma_e = \{X_e = 0, \quad Y \in \mathbb{R}\} \\ \omega_e^- &= \{-\gamma_e(Y) < X_e < 0, \quad Y \in \mathbb{R}\}, \end{aligned} \quad (\text{II.2.14})$$

and the differential operators are defined as follows for $\alpha_e = \chi'_e(y)$:

$$\begin{aligned}\nabla_e &= (\partial_{X_e}, -\alpha_e \partial_{X_e} - \partial_Y), & \nabla_e^\perp &= (\partial_Y + \alpha_e \partial_{X_e}, \partial_{X_e}), \\ \Delta_e &= \partial_{X_e}^2 + (\alpha \partial_{X_e} + \partial_Y)^2, & Q_e(\Psi, \tilde{\Psi}) &= \nabla_e^\perp \cdot \left((\nabla_e^\perp \Psi \cdot \nabla_e) \nabla_e^\perp \tilde{\Psi} \right).\end{aligned}\tag{II.2.15}$$

Note that the main equation in (II.2.13) is elliptic with respect to X_e and Y ; t and y are considered parameters. Although the analysis of the well-posedness of (II.2.13) is similar to one described for the western boundary layer, additional issues arise concerning the convergence of Ψ_e^n when $\varepsilon \rightarrow 0$. Indeed, the analysis of the problem (II.2.13) in the half-space reveals the lack of spectral gap, which prevents the decay far from the boundary, see Section II.8.1. To guarantee the convergence of the eastern boundary layer profile when $X_e \rightarrow \infty$ while keeping the same structure of the domain, we add some probabilistic information (ergodic properties).

Let $\varepsilon > 0$ and (P, Π, μ) be a probability space. For instance, P could be considered as the set of K -Lipschitz functions, with $K > 0$; Π the borelian σ -algebra of P , seen as a subset of $C_b(\mathbb{R}^2; \mathbb{R})$ and μ a the probability measure preserved by the translation group (τ_Y) acting on P . Then, the eastern boundary layer domain can be described as follows for $m \in P$:

$$\omega_e(m) = \left\{ (X_e, Y) \in \mathbb{R}^2 : X_e > -\gamma_e(m, Y) \right\},$$

where $\omega_e(m) = \omega_e^+(m) \cup \sigma_e \cup \omega_e^-(m)$ and $\omega_e^\pm(m) = \omega_e(m) \cap \{\pm X > 0\}$. Here γ_e are homogeneous and measure-preserving random process.

In this context, we are able to distinguish three components of Ψ_e^n with different asymptotic behavior far from the boundary for which we have obtained the following result:

Theorem II.2.2. *Let ω_e a domain defined as before and γ_e an ergodic stationary random process, K -Lipschitz almost surely, for some $K > 0$. Let $g_1, g_2, g_3 \in W^{2,\infty}(\mathbb{R}_+ \times [y_{\min}, y_{\max}] \times \mathbb{R})$, then there exist a unique measurable map $C^n(t, y)$ such that problem (II.2.13) has a unique solution $\Psi_e^n = \Psi_{\text{exp}}^n + \Psi_{\text{alg}}^n + \Psi_{\text{erg}}^n$ where*

1. $\left\| \Psi_{\text{erg}}^n \right\|_{L^q(\omega_e^+)} \xrightarrow{X_e \rightarrow +\infty} 0$, locally uniformly in Y , almost surely and in $L^q(P)$ for all finite q ,

2. there exist constants $\delta, C > 0$ such that

$$\|e^{\delta X_e} \Psi_{\text{exp}}^n\|_{L^\infty(\omega_e^+)} \leq C \left(\sum_{k=1}^3 \|g_k\|_{L^\infty} + \|C^n\|_{W^{2,\infty}} \right),\tag{II.2.16}$$

3. there exists a constant $C > 0$ such that

$$\|(1 + X_e)^{1/4} \Psi_{\text{alg}}^n\|_{L^\infty(\omega_e^+)} \leq C \left(\sum_{k=1}^3 \|g_k\|_{L^\infty} + \|C^n\|_{W^{2,\infty}} \right).\tag{II.2.17}$$

Moreover, Ψ_e satisfies

$$\|\Psi_e^n\|_{H_{\text{uloc}}^2(\omega_e)} < +\infty, \quad \text{almost surely.}\tag{II.2.18}$$

The proof of the above result also relies on the use of wall laws and follows the same ideas of Theorem II.2.1 for the western boundary layer. First, we apply Fourier analysis to problem (II.2.13) in the half-space, which hints directly to the singular behavior at low frequencies far from the boundary. We show that the properties involving Ψ_{exp}^n and Ψ_{alg}^n in Theorem II.2.2 remain true in a deterministic setting by following the same ideas used for the western boundary layer. Only the convergence of Ψ_{erg}^n is shown using the ergodic theorem. Then, we define the associated Poincaré-Steklov operator for boundary data in $H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$. Finally, we look for the solution of a problem equivalent to (II.2.13) defined in a domain in which a transparent boundary condition is prescribed. For the equivalent system, we derive energy estimates in H_{uloc}^2 which are then used to prove existence and uniqueness of the solution.

The eastern boundary layer not converging to zero or not doing it fast enough at infinity poses an issue when solving the problem at Ψ_w^n . In particular, the terms Ψ_{alg}^n and Ψ_{erg}^n influence the western boundary layer mainly through the nonlinear term. Adding ad hoc correctors allows us to show the following results for $\Psi^\varepsilon - \Psi_{\text{app}}$.

Theorem II.2.3. *Let Ψ^ε be the solution of problem (II.2.1) and Ψ_{app} defined as in (II.2.5). Moreover, let $\Psi_{\text{ini}}^\varepsilon$ be such that $\Psi_{\text{ini}}^\varepsilon|_{\Omega^\varepsilon} = \partial_n \Psi_{\text{ini}}^\varepsilon = 0$ and $\|\Psi_{\text{ini}}^\varepsilon - \Psi_{\text{app}}|_{t=0}\|_{H^1(\Omega^\varepsilon)} \rightarrow 0$. There exists C_∞ , such that if $\|\text{curl } \tau\|_{W^{2,\infty}} < C_\infty$, then*

$$\|\Psi^\varepsilon - \Psi_{\text{app}}\|_{L^2(0,T;H^2(\Omega^\varepsilon))} + \|\Psi^\varepsilon - \Psi_{\text{app}}\|_{L^\infty(0,T;H^1(\Omega^\varepsilon))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad \text{almost surely.} \quad (\text{II.2.19})$$

In the periodic setting, for which the boundary layer profiles decay exponentially when ε goes to zero, the bound becomes $O(\varepsilon^{1/2})$ for the H^1 estimate (see [53]). In the general random setting, the convergence rate is limited by the behavior of eastern boundary layer profiles. The convergence result in Theorem II.9 is obtained by computing energy estimates on $\Psi^\varepsilon - \Psi_{\text{app}}$. The accuracy of the estimates depends greatly on each element in the approximate solution, their interactions and contributions. Each component needs to be smooth enough, with proper controls on the corresponding derivatives.

II.3 Formal asymptotic expansion and first profiles

In this section, we construct the approximate solution $\Psi_{\text{app}}^\varepsilon$ for the singularly perturbed problem (II.2.1) employing a matched asymptotic expansion. Inner and outer expansions (boundary layers) are determined in the interior and the rough shores domain. Then, matching conditions at the interface are imposed to obtain an approximate global solution.

Let X and Y define the local variables obtained after scaling: $Y = y/\varepsilon$ while $X_w = \frac{x - \chi_w(y)}{\varepsilon}$, $X_e = \frac{\chi_e(y) - x}{\varepsilon}$. We seek for an approximate solution of (II.2.1) of the form

$$\Psi_{\text{app}}^\varepsilon(t, x, y) = \sum_{i=0}^N \varepsilon^i \left(\Psi_{\text{int}}^i(t, x, y) + \Psi_w^i(t, y, X_w, Y) + \Psi_e^i(t, y, X_e, Y) \right) + O(\varepsilon^{N+1}), \quad (\text{II.3.1})$$

where $\Psi_{\text{int}}^i(t, x, y)$ correspond to the interior terms, while Ψ_w^i and Ψ_e^i denote the western and eastern boundary layer profiles. Without loss of generality, we assume that the interior terms are zero outside $\bar{\Omega}$.

Since the boundary layer terms are expected not to have an effect far from the boundaries, we assume

$$\Psi_e^i \xrightarrow{X_e \rightarrow \infty} 0, \quad \Psi_w^i \xrightarrow{X_w \rightarrow \infty} 0. \quad (\text{II.3.2})$$

The approximate solution must additionally satisfy the boundary condition $\Psi_{app}^\varepsilon = 0$ at $\partial\Omega^\varepsilon$. Thus, we have

$$\Psi_w^i \Big|_{X_w = -\gamma_w(Y)} = 0, \quad \Psi_e^i \Big|_{X_e = -\gamma_e(Y)} = 0 \quad (\text{II.3.3})$$

From the homogeneous Neumann condition, we obtain the following conditions on the boundary layer profile:

$$\frac{\partial \Psi_w^i}{\partial n_w} \Big|_{X = -\gamma_w(Y)} = 0, \quad \frac{\partial \Psi_e^i}{\partial n_e} \Big|_{X = \gamma_e(Y)} = 0. \quad (\text{II.3.4})$$

There is no loss of generality in assuming $\Psi_w^i|_{\Omega_\varepsilon} = 0$ and $\Psi_e^i|_{\Omega_\varepsilon} = 0$. This condition directly gives (II.3.3) and (II.3.4).

Additional conditions are needed at the interfaces separating the interior domain and the boundary layer domains to guarantee the existence of the derivatives in the weak sense over the whole domain. Since the interior terms are zero outside $\bar{\Omega}$, they create discontinuities at the interfaces Σ_w and Σ_e . Then, boundary layer terms are added to cancel such discontinuities; see for instance [54, 66]. To guarantee the approximation is regular enough, we impose the condition:

$$\left[\partial_X^k \Psi_{app}^\varepsilon \right] \Big|_{\Sigma_w \cup \Sigma_e} = 0, \quad k = 0, \dots, 3. \quad (\text{II.3.5})$$

We have the following jump conditions on the boundary layer terms:

$$\left[\Psi_w^i(\cdot) \right] \Big|_{\sigma_w} = -\Psi_{int}^i(\cdot) \Big|_{x=\chi_w(y)} - \left[\Psi_e^i(\cdot) \right] \Big|_{\sigma_w}, \quad \left[\Psi_e^i(\cdot) \right] \Big|_{\sigma_e} = -\Psi_{int}^i(\cdot) \Big|_{x=\chi_e(y)} - \left[\Psi_w^i(\cdot) \right] \Big|_{\sigma_e}, \quad (\text{II.3.6})$$

and

$$\left[\partial_X^k \Psi_w^i(\cdot) \right] \Big|_{\sigma_w} = f_w^{i,k}, \quad \left[\partial_X^k \Psi_e^i(\cdot) \right] \Big|_{\sigma_e} = f_e^{i,k}, \quad (\text{II.3.7})$$

where the $f_w^{i,k}$, $k = 1, 2, 3$, depends on the Ψ_{int}^j and Ψ_j^e , $j \leq i$. Here, $f_e^{i,k}$ is chosen to be independent of Ψ_j^w , while still relying on the behavior of the interior profiles.

Plugging (II.3.1) into (II.2.1), and equating all terms of the same order in powers of ε provide a family of mathematical systems establishing the behavior of each one of the profiles in the ansatz.

To facilitate the comprehension, we compute some terms of the approximation Ψ_{app}^ε . We are particularly interested in the ones corresponding to $i \in \{0, 1\}$.

When $i = 0$, we obtain in the interior of the domain the so-called Sverdrup relation:

$$\partial_x \Psi_{int}^0 = \text{curl } \tau, \quad (\text{II.3.8})$$

for which only one boundary condition can be prescribed, either on Σ_e or on the Σ_w .

Remark II.3.1. *In the non-rough case, the boundary layer problems are described by linear ODEs. Mainly, we have*

$$-\partial_X \Psi_e - (1 + \alpha_e^2)^2 \partial_X^4 \Psi_e = 0, \quad (\text{II.3.9})$$

and

$$\partial_X \Psi_w - (1 + \alpha_w^2)^2 \partial_X^4 \Psi_w = 0.$$

Notice that there is an asymmetry between the coasts (see [45, 17]). Indeed, only one boundary condition can be lifted on the east boundary since there is only one root with non-negative real part, whereas the space of admissible (localized) boundary corrections is of dimension two on the western boundary. Consequently, Ψ_e must vanish at first order on the East coast, leaving the solution on the boundary layer at Γ_e to correct the trace of $\partial_n \Psi_e$. This phenomenon is still present in the rough case.

Since the eastern cannot bear a large boundary layer, see Remark II.3.1, it is frequent in the literature to choose Ψ_{int}^0 tangent to the boundary Σ_e , see for example [45, 53]. Hence, we take

$$\Psi_{int}(t, x, y) = \begin{cases} - \int_x^{\chi_e(y)} \text{curl} \tau(t, x', y) dx' & \text{in } \Omega, \\ 0 & \text{in } \Omega^\varepsilon \setminus \Omega. \end{cases} \quad (\text{II.3.10})$$

and consequently, at order ε^{-4} , the eastern boundary layer profile is $\Psi_e^0 \equiv 0$. At the West, we have the following system

$$Q_w(\Psi_w^0, \Psi_w^0) + \partial_{X_w} \Psi_w^0 - \Delta_w^2 \Psi_w^0 = 0, \quad \text{in } \omega_w^+ \cup \omega_w^- \quad (\text{II.3.11a})$$

$$[\Psi_w^0] |_{\sigma_w} = - [\Psi_{int}^0] |_{\Sigma_w}, \quad (\text{II.3.11b})$$

$$[\partial_{X_w}^k \Psi_w^0] |_{\sigma_w} = 0, \quad k = 1, \dots, 3, \quad (\text{II.3.11c})$$

$$\Psi_w^0 |_{X_w = -\gamma_w(Y)} = 0, \quad \frac{\partial \Psi_w^0}{\partial n_w} |_{X_w = -\gamma_w(Y)} = 0, \quad (\text{II.3.11d})$$

$$\Psi_w^0 \longrightarrow 0 \text{ when } X_w \rightarrow +\infty, \quad (\text{II.3.11e})$$

Henceforth, the jump condition (II.3.11b) is described by a function ϕ defined as follows

$$\phi = \int_{\chi_w(y)}^{\chi_e(y)} \text{curl} \tau(t, x', y) dx', \quad (\text{II.3.12})$$

which is a direct result of (II.3.10).

It remains to prove the well-posedness of the nonlinear problem (II.3.11). Since it is quite technical, we address the matter later in Section II.6. This step concludes the computations in the main order.

Now, let us compute the next step in the asymptotic expansion. Similarly to the first interior profile, Ψ_{int}^1 follows the equation

$$\partial_x \Psi_{int}^1 = 0, \quad (\text{II.3.13})$$

hence, $\Psi_{int}^1(t, x, y) = C^1(t, y)$. The lack of source term is related to the factor ε^{-3} multiplying $\partial_x \Psi$ in (II.2.1). Accordingly, the equation driving the behavior of the interior profile becomes nonhomogeneous when $i \geq 3$.

At order ε^{-3} , the eastern boundary layer function is described by the equations

$$\begin{aligned}
 -\partial_{X_e} \Psi_e^1 - \Delta_e^2 \Psi_e^1 &= 0, \quad \text{in } \omega_e^+ \cup \omega_e^-, \\
 [\Psi_e^1] |_{\sigma_e} &= -[\Psi_{int}^1] |_{\Sigma_e}, \\
 [\partial_{X_e}^k \Psi_e^1] |_{\sigma_e} &= 0, \quad k = 1, \dots, 3, \\
 \Psi_e^1 |_{X_e = -\gamma_e(Y)} &= 0, \quad \frac{\partial \Psi_e^1}{\partial n_e} |_{X_e = -\gamma_e(Y)} = 0, \\
 \Psi_e^1 &\longrightarrow 0 \quad \text{when } X_e \rightarrow +\infty.
 \end{aligned} \tag{II.3.14}$$

The space of admissible boundary corrections at the rough eastern domain remains insufficient to satisfy simultaneously the boundary conditions and the one at infinity. Beyond imposing conditions on Ψ_{int}^1 , ergodicity assumptions will be needed to guarantee the existence of a solution Ψ_e^1 of (II.3.14). This question is the main focus of Section II.8.

In the western boundary layer domain, Ψ_w^1 satisfies the following system

$$\begin{aligned}
 Q_w(\Psi_w^0, \Psi_w^1) + Q_w(\Psi_w^1, \Psi_w^0) + \partial_{X_w} \Psi_w^1 - \Delta_w^2 \Psi_w^1 &= F^1, \quad \text{in } \omega_w^+ \cup \omega_w^-, \\
 [\Psi_w^1] |_{\sigma_w} &= -[\Psi_{int}^1] |_{\Sigma_w} - [\Psi_e^1] |_{\sigma_w}, \\
 [\partial_{X_w} \Psi_w^1] |_{\sigma_w} &= -[\partial_x \Psi_{int}^1] |_{\Sigma_w}, \\
 [\partial_{X_w}^k \Psi_w^1] |_{\sigma_w} &= 0, \quad k = 2, 3, \\
 \Psi_w^1 |_{X_w = -\gamma_w(Y)} &= 0, \quad \frac{\partial \Psi_w^1}{\partial n_w} |_{X_w = -\gamma_w(Y)} = 0, \\
 \Psi_w^1 &\longrightarrow 0 \quad \text{when } X_w \rightarrow +\infty,
 \end{aligned} \tag{II.3.15}$$

where $F^1 = -(\nabla \Psi_{int}^0 \cdot \nabla_w) \Delta \Psi_w^0$. The existence of a solution of problem (II.3.15) can be shown by following the same reasoning used for Ψ_w^0 (see Section II.7).

In the next section, we provide a formal method of proof of well-posedness for the problems previously mentioned with its core ideas and some general computations.

II.4 Existence and uniqueness of the solution of an elliptical problem in a rough domain: methodology

In hopes of facilitating the comprehension of this work, we describe a general method to prove the existence and uniqueness of the solution of a general problem encompassing all the possible behaviors within the boundary layer; in particular, the nonlinear and linearized western boundary layer systems and the linear eastern boundary layer equations. For $\alpha \in \mathbb{R}$, we start with elliptic differential systems of the form

$$\mathcal{L}_\alpha(\Psi) + \mathcal{Q}_\alpha(\Psi, \tilde{\Psi}) = F, \tag{II.4.1}$$

where F is a regular enough source term with sufficient decay at infinity, \mathcal{L}_α is a fourth order elliptic linear differential operator and \mathcal{Q}_α is the nonlinear/quasilinear part of the equation. Let us consider for a fixed $\alpha \in \mathbb{R}$

$$\begin{aligned}
 \mathcal{L}_\alpha(\Psi) &= \pm \partial_X \Psi - \Delta_\alpha^2 \Psi \\
 \mathcal{Q}_\alpha(\tilde{\Psi}, \Psi) &= \frac{j}{2} \left((\nabla_\alpha^\perp \tilde{\Psi} \cdot \nabla_\alpha) \Delta_\alpha \Psi + (\nabla_\alpha^\perp \Psi \cdot \nabla_\alpha) \Delta_\alpha \tilde{\Psi} \right)
 \end{aligned}$$

for $j \in \{0, 1, 2\}$, and

$$\begin{aligned}\nabla_\alpha &= (\pm\partial_X, \partial_Y \mp \alpha\partial_X), & \nabla_\alpha^\perp &= (-\partial_Y \pm \alpha\partial_X, \pm\partial_X) \\ \Delta_\alpha &= \nabla_\alpha \cdot \nabla_\alpha = \partial_X^2 + (\partial_Y \mp \alpha\partial_X)^2, & \Delta_\alpha^2 &= \Delta_\alpha \Delta_\alpha = \left(\partial_X^2 + (\partial_Y \mp \alpha\partial_X)^2\right)^2.\end{aligned}$$

In the definition of \mathcal{L}_α , the factor multiplying ∂_X is linked to the definition of the local variables provided in Section II.3: positive for the western boundary layer and negative in the eastern boundary layer domain. Moreover, α corresponds to the derivative of the function describing the interface between the interior and rough domain (namely $\chi'(y)$), and, therefore, different on each side.

Let us suppose that equation (II.4.1) holds in a domain $\omega = \omega^+ \cup \sigma \cup \omega^-$, where

$$\begin{aligned}\omega^+ &= \{X > 0, \quad Y \in \mathbb{R}\}, & \sigma &= \{X = 0, \quad Y \in \mathbb{R}\} \\ \omega^- &= \{-\gamma(Y) < X < 0, \quad Y \in \mathbb{R}\},\end{aligned}$$

and γ is a positive Lipschitz function such that $\inf \gamma > 0$. Problem (II.4.1) is supplemented with the following jump and boundary conditions:

$$\begin{aligned}\left[\partial_X^k \Psi\right] \Big|_\sigma &= g_k, \quad k = 0, \dots, 3, \\ \Psi \Big|_{X=-\gamma(Y)} &= \frac{\partial \Psi}{\partial n} \Big|_{X=-\gamma(Y)} = 0.\end{aligned}\tag{II.4.2}$$

Here, n denotes the unit outward normal vector of γ and g_k are smooth functions.

Let us point out some difficulties related to the proof of existence and uniqueness of the solution of problem (II.4.1)-(II.4.2). First, we consider a domain ω that is not bounded in the tangential direction. Moreover, functions g_k do not decay as Y goes to infinity, so that standard energy estimates are inefficient. As a consequence, only locally integrable functions are considered, which leads to a completely different treatment of the energy estimates.

If the problem was set in ω^- , with Dirichlet boundary conditions at $\{X = 0\}$, one could build a solution Ψ adapting ideas of Ladyženskaya and Solonnikov for the case of Navier-Stokes flows in tubes [60]. The existence of the solution in [60] is proven using an a priori differential inequality on local energies. Unfortunately, this method relies heavily on the bounded direction hypotheses to make possible the application of the Poincaré inequality. Hence, this reasoning is not applicable in our setting.

Moreover, contrary to what happens for the Laplace equation, one cannot rely on maximum principles to get an L^∞ bound since we are dealing with a fourth-degree operator.

This problem has been overcome in the literature for the Stokes boundary layer flow in [59] and, recently, for highly rotating fluids in [61]. The main idea is to impose a so-called transparent boundary condition when the variable in the normal direction is equal to a certain value $M > 0$, see Figure II.2.

This transparent condition separates the original domain in two: a half-plane $\{X > M\}$ and a bumped region bounded in the tangential direction. The Dirichlet problem on the half-space $\{X > M\}$ is solved by means of Fourier analysis and pseudo-differential tools in Kato spaces. The problem on the bumped sub-domain is not suitable for a similar treatment due to the nature of boundary and the fact that Kato spaces are defined through truncations in space¹. Nevertheless, it is now suitable for the application of the Ladyženskaya and

¹Similar difficulties arise in [67] when studying water waves equations in locally uniform spaces.

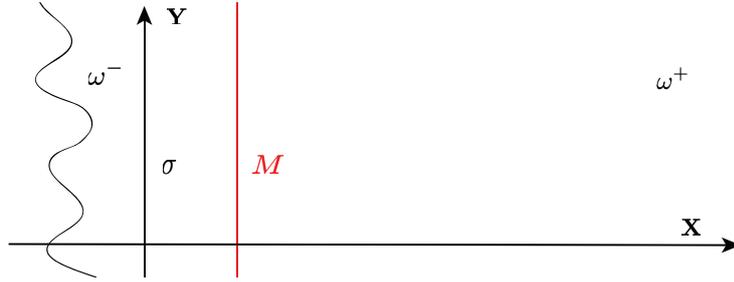


Figure II.2: Boundary layer domain ω with an artificial boundary at $X = M$.

Solonnikov method [60]. The remaining step consists of connecting both solutions on the artificial boundary.

II.4.1 Linear case.

If $j = 0$, the main steps of the proof are as follows:

- (L1) Prove existence and uniqueness of a solution of the linear system in a half-space with boundary data in $H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$.

$$\begin{cases} \mathcal{L}_\alpha(\Psi) = F, & \text{in } X > M, \\ \Psi|_{\sigma^M} = \psi_0, & \partial_X \Psi|_{\sigma^M} = \psi_1, \end{cases} \quad (\text{II.4.3})$$

where $\sigma^M = \{X = M, Y \in \mathbb{R}\}$. The solution is constructed by means of an integral representation using Fourier analysis. Indeed, we take the Fourier transform with respect to the tangential variable Y and do a thorough analysis of the characteristic equation of the resulting problem. If $F \neq 0$, we compute the fundamental solution using the Green function.

- (L2) Extend this well-posedness result to boundary data in $(\psi_0, \psi_1) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$ using ideas in [61]. A priori estimates on a solution of (II.4.3) are established in this scenario.
- (L3) Define the Poincaré-Steklov type operator for functions in $H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$ using the information recovered from the problem in the half-space $\{X > M\}$ and extend the result to the case when the boundary data belongs to a space of uniformly locally integrable functions. The Poincaré-Steklov operator associated to $\mathcal{L}_\alpha(\Psi)$ is a positive non-local boundary differential operator of the form

$$\begin{aligned} PS_\alpha : H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R}) &\rightarrow H_{\text{uloc}}^{-3/2}(\mathbb{R}) \times H_{\text{uloc}}^{-1/2}(\mathbb{R}) & (\text{II.4.4}) \\ \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} &\mapsto \begin{pmatrix} (1 + \alpha^2)\Delta_\alpha \Psi|_{\sigma^M} \\ -[(1 + \alpha^2)\partial_X \mp 2\alpha\partial_Y] \Delta_\alpha \Psi \pm \frac{\Psi}{2}|_{\sigma^M} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_2^\alpha(\psi_0, \psi_1, F) \\ \mathcal{A}_3^\alpha(\psi_0, \psi_1, F) \end{pmatrix} \end{aligned}$$

where the form of the differential operators \mathcal{A}_i^α , $i = 2, 3$, depend greatly on the solution determined in (L1) and is, therefore, particular to each case.

- (L4) Define an equivalent problem in a domain with transparent boundary condition $\omega^M = \omega \cup \{X = M\}$ and then, solve the problem

$$\begin{aligned}
\mathcal{L}_\alpha(\Psi^-) &= F, \quad \text{in } \omega^b \setminus \sigma^M, \\
\left[\partial_X^k \Psi^- \right] \Big|_\sigma &= g_k, \quad k = 0, \dots, 3, \\
(1 + \alpha^2) \Delta_\alpha \Psi^- \Big|_{\sigma^M} &= \mathcal{A}_2^\alpha \left(\Psi^- \Big|_{\sigma^M}, \partial_X \Psi^- \Big|_{\sigma^M}, F \right), \\
- \left[(1 + \alpha^2) \partial_X \mp 2\alpha \partial_Y \right] \Delta_\alpha \Psi^- \pm \frac{\Psi^-}{2} \Big|_{\sigma^M} &= \mathcal{A}_3^\alpha \left(\Psi^- \Big|_{\sigma^M}, \partial_X \Psi^- \Big|_{\sigma^M}, F \right), \\
\Psi^- \Big|_{X=-\gamma(Y)} &= \frac{\partial \Psi^-}{\partial n} \Big|_{X=-\gamma(Y)} = 0,
\end{aligned} \tag{II.4.5}$$

where ω^b refers to the rough ‘‘tubular’’ domain given by $\omega^b = \omega^M \setminus (\{X > M\} \times \mathbb{R})$ and \mathcal{A}_i^α , $i = 2, 3$ are the ones in (II.4.4). Note that for $M = 0$, $\omega^b = \omega^-$.

Proposition II.4.1. *Let $\gamma \in W^{2,\infty}(\mathbb{R})$ and $g_k \in L^\infty(\mathbb{R})$, for $k = 1, \dots, 3$. Assume that there exists $\delta_0 > 0$ such that $\|g_k\|_\infty < \delta_0$, for all $k = 0, \dots, 3$.*

- Let Ψ be a solution of (II.4.1)-(II.4.2) in ω such that $\Psi \in H_{\text{uloc}}^2(\omega)$. Then $\Psi|_{\omega^M}$ is a solution of (II.4.5), and for $X > M$, Ψ solves (II.4.3), with $\psi_0 := \Psi|_{X=M} \in H_{\text{uloc}}^{3/2}(\mathbb{R})$ and $\psi_1 := \partial_X \Psi|_{X=M} \in H_{\text{uloc}}^{1/2}(\mathbb{R})$.
- Conversely, let $\Psi^- \in H_{\text{uloc}}^2(\omega^M)$ be a solution of (II.4.5). Consider the solution $\Psi^+ \in H_{\text{uloc}}^2(\mathbb{R}_+^2)$ of (II.4.3). Setting

$$\Psi(X, \cdot) := \begin{cases} \Psi^-(X, \cdot) & \text{for } -\gamma(\cdot) < X < M, \\ \Psi^+(X, \cdot) & \text{for } X > M, \end{cases}$$

the function $\Psi \in H_{\text{loc}}^2(\omega)$ is a solution of the problem (II.4.1)-(II.4.2).

- (L5) Consequently, we focus our attention on the existence and uniqueness of solutions of the equivalent problem (II.4.5). To simplify the presentation, we replace in this paragraph the functions $\mathcal{A}_i^\alpha \left(\Psi^- \Big|_{\sigma^M}, -\partial_X \Psi^- \Big|_{\sigma^M}, F \right)$ by $\rho_i \in H_{\text{uloc}}^{3/2-i}(\mathbb{R})$, $i = 2, 3$. In fact, the Poincaré-Steklov operator is not local which hinders the application of the ideas in Ladyženskaya and Solonnikov [60], as seen in [59] and [52]. We will address in this difficulty in Section II.5.3. Showing the operators \mathcal{A}_i^α are well-defined depends on the Fourier representation of the solutions in the half-space, and consequently, on the definition of ∇_α and the domain. We leave the detailed discussion of each case for later. System (II.4.5) becomes

$$\begin{aligned}
\pm \partial_X \Psi^- - \Delta_\alpha^2 \Psi^- &= F, \quad \text{in } \omega^b \setminus \sigma^M, \\
\left[\partial_X^k \Psi^- \right] \Big|_\sigma &= g_k, \quad k = 0, \dots, 3, \\
(1 + \alpha^2) \Delta_\alpha \Psi^- \Big|_{\sigma^M} &= \rho_2, \\
- \left[(1 + \alpha^2) \partial_X \mp 2\alpha \partial_Y \right] \Delta_\alpha \Psi^- \mp \frac{\Psi^-}{2} \Big|_{\sigma^M} &= \rho_3, \\
\Psi^- \Big|_{X=-\gamma(Y)} &= \frac{\partial \Psi^-}{\partial n} \Big|_{X=-\gamma(Y)} = 0.
\end{aligned} \tag{II.4.6}$$

To facilitate the computations, we lift the conditions at the interface $\{X = 0\}$ by introducing the function

$$\Psi^L(X, Y) := \chi(X) \sum_{k=0}^3 g_k(Y) \frac{X^k}{k!}, \quad (\text{II.4.7})$$

where $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi \equiv 1$ near σ , and $\text{Supp}\chi \subset [0, \frac{M}{2}]$. Thus, $\Psi^L \equiv 0$ in ω^- and in ω^+ close to $X = M$. Additionally, it satisfies the jump conditions

$$\left[\partial_X^k \Psi^L \right] \Big|_\sigma = g_k, \quad k = 0, \dots, 3.$$

For $\tilde{\Psi} = \Psi^- - \Psi^L$, we have

$$\begin{aligned} \pm \partial_X \tilde{\Psi} - \Delta_\alpha^2 \tilde{\Psi} &= F^L, \quad \text{in } \omega^b \setminus \sigma^M, \\ (1 + \alpha^2) \Delta_\alpha \tilde{\Psi} \Big|_{\sigma^M} &= \rho_2, \\ - \left[(1 + \alpha^2) \partial_X \mp 2\alpha \partial_Y \right] \Delta_\alpha \tilde{\Psi} \mp \frac{\tilde{\Psi}}{2} \Big|_{\sigma^M} &= \rho_3, \\ \tilde{\Psi} \Big|_{X=-\gamma(Y)} &= \frac{\partial \tilde{\Psi}}{\partial n} \Big|_{X=-\gamma(Y)} = 0, \end{aligned} \quad (\text{II.4.8})$$

where the source term F^L depends also on g_k , $k = 0, \dots, 3$.

Since a priori estimates are needed, it is useful to write the weak formulation of (II.4.8).

Definition II.4.1. *Let \mathcal{V} be the space of functions $\varphi \in H^2(\overline{\omega^b})$ such that $\tilde{\Psi}|_\Gamma = \partial_n \tilde{\Psi}|_\Gamma = 0$ and $\text{Supp}\varphi$ is bounded. A function $\tilde{\Psi} \in H_{\text{loc}}^2(\omega^b)$ is a solution of (II.4.8) if it satisfies the homogeneous conditions at the rough boundary, and if, for all $\varphi \in \mathcal{V}$,*

$$\begin{aligned} \mp \int_{\omega^b} \partial_X \tilde{\Psi} \varphi - \int_{\omega^b} \Delta_\alpha \tilde{\Psi} \Delta_\alpha \varphi & \\ = \int_{\omega^b} F^L \varphi - \int_{\mathbb{R}} \left(\rho_3 \pm \frac{\tilde{\Psi}}{2} \right) \varphi \Big|_{X=M} dY - \int_{\mathbb{R}} \rho_2 \partial_X \varphi \Big|_{X=M} dY. & \end{aligned} \quad (\text{II.4.9})$$

Throughout this step, we will frequently be using the following technical lemma:

Lemma II.4.1. *Let U be a regular open set bounded at least in one direction. Then, for $f \in H^2(U)$ there exists a constant $C > 0$ such that*

$$\|f\|_{H^2(U)} \leq C \left(\|f\|_{L^2(U)} + \|\Delta_\alpha f\|_{L^2(U)} \right). \quad (\text{II.4.10})$$

If the function satisfies additionally that $f = \partial_n f = 0$ on some part of the boundary ∂U , the first term on the right-hand side of (II.4.10) is not longer needed for the inequality to hold.

We refer to Appendix II.E for a proof. Note that a direct result from Lemma II.4.1 is that controlling the L^2 -norm of $\Delta_\alpha f$ immediately provides $f \in H^2(U)$. This property is easily generalized to Kato spaces.

– *Energy estimates for (II.4.8).* We introduce, for all $n \in \mathbb{N}$, $k \in \mathbb{N}$

$$\begin{aligned}\omega_n &:= \omega^b \cap \{(X, Y) \in \mathbb{R}^2 : |Y| < n\}, & \omega_{k+1, k} &= \omega_{k+1} \setminus \omega_k, \\ \sigma_n^M &:= \{X = M, \text{ and } |Y| < n, Y \in \mathbb{R}\}, & \sigma_{k+1, k}^M &= \sigma_{k+1}^M \setminus \sigma_k^M, \\ \Gamma_n &= \{X = -\gamma(Y), |Y| < n\}.\end{aligned}\tag{II.4.11}$$

We consider the system (II.4.8) in ω_n

$$\begin{aligned}\pm \partial_X \tilde{\Psi}_n - \Delta_\alpha^2 \tilde{\Psi}_n &= F^L \\ (1 + \alpha^2) \Delta_\alpha \tilde{\Psi}_n \Big|_{\sigma^M} &= \rho_2 \\ - \left[(1 + \alpha^2) \partial_X \mp 2 \partial_Y \right] \Delta_\alpha \tilde{\Psi}_n \mp \frac{\tilde{\Psi}_n}{2} \Big|_{\sigma^M} &= \rho_3 \\ \tilde{\Psi}_n \Big|_{\Gamma_n} &= \partial_n \tilde{\Psi} \Big|_{\Gamma_n} = 0.\end{aligned}$$

In order to prove the existence of the solution of (II.4.5), we derive H_{uloc}^2 estimates on Ψ_n , uniform with respect to n . Then, passing to the limit when $n \rightarrow +\infty$, we achieve our goal. Indeed, taking $\tilde{\Psi}_n$ as a test function in (II.4.9), we obtain

$$\begin{aligned}\|\Delta_\alpha \tilde{\Psi}_n\|_{L^2(\omega_n)}^2 &= - \int_{\omega_n} F^L \tilde{\Psi}_n + \int_{\sigma_n^M} \rho_3 \tilde{\Psi}_n + \int_{\sigma_n^M} \rho_2 \partial_X \tilde{\Psi}_n \\ &\leq C_1 \sqrt{n} \left(\|F\|_{L^2_{\text{uloc}}(\omega^b)} + \sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})} \right) \|\tilde{\Psi}\|_{H^2(\omega_n)} \\ &\quad + C_2 \sqrt{n} \left(\|\rho_3\|_{L^2_{\text{uloc}}(\mathbb{R})} \|\tilde{\Psi}_n\|_{\sigma^M} \|L^2([-n, n])\right. \\ &\quad \left. + \|\rho_2\|_{L^2(\omega_n)} \|\partial_X \tilde{\Psi}_n\|_{\sigma^M} \|L^2([-n, n])\right),\end{aligned}\tag{II.4.12}$$

using the Cauchy-Schwarz and Poincaré inequalities over ω_n . Thus,

$$\begin{aligned}\|\Delta_\alpha \tilde{\Psi}_n\|_{L^2(\omega_n)}^2 &\leq \left(\delta + \|F\|_{L^2_{\text{uloc}}(\omega^b)}^2 \right) \left(\|\tilde{\Psi}_n\|_{X=M}^2 \|H^1([-n, n])\right. \\ &\quad \left. + \|\Psi\|_{L^2(\omega_n)}^2 \right) \\ &\quad + C_\delta n \left(\sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})}^2 + \|\rho_2\|_{L^2_{\text{uloc}}(\mathbb{R})}^2 + \|\rho_3\|_{L^2_{\text{uloc}}(\mathbb{R})}^2 \right)\end{aligned}\tag{II.4.14}$$

Notice that the first term on the r.h.s of the previous inequality can be absorbed by the one on the l.h.s for δ and F small enough. Then, using Poincaré inequality over the whole channel yields

$$E_n := \int_{\omega^b} |\Delta_\alpha \tilde{\Psi}_n|^2 \leq \int_{\omega_n} |\Delta_\alpha \tilde{\Psi}_n|^2 \leq C_0(g_0, \dots, g_3, \rho_2, \rho_3)n.\tag{II.4.15}$$

where constant C_0 depends on α and the size of the jumps and the values of the differential operators at the artificial boundary, as seen in (II.4.14). The existence of $\tilde{\Psi}_n$ in $H^2(\omega_n)$ follows.

Therefore, we resort to performing energy estimates on the system (II.4.8), following the strategy of Gérard-Varet and Masmoudi [59]. The idea is to use the quantity

$$E_k^n := \int_{\omega_k} |\Delta_\alpha \tilde{\Psi}_n|^2,$$

to derive an induction inequality on $(E_k^n)_{k \in \mathbb{N}}$, for all $n \in \mathbb{N}$. Hence, we consider $\varphi = \chi_k \Psi$, where $\chi_k \in C_0^\infty(\mathbb{R})$ is a cut-off function in the tangential variable such that $\text{Supp} \chi_k \subset [-k-1, k+1]$ and $\chi_n \equiv 1$ on $[-k, k]$ for $k \in \mathbb{N}$. Since the problem is defined in a two-dimensional domain, the support of $\nabla^j \chi_k$, $j = 1, \dots, 4$, is included in the reunion of two intervals of size 1.

Let us explain the overall strategy. We shall first derive the following inequality for all $k \in \{1, \dots, n\}$

$$E_k^n \leq C_1 \left((E_{k+1}^n - E_k^n) + \left(\sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})}^2 + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}(\mathbb{R})}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}(\mathbb{R})}^2 \right) (k+1) \right). \quad (\text{II.4.16})$$

Here, C_1 is a constant depending only on the characteristics of the domain.

Then, by backward induction on k , we deduce that

$$E_k^n \leq Ck, \quad \forall k \in \{k_0, \dots, n\},$$

where $k_0 \in \mathbb{N}$ is a large, but fixed integer (independent of n) and $E_{k_0}^n$ is bounded uniformly in n for a constant C depending on ω^b , g_k , $k = 0, \dots, 3$ and F . This provides the uniform boundness for a maximal energy of size k_0 . Since the derivation of energy estimates is invariant by translation on the tangential variable, we claim that

$$\sup_{a \in \mathcal{I}_{k_0}} \int_{\{(-1, M) \times a\} \cap \omega^b} |\Delta_\alpha \tilde{\Psi}_n|^2 \leq C. \quad (\text{II.4.17})$$

The set \mathcal{I}_{k_0} contains all the intervals of length $2k_0$ in $[-n, n]$ with extremities in \mathbb{Z} . Consequently, the uniform H_{uloc}^2 bound on Ψ^n is proved and an exact solution can be found by compactness. Indeed, by a diagonal argument, we can extract a subsequence $(\tilde{\Psi}_{r(n)})_{n \in \mathbb{N}}$ such that

$$\begin{aligned} \tilde{\Psi}_{r(n)} &\rightharpoonup \Psi && \text{weakly in } H^2(\omega_k), \\ \tilde{\Psi}_{r(n)} \Big|_{\sigma^M} &\rightharpoonup \Psi \Big|_{\sigma^M} && \text{weakly in } H^{3/2}(\sigma_k^M), \\ \partial_X \tilde{\Psi}_{r(n)} \Big|_{\sigma^M} &\rightharpoonup \partial_X \Psi \Big|_{\sigma^M} && \text{weakly in } H^{1/2}(\sigma_k^M), \end{aligned}$$

for all $k \in \mathbb{N}$. Of course, $\tilde{\Psi}$ is a solution of (II.4.8), and, consequently, $\Psi^- \in H_{\text{uloc}}^2(\omega^b)$ is solution of system (II.4.5).

To lighten notations in the subsequent proof, we shall denote E_k instead of E_k^n .

– *Deriving the inequality.* This part contains the proof of (II.4.16). Taking $\chi_k \tilde{\Psi}$ as

test function in (II.4.9) provides the following expression for the l.h.s.

$$\begin{aligned} \mp \int_{\omega^b} \partial_X \tilde{\Psi} \chi_k \tilde{\Psi} - \int_{\omega^b} \Delta_\alpha \tilde{\Psi} \Delta_\alpha (\chi_k \tilde{\Psi}) &= \pm \frac{1}{2} \int_{\mathbb{R}} \chi_k \Psi^2|_{X=M} - E_k \\ &\quad - 2 \int_{\omega^b} \Delta_\alpha \tilde{\Psi} \nabla_\alpha \chi_k \cdot \nabla_\alpha \tilde{\Psi} - \int_{\omega^b} \tilde{\Psi} \Delta_\alpha \tilde{\Psi} \partial_Y^2 \chi_k \\ &\quad - \int_{\omega_{k+1} \setminus \omega_k} \chi_k |\Delta_\alpha \tilde{\Psi}|^2. \end{aligned} \tag{II.4.18}$$

For the third term, we simply use the Cauchy–Schwarz and Poincaré inequalities:

$$\left| \int_{\omega^b} \Delta_\alpha \tilde{\Psi} \nabla_\alpha \chi_k \cdot \nabla_\alpha \tilde{\Psi} \right| \leq C \left(\int_{\omega_{k+1,k}} |\Delta_\alpha \tilde{\Psi}|^2 \right)^{1/2} \left(\int_{\omega_{k+1,k}} |\nabla_\alpha \tilde{\Psi}|^2 \right)^{1/2} \leq C(E_{k+1} - E_k). \tag{II.4.19}$$

In the same fashion, we find that $\int_{\omega^b} \tilde{\Psi} \Delta_\alpha \tilde{\Psi} \partial_Y^2 \chi_k$ and $\int_{\omega_{k+1} \setminus \omega_k} \chi_k |\Delta_\alpha \tilde{\Psi}|^2$ are also bounded by $C(E_{k+1} - E_k)$.

Gathering all boundary terms stemming from the biharmonic operator and the first term in the r.h.s. of (II.4.18) yields

$$- \int_{\mathbb{R}} \chi_k \left(\rho_3 \tilde{\Psi}|_{X=M} + \rho_2 \partial_X \tilde{\Psi}|_{X=M} \right).$$

The term above is bounded by

$$C \left(\|\rho_2\|_{L^2_{\text{uloc}}}^2 + \|\rho_3\|_{L^2_{\text{uloc}}}^2 \right) (E_{k+1} + (k+1))$$

where C depends only on M , α and on $\|\gamma\|_{W^{2,\infty}}$. The computation of this bound relies on the trace theorem and Young's inequality.

We are left with

$$\begin{aligned} \left| \int_{\omega^b} \chi_k F^L \tilde{\Psi} \right| &\leq C \left(\sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})} \right) E_{k+1}^{1/2} \sqrt{k+1} + \|F\|_{L^2_{\text{uloc}(\omega^b)}} E_{k+1}^{1/2}, \\ &\leq C_\nu \left(\sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})}^2 \right) (k+1) + (\nu + \|F\|_{L^2_{\text{uloc}(\omega^b)}}^2) E_{k+1}. \end{aligned}$$

Lastly, combining all the estimates and taking ν and $\|F\|_{L^2_{\text{uloc}(\omega^b)}}^2$ small enough give

$$E_k \leq C_1 ((E_{k+1} - E_k) + C_2(k+1)),$$

where C_1 is a constant independent of k and

$$C_2 := C_2(g_0, \dots, g_3, \rho_2, \rho_3) = \left(\sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})}^2 + \|\rho_2\|_{H^{m-1/2}_{\text{uloc}}}^2 + \|\rho_3\|_{H^{m-3/2}_{\text{uloc}}}^2 \right). \tag{II.4.20}$$

– *Induction.* Our goal is to show from (II.4.16) that there exists $k_0 \in \mathbb{N} \setminus \{0\}$, $C > 0$ such that, for all $n \in \mathbb{N}$

$$\int_{\omega_{k_0}} |\Delta_\alpha \tilde{\Psi}_n|^2 \leq C. \quad (\text{II.4.21})$$

From (II.4.16), we claim that induction on $n - k$ indicates there exists a positive constant C_3 depending only on C_0 , C_1 and C_2 appearing respectively in (II.4.15) and (II.4.21), such that, for all $k > k_0$,

$$E_k \leq C_3 C_2 (k + 1). \quad (\text{II.4.22})$$

Let us insist on the fact that C_3 is independent of n , k and will be adjusted in the course of the induction argument.

First, notice that thanks to (II.4.15), (II.4.22) is true for $k = n$ once $C_3 > C_0 C_2^{-1}$, recalling that $\tilde{\Psi}_n = 0$ on $\omega^b \setminus \omega_n$. We then assume that (II.4.22) holds for $n, n - 1, \dots, k + 1$, where k is a positive integer.

We prove (II.4.22) at rank k by contradiction. Assume that (II.4.22) does not hold at the rank k . Then, the induction implies

$$E_{k+1} - E_k < C_3 C_2.$$

Since $C_0, C_1 > 0$ are fixed and depend on α and $\|\gamma\|_{W^{2,\infty}}$ (see (II.4.15) for the definition of C_0), substituting the above inequality in (II.4.16) yields

$$C_3 C_2 (k + 1) < E_k \leq C_1 C_2 C_3 + C_1 C_2 (k + 1). \quad (\text{II.4.23})$$

Taking $C_3 \geq 2C_1$ and plugging it in (II.4.23) results in a contradiction for $k > k_0$, where $k_0 = 2C_1 + 1$. Therefore, (II.4.22) is true at the rank $k > k_0$. Moreover, since E_k is an increasing functional with respect to the value of k , we obtain that E_k is also bounded for $k \leq k_0$. It follows from (II.4.22), choosing $k = 2$, that there exists a constant $C > 0$, depending only on C_0, C_1, C_2, C_3 , and therefore only on $\alpha, \|\gamma\|_{L^\infty(\mathbb{R})}$ and on the norms on $g_k, k = 0, \dots, 3$ and $\rho_i, i = 2, 3$, such that,

$$E_{k_0} \leq E_{k_0+1} \leq C(k_0 + 1). \quad (\text{II.4.24})$$

Let us now consider the set \mathcal{I}_{k_0} of all segments contained in $\{(M, Y) : |Y| \leq n\}$ of length $2k_0$. As \mathcal{I}_{k_0} is finite, there exists an interval a in \mathcal{I}_{k_0} which maximizes

$$\left\{ \|\tilde{\Psi}_n\|_{H^2(\omega_a)} : a \in \mathcal{I}_{k_0} \right\},$$

where $\omega_a = \{x \in \omega^b : Y \in a\}$. We then shift $\tilde{\Psi}_n$ in such a manner that a is centered at 0. We call $\bar{\Psi}_n$ the shifted function. It is still compactly supported, but in ω_{2n} instead of ω_n :

$$\int_{\omega_{2n}} |\Delta_\alpha \bar{\Psi}_n|^2 = \int_{\omega_n} |\Delta_\alpha \tilde{\Psi}_n|^2 \quad \text{and} \quad \int_{\omega_{k_0}} |\Delta_\alpha \bar{\Psi}_n|^2 = \int_{\omega_a} |\Delta_\alpha \tilde{\Psi}_n|^2.$$

Analogously to E_k , we define \bar{E}_k . The arguments leading to the derivation of energy estimates are invariant by horizontal translation, and all constants depend

only on the parameter α and the norms on $g_k, \rho_i, i = 2, 3, F$ and γ , so (II.4.24) still holds when E_k is replaced by \bar{E}_k . On the other hand, \bar{E}_{k_0} maximizes $\|\tilde{\Psi}_n\|_{H^2(\omega_a)}^2$ on the set of intervals of length $2k_0$. This estimate being uniform, we can take k_0 large enough and obtain

$$\sup_{a \in I_{k_0}} \|\tilde{\Psi}_n\|_{H^2((0,-1) \times a) \cap \omega^b} < \infty,$$

which means that $\tilde{\Psi}_n$ is uniformly bounded in $H^2(\omega^b)$.

- *Uniqueness.* To establish uniqueness, we consider $\Psi = \Psi_1 - \Psi_2$, where $\Psi_i, i = 1, 2$, are solutions of the original problem. The goal is to show that the solution $\tilde{\Psi}$ of the following problem is identically zero.

$$\begin{aligned} \pm \partial_X \Psi - \Delta_\alpha^2 \Psi &= 0 \quad \text{in } \omega^b, \\ \Psi|_{X=-\gamma(Y)} = \partial_n \Psi|_{X=-\gamma(Y)} &= 0, \end{aligned} \tag{II.4.25}$$

We proceed similarly as in the ‘‘existence part’’ by multiplying the equation in (II.4.25) by $\Psi_k = \chi_k \Psi$ and integrating over ω^b . The resulting induction relation is

$$E_k \leq C(E_{k+1} - E_k).$$

Since $E_{k+1} - E_k$ is uniformly bounded in k , we obtain $E_k \leq C$ uniformly in k , meaning that the difference between two solutions belongs to H^2 . Hence, we can multiply the equation on Ψ by Ψ itself and integrate by parts, disregarding χ_k . This leads to

$$(1 - C\delta_0) \int_{\omega^b} |\Delta_\alpha \Psi|^2 \leq 0,$$

which provides the uniqueness result when $\delta_0 < C^{-1}$.

The values of ρ_2 and ρ_3 are later replaced by the corresponding non-local operators.

II.4.2 Nonlinear/ linearized problem.

If $j \in \{1, 2\}$, we proceed as follows:

(NL1) The well-posedness of the system on the half-space

$$\begin{cases} \mathcal{L}_\alpha(\Psi) + \mathcal{Q}_\alpha(\bar{\Psi}, \Psi) = F, & \text{in } X > M, \\ \Psi|_{\sigma_M} = \psi_0, \quad \partial_X \Psi|_{\sigma_M} = \psi_1, \end{cases} \tag{II.4.26}$$

for small enough but non-decaying boundary data ψ_0 and ψ_1 and source term F is shown by combining estimates of the linear problem for a certain source function \tilde{F} (steps (L1) and (L2)) with a fixed point argument. The problem is clearly nonlinear when $\bar{\Psi} = \Psi$. This corresponds to the case when $j = 1$ and the solution is obtained under a smallness assumption by applying a fixed point theorem in a space of exponentially decaying functions.

The linearized problem ($j = 2$), the solution Ψ is sought in a similar manner, with the particularity of only assuming $\bar{\Psi}$ is small enough.

(NL2) For any $(\rho_2, \rho_3) \in H_{\text{uloc}}^{-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{-3/2}(\mathbb{R})$ small enough, we introduce the function Ψ^- satisfying the following problem in the rough domain $\omega^b = \omega \setminus (\{X > M\} \times \mathbb{R})$

$$\begin{cases} \mathcal{L}_\alpha(\Psi^-) + \mathcal{Q}_\alpha(\bar{\Psi}^-, \Psi^-) = F, & \text{in } \omega^b \setminus \sigma^M \\ \left[\partial_X^k \Psi^- \right] |_\sigma = g_k, \quad k = 0, \dots, 3, \\ \Psi^- |_{X=-\gamma(Y)} = \frac{\partial \Psi^-}{\partial n} |_{X=-\gamma(Y)} = 0, \\ \mathcal{A}_i^- \left(\Psi^- |_{\sigma^M}, \partial_X \Psi^- |_{\sigma^M} \right) = \rho_i, \quad i = 2, 3. \end{cases} \quad (\text{II.4.27})$$

Here, \mathcal{A}_2 and \mathcal{A}_3 are the second and third-degree components of the Poincaré-Steklov operator, defined at the transparent boundary in the rough channel. The nonlinear/quasilinear nature of \mathcal{A}_3 depends on the choice of the function $\bar{\Psi}$ since it contains the boundary terms stemming from \mathcal{Q}_α .

The proof of existence and uniqueness of the solution of (II.4.27) follows the same ideas of (L5). The goal is to obtain uniform estimates on the quantity E_n^k by means of backward induction and then apply it to a translated channel to get a uniform local bound. The first obvious difference resides naturally in the induction relation. Here, the inequality is

$$E_k^n \leq \tilde{C}_1 (E_{k+1}^n - E_k^n)^{3/2} + C_1 (E_{k+1}^n - E_k^n) + C_2 (k+1), \quad \forall k \in \{1, \dots, n\}, \quad (\text{II.4.28})$$

where \tilde{C}_1 and C_1 are constants depending only on the domain, while C_2 is determined by the norms of g_k , $k = 0, \dots, 3$ and ρ_i , $i = 2, 3$. Relation (II.4.28) is obtained using a truncation over ω_k and energy estimates. The smallness assumption on the boundary data (resp. on $\bar{\Psi}$) is essential in the nonlinear (resp. linearized) case since it guarantees for the terms derived from \mathcal{Q}_α to be absorbed by the truncated energy on the r.h.s. In particular, for the linearized case, we have that $\tilde{C}_1 = 0$ in (II.4.28).

(NL3) Then, we will introduce the solution Ψ^+ of (II.4.26) with $\psi_0 = \Psi^- |_{X=M}$ and $\psi_1 = \partial_X \Psi^- |_{X=M}$ and connect the solutions Ψ^- and Ψ^+ at the transparent boundary. The strategy is to apply the implicit function theorem to a certain map

$$\mathcal{F} := \mathcal{F}(g_0, \dots, g_3, \rho_2, \rho_3),$$

to find a solution of $\mathcal{F} = 0$ in a neighborhood of zero. To do so, we first prove that \mathcal{F} is a \mathcal{C}^1 mapping in a neighborhood of the transparent boundary, which means, in turn, that higher regularity of the solution is needed.

(NL4) Once the regularity estimates have been computed, we define the mapping $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, where

$$\begin{aligned} \mathcal{F}_1(g_0, \dots, g_3, \rho_2, \rho_3) &= \mathcal{A}_2 \left(\Psi^+ |_{X=M}, \partial_X \Psi^+ |_{X=M}, F \right) - \rho_2, \\ \mathcal{F}_2(g_0, \dots, g_3, \rho_2, \rho_3) &= \mathcal{A}_3 \left(\Psi^+ |_{X=M}, \partial_X \Psi^+ |_{X=M}, F \right) - \rho_3. \end{aligned} \quad (\text{II.4.29})$$

The point will be to establish that for small enough g_k , $k = 0, \dots, 3$ the system

$$\begin{cases} \mathcal{F}_1(g_0, \dots, g_3, \rho_2, \rho_3) = 0, \\ \mathcal{F}_2(g_0, \dots, g_3, \rho_2, \rho_3) = 0, \end{cases}$$

has a unique solution, provided that $\mathcal{F}_i(0, \dots, 0) = 0$, for $i = 1, 2$. This result will be obtained via the implicit function theorem. When verifying that $d\mathcal{F}(0, \dots, 0)$ is an isomorphism of $H^{m-1/2}(\mathbb{R}) \times H^{m-3/2}(\mathbb{R})$, we need that the only solution of the linear problem (II.4.1)-(II.4.2), when $g_k \equiv 0$ for all k is $\Psi = 0$. This shows once again how intrinsically connected the linear and nonlinear/linearized problems are.

Therefore, the field Ψ defined by Ψ^\pm on each side of the transparent condition will be a solution of (II.4.1)-(II.4.2). The definitions of Ψ^\pm and tensors $\mathcal{A}_2^\pm(\cdot, \cdot, \cdot)$ and $\mathcal{A}_3^\pm(\cdot, \cdot, \cdot)$ provide that $[\partial_X^k \Psi] |_{X=M} = 0$, for $k = 0, \dots, 3$.

This section is a blueprint for the proofs in the remainder of the chapter, and we will refer to it profusely. Especially in the derivation of energy estimates, where only the terms different from the ones discussed above will be presented.

II.5 Western boundary layer: the linear case

This section is devoted to showing the well-posedness of the western boundary layer problems in a general regime. The western boundary layer plays a fundamental role in basin-scale wind-driven ocean circulation, and it has been long studied in several theoretical works, e.g., [68, 13]. In idealized ocean models with a flat bottom, this layer is required not only to balance the interior Sverdrup transport to close the gyre circulation, but also to dissipate the vorticity imposed by the wind-stress curl [69].

Note that while the boundary layer functions depend on (t, y) , these variables behave as parameters at a microscopic scale. On that account, they will be omitted from the boundary layer functions to lighten the notation when no confusion can arise.

We start by studying the linear problem

$$\begin{aligned} \partial_{X_w} \Psi_w - \Delta_w^2 \Psi_w &= 0, \quad \text{in } \omega_w^+ \cup \omega^- \\ \left[\partial_{X_w}^k \Psi_w \right] |_{X_w=0} &= g_k, \quad k = 0, \dots, 3, \\ \Psi_w |_{X=-\gamma(Y)} &= 0, \quad \frac{\partial \Psi_w}{\partial n_w} |_{X_w=-\gamma_w(Y)} = 0, \end{aligned} \tag{II.5.1}$$

where $g_k \in L^\infty(\mathbb{R})$, for all $k = 0, \dots, 3$.

Theorem II.5.1. *Let γ_w be a positive $W^{2,\infty}(\mathbb{R})$ function and ω_w be defined as before. Let $g_k \in L^\infty(\mathbb{R})$, for all $k = 0, \dots, 3$. Then, problem (II.5.1) has a unique solution Ψ_w in $H_{\text{uloc}}^2(\omega_w \setminus \sigma_w)$ and there exists positive constants $C, \delta > 0$ such that*

$$\|e^{\delta X_w} \Psi_w\|_{H_{\text{uloc}}^2(\omega_w)} \leq C \sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})}. \tag{II.5.2}$$

The proof of well-posedness of Theorem II.5.1 relies on the formulation of an equivalent system in a domain where transparent boundary conditions have been added at $X_w = M$, $M > 0$. We will be following the steps listed in Section II.4.1 for the linear case.

First, we show some preliminary results on a problem in the half-space. Then, we define the pseudo-differential operators of the Poincaré-Steklov type relating the solution in the half-space with the one in the rough domain at the “transparent” interface. Finally, we

restrict ourselves to the domain $\omega_w^b = \omega_w \cap \{X \leq M\}$ and solve an equivalent problem following the Ladyženskaya and Solonnikov method [60].

Throughout this section, we write X instead of X_w since the analysis is only focused on the western boundary layer; hence no confusion can arise.

II.5.1 The linear problem on the half-space

The main focus of this section is the analysis of the system

$$\begin{aligned} \partial_X \Psi_w - \Delta_w^2 \Psi_w &= F, \quad \text{in } \mathbb{R}_+^2 \\ \Psi_w|_{X=0} &= \psi_0, \quad \partial_X \Psi_w|_{X=0} = \psi_1. \end{aligned} \quad (\text{II.5.3})$$

Here, F is a function decaying exponentially as X_w goes to infinity, and we have considered $M = 0$ to facilitate the computations. The problem with a source term F is necessary for the subsequent study of the nonlinear problem describing the western boundary layer.

Note that if Ψ_w is a solution of (II.5.3), $\Psi_w(X - M, Y)$ is solution of the problem defined on $\{X > M\}$ with $M > 0$ as a consequence of the equation being invariant with respect to translations on X . Functional spaces of ψ_0 and ψ_1 are provided in the following theorem, which summarizes the main result of the section.

Theorem II.5.2. *Let $m \in \mathbb{N}$ such that $m \gg 1$. Let $\psi_0 \in H_{\text{uloc}}^{m+3/2}(\mathbb{R})$ and $\psi_1 \in H_{\text{uloc}}^{m+1/2}(\mathbb{R})$. Let F be such that $e^{\bar{\delta}X} F \in H_{\text{uloc}}^{m-2}(\mathbb{R}_+^2)$, for $\bar{\delta} \in \mathbb{R}_+^*$. Then, there exists a unique solution Ψ_w of system (II.5.3) satisfying*

$$\|e^{\delta X} \Psi_w\|_{H_{\text{uloc}}^{m+2}(\mathbb{R}_+^2)} \leq C \left(\|\psi_0\|_{H_{\text{uloc}}^{m+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m+1/2}(\mathbb{R})} + \|e^{\bar{\delta}X} F\|_{H_{\text{uloc}}^{m-2}(\mathbb{R}_+^2)} \right), \quad (\text{II.5.4})$$

for a constant C depending on α , $\delta < \bar{\delta}$.

Note that uniqueness consists of showing that if $F = 0$, $\psi_0 = 0$ and $\psi_1 = 0$, the only solution Ψ_w of (II.5.3) is $\Psi_w \equiv 0$. The proof of the result is rather easy and will be sketched in paragraph II.5.1.1. Consequently, the primary result will be the existence of a solution satisfying estimate (II.5.4). Similarly to [61], the existence results can be obtained by compactness arguments.

As the main equation is linear, we use a superposition principle to prove the desired result, meaning a solution of (II.5.3) is sought of the form

$$\Psi_w = \underline{\Psi}_w + \Psi_w^F,$$

where $\underline{\Psi}_w$ is the solution of a homogeneous linear problem

$$\begin{cases} \partial_X \underline{\Psi}_w - \Delta_w^2 \underline{\Psi}_w = 0, & \text{in } \mathbb{R}_+^2 \\ \underline{\Psi}_w|_{X=0} = \psi_0^*, & \partial_X \underline{\Psi}_w|_{X=0} = \psi_1^*, \end{cases} \quad (\text{II.5.5})$$

while, the function Ψ_w^F solves the equation

$$\partial_X \Psi_w^F - \Delta_w^2 \Psi_w^F = F, \quad \text{in } \mathbb{R}_+^2. \quad (\text{II.5.6})$$

Note that the boundary terms ψ_0^* and ψ_1^* are different from ψ_0 and ψ_1 . Indeed, it is convenient to construct the solution of (II.5.6) which does not satisfy homogeneous boundary conditions, and then lift the non-zero traces of Ψ_w^F and $\partial_X \Psi_w^F$ thanks to $\underline{\Psi}_w$.

First, we apply Fourier analysis when looking for the solution of homogeneous problem (II.5.5) with boundary conditions $\psi_0^* = \psi_0 - \Psi_w^F|_{X=0}$ and $\psi_1^* = \psi_1 - \partial_X \Psi_w^F|_{X=0}$. Then, we tackle the sub-problem regarding function Ψ_w^F . In this case, we disregard temporarily about boundary conditions and focus on the equation (II.5.6). Our goal in this step is to construct a solution by means of an integral representation involving the Green function.

II.5.1.1 Homogeneous linear problem

In order to prove the existence and uniqueness of the solution of problem (II.5.5) in $H_{\text{loc}}^2(\mathbb{R}_+^2)$, we first analyze the problem when the boundary data belongs to usual Sobolev spaces, see Proposition II.5.1, and then, extend the result to Kato spaces.

Proposition II.5.1. *Let $\psi_0^* \in H^{m+3/2}(\mathbb{R})$ and $\psi_1^* \in H^{m+1/2}(\mathbb{R})$. Then, the system*

$$\begin{cases} \partial_X \underline{\Psi}_w - \Delta_w^2 \underline{\Psi}_w = 0, & \text{in } \mathbb{R}_+^2 \\ \underline{\Psi}_w|_{X=0} = \psi_0^*, & \partial_X \underline{\Psi}_w|_{X=0} = \psi_1^*, \end{cases} \quad (\text{II.5.7})$$

admits a unique solution $\underline{\Psi}_w \in H^{m+2}(\mathbb{R}_+^2)$.

Proof. Existence. Let us illustrate the proof for $m = 0$. Given $\psi_0^* \in H^{3/2}(\mathbb{R}_+^2)$ and $\psi_1^* \in H^{1/2}(\mathbb{R}_+^2)$, we proceed with the construction of the fundamental solution by means of the Fourier transform. Applying the Fourier transform with respect to Y results in the following ODE problem

$$\begin{aligned} \partial_X \widehat{\underline{\Psi}}_w - (\partial_X^2 + (-\alpha \partial_X + i\xi)^2) \widehat{\underline{\Psi}}_w &= 0, & \text{in } \mathbb{R}_+^2 \\ \widehat{\underline{\Psi}}_w|_{X=0} &= \widehat{\psi}_0^*, & \partial_X \widehat{\underline{\Psi}}_w|_{X=0} = \widehat{\psi}_1^*, \end{aligned} \quad (\text{II.5.8})$$

where $\xi \in \mathbb{R}$ is the Fourier variable and $\widehat{\psi}_i^*$ is the Fourier transform of ψ_i^* , $i = 0, 1$. The corresponding characteristic polynomial is

$$P(\lambda) = -\lambda - (\lambda^2 + (\alpha\lambda + i\xi)^2). \quad (\text{II.5.9})$$

We are now interested in identifying possible degenerate cases using the relations between the coefficients and the roots of its characteristic equation.

Lemma II.5.1. *Let $P(\lambda)$ be the characteristic polynomial associated to the problem (II.5.5). Then, $P(\lambda)$ has four distinct complex roots λ_i^\pm , $i = 1, 2$. Moreover, when $\xi \neq 0$, $\Re(\lambda_i^+) > 0$ and $\Re(\lambda_i^-) < 0$.*

We refer the reader to Appendix II.A for a detailed proof of Lemma II.5.1.

The solutions of the problem resulting of applying the Fourier transform are linear combinations of $\exp(-\lambda_k^+ X)$ (with coefficients depending on ξ), where $(\lambda_k^+)_{k=1,2}$ are the complex-valued solutions of the characteristic polynomial satisfying $\Re(\lambda_k^+) > 0$. There exist $A_k^+ : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$\widehat{\underline{\Psi}}_w(X, \xi) = \sum_{k=1}^2 A_k^+(\xi) \exp(-\lambda_k^+(\xi)X), \quad \text{in } \mathbb{R}_+^2. \quad (\text{II.5.10})$$

Combining (II.5.10) with boundary conditions in (II.5.5), we have that coefficients A_1^+ , A_2^+ satisfy

$$\Lambda \begin{pmatrix} A_1^+(\xi) \\ A_2^+(\xi) \end{pmatrix} = \begin{pmatrix} \widehat{\psi}_0^* \\ -\widehat{\psi}_1^* \end{pmatrix}, \quad \text{where } \Lambda := \begin{pmatrix} 1 & 1 \\ \lambda_1^+ & \lambda_2^+ \end{pmatrix}. \quad (\text{II.5.11})$$

Note that the coefficients A_1^+ and A_2^+ are well-defined since Λ is invertible as a direct consequence of all the roots λ_k^+ , $k = 1, 2$ of (II.5.9) being simple.

It remains to check that the corresponding solution is sufficiently integrable, namely, $\|\underline{\Psi}_w\|_{H^{m+2}(\mathbb{R}_+^2)} < +\infty$. This assertion is equivalent to showing that for $0 \leq k \leq 2$

$$\int_{\mathbb{R}^+ \times \mathbb{R}} |\partial_X^k \widehat{\underline{\Psi}}_w|^2 d\xi dX + |\xi|^{2k} |\widehat{\underline{\Psi}}_w|^2 d\xi dX < +\infty. \quad (\text{II.5.12})$$

To that end, we need to investigate the behavior of λ_k^+ , A_k^+ for ξ close to zero and when $|\xi| \rightarrow \infty$. We gather the results in the following lemma, whose proof is postponed to Appendix II.B:

Lemma II.5.2. – As $\xi \rightarrow 0$,

$$\begin{aligned} \lambda_j^+(\xi) &= \bar{\lambda}_j^+ O(|\xi|), \quad \Re(\bar{\lambda}_j^+) > 0, \quad j = 1, 2, \\ A_j^+(\xi) &= a_{0,j} \widehat{\psi}_0^* + a_{1,j} \widehat{\psi}_1^* + O(|\widehat{\psi}_0^*| |\xi| + |\widehat{\psi}_1^*| |\xi|^2), \end{aligned}$$

where $\bar{\lambda}_j^+$, $a_{0,j}$ and $a_{1,j}$ depend continuously on α .

As $|\xi| \rightarrow +\infty$, we have the following asymptotic behavior for $j = 1, 2$ when

$$\begin{aligned} \lambda_j^+(\xi) &= p_1 |\xi| + (-1)^j p_0 |\xi|^{-\frac{1}{2}} + O(|\xi|^{-2}) \\ A_j^+(\xi) &= (-1)^j m_2 |\xi|^{3/2} + (-1)^j m_1 |\xi|^{1/2} + O(|\widehat{\psi}_0^*| + |\widehat{\psi}_1^*| |\xi|^{-1}), \end{aligned}$$

where $p_1 = \zeta^2 \mathbb{1}_{\xi > 0} + \bar{\zeta}^2 \mathbb{1}_{\xi < 0}$, $p_0 = \frac{1}{2} (i\zeta \mathbb{1}_{\xi > 0} + \bar{\zeta} \mathbb{1}_{\xi < 0})$, $m_1 = \frac{\widehat{\psi}_1^*}{|\zeta|^2} (\zeta \mathbb{1}_{\xi < 0} - i\bar{\zeta} \mathbb{1}_{\xi > 0})$ and $m_2 = \widehat{\psi}_0^* (i\zeta \mathbb{1}_{\xi > 0} - \bar{\zeta} \mathbb{1}_{\xi < 0})$. Here, ζ refers to the complex quantity $\sqrt{\frac{1-i\alpha}{\alpha^2+1}}$ satisfying $\Re(\zeta) > 0$.

Lemma II.5.2 is used to show (II.5.12) is in fact true for even larger values of k if ψ_0 and ψ_1 are regular enough. The detailed proof can be found on Appendix II.C.

Uniqueness. To show the uniqueness the solution, it is enough to solve (II.5.5) when $\psi_0^* = \psi_1^* = 0$ and verify $\Psi_w \equiv 0$. Applying the Fourier transform results in the following system

$$\Lambda \begin{pmatrix} A_1^+(\xi) \\ A_2^+(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where Λ is the invertible matrix previously defined. We conclude that $A_1 = A_2 = 0$, and thus $\widehat{\underline{\Psi}}(X, \xi) \equiv 0$. Since $\underline{\Psi}_w$ is an absolutely integrable function whose Fourier transform is identically equal to zero, then $\underline{\Psi}_w = 0$. \square

II.5.1.2 Non homogeneous problem

We begin the proof of the existence and uniqueness of the solution Ψ_w^F of (II.5.6) with the analysis of the equation

$$\partial_X \Psi_w^F - \Delta_w^2 \Psi_w^F = F, \quad \text{in } \mathbb{R}_+^2.$$

Our approach consists of constructing a particular solution of this equation, satisfying for some large enough m an estimate where the norm of F controls the norm of the solution in $L^\infty(H_{\text{uloc}}^{m-2})$. We look for a integral solution of the form

$$\Psi_w^F(X, \cdot) = \int_0^{+\infty} G(X - X', D) F(X', \cdot) dX', \quad (\text{II.5.13})$$

where G is the Green function verifying the equation

$$\partial_X G - \Delta_w^2 G = \delta_0. \quad (\text{II.5.14})$$

Here, $\delta_0(\cdot) = \delta(\cdot - X')$ denotes the Dirac delta function. In other words, G is the fundamental solution over \mathbb{R}^2 of the Fourier multiplier $L(X, \xi)$ for any $\xi \in \mathbb{R}$ and satisfies

$$L(X, \xi) G(X, \xi) = \delta_0(X).$$

Away from $X = 0$, $G(X, \xi)$ satisfies the homogeneous equation (II.5.5), see Section II.5.1.1.

For $X \neq 0$, $G(X, \xi)$ is a linear combination of $e^{-\lambda_i^\pm X}$, where $\lambda_i^\pm(\xi)$, $i = 1, 2$ are continuous functions of ξ and roots of the polynomial (II.5.9). We define G as follows

$$G = \begin{cases} \sum_{i=1}^2 B_i^- e^{-\lambda_i^- X}, & \text{in } X < 0 \\ \sum_{i=1}^2 B_i^+ e^{-\lambda_i^+ X}, & \text{in } X > 0, \end{cases} \quad B_i^\pm : \mathbb{R} \rightarrow \mathbb{C}, \quad k = 1, 2. \quad (\text{II.5.15})$$

Note that if G is considered discontinuous at $X = 0$, with the discontinuity modeled by a step function, then, $\partial_X G \propto \delta_0(X)$ and consequently, $\partial_X^k G \propto \delta_0^{(k)}(X)$, $k = 2, 3, 4$. However, (II.5.14) does not involve generalized functions beyond $\delta_0(X)$, and contains no derivatives of δ -functions. Thus, we conclude that $G(X, X')$ must be continuous throughout the domain and in particular at $X = 0$. $G(X, \cdot)$ is a C^2 function and $\partial^3 G / \partial X^3$ has a finite jump discontinuity of magnitude $-1/(1 + \alpha^2)^2$ at $X = X'$. More precisely, G satisfies

$$[\partial_X^k G]|_{X=0} = 0, \quad k = 0, 1, 2, \quad \text{and} \quad [\partial_X^3 G]|_{X=0} = -\frac{1}{(1 + \alpha^2)^2}.$$

Substituting (II.5.15) in the above interface conditions provides a linear system on the coefficients B_i^\pm .

$$\begin{cases} B_1^+ + B_2^+ - (B_1^- + B_2^-) = 0 \\ \lambda_1^+ B_1^+ + \lambda_2^+ B_2^+ - (\lambda_1^- B_1^- + \lambda_2^- B_2^-) = 0 \\ (\lambda_1^+)^2 B_1^+ + (\lambda_2^+)^2 B_2^+ - ((\lambda_1^-)^2 B_1^- + (\lambda_2^-)^2 B_2^-) = 0 \\ (\lambda_1^+)^3 B_1^+ + (\lambda_2^+)^3 B_2^+ - ((\lambda_1^-)^3 B_1^- + (\lambda_2^-)^3 B_2^-) = \frac{1}{(1 + \alpha^2)^2}. \end{cases} \quad (\text{II.5.16})$$

The determinant of the Vandermonde matrix associated to (II.5.16) is of the form

$$\bar{D} = (\lambda_1^+ - \lambda_2^+)(\lambda_1^+ - \lambda_1^-)(\lambda_2^+ - \lambda_1^-)(\lambda_1^+ - \lambda_2^-)(\lambda_2^+ - \lambda_2^-)(\lambda_1^- - \lambda_2^-).$$

Since all the λ_i^\pm are distinct (see Lemma II.5.1), \bar{D} is non-zero and system (II.5.16) has a unique set of solutions

$$\begin{aligned} B_1^+ &= - \frac{1}{(\alpha^2 + 1)^2 (\lambda_1^+ - \lambda_2^+) (\lambda_1^+ - \lambda_1^-) (\lambda_1^+ - \lambda_2^-)}, \\ B_2^+ &= \frac{1}{(\alpha^2 + 1)^2 (\lambda_1^+ - \lambda_2^+) (\lambda_2^+ - \lambda_1^-) (\lambda_2^+ - \lambda_2^-)}, \\ B_1^- &= \frac{1}{(\alpha^2 + 1)^2 (\lambda_1^+ - \lambda_1^-) (\lambda_2^+ - \lambda_1^-) (\lambda_1^- - \lambda_2^-)}, \\ B_2^- &= - \frac{1}{(\alpha^2 + 1)^2 (\lambda_1^+ - \lambda_2^-) (\lambda_2^+ - \lambda_2^-) (\lambda_2^- - \lambda_1^-)}. \end{aligned} \quad (\text{II.5.17})$$

This establishes the function G . Its asymptotic behavior as $|\xi| \rightarrow 0$ and $|\xi| \rightarrow \infty$ is summarized in the following lemma.

Lemma II.5.3. *We have*

$$G = \begin{cases} \sum_{j=1}^2 B_j^- e^{-\lambda_j^- X}, & X < 0 \\ \sum_{j=1}^2 B_j^+ e^{-\lambda_j^+ X}, & X > 0 \end{cases},$$

where the coefficients B_j^\pm are given by (II.5.17) and λ_j^\pm are the ones in Lemma II.5.2. Here, as $|\xi| \rightarrow 0$

$$\lambda_1^- = -|\xi|^4 + O(|\xi|^5), \quad \lambda_2^- = -\frac{1}{(\alpha^2 + 1)^{2/3}} + O(|\xi|).$$

If $|\xi| \rightarrow \infty$,

$$\lambda_j^-(\xi) = p_1 |\xi| + (-1)^j p_0 |\xi|^{-\frac{1}{2}} + O(|\xi|^{-2}),$$

where p_i , $i = 0, 1$, are defined as in Lemma II.5.2 and $\zeta' = \sqrt{\frac{-1-i\alpha}{\alpha^2+1}}$.

Asymptotic behavior:

- As $|\xi| \rightarrow 0$, we have that $B_j^\pm \rightarrow \bar{B}_j \in \mathbb{R}$ independent of α .
- For $|\xi| \gg 1$, we have $B_j^\pm = O(|\xi|^{-3/2})$.

We refer to Appendix II.D for a proof.

We proceed to rigorously prove that the field

$$\Psi_G^F(X, \cdot) = \int_0^{+\infty} G(X - X', D) F(X', \cdot) dX' = \int_0^{+\infty} \mathcal{F}_{\xi \rightarrow Y'}^{-1} (G(X - X', \cdot) \mathcal{F}_{Y' \rightarrow \xi} F(X', \cdot)) dX' \quad (\text{II.5.18})$$

is well-defined and satisfies (II.5.6). This is made more precise in the following lemma:

Lemma II.5.4. *Let F be smooth and compactly supported. The formula (II.5.13) defines a solution Ψ_w^F of (II.5.6) in $L_{loc}^\infty(\mathbb{R}_+, H^{m+2}(\mathbb{R}))$ for any $m \geq 0$.*

Proof. From the hypotheses, we can assert that $\widehat{F} = \widehat{F}(X', \xi)$ is in the Schwartz class with respect to $\xi \neq 0$, smooth and compactly supported in X' . We have additionally that $G(X - X', \xi)$ is smooth in $\xi \neq 0$, and continuous in X, X' . As a result, the function $J_{X, X'} : \xi \rightarrow G(X - X', \xi)\widehat{F}(X', \xi)$ belongs to $L^2((1 + |\xi|^2)^{m/2+1}d\xi) \times L^2((1 + |\xi|^2)^{m/2+1}d\xi)$ for any $X, X' \geq 0$ and is smooth in ξ and continuous in X, X' . At high frequencies, the functional satisfies

$$|J_{X, X'}(\xi)| \leq C|\xi|^N|\widehat{F}|,$$

where $N = -3/2$. This value comes from computing the B_i using Lemma II.5.3. We can conclude that it belongs to L^2 since \widehat{F} and its X' -derivatives are rapidly decreasing in ξ by definition of Schwartz class.

Furthermore, when $|\xi| \ll 1$, using once again the bounds derived in Lemma II.5.3,

$$G(X - X', \xi)\widehat{F}(X', \xi) = O(1).$$

Thus, Ψ_G^F defines a continuous function of X with values in $H^{m+2}(\mathbb{R}_+^2)$ for all $m \geq 0$. Moreover, the smoothness of F implies Ψ_G^F is smooth in X with values in the same space. It will be a solution of the problem due to classical results in the construction of Green functions.

Finally, as its Fourier transform is a linear combination of $e^{-\lambda_i(\xi)X}B_i(\xi)$, it also satisfies the linear problem treated in section II.5.1.1. \square

II.5.1.3 Bounds in Kato spaces

In this section we establish that Ψ_w is controlled by the norms of ψ_0, ψ_1 and F in $L^\infty(H_{\text{uloc}}^{m-2})$ for a large enough m . The proof follows [61].

Now we need to derive a representation formula for Ψ_w when $\psi_0 \in H_{\text{uloc}}^{3/2}(\mathbb{R})$ and $\psi_1 \in H_{\text{uloc}}^{1/2}(\mathbb{R})$ by using its Fourier transform. The critical point is to understand the action of the operators on L_{uloc}^2 functions.

Due to the form of the solution, the end goal will be to establish that for any $X > 0$, the kernel type $K_i^\pm = \mathcal{F}_{\xi \rightarrow Y}^{-1}(B_i^\pm(\xi)e^{-\lambda_i^\pm(\xi)X})$, $i = 1, 2$, defines an element of $L^1(\mathbb{R})$. The advantage of proving the latter is that $\Psi_i^\pm = K_i^\pm(\cdot, z) * \underline{\psi}_i$ will then be (at least) an L_{uloc}^1 function.

Lemma II.5.5. *Let $\underline{\psi}$ be $L_{\text{uloc}}^1(\mathbb{R})$. We define $\Psi_i^\pm(X, Y)$ by*

$$\Psi_i^\pm(X, \cdot) := \chi(D)P(D)e^{-\lambda_i^\pm(D)Z}\underline{\psi}, \quad \text{for } i = 1, 2, \quad (\text{II.5.19})$$

where $\chi = \chi(\xi) \in C_c^\infty(\mathbb{R})$ and $P = P(\xi) \in C^\infty(\mathbb{R})$ is a homogeneous polynomial of degree $k \geq 0$ in the same vicinity. Then, there exists C and $\delta > 0$ independent of $\underline{\psi}$ such that

$$\begin{aligned} \forall Z \geq 0, \quad & \|e^{\delta Z}\Psi_1^+\|_{L^\infty(\mathbb{R}_+^2)} + \|e^{\delta Z}\Psi_2^+\|_{L^\infty(\mathbb{R}_+^2)} \leq C\|\underline{\psi}\|_{L_{\text{uloc}}^1}, \\ \forall Z \leq 0, \quad & \|(1 + |Z|)^{k/4}\Psi_1^-\|_{L^\infty(\mathbb{R}_+^2)} + \|e^{\delta|Z|}\Psi_2^-\|_{L^\infty(\mathbb{R}_+^2)} \leq C\|\underline{\psi}\|_{L_{\text{uloc}}^1}. \end{aligned}$$

This lemma follows the same idea as [61, Lemma 7]. For the reader's convenience, we repeat the main ideas of the proof, thus making our exposition self-contained.

Proof. We introduce a partition of unity $(\varphi_q)_{q \in \mathbb{Z}}$ with $\varphi_q \in C_0^\infty(\mathbb{R})$, where $\text{Supp } \varphi_q \subset B(q, 2)$ for $q \in \mathbb{Z}$ and $\sup_q \|\varphi_q\|_{W^{k, \infty}} < +\infty$ for all k . We also introduce functions $\tilde{\varphi}_q \in C_0^\infty(\mathbb{R})$ such that $\tilde{\varphi}_q \equiv 1$ on $\text{Supp } \varphi_q$, and, say $\text{Supp } \tilde{\varphi}_q \subset B(q, 3)$. Then, for $i = 1, 2$

$$\Psi_i^\pm(Z, Y) = \sum_{q \in \mathbb{Z}} \chi(D) P(D) (\varphi_q \underline{\psi}) e^{-\lambda_i^\pm(D)Z} \quad (\text{II.5.20})$$

$$= \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}} K_i^\pm(Z, Y - Y') \underline{\psi}(Y') \varphi_q(Y') dY' \quad (\text{II.5.21})$$

$$= \sum_{q \in \mathbb{Z}} \int_{\mathbb{R}} K_{i,q}^\pm(Z, Y, Y') \varphi_q(Y') \underline{\psi}(Y') dY',$$

where

$$K_i^\pm(Z, Y) = \int_{\mathbb{R}} e^{iY \cdot \xi} \chi(\xi) P(\xi) e^{-\lambda_i^\pm(\xi)Z} d\xi, \quad K_{i,q}^\pm(Z, Y, Y') = K_i^\pm(Z, Y - Y') \tilde{\varphi}_q(Y').$$

We show that the following estimate holds:

Lemma II.5.6. *There exists $\delta > 0$ and for all $n \in \mathbb{N}$ a constant $C_n \geq 0$ such that for all $Y \in \mathbb{R}$, $Z > 0$ and $i = 1, 2$*

$$|K_i^+(Z, Y)| \leq C_n \frac{e^{-\delta Z}}{(1 + |Y|)^n}, \quad \text{for } i = 1, 2. \quad (\text{II.5.22})$$

When $Z < 0$, we have for all $Y \in \mathbb{R}$, for all $n \in \mathbb{N}$

$$|K_1^-(Z, Y)| \leq C_n \frac{|Z|^{(n-1-k)/4}}{|Z|^{n/4} + |Y|^n}, \quad |K_2^-(Z, Y)| \leq C_n \frac{e^{-\delta|Z|}}{(1 + |Y|^n)}. \quad (\text{II.5.23})$$

We finish the proof of the current lemma and then show the result in Lemma II.5.6.

Combining (II.5.20) and (II.5.22) when $i = 1, 2$ and $Z > 0$ yields

$$\begin{aligned} |\Psi_i^+(Z, Y)| &\leq C e^{-\delta Z} \left(\sum_{q \in \mathbb{Z}, |q-Y| \geq 4} \int |\varphi_q(Y') \underline{\psi}(Y')| dY' \frac{1}{|Y-q|^2-3} \right. \\ &\quad \left. + \sum_{q \in \mathbb{Z}, |q-Y| < 4} \int |\varphi_q(Y') \underline{\psi}(Y')| dY' \right) \\ &\leq C e^{-\delta Z} \|\underline{\psi}\|_{L_{\text{uloc}}^1}. \end{aligned} \quad (\text{II.5.24})$$

The latter also applies to Ψ_2^- . Furthermore, for $|Z| > 1$, we have

$$\begin{aligned} |\Psi_1^-(Z, Y)| &\leq C \left(\sum_{q \in \mathbb{Z}, |q-Y| \geq 4} \int |\varphi_q(Y') \underline{\psi}(Y')| dY' \frac{|Z|^{1-k/4}}{|Z|^{1/2} + |Y-q|^2-3} \right. \\ &\quad \left. + \sum_{q \in \mathbb{Z}, |q-Y| < 4} \int |\varphi_q(Y') \underline{\psi}(Y')| dY' \frac{1}{|Z|^{(1+k)/4}} \right) \\ &\leq C (|Z|^{1/4} + 1) |Z|^{-(1+k)/4} \|\underline{\psi}\|_{L_{\text{uloc}}^1} \leq C (|Z| + 1)^{-k/4} \|\underline{\psi}\|_{L_{\text{uloc}}^1}. \end{aligned} \quad (\text{II.5.25})$$

When $|Z| \leq 1$, a similar reasoning yields $|\Psi_1^-(Z, Y)| \leq C \|\underline{\psi}\|_{L_{\text{uloc}}^1}$.

Estimate in Lemma II.5.5 follows. \square

Proof of Lemma II.5.6. With the notation introduced in Lemma II.5.5, we follow the ideas in [61] to obtain the estimates. Let us consider $Z > 0$. Since λ_i^+ , $i = 1, 2$ are continuous and have non-vanishing real part on the support of χ , there exists a constant $\delta > 0$ such that $\Re(\lambda_i^+) \geq \delta$ for all $\xi \in \text{Supp } \chi$ and for $i = 1, 2$. When $|Y| \leq 1$, we obtain

$$|K_i^+(Z, Y)| \leq e^{-\delta Z} \|\chi P\|_{L^1}.$$

However, this estimate is not enough for greater values of Y . Let us define $\chi_i(Z, \xi) = \chi(\xi) \exp(-\lambda_i^+ Z)$ which is an $L^\infty(\mathbb{R}_+, \mathcal{S}(\mathbb{R}))$ function. It follows that for all $n_1, n_2 \in \mathbb{N}$

$$|\partial_\xi^{n_2} \chi_i(Z, \xi)| \leq C_{n_1, n_2} \frac{e^{-\delta Z}}{(1 + |\xi|^2)^{n_1}}.$$

Integrating by parts with respect to the frequency variable yields

$$\begin{aligned} Y^n K_i^+(Z, Y) &= \int_{\mathbb{R}} e^{iY \cdot \xi} D_\xi^n \left[\chi_i(\xi) P(\xi) e^{-\lambda_i^+(\xi) Z} \right] d\xi \\ &\leq e^{\delta Z} \int_{\mathbb{R}} e^{iY \cdot \xi} \sum_{m=0}^n \binom{n}{m-2} \partial_\xi^{n-m} \chi_i \partial_\xi^{m-2} (P(\xi) e^{-\lambda_i^+(\xi) Z}) d\xi. \end{aligned} \quad (\text{II.5.26})$$

Note that for any $k \in \{0, \dots, n\}$, $\partial_\xi^k P(\xi)$ remains a homogeneous polynomial. Thus, expression (II.5.26) is bounded by a linear combination of integrals of the form

$$\int_{\mathbb{R}} e^{iY \cdot \xi} e^{-\delta_n Z} \mathbb{1}_{\{|\xi| \leq C\}} d\xi,$$

and (II.5.22) follows. The estimate of K_2^- stems from the same ideas.

We proceed to compute a useful estimate on K_1^- . When $|Z| \leq 1$, we have

$$|K_1^-(Z, Y)| \leq \|\exp(-\delta|\xi|^4) \chi(\xi) P(\xi)\|_{L^1} < +\infty,$$

for all $Y \in \mathbb{R}$ and $k > -1$. Let us now consider $|Z| \geq 1$. By introducing the change of variables $\xi' = |Z|^{1/4} \xi$ and $Y' = |Z|^{-1/4} Y$, K can be rewritten as

$$K_1^-(Z, Y') = \frac{1}{|Z|^{(1+k)/4}} \int_{\mathbb{R}} e^{iY' \cdot \xi'} \chi \left(\frac{\xi'}{|Z|^{1/4}} \right) P(\xi') e^{-\lambda_1^- \left(\frac{\xi'}{|Z|^{1/4}} \right) |Z|} d\xi'.$$

Since $\lambda_1^- / |\xi|^4 \sim 1$ and does not vanish on the support of χ , there exists a positive constant δ such that $-\lambda_1^-(\xi) \leq -\delta|\xi|^4$ on $\text{Supp } \chi$. Therefore, for $|Y'| \leq 1$, it is easy to see that

$$|K_1^-(Z, Y)| \leq |Z|^{-(1+k)/4} \|\exp(-\delta|\xi'|^4) P(\xi')\|_{L^1}.$$

Now, for $|Y'| \geq 1$, we perform integration by parts and obtain for any $n \in \mathbb{N}$,

$$Y'^n K_1^-(Z, Y) = \frac{1}{|Z|^{(1+k)/4}} \int_{\mathbb{R}} e^{iY' \cdot \xi'} D_{\xi'}^n \left[\chi \left(\frac{\xi'}{|Z|^{1/4}} \right) P(\xi') \exp \left(-\lambda_1^- \left(\frac{\xi'}{|Z|^{1/4}} \right) |Z| \right) \right] d\xi'.$$

The main issues arise when the derivative acts on the exponential. Note that $\lambda_1^-(\xi) = |\xi|^4 \Lambda_1^-(\xi)$, where $\Lambda_1^- \in C^\infty(\mathbb{R})$ and $\Lambda_1^-(0) = 1$ therefore, for all $\xi' \in \mathbb{R}$, $Z < 0$,

$$\exp \left(-\lambda_1^- \left(\frac{\xi'}{|Z|^{1/4}} \right) |Z| \right) = \exp \left(-|\xi'|^4 \Lambda_1^- \left(\frac{\xi'}{|Z|^{1/4}} \right) \right).$$

We infer that for $\xi \in \text{Supp}\chi$,

$$\left| P(\xi') \partial_{\xi'}^n \left(\chi \left(\frac{\xi'}{|Z|^{1/4}} \right) e^{-\lambda_1^- \left(\frac{\xi'}{|Z|^{1/4}} \right) |Z|} \right) \right| \leq P_{3n}(\xi') e^{-\delta |\xi'|^4},$$

where P_{3n} denotes a polynomial on ξ' of degree $3n$. Hence, we have for all $n \in \mathbb{N}$

$$|Y^m K_1^-(Z, Y)| \leq C_n |Z|^{-(1+k)/4},$$

which provides in turn the following result for all $n \in \mathbb{N}$

$$|K_1^-(Z, Y)| \leq C \frac{|Z|^{-(1+k)/4}}{1 + |Y|^n} = C \frac{|Z|^{(n-1-k)/4}}{|Z|^{n/4} + |Y|^n}. \quad (\text{II.5.27})$$

Taking $n = 2$ in the previous inequality guarantees the convergence of integral controlling Ψ_1^- , completing the second estimate in the lemma. \square

Exponential decay is obtained at high frequencies by using the following result:

Lemma II.5.7. *Let $\chi \in C_c^\infty(\mathbb{R})$, with $\chi \equiv 1$ in a ball $B_r := B(0, r)$, for $r > 0$, and $P = P(\xi) \in C_b(B_r^c)$. For $\underline{\psi} \in L_{\text{uloc}}^2(\mathbb{R})$, we define $\Psi^i = \Psi^i(X, Y)$ by*

$$\Psi_i^\pm(X, Y) = (1 - \chi(D)) e^{-\lambda_i^\pm(D)Z} P(D) \underline{\psi}.$$

Then, for $Z \in \mathbb{R}$ and $\delta > 0$ small enough,

$$\|e^{\delta Z} \Psi^1\|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} + \|e^{\delta Z} \Psi^2\|_{L^\infty(\mathbb{R} \times \mathbb{R}^+)} \leq C \|\underline{\psi}\|_{H_{\text{uloc}}^N(\mathbb{R})}. \quad (\text{II.5.28})$$

The proof of the previous lemma follows almost exactly the one of Lemma 9 in [61], and consequently, it is not repeated here. The authors in [61] showed that for n large enough, and any $|Z| \neq 0$,

$$K_n(Z, Y) := \mathcal{F}^{-1} \left((1 + |\xi|^2)^{-n} (1 - \chi(\xi)) P(\xi) e^{\mp \lambda_i^\pm(\xi)Z} \right) \in L^1(\mathbb{R}).$$

Consequently, $\Psi^i = K_n \star ((1 - \partial_Y^2)^n \psi)$ is at least an element of L_{uloc}^2 when $\underline{\psi} \in H_{\text{uloc}}^N(\mathbb{R})$, for $N \geq 2n$. The choice of N is linked to the degree of polynomial $P(D)$, and thus, to the asymptotic behavior of the eigenvalues.

Proposition II.5.2. *Let $l \geq 0$ and F a compactly supported function of $H_{\text{uloc}}^N(\mathbb{R}_+^2)$, for $N \geq \left\lceil \frac{2l-1}{4} \right\rceil$. Then, the solution Ψ_w^F of (II.5.6) satisfies for $0 < \delta < \bar{\delta}$ and ,*

$$\|e^{\delta X} D_{X,Y}^l \Psi_w^F\|_{L^\infty} \leq C \|e^{\delta X} F\|_{L^\infty(H_{\text{uloc}}^N)}, \quad (\text{II.5.29})$$

where $D_{X,Y}$ is the differential operator with respect to the variables X and Y .

Proof. We distinguish between high and low frequencies. We introduce some $\chi = \chi(\xi) \in C_c^\infty(\mathbb{R})$ equal to 1 near $\xi = 0$.

Let Ψ_w^b denote the integral expression when ξ is in a vicinity of zero

$$\Psi_w^b = \int_0^{+\infty} I(X, X', \cdot) dX', \quad I(X, X', \cdot) = \chi(D) G(X - X', D) F(X', \cdot). \quad (\text{II.5.30})$$

Here,

$$\Psi_w^b = \int_0^X \chi(D)G(X - X', D)F(X', \cdot)dX' + \int_X^{+\infty} \chi(D)G(X - X', D)F(X', \cdot)dX'.$$

Let us first show (II.5.29). Using Lemmas II.5.3 and II.5.5, there exists a $\delta > 0$

$$\|G(X - X', D)F(X', \cdot)\|_{L^\infty(\mathbb{R})} \sim \begin{cases} e^{-\delta(X-X')} \|F(X', \cdot)\|_{L^1_{\text{uloc}}(\mathbb{R})}, & \text{if } X - X' > 0 \\ \|F(X', \cdot)\|_{L^1_{\text{uloc}}(\mathbb{R})}, & \text{if } X - X' < 0 \end{cases}.$$

Then, assuming F decays exponentially at rate $\delta < \bar{\delta}$

$$\begin{aligned} \left\| \int_0^X \chi(D)G(X - X', D)F(X', \cdot)dX' \right\|_{L^\infty(\mathbb{R})} &\leq C \sup_{Z \in \mathbb{R}_+} \|e^{\bar{\delta}Z} F(Z, \cdot)\|_{L^1_{\text{uloc}}(\mathbb{R})} \int_0^X e^{-\delta(X-X')} e^{-\bar{\delta}X'} dX' \\ &\leq C e^{-\delta X} \sup_{Z \in \mathbb{R}_+} \|e^{\bar{\delta}Z} F(Z, \cdot)\|_{L^1_{\text{uloc}}(\mathbb{R})}. \end{aligned}$$

Moreover,

$$\begin{aligned} \left\| \int_X^\infty \chi(D)G(X - X', D)F(X', \cdot)dX' \right\|_{L^\infty(\mathbb{R})} &\leq C \sup_{Z \in \mathbb{R}_+} \|e^{\bar{\delta}Z} F(Z, \cdot)\|_{L^1_{\text{uloc}}(\mathbb{R})} \int_X^\infty e^{-\bar{\delta}X'} dX' \\ &\leq C e^{-\delta X} \sup_{Z \in \mathbb{R}_+} \|e^{\bar{\delta}Z} F(Z, \cdot)\|_{L^1_{\text{uloc}}(\mathbb{R})}. \end{aligned}$$

Consequently,

$$\|e^{\delta X} \Psi_w^b\|_{L^\infty(\mathbb{R}_+^2)} \leq C \|e^{\bar{\delta}X} F\|_{L^\infty(L^1_{\text{uloc}}(\mathbb{R}_+^2))}. \quad (\text{II.5.31})$$

For high frequencies, we define

$$\Psi_w^\# = \int_0^{+\infty} J(\cdot, X, X')dX', \quad J(X, X', \cdot) = (1 - \chi(D))G(X - X', D)F(X', \cdot). \quad (\text{II.5.32})$$

On account of Lemma II.5.7, we have that for $m \geq 1$

$$\|J(X, X', \cdot)\|_{L^\infty(\mathbb{R})} \leq C e^{-\delta|X-X'|} \|e^{\bar{\delta}X'} F(X', \cdot)\|_{H^m_{\text{uloc}}(\mathbb{R})}, \quad l = 0, 1,$$

and, consequently,

$$\|\Psi_w^\#\|_{L^\infty(\mathbb{R}_+^2)} \leq C \int_0^{+\infty} e^{-\delta|X-X'|} dX' \|F\|_{L^\infty(L^2_{\text{uloc}}(\mathbb{R}))} \leq C e^{-\delta X} \|e^{\bar{\delta}X'} F(X', \cdot)\|_{L^\infty(L^2_{\text{uloc}}(\mathbb{R}_+^2))}, \quad (\text{II.5.33})$$

which combined with (II.5.31) provides (II.5.29) for $l = 0$.

It remains to show the result for the derivatives of Ψ_w^F . At low frequencies, the coefficients associated to $e^{-\lambda_i^+ X}$, $i = 1, 2$ and $e^{\lambda_2^- X}$ satisfy the same properties of exponential decay. The terms containing $e^{\lambda_1^- X}$ converge to a constant or decay to zero with polynomial weight. In particular, $D_{X,Y}^{4k+l} \Psi_w^b$ decays at a rate $O(X^{1-l/4})$, for $l = 1, \dots, 3$. Hence, following the same reasoning as for $l = 0$ we have

$$\|e^{\delta X} D_{X,Y}^l \Psi_w^b\|_{L^\infty(\mathbb{R}_+^2)} \leq C \|e^{\bar{\delta}X} F\|_{L^\infty(L^1_{\text{uloc}}(\mathbb{R}_+^2))}. \quad (\text{II.5.34})$$

At high frequencies, applying the differential operator D to Ψ_w adds at most a $|\xi|$ factor. Consequently, the l -th derivative at high frequencies behaves like $|\xi|^{l-3/2} \exp(-|\xi||Z|)$. We have that $P(D) = |\xi|^{l-3/2-2n} \in C_c(\mathbb{R} \setminus B(0, r))$, $r > 0$ if $l - 3/2 - 2n < -1$. Lemma II.5.7 gives for $N \geq 2 \left\lceil \frac{2l-1}{4} \right\rceil$

$$\|D_{X,Y}^l \Psi_w^\# \|_{L^\infty(\mathbb{R}_+^2)} \leq C \int_0^{+\infty} e^{-\delta|X-X'|} dX' \|F\|_{L^\infty(H_{\text{uloc}}^N(\mathbb{R}))} \leq C e^{-\delta X} \|e^{\delta X'} F(X', \cdot)\|_{L^\infty(H_{\text{uloc}}^N(\mathbb{R}))}. \quad (\text{II.5.35})$$

Finally, (II.5.29) results from gathering (II.5.34) and (II.5.35). \square

II.5.1.4 Proof of Theorem II.5.2

In previous sections, we have constructed the solutions $\underline{\Psi}_w$ and Ψ_w^F for the subproblems (II.5.5) and (II.5.6), respectively. This paragraph deals with the connection to the solution of (II.5.3).

The remarks following Theorem II.5.2 justify the existence of such a solution for smooth and compactly supported data, and it belongs to H_{uloc}^{m+2} , for $m \gg 1$. We will now focus on retrieving estimate (II.5.4).

Let us consider $\underline{\Psi}_w = \Psi_w - \Psi_w^F$. From Section II.5.1, we know the solution of the problem (II.5.5) will be well-defined for $\psi_0^* = \psi_0 - \Psi_w^F|_{X=0}$ and $\psi_1^* = \psi_1 - \partial_X \Psi_w^F|_{X=0}$ regular enough. Formal solutions of the homogeneous linear with zero source term and inhomogeneous Dirichlet data are given by the equation (II.5.10). Using Lemma II.5.2 and Lemma II.5.3, we study the behavior of $\underline{\Psi}_w$ at low and high frequencies following the ideas of the previous section.

Lemma II.5.8. *Let $m_0 \gg 1$. Then, there exists $\delta > 0$ and $C > 0$ such that the solution $\underline{\Psi}_w$ of (II.5.5) satisfies the estimate*

$$\|e^{\delta X} \underline{\Psi}_w\|_{L^\infty(\mathbb{R}^2)} \leq C \left(\|\psi_0\|_{H_{\text{uloc}}^{m_0+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m_0+1/2}(\mathbb{R})} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m_0-2}(\mathbb{R}_+^2)} \right).$$

Proof. Here, we make use once again of the function $\chi \in C_c^\infty(\mathbb{R})$ equal to one in a vicinity of $\xi = 0$ and zero elsewhere. At low frequencies, the asymptotic behavior in Lemmas II.5.2 and II.5.3 paired with Lemma II.5.5 yield

$$\|\chi(D) \underline{\Psi}_w\|_{L^\infty} \leq C e^{-\delta X} \left(\|\psi_0\|_{L_{\text{uloc}}^2(\mathbb{R})} + \|\psi_1\|_{L_{\text{uloc}}^2(\mathbb{R})} + \|e^{\delta X} F\|_{L_{\text{uloc}}^2(\mathbb{R}_+^2)} \right). \quad (\text{II.5.36})$$

Computation of estimates of $\underline{\Psi}_w$ at high frequencies relies on Lemma II.5.6. From the asymptotic behavior listed in Lemma II.5.2 and II.5.3, the coefficient multiplying ψ^i behaves as $|\xi|^{3/2-i} e^{-|\xi||Z|}$, $i = 0, 1$. If $m_0 > k > 1$ and $\psi_i \in H_{\text{uloc}}^{3/2-i}(\mathbb{R})$, considering $P(\xi) = |\xi|^{3/2-k-i}$ and $\underline{\psi} = (1 - \Delta)^{k/2} \psi_i$ in Lemma II.5.5 gives

$$\|(1 - \chi(D)) \underline{\Psi}_w\|_{L^\infty(\mathbb{R})} \leq C e^{-\delta X} \left(\|\psi_0\|_{H_{\text{uloc}}^{m_0+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m_0+1/2}(\mathbb{R})} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m_0-2}(\mathbb{R}_+^2)} \right). \quad (\text{II.5.37})$$

Combining (II.5.36) and (II.5.37), and in view of the estimate (II.5.29) satisfied by Ψ_w^F , we have

$$\|e^{\delta X} \Psi_w\|_{L^\infty} \leq C \left(\|\psi_0\|_{H_{\text{uloc}}^{m_0+3/2}} + \|\psi_1\|_{H_{\text{uloc}}^{m_0+1/2}} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m_0-2}} \right).$$

\square

We are left with the task of determining the higher regularity bound (II.5.4). Taking the derivatives directly on (II.5.10) and (II.5.18), it is clear that by considering larger values of m_0

$$\|e^{\delta X} \underline{\Psi}_w\|_{W^{2,\infty}} \leq C \left(\|\psi_0\|_{H_{\text{uloc}}^{m_0+3/2}} + \|\psi_1\|_{H_{\text{uloc}}^{m_0+1/2}} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m_0-2}} \right). \quad (\text{II.5.38})$$

Hence, $\Psi_w = \underline{\Psi}_w + \Psi_w^F$ verifies

$$\|e^{\delta X} \Psi_w\|_{W^{2,\infty}} \leq C \left(\|\psi_0\|_{H_{\text{uloc}}^{m_0+3/2}} + \|\psi_1\|_{H_{\text{uloc}}^{m_0+1/2}} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m_0-2}} \right). \quad (\text{II.5.39})$$

From this, it may be concluded that $\Psi_w \in H_{\text{uloc}}^2(\omega^+)$.

Notice that for all $k \in \mathbb{Z}^2$ and all $m \geq 0$, there exists a constant $C_m > 0$ such that

$$\|\Psi_w\|_{H^{m+2}(B(k,1) \cap \omega_w^+)} \leq C \left(\|\psi_0\|_{H_{\text{uloc}}^{m+3/2}} + \|\psi_1\|_{H_{\text{uloc}}^{m+1/2}} + \|F\|_{H^{m-2}(B(k,2) \cap \omega_w^+)} + \|\Psi_w\|_{H^2(B(k,2) \cap \omega_w^+)} \right).$$

This is a classical local elliptic regularity result, see [70]. Using the previous inequality and (II.2.11) provides

$$\begin{aligned} \|e^{\delta X} \Psi_w\|_{H_{\text{uloc}}^{m_0+2}(\omega^+)} &= \sup_{k \in \mathbb{Z}^2} \|e^{\delta X} \Psi_w\|_{H^{m_0+2}(B(k+1) \cap \omega^+)} \\ &= C_{m_0} \sup_{k \in \mathbb{Z}^2} \left(\|\psi_0\|_{H_{\text{uloc}}^{m_0+3/2}} + \|\psi_1\|_{H_{\text{uloc}}^{m_0+1/2}} \right. \\ &\quad \left. + e^{\delta k} \|F\|_{H^{m_0-2}(B(k,2) \cap \omega_w^+)} + e^{\delta k} \|\Psi_w\|_{H^2(B(k,2) \cap \omega_w^+)} \right). \end{aligned}$$

From $\sup_{k \in \mathbb{Z}^2} e^{\delta k} \|F\|_{H^{m_0-2}(B(k,2) \cap \omega_w^+)} \leq C \|e^{\delta k} F\|_{H_{\text{uloc}}^{m_0-2}(\omega_w^+)}$ and (II.5.39), we obtain (II.5.4) in Theorem II.5.2 for $\bar{\delta} > \delta$.

II.5.2 Differential operators at the transparent boundary

This paragraph is devoted to the well-posedness of the Poincaré-Steklov type operators defined at the boundary $X = M$.

Providing explicit representations for the Poincaré-Steklov operator in terms of boundary data and the source term $F \neq 0$ is quite technical and exceeds the scope of this chapter. From now on, we are only interested in the case where $F = 0$. Once again, without loss of generality, we assume $M = 0$.

Using Proposition II.5.1 and the variational formulation of problem (II.5.7), we have the following result:

Definition II.5.1. *Let $\underline{\Psi}_w \in H^2(\mathbb{R}_+^2)$ be the unique weak solution of the Dirichlet problem (II.5.7) for $(\psi_0^*, \psi_1^*) \in H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$. Then, the biharmonic matrix-valued Poincaré-Steklov operator is defined by*

$$\begin{aligned} PS_w : H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) &\rightarrow H^{-1/2}(\mathbb{R}) \times H^{-3/2}(\mathbb{R}) \\ PS_w \begin{pmatrix} \psi_0^* \\ \psi_1^* \end{pmatrix} &:= \begin{pmatrix} (1 + \alpha^2) \Delta_w \underline{\Psi}|_{X=0} \\ -((1 + \alpha^2) \partial_X - 2\alpha \partial_Y) \Delta_w \underline{\Psi}|_{X=0} + \frac{\underline{\Psi}}{2} \Big|_{X=0} \end{pmatrix} = K_w * \begin{pmatrix} \psi_0^* \\ \psi_1^* \end{pmatrix}, \end{aligned} \quad (\text{II.5.40})$$

where K_w is the distributional kernel.

Let us derive the expression of the operator in the Fourier space. We know that the unique solution $\underline{\Psi}_w$ of (II.5.3) in $H^2(\mathbb{R}_+^2)$ of the linear problem (II.5.5) for boundary data $\psi_0^* \in H^{3/2}(\mathbb{R})$ and $\psi_1^* \in H^{1/2}(\mathbb{R})$ can be written as

$$\widehat{\underline{\Psi}}_w(X, \xi) = \sum_{i=1}^2 A_i^+(\xi) \exp(-\lambda_i^+(\xi)X),$$

where $A_i^+(\xi)$ and λ_i^+ , $i = 1, 2$ are the ones in Lemma II.5.2. Going forward and for simplicity of notation, we drop the + sign from both the coefficients and the eigenvalues in this subsection.

Then, taking the Fourier transform of PS_w with respect to Y provides the following has the Fourier representation at the “transparent” boundary

$$\widehat{PS}_w \begin{pmatrix} \widehat{\psi}_0^* \\ \widehat{\psi}_1^* \end{pmatrix} = \begin{pmatrix} (1 + \alpha^2) \sum_{i=1}^2 A_i(\xi) (\lambda_i(\xi)^2 + (\alpha\lambda_i(\xi) + i\xi)^2) \\ \sum_{i=1}^2 A_i(\xi) \left[((1 + \alpha^2)\lambda_i + 2\alpha i\xi) (\lambda_i^2 + (\alpha\lambda_i + i\xi)^2) + \frac{1}{2} \right] \end{pmatrix}. \quad (\text{II.5.41})$$

Plugging in the above equation the coefficients $A_i(\xi)$ computed in (II.5.11) yields

$$PS_w \begin{pmatrix} \widehat{\psi}_0^* \\ \widehat{\psi}_1^* \end{pmatrix} = M_w \begin{pmatrix} \widehat{\psi}_0^* \\ \widehat{\psi}_1^* \end{pmatrix}.$$

We investigate the behavior of the matrix $M_w = (m_{i,j})_{2 \leq i \leq 3, 0 \leq j \leq 1} \in M_2(\mathbb{C})$ for ξ close to zero and for $|\xi| \rightarrow \infty$. The results are gathered in the following lemma:

Lemma II.5.9. • *Behavior at low frequencies: when $|\xi| \ll 1$*

$$M_w = \begin{pmatrix} -(\alpha^2 + 1)^{2/3} + O(|\xi|) & -(\alpha^2 + 1)^{4/3} + O(|\xi|) \\ -\frac{1}{2} + O(|\xi|) & -8\alpha i (\alpha^2 + 1) |\xi|^1 + O(|\xi|^2) \end{pmatrix}.$$

• *Behavior at high frequencies: when $|\xi| \gg 1$*

$$M_w = \begin{pmatrix} \overline{m}_{2,0} |\xi|^2 + O(|\xi|^{-1/2}) & \overline{m}_{2,1} |\xi| + O(|\xi|^{-1/2}) \\ \overline{m}_{3,0} |\xi|^3 + O(|\xi|^{3/2}) & \overline{m}_{3,1} |\xi|^2 + O(|\xi|^{1/2}) \end{pmatrix},$$

where $\overline{m}_{i,j}$ is a complex quantity depending on α , for $i = 2, 3, j = 0, 1$. Notice that the value of this constant at $\xi \rightarrow +\infty$ differs from the one at $\xi \rightarrow -\infty$ (see Lemma II.5.2 and Appendix II.B).

The proof of this lemma is elementary and will be given in Appendix II.F.

We have an additional result for the matrix M_w :

Lemma II.5.10. *At all frequencies,*

$$\partial_\xi^N M_w(\xi) = O((1 + |\xi|)^{3-N}),$$

for $N \in \mathbb{N}$, $0 \leq N \leq 5$.

Then, $\nabla^N[(1 - \chi(\xi))M_w(\xi)] \in L^1(\mathbb{R})$ that for $N = 5$, there exists a constant $C > 0$ such that

$$\mathcal{F}^{-1}((1 - \chi(\xi))M_w) \leq \frac{C}{|Y|^5}.$$

The Poincaré-Steklov operator PS_w associated to \mathbb{R}_+^2 has been defined as a continuous operator from $H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ to $H^{-1/2}(\mathbb{R}) \times H^{-3/2}(\mathbb{R})$. Our aim here is to prove that it has a unique extension to the space $H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$.

Let us first show this general result:

Lemma II.5.11. *Let $s, s' \in \mathbb{R}$. If for $\Psi \in H^s(\mathbb{R})$, the differential operator $\bar{\mathcal{A}} : H^s(\mathbb{R}) \rightarrow H^{s'}(\mathbb{R})$ is continuous, then, there exists a unique continuous extension $\mathcal{A} : H_{\text{uloc}}^s(\mathbb{R}) \rightarrow H_{\text{uloc}}^{s'}(\mathbb{R})$.*

Proof. First, we recall the definition of $H_{\text{uloc}}^s(\mathbb{R})$, that is:

$$\psi \in H_{\text{uloc}}^s(\mathbb{R}) \quad \text{iff} \quad \|\psi\|_{H_{\text{uloc}}^s(\mathbb{R})} = \sup_{q \in \mathbb{Z}} \|\eta_q \Psi\|_{H^s(\mathbb{R})} < +\infty, \quad (\text{II.5.42})$$

where $(\eta_q)_{q \in \mathbb{Z}}$ is a partition of unity satisfying $\eta_q \in C_0^\infty(\mathbb{R})$ and $\text{Supp } \eta_q \subset B(q, 1)$ for $q \in \mathbb{Z}$ and $\sup_q \|\eta_q\|_{W^{k, \infty}} < +\infty$ for all k . Definition (II.5.42) is independent of the choice of the function η_q (see Lemma 7.1 in [67]). Let ψ^* be function of $H_{\text{uloc}}^s(\mathbb{R})$, we introduce the notation ψ_q^* to denote $\eta_q \psi^*$. Then, we have

$$\psi^* = \sum_{q \in \mathbb{Z}} \psi_q^*.$$

We are interested in verifying that $\mathcal{A}\psi^*$ belongs to $H_{\text{uloc}}^{s'}(\mathbb{R})$, that is the same, as showing that $\eta_{q'} \mathcal{A}\psi^* \in H^{s'}(\mathbb{R})$, $\forall q' \in \mathbb{Z}$. We have the following decomposition

$$\begin{aligned} \eta_{q'} \mathcal{A}\psi^* &= \sum_{q \in \mathbb{Z}} \eta_{q'} \mathcal{A}(\eta_q \psi^*) \\ &= \sum_{|q-q'| \leq 4} \eta_{q'} \mathcal{A}(\eta_q \psi^*) + \sum_{|q-q'| > 4} \eta_{q'} \mathcal{A}(\eta_q \psi^*). \end{aligned} \quad (\text{II.5.43})$$

The first term in r.h.s can be easily bounded as follows: if $\eta_q \psi^* \in H^s(\mathbb{R})$, we have $\mathcal{A}(\eta_q \psi^*) \in H^{s'}(\mathbb{R})$ and furthermore, $\eta_{q'} \mathcal{A}(\eta_q \psi^*) \in H^{s'}(\mathbb{R})$. Then,

$$\left\| \sum_{|q-q'| \leq 4} \eta_{q'} \mathcal{A}(\eta_q \psi^*) \right\|_{H^{s'}} \leq C \|\bar{\mathcal{A}}\|_{\mathcal{L}(H^s, H^{s'})} \sum_{|q-q'| \leq 4} \|\eta_q \psi^*\|_{H^s} \leq C \|\bar{\mathcal{A}}\|_{\mathcal{L}(H^s, H^{s'})} \|\psi^*\|_{H_{\text{uloc}}^s(\mathbb{R})} < +\infty.$$

For the remaining term in (II.5.43), we consider the kernel representation of the operators. We have for $d(\text{Supp } \eta_{q'}, \text{Supp } \eta_q) \geq 1$ and all $Y' \in \text{Supp } \eta_{q'}$,

$$\begin{aligned} \left| \bar{\mathcal{A}}(\psi_q^*)(Y') \right| &\leq \int_{|Y'-Y| \geq 1} \frac{1}{|Y'-Y|^m} |\psi_q^*(y)| dY \leq \frac{1}{|q-q'|^m} \|\psi_q^*\|_{L^2} \\ &\leq \frac{1}{|q-q'|^m} \|\psi^*\|_{L_{\text{uloc}}^2(\mathbb{R})} \leq \frac{1}{|q-q'|^m} \|\psi^*\|_{H_{\text{uloc}}^s(\mathbb{R})}, \end{aligned} \quad (\text{II.5.44})$$

for $m \geq 5$. Thus,

$$\left| \sum_{|q-q'|>4} \eta_{q'} \mathcal{A}(\psi_{q'}^*) \right| \leq \sum_{|q-q'|>4} \frac{1}{|q-q'|^m} \|\psi^*\|_{H_{\text{uloc}}^s(\mathbb{R})} < +\infty.$$

Consequently, $\mathcal{A}(\psi^*) \in H_{\text{uloc}}^{s'}(\mathbb{R})$ if $\psi^* \in H_{\text{uloc}}^s(\mathbb{R})$, which ends the proof. \square

Now, it is possible to link the solution of the (II.5.5) with $(\psi_0^*, \psi_1^*) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$ and $PS_w(\psi_0^*, \psi_1^*)$.

Proposition II.5.3. *Let $(\psi_0^*, \psi_1^*) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$, and let $\underline{\Psi}_w$ be the unique solution of (II.5.3) with $F = 0$ and boundary data $\underline{\Psi}_w|_{X=0} = \psi_0^*$ and $\partial_X \underline{\Psi}_w|_{X=0} = \psi_1^*$. Then, for all $\varphi \in C_0^\infty(\mathbb{R}_+^2)$*

$$\int_{\mathbb{R}_+^2} \partial_X \underline{\Psi}_w \varphi - \int_{\mathbb{R}_+^2} \Delta_w \underline{\Psi}_w \Delta_w \varphi = \left\langle \mathcal{A}_3[\psi_0^*, \psi_1^*] - \frac{\psi_0^*}{2}, \varphi|_{X=0} \right\rangle + \left\langle \mathcal{A}_2[\psi_0^*, \psi_1^*], \partial_X \varphi|_{X=0} \right\rangle. \quad (\text{II.5.45})$$

Namely, for $(\psi_0^*, \psi_1^*) \in H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$, the Poincaré-Steklov operator satisfies the condition

$$\langle \mathcal{A}_3[\psi_0^*, \psi_1^*], \psi_0^* \rangle + \langle \mathcal{A}_2[\psi_0^*, \psi_1^*], \psi_1^* \rangle \leq 0. \quad (\text{II.5.46})$$

The proof of (II.5.45) relies once again on defining a smooth function $\tilde{\chi}$, with $\tilde{\chi} = 1$ in an open set containing $\text{Supp} \varphi$ and using the kernel representation formulae of the boundary differential operators. Estimate (II.5.46) results from considering $\underline{\Psi}_w$ as test function in (II.5.45). The detailed verification is left to the reader.

This section ends with other useful estimates on the Poincaré-Steklov operator:

Proposition II.5.4. *Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\text{Supp} \varphi \subset B(Y_0, R)$, $R \geq 1$, and $(\psi_0^*, \psi_1^*) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$. Then, there exists a constant $C > 0$ such that the following property holds.*

$$|\langle \mathcal{A}_3[\psi_0^*, \psi_1^*], \varphi \rangle| + |\langle \mathcal{A}_2[\psi_0^*, \psi_1^*], \partial_X \varphi \rangle| \leq C\sqrt{R} \left(\|\varphi\|_{H^{3/2}(\mathbb{R})} + \|\partial_X \varphi\|_{H^{1/2}(\mathbb{R})} \right) \left(\|\psi_0^*\|_{H_{\text{uloc}}^{3/2}(\mathbb{R})} + \|\psi_1^*\|_{H_{\text{uloc}}^{1/2}(\mathbb{R})} \right) \quad (\text{II.5.47})$$

In particular, if $\psi_j \in H^{3/2-j}(\mathbb{R})$, $j = 0, 1$,

$$|\langle \mathcal{A}_3[\psi_0^*, \psi_1^*], \varphi \rangle| + |\langle \mathcal{A}_2[\psi_0^*, \psi_1^*], \partial_X \varphi \rangle| \leq C \left(\|\varphi\|_{H^{3/2}(\mathbb{R})} + \|\partial_X \varphi\|_{H^{1/2}(\mathbb{R})} \right) \left(\|\psi_0^*\|_{H^{3/2}(\mathbb{R})} + \|\psi_1^*\|_{H^{1/2}(\mathbb{R})} \right) \quad (\text{II.5.48})$$

Proof. This construction is adapted from [52]. We consider a truncation function $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $B(Y_0, R+1)$ and $\text{Supp} \chi \subset B(Y_0, R+2)$, and such that $\|\partial_Y^r \chi\|_\infty \leq C_r$, with C_r independent of R , for all $r \in \mathbb{N}$. For the terms

$$\int_{\mathbb{R}} K_{3,j} * ((1-\chi)\psi_j^*) \varphi, \quad \int_{\mathbb{R}} K_{2,j} * ((1-\chi)\psi_j^*) \partial_X \varphi,$$

where $|K_{i,j}(Y)| \leq C|Y|^{-5}$, $i = 2, 3$, $j = 0, 1$, for all $Y \in \mathbb{R}$, we use the following relation

$$\begin{aligned} & C \int_{\mathbb{R} \times \mathbb{R}} \frac{1}{|Y'|^5} |1 - \chi(Y - Y')| |\psi_j^*(Y - Y')| |\partial_X^{3-i} \varphi(Y)| dY dY' \\ & \leq C \int_{\mathbb{R}} |\partial_X^{3-i} \varphi(Y)| dY \left(\int_{|Y'| \leq 1} \frac{|\psi_j^*(Y - Y')|^2}{|Y'|^5} dY' \right)^{\frac{1}{2}} \left(\int_{|Y'| \leq 1} \frac{1}{|Y'|^5} dY' \right)^{\frac{1}{2}} \quad (\text{II.5.49}) \\ & \leq C \|\psi_j^*\|_{L^2_{\text{uloc}}(\mathbb{R})} \|\partial_X^{3-i} \varphi\|_{L^1} \leq C\sqrt{R} \|\psi_j^*\|_{L^2_{\text{uloc}}(\mathbb{R})} \|\partial_X^{3-i} \varphi\|_{L^2}. \end{aligned}$$

It remains to analyze the terms of the type

$$\langle \mathcal{F}^{-1}(m_{i,j} \widehat{\chi \psi_j^*}), \partial_X^{3-i} \varphi \rangle_{H^{-(2i-3)/2}, H^{(2i-3)/2}}.$$

Since $m_{i,j}(\xi)$, $i = 2, 3$, $j = 0, 1$ is a kernel satisfying $\text{Op}(m_{i,j}) : H^{3/2-j}(\mathbb{R}) \rightarrow H^{-i+3/2}(\mathbb{R})$, these terms are bounded by

$$C \|\chi \psi_j^*\|_{H^{3/2-j}(\mathbb{R})} \|\partial_X^{3-i} \varphi\|_{H^{i-3/2}(\mathbb{R})}$$

We proceed to prove the estimate

$$\|\chi \psi_j^*\|_{H^{3/2-j}(\mathbb{R})} \leq C\sqrt{R} \|\psi_j^*\|_{H^{3/2-j}(\mathbb{R})} \quad (\text{II.5.50})$$

but first let us recall the norm definition in fractional Sobolev spaces.

Definition II.5.2. *Let $s \in (0, 1)$ be a fractional exponent and ω be a general, possibly non-smooth, open set in \mathbb{R}^n . For any $p \in [1, +\infty)$, the fractional Sobolev space $W^{s,p}(\omega)$ is defined as follows*

$$W^{s,p}(\omega) := \left\{ u \in L^p(\omega) : \frac{|u(Y) - u(Y')|}{|Y - Y'|^{\frac{n}{p} + s}} \in L^p(\omega \times \omega) \right\}; \quad (\text{II.5.51})$$

i.e. an intermediary Banach space between $L^p(\omega)$ and $W^{1,p}(\omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\omega)} := \left(\int_{\omega} |u|^p dY + [u]_{W^{s,p}(\omega)}^p \right)^{1/p}. \quad (\text{II.5.52})$$

Here, the term $[u]_{W^{s,p}(\omega)}$ is the so-called Gagliardo (semi)norm of u defined as

$$[u]_{W^{s,p}(\omega)} = \left(\int_{\omega} \int_{\omega} \frac{|u(Y) - u(Y')|^p}{|Y - Y'|^{n+sp}} \right)^{1/p}. \quad (\text{II.5.53})$$

If $s = m + \eta$, where $m \in \mathbb{Z}$ and $\eta \in (0, 1)$. The space $W^{s,p}(\omega)$ consists of

$$W^{s,p}(\omega) := \left\{ u \in W^{m,p}(\omega) : D^{\zeta} u \in W^{\eta,p}(\omega) \text{ for any } \zeta \text{ such that } |\zeta| = m \right\}; \quad (\text{II.5.54})$$

which is a Banach space with respect to the norm

$$\|u\|_{W^{s,p}(\omega)} := \left(\int_{\omega} \|u\|_{W^{m,p}(\omega)}^p dY + \sum_{|\zeta|=m} \|D^{\zeta} u\|_{W^{\eta,p}(\omega)}^p \right)^{1/p}. \quad (\text{II.5.55})$$

Before proving the estimates, we introduce a cut-off function θ satisfying (II.2.10), which will allow us to use (II.2.11). Following the same ideas in [52, Lemma 2.26], we have that for a certain u_0

$$\|\chi u_0\|_{L^2(\mathbb{R})}^2 \leq \sum_{k \in \mathbb{Z}} \|(\tau_k \theta) \chi u_0\|_{L^2(\mathbb{R})}^2 \leq \|\chi\|_{\infty}^2 \sum_{\substack{k \in \mathbb{Z} \\ k \leq CR}} \|(\tau_k \theta) u_0\|_{L^2(\mathbb{R})}^2 \leq CR \|\chi\|_{\infty}^2 \sup_{k \in \mathbb{Z}} \|(\tau_k \theta) u_0\|_{L^2(\mathbb{R})}^2 \quad (\text{II.5.56})$$

To deal with the Gagliardo norm, notice that the denominator in (II.5.55) for $p = 2$ can be written as

$$\begin{aligned} & |\chi u_0(Y) - \chi u_0(Y')|^2 \\ &= \left(\sum_{k \in \mathbb{Z}} k \theta(Y) \chi(Y) u_0(Y) - \tau_k \theta(Y') \chi(Y') u_0(Y') \right)^2 \\ &= \sum_{\substack{k, l \in \mathbb{Z} \\ |k-l| \leq 3}} (\tau_k \theta(Y) \chi(Y) u_0(Y) - \tau_k \theta(Y') \chi(Y') u_0(Y')) (\tau_l \theta(Y) \chi(Y) u_0(Y) - \tau_l \theta(Y') \chi(Y') u_0(Y')) \\ &\quad + \sum_{\substack{k, l \in \mathbb{Z} \\ |k-l| > 3}} (\tau_k \theta(Y) \chi(Y) u_0(Y) - \tau_k \theta(Y') \chi(Y') u_0(Y')) (\tau_l \theta(Y) \chi(Y) u_0(Y) - \tau_l \theta(Y') \chi(Y') u_0(Y')). \end{aligned}$$

As a result of the assumptions on θ , for $|k - l| > 3$, we obtain that $\tau_k \theta(Y) \tau_l \theta(Y) = 0$ for all $Y \in \mathbb{R}$. Moreover, if $\tau_k(Y) \tau_l(Y') \neq 0$, then, $|x - y| \geq |k - l| - 2$. Also, the first sum above contains $O(R)$ nonzero terms. Hence, using the Cauchy–Schwarz inequality gives

$$\begin{aligned} & [\chi u]_{W^{s,p}(\omega)}^2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\chi u_0(Y) - \chi u_0(Y')|^2}{|Y - Y'|^3} dY dY' \\ &\leq CR \sup_{k \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|(\tau_k \theta \chi u(Y)) - (\tau_k \theta \chi u(Y'))|^2}{|Y - Y'|^3} dY dY' \\ &\quad + \sum_{\substack{k, l \in \mathbb{Z} \\ |k-l| > 3}} \frac{1}{(|k - l| - 2)^3} (\tau_k \theta(Y) \chi(Y) u_0(Y) - \tau_k \theta(Y') \chi(Y') u_0(Y')) (\tau_l \theta(Y) \chi(Y) u_0(Y) - \tau_l \theta(Y') \chi(Y') u_0(Y')) \\ &= I_1 + I_2. \end{aligned}$$

We have

$$|I_1| \leq CR \|\chi\|_{W^{1,\infty}}^2 \|u_0\|_{H_{\text{uloc}}^{1/2}(\mathbb{R})}^2, \quad \text{and} \quad |I_2| \leq C \|u_0\|_{L_{\text{uloc}}^2}^2.$$

Then, for $u_0 = \psi_1^*$ and $R > 1$, it follows that

$$\begin{aligned} \|\chi \psi_1^*\|_{H^{1/2}(\mathbb{R})}^2 &= \|\chi \psi_1^*\|_{L^2(\mathbb{R})}^2 + [\chi \psi_1^*]_{H^{1/2}}^2 \leq CR \|\psi_1^*\|_{H_{\text{uloc}}^{1/2}(\mathbb{R})}^2 + C \|\psi_1^*\|_{L_{\text{uloc}}^2}^2 \\ &\leq CR \|\psi_1^*\|_{H_{\text{uloc}}^{1/2}(\mathbb{R})}^2. \end{aligned}$$

The remaining term is dealt with in a similar manner

$$\begin{aligned} \|\chi \psi_0^*\|_{H^{3/2}(\mathbb{R})}^2 &= \|\chi \psi_0^*\|_{L^2(\mathbb{R})}^2 + \|\chi D \psi_0^*\|_{L^2(\mathbb{R})}^2 + [\chi D \psi_0^*]_{H^{1/2}}^2 \\ &\leq CR \|\psi_0^*\|_{H_{\text{uloc}}^{3/2}(\mathbb{R})}^2 + C \|D \psi_0^*\|_{L_{\text{uloc}}^2}^2 \\ &\leq CR \|\psi_0^*\|_{H_{\text{uloc}}^{3/2}(\mathbb{R})}^2. \end{aligned}$$

From (II.5.49) and (II.5.50) we obtain (II.5.47). The proof of inequality (II.5.48) is classical and follows from the Fourier representation of the differential operators. \square

II.5.3 The problem in the rough channel

The section is devoted to proving the existence and uniqueness of weak solutions for the linear problem (II.5.1) by studying an equivalent problem defined in a channel ω_w^b presenting a transparent boundary at the interface $\{X = M\}$, $M > 0$. Here, only an accurate representation of the solution of the problem linear problem at $\{X = M\}$ is needed in order to obtain a good approximation of the solution of the original problem while solving a similar set of equations in the rough channel (step (L5) in Section II.4.1). The linear problem (II.5.1) acts on the new system through the coupling conditions described employing the Poincaré-Steklov operator in (II.5.40). As before, we are going to consider the linear problem without a source term, i.e., $F = 0$ in (II.5.1).

We define the following problem equivalent to (II.5.1) in the bounded channel $\omega_w^b = \omega_w \cap \{X \leq M\}$, $M > 0$,

$$\begin{aligned} \partial_X \Psi_w - \Delta_w^2 \Psi &= 0 \quad \text{in } \omega_w^b \setminus \sigma_w, \\ \left[\partial_X^k \Psi_w \right] |_{\sigma_w} &= g_k, \quad k = 0, \dots, 3, \\ \left[(1 + \alpha^2) \Delta_w \Psi_w \right] |_{\sigma_w^M} &= \mathcal{A}_2 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \\ \left[- \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \Psi_w + \frac{\Psi_w}{2} \right] |_{\sigma_w^M} &= \mathcal{A}_3 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \\ \Psi_w |_{X=-\gamma_w(Y)} = \partial_n \Psi_w |_{X=-\gamma_w(Y)} &= 0. \end{aligned} \quad (\text{II.5.57})$$

The equivalence between the solution of (II.5.57) and the one of the original problem is given in the following lemma:

Lemma II.5.12. *Let $\gamma \in W^{2,\infty}(\mathbb{R})$ and $g_k \in L^\infty(\mathbb{R})$, for $k = 0, \dots, 3$.*

- *Let Ψ_w be a solution of (II.5.1) in ω_w such that $\Psi_w \in H_{\text{uloc}}^2(\omega)$. Then, $\Psi|_{\omega_w^b}$ is a solution of (II.5.57), and for $X > M$, Ψ solves the homogeneous equivalent of problem (II.5.1) defined on the half-space $M \times \mathbb{R}$, with $\psi_0 := \Psi_w|_{X=M} \in H_{\text{uloc}}^{3/2}(\mathbb{R})$ and $\psi_1 := \partial_X \Psi_w|_{X=M} \in H_{\text{uloc}}^{1/2}(\mathbb{R})$.*
- *Furthermore, let $\Psi_w^- \in H_{\text{uloc}}^2(\omega_w^b)$ and $\Psi_w^+ \in H_{\text{uloc}}^2(\mathbb{R}_+^2)$ be solutions of (II.5.57) and (II.5.1), respectively. Taking*

$$\Psi_w(X, \cdot) := \begin{cases} \Psi_w^-(X, \cdot) & \text{for } -\gamma_w(\cdot) < X < M, \\ \Psi_w^+(X, \cdot) & \text{for } X > M, \end{cases}$$

the function $\Psi \in H_{\text{loc}}^2(\omega)$ is a solution of the problem (II.5.1).

Note that Ψ_w^- solves (II.5.57) in the trace sense, and for all $\varphi \in C_c^\infty(\overline{\omega_w^b})$ satisfies

$$\begin{aligned} \int_{\omega_w^b} \partial_X \Psi_w \varphi - \int_{\omega_w^b} \Delta_w \Psi_w \Delta_w \varphi &= - \left\langle \mathcal{A}_3 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}] - \frac{\psi_0^*}{2}, \varphi|_{X=M} \right\rangle \\ &\quad - \left\langle \mathcal{A}_2 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \partial_X \varphi|_{X=M} \right\rangle. \end{aligned} \quad (\text{II.5.58})$$

The above result easily follows from Theorem II.5.2 and Proposition II.5.3. Consequently, we focus our attention on showing a well-posedness of problem (II.5.57) in the remainder of the section.

Proposition II.5.5. *Let $\gamma_w \in W^{2,\infty}(\mathbb{R})$ and $\omega_w^b = \omega_w \cap \{X \leq M\}$, $M > 0$. Assume the Poincaré-Steklov operators $\mathcal{A}_i : H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R}) \rightarrow H_{\text{uloc}}^{3/2-i}(\mathbb{R})$, $i = 2, 3$ satisfy the properties in Proposition II.5.3 and $g_k \in L^\infty(\mathbb{R})$, for $k = 0, \dots, 3$. Then, there exists a unique solution $\Psi_w \in H_{\text{uloc}}^2(\omega_w^b \setminus \sigma_w)$ satisfying*

$$\|\Psi_w\|_{H_{\text{uloc}}^2(\omega_w^b)} \leq C \sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})}, \quad (\text{II.5.59})$$

where $C > 0$ is a universal constant.

Proof. From now on, we lose the w index to simplify the notation when no confusion can arise.

Before stating the main ideas of the proof, we first lift the nonhomogeneous jump conditions at σ_w by introducing the function Ψ^L as in (II.4.7). Then, for $\tilde{\Psi} = \Psi - \Psi^L$, we have

$$\begin{aligned} \partial_X \tilde{\Psi} - \Delta_w^2 \tilde{\Psi} &= F^L \quad \text{in } \omega_w^b \setminus \sigma_w^M, \\ \left[(1 + \alpha^2) \Delta_w \tilde{\Psi} \right] \Big|_{\sigma_w^M} &= \mathcal{A}_2 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \quad (\text{II.5.60}) \\ \left[- \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \tilde{\Psi} + \frac{\tilde{\Psi}}{2} \right] \Big|_{\sigma_w^M} &= \mathcal{A}_3 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \\ \tilde{\Psi}|_{X=-\gamma_w(Y)} = \partial_n \tilde{\Psi}|_{X=-\gamma_w(Y)} &= 0, \end{aligned}$$

where F^L is a function depending on g_k , $k = 0, \dots, 3$. The truncation technique introduced by Ladyženskaya and Solonnikov [60] is used to prove the existence and uniqueness of the solution of system (II.5.60) by means of a local uniform bound on $\Delta_w \tilde{\Psi}_n$, where $\tilde{\Psi}_n$ is the solution of the problem

$$\begin{aligned} \partial_X \tilde{\Psi}_n - \Delta_w^2 \tilde{\Psi}_n &= F_n^L \quad \text{in } \omega_n \\ (1 + \alpha^2) \Delta_w \tilde{\Psi}_n \Big|_{\sigma_n^M} &= \mathcal{A}_2 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \\ - \left[(1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right] \Delta_w \tilde{\Psi}_n + \frac{\tilde{\Psi}_n}{2} \Big|_{\sigma_n^M} &= \mathcal{A}_3 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \quad (\text{II.5.61}) \\ \tilde{\Psi}_n|_{\Gamma_n} = \frac{\partial \tilde{\Psi}}{\partial n_w} \Big|_{\Gamma_n} &= 0, \end{aligned}$$

where ω_n, σ_n^M and Γ_n are the ones in (II.4.11). The problem on $\tilde{\Psi}_n$ has the following weak formulation: Let $\varphi \in C_0^\infty(\omega^b)$ such that

$$\varphi = 0 \quad \text{on } \omega^b \setminus \omega_n, \quad \varphi|_{\Gamma_n} = 0 \quad \text{and} \quad \partial_X \varphi|_{\Gamma_n} = 0. \quad (\text{II.5.62})$$

Then, the solution $\tilde{\Psi}_n \in H^2(\omega_n)$ of (II.5.61) satisfies

$$\begin{aligned} \int_{\omega^b} \partial_X \tilde{\Psi}_n \varphi - \int_{\omega^b} \Delta_w \tilde{\Psi}_n \Delta_w \varphi &= - \left\langle \mathcal{A}_3 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}] - \frac{\tilde{\Psi}_n}{2} \Big|_{X=M}, \varphi \Big|_{X=M} \right\rangle \\ &\quad - \left\langle \mathcal{A}_2 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \partial_X \varphi \Big|_{X=M} \right\rangle - \int_{\omega^b} F_n^L \varphi. \end{aligned} \quad (\text{II.5.63})$$

Taking $\tilde{\Psi}_n$ as test function gives

$$\begin{aligned} \int_{\omega^b} |\Delta_w \tilde{\Psi}_n|^2 &= \underbrace{\left(\left\langle \mathcal{A}_3 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \tilde{\Psi}_n \Big|_{X=M} \right\rangle + \left\langle \mathcal{A}_2 [\Psi_w|_{X=M}, \partial_X \Psi_w|_{X=M}], \partial_X \tilde{\Psi}_n \Big|_{X=M} \right\rangle \right)}_{\leq 0} \\ &\quad - \int_{\omega^b} F_n^L \tilde{\Psi}_n \\ &\leq C \sqrt{n} \sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})} \|\tilde{\Psi}_n\|_{H^2(\omega_n)}, \end{aligned} \quad (\text{II.5.64})$$

where the constant C only depends on $\|\gamma_w\|_{W^{2,\infty}}$. Then, applying Poincaré inequality we have

$$\int_{\omega_k} |\Delta_w \tilde{\Psi}_n|^2 \leq C_0 n, \quad (\text{II.5.65})$$

where $C_0 = C_0(g_0, \dots, g_3)$. The existence of $\tilde{\Psi}_n$ in $H^2(\omega^b)$ follows from Lemma II.4.1. Uniqueness is obtained by following similar arguments as the ones presented in (L5), see Section II.4.1. We work with the energy

$$E_k^n := \int_{\omega_k} |\Delta_w \tilde{\Psi}_n|^2, \quad (\text{II.5.66})$$

for which we prove an inequality of the type

$$E_k^n \leq C_1 \left(k + 1 + m \sup_{k \leq j \leq k+m} (E_{j+1}^n - E_j^n) + \frac{1}{m^{4-2\eta}} \sup_{j \geq k+m} (E_{j+1}^n - E_j^n) \right) \quad \text{for all } k \in \{m, \dots, n\}, \quad (\text{II.5.67})$$

for any $m > 1$ and $\eta \in]0, 2[$. The constant $C_1 > 0$ is uniform constant in n depending only on $\|\gamma\|_{W^{2,\infty}}$ and $\|g_j\|_{W^{2-j,\infty}}$, $j = 0, \dots, 3$. The bound in $H_{\text{uloc}}^2(\omega^b)$ is then obtained via a nontrivial induction argument.

The remaining of the overall strategy is the same as the one detailed in step (L5) in Section II.4.1. Consequently, as we advance, we only discuss in detail the computations of the estimates involving the nonlocal differential operators and their incidence on the induction argument.

- *Induction.* To shorten the notation in the following paragraphs, we write E_k and $\tilde{\Psi}$ instead of E_k^n and $\tilde{\Psi}_n$. Let us show by induction on $n - k$ that for m large enough, (II.5.67) amounts to

$$E_k \leq C_1 \left(k + 1 + m^3 + \frac{1}{m^{4-2\eta}} \sup_{j \geq k+m} (E_{j+1} - E_j) \right) \quad \text{for all } \forall k \leq n, \quad (\text{II.5.68})$$

where the positive constant C_2 depends only on C_0 and C_1 , appearing respectively in (II.5.65) and (II.5.67). The inequality is clearly true when $k = n$, as soon as $C_2 > C_0$. Let us now assume that

$$E_{k'} \leq C_2 \left(k + 1 + m^3 + \frac{1}{m^{4-2\eta}} \sup_{j \geq k'+m} (E_{j+1} - E_j) \right) \quad \text{for all } k \in \{ \leq k + 1, \dots, n \}, \quad (\text{II.5.69})$$

holds and show it remains true for index k . If it were false, we have

$$E_{k'} \geq C_2 \left(k + 1 + m^3 + \frac{1}{m^{4-2\eta}} \sup_{j \geq k'+m} (E_{j+1} - E_j) \right). \quad (\text{II.5.70})$$

Combining inequalities (II.5.69) and (II.5.70) implies for all $k + m \geq j \geq k$,

$$E_{j+1} - E_j \leq E_{j+1} - E_k \leq C_2(m + 1). \quad (\text{II.5.71})$$

Then, (II.5.67) yields

$$E_k \leq C_1 \left(k + 1 + C_2 m(m + 1) + \frac{1}{m^{4-2\eta}} \sup_{j \geq k+m} (E_{j+1} - E_j) \right) \quad \text{for all } k \in \{m, \dots, n\}. \quad (\text{II.5.72})$$

Note that if $C_2 > C_1$ and $C_1 C_2 m(m + 1) \leq C_2 m^3$ in (II.5.72) we have a contradiction. This is verified when $C_2 > C_1$ and m is large enough. Hence, inequality (II.5.70) is valid for all $k \leq n$. Since equation (II.5.67) is invariant by a horizontal translations (see (L5) in Section II.4.1), we obtain

$$E_{k+1} - E_k \leq C_1 \left(2 + m^3 + \frac{1}{m^{4-2\eta}} \sup_{j \in \mathbb{N}} (E_{j+1} - E_j) \right) \quad \text{for all } k \in \{m, \dots, n\},$$

for all k , so that for m large enough, we conclude that

$$\sup_{k \in \mathbb{N}} (E_{k+1}^n - E_k^n) \leq C_1 \left(\frac{2 + m^3}{1 - m^{-4+2\eta}} \right) = C < +\infty,$$

which is a H_{uloc}^2 bound on $\tilde{\Psi}_n$. Hence, we can extract a subsequence of $\tilde{\Psi}_n$ that converges weakly to some $\tilde{\Psi}_n$ satisfying (II.5.60). Existence follows from the ideas presented at the beginning of the current section.

- *Establishing the Saint-Venant estimate.* This paragraph contains the proof of (II.5.67). The main difficulty in computing estimates independent of the size of the support of $\tilde{\Psi}_n$ resides on the nonlocal nature of the Poincaré-Steklov operators.

Thanks to the representation formula of the Poincaré-Steklov operators, the above formulation makes sense for $\varphi \in H^2(\omega_w^b)$ satisfying (II.5.62). To establish the estimates of E_k , we first introduce the cut-off function $\chi_k(Y)$ supported in σ_{k+1}^M and identically equal to 1 on σ_k^M . Considering $\varphi = \chi_k \tilde{\psi}$, $k < n$, as a test function in (II.5.45) yields for elements in l.h.s an expression equivalent to (II.4.18). Namely,

$$\begin{aligned} E_k = & \left(\left\langle \mathcal{A}_3 \left[\tilde{\Psi}|_{X=M}, \partial_X \tilde{\Psi}|_{X=M} \right], \chi_k \tilde{\Psi}|_{X=M} \right\rangle + \left\langle \mathcal{A}_2 \left[\tilde{\Psi}|_{X=M}, \partial_X \tilde{\Psi}|_{X=M} \right], \chi_k \partial_X \tilde{\Psi}|_{X=M} \right\rangle \right) \\ & - \int_{\omega^b} F^L \chi_k \tilde{\Psi} + \text{commutator terms stemming from the bilaplacian.} \end{aligned} \quad (\text{II.5.73})$$

All commutator terms are bounded by $C(E_{k+1} - E_k)$. The proof involves applying Poincaré and Young inequalities similarly to (II.4.19). Moreover,

$$\left| \int_{\omega^b} F^L \chi_k \tilde{\Psi} \right| \leq C \left(\sum_{k=0}^3 \|g_k\|_{L^\infty} \right) \sqrt{k+1} E_{k+1}^{1/2} \leq C(g_0, \dots, g_3) \sqrt{k+1} E_{k+1}^{1/2}. \quad (\text{II.5.74})$$

It remains to handle the non-local terms, i.e., the Poincaré-Steklov operator. Drawing inspiration from [52] and [71], we introduce the auxiliary parameter $m \in \mathbb{N}_*$ appearing in (II.5.67) and the following decomposition for ψ_j

$$\psi_j = (\chi_k + (\chi_{k+m} - \chi_k) + (1 - \chi_{k+m})) \psi_j, \quad j = 0, 1. \quad (\text{II.5.75})$$

Then, for $i = 2, 3$, the transparent operators can be written as

$$\begin{aligned} & \langle \mathcal{A}_i [\tilde{\Psi}|_{X=M}, \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \rangle \\ &= \langle \mathcal{A}_i [\chi_k \tilde{\Psi}|_{X=M}, \chi_k \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \rangle \\ &+ \langle \mathcal{A}_i [\chi_k \tilde{\Psi}|_{X=M}, 1 - \chi_k \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \rangle \\ &+ \mathcal{A}_i [1 - \chi_k \tilde{\Psi}|_{X=M}, \chi_k \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \\ &+ \mathcal{A}_i [(1 - \chi_k) \tilde{\Psi}|_{X=M}, (1 - \chi_k) \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \\ &\leq \left| \mathcal{A}_i [(\chi_{k+m} - \chi_k) \tilde{\Psi}|_{X=M}, (\chi_{k+m} - \chi_k) \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \right| \\ &+ \left| \mathcal{A}_i [(1 - \chi_{k+m}) \tilde{\Psi}|_{X=M}, (1 - \chi_{k+m}) \partial_X \tilde{\Psi}|_{X=M}], \chi_k \partial_X^{3-i} \tilde{\Psi}|_{X=M} \right| \\ &= I_{i,1} + I_{i,2}. \end{aligned} \quad (\text{II.5.76})$$

The inequality in (II.5.76) results from considering the negativity condition satisfied by the transparent operators. For the term $I_{i,1}$ we use Proposition II.5.4 and the estimate

$$\|\chi_k \tilde{\Psi}\|_{H^2(\omega^b)} + \|\chi_k \tilde{\Psi}|_{X=M}\|_{H^{3/2}(\omega^b)} + \|\chi_k \partial_X \tilde{\Psi}|_{X=M}\|_{H^{1/2}(\omega^b)} \leq C E_{k+1}^{1/2}.$$

Then,

$$\begin{aligned} |I_{2,1}| + |I_{3,1}| &\leq \|A_2[(\chi_{k+m} - \chi_k) \psi_0, (\chi_{k+m} - \chi_k) \psi_1]\|_{H^{-1/2}} \left\| \chi_k \partial_X \chi_k \tilde{\Psi}|_{X=M} \right\|_{H^{1/2}} \\ &+ \|A_3[(\chi_{k+m} - \chi_k) \psi_0, (\chi_{k+m} - \chi_k) \psi_1]\|_{H^{-3/2}} \left\| \chi_k \chi_k \tilde{\Psi}|_{X=M} \right\|_{H^{3/2}} \\ &\leq C \left(\|\partial_X \chi_k \tilde{\Psi}|_{X=M}\|_{H^{1/2}(\mathbb{R})} + \|\chi_k \tilde{\Psi}|_{X=M}\|_{H^{3/2}(\mathbb{R})} \right) \\ &\quad \times \left(\|(\chi_{k+m} - \chi_k) \partial_X \tilde{\Psi}|_{X=M}\|_{H^{1/2}(\mathbb{R})} + \|(\chi_{k+m} - \chi_k) \tilde{\Psi}|_{X=M}\|_{H^{3/2}(\mathbb{R})} \right) \\ &\leq C(E_{m+k+1} - E_k)^{1/2} E_{k+1}^{1/2}. \end{aligned}$$

We are left with the task of finding bounds for $I_{i,2}$. Note that for $m \geq 2$, $\text{Supp} \chi_{k+1} \cap \text{Supp}(1 - \chi_{k+m}) = \emptyset$, so, for $i = 2, 3$

$$\begin{aligned} \langle \mathcal{A}_i [\tilde{\Psi}|_{X=M}, \partial_X \tilde{\Psi}|_{X=M}], \partial_X^{3-i} \chi_k \tilde{\Psi} \rangle &= \int_{\mathbb{R}} K_{i,0} * ((1 - \chi_{k+m}) \tilde{\Psi}|_{X=M}) \partial_X^{3-i} \chi_k \tilde{\Psi} \\ &+ \int_{\mathbb{R}} K_{i,1} * ((1 - \chi_{k+m}) \partial_X \tilde{\Psi}|_{X=M}) \partial_X^{3-i} \chi_k \tilde{\Psi}. \end{aligned} \quad (\text{II.5.77})$$

The convolution terms in (II.5.77) decay like $|Y|^{-5}$. We have the following estimate:

Lemma II.5.13. *For all $k \geq m$ and $\eta \in]0, 2[$*

$$\|K_{i,j} * (1 - \chi_{k+m})\psi_j\|_{L^2(\sigma_{k+1}^M)} \leq C_\eta m^{-2+\eta} \left(\sup_{j \geq k+m} (E_{j+m} - E_j) \right)^{1/2}. \quad (\text{II.5.78})$$

Proof. We use an idea of Gérard-Varet and Masmoudi [2010], that was later used in [52], to treat the large scales: we decompose the set $\sigma^M \setminus \sigma_{k+m}^M$ as

$$\sigma^M \setminus \sigma_{k+m}^M = \bigcup_{j \geq k+m} \sigma_{j+1}^M \setminus \sigma_j^M.$$

On every set $\sigma_{j+1}^M \setminus \sigma_j^M$, we bound the L^2 norm of $(1 - \chi_{k+m})\psi_i$, $i = 0, 1$, by $E_{k+m+1} - E_{k+m}$. Thus we work with the quantity

$$\sup_{j \geq k+m} (E_{j+1} - E_j),$$

which we expect to be bounded uniformly in n, k . Now, applying the Cauchy–Schwarz inequality yields,

$$\begin{aligned} & \int_{\sigma_{k+1}^M} dY \int_{\sigma^M \setminus \sigma_{k+m}^M} \frac{1}{|Y' - Y|^5} |\partial_X^i \tilde{\Psi}|_{X=M} dY' \\ & \leq C \left(\int_{\sigma_{k+1}^M} dY \int_{\sigma^M \setminus \sigma_{k+m}^M} \frac{1}{|Y' - Y|^{5+2\eta}} dY' \int_{\sigma^M \setminus \sigma_{k+m}^M} \frac{1}{|Y' - Y|^{5-2\eta}} |\partial_X^i \tilde{\Psi}|_{X=M}^2 dY' \right)^{1/2} \\ & \leq C_\eta m^{-2+\eta} \left(\int_{\sigma_{k+1}^M} \int_{\sigma^M \setminus \sigma_{k+m}^M} \frac{1}{|Y' - Y|^{5+2\eta}} dY' dY \times \sup_{j \geq k+m} (E_{j+1} - E_j) \right)^{1/2}. \end{aligned} \quad (\text{II.5.79})$$

The previous result is obtained using the following computations: for $Y \in \sigma_{k+1}^M$

$$\begin{aligned} \int_{\sigma_{k+1}^M \setminus \sigma_{k+m}^M} \frac{1}{|Y' - Y|^{5-2\eta}} |\partial_X^i \tilde{\Psi}|_{X=M}^2 dY' &= \sum_{j \geq k+m} \int_{\sigma_{j+1}^M \setminus \sigma_j^M} \frac{1}{|j - Y|^{5-2\eta}} |\partial_X^i \tilde{\Psi}|_{X=M}^2 dY' \\ &\leq C \sum_{j \geq k+m} (E_{j+1} - E_j) \frac{1}{|j - Y|^{5-2\eta}} \\ &\leq C \sup_{j \geq k+m} (E_{j+1} - E_j) \sum_{j \geq k+m} \frac{1}{|j - Y|^{5-2\eta}} \\ &\leq C \sup_{j \geq k+m} (E_{j+1} - E_j) \sum_{j-k \geq m+1} \frac{1}{|j - k|^{5-2\eta}} \\ &\leq C_\eta m^{-4+2\eta} \sup_{j \geq k+m} (E_{j+1} - E_j). \end{aligned} \quad (\text{II.5.80})$$

The series above correspond to the Hurwitz zeta function which is absolutely convergent for $\eta \in]0, 2[$. On the other hand,

$$\int_{\sigma_{k+1}^M} \int_{\sigma^M \setminus \sigma_{k+m}^M} \frac{1}{|Y' - Y|^{5+2\eta}} dY' dY \leq C \int_{\mathbb{R} \setminus [0,1]} \frac{dX}{X^{4+2\eta}} \leq C_\eta < +\infty. \quad (\text{II.5.81})$$

Estimate (II.5.78) is easily obtained from (II.5.79), (II.5.80) and (II.5.81). \square

Applying several times Lemma II.5.13 combined with (II.5.74) gives

$$E_k \leq C_{\eta,g} \left(\sqrt{k+1} E_{k+1}^{1/2} + E_{k+1}^{1/2} (E_{k+m+1} - E_k)^{1/2} + m^{-2+\eta} E_{k+1}^{1/2} \left(\sup_{j \geq k+m} (E_{j+1} - E_j) \right)^{1/2} \right), \quad (\text{II.5.82})$$

for $k \geq m \geq 1$. Since E_k is a monotonically increasing function with respect to k , we have

$$E_{k+1} \leq E_k + (E_{k+m+1} - E_k).$$

Furthermore,

$$E_{k+m+1} - E_k = \sum_{k \leq j \leq k+m} (E_{j+1} - E_j) \leq m \sup_{k \leq j \leq k+m} (E_{j+1} - E_j).$$

Taking $C = \max_{\eta \in [0,2[} C_\eta$ and using Young's inequality gives that for all $\nu > 0$ there exists C_ν , such that

$$E_k \leq \nu E_k + C_{\nu,g} \left(k + 1 + m \sup_{k \leq j \leq k+m} (E_{j+1} - E_j) + \frac{1}{m^{4-2\eta}} \sup_{j \geq k+m} (E_{j+1} - E_j) \right). \quad (\text{II.5.83})$$

Inequality (II.5.67) follows from choosing ν sufficiently small.

□

II.6 Nonlinear boundary layer formation near the western coast

This section is devoted to showing the well-posedness of the western boundary layer when the model presents an advection term. We study the problem

$$\begin{aligned} \partial_{X_w} \Psi_w + Q_w(\Psi_w, \Psi_w) - \Delta_w^2 \Psi_w &= 0, \quad \text{in } \omega_w^+ \cup \omega^- \\ [\Psi_w] |_{X_w=0} &= \phi, \\ \left[\partial_{X_w}^k \Psi_w \right] |_{X_w=0} &= 0, \quad k = 1, \dots, 3, \\ \Psi_w |_{X=-\gamma(Y)} &= 0, \quad \frac{\partial \Psi_w}{\partial n_w} |_{X_w=-\gamma_w(Y)} = 0. \end{aligned} \quad (\text{II.6.1})$$

where $\phi \in W^{2,\infty}(\mathbb{R})$ and the nonlinear term is given by

$$Q_w(\Psi_w, \tilde{\Psi}_w) = \nabla_w^\perp \Psi_w \cdot \nabla_w (\Delta_w \tilde{\Psi}_w) = \nabla_w^\perp \left[\left(\nabla_w^\perp \Psi_w \cdot \nabla_w \right) \nabla_w^\perp \tilde{\Psi}_w \right].$$

As before, in this section we will write X instead of X_w .

The proof of Theorem II.2.1 for the nonlinear problem under a smallness assumption follows the general scheme presented in Section II.4.2. There are three main parts in our analysis: showing the well-posedness of the nonlinear problem in the half-space; proving the existence, uniqueness and regularity of the solution in the rough channel; and, finally, connecting both solutions at the ‘‘transparent’’ interface.

Later, in Section II.7 special attention will be paid to linearized problems in the western boundary layer domain. In its general form, this kind of problem is crucial in constructing the approximate solution since it describes the behavior of higher-order western profiles and additional correctors.

II.6.1 Nonlinear problem in the half-space

In this section, the well-posedness of the system (II.4.26) in the half-space is established under a smallness assumption. Namely, we study the problem

$$\begin{cases} Q_w(\Psi_w, \Psi_w) + \partial_X \Psi_w - \Delta_w^2 \Psi_w = 0, & \text{in } X > M \\ \Psi_w|_{\sigma_w^M} = \psi_0, & \partial_X \Psi_w|_{\sigma_w^M} = \psi_1. \end{cases} \quad (\text{II.6.2})$$

We shall solve (II.6.2) by means of a fixed point theorem using the a priori estimated provided in Theorem II.5.2. Basically, we use a contraction mapping argument in a suitable Banach space which will be norm invariant under the transformations that preserve the set of solutions, mainly the translations with respect to the X variable. We introduce the functional spaces:

$$\mathcal{H}^m := \left\{ f \in H_{\text{uloc}}^m(\mathbb{R}_+^2) : \|e^{\delta X} f\|_{H_{\text{uloc}}^m} < +\infty \right\}, \quad m \geq 0, \quad (\text{II.6.3})$$

with the norm $\|f\|_{\mathcal{H}^m} = C_m \|e^{\delta X} f\|_{H_{\text{uloc}}^m}$. Here, $\delta > 0$ is the one in Theorem II.5.2 and the constant C_m is chosen so that if $g, f \in \mathcal{H}^{m+1}$

$$\|(\nabla_w^\perp f \cdot \nabla_w) \nabla_w^\perp g\|_{\mathcal{H}^{m-1}} \leq \|f\|_{\mathcal{H}^{m+1}} \|g\|_{\mathcal{H}^{m+1}}.$$

We show the following:

Proposition II.6.1. *Let $m \in \mathbb{N}$, $m \gg 1$. There is $\delta_0 > 0$ such that for all $\psi_0 \in H_{\text{uloc}}^{m+3/2}(\mathbb{R})$ and $\psi_1 \in H_{\text{uloc}}^{m+1/2}(\mathbb{R})$,*

$$\|\psi_0\|_{H_{\text{uloc}}^{m+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m+1/2}(\mathbb{R})} < \delta_0, \quad (\text{II.6.4})$$

the system

$$\begin{cases} Q_w(\Psi_w, \Psi_w) + \partial_X \Psi_w - \Delta_w^2 \Psi_w = 0, & \text{in } \mathbb{R}_+^2 \\ \Psi_w|_{X=0} = \psi_0, & \partial_X \Psi_w|_{X=0} = \psi_1. \end{cases} \quad (\text{II.6.5})$$

has a unique solution in \mathcal{H}^{m+2} .

Proof. For any functions $\psi_0 \in H_{\text{uloc}}^{m+3/2}(\mathbb{R})$ and $\psi_1 \in H_{\text{uloc}}^{m+1/2}(\mathbb{R})$, let the operator $T_{(\psi_0, \psi_1)}$ be defined as follows: given a function $\Psi \in \mathcal{H}^{m+2}$, set $T_{(\psi_0, \psi_1)}(\Psi) = \tilde{\Psi}$, where $\tilde{\Psi}$ is the solution of (II.5.3) when $F = -\nabla_w^\perp [(\nabla_w^\perp \Psi \cdot \nabla_w) \nabla_w^\perp \Psi]$. According to Theorem II.5.2, there exists a constant C_0 such that for all $\tilde{\Psi} \in \mathcal{H}^{m+2}$,

$$\|T_{(\psi_0, \psi_1)}(\Psi)\|_{\mathcal{H}^{m+2}} \leq C_0 \left(\|\psi_0\|_{H_{\text{uloc}}^{m+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m+1/2}(\mathbb{R})} + \|\Psi\|_{\mathcal{H}^{m+2}}^2 \right).$$

The previous inequality results from taking into account that when $\bar{\delta} = 2\delta$

$$\|e^{\bar{\delta} X} F\|_{H_{\text{uloc}}^{m-2}(\mathbb{R}^2)} \leq C_{\bar{\delta}} \|e^{\bar{\delta} X} (\nabla_w^\perp \Psi \cdot \nabla_w) \nabla_w^\perp \Psi\|_{H_{\text{uloc}}^{m-1}(\mathbb{R}^2)} < \|\Psi\|_{\mathcal{H}^{m+2}}^2.$$

Let us verify that $T_{(\psi_0, \psi_1)}$ is a strict contraction under the smallness assumption (II.6.4). This implies the function has a fixed point in a closed ball of \mathbb{R}_+^2 . Let $\delta_0 < 1/(4C_0^2)$, and suppose that (II.6.4) holds. Thanks to the assumption on δ_0 , there exists $R_0 > 0$ such that

$$C_0 (\delta_0 + R_0^2) \leq R_0. \quad (\text{II.6.6})$$

Hence, R_0 belongs to $[R_-, R_+]$, where

$$R_{\pm} = \frac{1}{2C_0} \left(1 \pm \sqrt{1 - 4\delta_0 C_0^2} \right).$$

Therefore $0 < R_- < (2C_0^{-1})$, and it is always possible to choose $2R_0C_0 < 1$. (II.6.4) and (II.6.6) imply

$$\|\Psi\|_{\mathcal{H}^{m+2}} \leq R_0 \implies \|T_{(\psi_0, \psi_1)}(\Psi)\|_{H_{\text{uloc}}^{m+1}} < R_0.$$

Now, let $\Psi_i \in \mathcal{H}^{m+2}$ be a function satisfying $T_{(\psi_0, \psi_1)}(\Psi_i) = \tilde{\Psi}_i$, where Ψ_i is the solution of (II.5.6) when $F^i = (\nabla_w^\perp \Psi_i \cdot \nabla_w) \nabla_w^\perp \Psi_i$, $i = 1, 2$. If $\|\Psi_1\|_{H_{\text{uloc}}^{m+2}} \leq R_0$, $\|\Psi_2\|_{H_{\text{uloc}}^{m+2}} \leq R_0$ and $\bar{\Psi} = T_{(\psi_0, \psi_1)}(\Psi_1) - T_{(\psi_0, \psi_1)}(\Psi_2)$, we have that $\bar{\Psi}$ is a solution of (II.5.6) with $\bar{\Psi}|_{X=0} = \partial_X \bar{\Psi}|_{X=0} = 0$ and source term $F^1 - F^2 = \nabla_w^\perp [(\nabla_w^\perp \Psi_1 \cdot \nabla_w) \nabla_w^\perp \Psi_1] - \nabla_w^\perp [(\nabla_w^\perp \Psi_2 \cdot \nabla_w) \nabla_w^\perp \Psi_2]$. Thus, using once again Theorem II.5.2,

$$\|T_{(\psi_0, \psi_1)}(\Psi^1) - T_{(\psi_0, \psi_1)}(\Psi^2)\|_{\mathcal{H}^{m+2}} \leq C_0 \|F^1 - F^2\|_{\mathcal{H}^{m-1}} \leq 2C_0 R_0 \|\Psi_1 - \Psi_2\|_{\mathcal{H}^{m+2}}.$$

Since $2C_0 R_0 < 1$, $T_{(\psi_0, \psi_1)}$ is a contraction over the ball of radius R_0 in \mathcal{H}^{m+2} . We can then assert that $T_{(\psi_0, \psi_1)}$ has a fixed point in \mathcal{H}^{m+2} as a result of Banach's fixed point theorem which concludes the proof of Proposition II.6.1. \square

Remark II.6.1. *We can retrieve the solution for $X > M$ when $M > 0$ thanks to the problem being invariant with respect to translations along the X -axis. Let Ψ_0 be the solution of (II.6.5). Then, the solution Ψ_M of (II.6.2) in the half-space $X > M$, $M > 0$, satisfies $\Psi_M = \Psi_0(X - M)$.*

II.6.2 The problem in the rough channel

The goal in this section is to prove, by the truncation technique employed in [59], the existence of a solution of problem

$$\begin{cases} Q_w(\Psi_w^-, \Psi_w^-) + \partial_X \Psi_w^- - \Delta_w^2 \Psi_w^- = 0, & \text{in } \omega^b \setminus \sigma_w \\ [\Psi_w^-]_{\sigma_w} = \phi, & \left[\partial_X^k \Psi_w^- \right]_{\sigma_w} = 0, \quad k = 1, \dots, 3, \\ \Psi_w^-|_{X=-\gamma_w(Y)} = \frac{\partial \Psi_w^-}{\partial n_w}|_{X=-\gamma_w(Y)} = 0, \\ \mathcal{A}_2[\Psi_w^-|_{\sigma_w^M}, \partial_X \Psi_w^-|_{\sigma_w^M}] = \rho_2, & \mathcal{A}_3[\Psi_w^-|_{\sigma_w^M}, \partial_X \Psi_w^-|_{\sigma_w^M}] = \rho_3, \end{cases} \quad (\text{II.6.7})$$

where $\omega_w^b = \omega_w \setminus (\{X > M\} \times \mathbb{R})$ denotes the rough channel. We recall that

$$\begin{aligned} \mathcal{A}_2[\Psi|_{\sigma_w^M}, \partial_X \Psi|_{\sigma_w^M}] &= (1 + \alpha^2) \Delta_w \Psi, \\ \mathcal{A}_3[\Psi|_{\sigma_w^M}, \partial_X \Psi|_{\sigma_w^M}] &= - \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \Psi + \partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \\ &\quad + (\nabla_w^\perp \Psi \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi + \frac{\Psi}{2}. \end{aligned}$$

This part corresponds to step (NL2) in Section II.4.2. Although the idea is the same as for the linear case, an important difference resides in working indirectly with the values of

the Poincaré-Steklov operators. Indeed, here we “join” the solutions obtained at both sides of the artificial boundary using the implicit function theorem. To do so, higher regularity estimates of the solution near σ_w^M are essential.

As for the tensors at $X = M$, since we will need to construct solutions in H_{uloc}^{m+2} and due to the form of the differential operators, we look for ρ_2 and ρ_3 in the $H_{\text{uloc}}^{m-1/2}(\mathbb{R})$ and $H_{\text{uloc}}^{m-3/2}(\mathbb{R})$, respectively. We then claim that the following result holds:

Proposition II.6.2. *Let $m \gg 1$ be arbitrary. There exists $\delta > 0$ such that for all $\phi \in W^{2,\infty}(\mathbb{R})$, $\rho_2 \in H_{\text{uloc}}^{m-1/2}(\mathbb{R})$ and $\rho_3 \in H_{\text{uloc}}^{m-3/2}(\mathbb{R})$ with $\|\phi\|_{W^{2,\infty}(\mathbb{R})} < \delta$, $\|\rho_2\|_{H_{\text{uloc}}^{m-1/2}(\mathbb{R})} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}(\mathbb{R})} < \delta$, system (II.6.7) has a unique solution $\Psi_w^- \in H_{\text{uloc}}^2(\omega_w^b \setminus \sigma_w)$.*

Moreover, $\Psi_w^- \in H_{\text{uloc}}^{m+2}((M', M) \times \mathbb{R})$, for all $M' \in]0, M[$ and

$$\|\Psi_w^-\|_{H_{\text{uloc}}^{m+2}((M', M) \times \mathbb{R})} \leq C_{M'} \left(\|\phi\|_{W^{2,\infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}} \right).$$

Proof. First, we will briefly discuss the existence and uniqueness of a solution in H^2 , as well as the validity of the estimate. Then, the regularity result will be presented. Throughout the proof, we will drop the w from the notation when there is no confusion.

Step 1. Existence and uniqueness of the solution. We look for the solution $\tilde{\Psi} = \Psi^- - \Psi^L$ of the problem

$$\begin{aligned} Q_w(\tilde{\Psi}, \tilde{\Psi} + \Psi^L) + Q_w(\Psi^L, \tilde{\Psi}) + \partial_X \tilde{\Psi} - \Delta_w^2 \tilde{\Psi} &= F^L \quad \text{in } \omega^b, \\ \mathcal{A}_2[\tilde{\Psi}|_{\sigma_w^M}, \partial_X \tilde{\Psi}|_{\sigma_w^M}] &= \rho_2, \quad \mathcal{A}_3[\tilde{\Psi}|_{\sigma_w^M}, \partial_X \tilde{\Psi}|_{\sigma_w^M}] = \rho_3, \\ \tilde{\Psi}|_{X=-\gamma(Y)} &= 0, \quad \frac{\partial \tilde{\Psi}}{\partial n_w}|_{X=-\gamma(Y)} = 0. \end{aligned} \quad (\text{II.6.8})$$

Here, Ψ^L is defined as in (II.4.7) for $g_0 = \phi$ and $g_k \equiv 0$ for $k = 1, 2, 3$. In (II.6.8), F^L denotes $\Delta_w^2(\Psi^L) - \partial_X(\Psi^L) - Q_w(\Psi^L, \Psi^L)$. Notice that thanks to the regularity and smallness assumptions on ϕ , we have,

$$\|F^L\|_{L^2_{\text{uloc}}(\omega^b)} \leq C \left(\|\phi\|_{W^{2,\infty}} + \|\phi\|_{W^{2,\infty}}^2 \right) \leq C \|\phi\|_{W^{2,\infty}}.$$

In the nequality above, the constant C depends on α and $\|\gamma_w\|_{W^{2,\infty}}$. Before computing the a priori estimates, we write the weak formulation of (II.6.8).

Definition II.6.1. *Let \mathcal{V} be the space of functions $\varphi \in C_0^\infty(\overline{\omega^b})$ such that $\text{Supp} \varphi \cap \partial\Omega = \emptyset$ and $\mathcal{D}_0^2(\omega^b)$ its completion for the norm $\|\Psi\| = \|\Delta_w \Psi\|_{L^2}$. Define for $(\Psi, \tilde{\Psi}, \varphi) \in \mathcal{D}_0^2 \times \mathcal{D}_0^2 \times \mathcal{V}$, the trilinear form $b(\Psi, \tilde{\Psi}, \varphi) = -\int_\Omega (\nabla_w^\perp \Psi \cdot \nabla_w) \nabla_w^\perp \tilde{\Psi} \cdot \nabla_w^\perp \varphi$. A function $\tilde{\Psi} \in H_{\text{uloc}}^2(\omega^b)$ is a solution of (II.6.8) if it satisfies the homogeneous conditions $\tilde{\Psi}|_{\Gamma_w} = \partial_{n_w} \tilde{\Psi}|_{\Gamma_w} = 0$ at the rough boundary, and if, for all $\varphi \in \mathcal{V}$, we get*

$$\begin{aligned} \int_{\omega_w^b} \partial_X \tilde{\Psi} \varphi + b(\tilde{\Psi}, \tilde{\Psi}, \varphi) - \int_{\omega_w^b} \Delta_w \tilde{\Psi} \Delta_w \varphi \\ = - \int_{\mathbb{R}} \left(\rho_3 - \frac{\tilde{\Psi}}{2} \Big|_{X=M} \right) \varphi|_{X=M} dY \\ + \int_{\mathbb{R}} \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \right) \Big|_{X=M} \partial_Y \tilde{\Psi} \Big|_{X=M} dY - \int_{\mathbb{R}} \rho_2|_{X=M} \partial_X \varphi|_{X=M} dY. \end{aligned} \quad (\text{II.6.9})$$

We consider the system (II.6.8) in ω_n

$$\begin{aligned} Q_w(\tilde{\Psi}_n, \tilde{\Psi}_n + \Psi_n^L) + Q_w(\Psi_n^L, \tilde{\Psi}_n) + \partial_X \tilde{\Psi}_n - \Delta_w^2 \tilde{\Psi}_n &= F_n^L \quad \text{in } \omega_n, \\ \tilde{\Psi}_n &= 0, \quad \text{in } \omega^b \setminus \omega_n, \\ \tilde{\Psi}_n|_{X=-\gamma_n(Y)} = \partial_n \tilde{\Psi}_n|_{X=-\gamma_n(Y)} &= 0, \\ \mathcal{A}_2[\tilde{\Psi}_n|_{\sigma_n^M}, \partial_X \tilde{\Psi}_n|_{\sigma_n^M}] &= \rho_2, \\ \mathcal{A}_3[\tilde{\Psi}_n|_{\sigma_n^M}, \partial_X \tilde{\Psi}_n|_{\sigma_n^M}] &= \rho_3. \end{aligned} \quad (\text{II.6.10})$$

The domain ω_n and its components are the same as in (II.4.11).

By taking $\tilde{\Psi}_n$ as a test function in (II.6.10), we get a first energy estimate on $\tilde{\Psi}_n$

$$\begin{aligned} \|\Delta_w \tilde{\Psi}_n\|_{L^2(\omega_n)}^2 &= b(\tilde{\Psi}_n, \Psi_n^L, \tilde{\Psi}_n) + \int_{\sigma_n^M} \rho_3 \tilde{\Psi}_n + \int_{\sigma_n^M} \rho_2 \partial_X \tilde{\Psi}_n - \int_{\omega_n} F^L \tilde{\Psi}_n \\ &\leq C_1 \|\Delta_w \phi\|_{L^\infty(\omega_n)} \|\nabla_w^\perp \tilde{\Psi}_n\|_{L^2(\omega_n)}^2 + C_2 \sqrt{n} \left(\|\rho_3\|_{H_{\text{uloc}}^{m-3/2}(\mathbb{R})} \|\tilde{\Psi}_n|_{X=M}\|_{L^2([-n,n])} \right. \\ &\quad \left. + \|\rho_2\|_{H^{m-1/2}(\omega_n)} \|\nabla_w \tilde{\Psi}_n|_{X=M}\|_{L^2([-n,n])} \right) + \|F^L\|_{L^2_{\text{uloc}}} \|\tilde{\Psi}_n\|_{L^2(\omega_n)} \end{aligned} \quad (\text{II.6.11})$$

using the Cauchy-Schwartz and Poincaré inequalities over ω_n . Moreover, we have

$$\|\Delta_w \tilde{\Psi}_n\|_{L^2(\omega_n)}^2 \leq C \left(\|\phi\|_{W^{2,\infty}} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} \right) \|\tilde{\Psi}_n\|_{H^2(\omega_n)}.$$

Notice that as a consequence of the smallness assumption on $\|\phi\|_{H^2(\omega^b)}$, the first term on the r.h.s of inequality (II.6.11) can be absorbed by the one on the l.h.s for δ small enough. Then, using Poincaré inequality over the whole channel yields

$$E_n := \int_{\omega^b} |\Delta_w \tilde{\Psi}_n|^2 \leq \int_{\omega_n} |\Delta_w \tilde{\Psi}_n|^2 \leq C_0 n, \quad (\text{II.6.12})$$

where constant C_0 depends on α , $\|\phi\|_{W^{2,\infty}}$, $\|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}$, $\|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}$ and $\|\gamma\|_{W^{2,\infty}}$. The existence of $\tilde{\Psi}_n$ in $H^2(\omega_n)$ follows.

Following the same reasoning as in the linear case ((L5) in Section II.4.1 and Section II.5.3), we establish an induction inequality on $(E_k^n)_{k \in \mathbb{N}}$ for all $n \in \mathbb{N}$. Recall that

$$E_k^n := \int_{\omega^b} \chi_k |\Delta_w \tilde{\Psi}_n|^2,$$

where $\chi_k \in C_0^\infty(\mathbb{R})$ is a cut-off function in the tangential variable such that $\text{Supp} \chi_k \subset [-k-1, k+1]$ and $\chi_n \equiv 1$ on $[-k, k]$ for $k \in \mathbb{N}$. The induction relation allows one to obtain a uniform bound on the E_k , from which we deduce a H_{uloc}^2 bound on $\tilde{\Psi}_n$ uniformly in n . From this, an exact solution follows by compactness, see (L5) in Section II.4.1.

Here, we show the inequality for all $k \in \{1, \dots, n\}$

$$E_k^n \leq C_1 \left((E_{k+1}^n - E_k^n)^{3/2} + (E_{k+1}^n - E_k^n) + \left(\|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}^2 + \|\phi\|_{W^{2,\infty}}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}^2 \right) (k+1) \right), \quad (\text{II.6.13})$$

where C_1 is a constant depending only on the characteristics of the domain. Then, by backwards induction on k , we deduce that

$$E_k^n \leq Ck, \quad \forall k \in \{k_0, \dots, n\},$$

where $k_0 \in \mathbb{N}$ is a large, but fixed integer (independent of n) and $E_{k_0}^n$ is bounded uniformly in n for a constant C depending on ω^b , ϕ and ρ_i , $i = 2, 3$. Then, we use the fact that the derivation of energy estimates is invariant by translation on the tangential variable to prove that the uniform boundness holds not only for a maximal energy of size k_0 , but for all k , similarly to (L5) in Section II.4.1.

To lighten notations in the subsequent proof, we shall denote E_k instead of E_k^n .

- *Energy estimates.* This part is devoted to the proof of (II.6.13). We carry out the energy estimate on the system (II.6.8), focusing on having constants uniform in n as explained before. Since the linear part of the equation has already been analyzed on Section II.5, we discuss in detail only the nonlinear terms. In fact, the main issue consists in handling the quadratic terms $Q_w(\tilde{\Psi}, \tilde{\Psi} + \Psi^L)$, $Q_w(\Psi^L, \tilde{\Psi})$, which justifies the presence of the $|\nabla_w^\perp \tilde{\Psi}|^2$ in one of the tensors at σ_w^M . Plugging $\varphi = \chi_k \tilde{\Psi}$ into the nonlinear terms of (II.6.9) gives

$$\begin{aligned} b(\tilde{\Psi} + \Psi^L, \tilde{\Psi}, \chi_k \tilde{\Psi}) &= - \int_{\omega} \chi_k (\nabla_w^\perp (\tilde{\Psi} + \Psi^L) \cdot \nabla_w) \nabla_w^\perp \tilde{\Psi} \cdot \nabla_w^\perp \tilde{\Psi} \\ &\quad - \int_{\omega} (\nabla_w^\perp (\tilde{\Psi} + \Psi^L) \cdot \nabla_w) \nabla_w^\perp \tilde{\Psi} \cdot (\nabla_w^\perp \chi_k) \tilde{\Psi}, \end{aligned} \quad (\text{II.6.14})$$

$$\begin{aligned} b(\tilde{\Psi}, \Psi^L, \chi_k \tilde{\Psi}) &= - \int_{\omega} \chi_k (\nabla_w^\perp \tilde{\Psi} \cdot \nabla_w) \nabla_w^\perp \Psi^L \cdot \nabla_w^\perp \tilde{\Psi} \\ &\quad - \int_{\omega} (\nabla_w^\perp \tilde{\Psi} \cdot \nabla_w) \nabla_w^\perp \Psi^L \cdot (\nabla_w^\perp \chi_k) \tilde{\Psi}, \end{aligned} \quad (\text{II.6.15})$$

To bound each one of the terms we will frequently use the Sobolev inequality for all $\omega' \subset \omega^b$,

$$\forall u \in H^1(\omega'), \quad u|_{\Gamma} = 0, \quad \|u\|_{L^q(\omega')} \leq C_q \|\nabla u\|_{L^2(\omega')}, \quad q \in [1, +\infty). \quad (\text{II.6.16})$$

The constant C_q does not depend on ω' . Let us now illustrate the procedure for the first term in the l.h.s. of (II.6.15). By using Cauchy-Schwarz inequality and the properties of χ_k , we find that

$$\begin{aligned} \left| \int_{\omega} \chi_k (\nabla_w^\perp \tilde{\Psi} \cdot \nabla_w) \nabla_w^\perp \Psi^L \cdot \nabla_w^\perp \tilde{\Psi} \right| &\leq C \|\phi\|_{W^{2,\infty}(\omega_{k+1})} \|\nabla_w^\perp \tilde{\Psi}\|_{L^2(\omega_{k+1})}^2 \\ &\leq C \|\phi\|_{W^{2,\infty}} E_{k+1}, \end{aligned}$$

where C is a strictly positive constant depending on α , M and $\|\gamma_w\|_{W^{2,\infty}}$. On the other hand,

$$\begin{aligned} - \int_{\omega} \chi_k (\nabla_w^\perp (\tilde{\Psi} + \Psi^L) \cdot \nabla_w) \nabla_w^\perp \tilde{\Psi} \cdot \nabla_w^\perp \tilde{\Psi} &= - \int_{\omega} \chi_k \nabla_w^\perp (\tilde{\Psi} + \Psi^L) \cdot \nabla_w \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \right) \\ &= \int_{\omega} \nabla_w \chi_k \cdot \nabla_w^\perp (\tilde{\Psi} + \Psi^L) \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \right) \\ &\quad - \int_{\mathbb{R}} \chi_k \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \partial_Y \tilde{\Psi} \right) \Big|_{X=M} dY. \end{aligned}$$

Proceeding as before, it can be easily checked that the first term on the right-hand side is bounded by $C(E_{k+1} - E_k)^{3/2} + C\|\phi\|_{W^{2,\infty}}(E_{k+1} - E_k)$.

Applying integration by parts on the second integral gives

$$\begin{aligned} - \int_{\mathbb{R}} \chi_k \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \partial_Y \tilde{\Psi} \right) \Big|_{X=M} dY &= \int_{\mathbb{R}} \chi_k \partial_Y \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \right) \Big|_{X=M} \tilde{\Psi} dY \quad (\text{II.6.17}) \\ &+ \int_{\mathbb{R}} \partial_Y \chi_k \left(\frac{|\nabla_w^\perp \tilde{\Psi}|^2}{2} \right) \Big|_{X=M} \tilde{\Psi} dY. \end{aligned}$$

The first term can be grouped with other boundary terms stemming from the bilaplacian, while the second is bounded by $C(E_{k+1} - E_k)^{3/2}$ as a consequence of the trace theorem.

There remains to consider the r.h.s of (II.6.14) and (II.6.15), i.e, the terms $\int_\omega (\nabla_w^\perp (\tilde{\Psi} + \Psi^L) \cdot \nabla_w) \nabla_w^\perp \tilde{\Psi} \cdot (\nabla_w^\perp \chi_k) \tilde{\Psi}$ and $\int_\omega (\nabla_w^\perp \tilde{\Psi} \cdot \nabla) \nabla_w^\perp \Psi^L \cdot (\nabla_w^\perp \chi_k) \tilde{\Psi}$ which are bounded by $C(E_{k+1} - E_k)^{3/2} + C\|\phi\|_{W^{1,\infty}}(E_{k+1} - E_k)$ and $C\|\phi\|_{W^{2,\infty}}(E_{k+1} - E_k)$, respectively.

The linear terms defined on ω^b satisfy (II.4.18) and are bounded by $C(E_{k+1} - E_k)$ as seen in (II.4.19). Lastly, we deduce

$$\int_{\omega_w^b} \partial_X \tilde{\Psi} \chi_k \tilde{\Psi} = \frac{1}{2} \int_{\omega_w^b} \chi_k \partial_X (|\tilde{\Psi}|^2) = \frac{1}{2} \int_{\mathbb{R}} \chi_k |\Psi|^2 \Big|_{X=M}. \quad (\text{II.6.18})$$

From collecting the boundary terms coming from the bilaplacian with (II.6.17) and (II.6.18), we get

$$- \int_{\mathbb{R}} \chi_k \left(\tilde{\Psi} \rho_3 + \partial_X \tilde{\Psi} \rho_2 \right) \Big|_{X=M}.$$

The term above is bounded for any $\delta > 0$ by

$$C \left(\|\rho_2\|_{L^2_{\text{uloc}}} + \|\rho_3\|_{L^2_{\text{uloc}}} \right) E_{k+1} + C' \left(\|\rho_2\|_{L^2_{\text{uloc}}} + \|\rho_3\|_{L^2_{\text{uloc}}} \right) (k+1),$$

where C, C' depend only on M, α and on $\|\gamma\|_{W^{2,\infty}}$. The computation of this bound relies on the trace theorem and Young's inequality. We are left with

$$\left| \int_{\omega_w^b} \chi_k F^L \tilde{\Psi} \right| \leq C\|\phi\|_{W^{2,\infty}} E_{k+1}^{1/2} \sqrt{k+1}.$$

The last bound is not optimal but it suffices for our purposes. Applying once again Young's inequality yields

$$\left| \int_{\omega_w^b} \chi_k F^L \tilde{\Psi} \right| \leq C\|\phi\|_{W^{2,\infty}} E_{k+1} + C'\|\phi\|_{W^{2,\infty}} (k+1).$$

For $\|\phi\|_{W^{2,\infty}}, \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}$ and $\|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}$, gathering all the terms provides the following inequality

$$E_k \leq C_1(E_{k+1} - E_k)^{3/2} + C_2\|\phi\|_{W^{2,\infty}}(E_{k+1} - E_k) + C_3 \left(\|\phi\|_{W^{2,\infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}} \right) (k+1). \quad (\text{II.6.19})$$

- *Induction.* We aim to deduce from (II.6.13) that there exists $k_0 \in \mathbb{N} \setminus \{0\}$, $C > 0$ such that, for all $n \in \mathbb{N}$

$$\int_{\omega_{k_0}} |\Delta_w \tilde{\Psi}_n|^2 \leq C. \quad (\text{II.6.20})$$

Let $C_2(\phi)$ and $C_3(\phi, \rho_2, \rho_3)$ denote the coefficients associated to second and third k -dependent terms in inequality (II.6.19). From (II.6.13), we prove by downward induction on k , that there exists a positive constant C_4 depending only on $C_0, C_1, C_2(\phi)$ and $C_3(\phi, \rho_2, \rho_3)$, appearing respectively in (II.6.12) and (II.6.19), such that, for all $k > k_0$,

$$E_k \leq C_4 C_3(\phi, \rho_2, \rho_3)(k + 1). \quad (\text{II.6.21})$$

Here, C_4 is independent of n, k .

Note that (II.6.21) is holds for $k = n$ if $C_4 > C_0(C_3(\phi, \rho_2, \rho_3))^{-1}$, remembering that $\tilde{\Psi}_n = 0$ on $\omega^b \setminus \omega_n$. We then assume that (II.6.21) holds for $n, n - 1, \dots, k + 1$, where k is a positive integer.

To obtain the contradiction that allows us to claim (II.6.21) holds at the rank k , we assume that (II.6.21) is no longer true for k . Then, the induction yields

$$E_{k+1} - E_k < C_4 C_3(\phi, \rho_2, \rho_3).$$

Substituting the above inequality in (II.6.19) gives

$$C_4 C_3(\phi, \rho_2, \rho_3) < E_k \leq C_1 C_4^{3/2} C_3(\phi, \rho_2, \rho_3)^{3/2} + C_2(\phi) C_4 C_3(\phi, \rho_2, \rho_3) + C_3(\phi, \rho_2, \rho_3)(k + 1) \quad (\text{II.6.22})$$

Even when the values of $C_0, C_1, C_2(\phi), C_3(\phi, \rho_2, \rho_3) > 0$ are fixed, C_4 can be conveniently chosen. Taking $C_4 \geq 2$ and plugging it in (II.6.22) results in a contradiction for $k > k_0$, where $k_0 = \lfloor C_1 C_4^{3/2} C_3(\phi, \rho_2, \rho_3)^{1/2} + C_2(\phi) C_4 \rfloor$. Consequently, (II.6.21) is true at the rank $k > k_0$ and it also holds when $k \leq k_0$, since E_k is increasingly monotonic with respect to k .

Remark II.6.2. *The reader can find a detailed description of the method for the Stokes problem in [59] and for the Stokes-Coriolis system, in [52]. The backward induction in our case is less involved than the works mentioned above since we are not dealing directly with a non-local, non-linear Dirichlet-to-Neumann operator.*

By taking into account the translation invariance of the energy estimates and considering k_0 sufficiently large, we conclude that $\tilde{\Psi}_n$ is uniformly bounded in $H^2(\omega^b)$. This implies in turn that $\tilde{\Psi}$ and consequently, Ψ belong to $H^2_{\text{uloc}}(\omega^b)$.

- *Uniqueness.* Let $\bar{\Psi} = \Psi_1 - \Psi_2$, where $\Psi_i, i = 1, 2$, are solutions of satisfying the

smallness condition $\|\Psi_i\|_{H_{\text{uloc}}^m} < \delta$. We show that the solution $\bar{\Psi}$ of the problem

$$\begin{aligned} Q_w(\bar{\Psi}, \Psi_1) + Q_w(\Psi_2, \bar{\Psi}) + \partial_X \bar{\Psi} - \Delta_w^2 \bar{\Psi} &= 0 \quad \text{in } \omega^b, \\ (1 + \alpha^2) \Delta_w \bar{\Psi} \Big|_{\sigma_w^M} &= 0, \end{aligned}$$

$$\left(- \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \bar{\Psi} + \partial_Y \left(\frac{\nabla_w^\perp \bar{\Psi} \nabla_w^\perp (\Psi_1 + \Psi_2)}{2} \right) \right) \quad (\text{II.6.23})$$

$$+ (\nabla_w^\perp \bar{\Psi} \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi_1 \quad (\text{II.6.24})$$

$$+ (\nabla_w^\perp \Psi_2 \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \bar{\Psi} + \frac{\bar{\Psi}}{2} \Big|_{\sigma_w^M} = 0,$$

$$\Psi \Big|_{X=-\gamma(Y)} = \frac{\partial \Psi}{\partial n_w} \Big|_{X=-\gamma(Y)} = 0, \quad (\text{II.6.25})$$

$$(\text{II.6.26})$$

is $\bar{\Psi} \equiv 0$. The smallness assumption on Ψ_i , $i = 1, 2$ leads to the following inequality on the truncated energies

$$E_k \leq (\|\Psi_1\|_{H_{\text{uloc}}^2} + \|\Psi_2\|_{H_{\text{uloc}}^2}) C_1 (E_{k+1} - E_k),$$

where the constant C_1 depends only on the characteristics of the domain. Since $\sup_k (E_{k+1} - E_k) \leq \|\bar{\Psi}\|_{H_{\text{uloc}}^2}^2$, it is possible to show that E_k is uniformly bounded in k .

Therefore, the difference between two solutions belongs to $H^2(\mathbb{R})$ and we can repeat the method but without χ_k . The smallness assumption on $\|\Psi_i\|_{H_{\text{uloc}}^2}$, $i = 1, 2$, ensures for a constant $C > 0$

$$(1 - C\delta) \int_{\omega^b} |\Delta \Psi|^2 \leq 0,$$

which provides the uniqueness result when $\delta < C^{-1}$.

Step 2. Regularity. Higher regularity estimates for the solution are necessary for the subsequent application of the implicit function theorem. The analysis of the interior regularity and the regularity up to the boundary starts with the case of $m = 1$. This case is later used to obtain higher regularity estimates through induction.

II.6.2.0.1 First order interior regularity

Theorem II.6.1. *For any, $\Psi \in H_{\text{loc}}^2(\omega_w^b)$ which is a solution of*

$$\partial_X \Psi + (\nabla_w^\perp \Psi \cdot \nabla_w) \Delta_w \Psi - \Delta_w^2 \Psi = 0 \quad \text{in } \omega^b \quad (\text{II.6.27})$$

then $\Psi \in H_{\text{loc}}^s(\omega_w^b)$, $s \in \mathbb{N}$, $s \geq 2$. More precisely, for any bounded open set ω' satisfying $\omega' \subset\subset \omega_w^b$,

$$\|\Psi\|_{H^s(\omega')} \leq C \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)}, \quad (\text{II.6.28})$$

where C is a constant depending on ω' .

Proof. Let Ψ be a solution of (II.6.27) belonging to $H_{\text{uloc}}^2(\omega_w^b)$ and $\omega' \subset\subset \omega_w^b$. Note that $(\nabla_w^\perp \Psi \cdot \nabla_w) \Delta_w \Psi = \nabla_w \cdot ((\nabla_w^\perp \Psi) \Delta_w \Psi)$ and $\nabla_w^\perp \Psi \in H^1(\omega')$. Therefore, by Sobolev embedding

$$\nabla_w^\perp \Psi \in L^p(\omega') \quad \text{for any } p \in [1, +\infty).$$

As $\Delta_w \Psi \in L^2(\omega')$, we have that $(\Delta_w \Psi) \nabla_w^\perp \Psi \in L^q(\omega')$ for any $q \in [1, 2)$ and, on that account

$$\nabla_w \cdot ((\nabla_w^\perp \Psi) \Delta_w \Psi) \in W^{-1,q}(\omega') \quad \text{for any } q \in [1, 2).$$

Taking into account $\partial_X \Psi \in H^1(\omega') \subset W^{-1,q}(\omega')$ and using (II.6.27) provide $\Delta_w(\Delta_w \Psi) \in W^{-1,q}(\omega')$ for any $q \in [1, 2)$. Therefore, by means of classical elliptic regularity arguments, we obtain $\Delta_w \Psi \in W^{1,q}(\omega')$ for any $q \in [1, 2)$. This implies in turn that $\Psi \in W^{3,q}(\omega')$ for any $q \in [1, 2)$ and

$$\|\Psi\|_{W^{3,q}(\omega')} \leq C \|\Psi\|_{H^2(\omega'')} \leq C \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)},$$

where $\omega' \subset \omega'' \subset \omega_w^b$ and C is a constant depending only on ω' . Consequently, $\nabla^\perp \Psi \in W^{2,q}(\omega')$ for any $q \in [1, 2)$ leading to $\nabla^\perp \Psi \in W^{1,p}(\omega')$ for any $p \in [1, +\infty)$. Similarly, $\Delta_w \Psi \in W^{1,q}(\omega')$ for any $q \in [1, 2)$ which gives that $\Delta_w \Psi \nabla^\perp \Psi \in W^{1,q}(\omega')$ for any $q \in [1, 2)$ and $\Psi \in W^{4,q}(\omega')$ for any $q \in [1, 2)$. Repeating the procedure results in $\Psi \in W^{s,q}(\omega')$ for any $s \in \mathbb{N}^*$. Therefore, $\Psi \in H^s(\omega')$ for any $s \in \mathbb{R}$ by Sobolev embedding. In particular, there exists C depending only on ω' such that

$$\|\Psi\|_{H^3(\omega')} \leq C \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)}.$$

□

II.6.2.0.2 Regularity up to the boundary Since we are only interested in the regularity near the artificial boundary, we can consider without loss of generality that the behavior at the interface at $X = 0$ and the rough boundary does not influence our analysis. We tackle our regularity analysis for $X > M'$, where $M' \in (0, M)$. To prove H_{uloc}^3 -regularity up to the boundary, we need to compute a priori estimates for $\partial_Y \Psi$ in H_{uloc}^2 . First, we are going to localize the equation near a fixed $k \in \mathbb{Z}$. Let $\tilde{\varphi}_k \in C_0^\infty(\mathbb{R})$ be equal to 1 in a neighborhood of $k \in \mathbb{Z}$, and such that the size of $\text{Supp} \tilde{\varphi}_k$ is bounded uniformly in k . Moreover, we set $\varphi_k = \tilde{\varphi}_k^2$.

The idea is to apply a finite difference operator with a step $h > 0$ in the direction parallel to the boundary, that is to say parallel to the Y -axis, and then, pass to the limit when h goes to zero. This shows that $\partial_Y(\varphi_k \Psi_w^-) \in H^2(\omega_w^b)$. Then, using the equations, it implies that $\partial_X(\varphi_k \Psi_w^-) \in H^2(\omega_w^b)$ and thus $\varphi_k \Psi_w^- \in H^3(\omega_w^b)$. From the arbitrariness of $k \in \mathbb{Z}$ and φ_k , and from the interior regularity provided for the case when $m = 1$, this in turn implies that $\Psi_w^- \in H_{\text{uloc}}^3(\omega_w^b)$. Going forward, to alleviate the notation, we omit the k -dependence of φ and we denote Ψ instead of Ψ_w^- . We define the finite difference operator δ_h as follows:

$$\delta_h u = \frac{\tau_h u - u}{h}, \quad \tau_h u(X, Y) = u(X, Y + h).$$

Then, for $\omega' \subset\subset \omega_w^b$, there exists a constant $C > 0$ such that $|h| < \text{dist}(\partial\omega, \omega')$, and $f \in W^{k,p}(\omega)$,

$$\begin{aligned} \|\delta_h f\|_{W^{k-1,p}(\omega')} &\leq C \|f\|_{W^{k,p}(\omega)}, \\ \lim_{h \rightarrow 0} \|\delta_h(f) - \partial_Y f\|_{W^{k-1,p}(\omega')} &= 0. \end{aligned}$$

Considering $\psi_h = \delta_h(\varphi_k \Psi)$ in (II.6.7) yields

$$\begin{aligned} \partial_X \psi_h - \Delta_w^2 \psi_h &= f_h \quad \text{in } \omega^b \cap \text{Supp} \varphi_k, \\ (1 + \alpha^2) \Delta_w \psi_h &= \rho_2^h, \\ - \left[(1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right] \Delta_w \psi_h + \frac{\psi_h}{2} \Big|_{X=M} &= \rho_3^h, \\ \psi_h = \partial_X \psi_h &= 0, \quad \text{on } (M', M) \times \partial \text{Supp} \varphi_k. \end{aligned}$$

Taking into account that φ_k is independent of X , we have that

$$\begin{aligned} f_h &= -\delta_h \left[4\varphi_k^{(3)} (\partial_Y - \alpha \partial_X) \Psi + 6\varphi_k'' \Delta_w \Psi + 4\varphi_k' (\partial_Y - \alpha \partial_X) \Delta_w \Psi + \varphi_k^{(4)} \Psi \right. \\ &\quad \left. - \nabla_w \Psi \cdot \left(3(0, \varphi_k') \Delta_w \Psi + 3\varphi_k'' \nabla_w \Psi + (0, \varphi_k^{(3)}) \Psi \right) \right] - \delta_h \left[(\nabla_w^\perp \Psi \cdot \nabla_w) \Delta_w (\varphi_k \Psi) \right], \\ \rho_2^h &= \delta_h [\varphi_k \rho_2] - (1 + \alpha^2) \delta_h \left[2\varphi_k' (\partial_Y - \alpha \partial_X) \Psi + \varphi_k'' \Psi \right], \\ \rho_3^h &= \delta_h [\varphi_k \rho_3] + \delta_h \left[\left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \left(2\partial_Y \varphi_k (\partial_Y - \alpha \partial_X) \Psi + \partial_Y^2 \varphi_k \Psi \right) \right. \\ &\quad \left. - \nabla_w^\perp \Psi \cdot \left(-\alpha \nabla_w (\Psi \partial_Y \varphi_k) + \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi (0, \partial_Y) \varphi_k \right) - \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right) \right] \\ &\quad + \delta_h \left[(\nabla_w^\perp \Psi \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi) \right] - \delta_h [2\alpha \Delta_w \Psi \partial_Y \varphi_k]. \end{aligned}$$

Let us now state some technical lemmas which are necessary to the proof.

Lemma II.6.1. *Let $\sigma_w^M = \{(M, Y) : Y \in \mathbb{R}\}$. Define $\sigma_k = \sigma_w^M \cap \text{Supp} \varphi$ where $\varphi \in C_0^\infty(\mathbb{R})$ and it is equal to 1 in a neighborhood of σ_w^M . Consider the functions $\Psi \in H_{\text{uloc}}^2(\omega^b)$, $\rho_2 \in H_{\text{uloc}}^{1/2}(\sigma_M)$ and $\rho_3 \in H_{\text{uloc}}^{-1/2}(\sigma_M)$. Then, for any $h \in \mathbb{R}$, we have the estimates*

$$\begin{aligned} \|\rho_3^h\|_{H^{-3/2}(\sigma_k)} &\leq C(1 + \|\varphi\|_{W^{4,\infty}}) (\|\rho_3\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)} + \|\rho_2\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \\ &\quad + \|\Psi\|_{H_{\text{uloc}}^2} (1 + \|\psi_h\|_{H_{\text{uloc}}^2} + \|\Psi\|_{H_{\text{uloc}}^2})), \\ \|\rho_2^h\|_{H^{-1/2}(\sigma_k)} &\leq C \left(\|\rho_2\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} + \|\Psi\|_{H_{\text{uloc}}^2} \right) \|\varphi\|_{W^{3,\infty}}, \\ \|f_h\|_{H^{-2}(\omega_w^b)} &\leq C (\|\Psi\|_{H_{\text{uloc}}^2} (1 + \|\Psi\|_{H_{\text{uloc}}^2} + \|\psi_h\|_{H_{\text{uloc}}^2} + \|\Psi_L\|_{H_{\text{uloc}}^2})) (1 + \|\varphi\|_{W^{3,\infty}(\omega_w^b)}), \end{aligned}$$

where C is a constant depending only on the domain ω_w^b .

Lemma II.6.2. *Consider the linear problem*

$$\begin{aligned} \partial_X z - \Delta_w^2 z &= f_1 \quad \text{in } \omega_k, \\ (1 + \alpha^2) \Delta_w z &= f_2, \\ - \left[(1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right] \Delta_w z + \frac{z}{2} \Big|_{\sigma_k} &= f_3, \\ z = \partial_X z = 0, &\quad \text{on } \partial \omega_k \setminus \sigma_k, \end{aligned} \tag{II.6.29}$$

where $\omega_k = \omega^b \cap \text{Supp} \varphi$, $\sigma_k = \partial \omega_k \cap \partial \omega_w^b$ and ν stands for the unit outer normal vector at the boundary. If $f_1 \in H^{-2}(\omega_k)$, $f_2 \in H^{-1/2}(\sigma_k)$, $f_3 \in H^{-3/2}(\sigma_k)$. Then, problem (II.6.29) has a unique solution $z \in H^2(\omega_k)$ and satisfies the estimate

$$\|z\|_{H^2(\omega_k)} \leq C \left(\|f_1\|_{H^{-2}(\omega_k)} + \|f_2\|_{H^{-1/2}(\sigma_k)} + \|f_3\|_{H^{-3/2}(\sigma_k)} \right). \tag{II.6.30}$$

The regularity up to the boundary and the fact that $\varphi\Psi \in H^3(\omega_w^b)$ is a consequence of Lemmas II.6.1 and II.6.2 for $z = \psi_h$. Combining these two Lemmas with the smallness assumption on $\|\Psi\|_{H^2_{\text{uloc}}(\omega_w^b)}$, we have that $\psi_h \in H^2(\omega_w^b)$ and, for any $h \in \mathbb{R}^*$

$$\|\psi_h\|_{H^2(\omega_w^b)} \leq C \left(\|\Psi\|_{H^2_{\text{uloc}}(\omega_w^b)} + \|\rho_2\|_{H^{1/2}_{\text{uloc}}(\sigma_k)} + \|\rho_3\|_{H^{1/2}_{\text{uloc}}(\sigma_k)} \right),$$

for some constant C depending on the bound on $\|\Psi\|_{H^2_{\text{uloc}}(\omega_w^b)}$ but independent of h . This implies that $\psi_h \in H^2(\omega_w^b)$ and therefore, $\partial_Y(\Psi\varphi) \in H^2(\omega_w^b)$. From the arbitrariness of φ and as $\Psi \in H^2(\omega_w^b)$, this means that $\partial_Y\Psi \in H^2_{\text{uloc}}(\omega_w^b)$. Hence, that $\partial_Y\partial_X\Psi \in H^{1/2}_{\text{uloc}}(\sigma_k)$. In particular $\partial_X\Psi \in H^{3/2}_{\text{uloc}}(\sigma_k)$. The first boundary condition of (II.6.7) yields

$$\partial_X^2\Psi = (1 + \alpha)^{-1} \left(-\partial_Y^2\Psi + 2\alpha\partial_Y\partial_X\Psi + (1 + \alpha)^{-1}\rho_2 \right),$$

which implies that $\partial_X^2\Psi \in H^{1/2}_{\text{uloc}}(\sigma_k)$. This result combined with the second boundary condition in (II.6.7) gives $\partial_X\nabla_w\Psi \in H^{-1/2}_{\text{uloc}}(\sigma_k)$ and, consequently, $\Delta_w(\partial_X\Psi) \in H^{-1/2}_{\text{uloc}}(\sigma_k)$. Using now the main equation in (II.6.7), we deduce that $\Delta_w^2\Psi \in H^{-1}_{\text{uloc}}(\omega_w^b)$ and $\partial_X^2\Delta_w\Psi \in H^{-1}_{\text{uloc}}(\omega_w^b)$. Hence, $\partial_X^2\Delta\Psi \in H^{-3/2}_{\text{uloc}}(\sigma_k)$ and $A_3[\partial_X\Psi] \in H^{-3/2}_{\text{uloc}}(\sigma_k)$. Let $\chi \in C_c^\infty(\omega_w^b)$, setting

$$\begin{aligned} f_1 &= \Delta_w^2(\chi\partial_X\Psi) - (\chi\partial_X\Psi), \\ f_2 &= (1 + \alpha^2)\Delta_w(\chi\partial_X\Psi), \\ f_3 &= -[(1 + \alpha^2)\partial_X - 2\alpha\partial_Y]\Delta_w(\chi\partial_X\Psi) + \frac{\chi\partial_X\Psi}{2}, \end{aligned}$$

yields that $\chi\partial_X\Psi$ is solution to a linear problem of the form (II.6.29) where $f_1 \in H^{-2}(\sigma_k)$, $f_2 \in H^{-1/2}(\sigma_k)$, $f_3 \in H^{-2}(\sigma_k)$. From Lemma II.6.2, $\chi\partial_X\Psi \in H^2(\omega_w^b)$ and paired with the arbitrariness of χ $\partial_X\Psi \in H^2(\omega_w^b)$. Finally, combining this with $\partial_Y\Psi \in H^2(\omega_w^b)$ gives $\Psi \in H^3(\omega_w^b)$. It remains to prove the results in Lemmas II.6.1 and II.6.2. Lemma II.6.2 is a standard elliptic regularity result. The main difficulty resides in the proof of the estimates in Lemma II.6.1 when handling the nonlinear terms.

Proof of Lemma II.6.2. Before computing the elliptic regularity estimates, let us first briefly comment on the existence of an appropriate weak solution z of the boundary-value problem (II.6.29). Note that the weak formulation associated with (II.6.29) is

$$\begin{aligned} \int_{\omega_k} \partial_X z \theta - \int_{\omega_k} \Delta_w z \Delta_w \theta &= \int_{\omega_k} f_1 \theta - \int_{\sigma_k} f_3 \theta|_{X=M} \\ &\quad - \int_{\sigma_k} f_2 \partial_X \theta|_{X=M} + \int_{\sigma_k} \frac{1}{2} z|_{X=M} \theta|_{X=M}, \quad \forall \theta \in H^2(\omega_k), \end{aligned} \quad (\text{II.6.31})$$

which can be written as follows:

$$A(z, \theta) + a(z, \theta) = (f_1, \theta) - \int_{\sigma_k} f_3 \theta|_{X=M} - \int_{\sigma_k} f_2 \partial_X \theta|_{X=M} + \int_{\sigma_k} \frac{1}{2} z|_{X=M} \theta|_{X=M}, \quad (\text{II.6.32})$$

where $A(z, \theta) = \int_{\omega_k} \partial_X z \theta - \int_{\omega_k} \Delta_w z \Delta_w \theta$ and $a(z, \theta)$ contains the integral boundary terms. Here, (\cdot, \cdot) refers to the usual product in L^2 . The differential operator A is a bilinear and

continuous form in $H^2(\omega_k) \times H^2(\omega_k)$, as well as, $H_0^2(\omega_k)$ -elliptic. The existence and uniqueness of a solution $z \in H^2(\omega_k)$ is guaranteed by applying [70, Theorem 3.1, Section 1.3.2]. Taking $\theta = z$ in (II.6.31) yields the expression

$$\int_{\omega_k} |\Delta_w z|^2 = - \int_{\omega_k} f_1 z + \int_{\sigma_k} f_3 z + \int_{\sigma_k} f_2 \partial_X z, \quad (\text{II.6.33})$$

which leads naturally to the estimate by using Cauchy-Schwarz inequality, the definition of the norm in dual spaces and the trace theorem

$$\int_{\omega_k} |\Delta_w z|^2 \leq C \left(\|f\|_{H^{-2}(\omega_k)} + \|f_2\|_{H^{-1/2}(\sigma_k)} + \|f_3\|_{H^{-3/2}(\sigma_k)} \right) \|z\|_{H^2(\omega_k)}.$$

Finally, we obtain the desired result

$$\|z\|_{H^2(\omega_k)} \leq C \left(\|f\|_{H^{-2}(\omega_k)} + \|f_2\|_{H^{-1/2}(\sigma_k)} + \|f_3\|_{H^{-3/2}(\sigma_k)} \right).$$

□

Proof of Lemma II.6.1. This proof is divided in three parts corresponding to each one of the estimates of ρ_3^h , ρ_2^h and f^h .

Estimate of ρ_3^h . We have

$$\begin{aligned} \rho_3^h &= \delta_h [\varphi_k \rho_3] + \delta_h \left[\left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \left(2\partial_Y \varphi_k (\partial_Y - \alpha \partial_X) \Psi + \partial_Y^2 \varphi_k \Psi \right) \right. \\ &\quad \left. - \nabla_w^\perp \Psi \cdot \left(-\alpha \nabla_w (\Psi \partial_Y \varphi_k) + \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi (0, \partial_Y) \varphi_k \right) - \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right) \right] \\ &\quad + \delta_h \left[(\nabla_w^\perp \Psi \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi) \right] - \delta_h [2\alpha \Delta_w \Psi \partial_Y \varphi_k]. \end{aligned}$$

The first term on the r.h.s satisfies

$$\|\delta_h [\varphi_k \rho_3]\|_{H^{-3/2}(\sigma_k)} \leq C \|\rho_3\|_{H_{\text{uloc}}^{-1/2}(\sigma_M)}, \quad (\text{II.6.34})$$

Let us give a closer look at the second term. Enclosed in brackets are terms involving at most one order derivative of Ψ multiplied by a derivative of φ_k , which has compact support on σ_k . Let us now recall that Ψ belongs to $H_{\text{uloc}}^{3/2}$ and $\partial_X \Psi$ belongs to $H_{\text{uloc}}^{1/2}$ at $X = M$.

Moreover, using the conditions at σ_w^M , we get

$$\|\partial_X^2 \Psi\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)} \leq C (\|\Psi\|_{H_{\text{uloc}}^2(\omega^b)} + \|\rho_2\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)}). \quad (\text{II.6.35})$$

Hence,

$$\begin{aligned} &\left\| \delta_h \left[\left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \left(2\partial_Y \varphi_k (\partial_Y - \alpha \partial_X) \Psi + \partial_Y^2 \varphi_k \Psi \right) \right] \right\|_{H^{-3/2}(\sigma_k)} \\ &\leq C \left(\|\rho_2\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \right) \|\varphi_k\|_{W^{4,\infty}}. \end{aligned} \quad (\text{II.6.36})$$

Now, observe that for $f \in H_{\text{uloc}}^{1/2}(\sigma_k, \mathbb{R}^2)$, $g \in H_{\text{uloc}}^{1/2}(\sigma_k, \mathbb{R}^2)$, $\theta \in C_c^\infty(\sigma_k, \mathbb{R}_+)$ and as a result of the Sobolev embedding $H^{1/2}(\mathbb{R}) \subset L^p(\mathbb{R})$, for any $p \in [1, +\infty)$, we have

$$\|f g \theta\|_{H^{-1/2}(\sigma_k)} \leq \|f g \theta\|_{L^p(\sigma_k)} \leq \|f\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \|g\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \|\theta\|_{L^\infty}. \quad (\text{II.6.37})$$

Consequently, using Trace theorem

$$\begin{aligned}
& \left\| \delta_h \left[-\nabla_w^\perp \Psi \cdot \left(-\alpha \nabla_w (\Psi \partial_Y \varphi_k) + \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi(0, \partial_Y) \varphi_k \right) \right] \right\|_{H^{-3/2}(\sigma_k)} \\
& \leq C \left(\|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \right) \|\varphi_k\|_{W^{2,\infty}(\omega_w^b)} \\
& \leq C \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)}^2 \|\varphi_k\|_{W^{2,\infty}(\omega_w^b)}.
\end{aligned} \tag{II.6.38}$$

Finally, note that

$$\begin{aligned}
(\partial_Y^2 \Psi) \varphi_k &= \partial_Y^2 (\Psi \varphi_k) - 2(\partial_Y \Psi \partial_Y \varphi_k) - (\Psi \partial_Y^2 \varphi_k), \\
(\partial_X^2 \Psi) \varphi_k &= \partial_X^2 (\Psi \varphi_k), \\
(\partial_{YX}^2 \Psi) \varphi_k &= \partial_{YX}^2 (\Psi \varphi_k) - (\partial_X \Psi \partial_Y \varphi_k).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \delta_h \left[- \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right) \right] \\
& = \delta_h \left[- \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \varphi_k \right) \right) \right] - \delta_h \left[(\partial_Y \varphi_k) \frac{|\nabla_w^\perp \Psi|^2}{2} \right].
\end{aligned} \tag{II.6.39}$$

The second term is a quadratic in Ψ , linear in φ_k and involves at most one derivative of Ψ and one derivative of φ_k . Therefore,

$$\begin{aligned}
\|\delta_h \left[(\partial_Y \varphi_k) \frac{|\nabla_w^\perp \Psi|^2}{2} \right]\|_{H^{-3/2}(\sigma_k)} &\leq C \left\| (\partial_Y \varphi_k) \frac{|\nabla_w^\perp \Psi|^2}{2} \right\|_{H^{-1/2}(\sigma_k)} \\
&\leq C \|\varphi_k\|_{W^{1,\infty}(\sigma_k)} \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right)^2.
\end{aligned} \tag{II.6.40}$$

The first term of (II.6.39) can be treated as follows

$$\begin{aligned}
\delta_h (|\nabla_w^\perp \Psi|^2 \varphi_k) &= \delta_h \left[\left((1 + \alpha^2) (\partial_X \Psi)^2 - 2\alpha \partial_X \Psi \partial_Y \Psi + (\partial_Y \Psi)^2 \right) \varphi_k \right] \\
&= 2(1 + \alpha^2) (\partial_X \psi_h) (\partial_X \Psi) - 2\alpha \partial_X \psi_h \partial_Y \Psi - 2\alpha \partial_X \Psi \partial_Y \psi_h + 2(\partial_Y \Psi) (\partial_Y \psi_h) \\
&+ F_1(\Psi, \delta_h \varphi_k, \varphi_k) + F_2(\Psi, \delta_h \Psi, \varphi_k),
\end{aligned} \tag{II.6.41}$$

where $F_1(\Psi, \delta_h \varphi_k)$ is a sum of terms quadratic in Ψ , linear in $\delta_h \varphi_k$ and involving at most one derivative of Ψ and $\delta_h \varphi_k$. $F_2(\Psi, \delta_h \Psi, \varphi_k)$, on the other hand, is a sum of terms linear in Ψ , linear in $\delta_h \Psi$ and linear in φ_k involving at most one derivative in Ψ , no derivative in $\delta_h \Psi$ and one derivative in φ_k . As a result of (II.6.37), we have

$$\|F_1(\Psi, \delta_h \varphi_k)\|_{H^{-1/2}(\sigma_k)} \leq C \left(\|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \right)^2 \|\varphi_k\|_{W^{1,\infty}}, \tag{II.6.42}$$

and

$$\begin{aligned}
\|F_2(\Psi, \delta_h \Psi, \varphi_k)\|_{H^{-1/2}(\sigma_k)} &\leq C \left(\|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \right) \|\delta_h \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \|\varphi_k\|_{W^{1,\infty}} \\
&\leq C \left(\|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \right)^2 \|\varphi_k\|_{W^{1,\infty}}.
\end{aligned} \tag{II.6.43}$$

The reminder of (II.6.41) is now easy to handle using (II.6.37)

$$\begin{aligned} & \|2(1 + \alpha^2)(\partial_X \psi_h)(\partial_X \Psi) - 2\alpha \partial_X \psi_h \partial_Y \Psi - 2\alpha \partial_X \Psi \partial_Y \psi_h + 2(\partial_Y \Psi)(\partial_Y \psi_h)\|_{H_{\text{uloc}}^{-1/2}(\sigma_k)} \\ & \leq C \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right) \left(\|\partial_X \psi_h\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\psi_h\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right). \end{aligned} \quad (\text{II.6.44})$$

Using (II.6.40), (II.6.41), (II.6.42), (II.6.43), and (II.6.44) one has

$$\begin{aligned} & \left\| \delta_h \left[- \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right) \right] \right\|_{H^{-3/2}(\sigma_k)} \\ & \leq \left\| \delta_h \left[\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right] \right\|_{H^{-3/2}(\sigma_k)} + C \|\varphi_k\|_{W^{1,\infty}(\sigma_k)} \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right)^2 \\ & \leq C \left\| \delta_h \left[\frac{|\nabla_w^\perp \Psi|^2}{2} \varphi_k \right] \right\|_{H^{-1/2}(\sigma_k)} + C \|\varphi_k\|_{W^{1,\infty}(\sigma_k)} \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right)^2 \\ & \leq C \left(\|\varphi_k\|_{W^{1,\infty}(\sigma_k)} \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right)^2 \right. \\ & \quad \left. + \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right) \left(\|\partial_X \psi_h\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} + \|\psi_h\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} \right) \right). \end{aligned}$$

Combining the above result with the Trace theorem gives

$$\left\| \delta_h \left[- \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right) \right] \right\|_{H^{-3/2}(\sigma_k)} \leq C \left(\|\psi_h\|_{H^2(\omega_w^b)} \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)} + \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)}^2 \right) \|\varphi_k\|_{W^{1,\infty}}.$$

It remains to tackle the two last terms of (II.6.34), starting with

$$\delta_h \left[(\nabla_w^\perp \Psi \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi) \right].$$

Since too many derivatives are involved in the above expression, this term cannot be controlled roughly by controlling $\delta_h[\nabla_w^\perp \Psi]$ by $|\nabla(\nabla_w^\perp \Psi)|$. To address this issue, we make $\delta_h(\varphi_k \Psi) = \psi_h$ appear, similarly as we did for (II.6.39). We claim

$$\begin{aligned} & \delta_h \left(([\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi) \right) \\ & = (\tau_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\delta_h(\varphi_k \Psi)) + (\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi) \\ & = (\tau_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\psi_h) + (\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi). \end{aligned} \quad (\text{II.6.45})$$

For the first term

$$\begin{aligned} & \|(\tau_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\psi_h)\|_{H^{-3/2}(\sigma_k)} \\ & \leq C \|(\tau_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\psi_h)\|_{L^1(\sigma_k)} \\ & \leq \left(\|\psi_h\|_{H^{3/2}(\sigma_k)} + \|\partial_X \psi_h\|_{H^{1/2}(\sigma_k)} \right) \left(\|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_k)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} \right). \end{aligned}$$

This inequality results from the Sobolev embedding $L^1(\sigma_k) \subset H^{-3/2}(\sigma_k)$. It can also be seen coming back to the definition of $H^{-3/2}(\sigma_k)$ as done below at (II.6.47)-(II.6.49). For the second term of (II.6.45), we have

$$\begin{aligned} & (\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi) \\ &= \left[(\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] \varphi_k + (\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left(\left[\left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \varphi_k \right] \Psi \right) \\ & \quad + \left[(\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \varphi_k \right] \left[\left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right]. \end{aligned} \tag{II.6.46}$$

Note that the second and third terms above are easier to deal with as they have fewer derivatives of Ψ . Indeed, it consists of the product of two terms: the first one involves $\delta_h[\nabla_w^\perp \Psi]$ while the second one contains at most a first order derivative of Ψ and is multiplied by a term with compact support. Accordingly, the latter belongs to $H^{1/2}(\sigma_k)$. Using that for $f \in H^{1/2}(\sigma_k)$ and $g \in H_0^{3/2}(\sigma_k)$, we have $g \in W^{1/2, \infty}$ and $fg \in H^{1/2}(\sigma_k)$. Thus,

$$\|fg\|_{H^{1/2}(\sigma_k)} \leq C \|f\|_{H^{1/2}(\sigma_k)} \|g\|_{H^{3/2}(\sigma_k)}. \tag{II.6.47}$$

For $f_1 \in H_{\text{uloc}}^{1/2}(\sigma_w^M)$, $f_2 \in H_{\text{uloc}}^{1/2}(\sigma_w^M)$, $\varphi_k \in C_c^\infty(\sigma_k)$, and $g \in H_0^{3/2}(\sigma_k)$, using (II.6.37) gives

$$\begin{aligned} \left| \int_{\mathbb{R}} (\delta_h f_1) f_2 \varphi_k g \right| &= \left| \int_{\mathbb{R}} (\delta_h f_1 \bar{\varphi}_k) f_2 \varphi_k g \right| \\ &\leq \|\delta_h(f_1 \bar{\varphi}_k)\|_{H^{-1/2}(\mathbb{R})} \|f_2 \varphi_k g\|_{H^{1/2}(\mathbb{R})} \\ &\leq C \|f_1 \bar{\varphi}_k\|_{H^{1/2}(\mathbb{R})} \|f_2\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} \|\varphi_k\|_{L^\infty(\mathbb{R})} \|g\|_{H^{3/2}(\sigma_k)} \\ &\leq C \|f_1\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} \|f_2\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} \|\varphi_k\|_{W^{1, \infty}(\mathbb{R})} \|g\|_{H^{3/2}(\sigma_k)}, \end{aligned} \tag{II.6.48}$$

where $\bar{\varphi}_k$ belongs to $C_0^\infty(\mathbb{R})$ and satisfies $\text{Supp}(\varphi_k) \subset \text{Supp}(\bar{\varphi}_k)$. This implies that

$$\|(\delta_h f_1) f_2 \varphi_k\|_{H^{-3/2}(\sigma_k)} \leq C \|f_1\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} \|f_2\|_{H_{\text{uloc}}^{1/2}(\sigma_k)} \|\varphi_k\|_{W^{1, \infty}(\mathbb{R})}. \tag{II.6.49}$$

Hence,

$$\begin{aligned} & (\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left(\left[\left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \varphi_k \right] \Psi \right) \\ & \quad + \left[(\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \varphi_k \right] \left[\left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] \|_{H^{-3/2}(\sigma_k)} \\ & \leq C (\|\Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)})^2 \|\varphi_k\|_{W^{2, \infty}}. \end{aligned} \tag{II.6.50}$$

Now, the first term in the r.h.s of (II.6.46) satisfies

$$\begin{aligned} & \left[(\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] \varphi_k \\ &= \left[\varphi_k \left([\nabla_w^\perp \delta_h \Psi] \cdot \nabla_w \right) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] \\ &= \left[([\nabla_w^\perp \delta_h(\varphi_k \Psi)] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] - \left[([\nabla_w^\perp \varphi_k] \delta_h(\Psi)) \cdot \nabla_w \right] \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \\ & \quad - \left[([\nabla_w^\perp [\delta_h(\varphi_k) \tau_h \Psi]] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right], \end{aligned} \tag{II.6.51}$$

where τ_h is the translation of amplitude h on the Y axis. Once again, the second and third terms above are easier to deal with as they include fewer derivatives and can be bounded similarly to (II.6.50). Now, we are left with analyzing the first term. We have

$$\left[(\nabla_w^\perp \delta_h(\varphi_k \Psi)) \cdot \nabla_w \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] = \left[(\nabla_w^\perp \psi_h \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right].$$

Similarly to the case of (II.6.50), using that $\psi_h \in H^{3/2}(\sigma_k)$ and $\partial_X \psi_h \in H^{1/2}(\sigma_k)$ yields

$$\begin{aligned} & \left\| \left[(\nabla_w^\perp \delta_h(\varphi_k \Psi)) \cdot \nabla_w \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] \right\|_{H^{-3/2}(\sigma_k)} \\ &= \left\| \left[(\nabla_w^\perp \psi_h \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi \right] \right\|_{H^{-3/2}(\sigma_k)} \\ &\leq C \left(\|\psi_h\|_{H^{3/2}(\sigma_k)} + \|\partial_X \psi_h\|_{H^{1/2}(\sigma_k)} \right) \left(\|\Psi\|_{H_{\text{uloc}}^{3/2}(\sigma_w^M)} + \|\partial_X \Psi\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} + \|\rho_h^2\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)} \right). \end{aligned} \quad (\text{II.6.52})$$

Finally, taking $f \in H_{\text{uloc}}^{-1/2}(\sigma_k)$, $l \in C_c^\infty(\sigma_k)$, and $g \in H^{3/2}(\sigma_k)$ we have

$$\begin{aligned} \left| \int_{\mathbb{R}} f g l \right| &\leq \|f\|_{H_{\text{uloc}}^{-1/2}(\sigma_k)} \|g l\|_{H^{1/2}(\sigma_k)} \\ &\leq \|f\|_{H_{\text{uloc}}^{-1/2}(\sigma_k)} \|g\|_{H^{3/2}(\sigma_k)} \|l\|_{W^{2,\infty}(\sigma_k)}. \end{aligned}$$

Then, we infer

$$\|f l\|_{H^{-3/2}(\sigma_k)} \leq \|f\|_{H_{\text{uloc}}^{-1/2}(\sigma_k)} \|l\|_{W^{2,\infty}(\sigma_k)}.$$

From (II.6.35) follows that

$$\begin{aligned} \|\Delta_w \Psi \partial_Y \varphi_k\|_{H^{-3/2}(\sigma_k)} &\leq C \|\Delta_w \Psi\|_{H_{\text{uloc}}^{-1/2}(\sigma_k)} \|\varphi_k\|_{W^{2,\infty}(\sigma_k)} \\ &\leq C \left(\|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)} + \|\rho_2\|_{H_{\text{uloc}}^{-1/2}(\sigma_k)} \right) \|\varphi_k\|_{W^{2,\infty}(\sigma_k)}. \end{aligned} \quad (\text{II.6.53})$$

Therefore, from (II.6.34), (II.6.38), (II.6.42), (II.6.51), (II.6.50), (II.6.52), (II.6.53) and applying the Trace theorem

$$\|\rho_3^h\|_{H^{-3/2}(\sigma_k)} \leq C(1 + \|\varphi_k\|_{W^{4,\infty}}) \left(\|\rho_2\|_{H_{\text{uloc}}^{-1/2}(\sigma_w^M)} + \|\Psi\|_{H_{\text{uloc}}^2} (1 + \|\psi_h\|_{H_{\text{uloc}}^2} + \|\Psi\|_{H_{\text{uloc}}^2}) \right),$$

which is the desired estimate.

II.6.2.0.3 Estimate of f_h . Let us recall the expression of f_h ,

$$\begin{aligned} f_h &= -\delta_h \left[4\varphi_k^{(3)} (\partial_Y - \alpha \partial_X) \Psi + 6\varphi_k'' \Delta_w \Psi + 4\varphi_k' (\partial_Y - \alpha \partial_X) \Delta_w \Psi + \varphi_k^{(4)} \Psi \right. \\ &\quad \left. - \partial_X \Psi \left(3\varphi_k' \Delta_w \Psi + \varphi_k^{(3)} \Psi \right) - 3\varphi_k'' \Delta_w \Psi \right] - \delta_h \left[(\nabla_w^\perp \Psi \cdot \nabla_w) \Delta_w (\varphi_k \Psi) \right]. \end{aligned} \quad (\text{II.6.54})$$

Let us analyze the first (large) term of (II.6.54). Note that all the terms enclosed in brackets that are not quadratic in Ψ involve at most a third order derivative of Ψ and are all proportional to a derivative of φ_k (hence compactly supported). This implies that

$$\begin{aligned} & \left\| \delta_h \left[4\varphi_k^{(3)} (\partial_Y - \alpha \partial_X) \Psi + 6\varphi_k'' \Delta_w \Psi + 4\varphi_k' (\partial_Y - \alpha \partial_X) \Delta_w \Psi + \varphi_k^{(4)} \Psi \right] \right\|_{H^{-2}(\omega_w^b)} \\ &\leq C \left\| 4\varphi_k^{(3)} (\partial_Y - \alpha \partial_X) \Psi + 6\varphi_k'' \Delta_w \Psi + 4\varphi_k' (\partial_Y - \alpha \partial_X) \Delta_w \Psi + \varphi_k^{(4)} \Psi \right\|_{H^{-1}(\omega_w^b)} \\ &\leq C \|\Psi\|_{H_{\text{uloc}}^2} \|\varphi_k\|_{W^{4,\infty}(\omega_w^b)}. \end{aligned} \quad (\text{II.6.55})$$

The quadratic terms between the brackets in the second term of (II.6.54) can be treated in an analogous manner

$$\begin{aligned} & \left\| \delta_h \left[-\partial_X \Psi \left(3\varphi'_k \Delta_w \Psi + \varphi_k^{(3)} \Psi \right) - 3\varphi_k'' \Delta_w \Psi \right] \right\|_{H^{-2}(\omega_w^b)} \\ & \leq C \left\| -\partial_X \Psi \left(3\varphi'_k \Delta_w \Psi + \varphi_k^{(3)} \Psi \right) - 3\varphi_k'' \Delta_w \Psi \right\|_{H^{-1}(\omega_w^b)} \\ & \leq C \|\Psi\|_{H_{\text{uloc}}^2}^2 \|\varphi_k\|_{W^{3,\infty}(\omega_w^b)}. \end{aligned} \quad (\text{II.6.56})$$

On the last term on the r.h.s of (II.6.54), we use the property

$$\delta_h \left[\left((\nabla_w^\perp \Psi) \cdot \nabla_w \right) \Delta_w (\varphi_k \Psi) \right] = \delta_h \left(\nabla_w \cdot \left(\Delta_w (\varphi_k \Psi) \nabla_w^\perp \Psi \right) \right),$$

by virtue of $\nabla_w \cdot \nabla_w^\perp (\delta_h \Psi) = 0$. Therefore,

$$\left\| \delta_h \left[\left((\nabla_w^\perp \Psi) \cdot \nabla_w \right) \Delta_w (\varphi_k \Psi) \right] \right\|_{H^{-2}(\omega_w^b)} \leq C \left\| \delta_h \left(\Delta_w (\varphi_k \Psi) \nabla_w^\perp \Psi \right) \right\|_{H^{-1}(\omega_w^b)},$$

and it suffices to estimate $\delta_h \left(\Delta_w (\varphi_k \Psi) \left(\nabla_w^\perp \Psi \right) \right)$ is the $H^{-1}(\omega_w^b)$ norm. We get

$$\begin{aligned} \delta_h \left(\Delta_w (\varphi_k \Psi) \left(\nabla_w^\perp \Psi \right) \right) &= \Delta_w (\varphi_k \Psi) \delta_h \left(\nabla_w^\perp \Psi \right) + \delta_h \Delta_w (\varphi_k \Psi) \tau_h \left(\nabla_w^\perp \Psi \right) \\ &= \Delta_w (\varphi_k \Psi) \delta_h \left(\nabla_w^\perp \Psi \right) + \Delta_w (\psi_h) \tau_h \left(\nabla_w^\perp \Psi \right). \end{aligned} \quad (\text{II.6.57})$$

The second term satisfies

$$\left\| \Delta_w (\psi_h) \tau_h \left(\nabla_w^\perp \Psi \right) \right\|_{H^{-1}(\omega_w^b)} \leq C \|\psi_h\|_{H^2(\omega_w^b)} \|\Psi\|_{H_{\text{uloc}}^2(\omega)}. \quad (\text{II.6.58})$$

The above estimate is obtained from applying the following result: *let $f \in H_{\text{uloc}}^1(\omega_w^b)$, $g \in L^2(\omega_w^b)$ with compact support, and $l \in H^1(\omega_w^b)$,*

$$\begin{aligned} \left| \int_{\mathbb{R}} f g l \right| &\leq \|g\|_{L^2(\omega_w^b)} \|f l\|_{L_{\text{uloc}}^2(\omega_w^b)} \leq C \|g\|_{L^2(\omega_w^b)} \|f\|_{L_{\text{uloc}}^4(\omega_w^b)} \|l\|_{L^4(\omega_w^b)}, \\ &\leq C \|g\|_{L^2(\omega_w^b)} \|f\|_{H_{\text{uloc}}^1(\omega_w^b)} \|l\|_{H^1(\omega_w^b)}, \end{aligned} \quad (\text{II.6.59})$$

hence,

$$\|f g\|_{H^{-1}(\omega_w^b)} \leq C \|g\|_{L^2(\omega_w^b)} \|f\|_{H_{\text{uloc}}^1(\omega_w^b)}. \quad (\text{II.6.60})$$

The first term in (II.6.57) satisfies

$$\Delta_w (\varphi_k \Psi) \delta_h \left(\nabla_w^\perp \Psi \right) = \varphi_k \Delta_w (\Psi) \delta_h \left(\nabla_w^\perp \Psi \right) + F(\delta_h \Psi, \Psi, \varphi_k), \quad (\text{II.6.61})$$

where $F(\delta_h \Psi, \Psi, \varphi_k)$ is a linear combination of functions depending linearly of Ψ , $\delta_h \Psi$ and φ_k , compactly supported and involving at most a first order derivative of $\delta_h \Psi$, a first order derivative of Ψ and a second order derivative in φ_k . Therefore, proceeding similarly as in (II.6.48) but in $H^1(\omega_w^b)$ norm, we have

$$\begin{aligned} \|F(\delta_h \Psi, \Psi, \varphi_k)\|_{H^{-1}(\omega_w^b)} &\leq C \|\delta_h \Psi\|_{H_{\text{uloc}}^1} \|\Psi\|_{H_{\text{uloc}}^2} \|\varphi_k\|_{W^{2,\infty}} \\ &\leq C \|\Psi\|_{H_{\text{uloc}}^2}^2 \|\varphi_k\|_{W^{2,\infty}}. \end{aligned} \quad (\text{II.6.62})$$

The first term of (II.6.61) can be estimated noting that $\delta_h(fg) = (\delta_h f)g + \delta_h g \tau_h f$ where τ_h denotes the translation operator of size h along the Y -axis. This gives

$$\begin{aligned} \varphi_k \Delta_w(\Psi) \delta_h \left(\nabla_w^\perp \Psi \right) &= \Delta_w(\Psi) \delta_h \left(\nabla_w^\perp (\Psi \varphi_k) \right) \\ &- \Delta_w(\Psi) \delta_h \left(\Psi \nabla_w^\perp \varphi_k \right) - \Delta_w(\Psi) \left(\delta_h \varphi_k \tau_h (\nabla_w^\perp \Psi) \right) \\ &= \Delta_w(\Psi) \left(\nabla_w^\perp \psi_h \right) - \Delta_w(\Psi) \delta_h \left(\Psi \nabla_w^\perp \varphi_k \right) - \Delta_w(\Psi) \left(\delta_h \varphi_k \tau_h (\nabla_w^\perp \Psi) \right). \end{aligned}$$

The last two terms are now easier to bound. Using again (II.6.59)-(II.6.60) for a function $f \in H^1(\omega_w^b)$ with compact support and $g \in L_{\text{uloc}}^2(\omega_w^b)$ leads to

$$\begin{aligned} \|\varphi_k \Delta_w(\Psi) \delta_h \left(\nabla_w^\perp \Psi \right)\|_{H_{\text{uloc}}^{-1}(\omega_w^b)} &\leq C \|\Delta_w \Psi\|_{L_{\text{uloc}}^2(\omega_w^b)} \|\nabla_w^\perp \psi_h\|_{H^1(\omega_w^b)} + \|\varphi_k\|_{W^{2,\infty}(\omega_w^b)} \|\Psi\|_{H^2(\omega_w^b)}^2 \\ &\leq C \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)} \left(\|\psi_h\|_{H^2(\omega_w^b)} + \|\varphi_k\|_{W^{2,\infty}(\omega_w^b)} \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)} \right). \end{aligned} \quad (\text{II.6.63})$$

Therefore, from equations (II.6.55), (II.6.56), (II.6.58), (II.6.62), and (II.6.63), we have

$$\|f_h\|_{H^{-2}(\omega_w^b)} \leq C \left(\|\Psi\|_{H_{\text{uloc}}^2} (1 + \|\Psi\|_{H_{\text{uloc}}^2} + \|\psi_h\|_{H_{\text{uloc}}^2}) \right) (1 + \|\varphi_k\|_{W^{4,\infty}(\omega_w^b)}).$$

II.6.2.0.4 Estimate of ρ_2^h . We have

$$\begin{aligned} \rho_2^h &= \delta_h[\varphi_k \rho_2] + (1 + \alpha^2) \delta_h \left[2\partial_Y \varphi_k (\partial_Y - \alpha \partial_X) \Psi + \partial_Y^2 \varphi_k \Psi \right] \\ &= \delta_h[\varphi_k \rho_2] + (1 + \alpha^2) \left[2\partial_Y \varphi_k (\partial_Y - \alpha \partial_X) \psi_h + \partial_Y^2 \varphi_k \psi_h \right] \\ &\quad + (1 + \alpha^2) \left[2\partial_Y \delta_h \varphi_k (\partial_Y - \alpha \partial_X) \Psi + \partial_Y^2 \delta_h \varphi_k \Psi \right]. \end{aligned}$$

Since $\|\delta_h f\|_{H^p(\sigma_k)} \leq C \|f\|_{H^{p+1}(\sigma_k)}$ for h sufficiently small, where C is independent of h , we have

$$\begin{aligned} \|\rho_2^h\|_{H^{-1/2}(\sigma_k)} &\leq C \left(\|\rho_2\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \|\varphi_k\|_{W^{1,\infty}} + \|\varphi_k\|_{W^{1,\infty}} \|\Psi\|_{H^{3/2}(\sigma_k)} \right. \\ &\quad \left. + \|\varphi_k\|_{W^{3,\infty}} \|\Psi\|_{H^{-1/2}(\sigma_k)} + \|\varphi_k\|_{W^{2,\infty}} \|\Psi\|_{H^{1/2}(\sigma_k)} \right) \\ &\leq C \left(\|\rho_2\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} \|\varphi_k\|_{W^{1,\infty}} + \|\varphi_k\|_{W^{3,\infty}} \|\Psi\|_{H^{3/2}(\sigma_k)} \right). \end{aligned}$$

Therefore, using the Trace theorem

$$\|\rho_2^h\|_{H^{-1/2}(\sigma_k)} \leq C \left(\|\rho_2\|_{H_{\text{uloc}}^{1/2}(\sigma_w^M)} + \|\Psi\|_{H_{\text{uloc}}^2} \right) \|\varphi_k\|_{W^{3,\infty}},$$

which is the desired estimate. \square

II.6.2.0.5 Higher interior regularity. We intend to iterate the argument used in the case $m = 1$, thereby deducing our solution in various higher Sobolev spaces. As before, we start with the interior regularity analysis.

Proposition II.6.3. *Let m be a nonnegative integer and $\Psi \in H_{\text{uloc}}^2(\omega_w^b)$ be a solution of the PDE*

$$\partial_X \Psi + (\nabla_w^\perp \Psi \cdot \nabla_w) \Delta_w \Psi - \Delta_w^2 \Psi = 0 \quad \text{in } \omega_w^b.$$

Then, $\Psi \in H_{\text{uloc}}^{m+2}(\omega_w^b)$ and for each $\omega' \subset\subset \omega$

$$\|\Psi\|_{H^{2+m}(\omega')} \leq c(\omega') \|\Psi\|_{H_{\text{uloc}}^2(\omega_w^b)}$$

This proposition is a direct consequence of Theorem II.6.1.

II.6.2.0.6 Regularity up to the boundary for $m > 1$. We now complete the proof of Theorem II.6.2 for $m \geq 2$ using an induction argument. Taking $m = 1$ as the base case, we assume that the theorem holds up to $m \in \mathbb{N}^*$ and, then, prove that it is true as well for $m + 1$. Again we localize the solution near a fixed $k \in \mathbb{Z}$ using $\tilde{\varphi}_k$ and apply a finite difference operator δ_h to show that

$$\|\delta_h(\varphi_k \Psi)\|_{H^{m+2}(\omega_w^b)} \leq C(\varphi_k) (\|\rho_2\|_{H_{\text{uloc}}^{m+1/2}} + \|\rho_3\|_{H_{\text{uloc}}^{m-1/2}}). \quad (\text{II.6.64})$$

Therefore, $\tilde{\varphi}_k \Psi$ belongs to $H^{m+3}(\omega_w^b)$ if $\rho_2 \in H_{\text{uloc}}^{m+1/2}(\omega_w^b)$ and $\rho_3 \in H_{\text{uloc}}^{m-1/2}(\omega_w^b)$ which are exactly the hypotheses of Theorem II.6.2 when adding one degree of regularity to m . From the interior regularity result given earlier, we have that $\Psi \in H_{\text{uloc}}^{m+3}(\omega_w^b)$. Hence, Theorem II.6.2 indeed holds for $m + 1$, which concludes the induction.

The proof of estimate (II.6.64) follows the same ideas as the ones presented for the case $m = 1$. First, we have the following result

Lemma II.6.3. *Consider the linear problem*

$$\begin{aligned} \partial_X z - \Delta_w^2 z &= f_1 \quad \text{in } \omega_k, \\ (1 + \alpha^2) \Delta_w z &= f_2, \\ \left[(1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right] \Delta_w z - \frac{z}{2} \Big|_{\sigma_k} &= f_3, \\ z = \partial_X z &= 0, \quad \text{on } \partial\omega_k \setminus \sigma_k, \end{aligned} \quad (\text{II.6.65})$$

where $\omega_k = \omega^b \cap \text{Supp} \varphi_k$, $\sigma_k = \partial\omega_k \cap \partial\omega_w^b$ and \mathbf{n} stands for the unit outer normal vector at the boundary. If $f_1 \in H^{m-2}(\omega_k)$, $f_2 \in H^{m-1/2}(\sigma_k)$, $f_3 \in H^{m-3/2}(\sigma_k)$. This problem has a unique solution $z \in H^{m+2}(\omega_k)$ and it satisfies the estimate

$$\|z\|_{H^{m+2}(\omega_k)} \leq C \left(\|f_1\|_{H^{m-2}(\omega_k)} + \|f_2\|_{H^{m-1/2}(\sigma_k)} + \|f_3\|_{H^{m-3/2}(\sigma_k)} \right). \quad (\text{II.6.66})$$

This lemma is a direct consequence of Lemma II.6.2. As before, we need the following result

Lemma II.6.4. *Let $\sigma_k = \sigma_w^M \cap \text{Supp} \varphi_k$ where $\varphi_k \in C_0^\infty(\mathbb{R})$ and is equal to 1 in a neighborhood of σ_w^M . Consider the functions $\Psi \in H_{\text{uloc}}^{m+2}(\omega)$, $\rho_2 \in H_{\text{uloc}}^{m-1/2}(\mathbb{R})$ and $\rho_3 \in H_{\text{uloc}}^{m-3/2}(\mathbb{R})$. Then, for any $h \in \mathbb{R}$ we have the estimates*

$$\begin{aligned} \|\rho_3^h\|_{H^{m-3/2}(\sigma_k)} &\leq C(1 + \|\varphi_k\|_{W^{m+4,\infty}}) (\|\rho_3\|_{H_{\text{uloc}}^{m-1/2}(\sigma_w^M)} + \|\Psi\|_{H_{\text{uloc}}^{m+2}}(1 + \|\psi_h\|_{H_{\text{uloc}}^{m+2}} + \|\Psi\|_{H_{\text{uloc}}^{m+2}})), \\ \|\rho_2^h\|_{H^{m-1/2}(\sigma_k)} &\leq C \left(\|\rho_2\|_{H_{\text{uloc}}^{m+1/2}(\sigma_w^M)} + \|\Psi\|_{H_{\text{uloc}}^l} \right) \|\varphi_k\|_{W^{l+3,\infty}}, \\ \|f_h\|_{H^{m-2}(\omega_w^b)} &\leq C (\|\Psi\|_{H_{\text{uloc}}^{m+2}}(1 + \|\Psi\|_{H_{\text{uloc}}^{m+2}} + \|\psi_h\|_{H_{\text{uloc}}^{m+2}} + \|\Psi_L\|_{H_{\text{uloc}}^{m+2}})) (1 + \|\varphi_k\|_{W^{l+1,\infty}(\omega_w^b)}), \end{aligned}$$

where C is a constant depending only on the domain ω_w^b .

Assume that Lemma II.6.3 and II.6.4 holds. Then, for $l \in \mathbb{N}^*$ and $l \geq 2$, (II.6.64) follows directly by induction on $\{2, \dots, l\}$. We now show a way to adapt the proof of II.6.4 to prove Lemma II.6.4.

Proof of Lemma II.6.4. Let assume that $\Psi \in H^l$ for $l \geq 3$ (the case $l = 2$ was shown in Lemma II.6.2). The estimate of the linear terms are exactly the same as in the proof of II.6.1. Therefore we are left with showing the regularity of the quadratic terms in f_h and ρ_3^h .

II.6.2.0.7 Estimate of ρ_3^h . We have

$$\begin{aligned} \rho_3^h &= \delta_h[\varphi_k \rho_3] + \delta_h \left[\left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \left(2\partial_Y \varphi_k (\partial_Y - \alpha \partial_X) \Psi + \partial_Y^2 \varphi_k \Psi \right) \right. \\ &\quad \left. - \nabla_w^\perp \Psi \cdot \left(-\alpha \nabla_w (\Psi \partial_Y \varphi_k) + \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi (0, \partial_Y) \varphi_k \right) - \left(\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right) \right] \\ &\quad + (\delta_h[\nabla_w^\perp \Psi] \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\varphi_k \Psi). \end{aligned}$$

All the terms can be estimated as before. In fact most of them are easier to compute as $H^s(\sigma_k)$ is an algebra for $s > 1/2$. For $f \in H^{l-3/2}(\sigma_k)$ with compact support and $g \in H_{\text{uloc}}^{l-3/2}(\sigma_k)$, we have

$$\|fg\|_{H^{l-3/2}(\sigma_k)} \leq C \|f\|_{H^{l-3/2}(\sigma_k)} \|g\|_{H^{l-3/2}(\sigma_k)}.$$

which replaces (II.6.37). Let us illustrate the case of $\delta_h \left[\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \right) \varphi_k \right]$. As before, this term can be decomposed as in (II.6.39) and the difficulty consists of estimating

$$\delta_h \left[\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \varphi_k \right) \right]$$

From the algebra property, it follows that

$$\begin{aligned} \left\| \delta_h \left[\partial_Y \left(\frac{|\nabla_w^\perp \Psi|^2}{2} \varphi_k \right) \right] \right\|_{H^{l-7/2}(\sigma_k)} &\leq C \|\nabla^\perp \Psi\|^2 \varphi_k \|_{H^{l-3/2}(\sigma_k)} \\ &\leq C \|\nabla^\perp \Psi\|_{H_{\text{uloc}}^{l-3/2}(\sigma_k)}^2 \|\varphi_k\|_{H^{l-3/2}(\sigma_k)}. \\ &\leq C \left(\|\partial_X \Psi\|_{H_{\text{uloc}}^{l-3/2}(\sigma_k)} + \|\Psi\|_{H_{\text{uloc}}^{l-1/2}(\sigma_k)} \right) \|\varphi_k\|_{H^{l-3/2}(\sigma_k)}. \end{aligned}$$

Note that such a direct estimate does not work for $l = 2$, which explains why this term was treated previously in a lengthier way, also involving the norm of ψ_h .

II.6.2.0.8 Estimate of ρ_3^h . Concerning the estimate of f_h the same can be done by assuming that $l \geq 3$ and noticing that $H^2(\omega_w^b) \in L^\infty(\omega_w^b)$. Therefore, for $f \in H^{l-1}(\omega_w^b)$, $g \in H^{l-2}(\omega_w^b)$ and $\theta \in C_c^\infty(\overline{\omega_w^b})$, we obtain

$$\|fg\theta\|_{H^{l-3}(\omega_w^b)} \leq \|f\|_{H^{l-1}(\omega_w^b)} \|g\|_{H^{l-2}(\omega_w^b)} \|\theta\|_{W^{l,\infty}(\omega_w^b)}.$$

Noticing then that $H_{\text{uloc}}^s(\omega_w^b)$ is an algebra as soon as $s > 1$, we have

$$\| [(-\partial_Y + \alpha \partial_X) \psi_h \partial_X + \partial_X \psi_h (\partial_Y - \alpha \partial_X)] \Delta_w \Psi \|_{H^{-2}(\omega_w^b)} \leq C \| \psi_h \|_{H^2(\omega_w^b)} \| \Psi \|_{H_{\text{uloc}}^2}^2.$$

all other terms of f_h can be dealt with exactly as in the proof of Lemma II.6.1 and this ends the proof of Lemma II.6.4. \square

This concludes the proof of Theorem II.6.2. \square

II.6.3 Connecting the solutions at the artificial boundary

In previous sections, we showed the existence and uniqueness of the solution on both the half-space and the bumped domain. It remains to connect both results at the artificial boundary $X = M$. This local analysis is based on the implicit function theorem and will allow us to establish the solution of the problem on the whole boundary layer domain.

On account of Theorem II.6.2, we know the solution Ψ^- of (II.6.7) is well-defined and satisfies the estimate

$$\| \Psi^- |_{X=M} \|_{H_{\text{uloc}}^{m+3/2}} + \| \partial_X \Psi^- |_{X=M} \|_{H_{\text{uloc}}^{m+1/2}} \leq C \left(\| \phi \|_{W^{2,\infty}} + \| \rho_2 \|_{H_{\text{uloc}}^{m-1/2}} + \| \rho_3 \|_{H_{\text{uloc}}^{m-3/2}} \right),$$

for some positive constant C depending on α , M and $\| \gamma_w \|_{W^{2,\infty}}$. The existence and uniqueness of the solution Ψ^+ of (II.6.2) with $\Psi^+ |_{X=M} = \Psi^- |_{X=M}$ and $\partial_X \Psi^+ |_{X=M} = \partial_X \Psi^- |_{X=M}$ is guaranteed by Proposition II.6.1 when $C \left(\| \phi \|_{W^{2,\infty}} + \| \rho_2 \|_{H_{\text{uloc}}^{m-1/2}} + \| \rho_3 \|_{H_{\text{uloc}}^{m-3/2}} \right)$ smaller than a certain quantity $\delta_0 > 0$.

Furthermore, $\mathcal{A}_2[\Psi^+ |_{X=M}, \partial_X \Psi^+ |_{X=M}]$ and $\mathcal{A}_3[\Psi^+ |_{X=M}, \partial_X \Psi^+ |_{X=M}]$ belong to $H_{\text{uloc}}^{m-1/2}$ and $H_{\text{uloc}}^{m-3/2}$, respectively. Thus, the mapping

$$\mathcal{F} : W^{2,\infty}(\omega_w) \times H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R}) \rightarrow H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$$

given by

$$\mathcal{F}(\phi, \rho_2, \rho_3) = \left(\mathcal{A}_2[\Psi^+ |_{X=M}, \partial_X \Psi^+ |_{X=M}] - \rho_2, \mathcal{A}_3[\Psi^+ |_{X=M}, \partial_X \Psi^+ |_{X=M}] - \rho_3 \right)$$

is well-defined. Note that, when ϕ, ρ_2 and ρ_3 are simultaneously equal to zero, $\mathcal{F}(0, 0, 0) = 0$ as a direct consequence of Theorem II.6.2. The main idea consists in applying the implicit function theorem \mathcal{F} to find a solution of $\mathcal{F}(\phi, \rho_2, \rho_3) = 0$ for ϕ in a vicinity of zero. Thus, we need to first check the following hypotheses:

- \mathcal{F} is continuously Fréchet differentiable;
- $(v_1, v_2) \mapsto d\mathcal{F}(0, 0, 0)(0, v_1, v_2)$ is a Banach space isomorphism on $H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$, where d is the differential with respect to ρ_2 and ρ_3 .

F is a C^1 mapping in a neighborhood of zero: Let $(\phi^0, \rho_2^0, \rho_3^0)$ and (ϕ, ρ_2, ρ_3) be in a vicinity of zero in the sense of the functional norm of $H_{\text{uloc}}^{m+3/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$. Let Ψ_0^\pm and Ψ^\pm be solutions of (II.6.2), (II.6.7) associated to $(\phi^0, \rho_2^0, \rho_3^0)$ and $(\phi + \phi^0, \rho_2 + \rho_2^0, \rho_3 + \rho_3^0)$

respectively. We introduce the functions $\psi^\pm = \Psi^\pm - \Psi_0^\pm$. Then, easy computations show that ψ^- satisfies

$$\begin{aligned}
 \partial_X \psi^- - \Delta_w^2 \psi^- + Q_w(\psi^-, \psi^- + \Psi_0^-) + Q_w(\Psi_0^-, \psi^-) &= 0, \quad \text{in } \omega_w^b \setminus \sigma_w \\
 [\psi^-] \Big|_{\sigma_w} = \phi, \quad \left[\partial_X^k \psi^- \right] \Big|_{\sigma_w} &= 0, \quad k = 1, 2, 3 \\
 (1 + \alpha^2) \Delta_w \psi^- \Big|_{\sigma_w^M} &= \rho_2, \\
 \left[- \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi^- + \partial_Y \left(\frac{\nabla_w^\perp \psi^- \cdot \nabla_w^\perp (\psi^- + \Psi_0^-)}{2} \right) \right. & \\
 + (\nabla_w^\perp \psi^- \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\psi^- + \Psi_0^-) & \\
 \left. + (\nabla_w^\perp \Psi_0^- \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \psi^- + \frac{\psi^-}{2} \right] \Big|_{\sigma_w^M} &= \rho_3, \\
 \psi^- \Big|_{X=-\gamma(Y)} = \partial_n \psi^- \Big|_{X=-\gamma(Y)} &= 0.
 \end{aligned} \tag{II.6.67}$$

Similar arguments to the ones of Theorem II.6.2 apply to (II.6.67). This leads to the estimate

$$\|\psi^-\|_{H_{\text{uloc}}^2(\omega_w)} + \|\psi^-\|_{X=M} + \|\partial_X \psi^-\|_{X=M} \leq C \left(\|\phi\|_{W^{2,\infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}} \right),$$

for $\|\phi^0\|_{H_{\text{uloc}}^2} + \|\rho_2^0\|_{H_{\text{uloc}}^{m-1/2}} + \|\rho_3^0\|_{H_{\text{uloc}}^{m-3/2}}$ and $\|\phi\|_{W^{2,\infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}$ small enough. Hence, the solution ψ^- belongs to $H_{\text{uloc}}^2(\omega_w^b)$ and to $H_{\text{uloc}}^{m+2}((M, M') \times \mathbb{R})$, for $M' > \sup(-\gamma_w)$. Moreover, we can assume

$$\psi^- = \psi_\ell^- + O(\|\phi\|_{W^{2,\infty}}^2 + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}^2),$$

where ψ_ℓ^- is the solution of

$$\begin{aligned}
 \partial_X \psi_\ell^- - \Delta_w^2 \psi_\ell^- + Q_w(\psi_\ell^-, \Psi_0^-) + Q_w(\Psi_0^-, \psi_\ell^-) &= 0, \quad \text{in } \omega_w^b \\
 [\psi_\ell^-] \Big|_{X=0} = \phi, \quad \left[\partial_X^k \psi_\ell^- \right] \Big|_{X=0} &= 0, \quad k = 1, 2, 3 \\
 (1 + \alpha^2) \Delta_w \psi_\ell^- \Big|_{X=M} &= \rho_2, \\
 \left[- \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi_\ell^- \right. & \\
 + (\nabla_w^\perp \psi_\ell^- \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi_0^- & \\
 \left. + (\nabla_w^\perp \Psi_0^- \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \psi_\ell^- + \frac{\psi_\ell^-}{2} \right] \Big|_{X=M} &= \rho_3, \\
 \psi_\ell^- \Big|_{X=-\gamma(Y)} = \partial_n \psi_\ell^- \Big|_{X=-\gamma(Y)} &= 0,
 \end{aligned} \tag{II.6.68}$$

Note that (II.6.68) is similar to system (II.6.67) but lacks the quadratic terms. Solution of problem (II.6.68) can be sought using once again Ladyžhenskaya and Solonnikov's truncated energy method, which provides a dependence on the ϕ and ρ_i , $i = 2, 3$, at the main order. An analogous conclusion can be drawn from applying Theorem II.5.2 to the problem defined

on the half space. In this case, we have $\psi^+ = \psi_\ell^+ + O(\|\phi\|_{W^{2,\infty}}^2 + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}^2)$. Here, ψ_ℓ^+ satisfies

$$\begin{aligned} Q_w(\psi_\ell^+, \Psi_0^+) + Q_w(\Psi_0^+, \psi_\ell^+) + \partial_X \psi_\ell^+ - \Delta_w^2 \psi_\ell^+ &= 0, \quad \text{in } X > M \\ \psi_\ell^+|_{X=M} &= \psi_\ell^-|_{X=M}, \\ \partial_X \psi_\ell^+|_{X=M} &= \partial_X \psi_\ell^-|_{X=M}. \end{aligned} \quad (\text{II.6.69})$$

From Theorem II.5.2, we infer that

$$\|\psi_\ell^+\|_{H_{\text{uloc}}^{m+2}} \leq C \left(\|\psi_\ell^+|_{X=M}\|_{H_{\text{uloc}}^{m+3/2}} + \|\partial_X \psi_\ell^+|_{X=M}\|_{H_{\text{uloc}}^{m+1/2}} \right) \leq C \left(\|\phi\|_{W^{2,\infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}} \right).$$

More details on the resolution of problems similar to (II.6.68) and (II.6.69) can be found in Section II.7. Furthermore,

$$\begin{aligned} \mathcal{F}_1(\phi + \phi^0, \rho_2 + \rho_2^0, \rho_3 + \rho_3^0) - \mathcal{F}_1(\phi^0, \rho_2^0, \rho_3^0) &= -\rho_2 + (1 + \alpha^2) \Delta_w \psi_\ell^+ \\ &\quad + O \left(\|\phi\|_{W^{2,\infty}}^2 + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}^2 \right), \\ \mathcal{F}_2(\phi + \phi^0, \rho_2 + \rho_2^0, \rho_3 + \rho_3^0) - \mathcal{F}_2(\phi^0, \rho_2^0, \rho_3^0) &= -\rho_3 + \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi_\ell^+ - \frac{\psi_\ell^+}{2} \\ &\quad + (\nabla_w^\perp \psi_\ell^+ \cdot \nabla_w) \partial_X \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi_0^+ \\ &\quad + (\nabla_w^\perp \Psi_0^+ \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \psi_\ell^+ \\ &\quad + O \left(\|\phi\|_{W^{2,\infty}}^2 + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}^2 \right). \end{aligned} \quad (\text{II.6.70})$$

We see that the Fréchet differential of \mathcal{F} at $(\phi^0, \rho_2^0, \rho_3^0)$ is defined by $L = (L_1, L_2)$ where

$$\begin{aligned} L_1 &= -\rho_2 + (1 + \alpha^2) \Delta_w \psi_\ell^+, \\ L_2 &= -\rho_3 - \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi_\ell^+ + \frac{\psi_\ell^+}{2} + (\nabla_w^\perp \psi_\ell^+ \cdot \nabla_w) \partial_X \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi_0^+ \\ &\quad + (\nabla_w^\perp \Psi_0^+ \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \psi_\ell^+. \end{aligned}$$

It is important to emphasize that (II.6.69) is a linear problem with respect to the perturbation function ψ_ℓ^+ . Therefore, it is possible to obtain the exact form of its solution by applying the same reasoning as the one in Section II.5.1. The use of Fourier analysis to (II.6.69) provides a solution ψ_ℓ^+ showing continuous dependence on Ψ_0^+ , as well as, on the boundary condition. Note that $\psi_\ell^+|_{X=M}$ and $\partial_X \psi_\ell^+|_{X=M}$ depend in turn on Ψ_0^- . Hence, ψ^\pm depend continuously on Ψ_0^\pm , and conversely on $\phi^0, \rho_2^0, \rho_3^0$. Therefore, \mathcal{F} is a C^1 function in a neighborhood of zero.

$d\mathcal{F}(0, 0, 0)$ is invertible: Since $d\mathcal{F}(0, 0, 0) = L^0(\cdot, \cdot)$, we consider the systems satisfied by

ψ_ℓ^\pm with $\Psi_0^\pm = 0$ and $\phi = 0$.

$$\left\{ \begin{array}{l} \partial_X \psi_\ell^- - \Delta_w^2 \psi_\ell^- = 0, \quad \text{in } X \leq M \\ (1 + \alpha^2) \Delta_w \psi_\ell^- \Big|_{X=M} = \rho_2, \\ \left[-((1 + \alpha^2) \partial_X - 2\alpha \partial_Y) \partial_X \Delta_w \psi_\ell^- + \frac{\psi_\ell^-}{2} \right] \Big|_{X=M} = \rho_3, \\ \psi_\ell^- \Big|_{X=-\gamma(Y)} = \partial_n \psi_\ell^- \Big|_{X=-\gamma(Y)} = 0, \end{array} \right. ,$$

and

$$\left\{ \begin{array}{l} \partial_X \psi_\ell^+ - \Delta_w^2 \psi_\ell^+ = 0, \quad \text{in } X > M \\ \psi_\ell^+ \Big|_{X=M} = \psi_\ell^- \Big|_{X=M}, \\ \partial_X \psi_\ell^+ \Big|_{X=M} = \partial_X \psi_\ell^- \Big|_{X=M}. \end{array} \right.$$

If $L^0(\rho_2, \rho_3) = (0, 0)$, then $\psi_\ell := \mathbb{1}_{X \leq M} \psi_\ell^- + \mathbb{1}_{X > M} \psi_\ell^+$ is a solution of the linear system in the whole western boundary layer domain

$$\begin{aligned} \partial_X \psi_\ell - \Delta_w^2 \psi_\ell &= 0, \quad \text{in } \omega_w \\ \psi_\ell \Big|_{X=-\gamma(Y)} = \partial_n \psi_\ell \Big|_{X=-\gamma(Y)} &= 0. \end{aligned} \quad (\text{II.6.71})$$

The existence and uniqueness of the solution $\psi_\ell \equiv 0$ of the linear elliptic problem (II.6.71) is guaranteed by Theorem II.5.1, and therefore $\rho_2 = 0$ and $\rho_3 = 0$. Consequently, $\ker d\mathcal{F}(0, 0, 0) = \{(0, 0)\}$, and $\ker d\mathcal{F}(0, 0, 0)$ is one-to-one. Solving the equation

$$L^0(\rho_2, \rho_3) = (\bar{\rho}_2, \bar{\rho}_3),$$

for a given $(\bar{\rho}_2, \bar{\rho}_3) \in H^{m-1/2}(\mathbb{R}) \times H^{m-3/2}(\mathbb{R})$ is equivalent to finding the solution of the problem

$$\begin{aligned} \partial_X \psi_\ell^- - \Delta_w^2 \psi_\ell^- &= 0, \quad \text{in } X \leq M \\ \left((1 + \alpha^2) \Delta_w \psi_\ell^-, - \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi_\ell^- + \frac{\psi_\ell^-}{2} \right) \Big|_{X=M} &= -(\bar{\rho}_2, \bar{\rho}_3) + PS_w(\psi_\ell^- \Big|_{X=M}, \partial_X \psi_\ell^- \Big|_{X=M}), \\ \psi_\ell^- \Big|_{X=-\gamma(Y)} = \partial_n \psi_\ell^- \Big|_{X=-\gamma(Y)} &= 0, \end{aligned} \quad (\text{II.6.72})$$

where PS_w denotes the Poincaré-Steklov operator

$$PS_w(\psi_0, \psi_1) : H_{\text{uloc}}^{m+3/2}(\mathbb{R}) \times H_{\text{uloc}}^{m+1/2}(\mathbb{R}) \rightarrow H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R}).$$

The existence of a unique solution $\psi_\ell^- \in H_{\text{uloc}}^2(\omega_w^b)$ of problem (II.6.72) follows from the ideas of the proof of Proposition II.5.5 in Section II.5.

The only point remaining concerns proving that ψ_ℓ^- is a H_{uloc}^{m+2} function in $[M', M] \times \mathbb{R}$ for all $\sup(-\gamma_w) < M' < M$. Therefore, we notice that in the domain $(M', M) \times \mathbb{R}$, the derivatives up to order k of ψ_ℓ^- satisfy a linear system similar to the one above. It follows that $\partial_Y^l \psi_\ell^- \in H_{\text{uloc}}^2([M', M] \times \mathbb{R})$ for all $l \leq k$. In particular, $\partial_Y^l \psi_\ell^- \Big|_{X=M} \in H_{\text{uloc}}^{3/2}([M', M] \times \mathbb{R})$ and $\partial_X \partial_Y^l \psi_\ell^- \Big|_{X=M} \in H_{\text{uloc}}^{1/2}([M', M] \times \mathbb{R})$, hence, $\psi_\ell^- \in H_{\text{uloc}}^{m+3/2}(\mathbb{R})$ and $\partial_n \psi_\ell^- \in H_{\text{uloc}}^{m+1/2}(\mathbb{R})$. Consequently,

$$(\rho_2, \rho_3) = \left((1 + \alpha^2) \Delta_w \psi_\ell^-, - \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi_\ell^- + \frac{\psi_\ell^-}{2} \right).$$

We can finally assert that $d\mathcal{F}(0,0,0)$ is an isomorphism of $H^{m-1/2}(\mathbb{R}) \times H^{m-3/2}(\mathbb{R})$. Using the implicit function theorem, we deduce that for all $\phi \in W^{2,\infty}$ in a neighborhood of zero, there exists $(\rho_2, \rho_3) \in H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$ such that $\mathcal{F}(\phi, \rho_2, \rho_3) = 0$. Let $\Psi_w := \mathbb{1}_{X \leq M} \Psi_w^- + \mathbb{1}_{X > M} \Psi_w^+$, where Ψ_w^-, Ψ_w^+ are the solutions of (II.6.7) and (II.6.2) associated to (ϕ, ρ_2, ρ_3) . By definition, the jump of Ψ_w across the transparent boundary $\{X = M\}$ is zero, and since $\mathcal{F}(\phi, \rho_2, \rho_3) = (0, 0)$,

$$\mathcal{A}_i \left[\Psi_w^-|_{X=M}, \partial_X \Psi_w^-|_{X=M} \right] = \rho_i = \mathcal{A}_i \left[\Psi_w^+|_{X=M}, \partial_X \Psi_w^+|_{X=M} \right], \quad i = 2, 3.$$

Since $\partial_X^k \Psi_w^+|_{X=M} = \partial_X^k \Psi_w^-|_{X=M}$, $k = 0, 1$ we deduce that

$$\begin{aligned} & \left((1 + \alpha^2) \Delta_w \Psi^-, -((1 + \alpha^2) \partial_X - 2\alpha \partial_Y) \Delta_w \Psi^- + \frac{\Psi^-}{2} \right) \Big|_{X=M} \\ &= \\ & \left((1 + \alpha^2) \Delta_w \Psi^+, -((1 + \alpha^2) \partial_X - 2\alpha \partial_Y) \Delta_w \Psi^+ + \frac{\Psi^+}{2} \right) \Big|_{X=M}. \end{aligned}$$

Accordingly, these operators are continuous across $\{X = M\} \times \mathbb{R}$, and therefore Ψ_w is a solution of the western boundary layer system in the whole domain ω_w . This concludes the proof of Theorem II.2.1.

II.7 Linearized problem

This section focuses on the well-posedness analysis of the linearized problems driving the higher-order western profiles of the approximate solution and the correctors needed to deal with the influence of the east boundary layer on the western side of Ω^ε .

We are interested in the system

$$\begin{aligned} \partial_X \Psi_w + Q_w(\Psi_w, \Psi_w^0) + Q_w(\Psi_w^0, \Psi_w) - \Delta_w^2 \Psi_w &= F, \quad \text{in } \omega_w^+ \cup \omega_w^- \\ [\partial_X^k \Psi_w]|_{X=0} &= g^k, \quad k = 0, \dots, 3 \\ \Psi_w|_{X=-\gamma_w(Y)} &= 0, \quad \frac{\partial \Psi_w}{\partial n_w} \Big|_{X=-\gamma_w(Y)} = 0, \end{aligned} \tag{II.7.1}$$

where $g_k \in L^\infty(\mathbb{R})$. Assume $\Psi_w^0 \in H_{\text{uloc}}^2(\omega_w^+ \cup \omega_w^-)$ and $F \in H_{\text{uloc}}^{-2}(\omega_w)$ are exponentially decaying functions. The existence and uniqueness of a solution of (II.7.1) depend additionally on a smallness hypothesis made on the first profile of the western boundary layer Ψ_w^0 .

The main result concerning problem (II.7.1) is summarized in the following theorem:

Theorem II.7.1. *Let ω_w defined as in (II.2.6) and γ_w be a bounded Lipschitz function. Suppose $g_k \in L^\infty(\mathbb{R})$, $k = 0, \dots, 3$, and $\delta > 0$ and $\Psi_0 \in H_{\text{uloc}}^2(\omega_w^+ \cup \omega_w^-)$ a function satisfying*

$$\|e^{\delta X} \Psi_w^0\|_{H_{\text{uloc}}^2(\omega_w^+ \cup \omega_w^-)} < \delta_0.$$

Additionally, suppose that $e^{\delta F} \in H_{\text{uloc}}^{m-2}(\omega_w)$. Then, for δ_0 small enough, (II.7.1) has a unique solution Ψ in $H_{\text{uloc}}^2(\omega_w^+ \cup \omega_w^-)$. Moreover, the following estimate holds

$$\|e^{\delta X} \Psi\|_{H_{\text{uloc}}^2(\omega_w^+ \cup \omega_w^-)} \leq C \left(\sum_{k=0}^3 \|g_k\|_{L^\infty(\mathbb{R})} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m-2}(\omega_w)} \right), \tag{II.7.2}$$

for a universal constant C . Here, δ is the one in Theorem II.5.2.

Note that the smallness condition on Ψ_w^0 is ensured by choosing ϕ small enough in (II.6.1). Theorem II.7.1 results from following the same ideas of the nonlinear case. As a result, we are going to discuss the relevant steps of the construction listed in Section II.4.2 but only emphasize the differences concerning the previous section. The details are left to the reader.

II.7.1 The problem on the half space

We study the following problem on the half-space

$$\begin{aligned} \partial_X \Psi_w + Q_w(\Psi, \Psi_w^0) + Q_w(\Psi_w^0, \Psi_w) - \Delta_w^2 \Psi_w &= F, \quad \text{in } \mathbb{R}_+^2 \\ \Psi_w|_{X=0} &= \psi_0, \quad \partial_n \Psi_w|_{X=0} = \psi_1, \end{aligned} \quad (\text{II.7.3})$$

where $\psi_0 \in H_{\text{uloc}}^{m-3/2}(\mathbb{R})$ and $\psi_1 \in H_{\text{uloc}}^{m-1/2}(\mathbb{R})$. The result in this section equivalent to Proposition II.6.1 is as follows:

Proposition II.7.1. *Let $m \in \mathbb{N}$, $m \gg 1$ and \mathcal{H}^m the functional space defined in (II.6.3). There is small constant $\delta_0 > 0$ such that for all $\psi_j \in H_{\text{uloc}}^{m-j+3/2}(\mathbb{R})$, $j = 0, 1$ and $\Psi_w^0 \in H_{\text{uloc}}^{m+2}$ such that for $\delta > 0$, $\|e^{\delta X} \Psi_w^0\|_{H_{\text{uloc}}^{m+2}} \leq \delta_0$ and*

$$\|\psi_0\|_{H_{\text{uloc}}^{m+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m+1/2}(\mathbb{R})} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m-2}} < \delta_0. \quad (\text{II.7.4})$$

Then, the problem

$$\begin{cases} \partial_X \Psi_w + Q_w(\Psi_w^0, \Psi_w) + Q_w(\Psi, \Psi_w^0) - \Delta_w^2 \Psi_w &= F, & \text{in } \mathbb{R}_+^2 \\ \Psi_w|_{X=0} &= \psi_0, & \partial_X \Psi_w|_{X=0} = \psi_1. \end{cases} \quad (\text{II.7.5})$$

has a unique solution in \mathcal{H}^{m+2} .

The strategy of proof is the same as to the one of Proposition II.6.1. Indeed, the mapping $T_{(\psi_0, \psi_1)} : H_{\text{uloc}}^{m+2} \rightarrow H_{\text{uloc}}^{m+2}$ such that $T_{(\psi_0, \psi_1)}(\Psi_w) = \tilde{\Psi}$ is the solution of (II.5.3) when $\tilde{F} = F - Q_w(\Psi_w^0, \Psi_w) - Q_w(\Psi, \Psi_w^0)$ satisfies the estimate for $\bar{\delta} = 2\delta$

$$\begin{aligned} \|T_{(\psi_0, \psi_1)}(\Psi_w)\|_{\mathcal{H}^{m+2}} &< C_1 \left(\|\psi_0\|_{H_{\text{uloc}}^{m+3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{m+2}(\mathbb{R})} + \|e^{\delta X} F\|_{H_{\text{uloc}}^{m-2}} \right. \\ &\quad \left. + \|\Psi_w^0\|_{\mathcal{H}^{m+2}} \|\tilde{\Psi}\|_{\mathcal{H}^{m+2}} \right), \end{aligned}$$

for $C_1 > 0$. When considering the functions Ψ^i , $i = 1, 2$, such that $T_{(\psi_0, \psi_1)}(\Psi_w^i) = \tilde{\Psi}_w^i$ are solutions of (II.7.1) for $\tilde{F}^i = F - Q_w(\Psi_w^i, \Psi_w^0) - Q_w(\Psi_w^0, \Psi_w^i)$, we can show that $T_{(\psi_0, \psi_1)}(\Psi^1) - T_{(\psi_0, \psi_1)}(\Psi^2)$ is a strict contraction in a ball of radius $R_0 > 0$. Indeed, it is always possible to choose $R_0 \in]\delta(\eta), \frac{1}{2C_1}[$ when $0 < \delta_0 \leq \delta(\eta) = \frac{\eta-1}{2C_1\eta-C_1}$, $\eta > 1$. The existence result is a consequence of applying the fixed point theorem in \mathcal{H}^{m+2} . Since the ideas of proof are the same of Proposition II.6.1, they are not repeated here.

II.7.2 The linearized problem on the rough domain

This section is devoted to the well-posedness of the problem in the rough channel $\omega^b = \omega_w \setminus \{X < M\}$

$$\begin{aligned}
\partial_X \Psi_w^- + Q_w(\Psi_w^-, \Psi_w^0) + Q_w(\Psi_w^0, \Psi_w^-) - \Delta_w^2 \Psi_w^- &= F, \quad \text{in } \omega_w, \quad \text{in } \omega^b \setminus \sigma_w \\
\left[\partial_X^k \Psi_w^- \right] |_{\sigma_w} &= g_k, \quad k = 0, \dots, 3, \\
\mathcal{A}_2[\Psi_w^- |_{\sigma_w^M}, \partial_X \Psi_w^- |_{\sigma_w^M}] &= \rho_2, \\
\tilde{\mathcal{A}}_3[\Psi_w^- |_{\sigma_w^M}, \partial_X \Psi_w^- |_{\sigma_w^M}] &= \rho_3, \\
\Psi_w^- |_{X=-\gamma_w(Y)} &= 0, \quad \frac{\partial \Psi_w^-}{\partial n_w} \Big|_{X=-\gamma_w(Y)} = 0,
\end{aligned} \tag{II.7.6}$$

where $g_k \in W^{m+2-k, \infty}(\mathbb{R})$ and ρ_2 and ρ_3 belong to $H_{\text{uloc}}^{m-1/2}(\mathbb{R})$ and $H_{\text{uloc}}^{m-3/2}(\mathbb{R})$, respectively. The source term F is a function of $H_{\text{uloc}}^{m-2}(\omega_w^b)$. The transparent operators are defined by

$$\begin{aligned}
\mathcal{A}_2[\Psi, \partial_X \Psi] &= (1 + \alpha^2) \Delta_w \Psi, \\
\tilde{\mathcal{A}}_3[\Psi, \partial_X \Psi] &= -((1 + \alpha^2) \partial_X - 2\alpha \partial_Y) \Delta_w \Psi \\
&\quad + (\nabla_w^\perp \Psi \cdot \nabla_w) ((1 + \alpha^2) \partial_X - \alpha \partial_Y) \Psi_w^0 + (\nabla_w^\perp \Psi_w^0 \cdot \nabla_w) ((1 + \alpha^2) \partial_X - \alpha \partial_Y) \Psi + \frac{\Psi}{2}.
\end{aligned} \tag{II.7.7}$$

We claim the following result:

Proposition II.7.2. *Let $m \gg 1$ be arbitrary and $g_k \in W^{m+2-k, \infty}(\mathbb{R})$, $k = 0, \dots, 3$. There exists $\delta > 0$ such that for $\delta > 0$, $F \in H_{\text{uloc}}^{m-2}(\omega_w^b)$, $\rho_2 \in H_{\text{uloc}}^{m-1/2}(\mathbb{R})$ and $\rho_3 \in H_{\text{uloc}}^{m-3/2}(\mathbb{R})$ satisfy*

$$\|F\|_{H_{\text{uloc}}^{m-2}(\mathbb{R})} + \|\Psi_w^0\|_{H_{\text{uloc}}^{m+2}(\mathbb{R})} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}(\mathbb{R})} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}(\mathbb{R})} < \delta_0. \tag{II.7.8}$$

Then, system (II.7.6) has a unique solution $\Psi_w \in H_{\text{uloc}}^2(\omega_w^b)$.

Moreover, $\Psi_w \in H_{\text{uloc}}^{m+2}((M', M) \times \mathbb{R})$, for all $M' \in]\sup(-\gamma_w), M[$ and

$$\|\Psi_w\|_{H_{\text{uloc}}^{m+2}((M', M) \times \mathbb{R})} \leq C_{M'} \left(\sum_{k=0}^3 \|g_k\|_{W^{m+2-k, \infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}(\mathbb{R})} + \|F\|_{H_{\text{uloc}}^{m-2}((M', M) \times \mathbb{R})} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}(\mathbb{R})} \right) \tag{II.7.9}$$

Proof. Note that Proposition II.7.2 is the equivalent for the linearized case of Proposition II.6.2. Consequently, to look up the solution of problem (II.7.6), similar arguments to the nonlinear case apply, see Section II.6.2. First, we discuss the existence and uniqueness of the solution in $H_{\text{uloc}}^2(\omega^b)$ and later, its regularity near the artificial boundary.

We work with the truncated energies

$$E_k^n = \int_{\omega_n} |\chi_k \Delta_w \tilde{\Psi}|^2, \tag{II.7.10}$$

where the truncation function $\chi_k \in C_0^\infty(\mathbb{R})$ is such that $\chi_k \equiv 1 \in [-k, k]$, $\text{Supp} \chi_k \subset [-k-1, k+1]$, and the derivatives $\chi_k^{(j)}$, $j = 1, \dots, 4$ are bounded uniformly in k . Moreover,

$\tilde{\Psi} = \Psi - \Psi^L$ is the solution of the system

$$\begin{aligned} Q_w(\tilde{\Psi}, \Psi_w^0) + Q_w(\Psi_w^0, \tilde{\Psi}) + \partial_X \tilde{\Psi} - \Delta_w^2 \tilde{\Psi} &= F_0^L \quad \text{in } \omega^b, \\ \mathcal{A}_2[\tilde{\Psi}|_{\sigma_w^M}, \partial_X \tilde{\Psi}|_{\sigma_w^M}] &= \rho_2, \quad \mathcal{A}_2[\tilde{\Psi}|_{\sigma_w^M}, \partial_X \tilde{\Psi}|_{\sigma_w^M}] = \rho_3, \\ \tilde{\Psi}|_{X=-\gamma(Y)} &= 0, \quad \frac{\partial \tilde{\Psi}}{\partial n_w} \tilde{\Psi}|_{X=-\gamma(Y)} = 0. \end{aligned} \quad (\text{II.7.11})$$

Function Ψ^L depends on the g_k and is defined as in (II.4.7) and for $F_0^L = F - Q_w(\Psi^L, \Psi_w^0) - Q_w(\Psi_w^0, \Psi^L)$, we have the estimate

$$\|F_0^L\|_{H_{\text{uloc}}^{m-2}(\omega^b)} \leq C \left(\|F\|_{H_{\text{uloc}}^{m-2}(\omega^b)} + \sum_{k=0}^3 \|g^k\|_{W^{2,\infty}(\mathbb{R})} \|\Psi_w^0\|_{H_{\text{uloc}}^{m+2}(\mathbb{R})} \right). \quad (\text{II.7.12})$$

We compute the following inequality for the sequence $(E_k^n)_{l \leq n, n \in \mathbb{R}}$

$$E_k^n \leq C_1 \left((E_{k+1}^n - E_k^n) + \left(\|\rho_2\|_{H_{\text{uloc}}^{m-1/2}}^2 + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}}^2 \right) (k+1) \right). \quad (\text{II.7.13})$$

This a priori estimate allows one to obtain a uniform bound on E_n^k which is used in turn to deduce a H_{uloc}^2 bound on $\tilde{\Psi}$ using backward induction on $n - k$ and later a compactness argument. This corresponds to step (NL2) in Section II.4.2. For more details we refer the reader to Section II.6.2.

From now on, we drop the n 's to lighten the notation and only address the particularities of the features unique to the linearized case.

First, we write the weak formulation of (II.7.11).

Definition II.7.1. Let \mathcal{V} and \mathcal{D}_0^2 be spaces of functions in Definition II.6.1 and $b(\Psi, \tilde{\Psi}, \varphi) = -\int_{\Omega} (\nabla_w^\perp \Psi \cdot \nabla_w) \nabla_w^\perp \tilde{\Psi} \cdot \nabla_w^\perp \varphi$ for $(\Psi, \tilde{\Psi}, \varphi) \in \mathcal{D}_0^2 \times \mathcal{D}_0^2 \times \mathcal{V}$. The solution $\tilde{\Psi} \in H_{\text{uloc}}^2(\omega^b)$ of (II.7.11) satisfies for all $\varphi \in \mathcal{V}$

$$\begin{aligned} \int_{\omega_w^b} \partial_X \tilde{\Psi} \varphi + b(\tilde{\Psi}, \tilde{\Psi}^0, \varphi) + b(\tilde{\Psi}^0, \tilde{\Psi}, \varphi) - \int_{\omega_w^b} \Delta_w \tilde{\Psi} \Delta_w \varphi \\ = \int_{\omega_w^b} F_0^L \varphi - \int_{\mathbb{R}} \left(\rho_3 - \frac{\tilde{\Psi}}{2} \Big|_{X=M^-} \right) \varphi|_{X=M^-} dY - \int_{\mathbb{R}} \rho_2 \partial_X \varphi|_{X=M^-} dY. \end{aligned}$$

- *Energy estimates.* The proof of (II.7.13) is as follows. Plugging $\varphi = \chi_k \tilde{\Psi}$ into the trilinear terms in (II.7.14), and proceeding similarly as before it is possible to show that

$$\left| b(\tilde{\Psi}, \Psi_w^0, \chi_k \tilde{\Psi}) + b(\Psi_w^0, \tilde{\Psi}, \chi_k \tilde{\Psi}) \right| \leq C \|\Psi_w^0\|_{H_{\text{uloc}}^2} E_{k+1} + C_1 \|\Psi_w^0\|_{H_{\text{uloc}}^2} (E_{k+1} - E_k) \quad (\text{II.7.14})$$

where C and C_1 depend on the domain. To obtain estimate II.7.14, we first write the terms as in (II.6.14) and (II.6.15). Then, combining the Sobolev inequality (II.6.16) with the definition of norm in Kato spaces leads to the desired result. The terms associated to the biharmonic operator are bounded by $C(E_{k+1} - E_k)$ as seen in (II.4.19). We are left with

$$\left| \int_{\omega_w^b} \chi_k F_0^L \tilde{\Psi} \right| \leq C \left(\|F\|_{H_{\text{uloc}}^{m-2}(\omega^b)} + \sum_{k=0}^3 \|g^k\|_{W^{2,\infty}(\mathbb{R})} \|\Psi_w^0\|_{H_{\text{uloc}}^{m-2}(\mathbb{R})} \right) E_{k+1}^{1/2} \quad (\text{II.7.15})$$

At the artificial boundary, we get

$$\begin{aligned} & - \int_{\mathbb{R}} \chi_k \left(\rho_3 \tilde{\Psi}|_{X=M^-} + \rho_2 \partial_X \tilde{\Psi}|_{X=M^-} \right) \\ & \leq C \left(\|\rho_2\|_{L^2_{\text{uloc}}} + \|\rho_3\|_{L^2_{\text{uloc}}} \right) E_{k+1} + C \left(\|\rho_2\|_{L^2_{\text{uloc}}} + \|\rho_3\|_{L^2_{\text{uloc}}} \right) (k+1), \end{aligned} \quad (\text{II.7.16})$$

where C depends only on M , α and on $\|\gamma\|_{W^{2,\infty}}$. The computation of this bound relies on the trace theorem and Young's inequality.

From (II.7.14), (II.7.15), (II.7.16) and the bound on the linear term, we obtain

$$\begin{aligned} E_k & \leq C'_1 (E_{k+1} - E_k) + C'_2 \left(\|F\|_{H^{m-2}(\omega^b)} + \sum_{k=0}^3 \|g^k\|_{W^{2,\infty}(\mathbb{R})} \|\Psi_w^0\|_{H^{m-2}(\mathbb{R})} \right)^2 E_{k+1} \\ & \quad + C'_4 \left(\|\rho_2\|_{H^{m-1/2}} + \|\rho_3\|_{H^{m-3/2}} \right) E_{k+1} + C'_4 \left(\|\rho_2\|_{H^{m-1/2}} + \|\rho_3\|_{H^{m-3/2}} \right) (k+1). \end{aligned}$$

Choosing δ_0 small enough, the previous expressions becomes for $C_1 > 0$

$$E_k \leq C_1 (E_{k+1} - E_k + C_2(\rho_2, \rho_3)(k+1)), \quad (\text{II.7.17})$$

where $C_2(\rho_2, \rho_3)$ refers to the term $\|\rho_2\|_{H^{m-1/2}} + \|\rho_3\|_{H^{m-3/2}}$.

- *Induction.* Performing backwards induction on (II.7.17) is easier than the nonlinear case. Indeed, considering that there exists a constant C_3 independent of n and k , such that

$$E_k > C_3 C_2(\rho_2, \rho_3)(k+1),$$

implies that $E_{k+1} - E_k < C_3 C_2(\rho_2, \rho_3)$. Furthermore, substitution on (II.7.17) gives

$$C_3 C_2(\rho_2, \rho_3)(k+1) < E_k \leq C_1 C_2(\rho_2, \rho_3) C_3 + C_1 C_2(\rho_2, \rho_3)(k+1). \quad (\text{II.7.18})$$

Taking $C_3 \geq 2C_1$ and plugging it in (II.7.18) provides a contradiction for $k > k_0$, where $k_0 = \lfloor C_3 \rfloor$. Therefore, (II.7.17) is true at the rank $k > k_0$ and also for $k \leq k_0$, since E_k is an increasing functional with respect to the value of k . The derivation of the energy estimates is invariant by horizontal translation, and all constants depend only on norms of ρ_i , $i = 2, 3$ and γ_w . Following the same reasoning in Step (L5) of Section II.4.1, it is possible to show that $\tilde{\Psi}$ is uniformly bounded in $H^2_{\text{uloc}}(\omega_w^b)$.

- *Uniqueness.* Let Ψ_i , $i = 1, 2$, are solutions of (II.7.6). To establish uniqueness, we need to show that the solution $\bar{\Psi} = \Psi_1 - \Psi_2$ of the system

$$\begin{aligned} Q_w(\bar{\Psi}, \Psi_w^0) + Q_w(\Psi_w^0, \bar{\Psi}) + \partial_X \bar{\Psi} - \Delta_w^2 \bar{\Psi} & = 0 \quad \text{in } \omega^b, \\ \bar{\Psi}|_{X=-\gamma(Y)} = \partial_n \bar{\Psi}|_{X=-\gamma(Y)} & = 0, \end{aligned} \quad (\text{II.7.19})$$

is identically zero under a smallness assumption on Ψ_w^0 .

Repeating the same reasoning of ‘‘existence part’’ provides the induction relation on the truncated energies $\int_{\omega_n} \chi_k |\Delta_w \bar{\Psi}|^2$

$$E_k \leq C_{\delta^0} (E_{k+1} + (E_{k+1} - E_k)).$$

Here, the constant depends not only on the domain, but also on $\|\Psi_w^0\|_{H^m_{\text{uloc}}}$. The term $E_{k+1} - E_k$ is uniformly bounded in k , hence, $E_k \leq C$ uniformly in k . As a consequence, repeating the method but this time without the truncation function leads to $\bar{\Psi} \equiv 0$.

The last step in the proof of Proposition II.7.2 corresponds to the higher regularity estimates near $X = M$. Interior and boundary regularity results analogous to that of Lemma II.6.2 can be easily obtained by following the same ideas. The major difficulties we encountered when performing the regularity analysis near the boundary for the first western profile came from the nonlinear terms, which are not present in the linearized case. Moreover, information on the behavior and H^{m+2} -regularity of Ψ_w^0 in the vicinity of the transparent boundary is available. These factors greatly simplify the computations, and, therefore, the detailed verification is left to the reader. □

II.7.3 Joining ω_w^- and ω_w^+

In this section we are concerned with finding a solution on the whole domain ω_w . Let Ψ_w^- be the unique solution of (II.7.6) satisfying the regularity estimate (II.7.9). As a consequence of standard trace properties, we have that $\Psi^-|_{X=M} \in H_{\text{uloc}}^{m+3/2}(\mathbb{R})$ and $\partial_X \Psi^-|_{X=M} \in H_{\text{uloc}}^{m+1/2}(\mathbb{R})$. Taking $\psi_0 = \Psi^-|_{X=M}$ and $\psi_1 = \partial_X \Psi^-|_{X=M}$ in (II.7.5) guarantees the existence a unique solution Ψ^+ for the problem in the half-space decaying exponentially far from the boundary having imposed smallness conditions on Ψ_w^0 and ρ_i , $i = 2, 3$. Additionally, we have that $\mathcal{A}_2[\Psi^+|_{X=M^+}, \partial_X \Psi^+|_{X=M^+}] \in H^{m-1/2}$, $\tilde{\mathcal{A}}_3[\Psi^+|_{X=M^+}, \partial_X \Psi^+|_{X=M^+}] \in H^{m-3/2}$. We define the mapping $\mathcal{F} : H_{\text{uloc}}^{m+2}(\omega_w) \times (W^{2,\infty}(\mathbb{R}))^4 \times H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R}) \rightarrow H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$ by

$$\mathcal{F}(\Psi_w^0, g_0, \dots, g_3, \rho_2, \rho_3) = \left(\mathcal{A}_2[\Psi^+|_{X=M^+}, \partial_X \Psi^+|_{X=M^+}] - \rho_2, \tilde{\mathcal{A}}_3[\Psi^+|_{X=M^+}, \partial_X \Psi^+|_{X=M^+}] - \rho_3 \right).$$

Once again, if Ψ is a C^3 function at $X = 0$ and $X = M$, and $F = 0$, we have that $\mathcal{F} = 0$. Showing that \mathcal{F} is a C^1 mapping starts by considering two points in the vicinity of zero using the functional norm defined in the domain of \mathcal{F} . Suppose these points have an analogous form as the ones in Section II.6.3. Let Ψ_0^\pm and Ψ^\pm be the solutions associated to the points $(\Psi_0^0, g_0^0, \dots, g_3^0, \rho_2^0, \rho_3^0)$ and $(\Psi_w^0 + \Psi_0^0, g_0 + g_0^0, \dots, g_3 + g_3^0, \rho_2 + \rho_2^0, \rho_3 + \rho_3^0)$, respectively. Function

$\psi^- = \Psi^- - \Psi_0^-$ satisfies

$$\begin{aligned} \partial_X \psi^- - \Delta_w^2 \psi^- + Q_w(\Psi_w^0 + \Psi_0^0, \psi^-) + Q_w(\psi^-, \Psi_w^0 + \Psi_0^0) \\ + Q_w(\Psi_w^0, \Psi_0^-) + Q_w(\Psi_0^-, \Psi_w^0) = 0, \quad \text{in } \omega_w^b \setminus \{\sigma_w\} \\ \left[\partial_X^k \psi^- \right] \Big|_{\sigma_w} = g_k, \quad k = 0, \dots, 3 \\ (1 + \alpha^2) \Delta_w \psi^- \Big|_{\sigma_w^M} = \rho_2, \end{aligned} \quad (\text{II.7.20})$$

$$\left(- \left((1 + \alpha^2) \partial_X - 2\alpha \partial_Y \right) \Delta_w \psi^- + \frac{\psi^-}{2} \right) \quad (\text{II.7.21})$$

$$+ (\nabla_w^\perp \Psi_w^0 + \Psi_0^0 \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \psi^- \quad (\text{II.7.22})$$

$$+ (\nabla_w^\perp \Psi_w^0 \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi_0^- \quad (\text{II.7.23})$$

$$+ (\nabla_w^\perp \psi^- \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) (\Psi_w^0 + \Psi_0^0)$$

$$+ (\nabla_w^\perp \Psi_0^- \cdot \nabla_w) \left((1 + \alpha^2) \partial_X - \alpha \partial_Y \right) \Psi_w^0 \Big|_{X=M} = \rho_3,$$

$$\psi^- \Big|_{X=-\gamma(Y)} = \partial_n \psi^- \Big|_{X=-\gamma(Y)} = 0,$$

and for small enough norms $\|e^{\delta X} \Psi_w^0\|_{H_{\text{uloc}}^2}$, $\|\Psi_0^0\|_{H_{\text{uloc}}^2}$, and $\|\Psi_0^-\|_{H_{\text{uloc}}^{m+2}}$, we deduce

$$\begin{aligned} \|\psi^-\|_{H_{\text{uloc}}^2(\omega^b)} + \|\psi^-\|_{X=M} \|H_{\text{uloc}}^{m+1/2}\| + \|\partial_X \psi^-\|_{X=M} \|H_{\text{uloc}}^{m+1/2}\| \\ \leq C \left(\sum_{k=0}^3 \|g_k\|_{W^{m+2-k, \infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}} \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}} \right). \end{aligned}$$

Proceeding similarly on the half-space, suppose $\psi_i = \partial_X^k \psi^+ \Big|_{\sigma_w^M}$, $i = 0, 1$ and a smallness condition on Ψ_w^0 . Then, the unique solution ψ^+ of

$$\begin{aligned} \partial_X \psi^+ - \Delta_w^2 \psi^+ \\ + Q_w(\Psi_w^0 + \Psi_0^0, \psi^+) + Q_w(\Psi_w^0, \Psi_0^+) + Q_w(\psi^+, \Psi_w^0 + \Psi_0^0) + Q_w(\Psi_0^+, \Psi_w^0) = 0, \quad \text{in } \{X > M\} \\ \psi^+ \Big|_{\sigma_w^M} = \psi_0, \quad \partial_X \psi^+ \Big|_{\sigma_w^M} = \psi_1, \end{aligned} \quad (\text{II.7.24})$$

fulfills the estimate

$$\|e^{\delta X} \psi^+\|_{H_{\text{uloc}}^{m+2}} \leq C \left(\sum_{k=0}^3 \|g_k\|_{W^{m+2-k, \infty}} + \|\rho_2\|_{H_{\text{uloc}}^{m-1/2}(\mathbb{R})} + \|\rho_3\|_{H_{\text{uloc}}^{m-3/2}(\mathbb{R})} \right),$$

provided that $\|e^{\delta X} \Psi_0^-\|_{H_{\text{uloc}}^{m+2}}$ is small. It is easily seen the Fréchet differential depends continuously on ψ^+ , and consequently, on ρ_2, ρ_3 and g_k , $k = 0, \dots, 3$.

$d\mathcal{F}(0, \dots, 0)$ is invertible: We consider the systems satisfied by ψ^\pm with $\Psi_w^0 = 0$ where the solutions are considered to be C^3 . Here, ψ^\pm are the unique solutions of (II.5.3) and (II.5.57). If $\mathcal{F}(0, \dots, 0, \rho_2, \rho_3) = 0$, $\psi := \mathbb{1}_{X \leq M} \psi^- + \mathbb{1}_{X > M} \psi^+$ is a solution of the linear system in ω_w boundary layer domain (II.5.1). From Section II.5, we know (II.5.1) without jumps at $X = 0$ has a unique solution $\psi \equiv 0$ and therefore $\rho_2 = 0$ and $\rho_3 = 0$. Consequently, $\ker d\mathcal{F}(0, \dots, 0) = \{(0, 0)\}$.

By Lemma II.7.2, ψ^- is a H_{uloc}^{m+2} function in $[M', M] \times \mathbb{R}$ for all $0 < M' < M$. Thus,

$$(\rho_2, \rho_3) = \left((1 + \alpha^2) \Delta_w \psi^-, -((1 + \alpha^2) \partial_X - 2\alpha \partial_Y) \Delta_w \psi^- + \frac{\psi^-}{2} \right).$$

We see that $d\mathcal{F}(0, \dots, 0)$ is an isomorphism of $H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$. The implicit function theorem guarantees the existence of $(\rho_2, \rho_3) \in H_{\text{uloc}}^{m-1/2}(\mathbb{R}) \times H_{\text{uloc}}^{m-3/2}(\mathbb{R})$ such that $\mathcal{F}(\Psi_w^0, g_0, \dots, g_3, \rho_2, \rho_3) = 0$ for all $g_k \in W^{m+2-k, \infty}(\mathbb{R})$ and $\Psi_w^0 \in H_{\text{uloc}}^{m+2}(\omega_w^b)$ near zero. Finally, we can assert that $\Psi_w := \mathbb{1}_{X \leq M} \Psi_w^- + \mathbb{1}_{X > M} \Psi_w^+$, where Ψ_w^-, Ψ_w^+ are the solutions of (II.5.3) and (II.5.57) associated to $(\Psi_w^0, g_0, \dots, g_3, \rho_2, \rho_3)$ is a solution of the western boundary layer system in the whole domain ω_w . Namely, it satisfies

$$\begin{aligned} \mathcal{A}_2 \left[\Psi_w^-|_{X=M}, \partial_X \Psi_w^-|_{X=M} \right] &= \mathcal{A}_2 \left[\Psi_w^+|_{X=M}, \partial_X \Psi_w^+|_{X=M} \right], \\ \tilde{\mathcal{A}}_3 \left[\Psi_w^-|_{X=M}, \partial_X \Psi_w^-|_{X=M} \right] &= \tilde{\mathcal{A}}_3 \left[\Psi_w^+|_{X=M}, \partial_X \Psi_w^+|_{X=M} \right], \end{aligned}$$

which implies in turn the continuity of the differential operators across $\{X = M\}$, and therefore Ψ_w at the transparent boundary. This completes the proof of Theorem II.7.1.

II.8 Eastern boundary layer

This section is devoted to the proof of Theorem II.2.2 concerning the well-posedness of problem

$$\begin{aligned} -\partial_{X_e} \Psi_e - \Delta_e^2 \Psi_e &= 0, \quad \text{in } \omega_e^+ \cup \omega_e^- \\ \left[\partial_{X_e}^k \Psi_e \right]_{X_e=0} &= \tilde{g}_k, \quad k = 0, \dots, 3, \\ \Psi_e|_{X_e=-\gamma_e(Y)} &= \frac{\partial \Psi_e}{\partial n_e} \Big|_{X_e=-\gamma_e(Y)} = 0, \end{aligned} \tag{II.8.1}$$

where $\tilde{g}_k \in L^\infty$, for $k = 0, \dots, 3$. Problem (II.8.1) describes the behavior of the eastern boundary layer profiles. Note that the system driving Ψ_e^0 is obtained by choosing $\tilde{g}_k \equiv 0$, $\forall k$.

Similar to the previous case of the western boundary layer, we rely on wall laws to show the existence and uniqueness of solutions to system (II.8.1) following the steps listed in Section II.4.1. In Section II.8.1, the analysis of the problem in the half-space exhibits an additional difficulty: the presence of degeneracy at low frequencies. The singular behavior is going to impact the convergence of Ψ_e when X_e goes infinity. As a result, a probabilistic setting and ergodicity properties are prescribed to prove the convergence of the solution.

In this context, we will distinguish three different behaviors of Ψ_e far from the boundary: Ψ_{exp} which decays exponentially to zero, Ψ_{erg} whose convergence to a specific constant is driven by the ergodic theorem and Ψ_{alg} , a function converging to zero at a polynomial rate when $\varepsilon \rightarrow 0$. The analysis of each one of these functions is going to be conducted separately in subsection II.8.1.1, and final results are summarized in Theorem II.2.2. The probabilistic scenario is only necessary for the analysis of Ψ_{erg} since the convergence of the remaining components is obtained using deterministic methods. We pay special attention to the link between the value of the limit of Ψ_{erg} far for the boundary and the choice of \tilde{g}_0 . Later, on Section II.9.1, we will see that this translates in a strong connection between the problems

driving the n -th eastern profile and Ψ_{int}^n . Finally, we will briefly discuss the equivalence of solutions of the problems in the half-space and the rough channel at the “transparent” boundary, and, hence provide a solution for problem (II.8.3) in the whole domain ω_e .

To simplify the notation, we will write X instead of X_e throughout this section.

II.8.1 The problem in the half-space

In this section, we will consider without loss of generality that $M' = 0$, to facilitate the computations. We can easily recuperate the solution when $M' \neq 0$ since the differential operators involved in the problem are translation-invariant. We are confronted with solving the following linear problem

$$\begin{aligned} -\partial_X \Psi_e - \Delta_e^2 \Psi_e &= 0, \quad \text{in } \mathbb{R}_+^2, \\ \Psi_e|_{X=0} &= \psi_0, \quad \partial_X \Psi_e|_{X=0} = \psi_1. \end{aligned} \quad (\text{II.8.2})$$

Taking the Fourier transform with respect to Y on the main equation results in the following fourth order characteristic polynomial with complex coefficients

$$P_e(\mu, \xi) = -\mu + (\mu^2 + (-\alpha\mu + i\xi)^2)^2. \quad (\text{II.8.3})$$

Note that by considering $\mu = -\lambda$ in (II.8.3), we obtain the characteristic polynomial corresponding to the linear boundary layer problem defined at the western domain. Some properties of the polynomial are stated in the following lemma:

Lemma II.8.1. *Let $P_e(\mu, \xi)$ be the characteristic polynomial of (II.8.2), then:*

1. $P_e(\mu, \xi)$ has four distinct roots μ_i^\pm , $i = 1, 2$, where $\Re(\mu_i^+) > 0$, $i = 1, 2$ and $\Re(\mu_i^-) < 0$. Moreover, $\mu_i^\pm = -\lambda^\mp$, where λ^\pm are the roots of (II.5.9).
2. (Low frequencies) As $|\xi| \rightarrow 0$, we have

$$\begin{aligned} \mu_1^+ &= \xi^4 + O(\xi^5), \quad \mu_2^+ = \frac{1}{(1 + \alpha^2)^{2/3}} + O(\xi), \\ \mu_i^- &= -\frac{1}{2(1 + \alpha^2)^{2/3}} + (-1)^i \frac{\sqrt{3}}{2(1 + \alpha^2)^{2/3}} + O(\xi), \quad i = 1, 2. \end{aligned}$$

3. (High frequencies) When $\xi \rightarrow \infty$, we have

$$\begin{aligned} \mu_i^+ &= \frac{1 + i\alpha \operatorname{sgn} \xi}{1 + \alpha^2} |\xi| + \frac{(-1)^i}{2} \sqrt{\frac{1 + i\alpha \operatorname{sgn} \xi}{1 + \alpha^2}} |\xi|^{-1/2} + O(|\xi|^{-3/2}), \quad i = 1, 2, \\ \mu_i^- &= \frac{-1 + i\alpha \operatorname{sgn} \xi}{1 + \alpha^2} |\xi| + \frac{(-1)^i}{2} \sqrt{\frac{-1 + i\alpha \operatorname{sgn} \xi}{1 + \alpha^2}} |\xi|^{-1/2} + O(|\xi|^{-3/2}). \end{aligned}$$

The proof of the previous results follows the same ideas of the ones of Lemmas II.5.1 and II.5.2. For more details, we refer the reader to Appendix II.A.

When $\xi \neq 0$, polynomial (II.8.3) has two roots with strictly positive real part noted by μ_1 and μ_2 , whose asymptotic behavior is summarized in Lemma II.8.1. The solutions of

$$\begin{aligned} -\partial_X \widehat{\Psi}_e + (\partial_X^2 + (\alpha \partial_X + i\xi)^2) \widehat{\Psi}_e &= 0, \quad \text{in } \mathbb{R}_+^2 \\ \widehat{\Psi}_e|_{X=0} &= \widehat{\psi}_0, \quad \partial_X \widehat{\Psi}_e|_{X=0} = \widehat{\psi}_1, \end{aligned}$$

are linear combinations of $\exp(-\mu_k(\xi)X)$ with coefficients also depending on ξ , where $(\mu_k)_{1 \leq k \leq 2}$ are the roots of (II.8.3) satisfying $\Re \mu > 0$. More precisely, they are of the form $\sum_{i=1}^2 A_k(\xi) \exp(-\mu_k(\xi)X)$, where $A_k : \mathbb{R} \rightarrow \mathbb{C}$, $k = 1, 2$. Substituting the expression of $\widehat{\Psi}_e$ on the boundary conditions provides the following linear system of equations

$$\begin{aligned} A_1 + A_2 &= \widehat{\psi}_0, \\ \mu_1 A_1 + \mu_2 A_2 &= -\widehat{\psi}_1. \end{aligned}$$

It is then clear that the matrix associated to this system is invertible when $\xi \neq 0$ since all the roots of the characteristic polynomial are simple. Thus, we obtain the coefficients

$$\begin{aligned} A_1 &:= A_1^0 \widehat{\psi}_0 + A_1^1 \widehat{\psi}_1 = \frac{\mu_2}{\mu_2 - \mu_1} \widehat{\psi}_0 + \frac{1}{\mu_2 - \mu_1} \widehat{\psi}_1, \\ A_2 &:= A_2^0 \widehat{\psi}_0 + A_2^1 \widehat{\psi}_1 = -\frac{\mu_1}{\mu_2 - \mu_1} \widehat{\psi}_0 - \frac{1}{\mu_2 - \mu_1} \widehat{\psi}_1. \end{aligned} \tag{II.8.4}$$

Lemma II.8.2 (Asymptotic behavior of the coefficients). • (Low frequencies). When $|\xi| \rightarrow 0$, we have

$$\begin{aligned} A_1(\xi) &= \widehat{\psi}_0 + (1 + \alpha^2)^{2/3} \widehat{\psi}_1 + O(|\widehat{\psi}_0| |\xi|^4 + |\widehat{\psi}_1| |\xi|), \\ A_2(\xi) &= -(1 + \alpha^2)^{2/3} \widehat{\psi}_1 + O(|\widehat{\psi}_0| |\xi|^4 + |\widehat{\psi}_1| |\xi|). \end{aligned}$$

• (High frequencies) As $|\xi|$ goes to infinity, we have

$$\begin{aligned} A_1(\xi) &= -\sqrt{1 - i\alpha \operatorname{sgn} \xi} |\xi|^{1/2} \widehat{\psi}_1 + \left(\frac{1}{\sqrt{1 - i\alpha \operatorname{sgn} \xi}} |\xi|^{3/2} + \frac{1}{2} \right) \widehat{\psi}_0 \\ &\quad + O(|\widehat{\psi}_0| |\xi|^{-3/2} + |\widehat{\psi}_1| |\xi|^{-5/2}), \\ A_2(\xi) &= \sqrt{1 - i\alpha \operatorname{sgn} \xi} |\xi|^{1/2} \widehat{\psi}_1 - \left(\frac{1}{\sqrt{1 - i\alpha \operatorname{sgn} \xi}} |\xi|^{3/2} - \frac{1}{2} \right) \widehat{\psi}_0 \\ &\quad + O(|\widehat{\psi}_0| |\xi|^{-3/2} + |\widehat{\psi}_1| |\xi|^{-5/2}). \end{aligned}$$

II.8.1.1 Behavior far from the boundary

In the previous section, we proved that $\widehat{\Psi}_e(X, \xi)$ was a linear combination of $A_i(\xi) e^{-\mu_i^+(\xi)X}$, where $(\mu_i)_{1 \leq i \leq 4}$ are the roots of the characteristic polynomial (II.8.3) such that $\Re(\mu_i^+) > 0$ and the form of coefficients is given in (II.8.4). The next step would be to derive a representation formula for Ψ_e , based on the Fourier transform results, in such a way that the formula still makes sense when $\psi_0 \in H_{\text{uloc}}^{3/2}(\mathbb{R})$ and $\psi_1 \in H_{\text{uloc}}^{1/2}(\mathbb{R})$. The behavior of Ψ_e as $X \rightarrow +\infty$ is crucial to understanding the approximation of the solution. Contrary to

the periodic setting (see [53]), the exponential decay does not hold in this case. Although the terms associated to $A_2(\xi)e^{-\mu_2^+(\xi)X}$ decay exponentially to zero (see paragraph II.8.4), when looking closely at expressions on (II.8.4) and the asymptotic behavior in (II.8.1), it is clear that $A_1(\xi)e^{-\mu_1(\xi)X}$ does not converge to zero at low frequencies when $X \rightarrow +\infty$ since $\mu_1(\xi) = O(|\xi|^4)$.

This section is devoted to proving the results in Theorem II.2.2. We have divided the proof into a sequence of lemmas, one for each component of the solution Ψ_e : Ψ_{exp} , Ψ_{alg} and Ψ_{erg} . Ψ_{exp} is a function decaying exponentially at infinity, while Ψ_{alg} refers to the part of Ψ_e converging at polynomial rate. Finally, Ψ_{erg} is a function whose convergence at $X \rightarrow +\infty$ is guaranteed by using ergodic properties in a probabilistic setting.

II.8.1.1.1 Behavior at high frequencies. Let Ψ_e^\sharp denote the eastern boundary layer at high frequencies. Following the ideas in [61, Lemma 9] coupled with the behavior of $\mu_i^+(\xi)$ at infinity yields an equivalent result to the one in Lemma II.5.7:

Lemma II.8.3. *Let $\psi_0, \psi_1 \in L_{\text{uloc}}(\mathbb{R})$. Then, the behavior of the Ψ_e at high frequencies denoted by Ψ_e^\sharp satisfies the estimate*

$$\left\| e^{\delta X} \Psi_e^\sharp \right\|_{L^\infty} \leq C \left(\|\psi_0\|_{L_{\text{uloc}}^1(\mathbb{R})} + \|\psi_1\|_{L_{\text{uloc}}^1(\mathbb{R})} \right),$$

for some constants $C, \delta > 0$

II.8.1.1.2 Low frequencies. This paragraph is devoted to the analysis of the behavior of the eastern boundary layer at low frequencies. Each component is enclosed in the corresponding lemma, starting with the deterministic ones (exponential and algebraic) followed by the probabilistic limit.

Our analysis starts with the component of Ψ_e decaying exponentially at infinity. Explicitly, we deal with the term whose Fourier transform satisfies

$$\widehat{\Psi_{\text{exp}}}(X, \xi) = \sum_k \chi(\xi) A_2^k(\xi) e^{-\mu_2^+(\xi)X} \widehat{\psi}_k, \tag{II.8.5}$$

where $A_2^k(\xi)$ is the coefficient of $A_2(\xi)$ associated to $\widehat{\psi}_k$ in (II.8.4) and χ is defined as in Lemma II.8.3.

Following closely the arguments in Lemma II.5.5 and [61, Lemma 7], we obtain the result:

Lemma II.8.4. *Let Ψ_{exp} defined as in (II.8.5) and $\psi_0, \psi_1 \in L_{\text{uloc}}(\mathbb{R})$. Then, there exist constants $\delta, C > 0$ independent of $\psi_i, i = 0, 1$ such that*

$$\|e^{\delta X} \Psi_{\text{exp}}\|_{L^\infty(\omega_e^+)} \leq C(\|\psi_0\|_{L_{\text{uloc}}^1(\mathbb{R})} + \|\psi_1\|_{L_{\text{uloc}}^1(\mathbb{R})}). \tag{II.8.6}$$

There remains to study the terms of A_1 . From (II.8.4), it is clear that $A_1^k \rightarrow \bar{A}_1^k$, for $k = 0, 1$ as $\xi \rightarrow 0$ where $\bar{A}_1^k \in \mathbb{R}$. Therefore, it is possible to rewrite each one of the terms as $A_1^k(\xi) = \bar{A}_1^k + \tilde{A}_1^k(\xi)$ with

$$\bar{A}_1^k(\xi) = \begin{cases} 1, & \text{for } k = 0 \\ (1 + \alpha^2)^{2/3}, & \text{for } k = 1 \end{cases}, \quad \tilde{A}_1^k(\xi) = \begin{cases} \frac{\mu_1}{\mu_2 - \mu_1}, & \text{for } k = 0 \\ \frac{1}{\mu_2 - \mu_1} - (1 + \alpha^2)^{2/3}, & \text{for } k = 1 \end{cases}.$$

The terms $\tilde{A}_1^k(\xi)f$, $k = 0, 1$ decay to zero far from the boundary at polynomial rate as a result of Lemma II.5.5. Notice that, at low frequencies, \tilde{A}_1^1 behaves as a constant, while \tilde{A}_1^2 does it similarly to a homogeneous polynomial of degree 1, which is a straightforward consequence of Lemma II.8.1.

We introduce the notation Ψ_{alg} for the part of Ψ_e decaying to zero algebraically when $X \rightarrow +\infty$, i.e., the one associated to the coefficients \tilde{A}_k , $k = 0, 1$. Its behavior is summarized in the following lemma:

Lemma II.8.5. *Let $\psi_k \in L^1_{\text{uloc}}(\mathbb{R})$, $k = 0, 1$. Then, there exists a constant C independent of ψ_i , $i = 0, 1$ such that*

$$\|X^{1/4}\Psi_{\text{alg}}\|_{L^\infty(\mathbb{R}^2)} \leq C(\|\psi_0\|_{L^1_{\text{uloc}}(\mathbb{R})} + \|\psi_1\|_{L^1_{\text{uloc}}(\mathbb{R})}). \quad (\text{II.8.7})$$

It is easy to check that the convergence of the terms $\tilde{A}_1^k f$ is not guaranteed for any choice of function f , $k = 0, 1$. Therefore, we require additional hypotheses, in this case, of ergodicity.

We recall for the reader's convenience the probability setting: for $\varepsilon > 0$, let (P, Π, μ) be a probability space where P is the set of K -Lipschitz functions, with $K > 0$; Π the borelian σ -algebra of P , seen as a subset of $C_b(\mathbb{R}^2; \mathbb{R})$ and μ a probability measure. Let $(\tau_Y)_{Y \in \mathbb{R}}$, the measure-preserving transformation group acting on P . We recall that there exists a function $F \in L^\infty(P)$ such that

$$\gamma_\varepsilon(m, Y) = F(\tau_Y m), \quad Y \in \mathbb{R}, \quad m \in P.$$

We define the stochastic derivative of F by

$$\partial_m F(m) := \gamma'_\varepsilon(m, 0) \quad \forall m \in P,$$

so that $\gamma'_\varepsilon(m, Y) = \partial_m F(\tau_Y m)$ for $(m, Y) \in P \times \mathbb{R}$. Then, the eastern boundary layer domain can be described as follows for all $m \in P$,

$$\omega_\varepsilon(m) = \left\{ (X, Y) \in \mathbb{R}^2 : X > -\gamma_\varepsilon(m, Y) \right\},$$

where $\omega_\varepsilon(m) = \omega_\varepsilon^+(m) \cup \sigma_\varepsilon \cup \omega_\varepsilon^-(m)$ and $\omega_\varepsilon^\pm(m) = \omega_\varepsilon(m) \cap \{\pm X > 0\}$. We assume γ_ε is a homogeneous and measure-preserving random process.

Lemma II.8.6. *Let $\chi \in C_0^\infty(\mathbb{R})$ and μ_1^+ defined as in Lemma II.8.1. Suppose that f is a stationary random function belonging to L^∞ . Then,*

$$\chi(D)e^{-\mu_1^+(D)X} f \rightarrow \mathbb{E}(f), \quad \text{as } X \rightarrow +\infty,$$

locally uniformly in Y , almost surely and in $L^p(P)$ for all finite p .

The hypothesis on f can be relaxed since we need f to be at least to L^1_{uloc} .

Proof. Birkhoff Ergodic Theorem (see, e.g., Theorem 4.1.2 of [72]) guarantees the existence of the following limit

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R f(m, -Y') dY' = \mathbb{E}(f), \quad \text{almost surely.} \quad (\text{II.8.8})$$

Furthermore, following the ideas in [63, Lemma 4.6] we have

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R f(m, Y - Y') dY' = \mathbb{E}(f),$$

uniformly locally in Y , and (II.8.8) also satisfies almost surely

$$\lim_{R \rightarrow \infty} \frac{1}{R} \int_0^R f(m, Y') dY' = \mathbb{E}(f).$$

From Lemma II.5.5, we know that for $|X| \geq 1$, $Y' \rightarrow K(X, Y - Y')$ is $L^1(\mathbb{R})$. Let $m(X) = \int_{\mathbb{R}} K(X, Y) dY$, we have

$$\int_{\mathbb{R}} K(X, Y) dY = \widehat{K}(X, 0) = \chi(0) e^{-\mu_1(0)X} = 1,$$

and, consequently, $m_0(X) = 1$, when $X > 0$. Note that $m(X) = 1$, for all $X > 0$. Then, proceeding exactly as in [63, Lemma 4.6]

$$\chi(D) e^{-\mu_1^+(D)X} = \mathbb{E}(f) + \int_{\mathbb{R}} K(X, Y' - Y) (f(Y - Y') - \mathbb{E}(f)) dY'. \quad (\text{II.8.9})$$

Integrating by parts leads to

$$\chi(D) e^{-\mu_1^+(D)X} = \mathbb{E}(f) - \int_{\mathbb{R}} \partial_{Y'} K(X, Y') \left(\int_0^{Y'} (f(Y - Y_1) - \bar{f}) dY_1 \right) dY'.$$

Therefore,

$$\begin{aligned} \chi(D) e^{-\mu_1^+(D)X} f &= \mathbb{E}(f) - \int_{\mathbb{R}} Y' \partial_{Y'} K(X, Y') \left(\frac{1}{Y'} \int_0^{Y'} (f_0(Y - Y_1) - \bar{f}) dY_1 \right) dY \\ &= \mathbb{E}(f) + J. \end{aligned}$$

Now, we focus our attention on controlling the term J . From Lemma II.5.5, we have the key estimates

$$\begin{aligned} |Y' \partial_{Y'} K(X, Y')| &\leq C \frac{X^{1/4} Y}{X^{3/4} + Y^3}, \quad \forall |X| \geq 1, \quad \forall Y' \in \mathbb{R}, \\ \int_{\mathbb{R}} Y' \partial_{Y'} K(X, Y') &= -1. \end{aligned} \quad (\text{II.8.10})$$

We conduct the analysis by distinguishing the behavior of J in $|Y'| \leq R$ from the one on $|Y'| > R$ for $R \gg 1$, denoted by J_1 and J_2 respectively. For all $\delta > 0$ and all $L > 0$ there exists $R > 0$ such that for all $|Y| \leq L$ and $|Y'| \geq R$

$$\left| \frac{1}{Y'} \int_0^{Y'} f(Y - Y_1) dY_1 - \mathbb{E}(f) \right| \leq \delta,$$

and $|J_1| \leq C\delta$. For $|Y'| \leq R$, we have that

$$\begin{aligned} |J_2| &= \left| \int_{|Y'| \leq R} Y' \partial_{Y'} K(X, Y') \frac{1}{Y'} \int_0^{Y'} (f(Y - Y_1) - \mathbb{E}(f)) dY_1 dY' \right| \\ &\leq C \|f\|_{L^1_{\text{uloc}}} \left| \int_{|Y'| \leq R} \frac{X^{1/4} Y}{X^{3/4} + Y^3} dY' \right| \\ &\leq C_R \|f\|_{L^1_{\text{uloc}}} |X|^{-1/4} \xrightarrow{X \rightarrow \infty} 0. \end{aligned}$$

Thus,

$$\lim_{X \rightarrow \infty} J(X, Y) = 0,$$

which allows us to conclude that

$$\lim_{X \rightarrow \infty} \chi(D) e^{-\mu_1^+(D)X} f = \mathbb{E}(f).$$

□

Combining the results in Lemmas II.8.2 and II.8.6, we can provide a definition for the limit of Ψ_e .

Definition II.8.1. *Let ψ_0, ψ_1 be stationary random functions belonging to $L^\infty(\omega_e)$. Then, the limit of Ψ_{erg} as $X \rightarrow +\infty$ is the measurable function $\bar{\phi} : P \rightarrow \mathbb{R}$ of the form*

$$\bar{\phi} = \mathbb{E}[\psi_0] + (1 + \alpha^2)^{2/3} \mathbb{E}[\psi_1]. \quad (\text{II.8.11})$$

II.8.1.1.3 Almost sure convergence of Ψ_{erg} . We show almost sure estimates in the stationary ergodic setting as in [73].

Proposition II.8.1. *Let Ψ_{erg} be the part of the solution Ψ_e of the system (II.2.13) whose convergence is guaranteed by ergodicity hypotheses. Then the following estimates hold:*

$$\left\| \Psi_{erg}(\cdot/\varepsilon) - \bar{\phi} \right\|_{L^2(\Omega^\varepsilon)} = o(1) \quad \text{almost surely as } \varepsilon \rightarrow 0.$$

Proof. This proof follows the ideas in [73] inspired from the works by Souganidis (see [74]). Let $\delta > 0$ be arbitrary. Then, according to Egorov's Theorem, there exist a measurable set $M_\delta \subset P$ and a number $X_\delta > 0$ such that

$$\begin{aligned} \left| \Psi_{erg}(m, X, 0) - \bar{\phi} \right| &\leq \delta, \quad \forall m \in M_\delta, \forall X > X_\delta, \\ \mu(M_\delta^c) &\leq \delta. \end{aligned}$$

Without loss of generality, we assume that $X_\delta \leq \varepsilon^{-1}(\chi_\varepsilon(y) - \chi_w(y))$. From Birkhoff's ergodic theorem, we have that for almost every m there exists $k_\delta > 0$ (depending on m) such that if $k > k_\delta$

$$A_\delta = A_\delta(m) := \{Y \in \mathbb{R}, \tau_Y m \in M_\delta^c \text{ satisfies: } |A_\delta \cap (-k, k)| \leq 4k\delta\}.$$

Indeed, when k goes to infinity,

$$\frac{1}{2k} \int_{-k}^k \mathbb{1}_{\tau_Y m \in M_\delta^c} \rightarrow \mu(M_\delta^c) \leq \delta.$$

Therefore, for almost all $m \in P$, there exists some k_δ such that for all $k > k_\delta$

$$|A_\delta \cap (-k, k)| = \left| \int_{-k}^k \mathbb{1}_{\tau_Y m \in M_\delta^c} \right| \leq 4k\delta.$$

For all $\varepsilon > 0$, we have

$$\begin{aligned}
\left\| \Psi_{\text{erg}}(\cdot/\varepsilon) - \bar{\phi} \right\|_{L^2(\Omega^\varepsilon)}^2 &= \int_{y_{\min}}^{y_{\max}} \int_{\Gamma_e}^{\Sigma_w} \left| \Psi_e^{m, \text{erg}} \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) - \bar{\phi} \right|^2 dx dy \\
&= \varepsilon^2 \int_{y_{\min}/\varepsilon}^{y_{\max}/\varepsilon} \int_{\gamma_e(Y)}^{(\chi_e(\varepsilon Y) - \chi_w(\varepsilon Y))/\varepsilon} \left| \Psi_{\text{erg}}(m, X, Y) - \bar{\phi} \right|^2 dX dY \\
&= \varepsilon^2 \int_{y_{\min}/\varepsilon}^{y_{\max}/\varepsilon} \int_{\gamma_e(Y)}^{(\chi_e(\varepsilon Y) - \chi_w(\varepsilon Y))/\varepsilon} \left| \Psi_{\text{erg}}(\tau_Y m, X, 0) - \bar{\phi} \right|^2 dX dY \\
&= \varepsilon^2 \int_{y_{\min}/\varepsilon}^{y_{\max}/\varepsilon} \int_{X_\delta}^{X_\delta} \left| \Psi_{\text{erg}}(\tau_Y m, X, 0) - \bar{\phi} \right|^2 dX dY \\
&+ \varepsilon^2 \int_{y_{\min}/\varepsilon}^{y_{\max}/\varepsilon} \int_{X_\delta}^{(\chi_e(\varepsilon Y) - \chi_w(\varepsilon Y))/\varepsilon} \mathbb{1}_{\tau_Y m \in M_\delta} \left| \Psi_{\text{erg}}(\tau_Y m, X, 0) - \bar{\phi} \right|^2 dX dY \\
&+ \varepsilon^2 \int_{y_{\min}/\varepsilon}^{y_{\max}/\varepsilon} \int_{X_\delta}^{(\chi_e(\varepsilon Y) - \chi_w(\varepsilon Y))/\varepsilon} \mathbb{1}_{\tau_Y m \in M_\delta^c} \left| \Psi_{\text{erg}}(\tau_Y m, X, 0) - \bar{\phi} \right|^2 dX dY \\
&= \sum_{j=1}^3 I_j.
\end{aligned}$$

Then,

$$I_1 \leq \varepsilon \left\| \Psi_{\text{erg}} \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) - \bar{\phi} \right\|_{L^2(\{x < x_\delta\} \cap \Omega^\varepsilon)}^2 \leq C_\delta \varepsilon \left(\|\Psi_{\text{erg}}\|_\infty + |\bar{\phi}| \right), \quad (\text{II.8.12})$$

where the constant C_δ depends on the random parameter m and $x_\delta = \varepsilon X_\delta$. Taking into account that if $\tau_Y m \in M_\delta$, $\Psi_{\text{erg}}(m, X, Y) = \Psi_{\text{erg}}(\tau_Y m, X, 0)$ and $X > X_\delta$,

$$I_2 \leq \varepsilon^2 \left(\frac{y_{\max} - y_{\min}}{\varepsilon} \right) \left(\frac{\chi_w(y) - \chi_e(y) - x_\delta}{\varepsilon} \right) \delta^2 \leq C \delta^2. \quad (\text{II.8.13})$$

As for the third integral, we know that $\Psi_e \in L^\infty((-\infty, a) \times \mathbb{R})$. Assuming $a = \bar{\chi}_e := \max_{y \in (y_{\min}, y_{\max})} \chi_e(y)$ and $\varepsilon < 1/k_\delta$, we have

$$I_3 \leq \varepsilon \|\chi_e - \chi_w\|_\infty \|\Psi_{\text{erg}} - \bar{\phi}\|_{L^\infty((-\infty, \bar{\chi}_e) \times \mathbb{R})}^2 \lambda \left(\varepsilon^{-1}(y_{\min}, y_{\max}) \cap A_\delta \right),$$

where λ denotes the Lebesgue measure. Then,

$$I_3 \leq C \delta \left(|\bar{\phi}|^2 + \|\Psi_{\text{erg}}\|_{L^\infty((-\infty, \bar{\chi}_e) \times \mathbb{R})}^2 \right). \quad (\text{II.8.14})$$

The first estimate of the lemma is obtained by combining (II.8.12), (II.8.13) and (II.8.14). \square

II.8.1.1.4 Connection between the choice of \tilde{g}_0 and the limit of Ψ_{erg} . In the formal construction of the approximate solution, it was important to reflect that boundary layers are not supposed to have any impact far from the boundary (II.3.2). In Section II.8.1.1, we showed that at low frequencies and far for the boundary the eastern boundary layer can be decomposed as

$$\Psi_e = \Psi_{\text{exp}} + \Psi_{\text{alg}} + \Psi_{\text{erg}},$$

where Ψ_{exp} and Ψ_{alg} converge to zero when $X \rightarrow +\infty$ at different rates (exponential and with a polynomial weight, respectively), while Ψ_{erg} converges almost surely to a quantity $\bar{\phi}$ once ergodicity assumptions have been added. It is obvious that the far field condition does not hold for $\bar{\phi} \neq 0$. In this paragraph, we explore how choosing \tilde{g}_0 wisely can make

$$\Psi_e \xrightarrow{X \rightarrow +\infty} 0 \quad \text{almost surely,}$$

which is the last result in Theorem II.2.2.

We are interested in the specific case when all \tilde{g}_k are given but \tilde{g}_0 in (II.8.1). Moreover, \tilde{g}_0 is supposed to be constant with respect to the boundary layer variables. Note that if Ψ_e is a solution of (II.8.1), by linearity of the problem, $\tilde{\Psi}_e = \Psi_e + \tilde{g}_0$ also satisfies the boundary layer problem. Passing to the limit as $X \rightarrow +\infty$ gives

$$\lim_{X \rightarrow +\infty} \tilde{\Psi}_e = \bar{\phi} + \tilde{g}_0.$$

Then, for $\tilde{g}_0 = -\bar{\phi}$, the limit of Ψ_{erg} as X_e goes to infinity equals zero and, in turn, $\tilde{\Psi}_e$ satisfies both (II.8.1) and condition at $+\infty$.

Let us now illustrate the procedure for the profile Ψ_e^1 . This function satisfies the system of equations

$$\begin{aligned} -\partial_{X_e} \Psi_e^1 - \Delta_e^2 \Psi_e^1 &= 0, \quad \text{in } \omega_e^+ \cup \omega_e^- \\ \left[\partial_{X_e}^k \Psi_e^1 \right] \Big|_{\sigma_e} &= - \left[\Psi_{int}^1 \right] \Big|_{x=\chi_e(y)}, \\ \left[\partial_{X_e} \Psi_e^1 \right] \Big|_{\sigma_e} &= \left[\partial_x \Psi_{int}^0 \right] \Big|_{x=\chi_e(y)}, \\ \left[\partial_{X_e}^k \Psi_e^1 \right] \Big|_{\sigma_e} &= 0, \quad k = 2, 3, \\ \Psi_e^1 \Big|_{X_e=-\gamma_e(Y)} &= \frac{\partial \Psi_e^1}{\partial n_e} \Big|_{X_e=-\gamma_e(Y)} = 0. \end{aligned} \tag{II.8.15}$$

The jump of its derivative at σ_e depends of Ψ_{int}^0 , and it is, therefore, known from the previous step. Here, $\Psi^1(t, y)$ is the solution of the equation

$$\partial_x \Psi^1 = 0 \quad \text{in } \Omega$$

Consequently, $\Psi^1(t, y) = C^1(t, y)$.

Let us consider a solution Ψ_e^1 of (II.8.15) and define the function

$$\tilde{\Psi}_e^1 = \begin{cases} \Psi_e^1 - \bar{\phi}, & \text{in } \omega_e^+ \\ \Psi_e^1, & \text{in } \omega_e^- \end{cases}, \tag{II.8.16}$$

where $\bar{\phi}$ is the value of the limit Ψ_e^1 when $X \rightarrow +\infty$. Note that (II.8.16) satisfies all the conditions in (II.8.15) but $[\tilde{\Psi}_e^1] \Big|_{\sigma_e} = -\bar{\phi}$. Moreover, $\tilde{\Psi}_e^1 \rightarrow 0$ almost surely far from the eastern boundary. Since the jump at the interface σ_e is linked to the interior profile, we have

$$\bar{\phi} = \left[\Psi_{int}^1 \right] \Big|_{x=\chi_e(y)} = C^1$$

Therefore, considering $C^1(t, y) = \bar{\phi}(t, y)$ provides a solution for the boundary layer (II.8.15) satisfying the far field condition. This interdependence between the eastern boundary layer and the interior profile determines the construction of the approximate solution. First, we solve the problem at the East with a general function \tilde{g}_0 to obtain the value of the limit far from the boundary. Then, this information is considered when computing the solution of the corresponding interior profile. Finally, the western boundary layer problem is addressed.

II.8.2 Transparent operators

This section deals with the well-posedness of the differential operators associated with the eastern boundary layer problem at the transparent boundary. We stick in our analysis to the same ideas developed in Section II.5.2 and, as a consequence, we summarize the main steps and go into detail only when the differences with the western boundary layer are notable.

Once again, without loss of generality, suppose that the artificial boundary lays at $X = 0$ and is defined as follows

Definition II.8.2. *Let $\Psi_e \in H^2(\mathbb{R}_+^2)$ be the unique weak solution of the Dirichlet problem (II.8.1) for $(\psi_0, \psi_1) \in H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$. Then, the biharmonic matrix-valued Poincaré-Steklov operator is given by*

$$PS_e : H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \rightarrow H^{-1/2}(\mathbb{R}) \times H^{-3/2}(\mathbb{R})$$

$$PS_e \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} := \begin{pmatrix} -(1 + \alpha^2) \Delta_e \Psi_e \Big|_{X=0} \\ \left[\left((1 + \alpha^2) \partial_X + 2\alpha \partial_Y \right) \Delta_e \Psi_e + \frac{\Psi_e}{2} \right] \Big|_{X=0} \end{pmatrix} = K_e * \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \quad (\text{II.8.17})$$

where K_e is the distributional kernel.

Substituting the Fourier representation of the solution Ψ_e on PS_e similarly to (II.5.41) results in an explicit formula for the Poincaré-Steklov operator. Indeed, we have $PS_e = (\mathcal{B}_2[\psi_0, \psi_1], \mathcal{B}_3[\psi_0, \psi_1])$ for

$$\mathcal{B}_k : H^{3/2} \times H^{1/2} \rightarrow H^{-3/2+k}$$

$$\mathcal{B}_k[\psi_0, \psi_1] := \mathcal{F}^{-1}(n_{k,0} \widehat{\psi_0} + n_{k,1} \widehat{\psi_1}), \quad k = 2, 3, \quad (\text{II.8.18})$$

where $n_{i,j}$ denotes the components of the matrix $M_e = \mathcal{F}^{-1}(K_e)$.

The asymptotic behavior of M_e at low and high frequencies is summarized in the following lemma:

Lemma II.8.7. • *When $|\xi| \ll 1$*

$$M_e = \begin{pmatrix} (\alpha^2 + 1) |\xi|^2 + O(|\xi|^3) & (\alpha^2 + 1)^{4/3} + O(|\xi|) \\ \frac{1}{2} + O(|\xi|) & -(\alpha^2 + 1)^{2/3} + O(|\xi|). \end{pmatrix}$$

• *When $|\xi| \rightarrow +\infty$, there exist complex constants $\bar{n}_{i,j}$, $i = 2, 3$, $j = 0, 1$ depending on the parameter α such that*

$$M_e = \begin{pmatrix} \bar{n}_{2,0} |\xi|^2 + O(|\xi|^{-1/2}) & \bar{n}_{2,1} |\xi| + O(|\xi|^{-1/2}) \\ \bar{n}_{3,0} |\xi|^3 + O(1) & \bar{n}_{3,1} |\xi|^2 + O(|\xi|^{-1/2}). \end{pmatrix}$$

For a proof of Lemma II.8.7, we refer the reader to Appendix II.F. Applying the same ideas of Section II.5.2, it is easily seen that at high and low frequencies our operator is well-defined and continuous in usual Sobolev spaces. Moreover, Lemma II.5.10 also holds for M_e .

Of course, we are interested in extending the definition of PS_e to the case of functions that are not square-integrable in \mathbb{R}^2 , but rather locally uniformly integrable. Generalizing the results above can be easily achieved by following the same reasoning of Section II.5.2. The differences in the definition of the operators at the western and eastern boundary layers do not impact the estimates. Consequently, for the convenience of the reader, we list the relevant results without proof.

The unique extension of \mathcal{B}_i , $i = 2, 3$ to Kato spaces is guaranteed by Lemma II.5.11. Moreover, we have the integral representation:

Proposition II.8.2. *Let $(\psi_0, \psi_1) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$, and let Ψ_e be the unique solution of (II.5.3) with $F = 0$ and boundary data $\Psi_e|_{X=0} = \psi_0$ and $\partial_X \Psi_e|_{X=0} = \psi_1$. Then, for all $\varphi \in C_0^\infty(\bar{\mathbb{R}}_+^2)$*

$$-\int_{\mathbb{R}_+^2} \partial_X \Psi_e \varphi - \int_{\mathbb{R}_+^2} \Delta_e \Psi_e \Delta_w \varphi = \left\langle \mathcal{B}_3[\psi_0, \psi_1] - \frac{\psi_0}{2}, \varphi|_{X=0} \right\rangle + \left\langle \mathcal{B}_2[\psi_0, \psi_1], \partial_X \varphi|_{X=0} \right\rangle. \quad (\text{II.8.19})$$

In particular, for $(\psi_0, \psi_1) \in H^{3/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ the Poincaré-Steklov operator satisfies

$$\langle \mathcal{B}_3[\psi_0, \psi_1], \psi_0 \rangle + \langle \mathcal{B}_2[\psi_0, \psi_1], \psi_1 \rangle \leq 0. \quad (\text{II.8.20})$$

It is possible to relate the solution of the (II.5.5) to $(\psi_0, \psi_1) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$ and $PS_w(\psi_0, \psi_1)$ by introducing a smooth function $\tilde{\chi}$, with $\tilde{\chi} = 1$ in an open set containing $\text{Supp} \varphi$ and the kernel representation formula of the boundary operators. Estimate (II.8.20) follows from taking $\Psi_e \in H^2(\mathbb{R}_+^2)$ as test function in (II.8.19) and using a density argument.

Similarly to the linear problem driving the behavior of the western boundary layer, we have that

Proposition II.8.3. *Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\text{Supp} \varphi \subset B(Y_0, R)$, $R \geq 1$, and $(\psi_0, \psi_1) \in H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R})$. Then, there exists a constant $C > 0$ such that the following property holds.*

$$|\langle \mathcal{B}_3[\psi_0, \psi_1], \varphi \rangle| + |\langle \mathcal{B}_2[\psi_0, \psi_1], \partial_X \varphi \rangle| \leq C\sqrt{R} \left(\|\varphi\|_{H^{3/2}(\mathbb{R})} + \|\partial_X \varphi\|_{H^{1/2}(\mathbb{R})} \right) \left(\|\psi_0\|_{H_{\text{uloc}}^{3/2}(\mathbb{R})} + \|\psi_1\|_{H_{\text{uloc}}^{1/2}(\mathbb{R})} \right). \quad (\text{II.8.21})$$

Moreover, if $\psi_j \in H^{3/2-j}(\mathbb{R})$, $j = 0, 1$,

$$|\langle \mathcal{B}_3[\psi_0, \psi_1], \varphi \rangle| + |\langle \mathcal{B}_2[\psi_0, \psi_1], \partial_X \varphi \rangle| \leq C \left(\|\varphi\|_{H^{3/2}(\mathbb{R})} + \|\partial_X \varphi\|_{H^{1/2}(\mathbb{R})} \right) \left(\|\psi_0\|_{H^{3/2}(\mathbb{R})} + \|\psi_1\|_{H^{1/2}(\mathbb{R})} \right). \quad (\text{II.8.22})$$

The proof of the proposition relies mainly on bounds on M_e which are used to compute estimates in fractional Sobolev spaces. Since the proof is very similar to the one in Lemma II.5.10, we refer the reader to Section II.5.2 for details.

II.8.3 Equivalent problem and estimates on the rough channel

In this section, we are concerned with proving the existence of weak solutions to the linear system driving the behavior of Ψ_e in ω_e . There difficulties we confronted before remain: the irregularities of γ_e prevents us from using the Fourier transform in the tangential direction and the domain ω_e^b is unbounded and, it is therefore impossible to rely on Poincaré type inequalities. Here, we follow the ideas presented in Step (L5) of Section II.4.1. We start by defining a problem equivalent to (II.8.1) yet posed in the bounded channel $\omega_e^b = \{-\gamma_e(Y) \leq X \leq M\} \times \mathbb{R}$, where a transparent boundary condition has been imposed at the interface $X = M$, $M \geq 0$. We have the system

$$\begin{aligned} -\partial_X \Psi_e^- - \Delta_e^2 \Psi_e^- &= F_e^L, \quad \text{in } \omega_e^b \setminus \sigma_e \\ [\Psi_e^-] |_{\sigma_e} &= -\bar{\phi}, \end{aligned} \tag{II.8.23}$$

$$[\partial_X^k \Psi_e^-] |_{\sigma_e} = \tilde{g}_k, \quad k = 1, \dots, 3, \tag{II.8.24}$$

$$(1 + \alpha^2) \Delta_e \Psi_e^- |_{\sigma_e^M} = \mathcal{B}_2 [\Psi_e^- |_{X=M}, \partial_X \Psi_e^- |_{X=M}],$$

$$-(1 + \alpha^2) \partial_X \Delta_e \Psi_e^- + 2\alpha \partial_Y \Delta_e \Psi_e^- + \frac{\Psi_e^-}{2} \Big|_{\sigma_e^M} = \mathcal{B}_3 [\Psi_e^- |_{X=M}, \partial_X \Psi_e^- |_{X=M}],$$

$$\Psi_e^- |_{X=-\gamma_e(Y)} = \frac{\partial \Psi_e^-}{\partial n_e} \Big|_{X=-\gamma_e(Y)} = 0,$$

where $\tilde{g}_k \in L^\infty(\mathbb{R})$, $k = 1, \dots, 3$ and $\bar{\phi}$ is constant function with respect to the boundary layer variables chosen as in Definition II.8.1. This guarantees the solution of (II.8.23) satisfies the far field condition (see Section II.8.1.1). Here, \mathcal{B}_k denotes the components of the Poincaré-Steklov operator. The following lemma states the equivalence between the solutions of the problems (II.8.23) and (II.8.1).

Lemma II.8.8. *Let $\gamma_e \in W^{2,\infty}(\mathbb{R})$ be an ergodic stationary random process, K -Lipschitz almost surely, for some $K > 0$ in a probability space (P, Π, μ) . Assume $\bar{\phi}$ is a constant function with respect to the macroscopic variables and $g_k \in L^\infty(\mathbb{R})$.*

- If Ψ_e is a solution of (II.5.1) in ω_e such that $\Psi_e \in H_{\text{uloc}}^2(\omega_e)$, then, $\Psi|_{\omega_e^b}$ is a solution of (II.8.23), and for $X > M$, Ψ solves problem (II.5.1), with $\psi_0 := \Psi_e|_{X=M} \in H_{\text{uloc}}^{3/2}(\mathbb{R})$ and $\psi_1 := \partial_X \Psi_e|_{X=M} \in H_{\text{uloc}}^{1/2}(\mathbb{R})$.
- Conversely, if $\Psi_e^- \in H_{\text{uloc}}^2(\omega_e^b)$ and $\Psi_e^+ \in H_{\text{uloc}}^2(\mathbb{R}_+^2)$ are solutions of (II.8.23) and (II.8.2), respectively; then, the function

$$\Psi_e(X, \cdot) := \begin{cases} \Psi_e^-(X, \cdot) & \text{for } -\gamma_e(\cdot) < X < M, \\ \Psi_e^+(X, \cdot) & \text{for } X > M, \end{cases}$$

belongs to $H_{\text{loc}}^2(\omega)$ and is a solution of the problem (II.5.1).

The proof of Lemma II.8.8 follows from combining the results in Section II.8.1 and Proposition II.8.3 and is, therefore, left to the reader.

Proposition II.8.4. *Let $\gamma_e \in W^{2,\infty}(\mathbb{R})$ and $\omega_e^b = \omega_e \cap \{X \leq M\}$, $M > 0$. Let $\mathcal{B}_i : H_{\text{uloc}}^{3/2}(\mathbb{R}) \times H_{\text{uloc}}^{1/2}(\mathbb{R}) \rightarrow H_{\text{uloc}}^{3/2-i}(\mathbb{R})$, $i = 2, 3$ be Poincaré-Steklov operators verifying Proposition II.8.3. Moreover, $\bar{\phi}$ is a constant function with respect to the boundary layer variables and $\tilde{g}_k \in L^\infty(\mathbb{R})$, for $k = 1, \dots, 3$. Then, there exists a unique solution $\Psi_e \in H_{\text{uloc}}^2(\omega_e^b \setminus \sigma_e)$ satisfying for some constant $C > 0$ the estimate*

$$\|\Psi_e\|_{H_{\text{uloc}}^2(\omega_e^b)} \leq C \left(\|\bar{\phi}\|_\infty + \sum_{k=1}^3 \|\tilde{g}_k\|_{L^\infty(\mathbb{R})} \right). \quad (\text{II.8.25})$$

It is worth noting that the above result does not need the ergodicity hypothesis.

Proof. To facilitate the computations, we lift the jump conditions at $X = 0$ in order to work with a C^3 function at the interface between the interior and rough domains. Namely, we analyze the existence and uniqueness of a solution $\tilde{\Psi}_e = \Psi_e^- - \Psi_e^L$ of the system

$$\begin{aligned} -\partial_X \tilde{\Psi}_e - \Delta_e^2 \tilde{\Psi}_e &= F_e^L \quad \text{in } \omega_e^b \\ (1 + \alpha^2) \Delta_e \tilde{\Psi}_e|_{\sigma_e^M} &= \mathcal{B}_2 [\tilde{\Psi}_e|_{X=M}, \partial_X \tilde{\Psi}_e|_{X=M}] \\ -(1 + \alpha^2) \partial_X \Delta_e \tilde{\Psi}_e + 2\alpha \partial_Y \Delta_e \tilde{\Psi}_e + \frac{\tilde{\Psi}_e}{2} \Big|_{\sigma_e^M} &= \mathcal{B}_3 [\tilde{\Psi}_e|_{X=M}, \partial_X \tilde{\Psi}_e|_{X=M}] \\ \tilde{\Psi}_e|_{X=-\gamma_e(Y)} &= \frac{\partial \tilde{\Psi}_e}{\partial n_e} \Big|_{X=-\gamma_e(Y)} = 0, \end{aligned}$$

where Ψ_e^L is defined as in (II.4.7) and $F_e^L = \Delta_e^2 \Psi_e^L + \partial_X \Psi_e^L$.

Adapting Definition II.4.1 provides the weak formulation: *A function $\Psi \in H_{\text{uloc}}^2(\omega_e^b)$ is a solution of (II.8.26) if it satisfies the homogeneous conditions $\tilde{\Psi}_e|_{\Gamma_e} = \partial_n \tilde{\Psi}_e|_{\Gamma_e} = 0$ at the rough boundary, and if, for all $\varphi \in \tilde{\mathcal{V}}$, we have*

$$\begin{aligned} \int_{\omega_e^b} \partial_X \tilde{\Psi}_e \varphi + \int_{\omega^b} \Delta_e \tilde{\Psi}_e \Delta_e \varphi &= - \int_{\omega^b} F_e^L \varphi \quad (\text{II.8.26}) \\ &- \left\langle \mathcal{B}_3 [\tilde{\Psi}_e|_{X=M}, \partial_X \tilde{\Psi}_e|_{X=M}] - \frac{\tilde{\Psi}_e}{2} \Big|_{X=M}, \varphi \Big|_{X=M} \right\rangle_{H_{\text{uloc}}^{-3/2}, H_{\text{uloc}}^{3/2}} \\ &- \left\langle \mathcal{B}_2 [\tilde{\Psi}_e|_{X=M}, \partial_X \tilde{\Psi}_e|_{X=M}], \partial_X \varphi \Big|_{X=M} \right\rangle_{H_{\text{uloc}}^{-1/2}, H_{\text{uloc}}^{1/2}}, \quad (\text{II.8.27}) \end{aligned}$$

where $\tilde{\mathcal{V}}$ is the space of functions $\varphi \in C_0^\infty(\overline{\omega_e^b})$ such that $\text{Supp} \varphi \cap \partial\Omega = \emptyset$ and $\mathcal{D}_0^2(\omega^b)$ its completion for the norm $\|\Psi\| = \|\Delta_e \Psi\|_{L^2}$. Note that (II.8.26) is quite similar to (II.8.19).

To prove the existence and uniqueness of a H_{uloc}^2 solution of problem (II.8.26), we use the method by Ladyženskaja and Solonnikov on the truncated energies

$$E_k^n := \int_{\omega_k} |\Delta_e^2 \tilde{\Psi}_e^n|^2,$$

where $\tilde{\Psi}_{e,n}^-$ is equal to $\tilde{\Psi}_e$ on ω_n and zero elsewhere. Here, ω_n is defined as in (II.4.11). Then, one applies the same reasoning on the translated channel to get a uniform local bound. The latter allows us to show that the maximal energy is uniformly bounded, and we can extract a convergent subsequence and obtain the desired result using a compactness argument.

The weak formulation (II.8.26) and the estimates of the Poincaré-Steklov operator for the eastern boundary layer are very similar to the ones in Section II.5.2. Then, it is not surprising to obtain the inequality

$$E_k^n \leq C_1 \left(k + 1 + m \sup_{k \leq j \leq k+m} (E_{j+1}^n - E_j^n) + \frac{1}{m^{4-2\eta}} \sup_{j \geq k+m} (E_{j+1}^n - E_j^n) \right) \quad \text{for all } k \in \{m, \dots, n\}, \quad (\text{II.8.28})$$

where C_1 is a constant depending on the characteristics of the domain and the jump functions when following the reasoning in Section II.5.3. It is clear that being (II.8.28) a key element in the reminder of the analysis, the results obtained in Section II.5.3 also apply to (II.8.23). The proof is left to the reader. Finally, consider $\Psi_e^- \in H_{\text{uloc}}^2(\omega_e^b)$ to be the unique solution of (II.8.1). Then, take $\psi_0 := \Psi_e^-|_{X=M} \in H_{\text{uloc}}^{3/2}(\mathbb{R})$ and $\psi_1 := \partial_X \Psi_e^-|_{X=M} \in H_{\text{uloc}}^{1/2}(\mathbb{R})$, then, there exists a unique solution $\Psi_e^+ \in H_{\text{uloc}}^2(\{X > M\} \times \mathbb{R})$. We have that

$$\mathcal{B}_k \left[\Psi_e^+|_{X=M}, \partial_X \Psi_e^+|_{X=M} \right] = \mathcal{B}_k \left[\Psi_e^-|_{X=M}, \partial_X \Psi_e^-|_{X=M} \right], \quad (\text{II.8.29})$$

and $\Psi_e^+|_{X=M} = \Psi_e^-|_{X=M}$, $\partial_X \Psi_e^+|_{X=M} = \partial_X \Psi_e^-|_{X=M}$. Thus, $\Psi^+ = \Psi_e^+ \mathbb{1}_{\{X > M\}} \times \mathbb{R} + \Psi_e^- \mathbb{1}_{\{M \leq X \leq -\gamma_e(Y)\}}$ is a H_{uloc}^2 solution of original problem defined in ω_e (II.8.1). \square

II.9 Convergence result

We are now ready to prove the convergence result stated in Theorem II.2.3. The general scheme of the proof is classical: we build an approximate solution of the fluid system and then show that the approximation is close to an exact solution through energy estimates.

II.9.1 Construction of the approximate solution

In this section, we justify the well-posedness of each one of the functions within the approximate solution of the 2d quasigeostrophic problem (II.1.1).

Let us recall that an approximate solution of problem (II.2.1) is defined as follows

Definition II.9.1. *A function $\Psi_{app} \in H^2(\Omega^\varepsilon)$ is an approximate solution to (II.2.1) if it satisfies the approximate equation*

$$\left\{ \begin{array}{l} \left(\partial_t + \nabla^\perp \Psi_{app} \cdot \nabla \right) (\Delta \Psi_{app} + \varepsilon^{-3} y) + \Delta \Psi_{app} - \Delta^2 \Psi_{app} = \varepsilon^{-3} \text{curl } \tau + r^\varepsilon, \quad \text{in } \Omega^\varepsilon \\ \Psi_{app}|_{\partial\Omega^\varepsilon} = \frac{\partial \Psi_{app}}{\partial n}|_{\partial\Omega^\varepsilon} = 0, \\ \Psi_{app}|_{t=0} = \Psi_{ini}. \end{array} \right. \quad (\text{II.9.1})$$

for some $r^\varepsilon \in L_t^\infty([0, T], H^{-2}(\Omega^\varepsilon))$ such that $r^\varepsilon = r_0^\varepsilon + r_1^\varepsilon + r_2^\varepsilon$, where $r_k^\varepsilon = o(1)$ in $L_t^\infty([0, T], H^{-k})$, $k = 0, 1, 2$.

- At main order in the interior of the domain, we get the Sverdrup relation

$$\partial_x \Psi_{int}^0 = \text{curl } \tau.$$

In Section II.3, it was discussed that Ψ_{int}^0 does not vanish on the whole boundary. Therefore, we introduce boundary layer corrections resulting from the balance between

$\varepsilon^{-3}\partial_x\Psi_{bl}^0$ and $Q_{bl}(\Psi_{bl}^0, \Psi_{bl}^0) - \Delta^2\Psi_{bl}^0$. Since it is possible to prescribe only one boundary condition, either on the eastern coast or on the western coast, the boundary condition for the Sverdrup equation is chosen such that $\Psi_{bl}|_{\Sigma_e} = 0$ in (II.3.10).

- The assumption on Ψ_{int}^0 determines that $\Psi_e^0 \equiv 0$. The system driving the behavior of Ψ_w^0 is given by (II.3.11). A smallness condition on curl τ guarantees existence and uniqueness of a solution Ψ_w^0 of (II.3.11) with exponential decay when $\varepsilon \rightarrow 0$ (Theorem II.2.1), see Section II.6.1.

- Let us now present the general form of the remaining profiles in the approximation.

The n -th interior profile $\Psi_{int}^n = \Psi_{int}^n(t, x, y)$ satisfies

$$\partial_x \Psi_{int}^n = F_n, \quad (\text{II.9.2})$$

where F_n depends on the Ψ_{int}^m , $m \leq n-1$. Note that $F_1 = F_2 = 0$ as a result of the perturbation being of order ε^{-3} and the interior part not having singularities. The terms $\Delta\Psi_{int}^{n-3}$, $\Delta\Psi_{int}^{n-3}$ and $(\nabla^\perp\Psi_{int}^i \cdot \nabla)\Delta\Psi_{int}^j$ form the source term F^n in (II.9.2), when $i+j+3=n$, as well as, . Note that Ψ_{int}^n does not meet the boundary conditions, and therefore boundary layer correctors must be defined. Following the direction of propagation of the equation in the main order, we choose $\Psi_{int}^n(t, x, y) = C_n(t, y) - \int_x^{\chi_e(y)} F_n(t, x', y) dx'$.

Function $C_n(t, y)$ is determined by the eastern profile Ψ_e^n satisfying the system

$$\left\{ \begin{array}{l} -\Delta_e^2 \Psi_e^n - \partial_{X_e} \Psi_e^n = G_n, \quad \text{in } \omega_e^- \cup \omega_w^+, \\ [\Psi_e^n] |_{\sigma_e} = -C_n, \\ \left[\partial_{X_e}^k \Psi_e^n \right] |_{\sigma_e} = (-1)^{k+1} \left[\partial_x \Psi_{int}^{n-k} \right] |_{x=\chi_e(y)}, \quad k = 0, \dots, 3, \\ \Psi_e^n |_{X_e=-\gamma_e(Y)} = 0, \quad \frac{\partial \Psi_e^n}{\partial n_e} |_{X_e=-\gamma_e(Y)} = 0, \end{array} \right. \quad (\text{II.9.3})$$

where G_n is conditioned by the behavior of Ψ_e^m , for all $m \leq n-1$. Singularities at low frequencies for (II.9.3) are a consequence of the nature of the main equation as stated in Remark II.3.1 and later discussed in Section II.8.1.1. This impacts the asymptotic behavior of the solution far from the boundary, i.e., in the West at macroscopic level. $C_n(t, y)$ is chosen such that the limit stemming from the Ergodic Theorem converges to zero when $X_e \rightarrow +\infty$. Theorem II.2.2 guarantees the well-posedness of system II.9.3 in ω_e since G_n is usually small, it can be considered as a perturbation parameter of II.8.1.

Similar equations are obtained for the western profiles but with additional interaction terms. Namely,

$$\left\{ \begin{array}{l} \partial_{X_w} \Psi_w^n + Q_w(\Psi_w^n, \Psi_w^0) + Q_w(\Psi_w^0, \Psi_w^n) - \Delta_w^2 \Psi_w^n = H_n, \quad \text{in } \omega_w^- \cup \omega_w^+, \\ \left[\partial_{X_w}^k \Psi_w^n \right] |_{\sigma_w} = - \left[\partial_x \Psi_{int}^{n-k} \right] |_{x=\chi_w(y)} - \left[\partial_{X_e}^k \Psi_e^n \right] |_{\sigma_w}, \quad k = 0, \dots, 3, \\ \Psi_w^n |_{X_w=-\gamma_w(Y)} = 0, \quad \frac{\partial \Psi_w^n}{\partial n_w} |_{X_w=-\gamma_w(Y)} = 0, \end{array} \right. \quad (\text{II.9.4})$$

where H_n depends on Ψ_w^m , Ψ_e^m and Ψ_{int}^m , for all $m \leq n - 1$. Problem (II.9.4) has two features that clearly distinguish it from (II.9.3): two linearized terms containing Ψ_w^0 and the influence of the eastern boundary layer function on the jump at the interface separating the interior domain from the western rough domain. As a consequence, the well-posedness of the solution of (II.9.4) depends not only on the behavior of interior profiles, but on the first profile of the western boundary layer as well as the solutions of the eastern boundary layer system. The complete analysis of (II.9.4) is presented on Section II.7.

To summarize, for the eastern profile Ψ_e^n to be well-posed, it fixes the value of the constant in Ψ_{int}^n . Its role is to correct at Σ_e , $\partial_X^k \Psi_{int}^{n-k}$, for $k = 1, 2, 3$. On the other hand, Ψ_w^n plays same role for the interior profiles and the jump of Ψ_e^n and its derivatives at Σ_w .

II.9.2 Computing the reminder

Once we have constructed the approximate solution, a natural question arises: how “far” is Ψ_{app}^ε from Ψ^ε , i.e., what is the error (in a suitable norm) when replacing Ψ^ε by Ψ_{app}^ε ? How far should we take the expansion?; i.e., what is the minimum value of n to satisfy a suitable energy estimate?

II.9.2.1 Building the correctors

II.9.2.1.1 The interior term. The ε^{-3} factor in the main equation of (II.2.1) dictates the asymptotic development must be taken at least till order $n = 3$ to deal with the remaining stemming from the substitution of Ψ_{int}^0 on the original equation. We are referring in particular to

$$\partial_t \Delta \Psi_{int}^0, \quad (\nabla^\perp \Psi_{int}^0 \cdot \nabla) \Delta \Psi_{int}^0, \quad \text{and} \quad \Delta^2 \Psi_{int}^0.$$

All subsequent error terms containing only interior profiles are $O(\varepsilon)$.

II.9.2.1.2 Western boundary layer profiles We start by stressing that far from the western boundary, all terms containing at least one Ψ_w^i decay exponentially as a result of Theorem II.2.1. Let us, therefore, focus our analysis on the western region. The error terms resulting from evaluating in $\partial_t \Delta \Psi_{app} + \Delta \Psi_{app}$ can be considered as part of r^ε . To illustrate this, take $\Delta \Psi_w^0$. We have

$$\|\Delta \Psi_w^0\|_{H^{-2}(\Omega^\varepsilon)} \leq C \|\Psi_w^0\|_{L^2(\Omega^\varepsilon)} \leq C \sqrt{\varepsilon}. \quad (\text{II.9.5})$$

Let us now analyze the corresponding elements in the advection term. Namely,

$$(\nabla^\perp \Psi_w^i \cdot \nabla) \Delta \Psi_{int}^j \quad (\text{II.9.6a})$$

$$(\nabla^\perp \Psi_{int}^i \cdot \nabla) \Delta \Psi_w^j \quad (\text{II.9.6b})$$

$$(\nabla^\perp \Psi_w^i \cdot \nabla) \Delta \Psi_w^j \quad (\text{II.9.6c})$$

$$(\nabla^\perp \Psi_w^i \cdot \nabla) \Delta \Psi_e^j \quad (\text{II.9.6d})$$

$$(\nabla^\perp \Psi_e^i \cdot \nabla) \Delta \Psi_w^j. \quad (\text{II.9.6e})$$

Note that there is a part of (II.9.6a)–(II.9.6c) that is used to compute Ψ_w^n as a component of G^n , $i + j \leq n$. We need to elucidate if the remaining components can be included in r^ε or if an additional corrector is needed. For (II.9.6a), we have far from the western boundary that

$$\begin{aligned} \left\| \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \Delta \Psi_{int}^0 \right\|_{H^{-1}(\Omega^\varepsilon)} &\leq C \left(\left\| \nabla^\perp \left(\Psi_w^0 \cdot \nabla \Delta \Psi_{int}^0 \right) \right\|_{H^{-1}(\Omega^\varepsilon)} + \left\| \Psi_w^0 D^2 \Psi_{int}^0 \right\|_{L^2(\Omega^\varepsilon)} \right) \\ &\leq C\sqrt{\varepsilon}. \end{aligned}$$

Hence, there is no need for an additional corrector for (II.9.6a). The same cannot be said for (II.9.6b) since

$$\left\| \left(\nabla^\perp \Psi_{int}^0 \cdot \nabla \right) \Delta \Psi_w^0 \right\|_{H^{-1}(\Omega^\varepsilon)} \leq C\varepsilon^{-1/2}. \quad (\text{II.9.7})$$

The “problematic” part in the previous term can be considered as part of the source term in the problem driving the behavior of Ψ_w^1 . Although the action of the ergodic part is corrected by the choice of the corresponding interior profile far from the boundary, the advection terms including the parts of Ψ_e^n must be taken into account in the West. For example, let us consider the term of the type (II.9.6d)

$$\begin{aligned} \left\| \varepsilon \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \Delta \Psi_e^1 \right\|_{H^{-1}(\Omega^\varepsilon)} &\leq C\varepsilon \left\| \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \nabla \Psi_e^1 \right\|_{L^2(\Omega^\varepsilon)} \\ &\leq C\varepsilon \left(\left\| \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \nabla \Psi_e^1 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) \leq \sqrt{\varepsilon}\})} \right. \\ &\quad \left. + \left\| \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \nabla \Psi_e^1 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) > \sqrt{\varepsilon}\})} \right). \end{aligned}$$

As a consequence of the decreasing behavior of Ψ_w^0 far from the boundary, the second term in the last inequality is exponentially small. For the first element, we have

$$\begin{aligned} \left\| \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \nabla \Psi_{alg}^1 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) > \sqrt{\varepsilon}\})} &\leq \left\| \nabla^\perp \Psi_w^0 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) > \sqrt{\varepsilon}\})} \left\| D^2 \Psi_{alg}^1 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) > \sqrt{\varepsilon}\})} \\ &\leq C\varepsilon^{-7/4}. \end{aligned} \quad (\text{II.9.8})$$

Moreover, from Lemma II.8.6, we know that $\Psi_{erg}^1 = o(1)$ and it easy to verify that $X_e^{k/4} \nabla_e^k \Psi_{erg}^1 = o(1)$. Proceeding similarly as before yields

$$\left\| \varepsilon \left(\nabla^\perp \Psi_w^0 \cdot \nabla \right) \Delta \Psi_{erg}^1 \right\|_{H^{-2}(\Omega^\varepsilon)} = o(\varepsilon).$$

Consequently, the first term needs to be corrected; this is possible when including it as part of the source term in the system dictating Ψ_w^2 .

For the term (II.9.6d) when $i = 1$ and $j = 0$, we decompose the domain as before when analyzing the interaction terms

$$\left\| \varepsilon \left(\nabla^\perp \Psi_{alg}^1 \cdot \nabla \right) \nabla \Psi_w^0 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) \leq \sqrt{\varepsilon}\})} = O(1), \quad (\text{II.9.9})$$

and

$$\left\| \varepsilon \left(\nabla^\perp \Psi_{erg}^1 \cdot \nabla \right) \nabla \Psi_w^0 \right\|_{L^2(\Omega^\varepsilon \cap \{x - \chi_w(y) \leq \sqrt{\varepsilon}\})} = o(\varepsilon^{-1/4}). \quad (\text{II.9.10})$$

Therefore, to deal with these advection terms, we add the functions $\varepsilon^{5/4}\Psi_{ew}^{\text{erg}}$ and $\varepsilon^{3/4}\Psi_{ew}^{\text{alg}}$ to the approximate solution: one for the term decaying algebraically, and the other, for Ψ_{erg}^1 . These functions fulfill the equations

$$\begin{cases} \partial_{X_w} \Psi_{ew} + Q_w(\Psi_w^0, \Psi_{ew}) - \Delta_w^2 \Psi_{ew} &= G^{\varepsilon,t,y}(X_w, Y), & \text{in } \omega_w, \\ \Psi_{ew}|_{X=-\gamma_w(Y)} &= \frac{\partial \Psi_{ew}}{\partial n_w}|_{X=-\gamma_w(Y)} = 0, \end{cases} \quad (\text{II.9.11})$$

where,

$$G^{\varepsilon,t,y}(X_w, Y) = \begin{cases} \left(\nabla_e^\perp \Psi_{\text{erg}}^1 \left(\frac{\chi_e(y) - \chi_w(y)}{\varepsilon} - X_w, Y \right) \cdot \nabla_w \right) \nabla_w \Psi_w^0(X_w, Y), & \text{for } \Psi_{ew} = \Psi_{ew}^{\text{erg}}, \\ \left(\nabla_e^\perp \Psi_{\text{alg}}^1 \left(\frac{\chi_e(y) - \chi_w(y)}{\varepsilon} - X_w, Y \right) \cdot \nabla_w \right) \nabla_w \Psi_w^0(X_w, Y), & \text{for } \Psi_{ew} = \Psi_{ew}^{\text{alg}}. \end{cases}$$

Note that Ψ_{ew} is C^3 at the interface σ_w . From Section II.7, (II.9.11) has a unique solution decaying exponentially when ε goes to zero. The remaining terms are small enough and can be considered a part of r^ε . Indeed, repeating the same reasoning as before, we have

$$\begin{aligned} \|\varepsilon \left(\nabla^\perp \Psi_{\text{int}}^1 \cdot \nabla \right) \Delta \Psi_w^0\|_{H^{-1}(\Omega^\varepsilon)} &= O(\varepsilon^{1/2}), \\ \|\varepsilon^2 \left(\nabla^\perp \Psi_e^1 \cdot \nabla \right) \Delta \Psi_w^1\|_{H^{-2}(\Omega^\varepsilon)} &\leq \varepsilon^2 \|\nabla^\perp \Psi_e^1 \cdot \nabla \Psi_w^1\|_{L^2(\Omega^\varepsilon)} \\ &= \leq \|\nabla_{X_e, Y}^\perp \Psi_e^1 \cdot \nabla_{X_w, Y} \Psi_w^1\|_{L^2(\Omega^\varepsilon)} = o(\varepsilon^{1/2}), \\ \|\varepsilon^2 \left(\nabla^\perp \Psi_e^1 \cdot \nabla \right) \Delta \Psi_w^1\|_{H^{-2}(\Omega^\varepsilon)} &\leq \left\| \left(\nabla_{X_e, Y}^\perp \Psi_e^1 \cdot \nabla_{X_w, Y} \right) \Psi_w^1 \right\|_{L^2(\Omega^\varepsilon)} = o(\varepsilon^{1/2}). \end{aligned}$$

II.9.2.1.3 The eastern boundary layer profiles The same analysis must be applied to Ψ_e^n , $i \geq 1$. The linear terms involving Ψ_e^1 are $o(\varepsilon)$. Indeed, as a consequence of Proposition II.8.1 we have

$$\|\varepsilon \partial_t \Delta \Psi_e^1\|_{H^{-2}(\Omega^\varepsilon)} \leq \varepsilon \|\partial_t \Psi_e^1\|_{L^2(\Omega^\varepsilon)} = o(\varepsilon). \quad (\text{II.9.12})$$

The estimate for the Laplacian can be computed in the same manner. From the previous paragraph, we know that advection terms containing Ψ_w^0 are exponentially small on the eastern domain. It remains to check the terms

$$\left(\nabla^\perp \Psi_{\text{int}}^i \cdot \nabla \right) \Delta \Psi_e^j, \quad (\text{II.9.13a})$$

$$\left(\nabla^\perp \Psi_e^i \cdot \nabla \right) \Delta \Psi_w^j, \quad (\text{II.9.13b})$$

$$\left(\nabla^\perp \Psi_e^i \cdot \nabla \right) \Delta \Psi_e^j. \quad (\text{II.9.13c})$$

(II.9.13a) satisfies

$$\begin{aligned} \left\| \varepsilon \left(\nabla^\perp \Psi_e^1 \cdot \nabla \right) \Delta \Psi_{\text{int}}^0 \right\|_{H^{-2}(\Omega^\varepsilon)} &\leq \varepsilon \left\| \nabla^\perp \Psi_e^1 \Delta \Psi_{\text{int}}^0 \right\|_{H^{-1}(\Omega^\varepsilon)} \\ &\leq \left\| \Psi_e^1 \Delta \Psi_{\text{int}}^0 \right\|_{L^2(\Omega^\varepsilon)} + \left\| \Psi_e^1 \Delta \Psi_{\text{int}}^0 \right\|_{L^2(\Omega^\varepsilon)} \\ &\leq C \left\| \Psi_e^1 \right\|_{L^2(\Omega^\varepsilon)} = o(1). \end{aligned}$$

Furthermore, applying the same idea to (II.9.13b) yields $\left\| \varepsilon \left(\nabla^\perp \Psi_{int}^0 \cdot \nabla \right) \Delta \Psi_e^1 \right\|_{H^{-2}(\Omega^\varepsilon)} = o(1)$. Lastly, (II.9.13c) follows

$$\begin{aligned} \left\| \varepsilon^2 \left(\nabla^\perp \Psi_e^1 \cdot \nabla \right) \Delta \Psi_e^1 \right\|_{H^{-2}(\Omega^\varepsilon)} &\leq C\varepsilon^2 \left\| \nabla^\perp \Psi_e^1 \otimes \nabla \Psi_e^1 \right\|_{L^2(\Omega^\varepsilon)} \\ &\leq C\varepsilon^2 \|\nabla \Psi_e^1\|_{L^4}^2 \leq C\sqrt{\varepsilon}. \end{aligned}$$

We conclude there is no need for correctors in the eastern domain.

II.9.2.1.4 Traces at Σ_e and Σ_w The traces of Ψ_{exp}^n at Σ_w and of Ψ_w^n at Σ_e are exponentially small. Indeed, they satisfy

$$\left\| \partial_X^k \Psi_{exp}^n \right\|_{L_t^\infty(H^{3/2-k}(\Sigma_e \cup \Sigma_w))} + \left\| \partial_X^k \Psi_w^n \right\|_{L_t^\infty(H^{3/2-k}(\Sigma_e \cup \Sigma_w))} = O\left(\exp\left(-\frac{\delta}{\varepsilon}\right)\right), \quad \lambda > 0.$$

Moreover, the traces of Ψ_{erg}^n and Ψ_{alg}^n are included in the jump conditions in (II.9.4) and therefore, they are lifted by the western profiles.

Note that thanks to hypotheses we made near y_{min} and y_{max} , the traces are zero for $y \in [y_{max} - \lambda, y_{max}] \cup [y_{min}, y_{min} + \lambda]$. We add a corrector to ψ^ε to lift the jump conditions of the form

$$\phi_1^\varepsilon(t, x, y) = \sum_k \frac{x^k}{k!} (\tilde{\sigma}_k^e \theta(x - \chi_w(y)) + \tilde{\sigma}_k^w \theta(x - \chi_w(y))), \quad (\text{II.9.14})$$

where $\tilde{\sigma}_k^w$ and $\tilde{\sigma}_k^e$ denote the values of the traces of the western and eastern profiles, respectively. Moreover, θ is a function belonging to $C_c^\infty([-\delta, \delta])$ for $\delta > 0$ small enough.

For the traces of $\partial_x^k \Psi_{int}^n$, $k = 0, \dots, 3$ at $\Sigma_w \cup \Sigma_e$ to be well-defined, we need correctors Ψ_w^i , Ψ_e^i , $n \leq i \leq n + 3$ satisfying (II.9.4) and (II.9.3), respectively.

II.9.3 Energy estimates

Let us consider the difference $\psi^\varepsilon = \Psi^\varepsilon - \Psi_{app}^\varepsilon$, with Ψ^ε satisfying (II.2.1) and Ψ_{app}^ε defined as in Section II.9.1. Moreover, let r^ε be the error resulting from the difference between the original solution. We have

$$\begin{aligned} \partial_t \Delta \psi^\varepsilon + \nabla^\perp \Psi^\varepsilon \cdot \nabla (\Delta \psi^\varepsilon) + \nabla^\perp \psi^\varepsilon \cdot \nabla (\Delta \Psi_{app}^\varepsilon) + \varepsilon^{-3} \partial_x \psi^\varepsilon + \Delta \psi^\varepsilon - \Delta^2 \psi^\varepsilon &= r^\varepsilon \text{ in } \Omega^\varepsilon \\ \psi^\varepsilon|_{\partial\Omega} &= \frac{\partial \psi^\varepsilon}{\partial n} \Big|_{\partial\Omega} = 0 \\ \psi^\varepsilon|_{t=0} &= (\Psi_{ini} - \Psi_{app}^\varepsilon)|_{t=0}, \end{aligned} \quad (\text{II.9.15})$$

where $\|(\Psi_{ini} - \Psi_{app}^\varepsilon)|_{t=0}\|_{L_t^\infty(H^1(\Omega^\varepsilon))} = O(\varepsilon)$, which results from hypotheses made on Ψ_{ini} . Moreover, $\|r^\varepsilon\|_{L_t^\infty(H^{-2}(\Omega^\varepsilon))} = o(1)$. The details on the computation of the remainder have been discussed in detail in Section II.9.1. It is clear that ψ^ε belongs to $H^2(\Omega^\varepsilon)$. Multiplying

the main equation on (II.9.15) and integrating over Ω^ε provides the following

$$\begin{aligned}
\int_{\Omega^\varepsilon} \partial_t \Delta \psi^\varepsilon \cdot \psi^\varepsilon &= - \int_{\Omega^\varepsilon} \partial_t \nabla \psi^\varepsilon \cdot \nabla \psi^\varepsilon = -\frac{1}{2} \frac{d}{dt} \int_{\Omega^\varepsilon} |\nabla \psi^\varepsilon|^2 \\
\int_{\Omega^\varepsilon} \nabla^\perp \Psi^\varepsilon \cdot \nabla (\Delta \psi^\varepsilon) \psi^\varepsilon &= - \int_{\Omega^\varepsilon} \left((\nabla^\perp \Psi^\varepsilon \cdot \nabla) \nabla^\perp \psi^\varepsilon \right) \cdot \nabla^\perp \psi^\varepsilon = -\frac{1}{2} \int_{\Omega^\varepsilon} \nabla^\perp \Psi^\varepsilon \cdot \nabla |\nabla^\perp \psi^\varepsilon|^2 = 0 \\
\int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla) \Delta \Psi_{app}^\varepsilon \psi^\varepsilon &= - \int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla) \nabla^\perp \Psi_{app}^\varepsilon \cdot \nabla^\perp \psi^\varepsilon = \int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla \Psi_{app}^\varepsilon) D^2 \psi^\varepsilon \\
\int_{\Omega^\varepsilon} \partial_x \psi^\varepsilon \psi^\varepsilon &= 0 \\
\int_{\Omega^\varepsilon} \Delta \psi^\varepsilon \psi^\varepsilon &= \int_{\partial \Omega^\varepsilon} \partial_n \psi^\varepsilon \psi^\varepsilon - \int_{\Omega^\varepsilon} |\nabla \psi^\varepsilon|^2 = - \int_{\Omega^\varepsilon} |\nabla \psi^\varepsilon|^2 \\
- \int_{\Omega^\varepsilon} \Delta^2 \psi^\varepsilon \psi^\varepsilon &= - \int_{\Omega^\varepsilon} |\Delta \psi^\varepsilon|^2.
\end{aligned}$$

We claim that

$$\frac{1}{2} \partial_t \|\nabla \psi^\varepsilon(t, \cdot)\|_{L^2}^2 + \|\nabla \psi^\varepsilon(t, \cdot)\|_{L^2}^2 + \|D^2 \psi^\varepsilon(t, \cdot)\|_{L^2}^2 \leq \left| \int_{\Omega^\varepsilon} r^\varepsilon \psi^\varepsilon(t, \cdot) \right| + \left| \int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla \Psi_{app}^\varepsilon) D^2 \psi^\varepsilon \right|. \quad (\text{II.9.16})$$

We proceed to analyze each one of the terms on the r.h.s. For the first term we have

$$\begin{aligned}
\left| \int_{\Omega^\varepsilon} r^\varepsilon \psi^\varepsilon(t, \cdot) \right| &\leq \|D^2 \psi^\varepsilon(t, \cdot)\|_{L^2(\Omega^\varepsilon)} \|r^\varepsilon\|_{H^{-2}} \\
&\leq \frac{1}{2} \left(\|D^2 \psi^\varepsilon(t, \cdot)\|_{L^2(\Omega^\varepsilon)}^2 + \|r^\varepsilon\|_{H^{-2}}^2 \right)
\end{aligned} \quad (\text{II.9.17})$$

We are left with bounding the term $\int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla \Psi_{app}^\varepsilon) D^2 \psi^\varepsilon$. The difficulty here comes from Ψ_{app}^ε at the boundary layer since

$$\nabla \Psi_{app}^\varepsilon = \nabla \Psi_{int} + \nabla \Psi_e^\varepsilon + \nabla \Psi_w^\varepsilon.$$

In particular, from $\nabla^\varepsilon \Psi_w = O(\varepsilon^{-1})$ in L^∞ since $|\nabla \Psi_e^\varepsilon| = \varepsilon \nabla \Psi_e^1 + O(\varepsilon)$, where $\nabla \Psi_e^1$, similarly to $\varepsilon \nabla \Psi_{int}$, is small and bounded in L^∞ . Hence, we focus our attention on

$$\left| \int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla \Psi_w^0) D^2 \psi^\varepsilon \right|.$$

From Hardy's inequality, we have the following

$$\left\| \frac{\nabla^\perp \psi^\varepsilon(t, \cdot)}{d(\mathbf{x}, \Gamma_w^\varepsilon)} \right\|_{L^2} \leq C \|D^2 \psi^\varepsilon(t, \cdot)\|_{L^2}.$$

Here, $d(\mathbf{x}, \Gamma_w^\varepsilon)$ denotes the distance from $\mathbf{x} = (x, y) \in \mathbb{R}^2$ to the western rough boundary. Thus,

$$\left| \int_{\Omega^\varepsilon} (\nabla^\perp \psi^\varepsilon \cdot \nabla \Psi_w^\varepsilon) D^2 \psi^\varepsilon \right| \leq C \|d(\mathbf{x}, \Gamma_w^\varepsilon) \nabla \Psi_w^0\|_{L^\infty} \|D^2 \psi^\varepsilon(t, \cdot)\|_{L^2}^2. \quad (\text{II.9.18})$$

Note that $d(\mathbf{x}, \Gamma_w) \nabla \Psi_w^0 \sim \frac{z}{\varepsilon} \exp(-\frac{z}{\varepsilon})$, where z denotes the distance to the boundary, and, therefore, it satisfies

$$\|d(\mathbf{x}, \Gamma_w^\varepsilon) \nabla^\perp \Psi_w^0\|_{L^\infty} \leq C_0,$$

where the small constant C_0 does not depend on ε . Hence, (II.9.18) can be absorbed by the diffusion term on the l.h.s. Plugging (II.9.17) and (II.9.18) in (II.9.16) and then applying the Grönwall's inequality complete the proof of Theorem II.2.3.

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Appendix

II.A Study of the roots of equation of the linear western boundary layer

This appendix is concerned with proving Lemma II.5.1, i.e., that equation (II.5.9)

$$P(\lambda) = -\lambda - (\lambda^2 + (\alpha_w \lambda + i\varepsilon)^2) = 0,$$

does not have purely imaginary nor multiple roots when $\xi \neq 0$.

The first part can be easily checked by considering $\lambda = iR$, $R \in \mathbb{R}$ on (II.5.9). This yields the following relation

$$iR = ((1 + \alpha_w)^2 R^2 + 2\alpha_w \xi R + \xi^2)^2.$$

Taking the imaginary part of the previous equality results in $R = 0$.

The second part of the analysis, although strenuous because (II.5.9) is a fourth-degree polynomial with complex coefficients, leads us to discard the cases of purely imaginary roots and multiple eigenvalues. We proceed as follows:

- First, we prove that for α_w and ξ equal to zero, all the roots of the characteristic equation are simple.
- Then, for $\xi \neq 0$, we show that there are no purely imaginary simple nor double roots.
- The remaining cases are analyzed through equations resulting from the relation between repeated roots of a polynomial and its derivatives.

The main ingredient in the analysis of the multiplicity of the solutions will be the classical lemma:

Lemma II.A.1. *Let \mathbb{K} be a commutative ring and $P(x) \in \mathbb{K}[x]$, a nonconstant polynomial of the form $P(x) = \sum_k a_k x^k$. Let $a \in \mathbb{K}$ be a multiple root of P . Denote by P' , the derivative of the polynomial P . Then, $P'(a) = 0$.*

In this case, $P'(x)$ is a third-degree polynomial, and its algebraic solution can be explicitly derived by using, for example, Cardano's method. We will prove that the form of the original equation excludes the possibility of $P'(x)$ having real roots. Moreover, a system of equations describing the behavior of the real and imaginary parts of the complex roots will be solved to find the relation α_w and ξ satisfy in the presence of a double root.

- We start by proving the result for the cases when $\alpha_w = 0$ or $\xi = 0$.

– Let us first consider $\xi = 0$. Then, equation (II.5.9) becomes

$$\lambda + (1 + \alpha_w^2)^2 \lambda^4 = 0. \quad (\text{II.A.1})$$

The roots of the above equation are

$$\lambda_k = -\frac{1}{(1 + \alpha_w^2)^{2/3}} e^{i\frac{2\pi k}{3}}, \text{ for } k = 0, 1, 2, \quad \lambda_3 = 0.$$

It is evident that these roots are simple for all $\alpha_w \in \mathbb{R}$, and that $\Re(\lambda_0) < 0$, $\Re(\lambda_1) = \Re(\lambda_2) < 0$.

– When $\alpha_w = 0$, a double root must satisfy the following system of equations

$$\begin{cases} \lambda + (\lambda^2 - \xi^2)^2 = 0 \\ 1 + 4\lambda(\lambda^2 - \xi^2) = 0. \end{cases} \quad (\text{II.A.2})$$

Combining both equations in (II.A.2) gives the cubic polynomial

$$\lambda^3 + \frac{1}{16} = 0, \quad (\text{II.A.3})$$

The values satisfying the above equation are of the form

$$\lambda_k = -\frac{1}{2\sqrt[3]{2}} e^{\frac{2\pi}{3}ik}, \text{ for } k = 0, 1, 2.$$

Since ξ is a real-valued quantity, we have that $\Im(\lambda) = 0$ as a result of combining (II.A.2.b) and (II.A.3). The latter does not hold for λ_1 and λ_2 . Moreover, a quick substitution of λ_0 in (II.A.2.b) results in a contradiction since $\lambda_0 \xi^2 \leq 0$ for $\xi \in \mathbb{R}$. Hence, equation (II.5.9) does not have repeated roots when $\alpha_w = 0$.

- We now analyze the case when $\alpha_w, \xi \in \mathbb{R}^*$. Let us start by rewriting equation (II.5.9) as follows

$$\frac{\lambda}{(1 + \alpha_w^2)^2} + \left(\left(\lambda + \frac{\alpha_w i \xi}{1 + \alpha_w^2} \right)^2 - \frac{\xi^2}{(1 + \alpha_w^2)^2} \right)^2 = 0. \quad (\text{II.A.4})$$

We then introduce the variable $\mu = \lambda + \frac{\alpha_w i \xi}{1 + \alpha_w^2}$. Due to the affine relation between μ and λ , we can assert that a repeated root of the problem will satisfy the following system

$$\begin{cases} \frac{\mu}{(1 + \alpha_w^2)^2} - \frac{\alpha_w i \xi}{(1 + \alpha_w^2)^3} + \left(\mu^2 - \frac{\xi^2}{(1 + \alpha_w^2)^2} \right)^2 = 0, \\ \frac{1}{4(1 + \alpha_w^2)^2} + \mu \left(\mu^2 - \frac{\xi^2}{(1 + \alpha_w^2)^2} \right) = 0. \end{cases} \quad (\text{II.A.5})$$

Lemma II.A.2. *Let $\alpha_w, \xi \in \mathbb{R}^*$ and $\mu \in \mathbb{C} \setminus \mathbb{R}$ be a solution of (II.A.5). Then, $|\xi| < \frac{\sqrt{3}}{2}(1 + \alpha_w^2)^{1/3}$. Furthermore, setting $a = -\frac{\xi^2}{(1 + \alpha_w^2)^2}$, $b = \frac{1}{4(1 + \alpha_w^2)^2}$ and*

$$\begin{aligned} A &= \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \\ B &= \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \end{aligned}$$

the solution μ belongs to $\{\mu_-, \mu_+\}$, for

$$\begin{aligned}\mu_+ &= -\frac{1}{2}(A+B) + \frac{\sqrt{3}i}{2}(A-B) \\ \mu_- &= -\frac{1}{2}(A+B) - \frac{\sqrt{3}i}{2}(A-B).\end{aligned}$$

Proof. Note that (II.A.5) is written in canonical form. The sign of the term $\frac{b^2}{4} + \frac{a^3}{27}$ determines the number of real roots of the cubic polynomial, going from three in the cases when it is non-positive to one, when $\frac{b^2}{4} + \frac{a^3}{27}$ is greater than zero. Let us prove that if a repeated root μ exists, it must forcibly have a non zero imaginary part.

Assuming $\mu \in \mathbb{R}$ and then taking the imaginary part of (II.A.5a) lead to

$$-\frac{\alpha_w \xi}{(1 + \alpha_w^2)^3} = 0, \quad (\text{II.A.6})$$

which contradicts the assumption that α_w and ξ are not equal zero. We have thus proved that μ must be a complex quantity with nonzero imaginary part for the characteristic equation to have a repeated root. Therefore, $\frac{b^2}{4} + \frac{a^3}{27}$ has to be positive which in turn, implies that A and B must be real quantities and the Fourier variable should satisfy $|\xi| < \frac{\sqrt{3}}{2}(1 + \alpha_w^2)^{1/3}$. \square

Lemma II.A.3. *Let $\mu \in \mathbb{C} \setminus \mathbb{R}$ be a solution of (II.A.5). Then, μ is also a root of the quadratic polynomial*

$$\frac{3}{4}\mu^2 - \frac{\alpha_w i \xi}{1 + \alpha_w^2} \mu + \frac{\xi^2}{4(1 + \alpha_w^2)^2} = 0. \quad (\text{II.A.7})$$

This equation is easily obtained by first, multiplying (II.A.5a) by μ and (II.A.5b) by $\left(\mu^2 - \frac{\xi^2}{(1 + \alpha_w^2)^2}\right)$ and then, subtracting the resulting equations .

Substituting $\mu_{\pm} = -\frac{1}{2}(A+B) \pm \frac{\sqrt{3}i}{2}(A-B)$ in (II.A.7) yields the following system for the real and imaginary parts of the solution

$$\begin{cases} \frac{3}{16}(A+B)^2 - \frac{9}{16}(A-B)^2 \pm \frac{\sqrt{3}}{2} \frac{\alpha_w \xi}{1 + \alpha_w^2} (A-B) + \frac{\xi^2}{4(1 + \alpha_w^2)^2} = 0 \\ \mp \frac{3\sqrt{3}}{8}(A+B)(A-B) + \frac{\alpha_w \xi}{2(1 + \alpha_w^2)}(A+B) = 0. \end{cases} \quad (\text{II.A.8})$$

The root λ cannot be purely imaginary, hence, $A+B \neq 0$ and the condition $A-B \neq 0$ is derived from the fact that μ must be complex. Dividing equation (II.A.8b) by $A+B$ gives

$$A-B = \pm \frac{4}{3\sqrt{3}} \frac{\alpha_w}{1 + \alpha_w^2} \xi. \quad (\text{II.A.9})$$

We check at once that $A + B$ is also proportional to ξ . From the relation $(A + B)^2 = (A - B)^2 + 4AB$ and the fact that $AB = -\frac{a}{3}$, we obtain

$$(A + B)^2 = \frac{4}{3(1 + \alpha_w^2)^2} \left(\frac{4\alpha_w^2}{9} + 1 \right) \xi^2. \quad (\text{II.A.10})$$

The term $A + B$ must additionally satisfy equation (II.A.8a). Taking into account that $\xi \neq 0$, the combination of (II.A.8a) and (II.A.10) provides the following condition for α_w

$$\frac{8\alpha_w^2 + 9}{18(\alpha_w^2 + 1)^2} = 0. \quad (\text{II.A.11})$$

The roots of the above equality are complex; hence, there is no real-valued α_w for which (II.5.9) has multiple roots. The same conclusion can be drawn from solving system (II.A.8) directly since complex or null $A + B$ terms contradict our previous assumptions.

Finally, we conclude (II.5.9) has four simple roots $(\lambda_i^\pm)_{i=1,2} \forall \xi, \alpha_w \in \mathbb{R}$ in $\mathbb{C} \setminus \mathbb{I}$, satisfying $\Re(\lambda_i^+) > 0$ and $\Re(\lambda_i^-) > 0$ if $\xi \neq 0$.

II.B Expansion of the eigenvalues at high frequencies

This section is devoted to high-frequency expansions of the main functions we work with, namely, λ_k and A_k .

In high frequencies, that is, for $|\xi| \gg 1$, by considering $\lambda = \xi\rho$, where $\rho \in \mathbb{C}$, we obtain

$$\xi^{-3}\rho + \left((1 + \alpha_w^2)\rho^2 + 2i\alpha_w\rho - 1 \right)^2 = 0. \quad (\text{II.B.1})$$

The above polynomial provides the following approximation of the solutions

$$\rho^+ = \frac{1 - i\alpha_w}{\alpha_w^2 + 1} + O(|\xi|^{-3/2}), \quad \rho^- = -\frac{1 + i\alpha_w}{\alpha_w^2 + 1} + O(|\xi|^{-3/2}). \quad (\text{II.B.2})$$

The definition of λ exhibits a clear relation between the sign of its real part and the sign of $\rho\xi$. In particular, λ has positive real part if and only if $\text{sgn}(\Re(\rho)) = \text{sgn}(\xi)$.

Since we have already proved that all roots of (II.5.9) are simple, we will provide a second term on the expansion of the solutions to make this assertion evident to the reader. Let $\bar{\rho}$ be a root of the polynomial $((1 + \alpha_w^2)\rho^2 + 2i\alpha_w\rho - 1)^2$ and $\rho = \bar{\rho} + \xi^{-\eta}\tilde{\rho} + O(\xi^{-3})$, where $\eta < 3$, we have

$$\xi^{-3}\bar{\rho} + \left(2(1 + \alpha_w^2)\bar{\rho}\tilde{\rho}\xi^{-\eta} + 2i\alpha_w\tilde{\rho}\xi^{-\eta} + O(\xi^{-2\eta}) \right)^2 + O(\xi^{-3-\eta}) = 0. \quad (\text{II.B.3})$$

This is the same as

$$\xi^{-3}\bar{\rho} + 4\tilde{\rho}^2\xi^{-2\eta} \left((1 + \alpha_w^2)\bar{\rho} + i\alpha_w + O(\xi^{-\eta}) \right)^2 + O(\xi^{-3-\eta}) = 0,$$

from which we conclude that $\eta = 3/2$ and

$$\tilde{\rho}^\pm = \pm \frac{i\sqrt{\rho}}{2\sqrt{(\alpha_w^2 + 1)\rho + i\alpha_w}}. \quad (\text{II.B.4})$$

Consequently, for $j = 1, 2$,

$$\begin{aligned} \rho_j^+ &= \frac{1 - i\alpha_w}{\alpha_w^2 + 1} + (-1)^j \frac{i\xi^{-\frac{3}{2}}}{2} \left(\frac{1 - i\alpha_w}{\alpha_w^2 + 1} \right) + O(\xi^{-3}), \\ \rho_j^- &= -\frac{1 + i\alpha_w}{\alpha_w^2 + 1} + (-1)^j \frac{i\xi^{-\frac{3}{2}}}{2} \sqrt{\frac{1 + i\alpha_w}{\alpha_w^2 + 1}} + O(\xi^{-3}). \end{aligned}$$

We now turn towards the expressions of the A_k 's which satisfy the linear system

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \hat{\psi}_0^* \\ \hat{\psi}_1^* \end{pmatrix}.$$

Thus,

$$A_1 = \frac{\lambda_2 \hat{\psi}_0^*}{\lambda_2 - \lambda_1} - \frac{\hat{\psi}_1^*}{\lambda_2 - \lambda_1}, \quad A_2 = -\frac{\lambda_1 \hat{\psi}_0^*}{\lambda_2 - \lambda_1} + \frac{\hat{\psi}_1^*}{\lambda_2 - \lambda_1}. \quad (\text{II.B.5})$$

High frequency expansions. At infinity, the sign of real part of the λ will depend on the sign of ξ , hence, for $\xi \rightarrow +\infty$

$$\begin{aligned} \lambda_j^+(\xi) &= \frac{1 - i\alpha_w}{\alpha_w^2 + 1} \xi + (-1)^j \frac{i\xi^{-\frac{1}{2}}}{2} \sqrt{\frac{1 - i\alpha_w}{\alpha_w^2 + 1}} + O(\xi^{-2}), \\ A_j^+(\xi) &= (-1)^j i \xi^{\frac{3}{2}} \sqrt{\frac{1 - i\alpha_w}{\alpha_w^2 + 1}} \psi_0^* + (-1)^{j-1} i \xi^{\frac{1}{2}} \sqrt{1 + i\alpha_w} \psi_1^* + O(|\psi_0^*| + |\xi|^{-5/2} |\psi_1^*|), \end{aligned}$$

and, when $\xi \rightarrow -\infty$

$$\begin{aligned} \lambda_j^+(\xi) &= \frac{1 + i\alpha_w}{\alpha_w^2 + 1} |\xi| + (-1)^j \frac{|\xi|^{-\frac{1}{2}}}{2} \sqrt{\frac{1 + i\alpha_w}{\alpha_w^2 + 1}} + O(\xi^{-2}), \\ A_j^+(\xi) &= (-1)^{j-1} |\xi|^{\frac{3}{2}} \sqrt{\frac{1 + i\alpha_w}{\alpha_w^2 + 1}} \psi_0^* + (-1)^j |\xi|^{\frac{1}{2}} \sqrt{1 - i\alpha_w} \psi_1^* + O(|\psi_0^*| + |\xi|^{-5/2} |\psi_1^*|). \end{aligned}$$

II.C Computations of the regularity estimates

We now focus our attention on the term $\int_{\mathbb{R}} |\xi|^{2k} |\widehat{\Psi}_w|^2 d\xi$, $k = 0, 1, 2$. First, we decompose the integral into two pieces, one on $\{|\xi| > \xi_0\}$ and $\{|\xi| \leq \xi_0\}$.

▷ On the set $|\xi| \leq \xi_0$,

$$\int_{|\xi| \leq \xi_0} \int_0^\infty |\xi|^{2k} |\widehat{\Psi}_w|^2 d\xi dX \leq C \sum_{j=1}^2 \int_{|\xi| \leq \xi_0} |\xi|^{2k} \frac{|A_j(\xi)|^2}{2\Re(\lambda_j)}. \quad (\text{II.C.1})$$

Using (II.5.11) and Lemma II.5.2

$$\int_{|\xi| \leq \xi_0} \int_0^\infty |\xi|^{2k} |\widehat{\Psi}_w|^2 d\xi dX \leq C \int_{|\xi| \leq \xi_0} (|\hat{\psi}_0^*|^2 + |\hat{\psi}_1^*|^2) < +\infty.$$

▷ We now analyze the case when $|\xi| > \xi_0$. We are only illustrating the case when $\xi > 0$ since the negative case can be obtained in the same manner. For $k = 0, 1, 2$, we have

$$|\xi|^{2k} \int_0^\infty |\widehat{\Psi}_w|^2 dX = |\xi|^{2k} \left(\sum_{1 \leq l, m \leq 2} A_l^+ \bar{A}_m^+ \frac{1}{\lambda_l^+ + \bar{\lambda}_m^+} \right). \quad (\text{II.C.2})$$

From Lemma II.5.2, we have that $\lambda_l^+ + \bar{\lambda}_m^+ = a|\xi| + b_{lm}|\xi|^{-1/2} + O(|\xi|^{-2})$, $a = \Re(\zeta^2)$ and $b_{lm} \in \mathbb{R} \cup \mathbb{R}$, $l, m \in \{1, 2\}$. Hence,

$$\frac{1}{\lambda_k + \bar{\lambda}_l} = \frac{1}{a|\xi|} \left(1 - \frac{b_{lm}|\xi|^{-3/2}}{a} \right) + O(\xi^{-3}), \quad k, l = 1, 2.$$

As a consequence,

$$|\xi|^{2k} \int_0^\infty |\widehat{\Psi}_w|^2 dX = \frac{|\xi|^{2k-1}}{a} \left(|A_1^+ + A_2^+|^2 - \frac{\xi^{-3/2}}{a} \sum_{l, m} b_{lm} A_l^+ \bar{A}_m^+ \right) + O\left((|\widehat{\psi}_0^*| + |\widehat{\psi}_1^*|)|\xi|^{2k-1}\right).$$

Here, $|A_1^+ + A_2^+|^2 = |\widehat{\psi}_0^*|^2$ and $\sum_{l, m} b_{lm} A_l^+ \bar{A}_m^+ = O(|\xi|^3 |\widehat{\psi}_0^*|^2 + |\xi| |\widehat{\psi}_1^*|^2)$.

Asymptotic expansions in Lemma II.5.2 lead to

$$\lambda_l^+ + \bar{\lambda}_m^+ = \begin{cases} 2\Re(\zeta^2)\xi + (-1)^l \xi^{-1/2} \Im(\zeta) + O(\xi^{-2}), & \text{for } l = m, \\ 2\Re(\zeta^2)\xi + (-1)^{l-1} i \xi^{-1/2} \Re(\zeta) + O(\xi^{-2}), & \text{for } l \neq m, \end{cases}$$

for $\zeta = \sqrt{\frac{1-i\alpha_w}{\alpha_w^2+1}}$, and

$$A_l^+ \bar{A}_m^+ = \begin{cases} \left(\frac{1}{4} + i\Im(\zeta)\xi^{3/2} + |\zeta|^2 \xi^3 \right) |\widehat{\psi}_0^*|^2 - \frac{\xi}{|\zeta|^2} |\widehat{\psi}_1^*|^2 + \frac{(-1)^{l-1}}{|\zeta|^2} \left(i\Im(\zeta)\xi^{1/2} - 2\Re(\zeta^2)\xi^2 \right) |\widehat{\psi}_0^* \widehat{\psi}_1^*|, & l = m \\ \left(\frac{1}{4} + i\Im(\zeta)\xi^{3/2} + |\zeta|^2 \xi^3 \right) |\widehat{\psi}_0^*|^2 - \frac{\xi}{|\zeta|^2} |\widehat{\psi}_1^*|^2 + \frac{(-1)^{l-1}}{|\zeta|^2} \left(i\Re(\zeta)\xi^{1/2} - 2\Im(\zeta^2)\xi^2 \right) |\widehat{\psi}_0^* \widehat{\psi}_1^*|, & l \neq m. \end{cases}$$

These expressions could mistakenly lead to a need for more robust results on regularity at the boundary. We will show that nontrivial cancellations occur for the usual elliptic regularity result to hold in this case. All sums of the eigenvalues are of the type $a\xi + b_{lm}\xi^{-1/2} + O(\xi^{-2})$, $a = \Re(\zeta^2)$ and $b_{lm} \in \mathbb{R} \cup \mathbb{I}$, $l, m \in \{1, 2\}$. Thus,

$$\frac{1}{\lambda_k + \bar{\lambda}_l} = \frac{1}{a\xi} \left(1 - \frac{b_{lm}|\xi|^{-3/2}}{a} + \frac{b_{lm}^2 \xi^{-3}}{a^2} \right) + O(\xi^{-9/2}), \quad k, l = 1, 2,$$

where

$$b_{lm} = \begin{cases} (-1)^l \Im(\zeta), & l = m, \\ (-1)^{l-1} i \Re(\zeta), & l \neq m. \end{cases}$$

Substituting the above expression in (II.C.2) leads to

$$|\xi|^{2k} \int_0^\infty |\widehat{\Psi}_w|^2 dX = \frac{|\xi|^{2k-1}}{a} \left(|A_1 + A_2|^2 - \frac{|\xi|^{-3/2}}{a} \sum_{l, m} b_{lm} A_l \bar{A}_m + \frac{\xi^{-3}}{a^2} \sum_{l, m} b_{lm}^2 A_l \bar{A}_m \right) + O(\xi^{2k-9/2}). \quad (\text{II.C.3})$$

Moreover,

$$\begin{aligned} |A_1 + A_2|^2 &= |\psi_0^*|^2, \\ \sum_{l,m} b_{lm} A_l \bar{A}_m &= A_2 \bar{A}_2 \Im(\zeta) + A_1 \bar{A}_1 (-\Im(\zeta)) - i A_2 \bar{A}_1 \Re(\zeta) + i A_1 \bar{A}_2 \Re(\zeta) \\ &= 2\sqrt{\xi} \left(2\xi^{3/2} \Im(\zeta) - 1 \right) \psi_0^* \psi_1^*, \end{aligned} \quad (\text{II.C.4})$$

$$\begin{aligned} \sum_{l,m} b_{lm}^2 A_l \bar{A}_m &= \Im(\zeta)^2 \left(|A_1|^2 + |A_2|^2 \right) - \Re(\zeta)^2 \left(A_1 \bar{A}_2 + \bar{A}_1 A_2 \right) \\ &= \frac{\Im(\zeta)^2}{2} \left(\frac{4}{|\zeta|^2} \xi |\psi_1^*|^2 + \left(4i\xi^{3/2} \Im(\zeta) + 4\xi^3 |\zeta|^2 + 1 \right) |\psi_0^*|^2 \right) \\ &\quad + \frac{2\Re(\zeta)^3 \left(-4\xi^{3/2} \Im(\zeta) + i \right)}{|\zeta|^2} \sqrt{\xi} \psi_0^* \psi_1^*. \end{aligned} \quad (\text{II.C.5})$$

Combining the equations (II.C.4) and (II.C.3) we obtain,

$$|\xi|^{2k} \int_0^\infty |\hat{\Psi}_w|^2 dX = O \left(|\xi|^{2k-1} |\psi_0^*|^2 + |\xi|^{2k-3} |\psi_1^*|^2 \right).$$

To complete the proof of (II.5.12), it remains to check the behavior of the derivatives of $\hat{\Psi}_w$ with respect to X up to the second order. Each derivation adds a factor $(-1) \left(\lambda_l^+ + \bar{\lambda}_m^+ \right)$. It is clear the expression is bounded when $|\xi| \leq \xi_0$. Moreover, simple computations show $|\lambda_l^+ + \bar{\lambda}_m^+|^k = O(|\xi|^k)$ for $1 \leq l, m \leq 2$ and $k = 0, 1, 2$ when $|\xi| > \xi_0$. Note that the term $\int_{\{|\xi| > \xi_0\} \times \mathbb{R}^+} |\partial_X^k \hat{\Psi}_w| dX d\xi$ will behave asymptotically as $\int_{\{|\xi| > \xi_0\} \times \mathbb{R}^+} |\xi|^{2k} |\hat{\Psi}_w| dX d\xi$, and therefore, its boundedness depends on the regularity of the functions ψ_0^* and ψ_1^* .

II.D Asymptotic behavior of the Green function coefficients

This section is devoted to the proof of Lemma II.5.3 dealing with low and high-frequency expansions of the coefficients of the Green function (II.5.15). The form of the coefficients

$$\begin{aligned} B_1^+ &= -\frac{1}{(\alpha^2 + 1)^2 \left(\lambda_1^+ - \lambda_2^+ \right) \left(\lambda_1^+ - \lambda_1^- \right) \left(\lambda_1^+ - \lambda_2^- \right)}, \\ B_2^+ &= \frac{1}{(\alpha^2 + 1)^2 \left(\lambda_1^+ - \lambda_2^+ \right) \left(\lambda_2^+ - \lambda_1^- \right) \left(\lambda_2^+ - \lambda_2^- \right)}, \\ B_1^- &= \frac{1}{(\alpha^2 + 1)^2 \left(\lambda_1^+ - \lambda_1^- \right) \left(\lambda_2^+ - \lambda_1^- \right) \left(\lambda_1^- - \lambda_2^- \right)}, \\ B_2^- &= -\frac{1}{(\alpha^2 + 1)^2 \left(\lambda_1^+ - \lambda_2^- \right) \left(\lambda_2^+ - \lambda_2^- \right) \left(\lambda_2^- - \lambda_1^- \right)}. \end{aligned}$$

combined with the asymptotic behavior of the eigenvalues stemming from the characteristic equation provide the desired results. Let us now illustrate this for each case.

Low frequencies: When $|\xi| \ll 1$, we have

$$\lambda_1^- = -|\xi|^4 + O(|\xi|^5), \quad \lambda_2^- = -\frac{1}{(\alpha^2 + 1)^{2/3}} + O(|\xi|), \quad \lambda_j^+ = \frac{1 + (-1)^j i \sqrt{3}}{2(\alpha^2 + 1)^{2/3}} + O(|\xi|), \quad j = 1, 2.$$

Consequently,

$$B_j^+ = B_2^- = \frac{1}{3} + O(|\xi|), \quad j = 1, 2, \quad B_1^- = 1 + O(|\xi|).$$

High frequencies: As $|\xi| \rightarrow +\infty$, the eigenvalues behave for $j = 1, 2$ as

$$\lambda_j^- = -\frac{i|\xi|}{\alpha + i} + (-1)^j \frac{|\xi|^{-1/2}}{2\sqrt{1 - i\alpha}} + O(|\xi|^{-3/2}), \quad \lambda_j^+ = -\frac{i\xi}{\alpha - i} + (-1)^j \frac{\sqrt{\frac{1}{\xi}}}{2\sqrt{-1 - i\alpha}} + O(|\xi|^{-3/2}).$$

II.E Proof of Lemma II.4.1

Note that we have the following Poincaré inequality for $H_0^2(U)$:

$$\|f\|_{H_0^2(U)} \leq C \|D^2 f\|_{L^2(U)}.$$

The previous result is obtained by chaining the Poincaré inequality for f with the Poincaré inequality for Df . We considered the norm $\|f\|_*^2 = \|D^2 f\|_{L^2(U)}^2$ on $H_0^2(U)$ which is equivalent to the standard $H_0^2(U)$ norm.

We claim that

$$\|\Delta f\|_{L^2(U)} = \|D^2 f\|_{L^2(U)} = \|f\|_*$$

for any $f \in H_0^2(U)$.

Indeed, let us consider $f \in C_0^\infty(U)$. Then integration by parts and commutativity of partial derivatives for smooth functions implies

$$\int_U f_{x_i x_i} f_{x_j x_j} dx = - \int_U f_{x_i} f_{x_j x_j x_i} dx = - \int_U f_{x_i} f_{x_j x_i x_j} dx = \int_U f_{x_i x_j} f_{x_i x_j} dx,$$

for all $1 \leq i, j \leq n$. Summing over all i and j yields

$$\|\Delta f\|_{L^2(U)} = \|D^2 f\|_{L^2(U)},$$

for all $f \in C_0^\infty(U)$. Since $C_0^\infty(U)$ is dense in $H_0^2(U)$, passing to limits we find that

$$\|\Delta f\|_{L^2(U)} = \|D^2 f\|_{L^2(U)} \quad \text{for all } f \in H_0^2(U).$$

This gives the desired equality of norms. Hence, in $H_0^2(U)$ we have

$$\|f\|_{H_0^2(U)} \leq C \|\Delta f\|_{L^2(U)}.$$

Let us now show that

$$\|\Delta f\|_{L^2(U)} \leq C \|\Delta_\alpha f\|_{L^2(U)}, \tag{II.E.1}$$

where $C > 0$ is a constant depending on $\alpha \in \mathbb{R}$.

As a result of Plancherel theorem, we have that

$$\|\Delta_\alpha f\|_{L^2(U)}^2 = \|\widehat{\Delta_\alpha f}\|_{L^2(U)}^2, \quad \text{and} \quad \|\Delta f\|_{L^2(U)}^2 = \|\widehat{\Delta f}\|_{L^2(U)}^2.$$

Let (ξ_1, ξ_2) be the Fourier variables associated to X and Y , respectively. We have

$$\begin{aligned} \|\widehat{\Delta_\alpha f}\|_{L^2(U)}^2 &= \int_U |\xi_1^2 + (\xi_2 \pm \alpha \xi_1)^2| |\widehat{f}|^2 \\ &\geq \int_U |\xi_1^2 + \xi_2^2 - \alpha^2 \xi_1^2| |\widehat{f}|^2 \\ &\geq (1 - \alpha^2)^2 \|\widehat{\Delta f}\|_{L^2(U)}^2 \end{aligned} \tag{II.E.2}$$

Here, we have used that $(a + b)^2 \geq a^2 - b^2$, for $a, b \in \mathbf{R}$. Therefore, if $\Delta_\alpha f \in L^2(U)$, we have that $\Delta f \in L^2(U)$.

For non-homogeneous Dirichlet boundary data, a similar result can be obtained by applying the Poincaré-Wirtinger inequality. The additional term in the estimate is introduced to control the boundary terms. The proof is left to the reader.

Other results. In applications, it is important to find different norms which are equivalent to $\|\cdot\|_{H^2}$. Let U be a bounded subset of \mathbb{R}^2 . Then, the scalar product

$$(u, v)_\alpha = \int_U \langle D_\alpha^2 u, D_\alpha^2 v \rangle_F, \tag{II.E.3}$$

defines a norm on $H_0^2(U)$ for

$$D_\alpha^2 = \begin{pmatrix} (1 + \alpha^2)\partial_X^2 & \pm\alpha\partial_{XY}^2 \\ \pm\alpha\partial_{XY}^2 & \partial_Y^2 \end{pmatrix} \tag{II.E.4}$$

and the Frobenius inner product.

Remark II.E.1. *The Frobenius norm on $M_{m,n}(K)$ is derived from the scalar or standard Hermitian product on this space, namely*

$$(A, B) \in M_{m,n}(K)^2 \mapsto \langle A, B \rangle = \text{tr}(A^* B) = \text{tr}(B A^*), \tag{II.E.5}$$

where A^* denotes the conjugate transpose of A et tr is the trace.

Let u be a function of $H_0^2(U)$, we have that

$$\frac{1}{1 + \alpha^2} \|u\|_\alpha \leq \|u\|_{H_0^2(U)} \leq \max(1 + \alpha^2, 2\alpha^2) \|u\|_\alpha. \tag{II.E.6}$$

II.F Proof of Lemmas II.5.9 and II.8.7

Proof of Lemma II.5.9. The Fourier multiplier $M_w = (m_{i,j})_{2 \leq i \leq 3, 0 \leq j \leq 1} \in M_2(\mathbb{C})$ has the components

$$\begin{aligned} m_{2,0} &= -(\alpha_w^2 + 1) \left((\alpha_w^2 + 1) \lambda_1 \lambda_2 + \xi^2 \right), \\ m_{2,1} &= -(\alpha_w^2 + 1) \left((\alpha_w^2 + 1) \lambda_1 + (\alpha_w^2 + 1) \lambda_2 + 2i\alpha_w \xi \right), \\ m_{3,0} &= -\frac{1}{2} \left(2(\alpha_w^2 + 1) \lambda_1 \lambda_2 \left((\alpha_w^2 + 1) \lambda_1 + (\alpha_w^2 + 1) \lambda_2 + 4i\alpha_w \xi \right) + 4i\alpha_w \xi^3 - 1 \right), \\ m_{3,1} &= \left(5\alpha_w^2 + 1 \right) \xi^2 \\ &\quad + (\alpha_w^2 + 1) \left(-\lambda_1 \left((\alpha_w^2 + 1) \lambda_2 + 4i\alpha_w \xi \right) - \lambda_2 \left((\alpha_w^2 + 1) \lambda_2 + 4i\alpha_w \xi \right) - (\alpha_w^2 + 1) \lambda_1^2 \right). \end{aligned} \tag{II.F.1}$$

Recall that λ_i are the roots of the characteristic equation

$$P(\lambda) = -\lambda - (\lambda^2 + (\alpha\lambda + i\xi)^2) = 0,$$

satisfying $\Re(\lambda_i) > 0$.

Expressions in (II.F.1) together with the asymptotic expansions in Lemma II.5.2 are the core ingredients of the proof.

- At low frequencies, the eigenvalues are complex conjugate constants depending on the parameter α_w . In particular, substituting $\lambda_1(\xi) = \frac{1 - i\sqrt{3}}{2(\alpha_w^2 + 1)^{2/3}} + O(|\xi|)$, $\lambda_2 = \bar{\lambda}_1$ in (II.F.1) provides

$$\begin{aligned} m_{2,0} &= -(\alpha_w^2 + 1)^2 |\lambda_1|^2 + O(|\xi|) = -(\alpha_w^2 + 1)^{2/3} |\lambda_1|^2 + O(|\xi|), \\ m_{2,1} &= -2(\alpha_w^2 + 1)^2 \Re(\lambda_1) + O(|\xi|) = -(\alpha_w^2 + 1)^{4/3} |\lambda_1|^2 + O(|\xi|), \\ m_{3,0} &= -2(\alpha_w^2 + 1)^2 |\lambda_1|^2 \Re(\lambda_1) + \frac{1}{2} + O(|\xi|) = -\frac{1}{2} + O(|\xi|), \\ m_{3,1} &= -(\alpha_w^2 + 1)^2 \left(3\Re(\lambda_1)^2 - \Im(\lambda_1)^2 \right) + O(|\xi|) = O(|\xi|). \end{aligned}$$

To compute a more precise value of $m_{3,1}$, we take into account that $\lambda_j = \frac{1 + (-1)^j i\sqrt{3}}{2(\alpha_w^2 + 1)^{2/3}} + \frac{4i\alpha|\xi|}{3\alpha^2 + 3} + O(|\xi|^2)$ and do not neglect the coefficients of ξ .

- When $\xi \rightarrow +\infty$, the roots behave as $\lambda_j^+(\xi) = \zeta^2 \xi + (-1)^j \frac{i\xi^{-1/2}}{2} \zeta + O(|\xi|^{-2})$, where $\zeta = \sqrt{\frac{1+i\alpha_w}{\alpha_w^2 + 1}}$. This yields the matrix,

$$M_w = \begin{pmatrix} -2(1 + i\alpha_w)\xi^2 + O(\xi^{-1/2}) & -2(1 + 2i\alpha_w)(\alpha_w^2 + 1)\xi + O(\xi^{-1/2}) \\ 2\left(7 + 2i\alpha_w - \frac{8(1 + i\alpha_w)}{\alpha_w^2 + 1}\right)\xi^3 + O(1) & -2(1 - 8\alpha_w^2 + 7i\alpha_w)\xi^2 + O(\xi^{1/2}) \end{pmatrix}.$$

Let us illustrate the proof of the above result. We see that

$$\begin{aligned}
m_{2,0} &= -(\alpha_w^2 + 1) \left((\alpha_w^2 + 1) \zeta^4 + 1 \right) \xi^2 + O(\xi^{-1/2}) \\
&= -(\alpha_w^2 + 1) \left(\frac{1 + 2i\alpha_w - \alpha_w^2}{1 + \alpha_w^2} + 1 \right) + O(\xi^{-1/2}) = -2(1 + i\alpha_w) + O(\xi^{-1/2}), \\
m_{2,1} &= -2(\alpha_w^2 + 1) \left((\alpha_w^2 + 1) \zeta^2 + i\alpha_w \right) \xi + O(\xi^{-1/2}) = -2(\alpha_w^2 + 1)(1 + 2i\alpha_w) + O(\xi^{-1/2}), \\
m_{3,0} &= -2 \left((\alpha_w^2 + 1)^2 \zeta^6 + 2(\alpha_w^2 + 1) i\alpha_w \zeta^4 + i\alpha_w \right) \xi^3 + O(1) \\
&= -2 \left(\frac{(1 + i\alpha_w)^2}{1 + \alpha_w^2} (1 + 3i\alpha_w) + i\alpha_w \right) \xi^3 + O(\xi^{3/2}) = 2 \left(7 + 2i\alpha_w - \frac{8(1 + i\alpha_w)}{\alpha_w^2 + 1} \right) \xi^3 + O(1), \\
m_{3,1} &= \left(5\alpha_w^2 + 1 - \left(3(\alpha_w^2 + 1)^2 \zeta^4 + 8i\alpha_w (\alpha_w^2 + 1) \zeta^2 \right) \right) \xi^2 + O(\xi^{1/2}) \\
&= \left(5\alpha_w^2 + 1 - \left(3(1 + \alpha_w i)^2 + 8i\alpha_w (1 + \alpha_w i) \right) \right) \xi^2 + O(\xi^{1/2}) \\
&= -2(1 - 8\alpha_w^2 + 7i\alpha_w) \xi^2 + O(\xi^{1/2}).
\end{aligned}$$

The matrix values when $\xi \rightarrow -\infty$ can be computed in the same manner.

□

Proof of Lemma II.8.7. In the eastern boundary layer domain, the matrix of Fourier multipliers $M_e = (n_{i,j})_{2 \leq i \leq 3, 0 \leq j \leq 1} \in M_2(\mathbb{C})$ is formed by the elements

$$\begin{aligned}
n_{2,0} &= (\alpha_e^2 + 1) \left((\alpha_e^2 + 1) \mu_1 \mu_2 + \xi^2 \right), \\
n_{2,1} &= (\alpha_e^2 + 1) \left((\alpha_e^2 + 1) \mu_1 + (\alpha_e^2 + 1) \mu_2 - 2i\alpha_e \xi \right), \\
n_{3,0} &= -\frac{1}{2} \left(2(\alpha_e^2 + 1)^2 \mu_1 \mu_2 (\mu_1 + \mu_2) + 4i\alpha_e \xi^3 - 1 \right), \\
n_{3,1} &= - \left((\alpha_e^2 + 1)^2 (\mu_1^2 + \mu_2 \mu_1 + \mu_2^2) + (3\alpha_e^2 - 1) \xi^2 \right),
\end{aligned} \tag{II.F.2}$$

where μ_i , $i = 1, 2$ are the complex roots of positive real part of the equation

$$P_e(\mu, \xi) = -\mu + (\mu^2 + (-\alpha_e \mu + i\xi)^2)^2.$$

- When $|\xi| \gg 1$, the eigenvalues behave like (see Lemma II.8.1)

$$\mu_1 = |\xi|^4 + O(|\xi|^5), \quad \mu_2 = \frac{1}{(\alpha^2 + 1)^{2/3}} + O(|\xi|).$$

We obtain immediately that

$$\begin{aligned}
n_{2,0} &= (\alpha_e^2 + 1) \xi^2 + O(\xi^4), \quad n_{2,1} = (\alpha_e^2 + 1)^{4/3} + O(\xi), \\
n_{3,0} &= \frac{1}{2} + O(\xi), \quad n_{3,1} = -(\alpha_e^2 + 1)^2 (\alpha^2 + 1)^{-4/3} + O(\xi) = -(\alpha^2 + 1)^{2/3} + O(\xi).
\end{aligned}$$

- Similarly to the proof of Lemma II.5.9, we only discuss the case when $\xi \rightarrow +\infty$, since the behavior at $-\infty$ results from applying the same reasoning.

The high frequency expansion of the eigenvalues are of the form

$$\mu_j(\xi) = \zeta_e^2 \xi + \frac{(-1)^j}{2} \xi^{-\frac{1}{2}} \zeta_e + O(\xi^{-3/2}), \quad \zeta_e = \sqrt{\frac{1 - i\alpha_e}{\alpha_e^2 + 1}}.$$

We have

$$M_e = \begin{pmatrix} 2(1 - i\alpha_e)\xi^2 + O(\xi^{-1/2}) & 2(1 - 2i\alpha_e)(\alpha_e^2 + 1)\xi + O(\xi^{-1/2}) \\ 2\left(3 - 2i\alpha_e - \frac{4(1 - i\alpha_e)}{\alpha_e^2 + 1}\right)\xi^3 + O(1) & -2(1 - 3i\alpha_e)\xi^2 + O(\xi^{-1/2}) \end{pmatrix}.$$

Indeed, substituting the formulae of the eigenvalues at high frequencies in (II.F.2) gives

$$\begin{aligned} n_{2,0} &= (\alpha_e^2 + 1) \left((\alpha_e^2 + 1) \zeta_e^4 + 1 \right) \xi^2 + O(\xi^{-1/2}) = 2(1 - i\alpha_e) + O(\xi^{-1/2}), \\ n_{2,1} &= -2(\alpha_e^2 + 1) \left((\alpha_e^2 + 1) \zeta_e^2 - i\alpha_e \right) \xi + O(\xi^{-1/2}) = 2(1 - 2i\alpha_e)(\alpha_e^2 + 1) + O(\xi^{-1/2}), \\ n_{3,0} &= -2 \left((\alpha_e^2 + 1)^2 \zeta_e^6 + i\alpha_e \right) \xi^3 + O(\xi^{3/2}) = -2 \left(\frac{2i\alpha^3 - 3\alpha^2 - 2i\alpha + 1}{1 + \alpha_e^2} \right) \xi^3 + O(1) \\ &= 2 \left(3 - 2i\alpha_e - \frac{4(1 - i\alpha_e)}{\alpha_e^2 + 1} \right) \xi^3 + O(1), \\ n_{3,1} &= - \left(3(\alpha_e^2 + 1)^2 \zeta_e^4 + 3\alpha_e^2 - 1 \right) \xi^2 + O(\xi^{-1/2}) = - \left(3(1 - i\alpha_e)^2 + 3\alpha_e^2 - 1 \right) \xi^2 + O(\xi^{-1/2}) \\ &= -2(1 - 3i\alpha_e)\xi^2 + O(\xi^{-1/2}). \end{aligned}$$

□

III – Traveling waves for the porous medium equation in the incompressible limit: asymptotic behavior and nonlinear stability

Contents

III.1 Introduction	152
III.2 Traveling waves for the Hele-Shaw system and the porous media equation	157
III.2.1 TW for the limit Hele-Shaw system	157
III.2.2 Qualitative properties of travelling waves for the porous medium equation	159
III.2.3 Phase portrait of N_γ and further consequences	163
III.2.4 Quantitative bounds for the profiles N_γ	169
III.3 Stability of the profiles N_γ	177
III.3.1 Overall strategy	177
III.3.2 Control of the nonlinear terms and long-time behavior	183
III.4 Proofs of some technical results	188
III.4.1 Derivation of the equation on u	188
III.4.2 Structure of the linearized system: Lemmas III.3.1 and equality (III.3.18)	189
III.4.3 Properties of the weights \bar{w}_0 and \bar{w} : Lemma III.3.2 and estimate (III.3.19)	190
III.4.4 Proof of the weighted Poincaré inequality	194

In this chapter, we analyze the behavior of monotone traveling waves of a one-dimensional porous medium equation modeling mechanical properties of living tissues. We are interested in the asymptotics where the pressure governing the diffusion process and limits the creation of new cells, becomes very stiff, and the porous medium equation degenerates towards a free boundary problem of Hele-Shaw type. This is the so-called *incompressible limit*. The solutions of the limit Hele-Shaw problem then couple “free dynamics” with zero pressure, and “incompressible dynamics” with positive pressure and constant density. In the first part

of the work, we provide a refined description of the traveling waves for the porous medium equation in the vicinity of the transition between the free domain and the incompressible domain. The second part of the study is devoted to the analysis of the stability of the traveling waves. We prove that the linearized system enjoys a spectral gap property in suitable weighted L^2 spaces, and we give quantitative estimates on the rate of decay of solutions. The nonlinear terms are treated perturbatively, using an L^∞ control stemming from the maximum principle. As a consequence, we prove that traveling waves are stable under small perturbations.

III.1 Introduction

This chapter is devoted to the asymptotic analysis and the stability of traveling waves for the porous medium equation (PME). More precisely, let us consider the following nonlinear parabolic equation

$$\partial_t n - \partial_x(n \partial_x p(n)) = n \Phi(p(n)), \quad (\text{III.1.1})$$

endowed with the boundary conditions

$$\lim_{x \rightarrow \pm\infty} n(t, x) = n_\pm,$$

where n_\pm are constant stationary states of the equation. This equation has been introduced in the literature to model tissue growth and, particularly, in the propagation of tumors (see for instance [85, 133, 112]). The left-hand side corresponds to the Porous Medium Equation (PME): the density of cells, n , is transported by a velocity given by the Darcy law $v = -\partial_x p$ where $p = p(n)$ denotes the mechanical pressure. The right-hand side models the cell proliferation in the medium, proliferation which is limited by the pressure. Hence, the function Φ is usually taken as a decreasing function of the pressure and is such that $\Phi(p_M) = 0$ for some $p_M > 0$ called the *homeostatic pressure*. In this study, we shall assume for simplicity that

$$p(n) = p_\gamma(n) = n^\gamma \text{ with } \gamma > 1, \quad \Phi(p) = 1 - p. \quad (\text{III.1.2})$$

In other words, the function Φ becomes negative above the threshold pressure $p_M = p_\gamma(n_M) = 1$, which means that cells are destroyed above the *maximal packing density* $n_M = 1$. We will also pick $n_- = n_M = 1$, and $n_+ = 0$.

This study aims to analyze the behavior of traveling waves (TWs) solutions of (III.1.1) when the parameter γ appearing in the equation of state (III.1.2) tends to $+\infty$. For $\Phi(p) = 0$, *i.e.* without the reaction term in the equation, this limit $\gamma \rightarrow +\infty$ is referred as the *mesa limit* and has been studied for instance by Caffarelli and Friedman [89]. In this paper, the authors consider an initial datum larger than 1 on a nontrivial set and show that this upper part exceeding 1 collapses at $t = 0^+$ to $\{n = 1\}$. This phenomenon is due to the blow up of the diffusivity $np'_\gamma(n) = \gamma n^\gamma \rightarrow +\infty$ when $n > 1$. The singular limit $\gamma \rightarrow +\infty$ for solutions of the PME is then called the “mesa” limit in reference to the shape of the target density $n_\infty \in [0, 1]$ which is similar to the flat-topped mountains. In the presence of a growth source term Φ , the limit $\gamma \rightarrow +\infty$ has been first tackled by Perthame *et al.* in [85]. As in the previous case, the blow-up of the pressure as $\gamma \rightarrow +\infty$ when $n > 1$ forces the limit density

to lie in $[0, 1]$. The sequence $(n_\gamma)_{\gamma>1}$ of weak solutions to (III.1.1) is then shown to converge (for a suitable topology) towards a weak solution of the following Hele-Shaw system

$$\begin{cases} \partial_t n - \partial_x(n\partial_x p) = n\Phi(p), & \text{(III.1.3a)} \\ 0 \leq n \leq 1, \quad (1-n)p = 0, \quad p \geq 0, & \text{(III.1.3b)} \\ p(\partial_x^2 p + \Phi(p)) = 0. & \text{(III.1.3c)} \end{cases}$$

The transition between equation (III.1.1) and system (III.1.3) is usually called the *incompressible limit* in reference to the fact that, when the solution n of (III.1.3) reaches 1, it is blocked to this maximal value (the combination of the mass equation (III.1.3a) with the complementary relation (III.1.3b) yields formally $\partial_t n = 0$ in $\{n = 1\}$) and the medium cannot be further compressed. Beyond the physical and biological relevancy of system (III.1.1) seen as an approximation of (III.1.3), Mellet *et al.* [88] have shown that the incompressible limit can provide crucial qualitative information on the solutions of the Hele-Shaw system (III.1.3), like the regularity of the free boundary $\partial\{n = 1\}$.

To finish with the incompressible limit, let us mention that this type of singular limit has been studied in other frameworks: for other singular equations of state [86], in the case of coupling with the dynamics of nutrients [112], in the case of more than one type of cancerous cell as seen in [113, 115, 120], when the Darcy law is replaced by the Brinkman equation [114] or the Navier-Stokes equations [116].

Up to our knowledge, the issue of TWs solutions to (III.1.3) remains rare in the literature (see [121] when nutrients are considered), even when the topic was intensively studied for nonlinear reaction-diffusion equations like (III.1.1). Indeed, TWs as a class of special solutions have been shown to provide valuable information on general solutions of these reaction-diffusion equations (see the books [122] and [123]). Most of the results concern the long-term behavior (convergence to TWs, asymptotic rate of propagation of disturbances) or the behavior close to interfaces of general solutions.

Regarding the issue of interfaces, Gilding and Kersner study in [117] the existence of *sharp* (or *finite*) TWs whose support is bounded on one side in case of nonlinear degenerate diffusion, and deduce a result about the existence of an interface $\partial\{n = 0\}$ for general solutions. In [124], TWs are used to study the regularity of the general solutions near the free boundary $\partial\{n = 0\}$, as well as for the derivation of the interface motion. The essential tools of the analysis are then: the continuity of the flux across the interface and a comparison principle bracketing a general solution between two TWs.

Concerning the long-time behavior of solutions to reaction-diffusion scalar equations like (III.1.1), let us mention two types of results related to the nature of the wave-front. For *sharp fronts*, that is, TWs with support bounded from above (or below), Kamin and Rosenau prove in [125] that initial data decaying sufficiently fast at infinity converge (in a specific sense) towards a sharp TW. The techniques they employ are inspired by L^1 -stability theory of shock waves for viscous conservation laws (see for instance [126]): use of comparison principle (already mentioned above), derivation of L^1 conservation, and contraction principles with an exponential weight. It is worth pointing out that this result cannot be extended to *smooth fronts*, *i.e.* TWs that do not vanish and remain smooth on \mathbb{R} . Indeed the weight used in [125] is specific to the critical speed c^* at which the sharp fronts travel (see Theorem III.2.1 below) and is not suited for the smooth fronts propagating at speed $c > c^*$. To our knowledge, the only result dealing with smooth fronts is a spectral stability result obtained recently by

Leyva and Plaza in [110]. In their work, the difficulties associated with the degeneracy of the diffusion term are overcome with the derivation of a kind relative entropy estimate with a well-suited exponential weight.

In this chapter, the study of smooth TWs of (III.1.1) as $\gamma \rightarrow +\infty$ can be seen as a first step in the analysis of the free boundary $\partial\{n = 1\}$ for the limit Hele-Shaw system (III.1.3). Our contributions are twofold: we first give a qualitative and quantitative description (in terms of γ) of smooth TWs of (III.1.1) and show the convergence towards TWs of (III.1.3) that are discontinuous at the interface $\partial\{n = 1\}$; we also study the nonlinear asymptotic stability of the smooth TWs for small (quantified in terms of γ) general perturbations of these wave-fronts.

As in [124], our analysis relies strongly on the control of the flux around the interface (passage to the limit as $\gamma \rightarrow +\infty$, determination of the transmission conditions across the interface on the limit system); and the comparison principle (quantitative behavior of TWs as $\gamma \rightarrow +\infty$, control of general solutions lying between two TWs). Compared to the stability analysis of Leyva and Plaza [110], we have to deal with additional nonlinear contributions that we treat in a perturbative manner and control thanks to a Poincaré-type inequality. This latter also allows us to get a decay rate of the perturbation as $t \rightarrow +\infty$.

Statement of main results

In this chapter, we focus on traveling waves solutions of (III.1.1)-(III.1.2), that is solutions n_γ such that $n_\gamma(t, x) = N_\gamma(x - ct)$ where N_γ is the wave profile, $\xi = x - ct$ is the wave coordinate and c is the speed of propagation of the wave. The profile N_γ is then solution to the differential equation:

$$-cN'_\gamma - \gamma(N_\gamma^\gamma N'_\gamma)' = N_\gamma(1 - N_\gamma^\gamma). \quad (\text{III.1.4})$$

The above equation admits two equilibrium states: $N \equiv 0$ (unstable) and $N \equiv 1$ (stable), and we seek therefore wavefronts N_γ connecting these two states:

$$\lim_{\xi \rightarrow -\infty} N_\gamma(\xi) = 1, \quad \lim_{\xi \rightarrow +\infty} N_\gamma(\xi) = 0. \quad (\text{III.1.5})$$

The existence and uniqueness (up to a shift) of a monotone (decreasing) solution to (III.1.4)-(III.1.5), as well as the asymptotic behavior of N_γ close to $\pm\infty$, were previously investigated by Gilding and Kersner [117] for c larger than a threshold velocity $c_\gamma^* > 0$ (see below Theorem III.2.1 for a precise statement). In the present study, we intend to analyze further the behavior of N_γ and $P_\gamma(\xi) = (N_\gamma(\xi))^\gamma$, the associated pressure profile, with respect to the parameter γ . Our first main result concerns the qualitative and quantitative behaviors as $\gamma \rightarrow +\infty$.

Theorem III.1.1. *Let $\gamma > 1$ sufficiently large, $c > 1$ be fixed, independent of γ , and let N_γ be the solution of (III.1.4)-(III.1.5) such that $P_\gamma(0) = \frac{1}{\gamma}$. Then the following properties hold true.*

- *There exist $\xi_\gamma^-, \tilde{\xi}_\gamma$ with $\xi_\gamma^- = O\left(\frac{1}{\sqrt{\gamma}}\right) < 0 < \tilde{\xi}_\gamma = O\left(\frac{1}{\gamma}\right)$, such that the profile (N_γ, P_γ) satisfies*

- in the congested zone $\xi < \xi_\gamma^-$, the density N_γ converges uniformly to 1: there exists a constant $C > 0$ depending only on c such that

$$\left(\frac{C}{\sqrt{\gamma}}\right)^{\frac{1}{\gamma}} \leq N_\gamma(\xi) \leq 1 \quad \forall \xi \leq \xi_\gamma^-, \quad (\text{III.1.6})$$

and there exist constants $C' \geq C > 0$ independent of γ such that

$$1 - \left(1 - \frac{C'}{\sqrt{\gamma}}\right) e^{(1-C\gamma^{-1/2})\xi} \leq P_\gamma(\xi) \leq 1 - \left(1 - \frac{C}{\sqrt{\gamma}}\right) e^\xi \quad \forall \xi \leq \xi_\gamma^-; \quad (\text{III.1.7})$$

- in the intermediate region $\xi \in [\xi_\gamma^-, \tilde{\xi}_\gamma]$, N'_γ takes exponentially large values with respect to γ :

$$\|N'_\gamma\|_{L^\infty(\xi_\gamma^-, \tilde{\xi}_\gamma)} = O\left(\left(1 - \frac{1}{2c}\right)^{-\gamma}\right), \quad (\text{III.1.8})$$

while the pressure P_γ converges uniformly to 0 as $\gamma \rightarrow +\infty$: there exists $\delta \in (0, 1 - c^{-1})$, independent of γ such that

$$\left(1 - \frac{1}{c} - \delta\right)^\gamma \leq P_\gamma(\xi) \leq \frac{C}{\sqrt{\gamma}} \quad \forall \xi \in [\xi_\gamma^-, \tilde{\xi}_\gamma]; \quad (\text{III.1.9})$$

- in the free zone $\xi > \tilde{\xi}_\gamma$, the pressure P_γ takes exponentially small values (wrt γ): $P_\gamma(\xi) \leq \left(1 - \frac{1}{2c}\right)^\gamma$ and N_γ decreases exponentially to 0 as $\xi \rightarrow +\infty$: there exists $\delta > 0$ independent of γ , such that for γ large enough

$$\left(1 - \frac{1}{c} - \delta\right) \exp\left(-\left(\frac{1}{c} + \delta\right)\xi\right) \leq N_\gamma(\xi) \leq \left(1 - \frac{1}{c} + \delta\right) \exp\left(-\frac{1}{2c}\xi\right) \quad \forall \xi > \tilde{\xi}_\gamma; \quad (\text{III.1.10})$$

- As $\gamma \rightarrow +\infty$, there exists $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ such that $N_\gamma \rightarrow N_{HS}$ in $L^p_{loc}(\mathbb{R})$ and $P_\gamma \rightarrow P_{HS}$ in $W^{1,p}_{loc}(\mathbb{R})$ for any $p \in [1, \infty[$, and (N_{HS}, P_{HS}) is a wave-front profile of the Hele-Shaw equations (III.1.3) such that $P_{HS}(\xi) = (1 - e^\xi)\mathbf{1}_{\xi \leq 0}$, $\lim_{\xi \rightarrow 0^+} N_{HS} = 1 - \frac{1}{c}$.

Remark III.1.1. Concerning the convergence of (N_γ, P_γ) towards (N_{HS}, P_{HS}) , a key ingredient of our proof is the uniform control of the flux $J_\gamma = cN_\gamma + N_\gamma P'_\gamma$ which is such $J'_\gamma = -N_\gamma(1 - P_\gamma) \in [-1, 0]$. The control of J_γ implies in particular the control of P'_γ and thus yields the uniform convergence of $(P_\gamma)_\gamma$. It is important to note that this uniform convergence of (P_γ) is uncorrelated to the convergence of $(N_\gamma)_\gamma$. Indeed, we have $P'_\gamma = \gamma N_\gamma^{\gamma-1} N'_\gamma$ but the pre-factor $\gamma N_\gamma^{\gamma-1}$ which tends to 0 on a half-space, prevents us to get a uniform bound on N'_γ . Actually this derivative blows up as it can be observed on (III.1.8). The uniform convergence of the flux J_γ is also crucial to determine the value of N_{HS} on the right side of the interface $\xi = 0$. Since then $J_{HS}(0) = c + \lim_{\xi \rightarrow 0^-} P'_{HS}(\xi) = c - 1$, we deduce that $\lim_{\xi \rightarrow 0^+} N_{HS} = c^{-1} J_{HS}(0) = 1 - \frac{1}{c}$.

Remark III.1.2. *A legitimate question is the possible extension of the previous result to more general pressure laws (as for instance the singular potentials considered in [86] or [186]) and reaction terms Φ . Our analysis actually starts with the results obtained by Gilding and Kersner [117]. In particular in [117], the determination of the critical speed $c^* = c_\gamma^*$ is specific to the pressure law $p_\gamma(n) = n^\gamma$. To our knowledge, the explicit characterization of c^* has not been tackled in the literature, more precisely we would need an upper bound on c^* independent of the parameter characterizing the incompressible limit. The extension of [117] to the case of more general pressures and reaction terms is therefore out of the scope of the present chapter but there is a reasonable hope for a generalization of the previous theorem once the existence of a profile N_γ for a fixed speed $c > c^*$ (independent of parameter γ) is ensured.*

We believe that several steps of our strategy could be extended to other pressure laws (analysis of the phase portrait of the traveling wave and consequences, design of appropriate weights for the coercivity of the linearized operator, etc.) However, in several instances some quantitative arguments rely heavily on fine properties of N_γ (e.g. the description of the transition zone). It is unavoidable that such properties will depend on the exact nature of the pressure law, and that a case by case analysis needs to be performed.

Our second result is dedicated to the analysis of stability of the wavefront N_γ in weighted Sobolev spaces. To that end, we introduce the weight

$$W(\xi) := N_\gamma(\xi)^\gamma \exp\left(\int_{\xi^-}^{\xi} \frac{c}{\gamma N_\gamma^\gamma}\right).$$

Note that W has a double exponential growth as $\xi \rightarrow +\infty$, and a (slow) exponential decay as $\xi \rightarrow -\infty$. Therefore, W will provide a very good control of the difference $n_\gamma(t, x) - N_\gamma(x - ct)$ in the free zone $x - ct > 0$.

Our result is the following:

Theorem III.1.2. *There exists constants $\eta_1, \eta_2 \in]0, 1[$, depending only on $c > 1$, such that the following result holds.*

Let $\gamma > 1$ be fixed, sufficiently large. We make the following assumptions on the initial data n_γ^0 :

(H1) *n_γ^0 lies between two shifts of N_γ , i.e. there exists $h > 0$ such that $n_\gamma^0(x) \in [N_\gamma(x + h), N_\gamma(x - h)]$ for all $x \in \mathbb{R}$;*

(H2) *The difference $n_\gamma^0 - N_\gamma$ is sufficiently decaying, namely*

$$\int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x) dx < \infty.$$

Let n_γ be the solution of (III.1.1) associated with n_γ^0 .

Then there exists a constant $c_\gamma > 0$, $c_\gamma = O(\eta_1^\gamma)$, such that if $|h| \leq \eta_2^\gamma$, the following inequality holds:

$$\int_{\mathbb{R}} (n_\gamma(t, x) - N_\gamma(x - ct))^2 W(x - ct) dx \leq e^{-c_\gamma t} \int_{\mathbb{R}} \left(n_\gamma^0(x) - N_\gamma(x)\right)^2 W(x) dx \quad \forall t \geq 0.$$

Moreover, setting $u_\gamma(t, x) := (n_\gamma(t, x) - N_\gamma(x - ct))/N'_\gamma(x - ct)$, we have the additional dissipation of energy:

$$\gamma \int_0^\infty \int_{\mathbb{R}} (\partial_x u_\gamma(t, x))^2 (N_\gamma^\gamma (N'_\gamma)^2 W)(x - ct) \, dx \, dt \leq \int_{\mathbb{R}} (n_\gamma^0(x) - N_\gamma(x))^2 W(x) \, dx.$$

Let us give a short sketch of proof of the above result. An important feature of equation (III.1.1) lies in the fact that its linearization around $N_\gamma(x - ct)$ is spectrally stable in suitable weighted Sobolev spaces. This property has been identified recently by Leyva and Plaza [110], using Sobolev spaces with an exponential weight. Here, we work with different weights, which we believe follow more closely the structure of the equation, see Lemma III.3.1 and subsection III.4.2, and which give a better control in the congested zone. One crucial point of our analysis lies in the derivation of a new weighted Poincaré inequality associated with this weight, see Proposition III.3.1. This allows us to prove a spectral gap property, leading to the exponential decay announced in the above Theorem. Once the dissipation properties of the linearized equation have been identified and quantified, we perform the nonlinear estimates by treating the quadratic terms as perturbations. In this regard, assumption (H1) allows us to have a uniform L^∞ control on $n_\gamma(t, x) - N_\gamma(x - ct)$, thanks to the parabolic nature of the equation.

Note that the rate of decay c_γ of the energy is exponentially small. This is linked to the exponential blow-up of N'_γ in the transition zone, see Theorem III.1.1. This blow-up also imposes a strong limitation on the admissible size of the perturbation in L^∞ , and thereby on the size of h . It is not clear whether this assumption could be substantially lowered, taking for instance initial perturbations that would be algebraically - but not exponentially - small. Indeed, it is possible that the strong variations of N_γ in the transition zone destabilize the flow.

Our study is organized as follows. In Section III.2, we describe traveling fronts for both systems (III.1.1) and (III.1.3) and give a refined behavior of the profile N_γ in the transition zone between the congested region and the free region. Next, we prove in Section III.3 the asymptotic stability of the profile N_γ (γ being fixed) for some L^2 -weighted norm. Finally, we have postponed in Section III.4 the proofs of some technical lemmas used in Section III.3.

III.2 Traveling waves for the Hele-Shaw system and the porous media equation

This section is devoted to studying the existence and properties of traveling fronts of both systems: Hele-Shaw and the mechanical model of tumor growth with “stiff pressure law” depending on the parameter γ . For the latter, an asymptotic expansion of this type of solution will be computed.

III.2.1 TW for the limit Hele-Shaw system

We look for traveling wave-type solutions of the form $(n, p) = (N, P)(x - ct)$, where $c > 0$ is a constant representing the traveling wave speed and N, P are real nonnegative functions.

We may assume that $c > 0$, since for $c = 0$ we find again the stationary solutions, and the case $c < 0$ can be reduced to $c > 0$ by reflection.

Lemma III.2.1. *Let $c > 1$ be arbitrary, and let ξ denote the traveling wave variable $\xi = x - ct$.*

1. Define the profile $(N_{HS}, P_{HS}) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ by

$$P_{HS}(\xi) = \begin{cases} 0 & \text{if } \xi > 0, \\ 1 - e^\xi & \text{if } \xi < 0, \end{cases} \quad N_{HS}(\xi) = \begin{cases} \left(1 - \frac{1}{c}\right) e^{-\frac{\xi}{c}} & \text{if } \xi > 0, \\ 1 & \text{if } \xi < 0. \end{cases} \quad (\text{III.2.1})$$

Then $(N_{HS}, P_{HS})(x-ct)$ is a traveling wave moving at speed c solution of the Hele-Shaw system

$$cN' + (NP')' + N\Phi(P) = 0, \quad (\text{III.2.2})$$

$$0 \leq N \leq 1, \quad (1 - N)P = 0, \quad P \geq 0, \quad (\text{III.2.3})$$

$$P(P'' + \Phi(P)) = 0. \quad (\text{III.2.4})$$

2. Let $(N, P) \in L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ be a traveling wave profile moving at speed c of the Hele-Shaw system (III.1.3). Then there exists $\xi_0 \in \mathbb{R}$ such that $(N, P) = (N_{HS}, P_{HS})(\cdot - \xi_0)$.

Remark III.2.1. • The Lipschitz regularity assumption on P ensures that the term $P'N$ is well-defined, as a product of two L^∞ functions.

• An important feature of the analysis is the continuity of the flux $(c + P')N$ on \mathbb{R} (and in particular at the transition point ξ_0). This property will determine the value of $N(\xi_0^+)$.

Proof. It is easily checked that (N_{HS}, P_{HS}) is a solution of (III.2.2)-(III.2.4). Hence the difficulty is to prove that all solutions are equal to (N_{HS}, P_{HS}) (up to a translation). As emphasized in Remark III.2.1, the flux $J = cN + NP'$ satisfies $J' = -N\Phi(P) \in [-1, 0]$. Hence J is Lipschitz continuous and decreasing. Using the values of N, P at $\pm\infty$, we find that $J(-\infty) = c$, $J(+\infty) = 0$, and therefore $0 \leq J \leq c$ a.e.

Since P is Lipschitz continuous, the set $\{P > 0\}$ is a countable union of disjoint open intervals, say $\cup_{j \in \mathcal{J}} (a_j, b_j)$. On any such interval (a_j, b_j) , we have $N = 1$ and

$$-P'' = 1 - P, \quad \forall \xi \in (a_j, b_j).$$

Hence there exist C_j^\pm such that

$$P(\xi) = 1 + C_j^+ e^\xi + C_j^- e^{-\xi} \quad \forall \xi \in (a_j, b_j).$$

Note that the case $b_j = +\infty$ is excluded, since $N(+\infty) = 0$, and that $C_j^- = 0$ if $a_j = -\infty$. Furthermore, on any interval (a_j, b_j) , we have $J = c + P' \in [0, c]$, and $J' = P'' \leq 0$. Hence P is non-increasing and concave on (a_j, b_j) . If $a_j, b_j \in \mathbb{R}$, we have additionally $P(a_j) = P(b_j) = 0$, since $a_j, b_j \in \partial\{P > 0\}$. This entails that $P(\xi) = 0$ for all $\xi \in (a_j, b_j)$, which is absurd. Hence \mathcal{J} is a singleton and there exists $\xi_0 \in \mathbb{R}$ such that $\{P > 0\} = (-\infty, \xi_0)$. Furthermore, since $P(\xi_0) = 0$, we find that

$$P(\xi) = 1 - e^{\xi - \xi_0} \quad \forall \xi < \xi_0. \quad (\text{III.2.5})$$

Let us now consider the free-phase, i.e. the set $\{P = 0\} = [\xi_0, +\infty[$. In (the interior of) this interval, the equation becomes

$$cN' = -N, \quad \forall \xi > \xi_0.$$

The solution of the above linear equation is of the form

$$N(\xi) = C \exp\left(-\frac{\xi - \xi_0}{c}\right).$$

We infer that in $(\xi_0, +\infty)$, $J = cC \exp\left(-\frac{\xi - \xi_0}{c}\right)$. By continuity of J at $\xi = \xi_0$, we obtain

$$c - 1 = J(\xi_0^-) = J(\xi_0^+) = cC.$$

Thus $C = (c - 1)/c$, and we find that $(N, P) = (N_{HS}, P_{HS})(\cdot - \xi_0)$. □

III.2.2 Qualitative properties of travelling waves for the porous medium equation

Let us now consider traveling waves for the porous medium equation (III.1.1). We are interested in the behavior of such profiles in the limit $\gamma \rightarrow +\infty$, with a fixed velocity $c > 0$. In the following two subsections, we aim to derive qualitative and quantitative information on the profiles when $\gamma \gg 1$.

The existence of a profile N_γ solution to (III.1.4)-(III.1.5) is ensured by a former study of Gilding and Kersner [117]. More precisely, as a particular case of [117], one has the following result.

Theorem III.2.1 (Gilding & Kersner [117]). *Let $c_\gamma^* = \sqrt{\frac{\gamma}{\gamma+1}}$.*

1. *System (III.1.4)-(III.1.5) has a unique solution N_γ (up to a shift) for every $c \geq c_\gamma^*$ and no solution for $c < c_\gamma^*$.*
2. *When $c = c_\gamma^*$, N_γ is a sharp front, i.e. the support of N_γ is bounded above, and, modulo translation,*

$$N_\gamma(\xi) = \begin{cases} (1 - \exp(c\xi))^{1/\gamma} & \text{for } \xi < 0, \\ 0 & \text{for } \xi \geq 0. \end{cases}$$

3. *When $c > c_\gamma^*$, N_γ is positive, strictly monotonic and satisfies*

$$(\ln(1 - N_\gamma))'(\xi) \rightarrow \sqrt{1 + \frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma} = \frac{1}{\sqrt{1 + \frac{c^2}{4\gamma^2} + \frac{c}{2\gamma}}} \quad \text{as } \xi \rightarrow -\infty, \quad (\text{III.2.6})$$

and

$$(\ln(N_\gamma))'(\xi) \rightarrow -\frac{1}{c}, \quad \text{as } \xi \rightarrow +\infty.$$

The above theorem guarantees the existence (and the uniqueness up to a shift) of a TW N_γ for all $c \geq c_\gamma^* = \sqrt{\frac{\gamma}{\gamma+1}}$; the smoothness of N_γ when $c > c_\gamma^*$; the monotonically decreasing behavior of N_γ and its boundness on \mathbb{R} . Notice that the sharp front with minimal speed $c = c_\gamma^*$ it is only Hölder continuous with exponent $1/\gamma$ at $\xi = 0$. The fact of $N_\gamma^{\gamma+1}$ being continuously differentiable in the whole domain means this traveling wave is a weak solution in the usual sense, while from the physics perspective, it indicates the presence of continuous flux.

Theorem III.2.1 is adapted from Theorem 1 in [117], and therefore, we refer to this work for detailed proof.

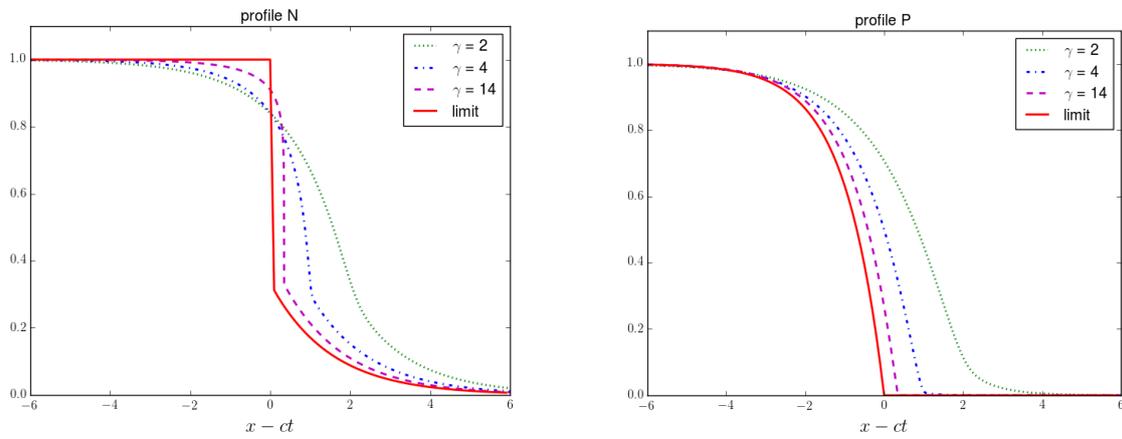


Figure III.1: Density and pressure profiles for finite values of γ and limit profiles, $c = 1.5$.

From now on, we pick a velocity $c > 1$ independent of γ , so that $c > c_\gamma^*$ ¹. We also fix the shift in N_γ by imposing

$$N_\gamma(0) = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma}}. \quad (\text{III.2.7})$$

The goal of this subsection is to prove the following result:

Proposition III.2.1. *Let $c > 1$ and let (N_γ, P_γ) , $P_\gamma := p_\gamma(N_\gamma)$, be the unique bounded weak solution to (III.1.4) satisfying (III.2.7). Let $(N_{HS}, P_{HS}) \in L^\infty \cap W^{1,\infty}(\mathbb{R})$ be the reference traveling wave solution moving with speed c of the Hele-Shaw system, see (III.2.1).*

1. *The following convergence properties hold:*

- *Weak-star convergence:*

$$N_\gamma \rightharpoonup N_{HS} \quad \text{in } w^* - L^\infty, \quad P_\gamma \rightharpoonup P_{HS} \quad \text{in } w^* - W^{1,\infty};$$

¹All the results of this chapter remain true with little or no modification if the velocity c_γ depends on γ in such a way that $c_\gamma \rightarrow \bar{c}$ with $\bar{c} > 1$. However for the sake of readability we have chosen $c_\gamma \equiv c > 1$.

- for any compact set $K \subset \mathbb{R}$

$$P_\gamma \rightarrow P_{HS} \quad \text{in } C(K);$$

- $N_\gamma \rightarrow 1$ uniformly on \mathbb{R}_- and $P'_\gamma \rightarrow P'_{HS}$ uniformly in $\mathcal{C}(]-\infty, 0])$.

2. Pointwise bounds for P_γ on \mathbb{R}_- : setting $\lambda = (-c + \sqrt{c^2 + 4})/2$, we have,

$$1 - \left(1 - \frac{1}{\gamma}\right) e^{\lambda \xi} \leq P_\gamma \leq 1 - \left(1 - \frac{1}{\gamma}\right) e^\xi, \quad \forall \xi \leq 0; \quad (\text{III.2.8})$$

The rest of this subsection is devoted to the proof of Proposition III.2.1.

L^∞ **bounds.** From the maximum packing constraint, we know that $0 \leq P_\gamma \leq p_M = 1$. Then, standard comparison provides

$$0 \leq N_\gamma \leq \lim_{\gamma \rightarrow +\infty} (p_M)^{1/\gamma} = 1.$$

From Theorem III.2.1, we know that $N'_\gamma \leq 0$, which combined with the definition of the stiff pressure yields $P'_\gamma \leq 0$.

Therefore there exist $(N, P) \in L^\infty \times L^\infty(\mathbb{R})$ such that up to the extraction of a subsequence, $N_\gamma \rightharpoonup N$, $P_\gamma \rightharpoonup P$ in $w^* - L^\infty(\mathbb{R})$. Furthermore, N, P are non-increasing. The choice of shift (III.2.7) implies that $N_\gamma(0) \rightarrow 1$, $P_\gamma(0) \rightarrow 0$. Hence, since N_γ is non-increasing N_γ converges uniformly towards 1 on $]-\infty, 0]$, and P_γ converges uniformly towards zero on $[0, +\infty[$. It follows that $N(\xi) = 1$ for $\xi < 0$ and $P(\xi) = 0$ for $\xi > 0$.

Strong convergence of P_γ and J_γ . Define the flux $J_\gamma := cN_\gamma + \gamma N'_\gamma N_\gamma^\gamma = cN_\gamma + N_\gamma P'_\gamma$. We observe that equation (III.1.4) can be written as

$$J'_\gamma = -N_\gamma(1 - N_\gamma^\gamma),$$

so that J_γ is decreasing on \mathbb{R} . Combining the latter with the L^∞ bounds on N_γ yields

$$\begin{aligned} -1 &\leq J'_\gamma \leq 0, \\ 0 &= J_\gamma(+\infty) \leq J_\gamma \leq J_\gamma(-\infty) = c. \end{aligned} \quad (\text{III.2.9})$$

In particular, $cN_\gamma + N_\gamma P'_\gamma \geq 0$. Since we already know that P_γ is non-increasing, it follows that

$$-c \leq P'_\gamma \leq 0, \quad 0 \leq P_\gamma \leq 1. \quad (\text{III.2.10})$$

From inequality (III.2.9) (resp. (III.2.10)) and Ascoli's theorem, J_γ (resp. P_γ) converges strongly, up to a subsequence, in $\mathcal{C}(K)$ for any compact set $K \subset \mathbb{R}$. Note also that $P'_\gamma \xrightarrow{*} P'$ in $L^\infty(\mathbb{R})$; since N_γ converges uniformly towards 1 on \mathbb{R}_- , we find that $J = cN + NP'$ on $]-\infty, 0]$.

The exact same cannot be done with N_γ . Indeed, from (III.1.4) and (III.2.9), we can deduce the following bounds for N_γ

$$-c \frac{N_\gamma}{N_\gamma^\gamma \gamma} \leq N'_\gamma \leq \frac{c(1 - N_\gamma)}{N_\gamma^\gamma \gamma}. \quad (\text{III.2.11})$$

Note that obtaining an L^∞ bound implies controlling $N_\gamma^{1-\gamma}\gamma^{-1}$ in L^∞ over any compact on \mathbb{R} when $\gamma \rightarrow +\infty$. This is impossible as $N_\gamma \in (0, 1)$. In fact, we show in what follows that N is discontinuous in $\xi = 0$.

Passing to the limit in equation (III.1.4). We can write the diffusion term as

$$(N_\gamma P'_\gamma)' = (\gamma N'_\gamma N_\gamma) = \frac{\gamma}{\gamma+1} (N_\gamma^{\gamma+1})'' = \frac{\gamma}{\gamma+1} (P_\gamma N_\gamma)'.$$

Since P_γ converges strongly in $\mathcal{C}(K)$ for all compact set $K \subset \mathbb{R}$, while N_γ converges weakly-* in $L^\infty(\mathbb{R})$, we can pass to the (weak) limit in equation (III.1.4).

We obtain that (N, P) satisfies the following equation in the sense of distributions

$$-cN' - (NP)'' = N(1 - P). \quad (\text{III.2.12})$$

The same argument also shows that $J = cN + (NP)'$ on \mathbb{R} .

Limit in the free-phase ($\xi > 0$). We recall that $P = 0$ in \mathbb{R}_+ . Hence, in $(0, +\infty)$, equation (III.2.12) becomes

$$-cN' = N.$$

We recognize the ODE satisfied by N_{HS} in the free-phase in the Hele-Shaw system. It follows that

$$N(\xi) = C \exp\left(-\frac{\xi}{c}\right) \quad \forall \xi > 0,$$

for some $C > 0$.

Limit in the congested phase ($\xi < 0$). We recall that $N = 1$ on $] -\infty, 0[$. Inserting this information into (III.2.12), the following elliptic equation (complementarity equation) is obtained

$$P'' + (1 - P) = 0, \quad \text{in } \mathcal{D}'((-\infty, 0)). \quad (\text{III.2.13})$$

From $P(0) = 0$ (recall that P is continuous), it follows that $P(\xi) = 1 - e^\xi$ for $\xi \in \mathbb{R}_-$.

(N,P) satisfies (III.2.3). We know that $P = 0$ on $[0, +\infty)$ and $N = 1$ on \mathbb{R}_- ; hence, $P(1 - N) = 0$ on \mathbb{R} as in (III.2.3).

Jump relation at $\xi = 0$. We recall that the flux $J = cN + (NP)'$ is continuous on \mathbb{R} , and in particular at $\xi = 0$. Thus,

$$\lim_{\xi \rightarrow 0^+} N(\xi) = 1 - \frac{1}{c}. \quad (\text{III.2.14})$$

Gathering all the information, we find that $(N, P) = (N_{HS}, P_{HS})$. Furthermore, since the limit is uniquely identified, we deduce that the whole sequence (N_γ, P_γ) converges (in the sense given above).

Sub- and super-solution for P_γ on \mathbb{R}_- . Using (III.2.10), it follows that

$$-P''_\gamma N_\gamma = N_\gamma(1 - P_\gamma) + (c + P'_\gamma)N'_\gamma \leq N_\gamma(1 - P_\gamma),$$

whence

$$-P''_\gamma \leq 1 - P_\gamma \quad \text{on } \mathbb{R}.$$

Now, let $\xi_1 \in \mathbb{R}$ be arbitrary, and let $P_1 := P_\gamma(\xi_1)$. We have for $P_+ := 1 - (1 - P_1)e^{\xi - \xi_1}$ that $-P_+'' = 1 - P_+$. Furthermore,

$$-(P_\gamma - P_+)'' \leq 1 - (P_\gamma - P_+) \text{ on } (-\infty, \xi^*).$$

It follows from the maximum principle that $P_\gamma \leq P_+$ for $\xi \leq \xi^*$. In particular, taking $\xi_1 = 0$ and $P_1 = 1/\gamma$,

$$0 \leq P_\gamma(\xi) \leq 1 - \left(1 - \frac{1}{\gamma}\right) e^\xi \quad \forall \xi \leq 0. \quad (\text{III.2.15})$$

In a similar fashion, recalling that $\gamma P_\gamma \geq 1$ on \mathbb{R}_- and $P_\gamma' \leq 0$, we have

$$-P_\gamma'' = 1 - P_\gamma + \frac{cP_\gamma'}{\gamma P_\gamma} + \frac{(P_\gamma')^2}{\gamma P_\gamma} \geq 1 - P_\gamma + cP_\gamma'.$$

Arguing as before, we define $P_-(\xi) = 1 - \left(1 - \frac{1}{\gamma}\right) e^{\lambda \xi}$, where λ is the positive root of $\lambda^2 + c\lambda - 1 = 0$ (i.e. $\lambda = (-c + \sqrt{c^2 + 4})/2$). By definition of λ , P_- satisfies

$$-P_-'' = 1 - P_- + cP_-', \quad \lim_{\xi \rightarrow -\infty} P_-(\xi) = 1, \quad P_-(0) = \frac{1}{\gamma}.$$

We infer that

$$1 - \left(1 - \frac{1}{\gamma}\right) e^{\lambda \xi} \leq P_\gamma(\xi) \quad \forall \xi \leq 0. \quad (\text{III.2.16})$$

Uniform convergence of the flux and of P_γ' on \mathbb{R}_- .

We recall that $J_\gamma' = -N_\gamma(1 - P_\gamma)$. The pointwise bounds on P_γ imply that

$$|J_\gamma'| \leq e^{\lambda \xi} \quad \forall \xi \leq 0, \quad \forall \gamma > 0.$$

It follows immediately that J_γ converges towards J_{HS} uniformly in $\mathcal{C}(\mathbb{R}_-)$. Since

$$P_\gamma' = \frac{J_\gamma}{N_\gamma} - c,$$

we infer that P_γ' also converges uniformly towards P_{HS}' in $\mathcal{C}(\mathbb{R}_-)$. This concludes the proof of Proposition III.2.1. \square

III.2.3 Phase portrait of N_γ and further consequences

In this subsection, we derive other properties of the family $(N_\gamma)_{\gamma > 0}$, which will be useful in our stability analysis. These properties rely crucially on the analysis of the phase portrait of N_γ .

In order to plot the phase portrait of N_γ , we use the results of [97], together with the following remark: using equation (III.1.4), we have

$$\begin{aligned} \frac{dN_\gamma'}{dN_\gamma} &= \frac{dN_\gamma'}{d\xi} \frac{d\xi}{dN_\gamma} \\ &= -\frac{1}{\gamma N_\gamma^\gamma N_\gamma'} \left[cN_\gamma' + \gamma^2 (N_\gamma')^2 N_\gamma^{\gamma-1} + N_\gamma(1 - N_\gamma^\gamma) \right]. \end{aligned}$$

Hence dN'_γ/dN_γ vanishes if and only if $N_\gamma^\gamma(1-N_\gamma^\gamma) \leq c^2/(4\gamma^2)$ and $N'_\gamma \in \{Q_-(N_\gamma), Q_+(N_\gamma)\}$, where

$$Q_\pm(N_\gamma) = \frac{1}{2\gamma^2 N_\gamma^{\gamma-1}} \left(-c \pm \sqrt{c^2 - 4\gamma^2 N_\gamma^\gamma (1 - N_\gamma^\gamma)} \right). \quad (\text{III.2.17})$$

Note that the curves $\Gamma_\pm = \{(N, Q_\pm(N)), N \in (0, 1)\}$ each consist of two branches, for $N \in (0, N_1)$ and $N \in (N_2, 1)$. The points N_i are the roots of the discriminant, i.e. $N_i^\gamma(1-N_i^\gamma) = c^2/(4\gamma^2)$. The curves Γ_+ and Γ_- intersect at $N = N_1$ and at $N = N_2$. A straightforward analysis shows that

$$N_1 = 1 - \frac{2 \ln \gamma}{\gamma} + o\left(\frac{\ln \gamma}{\gamma}\right), \quad N_2 = 1 - \frac{c^2}{4\gamma^3} + o\left(\frac{1}{\gamma^3}\right),$$

with

$$Q_\pm(N_1) \sim -\frac{2}{c}, \quad Q_\pm(N_2) \sim -\frac{c}{2\gamma^2}.$$

Furthermore, $Q_+(N) \sim -\frac{N}{c}$ for $N \ll 1$, while $Q_-(N) \rightarrow -\infty$ as $N \rightarrow 0$, and $Q_+(N) \sim -\gamma(1-N)/c$ for $1-N \ll 1$, while $Q_-(1) = -c/\gamma^2$.

Note also that with the normalisation of the previous section, i.e. $N_\gamma(0) = \gamma^{-1/\gamma}$, we have $N_\gamma(0) \in [N_1, N_2]$.

Now, let us denote by \mathcal{T} (resp. \mathcal{S}) the interior region between the curves Γ_- and Γ_+ for $0 < N < N_1$ (resp. $N_2 < N < 1$). We also denote by Γ the curve (N_γ, N'_γ) , which we orientate in the direction of growing N_γ . We make the following observations:

- (i) For all $N_\gamma \in (N_1, N_2)$, $dN'_\gamma/dN_\gamma \geq 0$;
- (ii) For all $N_\gamma \in (0, N_1)$ (resp. $N_\gamma \in (N_2, 1)$)
 $dN'_\gamma/dN_\gamma < 0$ iff $(N_\gamma, N'_\gamma) \in \mathcal{T}$ (resp. $(N_\gamma, N'_\gamma) \in \mathcal{S}$);
- (iii) If Γ crosses one of the curves Γ_\pm , then $dN'_\gamma/dN_\gamma = 0$ at the crossing point and therefore the tangent to Γ at the crossing point is horizontal;
- (iv) $\frac{dQ_\pm}{dN} \geq 0$ for all $N \in (N_2, 1)$;
- (v) $\frac{dQ_\pm}{dN} \leq 0$ for all $N \in (0, N_1)$;
- (vi) When $\xi \rightarrow -\infty$, we have $N_\gamma(\xi) \rightarrow 1$, and $N'_\gamma(\xi) \sim -\left(\sqrt{1 + \frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma}\right)(1 - N_\gamma(\xi))$.

The proof of all items is easy and left to the reader, except for (v), which we prove below. Note that (vi) is a consequence of (III.2.6). It follows from (vi) that for N_γ in a neighborhood of 1 (the size of which depends on γ), the curve Γ is above Γ_+ . Furthermore, (iii) and (iv) imply that if the curve Γ intersects the region \mathcal{S} , then it cannot exit \mathcal{S} . It follows that Γ lies strictly above Γ_+ for all $N \in (N_2, 0)$. Consequently, for all $N_\gamma \in (N_1, 1)$, $dN'_\gamma/dN_\gamma \geq 0$.

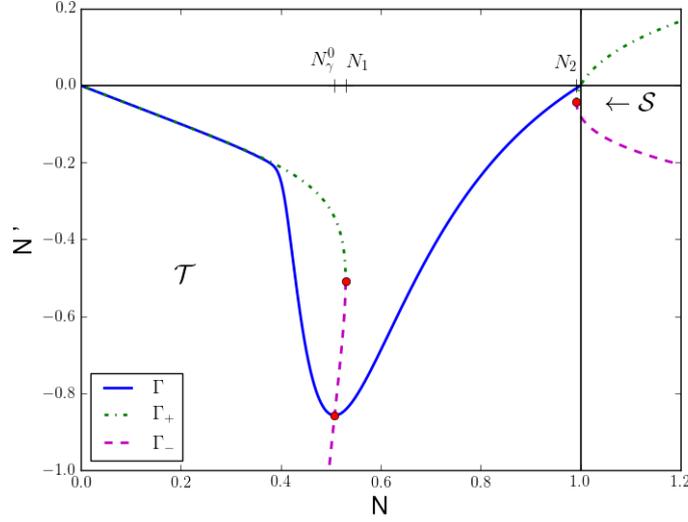


Figure III.1: Trajectory Γ in the phase plane (N, N') , $c = 2$, $\gamma = 5$.

Let us now prove that $dQ_+/dN \leq 0$ for all $N \in (0, N_1)$ (the inequality for Q_- is easier and left to the reader). We have, setting $P = N^\gamma$,

$$\begin{aligned}
 \frac{dQ_+}{dN} &= \frac{d}{dN} \left[-\frac{c}{2\gamma^2 N^{\gamma-1}} \left(1 - \sqrt{1 - \frac{4\gamma^2}{c^2} N^\gamma (1 - N^\gamma)} \right) \right] \\
 &= -\frac{c}{2\gamma^2 N^{\gamma-1}} \left[-\frac{\gamma-1}{N} \left(1 - \sqrt{1 - \frac{4\gamma^2}{c^2} N^\gamma (1 - N^\gamma)} \right) + \frac{2\gamma^3}{c^2} \frac{N^{\gamma-1} (1 - 2N^\gamma)}{\sqrt{1 - \frac{4\gamma^2}{c^2} N^\gamma (1 - N^\gamma)}} \right] \\
 &= -\frac{c}{2\gamma^2 N^\gamma} \left[-\frac{4\gamma^2 (\gamma-1) P (1-P)}{c^2 (1 + \sqrt{1 - \frac{4\gamma^2}{c^2} P (1-P)})} + \frac{2\gamma^3 P (1-2P)}{c^2 \sqrt{1 - \frac{4\gamma^2}{c^2} P (1-P)}} \right] \\
 &= -\frac{\gamma(1-2P) - (\gamma-2+2P)\sqrt{1 - \frac{4\gamma^2}{c^2} P (1-P)}}{c(1 + \sqrt{1 - \frac{4\gamma^2}{c^2} P (1-P)})\sqrt{1 - \frac{4\gamma^2}{c^2} P (1-P)}}.
 \end{aligned}$$

Note that for $0 < N < N_1$, $P = O(1/\gamma^2)$. In this regime, it can be easily checked that the numerator of the right-hand side is positive, and therefore $dQ_+/dN < 0$ for all $N \in (0, N_1)$. This completes the proof of (v).

We deduce that for $N \in (0, N_1)$, the curve Γ can cross Γ_+ at most once, as Γ exits the region \mathcal{T} . This completes the proof of (v). Now, let us argue by contradiction and assume that there exists $N_\gamma^0 \in (0, N_1)$ such that $(N_\gamma^0, (N_\gamma^0)') \in \Gamma$ lies above Γ_+ . Then there are two possibilities:

- either (N_γ, N_γ') is above Γ_+ for all $N_\gamma \in (0, N_1)$. In that case, $dN_\gamma'/dN_\gamma \geq 0$ for all $N_\gamma \in (0, N_1)$. Since $(0, 0) \in \Gamma$ and $N_\gamma' \leq 0$, it follows that $N_\gamma' = 0$ for all $N_\gamma \in (0, N_1)$, which contradicts Theorem III.2.1.

- or there exists $N_\gamma^1 \in (0, N_\gamma^0)$ such that $(N_\gamma^1, (N_\gamma^1)') \in \Gamma \cap \mathcal{T}$. In that case, since Γ and Γ_+ intersect at most once, there exists $N_3 \in (0, N_1)$ such that for all $N_\gamma \in (0, N_3)$, $(N_\gamma, N_\gamma') \in \Gamma \cap \mathcal{T}$ and for $N_\gamma \in (N_3, N_1)$, (N_γ, N_γ') is above Γ_+ . Hence N_γ' reaches a minimum for $N_\gamma = N_3$, and the value of this minimum is $Q_+(N_3) \geq Q_+(N_1) \sim -2/c$. Thus N_γ' is bounded in L^∞ . Using Ascoli's theorem, we infer that N_γ converges uniformly on $\mathcal{C}(K)$ for any compact set $K \subset \mathbb{R}$ as $\gamma \rightarrow \infty$. Since N_{HS} is discontinuous at $\xi = 0$, we have reached a contradiction.

We conclude that (N_γ, N_γ') remains below Γ_+ for all $N_\gamma \in (0, N_1)$, and therefore Γ does not cross Γ_+ . Using once again the fact that $\min N_\gamma'$ must blow up as $\gamma \rightarrow \infty$, we infer that Γ and Γ_- intersect exactly once, at a point where $N_\gamma = N_\gamma^0 \in (0, N_1)$, and N_γ^0 is such that $Q_-(N_\gamma^0) \rightarrow -\infty$ as $\gamma \rightarrow +\infty$. For all $N_\gamma \in (0, N_\gamma^0)$, $dN_\gamma'/dN_\gamma \leq 0$, and for $N_\gamma \in (N_\gamma^0, 1)$, $dN_\gamma'/dN_\gamma \geq 0$. Thus we obtain the phase portrait drawn in Figure III.1.

Let us now go back to the analysis of $\xi \in \mathbb{R} \mapsto N_\gamma(\xi)$. There exists a unique $\xi_\gamma^0 \in \mathbb{R}$ such that $N_\gamma(\xi_\gamma^0) = N_\gamma^0$. Note that dN_γ'/dN_γ and N_γ'' have opposite signs. Hence, N_γ is concave on $(-\infty, \xi_\gamma^0)$ and convex on $(\xi_\gamma^0, +\infty)$. We are now ready to prove the following Lemma:

Lemma III.2.2. *We normalize the function N_γ so that $N_\gamma(0) = \gamma^{-1/\gamma}$. We have the following properties:*

- $\xi_\gamma^0 > 0$ and $\lim_{\gamma \rightarrow \infty} \xi_\gamma^0 = 0$;
- $\sup_{\gamma > 0} \sup_{\xi < 0} |N_\gamma'(\xi)| < +\infty$ and $\|N_\gamma'\|_{L^\infty(\mathbb{R})} = -Q_-(N_\gamma^0) \rightarrow +\infty$ as $\gamma \rightarrow \infty$;
- $\lim_{\gamma \rightarrow \infty} N_\gamma^0 = 1 - c^{-1}$;
- For γ large enough, for all $\xi \geq \xi_\gamma^0$,

$$0 \leq N_\gamma(\xi) \leq N_\gamma^0 \exp\left(-\frac{1}{2c}(\xi - \xi_0)\right);$$

- $P_\gamma' \rightarrow P_{HS}'$ and $N_\gamma \rightarrow N_{HS}$ in $L_{loc}^p(\mathbb{R})$ for all $p \in [1, +\infty[$;
- Let $\xi_\gamma^* > \xi_\gamma^0$ such that $N_\gamma'(\xi_\gamma^*) = -\frac{1}{c}\left(1 - \frac{1}{c}\right)$. Then $\xi_\gamma^* \rightarrow 0$ and $N_\gamma(\xi_\gamma^*) \rightarrow 1 - c^{-1}$ as $\gamma \rightarrow \infty$.

Proof. First step: Upper-bound on N_γ^0 and on ξ_γ^0 .

The analysis of the phase portrait entails immediately that $\|N_\gamma'\|_{L^\infty(\mathbb{R})} = -Q_-(N_\gamma^0)$. As recalled above, if $Q_-(N_\gamma^0)$ remains bounded, then N_γ converges strongly in $\mathcal{C}(K)$ for any compact set K , which is absurd since N_{HS} is discontinuous. Hence $Q_-(N_\gamma^0)$ must blow up. Since

$$-\frac{c}{\gamma^2(N_\gamma^0)^{\gamma-1}} \leq Q_-(N_\gamma^0) \leq -\frac{c}{2\gamma^2(N_\gamma^0)^{\gamma-1}},$$

we deduce that $(N_\gamma^0)^\gamma = o(\gamma^{-2}) \ll \gamma^{-1} = N_\gamma(0)^\gamma$. Thus $\xi_\gamma^0 > 0$.

Since the flux J_γ is decreasing on \mathbb{R} , it follows that $J_\gamma(\xi_\gamma^0) \leq J_\gamma(0)$. Now

$$J_\gamma(\xi_\gamma^0) = cN_\gamma^0 + \gamma(N_\gamma^0)^\gamma Q_-(N_\gamma^0) = \left(c + O\left(\frac{1}{\gamma}\right)\right) N_\gamma^0, \quad (\text{III.2.18})$$

and

$$J_\gamma(0) = \gamma^{-1/\gamma} \left(c + P'_\gamma(0) \right).$$

Thanks to the sub- and super-solutions for P_γ on \mathbb{R}_- , we know that for all $\xi < 0$,

$$\left(1 - \frac{1}{\gamma} \right) (1 - e^{\lambda\xi}) \leq P_\gamma - P_\gamma(0) \leq \left(1 - \frac{1}{\gamma} \right) (1 - e^\xi),$$

where $\lambda = (\sqrt{c^2 + 4} - c)/2$. Hence $P'_\gamma(0) \in [-(1 - \gamma^{-1}), (1 - \gamma^{-1})\lambda]$. We deduce that $J(0) \leq c - (1 - \gamma^{-1})\lambda$, and therefore

$$N_\gamma^0 \leq \left(c + O\left(\frac{1}{\gamma}\right) \right)^{-1} \left(c - \left(1 - \frac{1}{\gamma} \right) \lambda \right) \leq 1 - \frac{\lambda}{c} + O\left(\frac{1}{\gamma}\right). \quad (\text{III.2.19})$$

Hence $\limsup_{\gamma \rightarrow \infty} N_\gamma^0 \leq (c - \lambda)/c < 1$.

The bound on $P'_\gamma(0)$ also implies the boundedness of N'_γ on \mathbb{R}_- . Indeed, since $\xi_\gamma^0 > 0$, N'_γ is decreasing and negative on \mathbb{R}_- , and

$$\sup_{\xi < 0} |N'_\gamma(\xi)| = |N'_\gamma(0)| = -\frac{P'_\gamma(0)}{\gamma N_\gamma(0)^{\gamma-1}} = O(1).$$

Let us now address the upper-bound on ξ_γ^0 . We recall that N_γ is concave on $(-\infty, \xi_\gamma^0)$. Consequently, for all $\xi \in (0, \xi_\gamma^0)$,

$$0 \leq N_\gamma(\xi) \leq N_\gamma(0) + N'_\gamma(0)\xi.$$

In particular, taking $\xi = \xi_\gamma^0$, we deduce that

$$\xi_\gamma^0 \leq -\frac{N_\gamma(0)}{N'_\gamma(0)} = -\frac{\gamma P_\gamma(0)}{P'_\gamma(0)} = -\frac{1}{P'_\gamma(0)},$$

since $P_\gamma(0) = \gamma^{-1}$ by choice of our normalization. We deduce in particular that

$$0 \leq \xi_\gamma^0 \leq \frac{1}{\lambda(1 - \gamma^{-1})}.$$

Second step: Super-solution for N_γ on $(\xi_\gamma^0, +\infty)$.

We recall that N_γ is convex on $(\xi_\gamma^0, +\infty)$. As a consequence, using the equation on N_γ , we have

$$-cN'_\gamma = N_\gamma(1 - N_\gamma^\gamma) + \gamma N_\gamma'' N_\gamma^\gamma + \gamma^2 (N'_\gamma)^2 N_\gamma^{\gamma-1} \geq N_\gamma(1 - (N_\gamma^0)^\gamma) \quad \forall \xi \in (\xi_\gamma^0, +\infty). \quad (\text{III.2.20})$$

The Gronwall Lemma then implies that

$$N_\gamma(\xi) \leq N_\gamma^0 \exp\left(-\frac{1 - (N_\gamma^0)^\gamma}{c}(\xi - \xi_\gamma^0)\right) \quad \forall \xi \geq \xi_\gamma^0. \quad (\text{III.2.21})$$

Recalling (III.2.19), we deduce that for γ large enough, for all $\xi \in (\xi_\gamma^0, +\infty)$,

$$N_\gamma(\xi) \leq N_\gamma^0 \exp\left(-\frac{1}{2c}(\xi - \xi_\gamma^0)\right). \quad (\text{III.2.22})$$

Third step: Strong convergence of P'_γ and N_γ .

We start with an L^2 bound for P'_γ . From (III.1.4), P_γ is solution to

$$-c_\gamma P'_\gamma - \gamma P_\gamma P''_\gamma - (P'_\gamma)^2 = \gamma P_\gamma(1 - P_\gamma). \quad (\text{III.2.23})$$

Integrating Equation (III.2.23) over \mathbb{R} gives

$$(\gamma - 1) \int_{\mathbb{R}} |P'_\gamma(\xi)|^2 d\xi = -c_\gamma + \gamma \int_{\mathbb{R}} P_\gamma(1 - P_\gamma) d\xi.$$

Hence, we get the following inequality

$$\|P'_\gamma\|_{L^2(\mathbb{R})}^2 \leq \frac{\gamma}{\gamma - 1} \left(\|P_\gamma\|_{L^1(\mathbb{R}_+)} + \|1 - P_\gamma\|_{L^1(\mathbb{R}_-)} \right).$$

The right-hand side is uniformly bounded with respect to γ thanks to sub-solution for P_γ on \mathbb{R}_- (see Proposition III.2.1) and to the upper-bound for N_γ on $(\xi_\gamma^0, +\infty)$ (see (III.2.22)). On the interval $(0, \xi_\gamma^0)$, we simply use the fact that ξ_γ^0 is bounded and $P_\gamma \leq P_\gamma(0)$. Hence, $(P'_\gamma)_{\gamma>1}$ is bounded in $L^2(\mathbb{R})$.

We now show an additional strong convergence of $(P'_\gamma)_\gamma$ in L^2 . Going back to Equation (III.2.23) and taking into account that $(P'_\gamma)_\gamma$ is uniformly bounded in $L^2(\mathbb{R})$, we have for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} \psi P_\gamma [P''_\gamma + (1 - P_\gamma)] d\xi = -\frac{1}{\gamma} \int_{\mathbb{R}} \psi [c_\gamma P'_\gamma + (P'_\gamma)^2] d\xi \rightarrow 0 \quad \text{as } \gamma \rightarrow +\infty.$$

Hence, by integration by parts in the left-hand side:

$$\int_{\mathbb{R}} \psi (P'_\gamma)^2 d\xi + \int_{\mathbb{R}} \psi' P_\gamma P'_\gamma d\xi + \int_{\mathbb{R}} \psi P_\gamma(1 - P_\gamma) d\xi \rightarrow 0 \quad \text{as } \gamma \rightarrow +\infty.$$

From the previous bounds, it is clear that

$$\int_{\mathbb{R}} \psi' P_\gamma P'_\gamma d\xi + \int_{\mathbb{R}} \psi P_\gamma(1 - P_\gamma) d\xi \xrightarrow{\gamma \rightarrow +\infty} \int_{\mathbb{R}} \psi' P_{HS} P'_{HS} d\xi + \int_{\mathbb{R}} \psi P_{HS}(1 - P_{HS}) d\xi = - \int_{\mathbb{R}} \psi (P')^2,$$

using the complementary equation (III.2.13). Finally

$$\int_{\mathbb{R}} \psi (P'_\gamma)^2 d\xi \rightarrow \int_{\mathbb{R}} \psi (P'_{HS})^2 d\xi \quad \text{as } \gamma \rightarrow +\infty,$$

which means that $(P'_\gamma)_\gamma$ converges strongly in $L^2_{\text{loc}}(\mathbb{R})$ to P'_{HS} .

We then recall that $J_\gamma = N_\gamma(c + P'_\gamma)$. Since $(P'_\gamma)_\gamma$ converges in L^2_{loc} , there exists a subsequence (which we still denote by P'_γ) which also converges almost everywhere. Recall that $(J_\gamma)_\gamma$ converges in $\mathcal{C}(K)$ for any compact set $K \subset \mathbb{R}$. Therefore $(N_\gamma)_\gamma$ converges almost everywhere - up to a subsequence - on any set of the form $\cap_{\gamma>0} \{c + P'_\gamma \geq \delta\}$, with $\delta > 0$.

Let K be a compact set in \mathbb{R} , and let $M = \sup_K J_{HS} < c$, $m = \inf_K J_{HS} > 0$. There exists $\gamma_K > 0$ such that for $\gamma \geq \gamma_K$, $J_\gamma \in [m/2, (c + M)/2]$. Since $J_\gamma \leq c + P'_\gamma$, we deduce that $c + P'_\gamma \geq m/2$ on K for $\gamma \geq \gamma_K$. Whence $(N_\gamma)_\gamma$ converges almost everywhere on K , up to a subsequence. Since N_γ is bounded in L^∞ , Lebesgue's dominated convergence theorem

implies that $(N_\gamma)_\gamma$ converges towards N_{HS} in $L^p(K)$ for any $p \in [1, +\infty[$. Note that the limit is uniquely identified. Hence the whole sequence $(N_\gamma)_\gamma$ converges in L^p_{loc} .

Fourth step: Convergence of ξ_γ^0 and N_γ^0 .

We argue by contradiction and assume that $\limsup_{\gamma \rightarrow \infty} \xi_\gamma^0 > 0$. Then there exists $\bar{\xi} > 0$ such that $\bar{\xi} \leq \xi_\gamma^0$ for a subsequence. We recall that $N''_\gamma \leq 0$ on $(0, \bar{\xi})$. Passing to the limit in the sense of distributions along this subsequence, we obtain that $N''_{HS} \leq 0$ in $\mathcal{D}'((0, \bar{\xi}))$, which is absurd. Thus $\xi_\gamma^0 \rightarrow 0$ as $\gamma \rightarrow \infty$.

Let us now go back to (III.2.18). We recall that J_γ converges uniformly towards J_{HS} , and that $J_{HS}(0) = c - 1$. It follows that $\lim_{\gamma \rightarrow \infty} N_\gamma^0 = 1 - c^{-1}$.

Fifth step: Asymptotic behavior of ξ_γ^ and $N_\gamma(\xi_\gamma^*)$.*

First, notice that ξ_γ^* is well-defined since N'_γ is increasing on $(\xi_\gamma^0, +\infty)$, with $N'_\gamma(+\infty) = 0$ and $\lim_{\gamma \rightarrow \infty} N'_\gamma(\xi_\gamma^0) = -\infty$. Furthermore, since $N_\gamma(\xi_\gamma^*) \leq N_\gamma^0$, $\limsup_{\gamma \rightarrow \infty} N_\gamma(\xi_\gamma^*) \leq 1 - c^{-1}$.

In order to prove that $\lim_{\gamma \rightarrow \infty} \xi_\gamma^* = 0$, we argue once again by contradiction and we assume that $\limsup_{\gamma \rightarrow \infty} \xi_\gamma^* > 0$. Thus there exists $\delta > 0$ such that $\xi_\gamma^* \geq \delta$ along a subsequence. By monotony of N'_γ , we know that $N'_\gamma(\xi) \leq N'_\gamma(\xi_\gamma^*) = -c^{-1}(1 - c^{-1})$ for all $\xi \in (\xi_\gamma^0, \delta) \subset (\xi_\gamma^0, \xi_\gamma^*)$. Thus, passing to the weak limit, we find that there exists a non-empty open interval included in $(0, +\infty)$ on which $N'_{HS} \leq -c^{-1}(1 - c^{-1})$. This contradicts the explicit expression of N'_{HS} , namely $N'_{HS} = -c^{-1}(1 - c^{-1})e^{-\xi/c}$; and therefore $\lim_{\gamma \rightarrow \infty} \xi_\gamma^* = 0$.

The convergence of $N_\gamma(\xi_\gamma^*)$ towards $1 - c^{-1}$ follows from the same arguments as the one of N_γ^0 : we note that

$$J_\gamma(\xi_\gamma^*) = cN_\gamma(\xi_\gamma^*) - \gamma \frac{1}{c} \left(1 - \frac{1}{c}\right) N_\gamma(\xi_\gamma^*)^\gamma.$$

Since $\limsup_{\gamma \rightarrow \infty} N_\gamma(\xi_\gamma^*) < 1$, the second term in the right-hand side converges towards zero exponentially fast. We also recall that by uniform convergence of J_γ , $J_\gamma(\xi_\gamma^*) \rightarrow J_{HS}(0) = c - 1$ as $\gamma \rightarrow \infty$. Hence $\lim_{\gamma \rightarrow \infty} N_\gamma(\xi_\gamma^*) = 1 - c^{-1}$. □

III.2.4 Quantitative bounds for the profiles N_γ

In order to prove our quantitative stability result in Theorem III.1.2, we will need some quantitative information on the asymptotic behavior of N_γ and its derivatives (e.g., the size of $\|N'_\gamma\|_{L^\infty}$). This subsection is devoted to the proof of such bounds. More precisely, we prove the following result:

Lemma III.2.3. *There exists a constant $C > 1$, depending only on c , such that the following properties hold, for any $\gamma > 0$:*

$$\begin{aligned} \sup_{0 < |h| \leq 1} \sup_{x \in \mathbb{R}} \frac{1}{|h|} \frac{|N_\gamma(x+h) - N_\gamma(x)|}{N_\gamma(x)} + \left\| \frac{N'_\gamma}{N_\gamma} \right\|_\infty &\leq C^\gamma, \\ \sup_{0 < |h| \leq 1} \sup_{x \in \mathbb{R}} \frac{1}{|h|} \left| \frac{N_\gamma(x+h) - N_\gamma(x)}{N'_\gamma(x)} \right| &\leq C^\gamma, \quad \sup_{\xi < 0} \left| \frac{1 - P_\gamma}{P'_\gamma} \right| \leq C. \end{aligned}$$

Proof. Bound on N'_γ/N_γ in the free zone $\xi > \xi_\gamma^$.*

We set $L_\gamma := \frac{N'_\gamma}{N_\gamma} + c^{-1}$. Using the equation and the convexity of N_γ in $\xi > \xi_\gamma^*$, see (III.2.20), we have

$$-cN'_\gamma(\xi) \geq N_\gamma(\xi)(1-(N_\gamma(\xi))^\gamma) \implies L_\gamma(\xi) = \frac{N'_\gamma(\xi)}{N_\gamma(\xi)} + \frac{1}{c} \leq \frac{(N_\gamma(\xi))^\gamma}{c} \leq \frac{(N_\gamma(\xi_\gamma^*))^\gamma}{c} \quad \forall \xi \geq \xi_\gamma^*.$$

Furthermore, since $N'_\gamma(\xi_\gamma^*) = -c^{-1}(1-c^{-1})$ and $N_\gamma(\xi_\gamma^*) \rightarrow 1-c^{-1}$, we immediately infer that $L_\gamma(\xi_\gamma^*)$ vanishes as $\gamma \rightarrow \infty$. We now derive an equation for L_γ in order to obtain a lower bound on L_γ . We have, using the equation on N_γ ,

$$\begin{aligned} L'_\gamma &= \frac{N''_\gamma}{N_\gamma} - \frac{(N'_\gamma)^2}{N_\gamma^2} \\ &= -\frac{1}{\gamma N_\gamma^{\gamma+1}} \left(N_\gamma(1-N_\gamma^\gamma) + cN'_\gamma + \gamma^2(N'_\gamma)^2 N_\gamma^{\gamma-1} \right) - \frac{(N'_\gamma)^2}{N_\gamma^2} \\ &= -\frac{cL_\gamma}{\gamma N_\gamma^\gamma} + \frac{1}{\gamma} - (\gamma^2 + 1) \left(L_\gamma - \frac{1}{c} \right)^2. \end{aligned}$$

Thus L_γ satisfies the differential equation

$$L'_\gamma + \left[(\gamma^2 + 1)L_\gamma + \frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma^2 + 1)}{c} \right] L_\gamma = \frac{1}{\gamma} - \frac{\gamma^2 + 1}{c^2}.$$

Note that the coefficient $\frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma^2+1)}{c}$ is exponentially large in the free zone, and drives a strong convergence of L_γ towards zero. Thus the whole idea is to prove that the quadratic term $(\gamma^2 + 1)L_\gamma^2$ does not perturb the linear behavior. This easily follows from a bootstrap argument. Indeed, note that at $\xi = \xi_\gamma^*$, for γ large enough

$$(\gamma^2 + 1)L_\gamma + \frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma^2 + 1)}{c} > \frac{c}{2\gamma N_\gamma^\gamma}. \quad (\text{III.2.24})$$

Thus by continuity, this property remains true on a non-empty open interval on the right of ξ_γ^* . Let

$$\xi_{\max} := \sup\{\xi > \xi_\gamma^*, \text{ (III.2.24) holds on } (\xi_\gamma^*, \xi)\}.$$

Then $\xi_{\max} > \xi_\gamma^*$, and on the interval $(\xi_\gamma^*, \xi_{\max})$, we have

$$L'_\gamma + \frac{c}{2\gamma N_\gamma^\gamma} L_\gamma \geq \frac{1}{\gamma} - \frac{\gamma^2 + 1}{c^2} \geq -\frac{\gamma^2 + 1}{c^2}.$$

The Gronwall Lemma then implies that for all $\xi \in (\xi_\gamma^*, \xi_{\max})$,

$$L_\gamma(\xi) \geq L_\gamma(\xi_\gamma^*) \exp\left(-\int_{\xi_\gamma^*}^{\xi} \frac{c}{2\gamma N_\gamma^\gamma}\right) - \frac{\gamma^2 + 1}{c^2} \int_{\xi_\gamma^*}^{\xi} \exp\left(-\int_{\xi'}^{\xi} \frac{c}{2\gamma N_\gamma^\gamma}\right) d\xi'.$$

Now, we recall that for $\xi > \xi_\gamma^*$, for γ sufficiently large,

$$N_\gamma(\xi) \leq N_\gamma(\xi_\gamma^*) \leq 1 - \frac{1}{2c}.$$

Thus for all $\xi \in (\xi_\gamma^*, \xi_{\max})$,

$$\begin{aligned} L_\gamma(\xi) &\geq -|L_\gamma(\xi_\gamma^*)| \exp\left(-(\xi - \xi_\gamma^*) \frac{c}{2\gamma} \left(1 - \frac{1}{2c}\right)^{-\gamma}\right) \\ &\quad - \frac{\gamma^2 + 1}{c} \int_{\xi_\gamma^*}^{\xi} \exp\left(-(\xi - \xi') \frac{c}{2\gamma} \left(1 - \frac{1}{2c}\right)^{-\gamma}\right) d\xi' \\ &\geq -|L_\gamma(\xi_\gamma^*)| \exp\left(-(\xi - \xi_\gamma^*) \frac{c}{2\gamma} \left(1 - \frac{1}{2c}\right)^{-\gamma}\right) - \frac{2\gamma(\gamma^2 + 1)}{c^2} \left(1 - \frac{1}{2c}\right)^\gamma. \end{aligned}$$

Note that the right-hand side of the above inequality converges uniformly towards zero. In particular, for γ sufficiently large, $L_\gamma(\xi) \geq -1$ for all $\xi \in (\xi_\gamma^*, \xi_{\max})$. It follows that

$$(\gamma^2 + 1)L_\gamma + \frac{c}{\gamma N_\gamma^\gamma} - \frac{2(\gamma^2 + 1)}{c} \geq \frac{c}{\gamma N_\gamma^\gamma} - \frac{(c+2)(\gamma^2 + 1)}{c} \geq \frac{3c}{4\gamma N_\gamma^\gamma} \quad \forall \xi \in (\xi_\gamma^*, \xi_{\max}).$$

By a bootstrap argument, we deduce that $\xi_{\max} = +\infty$. This implies, in particular, that $L_\gamma \rightarrow 0$ uniformly on $(\xi_\gamma^*, +\infty)$.

Remark III.2.2. *The uniform convergence of L_γ towards zero yields the existence of sub-solutions of N_γ in the zone $\xi > \xi_\gamma^*$. Indeed, let $\delta > 0$ be arbitrary. Then for γ large enough, $L_\gamma \geq -\delta$, and therefore $\frac{N'_\gamma}{N_\gamma} \geq -(c^{-1} + \delta)$. By the Gronwall Lemma, we obtain*

$$N_\gamma(\xi) \geq N_\gamma(\xi_\gamma^*) \exp\left(-\left(\frac{1}{c} + \delta\right)(\xi - \xi_\gamma^*)\right). \quad (\text{III.2.25})$$

Bound on N'_γ/N_γ and on the first difference quotient in L^∞ .

We distinguish between $\xi < \xi_\gamma^*$ and $\xi > \xi_\gamma^*$ and we write, for γ sufficiently large,

$$\begin{aligned} \left\| \frac{N'_\gamma}{N_\gamma} \right\|_\infty &= \max\left(\sup_{\xi < \xi_\gamma^*} \frac{|N'_\gamma|}{N_\gamma}, \sup_{\xi > \xi_\gamma^*} \left|L_\gamma - \frac{1}{c}\right|\right) \\ &\leq \max\left(\frac{1}{N_\gamma(\xi_\gamma^*)} \|N'_\gamma\|_\infty, \frac{1}{c} + 1\right) \\ &\leq C|Q_-(N_\gamma^0)| \leq \left(1 - \frac{1}{2c}\right)^{-\gamma}. \end{aligned}$$

Let us now consider the difference quotient

$$\frac{1}{|h|} \frac{|N_\gamma(x+h) - N_\gamma(x)|}{N_\gamma(x)}.$$

We will need to distinguish several cases:

- Case $x < \xi_\gamma^*$: in that case, $N_\gamma(x) \geq N_\gamma(\xi_\gamma^*) \rightarrow 1 - c^{-1}$, and therefore the difference quotient is bounded by $C\|N'_\gamma\|_\infty$.
- Case $x > \xi_\gamma^*$:

- Sub-case $h > 0$: we write $N_\gamma(x+h) - N_\gamma(x) = \int_0^h N'_\gamma(x+y) dy$, and we recall that since L_γ is uniformly bounded, $|N'_\gamma| \leq CN_\gamma$ for some constant C in $(\xi_\gamma^*, +\infty)$. Using the monotony of N_γ , we deduce that the difference quotient is bounded.
- Sub-case $h < 0$ and $x+h > \xi_\gamma^*$: an argument similar to the sub-case $h > 0$ applies. In that case, we obtain, using a variant of Remark III.2.2,

$$\frac{1}{|h|} \frac{|N_\gamma(x+h) - N_\gamma(x)|}{N_\gamma(x)} \leq C \frac{N_\gamma(x+h)}{N_\gamma(x)} \leq C.$$

- Sub-case $x+h \leq \xi_\gamma^*$: in that case, note that $x = x+h-h \leq \xi_\gamma^* + 1$ since $|h| \leq 1$. Hence $N_\gamma(x) \geq N_\gamma(\xi_\gamma^* + 1)$, which is uniformly bounded from below thanks to (III.2.25). Thus the difference quotient is bounded by $C\|N'_\gamma\|_\infty$.

Gathering these results, we obtain the bounds announced in the Lemma.

Bound on $(1 - P_\gamma)/P'_\gamma$ on \mathbb{R}_- .

Let $M_\gamma := (1 - P_\gamma)/P'_\gamma$. According to Proposition III.2.1, $M_\gamma \rightarrow (1 - P_{HS})/P'_{HS} = -1$ locally uniformly on \mathbb{R}_- . So, for γ sufficiently large, $M_\gamma(\xi) \in [-3/2, -1/2]$ for all $\xi \in [-1, 0]$. Furthermore we know that

$$\frac{N'_\gamma}{P'_\gamma}(\xi) = \frac{1}{\gamma(N_\gamma(\xi))^{\gamma-1}} \rightarrow \frac{1}{\gamma}, \quad \lim_{\xi \rightarrow -\infty} \frac{1 - P_\gamma(\xi)}{1 - N_\gamma(\xi)} = \lim_{N \rightarrow 1^-} \frac{1 - N^\gamma}{1 - N} = \gamma,$$

so that, thanks to Theorem III.2.1,

$$M_\gamma(\xi) = \frac{N'_\gamma(\xi)}{P'_\gamma(\xi)} \frac{1 - N_\gamma(\xi)}{N'_\gamma(\xi)} \frac{1 - P_\gamma(\xi)}{1 - N_\gamma(\xi)} \rightarrow - \left(\sqrt{1 + \frac{c^2}{4\gamma^2}} - \frac{c}{2\gamma} \right)^{-1}, \quad \text{as } \xi \rightarrow -\infty.$$

Now, let us consider the interval $(-\infty, -1]$. There are two possibilities:

- either $M_\gamma(\xi) \in [M_\gamma(-1), M_\gamma(-\infty)]$ for all $\xi \in (-\infty, -1]$. In that case, for γ sufficiently large, $M_\gamma(\xi) \in [-3/2, -1/2]$ for all $\xi \in (-\infty, -1]$;
- or M_γ takes values outside the interval $[M_\gamma(-1), M_\gamma(-\infty)]$. In that case M_γ reaches a local extremum at some $\xi_M \in (-\infty, -1)$, and therefore $M'_\gamma(\xi_M) = 0$.

Let us compute M'_γ . Using the equation satisfied by P_γ (III.2.23), we have

$$\begin{aligned} M'_\gamma &= -1 - \frac{P''_\gamma(1 - P_\gamma)}{(P'_\gamma)^2} \\ &= -1 + \frac{1 - P_\gamma}{(P'_\gamma)^2} \left(1 - P_\gamma + \frac{cP'_\gamma}{\gamma P_\gamma} + \frac{(P'_\gamma)^2}{\gamma P_\gamma} \right) \\ &= -1 + M_\gamma^2 + c \frac{M_\gamma}{\gamma P_\gamma} + \frac{1 - P_\gamma}{\gamma P_\gamma}. \end{aligned}$$

At $\xi = \xi_M$, the right-hand side vanishes, and therefore

$$M_\gamma(\xi_M) = \frac{1}{2} \left(-\frac{c}{\gamma P_\gamma(\xi_M)} \pm \sqrt{4 + \frac{c^2}{\gamma^2 P_\gamma(\xi_M)^2} - 4 \frac{1 - P_\gamma(\xi_M)}{\gamma P_\gamma(\xi_M)}} \right).$$

Note that, thanks to (III.2.8), $P_\gamma(\xi_M) \geq P_\gamma(-1) \geq 1 - e^{-\lambda} > 0$. Hence $M_\gamma(\xi_M) = \pm 1 + O(\gamma^{-1})$. Recalling that $M_\gamma < 0$ on \mathbb{R}_- , we deduce that $M_\gamma(\xi_M) = -1 + O(\gamma^{-1})$.

Once again, for γ sufficiently large, we find that $M_\gamma(\xi) \in [-3/2, -1/2]$ for all $\xi \in (-\infty, -1]$.

Hence in all cases, we deduce that for γ sufficiently large,

$$-\frac{3}{2} \leq \frac{1 - P_\gamma}{P'_\gamma} \leq -\frac{1}{2} \quad \forall \xi \in \mathbb{R}_-. \quad (\text{III.2.26})$$

Note that these bounds (which are stronger than what is announced in the statement of the Lemma) imply in particular the following inequalities, which are easy consequences of the Gronwall Lemma: for all $\xi \leq \xi' \leq 0$, for γ large enough,

$$(1 - P_\gamma(\xi)) \exp(-2(\xi' - \xi)) \leq 1 - P_\gamma(\xi') \leq (1 - P_\gamma(\xi)) \exp\left(-\frac{2}{3}(\xi' - \xi)\right). \quad (\text{III.2.27})$$

Bound on the second difference quotient.

We now address the bound on

$$\sup_{0 < |h| \leq 1} \sup_{x \in \mathbb{R}} \frac{1}{|h|} \left| \frac{N_\gamma(x+h) - N_\gamma(x)}{N'_\gamma(x)} \right|.$$

Once again, we will need to distinguish between several zones. First, note that

$$\frac{1}{|h|} \left| \frac{N_\gamma(x+h) - N_\gamma(x)}{N'_\gamma(x)} \right| = \frac{1}{|h|} \left| \frac{N_\gamma(x+h) - N_\gamma(x)}{N_\gamma(x)} \right| \left| \frac{N_\gamma(x)}{N'_\gamma(x)} \right|.$$

Hence for $x > -2$, this difference quotient is bounded by

$$\sup_{x \in \mathbb{R}} \sup_{0 < |h| \leq 1} \frac{1}{|h|} \left| \frac{N_\gamma(x+h) - N_\gamma(x)}{N_\gamma(x)} \right| \sup_{x > -2} \frac{N_\gamma(x)}{|N'_\gamma(x)|}.$$

For $x > \xi_\gamma^*$, $N_\gamma/N'_\gamma = (L_\gamma - c^{-1})^{-1}$, and we recall that L_γ converges uniformly towards zero on $(\xi_\gamma^*, +\infty)$. Hence N_γ/N'_γ is uniformly bounded on $(\xi_\gamma^*, +\infty)$. And looking at the variations of N'_γ , we infer that

$$\sup_{x \in (-2, \xi_\gamma^*)} \frac{N_\gamma(x)}{|N'_\gamma(x)|} \leq \max\left(\frac{1}{|N'_\gamma(-2)|}, \frac{1}{|N'_\gamma(\xi_\gamma^*)|}\right) \leq C\gamma.$$

Thus

$$\sup_{0 < |h| \leq 1} \sup_{x \in (-2, +\infty)} \frac{1}{|h|} \left| \frac{N_\gamma(x+h) - N_\gamma(x)}{N'_\gamma(x)} \right| \leq \gamma C^\gamma \leq C_1^\gamma,$$

for some constant $C_1 > C$.

We now consider the interval $(-\infty, -2)$. Since $|h| \leq 1$, we have $x+h \leq -1$. Hence x and $x+h$ are in the congested zone. We write

$$\frac{1}{h} \frac{N_\gamma(x+h) - N_\gamma(x)}{N'_\gamma(x)} = \int_0^1 \frac{N'_\gamma(x+\tau h)}{N'_\gamma(x)} d\tau.$$

Recall that $N'_\gamma = \gamma^{-1} P'_\gamma N_\gamma^{-(\gamma-1)}$. Hence

$$\frac{N'_\gamma(x + \tau h)}{N'_\gamma(x)} = \frac{P'_\gamma(x + \tau h)}{P'_\gamma(x)} \frac{N_\gamma(x)^{\gamma-1}}{N_\gamma(x + h)^{\gamma-1}}.$$

Note that $N_\gamma^{\gamma-1} = P_\gamma/N_\gamma$ is uniformly bounded from above and from below on $(-\infty, -1)$. Thus we focus on the quotient $P'_\gamma(x + \tau h)/P'_\gamma(x)$, which we further decompose as

$$\frac{P'_\gamma(x + \tau h)}{P'_\gamma(x)} = \frac{P'_\gamma(x + \tau h)}{1 - P_\gamma(x + \tau h)} \frac{1 - P_\gamma(x + \tau h)}{1 - P_\gamma(x)} \frac{1 - P_\gamma(x)}{P'_\gamma(x)} = \frac{M_\gamma(x)}{M_\gamma(x + \tau h)} \frac{1 - P_\gamma(x + \tau h)}{1 - P_\gamma(x)}.$$

Using (III.2.27) and (III.2.26), we deduce that

$$\left| \frac{P'_\gamma(x + \tau h)}{P'_\gamma(x)} \right| \leq C e^{2|h|}.$$

Hence

$$\sup_{0 < |h| \leq 1} \sup_{x \leq -2} \frac{1}{|h|} \left| \frac{N_\gamma(x + h) - N_\gamma(x)}{N'_\gamma(x)} \right| \leq C.$$

□

Our nonlinear stability result will hold in weighted Sobolev spaces. The weights will depend on the function N_γ and its derivative, and therefore will have abrupt changes in the transition zone $(0, \xi_\gamma^*)$. In order to monitor precisely these changes, we introduce two additional abscissa ξ_γ^- and $\tilde{\xi}_\gamma$, which we define as follows:

Definition III.2.1 (Definition of ξ_γ^- and $\tilde{\xi}_\gamma$).

- The abscissa $\xi_\gamma^- \in \mathbb{R}$ is the unique point where

$$P_\gamma(\xi_\gamma^-) = \left(\frac{c^3}{(c-1)(\gamma+1)} \right)^{1/2}. \quad (\text{III.2.28})$$

- The abscissa $\tilde{\xi}_\gamma \in \mathbb{R}$ is the unique point such that $N_\gamma(\tilde{\xi}_\gamma) \in (0, N_\gamma^0)$ and

$$N'_\gamma(\tilde{\xi}_\gamma) = -\frac{c-1}{4\gamma^2 N_\gamma(\tilde{\xi}_\gamma)^{\gamma-1}}.$$

Remark III.2.3. • Note that ξ_γ^- is well-defined by monotony of P_γ , and $\xi_\gamma^- < 0$ since $P_\gamma(\xi_\gamma^-) > P_\gamma(0)$;

- The definition of $\tilde{\xi}_\gamma$ is a little more intricate. We recall that for all $N_\gamma \in (0, N_\gamma^0)$, $Q_-(N) < N'_\gamma < 0$, where Q_- is defined in (III.2.17) and $dN'_\gamma/dN_\gamma \leq 0$ for all $N_\gamma \in (0, N_\gamma^0)$; we refer to the analysis of the phase portrait in the previous subsection.

Now, define $\tilde{Q}(N)$ by

$$\tilde{Q}(N) := -\frac{c-1}{4\gamma^2 N^{\gamma-1}}.$$

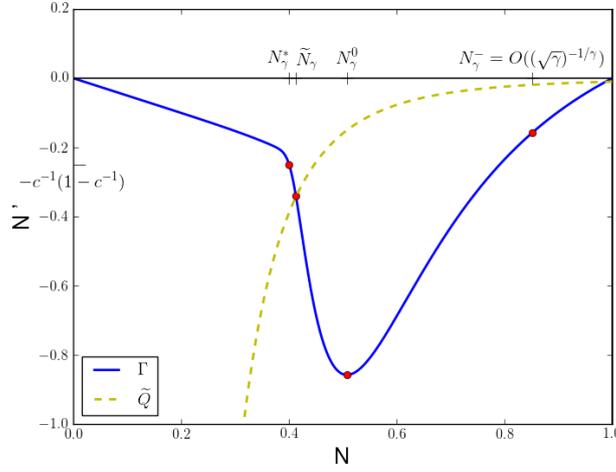


Figure III.1: Definition of the point \tilde{N}_γ in the phase plane (N, N') , $c = 2$, $\gamma = 5$.

It is clear from the definition of \tilde{Q} and Q_- that $Q_- < \tilde{Q}$ for all $N \in (0, N_\gamma^0)$, and \tilde{Q} is monotone increasing on that interval. Consequently, the curve (N_γ, N'_γ) intersects the curve $(N, \tilde{Q}(N))$ exactly once on the interval $(0, N_\gamma^0)$ (see Figure III.1). We denote the abscissa of the intersection point as \tilde{N}_γ , and $\tilde{\xi}_\gamma$ is defined implicitly as $N_\gamma(\tilde{\xi}_\gamma) = \tilde{N}_\gamma$.

Let us now give some properties of $\tilde{\xi}_\gamma$ and ξ_γ^- , which will be used in the next section:

Lemma III.2.4 (Properties of $\tilde{\xi}_\gamma$ and ξ_γ^-). *For γ large enough, the following properties hold:*

- $\xi_\gamma^- < 0 < \xi_\gamma^0 < \tilde{\xi}_\gamma < \xi_\gamma^*$. As a consequence, $\lim_{\gamma \rightarrow \infty} N_\gamma(\tilde{\xi}_\gamma) = 1 - c^{-1}$;
- $\xi_\gamma^- = O(\gamma^{-1/2})$, and $\tilde{\xi}_\gamma = O(\gamma^{-1})$;
- $P'_\gamma \leq -C\gamma^{-1}$ for all $\xi \in (\xi_\gamma^-, \tilde{\xi}_\gamma)$.

Proof. Relative positions of $\xi_\gamma^0, \tilde{\xi}_\gamma, \xi_\gamma^$.*

By definition of $\tilde{\xi}_\gamma$, $N_\gamma(\tilde{\xi}_\gamma) < N_\gamma^0$, and thus $\xi_\gamma^0 < \tilde{\xi}_\gamma$. Furthermore, we recall that N'_γ is monotone increasing on $(\xi_\gamma^0, +\infty)$, and

$$N'_\gamma(\tilde{\xi}_\gamma) = -\frac{c-1}{4\gamma^2 N_\gamma(\tilde{\xi}_\gamma)^{\gamma-1}} \leq -\frac{c-1}{4\gamma^2 (N_\gamma^0)^{\gamma-1}} \rightarrow -\infty.$$

Whence $N'_\gamma(\tilde{\xi}_\gamma) < N'_\gamma(\xi_\gamma^*)$, and therefore $\tilde{\xi}_\gamma < \xi_\gamma^*$. The limit of $N_\gamma(\tilde{\xi}_\gamma)$ follows from the monotony of N_γ and the fact that $\lim_{\gamma \rightarrow \infty} N_\gamma(\xi_\gamma^0) = \lim_{\gamma \rightarrow \infty} N_\gamma(\xi_\gamma^*) = 1 - c^{-1}$ (see Lemma III.2.2).

Size of ξ_γ^- .

First, considering the sub-solution for P_γ , we see that $\xi_\gamma^- > -1$. Using (III.2.26), we recall that P'_γ is bounded away from zero on $(-1, 0)$, for γ large enough. Thus,

$$\frac{|P_\gamma(\xi_\gamma^-) - P_\gamma(0)|}{\sup_{(\xi_\gamma^-, 0)} |P'_\gamma|} \leq |\xi_\gamma^-| \leq \frac{|P_\gamma(\xi_\gamma^-) - P_\gamma(0)|}{\inf_{(\xi_\gamma^-, 0)} |P'_\gamma|},$$

and thus

$$\frac{C^{-1}}{\sqrt{\gamma}} \leq |\xi_\gamma^-| \leq \frac{C}{\sqrt{\gamma}}.$$

Lower-bound for $|P'_\gamma|$ on $(\xi_\gamma^-, \tilde{\xi}_\gamma)$ and size of $\tilde{\xi}_\gamma$.

Let us introduce yet another intermediate point ξ_γ^{int} such that $N_\gamma(\xi_\gamma^{\text{int}}) = 1 - (2c)^{-1}$. We recall that $N_\gamma(\xi_\gamma^0) \rightarrow 1 - c^{-1}$, and therefore $\xi_\gamma^{\text{int}} \in (0, \xi_\gamma^0)$ for γ large enough. Now, for $\xi \in (\xi_\gamma^-, \xi_\gamma^{\text{int}})$, we have $N_\gamma(\xi) \in [1 - (2c)^{-1}, 1]$, and

$$P'_\gamma = \frac{J_\gamma - cN_\gamma}{N_\gamma} \leq J_\gamma - c + \frac{1}{2}.$$

We recall that $J_\gamma(\xi) \rightarrow c - 1$ uniformly on that interval. Thus $P'_\gamma \leq -C < 0$ on $(\xi_\gamma^-, \xi_\gamma^{\text{int}})$ for γ sufficiently large, for some uniform constant C .

In particular, since $P_\gamma(\xi_\gamma^{\text{int}}) = (1 - (2c)^{-1})^\gamma$ is exponentially small, it follows that

$$\xi_\gamma^{\text{int}} \leq \frac{|P_\gamma(\xi_\gamma^{\text{int}}) - P_\gamma(0)|}{\inf_{[0, \xi_\gamma^{\text{int}}]} |P'_\gamma|} \leq \frac{C}{\gamma}.$$

Let us now consider the intervals $(\xi_\gamma^{\text{int}}, \xi_\gamma^0)$ and $(\xi_\gamma^0, \tilde{\xi}_\gamma)$. Using the notations introduced in the previous subsection, it is easily checked that $N_\gamma(\xi_\gamma^{\text{int}}) \leq N_1$. As a result, using the phase portrait of N_γ (see Figure III.1), $N'_\gamma \leq Q_-(N_\gamma)$ for all $\xi \in (\xi_\gamma^{\text{int}}, \xi_\gamma^0)$. In particular,

$$\begin{aligned} P'_\gamma &= \gamma N'_\gamma(\xi) (N_\gamma(\xi))^{\gamma-1} \\ &\leq \gamma (N_\gamma(\xi))^{\gamma-1} \times \frac{1}{2\gamma^2 (N_\gamma(\xi))^{\gamma-1}} (-c - \sqrt{c^2 - 4\gamma^2 (N_\gamma(\xi))^\gamma (1 - (N_\gamma(\xi))^\gamma)}) \\ &\leq -\frac{c}{2\gamma} \quad \forall \xi \in (\xi_\gamma^{\text{int}}, \xi_\gamma^0). \end{aligned}$$

For $\xi \in (\xi_\gamma^0, \tilde{\xi}_\gamma)$, the argument is similar. On this interval, $N'_\gamma \geq Q_-(N_\gamma)$, but $N'_\gamma \leq \tilde{Q}(N_\gamma)$ by definition of $\tilde{\xi}_\gamma$. Thus

$$P'_\gamma(\xi) \leq \gamma (N_\gamma(\xi))^{\gamma-1} \times \left(-\frac{c-1}{4\gamma^2 (N_\gamma(\xi))^{\gamma-1}} \right) \leq -\frac{c-1}{4\gamma} \quad \forall \xi \in (\xi_\gamma^0, \tilde{\xi}_\gamma).$$

We obtain the desired lower bound on $|P'_\gamma|$ on $(\xi_\gamma^{\text{int}}, \tilde{\xi}_\gamma)$. It follows that

$$\tilde{\xi}_\gamma - \xi_\gamma^{\text{int}} \leq \frac{|P_\gamma(\tilde{\xi}_\gamma) - P_\gamma(\xi_\gamma^{\text{int}})|}{\inf_{(\xi_\gamma^{\text{int}}, \tilde{\xi}_\gamma)} |P'_\gamma|} \leq C\gamma \left(1 - \frac{1}{2c} \right)^\gamma = o(\gamma^{-1}).$$

Hence $\tilde{\xi}_\gamma$ and ξ_γ^{int} are exponentially close. The estimate on $\tilde{\xi}_\gamma$ follows. \square

Let us conclude this section by saying a few words about the proof of Theorem III.1.1. The sizes and signs of ξ_γ^- and $\tilde{\xi}_\gamma$ are given in Lemma III.2.4. Inequality (III.1.6) follows from the monotony of N_γ and from the definition of ξ_γ^- . Let us say few words about the inequality claimed in (III.1.7). Actually, the reader may check that the derivation of sub- and super-solutions on \mathbb{R}_- made in (III.2.15)-(III.2.16) can be easily adapted to the interval $(-\infty, \xi_\gamma^-]$, using the fact that $P_\gamma(\xi_\gamma^-) = O(\gamma^{-1/2})$ and $\gamma P_\gamma \geq C\sqrt{\gamma}$ on $(-\infty, \xi_\gamma^-]$. It follows that

$$1 - \left(1 - P_\gamma(\xi_\gamma^-)\right) e^{\mu_\gamma \xi} \leq P_\gamma(\xi) \leq 1 - \left(1 - P_\gamma(\xi_\gamma^-)\right) e^\xi,$$

where μ_γ is the positive root of $\mu^2 + \frac{C}{\sqrt{\gamma}}\mu - 1 = 0$. It is easily checked that $\mu = 1 - O(\gamma^{-1/2})$, which leads to inequality (III.1.7).

The size of $\|N'_\gamma\|_\infty$ in the intermediate region $(\xi_\gamma^-, \tilde{\xi}_\gamma)$ is an easy consequence of Lemma III.2.2, and the bounds on the pressure in that zone follow from the monotony of P_γ , the definitions of ξ_γ^- and the asymptotic behavior of $N_\gamma(\tilde{\xi}_\gamma)$ (see Lemma III.2.4).

Eventually, the lower and bounds on N_γ in the free zone follow from (III.2.25) and (III.2.22) respectively.

The convergence properties for N_γ, P_γ at the end of Theorem III.1.1 are a consequence of Lemma III.2.2 and Proposition III.2.1.

III.3 Stability of the profiles N_γ

The goal of this section is to prove that the solution of the equation

$$\partial_t n_\gamma - \gamma \partial_x (n_\gamma^\gamma \partial_x n_\gamma) = n_\gamma (1 - n_\gamma^\gamma) \quad (\text{III.3.1})$$

associated to an initial datum that lies between two shift of the profile N_γ , converges (in a sense specified below) towards N_γ as $t \rightarrow +\infty$. After a presentation of the general strategy, we enter in the details of the two main steps of the demonstration: the analysis of the linearized system and, next, the control of the nonlinear contributions. To keep the presentation as seamless as possible, we have postponed the proof of some technical lemmas to the next section.

This section contains rather technical ingredients. Therefore, in order to alleviate the notation as much as possible, *we will systematically drop the dependency with respect to γ* in the computations and proofs: N_γ will be denoted by N , n_γ will be denoted by n , etc. We only keep track of this dependency in the statement of our main result.

In the whole section, for all weights and coefficients $f(t, x)$ that only depend on $\xi = x - ct$, we denote $f(t, x) = \bar{f}(x - ct)$.

III.3.1 Overall strategy

We define here our notion of stability and convergence towards the profile N_γ . We introduce a weight

$$\bar{w}_0(\xi) = KN_\gamma^\gamma(\xi)(N'_\gamma(\xi))^2 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N_\gamma(z)} dz\right), \quad (\text{III.3.2})$$

with a normalization constant K chosen so that $\bar{w}_0(\xi_\gamma^-) = 1$. We will prove that for sufficiently small and decaying initial data,

$$\int_{\mathbb{R}} \left| \frac{n_\gamma(t, x) - N_\gamma(x - ct)}{N'_\gamma(x - ct)} \right|^2 \bar{w}_0(x - ct) dx \rightarrow 0 \quad \text{as } t \rightarrow +\infty. \quad (\text{III.3.3})$$

The result is summarized in the following theorem.

Theorem III.3.1. *There exists $\eta \in]0, 1[$ such that the following result holds. Let $\gamma > 1$ be fixed, sufficiently large. Let us assume that n_γ^0 lies between two shifts of N_γ , i.e. there exists $h > 0$ such that $n_\gamma^0(x) \in [N_\gamma(x + h), N_\gamma(x - h)]$ for all $x \in \mathbb{R}$. Let n_γ be the solution of (III.1.1) associated with n_γ^0 and*

$$u_\gamma(t, x) := \frac{n_\gamma(t, x) - N_\gamma(x - ct)}{N'_\gamma(x - ct)}.$$

Assume that

$$\int_{\mathbb{R}} |u_\gamma(0, x)|^2 \bar{w}_0(x) dx < +\infty.$$

Then there exists a constant $c_\gamma > 0$, decreasing exponentially with γ , such that if $h \leq \eta^\gamma$, the following inequalities hold

$$\begin{aligned} \int_{\mathbb{R}} |u_\gamma(t, x)|^2 \bar{w}_0(x - ct) dx &\leq e^{-c_\gamma t} \int_{\mathbb{R}} |u_\gamma(0, x)|^2 \bar{w}_0(x) dx \quad \forall t \geq 0, \\ \gamma \int_0^\infty \int_{\mathbb{R}} |\partial_x u_\gamma(t, x)|^2 N_\gamma^\gamma(x - ct) \bar{w}_0(x - ct) dx dt &\leq \int_{\mathbb{R}} |u(0, x)|^2 \bar{w}_0(x) dx. \end{aligned} \quad (\text{III.3.4})$$

Note that this statement is merely a rephrasing of Theorem III.1.2 in terms of the unknown u_γ . We emphasize that u_γ is a natural variable when linearizing equation (III.3.1) around $N_\gamma(x - ct)$. Indeed, since equation (III.3.1) has constant coefficients and since $N_\gamma(x - ct)$ is a particular solution of the equation, it is classical that $\partial_x N_\gamma(x - ct)$ is a solution of the linearized equation around $N_\gamma(x - ct)$ (and we also recall that $\partial_x N_\gamma$ does not vanish on \mathbb{R}). Moreover, $n_\gamma(t, x) - N_\gamma(x - ct)$ is also a solution of the linearized equation, up to a quadratic remainder which we will treat perturbatively. Therefore working with energies depending on u_γ is similar to deriving relative entropies for the system.

The result relies on two main estimates: a L^∞ control on $n - N$ (almost immediate, see below) and a more complicated L^2 weighted estimate on the variable u . Indeed, an easy computation (see subsection III.4.1) shows that u satisfies the equation

$$\partial_t u + b \partial_x u - a \partial_x^2 u = \frac{\gamma}{\gamma + 1} \frac{\partial_x^2 G(u)}{N'(x - ct)} - \frac{G(u)}{N'(x - ct)}, \quad (\text{III.3.5})$$

with $a = \bar{a}(x - ct)$, $b = \bar{b}(x - ct)$ and

$$\bar{a} := \gamma N^\gamma, \quad \bar{b} := -2\gamma \frac{(N^\gamma N')'}{N'} = -2\gamma^2 N^{\gamma-1} N' - 2\gamma N^\gamma \frac{N''}{N'},$$

and

$$G(u) := n^{\gamma+1} - N^{\gamma+1}(x - ct) - (\gamma + 1) N^\gamma(x - ct)(n - N(x - ct)). \quad (\text{III.3.6})$$

Let us make a few remarks before exposing the main ingredients of the proof. First, we emphasize that all unknowns and coefficients depend on γ (i.e. b, a, u, G, N). As mentioned above, we chose not to make this dependency explicit in our notation. Second, equation (III.3.5) has a structure of the type

$$\partial_t u + \mathcal{L}u = \mathcal{G}[u],$$

where \mathcal{L} is a linear operator, corresponding to the linearization of equation (III.3.1) around $n_\gamma = N_\gamma$, and $\mathcal{G}[u]$ is a quadratic operator in the sense of (III.3.8).

Quite classically, the core of our proof relies on the two following observations:

- The linear operator \mathcal{L} is coercive in some weighted H^1 space. More precisely, there exists a weight \bar{w} and a constant $\delta = \delta_\gamma > 0$ with the following property: for any $v \in \mathcal{C}_c^2(\mathbb{R})$,

$$\int_{\mathbb{R}} (\bar{b}\partial_\xi v - \bar{a}\partial_\xi^2 v)v\bar{w} \geq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a}\bar{w} + \frac{\delta}{2} \int_{\mathbb{R}} |v|^2 e^{\sqrt{\gamma}\xi} - \frac{c}{2} \int_{\mathbb{R}} |v|^2 \partial_\xi \bar{w}. \quad (\text{III.3.7})$$

Note that the last term will enter the time derivative of the energy $\int |u|^2 w$ when we perform energy estimates.

This type of coercivity property had been identified by Leyva and Plaza in [110], without the L^2 term $\int_{\mathbb{R}} |v|^2 e^{\sqrt{\gamma}\xi}$, which will play a crucial role in the energy estimates.

- The nonlinear term $\mathcal{G}[u]$ satisfies

$$|\mathcal{G}[u]| \leq C_\gamma |u| (|u| + |\partial_x u|). \quad (\text{III.3.8})$$

Hence, if $\|u\|_{L^\infty}$ is small enough, we can hope to absorb this term in the energy dissipation provided by the coercivity of \mathcal{L} .

The remainder of the section is devoted to a more rigorous statement and to the proofs of the above heuristic arguments. Concerning the smallness of the L^∞ bound, a possible strategy could be to differentiate equation (III.3.5) with respect to x and to derive uniform, high regularity bounds on u . This strategy is likely to succeed. However, it will probably come at a high technical cost. Consequently, to simplify the proof and the presentation, we chose here to take advantage of the parabolic structure of the equation and use the comparison principle (or maximum principle), which immediately implies an L^∞ bound on n and u .

Remark III.3.1. *Let us mention by anticipation that the constant δ in (III.3.7) will be small, while the constant C_γ in (III.3.8) will be very large. Whence we will need $\|u\|_{L^\infty}$ to be very small (in fact, exponentially small) to treat the quadratic term as a perturbation. This is related to the strong singularities in N'_γ which were highlighted in the previous section (recall that $\|N'_\gamma\|_{L^\infty}$ blows up exponentially, see Lemmas III.2.2 and III.2.3).*

Let us now present the main ideas of the proof.

Structure of the linearized system - weighted L^2 estimate

We start from a reference weight \bar{w}_0 , which is defined as the solution of the differential equation

$$\begin{cases} (\bar{a}\bar{w}_0)'(\xi) + (\bar{b}(\xi) - c)\bar{w}_0 = 0 & \text{for } \xi \in \mathbb{R}, \\ \bar{w}_0(\xi_0) = 1 & \text{for some } \xi_0 \in \mathbb{R}. \end{cases} \quad (\text{III.3.9})$$

This weight is identical to the one identified by Leyva and Plaza in [110, Section 3.1], although our derivation differs from theirs, see subsection III.4.3. For this weight \bar{w}_0 , we have the following

Lemma III.3.1 (Stability estimates for the linearized system). *Let u be a smooth solution to*

$$\partial_t u + b\partial_x u - a\partial_x^2 u = S, \quad (\text{III.3.10})$$

where S is a general source term. The following equality holds, with $w_0(t, x) = \bar{w}_0(x - ct)$

$$\begin{aligned} & \int_{\mathbb{R}} |u(t, x)|^2 w_0(t, x) dx + 2 \int_0^t \int_{\mathbb{R}} a(s, x) (\partial_x u(s, x))^2 w_0(s, x) dx ds \\ &= \int_{\mathbb{R}} |u^0(x)|^2 \bar{w}_0(x) dx + 2 \int_0^t \int_{\mathbb{R}} S(s, x) u(s, x) w_0(s, x) dx ds. \end{aligned} \quad (\text{III.3.11})$$

Furthermore the weight \bar{w}_0 fulfills the following properties:

Lemma III.3.2 (Asymptotic behaviors of \bar{w}_0). *The solution of (III.3.9) with $\xi_0 = \xi_\gamma^-$ is given by*

$$\bar{w}_0(\xi) = KN^\gamma(\xi)(N'(\xi))^2 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\bar{a}(z)} dz\right), \quad (\text{III.3.12})$$

where the normalization constant K is chosen so that $\bar{w}_0(\xi_\gamma^-) = 1$. We find that $K \propto \gamma^{3/2}$. Consequently \bar{w}_0 has the following asymptotic behaviors:

- as $\xi \rightarrow +\infty$, \bar{w}_0 has a double exponential growth: there exist $C_1, C_2, C > 0$ independent of γ such that for all $\xi \geq C$,

$$\exp(\exp(C_1\gamma\xi)) \leq \bar{w}_0 \leq \exp(\exp(C_2\gamma\xi)); \quad (\text{III.3.13})$$

- as $\xi \rightarrow -\infty$, \bar{w}_0 decreases exponentially to 0: there exists $C > 0$ independent of γ such that for all $\xi \leq -C$,

$$C^{-1} \frac{K}{\gamma^2} \exp\left(2\left(1 + \frac{C}{\sqrt{\gamma}}\right)\xi\right) \leq \bar{w}_0 \leq C \frac{K}{\gamma^2} \exp\left(2\left(1 - \frac{C}{\sqrt{\gamma}}\right)\xi\right). \quad (\text{III.3.14})$$

Spectral gap and Poincaré inequality

However, the sole weight w_0 is not entirely sufficient to have an exponential decay in time of the energy $\int_{\mathbb{R}} |u|^2 w_0$. Indeed, in order to prove such an exponential decay, we need a Poincaré inequality of the type

$$\int_{\mathbb{R}} |v|^2 \bar{w}_0 \leq C_\gamma \int_{\mathbb{R}} (\partial_x v)^2 \bar{a} \bar{w}_0 \quad \forall v \in \mathcal{C}_c^1(\mathbb{R}).$$

In other words, we need to prove a spectral gap inequality. To the best of our knowledge, such an inequality does not hold for the weight \bar{w}_0 . However, we are able to prove a variant of such an inequality, with an additional L^2 term in the right-hand side:

Proposition III.3.1 (Weighted Poincaré-type inequality). *Let \bar{w}_0 be reference weight which is defined as the solution of the differential equation*

$$\begin{cases} (\bar{a}\bar{w}_0)'(\xi) + (\bar{b}(\xi) - c)\bar{w}_0 = 0 & \text{for } \xi \in \mathbb{R}, \\ \bar{w}_0(\xi_0) = 1 & \text{for some } \xi_0 \in \mathbb{R}, \end{cases} \quad (\text{III.3.15})$$

and $v \in C_c^1(\mathbb{R})$.

Then there exists a constant \bar{C} independent of γ and a constant $C_\gamma \leq C^\gamma$ such that

$$\int_{-\infty}^{\xi_\gamma^-} v^2 \gamma N_\gamma^\gamma \bar{w}_0 d\xi + \int_{\xi}^{+\infty} v^2 \frac{1}{\gamma N_\gamma^\gamma} \bar{w}_0 d\xi \leq \bar{C} \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a} \bar{w}_0 d\xi + C_\gamma \int_{\mathbb{R}} v^2 e^{\sqrt{\gamma}\xi} d\xi. \quad (\text{III.3.16})$$

In particular, there exists a constant c_γ , satisfying $c_\gamma \geq \eta^\gamma$ for some $\eta \in]0, 1[$ independent of γ , such that

$$c_\gamma \int_{\mathbb{R}} v^2 \bar{w}_0 d\xi \leq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a} \bar{w}_0 d\xi + \int_{\mathbb{R}} |v|^2 \exp(\sqrt{\gamma}\xi) d\xi.$$

Remark III.3.2. • We recall that we defined ξ_γ^- so that $P_\gamma(\xi_\gamma^-) = \left(\frac{c^3}{(c-1)(\gamma+1)}\right)^{1/2}$. Consequently, in the first integral of (III.3.16), the term γN_γ^γ is bounded from below by $C\sqrt{\gamma}$.

- In a similar way, for $\xi > \tilde{\xi}$, we have $N_\gamma \leq 1 - (2c)^{-1}$, so that the term $\frac{1}{\gamma N_\gamma^\gamma}$ in the second integral in the left-hand side of (III.3.16) is exponentially large.
- We stated this result for $v \in C_c^1(\mathbb{R})$, but the result can be extended to v in suitable weighted Sobolev spaces by a classical density argument.
- Let us give a few motivations for the weight $e^{\sqrt{\gamma}\xi}$ in the right-hand side. We actually have some freedom in the choice of the coefficient of the exponential that we take equal to $\alpha_\gamma = \sqrt{\gamma}$. We could a priori take a larger coefficient α_γ with respect to γ . However α_γ must satisfy a number of conditions. First, an important feature is that the growth (resp. decay) of this weight as $\xi \rightarrow +\infty$ (resp. $\xi \rightarrow -\infty$) is lower (resp. stronger) than the one of \bar{w}_0 . Moreover, the energy dissipation provides a very good control of the energy in the zone $\xi < \xi_\gamma^-$ and $\xi > \tilde{\xi}_\gamma$, as we can see in inequality (III.3.16). The additional term $\int_{\mathbb{R}} v^2 e^{\sqrt{\gamma}\xi}$ is only needed in the transition zone $(\xi_\gamma^-, \tilde{\xi}_\gamma)$, as we shall see in the course of the proof. Our choice $\alpha_\gamma = \sqrt{\gamma}$ is actually motivated by the need to control, uniformly with respect to γ , the exponential $\exp(\alpha_\gamma \xi_\gamma^-)$ (see in particular (III.4.11)). Since $\xi_\gamma^- = O(\gamma^{-1/2})$, it leads us to set $\alpha_\gamma = \sqrt{\gamma}$.
- The proof of Proposition III.3.1 relies on the quantitative estimations of Lemma III.2.3, and will be performed in subsection III.4.4.

As a consequence, if we are able to have an additional lower-order dissipation term in the energy estimate (the term $\int |v|^2 e^{\sqrt{\gamma}\xi}$ in the right-hand side), the exponential decay of

the energy for the linearized system will follow. In order to get this extra dissipation, it turns out that it is sufficient to modulate slightly the weight \bar{w}_0 . More precisely, we define $\bar{w} = \bar{w}_0 \bar{\phi}$, where

$$\begin{aligned} \bar{\phi}(-\infty) &= 2, \\ \bar{\phi}' &= -\frac{\delta_\gamma}{\sqrt{\gamma}} \exp(\sqrt{\gamma}\xi) (\bar{a}\bar{w}_0)^{-1}, \end{aligned} \quad (\text{III.3.17})$$

and the constant $\delta_\gamma > 0$ is chosen such that $\bar{\phi}(+\infty) \geq 1$.

Lemma III.3.2 ensures that $\bar{\phi}' \in L^1(\mathbb{R})$, and therefore $\bar{\phi}$ is well-defined and monotonous. Note that since $1 \leq \bar{\phi} \leq 2$ by construction, the weights \bar{w}_0 and \bar{w} are equivalent. However, choosing \bar{w} gives us the following additional control:

Lemma III.3.3. *Under the same assumptions and notation as in Lemma III.3.1, we have*

$$\begin{aligned} \int_{\mathbb{R}} |u(t, x)|^2 w(t, x) dx + 2 \int_0^t \int_{\mathbb{R}} (\partial_x u)^2 a w + \delta_\gamma \int_0^t \int_{\mathbb{R}} |u(s, x)|^2 \exp(\sqrt{\gamma}(x - cs)) dx ds \\ = \int_{\mathbb{R}} |u^0|^2 \bar{w} dx + \int_0^t \int_{\mathbb{R}} S(s, x) u(s, x) w(s, x) dx ds, \end{aligned} \quad (\text{III.3.18})$$

with

$$\delta_\gamma = \delta_0 \gamma^{-1/2} \quad (\text{III.3.19})$$

, for some constant δ_0 independent of γ .

Furthermore, there exists $\eta_1 \in]0, 1[$ and a constant $c_\gamma \geq \eta_1^\gamma$ such that for all $v \in \mathcal{C}_c^1(\mathbb{R})$,

$$c_\gamma \int_{\mathbb{R}} |v|^2 \bar{w} \leq 2 \int_{\mathbb{R}} (\partial_x v)^2 \bar{a} \bar{w} + \delta_\gamma \int_{\mathbb{R}} |v(\xi)|^2 \exp(\sqrt{\gamma}\xi) d\xi.$$

Definition III.3.1. *In the rest of the chapter, we set*

$$\mathcal{D}_\gamma(t) := 2 \int_{\mathbb{R}} (\partial_x u(t, x))^2 a(t, x) w(t, x) dx + \delta_\gamma \int_{\mathbb{R}} |u(t, x)|^2 \exp(\sqrt{\gamma}(x - ct)) dx,$$

which is the total dissipation term.

Gathering the above results, we see that any solution of the linearized equation (III.3.10) with $S = 0$, with an initial data such that $\int_{\mathbb{R}} |u_0|^2 \bar{w}_0 < \infty$, decays exponentially (at a rate c_γ) as $t \rightarrow \infty$.

L^∞ estimate

In order to prove that the dynamics of the nonlinear equation (III.3.5) is driven by the linearized part of the equation, and that the nonlinear term in the right-hand side of (III.3.5) can be treated perturbatively, we will need a last ingredient, which is a direct consequence of the comparison principle:

Lemma III.3.4 (L^∞ estimate). *Let h be small enough and assume that n^0 lies between two shifts of the reference profile N :*

$$N(x + h) \leq n^0(x) \leq N(x - h).$$

Then, for all $t \geq 0$, for all $x \in \mathbb{R}$,

$$N(x + h - ct) \leq n(t, x) \leq N(x - h - ct).$$

From Lemma III.2.3, we have

$$\|u\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} + \left\| \frac{n - N(x - ct)}{N(x - ct)} \right\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq C^\gamma h, \quad (\text{III.3.20})$$

where C is a positive constant independent of γ .

Equipped with this estimate and the control in L^∞ from Lemma III.3.4, we can control the nonlinear contributions and deduce an exponential decay of the L^2 weighted norm as $t \rightarrow +\infty$, as stated in Proposition III.3.1. The next subsection is devoted to the control of the nonlinear terms. We then give a proof of Theorem III.1.2 at the end of section III.3.

III.3.2 Control of the nonlinear terms and long-time behavior

We now address the proof of Theorem III.3.1 using the tools described above. Let u be a smooth solution to (III.3.5), we get by applying (III.3.18):

$$\begin{aligned} & \int_{\mathbb{R}} |u|^2 w_0 \phi \, dx + 2 \int_0^t \int_{\mathbb{R}} a(\partial_x u)^2 w_0 \phi \, dx ds + \int_0^t \int_{\mathbb{R}} |u|^2 \delta_\gamma \exp(\sqrt{\gamma}(x - cs)) \, dx ds \\ &= \int_{\mathbb{R}} |u_0| \bar{w} + \frac{\gamma}{\gamma + 1} \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 G(u)}{\partial_x N} u w_0 \phi \, dx ds - \int_0^t \int_{\mathbb{R}} \frac{G(u)}{\partial_x N} u w_0 \phi \, dx ds. \end{aligned} \quad (\text{III.3.21})$$

Observe that the first term of the right-hand side comes from the nonlinear diffusion while the second comes from the reaction term. We also recall that

$$G(u) = (N + u \partial_x N)^{\gamma+1} - N^{\gamma+1} - (\gamma + 1) N^\gamma u \partial_x N.$$

First, let us estimate $G(u)$.

Lemma III.3.5. *Assume that*

$$\left\| u \frac{\partial_x N}{N} \right\|_\infty = \left\| \frac{n(t, x) - N(x - ct)}{N(x - ct)} \right\|_\infty \leq \frac{1}{\gamma}.$$

Then

$$|G(u)| \leq C \gamma^2 (u \partial_x N)^2 N^{\gamma-1}, \quad (\text{III.3.22})$$

$$\begin{aligned} |\partial_x G(u)| &\leq C \gamma^3 \partial_x N N^{\gamma-2} (u \partial_x N)^2 + C \gamma^2 N^{\gamma-1} (u \partial_x N) \partial_x (u \partial_x N) \\ &\leq C \gamma^3 \partial_x N N^{\gamma-2} (u \partial_x N)^2 + C \gamma^2 N^{\gamma-1} (\partial_x N)^2 |u| |\partial_x u| \\ &\quad + C \gamma u^2 |\partial_x N|^2 N^{-1} + C \gamma u^2 |\partial_x N|, \end{aligned} \quad (\text{III.3.23})$$

for some constant $C > 0$ independent of γ .

Proof. The first estimate can be easily proved by writing

$$G(u) = N^{\gamma+1} g\left(\frac{u\partial_x N}{N}\right),$$

where $g(X) = (1+X)^{\gamma+1} - 1 - (\gamma+1)X$. A Taylor expansion at order two close to $X = 0$ shows that if $|X| \leq \gamma^{-1}$,

$$|g(X)| \leq \frac{1}{2}\gamma(\gamma+1) \left(1 + \frac{1}{\gamma}\right)^{\gamma-1} |X|^2 \lesssim \gamma^2 |X|^2.$$

Estimate (III.3.22) follows. We also know by convexity that $G(u) \geq 0$.

For the second estimate, we differentiate G and get

$$\begin{aligned} \partial_x G(u) &= (\gamma+1)\partial_x N \left((N + u\partial_x N)^\gamma - N^\gamma - \gamma N^{\gamma-1} u\partial_x N \right) \\ &\quad + (\gamma+1)\partial_x(u\partial_x N) \left((N + u\partial_x N)^\gamma - N^\gamma \right). \end{aligned}$$

Reasoning as before, we infer that

$$|\partial_x G(u)| \leq C(\gamma+1)\gamma^2 |\partial_x N| |u\partial_x N|^2 N^{\gamma-2} + C(\gamma+1)\gamma |\partial_x(u\partial_x N)| |u\partial_x N| N^{\gamma-1}.$$

To obtain the last set of inequalities, we use the equation on N , and we recall that

$$\gamma \partial_x^2 N N^\gamma = -c\partial_x N - \gamma^2 (\partial_x N)^2 N^{\gamma-1} - N(1 - N^\gamma),$$

which concludes the proof of the lemma. \square

Lemma III.3.6 (Control of the nonlinear reaction term). *There exists a constant $\eta_2 \in]0, 1[$ such that if*

$$\left\| \frac{u\partial_x N}{N} \right\|_\infty \leq \eta_2^\gamma,$$

then the following inequality holds

$$\left| \int_{\mathbb{R}} \frac{G(u)}{\partial_x N} u w_0 \phi \, dx \right| \leq \frac{1}{4} \mathcal{D}_\gamma. \quad (\text{III.3.24})$$

Proof. Using Lemma III.3.5, we have

$$\left| \int_{\mathbb{R}} \frac{G(u)}{\partial_x N} u w_0 \phi \, dx \right| \leq C\gamma^2 \int_{\mathbb{R}} |u|^3 |\partial_x N| N^{\gamma-1} w_0 \phi \, dx,$$

that we want to absorb in the left-hand side of the equality (III.3.21) thanks to the diffusion and damping terms:

$$\mathcal{D}_\gamma = 2 \int_{\mathbb{R}} a(\partial_x u)^2 w_0 \phi \, dx + \int_{\mathbb{R}} \delta_\gamma \exp(\sqrt{\gamma}\xi) u^2 \, dx.$$

Recalling Proposition III.3.1, we observe that it suffices to have

$$\gamma^2 \left\| \frac{u\partial_x N}{N} \right\|_\infty \leq \frac{c_\gamma}{4},$$

which concludes the proof, choosing $\eta_2 < \eta_1$. \square

Lemma III.3.7 (Control of the nonlinear diffusion term). *There exists a constant $\eta_2 \in]0, 1[$ such that if*

$$\|u\|_\infty + \left\| \frac{u \partial_x N}{N} \right\|_\infty \leq \eta_2^\gamma,$$

then the following inequality holds

$$\left| \int_{\mathbb{R}} \frac{\partial_x^2(G(u))}{\partial_x N} u w_0 \phi \, dx \right| \leq \frac{1}{4} \mathcal{D}_\gamma. \quad (\text{III.3.25})$$

Proof. Integrating by parts the nonlinear term stemming from the diffusion, we have

$$\int_{\mathbb{R}} \frac{\partial_x^2(G(u))}{\partial_x N} u w_0 \phi \, dx = - \int_{\mathbb{R}} \partial_x G(u) \partial_x u \left(\frac{w_0 \phi}{\partial_x N} \right) - \int_{\mathbb{R}} \partial_x G(u) u \partial_x \left(\frac{w_0 \phi}{\partial_x N} \right). \quad (\text{III.3.26})$$

• We first address the first term in the right-hand side of (III.3.26), using the estimate on $\partial_x G(u)$ from Lemma III.3.5. It follows that

$$\begin{aligned} \left| \int_{\mathbb{R}} \partial_x G(u) \partial_x u \left(\frac{w_0 \phi}{\partial_x N} \right) \right| &\leq C \int_{\mathbb{R}} \gamma^3 N^{\gamma-2} |u|^2 |\partial_x u| (\partial_x N)^2 w_0 \phi \\ &\quad + C \int_{\mathbb{R}} \gamma^2 |u| |\partial_x u|^2 N^{\gamma-1} |\partial_x N| w_0 \phi \\ &\quad + C \int_{\mathbb{R}} \gamma |u|^2 |\partial_x u| \frac{|\partial_x N|}{N} w_0 \phi \\ &\quad + C \int_{\mathbb{R}} \gamma |u|^2 |\partial_x u| w_0 \phi \\ &= \sum_{i=1}^4 I_i. \end{aligned}$$

We then address each term I_i separately. We start with the term I_2 , for which we simply write

$$I_2 \leq C \gamma \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} \int_{\mathbb{R}} (\partial_x u)^2 a w_0 \phi,$$

which is smaller than $\mathcal{D}_\gamma/16$, provided $\|u \partial_x N/N\|_\infty \leq (16C\gamma)^{-1}$.

For all other terms, we first perform a Cauchy-Schwarz inequality. We have, recalling that $\bar{a} = \gamma N^\gamma$,

$$\begin{aligned} I_1 &\leq C \left(\int_{\mathbb{R}} (\partial_x u)^2 a w \right)^{1/2} \left(\gamma^5 \int_{\mathbb{R}} |u|^4 N^{\gamma-4} w_0 \phi \right)^{1/2}, \\ I_3 &\leq C \left(\int_{\mathbb{R}} (\partial_x u)^2 a w \right)^{1/2} \left(\int_{\mathbb{R}} \gamma |u|^4 \frac{(\partial_x N)^2}{N^{\gamma+2}} w_0 \phi \right)^{1/2}, \\ I_4 &\leq C \left(\int_{\mathbb{R}} (\partial_x u)^2 a w \right)^{1/2} \left(\int_{\mathbb{R}} \gamma |u|^4 N^{-\gamma} w_0 \phi \right)^{1/2}. \end{aligned}$$

We then bound each integral with $|u|^4$ in the right-hand side by using the Poincaré inequality from Proposition III.3.1 and the L^∞ estimate on u . The simplest term is I_1 , for which we have

$$I_1 \leq C \gamma^{5/2} \|u\|_{L^\infty} \left(\int_{\mathbb{R}} (\partial_x u)^2 a w \right)^{1/2} \left(\int_{\mathbb{R}} |u|^2 w_0 \phi \right)^{1/2} \leq C \gamma^{5/2} \|u\|_{L^\infty} c_\gamma^{-1/2} \mathcal{D}_\gamma.$$

Concerning the term I_3 , we have

$$\int_{\mathbb{R}} \gamma |u|^4 \frac{(\partial_x N)^2}{N^{\gamma+2}} w_0 \phi \leq \gamma \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty}^2 \int_{\mathbb{R}} |u|^2 \frac{w_0 \phi}{N^\gamma}.$$

Using Proposition III.3.1, we have

$$\begin{aligned} \int_{\mathbb{R}} |u|^2 \frac{w_0 \phi}{N^\gamma} &= \int_{\xi > \tilde{\xi}} |u|^2 \frac{w_0 \phi}{N^\gamma} + \int_{\xi < \tilde{\xi}} |u|^2 \frac{w_0 \phi}{N^\gamma} \\ &\leq \gamma \int_{\mathbb{R}} (\partial_x u)^2 a w_0 \phi + \gamma C_\gamma \int_{\mathbb{R}} u^2 e^{\sqrt{\gamma}(x-ct)} dx \\ &\quad + \frac{1}{N(\tilde{\xi})^\gamma} \int_{\xi < \tilde{\xi}} |u|^2 w_0 \phi \\ &\leq C'_\gamma \mathcal{D}_\gamma, \end{aligned}$$

for some exponentially large constant C'_γ . The above inequality also allows us to bound I_4 . Thus, provided

$$\|u\|_{L^\infty} \leq \delta \inf \left(c_\gamma^{1/2} \delta^{-5/2}, (C'_\gamma)^{-1/2} \gamma^{-1/2} \right), \quad \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} \leq \delta (C'_\gamma)^{-1/2} \gamma^{-1/2},$$

for some small constant δ independent of γ , we infer that

$$\left| \int_{\mathbb{R}} \partial_x G(u) \partial_x u \left(\frac{w_0 \phi}{\partial_x N} \right) \right| \leq \frac{1}{8} \mathcal{D}_\gamma.$$

• Let us now consider the second term in the right-hand side of (III.3.26). Computing the weight in the right-hand side and using the definitions of ϕ , w and the equation satisfied by N , we find

$$\begin{aligned} \partial_\xi \left(\frac{\bar{w}_0 \bar{\phi}}{\partial_\xi N} \right) &= \partial_\xi \bar{\phi} \frac{\bar{w}_0}{\partial_\xi N} \\ &\quad + K \bar{\phi} \partial_\xi (\partial_\xi N N^\gamma) \exp \left(\int_{\xi_\gamma^-}^\xi \frac{c}{\bar{a}} \right) \\ &\quad + K \bar{\phi} \partial_\xi N N^\gamma \frac{c}{\bar{a}} \exp \left(\int_{\xi_\gamma^-}^\xi \frac{c}{\bar{a}} \right) \\ &= -\frac{\delta_\gamma}{\sqrt{\gamma}} \frac{\exp(\sqrt{\gamma} \xi)}{\gamma N^\gamma \partial_x N} - \frac{K}{\gamma} \bar{\phi} N (1 - N^\gamma) \exp \left(\int_{\xi_\gamma^-}^\xi \frac{c}{\bar{a}} \right) \\ &=: W_1 + W_2. \end{aligned}$$

We then use the estimate on $\partial_x G(u)$ from Lemma III.3.5, treating W_1 and W_2 separately. We have, concerning the terms with W_1 ,

$$\begin{aligned} &\left| \int_{\mathbb{R}} \partial_x G(u) u W_1 \right| \\ &\leq C \frac{\delta_\gamma}{\sqrt{\gamma}} \left[\int_{\mathbb{R}} \gamma^2 |u|^3 \frac{(\partial_x N)^2}{N^2} e^{\sqrt{\gamma} \xi} + \int_{\mathbb{R}} \gamma \frac{|\partial_x N|}{N} |u|^2 |\partial_x u| e^{\sqrt{\gamma} \xi} \right] \\ &+ C \frac{\delta_\gamma}{\sqrt{\gamma}} \int_{\mathbb{R}} |u|^3 N^{-\gamma} \left(1 + \frac{|\partial_x N|}{N} \right) e^{\sqrt{\gamma} \xi}. \end{aligned}$$

Using a Cauchy-Schwarz inequality for the second integral, we get

$$\begin{aligned}
 & \left| \int_{\mathbb{R}} \partial_x G(u) u W_1 \right| \\
 & \leq C \delta_\gamma \gamma^{3/2} \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} \left\| \frac{\partial_x N}{N} \right\|_{L^\infty} \int_{\mathbb{R}} u^2 e^{\sqrt{\gamma} \xi} \\
 & \quad + C \delta_\gamma \left(\int_{\mathbb{R}} (\partial_x u)^2 a w \right)^{1/2} \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} \left(\int_{\mathbb{R}} \frac{u^2}{N^\gamma w_0} e^{2\sqrt{\gamma} \xi} \right)^{1/2} \\
 & \quad + C \frac{\delta_\gamma}{\sqrt{\gamma}} \left(\|u\|_{L^\infty} + \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} \right) \int_{\mathbb{R}} \frac{u^2}{N^\gamma} e^{\sqrt{\gamma} \xi}.
 \end{aligned}$$

We then observe that thanks to the growth and decay properties of the weight w_0 at $\pm\infty$, there exists a constant C_γ such that

$$e^{\sqrt{\gamma} \xi} \leq C_\gamma w(\xi) \quad \forall \xi \in \mathbb{R}. \quad (\text{III.3.27})$$

However, because of the strong degeneracy and singular behavior in the transition zone near ξ_γ^0 , the constant C_γ might be exponentially large. It follows that

$$\int_{\mathbb{R}} u^2 e^{\sqrt{\gamma} \xi} \leq C_\gamma \int_{\mathbb{R}} u^2 w \leq C_\gamma c_\gamma^{-1} \mathcal{D}_\gamma.$$

Concerning the integral $\int_{\mathbb{R}} \frac{u^2}{N^\gamma} e^{\sqrt{\gamma} \xi}$, using Proposition III.3.1 together with (III.3.27) yields a bound $C_\gamma c_\gamma^{-1} \mathcal{D}_\gamma$ for this term, with a possibly different constant C_γ . The last term is treated in the same fashion. We obtain

$$\left| \int_{\mathbb{R}} \partial_x G(u) u W_1 \right| \leq C_\gamma \left(\left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} + \|u\|_{L^\infty} \right) \mathcal{D}_\gamma \leq \frac{1}{8} \mathcal{D}_\gamma,$$

provided $\|u\|_{L^\infty}$ and $\|u \partial_x N / N\|_{L^\infty}$ are small enough (exponentially small with γ).

There remains to address the terms containing W_2 . We have, using Lemma III.3.5 and recalling the expression of the weight \bar{w}_0 from Lemma III.3.2,

$$\begin{aligned}
 & \left| \int_{\mathbb{R}} \partial_x G(u) u W_2 \right| \\
 & \leq C \gamma^2 \int_{\mathbb{R}} \frac{|\partial_x N|}{N} |u|^3 w + C \gamma \int_{\mathbb{R}} u^2 |\partial_x u| w + \int_{\mathbb{R}} |u|^3 \frac{w}{N^\gamma} + \int_{\mathbb{R}} \frac{|u|^3 (1 - N^\gamma)}{|\partial_x N| N^{\gamma-1}} w.
 \end{aligned}$$

Using the same arguments as above, we find that the first three terms are bounded by

$$\left(\gamma^2 \left\| u \frac{\partial_x N}{N} \right\|_{L^\infty} c_\gamma^{-1} + \gamma \|u\|_{L^\infty} C_\gamma + \gamma \|u\|_{L^\infty} c_\gamma^{-1} C_\gamma \right) \mathcal{D}_\gamma,$$

and can be absorbed in \mathcal{D}_γ under the assumptions of the Lemma, provided η is sufficiently small.

There remains the last term, which has an additional singularity in the congested zone because of the $\partial_x N$ factor in the denominator (note that in the free zone $\xi > \tilde{\xi}_\gamma^*$, $|\partial_x N| \gtrsim N$, so that this singularity can be treated thanks to the weighted Poincaré inequality from

Proposition III.3.1.) However this singularity is compensated by the factor $(1 - N^\gamma)$ in the numerator. More precisely, we have, for $\xi \leq \tilde{\xi}_\gamma$ and using Lemma III.2.3,

$$\left| \frac{1 - N^\gamma}{\partial_x N} \right| = \gamma \left| \frac{1 - P}{P'} \right| P^{1-\frac{1}{\gamma}} \lesssim \gamma.$$

Hence

$$\int_{x-ct \leq \tilde{\xi}_\gamma} \frac{|u|^3(1 - N^\gamma)}{|\partial_x N| N^{\gamma-1}} w \leq C\gamma \|u\|_{L^\infty} \int_{x-ct < 0} |u|^2 w \leq C\gamma \|u\|_{L^\infty} c_\gamma^{-1} \mathcal{D}_\gamma.$$

Gathering all the terms, we obtain the inequality announced in the Lemma. \square

Let us now complete the proof of Theorem III.1.2. First, we choose h so that $h \leq \eta_2/C$, where $\eta_2 \in]0, 1[$ is the constant appearing in Lemmas III.3.6 and III.3.7, and C is the constant in (III.3.20). Then Lemma III.3.4 entails that

$$\|u\|_\infty + \left\| u \frac{\partial_x N}{N} \right\|_\infty \leq \eta_2^\gamma.$$

It follows from Lemmas III.3.6 and III.3.7 that the sum of the two nonlinear terms in the right-hand side of (III.3.21) is bounded by $\int_0^t \mathcal{D}_\gamma/2$. Therefore, we obtain for all $t \geq 0$,

$$\int_{\mathbb{R}} |u|^2 w + \frac{1}{2} \int_0^t \mathcal{D}_\gamma \leq \int_{\mathbb{R}} |u_0|^2 w.$$

Letting $t \rightarrow \infty$, we obtain the control of the diffusion announced in Theorem III.1.2. Now, applying the Poincaré inequality from Proposition III.3.1 and Lemma III.3.3, we get, for all $t \geq 0$,

$$\int_{\mathbb{R}} |u|^2 w + \frac{c_\gamma}{2} \int_0^t |u|^2 w \leq \int_{\mathbb{R}} |u_0|^2 w.$$

The exponential decay with a rate $c_\gamma/2$ (which we rename c_γ) follows easily from the Gronwall Lemma.

III.4 Proofs of some technical results

III.4.1 Derivation of the equation on u

In this subsection, we prove that u defined by

$$u(t, x) = \frac{n(t, x) - N(x - ct)}{\partial_x N(x - ct)},$$

is a solution of (III.3.21). As in the previous section, we omit the dependency of γ for simplicity.

First, we recall that n and $N(x - ct)$ are both solutions of (III.3.1), in which we rewrite the diffusion term as

$$\gamma \partial_x (n^\gamma \partial_x n) = \frac{\gamma}{\gamma + 1} \partial_{xx} n^{\gamma+1}.$$

Recalling the definition of $G(u)$ from (III.3.6), we write (omitting $x - ct$ in the argument of N)

$$n^{\gamma+1} - N^{\gamma+1} = (\gamma + 1)N(n - N) + G(u).$$

Introducing $\nu(t, x) = n(t, x) - N(x - ct)$, we find that ν is a solution of

$$\partial_t \nu - \gamma \partial_x^2 (N^\gamma \nu) - \nu (1 - (\gamma + 1)N^\gamma) = \frac{\gamma}{\gamma + 1} \partial_x^2 (G(u)) - G(u). \quad (\text{III.4.1})$$

Observe that, from (III.1.4), $\partial_x N(x - ct)$ is a (negative) solution of the linearized equation

$$\partial_t \partial_x N(x - ct) - \gamma \partial_x^2 (N^\gamma \partial_x N(x - ct)) - (1 - (\gamma + 1)N^\gamma) \partial_x N(x - ct) = 0.$$

Let us now set $u(t, x) := \nu(t, x) / \partial_x N(x - ct) = (n(t, x) - N(x - ct)) / \partial_x N(x - ct)$ and compute the equation satisfied by u . Using the identity

$$\begin{aligned} \partial_x^2 u &= \partial_x \left(\frac{\partial_x \nu}{\partial_x N} - \frac{\nu \partial_x^2 N}{(\partial_x N)^2} \right) \\ &= \frac{\partial_x^2 \nu}{\partial_x N} - \frac{\nu}{(\partial_x N)^2} \partial_x^3 N - 2 \frac{\partial_x^2 N}{\partial_x N} \partial_x u, \end{aligned}$$

we infer that

$$\partial_t u + b \partial_x u - a \partial_x^2 u = \frac{\gamma}{\gamma + 1} \frac{\partial_x^2 G(u)}{N'(x - ct)} - \frac{G(u)}{N'(x - ct)}, \quad (\text{III.4.2})$$

where $a(t, x) = \bar{a}(x - ct)$, $b(t, x) = \bar{b}(x - ct)$ and

$$\bar{b} := -2\gamma \partial_x N^\gamma - 2\gamma N^\gamma \frac{\partial_x^2 N}{\partial_x N}, \quad \bar{a} := \gamma N^\gamma. \quad (\text{III.4.3})$$

III.4.2 Structure of the linearized system: Lemmas III.3.1 and equality (III.3.18)

Proof of Lemma III.3.1. Multiplying (III.4.2) by $2uw_0$ and integrating on \mathbb{R} , we obtain formally

$$\frac{d}{dt} \int_{\mathbb{R}} |u|^2 w_0 - \int_{\mathbb{R}} |u|^2 \partial_t w_0 + \int_{\mathbb{R}} 2u \partial_x u (bw_0 + \partial_x (aw_0)) + 2 \int_{\mathbb{R}} aw_0 (\partial_x u)^2 = 2 \int_{\mathbb{R}} Suw_0.$$

Integrating by parts the middle term gives

$$\begin{aligned} &\int_{\mathbb{R}} 2u \partial_x u (bw_0 + \partial_x (aw_0)) \\ &= - \int_{\mathbb{R}} |u|^2 \partial_x (bw_0 + \partial_x (aw_0)). \end{aligned}$$

Gathering all the terms, we have

$$\frac{d}{dt} \int_{\mathbb{R}} |u|^2 w_0 - \int_{\mathbb{R}} |u|^2 [\partial_t w_0 + \partial_x (bw_0 + \partial_x (aw_0))] + \int_{\mathbb{R}} aw_0 (\partial_x u)^2 = 2 \int_{\mathbb{R}} Suw_0.$$

Now, let us look at the term between brackets. As $w_0 = \bar{w}_0(x - ct)$,

$$\begin{aligned} [\partial_t w_0 + \partial_x (bw_0 + \partial_x (aw_0))] &= -c\bar{w}'_0 + (\bar{b}\bar{w}_0 + (\bar{a}\bar{w}_0)'), \\ &= \partial_x \left((\bar{a}\bar{w}_0)' + (\bar{b} - c)\bar{w}_0 \right), \\ &= 0, \end{aligned}$$

from the definition of \bar{w}_0 . This implies

$$\frac{d}{dt} \int_{\mathbb{R}} |u|^2 w_0 + 2 \int_{\mathbb{R}} a w_0 (\partial_x u)^2 = 2 \int_{\mathbb{R}} S u w_0,$$

and therefore, integrating with respect to t ,

$$\begin{aligned} \int_{\mathbb{R}} |u(t, x)|^2 w_0(t, x) dx + 2 \int_0^t \int_{\mathbb{R}} a(s, x) w_0(s, x) (\partial_x u(s, x))^2 dx ds \\ = \int_{\mathbb{R}} |u^0(x)|^2 w_0(0, x) dx + 2 \int_0^t \int_{\mathbb{R}} S u w_0. \end{aligned}$$

Note that $w_0(0, x) = \bar{w}_0(x)$. Hence we obtain the identity announced in the Lemma. \square

Proof of equality (III.3.18). This proof is very similar to that of Lemma III.3.1. Observe first that for $w = w_0 \phi$, one has

$$\begin{aligned} -[\partial_t w + \partial_x (b w + \partial_x (a w))](t, x) &= (c \bar{w}_0 \bar{\phi} - \bar{b} \bar{w}_0 \bar{\phi} - (\bar{a} \bar{w}_0 \bar{\phi})')'(x - ct), \\ &= -(\bar{a} \bar{w}_0 \bar{\phi}')'(x - ct). \end{aligned}$$

Note that from the definition of ϕ , $\bar{a} \bar{w}_0 \bar{\phi}' = -\frac{\delta_\gamma}{\sqrt{\gamma}} \exp(\sqrt{\gamma} \xi)$, hence

$$-[\partial_t w + \partial_x (b w + \partial_x (a w))](t, x) = \delta_\gamma \exp(\sqrt{\gamma} \xi).$$

Proceeding exactly as in the proof of Lemma III.3.1, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |u|^2 w - \int_{\mathbb{R}} |u|^2 [\partial_t w + \partial_x (b w + \partial_x (a w))] + 2 \int_{\mathbb{R}} a w (\partial_x u)^2 &= 2 \int_{\mathbb{R}} S u w, \\ \frac{d}{dt} \int_{\mathbb{R}} |u|^2 w + \delta_\gamma \int_{\mathbb{R}} |u(t, x)|^2 \exp(\sqrt{\gamma}(x - cs)) dx + 2 \int_{\mathbb{R}} a w (\partial_x u)^2 &= 2 \int_{\mathbb{R}} S u w, \end{aligned}$$

and therefore, integrating again with respect to t , we obtain, for all $t \geq 0$,

$$\begin{aligned} \int_{\mathbb{R}} |u(t, x)|^2 w(t, x) dx + \delta_\gamma \int_{\mathbb{R}_+} \int_{\mathbb{R}} |u(s, x)|^2 \exp(\sqrt{\gamma}(x - cs)) dx \\ + 2 \int_{\mathbb{R}_+} \int_{\mathbb{R}} a(s, x) w(s, x) (\partial_x u(s, x))^2 dx ds = \int_{\mathbb{R}} |u^0(x)|^2 \bar{w}(x) dx + 2 \int_0^t \int_{\mathbb{R}} S u w. \end{aligned}$$

The Poincaré inequality stated in Lemma III.3.3 is an easy consequence of Proposition III.3.1 and of the equivalence between the weights \bar{w} and \bar{w}_0 . \square

III.4.3 Properties of the weights \bar{w}_0 and \bar{w} : Lemma III.3.2 and estimate (III.3.19)

Proof of Lemma III.3.2. Let us rewrite equation (III.3.9) as

$$(\bar{a} \bar{w}_0)' + \frac{\bar{b} - c}{\bar{a}} (\bar{a} \bar{w}_0) = 0,$$

which yields, since $\bar{w}_0(\xi_\gamma^-) = 1$,

$$\bar{a} \bar{w}_0(\xi) = \bar{a}(\xi_\gamma^-) \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c - \bar{b}}{\bar{a}}\right).$$

We recall that

$$\bar{a} = \gamma N^\gamma, \quad \bar{b} = -2\gamma^2 N^{\gamma-1} \partial_x N - 2\gamma N^\gamma \frac{\partial_x^2 N}{\partial_x N},$$

hence

$$\begin{aligned} - \int_{\xi_\gamma^-}^{\xi} \frac{\bar{b}(z)}{\bar{a}(z)} dz &= 2 \int_{\xi_\gamma^-}^{\xi} \left[\frac{(N^\gamma)'(z)}{N^\gamma(z)} + \frac{N''(z)}{N'(z)} \right] dz \\ &= 2 \ln \left(\frac{(N(\xi))^\gamma}{(N(\xi_\gamma^-))^\gamma} \right) + 2 \ln \left(\frac{|N'(\xi)|}{|N'(\xi_\gamma^-)|} \right), \end{aligned}$$

and therefore

$$\bar{w}_0(\xi) = \frac{\bar{a}(\xi_\gamma^-)}{\bar{a}(\xi)} \exp \left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\bar{a}} \right) \exp \left(- \int_{\xi_\gamma^-}^{\xi} \frac{\bar{b}}{\bar{a}} \right) \quad (\text{III.4.4})$$

$$= \frac{1}{(N(\xi_\gamma^-))^\gamma (N'(\xi_\gamma^-))^2} (N(\xi))^\gamma (N'(\xi))^2 \exp \left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\bar{a}} dz \right). \quad (\text{III.4.5})$$

Therefore we find the expression announced in Lemma III.3.2, with a normalization constant

$$K := \frac{1}{(N(\xi_\gamma^-))^\gamma (N'(\xi_\gamma^-))^2}.$$

Let us now estimate K . We recall that ξ_γ^- is defined in (III.2.1) Since $N' = \gamma^{-1} P' P^{\frac{1}{\gamma}-1}$, it also follows that

$$N'(\xi_\gamma^-) = \frac{1}{\gamma} P'(\xi_\gamma^-) \left(\frac{(c-1)(\gamma+1)}{c^3} \right)^{\frac{1}{2}-\frac{1}{2\gamma}},$$

and thus

$$K = \left(\frac{c^3}{(c-1)(\gamma+1)} \right)^{\frac{1}{2}-\frac{1}{\gamma}} \frac{\gamma^2}{P'(\xi_\gamma^-)^2}.$$

The sub- and super-solutions for P (see Proposition III.2.1) entail that $P'(\xi_\gamma^-)$ is bounded from above and below. Hence K is of order $\gamma^{3/2}$.

For $\xi \geq \tilde{\xi}$, we know by (III.1.10) that there exist $0 < A_1 < A_2 < 1$ (close to $1 - c^{-1}$) such that

$$A_1 e^{-\frac{2}{c}\xi} \leq N(\xi) \leq A_2 e^{-\frac{\xi}{c}}. \quad (\text{III.4.6})$$

Moreover, remember that $L_\gamma = \frac{N'}{N} + \frac{1}{c}$ converges uniformly to 0 on $[\xi^*, +\infty)$ as $\gamma \rightarrow +\infty$ (see the proof of Lemma III.2.3). Hence, for any $\eta > 0$, there exists γ_0 such that for all $\gamma > \gamma_0$, $N'/N \in [-1/c - \eta, -1/c + \eta]$ for all $\xi \in [\xi^*, +\infty)$. By (III.4.6), we deduce that

$$\tilde{A}_1 e^{-\frac{2}{c}\xi} \leq |N'(\xi)| \leq \tilde{A}_2 e^{-\frac{\xi}{c}}, \quad (\text{III.4.7})$$

with $\tilde{A}_{1,2} \in (0, 1)$. As a consequence \bar{w}_0 has a double exponential growth as $\xi \rightarrow +\infty$:

$$\begin{aligned} &K_1 A_1^\gamma \tilde{A}_1^2 \exp \left(-2 \frac{\gamma+2}{c} \xi \right) \exp \left(\frac{c^2}{\gamma^2} A_2^{-\gamma} \left(\exp \left(\frac{\gamma}{c} \xi \right) - \exp \left(\frac{\gamma}{c} \xi_\gamma^- \right) \right) \right) \\ &\leq \bar{w}_0 \\ &\leq K_2 A_2^\gamma \tilde{A}_2^2 \exp \left(-\frac{\gamma+2}{c} \xi \right) \exp \left(\frac{c^2}{2\gamma^2} A_1^{-\gamma} \left(\exp \left(\frac{2\gamma}{c} \xi \right) - \exp \left(\frac{2\gamma}{c} \xi_\gamma^- \right) \right) \right), \end{aligned} \quad (\text{III.4.8})$$

with $K_{1,2} \propto \gamma^{3/2}$. The estimate from the Lemma follows.

For $\xi \rightarrow -\infty$, we have using (III.2.6) and denoting $\tilde{\lambda} = \sqrt{1 + \frac{c^2}{4\gamma^2} - \frac{c}{2\gamma}}$,

$$\bar{w}_0 = KN^\gamma(N')^2 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma}\right) \underset{\xi \rightarrow -\infty}{\sim} K\tilde{\lambda}^2 P(1-N)^2 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma P}\right).$$

Recalling estimates (III.1.7), we get

$$\frac{C}{\sqrt{\gamma}}(\xi - \xi_\gamma^-) \leq \int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma P} \leq \frac{c}{\gamma}(\xi - \xi_\gamma^-),$$

and, for $|\xi| \leq C$ with C independent of γ

$$\frac{C^{-1}}{\gamma} \left(1 - \frac{C'}{\sqrt{\gamma}}\right) \exp(\xi) \leq 1 - N \leq \frac{C}{\gamma} \left(1 - \frac{C'}{\sqrt{\gamma}}\right) \exp\left(\left(1 - \frac{C}{\sqrt{\gamma}}\right)\xi\right).$$

Gathering the terms, we obtain

$$C^{-1} \frac{K}{\gamma^2} \exp\left(2\left(1 + \frac{C}{\sqrt{\gamma}}\right)\xi\right) \leq \bar{w}_0 \leq C \frac{K}{\gamma^2} \exp\left(2\left(1 - \frac{C}{\sqrt{\gamma}}\right)\xi\right),$$

and we deduce the result announced in Lemma III.3.2. □

Estimate of the constant δ_γ in \bar{w} . Let

$$\bar{\psi} := \bar{\phi}' \bar{a} \bar{w}_0 = -\frac{\delta_\gamma}{\sqrt{\gamma}} \exp(\sqrt{\gamma}\xi).$$

Using the previous Lemma and recalling (III.4.6), we observe that the double exponential growth of \bar{w}_0 dominates the growth in $\bar{\psi}$ as $\xi \rightarrow +\infty$. On the other hand, for $\xi \leq -C$, we have

$$\frac{\bar{\psi}(\xi)}{\bar{a}(\xi)\bar{w}_0(\xi)} = -\frac{\delta_\gamma}{\sqrt{\gamma}} \frac{\exp(\sqrt{\gamma}\xi)}{\gamma(\bar{N}(\xi))^\gamma \bar{w}_0(\xi)} \geq -C \frac{\delta_\gamma}{\gamma^3} \exp\left(\left(\sqrt{\gamma} - 2\left(1 + \frac{C}{\sqrt{\gamma}}\right)\right)\xi\right).$$

Hence, for γ large enough $\sqrt{\gamma} - 3 > 0$ and $\frac{\bar{\psi}(\xi)}{\bar{a}(\xi)\bar{w}_0(\xi)}$ decreases exponentially to 0 as $\xi \rightarrow -\infty$.

We conclude then to the integrability of $\frac{\bar{\psi}}{\bar{a}\bar{w}_0}$ on \mathbb{R} .

Let us now study the behavior of ϕ . For that purpose, we analyze separately the different regions according to the value of ξ .

- for $\xi > \tilde{\xi}$: using the estimates (III.4.6)-(III.4.8), we infer that

$$\begin{aligned} |\phi'(\xi)| &= -\frac{\bar{\psi}(\xi)}{\bar{a}(\xi)\bar{w}_0(\xi)} \\ &\leq C \frac{\delta_\gamma}{\gamma^{3/2}K} A_1^{-2\gamma} \exp\left(\left(\sqrt{\gamma} + \frac{4(\gamma+1)}{c}\right)\xi\right) \exp\left(-A_2^{-\gamma} \left(\exp\left(\frac{\gamma}{c}\xi\right) - \exp\left(\frac{\gamma}{c}\xi_\gamma^-\right)\right)\right) \\ &\leq C \frac{\delta_\gamma}{\gamma^{3/2}K} A_1^{-2\gamma} \exp\left(5\frac{\gamma}{c}\xi\right) \exp\left(-A_2^{-\gamma} \left(\exp\left(\frac{\gamma}{c}\xi\right) - \exp\left(\frac{\gamma}{c}\xi_\gamma^-\right)\right)\right). \end{aligned}$$

Hence, integrating and using the asymptotic values of ξ_γ^- and $\tilde{\xi}$ from Lemma III.2.4, we get

$$\begin{aligned} \int_{\tilde{\xi}}^{\infty} |\bar{\phi}'| &\leq C \frac{\delta_\gamma}{\gamma^4} A_1^{-2\gamma} A_2^\gamma \exp\left(-A_2^{-\gamma} \left(\exp\left(\frac{\gamma}{c} \tilde{\xi}\right) - \exp\left(\frac{\gamma}{c} \xi_\gamma^-\right)\right)\right) \\ &\leq C \frac{\delta_\gamma}{\gamma^4} A_1^{-2\gamma} A_2^\gamma \exp(-CA_2^{-\gamma}). \end{aligned} \quad (\text{III.4.9})$$

- for the intermediate region $\xi \in [\xi^-, \tilde{\xi}]$, we write

$$\begin{aligned} \int_{\xi^-}^{\tilde{\xi}} |\bar{\phi}'(z)| dz &= \frac{\delta_\gamma}{\sqrt{\gamma}} \int_{\xi^-}^{\tilde{\xi}} \frac{e^{\sqrt{\gamma}z}}{\gamma(N(z))^\gamma} \times \frac{1}{K(N(z))^\gamma (N'(z))^2 \exp\left(\int_{\xi^-}^z \frac{c}{\gamma N^\gamma}\right)} dz \\ &= \frac{\delta_\gamma}{\gamma^{-1/2} K} \int_{\xi^-}^{\tilde{\xi}} \frac{e^{\sqrt{\gamma}z}}{(P'(z))^2 (N(z))^2} \exp\left(-\int_{\xi^-}^z \frac{c}{\gamma N^\gamma}\right) dz. \end{aligned}$$

Now, using Lemma III.2.4 and the definition of ξ^- , we have in this region

$$|P'(\xi)| \geq \frac{C}{\gamma}, \quad N(\xi) \geq N(\tilde{\xi}) > 1 - \frac{2}{c}, \quad (N(\xi))^\gamma \leq P(\xi^-) = O\left(\frac{1}{\sqrt{\gamma}}\right),$$

and thus

$$\begin{aligned} \int_{\xi^-}^{\tilde{\xi}} |\phi'(z)| dz &\leq C \frac{\delta_\gamma}{\gamma^{-5/2} K} e^{\sqrt{\gamma}\tilde{\xi}} \int_{\xi^-}^{\tilde{\xi}} \exp\left(-\frac{C}{\gamma^{1/2}} z\right) dz \\ &\leq C \gamma^{3/2} \delta_\gamma \left(\exp\left(-\frac{C\xi_\gamma^-}{\sqrt{\gamma}}\right) - \exp\left(-\frac{C\tilde{\xi}}{\sqrt{\gamma}}\right)\right), \\ &\leq C \gamma^{1/2} \delta_\gamma, \end{aligned} \quad (\text{III.4.10})$$

where we have used the fact that $\tilde{\xi} = O(\gamma^{-1})$, $\xi_\gamma^- = O(\gamma^{-\frac{1}{2}})$ (cf Lemma III.2.4).

- for $\xi < \xi_\gamma^-$, we use the fact that

$$\frac{1-P}{P'} \xrightarrow{\gamma \rightarrow +\infty} -1 \quad \text{uniformly on } \mathbb{R}_-,$$

and the control (III.2.8)

$$1 - P \geq \left(1 - \frac{1}{\gamma}\right) e^\xi \quad \forall \xi < 0,$$

to infer that

$$\begin{aligned} -\int_{-\infty}^{\xi^-} \phi'(z) dz &= \frac{\delta_\gamma}{\gamma^{-1/2} K} \int_{-\infty}^{\xi^-} \frac{e^{\sqrt{\gamma}z}}{(P'(z))^2 (N(z))^2} \exp\left(\int_z^{\xi^-} \frac{c}{\gamma N^\gamma}\right) dz \\ &\leq C \frac{\delta_\gamma}{\gamma^{-1/2} K} \int_{-\infty}^{\xi^-} \frac{e^{\sqrt{\gamma}z}}{(1-P(z))^2} e^{\frac{c}{\sqrt{\gamma}}(\xi^- - z)} dz \\ &\leq C \frac{\delta_\gamma}{\gamma^{-1/2} K} \int_{-\infty}^{\xi^-} e^{\sqrt{\gamma}z} e^{-2z} e^{\frac{c}{\sqrt{\gamma}}(\xi^- - z)} dz \\ &\leq C \frac{\delta_\gamma}{\gamma^{-1/2} K} \int_{-\infty}^{\xi^-} e^{\frac{\sqrt{\gamma}}{2}z} dz \\ &\leq C \gamma^{-3/2} \delta_\gamma, \end{aligned} \quad (\text{III.4.11})$$

thanks to the fact that $\xi^- = O(\gamma^{-1/2})$ (cf Lemma III.2.4). Combining (III.4.9)-(III.4.10)-(III.4.11), there exists δ_0 bounded away from 0 such that for $\delta_\gamma = \delta_0 \gamma^{-1/2}$, $|\int_{\mathbb{R}} \phi'| \leq 1$. \square

III.4.4 Proof of the weighted Poincaré inequality

Proof of Proposition III.3.1. To lighten the notations, we forget in what follows the notation $\bar{\cdot}$ when it is clear that we work with functions of variable ξ . Formally, we have the following inequalities, for any $\rho \in \mathcal{C}^2(\mathbb{R})$ and $v \in \mathcal{C}_c^1(\mathbb{R})$

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} (\partial_\xi(v\rho))^2 d\xi \\ &= \int_{\mathbb{R}} [\rho^2(\partial_\xi v)^2 + v^2(\partial_\xi \rho)^2 d\xi + 2 \int_{\mathbb{R}} u \partial_\xi v \rho \partial_\xi \rho] d\xi \\ &= \int_{\mathbb{R}} [\rho^2(\partial_\xi v)^2 + v^2(\partial_\xi \rho)^2] d\xi - \int_{\mathbb{R}} v^2 \partial_\xi(\rho \partial_\xi \rho) d\xi \\ &= \int_{\mathbb{R}} \rho^2(\partial_\xi v)^2 d\xi - \int_{\mathbb{R}} v^2 \rho \partial_\xi^2 \rho d\xi. \end{aligned}$$

Note that when ρ is positive and strictly convex, we obtain a Poincaré inequality. We want to apply this inequality with $\rho := (\bar{a}\bar{w}_0)^{1/2}$. However, the weight ρ is not convex on \mathbb{R} and we cannot guarantee the sign of the second integral. Let us compute the derivatives of ρ . Using (III.3.12), we have

$$\rho = -\sqrt{\gamma K} N' N^\gamma \exp\left(\int_{\xi_\gamma^-}^\xi \frac{c}{2\gamma N(\xi)^\gamma}\right),$$

and thus

$$\rho'(\xi) = \sqrt{\gamma K} \exp\left(\int_{\xi_\gamma^-}^\xi \frac{c}{2\gamma N(\xi)^\gamma}\right) \left[-(N'(\xi)N(\xi)^\gamma)' - \frac{c}{2\gamma} N'(\xi)\right].$$

We recall that

$$-cN' - \gamma(N'N^\gamma)' = N(1 - N^\gamma),$$

so that

$$\rho'(\xi) = \sqrt{\gamma K} \exp\left(\int_{\xi_\gamma^-}^\xi \frac{c}{2\gamma N^\gamma(z)}\right) \left[\frac{1}{\gamma} N(\xi)(1 - N^\gamma(\xi)) + \frac{c}{2\gamma} N'(\xi)\right].$$

Differentiating once again and using the equation on N , we get

$$\begin{aligned} \rho''(\xi) &= \sqrt{\frac{K}{\gamma}} \exp\left(\int_{\xi_\gamma^-}^\xi \frac{c}{2\gamma N^\gamma(z)} dz\right) \left[N'(1 - (\gamma + 1)N^\gamma) + \frac{c}{2} N'' + \frac{c}{2\gamma N^{\gamma-1}}(1 - N^\gamma) + \frac{c^2}{4\gamma} \frac{N'}{N^\gamma}\right] \\ &= \sqrt{\frac{K}{\gamma}} \exp\left(\int_{\xi_\gamma^-}^\xi \frac{c}{2\gamma N^\gamma(z)} dz\right) \left[N'(1 - (\gamma + 1)N^\gamma) \right. \\ &\quad \left. - \frac{c}{2\gamma N^\gamma} (cN' + \gamma^2(N')^2 N^{\gamma-1} + N(1 - N^\gamma)) + \frac{c}{2\gamma N^{\gamma-1}}(1 - N^\gamma) + \frac{c^2}{4\gamma} \frac{N'}{N^\gamma}\right] \\ &= -\sqrt{\frac{K}{\gamma}} N' \exp\left(\int_{\xi_\gamma^-}^\xi \frac{c}{2\gamma N^\gamma(z)} dz\right) \left[(\gamma + 1)N^\gamma + \frac{c^2}{4\gamma N^\gamma} + c\gamma \frac{N'}{2N} - 1\right]. \end{aligned}$$

Note that $\rho'' \geq 0$ provided the term in brackets is non-negative. The bracketed term is a sum of four terms, among which the first two are positive, and the last two are negative. Furthermore,

$$\frac{\gamma+1}{c}N^\gamma + \frac{c}{4\gamma N^\gamma} = \left(\sqrt{\frac{\gamma+1}{c}}N^\gamma - \sqrt{\frac{c}{4\gamma N^\gamma}} \right)^2 + \sqrt{\frac{\gamma+1}{\gamma}} \geq 1,$$

so that

$$\rho'' \geq -\sqrt{\frac{K}{\gamma}}N' \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{2\gamma N^\gamma}\right) \left[(\gamma+1)\left(1 - \frac{1}{c}\right)N^\gamma + \frac{c(c-1)}{4\gamma N^\gamma} + \frac{1}{4\gamma} + c\gamma\frac{N'}{2N} \right]. \quad (\text{III.4.12})$$

Thus the only zone where ρ'' is non-positive is the region where the last term in the above bracket is not dominated by others. Decomposing the domain in three zones, we have

- $\xi \geq \tilde{\xi}$: using the notations of Section III.2, we have

$$N'(\xi) \geq \tilde{Q}(N) = -\frac{c-1}{4\gamma^2(N(\xi))^{\gamma-1}},$$

so that

$$c\gamma\frac{N'}{2N} < \frac{c(c-1)}{8\gamma N^\gamma} \quad \forall \xi \geq \tilde{\xi}.$$

Recalling the expressions of ρ and \bar{w}_0 , we infer that for $\xi > \tilde{\xi}$

$$\begin{aligned} \rho\rho'' &\geq C\sqrt{\gamma K}\sqrt{\frac{K}{\gamma}}(N')^2N^\gamma \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma(z)}dz\right) \frac{C}{\gamma N^\gamma} \\ &\geq \frac{C}{\gamma N^\gamma}w_0. \end{aligned}$$

- for $\xi \leq \xi_\gamma^-$, we have $P \geq P(\xi_\gamma^-) = \left(\frac{c^3}{(c-1)(\gamma+1)}\right)^{1/2}$ while $P' \in [-c, 0]$. Hence, we ensure that

$$-cP' \leq c^2 \leq (\gamma+1)\left(1 - \frac{1}{c}\right)P^2 \quad \forall \xi \leq \xi_\gamma^-,$$

and therefore

$$-c\gamma\frac{N'}{2N} \leq \frac{(\gamma+1)}{2}\left(1 - \frac{1}{c}\right)N^\gamma \quad \forall \xi \leq \xi_\gamma^-.$$

Let us mention that this inequality is precisely the property that lead us to the normalization (III.2.1). We deduce then

$$\begin{aligned} \rho\rho'' &\geq K(N')^2N^{2\gamma} \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma(z)}dz\right) \frac{(\gamma+1)}{2}\left(1 - \frac{1}{c}\right) \\ &\geq C\gamma N^\gamma w_0. \end{aligned}$$

- for intermediate region, i.e., $\xi \in [\xi_\gamma^-, \tilde{\xi}]$, we can always bound the negative contribution as follows

$$\rho(\rho'')_- \leq \frac{c\gamma}{2}K|N'|^3 \exp\left(\int_{\xi_\gamma^-}^{\xi} \frac{c}{\gamma N^\gamma(z)}dz\right) N^{\gamma-1} \leq C_\gamma e^{\sqrt{\gamma}\xi},$$

for $\xi \in (\xi_\gamma^-, \tilde{\xi})$, where

$$C_\gamma \leq C\gamma^{5/2} \|N'\|_{L^\infty(\xi_\gamma^-, \tilde{\xi})}^3 \exp\left(\int_{\xi^-}^{\tilde{\xi}} \frac{1}{\gamma N^\gamma}\right).$$

Let us now evaluate the integral in the argument of the exponential. Using Lemma III.2.4, we recall that $|P'| = \gamma|N'|N^{\gamma-1} \geq C\gamma^{-1}$ on $(\xi^-, \tilde{\xi})$. Hence

$$\begin{aligned} \int_{\xi^-}^{\tilde{\xi}} \frac{1}{\gamma N^\gamma} &= \int_{\xi^-}^{\tilde{\xi}} \frac{|N'|}{\gamma|N'|N^\gamma} \\ &\leq C\gamma \int_{\xi^-}^{\tilde{\xi}} \frac{|N'|}{N} \\ &\leq C\gamma \ln\left(\frac{N(\xi^-)}{N(\tilde{\xi})}\right) \leq C\gamma. \end{aligned}$$

Thus $C_\gamma \leq C\gamma$ for some constant $C > 1$ independent of γ .

Gathering all the terms, we obtain

$$\int_{\xi \leq \xi_\gamma^-} v^2 \rho \rho'' + \int_{\xi \geq \tilde{\xi}} v^2 \rho \rho'' \leq \int_{\mathbb{R}} (\partial_\xi v)^2 \bar{a} \bar{w}_0 + \int_{\xi_\gamma^-}^{\tilde{\xi}} v^2 \rho(\rho'')_-.$$

Replacing $\rho \rho''$ by their lower bounds on $(-\infty, 0)$ and on $(\tilde{\xi}, +\infty)$, we obtain the inequality announced in the Proposition. □

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Glossary

acoustic wave An acoustic wave is a mechanical or longitudinal wave resulting from fluctuations in the pressure field. They propagate as a result of the compression and decompression through a medium. [13](#)

advective time scale It refers to the time required for the bulk of a fluid to travel characteristic distance L at speed U . It is, therefore, determined by dimensional constraints. Let t_a denote the time scale for advection, then $t_a = \frac{L}{U}$. [15](#)

Bernoulli numbers These quantities first appeared in *Ars Conjectandi*, a treatise published in 1713, by Jakob Bernoulli (1654-1705). They usually can be found in the expansion of known functions such as $\tan(x)$, $\tanh(x)$, among others. They are related to Riemann's zeta function ζ through the formula

$$B_n = -n\zeta(1-n) \quad \text{for } n \leq 1,$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, if $\text{Re}(s) > 1$ and B_n denotes the Bernoulli number. [3](#)

beta-plane approximation It corresponds to the linearization of the Coriolis parameter with respect to a certain latitude θ when the scale of the motion be sufficiently small in north-south extent. Then, $f \sim f_0 + \beta y$ where $\beta y \ll f_0$. [15](#)

boundary layer Thin layer of viscous fluid close to the solid surface of a wall in contact with a moving stream in which the effects of viscosity are comparable in magnitude with those due to the inertia force, and therefore, cannot be neglected. [1](#), [3](#), [4](#)

chemotaxis Chemotaxis in microbiology refers to the migration of cells in relation to chemical agents, such as a growth factor. [27](#)

Coriolis force Force produced by the complementary acceleration due to the Earth's rotation. It determines the general direction of winds and ocean currents. Its traditional approximation is known as the Coriolis parameter and is defined by $f = 2|\Omega| \sin \theta$ where θ is latitude and Ω corresponds to the angular velocity. At mid-latitudes, a typical value of f is 10^4s^{-1} . [5](#), [15](#)

Ekman layer The wind exerts stress on the ocean surface proportional to the square of the wind speed and in the direction of the wind, setting the surface water in motion.

This motion extends to a depth of about 100 meters in what is called the Ekman layer. Within the oceanic Ekman layer, the wind stress is balanced by the Coriolis force and frictional forces. [4](#), [16](#)

Ekman spiral Theoretical displacement of current direction by the Coriolis effect, given a steady wind blowing over an ocean of infinite depth, extent, and uniform eddy viscosity. It refers to the movement of an average water particle within the Ekman layer: to the right of the wind direction in the Northern Hemisphere and its left in the Southern Hemisphere. [4](#)

Froude number Dimensionless number defined as the ratio of the flow inertia to the external field. It is based on the speed-length ratio defined by $Fr = \frac{U}{\sqrt{gL}}$, where U is the local flow velocity, g is the local external field, and L is a characteristic length. It is used to determine the resistance of a partially submerged object moving through water. [15](#)

Gamma function Complex function denoted by Γ commonly used extension of the factorial function to complex numbers. For every complex number z , satisfying $\Re(z) > 0$, the Gamma function is defined via a convergent improper integral as follows:

$$\Gamma : z \mapsto \int_0^{+\infty} t^{z-1} e^{-t} dt$$

. [3](#)

geostrophic approximations The assumption that the horizontal wind behaves as the geostrophic wind, i.e., a wind for which the Coriolis acceleration exactly balances the horizontal pressure force. [5](#)

geostrophic degeneracy It refers to the geostrophic approximation inability to provide a complete dynamical determination of motion. The geostrophic approximation fails to near equatorial regions, see [\[17\]](#). [17](#), [24](#)

geostrophic equilibrium The term *geostrophy* comes from Greek *ge*-Earth + *strophikos*-turned. It refers to the balance between the horizontal pressure gradient and the Coriolis force. [5](#)

glioma Gliomas or glial tumors are all brain and spinal cord tumors arising from the neuronal support tissue or glia. [27](#)

haptotaxis It refers to the moment cells encounter immobilized ligands, such as extracellular matrix proteins, and these ligands promote directional migration. [27](#)

homeostatic pressure It is the value of the pressure for which cell division and death achieve an equilibrium. It is a measure of the competition between a tumor and the surrounding tissue. [26](#), [32](#)

inertia-gravity waves Also known as *Poincaré waves*, they are waves traveling at speed \sqrt{Hg} , with H denoting the ocean depth and g , the acceleration of gravity. In the deep ocean, they are about 5-10 times slower than acoustic waves. [5](#)

Mach number Dimensionless quantity describing the ratio between the velocity of a fluid and the velocity of sound in that fluid. It is usually denoted by M and defined as follows

$$M = \frac{U}{c},$$

where U is the local flow velocity to the boundaries, and c is the speed of sound in the medium. Low values of the Mach number indicate that high-velocity variations are required to produce a change in density. Thus the flow tends to be incompressible. Experimentally, the incompressible behavior of the flow occurs for $M < 0.3$. 2

Munk layer Boundary layer characterized by a balance of the meridional transport of planetary vorticity and the viscous dissipation. It is responsible for the western intensification phenomenon. 16, 19

no-slip condition Condition stating that in the absence of retarding frictional forces, the fluid will have zero velocity relative to the boundary at a solid surface. 2, 3, 15

reduced-gravity model Model based on the assumption that the gravitational field of a given environment is smaller than that of the Earth. 4

Reynolds number Dimensionless number describing the ratio between inertial and viscous forces. It used to characterize a flow in laminar (low Reynolds number), transient, turbulent (large Reynolds number). It is defined by $Re = \frac{UL}{\nu}$, where where U is the local flow velocity, L is a characteristic length and ν is the kinematic viscosity. 2, 3, 15

Rosby number In geophysical fluid dynamics, it is a dimensionless number representing the ratio between the forces of inertia and the forces due to the rotation, which characterizes the movement of a fluid in a rotating frame. It is computed using the formula

$$Ro = \frac{U}{fL},$$

where U is the local flow velocity, L is a characteristic length, and f is the Coriolis parameter. 2, 15

Rosby waves Also known as planetary waves, they are oscillatory movements of the atmosphere or ocean of long wavelength. They are a consequence of variations of the Coriolis force. 18

Stirling formula It is an approximation for factorials leading to accurate results. The usual version of the formula is

$$\ln n! = n \ln n - n + O(\ln n).$$

Here the big O notation is considered as $n \rightarrow \infty$. It can be also found written as $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$, where e is the Euler's number. 3

stratified fluid Fluid with density variations in the vertical direction, see [stratification](#). 5

Sverdrup balance Theoretical relationship between the wind stress at the ocean surface and the vertically integrated transport of ocean water resulting from the Ekman pumping/suction. 4

vertical momentum balance The momentum equation is a statement of Newton's Second Law and relates the sum of the forces acting on a fluid to its acceleration. Since the velocity field is a vector, it is possible to obtain the independent equations for the horizontal and vertical components. The vertical balance equation is given by

$$\frac{Dw}{Dt} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z},$$

where g is the effective gravity acceleration, p is the pressure and ρ denotes the density of the fluid . 13

western intensification At the beta plane, a zonal asymmetry appears in the circulation and consistently at sea level. Indeed, to the east, the cyclonic beta effect partially compensates for the anticyclonic wind stress so that less bottom friction is needed for the flow to reach a vorticity balance. As friction is proportional to vorticity, this means that the flow must slow down. Reversely, to the west, both the beta effect and the wind stress impart anticyclonic vorticity. Hence the bottom friction must be enhanced to balance both terms. This is done through the intensification of the northward flow. As the basin is closed, by continuity, the northward and southward transports must be equal, which implies a narrow western boundary northward flow and a wide interior southward flow. 4, 6

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Sujet : Étude asymptotique de deux problèmes de la mécanique de fluides

Résumé : Ce manuscrit porte sur l'analyse asymptotique de deux problèmes provenant de la mécanique des fluides : l'effet des rugosités sur le comportement des courants océaniques, et la description mathématique des phénomènes de congestion dans la croissance tumorale. Tout d'abord, nous analysons l'impact des irrégularités du littoral sur le mouvement océanique entraîné par le vent, lorsque la géométrie des côtes ne suit pas un modèle spatial précis. Cette hypothèse a deux conséquences principales dans l'étude du problème singulièrement perturbé qu'est le modèle quasigeostrophique 2D : les équations de couches limites sont définies dans des domaines infinis avec des données aux bords non décroissantes ; et la couche limite à l'Est présente des problèmes de convergence loin de la frontière. Nous montrons le caractère bien posé de ces problèmes dans des espaces de Sobolev non locaux en nous servant des propriétés d'ergodicité et de l'analyse pseudo-différentielle. Nous construisons une solution approchée du problème original et analysons sa convergence. Dans la deuxième partie de ce travail, nous étudions une équation du milieu poreux (PME) unidimensionnelle modélisant les propriétés mécaniques de la croissance tumorale. Nous nous intéressons à la limite singulière "loi de pression rigide" lorsque le PME dégénère vers un problème à frontière libre de type Hele-Shaw. Nous proposons une description précise des ondes progressives au voisinage de la transition entre le domaine libre à pression nulle et le domaine incompressible à pression positive. Nous effectuons après une analyse de stabilité des ondes progressives.

Mots clés : mécanique des fluides - analyse asymptotique - couches limites - modèle quasigeostrophique - équation du milieu poreux - ondes progressives - limite d'incompressibilité - équations de Hele-Shaw -croissance tumorale

Subject : Asymptotic study of two problems in fluid mechanics

Abstract: This manuscript deals with the asymptotic analysis of two problems arising in fluid mechanics: the effect of roughness on oceanic motion taking as a starting point the single-layered quasigeostrophic equation and the mathematical description of congestion phenomena in tumor growth. First, we are interested in the impact of the irregularities of the coastline on wind-driven oceanic motion when the geometry of the coasts does not follow a specific spatial pattern. The assumption on the roughness has two main consequences in the asymptotic analysis of the quasigeostrophic model: the governing boundary layer equations are defined in infinite domains with not-decaying boundary data, and the eastern boundary layer exhibits convergence issues far from the boundary. We establish the well-posedness of the solution of the boundary layer profiles in nonlocalized Sobolev space by adding ergodicity properties and using pseudo-differential analysis. We construct an approximate solution to the original problem and analyze its convergence. In the second part of this work, we study a one-dimensional porous medium equation (PME) modeling the mechanical properties of tumor growth. We are interested in the singular "stiff pressure law" limit when the PME degenerates towards a free boundary problem of Hele-Shaw type. We provide a refined description of the traveling waves in the vicinity of the transition between the free domain with zero pressure and the congested domain with positive pressure and then perform a stability analysis of the traveling waves.

Keywords : fluid mechanics - asymptotic analysis- boundary layers - quasigeostrophic model -porous medium equation- traveling waves-incompressible limit -Hele-Shaw equations -tumor growth