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L'UNIVERSITÉ DE RENNES 1

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*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*
Spécialité : *MI : Mathématiques et leurs Interactions*

Par

Louis EMERALD

Modèles de type dispersion complète en océanographie côtière

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Table of Contents

0	Introduction	7
0.1	Le modèle des vagues	9
0.1.1	Hypothèses de modélisation et équations de Bernoulli avec surface libre	9
0.1.2	Dérivation des équations des vagues adimensionnelles	13
0.2	Les modèles bidirectionnels d'eau peu profonde	18
0.2.1	Construction d'approximations de l'opérateur de Dirichlet-Neumann	18
0.2.2	Les modèles bidirectionnels classiques d'eau peu profonde	21
0.2.2.1	Justification du modèle de Saint-Venant	22
0.2.2.2	Justification du modèle de Green-Naghdi	24
0.2.2.3	Justification d'un modèle de Boussinesq	27
0.2.2.4	Comparaison de modèles dans la propagation de tsunamis générés par des glissements de terrains	30
0.2.3	Les modèles bidirectionnels de type dispersion complète	32
0.2.3.1	Les modèles de Whitham-Boussinesq	33
0.2.3.2	Les modèles de Whitham-Green-Naghdi	39
0.3	Les modèles unidirectionnels d'eau peu profonde	43
0.3.1	Construction d'invariants de Riemann et dérivation de modèles unidirectionnels	43
0.4	Algorithme de la forme normale de Birkhoff et obtention de modèles unidirectionnels	47
0.5	Conclusion et perspectives	53
0.5.1	Conclusion	53
0.5.2	Perspectives	54

1	Rigorous derivation from the water waves equations of some full dispersion shallow water models	57
1.1	Introduction	57
1.1.1	Motivations	57
1.1.2	Consistency problem	58
1.1.3	Outline	66
1.2	Shallow water approximations	67
1.2.1	Shallow water approximations of the velocity potential in the flat strip	67
1.2.2	Shallow water approximations of the vertically averaged horizontal component of the velocity	70
1.3	Derivation and consistency of a first full dispersion Green-Naghdi system .	75
1.3.1	Formal Derivation	75
1.3.2	Consistency with respect to the water waves system	76
1.4	Derivation and consistency of a second full dispersion Green-Naghdi system	79
1.4.1	Formal Derivation	79
1.4.2	Consistency with respect to the water waves equations	81
1.5	Consistency of other full dispersion models appearing in the literature . . .	90
1.5.1	Full dispersion Green-Naghdi system	90
1.5.2	Full dispersion Boussinesq systems	92
1.6	Technical tools	94
2	On a well-posedness result for a class of non-local quasi-linear systems and its application to the justification of Whitham-Boussinesq systems	99
2.1	Introduction	99
2.1.1	Motivations and main results	99
2.1.2	Outline	105
2.2	Energy estimates	106
2.2.1	Symmetrization	106
2.2.2	Energy estimates of order 0	109
2.2.3	Energy estimates of higher order	113
2.3	Local well-posedness and stability	115
2.3.1	Well-posedness of the linearized systems	115
2.3.2	Well-posedness of the non-linear systems	118
2.3.3	Blow up criterion	120

2.4	Stability	121
2.5	Full justification of a class of Whitham-Boussinesq systems	122
2.6	Technical Tools	124
3	Rigorous derivation of the Whitham equations from the water waves equations in the shallow water regime	125
3.1	Introduction	125
3.1.1	Motivations	125
3.1.2	Main results	128
3.1.3	Outline	135
3.2	Derivation of the Whitham equations from the Riemann invariants of a Whitham-Boussinesq system	136
3.2.1	Diagonalization of the Whitham-Boussinesq equations	137
3.2.2	From the diagonalized Whitham-Boussinesq equations to the Whitham equations	143
3.3	Decoupling the water waves equations into two Whitham equations	150
3.3.1	From the water waves equations to a Whitham-Boussinesq system	151
3.3.2	Application of Birkhoff's algorithm	154
3.3.3	From the Hamiltonian of the Whitham-Boussinesq system under normal form to two decoupled Whitham equations	164
3.4	Technical Tools	173
4	Comparison of models for the simulation of landslide generated tsunamis	177
4.1	Introduction	178
4.2	Description of the study	179
4.2.1	The test case	179
4.2.2	The models and their numerical resolution	180
4.2.2.1	OpenFOAM solver of the free surface incompressible Navier-Stokes model	180
4.2.2.2	Hydra solver of the shallow water model	181
4.2.2.3	AVALANCHE solver of the Boussinesq model	182
4.3	Comparison of models	183
4.3.1	The methodology	183
4.3.1.1	Strategy of computation of the bottom displacement	183

TABLE OF CONTENTS

4.3.1.2	Simulated data based and treatment	184
4.3.2	The numerical results	186
4.3.2.1	Comparison in the wave generation zone	186
4.3.2.2	Comparison in the wave propagation zone	189
4.3.2.3	Quantities of interest and correlations between the models	191
4.3.2.4	A shallow water/Boussinesq mixed model	193
4.4	Inverse problem	194
4.4.1	The methodology	195
4.4.1.1	Modeling assumptions	195
4.4.1.2	Numerical strategy	196
4.4.2	The numerical results	197
4.4.2.1	Gaussian in translation	198
4.4.2.2	Gaussian in translation with decay	198
4.5	Conclusion	199

Bibliography	201
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Chapitre 0

Introduction

Les équations générales de la dynamique des vagues surfaciques en océanographie côtière, les équations des vagues, sont d'une complexité telle que l'étude qualitative des solutions et la résolution numérique de ces équations est probablement impossible avec les outils mathématiques actuels. L'un des enjeux du domaine est donc de construire de nouveaux modèles, simplifiés, mais qui restent valables dans des régimes de comportements physiques particuliers. Un de ces régimes est le régime d'eau peu profonde. Il est caractérisé par des vagues dont la longueur d'onde horizontale est grande par rapport à la profondeur de l'eau. C'est le cas des tsunamis par exemple, dont les longueurs d'ondes peuvent atteindre plusieurs centaines de kilomètres, alors que la profondeur de l'eau plafonne à quelques kilomètres. Dans ce cadre, nous connaissons de nombreux modèles bidirectionnels et unidirectionnels dont les solutions approchent celles des équations des vagues avec une plus ou moins grande précision. Nous pouvons mentionner, par exemple, les équations de Saint-Venant, les équations de Boussinesq, les équations de Serre-Green-Naghdi, les équations de Korteweg-de-Vries et bien d'autres encore. Tous les modèles précédemment cités sont appelés des modèles d'eau peu profonde. Ils ont été rigoureusement justifiés en tant qu'approximation des équations des vagues dans le régime d'eau peu profonde ou des sous-régimes de ce dernier. Cependant, il existe de nombreuses situations dans lesquelles les vagues changent radicalement de comportement au cours de leur propagation, passant par exemple d'un régime de petite profondeur d'eau à un autre régime comme celui de faible nonlinéarité. Ce dernier est caractérisé par des vagues dont l'amplitude surfacique est petite par rapport à la profondeur de l'eau. Une situation de ce type a été mise en évidence dans [BB93 ; Din94], où les auteurs ont étudié le changement de comportement dans la propagation d'une vague de petite amplitude et de grande longueur d'onde lorsqu'elle passe

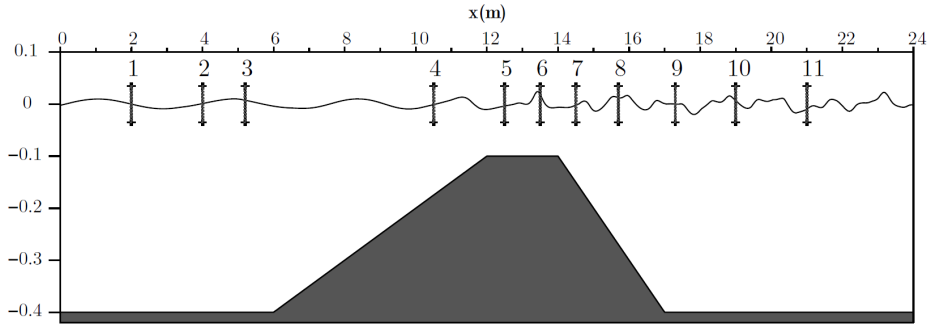


FIGURE 1 – Changement de comportement dans la propagation d’une vague passant au-dessus d’un mont marin [Lan13b]

dans une zone où le fond marin forme un mont (figure 1). Lorsque la vague arrive au-dessus du mont, il y a création d’harmoniques de hautes fréquences qui sont ensuite relâchées librement une fois le mont passé, ce qui change le comportement de la propagation. Dans ce cas de figure, les modèles classiques d’eau peu profonde échouent à approcher les équations des vagues dans la deuxième partie de la propagation [Lan13b]. Dans le but d’améliorer la robustesse des modèles d’eau peu profonde, de nombreuses modifications de ces dernières ont été proposées dans la littérature. Elles jouent toutes sur le caractère dispersif des modèles, de telle sorte que la relation de dispersion associée se rapproche au mieux de celle des équations des vagues. Même si ce ne sera pas l’objet de notre étude, on peut mentionner dans ce cadre les équations de Boussinesq ou de Green-Naghdi à un ou plusieurs paramètres [Lan13b]. Dans cette thèse nous nous focaliserons sur les modifications des modèles d’eau peu profonde dont la relation de dispersion associée est exactement celle des équations des vagues. On les appelle les modèles de type dispersion complète. Historiquement, le premier modèle de type dispersion complète, appelé les équations de Whitham, a été introduit par G. Whitham en 1967 dans [Whi67]. C’est une modification du modèle d’eau peu profonde unidirectionnel, les équations de Korteweg-de Vries. Son but était d’étudier deux phénomènes majeurs de l’océanographie côtière, les vagues surplombantes et les vagues de Stokes d’amplitude maximales, c’est-à-dire des vagues périodiques, non régulières, se propageant au-dessus d’une eau de profondeur moyenne constante. Par la suite, d’autres modèles de type dispersion complète ont été introduit dans la littérature, comme les équations de Whitham-Boussinesq ou de Whitham-Green-Naghdi, et ont fait l’objet d’une recherche active. L’un des buts principaux de cette thèse est d’apporter une contribution à la justification rigoureuse de ces modèles. C’est-à-dire de montrer que les équations sont

proches de celles des équations de vagues (voir [Eme20 ; Eme21b]), qu'elles sont bien-posées localement, au sens où pour toute condition initiale donnée, suffisamment régulière et vérifiant l'hypothèse de non-cavitation, il existe une unique solution dans des espaces bien choisis ([Eme21a] en préparation), et que ses solutions sont proches de celles des équations des vagues ([Eme21b ; Eme21a]). Le but étant aussi de caractériser précisément la précision de ces modèles.

La première section de cette introduction 0.1 est dédiée à l'établissement du modèle général de l'océanographie côtière, les équations des vagues. Dans la deuxième section 0.2, nous présentons les résultats clés, permettant de dériver et justifier des modèles bidirectionnels simplifiés, dans le régime asymptotique d'eau peu profonde. Nous présenterons à la fois les modèles bidirectionnels classiques, et les modèles de type dispersion complète. Dans la troisième et dernière section 0.3, nous présentons deux méthodes de dérivation des équations de type dispersion complète historiques, les équations de Whitham.

0.1 Le modèle des vagues

Le but de cette première partie de l'introduction est de définir les hypothèses de modélisation et de voir comment, de celles-ci, on obtient les équations des vagues adimensionnelles.

0.1.1 Hypothèses de modélisation et équations de Bernoulli avec surface libre

On veut décrire le mouvement d'un fluide irrotationnel, soumis à la gravité, délimité par un fond fixe, et une surface le séparant d'un autre fluide de densité négligeable. On pourra supposer que le fluide soumis à la gravité est l'eau, et que le fluide de densité négligeable est l'air. Dans ce cadre, nous formulons les hypothèses de modélisation suivantes :

- (H1) L'eau est homogène et non visqueuse.
- (H2) L'eau est incompressible.
- (H3) Le mouvement de l'eau est irrotationnel. On suppose donc qu'il n'y a pas de tourbillon. Cette hypothèse est souvent faite en océanographie côtière.
- (H4) La surface de l'eau et le fond marin peuvent être paramétrés par des graphes de fonctions. On supposera en fait dans cette thèse que le fond marin est plat.

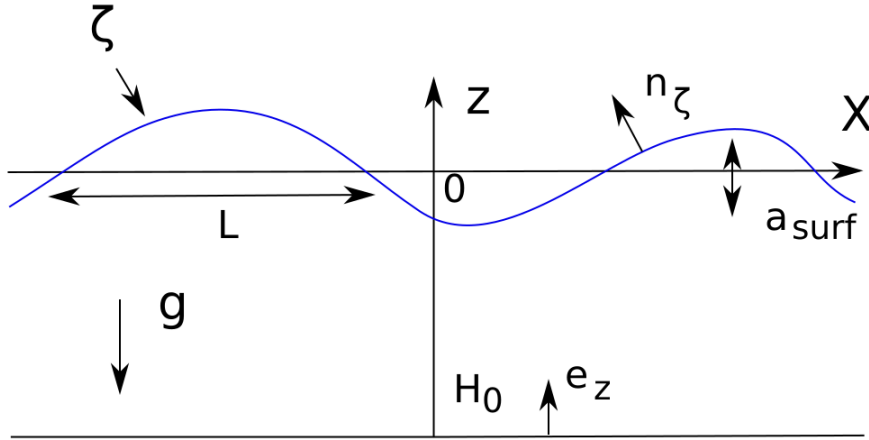


FIGURE 2 – Système étudié et notations.

- (H5) Les particules d'eau ne traversent pas le fond marin.
- (H6) Les particules d'eau ne traversent pas l'interface eau/air.
- (H7) On suppose qu'il n'y a pas de tension de surface et que la pression extérieure est constante.
- (H8) L'eau est au repos à l'infini.
- (H9) La profondeur de l'eau est toujours minorée par une constante strictement positive. Ceci est une restriction sérieuse pour l'océanographie côtière, mais elle est indispensable pour l'analyse mathématique du système. On appelle cette hypothèse, l'hypothèse de non-cavitation.

Voyons maintenant comment traduire mathématiquement ces hypothèses. Mais avant cela, il convient de poser des notations.

Notation 0.1.1. — *Nous séparerons la variable spatiale en deux parties. Nous noterons X la variable spatiale longitudinale, $X \in \mathbb{R}^d$, où $d = 1, 2$. Et nous noterons z la variable spatiale transversale, $z \in \mathbb{R}$.*

- *Le domaine occupé par l'eau au temps t sera noté Ω_t .*
- *La vitesse de la particule de fluide localisée en $(X, z) \in \Omega_t$ au temps t sera noté $U(t, X, z) \in \mathbb{R}^{d+1}$.*
- *La pression de l'eau sera notée $P(t, X, z) \in \mathbb{R}$.*

- La gravité sera notée $-g\vec{e}_z$, avec g l'intensité de la pesanteur et \vec{e}_z le vecteur normal unitaire vertical ascendant.
- La densité de l'eau sera notée ρ .

Pour traduire les hypothèses (H1) et (H2), il faut savoir qu'il existe deux types d'équations qui permettent de modéliser la dynamique d'un fluide homogène incompressible, les équations de Navier-Stokes et les équations d'Euler. Ce qui les différencie, c'est la prise en compte ou non de la viscosité. Ici, on considère un fluide non visqueux, donc ces hypothèses se traduisent mathématiquement par le fait que le champ de vitesse du fluide, U , et la pression du fluide P vérifient les équations d'Euler incompressibles :

$$(H1) \text{ et } (H2) \implies \begin{cases} \partial_t U + (U \cdot \nabla_{X,z})U = -\frac{1}{\rho}\nabla_{X,z}P - g\vec{e}_z & \text{dans } \Omega_t, \\ \nabla_{X,z} \cdot U = 0 & \text{dans } \Omega_t. \end{cases} \quad (1)$$

Ici $\nabla_{X,z}$ désigne le gradient en espace, i.e.

$$\nabla_{X,z} := \begin{cases} (\partial_x, \partial_z)^T & \text{si } d = 1, \\ (\partial_x, \partial_y, \partial_z)^T & \text{si } d = 2. \end{cases}$$

L'hypothèse (H3), d'irrotationalité du mouvement de l'eau, nous dit que le champ de vitesse U dérive d'un potentiel, qu'on notera par la suite Φ , défini à une constante additive près, dépendante du temps, i.e. $U = \nabla_{X,z}\Phi$. Ceci, couplé avec $\nabla_{X,z} \cdot U = 0$, nous donne $\Delta_{X,z}\Phi = 0$, où $\Delta_{X,z}$ est la laplacien dans \mathbb{R}^{d+1} .

On peut donc réécrire les équations d'Euler (1) de la manière suivante :

$$\partial_t \nabla_{X,z}\Phi + (\nabla_{X,z}\Phi \cdot \nabla_{X,z})\nabla_{X,z}\Phi = -\frac{1}{\rho}\nabla_{X,z}P - g\vec{e}_z \text{ dans } \Omega_t.$$

De plus, étant donné que $\nabla_{X,z}\Phi$ est irrotationnel, nous avons l'identité

$$(\nabla_{X,z}\Phi \cdot \nabla_{X,z})\nabla_{X,z}\Phi = \frac{1}{2}\nabla_{X,z}|\nabla_{X,z}\Phi|^2.$$

D'où

$$\nabla_{X,z}\partial_t\Phi + \frac{1}{2}\nabla_{X,z}|\nabla_{X,z}\Phi|^2 = -\frac{1}{2}\nabla_{X,z}P - g\vec{e}_z \text{ dans } \Omega_t. \quad (2)$$

En intégrant (2), tout en considérant l'hypothèse (H7), on obtient que Φ et P vérifient les

équations de Bernoulli incompressibles dans le domaine de l'eau Ω_t :

$$\begin{cases} \partial_t \Phi + \frac{1}{2} |\nabla_{X,z} \Phi|^2 + gz = -\frac{1}{\rho} (P - P_{\text{atm}}), \\ \Delta_{X,z} \Phi = 0, \end{cases} \quad (3)$$

où P_{atm} est la pression atmosphérique, et où la première équation est définie à une constante additive près, dépendante du temps (qui peut être choisie nulle par le choix correspondant de la constante additive de Φ).

L'hypothèse (H4) nous donne l'existence d'une fonction $\zeta : (t, X) \in [0, T[\times \mathbb{R}^d \rightarrow \zeta(t, X) \in \mathbb{R}$, où $T > 0$, telle que pour tout temps $t \in [0, T[$, le domaine de l'eau Ω_t s'écrive

$$\Omega_t = \{(X, z) \in \mathbb{R}^{d+1}; -H_0 < z < \zeta(t, X)\},$$

où H_0 est une constante de référence pour la profondeur de l'eau.

Notons $\vec{n}_b(X, z)$ le vecteur normal unitaire du fond marin, entrant dans le domaine Ω_t . On a $\vec{n}_b(X, z) = \vec{e}_z$. Et notons \vec{n}_ζ le vecteur normal unitaire de la surface de l'eau, sortant du domaine Ω_t (voir figure 2). Montrons que les hypothèses (H5) et (H6) se traduisent par respectivement

$$\partial_z \Phi = 0 \quad \text{sur} \quad \{z = -H_0\}, \quad (4)$$

$$\text{et} \quad \partial_t \zeta - \sqrt{1 + |\nabla_X \zeta|^2} \nabla_{X,z} \Phi \cdot \vec{n}_\zeta = 0 \quad \text{sur} \quad \{z = \zeta(t, X)\}. \quad (5)$$

Ici ∇_X est le gradient longitudinale.

De manière générale, notons Γ_t l'hypersurface d'équation $\gamma(t, X, z) = 0$, et $M(t) = (X(t), z(t))$ la position d'une particule de fluide au temps t . Cette particule est sur Γ_t si et seulement si $\gamma(t, M(t)) = 0$ et reste sur Γ_t si $\frac{d}{dt} \gamma(t, M(t)) = 0$. Ce qui est équivalent à

$$\partial_t \gamma + \frac{d}{dt} M(t) \cdot \nabla_{X,z} \gamma = 0.$$

Et puisque par définition $\frac{d}{dt} M(t) = U = \nabla_{X,z} \Phi$, il reste à prendre $\gamma = z - \zeta(t, X)$ ou $\gamma = z - H_0$ pour obtenir le résultat.

L'hypothèse (H8) se traduit naturellement par

$$\forall t \in [0, T[, \lim_{(X,z) \in \Omega_t, |X| \rightarrow +\infty} |\zeta(t, X)| + |\nabla_{X,z} \Phi(t, X, z)| = 0. \quad (6)$$

L'hypothèse (H9) nous donne l'existence d'une constante $h_{\min} > 0$ telle que

$$\forall(t, X) \in [0, T[\times \mathbb{R}^d, H_0 + \zeta(t, X) \geq h_{\min}. \quad (7)$$

Finalement, le potentiel de vitesse Φ , la pression P et la fonction définissant la surface de l'eau ζ , vérifient ce qu'on appelle les équations de Bernoulli avec surface libre, i.e.

$$\left\{ \begin{array}{l} \partial_t \Phi + \frac{1}{2} |\nabla_{X,z} \Phi|^2 + gz = -\frac{1}{\rho} (P - P_{\text{atm}}) \quad \text{dans } \Omega_t, \\ \Delta_{X,z} \Phi = 0 \quad \text{dans } \Omega_t, \\ H_0 + \zeta(t, X) \geq h_{\min}, \\ \Omega_t = \{(X, z) \in \mathbb{R}^{d+1}; -H_0 < z < \zeta(t, X)\}, \\ \partial_z \Phi = 0 \quad \text{sur } \{z = -H_0\}, \\ \partial_t \zeta - \sqrt{1 + |\nabla_X \zeta|^2} \nabla_{X,z} \Phi \cdot \vec{n}_\zeta = 0 \quad \text{sur } \{z = \zeta(t, X)\}, \\ \lim_{(X,z) \in \Omega_t, |X| \rightarrow +\infty} |\zeta(t, X, z)| + |\nabla_{X,z} \Phi(t, X, z)| = 0, \\ P = P_{\text{atm}} \quad \text{sur } \{z = \zeta(t, X)\}. \end{array} \right. \quad (8)$$

0.1.2 Dérivation des équations des vagues adimensionnelles

Dans son article [Zak68], Zakharov a remarqué qu'il était possible de reconstruire entièrement le potentiel de vitesse, Φ , dans tout le domaine de l'eau, Ω_t , à partir des seules données de la fonction ζ et du potentiel de vitesse à la surface de l'eau, que l'on notera ψ ; ceci à condition de pouvoir résoudre le problème harmonique suivant :

$$\left\{ \begin{array}{l} \Delta_{X,z} \Phi = 0 \quad \text{dans } \Omega_t, \\ \Phi|_{z=\zeta} = \psi, \quad \partial_z \Phi|_{z=-H_0} = 0. \end{array} \right. \quad (9)$$

La résolution du problème (9) est possible si on suppose suffisamment de régularité sur ζ et ψ , et que ces dernières vérifient les hypothèses (6) et (7).

De plus, si on connaît Φ , alors on retrouve le champ de vitesse U via l'identité $U = \nabla_{X,z} \Phi$. Et les équations d'Euler (1) nous permettent d'obtenir la pression P .

L'objectif de cette sous-section sera donc de déterminer deux équations d'évolutions en ζ et ψ .

Pour cela, dans leurs papiers [CS93; CSS92], Craig, Sulem, et Sulem ont utilisé l'opérateur de Dirichlet-Neumann.

Definition 0.1.2. On définit l'opérateur de Dirichlet-Neumann par

$$\mathcal{G}[\zeta] : \psi \rightarrow \sqrt{1 + |\nabla_X \zeta|^2} \nabla_{X,z} \Phi \cdot \vec{n}_\zeta|_{z=\zeta} = -\nabla_X \zeta \cdot \nabla_X \Phi|_{z=\zeta} + \partial_z \Phi|_{z=\zeta},$$

où Φ résoud le problème harmonique (9).

L'opérateur de Dirichlet-Neumann est linéaire en ψ mais non-linéaire en ζ .

En considérant cet opérateur, l'équation cinématique (5) se réécrit

$$\partial_t \zeta - \mathcal{G}[\zeta] \psi = 0. \quad (10)$$

Aussi, en utilisant les identités suivantes :

$$\begin{cases} (\partial_t \Phi)|_{z=\zeta} = \partial_t \psi - (\partial_z \Phi)|_{z=\zeta} \partial_t \zeta, \\ (\nabla_{X,z} \Phi)|_{z=\zeta} = \nabla_X \psi - (\partial_z \Phi)|_{z=\zeta} \nabla_X \zeta, \\ (\partial_z \Phi)|_{z=\zeta} = \frac{\mathcal{G}[\zeta] \psi + \nabla_X \zeta \cdot \nabla_X \psi}{1 + |\nabla_X \zeta|^2}, \end{cases}$$

et après quelques calculs simples, l'équation d'évolution en Φ du système de Bernoulli incompressible (3) devient, après intégration,

$$\partial_t \psi + g\zeta + \frac{1}{2} |\nabla_X \psi|^2 - \frac{(\mathcal{G}[\zeta] \psi + \nabla_X \zeta \cdot \nabla_X \psi)^2}{2(1 + |\nabla_X \zeta|^2)} = 0. \quad (11)$$

La concaténation des deux équations (10) et (11) forme les équations des vagues :

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta] \psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2} |\nabla_X \psi|^2 - \frac{(\mathcal{G}[\zeta] \psi + \nabla_X \zeta \cdot \nabla_X \psi)^2}{2(1 + |\nabla_X \zeta|^2)} = 0. \end{cases} \quad (12)$$

Ce système possède une structure hamiltonienne. En effet, en posant

$$H(\zeta, \psi) := K(\zeta, \psi) + E(\zeta) := \frac{1}{2} \int_{\mathbb{R}^d} \psi \mathcal{G}[\zeta] \psi dX + \frac{1}{2} \int_{\mathbb{R}^d} g\zeta^2 dX, \quad (13)$$

où ρK est l'énergie cinétique et ρE est l'énergie potentielle, les équations des vagues sont

équivalentes à

$$\begin{cases} \partial_t \zeta = \delta_\psi H, \\ \partial_t \psi = -\delta_\zeta H, \end{cases}$$

où δ_ψ et δ_ζ sont des dérivées fonctionnelles.

L'étude qualitative des équations des vagues (12) présente un problème de taille. Il est possible d'exhiber des solutions aux propriétés qualitatives complètement différentes, dépendant des caractéristiques physiques du flot. Par exemple, toutes choses égales par ailleurs, plus la profondeur de l'eau est élevée, plus les effets de dispersion seront importants. Et plus l'amplitude des vagues sera grande, plus les effets de nonlinéarités seront grands. C'est pour cela que l'on va chercher à adimensionnaliser ces équations en comparant chaque variable et chaque fonction à des grandeurs caractéristiques physiques de même dimension. Parmi celles-ci, il y a la profondeur d'eau caractéristique, la longueur d'onde caractéristique, et l'amplitude caractéristique des vagues. On les notera respectivement H_0 , L et a_{surf} (voir figure 2). De ces comparaisons apparaissent deux paramètres sans dimension qui seront de grande importance dans cette thèse :

$$\mu := \frac{H_0^2}{L^2} \quad \text{et} \quad \varepsilon := \frac{a_{\text{surf}}}{H_0}. \quad (14)$$

Le premier paramètre, μ , est appelé le paramètre de petite profondeur d'eau, et le second paramètre, ε , est appelé le paramètre de nonlinéarité.

Dans cette thèse nous travaillerons dans le régime asymptotique de petite profondeur d'eau et deux sous-régimes associés. Donnons leurs définitions ici.

Definition 0.1.3. *On dit que le système est dans le régime d'eau peu profonde si les paramètres μ et ε sont dans l'ensemble*

$$\mathcal{A} := \{(\mu, \varepsilon); \quad 0 \leq \mu \leq 1, \quad 0 \leq \varepsilon \leq 1\}. \quad (15)$$

On dit que le système est dans le régime longues vagues si les paramètres μ et ε sont dans l'ensemble

$$\mathcal{A}_{\text{LV}} := \{(\mu, \varepsilon); \quad 0 \leq \mu \leq 1, \quad 0 \leq \varepsilon \leq 1, \quad \varepsilon = O(\mu)\}. \quad (16)$$

On dit que le système est dans le régime de faible non-linéarité si les paramètres μ et ε sont dans l'ensemble

$$\mathcal{A}_{\text{FN}} := \{(\mu, \varepsilon); \mu = 1, 0 \leq \varepsilon \leq 1\}. \quad (17)$$

En prenant en compte ces considérations, et après remise à l'échelle adaptée [Lan13b], les équations des vagues deviennent

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla_X \psi|^2 - \frac{\mu \varepsilon (\mathcal{G}^\mu[\varepsilon \zeta] \psi + \varepsilon \nabla_X \zeta \cdot \nabla_X \psi)^2}{2(1 + \mu \varepsilon^2 |\nabla_X \zeta|^2)} = 0, \end{cases} \quad (18)$$

où l'opérateur de Dirichlet-Neumann est maintenant défini par

$$\mathcal{G}^\mu[\varepsilon \zeta] : \psi \rightarrow \sqrt{1 + \varepsilon^2 |\nabla_X \zeta|^2} \nabla_{X,z} \Phi \cdot \vec{n}_\zeta^\mu|_{z=\varepsilon \zeta} = -\mu \varepsilon \nabla_X \zeta \cdot \nabla_X \Phi|_{z=\zeta} + \partial_z \Phi|_{z=\zeta}, \quad (19)$$

et Φ vérifie le problème harmonique suivant :

$$\begin{cases} \Delta_{X,z}^\mu \Phi := \mu \Delta_X \Phi + \partial_z^2 \Phi = 0 \text{ dans } \Omega_t = \{(X, z); -1 < z < \varepsilon \zeta(t, X)\}, \\ \Phi|_{z=\varepsilon \zeta} = \psi, \quad \partial_z \Phi|_{z=-1} = 0. \end{cases} \quad (20)$$

La question du caractère bien posé local des équations des vagues a donné lieu à une recherche active. Nous présentons ici un résultat caractérisant le temps d'existence des solutions en fonction des paramètres μ et ε (voir [Lan05; AL08b; Igu07; Lan13b]).

Mais tout d'abord, voici quelques notations et définitions que nous utiliserons tout au long de l'introduction.

Notations 0.1.4. — Pour tout $s \geq 0$ nous noterons $H^s(\mathbb{R}^d)$ l'espace de Sobolev d'ordre s associé à l'espace $L^2(\mathbb{R}^d)$.

— Pour tout $s \geq 1$ nous noterons $\dot{H}^s(\mathbb{R}^d) := \{f \in L^2_{\text{loc}}(\mathbb{R}^d), \nabla f \in H^{s-1}(\mathbb{R}^d)\}$ l'espace de Beppo-Levi d'ordre s .

— La norme associée à l'espace $L^2(\mathbb{R}^d)$ sera noté $|\cdot|_2$.

— Notons $\Lambda^s := (1 - \Delta)^{s/2}$, où Δ est l'opérateur de Laplace dans \mathbb{R}^d . La norme associée à l'espace $H^s(\mathbb{R}^d)$ sera notée $|\cdot|_{H^s} := |\Lambda^s \cdot|_2$. De plus, la semi-norme associée à l'espace $\dot{H}^{s+1}(\mathbb{R}^d)$ sera $|\cdot|_{\dot{H}^{s+1}} := |\nabla(\cdot)|_{H^s}$.

— Nous utiliserons la notation $C(\circ)$ pour désigner une constante dépendante de manière croissante de \circ .

Definition 0.1.5. Soit $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ une distribution tempérée. Soit \widehat{u} sa transformée de Fourier. Soit $F : \mathbb{R}^d \rightarrow \mathbb{R}$ une fonction lisse avec décroissance polynomiale. Alors l'action du multiplicateur de Fourier associé à la fonction $F(\xi)$, noté $F(D)$ (ou noté F quand aucune confusion n'est possible), est défini par la formule :

$$\widehat{F(D)u}(\xi) = F(\xi)\widehat{u}(\xi).$$

Et maintenant, le résultat.

Théorème 0.1.6. Soit $t_0 > d/2$, $N \geq t_0 + \max(t_0, 2) + 3/2$, et $(\mu, \varepsilon) \in \mathcal{A}$. Soit $U_0 = (\zeta_0, \psi_0) \in E_0^N$ vérifiant l'hypothèse de non-cavitation (7), où pour $T \in \mathbb{R}^+$

$$E_T^N := \{U \in C([0, T], H^{t_0+2} \times \dot{H}^2(\mathbb{R}^d)); \mathcal{E}^N(U(\cdot)) \in L^\infty([0, T])\} \quad (21)$$

avec $\mathcal{E}^N(U)$ la norme d'énergie de U d'ordre N . Formellement,

$$\mathcal{E}^N(U) \text{ " " } \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq N} |\partial^\alpha \zeta|_2^2 + \left| \frac{|D|}{(1 + \sqrt{\mu}|D|)^{1/2}} \partial^\alpha \psi \right|_2^2. \quad (22)$$

Alors, il existe $T > 0$ et une unique solution $U \in E_{T/\varepsilon}^N$ des équations des vagues (18) avec pour conditions initiales U_0 .

De plus,

$$\frac{1}{T} = c_1 \text{ et } \sup_{t \in [0, T/\varepsilon]} \mathcal{E}^N(U(t)) = c_2,$$

où $c_j = C(\frac{1}{h_{\min}}, \mathcal{E}^N(U_0))$, pour $j = 1, 2$.

Remarque 0.1.7. — Le temps d'existence des solutions est d'ordre $1/\varepsilon$ et est indépendant du paramètre de petite profondeur d'eau μ .

- Pour une définition rigoureuse de la norme d'énergie, voir le Chapitre 4, Section 4.1, du livre [Lan13b].
- Contrairement au Théorème 4.16, énoncé dans [Lan13b], le Théorème 0.1.6 ne fait pas intervenir d'hypothèse d'hyperbolicité sur les conditions initiales (critère de Rayleigh-Taylor). Cela est dû au fait qu'on se place ici dans le cadre d'un fond marin plat (voir la section 4.3.5 dans [Lan13b]).

0.2 Les modèles bidirectionnels d'eau peu profonde

Dans cette deuxième partie de l'introduction, nous présentons la construction d'approximations de l'opérateur de Dirichlet-Neumann (19). Nous établissons ensuite les modèles classiques bidirectionnels en eau peu profonde que sont les équations de Saint-Venant, de Green-Naghdi et de Boussinesq. Nous présentons certains résultats du Chapitre 4, dans lequel nous comparons ces modèles dans le cadre de la propagation de tsunamis générés par des glissements de terrains. Nous présentons ensuite les résultats principaux du Chapitre 1 et du Chapitre 2, portant sur des éléments clés de la justification des modèles de Whitham-Green-Naghdi et de Whitham-Boussinesq.

0.2.1 Construction d'approximations de l'opérateur de Dirichlet-Neumann

L'objectif de cette sous-section est de construire des développements asymptotiques en μ et ε (voir (14)) de l'opérateur de Dirichlet-Neumann (19).

La principale difficulté dans la manipulation des équations des vagues vient de l'opérateur de Dirichlet-Neumann, et plus précisément de la dépendance en temps et espace du domaine de l'eau Ω_t , dans lequel le problème harmonique (20) est posé. Nous allons, à l'aide d'un difféomorphisme, transposer cette dépendance en temps/espace du domaine, en une dépendance en temps/espace de l'opérateur laplacien.

Definition 0.2.1. Soit $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ ($t_0 > d/2$) vérifiant l'hypothèse de non-cavitation (7). On définit le difféomorphisme qui envoie la bande $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ sur le domaine de l'eau Ω_t

$$\begin{aligned} \Sigma_t : \mathcal{S} := \mathbb{R}^d \times (-1, 0) &\rightarrow \Omega_t := \{(X, z) \in \mathbb{R}^{d+1} : -1 < z < \varepsilon\zeta(t, X)\} \\ (X, z) &\mapsto (X, z + \varepsilon(z + 1)\zeta(t, X)). \end{aligned}$$

Notons ϕ le potentiel de vitesse dans la bande \mathcal{S} , $\phi = \Phi \circ \Sigma_t$. Il satisfait le problème elliptique

$$\begin{cases} \nabla_{X,z}^\mu \cdot P(\Sigma_t) \nabla_{X,z}^\mu \phi = 0 & \text{dans } \mathcal{S}, \\ \phi|_{z=0} = \psi, \quad \partial_z \phi|_{z=-1} = 0, \end{cases} \quad (23)$$

où ∇^μ est l'opérateur défini par $\nabla_{X,z}^\mu = (\sqrt{\mu} \nabla_X^T, \partial_z)^T$,

et $P(\Sigma_t) = \begin{pmatrix} (1 + \varepsilon\zeta)\text{Id} & -\sqrt{\mu}\varepsilon(z+1)\nabla_X\zeta \\ -\sqrt{\mu}\varepsilon(z+1)\nabla_X\zeta^T & \frac{1+\mu\varepsilon^2(z+1)^2|\nabla_X\zeta|^2}{1+\varepsilon\zeta} \end{pmatrix}$. Le domaine sur lequel le nouveau problème elliptique (23) est posé, est indépendant du temps.

À partir de la connaissance de ϕ , on peut définir une nouvelle quantité.

Definition 0.2.2. *Soit $s \geq 0, t_0 > d/2$, $\psi \in \dot{H}^{s+3/2}(\mathbb{R}^d)$ et $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ vérifiant l'hypothèse de non-cavitation (7). Soit ϕ le potentiel de vitesse dans la bande \mathcal{S} , satisfaisant le problème elliptique (23). Soit aussi h la profondeur de l'eau, $h = 1 + \varepsilon\zeta$. On définit la moyenne verticale de la composante horizontale du champ de vitesse $\bar{V}[\varepsilon\zeta]\psi$ (noté \bar{V} quand aucune confusion n'est possible) par la formule*

$$\bar{V} = \frac{1}{h} \int_{-1}^0 [h\nabla\phi - \varepsilon(z+1)\nabla\zeta\partial_z\phi] dz. \quad (24)$$

Dans [Lan13b], l'auteur montre qu'il existe une relation directe entre l'opérateur de Dirichlet-Neumann et \bar{V} . On a

$$\frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi = -\nabla \cdot (h\bar{V}). \quad (25)$$

Ainsi, le redressement du domaine de l'eau nous permet d'exprimer l'opérateur de Dirichlet-Neumann à partir des quantités ζ et ϕ de manière simple, à travers (24) et (25).

L'idée, pour construire les modèles d'eau peu profonde bidirectionnels classiques, est de construire un développement asymptotique en μ , de la quantité ϕ . Par la relation (24), nous obtiendrons ainsi une expression simplifiée de \bar{V} (et donc de l'opérateur de Dirichlet-Neumann). Ce travail a été effectué dans le Chapitre 2 du livre [Lan13b]. L'auteur obtient les résultats suivant.

Proposition 0.2.3. *Soit $t_0 > d/2$, $s \geq 0$ et $\zeta \in H^{s+4}(\mathbb{R}^d)$ vérifiant l'hypothèse (7) de non-cavitation. Soit aussi $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$ et \bar{V} défini par (24). On a les estimations suivantes :*

$$|\bar{V} - \nabla\psi|_{H^s} \leq \mu C\left(\frac{1}{h_{\min}}, |\zeta|_{H^{\max(t_0+1, s+2)}}\right) |\nabla\psi|_{H^{s+2}}, \quad (26)$$

$$|\bar{V} - \nabla\psi - \frac{\mu}{3h} \nabla(h^3 \Delta\psi)|_{H^s} \leq \mu^2 C\left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+4}}\right) |\nabla\psi|_{H^{s+4}}, \quad (27)$$

$$|\bar{V} - \nabla\psi - \frac{\mu}{3h} \nabla(h^3 \nabla \cdot \bar{V})|_{H^s} \leq \mu^2 C\left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+4}}\right) |\nabla\psi|_{H^{s+4}}. \quad (28)$$

Les estimations (26) et (28) montrent que dans la recherche de modèles approchant les équations des vagues à l'ordre μ ou μ^2 , on peut changer l'inconnue ψ dans les équations, par \bar{V} , quitte à rajouter des termes. Pour construire des modèles ayant de bonnes propriétés mathématiques, notamment de caractère bien posé, nous considérerons des modèles en ζ et \bar{V} . Nous verrons cela dans la sous-section suivante 0.2.2.

Les premiers résultats originaux de cette thèse, développés dans le Chapitre 1 (papier à paraître dans le journal *Journal on Mathematical Analysis* (SIMA) [Eme20]), portent sur la construction d'approximations de la quantité ϕ , et donc de \bar{V} , qui restent cohérentes lors d'un changement de régime de comportement dans la dynamique des vagues [Eme20]. Dans la proposition suivante, nous présentons quelques unes des approximations obtenues.

Proposition 0.2.4. *Soit $s \geq 0$, et $\zeta \in H^{s+4}(\mathbb{R}^d)$ vérifiant l'hypothèse de non-cavitation (7). Soit $\psi \in \dot{H}^{s+5}(\mathbb{R}^d)$, et \bar{V} défini par (24). Soit aussi F_1^μ, F_2^μ et F_3^μ trois multiplicateurs de Fourier définis respectivement par*

$$F_1^\mu := \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}, \quad F_2^\mu := \frac{3}{\mu|D|^2} \left(1 - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\right) \quad \text{et} \quad F_3^\mu := F_2^\mu (F_1^\mu)^{-1}, \quad (29)$$

où $(F_1^\mu)^{-1}$ est le multiplicateur de Fourier inverse de F_1^μ , associé à la fonction $1/F_1^\mu$. On a les estimations suivantes :

$$|\bar{V} - F_1^\mu \nabla \psi|_{H^s} \leq \mu \varepsilon C \left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+3}} \right) |\nabla \psi|_{H^{s+3}}, \quad (30)$$

$$|\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla (h^3 F_2^\mu \Delta \psi)|_{H^s} \leq \mu^2 \varepsilon C \left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+3}} \right) |\nabla \psi|_{H^{s+4}}, \quad (31)$$

$$|\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla (h^3 F_3^\mu \nabla \cdot \bar{V})|_{H^s} \leq \mu^2 \varepsilon C \left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+4}} \right) |\nabla \psi|_{H^{s+4}}, \quad (32)$$

Remarque 0.2.5. — *Les approximations de la Proposition 0.2.4 sont non-locales. Elles font intervenir des multiplicateurs de Fourier. C'est d'ailleurs de par ces multiplicateurs de Fourier qu'elles diffèrent des approximations de la Proposition 0.2.3.*

— *Les approximations de la Proposition 0.2.4 sont exactes au niveau linéaire (quand on pose $\varepsilon = 0$). Ceci n'est pas le cas pour celles de la Proposition 0.2.3.*

Lorsqu'on se place dans le régime longues vagues ($\varepsilon = O(\mu)$), l'estimation (30) est précise à l'ordre μ^2 , alors que l'estimation (26) n'est précise qu'à l'ordre μ . De même, les estimations (31) et (32) sont précises à l'ordre μ^3 alors que les estimations (27)

et (28) ne sont précises qu'à l'ordre μ^2 . On gagne donc un ordre sur μ à nombre de termes et de dérivées (presque) égales.

- Dans le régime de faible non-linéarité, formellement lorsque $\mu = 1$ et $\varepsilon \ll 1$, les approximations de la Proposition 0.2.3 ne donnent aucune information, alors que celles de la Proposition 0.2.4 restent valables puisque $\varepsilon \ll 1$.

Nous verrons, dans la Sous-section 0.2.3, que les modèles que l'on obtient à partir des estimations de la Proposition 0.2.4 sont exacts au niveau linéaire. On les appelle les modèles de type dispersion complète.

0.2.2 Les modèles bidirectionnels classiques d'eau peu profonde

Dans cette sous-section nous présentons les modèles bidirectionnels classiques d'eau peu profonde que l'on obtient via les approximations de la Proposition 0.2.3.

Définissons à ces fins la notion de cohérence des équations des vagues par rapport à un modèle asymptotique dans le régime d'eau peu profonde.

Definition 0.2.6. Soit \mathcal{R} un ensemble définissant un régime de comportement physique. Notons (A) et (A') deux modèles asymptotiques de la forme :

$$(A) \quad \begin{cases} \partial_t \zeta + \mathcal{N}_{(A)}^1(\zeta, \psi) = 0, \\ \partial_t \psi + \mathcal{N}_{(A)}^2(\zeta, \psi) = 0, \end{cases} \quad (A') \quad \begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((\text{Id} + \mu T_{(A')}[h])\bar{V}) + \mathcal{N}_{(A')}^3(\zeta, \bar{V}) = 0, \end{cases}$$

où $\mathcal{N}_{(A)}^1$, $\mathcal{N}_{(A)}^2$ et $\mathcal{N}_{(A')}^3$ sont des opérateurs non-linéaires qui dépendent respectivement du modèle asymptotique (A) , (A) et (A') . Et où $T_{(A')}$ est un opérateur, non-linéaire en $h = 1 + \varepsilon\zeta$, et linéaire en \bar{V} , qui dépend du modèle asymptotique (A') .

On dit que les équations des vagues sont cohérentes à l'ordre $O(f(\mu, \varepsilon))$, où $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ est une fonction strictement positive, avec respectivement (A) ou (A') dans le régime \mathcal{R} s'il existe $n \in \mathbb{N}$ et $T > 0$ tels que pour tout $s \geq 0$ et $(\mu, \varepsilon) \in \mathcal{R}$, et pour toute solution $(\zeta, \psi) \in C([0, \frac{T}{\varepsilon}]; H^{s+n}(\mathbb{R}^d) \times \dot{H}^{s+n+1}(\mathbb{R}^d))$ des équations des vagues (18) on ait respectivement

$$\begin{cases} \partial_t \zeta + \mathcal{N}_{(A)}^1(\zeta, \psi) = f(\mu, \varepsilon)R_1, \\ \partial_t \psi + \mathcal{N}_{(A)}^2(\zeta, \psi) = f(\mu, \varepsilon)R_2, \end{cases} \quad \text{ou} \quad \begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((\text{Id} + \mu T_{(A')}[h])\bar{V}) + \mathcal{N}_{(A')}^3(\zeta, \bar{V}) = f(\mu, \varepsilon)R_3, \end{cases}$$

où \bar{V} est défini par ζ et ψ à travers (23) et (24), et avec respectivement $|R_1|_{H^s}, |R_2|_{H^s} \leq$

$C(\frac{1}{h_{\min}}, |\zeta|_{H^{s+n}}, |\nabla\psi|_{H^{s+n}})$ ou $|R_3|_{H^s} \leq C(\frac{1}{h_{\min}}, |\zeta|_{H^{s+n}}, |\nabla\psi|_{H^{s+n+1}})$ sur $[0, \frac{T}{\varepsilon}]$ uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{R}$.

0.2.2.1 Justification du modèle de Saint-Venant

Le modèle de Saint-Venant a été introduit par Saint-Venant en 1871. Le but étant d'étudier l'écoulement d'un fluide quasi-unidirectionnel dans un canal ou un cours d'eau. Ce modèle fait, encore aujourd'hui, l'objet d'une recherche active, et est souvent utilisé dans les systèmes de prévention d'inondations, dans l'analyse des ruptures de barrages, dans la modélisation du ruissellement des eaux pluviales, etc.

La justification du modèle de Saint-Venant a fait l'objet de nombreux articles, voir [Ovs74; Ovs76; KN79; Kan86] dans le cas de données analytiques. Dans ce qui suit, nous regarderons le cas de données dans les espaces de Sobolev, voir [Igu09; AL08b; Lan13b].

En utilisant, entre autre, l'estimation (26), l'auteur de [Lan13b] obtient le résultat suivant (voir la Proposition 5.2).

Proposition 0.2.7. *Dans le régime d'eau peu profonde \mathcal{A} (voir (15)), les équations des vagues (18) sont cohérentes à l'ordre $O(\mu)$ avec le système*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t \bar{V} + \nabla \zeta + \varepsilon(\bar{V} \cdot \nabla)\bar{V} = 0, \end{cases} \quad (33)$$

au sens de la Définition 0.2.6, avec $n = 4$.

Le système (33) s'appelle le système (ou les équations) de Saint-Venant. Ce sont des équations non-dispersives.

Remarque 0.2.8. — *Le système de Saint-Venant (33) ne dépend pas du paramètre de petite profondeur d'eau μ .*

— *De manière générale, on dit qu'un système est de type Saint-Venant si c'est un système bidirectionnel approchant les équations des vagues à l'ordre $O(\mu)$ dans le régime de petite profondeur d'eau \mathcal{A} .*

Le système de Saint-Venant a une structure d'équations quasi-linéaires hyperboliques symétrisables. Il existe une littérature riche sur le caractère bien posé local de ce type d'équations (voir par exemple [Mét08]). Les résultats suivants proviennent de [Lan13b].

Proposition 0.2.9. *Soit $t_0 > d/2$, $s \geq t_0 + 1$. Soit $\zeta_0 \in H^s(\mathbb{R}^d)$ vérifiant l'hypothèse de non-cavitation (7). Soit aussi $\bar{V}_0 \in H^s(\mathbb{R}^d)^d$. Il existe $T > 0$ tel que $\forall \varepsilon \geq 0$ les équations de Saint-Venant (33) admettent une unique solution $(\zeta, \bar{V}) \in C([0, T/\varepsilon], H^s(\mathbb{R}^d)^{1+d})$ avec pour conditions initiales (ζ_0, \bar{V}_0) .*

De plus,

$$\frac{1}{T} = c_1 \quad \text{et} \quad \sup_{0 \leq t \leq T/\varepsilon} |\zeta(t)|_{H^s} + |\bar{V}(t)|_{H^s} = c_2,$$

avec $c_j = C(\frac{1}{h_{\min}}, |\zeta_0|_{H^s}, |\bar{V}_0|_{H^s})$ pour $j = 1, 2$.

Remarque 0.2.10. *L'unicité dans la Proposition 0.2.9 nous permet de définir le temps T_{\max}/ε , supremum des temps $T/\varepsilon > 0$ tels que le problème de Cauchy associé aux équations (33) ait une solution $(\zeta, \bar{V}) \in C^0([0, T/\varepsilon], H^s(\mathbb{R}^d)^{1+d})$ vérifiant l'hypothèse de non-cavitation (7). Plus précisément, on a le critère d'explosion suivant :*

$$T_{\max} < \infty \implies \lim_{t \rightarrow T_{\max}/\varepsilon} (|\zeta(t)|_{W^{1,\infty}} + |\bar{V}(t)|_{W^{1,\infty}}) = \infty.$$

En particulier, T_{\max} et la solution maximale ne dépendent pas de l'indice de régularité s .

On a aussi la propriété de stabilité suivante.

Proposition 0.2.11. *Posons les mêmes hypothèses que celles de la Proposition 0.2.9. Supposons aussi qu'il existe $(\tilde{\zeta}, \tilde{\bar{V}}) \in C([0, T/\varepsilon], H^s(\mathbb{R}^d)^{1+d})$ tel que*

$$\begin{cases} \partial_t \tilde{\zeta} + \nabla \cdot (\tilde{h} \tilde{\bar{V}}) = r, \\ \partial_t \tilde{\bar{V}} + \nabla \tilde{\zeta} + \varepsilon (\tilde{\bar{V}} \cdot \nabla) \tilde{\bar{V}} = R, \end{cases} \quad \text{où } \tilde{h} = 1 + \varepsilon \tilde{\zeta},$$

avec $\tilde{R} := (r, R) \in L^\infty([0, T/\varepsilon], H^{s-1}(\mathbb{R}^d)^{1+d})$. Alors, pour tout $t \in [0, T/\varepsilon]$, l'erreur par rapport aux solutions (ζ, \bar{V}) données par la Proposition 0.2.9 vérifie

$$|\mathbf{e}|_{L^\infty([0,t], H^{s-1})} \leq C(c_1, c_2, |\tilde{\zeta}, \tilde{\bar{V}}, \tilde{R}|_{L^\infty([0,t], H^s)}) (|\mathbf{e}|_{t=0}|_{H^{s-1}} + t|\tilde{R}|_{L^\infty([0,t], H^{s-1})}),$$

où $\mathbf{e} = (\zeta, \bar{V}) - (\tilde{\zeta}, \tilde{\bar{V}})$ et c_1, c_2 sont comme dans la Proposition 0.2.9.

La concaténation des résultats du Théorème 0.1.6, et des Propositions 0.2.7, 0.2.9 et 0.2.11, nous donne le théorème de justification (convergence) suivant, du modèle de Saint-Venant.

Théorème 0.2.12. *Soit $N > d/2 + 6$ et $(\mu, \varepsilon) \in \mathcal{A}$ (voir (15)). Soit aussi $U_0 = (\zeta_0, \psi_0) \in E_0^N$ (voir (21)) vérifiant l'hypothèse de non-cavitation (7).*

Alors, il existe $T > 0$ qui ne dépend pas des paramètres $(\mu, \varepsilon) \in \mathcal{A}$ et tel que :

1. *Les équations des vagues (18) admettent une unique solution $U = (\zeta, \psi) \in E_{T/\varepsilon}^N$ (voir (21)) avec pour conditions initiales U_0 , et à laquelle on associe à travers (23) et (24) une vitesse moyenne $\bar{V} \in C([0, T/\varepsilon], H^{N-3}(\mathbb{R}^d)^d)$.*
2. *Il existe une unique solution $(\zeta_{\text{SW}}, \bar{V}_{\text{SW}}) \in C([0, T/\varepsilon], H^{N-3}(\mathbb{R}^d)^{1+d})$ aux équations de Saint-Venant (33) avec pour conditions initiales $(\zeta_0, \bar{V}_0 = \bar{V}|_{t=0})$.*
3. *L'estimation d'erreur suivante est vérifiée pour tout $0 \leq t \leq T/\varepsilon$:*

$$|\mathfrak{e}_{\text{SW}}|_{L^\infty([0, t] \times \mathbb{R}^d)} \leq \mu t C\left(\frac{1}{h_{\min}}, \mathcal{E}^N(U_0)\right),$$

où $\mathfrak{e}_{\text{SW}} = (\zeta, \bar{V}) - (\zeta_{\text{SW}}, \bar{V}_{\text{SW}})$ et où $\mathcal{E}^N(U_0)$ est formellement défini par (22).

0.2.2.2 Justification du modèle de Green-Naghdi

Le modèle de (Serre-)Green-Naghdi a été introduit en dimension $d = 1$ par Serre en 1953 [Ser53], puis redécouvert par Su et Gardner en 1969 [SG69]. En 1976, Green et Naghdi ont établi ce modèle en dimension $d = 2$ [GN76].

La justification du modèle de Green-Naghdi a d'abord été effectuée en dimension $d = 1$ dans [Mak86] pour des données analytiques et dans [Li06] pour des données dans les espaces de Sobolev. Dans ce qui suit nous regarderons le cas $d \geq 1$ dans les espaces de Sobolev, voir [Lan13b].

En utilisant, entre autre, les estimations (27) et (28), l'auteur de [Lan13b] obtient le résultat suivant (voir la Proposition 5.8).

Proposition 0.2.13. *Dans le régime d'eau peu profonde \mathcal{A} (voir (15)), les équations des vagues (18) sont cohérentes à l'ordre $O(\mu^2)$ avec le système*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ (\text{Id} - \frac{\mu}{3h} \nabla (h^3 \nabla \cdot [\circ])) \partial_t \bar{V} + \nabla \zeta + \varepsilon (\bar{V} \cdot \nabla) \bar{V} - \frac{\mu \varepsilon}{3h} \nabla [h^3 ((\bar{V} \cdot \nabla)(\nabla \cdot \bar{V}) - (\nabla \cdot \bar{V})^2)] = 0, \end{cases} \quad (34)$$

au sens de la Définition 0.2.6, avec $n = 6$.

Le système (34) s'appelle le système (les équations) de Green-Naghdi.

Remarque 0.2.14. *De manière générale, on dit qu'un système est de type Green-Naghdi si c'est un système bidirectionnel approchant les équations des vagues à l'ordre $O(\mu^2)$ dans le régime de petite profondeur d'eau \mathcal{A} . Dans [Lan13b], l'auteur dérive des systèmes de type Green-Naghdi dont la relation de dispersion approche celle des équations des vagues à un ordre plus élevé que celle du système (34).*

Dans [Li06], l'auteur a prouvé le caractère bien-posé local de ces équations dans le cas de la dimension $d = 1$. Dans [AL08a] les auteurs ont utilisé un procédé de Nash-Moser pour prouver l'existence et l'unicité de solutions avec une perte de dérivées, pour $d \geq 2$. Dans [FI15], les auteurs ont prouvé le caractère bien posé local des équations de Green-Naghdi dans un cadre général, dans lequel on considère un fond marin non plat et dépendant du temps. Une conséquence directe de leurs travaux est le résultat suivant.

Proposition 0.2.15. *Soit $t_0 > d/2$, $s \geq t_0 + 1$ et $(\mu, \varepsilon) \in \mathcal{A}$ (voir (15)). Soit $\zeta_0 \in H^s(\mathbb{R}^d)$ vérifiant l'hypothèse de non-cavitation (7). Soit aussi $\bar{V}_0 \in X^s$, où*

$$X^s := \{V \in H^s(\mathbb{R}^d)^d, |V|_{X^s} < \infty\} \quad \text{avec} \quad |V|_{X^s}^2 = |V|_{H^s}^2 + \mu |\nabla \cdot V|_{H^s}^2. \quad (35)$$

Il existe $T > 0$ tel que les équations de Green-Naghdi (34) admettent une unique solution $(\zeta, \bar{V}) \in C([0, T/\varepsilon], H^s(\mathbb{R}^d) \times X^s)$, avec pour conditions initiales (ζ_0, \bar{V}_0) .

De plus,

$$\frac{1}{T} = c_1 \quad \text{et} \quad \sup_{0 \leq t \leq T/\varepsilon} |(\zeta(t), \bar{V}(t))|_{H^s \times X^s} = c_2,$$

avec $c_j = C(\frac{1}{h_{\min}}, |\zeta_0|_{H^s}, |\bar{V}_0|_{X^s})$ pour $j = 1, 2$ uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}$.

Remarque 0.2.16. *Il est important de remarquer que le temps d'existence des solutions dans la Proposition 0.2.15 est indépendant du paramètre de petite profondeur d'eau μ . En effet, en tant que modèle asymptotique approchant les équations des vagues lorsque $\mu \ll 1$, on demande à ce que le temps d'existence des solutions du système de Green-Naghdi ne tende pas vers 0 lorsque μ tend vers 0.*

Remarque 0.2.17. *L'unicité dans la Proposition 0.2.15 nous permet de définir le temps T_{\max}/ε , supremum des temps $T/\varepsilon > 0$ tels que le problème de Cauchy associé aux équations (34) ait une solution $(\zeta, \bar{V}) \in C^0([0, T/\varepsilon], H^s(\mathbb{R}^d) \times X^s)$ vérifiant l'hypothèse de non-*

cavitation (7). Plus précisément, on a le critère d'explosion suivant :

$$T_{\max} < \infty \implies \lim_{t \rightarrow T_{\max}/\varepsilon} (|\zeta|_{L^\infty([0,t],H^{t_0+1})} + |\bar{V}|_{L^\infty([0,t],X^{t_0+1})}) = \infty.$$

En particulier, T_{\max} et la solution maximale ne dépendent pas de l'indice de régularité s .

On a aussi la propriété de stabilité suivante.

Proposition 0.2.18. *Posons les mêmes hypothèses que celles de la Proposition 0.2.15. Supposons aussi qu'il existe $(\tilde{\zeta}, \tilde{V}) \in C([0, T/\varepsilon], H^s(\mathbb{R}^d) \times X^s)$ tel que*

$$\begin{cases} \partial_t \tilde{\zeta} + \nabla \cdot (\tilde{h} \tilde{V}) = r, \\ (\text{Id} - \frac{\mu}{3h} \nabla (\tilde{h}^3 \nabla \cdot [\circ])) \partial_t \tilde{V} + \nabla \zeta + \varepsilon (\tilde{V} \cdot \nabla) \tilde{V} - \frac{\mu \varepsilon}{3h} \nabla [\tilde{h}^3 ((\tilde{V} \cdot \nabla)(\nabla \cdot \tilde{V}) - (\nabla \cdot \tilde{V})^2)] = R, \end{cases}$$

avec $\tilde{R} := (r, R) \in L^\infty([0, T/\varepsilon], H^s(\mathbb{R}^d) \times Y^s)$, où

$$Y^s := \{u \in (X^0)'; |u|_{Y^s} < \infty\} \quad \text{avec} \quad |u|_{Y^s} = \sum_{|k|=0}^s |\partial^k u|_{(X^0)'}$$

Alors, pour tout $t \in [0, T/\varepsilon]$, l'erreur par rapport aux solutions (ζ, \bar{V}) données par la Proposition 0.2.15 vérifie

$$|\mathbf{e}|_{L^\infty([0,t],H^s \times X^s)} \leq C(c_1, c_2, |(\tilde{\zeta}, \tilde{V})|_{L^\infty([0,t],H^s \times X^s)}) (|\mathbf{e}|_{t=0}|_{H^s \times X^s} + t|\tilde{R}|_{L^\infty([0,t],H^s \times Y^s)}),$$

où $\mathbf{e} = (\zeta, \bar{V}) - (\tilde{\zeta}, \tilde{V})$ et c_1, c_2 sont comme dans la Proposition 0.2.15.

La concaténation des résultats du Théorème 0.1.6, et des Propositions 0.2.13, 0.2.15 et 0.2.18, nous donne le théorème de justification (convergence) suivant, du modèle de Green-Naghdi.

Théorème 0.2.19. *Soit $N \in \mathbb{N}$ tel que $N > d/2 + 8$ et $(\mu, \varepsilon) \in \mathcal{A}$ (voir (15)). Soit aussi $U_0 = (\zeta_0, \psi_0) \in E_0^N$ (voir (21)) vérifiant l'hypothèse de non-cavitation (7).*

Il existe $T > 0$ qui ne dépend pas des paramètres $(\mu, \varepsilon) \in \mathcal{A}$ et tel que :

1. *Les équations des vagues (18) admettent une unique solution $U = (\zeta, \psi) \in E_{T/\varepsilon}^N$ (voir (21)) avec pour conditions initiales U_0 , et à laquelle on associe à travers (23) et (24) une vitesse moyenne $\bar{V} \in C([0, T/\varepsilon], X^{N-4})$.*
2. *Il existe une unique solution $(\zeta_{\text{GN}}, \bar{V}_{\text{GN}}) \in C([0, T/\varepsilon], H^{N-4}(\mathbb{R}^d) \times X^{N-4})$ aux équations de Green-Naghdi (34) avec pour conditions initiales $(\zeta_0, \bar{V}_0 = \bar{V}|_{t=0})$.*

3. L'estimation d'erreur suivante est vérifiée pour tout $0 \leq t \leq T/\varepsilon$:

$$|\mathbf{e}_{\text{GN}}|_{L^\infty([0,t] \times \mathbb{R}^d)} \leq \mu^2 t C\left(\frac{1}{h_{\min}}, \mathcal{E}^N(U_0)\right),$$

où $\mathbf{e}_{\text{GN}} = (\zeta, \bar{V}) - (\zeta_{\text{GN}}, \bar{V}_{\text{GN}})$ et où $\mathcal{E}^N(U_0)$ est formellement défini par (22).

0.2.2.3 Justification d'un modèle de Boussinesq

Historiquement, le premier modèle de dynamique des vagues, prenant en compte à la fois les effets de non-linéarité et les effets de dispersion, a été établi par Boussinesq en 1872. Aujourd'hui, une multitude de ces modèles ont été obtenus dans la littérature. Nous en présentons un ici.

En enlevant les termes d'ordre $O(\mu^2 + \mu\varepsilon)$ du système de Green-Naghdi, et en utilisant le fait que, \bar{V} , défini par la relation (0.2.2), vérifie

$$\begin{aligned} \nabla \nabla \cdot \partial_t \bar{V} &= \Delta \partial_t \bar{V} + O(\mu), \\ \text{rot}(\bar{V}) &= O(\mu), \end{aligned}$$

nous obtenons le système

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ (\text{Id} - \frac{\mu}{3} \Delta) \partial_t \bar{V} + \nabla \zeta + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) = 0. \end{cases} \quad (36)$$

On a la proposition de cohérence de ce modèle suivante, extrait de [Lan13b].

Proposition 0.2.20. *Dans le régime de petite profondeur d'eau \mathcal{A} , les équations des vagues (18) sont cohérentes à l'ordre $O(\mu^2 + \mu\varepsilon)$ avec le système (36) au sens de la Définition 0.2.6.*

Le système (36) est un système de Boussinesq.

Remarque 0.2.21. — *Dans le régime de longues vagues \mathcal{A}_{LV} , où $\varepsilon = O(\mu)$, les équations des vagues sont cohérentes à l'ordre $O(\mu^2)$ avec le système de Boussinesq (36) au sens de la définition 0.2.6.*

— *De manière générale, on peut définir un système de Boussinesq comme un système bidirectionnel approchant les équations des vagues à l'ordre $O(\mu^2)$ dans le régime longue vague. On peut notamment citer [BCS02; BCL05; Lan13b] où les auteurs*

ont dérivé des modèles de Boussinesq dont la relation de dispersion approche celle des équations des vagues à un ordre plus élevé que celle du système (36).

Dans [SX12], les auteurs prouvent le caractère bien posé local du système (36). Voici le résultat.

Proposition 0.2.22. *Soit $t_0 > d/2$, $s \geq t_0 + 2$ et $(\mu, \varepsilon) \in \mathcal{A}_{LV}$ (voir (16)). Soit $\zeta_0 \in X_1^s(\mathbb{R}^d)$, où pour $k \in \mathbb{N}$*

$$X_k^s := \{V \in H^s(\mathbb{R}^d); |V|_{X_k^s} < \infty\} \quad \text{avec} \quad |V|_{X_k^s}^2 = |V|_{H^s}^2 + \mu^k |V|_{H^{s+k}}^2,$$

avec ζ_0 vérifiant l'hypothèse de non-cavitation (7). Soit aussi $\bar{V}_0 \in X_2^s(\mathbb{R}^d)^d$.

Il existe $c_0 = C(\frac{1}{h_{\min}})$ tel que pour tout $\varepsilon \leq \varepsilon_0 = \frac{1-h_{\min}}{c_0(|\zeta_0|_{X_1^s} + |\bar{V}_0|_{X_2^s})}$, il existe $T > 0$ tel que les équations de Boussinesq (36) admettent une unique solution $(\zeta, \bar{V}) \in C([0, T/\varepsilon], X_1^s(\mathbb{R}^d) \times X_2^s(\mathbb{R}^d)^d)$, avec pour conditions initiales (ζ_0, \bar{V}_0) .

De plus,

$$\frac{1}{T} = c_1 \quad \text{et} \quad \sup_{0 \leq t \leq T/\varepsilon} |(\zeta(t), \bar{V}(t))|_{X_1^s \times X_2^s} = c_2,$$

avec $c_j = C(\frac{1}{h_{\min}}, |(\zeta_0, \bar{V}_0)|_{X_1^s \times X_2^s})$ pour $j = 1, 2$, uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}_{LV}$.

Remarque 0.2.23. *La Remarque 0.2.16 sur l'indépendance en μ du temps d'existence des solutions des équations de Green-Naghdi (33) est aussi valable dans le cas des équations de Boussinesq (36).*

Remarque 0.2.24. *Dans [Bur16], l'auteur prouve un résultat de caractère bien-posé local pour les équations de Boussinesq (36), sans avoir à supposer que l'hypothèse de non-cavitation (7) soit vérifiée par la condition initiale ζ_0 . Cependant, pour cela, il doit se placer dans un régime de comportement particulier, non physique, appelé le régime de Korteweg-de Vries, dans lequel $\mu = \varepsilon$.*

Remarque 0.2.25. *L'unicité dans la Proposition 0.2.22 nous permet de définir le temps T_{\max}/ε , supremum des temps $T/\varepsilon > 0$ tels que le problème de Cauchy associé aux équations (36) ait une solution $(\zeta, \bar{V}) \in C^0([0, T/\varepsilon], X_1^s(\mathbb{R}^d) \times X_2^s(\mathbb{R}^d)^d)$ vérifiant l'hypothèse de non-cavitation (7). Plus précisément, on a le critère d'explosion suivant :*

$$T_{\max} < \infty \implies \lim_{t \rightarrow T_{\max}/\varepsilon} (|\zeta|_{L^\infty([0, t], X_1^{t_0+2})} + |\bar{V}|_{L^\infty([0, t], X_2^{t_0+2})}) = \infty.$$

En particulier, T_{\max} et la solution maximale ne dépendent pas de l'indice de régularité s .

On a aussi la propriété de stabilité suivante.

Proposition 0.2.26. *Posons les mêmes hypothèses que celles de la Proposition 0.2.22. Supposons aussi qu'il existe $(\tilde{\zeta}, \tilde{V}) \in C([0, T/\varepsilon], X_1^s(\mathbb{R}^d) \times X_2^s(\mathbb{R}^d)^d)$ tel que*

$$\begin{cases} \partial_t \tilde{\zeta} + \nabla \cdot (\tilde{h} \tilde{V}) = r, \\ (\text{Id} - \frac{\mu}{3} \Delta) \partial_t \tilde{V} + \nabla \zeta + \varepsilon (\tilde{V} \cdot \nabla) \tilde{V} = R, \end{cases}$$

avec $\tilde{R} := (r, R) \in L^\infty([0, T/\varepsilon], Y_1^s(\mathbb{R}^d) \times Y_2^s(\mathbb{R}^d)^d)$, où pour $k \in \mathbb{N}$

$$Y_k^s := \{u \in (X_k^0)'; |u|_{Y_k^s} < \infty\} \quad \text{avec} \quad |u|_{Y_k^s} = \sum_{|k|=0}^s |\partial^k u|_{(X_k^0)'}$$

Alors, pour tout $t \in [0, T/\varepsilon]$, l'erreur par rapport aux solutions (ζ, \bar{V}) données par la Proposition 0.2.22 vérifie

$$|\mathbf{e}|_{L^\infty([0, t], X_1^s \times X_2^s)} \leq C(c_1, c_2, |(\tilde{\zeta}, \tilde{V})|_{L^\infty([0, t], X_1^s \times X_2^s)} (|\mathbf{e}|_{t=0}|_{X_1^s \times X_2^s} + t|\tilde{R}|_{L^\infty([0, t], Y_1^s \times Y_2^s)}),$$

où $\mathbf{e} = (\zeta, \bar{V}) - (\tilde{\zeta}, \tilde{V})$ et c_1, c_2 sont comme dans la Proposition 0.2.22.

La concaténation des résultats du Théorème 0.1.6, et des Propositions 0.2.20, 0.2.20 et 0.2.26, nous donne le théorème de justification (convergence) suivant, du modèle de Boussinesq (36).

Théorème 0.2.27. *Soit $N \in \mathbb{N}$ tel que $N > d/2 + 8$ et $(\mu, \varepsilon) \in \mathcal{A}_{\text{LV}}$ (voir (16)). Soit aussi $U_0 = (\zeta_0, \psi_0) \in E_0^N$ (voir (21)) vérifiant l'hypothèse de non-cavitation (7).*

Alors il existe $T > 0$ qui ne dépend pas des paramètres $(\mu, \varepsilon) \in \mathcal{A}_{\text{LV}}$ et tel que :

1. *Les équations des vagues (18) admettent une unique solution $U = (\zeta, \psi) \in E_{T/\varepsilon}^N$ (voir (21)) avec pour conditions initiales U_0 , et à laquelle on associe à travers (23) et (24) une vitesse moyenne $\bar{V} \in C([0, T/\varepsilon], X_2^{N-4})$.*
2. *Il existe une unique solution $(\zeta_B, \bar{V}_B) \in C([0, T/\varepsilon], X_1^{N-4} \times X_2^{N-4})$ aux équations de Boussinesq (36) avec pour conditions initiales $(\zeta_0, \bar{V}_0 = \bar{V}|_{t=0})$.*
3. *L'estimation d'erreur suivante est vérifiée pour tout $0 \leq t \leq T/\varepsilon$:*

$$|\mathbf{e}_B|_{L^\infty([0, t] \times \mathbb{R}^d)} \leq \mu^2 t C \left(\frac{1}{h_{\min}}, \mathcal{E}^N(U_0) \right),$$

où $\mathbf{e}_B = (\zeta, \bar{V}) - (\zeta_B, \bar{V}_B)$ et où $\mathcal{E}^N(U_0)$ est formellement défini par (22).

0.2.2.4 Comparaison de modèles dans la propagation de tsunamis générés par des glissements de terrains

Le travail du Chapitre 4 provient d'une collaboration avec E. Audusse¹, J.G Caldas Steinstraesser², P. Heinrich³, A. Paris⁴ et M. Parisot⁵, lors du CEMRACS 2019. Le résultat de ce travail a été publié dans le journal ESAIM : PROCEEDINGS AND SURVEYS en 2021, [Aud+21]. Cet article s'inscrit dans l'étude de la propagation de tsunamis générés par des glissements de terrains. Le but étant de savoir si les modèles bidirectionnels d'eau peu profonde, présentés dans les sous-sections précédentes, sont efficaces dans la modélisation de ce genre de phénomène. On se placera en dimension horizontale $d = 1$.

Les modèles de Saint-Venant et de Boussinesq peuvent se généraliser dans le cas où le fond marin est non plat et dépend du temps. Il est nécessaire d'utiliser ces généralisations lorsqu'il y a un glissement de terrain. Dans cette sous-section, nous considérerons les équations sous leur forme dimensionnelle.

Voici les équations de Saint-Venant dimensionnelles sous leur forme conservative :

$$\begin{cases} \partial_t h + \partial_x(h\bar{V}) = 0, \\ \partial_t(h\bar{V}) + \partial_x(h\bar{V}^2 + \frac{g}{2}h^2) = gh\partial_x b. \end{cases} \quad (37)$$

Ici, g est l'intensité de la pesanteur, la surface du fond marin est représentée par le graphe de la fonction b , h est la profondeur de l'eau, elle s'écrit $h(t, x) = H_0 + \zeta(t, x) - b(t, x)$ et \bar{V} est la composante horizontale du champ de vitesse, moyennée sur la profondeur de l'eau.

Voici les équations de Boussinesq dimensionnelles

$$\begin{cases} \partial_t \zeta + \partial_x(h\bar{V}) = \partial_t d, \\ \partial_t \bar{V} + \bar{V}\partial_x \bar{V} = -g\partial_x \zeta - \frac{d^3}{3}\partial_x^2(\partial_t \bar{V}) - \frac{d}{2}\partial_t \bar{V}\partial_x^2 \bar{V} + d\partial_x d\partial_x(\partial_t \bar{V}). \end{cases} \quad (38)$$

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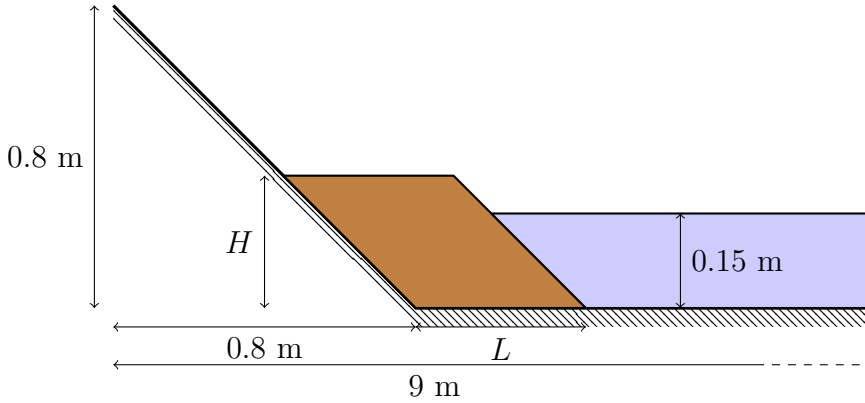
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Ici, la surface de l'eau est représentée par le graphe de la fonction ζ , et $d = H_0 - b$.

Nous comparons ces deux modèles dans le cas test suivant, inspiré du papier [VSK14].



On considère que le terrain est initialement un parallélogramme de hauteur H et de longueur L et qu'il glisse le long d'une pente inclinée à 45° .

Tout d'abord, remarquons que l'hypothèse de non-cavitation (7) n'est pas respectée, il y a des zones sèches. La justification des modèles bidirectionnels classiques dans ce cas a fait l'objet de plusieurs travaux (voir notamment [LM18]). Dans le cas des équations de Saint-Venant (37), nous utilisons le code de résolution Hydra, développé par M. Parisot, qui se base sur les papiers [Bou04; Aud+04]. Dans le cas des équations de Boussinesq (38), nous utilisons le code de résolution AVALANCHE développé par le CEA pour simuler des tsunamis. Il a notamment été utilisé pour étudier des tsunamis générés par des glissement de terrains subaériens et sous-marins [HP00; Héb+02; Le +03; Rod+13; Pou+17; Par+9a; Par+9b].

Pour résoudre numériquement les systèmes (37) et (38), il est nécessaire de connaître la dynamique du fond marin. Pour cela, nous utilisons le code de résolution OpenFOAM, qui nous permet de résoudre les équations de Navier-Stokes multi-couches incompressibles. Le terrain est considéré ici comme un fluide. Une fois la dynamique du fond marin déterminée, nous l'utilisons comme variable d'entrée pour les systèmes (37) et (38).

Dans tous les cas tests (en faisant varier H et L), on peut distinguer principalement deux phénomènes. Le premier est la génération de la vague, qui se caractérise par la poussée de l'eau par le terrain glissant. Le deuxième est la propagation de la vague; une fois l'eau poussée par le terrain, la vague générée ne fait que se propager et n'interagit plus avec le terrain.

La comparaison du modèle de Boussinesq (38), avec le modèle de Navier-Stokes multi-

couches incompressible, dans les cas tests étudiés, nous montre que ce modèle n'est pas adapté pour reproduire la génération de la vague. Le modèle de Saint-Venant est, lui, plus adéquat dans ce cas. Pour ce qui est de la propagation de la vague, les conclusions sont inversées, sauf dans les cas où le glissement de terrain est très important. Dans ce dernier cas, les deux modèles ne permettent pas de reproduire l'amplitude de la vague, mais contrairement au modèle de Boussinesq, le modèle de Saint-Venant réussit à reconstruire la vitesse de propagation.

Pour améliorer les résultats numériques, on propose donc une stratégie de changement de modèle entre la zone où la vague est engendrée, dans laquelle on utilise le modèle de Saint-Venant, et la zone où la vague se propage, dans laquelle on utilise le modèle de Boussinesq.

Une autre stratégie aurait été d'utiliser la généralisation du modèle de Green-Naghdi aux fonds marins non plats et dépendant du temps, mais il semble qu'il faille des codes plus robustes. En effet, à cause des zones sèches et des retours de vagues, les codes de résolution à notre disposition étaient instables numériquement.

Nous mettons aussi en évidence que la dynamique des vagues engendrées par les glissements de terrain est directement corrélée à l'énergie transférée à l'eau.

Dernièrement, nous étudions la reconstruction d'un glissement de terrain à partir de la connaissance de la vague générée, que nous formulons comme un problème de minimisation. Considérant une vague de référence, nous recherchons le glissement de terrain qui minimise l'erreur entre cette vague, et la vague calculée par un des modèles, (37) ou (38). Nous nous limitons au cas où le glissement de terrain est supposée être une somme finie de gaussiennes. Même si notre analyse est trop limitée pour conclure, notre stratégie numérique est validée dans des cas simples.

0.2.3 Les modèles bidirectionnels de type dispersion complète

Dans cette sous-section nous présentons les modèles bidirectionnels de type dispersion complète que sont les modèles de Whitham-Boussinesq et de Whitham-Green-Naghdi, et plus particulièrement les résultats principaux obtenus dans les Chapitre 1 et 2 [Eme20; Eme21b], sur la justification de ces modèles.

Pour comprendre ce qui définit un modèle d'eau peu profonde de type dispersion complète, revenons aux équations des vagues (18), et étudions le linéarisé de ces équations autour de l'état de repos $(\zeta, \psi) = (0, 0)$. Pour cela, remplaçons le paramètre de non-linéarité,

ε , par 0 dans les équations. On obtient

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}^\mu[0] \psi = 0, \\ \partial_t \psi + \zeta = 0. \end{cases}$$

Or, une simple analyse de Fourier nous permet de résoudre le problème harmonique (20) dans le cas où $\varepsilon \zeta = 0$. On obtient alors l'expression suivante de $\frac{1}{\mu} \mathcal{G}^\mu[0] \psi$:

$$\frac{1}{\mu} \mathcal{G}^\mu[0] \psi = -\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} \Delta \psi := -F_1^\mu \Delta \psi.$$

Ainsi, le linéarisé des équations des vagues (18) autour de l'état de repos s'écrit

$$\begin{cases} \partial_t \zeta + F_1^\mu \Delta \psi = 0, \\ \partial_t \psi + \zeta = 0. \end{cases} \quad (39)$$

On dit qu'un modèle d'eau peu profonde est de type dispersion complète lorsque son linéarisé autour de l'état de repos est le même que celui des équations des vagues, à savoir (39). Entre autres propriétés, un modèle de type dispersion complète aura la même relation de dispersion que celle des équations des vagues, c'est-à-dire

$$\omega(\xi)^2 = \frac{1}{\sqrt{\mu}} |\xi| \tanh(\sqrt{\mu}|\xi|) = F_1^\mu(\xi) |\xi|^2,$$

où $\omega(\xi)$ est la fréquence angulaire associée au vecteur d'onde ξ .

Remarque 0.2.28. *Nous avons mentionné dans la remarque 0.2.5 que les approximations de la quantité \bar{V} , dans la Proposition 0.2.4, étaient exactes au niveau linéaire. Au vu de la précédente caractérisation des modèles de type dispersion complète, cette propriété est nécessaire pour construire des modèles de ce type approchant les équations des vagues.*

0.2.3.1 Les modèles de Whitham-Boussinesq

En utilisant l'estimation (30) comme point angulaire, nous obtenons, dans le Chapitre 1 [Eme20], les deux propositions suivantes.

Proposition 0.2.29. *Soit F_1^μ , le multiplicateur de Fourier défini en (29). Dans le régime d'eau peu profonde \mathcal{A} (voir (15)), les équations des vagues (18) sont cohérentes à l'ordre*

$O(\mu\varepsilon)$ avec le système

$$\begin{cases} \partial_t \zeta + F_1^\mu \Delta \psi + \varepsilon F_1^\mu \nabla \cdot (\zeta F_1^\mu \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |F_1^\mu \nabla \psi|^2 = 0, \end{cases} \quad (40)$$

au sens de la Définition 0.2.6, avec $n = 4$.

De plus, le système (40) possède une structure Hamiltonienne, avec pour Hamiltonien

$$H_{\text{WB}} = \frac{1}{2} \int_{\mathbb{R}^d} (\zeta^2 + |(F_1^\mu)^{1/2} [\nabla \psi]|^2) dX + \frac{\varepsilon}{2} \int_{\mathbb{R}^d} \zeta |F_1^\mu [\nabla \psi]|^2 dX.$$

Proposition 0.2.30. Soit F_1^μ , le multiplicateur de Fourier défini en (29). Dans le régime d'eau peu profonde \mathcal{A} (voir (15)), les équations des vagues (18) sont cohérentes à l'ordre $O(\mu\varepsilon)$ avec le système

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ \partial_t \bar{V} + F_1^\mu \nabla \zeta + \frac{\varepsilon}{2} F_1^\mu \nabla (|\bar{V}|^2) = 0, \end{cases} \quad (41)$$

au sens de la Définition 0.2.6, avec $n = 4$.

Les modèles de Whitham-Boussinesq sont définis comme étant des modèles d'eau peu profonde de type dispersion complète pour lesquels la précision par rapport aux équations des vagues (18) dans le régime de petite profondeur d'eau (16) est d'ordre $O(\mu\varepsilon)$. En ce sens, les modèles (40) et (41) sont des modèles de Whitham-Boussinesq.

Remarque 0.2.31. Grâce aux techniques du Chapitre 1 [Eme20], on peut en fait montrer que tous les modèles du type

$$\begin{cases} \partial_t \zeta + F_1^\mu \Delta \psi + \varepsilon G_1^\mu \nabla \cdot (\zeta G_2^\mu \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |G_1^\mu [\nabla \psi]| |G_2^\mu [\nabla \psi]| = 0, \end{cases} \quad (42)$$

avec G_1^μ et G_2^μ vérifiant pour tout $\xi \in \mathbb{R}^d$,

$$G_1^\mu(\xi) = 1 + O(\mu|\xi|^2), \quad G_2^\mu(\xi) = 1 + O(\mu|\xi|^2),$$

sont des modèles de Whitham-Boussinesq.

Les Propositions 0.2.29 et 0.2.30 donnent pour la première fois une caractérisation précise de l'ordre d'approximation des modèles Whitham-Boussinesq.

Remarque 0.2.32. Ici, contrairement au cas classique, on a deux types de systèmes, un dont les inconnues sont ζ et ψ (40), et un autre dont les inconnues sont ζ et \bar{V} (41).

Si, dans (40), on remplace le multiplicateur de Fourier F_1^μ par $1 + \frac{\mu}{3}\Delta$, on obtient un système de Boussinesq mal posé, ce qui explique pourquoi ce dernier n'est pas discuté dans ce document.

Remarque 0.2.33. Les systèmes de Whitham-Boussinesq en ζ et ψ sont généralement écrits dans la littérature à partir des variables ζ et $v := \nabla\psi$. Pour cela, il faut au préalable appliquer l'opérateur ∇ à l'équation en ψ . Prenons l'exemple du système (40), il peut se réécrire de la façon suivante :

$$\begin{cases} \partial_t \zeta + F_1^\mu \nabla \cdot v + \varepsilon F_1^\mu \nabla \cdot (\zeta F_1^\mu[v]) = 0, \\ \partial_t v + \nabla \zeta + \varepsilon (F_1^\mu[v] \cdot \nabla) F_1^\mu[v] = 0, \end{cases}$$

où nous avons utilisé que $F_1^\mu[\nabla\psi]$ est irrotationnel, et que dans ce cas, nous avons la relation

$$\nabla(|F_1^\mu[\nabla\psi]|^2) = 2(F_1^\mu[v] \cdot \nabla)F_1^\mu[v].$$

Dans la suite de cette sous-section, nous écrirons les systèmes de Whitham-Boussinesq en ζ et ψ de cette manière.

Le deuxième point pour justifier les modèles de Whitham-Boussinesq est de prouver leur caractère bien posé local, avec un temps d'existence des solutions d'ordre $1/\varepsilon$ et indépendant de μ .

Le caractère bien-posé de certains systèmes de Whitham-Boussinesq ont été étudiés dans [Din19; Din20; DST19; KP18]. Il convient de porter une attention particulière à [DST19] où les auteurs ont prouvé le caractère bien-posé local du système (40) en dimension 2, et le caractère bien-posé global en dimension 1. Cependant, les techniques utilisées dans ces travaux ne permettent pas d'établir l'existence et le contrôle des solutions indépendamment du paramètre μ .

Dans le Chapitre 2, nous obtenons le caractère bien-posé local d'une classe de systèmes de Whitham-Boussinesq quasi-linéaires, avec un temps d'existence d'ordre $1/\varepsilon$ et indépendant de μ . Avant de présenter le résultat, rappelons la définition de l'ordre d'un multiplicateur de Fourier.

Definition 0.2.34. On dit qu'un multiplicateur de Fourier G est d'ordre $s \in \mathbb{R}$ si $\xi \in$

$\mathbb{R}^d \mapsto G(\xi) \in \mathbb{R}$ est lisse et vérifie

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \quad \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^{|\beta| - s} |\partial^\beta G(\xi)| < \infty.$$

Notation 0.2.35. Nous noterons $\mathcal{S}'(\mathbb{R}^d)$ l'ensemble des distributions tempérées.

Et maintenant, voici le résultat.

Théorème 0.2.36. Soient $s > d/2 + 1$, $h_{\min} > 0$ et $M \geq 0$. Soit F_1^μ , le multiplicateur de Fourier défini en (29). Soit aussi G^μ un multiplicateur de Fourier d'ordre 0 de la forme $G^\mu = G(\sqrt{\mu}D)$ où G est un multiplicateur de Fourier indépendant de μ . Supposons de plus que pour tout $\xi \in \mathbb{R}^d$, $|G^\mu(\xi)| \leq |F_1^\mu(\xi)|$.

Il existe $T > 0$ et $C > 0$ tels que pour tout $(\mu, \varepsilon) \in \mathcal{A}$ (voir (15)), et $(\zeta_0, v_0) \in X^s(\mathbb{R}^d)$, où

$$X^s(\mathbb{R}^d) := \{(\zeta, v) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)^d, \quad |(\zeta, v)|_{X^s} < +\infty\},$$

avec $|(\zeta, v)|_{X^s} := |\zeta|_{H^s} + |(F_1^\mu)^{1/2}[v]|_{H^s}$, vérifiant $|(\zeta_0, v_0)|_{X^s} \leq M$ et l'hypothèse de non-cavitation (7), les équations de Whitham-Boussinesq

$$\begin{cases} \partial_t \zeta + F_1^\mu \nabla \cdot v + \varepsilon G^\mu \nabla \cdot (\zeta G^\mu[v]) = 0, \\ \partial_t v + \nabla \zeta + \varepsilon (G^\mu[v] \cdot \nabla) G^\mu[v] = 0, \end{cases} \quad (43)$$

admettent une unique solution $(\zeta, v) \in C^0([0, T/\varepsilon], X^s(\mathbb{R}^d))$, avec pour conditions initiales (ζ_0, v_0) .

De plus, nous avons le contrôle suivant sur (ζ, v) :

$$|(\zeta, v)|_{L^\infty([0, T/\varepsilon], X^s)} \leq C |(\zeta_0, v_0)|_{X^s}.$$

Remarque 0.2.37. — Les systèmes de la forme (43), où G^μ vérifie les hypothèses du Théorème 0.2.36, ont une structure d'équations quasi-linéaires semblable aux équations de Saint-Venant (33), à la différence près que ces systèmes dépendent d'opérateurs non-locaux (F_1^μ et G^μ). Les techniques utilisées pour démontrer le Théorème 0.2.36 sont donc inspirées de celles utilisées pour démontrer le Théorème 0.2.9.

— Si, dans (43), on remplace G^μ par F_1^μ , on retrouve le système (40) (à l'application

de l'opérateur ∇ près, voir la Remarque 0.2.33). Dû aux propriétés de régularisation de F_1^μ , ce dernier système a une structure d'équations semi-linéaires.

- Dans le Chapitre 2, nous démontrons un résultat plus général de caractère bien-posé local de systèmes quasi-linéaires non-locaux. Le Théorème 0.2.36 n'est qu'un cas particulier de ce résultat.

Remarque 0.2.38. L'unicité dans la Proposition 0.2.36 nous permet de définir le temps T_{\max}/ε , supremum des temps $T/\varepsilon > 0$ tels que le problème de Cauchy associé aux équations (43) ait une solution $(\zeta, v) \in C^0([0, T], X^s(\mathbb{R}^d))$ vérifiant l'hypothèse de non-cavitation (7). Plus précisément, soit $t_0 > d/2$, on a le critère d'explosion suivant :

$$T_{\max} < \infty \implies \lim_{t \rightarrow T_{\max}/\varepsilon} |(\zeta, v)|_{L^\infty([0, t], X^{t_0+1})} = \infty.$$

En particulier, T_{\max} et la solution maximale ne dépendent pas de l'indice de régularité s .

On a aussi la propriété de stabilité suivante.

Proposition 0.2.39. Posons les mêmes hypothèses que celles du Théorème 0.2.36 et supposons aussi qu'il existe $(\tilde{\zeta}, \tilde{v}) \in C([0, T/\varepsilon], X^s(\mathbb{R}^d))$ solutions du système

$$\begin{cases} \partial_t \tilde{\zeta} + F_1^\mu \nabla \cdot \tilde{v} + \varepsilon G^\mu \nabla \cdot (\tilde{\zeta} G^\mu [\tilde{v}]) = r, \\ \partial_t \tilde{v} + \nabla \tilde{\zeta} + \varepsilon (G^\mu [\tilde{v}] \cdot \nabla) G^\mu [\tilde{v}] = R, \end{cases}$$

où $\tilde{R} := (r, R) \in L^\infty([0, T/\varepsilon], X^{s-1}(\mathbb{R}^d))$. Alors, pour tout $t \in [0, T/\varepsilon]$, l'erreur par rapport aux solutions (ζ, v) données par le Théorème 0.2.36 vérifie

$$|\mathbf{e}|_{L^\infty([0, t], X^{s-1})} \leq C \left(\frac{1}{h_{\min}}, |(\zeta, v)|_{L^\infty([0, t], X^s)}, |(\tilde{\zeta}, \tilde{v})|_{L^\infty([0, t], X^s)} \right) (|\mathbf{e}|_{X^{s-1}}|_{t=0} + t |\tilde{R}|_{L^\infty([0, t], X^{s-1})}),$$

où $\mathbf{e} = (\zeta, v) - (\tilde{\zeta}, \tilde{v})$.

La concaténation des résultats de la Proposition 0.2.29 (et de la Remarque 0.2.31), du Théorème 0.2.36 et de la Proposition 0.2.39 nous donne le théorème de justification (convergence) suivant, des modèles de Whitham-Boussinesq (42).

Théorème 0.2.40. Soient $(\mu, \varepsilon) \in \mathcal{A}$ et $s > d/2$. Soit aussi G^μ vérifiant les hypothèses du Théorème 0.2.36. Il existe $T > 0$ indépendant de (μ, ε) , tel que pour tout $T' > 0$ et toute solution (ζ, ψ) des équations des vagues (18) vérifiant l'hypothèse de non-cavitation (7) et

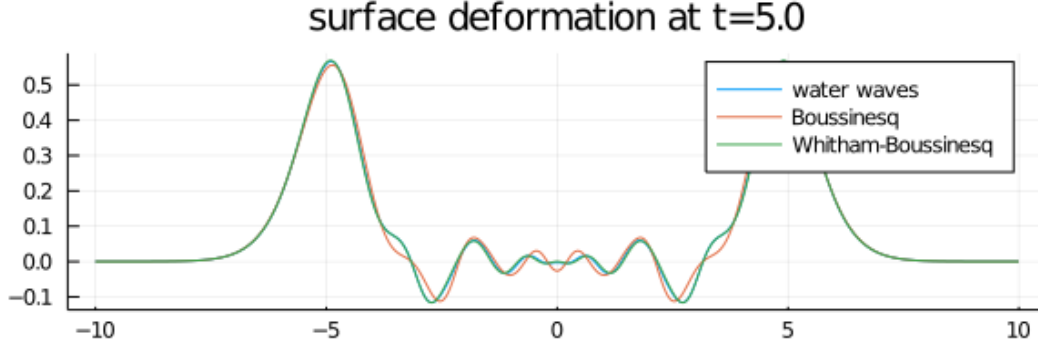


FIGURE 3 – Paramètres : $\mu = 0.1$, $\varepsilon = 0.1$, $N_x = 2^{12}$, $dt = 10^{-3}$.

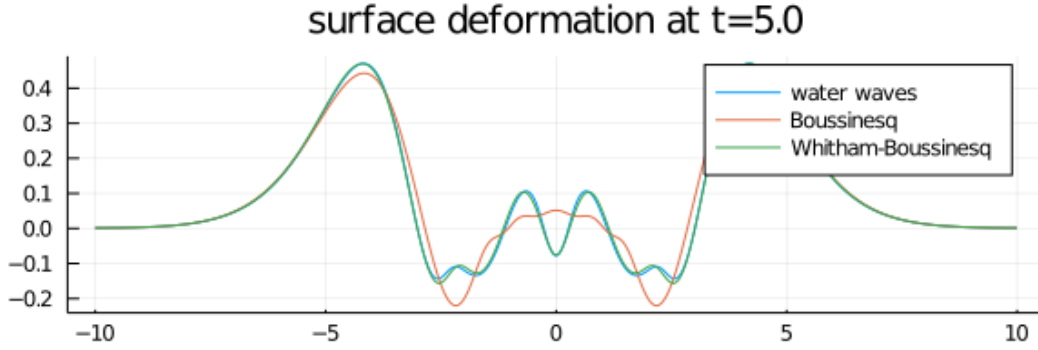


FIGURE 4 – Paramètres : $\mu = 0.5$, $\varepsilon = 0.1$, $N_x = 2^{12}$, $dt = 10^{-3}$.

$U = (\zeta, \nabla\psi) \in C^0([0, T'/\varepsilon], H^{s+4}(\mathbb{R}^d))$, il existe une unique solution $U_{WB} = (\zeta_{WB}, v_{WB}) \in C^0([0, T/\varepsilon], X^{s+3}(\mathbb{R}^d))$ des équations de Whitham-Boussinesq (43) ayant pour conditions initiales $(\zeta|_{t=0}, \nabla\psi|_{t=0})$. De plus, pour tout temps $t \in [0, \min(T, T')/\varepsilon]$, on a

$$|\mathbf{e}|_{L^\infty([0,t], X^s)} \leq \mu \varepsilon t C \left(\frac{1}{h_{\min}}, |U|_{L^\infty([0,t], H^{s+4})} \right),$$

où $\mathbf{e} = U - U_{WB}$.

Illustrons ce résultat avec quelques simulations. Plaçons-nous dans le cas où $d = 1$ et prenons pour condition initiale $(\zeta_0(x) = e^{-x^4}, \nabla\psi_0 = 0)$. Dans les figures 3,4 et 5, nous comparons l'élévation de la surface obtenue par résolution des équations des vagues avec les élévations obtenues par résolution des équations de Whitham-Boussinesq (43) où $G_\mu = (F_1^\mu)^{-1/2}$, et par résolution d'un système de Boussinesq.

On retrouve bien, dans les trois figures, une meilleure précision du modèle de Whitham-Boussinesq en comparaison avec le modèle de Boussinesq. Même lorsque la précision théorique est la même pour les deux modèles, par exemple lorsque $\mu = \varepsilon = 0.1$ (figure 3)

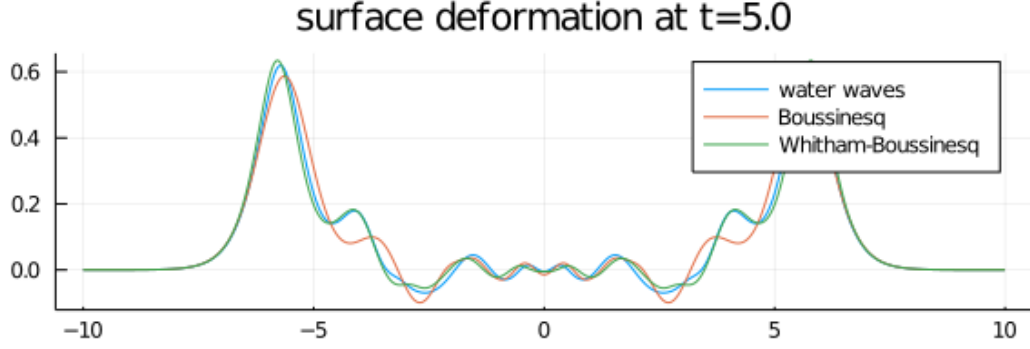


FIGURE 5 – Paramètres : $\mu = 0.1$, $\varepsilon = 0.5$, $N_x = 2^{12}$, $dt = 10^{-3}$.

le modèle de Whitham-Boussinesq semble plus précis.

Lorsque μ n'est pas petit, par exemple $\mu = 0.5$ et $\varepsilon = 0.1$ (figure 4), le modèle de Boussinesq ne semble approcher que le profil de la première vague générée. Le modèle de Whitham-Boussinesq reste quant à lui, une bonne approximation de la solution dans sa globalité puisque ε est petit.

Remarque 0.2.41. *Le code de résolution numérique des équations des vagues développé et utilisé dans cette thèse est basé sur une méthode de transformation conforme du domaine de résolution des équations Ω_t (voir [CC99; Dya+96; VDD14; LHC04]).*

Les codes de résolution du système de Boussinesq et du système de Whitham-Boussinesq proviennent de la librairie Julia "ShallowWaterModels" développé par V. Duchêne⁶.

0.2.3.2 Les modèles de Whitham-Green-Naghdi

En utilisant les estimations (31) et (32) comme points angulaires, nous obtenons, dans le Chapitre 1 [Eme20], les deux propositions suivantes.

Proposition 0.2.42. *Soit F_2^μ le multiplicateur de Fourier défini en (29). Dans le régime d'eau peu profonde \mathcal{A} (voir (15)), les équations des vagues (18) sont cohérentes à l'ordre $O(\mu^2\varepsilon)$ avec le système*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \nabla \psi) + \frac{\mu}{6} \left(\Delta (F_2^\mu [h^3 \Delta \psi]) + \Delta (h^3 F_2^\mu [\Delta \psi]) \right) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\mu \varepsilon}{2} h^2 (F_2^\mu [\Delta \psi]) \Delta \psi = 0, \end{cases} \quad (44)$$

au sens de la Définition 0.2.6, avec $n = 5$.

⁶. Université de Rennes 1, IRMAR, CNRS

De plus, le système (44) possède une structure Hamiltonienne, avec pour Hamiltonien

$$H_{\text{GN1}} = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dx + \frac{1}{2} \int_{\mathbb{R}^d} h \nabla \psi \cdot \nabla \psi dX + \frac{\mu}{6} \int_{\mathbb{R}^d} \nabla (h^3 F_2^\mu [\nabla \cdot \nabla \psi]) \cdot \nabla \psi dX.$$

Remarque 0.2.43. Dans le système (44), si on remplace F_2^μ par l'identité, on se retrouve avec un système de type Green-Naghdi (voir Remarque 0.2.14) en ζ et ψ , qui est mal posé.

Proposition 0.2.44. Soit F_3^μ le multiplicateur de Fourier défini en (29). Soit aussi l'opérateur T défini par la formule

$$T[h]V := -\frac{1}{6h} (\nabla (h^3 F_3^\mu [\nabla \cdot V]) + \nabla (F_3^\mu [h^3 \nabla \cdot V])).$$

Dans le régime d'eau peu profonde \mathcal{A} (voir (15)), les équations des vagues (18) sont cohérentes à l'ordre $O(\mu^2 \varepsilon)$ avec le système

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ \partial_t ((\text{Id} + \mu T[h]) \bar{V}) + \nabla \zeta + \frac{\varepsilon}{2} \nabla (|\bar{V}|^2) - \frac{\mu \varepsilon}{6} \nabla \left(\frac{1}{h} \bar{V} \cdot \nabla (h^3 F_3^\mu [\nabla \cdot \bar{V}] + F_3^\mu [h^3 \nabla \cdot \bar{V}]) \right) \\ - \frac{\mu \varepsilon}{2} \nabla (h^2 F_3^\mu [\nabla \cdot \bar{V}] \nabla \cdot \bar{V}) = 0. \end{cases} \quad (45)$$

au sens de la Définition 0.2.6, avec $n = 6$.

De plus, si l'opérateur $I[h] := h(\text{Id} + \mu T[h])$ est inversible, le système (45) possède une structure Hamiltonienne, avec pour Hamiltonien

$$H_{\text{GN2}} = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2} \int_{\mathbb{R}^d} h I[h]^{-1} [h \nabla \psi] \cdot \nabla \psi dX.$$

On définit dans cette thèse les modèles dits de Whitham-Green-Naghdi comme des modèles de type dispersion complète approchant les équations des vagues (18) dans le régime de petite profondeur d'eau (15) à l'ordre $O(\mu^2 \varepsilon)$. En ce sens, les systèmes (44) et (45) sont des systèmes de Whitham-Green-Naghdi.

Remarque 0.2.45. — Dans [DIT16], les auteurs ont introduit un système de type dispersion complète multi-couches, proche du système (45), et avaient conjecturé que leur modèle était cohérent avec les équations des vagues à l'ordre $O(\mu^2 \varepsilon)$.

— La remarque 0.2.32 est aussi valable ici.

— Dans le régime longues vagues \mathcal{A}_{LV} , les équations des vagues (18) sont cohérentes à

l'ordre $O(\mu^3)$ avec les systèmes (44) et (45).

- En utilisant les techniques du Chapitre 1, on pourrait montrer que les équations des vagues (18) sont cohérentes à l'ordre $O(\mu^2\varepsilon)$ avec tous les systèmes du type

$$\begin{cases} \partial_t \zeta + F_1^\mu \Delta \psi + \nabla \cdot G_1^\mu [(h-1)G_1^\mu [\nabla \psi]] - \frac{\mu}{3} \Delta G_2^\mu [(h^3-1)G_2^\mu [\Delta \psi]] = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |G_1^\mu [\nabla \psi]|^2 + \frac{\mu \varepsilon}{2} h^2 G_2^\mu [\Delta \psi]^2 = 0, \end{cases} \quad (46)$$

où G_1^μ, G_2^μ sont deux multiplicateurs de Fourier vérifiant, pour tout $\xi \in \mathbb{R}^d$,

$$G_1^\mu(\xi) = 1 + O(\mu^2 |\xi|^4), \quad G_2^\mu(\xi) = 1 + O(\mu |\xi|^2).$$

La liberté que l'on a sur les multiplicateurs de Fourier pourrait nous servir à exhiber des systèmes de Whitham-Green-Naghdi bien posés localement. Formellement, si on impose à G_1^μ et $\sqrt{\mu} \nabla \cdot G_2^\mu$ d'être d'ordre 0 et de vérifier pour tout $\xi \in \mathbb{R}^d$, $|G_1^\mu(\xi)|^2, |\sqrt{\mu} \xi |G_2^\mu(\xi)|^2 \leq |F_1^\mu(\xi)|$, on obtient des systèmes quasi-linéaires non-locaux, similaires à (43), et les techniques du Chapitre 2 pourraient alors s'appliquer.

- D'un point de vue pratique, il semble plus facile de résoudre numériquement les systèmes de Whitham-Green-Naghdi dont les inconnues sont ζ et ψ . En effet, contrairement aux systèmes en ζ et \bar{V} , il n'y a pas d'opérateur à inverser.

Comparons la précision du modèle de Green-Naghdi et d'un modèle de Whitham-Green-Naghdi via quelques simulations. Plaçons-nous dans le cas $d = 1$ et prenons comme condition initiale ($\zeta_0 = e^{-x^4}, \nabla \psi_0 = 0$). En utilisant le code de résolution des équations des vagues mentionné dans la Remarque 0.2.41 et les codes de résolutions des systèmes de Green-Naghdi et de Whitham-Green-Naghdi de la librairie "ShallowWaterModels" mentionnée dans la même remarque, nous obtenons les figures 6, 7 et 8.

On retrouve les mêmes conclusions que dans la comparaison des modèles de Boussinesq et de Whitham-Boussinesq (voir les figures 3, 4 et 5)

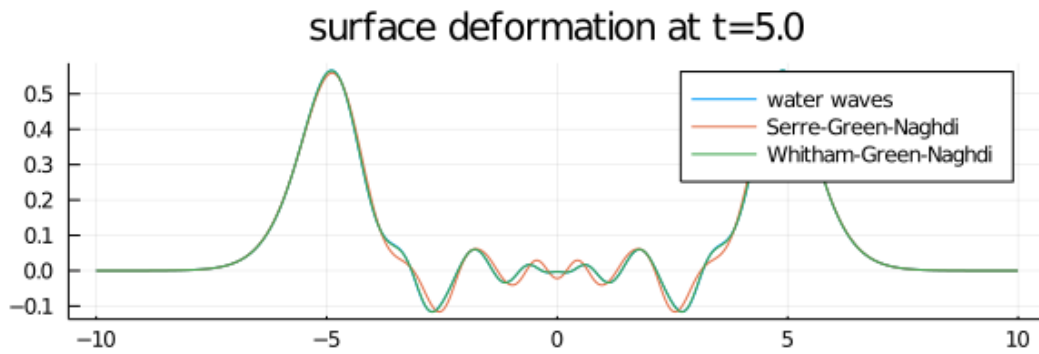


FIGURE 6 – Paramètres : $\mu = 0.1$, $\varepsilon = 0.1$, $N_x = 2^{12}$, $dt = 10^{-3}$.

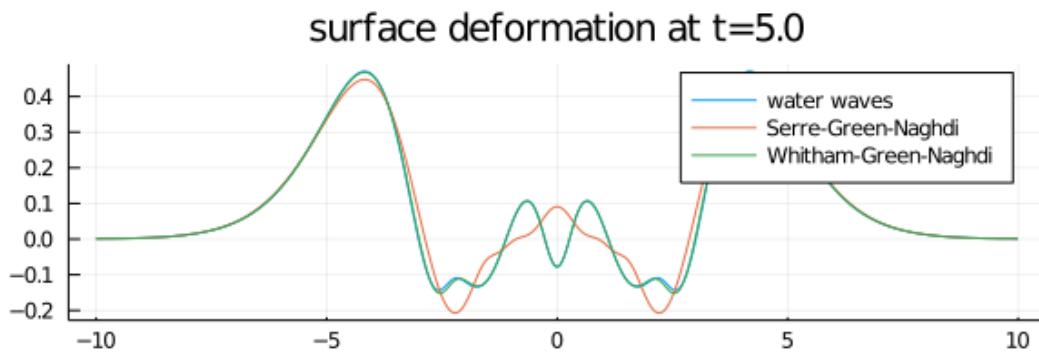


FIGURE 7 – Paramètres : $\mu = 0.5$, $\varepsilon = 0.1$, $N_x = 2^{12}$, $dt = 10^{-3}$.

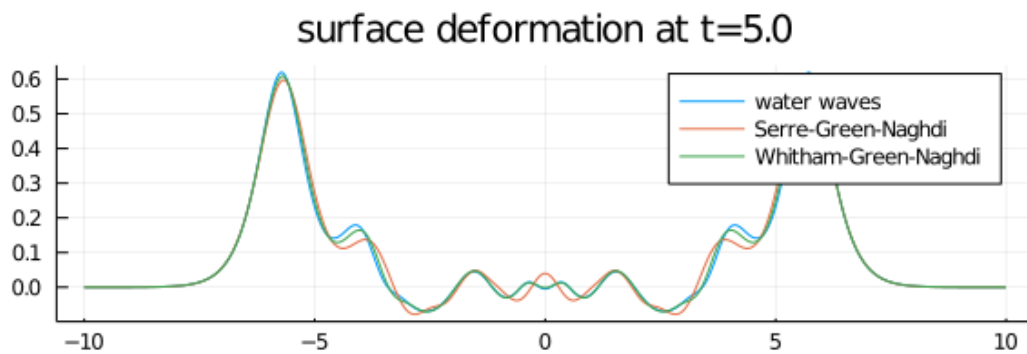


FIGURE 8 – Paramètres : $\mu = 0.1$, $\varepsilon = 0.5$, $N_x = 2^{12}$, $dt = 10^{-3}$.

0.3 Les modèles unidirectionnels d'eau peu profonde

Plaçons-nous dans le cas où $d = 1$. Dans tous les modèles présentés précédemment, si on néglige tous les termes d'ordre $O(\mu)$ et $O(\varepsilon)$ nous obtenons un système du type

$$\begin{cases} \partial_t \zeta + \partial_x \bar{V} = 0, \\ \partial_t \bar{V} + \partial_x \zeta = 0. \end{cases}$$

En particulier, la fonction ζ vérifie une simple équation des ondes

$$\partial_t^2 \zeta - \partial_x^2 \zeta = 0.$$

Cela suggère que pour toute condition initiale donnée, en dimension horizontale $d = 1$, la surface de l'eau se décompose en deux vagues se propageant dans des directions opposées. L'objectif d'un modèle de propagation unidirectionnel est de décrire une de ces deux vagues.

Dans cette dernière partie de l'introduction, nous présentons la dérivation des équations de Hopf à partir des invariants de Riemann des équations de Saint-Venant. Puis nous présentons le résultat du Chapitre 2 (papier à paraître au journal *Nonlinearity* [Eme21b]), portant sur la dérivation des équations de Whitham à partir de la construction de "quasi"-invariants de Riemann d'un système de Whitham-Boussinesq bien choisi. Cette méthode suppose les conditions initiales bien préparées, de telle sorte à isoler une des deux directions de propagation.

Ensuite, nous présentons la méthode de Bambusi [Bam20] sur la dérivation des équations de Korteweg-de Vries, utilisant une généralisation de l'algorithme de la forme normale de Birkhoff. Enfin, nous présentons le résultat du Chapitre 3 utilisant une adaptation de cette méthode pour montrer que les deux vagues se propageant dans des directions opposées peuvent être correctement décrites par un système de deux équations de Whitham. Ce dernier système est donc adapté à un mode de propagation bidirectionnel.

0.3.1 Construction d'invariants de Riemann et dérivation de modèles unidirectionnels

Plaçons-nous en dimension horizontale $d = 1$.

On présente ici une méthode pour obtenir les équations de Hopf depuis le système de

Saint-Venant. Soient ζ et \bar{V} solutions des équations de Saint-Venant (33). Les quantités

$$u^+ = \frac{\sqrt{h} - 1}{\varepsilon} + \frac{\bar{V}}{2}, \quad u^- = -\frac{\sqrt{h} - 1}{\varepsilon} + \frac{\bar{V}}{2},$$

sont solutions du système

$$\begin{cases} \partial_t u^+ + (\varepsilon \frac{3u^+ + u^-}{2} - 1) \partial_x u^+ = 0, \\ \partial_t u^- + (\varepsilon \frac{3u^- + u^+}{2} - 1) \partial_x u^- = 0. \end{cases}$$

Les quantités u^+ et u^- sont appelés les invariants de Riemann du système de Saint-Venant.

En imposant aux conditions initiales ζ_0 et \bar{V}_0 de satisfaire la relation $u^-(0) = 0$, On obtient $u^- \equiv 0$. Et alors u^+ est solution de l'équation de Hopf

$$\partial_t u^+ + (1 + \frac{3\varepsilon}{2} u^+) \partial_x u^+ = 0.$$

Dans le Chapitre 3, on s'inspire de ce simple procédé pour déduire les équations de Whitham depuis un système de Whitham-Boussinesq bien choisi. Voici les résultats obtenus.

Proposition 0.3.1. *Soit F_1^μ le multiplicateur de Fourier défini en (29). Il existe $n \in \mathbb{N}^*$ tel que pour tout $(\mu, \varepsilon) \in \mathcal{A}$ (voir (15)) et $\zeta \in L_t^\infty H_x^{\alpha+n}(\mathbb{R})$ et $\psi \in L_t^\infty \dot{H}_x^{\alpha+n}(\mathbb{R})$ solutions des équations des vagues (18) vérifiant l'hypothèse de non-cavitation (7), il existe $R_1, R_2 \in H^\alpha(\mathbb{R})$ uniformément bornés en (μ, ε) de telle sorte que les quantités*

$$u^+ = \frac{\sqrt{h} - 1}{\varepsilon} + \frac{1}{2}(F_1^\mu)^{-1/2}[v], \quad u^- = -\frac{\sqrt{h} - 1}{\varepsilon} + \frac{1}{2}(F_1^\mu)^{-1/2}[v], \quad (47)$$

où $v = F_1^\mu[\partial_x \psi]$ et $h = 1 + \varepsilon \zeta$, satisfont les équations

$$\begin{cases} \partial_t u^+ + (\varepsilon \frac{3u^+ + u^-}{2} + 1)(F_1^\mu)^{1/2} \partial_x u^+ = \mu \varepsilon R_1, \\ \partial_t u^- + (\varepsilon \frac{u^+ + 3u^-}{2} - 1)(F_1^\mu)^{1/2} \partial_x u^- = \mu \varepsilon R_2. \end{cases}$$

Remarque 0.3.2. *Les quantités u^+ et u^- peuvent être vues comme des invariants de*

Riemann approchés à l'ordre $O(\mu\varepsilon)$, des équations de Whitham-Boussinesq

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon \partial_x (\zeta \partial_x v) = 0, \\ \partial_t v + F_1^\mu \partial_x \zeta + \varepsilon \partial_x v \partial_x^2 v = 0, \end{cases}$$

où $v = F_1^\mu[\partial_x \psi]$, pour lesquels nous connaissons l'obtention rigoureuse (voir Proposition 0.2.29 et Remarque 0.2.31).

Corollary 0.3.3. *Il existe $n \in \mathbb{N}^*$ tel que pour tout $\alpha \geq 0$ et tout $(\mu, \varepsilon) \in \mathcal{A}$, nous avons ce qui suit. Considérons le problème de Cauchy associé aux équations des vagues (18) avec pour conditions initiales $(\zeta_0, \psi_0) \in H^{\alpha+n}(\mathbb{R}) \times \dot{H}^{\alpha+n+1}(\mathbb{R})$ vérifiant l'hypothèse de non-cavitation (7), et notons (ζ, ψ) la solution correspondante. Soit $(u_{e,0}^+, u_{e,0}^-)$ défini par les formules (47) appliquées à (ζ_0, ψ_0) . Il existe une unique solution (u_e^+, u_e^-) au système*

$$\begin{cases} \partial_t u_e^+ + (\varepsilon \frac{3u_e^+ + u_e^-}{2} + 1)(F_1^\mu)^{1/2} \partial_x u_e^+ = 0, \\ \partial_t u_e^- + (\varepsilon \frac{u_e^+ + 3u_e^-}{2} - 1)(F_1^\mu)^{1/2} \partial_x u_e^- = 0, \end{cases} \quad (48)$$

avec pour condition initiale $(u_{e,0}^+, u_{e,0}^-)$. Elle vérifie la propriété suivante : notons ζ_c et ψ_c les quantités suivantes

$$\zeta_c = \frac{1}{\varepsilon} \left[\left(\frac{\varepsilon}{2} (u_e^+ - u_e^-) + 1 \right)^2 - 1 \right] \quad \text{et} \quad \psi_c = \int_0^x (F_1^\mu)^{-1/2} [u_e^+ + u_e^-] dx. \quad (49)$$

Pour tout temps $t \in [0, \frac{T}{\varepsilon}]$, on a

$$|(\zeta - \zeta_c, \psi - \psi_c)|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq \mu \varepsilon C t,$$

où $C, T^{-1} = C(\frac{1}{h_{\min}}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$, uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}$.

Proposition 0.3.4. *Avec les mêmes hypothèses et les mêmes notations que celles de la Proposition 0.3.1, si $u^-(0) = O(\mu)$, alors il existe $T_1 > 0$ tel que pour tout temps $t \in [0, T_1/\varepsilon]$, $u^-(t) = O(\mu)$. De plus, pour ces temps, u^+ est solution des équations de Whitham à un terme de reste près d'ordre $O(\mu\varepsilon)$, i.e.*

$$\partial_t u^+ + (F_1^\mu)^{1/2} [\partial_x u^+] + \frac{3\varepsilon}{2} u^+ \partial_x u^+ = O(\mu\varepsilon).$$

Si, plutôt, $u^+(0) = O(\mu)$, alors il existe $T_2 > 0$ tel que pour tout temps $t \in [0, T_2/\varepsilon]$, $u^+(t) = O(\mu)$. De plus, pour ces temps, u^- est solution des équations de Whitham à

propagation opposée à un terme de reste près d'ordre $O(\mu\varepsilon)$, i.e.

$$\partial_t u^- - (F_1^\mu)^{1/2}[\partial_x u^-] + \frac{3\varepsilon}{2}u^- \partial_x u^- = O(\mu\varepsilon). \quad (50)$$

Théorème 0.3.5. (Convergence) Dans le Corollaire 0.3.3, si $u_e^-(0) = O(\mu)$, alors on peut remplacer ζ_c et ψ_c par

$$\zeta_{\text{Wh},+} = \frac{1}{\varepsilon} \left[\left(\frac{\varepsilon}{2}u^+ + 1 \right)^2 - 1 \right] \quad \text{et} \quad \psi_{\text{Wh},+} = \int_0^x (F_1^\mu)^{-1/2}[u^+]dx,$$

où $u^+ \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}))$ est solution des équations de Whitham

$$\partial_t u^+ + (F_1^\mu)^{1/2}[\partial_x u^+] + \frac{3\varepsilon}{2}u^+ \partial_x u^+ = 0, \quad (51)$$

et obtenir pour tout temps $t \in [0, \frac{T}{\varepsilon}]$,

$$|(\zeta - \zeta_{\text{Wh},+}, \psi - \psi_{\text{Wh},+})|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(|u_e^-(0)|_{H^{\alpha+1}} + \mu\varepsilon t),$$

où $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$, uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}$.

Si, plutôt, $u_e^+(0) = O(\mu)$, alors on peut remplacer ζ_c et ψ_c par

$$\zeta_{\text{Wh},-} = \frac{1}{\varepsilon} \left[\left(\frac{\varepsilon}{2}u^- - 1 \right)^2 - 1 \right] \quad \text{et} \quad \psi_{\text{Wh},-} = \int_0^x (F_1^\mu)^{-1/2}[u^-]dx,$$

où $u^- \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}))$ est solution des équations de Whitham exactes à propagation opposée

$$\partial_t u^- - (F_1^\mu)^{1/2}[\partial_x u^-] + \frac{3\varepsilon}{2}u^- \partial_x u^- = 0,$$

et obtenir pour tout temps $t \in [0, \frac{T}{\varepsilon}]$,

$$|(\zeta - \zeta_{\text{Wh},-}, \psi - \psi_{\text{Wh},-})|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(|u_e^+(0)|_{H^{\alpha+1}} + \mu\varepsilon t),$$

où $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$, uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}$.

Remarque 0.3.6. Si on suppose que $u_e^-(0) = O(\mu\varepsilon)$ alors une traduction textuelle du Théorème 0.3.5 est la suivante. Les solutions des équations des vagues associées à des conditions initiales vérifiant $u_e^-(0) = O(\mu\varepsilon)$ peuvent être approchées à l'ordre $\mu\varepsilon + (\mu\varepsilon)t$

dans le régime de petite profondeur d'eau, pendant un temps d'ordre ε^{-1} , par une solution des équations de Whitham (51).

On a bien sûr un résultat analogue en supposant plutôt $u_e^+(0) = O(\mu\varepsilon)$.

0.4 Algorithme de la forme normale de Birkhoff et obtention de modèles unidirectionnels

Dans son article [Bam20], Bambusi développe une généralisation de l'algorithme de la forme normale de Birkhoff pour obtenir les équations de Korteweg-de Vries (KdV) depuis les équations des vagues. Nous résumons le procédé ici. Plaçons-nous, tout d'abord, dans le régime asymptotique de KdV

$$\mathcal{A}_{\text{KdV}} := \{(\mu, \varepsilon) \in \mathcal{A}; \varepsilon = \mu\},$$

Le but du procédé est de découpler les équations des vagues à un ordre $O(\varepsilon^2)$ en se servant de la structure hamiltonienne de ces dernières, et plus précisément en transformant son hamiltonien en forme normale.

Partons de l'hamiltonien des équations des vagues (13) sous sa forme adimensionnelle

$$H(\zeta, \psi) = \frac{1}{2} \int_{\mathbb{R}} \zeta^2 dx + \frac{1}{2} \int_{\mathbb{R}} \psi \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta] \psi dx, \quad (52)$$

où $\mathcal{G}^\mu[\varepsilon\zeta]\psi$ est défini en (19). Cet hamiltonien est associé au tenseur de Poisson $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Rappelons que les équations des vagues s'écrivent alors

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = J \nabla_{\zeta, \psi} H,$$

où $\nabla_{\zeta, \psi} = (\delta_\zeta, \delta_\psi)^T$, avec δ_ζ et δ_ψ les dérivées fonctionnelles par rapport à respectivement ζ et ψ .

Via la relation (25) et la Proposition 0.2.3, on peut montrer que l'hamiltonien H admet un développement asymptotique à l'ordre $O(\varepsilon^2)$ de la forme

$$H = H_0 + \varepsilon H_1 + O(\varepsilon^2),$$

où

$$H_0(\zeta, \psi) = \frac{1}{2} \int_{\mathbb{R}} \zeta^2 dx + \frac{1}{2} \int_{\mathbb{R}} \partial_x \psi^2 dx$$

et $H_1(\zeta, \psi) = -\frac{1}{6} \int_{\mathbb{R}} (\partial_x^2 \psi)^2 dx + \frac{1}{2} \int_{\mathbb{R}} \zeta (\partial_x \psi)^2 dx.$

On commence par découpler la partie linéaire définie par H_0 en posant le changement de variables

$$r := \frac{\zeta + \partial_x \psi}{2}, \quad s := \frac{\zeta - \partial_x \psi}{2}. \quad (53)$$

On se ramène alors à travailler sur l'hamiltonien

$$\begin{aligned} \widetilde{H}(r, s) &= \widetilde{H}_0(r, s) + \varepsilon \widetilde{H}_1(r, s) \\ &:= \int_{\mathbb{R}} (r^2 + s^2) dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^3 + s^3) dx - \frac{\varepsilon}{6} \int_{\mathbb{R}} ((\partial_x r)^2 + (\partial_x s)^2) dx \\ &\quad - \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^2 s + r s^2) dx + \frac{\varepsilon}{6} \int_{\mathbb{R}} (\partial_x r)(\partial_x s) dx, \end{aligned}$$

associé au tenseur de Poisson $\widetilde{J} = \begin{pmatrix} -\frac{\partial_x}{2} & 0 \\ 0 & \frac{\partial_x}{2} \end{pmatrix}.$

Tout d'abord, on peut remarquer que \widetilde{H}_0 est sous forme normale, c'est-à-dire que les équations hamiltoniennes associées sont découplées. C'était l'intérêt de poser le changement de variables (53). De plus, \widetilde{H}_1 se décompose en

$$\widetilde{H}_1 = Z_1 + W_1,$$

où $Z_1 = \frac{1}{2} \int_{\mathbb{R}} (r^3 + s^3) dx - \frac{1}{6} \int_{\mathbb{R}} ((\partial_x r)^2 + (\partial_x s)^2) dx$ est sous forme normale. Seul W_1 donne lieu à des termes couplés.

L'idée de l'algorithme de la forme normale de Birkhoff est de conjuguer \widetilde{H} par le flot associé à une fonction lisse bien choisie, de telle sorte qu'à l'ordre $O(\varepsilon^2)$, son développement asymptotique soit sous forme normale.

Soit donc $G(p, q)$ une fonction lisse. Considérons les équations hamiltoniennes associées

$$\partial_t \begin{pmatrix} p \\ q \end{pmatrix} = \widetilde{J} \nabla_{p,q} G(p, q), \quad (54)$$

où $\nabla_{p,q} = (\delta_p, \delta_q)^T$ avec δ_p et δ_q les dérivées fonctionnelles par rapport à respectivement p et q . Notons alors F_G^t le flot associé à (54).

Définissons aussi le crochet de Poisson de deux fonctions lisses $G_1(p, q)$ et $G_2(p, q)$ par la formule

$$\{G_1, G_2\} := (\nabla_{p,q} G_1, \tilde{J} \nabla_{p,q} G_2),$$

où (\cdot, \cdot) désigne le produit scalaire dans $L^2(\mathbb{R}^d)^2$.

On peut montrer que

$$\tilde{H} \circ F_G^\varepsilon = \tilde{H}_0 + \varepsilon \{\tilde{H}_0, G\} + \varepsilon \tilde{H}_1 + O(\varepsilon^2),$$

(voir [Bam20]). Si G est solution de l'équation homologique

$$\{\tilde{H}_0, G\} + W_1 = 0, \tag{55}$$

on obtient donc

$$\tilde{H} \circ F_G^\varepsilon = \tilde{H}_0 + \varepsilon Z_1 + O(\varepsilon^2). \tag{56}$$

De plus, les quantités $\begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} := F_G^\varepsilon \begin{pmatrix} r \\ s \end{pmatrix}$ vérifient les équations hamiltoniennes associées à $\tilde{H}_0 + \varepsilon Z_1$ que sont

$$\begin{cases} \partial_t \tilde{r} + (1 + \frac{\varepsilon}{6} \partial_x^2) \partial_x \tilde{r} + \frac{3\varepsilon}{2} \tilde{r} \partial_x \tilde{r} = 0, \\ \partial_t \tilde{s} - (1 + \frac{\varepsilon}{6} \partial_x^2) \partial_x \tilde{s} - \frac{3\varepsilon}{2} \tilde{s} \partial_x \tilde{s} = 0, \end{cases} \tag{57}$$

à des termes de reste près d'ordre $O(\varepsilon^2)$. Les deux équations de (57) sont des équations de KdV.

Dans son article [Bam20], Bambusi montre qu'il existe une solution de l'équation homologique (55), mais que cette solution n'engendre pas de flot. Il faut donc un argument supplémentaire pour rendre le raisonnement précédent rigoureux. Son idée est de remplacer le flot par ce que serait son développement asymptotique s'il existait.

Si une fonction lisse G engendre un flot F_G^t , alors ce dernier admet le développement

asymptotique à l'ordre $O(\varepsilon^2)$ suivant

$$F_G^\varepsilon = \text{Id} + \varepsilon G + O(\varepsilon^2).$$

Bambusi montre que dans le raisonnement précédent, on peut remplacer le flot associé à la solution G de l'équation (55) par $\text{Id} + \varepsilon G$ et obtenir (56)-(56).

Remarque 0.4.1. *Grâce à cette méthode, Bambusi montre que les solutions des équations des vagues peuvent être approchées à l'ordre ε sur un temps d'ordre $1/\varepsilon$ dans le régime de KdV, par deux vagues se propageant dans des directions opposées, solutions d'équations de KdV. Il améliore ainsi les résultats obtenus par Schneider et Wayne dans [SW00], et ceux obtenus par Bona, Colin et Lannes dans [BCL05]. En effet, le résultat ne fait pas apparaître de croissance résiduelle. Il suppose en revanche un contrôle des solutions de KdV dans les espaces de Sobolev $W^{\alpha,1}(\mathbb{R})$ (voir Notations 0.4.2).*

Dans le Chapitre 3, nous nous plaçons dans le régime asymptotique de petite profondeur d'eau \mathcal{A} (voir (15)) et adaptons la méthode de Bambusi en partant d'un développement asymptotique de l'hamiltonien des équations des vagues (52) à l'ordre $O(\mu\varepsilon)$, obtenu à partir de (30). Une des difficultés supplémentaires étant de gérer les termes non-locaux présents dans ce développement. Voici les résultats obtenus.

Notations 0.4.2. — *Pour tout $\alpha \geq 0$ nous noterons $W^{\alpha,1}(\mathbb{R})$ l'espace de Sobolev d'ordre α associé à $L^1(\mathbb{R})$. Nous noterons les normes associées $|\cdot|_{W^{\alpha,1}}$.*

— *Pour tout $\alpha \geq 0$ nous noterons $\dot{W}^{\alpha+1,1}(\mathbb{R}) := \{f \in L^1_{\text{loc}}(\mathbb{R}), \partial_x f \in W^{\alpha,1}(\mathbb{R})\}$ l'espace de Beppo-Levi d'ordre α associé à $L^1(\mathbb{R})$. Les semi-normes associés sont $|\cdot|_{\dot{W}^{\alpha+1,1}} := |\partial_x(\cdot)|_{W^{\alpha,1}}$.*

Definition 0.4.3. — *Soit $\alpha \geq 0$. On définit la transformation $\mathcal{T}_1 : H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R}) \rightarrow H^\alpha(\mathbb{R}) \times \dot{H}^{\alpha+1}(\mathbb{R})$ par la formule*

$$\mathcal{T}_1(\zeta, v) = \begin{pmatrix} \zeta \\ \int_0^x v(y) dy \end{pmatrix}.$$

— *Soit $\alpha \geq 0$. Soit $(F_1^\mu)^{-1/2}$ l'inverse du multiplicateur de Fourier $(F_1^\mu)^{1/2}$. On définit*

la transformation $\mathcal{T}_D : H^{\alpha+1}(\mathbb{R}) \times H^{\alpha+1}(\mathbb{R}) \rightarrow H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R})$ par la formule

$$\mathcal{T}_D(r, s) = \begin{pmatrix} r + s \\ (F_1^\mu)^{-1/2}[r - s] \end{pmatrix}.$$

— On définit l'opérateur $\partial^{-1} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ par la formule

$$\partial^{-1}u(y) = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(y - y_1)u(y_1)dy_1.$$

— Soit $\alpha \in \mathbb{N}$. On définit la transformation $\mathcal{T}_B : W^{\alpha+1,1}(\mathbb{R}) \times W^{\alpha+1,1}(\mathbb{R}) \rightarrow W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})$ par la formule

$$\mathcal{T}_B(r, s) = \begin{pmatrix} r + \frac{\varepsilon}{4}\partial_x(r)\partial^{-1}(s) + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}s^2 \\ s + \frac{\varepsilon}{4}\partial_x(s)\partial^{-1}(r) + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}r^2 \end{pmatrix}.$$

Théorème 0.4.4. Soit $\alpha \in \mathbb{N}$ et $(\mu, \varepsilon) \in \mathcal{A}$. Soit F_1^μ le multiplicateur de Fourier défini en (29). Soient $r, s \in C^1([0, \frac{T}{\varepsilon}], W^{\alpha+7,1}(\mathbb{R}))$ solutions des équations de Whitham découplées

$$\begin{cases} \partial_t r + (F_1^\mu)^{1/2}[\partial_x r] + \frac{3\varepsilon}{2}r\partial_x r = 0, \\ \partial_t s - (F_1^\mu)^{1/2}[\partial_x s] - \frac{3\varepsilon}{2}s\partial_x s = 0. \end{cases} \quad (58)$$

Soit aussi H l'hamiltonien des équations des vagues (52) et $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ le tenseur de Poisson associé. Alors les quantités

$$\begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} := \mathcal{T}_I(\mathcal{T}_D(\mathcal{T}_B(r, s))), \quad (59)$$

satisfont

$$\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} = J\nabla(H_{\text{WW}})(\zeta_{\text{Wh}}, \psi_{\text{Wh}}) + (\mu\varepsilon + \varepsilon^2)R, \quad \forall t \in [0, \frac{T}{\varepsilon}],$$

où $|R|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(|r|_{W^{\alpha+7,1}}, |s|_{W^{\alpha+7,1}})$, uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}$.

Corollary 0.4.5. Il existe $n \in \mathbb{N}^*$ tel que pour tout $\alpha \geq 0$ et tout $(\mu, \varepsilon) \in \mathcal{A}$, on ait ce qui suit. Considérons le problème de Cauchy associé aux équations des vagues (18) avec pour conditions initiales $(\zeta_0, \psi_0) \in W^{\alpha+n,1}(\mathbb{R}) \times \dot{W}^{\alpha+n+1,1}(\mathbb{R})$ vérifiant l'hypothèse

de non-cavitation (7), et notons (ζ, ψ) la solution correspondante. Il existe une solution (r, s) du système Hamiltonien (58) qui vérifie la propriété suivante : notons $(\zeta_{\text{Wh}}, \psi_{\text{Wh}})$ la solution définie par (59), alors pour tout temps $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$, on a

$$|(\zeta - \zeta_{\text{Wh}}, \psi - \psi_{\text{Wh}})|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t),$$

où $C, T^{-1} = C(\frac{1}{h_{\min}}, |\zeta_0|_{W^{\alpha+n}}, |\psi_0|_{\dot{W}^{\alpha+n+1}})$, uniformément par rapport à $(\mu, \varepsilon) \in \mathcal{A}$.

Remarque 0.4.6. — Dans le Corollaire 0.4.5, les conditions initiales (r_0, s_0) associées à (r, s) sont

$$(r_0, s_0) = \mathcal{T}_B^{\text{inv}}(\mathcal{T}_D^{-1}(\mathcal{T}_I^{-1}(\zeta_0, \psi_0))), \quad (60)$$

où la transformation $\mathcal{T}_B^{\text{inv}} : W^{\beta+1,1}(\mathbb{R}) \times W^{\beta+1,1}(\mathbb{R}) \rightarrow W^{\beta,1}(\mathbb{R}) \times W^{\beta,1}(\mathbb{R})$, $\beta \geq 0$, est définie par la formule

$$\mathcal{T}_B^{\text{inv}} = \text{Id} - \varepsilon \widetilde{\mathcal{T}}_B,$$

avec $\mathcal{T}_B = \text{Id} + \varepsilon \widetilde{\mathcal{T}}_B$.

- Le Corollaire 0.4.5 nous dit que les solutions des équations des vagues peuvent être approchées à l'ordre $\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t$ dans le régime de petite profondeur d'eau, pour un temps d'ordre $(\max(\mu, \varepsilon))^{-1}$, par deux vagues se propageant dans des directions opposées, solutions d'équations de Whitham. Le ε^2 dans l'ordre de précision provient des effets de couplage entre les deux vagues se propageant dans des directions opposées.
- Dans le régime de faible non-linéarité \mathcal{A}_{FN} (17) ($\varepsilon \leq \mu = 1$), on obtient une précision d'ordre ε sur un temps 1. Si ε est petit, les solutions des équations de Whitham donnent donc une bonne approximation des solutions des équations des vagues, même si μ n'est pas petit.

Remarque 0.4.7. Dans le tableau ci-dessous nous présentons les ordres de précision relatifs au modèle de KdV et de Whitham dans les cas de propagations unidirectionnelles et bidirectionnelles.

	vagues unidirectionnelles sur un temps ε^{-1}	vagues bidirectionnelles sur un temps $(\max(\mu, \varepsilon))^{-1}$
KdV	$(\mu\varepsilon + \mu^2)t$	$\varepsilon^2 + (\mu\varepsilon + \mu^2 + \varepsilon^2)t$
Whitham	$(\mu\varepsilon)t$	$\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t$

Le résultat sur le modèle de KdV, dans le cas bidirectionnel, provient de la méthode de Bambusi présentée précédemment, en se plaçant dans le régime d'eau peu profonde \mathcal{A} (voir (15)). Le résultat pour ce même modèle, dans le cas unidirectionnel provient de la Section 7.3.2 dans [Lan13b].

- On peut remarquer que dans le régime KdV ($\mu = \varepsilon$), on retrouve les mêmes ordres de précision pour les deux modèles.
- Dans le régime d'eau peu profonde, si μ n'est pas petit, les équations de KdV ne sont pas appropriées pour approcher les équations des vagues, et cela, même si $\varepsilon \ll 1$ (contrairement aux équations de Whitham).
- La précision des équations de Whitham dans le cas unidirectionnel est meilleure que dans le cas bidirectionnel. Cela est dû aux effets de couplage entre les deux vagues dans ce dernier cas.
- Le modèle de Whitham admet la possibilité d'un changement de comportement dans la propagation des vagues, qui pourrait passer d'un régime longues vagues où $\varepsilon = O(\mu)$ avec $\mu \ll 1$, à un régime de faible non-linéarité où seulement $\varepsilon \ll 1$ et μ n'est pas petit.

0.5 Conclusion et perspectives

0.5.1 Conclusion

Dans le Chapitre 1 [Eme20], nous prouvons la cohérence des équations des vagues (18), dans le régime de petite profondeur d'eau \mathcal{A} , à l'ordre $O(\mu\varepsilon)$ et $O(\mu^2\varepsilon)$ respectivement, du modèle de Whitham-Boussinesq (Proposition 0.2.29 et Proposition 0.2.30), et du modèle de Whitham-Green-Naghdi (Proposition 0.2.42 et Proposition 0.2.44), donnant ainsi pour la première fois une caractérisation non triviale de l'ordre de précision des modèles bidirectionnels de type dispersion complète.

Dans le Chapitre 2, nous prouvons le caractère bien-posé local d'une classe de systèmes quasi-linéaires symétrisables non locaux. Puis nous appliquons ce résultat à une classe de systèmes de Whitham-Boussinesq quasi-linéaires symétrisables pour obtenir leur caractère bien posé local avec un temps d'existence des solutions indépendant du paramètre de petite profondeur d'eau μ (Théorème 0.2.36). Nous démontrons aussi un résultat de stabilité (Proposition 0.2.39) puis de justification (Théorème 0.2.40) de cette classe de modèles de

Whitham-Boussinesq.

Dans le Chapitre 3 [Eme21b], nous prouvons la justification des équations de Whitham dans le régime de petite profondeur d'eau, dans deux cas différents. Le premier suppose les conditions initiales des équations des vagues bien préparées et adaptées à un mode de propagation unidirectionnel. Il donne la justification des équations de Whitham comme modèle d'eau peu profonde avec une précision $\mu\varepsilon t$ pour un temps d'ordre ε^{-1} (Theorem 0.3.5). Le deuxième cas suppose un contrôle des conditions initiales des équations des vagues dans les espaces de Sobolev $W^{\alpha,1}(\mathbb{R})$, et est adapté à un mode de propagation bidirectionnel. Il donne la justification des équations de Whitham avec une précision $\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t$ pour un temps d'ordre $(\max(\mu, \varepsilon))^{-1}$ (Corollary 0.4.5). Ces deux résultats donnent, pour la première fois, une justification complète et précise des équations de Whitham dans un régime asymptotique physique, le régime de petite profondeur d'eau.

Dans le Chapitre 4 [Aud+21], nous analysons numériquement la cohérence des modèles de Saint-Venant et de Boussinesq, généralisés dans le cas de fonds marins non plats et dépendant du temps, dans le cadre de tsunamis engendrés par des glissements de terrain. À partir de ces analyses, nous proposons une stratégie de changement de modèle, passant du modèle de Saint-Venant lors de la génération de la vague, au modèle de Boussinesq lors de la propagation, permettant ainsi d'améliorer les résultats numériques. Nous établissons aussi des corrélations entre des grandeurs physiques caractéristiques du glissement de terrain et la vitesse de propagation ou l'amplitude des vagues générées. Et enfin, à partir de la seule connaissance d'une vague générée par un glissement de terrain, nous étudions un problème inverse permettant de reconstruire ce dernier dans des cas simples.

0.5.2 Perspectives

Plusieurs pistes d'approfondissements des résultats obtenus dans cette thèse sont possibles. Nous pourrions poursuivre la justification des modèles de Whitham-Green-Naghdi. Dans la Remarque 0.2.45, nous avons évoqué qu'il serait sûrement possible d'utiliser les techniques du Chapitre 2, pour obtenir un résultat de caractère bien-posé local, avec un temps d'existence d'ordre ε^{-1} , indépendant du paramètre de petite profondeur d'eau μ , des solutions des systèmes de Whitham-Green-Naghdi (46). Il suffirait, ensuite, d'obtenir un résultat de stabilité de ces systèmes, pour obtenir un théorème de convergence.

Dans [DST19], les auteurs ont prouvé un résultat de caractère bien-posé global, dans

le cas de la dimension $d = 1$, pour le système (40). Les techniques de cet article utilisent principalement la structure semi-linéaire des équations. Si on suppose les multiplicateurs de Fourier G_1^μ et G_2^μ suffisamment régularisant, les systèmes (46) auraient alors une structure semi-linéaire. Il serait probablement possible d'utiliser les techniques de [DST19], pour obtenir le caractère bien-posé global de ces systèmes, dans le cas de la dimension $d = 1$.

On pourrait aussi s'intéresser aux dérivations de modèles bidirectionnels de type dispersion complète dans le cas où le fond marin n'est pas plat.

Mentionnons aussi qu'un modèle de type Whitham-Green-Naghdi multi-couches a été introduit dans [DIT16]. On pourrait donc essayer d'étendre la dérivation rigoureuse des modèles de type dispersion complète dans le cas d'océans stratifiés.

Dans cette thèse, on tend à prouver que les modèles de type dispersion complète sont plus robustes que les modèles bidirectionnels classiques d'eau peu profonde. En considérant aussi les généralisations possibles des modèles de Whitham-Boussinesq et de Whitham-Green-Naghdi aux fonds marins non-plat, ou aux eaux stratifiées, on pourrait évaluer le gain obtenu dans des cas concrets. Par exemple, dans le cadre de la génération de tsunamis générés par des glissements de terrain, étudié au Chapitre 4, nous utilisons une stratégie de changement de modèle. Dans la zone de génération de la vague, nous utilisons les équations de Saint-Venant, et dans la zone de propagation, nous utilisons le modèle de Boussinesq. On pourrait remplacer ce dernier modèle par un système de type dispersion complète. Surtout que nous montrons que dans le cas où l'énergie transférée à l'eau par le glissement de terrain dépasse un certain seuil, les deux modèles de Saint-Venant et de Boussinesq semblent échouer à reproduire la propagation du tsunami. Il serait intéressant de voir si c'est aussi le cas pour les modèles de Whitham-Boussinesq et de Whitham-Green-Naghdi.

Dans tous les modèles, bidirectionnels et unidirectionnels, de type dispersion complète, on pourrait essayer d'adapter les démonstrations au cas où il y aurait de la tension de surface.

Dans la littérature, des modèles de Whitham d'ordre supérieur ont été introduits (voir par exemple [MKD15]). On pourrait se demander si les techniques du Chapitre 3 pourraient être utilisées pour les dériver rigoureusement en partant d'un système de Whitham-Green-Naghdi bien choisi, dont l'ordre de précision est $O(\mu^2\varepsilon)$. Cependant, il ne semble pas possible d'obtenir l'ordre de précision caractérisant ces modèles, puisque des termes d'ordre $O(\mu\varepsilon)$ apparaîtraient dans les procédés.

Chapitre 1

Rigorous derivation from the water waves equations of some full dispersion shallow water models

Article soumis et accepté au journal Journal on Mathematical Analysis (SIMA) [Eme20].

Abstract: In order to improve the frequency dispersion effects of irrotational shallow water models in coastal oceanography, several full dispersion versions of classical models have been formally derived in the literature. The idea, coming from G. Whitham in [Whi67], was to modify them so that their dispersion relation is the same as the water waves equations. In this paper we construct new shallow water approximations of the velocity potential then deducing ones on the vertically averaged horizontal component of the velocity. We make use of them to derive rigorously from the water waves equations two new Hamiltonian full dispersion models. This provides for the first time non-trivial precision results characterizing the order of approximation of the full dispersion models. They are non-trivial in the sense that they are better than the ones for the corresponding classical models.

1.1 Introduction

1.1.1 Motivations

In this work, we consider full dispersion models for the propagation of surface waves in coastal oceanography. It is a class of irrotational shallow water models, each model having the same dispersion relation as the water waves equations. The first nonlinear full dispersion model appearing in the literature was introduced formally by Whitham [Whi67 ;

Whi74] to study wave breaking phenomena [Hur17] and Stokes waves of extreme amplitude [EW19]. It is a modification of the Korteweg–de Vries (KdV) equations called the Whitham equations, see [Eme21b] for a rigorous derivation of the latter equations and a comparison with the KdV equations. Later on, the same kind of formal modifications has been made on other standard shallow water models such as the Boussinesq or Green-Naghdi systems, thus creating a whole class of full dispersion models. The motivation was to widen the range of validity of the shallow water models, see Section 5.3 of [Lan13b], and to study the propagation of waves above obstacles, a situation where there is creation of high harmonics which are then freely released, see [BB93; Din94; Lan13b].

The models obtained by modifying the Boussinesq system, generally called Whitham-Boussinesq systems, have been the subject of active research, see [Car18; DDK19] for comparative studies, [Din19; Din20; DST19; KP18] for the well-posedness theory, [DN19; NW19] for some works on solitary waves solutions, and [HP16] for a study on modulational instability (this list is not exhaustive, see also [Pan19; VP16]). In the case of the modified Green-Naghdi systems, see [DIT16] for a fully justified two-layer one.

However at the best of the author’s knowledge no direct derivation of these models from the water waves equations has been done. In this paper we provide asymptotic approximations of the Dirichlet-to-Neumann operator. Then we use them to derive two different Hamiltonian full dispersion systems (see (1.9) and (1.10)) and justify them in the sense of consistency (see definition 1.1.10) of the water waves equations with these two models. Subsequently we deduce from them an improved precision result with respect to the one already known for the different full dispersion models appearing in the literature.

1.1.2 Consistency problem

Throughout this paper d will be the dimension of the horizontal variable (denoted $X \in \mathbb{R}^d$).

The starting point of this study is the adimensional water waves problem, that is ($d = 1, 2$)

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}^\mu \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\varepsilon}{\mu} \frac{(\mathcal{G}^\mu \psi + \varepsilon \mu \nabla \zeta \cdot \nabla \psi)^2}{2(1 + \varepsilon^2 \mu |\nabla \zeta|^2)} = 0. \end{cases} \quad (1.1)$$

Here

- ∇ is the horizontal gradient, i.e.

$$\nabla := \begin{cases} \partial_x, & \text{when } d = 1, \\ (\partial_x, \partial_y)^T, & \text{when } d = 2. \end{cases}$$

We will also denote by Δ the Laplacian in \mathbb{R}^d .

- The free surface elevation is the graph of ζ , which is a function of time t and horizontal space $X \in \mathbb{R}^d$.
- $\psi(t, X)$ is the trace at the surface of the velocity potential.
- \mathcal{G}^μ is the Dirichlet-to-Neumann operator defined later in Definition 1.1.4.

Moreover every variable and function in (1.1) is compared with physical characteristic parameters of the same dimension. Among those are the characteristic water depth H_0 , the characteristic wave amplitude a_{surf} and the characteristic wavelength L . From these comparisons appear two adimensional parameters of main importance :

- $\mu := \frac{H_0^2}{L^2}$: the shallow water parameter,
- $\varepsilon := \frac{a_{\text{surf}}}{H_0}$: the nonlinearity parameter.

We refer to [Lan13b] for details on the derivation of these equations.

Before giving the main definitions of this section, here are two assumptions maintained throughout this paper.

Hypothesis 1.1.1. — *A fundamental hypothesis in this study will be the lower boundedness by a positive constant of the water depth (non-cavitation assumption) :*

$$\exists h_{\min} > 0, \forall X \in \mathbb{R}^d, \quad h := 1 + \varepsilon\zeta(t, X) \geq h_{\min}. \quad (1.2)$$

- *We suppose that the bottom of the sea is flat. The water domain is then defined by $\Omega_t := \{(X, z) \in \mathbb{R}^{d+1} : -1 < z < \varepsilon\zeta(X)\}$.*

In what follows we need some notations on the functional framework of this paper.

Notations 1.1.2. — *For any $s \geq 0$ we will denote $H^s(\mathbb{R}^d)$ the Sobolev space of order s in $L^2(\mathbb{R}^d)$.*

- *For any $s \geq 1$ we will denote $\dot{H}^s(\mathbb{R}^d) := \{f \in L^2_{\text{loc}}(\mathbb{R}^d), \nabla f \in H^{s-1}(\mathbb{R}^d)\}$ the Beppo-Levi space of order s .*
- *The $L^2(\mathbb{R}^d)$ norm will be written $|\cdot|_2$. The $L^2(\mathcal{S})$ norm, where $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ (see Definition 1.1.3), will be denoted $\|\cdot\|_2$.*

- Denoting $\Lambda^s := (1 - \Delta)^{s/2}$, where Δ is the Laplace operator in \mathbb{R}^d , the $H^s(\mathbb{R}^d)$ norm will be $|\cdot|_{H^s} := |\Lambda^s \cdot|_2$. Moreover the seminorm associated with $\dot{H}^{s+1}(\mathbb{R}^d)$ will be $|\cdot|_{\dot{H}^{s+1}} := |\nabla(\cdot)|_{H^s}$.

From the trace at the surface of the velocity potential ψ , we can reconstruct the velocity potential, denoted Φ , on the whole water domain Ω_t through the elliptic problem

$$\begin{cases} \Delta \Phi = 0 & \text{in } \Omega_t, \\ \Phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_z \Phi|_{z=-1} = 0, \end{cases}$$

(see [Lan13b]). However the domain on which this problem is posed is time-dependent. That is why we define the following diffeomorphism.

Definition 1.1.3. Let $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ ($t_0 > d/2$) such that (1.2) is satisfied. We define the time-dependent diffeomorphism mapping the flat strip $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ onto the water domain Ω_t

$$\begin{aligned} \Sigma_t : \mathcal{S} := \mathbb{R}^d \times (-1, 0) &\rightarrow \Omega_t := \{(X, z) \in \mathbb{R}^{d+1} : -1 < z < \varepsilon\zeta(X)\} \\ (X, z) &\mapsto (X, z + \varepsilon(z+1)\zeta(X)). \end{aligned} \quad (1.3)$$

Let ϕ be the velocity potential in the flat strip \mathcal{S} , $\phi = \Phi \circ \Sigma_t$. It satisfies the elliptic problem

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu \phi = 0 & \text{in } \mathcal{S}, \\ \phi|_{z=0} = \psi, \quad \partial_z \phi|_{z=-1} = 0, \end{cases} \quad (1.4)$$

where ∇^μ is the $(d+1)$ -gradient operator defined by $\nabla^\mu = (\sqrt{\mu} \nabla^T, \partial_z)^T$,

and $P(\Sigma_t) = \begin{pmatrix} (1 + \varepsilon\zeta)I_d & -\sqrt{\mu}\varepsilon(z+1)\nabla\zeta \\ -\sqrt{\mu}\varepsilon(z+1)\nabla\zeta^T & \frac{1 + \mu\varepsilon^2(z+1)^2|\nabla\zeta|^2}{1 + \varepsilon\zeta} \end{pmatrix}$. The domain on which the elliptic problem (1.4) is posed is time-independent.

We can now define the Dirichlet-to-Neumann operator \mathcal{G}^μ , see the chapters 2 and 3 in [Lan13b].

Definition 1.1.4. Let $s \geq 0$, $t_0 > d/2$, $\psi \in \dot{H}^{s+3/2}(\mathbb{R}^d)$ and $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let ϕ be the velocity potential in the flat strip \mathcal{S} satisfying the elliptic

problem (1.4). Let also denote by h the water depth, $h = 1 + \varepsilon\zeta$. We define the vertically averaged horizontal velocity $\bar{V}[\varepsilon\zeta]\psi$ (denoted \bar{V} when no confusion is possible) by the formula

$$\bar{V} = \frac{1}{h} \int_{-1}^0 [h\nabla\phi - \varepsilon(z+1)\nabla\zeta\partial_z\phi] dz. \quad (1.5)$$

The Dirichlet-to-Neumann operator $\mathcal{G}^\mu[\varepsilon\zeta]$ (denoted \mathcal{G}^μ when no confusion is possible) is then defined as

$$\begin{aligned} \mathcal{G}^\mu &: \dot{H}^{s+3/2}(\mathbb{R}^d) \rightarrow H^{s+1/2}(\mathbb{R}^d) \\ &\psi \quad \mapsto -\mu\nabla \cdot (h\bar{V}). \end{aligned} \quad (1.6)$$

Before stating the first result of this paper, we recall the definition of a Fourier multiplier.

Definition 1.1.5. Let $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a tempered distribution, let \hat{u} be its Fourier transform. Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function with polynomial decay. Then the Fourier multiplier associated with $F(\xi)$ is denoted $F(D)$ (denoted F when no confusion is possible) and defined by the formula :

$$\widehat{F(D)u}(\xi) = F(\xi)\hat{u}(\xi).$$

We also need other notations.

Notations 1.1.6. All the results of this paper will use the following notations, where $C(\circ)$ means a constant depending on \circ .

Let $t_0 > d/2$, $s \geq 0$, and $\mu_{\max} > 0$. Given sufficiently regular ζ and ψ satisfying (1.2) we will write

- $M_0 := C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0}})$.
- $M(s) := C(M_0, |\zeta|_{H^{\max(t_0+1, s)}})$.
- $N(s) := C(M(s), |\nabla\psi|_{H^s})$.

Remark 1.1.7. In this paper, the notation t_0 is for a real number larger than $d/2$. However, it is not to be taken too large, we can consider $d/2 < t_0 \leq 2$. So that, when $s \geq 3$, $M(s)$ is in fact $M(s) := C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^s})$.

The first result of this paper provides asymptotic expansions of the velocity potential in the flat strip at order $O(\mu\varepsilon)$ or $O(\mu^2\varepsilon)$ with estimations of error.

Proposition 1.1.8. *Let $s \geq 0$, and $\zeta \in H^{s+2}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let also $\psi \in \dot{H}^{s+4}(\mathbb{R}^d)$ and $F_0 = \frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}$ be a Fourier multiplier depending on the transversal variable $z \in [-1, 0]$. We have the following estimates :*

$$\begin{cases} \|\Lambda^s \nabla^\mu(\phi - F_0 \psi)\|_2 \leq \mu \varepsilon M(s+2) |\nabla \psi|_{H^{s+2}}, \\ \|\Lambda^s \nabla^\mu(\phi - (F_0 \psi - \mu \varepsilon \zeta(1+h)(\frac{z^2}{2} + z)\Delta \psi))\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla \psi|_{H^{s+3}}, \\ \|\Lambda^s \nabla^\mu(\phi - (\psi + h^2(F_0 - 1)\psi))\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla \psi|_{H^{s+3}}. \end{cases}$$

From this proposition we get asymptotic expansions of the vertically averaged horizontal velocity (which implies ones on the Dirichlet-Neumann operator through (1.6)) with estimations of error. In addition we get an approximation of the velocity potential at the surface expressed in terms of the vertically averaged horizontal velocity at order $O(\mu^2 \varepsilon)$.

Proposition 1.1.9. *Let $s \geq 0$, and $\zeta \in H^{s+5}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+6}(\mathbb{R}^d)$, and \bar{V} be as in (1.5). Let also F_1, F_2 and F_3 be the three Fourier multipliers defined respectively by*

$$F_1 := \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}, \quad F_2 := \frac{3}{\mu|D|^2} \left(1 - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\right) \quad \text{and} \quad F_3 := F_2 F_1^{-1}, \quad (1.7)$$

where F_1^{-1} is the inverse Fourier multiplier of F_1 , associated with the function $1/F_1$. The following estimates hold :

$$\begin{cases} |\bar{V} - \nabla \psi|_{H^s} \leq \mu M(s+2) |\nabla \psi|_{H^{s+2}}, \\ |\bar{V} - F_1 \nabla \psi|_{H^s} \leq \mu \varepsilon M(s+3) |\nabla \psi|_{H^{s+3}}, \\ |\bar{V} - F_1 \nabla \psi - \frac{\mu \varepsilon}{3} [h \nabla \zeta \Delta \psi + \nabla(\zeta(1+h)\Delta \psi)]|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}}, \\ |\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla(h^3 F_2 \Delta \psi)|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}}, \\ |\bar{V} - \nabla \psi - \frac{\mu}{6h} \nabla(h^3 F_2 [\Delta \psi] + F_2 [h^3 \Delta \psi])|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}}, \\ |\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla(h^3 F_3 \nabla \cdot \bar{V})|_{H^s} \leq \mu^2 \varepsilon M(s+4) |\nabla \psi|_{H^{s+4}}, \\ |\bar{V} - \nabla \psi - \frac{\mu}{6h} \nabla(h^3 F_3 [\nabla \cdot \bar{V}] + F_3 [h^3 \nabla \cdot \bar{V}])|_{H^s} \leq \mu^2 \varepsilon M(s+5) |\nabla \psi|_{H^{s+5}}. \end{cases} \quad (1.8)$$

To state the next result of this paper we need to define the notion of consistency of the water waves equations (1.1) with respect to a given model in the shallow water asymptotic regime at a certain order in μ and ε .

Definition 1.1.10. (*Consistency*)

Let $\mu_{\max} > 0$. Let $\mathcal{A} \subset \{(\varepsilon, \mu), 0 \leq \varepsilon \leq 1, 0 \leq \mu \leq \mu_{\max}\}$ be the shallow water asymptotic regime. We denote by (A) and (A') two asymptotic models of the following form :

$$(A) \quad \begin{cases} \partial_t \zeta + \mathcal{N}_{(A)}^1(\zeta, \psi) = 0, \\ \partial_t \psi + \mathcal{N}_{(A)}^2(\zeta, \psi) = 0, \end{cases}, \quad (A') \quad \begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((Id + \mu T_{(A')}[h])\bar{V}) + \mathcal{N}_{(A')}^3(\zeta, \bar{V}) = 0, \end{cases}$$

where $\mathcal{N}_{(A)}^1$, $\mathcal{N}_{(A)}^2$ and $\mathcal{N}_{(A')}^3$ are nonlinear operators that depend respectively on the asymptotic model (A) , (A) and (A') . And $T_{(A')}$ is an operator nonlinear in h and linear in \bar{V} which depends on the asymptotic model (A') .

We say that the water waves equations are consistent at order $O(\mu^k \varepsilon^l)$ with respectively (A) or (A') in the regime \mathcal{A} if there exists $n \in \mathbb{N}$ and $T > 0$ such that for all $s \geq 0$ and $p = (\varepsilon, \mu) \in \mathcal{A}$, and for every solution $(\zeta, \psi) \in C([0, \frac{T}{\varepsilon}]; H^{s+n}(\mathbb{R}^d) \times \dot{H}^{s+n+1}(\mathbb{R}^d))$ to the water waves equations (1.1) one has respectively

$$\begin{cases} \partial_t \zeta + \mathcal{N}_{(A)}^1(\zeta, \psi) = \mu^k \varepsilon^l R_1, \\ \partial_t \psi + \mathcal{N}_{(A)}^2(\zeta, \psi) = \mu^k \varepsilon^l R_2, \end{cases} \quad \text{or} \quad \begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((Id + \mu T_{(A')}[h])\bar{V}) + \mathcal{N}_{(A')}^3(\zeta, \bar{V}) = \mu^k \varepsilon^l R_3, \end{cases}$$

where \bar{V} is defined by ζ and ψ through (1.5), and with respectively $|R_1|_{H^s}, |R_2|_{H^s} \leq N(s+n)$ or $|R_3|_{H^s} \leq N(s+n+1)$ on $[0, \frac{T}{\varepsilon}]$.

For sufficiently regular initial data satisfying hypothesis (1.2), the existence and uniqueness of a solution of the water waves equations with existence time of order $1/\varepsilon$ independent of μ and with the regularity we want is given by the theorem 4.16 in [Lan13b].

We now state our consistency results.

Proposition 1.1.11. Let F_1 and F_2 be the Fourier multipliers of (1.7). The water waves equations (1.1) are consistent at order $O(\mu^2 \varepsilon)$ in the shallow water regime \mathcal{A} (see Definition 1.1.10) with

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\nabla\psi) + \frac{\mu}{6} \left(\Delta(F_2[h^3\Delta\psi]) + \Delta(h^3F_2[\Delta\psi]) \right) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla\psi|^2 - \frac{\mu\varepsilon}{2} h^2(F_2[\Delta\psi])\Delta\psi = 0, \end{cases} \quad (1.9)$$

with $n = 5$.

Proposition 1.1.12. Let F_3 be the Fourier multiplier of (1.7). Let $T[h]V := -\frac{1}{6h}(\nabla(h^3F_3[\nabla \cdot V]) + \nabla(F_3[h^3\nabla \cdot V]))$. The water waves equations are consistent at order $O(\mu^2 \varepsilon)$ in the

shallow water regime \mathcal{A} with

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((\text{Id} + \mu T[h])\bar{V}) + \nabla \zeta & + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6} \nabla(\frac{1}{h}\bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])) \\ & - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3[\nabla \cdot \bar{V}]\nabla \cdot \bar{V}) = 0. \end{cases} \quad (1.10)$$

with $n = 6$.

Setting $F_3 = \text{Id}$ in (1.9) we recover the Green-Naghdi system introduced in [CG94; Whi67]. Setting $F_3 = \text{Id}$ in (1.10) we recover the classical Green-Naghdi system which has been proved to be consistent with precision $O(\mu^2)$ in [Lan13b; LB09]. This is why we refer to (1.9) and (1.10) as full dispersion Green-Naghdi systems.

Remark 1.1.13. *The two full dispersion Green-Naghdi systems (1.9) and (1.10) enjoy a canonical Hamiltonian formulation (see (1.46) and (1.55)).*

From these two previous propositions we are able to give results on the consistency with respect to water waves equations (1.1) of most of the full dispersion systems appearing in the literature. We give the examples of two systems that kept the author attention for their mathematical properties. The first one is a single layer, two dimensional generalisation with no surface tension of the model introduced in [DIT16] to study high-frequency Kelvin-Helmholtz instabilities. That is

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t(\bar{V} - \frac{\mu}{3h} \nabla(\sqrt{F_3} h^3 \sqrt{F_3}[\nabla \cdot \bar{V}])) + \nabla \zeta & + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{3} \nabla(\frac{1}{h}\bar{V} \cdot \nabla(\sqrt{F_3} h^3 \sqrt{F_3}[\nabla \cdot \bar{V}])) \\ & - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3[\nabla \cdot \bar{V}]\nabla \cdot \bar{V}) = 0. \end{cases} \quad (1.11)$$

Proposition 1.1.14. *The water waves equations are consistent at order $O(\mu^2\varepsilon)$ in the shallow water regime \mathcal{A} with the system (1.11) with $n = 6$.*

For the same reason as system (1.10), we refer to (1.11) as a full dispersion Green-Naghdi system.

The second one is a Whitham-Boussinesq system studied in [DST19]. The authors proved a local well-posedness result in dimension 2 and a global well-posedness result for

small data in dimension 1. This system is

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon F_1 \nabla \cdot (\zeta F_1 \nabla \psi) = 0, \\ \partial_t \nabla \psi + \nabla \zeta + \frac{\varepsilon}{2} \nabla ((F_1 |\nabla \psi|)^2) = 0. \end{cases} \quad (1.12)$$

With the definition we gave of consistency (see Definition 1.1.10) we have the following proposition.

Proposition 1.1.15. *The water waves equations (1.1) are consistent at order $O(\mu\varepsilon)$ in the shallow water regime \mathcal{A} with the system*

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon F_1 \nabla \cdot (\zeta F_1 \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} (F_1 |\nabla \psi|)^2 = 0, \end{cases}$$

with $n = 4$.

But one can easily adapt Definition 1.1.10 to say that the water waves equations (1.1) are consistent at order $O(\mu\varepsilon)$ with system (1.12).

We can also derive from the water waves equations (1.1) a Whitham-Boussinesq system with unknowns ζ and \bar{V} .

Proposition 1.1.16. *The water waves equations (1.1) are consistent at order $O(\mu\varepsilon)$ in the shallow water regime \mathcal{A} with the system*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ \partial_t \bar{V} + F_1 \nabla \zeta + \frac{\varepsilon}{2} F_1 \nabla (|\bar{V}|^2) = 0, \end{cases} \quad (1.13)$$

with $n = 4$.

Remark 1.1.17. *At the best of the author's knowledge, in the case of the full dispersion Green-Naghdi systems, before this work it was only known that the water waves equations are consistent in the shallow water regime with respect to system (1.11) at order $O(\mu^2)$ at worse (Proposition 5.7 in [DIT16]), that is the same precision as the one of the classical Green-Naghdi models, see chapter 5 in [Lan13b]. The use of Proposition 1.1.9 allow us to improve the precision order by a factor ε , as conjectured in [DIT16]. So that in a situation in which $\varepsilon \sim \mu$ (long wave regime) we gain a power of μ , i.e. the full dispersion*

Green-Naghdi systems are precise at order $O(\mu^3)$ in the long wave regime, in the sense of consistency. Moreover even if μ is not so small, the latter systems stay good approximations of the water waves equations as long as ε is small enough, making them more robust than the corresponding classical Green-Naghdi models.

The case of the Whitham-Boussinesq systems is a bit more subtle. Indeed, using the same argument as in the proof of Proposition 5.7 in [DIT16], one would obtain, in the shallow water regime, a precision order of $O(\mu^2 + \mu\varepsilon)$ for these systems, that is the same as the one of the Boussinesq models, see chapter 5 in [Lan13b]. In this paper we prove that the precision order of the Whitham-Boussinesq systems is in fact $O(\mu\varepsilon)$. So that the improvement can only be seen in a regime in which $\varepsilon \ll \mu$. It still makes them more robust than the Boussinesq models for the same reason as the full dispersion Green-Naghdi systems.

Throughout this paper we will use abundantly the product estimates of Proposition 1.6.1 and the quotient estimates of Proposition 1.6.2. When it is obvious, we will not mention the use of these classical results.

1.1.3 Outline

In Section 1.2 we prove Proposition 1.1.8 and Proposition 1.1.9. Subsection 1.2.1 is dedicated to the proof of Proposition 1.1.8. In Subsection 1.2.2 we make use of the latter proposition to prove Proposition 1.1.9.

In Section 1.3 we focus on system (1.9). First, in Subsection 1.3.1, we derive formally (1.9) from Hamilton's equations associated with an approximated Hamiltonian based on Proposition 1.1.9. In Subsection 1.3.2 we prove rigorously the consistency of the water waves equations with system (1.9), i.e. we prove Proposition 1.1.11.

In Section 1.4 we focus on system (1.10) and do the same process as for system (1.9). In Subsection 1.4.1 we get a second approximated Hamiltonian of the water waves system (1.1) and derive the Hamilton equations associated with, giving (1.10). In subsection 1.4.2 we prove Proposition 1.1.12.

In Section 1.5 we prove the consistency of the water waves equations with the systems (1.11), (1.12) and (1.13). In Subsection 1.5.1 we make use of Proposition 1.1.11 to prove Proposition 1.1.14. And in Subsection 1.5.2 we use the Propositions 1.1.11 and 1.1.12 to prove the Propositions 1.1.15 and 1.1.16.

1.2 Shallow water approximations

In this section we construct some shallow water approximations of ϕ , the velocity potential written in the time-independent domain $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ (see (1.4)). Then we deduce ones on the vertically averaged horizontal velocity \bar{V} (see Definition 1.1.4) in terms of the trace at the surface of the velocity potential ψ . We also construct an approximation of the latter quantity in terms of \bar{V} . For these purposes, we use a method similar to the one developed in Chapter 5 in [Lan13b].

1.2.1 Shallow water approximations of the velocity potential in the flat strip

We prove here Proposition 1.1.8 which we recall here for the sake of clarity.

Proposition 1.2.1. *Let $s \geq 0$, and $\zeta \in H^{s+2}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let also $\psi \in \dot{H}^{s+4}(\mathbb{R}^d)$ and $F_0 = \frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}$ be a Fourier multiplier depending on the transversal variable $z \in [-1, 0]$. We have the following estimates :*

$$\begin{cases} \|\Lambda^s \nabla^\mu (\phi - F_0 \psi)\|_2 \leq \mu \varepsilon M(s+2) |\nabla \psi|_{H^{s+2}}, \\ \|\Lambda^s \nabla^\mu (\phi - (F_0 \psi - \mu \varepsilon \zeta (1+h) (\frac{z^2}{2} + z) \Delta \psi))\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla \psi|_{H^{s+3}}, \\ \|\Lambda^s \nabla^\mu (\phi - (\psi + h^2 (F_0 - 1) \psi))\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla \psi|_{H^{s+3}}. \end{cases}$$

Démonstration. By definition ϕ satisfies an elliptic problem in the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$. This problem is written in term of the velocity potential at the surface ψ :

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu \phi = 0, \\ \phi|_{z=0} = \psi \quad , \quad \partial_z \phi|_{z=-1} = 0, \end{cases} \quad (1.14)$$

where ∇^μ stands for the $(d+1)$ -gradient operator defined by $\nabla^\mu = (\sqrt{\mu} \nabla^T, \partial_z)^T$,

$$\text{and } P(\Sigma_t) = \begin{pmatrix} (1 + \varepsilon \zeta) I_d & -\sqrt{\mu} \varepsilon (z+1) \nabla \zeta \\ -\sqrt{\mu} \varepsilon (z+1) \nabla \zeta^T & \frac{1 + \mu \varepsilon^2 (z+1)^2 |\nabla \zeta|^2}{1 + \varepsilon \zeta} \end{pmatrix}.$$

To prove Proposition 1.2.1, the idea is to do a multi-scale expansion for the solution of the elliptic problem (1.14).

Let us remark that multiplying the elliptic equation of (1.14) by the depth $h = 1 + \varepsilon \zeta$

allow us to decompose it into two parts :

$$h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\phi = (\partial_z^2\phi + \mu\Delta\phi) + \mu\varepsilon A(\nabla, \partial_z)[\phi], \quad (1.15)$$

where $A(\nabla, \partial_z)$ is an operator defined as follow

$$\begin{aligned} A(\nabla, \partial_z)[\phi] &= \nabla \cdot (\zeta\nabla\phi) + \zeta\nabla \cdot ((1 + \varepsilon\zeta)\nabla\phi) + \varepsilon|\nabla\zeta|^2\partial_z((z + 1)^2\partial_z\phi) \\ &\quad - (1 + \varepsilon\zeta)(z + 1)\nabla \cdot (\nabla\zeta\partial_z\phi) - (1 + \varepsilon\zeta)\nabla\zeta \cdot \partial_z((z + 1)\nabla\phi). \end{aligned} \quad (1.16)$$

In the elliptic problem (1.14) let us only consider the part which is not of order $\mu\varepsilon$ and denote ϕ_0 its solution. By a Fourier analysis we have

$$\phi_0 = \frac{\cosh((z + 1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)}\psi = F_0\psi. \quad (1.17)$$

Thus ϕ_0 is defined as a bounded Fourier multiplier applied to the trace at the surface of the velocity potential. Then the quantity $u_0 := \phi - \phi_0$ solves the elliptic problem

$$\begin{cases} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu u_0 = -\mu\varepsilon R, \\ u_0|_{z=0} = 0, \quad \partial_z u_0|_{z=-1} = 0, \end{cases}$$

where $R_0 = \frac{1}{\mu\varepsilon}h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu\phi_0 = A(\nabla, \partial_z)[\phi_0]$. Applying the classical Lemma 1.6.3 on u_0 , we get

$$\|\Lambda^s\nabla^\mu u_0\|_2 \leq \mu\varepsilon M(s + 1)\|\Lambda^s R_0\|_2.$$

It only remains to estimate R_0 . To do so, we use the following estimations on F_0 (where $a \lesssim b$ means there exists a constant $C > 0$ independent of μ such that $a \leq Cb$)

$$\|\Lambda^s(F_0 - 1)\psi\|_2 \lesssim \mu|\nabla\psi|_{H^{s+1}}, \quad \|\Lambda^s\partial_z F_0\psi\|_2 \lesssim \mu|\nabla\psi|_{H^{s+1}}, \quad \|\Lambda^s\partial_z^2 F_0\psi\|_2 \lesssim \mu|\nabla\psi|_{H^{s+1}},$$

stemming from the existence of $C > 0$ such that for any $z \in [-1, 0]$, $\xi \in \mathbb{R}^d$

$$|F_0(z, \xi) - 1| + |\partial_z F_0(z, \xi)| + |\partial_z^2 F_0(z, \xi)| \leq C\mu|\xi|^2. \quad (1.18)$$

We get

$$\|\Lambda^s \nabla^\mu u_0\|_2 \leq \mu \varepsilon M(s+1) \|\Lambda^s R_0\|_2 \leq \mu \varepsilon M(s+2) \|\nabla \psi\|_{H^{s+2}}.$$

This is the first estimate of Proposition 1.2.1.

We define now ϕ_1 , the solution of the following problem :

$$\begin{cases} \partial_z^2 \phi_1 = -A(\nabla, \partial_z)[\phi_0], \\ \phi_1|_{z=0} = 0 \quad , \quad \partial_z \phi_1|_{z=-1} = 0, \end{cases} \quad (1.19)$$

so that

$$h \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu (\phi_0 + \mu \varepsilon \phi_1) = -\mu^2 \varepsilon \Delta \phi_1 - \mu^2 \varepsilon A(\nabla, \partial_z)[\phi_1] = O(\mu \varepsilon).$$

To solve (1.19), we integrate two times with respect to the transversal variable z . Because of the complexity of the expression of ϕ_1 , we simplify the result obtained using the Fourier multipliers approximations of Proposition 1.6.4. After computations we get a simple approximation of ϕ_1 :

$$\phi_1 = -\zeta(1+h) \left(\frac{z^2}{2} + z \right) \Delta \psi + O(\mu).$$

It justifies the introduction of the quantity ϕ_{app} defined by the formula

$$\phi_{\text{app}} = F_0 \psi - \mu \varepsilon \zeta(1+h) \left(\frac{z^2}{2} + z \right) \Delta \psi. \quad (1.20)$$

The first estimation of Proposition 1.6.4 motivates the introduction of another quantity, denoted $\tilde{\phi}_{\text{app}}$, defined by the formula

$$\tilde{\phi}_{\text{app}} = \psi + h^2 (F_0 - 1) \psi. \quad (1.21)$$

We prove that ϕ_{app} and $\tilde{\phi}_{\text{app}}$ are shallow water approximations of the velocity potential in the flat strip, ϕ .

We begin by computing $h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \tilde{\phi}_{\text{app}}$. We have

$$\begin{aligned} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \tilde{\phi}_{\text{app}} &= \mu(h^2 - 1)(F_0 - 1)\Delta\psi + \mu\Delta((h^2 - 1)(F_0 - 1)\psi) \\ &\quad + \mu\varepsilon A(\nabla, \partial_z)(F_0 - 1)\psi + \mu\varepsilon A(\nabla, \partial_z)(h^2 - 1)(F_0 - 1)\psi, \end{aligned}$$

(see (1.16) for the definition of the operator A). We estimate it thanks to the product estimates of Proposition 1.6.1 and (1.18). We get

$$\|\Lambda^s h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \tilde{\phi}_{\text{app}}\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla\psi|_{H^{s+3}}. \quad (1.22)$$

Let us define now the function $\tilde{u} := \phi - \tilde{\phi}_{\text{app}}$, where ϕ is solution of (1.14). It solves the following elliptic problem

$$\begin{cases} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \tilde{u} = -\mu^2 \varepsilon R, \\ \tilde{u}|_{z=0} = 0, \quad \partial_z \tilde{u}|_{z=-1} = 0, \end{cases}$$

where $R = \frac{1}{\mu^2 \varepsilon} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu \tilde{\phi}_{\text{app}}$. So (1.22) gives a control on the remainder and we get from Lemma 1.6.3 one of the wanted estimations :

$$\|\Lambda^s \nabla^\mu (\phi - \tilde{\phi}_{\text{app}})\|_2 = \|\Lambda^s \nabla^\mu \tilde{u}\|_2 \leq \mu^2 \varepsilon M(s+1) \|\Lambda^s R\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla\psi|_{H^{s+3}}. \quad (1.23)$$

Proceeding similarly as for (1.23) we get the estimates on ϕ_{app} defined by (1.20)

$$\|\Lambda^s \nabla^\mu (\phi - \phi_{\text{app}})\|_2 \leq \mu^2 \varepsilon M(s+2) |\nabla\psi|_{H^{s+3}}, \quad (1.24)$$

which is the last estimate we needed to prove. \square

1.2.2 Shallow water approximations of the vertically averaged horizontal component of the velocity

In this subsection we prove Proposition 1.1.9, which we recall here for the sake of clarity.

Proposition 1.2.2. *Let $s \geq 0$, and $\zeta \in H^{s+5}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+6}(\mathbb{R}^d)$, and \bar{V} be as in (1.5). Let also F_1, F_2 and F_3 be the three Fourier multipliers*

defined respectively by

$$F_1 := \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}, \quad F_2 := \frac{3}{\mu|D|^2} \left(1 - \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\right) \quad \text{and} \quad F_3 := F_2 F_1^{-1}, \quad (1.25)$$

where F_1^{-1} is the inverse Fourier multiplier of F_1 , associated with the function $1/F_1$. The following estimates hold :

$$|\bar{V} - \nabla\psi|_{H^s} \leq \mu M(s+2) |\nabla\psi|_{H^{s+2}}, \quad (1.26)$$

$$|\bar{V} - F_1 \nabla\psi|_{H^s} \leq \mu \varepsilon M(s+3) |\nabla\psi|_{H^{s+3}}, \quad (1.27)$$

$$|\bar{V} - F_1 \nabla\psi - \frac{\mu \varepsilon}{3} [h \nabla \zeta \Delta \psi + \nabla(\zeta(1+h)\Delta \psi)]|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla\psi|_{H^{s+4}}, \quad (1.28)$$

$$|\bar{V} - \nabla\psi - \frac{\mu}{3h} \nabla(h^3 F_2 \Delta \psi)|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla\psi|_{H^{s+4}}, \quad (1.29)$$

$$|\bar{V} - \nabla\psi - \frac{\mu}{6h} \nabla(h^3 F_2 [\Delta \psi] + F_2 [h^3 \Delta \psi])|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla\psi|_{H^{s+4}}, \quad (1.30)$$

$$|\bar{V} - \nabla\psi - \frac{\mu}{3h} \nabla(h^3 F_3 \nabla \cdot \bar{V})|_{H^s} \leq \mu^2 \varepsilon M(s+4) |\nabla\psi|_{H^{s+4}}, \quad (1.31)$$

$$|\bar{V} - \nabla\psi - \frac{\mu}{6h} \nabla(h^3 F_3 [\nabla \cdot \bar{V}] + F_3 [h^3 \nabla \cdot \bar{V}])|_{H^s} \leq \mu^2 \varepsilon M(s+5) |\nabla\psi|_{H^{s+5}}. \quad (1.32)$$

Démonstration. The estimate (1.26) comes from Proposition 3.37 in [Lan13b]. We decompose the proof of the other estimates into three different parts.

Part 1 : Here we prove the estimate (1.27). From Proposition 1.2.1, we know that

$$\|\Lambda^s \nabla^\mu (\phi - \phi_0)\|_2 \leq \mu \varepsilon M(s+2) |\nabla\psi|_{H^{s+2}},$$

where $\phi_0 = F_0 \psi := \frac{\cosh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)} \psi$.

Replacing ϕ by ϕ_0 in the definition of \bar{V} (see (1.5)), we define the quantity \bar{V}_0 , i.e.

$$\bar{V}_0 := \frac{1}{h} \int_{-1}^0 [h \nabla \phi_0 - \varepsilon(z+1) \nabla \zeta \partial_z \phi_0] dz$$

But for any function u sufficiently regular and vanishing at $z = 0$ we have, thanks to Jensen inequality and Poincaré inequality (page 40 of [Lan13b]) :

$$\begin{aligned} \left| \int_{-1}^0 u dz \right|_{H^s}^2 &= \int_{\mathbb{R}^d} \left| \Lambda^s \int_{-1}^0 u dz \right|^2 dX \leq \int_{\mathbb{R}^d} \left(\int_{-1}^0 |\Lambda^s u| dz \right)^2 dX \leq \int_{-1}^0 \int_{\mathbb{R}^d} |\Lambda^s u|^2 dX dz \\ &\leq \|\Lambda^s \partial_z u\|_2^2 \leq \|\Lambda^s \nabla^\mu u\|_2^2. \end{aligned} \quad (1.33)$$

Applying the latter inequality to $\bar{V} - \bar{V}_0$ we get

$$|\bar{V} - \bar{V}_0|_{H^s} \leq M(s+1) \|\Lambda^{s+1} \nabla^\mu(\phi - \phi_0)\|_2 \leq \mu \varepsilon M(s+3) |\nabla \psi|_{H^{s+3}}. \quad (1.34)$$

Then remarking the following equality

$$F_1 \nabla \psi = \frac{1}{h} \int_{-1}^0 h \nabla \phi_0 \, dz = \bar{V}_0 + \frac{1}{h} \int_{-1}^0 \varepsilon(z+1) \nabla \zeta \partial_z \phi_0 \, dz,$$

we get from (1.18)

$$|\bar{V} - F_1 \nabla \psi|_{H^s} \leq |\bar{V} - \bar{V}_0|_{H^s} + \left| \frac{1}{h} \int_{-1}^0 \varepsilon(z+1) \nabla \zeta \partial_z \phi_0 \, dz \right|_{H^s} \leq \mu \varepsilon M(s+3) |\nabla \psi|_{H^{s+3}}. \quad (1.35)$$

This is the second estimate.

Part 2 : Here we prove the estimate (1.28). Replacing ϕ by ϕ_{app} in (1.5) (see (1.20) for the definition of ϕ_{app}), we define the quantity

$$\bar{V}_{\text{app},0} := \frac{1}{h} \int_{-1}^0 [h \nabla \phi_{\text{app}} - (z \nabla h + \varepsilon \nabla \zeta) \partial_z \phi_{\text{app}}] \, dz. \quad (1.36)$$

Using again (1.33) and Proposition 1.2.1 we get

$$|\bar{V} - \bar{V}_{\text{app},0}|_{H^s} \leq M(s+1) \|\Lambda^{s+1} \nabla^\mu(\phi - \phi_{\text{app}})\|_2 \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}}. \quad (1.37)$$

Moreover, by computations we have

$$\bar{V}_{\text{app},0} = F_1 \nabla \psi + \varepsilon \frac{\nabla \zeta}{h} ((F_1 - 1)\psi) + \frac{\mu \varepsilon}{3} \left[\frac{\nabla \zeta}{h} \varepsilon \zeta (1+h) \Delta \psi + \nabla(\zeta(1+h) \Delta \psi) \right]. \quad (1.38)$$

We simplify the expression of $\bar{V}_{\text{app},0}$ using the third estimate of Proposition 1.6.4 and define

$$\bar{V}_{\text{app}} := F_1 \nabla \psi + \frac{\mu \varepsilon}{3} [h \nabla \zeta \Delta \psi + \nabla(\zeta(1+h) \Delta \psi)]. \quad (1.39)$$

By the triangular inequality we get

$$|\bar{V} - \bar{V}_{\text{app}}|_{H^s} \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}}.$$

Part 3 : Here we prove the estimate (1.29). Replacing ϕ by $\tilde{\phi}_{\text{app}}$ in (1.5) (see (1.21) for the definition of $\tilde{\phi}_{\text{app}}$), we define the quantity

$$\tilde{\bar{V}}_{\text{app}} := \frac{1}{h} \int_{-1}^0 [h \nabla \tilde{\phi}_{\text{app}} - (z \nabla h + \varepsilon \nabla \zeta) \partial_z \tilde{\phi}_{\text{app}}] dz. \quad (1.40)$$

Using (1.33) and Proposition 1.2.1 we get

$$|\bar{V} - \tilde{\bar{V}}_{\text{app}}|_{H^s} \leq M(s+1) \|\Lambda^{s+1} \nabla^\mu (\phi - \tilde{\phi}_{\text{app}})\|_2 \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}}. \quad (1.41)$$

Moreover, by computations we have

$$\tilde{\bar{V}}_{\text{app}} = \nabla \psi + \frac{\mu}{3h} \nabla (h^3 F_2 \Delta \psi). \quad (1.42)$$

Part 4 : Here we prove the estimate (1.31). Using the error estimates on $\tilde{\bar{V}}_{\text{app}}$ (1.41) and the upper bound $F_2(\sqrt{\mu}|\xi|) \leq \frac{1}{1+\frac{\mu|\xi|^2}{3}}$ we get

$$\begin{aligned} & |\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla (h^3 F_2 \nabla \cdot F_1^{-1} \bar{V})|_{H^s} \\ & \leq |\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla (h^3 F_2 \nabla \cdot \nabla \psi)|_{H^s} + |\frac{\mu}{3h} \nabla (h^3 F_2 \nabla \cdot (F_1^{-1} \bar{V} - \nabla \psi))|_{H^s} \\ & \leq \mu^2 \varepsilon M(s+3) |\nabla \psi|_{H^{s+4}} + \mu M_0 |F_1^{-1} \bar{V} - \nabla \psi|_{H^s}. \end{aligned}$$

But $\frac{1}{F_1(\sqrt{\mu}|\xi|)} \leq 1 + \sqrt{\mu}|\xi|$. So

$$|F_1^{-1} \bar{V} - \nabla \psi|_{H^s} \leq |\bar{V} - F_1 \nabla \psi|_{H^{s+1}}.$$

Using (1.35) we have

$$|\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla (h^3 F_2 \nabla \cdot F_1^{-1} \bar{V})|_{H^s} \leq \mu^2 \varepsilon M(s+4) |\nabla \psi|_{H^{s+4}}. \quad (1.43)$$

Part 5 : Here we prove the estimate (1.30). We have

$$\begin{aligned} & |\bar{V} - \nabla\psi - \frac{\mu}{6h}\nabla(h^3F_2[\Delta\psi] + F_2[h^3\Delta\psi])|_{H^s} \\ \leq & |\bar{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3F_2\Delta\psi)|_{H^s} + |\frac{\mu}{3h}\nabla(h^3F_2[\Delta\psi]) - \frac{\mu}{6h}\nabla(h^3F_2[\Delta\psi] + F_2[h^3\Delta\psi])|_{H^s} \end{aligned}$$

The first term of the right hand size is estimated using (1.41). For the second term, we remark that

$$\nabla(h^3F_2[\Delta\psi]) - \frac{1}{2}\nabla(h^3F_2[\Delta\psi] + F_2[h^3\Delta\psi]) = -\frac{1}{2}\nabla([F_2 - 1, h^3 - 1]\Delta\psi),$$

where $[\cdot, \cdot]$ stands for the commutator. Using the commutator estimates of Proposition 1.6.6 with $F_2 - 1$ of order 2 and $\mathcal{N}^2(F_2 - 1) \lesssim \mu$ (see Definition 1.6.5 for the definition of $\mathcal{N}^2(F_2 - 1)$) we get

$$\begin{aligned} |\nabla([F_2 - 1, h^3 - 1]\Delta\psi)|_{H^s} & \leq |[(F_2 - 1)\Lambda^{s+1}, h^3 - 1]\Delta\psi|_2 + |[\Lambda^{s+1}, h^3 - 1](F_2 - 1)\Delta\psi|_2 \\ & \leq \mu\varepsilon \left| \frac{h^3 - 1}{\varepsilon} \right|_{H^{s+3}} |\nabla\psi|_{H^{s+3}} \\ & \leq \mu\varepsilon M(s+3) |\nabla\psi|_{H^{s+3}}. \end{aligned}$$

At the end we get

$$|\bar{V} - \nabla\psi - \frac{\mu}{6h}\nabla(h^3F_2[\Delta\psi] + F_2[h^3\Delta\psi])|_{H^s} \leq \mu^2\varepsilon M(s+3) |\nabla\psi|_{H^{s+4}}.$$

Part 6 : Here we prove the estimate (1.32). The reasoning is the same as the proof of estimate (1.30). We have

$$\begin{aligned} & |\bar{V} - \nabla\psi - \frac{\mu}{6h}\nabla(h^3F_3[\nabla \cdot \bar{V}] + F_3[h^3\nabla \cdot \bar{V}])|_{H^s} \\ \leq & |\bar{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3F_3\nabla \cdot \bar{V})|_{H^s} + |\frac{\mu}{6h}\nabla([F_3 - 1, h^3 - 1]\Delta\psi)|_{H^s}, \end{aligned}$$

Using (1.43) and the commutator estimates of Proposition 1.6.6 with $F_3 - 1$ of order 2 and $\mathcal{N}^2(F_3 - 1) \lesssim \mu$ we get

$$|\bar{V} - \nabla\psi - \frac{\mu}{6h}\nabla(h^3F_3[\nabla \cdot \bar{V}] + F_3[h^3\nabla \cdot \bar{V}])|_{H^s} \leq \mu^2\varepsilon M(s+4) |\bar{V}|_{H^{s+3}}.$$

It only remains to use (1.26) which implies

$$|\bar{V}|_{H^{s+3}} \leq M(s+5)|\nabla\psi|_{H^{s+5}}.$$

This concludes the proof. \square

1.3 Derivation and consistency of a first full dispersion Green-Naghdi system

1.3.1 Formal Derivation

Let H be the Hamiltonian of the Zakharov/Craig-Sulem's formulation of the water waves problem (1.1) (see [Zak68]) :

$$H(\zeta, \psi) = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2\mu} \int_{\mathbb{R}^d} \psi \mathcal{G}^\mu \psi dX, \quad (1.44)$$

where ζ is the surface elevation, ψ is the velocity potential at the surface and \mathcal{G}^μ is the Dirichlet-to-Neumann operator. Let us recall that the expression of the Hamilton equations derived from an Hamiltonian, here $H(\zeta, \psi)$, is

$$\begin{cases} \partial_t \zeta = \delta_\psi H, \\ \partial_t \psi = -\delta_\zeta H, \end{cases}$$

where δ_ζ and δ_ψ are functional derivatives.

Using the definition of \mathcal{G}^μ in term of \bar{V} (1.6) and a formal integration by parts, we get

$$H(\zeta, \psi) = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2} \int_{\mathbb{R}^d} h \bar{V}[\varepsilon \zeta] \psi \cdot \nabla \psi dX. \quad (1.45)$$

Replacing \bar{V} by one of its approximation $\tilde{\bar{V}}_{\text{app}}$ (1.42), we obtain an approximation of the Hamiltonian at order $O(\mu^2 \varepsilon)$ (see (1.41)), denoted H_{app}

$$H_{\text{app}} = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2} \int_{\mathbb{R}^d} h \nabla \psi \cdot \nabla \psi dX + \frac{\mu}{6} \int_{\mathbb{R}^d} \nabla (h^3 F_2[\nabla \cdot \nabla \psi]) \cdot \nabla \psi dX. \quad (1.46)$$

Now let us differentiate this approximated Hamiltonian in the sense of functional derivatives with respect to ψ and ζ , we get

$$\begin{cases} \delta_\psi H_{\text{app}} = -\nabla \cdot (h\nabla\psi) - \frac{\mu}{6}\Delta(h^3F_2[\Delta\psi] + F_2[h^3\Delta\psi]), \\ \delta_\zeta H_{\text{app}} = \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\mu\varepsilon}{2}h^2F_2[\Delta\psi]\Delta\psi. \end{cases} \quad (1.47)$$

We can now write down the Hamilton equations on the approximated Hamiltonian H_{app} (see (1.46))

$$\begin{cases} \partial_t\zeta = -\nabla \cdot (h\nabla\psi) - \frac{\mu}{6}\Delta(F_2[h^3\Delta\psi] + h^3F_2[\Delta\psi]), \\ \partial_t\psi = -\zeta - \frac{\varepsilon}{2}|\nabla\psi|^2 + \frac{\mu\varepsilon}{2}h^2F_2[\Delta\psi]\Delta\psi. \end{cases} \quad (1.48)$$

Remark 1.3.1. — *This system is the full dispersion equivalent of a Green-Naghdi system with variables (ζ, ψ) (set $F_2 = \text{Id}$ to get the latter), see [CG94] and [Whi67]. The latter is never studied because of its ill-posedness at the linear level. But for (1.48) the ill-posedness is not clear. Indeed, by construction when linearizing this system around the rest state, we obtain the same system as the linearized water waves equations, that is*

$$\begin{cases} \partial_t\zeta + F_1\Delta\psi = 0, \\ \partial_t\psi + \zeta = 0, \end{cases}$$

which is well-posed in Sobolev spaces.

— *The system (1.48) is Hamiltonian by construction. Hence smooth solutions preserve energy H_{app} in addition to mass $\int_{\mathbb{R}^d}\zeta$ and momentum $\int_{\mathbb{R}^d}\zeta\nabla\psi$.*

1.3.2 Consistency with respect to the water waves system

We now prove Proposition 1.1.11, on the consistency at order $O(\mu^2\varepsilon)$ with respect to the water waves system (see Definition 1.1.10) of the first full dispersion system of Green-Naghdi type derived in the previous subsection (1.48). We recall the proposition here.

Proposition 1.3.2. *Let F_1 and F_2 be as in (1.2.2). The water waves equations (1.1) are consistent at order $O(\mu^2\varepsilon)$ in the shallow water regime \mathcal{A} (see Definition 1.1.10) with the*

following full dispersion Hamiltonian Green-Naghdi system

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \nabla \psi) + \frac{\mu}{6} \Delta (\mathbb{F}_2[h^3 \Delta \psi] + h^3 \mathbb{F}_2[\Delta \psi]) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\mu \varepsilon}{2} h^2 \mathbb{F}_2[\Delta \psi] \Delta \psi = 0, \end{cases} \quad (1.49)$$

with $n = 5$.

Démonstration. Let ζ and ψ be the solutions of the water waves system (1.1). Using the notations of Definition 1.1.10 we have in our case

$$\begin{cases} \mathcal{N}_{(A)}^1(\zeta, \psi) := \nabla \cdot (h \nabla \psi) + \frac{\mu}{6} \Delta (\mathbb{F}_2[h^3 \Delta \psi] + h^3 \mathbb{F}_2[\Delta \psi]), \\ \mathcal{N}_{(A)}^2(\zeta, \psi) := \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\mu \varepsilon}{2} h^2 \mathbb{F}_2[\Delta \psi] \Delta \psi. \end{cases}$$

We will prove that

$$\begin{cases} |\partial_t \zeta + \nabla \cdot (h \nabla \psi) + \frac{\mu}{6} \Delta (\mathbb{F}_2[h^3 \Delta \psi] + h^3 \mathbb{F}_2[\Delta \psi])|_{H^s} \leq \mu^2 \varepsilon N(s+5), \\ |\partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\mu \varepsilon}{2} h^2 \mathbb{F}_2[\Delta \psi] \Delta \psi|_{H^s} \leq \mu^2 \varepsilon N(s+4). \end{cases} \quad (1.50)$$

Step 1 : We prove the first estimate of (1.50).

Using the definition of the Dirichlet-to-Neumann operator \mathcal{G}^μ in term of the vertically averaged horizontal velocity \bar{V} (1.6), we know that the water waves solutions (ζ, ψ) satisfy

$$\partial_t \zeta + \nabla \cdot (h \bar{V}) = 0.$$

It only remains to use the fifth estimate of Proposition 1.2.2 to get the result.

Step 2 : We now prove the second estimate of (1.50).

We know that the solutions of the water waves system (ζ, ψ) satisfy

$$\partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\mu \varepsilon (\frac{1}{\mu} \mathcal{G}^\mu \psi + \varepsilon \nabla \zeta \cdot \nabla \psi)^2}{1 + \varepsilon^2 \mu |\nabla \zeta|^2} = 0.$$

Using the quotient estimates of Proposition 1.6.2, the product estimates of Proposition

1.6.1 and Proposition 1.6.7 we get

$$\begin{aligned}
& \left| \frac{(\frac{1}{\mu}\mathcal{G}^\mu\psi + \varepsilon\nabla\zeta \cdot \nabla\psi)^2}{1 + \varepsilon^2\mu|\nabla\zeta|^2} - (\frac{1}{\mu}\mathcal{G}^\mu\psi + \varepsilon\nabla\zeta \cdot \nabla\psi)^2 \right|_{H^s} \\
&= \left| \frac{\mu\varepsilon^2|\nabla\zeta|^2(\frac{1}{\mu}\mathcal{G}^\mu\psi + \varepsilon\nabla\zeta \cdot \nabla\psi)^2}{1 + \varepsilon^2\mu|\nabla\zeta|^2} \right|_{H^s} \\
&\leq \mu\varepsilon^2 M(s+1) \left| (\frac{1}{\mu}\mathcal{G}^\mu\psi + \varepsilon\nabla\zeta \cdot \nabla\psi)^2 \right|_{H^s} \\
&\leq \mu\varepsilon^2 M(s+1) \left| \frac{1}{\mu}\mathcal{G}^\mu\psi + \varepsilon\nabla\zeta \cdot \nabla\psi \right|_{H^{s+1}}^2 \\
&\leq \mu\varepsilon^2 M(s+1) \left(\left| \frac{1}{\mu}\mathcal{G}^\mu\psi \right|_{H^{s+1}} + \varepsilon|\zeta|_{H^{s+2}}|\nabla\psi|_{H^{s+1}} \right)^2 \\
&\leq \mu\varepsilon^2 C(M(s+3), |\nabla\psi|_{H^{s+2}}).
\end{aligned}$$

So up to $O(\mu^2\varepsilon)$ terms, we can replace the second equation of (1.1) by a simpler one, i.e. there exists $R_1 \in H^s(\mathbb{R}^d)$ such that $|R_1|_{H^s} \leq C(M(s+3), |\nabla\psi|_{H^{s+2}})$ and

$$\partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\mu\varepsilon}{2}(-\nabla \cdot (h\bar{V}) + \varepsilon\nabla\zeta \cdot \nabla\psi)^2 = \mu^2\varepsilon R_1. \quad (1.51)$$

Using the first estimate of Proposition 1.2.2 (which also implies $|\bar{V}|_{H^s} \leq M(s+2)|\nabla\psi|_{H^{s+2}}$) and the product estimates of Proposition 1.6.1 we have

$$\begin{aligned}
& |(-\nabla \cdot (h\bar{V}) + \varepsilon\nabla\zeta \cdot \nabla\psi)^2 - (-\nabla \cdot (h\nabla\psi) + \varepsilon\nabla\zeta \cdot \nabla\psi)^2|_{H^s} \\
&\leq |\nabla \cdot (h(\bar{V} - \nabla\psi))|_{H^{s+1}} |-\nabla \cdot (h\bar{V}) + 2\varepsilon\nabla\zeta \cdot \nabla\psi - \nabla \cdot (h\nabla\psi)|_{H^{s+1}} \\
&\leq \mu C(M(s+4), |\nabla\psi|_{H^{s+4}}).
\end{aligned}$$

Hence up to $O(\mu^2\varepsilon)$ terms, we can replace (1.51) by a simpler one, i.e. there exists $R_2 \in H^s(\mathbb{R}^d)$ such that $|R_2|_{H^s} \leq C(M(s+4), |\nabla\psi|_{H^{s+4}})$ and

$$\partial_t\psi + \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\mu\varepsilon}{2}h^2(\Delta\psi)^2 = \mu^2\varepsilon R_2. \quad (1.52)$$

Now it only remains to use the estimates on F_2 of Proposition 1.6.4 to get

$$|h^2F_2[\Delta\psi]\Delta\psi - h^2(\Delta\psi)^2|_{H^s} \leq \mu C(M(s+2), |\nabla\psi|_{H^{s+3}}).$$

So there exists $R_3 \in H^s(\mathbb{R}^d)$ such that $|R_3| \leq C(M(s+4), |\nabla\psi|_{H^{s+4}})$ and

$$\partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla\psi|^2 - \frac{\mu\varepsilon}{2} h^2 F_2[\Delta\psi] \Delta\psi = \mu^2 \varepsilon R_3.$$

Thus we proved the consistency of the water waves equations (1.1) at order $O(\mu^2\varepsilon)$ in the shallow water regime with the system (1.48) with $n = 5$. \square

1.4 Derivation and consistency of a second full dispersion Green-Naghdi system

1.4.1 Formal Derivation

In this subsection we explain formally how to derive the second full dispersion Green-Naghdi system (1.10) from the Hamiltonian of the water waves equations.

From the last estimate of Proposition 1.2.2, we know that

$$h\nabla\psi = h\bar{V} - \frac{\mu}{6} \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}]) + O(\mu^2\varepsilon). \quad (1.53)$$

We define the two operators $T[h]V := -\frac{1}{6h} \nabla(h^3 F_3[\nabla \cdot V] + F_3[h^3 \nabla \cdot V])$ and $I[h]V := h(V + \mu T[h]V)$. So that the previous approximation of $\nabla\psi$ can be written :

$$h\nabla\psi = I[h]\bar{V} + O(\mu^2\varepsilon). \quad (1.54)$$

We suppose $I[h]$ invertible and do formally all the computations with this assumption.

Let H be the Hamiltonian of the Zakharov/Craig-Sulem formulation (1.45) :

$$H = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2} \int_{\mathbb{R}^d} h\bar{V} \cdot \nabla\psi dX,$$

Using (1.54) an approximated Hamiltonian would be :

$$H_{\text{app}} = \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 dX + \frac{1}{2} \int_{\mathbb{R}^d} hI[h]^{-1}[h\nabla\psi] \cdot \nabla\psi dX. \quad (1.55)$$

We differentiate this Hamiltonian in the sense of functional derivative.

Through some computations we first get the derivative of H_{app} with respect to ψ

$$\delta_\psi H_{\text{app}} = -\nabla \cdot (hI[h]^{-1}[h\nabla\psi]).$$

To compute the derivative of H_{app} with respect to ζ we use the formula

$$I[h]\partial_\zeta(I[h]^{-1})I[h]V = -\partial_\zeta I[h]V. \quad (1.56)$$

From (1.56) and the fact that $I[h]^{-1}$ is a symmetric operator, we can compute the derivative in ζ of H_{app} . We get

$$\begin{aligned} \delta_\zeta H_{\text{app}} = & \zeta + \frac{\varepsilon}{2}I[h]^{-1}[h\nabla\psi] \cdot \nabla\psi - \frac{\varepsilon}{2}I[h]^{-1}[h\nabla\psi] \cdot I[h]^{-1}[h\nabla\psi] \\ & + \frac{\mu\varepsilon}{4}\nabla \cdot (I[h]^{-1}[h\nabla\psi])h^2F_3[\nabla \cdot (I[h]^{-1}[h\nabla\psi])] + \frac{\mu\varepsilon}{4}\nabla \cdot (F_3[I[h]^{-1}[h\nabla\psi]])h^2\nabla \cdot (I[h]^{-1}[h\nabla\psi]) \\ & + \frac{\varepsilon}{2}I[h]^{-1}[h\nabla\psi] \cdot \nabla\psi. \end{aligned}$$

Then we define $\bar{V}_{\text{app}} := (h(Id + \mu T[h]))^{-1}[h\nabla\psi] = I[h]^{-1}[h\nabla\psi]$. This new quantity approximates \bar{V} at order $O(\mu^2\varepsilon)$, i.e. $\bar{V}_{\text{app}} = \bar{V} + O(\mu^2\varepsilon)$. The two functional derivatives $\delta_\psi H_{\text{app}}$ and $\delta_\zeta H_{\text{app}}$ allow us to obtain the equations of movement in ζ and \bar{V}_{app} .

First the conservation of mass :

$$\begin{aligned} \partial_t \zeta = \delta_\psi H_{\text{app}} & \iff \partial_t \zeta = -\nabla \cdot (hI[h]^{-1}[h\nabla\psi]) \\ & = -\nabla \cdot (h\bar{V}_{\text{app}}). \end{aligned}$$

And next the conservation of momentum :

$$\begin{aligned} \partial_t \psi & = -\delta_\zeta H_{\text{app}} \\ \iff \partial_t \psi & = -\zeta - \frac{\varepsilon}{2}\nabla\psi \cdot I[h]^{-1}[h\nabla\psi] + \frac{\varepsilon}{2}I[h]^{-1}[h\nabla\psi] \cdot I[h]^{-1}[h\nabla\psi] \\ & + \frac{\mu\varepsilon}{4}\nabla \cdot (I[h]^{-1}[h\nabla\psi])h^2F_3[\nabla \cdot (I[h]^{-1}[h\nabla\psi])] \\ & + \frac{\mu\varepsilon}{4}\nabla \cdot (F_3[I[h]^{-1}[h\nabla\psi]])h^2\nabla \cdot (I[h]^{-1}[h\nabla\psi]) - \frac{\varepsilon}{2}I[h]^{-1}[h\nabla\psi] \cdot \nabla\psi. \end{aligned}$$

Then applying ∇ , we obtain

$$\begin{aligned} \partial_t((Id + \mu T[h])\bar{V}_{\text{app}}) &= -\nabla\zeta - \frac{\varepsilon}{2}\nabla(|\bar{V}_{\text{app}}|^2) + \frac{\mu\varepsilon}{6}\nabla\left(\frac{1}{h}\bar{V}_{\text{app}} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}_{\text{app}}] + F_3[h^3 \nabla \cdot \bar{V}_{\text{app}}])\right) \\ &\quad + \frac{\mu\varepsilon}{2}\nabla(h^2 F_3[\nabla \cdot \bar{V}_{\text{app}}]\nabla \cdot \bar{V}_{\text{app}}). \end{aligned}$$

Thus we get the second full dispersion Green-Naghdi model (1.10) that we recall here

$$\begin{cases} \partial_t\zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((Id + \mu T[h])\bar{V}) + \nabla\zeta + \frac{\varepsilon}{2}\nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6}\nabla\left(\frac{1}{h}\bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \\ \quad - \frac{\mu\varepsilon}{2}\nabla(h^2 F_3[\nabla \cdot \bar{V}]\nabla \cdot \bar{V}) = 0. \end{cases}$$

One can replace $I[h]\bar{V}$ in the above reasoning by any approximation of it of order $O(\mu\varepsilon^2)$. It would give another full dispersion Green-Naghdi system enjoying an Hamiltonian structure.

1.4.2 Consistency with respect to the water waves equations

Taking the same notations and definitions as in Subsection 3.2, we prove here Proposition 1.1.12. For the sake of clarity we recall it here.

Proposition 1.4.1. *Let F_3 be the Fourier multiplier of (1.25). Let $T[h]V := -\frac{1}{6h}\nabla(h^3 F_3[\nabla \cdot V] + F_3[h^3 \nabla \cdot V])$. The water waves equations (1.1) are consistent at order $O(\mu^2\varepsilon)$ in the shallow water regime \mathcal{A} with*

$$\begin{cases} \partial_t\zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t((Id + \mu T[h])\bar{V}) + \nabla\zeta + \frac{\varepsilon}{2}\nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6}\nabla\left(\frac{1}{h}\bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \\ \quad - \frac{\mu\varepsilon}{2}\nabla(h^2 F_3[\nabla \cdot \bar{V}]\nabla \cdot \bar{V}) = 0, \end{cases} \tag{1.57}$$

with $n = 6$.

Démonstration. Let ζ and ψ (so \bar{V} through (1.5)) be the solutions of the water waves

system (1.1). Using the notations of Definition 1.1.10 we have in our case

$$\begin{aligned} \mathcal{N}_{(A')}^3(\zeta, \bar{V}) := & \mu \partial_t(T[h]\bar{V}) + \nabla \zeta + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6} \nabla\left(\frac{1}{h} \bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \\ & - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3[\nabla \cdot \bar{V}] \nabla \cdot \bar{V}). \end{aligned} \quad (1.58)$$

We need to prove

$$\begin{aligned} |\partial_t \bar{V} + \nabla \cdot (h \nabla \psi) + \mu \partial_t(T[h]\bar{V}) + \nabla \zeta + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6} \nabla\left(\frac{1}{h} \bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \\ - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3[\nabla \cdot \bar{V}] \nabla \cdot \bar{V})|_{H^s} \leq \mu^2 \varepsilon N(s+7) \end{aligned} \quad (1.59)$$

Step 1 : In this first step, we focus on the terms of the second equation of (1.57) which are not differentiated in time and prove that there exists $R_3 \in H^s(\mathbb{R}^d)$ such that $|R_3|_{H^s} \leq N(s+6)$ and

$$\begin{aligned} \partial_t \nabla \psi + \nabla \zeta + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6} \nabla\left(\frac{1}{h} \bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \\ - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3[\nabla \cdot \bar{V}] \nabla \cdot \bar{V}) = \mu^2 \varepsilon R_3 \end{aligned} \quad (1.60)$$

Taking (ζ, ψ) solutions of the water waves system (1.1) we proved in Subsection 3.2 (see (1.52)) that there exists a remainder $R_2 \in H^s(\mathbb{R}^d)$ such that $|R_2|_{H^s} \leq N(s+4)$ and

$$\partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 - \frac{\mu\varepsilon}{2} h^2 (\Delta \psi)^2 = \mu^2 \varepsilon R_2. \quad (1.61)$$

Applying ∇ to (1.61), then using $|(F_3 - 1)\psi|_{H^\alpha} \lesssim \mu |\nabla \psi|_{H^{\alpha+1}}$ for any $\alpha \geq 0$ (see Proposition 1.6.4), and the first estimate of Proposition 1.2.2, we get the existence of $R_3 \in H^s(\mathbb{R}^d)$ such that $|R_3|_{H^s} \leq N(s+5)$ and

$$\partial_t \nabla \psi + \nabla \zeta + \frac{\varepsilon}{2} \nabla(|\nabla \psi|^2) - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3[\nabla \cdot \bar{V}] \nabla \cdot \bar{V}) = \mu^2 \varepsilon R_3.$$

Moreover

$$\begin{aligned}
& \left| \frac{\varepsilon}{2} \nabla(|\nabla\psi|^2) - \left(\frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6} \nabla\left(\frac{1}{h} \bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \right) \right|_{H^s} \\
& \leq \left| \frac{\varepsilon}{2} \nabla(|\nabla\psi|^2) - \frac{\varepsilon}{2} \nabla\left(|\bar{V} - \frac{\mu}{6h} \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])|^2\right) \right|_{H^s} \\
& \quad + \frac{\mu^2\varepsilon}{2} \left| \nabla\left(\frac{1}{6^2 h^2} |\nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])|^2\right) \right|_{H^s} \\
& := I_1 + I_2.
\end{aligned}$$

Remark that the following inequality on F_3 holds :

$$|F_3(\sqrt{\mu}\xi)| = \left| \frac{3}{\mu|\xi|^2} \left(\frac{\sqrt{\mu}|\xi|}{\tanh(\sqrt{\mu}|\xi|)} - 1 \right) \right| \leq \frac{1}{1 + \frac{\sqrt{\mu}|\xi|}{3}}. \quad (1.62)$$

Using product estimates 1.6.1, (1.62) and the first estimate of Proposition 1.2.2, we get

$$\begin{aligned}
\left| \nabla\left(\frac{1}{6^2 h^2} |\nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])|^2\right) \right|_{H^s} & \leq M(s+1) \left| \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}]) \right|_{H^{s+1}}^2 \\
& \leq M(s+1) \left| \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}]) \right|_{H^{s+2}}^2 \\
& \leq M(s+3) \left| F_3[\nabla \cdot \bar{V}] \right|_{H^{s+3}} + \left| F_3[h^3 \nabla \cdot \bar{V}] \right|_{H^{s+3}} \\
& \leq M(s+3) |\bar{V}|_{H^{s+3}} \leq M(s+5) |\nabla\psi|_{H^{s+5}},
\end{aligned}$$

which gives the bound on I_2 .

Now we use the last estimate of Proposition 1.2.2 and product estimates 1.6.1 to estimate the first term I_1 :

$$\begin{aligned}
& \left| \frac{\varepsilon}{2} \nabla(|\nabla\psi|^2) - \frac{\varepsilon}{2} \nabla\left(|\bar{V} - \frac{\mu}{6h} \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])|^2\right) \right|_{H^s} \\
& \leq \frac{\varepsilon}{2} \left| \nabla\psi - \left(\bar{V} - \frac{\mu}{6h} \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}]) \right) \right|_{H^{s+1}} \\
& \quad \times \left| \nabla\psi + \bar{V} - \frac{\mu}{6h} \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}]) \right|_{H^{s+2}} \\
& \leq \mu^2 \varepsilon^2 M(s+6) |\nabla\psi|_{H^{s+6}}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \left| \frac{\varepsilon}{2} \nabla(|\nabla\psi|^2) - \left(\frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{6} \nabla\left(\frac{1}{h} \bar{V} \cdot \nabla(h^3 F_3[\nabla \cdot \bar{V}] + F_3[h^3 \nabla \cdot \bar{V}])\right) \right) \right|_{H^s} \\
& \leq \mu^2 \varepsilon C(M(s+6), |\nabla\psi|_{H^{s+6}}).
\end{aligned}$$

And we proved (1.60).

Step 2 : To prove the consistency of the water waves equations (1.1) with respect to the second full dispersion Green-Naghdi system (1.57) at order $O(\mu^2\varepsilon)$ it only remains to focus on the term differentiated in time and show that

$$|\partial_t(\nabla\psi - (\bar{V} - \frac{\mu}{6h}\nabla(h^3F_3[\nabla \cdot \bar{V}] + F_3[h^3\nabla \cdot \bar{V}])))|_{H^s} \leq \mu^2\varepsilon N(s+7). \quad (1.63)$$

What we will prove is in fact

$$\begin{aligned} & |\partial_t(\nabla\psi - (\bar{V} - \frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot \bar{V}])))|_{H^s} \\ & + |\partial_t(\frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot \bar{V}]) - \frac{\mu}{6h}\nabla(h^3F_3[\nabla \cdot \bar{V}] + F_3[h^3\nabla \cdot \bar{V}])))|_{H^s} \leq \mu^2\varepsilon N(s+7). \end{aligned}$$

In this second step we estimate

$$|\partial_t(\nabla\psi - (\bar{V} - \frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot \bar{V}])))|_{H^s}. \quad (1.64)$$

In that end we denote $\tilde{u} = \phi - \tilde{\phi}_{\text{app}}$, where ϕ is defined in Definition 1.1.4 and $\tilde{\phi}_{\text{app}}$ in (1.21).

Step 2.1 : Here we find a control on \tilde{u} and prove

$$\|\Lambda^s \nabla^\mu \partial_t \tilde{u}\|_2 \leq \mu^2\varepsilon N(s+5). \quad (1.65)$$

By definition of ϕ and $\tilde{\phi}_{\text{app}}$ (see Definition 1.1.4) we know that \tilde{u} solves an elliptic problem :

$$\begin{cases} h\nabla^\mu \cdot P(\Sigma_t)\nabla^\mu(\tilde{u}) = -\mu^2\varepsilon R, \\ \tilde{u}|_{z=0} = 0, \quad \partial_z \tilde{u}|_{z=-1} = 0, \end{cases} \quad (1.66)$$

where

$$\begin{aligned} R = \frac{1}{\mu^2\varepsilon} \Big[& \mu(h^2 - 1)(F_0 - 1)\Delta\psi + \mu\Delta((h^2 - 1)(F_0 - 1)\psi) \\ & + \mu\varepsilon A(\nabla, \partial_z)(F_0 - 1)\psi + \mu\varepsilon A(\nabla, \partial_z)(h^2 - 1)(F_0 - 1)\psi \Big]. \end{aligned} \quad (1.67)$$

Using Proposition 1.6.4 for F_0 , and the product estimates of Proposition 1.6.1, we have

the following estimation of the remainder R :

$$\|\Lambda^s R\|_2 \leq M(s+2) \|\nabla \psi\|_{H^{s+3}}. \quad (1.68)$$

Moreover we can differentiate in time the elliptic equation in (1.66) as follow

$$\begin{aligned} \partial_t(\nabla^\mu \cdot P(\Sigma_t) \nabla^\mu(\tilde{u})) &= -\mu^2 \varepsilon \partial_t R \\ \iff \nabla^\mu \cdot \partial_t(P(\Sigma_t)) \nabla^\mu \tilde{u} + \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu \partial_t \tilde{u} &= -\mu^2 \varepsilon \partial_t R. \end{aligned}$$

where here $R = (1.67)/h$ (we use the same notation for both remainders, thanks to the quotient estimates of Proposition 1.6.2, the previous estimation holds).

It invites us to denote $v := \partial_t \tilde{u}$ and decompose it into $v := v_1 + v_2$ where v_1 satisfy one elliptic problem

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu v_1 = -\mu^2 \varepsilon \partial_t R, \\ v_1|_{z=0} = 0, \quad \partial_z v_1|_{z=-1} = 0, \end{cases} \quad (1.69)$$

and v_2 satisfy, for $g := \partial_t P(\Sigma_t) \nabla^\mu \tilde{u}$ (see Definition 1.1.4 for the expression of $P(\Sigma_t)$), a second elliptic problem

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu v_2 = -\nabla^\mu \cdot g, \\ v_2|_{z=0} = 0, \quad v_2|_{z=-1} = -e_z \cdot g|_{z=-1}. \end{cases} \quad (1.70)$$

Thanks to the Lemma 1.6.3 we have a control on v_1 given by

$$\|\Lambda^s \nabla^\mu v_1\|_2 \leq \mu^2 \varepsilon M(s+1) \|\Lambda^s \partial_t R\|_2. \quad (1.71)$$

And having an explicit form of R we can easily find an estimation of $\partial_t R$:

$$\|\Lambda^s \partial_t R\|_2 \leq C(M(s+2), |\partial_t \zeta|_{H^{s+3}}) \|(\nabla \psi, \partial_t \nabla \psi)\|_{H^{s+3}}. \quad (1.72)$$

Then using the water waves equations (1.1) we obtain estimates on the partial derivatives in time of ζ and $\nabla \psi$.

Lemma 1.4.2. *Let $s \geq 0$, and $\zeta \in H^{s+4}(\mathbb{R}^d)$. Let $\psi \in \dot{H}^{s+4}(\mathbb{R}^d)$.*

The two estimations hold :

$$\begin{cases} |\partial_t \zeta|_{H^{s+2}} \leq N(s+4), \\ |\partial_t \nabla \psi|_{H^{s+2}} \leq N(s+4). \end{cases}$$

Démonstration. For both estimations the tools are the same. We use the water waves equations (1.1) and Proposition 1.6.7.

We first prove the inequality on $\partial_t \zeta$. Denoting $s' = s + 2$ and \mathfrak{P} the Fourier multiplier defined as $\mathfrak{P} := \frac{|D|}{(1+\sqrt{\mu}|D|)^{1/2}}$ (we recall that $|D|$ means $|\xi|$ in Fourier space) we have

$$|\partial_t \zeta|_{H^{s'}} = \left| \frac{1}{\mu} \mathcal{G}^\mu [\varepsilon \zeta] \psi \right|_{H^{s'}} \leq N(s' + 2).$$

On the other hand for $\partial_t \nabla \psi$ we get

$$\begin{aligned} |\partial_t \psi|_{H^{s'}} &\leq |\zeta|_{H^{s'}} + \|\nabla \psi\|_{H^{s'}}^2 + \left| \frac{(\frac{1}{\sqrt{\mu}} \mathcal{G}^\mu \psi + \varepsilon \sqrt{\mu} \nabla \zeta \cdot \nabla \psi)^2}{1 + \varepsilon^2 \mu |\nabla \zeta|^2} \right|_{H^{s'}} \\ &\leq |\zeta|_{H^{s'}} + \|\nabla \psi\|_{H^{s'}}^2 + C(\mu_{\max}, \frac{1}{h_{\min}}, \|\nabla \zeta\|_{H^{s'}}) \left| (\frac{1}{\sqrt{\mu}} \mathcal{G}^\mu \psi + \varepsilon \sqrt{\mu} \nabla \zeta \cdot \nabla \psi)^2 \right|_{H^{s'}} \\ &\leq |\zeta|_{H^{s'}} + \|\nabla \psi\|_{H^{s'}}^2 + C(\mu_{\max}, \frac{1}{h_{\min}}, |\zeta|_{H^{s'+1}}, \left| \frac{1}{\sqrt{\mu}} \mathcal{G}^\mu \psi \right|_{H^{s'}}, |\nabla \zeta \cdot \nabla \psi|_{H^{s'}}) \\ &\leq N(s' + 1). \end{aligned}$$

Thus

$$|\partial_t \nabla \psi|_{H^{s'}} \leq |\partial_t \psi|_{H^{s'+1}} \leq N(s' + 2).$$

□

Using the previous lemma, mixed with (1.71) and (1.72) we get the control on v_1 :

$$\|\Lambda^s \nabla^\mu v_1\|_2 \leq \mu^2 \varepsilon N(s+5).$$

To get the control on v_2 we use a classical result for solutions of elliptic problems such as (1.70). It is the Lemma 1.6.8. We get

$$\|\Lambda^s \nabla^\mu v_2\|_2 \leq M(s+1) \|\Lambda^s g\|_2.$$

But \tilde{u} solves an elliptic problem for which we can use Lemma 1.6.3. Using also Lemma 1.4.2 and (1.68) we get

$$\|\Lambda^s \nabla^\mu v_2\|_2 \leq C(M(s+1), |\partial_t \zeta|_{H^{s+3}}) \mu^2 \varepsilon M(s+1) \|\Lambda^s R\|_2 \leq \mu^2 \varepsilon N(s+5) \quad (1.73)$$

At the end, joining together the control on v_1 (1.71) and the one on v_2 (1.73) we get (1.65).
Step 2.2 : We can now give the control on (1.64). To do that we will first use (1.65) to prove the following lemma.

Lemma 1.4.3. *Let $s \geq 0$, and $\zeta \in H^{s+6}(\mathbb{R}^d)$ be such that (1.2) is satisfied. Let $\psi \in \dot{H}^{s+7}(\mathbb{R}^d)$, and \bar{V} be as in (1.5). Let also F_1 and F_2 be the Fourier multipliers of (1.25). The following estimates hold :*

$$\begin{cases} |\partial_t(\bar{V} - F_1 \nabla \psi)|_{H^s} \leq \mu \varepsilon N(s+4), \\ |\partial_t(\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla(h^3 F_2[\Delta \psi]))|_{H^s} \leq \mu^2 \varepsilon N(s+6). \end{cases} \quad (1.74)$$

Démonstration. We begin by proving the second inequality. Let R be the quantity defined by $\mu^2 \varepsilon R = \bar{V} - \tilde{V}_{\text{app}}$ where

$$\begin{cases} \bar{V} := \frac{1}{h} \int_{-1}^0 [h \nabla \phi - (z \nabla h + \varepsilon \nabla \zeta) \partial_z \phi] dz, \\ \tilde{V}_{\text{app}} := \frac{1}{h} \int_{-1}^0 [h \nabla \tilde{\phi}_{\text{app}} - (z \nabla h + \varepsilon \nabla \zeta) \partial_z \tilde{\phi}_{\text{app}}] dz = \nabla \psi + \frac{\mu}{3h} \nabla(h^3 F_2[\Delta \psi]). \end{cases} \quad (1.75)$$

We have the following equality

$$\begin{aligned} & |\partial_t(\bar{V} - \nabla \psi - \frac{1}{h} \nabla(h^3(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1)\psi))|_{H^s} = \mu^2 \varepsilon |\partial_t R|_{H^s} \\ & = |\int_{-1}^0 (\nabla \partial_t \tilde{u} - \frac{1}{h} (z \nabla h + \varepsilon \nabla \zeta) \partial_z \partial_t \tilde{u}) dz - \int_{-1}^0 \partial_t(\frac{z \nabla h + \varepsilon \nabla \zeta}{h}) \partial_z \tilde{u} dz|_{H^s}. \end{aligned}$$

So that Poincaré's inequality (page 40 of [Lan13b]) gives us

$$\mu^2 \varepsilon |\partial_t R|_{H^s} \leq \|\Lambda^{s+1} \nabla^\mu \partial_t \tilde{u}\|_2 \leq \mu^2 \varepsilon N(s+6),$$

which is equivalent to

$$|\partial_t(\bar{V} - \nabla \psi - \frac{\mu}{3h} \nabla(h^3 F_2[\Delta \psi]))|_{H^s} \leq \mu^2 \varepsilon N(s+6). \quad (1.76)$$

To prove the first inequality of (1.74) we just need to do the same reasoning from Step 2.1 to this point but taking instead $\tilde{u} = \phi - \phi_0$, where ϕ_0 is defined in (1.17). \square

Having in mind this lemma, we decompose (1.64) in two parts :

$$\begin{aligned} & |\partial_t(\bar{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3F_2F_1^{-1}[\nabla \cdot \bar{V}])))|_{H^s} \\ & \leq |\partial_t(\frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot (\bar{V} - F_1\nabla\psi)]))|_{H^s} + |\partial_t(\bar{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3F_2[\Delta\psi]))|_{H^s}. \end{aligned}$$

The bound on the second term is given by the second inequality of Lemma 1.4.3.

The first term can be decomposed in three parts :

$$\begin{aligned} & |\partial_t(\frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot (\bar{V} - F_1\nabla\psi)]))|_{H^s} \\ & \leq |\frac{\mu\varepsilon}{3h^2}\partial_t\zeta\nabla(h^3F_3[\nabla \cdot (\bar{V} - F_1\nabla\psi)])|_{H^s} + |\frac{\mu\varepsilon}{3h}\nabla(3h^2\partial_t\zeta F_3[\nabla \cdot (\bar{V} - F_1\nabla\psi)])|_{H^s} \\ & + |\frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot \partial_t(\bar{V} - F_1\nabla\psi)])|_{H^s}. \end{aligned}$$

Each of these terms are bounded using Lemma 1.4.2, the second estimate of Proposition 1.2.2 and the first inequality of Lemma 1.4.3.

At the end we get what we wanted to prove in this step 2 :

$$|\partial_t(\bar{V} - \nabla\psi - \frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot \bar{V}])))|_{H^s} \leq \mu^2\varepsilon N(s+6).$$

Step 3 : The last step is to bound

$$|\partial_t(\frac{\mu}{3h}\nabla(h^3F_3[\nabla \cdot \bar{V}]) - \frac{\mu}{6h}\nabla(h^3F_3[\nabla \cdot \bar{V}] + F_3[h^3\nabla \cdot \bar{V}])))|_{H^s} \quad (1.77)$$

The main key is the commutator estimates of Proposition 1.6.6. We decompose (1.77) into

three parts :

$$\begin{aligned}
& \left| \partial_t \left(\frac{\mu}{3h} \nabla (h^3 F_3 [\nabla \cdot \bar{V}]) - \frac{\mu}{6h} \nabla (h^3 F_3 [\nabla \cdot \bar{V}] + F_3 [h^3 \nabla \cdot \bar{V}]) \right) \right|_{H^s} \\
& \leq \mu \varepsilon \left| \frac{\partial_t \zeta}{h^2} [\nabla (h^3 - 1) (F_3 - 1) [\nabla \cdot \bar{V}] - (F_3 - 1) [h^3 - 1 \nabla \cdot \bar{V}]] \right|_{H^s} \\
& + \mu \varepsilon \left| \frac{1}{h} \nabla (\partial_t \zeta h^2 (F_3 - 1) [\nabla \cdot \bar{V}] - (F_3 - 1) [\partial_t \zeta h^2 \nabla \cdot \bar{V}]) \right|_{H^s} \\
& + \mu \varepsilon \left| \frac{1}{h} \nabla \left(\frac{h^3 - 1}{\varepsilon} (F_3 - 1) [\nabla \cdot \partial_t \bar{V}] - (F_3 - 1) \left[\frac{h^3 - 1}{\varepsilon} \nabla \cdot \partial_t \bar{V} \right] \right) \right|_{H^s} \\
& := T_1 + T_2 + T_3.
\end{aligned}$$

Using Lemma 1.4.2 we get

$$\begin{cases} T_1 \leq \mu \varepsilon M(s+4) | [F_3 - 1, h^3 - 1] \nabla \cdot \bar{V} |_{H^{s+1}}, \\ T_2 \leq \mu \varepsilon M_0 | [F_3 - 1, \partial_t \zeta h^2] \nabla \cdot \bar{V} |_{H^{s+1}}, \\ T_3 \leq \mu \varepsilon M_0 | [F_3 - 1, \frac{h^3-1}{\varepsilon}] \nabla \cdot \partial_t \bar{V} |_{H^{s+1}}. \end{cases}$$

But using the fact that for any $s \geq 0$ the operator Λ^s and $F_3 - 1$ commute we have

$$\begin{cases} | [F_3 - 1, h^3 - 1] \nabla \cdot \bar{V} |_{H^{s+1}} & \leq | [(F_3 - 1) \Lambda^{s+1}, h^3 - 1] \nabla \cdot \bar{V} |_2 \\ & + | [\Lambda^{s+1}, h^3 - 1] (F_3 - 1) [\nabla \cdot \bar{V}] |_2, \\ | [F_3 - 1, \partial_t \zeta h^2] \nabla \cdot \bar{V} |_{H^{s+1}} & \leq | [(F_3 - 1) \Lambda^{s+1}, \partial_t \zeta h^2] \nabla \cdot \bar{V} |_2 \\ & + | [\Lambda^{s+1}, \partial_t \zeta h^2] (F_3 - 1) [\nabla \cdot \bar{V}] |_2, \\ | [F_3 - 1, \frac{h^3-1}{\varepsilon}] \nabla \cdot \partial_t \bar{V} |_{H^{s+1}} & \leq | [(F_3 - 1) \Lambda^{s+1}, \frac{h^3-1}{\varepsilon}] \nabla \cdot \partial_t \bar{V} |_2 \\ & + | [\Lambda^{s+1}, \frac{h^3-1}{\varepsilon}] (F_3 - 1) [\nabla \cdot \partial_t \bar{V}] |_2. \end{cases}$$

We already proved all the elements we need to bound these terms. For the sake of clarity we list them here. Using the commutator estimates of Proposition 1.6.6 with $F_3 - 1$ of order 2 and $\mathcal{N}^2(F_3 - 1) \lesssim \mu$ (see Definition 1.6.5 for the definition of $\mathcal{N}^2(F_3 - 1)$), Lemma 1.4.2, the first inequality of Proposition 1.2.2 and the first inequality of Lemma 1.4.3 we

obtain

$$\begin{cases} T_1 \leq \mu^2 \varepsilon M(s+5) |\nabla \psi|_{H^{s+5}} \leq N(s+5), \\ T_2 \leq \mu^2 \varepsilon M(s+5) |\nabla \psi|_{H^{s+5}} \leq N(s+5), \\ T_3 \leq \mu^2 \varepsilon N(s+7). \end{cases}$$

It finishes the step 3 and the proof of Proposition 1.4.1, i.e. we proved the consistency of the water waves at order $O(\mu^2 \varepsilon)$ in the shallow water regime with the full dispersion Green-Naghdi system (1.57) (with $n = 6$). \square

Remark 1.4.4. *The $n = 6$ regularity asked for deriving (1.57) appeared only in the last step of the proof when we wanted to pass from a non-symmetric system to a symmetric one. Only $n = 5$ is asked for the solutions of the water waves equations (1.1) to prove the consistency at order $O(\mu^2 \varepsilon)$ with respect to the system*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ \partial_t (\bar{V} - \frac{\mu}{3h} \nabla (h^3 F_3[\nabla \cdot \bar{V}])) + \nabla \zeta + \frac{\varepsilon}{2} \nabla (|\bar{V}|^2) - \frac{\mu \varepsilon}{3} \nabla (\frac{1}{h} \bar{V} \cdot \nabla (h^3 F_3[\nabla \cdot \bar{V}])) \\ - \frac{\mu \varepsilon}{2} \nabla (h^2 F_3[\nabla \cdot \bar{V}] \nabla \cdot \bar{V}) = 0. \end{cases} \quad (1.78)$$

However the system (1.78) does not have a Hamiltonian formulation.

1.5 Consistency of other full dispersion models appearing in the literature

1.5.1 Full dispersion Green-Naghdi system

To derive (1.57) we chose a kind of symmetrization (see (1.53)) which were naturally induced by an analogy with the one appearing in the first full dispersion Green-Naghdi system (1.48) we derived in this paper. Another kind of symmetrization appears in the literature for a full dispersion Green-Naghdi system, see [DIT16]. They introduced a two-layer full dispersion Green-Naghdi system with surface tension in order to study high-frequency Kelvin-Helmholtz instabilities. In dimension $d = 2$, without surface tension,

their system for a one-layer fluid is

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t (\bar{V} - \frac{\mu}{3h} \nabla(\sqrt{F_3} h^3 \sqrt{F_3} [\nabla \cdot \bar{V}])) + \nabla \zeta + \frac{\varepsilon}{2} \nabla(|\bar{V}|^2) - \frac{\mu\varepsilon}{3} \nabla(\frac{1}{h} \bar{V} \cdot \nabla(\sqrt{F_3} h^3 \sqrt{F_3} [\nabla \cdot \bar{V}])) \\ - \frac{\mu\varepsilon}{2} \nabla(h^2 F_3 [\nabla \cdot \bar{V}] \nabla \cdot \bar{V}) = 0. \end{cases} \quad (1.79)$$

Proposition 1.5.1. *Let F_3 be the Fourier multiplier of (1.25). The water waves equations are consistent at order $O(\mu^2\varepsilon)$ in the shallow water regime \mathcal{A} with the system (1.79) with $n = 6$.*

Démonstration. We will only do a formal proof. The rigorous one would use the same tools as the proof of Proposition 1.1.12 (see Subsection 1.4.2).

It is easy to see that

$$2\sqrt{F_3}[h^3\sqrt{F_3}[V]] = h^3F_3[V] + F_3[h^3V] + O(\mu). \quad (1.80)$$

It only remains to use Proposition 1.1.12 together with the estimates on F_3 of Proposition 1.6.4 to get the result. \square

The difference between (1.10) and (1.79) in the mathematical point of view is of importance. Indeed the operator

$$h(\text{Id} - \frac{\mu}{3h} \nabla(\sqrt{F_3} h^3 \sqrt{F_3} [\nabla \cdot \circ]))$$

is invertible because one can decompose it in the following way :

$$h\text{Id} + \mu(\frac{h}{\sqrt{3}} \sqrt{F_3} \nabla \cdot)^* h(\frac{h}{\sqrt{3}} \sqrt{F_3} \nabla \cdot), \quad (1.81)$$

where $*$ is the adjoint in L^2 . It gives the positiveness of the operator and the coercivity of the bilinear form associated with. The Lax-Milgram theorem concludes [DIT16].

However we do not have a similar decomposition as (1.81) for the operator $h(\text{Id} - \frac{\mu}{6h} \nabla(h^3 F_3 [\nabla \cdot \circ] + F_3 [h^3 \nabla \cdot \circ]))$. To ensure the invertibility it seems that we need an additionnal hypothesis on the smallness of $\varepsilon\zeta$.

1.5.2 Full dispersion Boussinesq systems

In the literature several Whitham-Boussinesq systems (or full dispersion Boussinesq systems) are introduced, see [Car18; Din19; Din20; DDK19; DN19; DST19; HP16; NW19; VP16]. We pay a particular attention to the one studied in [DST19] for which the authors proved a local well-posedness result in dimension $d = 2$ and a global well-posedness result for small data in dimension $d = 1$. We recall it for $d = 2$

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon F_1 \nabla \cdot (\zeta F_1 \nabla \psi) = 0, \\ \partial_t \nabla \psi + \nabla \zeta + \frac{\varepsilon}{2} \nabla (F_1 |\nabla \psi|^2) = 0. \end{cases} \quad (1.82)$$

Proposition 1.5.2. *The water waves equations (1.1) are consistent at order $O(\mu\varepsilon)$ in the shallow water regime \mathcal{A} with the following system*

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon F_1 \nabla \cdot (\zeta F_1 \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} (F_1 |\nabla \psi|^2) = 0, \end{cases} \quad (1.83)$$

with $n = 4$.

Démonstration. Again we will only do a formal proof. The rigorous one would use the same tools as the one of Proposition 1.1.11.

To do so let us use the fact that we proved the consistency of the water waves equations at order $O(\mu^2\varepsilon)$ with system (1.9), we discard all the terms of order $O(\mu\varepsilon)$ of the latter. We obtain a formal consistency of the water waves system at order $O(\mu\varepsilon)$ with the system

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon \nabla \cdot (\zeta \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} |\nabla \psi|^2 = 0, \end{cases}$$

(We used the identity $\Delta \psi + \frac{\mu}{3} \Delta F_2 \Delta \psi = F_1 \Delta \psi$).

It only remains to have in mind the estimates on F_1 of Proposition 1.6.4 to see that taking (ζ, ψ) solutions of the water waves system (1.1), one has

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon F_1 \nabla \cdot (\zeta F_1 \nabla \psi) = O(\mu\varepsilon), \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} (F_1 |\nabla \psi|^2) = O(\mu\varepsilon). \end{cases}$$

This concludes the formal demonstration. □

Remark 1.5.3. — We could easily adapt Definition 1.1.10 to match with system (1.82). And say that the water waves equations (1.1) are consistent with this Whitham-Boussinesq system at order $O(\mu\varepsilon)$.

- Using the same tools we could prove the consistency at order $O(\mu\varepsilon)$ of the water waves equations (1.1) with the other Whitham-Boussinesq systems of the literature.
- As the formal proof of Proposition 1.5.2 makes clear, the water waves equations are consistent at order $O(\mu\varepsilon)$ with every systems

$$\begin{cases} \partial_t \zeta + F_1 \Delta \psi + \varepsilon G_1 \nabla \cdot (\zeta G_2 \nabla \psi) = 0, \\ \partial_t \nabla \psi + \nabla \zeta + \frac{\varepsilon}{2} G_3 \nabla (|G_4 \nabla \psi|^2) = 0. \end{cases}$$

where the Fourier multipliers G_1, G_2, G_3 and G_4 are approximations of identity of the type $G_i = 1 + O(\mu)$. However, the well-posedness properties of the system will depend on the characteristics of the Fourier multipliers and in particular the order of their symbol (Definition 1.6.5). We postpone the study of the well-posedness of such systems to a future work.

With the same reasoning but starting from (1.57) we can also derive a Whitham-Boussinesq system with unknowns ζ and \bar{V} .

Proposition 1.5.4. *The water waves equations (1.1) are consistent at order $O(\mu\varepsilon)$ in the shallow water regime \mathcal{A} with the system*

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ \partial_t \bar{V} + F_1 \nabla \zeta + \frac{\varepsilon}{2} F_1 \nabla (|\bar{V}|^2) = 0, \end{cases} \quad (1.84)$$

with $n = 4$.

Démonstration. The water waves equations (1.1) are consistent at order $O(\mu^2\varepsilon)$ with system (1.57). Discarding the terms of order $O(\mu\varepsilon)$ of the latter we get

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h \bar{V}) = 0, \\ \partial_t (\bar{V} - \frac{\mu}{6h} \nabla (h^3 F_3 [\nabla \cdot V] + F_3 [h^3 \nabla \cdot V])) + \nabla \zeta + \frac{\varepsilon}{2} \nabla (|\bar{V}|^2) = O(\mu\varepsilon). \end{cases}$$

Then using (1.63) we obtain

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\bar{V}) = 0, \\ \partial_t \nabla \psi + \nabla \zeta + \frac{\varepsilon}{2} \nabla (|\bar{V}|^2) = O(\mu\varepsilon). \end{cases}$$

It only remains to apply F_1 to the second equation and to use the first estimate of Lemma 1.4.3 to get the result. \square

1.6 Technical tools

Proposition 1.6.1. (*Product estimates*)

1. Let $t_0 > d/2$, $s \geq -t_0$ and $f \in H^s \cap H^{t_0}(\mathbb{R}^d)$, $g \in H^s(\mathbb{R}^d)$. Then $fg \in H^s(\mathbb{R}^d)$ and

$$|fg|_{H^s} \lesssim |f|_{H^{\max(t_0, s)}} |g|_{H^s}$$

2. Let $s_1, s_2 \in \mathbb{R}$ be such that $s_1 + s_2 \geq 0$. Then for all $s \leq s_j$ ($j = 1, 2$) and $s < s_1 + s_2 - d/2$, and all $f \in H^{s_1}(\mathbb{R}^d)$, $g \in H^{s_2}(\mathbb{R}^d)$, one has $fg \in H^s(\mathbb{R}^d)$ and

$$|fg|_{H^s} \lesssim |f|_{H^{s_1}} |g|_{H^{s_2}}$$

Démonstration. See Appendix B.1 in [Lan13b]. \square

Proposition 1.6.2. (*Quotient estimates*) Let $t_0 > d/2$, $s \geq -t_0$ and $c_0 > 0$. Also let $f \in H^s(\mathbb{R}^d)$ and $g \in H^s \cap H^{t_0}(\mathbb{R}^d)$ be such that for all $X \in \mathbb{R}^d$, one has $1 + g(X) \geq c_0$. Then $\frac{f}{1+g}$ belongs to $H^s(\mathbb{R}^d)$ and

$$\left| \frac{f}{1+g} \right|_{H^s} \leq C \left(\frac{1}{c_0}, |g|_{H^{\max(t_0, s)}} \right) |f|_{H^s}$$

Démonstration. See Appendix B.1 in [Lan13b]. \square

Lemma 1.6.3. Let $P(\Sigma_t)$ be as in Definition 1.1.4. Let $h \in L_z^2 H_X^s((-1, 0) \times \mathbb{R}^d)$ and $u \in L_z^2 H_X^{s+1} \cap H_z^1 H_X^s((-1, 0) \times \mathbb{R}^d)$ ($s \geq 0$) solve the boundary value problem

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t) \nabla^\mu u = h, \\ u|_{z=0} = 0, \quad \partial_z u|_{z=-1} = 0. \end{cases}$$

Then one has

$$\|\Lambda^s \nabla^\mu u\|_2 \leq M(s+1) \|\Lambda^s h\|_2.$$

Démonstration. See Lemma 3.43 in [Lan13b]. \square

Proposition 1.6.4. *Let $s \geq 0$, $z \in (-1, 0)$ and ψ such that $\nabla\psi \in H^{s+1}(\mathbb{R}^d)$, then we have the following estimations*

$$\left\{ \begin{array}{l} |(\frac{1-F_0}{\mu|D|^2} + \frac{z^2}{2} + z)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \quad , \quad |(1 - (z+1)^2 F_0 + z^2 + 2z)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \\ |(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \quad , \quad |(\frac{z+1}{\sqrt{\mu}|D|} \frac{\sinh((z+1)\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu}|D|)} - (z+1)^2)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \\ |(F_2 - 1)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \quad , \quad |(F_3 - 1)\psi|_{H^s} \lesssim \mu|\nabla\psi|_{H^{s+1}} \end{array} \right.$$

An estimation of order $O(\mu^2)$ for $\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}$ will also be useful. If $\nabla\psi \in H^{s+3}$ then

$$|(\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|} - 1 + \frac{1}{3}\mu|D|^2)\psi|_{H^s} \lesssim \mu^2|\nabla\psi|_{H^{s+3}}$$

Démonstration. All the proofs are similar and the main key is the Taylor-Lagrange formula. All these estimations except the last one are in the same form where G is a smooth function on $(0, +\infty)$, continuous in 0.

$$|(G(\sqrt{\mu}|D|) - G(0))\psi|_{H^s} \leq \mu|\nabla\psi|_{H^{s+1}} \iff |(G(\sqrt{\mu}|\xi|) - G(0))\hat{\psi}|_{H^s} \leq \mu|\nabla\psi|_{H^{s+1}}$$

For the last one it would be

$$\begin{aligned} & |(G(\sqrt{\mu}|D|) - G(0) - \mu|D|^2 G''(0))\psi|_{H^s} \leq \mu^2|\nabla\psi|_{H^{s+3}} \\ \iff & |(G(\sqrt{\mu}|\xi|) - G(0) - \mu|\xi|^2 G''(0))\hat{\psi}|_2 \leq \mu^2|\nabla\psi|_{H^{s+3}} \end{aligned}$$

If we succeed in proving that the second derivative of G is bounded in $[0, +\infty)$, and that $G'(0) = 0$ then we can use the Taylor-Lagrange formula stating that for all $x \in [0, +\infty)$ there exists $\theta \in [0, 1]$ such that

$$G(x) - G(0) = \frac{x^2}{2} G''(\theta x)$$

then the boundedness of G'' allow us to write

$$|G(x) - G(0)| \leq |G''|_{\infty} x^2$$

Replacing x by $\sqrt{\mu}|\xi|$ in the last inequality we obtain

$$|(G(\sqrt{\mu}|\xi|) - G(0))\hat{\psi}|_2 \leq \mu|\xi|^2\hat{\psi}|_2 \leq \mu|\nabla\psi|_{H^1}$$

Thus it is sufficient to prove the boundedness in $C^2([0, +\infty))$ of G and the fact that $G'(0) = 0$ for the following functions :

$$\begin{cases} G_1(x) = \frac{1}{x^2} \left(1 - \frac{\cosh((z+1)x)}{\cosh(x)}\right), & G_2(x) = 1 - (z+1)^2 \frac{\cosh((z+1)x)}{\cosh(x)} \\ G_3(x) = \frac{\tanh(x)}{x}, & G_4(x) = \frac{z+1}{x} \frac{\sinh((z+1)x)}{\cosh(x)} \\ G_5(x) = \frac{3}{\mu x^2} \left(1 - \frac{\tanh(x)}{x}\right), & G_6(x) = \frac{3}{x \tanh(x)} - \frac{3}{x^2}, \end{cases}$$

where $z \in [-1, 0]$, which is given by the limited development of the functions \sinh , \cosh and \tanh around 0. \square

Definition 1.6.5. We say that a Fourier multiplier $F(D)$ is of order s ($s \in \mathbb{R}$) and write $F \in \mathcal{S}^s$ if $\xi \in \mathbb{R}^d \mapsto F(\xi) \in \mathbb{C}$ is smooth and satisfies

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta|-s} |\partial^\beta F(\xi)| < \infty.$$

We also introduce the seminorm

$$\mathcal{N}^s(F) = \sup_{\beta \in \mathbb{N}^d, |\beta| \leq 2+d+\lceil \frac{d}{2} \rceil} \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta|-s} |\partial^\beta F(\xi)|.$$

Proposition 1.6.6. Let $t_0 > d/2$, $s \geq 0$ and $F \in \mathcal{S}^s$. If $f \in H^s \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d)$,

$$|[F(D), f]g|_2 \leq \mathcal{N}^s(F) \|f\|_{H^{\max(t_0+1, s)}} \|g\|_{H^{s-1}}.$$

Démonstration. See Appendix B.2 in [Lan13b] for a proof of this proposition. \square

Proposition 1.6.7. Let $s \geq 0$. Let $\zeta \in H^{s+3}(\mathbb{R}^d)$ be such that (1.2) is satisfied and

$\psi \in \dot{H}^{s+3}(\mathbb{R}^d)$. Then one has

$$\begin{cases} |\frac{1}{\mu}\mathcal{G}^\mu\psi|_{H^{s+1}} \leq M(s+3)|\nabla\psi|_{H^{s+2}}, \\ |\frac{1}{\sqrt{\mu}}\mathcal{G}^\mu\psi|_{H^{s+1}} \leq \mu^{1/4}M(s+2)|\nabla\psi|_{H^{s+1}}. \end{cases}$$

Démonstration. This is a direct consequence of theorem 3.15 in [Lan13b]. \square

Lemma 1.6.8. *Let $t_0 > d/2$, $s \geq 0$. Let $P(\Sigma_t)$ be as in definition 1.1.4. Let $g(X, z)$ be a function on $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$ sufficiently regular such that its trace at $z = -1$ makes sense. Let u solve the boundary value problem*

$$\begin{cases} \nabla^\mu \cdot P(\Sigma_t)\nabla^\mu u = -\nabla^\mu \cdot g, \\ u|_{z=0} = 0, \quad v_2|_{z=-1} = -e_z \cdot g|_{z=-1}. \end{cases}$$

Then one has

$$\|\Lambda^s \nabla^\mu u\|_2 \leq M(s+1)\|\Lambda^s g\|_2.$$

Démonstration. See Lemma 2.38 in [Lan13b]. \square

Chapitre 2

On a well-posedness result for a class of non-local quasi-linear systems and its application to the justification of Whitham-Boussinesq systems

Article in preparation.

Abstract: In this paper we prove a local well-posedness result for a class of non-local quasi-linear hyperbolic systems involving Fourier multipliers. A direct consequence of this result is the local well-posedness of a class of Whitham-Boussinesq systems with existence time independent of the shallow water parameter. It allows us to prove the rigorous justification of the latter systems as approximations of the general water waves system on the relevant time scale.

2.1 Introduction

2.1.1 Motivations and main results

In this work we prove a well-posedness result and a stability result for a class of non-local quasi-linear systems involving Fourier multipliers. We apply these results to prove the rigorous justification of a specific class of irrotational shallow water models in coastal oceanography, the Whitham-Boussinesq systems. The latter systems have been the subject of active research, see [Car18; DDK19] for comparative studies, see [DN19; NW19] for some works on solitary waves solutions and [HP16] for a work on modulational

instability. It has been recently proved in [Eme20] that these models are approximations of order $O(\mu\varepsilon)$ of the general water waves model, where μ is the shallow water parameter and ε is the nonlinear parameter (see Definition 2.1.8), in the sense of consistency (see Proposition 2.1.10). The previous property is the first step to prove the full justification of a model for surface gravity waves. The second step is to prove the local well-posedness of the model for sufficiently regular data, with an existence time independent of μ . The well-posedness of the Whitham-Boussinesq systems has been studied in [Din19; Din20; DST19; KP18]. We pay special attention to [DST19] where the authors proved the local well-posedness in dimension 2 and global well-posedness in dimension 1 of a particular Whitham-Boussinesq system which presents a semi-linear structure. Yet, due to the strong use of dispersive estimates of the system, the time of existence and control of solutions (and their derivatives) is not uniform with respect to μ . In this paper we prove the local well-posedness with an existence time independent of μ of a class of Whitham-boussinesq systems which present a quasi-linear structure. We also prove a specific stability result for those systems, concluding their full justification.

Before writing down these models, we recall what is a Fourier multiplier of a certain order $s \in \mathbb{R}$.

Definition 2.1.1. For $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ a tempered distribution, let \widehat{u} be its Fourier transform. Let $G : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth function with polynomial growth. Then the Fourier multiplier associated with $G(\xi)$ is denoted G and defined by the formula :

$$\widehat{G[u]}(\xi) = G(\xi)\widehat{u}(\xi).$$

Definition 2.1.2. We say that a Fourier multiplier G is of order $s \in \mathbb{R}$ if $\xi \in \mathbb{R}^d \mapsto G(\xi) \in \mathbb{R}$ is smooth and satisfies

$$\forall \xi \in \mathbb{R}^d, \forall \beta \in \mathbb{N}^d, \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^{|\beta|-s} |\partial^\beta G(\xi)| < \infty.$$

Definition 2.1.3. We say that a couple of Fourier multipliers (G_1, G_2) is admissible if

- The Fourier multipliers G_1 and G_2 are of order 0.
- The Fourier multiplier G_1 is defined by a positive function G_1 .
- For all $\xi \in \mathbb{R}^d$, we have $|G_2(\xi)| \leq |G_1(\xi)|$.

Let $d = 1, 2$. The models we consider in this paper are

$$\begin{cases} \partial_t \zeta + (G_1)^2 \nabla \cdot v + \varepsilon G_2 \nabla \cdot (\zeta G_2[v]) = 0, \\ \partial_t v + \nabla \zeta + \varepsilon (G_2[v] \cdot \nabla) G_2[v] = 0, \end{cases} \quad (2.1)$$

where (G_1, G_2) is admissible, ζ and v are functions of time t and space $x \in \mathbb{R}^d$ valued in respectively \mathbb{R} and \mathbb{R}^d .

Denoting $U := \begin{pmatrix} \zeta \\ v \end{pmatrix}$, we can write the systems (2.1) under their matricial form

$$\partial_t U + \sum_{j=1}^d A_j(U) [\partial_j U] = 0, \quad (2.2)$$

where

$$\begin{cases} A_1(U)[\circ] = \begin{pmatrix} \varepsilon G_2[G_2[v_1]\circ] & (G_1)^2[\circ] + \varepsilon G_2[\zeta G_2[\circ]] & 0 \\ 1 & \varepsilon G_2[v_1]G_2[\circ] & 0 \\ 0 & 0 & \varepsilon G_2[v_1]G_2[\circ] \end{pmatrix}, \\ A_2(U)[\circ] = \begin{pmatrix} \varepsilon G_2[G_2[v_2]\circ] & 0 & (G_1)^2[\circ] + \varepsilon G_2[\zeta G_2[\circ]] \\ 0 & \varepsilon G_2[v_2]G_2[\circ] & 0 \\ 1 & 0 & \varepsilon G_2[v_2]G_2[\circ] \end{pmatrix}. \end{cases} \quad (2.3)$$

The natural functional setting is given by the energy norms.

Definition 2.1.4. — We denote by $\mathcal{S}'(\mathbb{R}^d)$ the set of tempered distributions.

- We denote by respectively $|\cdot|_2$ and $(\cdot, \cdot)_2$, the norm and the scalar product in $L^2(\mathbb{R}^d)$.
- Let $s \geq 0$. We denote by $H^s(\mathbb{R}^d)$ the Sobolev spaces of order s in $L^2(\mathbb{R}^d)$. Denoting $\Lambda^s := (1 - \Delta)^{s/2}$, where Δ is the Laplace operator in \mathbb{R}^d , the norm associated with $H^s(\mathbb{R}^d)$ is $|\cdot|_{H^s} := |\Lambda^s \cdot|_2$.
- Let G_1 be a Fourier multiplier of order 0, defined by positive function G_1 . Let also

$s \geq 0$. We define the Banach spaces $X^s(\mathbb{R}^d)$ and $Y^s(\mathbb{R}^d)$ by

$$\begin{aligned} X^s(\mathbb{R}^d) &:= \left\{ U = \begin{pmatrix} \zeta \\ v \end{pmatrix} \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)^d, \quad |U|_{X^s} < +\infty \right\}, \\ Y^s(\mathbb{R}^d) &:= \left\{ U = \begin{pmatrix} \zeta \\ v \end{pmatrix} \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)^d, \quad |U|_{Y^s} < +\infty \right\}, \end{aligned}$$

where $|U|_{X^s} := |\zeta|_{H^s} + |G_1[v]|_{H^s}$ and $|U|_{Y^s} := |\zeta|_{H^s} + |G_1^{-1}[v]|_{H^s}$. These are the energy norms associated with system (2.1).

The systems (2.1) are symmetrizable quasi-linear hyperbolic. The main key to prove the local well-posedness of such systems is the following energy estimates on the linearized system.

Proposition 2.1.5. *Let $s \geq 0$, $t_0 > d/2$, $T > 0$ and $0 \leq \varepsilon \leq 1$. Let also $\underline{U} = \begin{pmatrix} \zeta \\ v \end{pmatrix} \in W^{1,\infty}([0, T/\varepsilon], X^{t_0}(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^{\max(t_0+1, s)}(\mathbb{R}^d))$ for which there exists $h_{\min} > 0$ such that for all $(t, x) \in [0, T/\varepsilon] \times \mathbb{R}^d$*

$$1 + \varepsilon \zeta \geq h_{\min}. \quad (2.4)$$

Let also G_1 and G_2 be two Fourier multipliers of order 0 satisfying for all $\xi \in \mathbb{R}^d$, $|G_2(\xi)| \leq |G_1(\xi)|$. If $U = \begin{pmatrix} \zeta \\ v \end{pmatrix} \in W^{1,\infty}([0, T/\varepsilon], X^s(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^{s+1}(\mathbb{R}^d))$ satisfies the system

$$\partial_t U + \sum_{j=1}^d A_j(\underline{U})[\partial_j U] = \varepsilon R,$$

where $R \in L^\infty([0, T/\varepsilon], X^s(\mathbb{R}^d))$, and for $j = 1, 2$, $A_j(\underline{U})$ is defined by (2.3), we have for any $t \in [0, T/\varepsilon]$,

$$|U|_{X^s} \leq \kappa_0 e^{\varepsilon \lambda_s t} |U|_{X^s}|_{t=0} + \varepsilon \nu_s \int_0^t |R(t')|_{X^s} dt', \quad (2.5)$$

where $\lambda_s, \nu_s := C(\frac{1}{h_{\min}}, T, |\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{\max(t_0+1, s)}})$ and $\kappa_0 := C(\frac{1}{h_{\min}}, |U|_{X^{t_0}}|_{t=0})$.

Using a Picard iteration scheme and the regularization method from Chapter 7 in [Mét08] we get the following well-posedness result on the systems (2.1).

Theorem 2.1.6. *Let $s > d/2 + 1$, $h_{\min} > 0$ and $M > 0$. Let also (G_1, G_2) be a couple of admissible Fourier multipliers. There exist $T > 0$ and $C > 0$ such that for all $\varepsilon \in [0, 1]$, $U_0 \in X^s(\mathbb{R}^d)$ with $|U_0|_{X^s} \leq M$ and satisfying (2.4), there exists a unique solution $U \in C^0([0, T/\varepsilon], X^s(\mathbb{R}^d))$ of the Cauchy problem*

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = 0, \\ U|_{t=0} = U_0. \end{cases}$$

Moreover $|U|_{L^\infty([0, T/\varepsilon], X^s)} \leq C|U_0|_{X^s}$.

We also have the following stability result.

Proposition 2.1.7. *Let the assumptions of Theorem 2.1.6 be satisfied and assume also that there exists $\tilde{U} \in C([0, T/\varepsilon], X^s(\mathbb{R}^d))$ solution of*

$$\partial_t \tilde{U} + \sum_{j=1}^d A_j(\tilde{U}) \partial_j \tilde{U} = \tilde{R},$$

where $\tilde{R} \in L^\infty([0, T/\varepsilon], X^{s-1}(\mathbb{R}^d))$. Then, the error with respect to the solution $U \in C^0([0, T^*/\varepsilon], X^s(\mathbb{R}^d))$ given by Theorem 2.1.6 satisfies for all times $t \in [0, \min(T, T^*)/\varepsilon]$,

$$|\mathbf{e}|_{L^\infty([0, t], X^{s-1})} \leq C \left(\frac{1}{h_{\min}}, |U|_{L^\infty([0, t], X^s)}, |\tilde{U}|_{L^\infty([0, t], X^s)} \right) (|\mathbf{e}|_{X^{s-1}}|_{t=0} + t|\tilde{R}|_{L^\infty([0, t], X^{s-1})}),$$

where $\mathbf{e} = U - \tilde{U}$.

In the context of irrotational shallow water systems, we recall the notations and the physical meaning of each element of the systems (2.1) (see [Eme20]).

- The free surface elevation is the graph of ζ , which is a function of time t and horizontal space $x \in \mathbb{R}^d$.
- $v(t, x)$ is the gradient of the trace at the surface of the velocity potential.

Moreover every variable and function in (2.1) is compared with physical characteristic parameters of the same dimension. Among those are the characteristic water depth H_0 , the characteristic wave amplitude a_{surf} and the characteristic wavelength L .

Definition 2.1.8. *From the above physical characteristic parameters, we define two*

adimensional paramaters of main importance :

$$\mu := \frac{H_0^2}{L^2}, \quad \text{and} \quad \varepsilon := \frac{a_{\text{surf}}}{H_0}.$$

The first parameter, μ is called the shallow water parameter, $\mu \in [0, 1]$. The second parameter ε is called the nonlinearity parameter, $\varepsilon \in [0, 1]$.

Notation 2.1.9. From now on the Fourier multipliers G_1 and G_2 depend on μ and are of the form $G_k^\mu = G_k(\sqrt{\mu}D)$, $k = 1, 2$ where G_k is a Fourier multiplier independent of μ .

Proposition 1.15 in [Eme20] can be easily extended to the following result.

Proposition 2.1.10. Let $(\mu, \varepsilon) \in [0, 1]^2$ and $s \geq 0$. In (2.1), let $G_1^\mu := \sqrt{\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}}$. Let also G_2^μ be a Fourier multiplier such that for all $\xi \in \mathbb{R}^d$, $|G_2^\mu(\xi)| \leq |G_1^\mu(\xi)|$, and satisfying for any $s' \geq 0$, $w \in H^{s'+2}(\mathbb{R}^d)$,

$$|G_2^\mu[w] - w|_{H^{s'}} \lesssim \mu |w|_{H^{s'+2}}.$$

Then any classical solution (ζ, ψ) of the water waves equations satisfying the non-cavitation hypothesis (2.4), with $U = (\zeta, \nabla\psi) \in C^0([0, T/\varepsilon], H^{s+4}(\mathbb{R}^d)^{1+d})$, satisfy the system (2.1) up to a remainder term of order $O(\mu\varepsilon)$, i.e.

$$\partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = \mu\varepsilon R,$$

where $|R|_{H^s} \leq C(\frac{1}{h_{\min}}, |\zeta|_{H^{s+4}}, |\nabla\psi|_{H^{s+4}})$, uniformly with respect to $(\mu, \varepsilon) \in [0, 1]^2$.

We say that the water waves equations are consistent at order $O(\mu\varepsilon)$ in the shallow water regime with the systems (2.1).

Remark 2.1.11. — Within the assumptions of Proposition 2.1.10, the systems (2.1) are Whitham-Boussinesq systems (see [Eme20]).

- Taking $G_2^\mu = (G_1^\mu)^2$ we get the system studied in [DST19] for which the authors proved a local well-posedness result in dimension $d = 2$ and a global well-posedness result when $d = 1$. The proofs use the regularizing effect of $(G_1^\mu)^2$ which gives to the system a semi-linear structure.
- The Theorem 2.1.6 allows us to take, for example, $G_2^\mu = G_1^\mu$, thus having a local well-posedness result for a Whitham-Boussinesq system with a quasi-linear structure.

— When $G_2^\mu = Id$, the system (2.1) is a Whitham-Boussinesq system. However the question of local well-posedness for such systems is still open.

From Theorem 2.1.6, Proposition 2.1.7 and Proposition 2.1.10 we get the full justification of a class of Whitham-Boussinesq systems.

Theorem 2.1.12. *Let $(\mu, \varepsilon) \in [0, 1]^2$ and $s > d/2$. Let $G_1^\mu := \sqrt{\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}}$. Let also G_2^μ be a Fourier multiplier of order 0 such that for all $\xi \in \mathbb{R}^d$, $|G_2^\mu(\xi)| \leq |G_1^\mu(\xi)|$, and satisfying for any $s' \geq 0$ and $w \in H^{s'+2}(\mathbb{R}^d)$,*

$$|G_2^\mu[w] - w|_{H^{s'}} \lesssim \mu|w|_{H^{s'+2}}.$$

There exists $T > 0$ independent of the $(\mu, \varepsilon) \in [0, 1]^2$ such that for any $T' > 0$ and any classical solutions (ζ, ψ) of the water waves equations satisfying the non-cavitation assumption (2.4) with $U = (\zeta, \nabla\psi) \in C^0([0, T'/\varepsilon], H^{s+4}(\mathbb{R}^d))$, there exists a unique classical solution $U_{\text{WB}} = (\zeta_{\text{WB}}, v_{\text{WB}}) \in C^0([0, T/\varepsilon], X^{s+3}(\mathbb{R}^d))$ with initial data $(\zeta|_{t=0}, \nabla\psi|_{t=0})$ of the Whitham-Boussinesq systems (2.1) and one has for all times $t \in [0, \min(T, T')/\varepsilon]$,

$$|\mathfrak{e}|_{L^\infty([0,t], X^s)} \leq \mu\varepsilon t C\left(\frac{1}{h_{\min}}, |U|_{L^\infty([0,t], H^{s+4})}\right),$$

where $\mathfrak{e} = U - U_{\text{WB}}$.

Remark 2.1.13. *Regular solutions of the water waves equations as in Theorem 2.1.12 are provided by Theorem 4.16 in [Lan13b].*

2.1.2 Outline

Section 2.2 is dedicated to the proof of Proposition 2.1.5. In Subsection 2.2.1 we focus on the symmetrization of the systems (2.1). In Subsection 2.2.2, we prove the energy estimates of Proposition 2.1.5 in the case of $s = 0$. Finally in Subsection 2.2.3, we prove the general case $s \geq 0$.

Section 2.3 is dedicated to the proof of Theorem 2.1.6. In Subsection 2.3.1 we prove a local well-posedness result for the systems (2.1) linearized around a sufficiently regular state. In Subsection 2.3.2 we focus on the proof of Theorem 2.1.6. In Subsection 2.3.3 we establish a blow-up criterion for the local well-posedness of the systems (2.1).

In Section 2.4 we prove Proposition 2.1.7.

In Section 2.5 we prove Theorem 2.1.12.

2.2 Energy estimates

This section is dedicated to the proof of Proposition 2.1.5.

2.2.1 Symmetrization

In this subsection we focus on the symmetrization of the systems (2.2).

Property 2.2.1. *Let $s \geq 0$ and $t_0 > d/2$. For any $U \in X^{t_0+1}(\mathbb{R}^d)$, let*

$$S_0(U)[\circ] := \begin{pmatrix} 1 & 0 \\ 0 & ((G_1)^2[\circ] + \epsilon G_2[\zeta G_2[\circ]])Id \end{pmatrix}, \quad (2.6)$$

be an operator defined in $X^0(\mathbb{R}^d)$. Applying the latter operator to system (2.2) we get

$$S_0(U)[\partial_t U] + \sum_{j=1}^d \tilde{A}_j(U)[\partial_j U] = \epsilon \sum_{j=1}^d F_j(U)[\partial_j U], \quad (2.7)$$

where for $j = 1, 2$, $\tilde{A}_j(U)$ is a symmetric operator defined by

$$\tilde{A}_j(U) = \frac{B_j(U) + B_j(U)^*}{2},$$

with

$$B_1(U)[\circ] = \begin{pmatrix} B_1^{1,1}(U)[\circ] & B_1^{1,2}(U)[\circ] & 0 \\ B_1^{1,2}(U)[\circ] & B_1^{2,2}(U)[\circ] & 0 \\ 0 & 0 & B_1^{2,2}(U)[\circ] \end{pmatrix},$$

where

$$\begin{cases} B_1^{1,1}(U)[\circ] = \epsilon G_2[G_2[v_1]\circ], \\ B_1^{1,2}(U)[\circ] = (G_1)^2[\circ] + \epsilon G_2[\zeta G_2[\circ]], \\ B_1^{2,2}(U)[\circ] = \epsilon (G_1)^2[G_2[v_1]G_2[\circ]] + \epsilon^2 G_2[\zeta G_2[G_2[v_1]G_2[\circ]]], \end{cases}$$

and

$$B_2(U)[\circ] = \begin{pmatrix} B_2^{1,1}(U)[\circ] & 0 & B_2^{1,3}(U)[\circ] \\ 0 & B_2^{2,2}(U)[\circ] & 0 \\ B_2^{1,3}(U)[\circ] & 0 & B_2^{2,2}(U)[\circ] \end{pmatrix},$$

where

$$\begin{cases} B_2^{1,1}(U)[\circ] = \epsilon G_2[G_2[v_2]\circ], \\ B_2^{1,3}(U)[\circ] = (G_1)^2[\circ] + \epsilon G_2[\zeta G_2[\circ]], \\ B_2^{2,2}(U)[\circ] = \epsilon(G_1)^2[G_2[v_2]G_2[\circ]] + \epsilon^2 G_2[\zeta G_2[G_2[v_2]G_2[\circ]]]. \end{cases}$$

Also, $*$ stands for the adjoint in $L^2(\mathbb{R}^d)$.

And finally, for $j = 1, 2$,

$$F_j(U)[\circ] = -\frac{B_j(U)[\circ] - B_j(U)^*[\circ]}{2\epsilon}.$$

Démonstration. Applying the matricial operator $S_0(U)[\circ] := \begin{pmatrix} 1 & 0 \\ 0 & ((G_1)^2[\circ] + \epsilon G_2[\zeta G_2[\circ]])\text{Id} \end{pmatrix}$ to the system (2.2), we get

$$S_0(U)[\partial_t U] + \sum_{j=1}^d B_j(U)[\partial_j U] = 0.$$

Then we write

$$S_0(U)\partial_t U + \sum_{j=1}^d \frac{B_j(U) + B_j(U)^*}{2}[\partial_j U] = -\sum_{j=1}^d \frac{B_j(U) - B_j(U)^*}{2}[\partial_j U].$$

□

Proposition 2.2.2. *Let $s \geq 0$ and $t_0 > d/2$. For any $\underline{U} \in X^{\max(t_0+1, s)}(\mathbb{R}^d)$ and any $U \in X^s(\mathbb{R}^d)$, for $j = 1, 2$, we have*

$$|F_j(\underline{U})[\partial_j U]|_{Y^s} \leq C(|\underline{U}|_{X^{\max(t_0+1, s)}})|U|_{X^s}. \quad (2.8)$$

It makes it a term of order 0 with respect to the energy norm.

Démonstration. First remark that $(B_1^{1,2})^* = B_1^{1,2}$ and $(B_2^{1,3})^* = B_2^{1,3}$. So that we need to

estimate the terms of $F_j(\underline{U})$, $j = 1, 2$, defined by $B_1^{1,1}$, $B_1^{2,2}$, $B_2^{1,1}$ and $B_2^{2,2}$. We only do the proof for $B_1^{2,2}(\underline{U})[\partial_1 v_2] - (B_1^{2,2}(\underline{U}))^*[\partial_1 v_2]$ as the estimates of the other terms would use the same tools. For the first term, we have

$$\begin{aligned} & |(\mathbf{G}_1)^{-1}[(\mathbf{G}_1)^2[\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\partial_1 v_2]] - \mathbf{G}_2[\mathbf{G}_2[\underline{v}_1](\mathbf{G}_1)^2[\partial_1 v_2]]]|_{H^s} \\ & \leq |\mathbf{G}_1[\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\partial_1 v_2]] - (\mathbf{G}_1)^{-1}\mathbf{G}_1\mathbf{G}_2[\mathbf{G}_2[\underline{v}_1]\mathbf{G}_1[\partial_1 v_2]]|_{H^s} + |(\mathbf{G}_1)^{-1}\mathbf{G}_2[[\mathbf{G}_1, \mathbf{G}_2[\underline{v}_1]]\mathbf{G}_1[\partial_1 v_2]]|_{H^s} \\ & := I_1 + I_2. \end{aligned}$$

But

$$I_1 = |[\mathbf{G}_1, \mathbf{G}_2[\underline{v}_1]]\mathbf{G}_2[\partial_1 v_2] - [\mathbf{G}_2, \mathbf{G}_2[\underline{v}_1]]\mathbf{G}_1[\partial_1 v_2]|_{H^s},$$

so that using the commutator estimates of Proposition 2.6.3 and the assumption $\mathbf{G}_2 \leq \mathbf{G}_1$ with \mathbf{G}_1 and \mathbf{G}_2 of order 0, we get

$$I_1 \lesssim |\mathbf{G}_2[\underline{v}_1]|_{H^{\max(t_0+1, s)}} |\mathbf{G}_1[\partial_1 v_2]|_{H^{s-1}} \lesssim |\underline{U}|_{X^{\max(t_0+1, s)}} |U|_{X^s}.$$

Using the same tools, we get

$$I_2 \lesssim |\underline{U}|_{X^{\max(t_0+1, s)}} |U|_{X^s}.$$

For the second term, using the same tools in addition with the product estimates of Proposition 2.6.1, we have

$$\begin{aligned} & |(\mathbf{G}_1)^{-1}[\mathbf{G}_2[\underline{\zeta}\mathbf{G}_2[\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\partial_1 v_2]]] - \mathbf{G}_2[\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\underline{\zeta}\mathbf{G}_2[\partial_1 v_2]]]|_{H^s} \\ & \leq |\underline{\zeta}\mathbf{G}_2[\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\partial_1 v_2]] - \mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\underline{\zeta}\mathbf{G}_2[\partial_1 v_2]]|_{H^s} \\ & \leq | - [\mathbf{G}_2, \underline{\zeta}](\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\partial_1 v_2]) + [\mathbf{G}_2, \mathbf{G}_2[\underline{v}_1]](\underline{\zeta}\mathbf{G}_2[\partial_1 v_2]) |_{H^s} \\ & \leq |\underline{\zeta}|_{H^{\max(t_0+1, s)}} |\mathbf{G}_2[\underline{v}_1]\mathbf{G}_2[\partial_1 v_2]|_{H^{s-1}} + |\mathbf{G}_2[\underline{v}_1]|_{H^{\max(t_0+1, s)}} |\underline{\zeta}\mathbf{G}_2[\partial_1 v_2]|_{H^{s-1}} \\ & \leq |\underline{U}|_{X^{\max(t_0+1, s)}}^2 |U|_{X^s}. \end{aligned}$$

□

2.2.2 Energy estimates of order 0

In this subsection we prove the energy estimates of order 0 on the systems (2.2) linearized around a sufficiently regular state.

Proposition 2.2.3. *Let $t_0 > d/2$ and $T > 0$. Let $\underline{U} \in W^{1,\infty}([0, T/\varepsilon], X^{t_0}(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^{t_0+1}(\mathbb{R}^d))$ be such that (2.4) is satisfied. For all $U \in W^{1,\infty}([0, T/\varepsilon], X^0(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^1(\mathbb{R}^d))$ which satisfies the system*

$$\partial_t U + \sum_{j=1}^d A_j(\underline{U})[\partial_j U] = \varepsilon R, \quad (2.9)$$

where $R \in L^\infty([0, T/\varepsilon], X^0(\mathbb{R}^d))$, and for $j = 1, 2$, $A_j(\underline{U})$ is defined by (2.3), we have for any $t \in [0, T/\varepsilon]$,

$$|U|_{X^0} \leq \kappa_0 e^{\varepsilon \lambda_0 t} |U|_{X^0}|_{t=0} + \varepsilon \nu_0 \int_0^t |R(t')|_{X^0} dt', \quad (2.10)$$

where $\lambda_0, \nu_0 := C(\frac{1}{h_{\min}}, T, |\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{t_0+1}})$ and $\kappa_0 := C(\frac{1}{h_{\min}}, |\underline{U}|_{X^{t_0}}|_{t=0})$.

To prove this result we need some properties on the symmetrizer $S_0(\underline{U})$.

Lemma 2.2.4. *Let $t_0 > d/2$. Let $\underline{U} \in X^{t_0}(\mathbb{R}^d)$ be such that (2.4) is satisfied. The symmetrizer $S_0(\underline{U})$ satisfies for any $U \in X^0(\mathbb{R}^d)$*

$$(S_0(\underline{U})U, U)_2 \geq |\zeta|_2^2 + h_{\min} |G_1[v]|_2^2 \geq h_{\min} |U|_{X^0}^2. \quad (2.11)$$

and

$$\begin{cases} |S_0(\underline{U})[U]|_{Y^0} \leq C(|\underline{U}|_{X^{t_0}})|U|_{X^0}, \\ (S_0(\underline{U})[U], U)_2 \leq C(|\underline{U}|_{X^{t_0}})|U|_{X^0}^2. \end{cases} \quad (2.12)$$

Démonstration. Given the definition of $S_0(\underline{U})$ (see (2.6)), the assumption $G_2 \leq G_1$ and the Sobolev embedding $H^{t_0} \subset L^\infty$, the estimates of (2.12) are obvious. We prove here the inequality (2.11).

$$\begin{aligned} (S_0(\underline{U})U, U)_2 &= |\zeta|_2^2 + ((G_1)^2[v] + G_2[\varepsilon \zeta G_2[v]], v)_2 \\ &= |\zeta|_2^2 + (((G_1)^2 - (1 - h_{\min})(G_2)^2)[v], v)_2 + ((1 - h_{\min})(G_2)^2[v] + G_2[\varepsilon \zeta G_2[v]], v)_2 \end{aligned}$$

But $G_2 \leq G_1$, $\varepsilon \underline{\zeta} \geq h_{\min} - 1$ so :

$$\begin{aligned} (S_0(\underline{U})U, U)_2 &\geq |\zeta|_2^2 + h_{\min}|G_1[v]|_2^2 - (h_{\min} - 1)|G_2[v]|_2^2 + (h_{\min} - 1)|G_2[v]|_2^2 \\ &\geq |\zeta|_2^2 + h_{\min}|G_1[v]|_2^2. \end{aligned}$$

□

Lemma 2.2.5. *With the same assumptions as in Proposition 2.2.3, we have the following estimates*

$$\frac{d}{dt}(S_0(\underline{U})[U], U)_2 \leq \varepsilon C(|\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{t_0+1}})(S_0(\underline{U})[U], U)_2 + \varepsilon C(|\underline{U}|_{L_t^\infty X^{t_0}})|R|_{X^0} \sqrt{(S_0(\underline{U})[U], U)_2}.$$

Démonstration. Using the self-adjointness of $S_0(\underline{U})$, we have

$$\frac{1}{2} \frac{d}{dt}(S_0(\underline{U})[U], U)_2 = \frac{1}{2}((\partial_t S_0(\underline{U})))[U], U)_2 + (S_0(\underline{U})[\partial_t U], U)_2,$$

where

$$\partial_t S_0(\underline{U})[\circ] = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon G_2[(\partial_t \underline{\zeta})G_2[\circ]]I_d \end{pmatrix}.$$

Then using the symmetrization (2.7), we get

$$\frac{1}{2} \frac{d}{dt}(S_0(\underline{U})[U], U)_2 = \frac{1}{2}((\partial_t S_0(\underline{U})))[U], U)_2 - \sum_{j=1}^d (\tilde{A}_j(\underline{U})[\partial_j U], U)_2 + \varepsilon (S_0(\underline{U})[R], U)_2 + \varepsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U)_2$$

But using the symmetry of \tilde{A}_j , $j = 1, 2$, we have

$$\begin{aligned} (\tilde{A}_j(\underline{U})[\partial_j U], U)_2 &= (\partial_j U, \tilde{A}_j(\underline{U})[U])_2 \\ &= -(U, (\partial_j \tilde{A}_j(\underline{U})))[U])_2 - (U, \tilde{A}_j(\underline{U})[\partial_j U])_2, \end{aligned}$$

where

$$(\partial_1 \tilde{A}_1(\underline{U}))[\circ] = \frac{(\partial_1 B_1(\underline{U}))[\circ] + (\partial_1 (B_1(\underline{U})^*))[\circ]}{2},$$

with

$$(\partial_1 B_1(\underline{U}))[\circ] = \begin{pmatrix} (\partial_1 B_1^{1,1}(\underline{U}))[\circ] & (\partial_1 B_1^{1,2}(\underline{U}))[\circ] & 0 \\ (\partial_1 B_1^{1,2}(\underline{U}))[\circ] & (\partial_1 B_1^{2,2}(\underline{U}))[\circ] & 0 \\ 0 & 0 & (\partial_1 B_1^{2,2}(\underline{U}))[\circ] \end{pmatrix},$$

where

$$\begin{cases} (\partial_1 B_1^{1,1}(\underline{U}))[\circ] = \varepsilon G_2[G_2[\partial_1 \underline{v}_1]\circ], \\ (\partial_1 B_1^{1,2}(\underline{U}))[\circ] = \varepsilon G_2[(\partial_1 \underline{\zeta})G_2[\circ]], \\ (\partial_1 B_1^{2,2}(\underline{U}))[\circ] = \varepsilon(G_1)^2[G_2[\partial_1 \underline{v}_1]G_2[\circ]] + \varepsilon^2 G_2[(\partial_1 \underline{\zeta})G_2[G_2[\underline{v}_1]G_2[\circ]]] + \varepsilon^2 G_2[\underline{\zeta}G_2[G_2[\partial_1 \underline{v}_1]G_2[\circ]]]. \end{cases}$$

And

$$(\partial_2 \tilde{A}_2(\underline{U}))[\circ] = \frac{(\partial_2 B_2(\underline{U}))[\circ] + (\partial_2 (B_2(\underline{U})^*))}{2},$$

with

$$(\partial_2 B_2(\underline{U}))[\circ] = \begin{pmatrix} (\partial_2 B_2^{1,1}(\underline{U}))[\circ] & 0 & (\partial_2 B_2^{1,3}(\underline{U}))[\circ] \\ 0 & (\partial_2 B_2^{2,2}(\underline{U}))[\circ] & 0 \\ (\partial_2 B_2^{1,3}(\underline{U}))[\circ] & 0 & (\partial_2 B_2^{2,2}(\underline{U}))[\circ] \end{pmatrix},$$

where

$$\begin{cases} (\partial_2 B_2^{1,1}(\underline{U}))[\circ] = \varepsilon G_2[G_2[\partial_2 \underline{v}_2]\circ], \\ (\partial_2 B_2^{1,3}(\underline{U}))[\circ] = \varepsilon G_2[(\partial_2 \underline{\zeta})G_2[\circ]], \\ (\partial_2 B_2^{2,2}(\underline{U}))[\circ] = \varepsilon(G_1)^2[G_2[\partial_2 \underline{v}_2]G_2[\circ]] + \varepsilon^2 G_2[(\partial_2 \underline{\zeta})G_2[G_2[\underline{v}_2]G_2[\circ]]] + \varepsilon^2 G_2[\underline{\zeta}G_2[G_2[\partial_2 \underline{v}_2]G_2[\circ]]]. \end{cases}$$

For any $j = 1, 2$, $(\partial_j (B_j(\underline{U})^*))[\circ]$ is also easily computed and have the same mathematical structure as $(\partial_j B_j(\underline{U}))[\circ]$.

So

$$\frac{1}{2} \frac{d}{dt} (S_0(\underline{U})U, U)_2 = \frac{1}{2} ((\partial_t S_0(\underline{U}) + \sum_{j=1}^d \partial_j A_j(\underline{U}))[\underline{U}], U)_2 + \varepsilon (S_0(\underline{U})[R], U)_2 + \varepsilon \sum_{j=1}^d (F_j(\underline{U})[\partial_j U], U)_2. \quad (2.13)$$

The term $\varepsilon(S_0(\underline{U})[R], U)_2$ is easily estimated by Cauchy-Schwarz inequality and (2.12) :

$$\varepsilon(S_0(\underline{U})[R], U)_2 \leq \varepsilon |S_0(\underline{U})[R]|_{Y^0} |U|_{X^0} \leq \varepsilon C(|\underline{U}|_{L_t^\infty X^{t_0}}) |R|_{X^0} |U|_{X^0}. \quad (2.14)$$

The term $\varepsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U)_2$ is estimated using the Proposition 2.2.2. Indeed, we have

$$\varepsilon \sum_{j=1}^d (F_j(U)[\partial_j U], U)_2 \leq \varepsilon \sum_{j=1}^d |F_j(U)[\partial_j U]|_{Y^0} |U|_{X^0} \leq \varepsilon C(|\underline{U}|_{L_t^\infty X^{t_0+1}}) |U|_{X^0}^2. \quad (2.15)$$

Then, the result comes from the following estimates

$$|((\partial_t S_0(\underline{U}) + \sum_{j=1}^d \partial_j A_j(\underline{U}))[U], U)_2| \leq \varepsilon C(|\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{t_0+1}}) |U|_{X^0}^2. \quad (2.16)$$

Here are some examples of controls needed to get the latter estimates :

$$\left\{ \begin{array}{l} |(\varepsilon G_2[(\partial_t \underline{\zeta}) G_2[v], v])_2| \leq \varepsilon |\partial_t \underline{\zeta}|_{L_t^\infty H^{t_0}} |G_1[v]|_2^2 \leq \varepsilon |\partial_t \underline{\zeta}|_{L_t^\infty H^{t_0}} |U|_{X^0}^2, \\ |(\varepsilon G_2[(\partial_t \underline{\zeta}) G_2[\zeta], v_1])_2| \leq \varepsilon |\underline{\zeta}|_{L_t^\infty H^{t_0+1}} |\zeta|_2 |G_1[v_1]|_2 \leq \varepsilon |\underline{\zeta}|_{L_t^\infty H^{t_0+1}} |U|_{X^0}^2, \\ |(\varepsilon^2 G_2[\partial_t \underline{\zeta} G_2[G_2[v_1] G_2[v_1]], v_1])_2| \leq \varepsilon^2 |\underline{\zeta}|_{L_t^\infty H^{t_0+1}} |G_2[G_2[v_1] G_2[v_1]]|_2 |G_2[v_1]|_2 \\ \leq \varepsilon^2 |\underline{\zeta}|_{L_t^\infty H^{t_0+1}} |G_1[v_1]|_{L_t^\infty H^{t_0}} |G_1[v_1]|_2^2 \leq \varepsilon^2 |\underline{\zeta}|_{L_t^\infty H^{t_0+1}} |v_1|_{L_t^\infty X^{t_0}} |U|_{X^0}^2, \end{array} \right.$$

where we used the assumption $G_2 \leq G_1$ and the boundedness of the latter Fourier multiplier in $H^s(\mathbb{R}^d)$. The others use the same tools.

Combining (2.13), (2.14), (2.15), (2.16) and (2.11) we get

$$\frac{d}{dt} (S_0(\underline{U})[U], U)_2 \leq \varepsilon C\left(\frac{1}{h_{\min}}, |\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{t_0+1}}\right) (S_0(\underline{U})[U], U)_2 + \varepsilon C\left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{t_0}}\right) |R|_{X^0} \sqrt{(S_0(\underline{U})[U], U)_2}$$

□

We now have all the elements needed to prove the Proposition 2.2.3.

Démonstration. We proved that if $U \in W^{1,\infty}([0, T/\varepsilon], X^0(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^1(\mathbb{R}^d))$, we

have

$$\begin{aligned} & \sqrt{(S_0(\underline{U})[U], U)_2} \frac{d}{dt} \sqrt{(S_0(\underline{U})[U], U)_2} = \frac{1}{2} \frac{d}{dt} (S_0(\underline{U})[U], U)_2 \\ & \leq \varepsilon C \left(\frac{1}{h_{\min}}, |\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{t_0+1}} \right) (S_0(\underline{U})[U], U)_2 + \varepsilon C \left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{t_0}} \right) |R|_{X^0} \sqrt{(S_0(\underline{U})[U], U)_2}. \end{aligned}$$

Dividing by $\sqrt{(S_0(\underline{U})[U], U)_2}$, we get

$$\frac{d}{dt} \sqrt{(S_0(\underline{U})[U], U)_2} \leq \varepsilon C \left(\frac{1}{h_{\min}}, |\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{t_0+1}} \right) \sqrt{(S_0(\underline{U})[U], U)_2} + \varepsilon C \left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{t_0}} \right) |R|_{X^0}.$$

We can integrate this inequality in time between 0 and t to get

$$\sqrt{(S_0(\underline{U})[U], U)_2} \leq e^{\varepsilon \lambda_0 t} \sqrt{(S_0(\underline{U})[U], U)_2}|_{t=0} + \varepsilon C \left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{t_0}} \right) \int_0^t e^{\varepsilon(t-t')\lambda_0} |R(t')|_{X^0}^2 dt'.$$

And using Lemma 2.2.4, we have

$$|U|_{X^0} \leq \kappa_0 e^{\varepsilon \lambda_0 t} |U|_{X^0}|_{t=0} + \varepsilon \nu_0 \int_0^t |R(t')|_{X^0} dt'.$$

□

2.2.3 Energy estimates of higher order

We have now everything we need to prove Proposition 2.1.5 which we recall here for the sake of clarity.

Proposition 2.2.6. *Let $s \geq 0$, $t_0 > d/2$ and $T > 0$. Let $\underline{U} \in W^{1,\infty}([0, T/\varepsilon], X^{t_0}(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^{\max(t_0+1, s)}(\mathbb{R}^d))$ be such that (2.4) holds. Let also (G_1, G_2) be a couple of admissible Fourier multipliers. If $U \in W^{1,\infty}([0, T/\varepsilon], X^s(\mathbb{R}^d)) \cap L^\infty([0, T/\varepsilon], X^{s+1}(\mathbb{R}^d))$ satisfies the system*

$$\partial_t U + \sum_{j=1}^d A_j(\underline{U})[\partial_j U] = \varepsilon R, \quad (2.17)$$

where $R \in L^\infty([0, T/\varepsilon], X^s(\mathbb{R}^d))$, and for $j = 1, 2$, $A_j(\underline{U})$ is defined by (2.3), we have for

any $t \in [0, T/\varepsilon]$,

$$|U|_{X^s} \leq \kappa_0 e^{\varepsilon \lambda_s t} |U|_{X^s}|_{t=0} + \varepsilon \nu_s \int_0^t |R(t')|_{X^s} dt', \quad (2.18)$$

where $\lambda_s, \nu_s := C(\frac{1}{h_{\min}}, T, |\underline{U}|_{W_t^{1,\infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{\max(t_0+1,s)}})$ and $\kappa_0 := C(\frac{1}{h_{\min}}, |\underline{U}|_{X^{t_0}}|_{t=0})$.

We begin by proving the following lemma.

Lemma 2.2.7. *With the same assumptions as Proposition 2.2.6, there exists $R_{(s)} \in L^\infty([0, T/\varepsilon], X^0(\mathbb{R}^d))$ such that*

$$S_0(\underline{U}) \partial_t \Lambda^s U + \sum_{j=1}^d A_j(\underline{U}) \partial_j \Lambda^s U = \varepsilon R_{(s)},$$

with $\Lambda^s := (1 - \Delta)^{s/2}$ and

$$|R_{(s)}|_{X^0} \leq C\left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{\max(t_0+1,s)}}\right) (|U|_{X^s} + |R|_{X^s}).$$

Démonstration. We know that

$$\partial_t U + \sum_{j=1}^d A_j(\underline{U}) \partial_j U = \varepsilon R.$$

Applying the operator $\Lambda^s = (1 - \Delta)^{s/2}$ to this equation, we get

$$\partial_t \Lambda^s U + \sum_{j=1}^d A_j(\underline{U}) [\partial_j \Lambda^s U] = \varepsilon \Lambda^s R - \sum_{j=1}^d [\Lambda^s, A_j(\underline{U})] [\partial_j U].$$

But because Λ^s commutes with G_1 and G_2 , we have

$$\left\{ \begin{array}{l} [\Lambda^s, A_1(\underline{U})][\circ] = \begin{pmatrix} \varepsilon G_2[[\Lambda^s, G_2[\underline{v}_1]][\circ]] & \varepsilon G_2[[\Lambda^s, \underline{\zeta}]G_2[\circ]] & 0 \\ 0 & \varepsilon[\Lambda^s, G_2[\underline{v}_1]]G_2[\circ] & 0 \\ 0 & 0 & \varepsilon[\Lambda^s, G_2[\underline{v}_1]]G_2[\circ] \end{pmatrix}, \\ [\Lambda^s, A_2(\underline{U})][\circ] = \begin{pmatrix} \varepsilon G_2[[\Lambda^s, G_2[\underline{v}_2]][\circ]] & 0 & \varepsilon G_2[[\Lambda^s, \underline{\zeta}]G_2[\circ]] \\ 0 & \varepsilon[\Lambda^s, G_2[\underline{v}_2]]G_2[\circ] & 0 \\ 0 & 0 & \varepsilon[\Lambda^s, G_2[\underline{v}_2]]G_2[\circ] \end{pmatrix}. \end{array} \right. \quad (2.19)$$

So that using the commutator estimates of Proposition 2.6.3, we get

$$|\sum_{j=1}^d [\Lambda^s, A_j(\underline{U})][\partial_j U]|_{X^0} \leq \varepsilon C(|\underline{U}|_{L_t^\infty X^{\max(t_0+1, s)}})|U|_{X^s}.$$

At the end, denoting $R_{(s)} = \Lambda^s R - \frac{1}{\varepsilon} \sum_{j=1}^d [\Lambda^s, A_j(\underline{U})][\partial_j U]$, we get the result. □

We now prove Proposition 2.2.6.

Démonstration. Using the energy estimates of Proposition 2.2.3 on $\Lambda^s U$ and Lemma 2.2.7 we get

$$\begin{aligned} |\Lambda^s U|_{X^0} &\leq \kappa_0 e^{\varepsilon \lambda_0 t} |\Lambda^s U|_{X^s}|_{t=0} + \varepsilon \nu_0 \int_0^t |R_{(s)}(t')|_{X^0} dt' \\ \implies |U|_{X^s} &\leq \kappa_0 e^{\varepsilon \lambda_0 t} |U|_{X^s}|_{t=0} + \varepsilon \nu_0 \int_0^t C\left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{\max(t_0+1, s)}}\right) |U|_{X^s} dt' \\ &\quad + \varepsilon \nu_0 \int_0^t C\left(\frac{1}{h_{\min}}, |\underline{U}|_{L_t^\infty X^{\max(t_0+1, s)}}\right) |R(t')|_{X^s} dt', \end{aligned}$$

And using Gronwall's lemma, we get

$$\begin{aligned} |U|_{X^s} &\leq (\kappa_0 e^{\varepsilon \lambda_0 t} |U|_{X^s}|_{t=0} + \varepsilon \nu_0 \int_0^t C(|\underline{U}|_{L_t^\infty X^{\max(t_0+1, s)}}) |R(t')|_{X^s} dt') e^{\varepsilon \nu_0 C(|\underline{U}|) t} \\ &\leq \kappa_0 e^{\varepsilon \lambda_s t} |U|_{X^s}|_{t=0} + \varepsilon \nu_s \int_0^t |R(t')|_{X^s} dt', \end{aligned}$$

where $\nu_s, \lambda_s := C(T, \frac{1}{h_{\min}}, |\underline{U}|_{W_t^{1, \infty} X^{t_0}}, |\underline{U}|_{L_t^\infty X^{\max(t_0+1, s)}})$ □

2.3 Local well-posedness and stability

In this section we follow the method in the Chapter 7 of [Mét08] which uses a regularization method to prove the well-posedness of symmetrizable quasi-linear hyperbolic systems.

2.3.1 Well-posedness of the linearized systems

In this subsection we study the local well-posedness of the systems (2.2) linearized around a sufficiently regular state.

Theorem 2.3.1. *Let $s > d/2 + 1$, $h_{\min} > 0$ and $0 \leq \epsilon \leq 1$. Let also (G_1, G_2) be a couple of admissible Fourier multipliers. Let $\underline{U} \in W^{1,\infty}([0, T/\epsilon], X^{s-1}(\mathbb{R}^d)) \cap L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ be such that (2.4) is satisfied. Let $R \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ and $U_0 \in X^s(\mathbb{R}^d)$. The Cauchy problem*

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(\underline{U})[\partial_j U] = \epsilon R, \\ U|_{t=0} = U_0, \end{cases} \quad (2.20)$$

where $A_j(\underline{U})$ is defined by (2.3), has a unique solution in $C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$. Moreover, the solution satisfies the energy estimates (2.18).

As said previously, to prove this result, we use a regularization method.

Let $J_\alpha = (1 - \alpha\Delta)^{-1/2}$, $\alpha \in]0, 1]$, be a regularizing Fourier multiplier. We have the following properties :

- For $\alpha > 0$, J_α is a regularizing operator of order -1.
- For any $s \geq 0$, the family $\{J_\alpha, \alpha \in]0, 1]\}$ is uniformly bounded in $X^s(\mathbb{R}^d)$.
- For any $s \geq 0$, and for all $v \in X^s(\mathbb{R}^d)$, $J_\alpha v \rightarrow v$ in $X^s(\mathbb{R}^d)$ as $\alpha \rightarrow 0$.

We decompose the proof of Theorem 2.3.1 into several lemmas.

Lemma 2.3.2. *With the same assumptions as in Theorem 2.3.1, the Cauchy problem (2.20) has a weak solution $U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$.*

Démonstration. We consider the Cauchy problem

$$\begin{cases} \partial_t U_\alpha + \sum_{j=1}^d A_j(\underline{U})\partial_j J_\alpha U_\alpha = \epsilon R, \\ U_\alpha|_{t=0} = U_0. \end{cases} \quad (2.21)$$

For $j = 1, 2$ the operator $A_j(\underline{U})\partial_j J_\alpha$ is bounded in $X^s(\mathbb{R}^d)$, so that the Cauchy-Lipschitz theorem gives the existence of a solution $U_\alpha \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$. Because J_α is a Fourier multiplier of order 0 uniformly in α , and is bounded in $X^s(\mathbb{R}^d)$ uniformly in α , we can get the same energy estimates for (2.21) as the ones for (2.9). It implies that the sequence U_α is bounded in $L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$.

Now recall that $L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ is the dual of $L^1([0, T/\epsilon], Y^{-s}(\mathbb{R}^d))$. So by the weak* compactness of the closed balls of the dual of a normed space, there exists a subsequence, still denoted U_α , which converges weak* as $\alpha \rightarrow 0$ in $L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$

to an element U . Passing to the limit in the sense of distributions in (2.21) we get $U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ is a weak solution of (2.20).

It still remains to prove that $U|_{t=0} = U_0$ makes sense. Remark that from the equation $\partial_t U \in L^\infty([0, T/\epsilon], X^{s-1}(\mathbb{R}^d))$. So by the Sobolev embedding, $U \in C^0([0, T/\epsilon], X^{s-1}(\mathbb{R}^d))$, and it makes sense to take the trace at $t = 0$ of U in $X^{s-1}(\mathbb{R}^d)$. \square

Lemma 2.3.3. *Let $R \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ and suppose that $U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ satisfies the linearized system (2.20) with $U_0 = U|_{t=0} \in X^s(\mathbb{R}^d)$. Then $U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ and satisfies the energy estimates (2.18).*

Démonstration. Applying J_α to the system (2.20), we get

$$\partial_t J_\alpha U + \sum_{j=1}^d A_j(\underline{U})[\partial_j J_\alpha U] = \epsilon J_\alpha R - \sum_{j=1}^d [J_\alpha, A_j(\underline{U})][\partial_j U]. \quad (2.22)$$

We denote $R_{(\alpha)} = J_\alpha - \frac{1}{\epsilon} \sum_{j=1}^d [J_\alpha, A_j(\underline{U})][\partial_j U]$. We easily see that $R_{(\alpha)} \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$ using the same argument as for the proof of Lemma 2.2.7 and the fact that J_α is of order 0.

Moreover from the density of $X^{s+1}(\mathbb{R}^d)$ in $X^s(\mathbb{R}^d)$, we get

$$[J_\alpha, A_j(\underline{U})][\partial_j U] \rightarrow 0 \text{ in } L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d)),$$

as $\alpha \rightarrow 0$. It implies that $R_{(\alpha)} \rightarrow R$ in $L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$.

We know that $J_\alpha U$ is in $L^\infty([0, T/\epsilon], X^{s+1}(\mathbb{R}^d))$, so using the equation (2.22), we get $\partial_t J_\alpha U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$. The Sobolev embedding in dimension 1 gives $J_\alpha U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$. Using the energy estimates of order s (2.18) on $J_\alpha U - J_{\alpha'} U$ we get that $(J_\alpha U)_{\alpha \geq 0}$ is a Cauchy sequence in $C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ as $\alpha \rightarrow 0$. So $J_\alpha U$ converges in $C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$. But $J_\alpha U$ converges to U in $L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$. Thus $J_\alpha U \rightarrow U$ in $C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ as $\alpha \rightarrow 0$.

Using again the energy estimates of order s (2.18) but this time on $J_\alpha U$ and passing to the limit $\alpha \rightarrow 0$, we get that U satisfies the energy estimates of order s . \square

We now prove Theorem 2.3.1.

Démonstration. The two previous lemmas give the existence of a solution $U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ which satisfies the energy estimates (2.18).

It only remains to prove the uniqueness. Taking two solutions U_1 and U_2 with the same initial condition $U_0 \in X^s(\mathbb{R}^d)$, the difference $V = U_1 - U_2$ satisfies the system

$$\begin{cases} \partial_t V + \sum_{j=1}^d A_j(\underline{U}) \partial_j V = 0, \\ V|_{t=0} = 0. \end{cases}$$

The energy estimates on V give $V = 0$. □

2.3.2 Well-posedness of the non-linear systems

This subsection is dedicated to the proof of Theorem 2.1.6 which we recall here for the sake of clarity.

Theorem 2.3.4. *Let $s > d/2 + 1$, $h_{\min} > 0$ and $M > 0$. Let also (G_1, G_2) be a couple of admissible Fourier multipliers. There exist $T > 0$ and $C > 0$ such that for all $\epsilon \in [0, 1]$, $U_0 \in X^s(\mathbb{R}^d)$ with $|U_0|_{X^s} \leq M$ and satisfying (2.4), there exists a unique solution $U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ of the Cauchy problem*

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = 0, \\ U|_{t=0} = U_0. \end{cases} \quad (2.23)$$

Moreover $|U|_{L^\infty([0, T/\epsilon], X^s)} \leq C|U_0|_{X^s}$.

Démonstration. Consider the iterative scheme $U_0(t, x) = U_0(x)$ and for $n \in \mathbb{N}$

$$\begin{cases} \partial_t U_{n+1} + \sum_{j=1}^d A_j(U_n) \partial_j U_{n+1} = 0, \\ U_{n+1}|_{t=0} = U_0. \end{cases} \quad (2.24)$$

.

Lemma 2.3.5. *There exists $T > 0$ as in Theorem 2.3.4 such that the sequences U_n and $\partial_t U_n$ are well defined and are bounded in respectively $C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ and $C^0([0, T/\epsilon], X^{s-1}(\mathbb{R}^d))$.*

Démonstration. We prove by induction that there exists $C_1, C_2 > 0$ and $T > 0$ as in Theorem 2.3.4 for all $n \in \mathbb{N}$

$$\sup_{t \in [0, T/\epsilon]} |U_n|_{X^s} \leq C_1 |U_0|_{X^s}, \quad \sup_{t \in [0, T/\epsilon]} |\partial_t U_n|_{X^{s-1}} \leq C_2 |U_0|_{X^s}, \quad \inf_{t \in [0, T/\epsilon], x \in \mathbb{R}^d} (1 + \epsilon \zeta_n(t, x)) \geq h_{\min}/2.$$

By Theorem 2.3.1 U_{n+1} is well-defined and satisfies (2.18). Specifically, on the time interval $[0, T/\epsilon]$ we have

$$|U_{n+1}|_{X^s} \leq \kappa_0 e^{\epsilon \lambda_n t} |U_0|_{X^s}.$$

where $\lambda_n = C(T, |U_n|_{W_t^{1,\infty} X^{t_0}}, |U_n|_{L_t^\infty X^{\max(t_0+1, s)}})$ and $\kappa_0 = C(|U_0|_{X^{t_0}}|_{t=0})$. Also, using the equation and the product estimates of Proposition 2.6.1,

$$|\partial_t U_{n+1}|_{X^{s-1}} \leq \tilde{C}(|U_n|_{X^s}) |U_{n+1}|_{X^s}.$$

Moreover

$$\epsilon \zeta_{n+1}(t, x) = \epsilon \zeta_0(x) + \epsilon \int_0^t \partial_t \zeta_{n+1}(T, x) dt'.$$

But from the Sobolev embedding there exists $C_s > 0$ such that

$$|\partial_t \zeta_{n+1}(t, x)| \leq C_s |\partial_t \zeta|_{L_t^\infty X^{s-1}}.$$

So

$$1 + \epsilon \zeta_{n+1} \geq h_{\min} - TC_s |\partial_t \zeta|_{L_t^\infty X^{s-1}}.$$

Let $C_1 > \kappa_0$. Let C_2 be such that $\tilde{C}(C_1 |U_0|_{X^s}) C_1 \leq C_2$. And let T be sufficiently small so that, $\kappa_0 e^{\lambda T} \leq C_1$ where $\lambda_n \leq \lambda = C(T, C_1 |U_0|_{X^s}, C_2 |U_0|_{X^s})$, and $TC_s C_2 |U_0|_{X^s} \leq h_{\min}/2$. \square

Lemma 2.3.6. *The sequence U_n is a Cauchy sequence in $C^0([0, T/\epsilon], X^0(\mathbb{R}^d))$.*

Démonstration. Let $V_n := U_{n+1} - U_n$. For $n \geq 1$, it satisfies

$$\begin{cases} \partial_t V_n + \sum_{j=1}^d A_j(U_n) \partial_j V_n = \epsilon R_n, \\ V_{n+1}|_{t=0} = 0, \end{cases}$$

where

$$R_n = -\frac{1}{\epsilon} \sum_{j=1}^d (A_j(U_n) - A_j(U_{n-1})) \partial_j U_n.$$

But from the expression of A_j (2.3) and the uniform bounds of the sequence U_n (see Lemma 2.3.5) and the product estimates of Proposition 2.6.1 it is easy to see that there exists a constant $M > 0$ independent of n such that

$$|R_n|_{X^0} \leq M|V_{n-1}|_{X^0}.$$

And from the energy estimates (2.18) and the uniform bounds of U_n and $\partial_t U_n$ (see again Lemma 2.3.5), there exists a constant $M > 0$ independent of n such that

$$|V_n|_{X^0} \leq \epsilon M \int_0^t |V_{n-1}(t')|_{X^0} dt'.$$

So

$$|V_n|_{X^0} \leq \frac{M^n t^n}{n!} \sup_{t \in [0, T/\epsilon]} |V_0|_{X^0}.$$

Thus, the series $\sum V_n$ converges in $C^0([0, T/\epsilon], X^0(\mathbb{R}^d))$. \square

From Lemma 2.3.6, the sequence U_n converges in $C^0([0, T/\epsilon], X^0(\mathbb{R}^d))$. From Lemma 2.3.5, the sequence U_n is uniformly bounded in $C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$. So for any $s' < s$, U_n converges in $C^0([0, T/\epsilon], X^{s'}(\mathbb{R}^d))$. Take $s' > t_0 + 1$ and denote by U the limit. The sequences U_n , $\partial_t U_n$ and for $j = 1, 2$, $\partial_j U_n$ converge uniformly in C^0 to respectively U , $\partial_t U$ and $\partial_j U$. Hence U is solution to (2.2). Moreover, from Lemma 2.3.5, $U \in L^\infty([0, T/\epsilon], X^s(\mathbb{R}^d))$, $\partial_t U \in L^\infty([0, T/\epsilon], X^{s-1}(\mathbb{R}^d))$, and U satisfies the estimates of Theorem 2.3.4. So we can consider U as a solution of the linearized system (2.20) taking \underline{U} as U . The Theorem 2.3.1 gives $U \in C^0([0, T/\epsilon], X^s(\mathbb{R}^d))$ and its uniqueness as a solution of the Cauchy problem (2.23). This concludes the proof of Theorem 2.3.4. \square

2.3.3 Blow up criterion

From Theorem 2.3.4, one can define the maximal time existence $T^* > 0$ of the solution $U \in C^0([0, T^*/\epsilon], X^s(\mathbb{R}^d))$ of the Cauchy problem (2.23) associated to an initial condition $U_0 \in X^s(\mathbb{R}^d)$ such that (2.4) holds.

Proposition 2.3.7. *We have*

$$T^* < +\infty \implies \lim_{t \rightarrow T^*/\epsilon} \|U\|_{L^\infty([0, t], X^s)} = +\infty.$$

Démonstration. Suppose that there exists $M > 0$ such that

$$|U|_{L^\infty([0, T^*/\varepsilon], X^s)} = M.$$

Then from Theorem 2.3.4, there exists $T > 0$ such that for any $\beta > 0$, and $T_\beta = T^* - \beta$, the Cauchy problem (2.23) with initial condition $U(T_\beta)$ has a unique solution in $C^0([T_\beta/\varepsilon, T_1/\varepsilon], X^s(\mathbb{R}^d))$ with $T_1 = T_\beta + T > T^*$. Taking $\beta = T/2$, by uniqueness, U has an extension $\tilde{U} \in C^0([0, T_1/\varepsilon], X^s(\mathbb{R}^d))$ solution of the Cauchy problem (2.23). Thus, necessarily, $T^* = +\infty$. If not, it contradicts the definition of T^* . \square

2.4 Stability

In this section we prove the stability result of Proposition 2.1.7.

Proposition 2.4.1. *Let the assumptions of Theorem 2.3.4 be satisfied and assume also that there exists $\tilde{U} \in C^0([0, T/\varepsilon], X^s(\mathbb{R}^d))$ solution of*

$$\partial_t \tilde{U} + \sum_{j=1}^d A_j(\tilde{U}) \partial_j \tilde{U} = \tilde{R},$$

where $\tilde{R} \in L^\infty([0, T/\varepsilon], X^{s-1}(\mathbb{R}^d))$. Then, the error with respect to the solution $U \in C^0([0, T^/\varepsilon], X^s(\mathbb{R}^d))$ given by Theorem 2.3.4 and Proposition 2.3.7 satisfies for all $t \in [0, \min(T, T^*)/\varepsilon]$,*

$$|\mathbf{e}|_{L^\infty([0, t], X^{s-1})} \leq C \left(\frac{1}{h_{\min}}, |U|_{L^\infty([0, t], X^s)}, |\tilde{U}|_{L^\infty([0, t], X^s)} \right) (|\mathbf{e}|_{X^{s-1}}|_{t=0} + t |\tilde{R}|_{L^\infty([0, t], X^{s-1})}),$$

where $\mathbf{e} = U - \tilde{U}$.

Démonstration. We know that

$$\begin{cases} \partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = 0, \\ \partial_t \tilde{U} + \sum_{j=1}^d A_j(\tilde{U}) \partial_j \tilde{U} = \tilde{R}. \end{cases} \quad (2.25)$$

Subtracting both equations we get

$$\partial_t \mathbf{e} + \sum_{j=1}^d A_j(U) \partial_j \mathbf{e} = \epsilon F, \quad (2.26)$$

where

$$F = -\frac{1}{\epsilon} \tilde{R} - \frac{1}{\epsilon} \sum_{j=1}^d (A_j(U) - A_j(\tilde{U})) \partial_j \tilde{U}.$$

We can easily estimate F using the product estimates of Proposition 2.6.1 ($s - 1 > d/2$) :

$$|F|_{X^{s-1}} \leq \frac{1}{\epsilon} |\tilde{R}|_{X^{s-1}} + |\tilde{U}|_{X^s} |\mathbf{e}|_{X^{s-1}}.$$

We use the energy estimates of Proposition 2.2.6 on (2.26) to get

$$|\mathbf{e}|_{X^{s-1}} \leq \kappa_0 e^{\epsilon \lambda_{s-1} t} |\mathbf{e}|_{X^{s-1}}|_{t=0} + \nu_{s-1} \int_0^t |\tilde{R}(t')|_{X^{s-1}} dt' + \epsilon \nu_{s-1} \int_0^t |\tilde{U}|_{X^s} |\mathbf{e}|_{X^{s-1}} dt'.$$

Using Gronwall's lemma, we then have

$$|\mathbf{e}|_{X^{s-1}} \leq (\kappa_0 e^{\epsilon \lambda_{s-1} t} |\mathbf{e}|_{X^{s-1}}|_{t=0} + \nu_{s-1} t |\tilde{R}(t')|_{L^\infty([0,t], X^{s-1})} dt') e^{\epsilon \nu_{s-1} |\tilde{U}|_{L^\infty([0,t], X^s)} t}.$$

It only remains to see that using the equation on U of (2.25) and the product estimates of Proposition 2.6.1, we have for all times $t \in [0, T/\epsilon]$.

$$|\partial_t U|_{X^{s-1}} \leq C(|U|_{X^s}),$$

to get the result. □

2.5 Full justification of a class of Whitham-Boussinesq systems

In this section we prove the full justification of a class of Whitham-Boussinesq systems, Theorem 2.1.12.

Theorem 2.5.1. *Let $(\mu, \epsilon) \in [0, 1]^2$, $s > d/2$. Let $G_1^\mu := \sqrt{\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}}$. Let also G_2^μ (see Notation 2.1.9) be a Fourier multiplier of order 0 such that for all $\xi \in \mathbb{R}^d$, $|G_2^\mu(\xi)| \leq |G_1^\mu(\xi)|$,*

and satisfying for any $s' \geq 0$ and $w \in H^{s'+2}(\mathbb{R}^d)$,

$$|G_2^\mu[w] - w|_{H^{s'}} \lesssim \mu|w|_{H^{s'+2}}.$$

There exists $T > 0$ independent of $(\mu, \epsilon) \in [0, 1]^2$ such that for any $T' > 0$ and any classical solutions (ζ, ψ) of the water waves equations satisfying the non-cavitation assumption (2.4) and $U = (\zeta, \nabla\psi) \in C^0([0, T'/\epsilon], H^{s+4}(\mathbb{R}^d))$, there exists a unique classical solution $U_{\text{WB}} = (\zeta_{\text{WB}}, v_{\text{WB}}) \in C^0([0, T/\epsilon], X^{s+3}(\mathbb{R}^d))$ with initial data $(\zeta|_{t=0}, \nabla\psi|_{t=0})$ of the Whitham-Boussinesq systems (2.1) and one has for all times $t \in [0, \min(T, T')/\epsilon]$,

$$|\mathfrak{e}|_{L^\infty([0,t], X^s)} \leq \mu\epsilon t C\left(\frac{1}{h_{\min}}, |U|_{L^\infty([0,t], H^{s+4})}\right),$$

where $\mathfrak{e} = U - U_{\text{WB}}$.

Démonstration. It is important to remark first that the Fourier multipliers G_1^μ and G_2^μ are uniformly bounded in $\mu \in [0, 1]$ in the $X^s(\mathbb{R}^d)$ norms. Moreover $\mathcal{N}^0(G_1^\mu)$ and $\mathcal{N}^0(G_2^\mu)$ (see Definition 2.6.2) are also bounded uniformly in μ . So that when considering Fourier multipliers of the form G_1^μ and G_2^μ (see Notation 2.1.9), all the estimates proved in this paper holds uniformly in μ , including the energy estimates of Proposition 2.2.6. In particular, the existence time in Theorem 2.3.4 is independent of μ .

Now, from the embedding $H^{s+4}(\mathbb{R}^d) \subset X^{s+3}(\mathbb{R}^d)$ and the Theorem 2.3.4 we have the existence and uniqueness of U_{WB} with T independent of μ . From Proposition 2.1.10, we know that U satisfies

$$\partial_t U + \sum_{j=1}^d A_j(U) \partial_j U = \mu\epsilon R,$$

where $|R|_{H^s} \leq C\left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+4}}, |\nabla\psi|_{H^{s+4}}\right)$.

From the stability result of Proposition 2.4.1 and the control on $|U_{\text{WB}}|_{L^\infty([0, T/\epsilon], X^s)}$ by $|U_0|_{X^s}$ we have from Theorem 2.1.6, we know that for all times $t \in [0, \min(T, T')/\epsilon]$, one has

$$|\mathfrak{e}|_{L^\infty([0,t], X^s)} \leq \mu\epsilon t C(|U|_{L^\infty([0,t], X^{s+1})}) |R|_{L^\infty([0,t], X^s)}.$$

But

$$|R|_{L^\infty([0,t],X^s)} \leq |R|_{L^\infty([0,t],H^s)} \leq C\left(\frac{1}{h_{\min}}, |\zeta|_{H^{s+4}}, |\nabla\psi|_{H^{s+4}}\right).$$

Hence the result holds. \square

2.6 Technical Tools

Proposition 2.6.1. (*Product estimates*) Let $t_0 > d/2$, $s \geq -t_0$ and $f \in H^s \cap H^{t_0}(\mathbb{R}^d)$, $g \in H^s(\mathbb{R}^d)$. Then $fg \in H^s(\mathbb{R}^d)$ and

$$|fg|_{H^s} \lesssim |f|_{H^{\max(t_0,s)}} |g|_{H^s}$$

Démonstration. See Appendix B.1 in [Lan13b]. \square

Definition 2.6.2. For a Fourier multiplier F of order α ($\alpha \in \mathbb{R}$) we define the seminorm

$$\mathcal{N}^\alpha(F) = \sup_{\beta \in \mathbb{N}^d, |\beta| \leq 2+d+\lceil \frac{d}{2} \rceil} \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta|-\alpha} |\partial^\beta F(\xi)|.$$

Proposition 2.6.3. Let $t_0 > d/2$, $\alpha \geq 0$, $s \geq 0$ and $F \in \mathcal{S}^\alpha$. If $f \in H^{\alpha+s} \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{\alpha+s-1}(\mathbb{R}^d)$,

$$\|[F, f]g\|_{H^s} \leq \mathcal{N}^\alpha(F) \|f\|_{H^{\max(t_0+1, \alpha+s)}} \|g\|_{H^{\alpha+s-1}}.$$

Démonstration. See Appendix B.2 in [Lan13b] for a proof of this proposition in the case $s = 0$.

If $s > 0$, we define $\Lambda^s := (1 - \Delta)^{s/2}$. We have

$$\|[F, f]g\|_{H^s} = \|[F\Lambda^s, f]g - [\Lambda^s, f]F[g]\|_{L^2} \leq \|[F\Lambda^s, f]g\|_{L^2} + \|[\Lambda^s, f]F[g]\|_{L^2}.$$

Using the L^2 commutator estimates, knowing that $F\Lambda^s \in \mathcal{S}^{\alpha+s}$, we get the result. \square

Chapitre 3

Rigorous derivation of the Whitham equations from the water waves equations in the shallow water regime

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Abstract: We derive the Whitham equations from the water waves equations in the shallow water regime using two different methods, thus obtaining a direct and rigorous link between these two models. The first one is based on the construction of approximate Riemann invariants for a Whitham-Boussinesq system and is adapted to unidirectional waves. The second one is based on a generalisation of Birkhoff's normal form algorithm for almost smooth Hamiltonians and is adapted to bidirectional propagation. In both cases we clarify the improved accuracy on the fully dispersive Whitham model with respect to the long wave Korteweg-de Vries approximation.

3.1 Introduction

3.1.1 Motivations

In this paper we work on a unidirectional model for surface waves in coastal oceanography, the Whitham equation, introduced by Whitham in [Whi67] and [Whi74]. This equation, having the same dispersion relation as the general water waves model, is a full

dispersion modification of the Korteweg-de Vries equation (KdV). As such one expects an improved accuracy and range of validity as a model for surface waves compared to the long wave KdV approximation. More precisely, we anticipate that in the presence of weak nonlinearities, the Whitham equation stays close to the water waves equations, even if the dispersion effects are not small, as obtained in [Eme20] for the bidirectional full dispersion systems.

The Whitham model has weaker dispersive effects compared to the KdV model and thus allows for wavebreaking and Stokes waves of maximal amplitude to occur, as one would expect of a model for surface waves in coastal oceanography. This was the historical reason for which Whitham introduced it. A rigorous proof of the existence of the wavebreaking phenomenon has been established by Hur [Hur17] and Saut & Wang [SW20]. Ehrnström & Wahlén [EW19] and Truong *et al.* [TWW20] proved the existence of Stokes waves of maximal amplitude.

Besides [Hur17; SW20; EW19; TWW20] we emphasize some other works on the Whitham equation. Klein *et al.* [Kle+18] compared rigorously the solutions of the Whitham equation with those of the KdV equation and exhibited three different regimes of behaviour : scattering for small initial data, finite time blow-up and a KdV long wave regime. Ehrnström & Wang [EW20] proved an enhanced existence time for solutions of the Whitham equation associated with small initial data. Ehrnström & Kalisch [EK09] proved the existence of periodic traveling-waves solutions. Sanford *et al.* [San+14] established that large-amplitude periodic traveling-wave solutions are unstable, while those of small-amplitude are stable if their wavelength is long enough. Denoting by μ the shallow water parameter and by ε the nonlinearity parameter, Klein *et al.* [Kle+18] proved the validity of the Whitham equation in the KdV regime $\{\mu = \varepsilon\}$ by obtaining that the Whitham equation approximate the KdV equation at a precision of order ε^2 . Moldabayev *et al.* [MKD15] formally derived the Whitham equation from the Hamiltonian of the water waves equations in the unusual regime $\{\varepsilon = e^{-\frac{\kappa}{\mu^{\nu/2}}}\}$, where κ and ν are parameters numerically inferred from the solitary waves of the Whitham model. They also assumed the initial data of the water waves equations prepared to generate unidirectional waves. In this work we propose to expand the range of validity of the Whitham equations in the general shallow water regime, using two different methods.

The first method is based on an adaptation of the one used to derive the inviscid Burgers equations from the Nonlinear Shallow Water system using the Riemann invariants of the latter and requires, as in [MKD15], that initial data are prepared to generate

unidirectional waves. We show from it that the Whitham equation can be derived from the water waves equations at the order of precision $\mu\varepsilon$, which is the same order as for the Whitham-Boussinesq equations when considering general initial data (see [Eme20] for a rigorous derivation of the latter equations in the shallow water regime). From this derivation, we prove that solutions of the water waves equations, associated with well-prepared initial data, can be approximated in the shallow water regime, to within order $\mu\varepsilon t$ in a time scale of order ε^{-1} , by a right or left propagating wave solving a Whitham-type equation.

The second method does not need well-prepared initial data. It is based on a generalization of Birkhoff's normal form algorithm for almost smooth functions (see Definition 3.3.14) developed by Bambusi in [Bam20] to show that the solutions of the water waves equations can be approximated in the KdV regime, to within order ε in a time scale of order ε^{-1} , by two counter-propagating waves solving KdV-type equations, thus improving both the results obtained by Schneider and Wayne in [SW00], and those obtained by Bona, Colin and Lannes in [BCL05]. If we transpose Bambusi's result in the shallow water regime, we get a precision of order $\varepsilon^2 + (\mu^2 + \mu\varepsilon + \varepsilon^2)t$ in a time scale of order $(\max(\mu, \varepsilon))^{-1}$, which tells us that the two KdV-type equations are not appropriate to approximate the water waves equations when μ is not small. Bambusi's method uses the local structure of the KdV equation. We adapt it to the nonlocal Whitham equation and prove that the solutions of the water waves equations can be approximated in the shallow water regime, to within order $\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t$ in a time scale of order $(\max(\mu, \varepsilon))^{-1}$, by two counter-propagating waves solving Whitham-type equations. In addition, we express explicitly the loss of regularity needed to derive the two uncoupled Whitham-type equations solved by the counter-propagating waves.

We now compare the results from the first method and Bambusi's method for the KdV and the Whitham equations. In the table below, we summarize the different orders of precision for unidirectional and bidirectional waves :

	unidirectional waves on time scale ε^{-1}	bidirectional waves on time scale $(\max(\mu, \varepsilon))^{-1}$
KdV	$(\mu\varepsilon + \mu^2)t$	$\varepsilon^2 + (\mu\varepsilon + \mu^2 + \varepsilon^2)t$
Whitham	$(\mu\varepsilon)t$	$\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t$

One can refer to Section 7.3.2 in [Lan13b] for the order of precision of the KdV model in the case of unidirectional waves.

In the KdV regime ($\mu = \varepsilon$), we get the same order of precision for the two models in the same time scale.

In the general shallow water regime, if μ is not small, the KdV equations are not appropriate to approximate the water waves equations. In the presence of weak nonlinearities ($\varepsilon \ll 1$), the Whitham equations succeed in doing so. The latter model is also more robust as it allows a possible change of dynamics in the wave propagation, which could for example pass from the long wave regime ($\varepsilon \sim \mu$, $\mu \ll 1$) to the weakly nonlinear regime ($\varepsilon \ll 1$).

In [Bon+18], the authors formally derived a higher order KdV type equation adapted to unidirectional waves. We could probably make this derivation rigorous using the techniques of the Section 7.3.1 in [Lan13b] and get, in the KdV regime, a precision of order $\varepsilon^3 t$ in a time scale ε^{-1} . In [KMV17], the authors formally derived a higher order Whitham equation adapted to unidirectional waves. However, it is not clear that the techniques of the first method of the present paper are sufficient to derive rigorously such an equation with enough accuracy. Instead of the Whitham-Boussinesq systems, one would work on the Whitham-Green-Naghdi systems, which have been proved in [Eme20] to be consistent with the water waves equations in the shallow water regime, at the order of precision $\mu^2 \varepsilon$. But with the method presented here, one would get terms of order $\mu \varepsilon$ in the process, and thus only get the same order of precision as the Whitham equation.

In [Bam20], Bambusi rigorously derived a system of two counter-propagating higher order KdV-type equations which is precise in the KdV regime, at the order $\varepsilon^3 + \varepsilon^3 t$ in a time scale $(\max(\mu, \varepsilon))^{-1}$. However the generalisation of this method presented in this paper is not suitable to derive two counter-propagating higher order Whitham-type equations with enough accuracy in the shallow water regime. Indeed, one would get again terms of order $\mu \varepsilon$ in the process, and only get the same order of precision as a system of two counter-propagating Whitham-type equations.

3.1.2 Main results

The starting point of our study is the water waves equations :

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} (\partial_x \psi)^2 - \frac{\mu \varepsilon}{2} \frac{(\frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi + \varepsilon \partial_x \zeta \cdot \partial_x \psi)^2}{1 + \varepsilon^2 \mu (\partial_x \zeta)^2} = 0. \end{cases} \quad (3.1)$$

Here

- The free surface elevation is the graph of ζ , which is a function of time t and horizontal space $x \in \mathbb{R}$.
- $\psi(t, x)$ is the trace at the surface of the velocity potential.
- \mathcal{G}^μ is the Dirichlet-Neumann operator defined later in Definition 3.1.4.

Every variable and function in (3.2) is compared with physical characteristic parameters of the same dimension. Among those are the characteristic water depth H_0 , the characteristic wave amplitude a_{surf} and the characteristic wavelength L . From these comparisons appear two adimensional parameters of main importance :

- $\mu := \frac{H_0^2}{L^2}$: the shallow water parameter,
- $\varepsilon := \frac{a_{\text{surf}}}{H_0}$: the nonlinearity parameter.

We refer to [Lan13b] for details on the derivation of these equations.

In [Zak68], Zakharov proved that the water waves system (3.1) enjoys an Hamiltonian formulation. Let H_{WW} be defined by

$$H_{\text{WW}} := \frac{1}{2} \int_{\mathbb{R}} \zeta^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} \psi \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi \, dx. \quad (3.2)$$

Then (3.1) is equivalent to the Hamilton equations

$$\begin{cases} \partial_t \zeta = \delta_\psi H_{\text{WW}}, \\ \partial_t \psi = -\delta_\zeta H_{\text{WW}}, \end{cases}$$

where δ_ζ and δ_ψ are functional derivatives.

Before giving the main definitions of this section, here is one assumption maintained throughout this paper.

Hypothesis 3.1.1. *A fundamental hypothesis is the lower boundedness by a positive constant of the water depth (non-cavitation assumption) :*

$$\exists h_{\min} > 0, \forall x \in \mathbb{R}, \quad h := 1 + \varepsilon \zeta(t, x) \geq h_{\min}. \quad (3.3)$$

Moreover, we will always work in the shallow water regime which we define here.

Definition 3.1.2. *Let $\mu_{\max} > 0$, we define the shallow water regime $\mathcal{A} := \{(\mu, \varepsilon), \quad 0 \leq \mu \leq \mu_{\max}, \quad 0 \leq \varepsilon \leq 1\}$.*

In what follows we need some notations on the functional framework of this paper.

Notations 3.1.3. — For any $\alpha \geq 0$ we will respectively denote $H^\alpha(\mathbb{R})$, $W^{\alpha,1}(\mathbb{R})$ and $W^{\alpha,\infty}(\mathbb{R})$ the Sobolev spaces of order α in respectively $L^2(\mathbb{R})$, $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$. We will denote their associated norms by $|\cdot|_{H^\alpha}$, $|\cdot|_{W^{\alpha,1}}$ and $|\cdot|_{W^{\alpha,\infty}}$.

— For any $\alpha \geq 0$ we will denote $\dot{H}^{\alpha+1}(\mathbb{R}) := \{f \in L^2_{\text{loc}}(\mathbb{R}), \partial_x f \in H^\alpha(\mathbb{R})\}$ and $\dot{W}^{\alpha+1,1}(\mathbb{R}) := \{f \in L^1_{\text{loc}}(\mathbb{R}), \partial_x f \in W^{\alpha,1}(\mathbb{R})\}$ the Beppo-Levi spaces of order α . Their associated seminorms are respectively $|\cdot|_{\dot{H}^{\alpha+1}} := |\partial_x(\cdot)|_{H^\alpha}$ and $|\cdot|_{\dot{W}^{\alpha+1,1}} := |\partial_x(\cdot)|_{W^{\alpha,1}}$.

Now we define the Dirichlet-Neumann operator.

Definition 3.1.4. For all ζ and ψ sufficiently smooth, let ϕ be the velocity potential solving the elliptic problem

$$\begin{cases} (\mu\partial_x^2 + \partial_z^2)\phi = 0, \\ \phi|_{z=\varepsilon\zeta} = \psi, \quad \partial_z\phi|_{z=-1} = 0. \end{cases}$$

where $z \in (-1, \varepsilon\zeta)$ is the vertical space variable.

We define the Dirichlet-Neumann operator by the formula

$$\mathcal{G}^\mu[\varepsilon\zeta]\psi = \sqrt{1 + \varepsilon^2(\partial_x\zeta)^2}\partial_{\mathbf{n}^\mu}\phi|_{\varepsilon\zeta},$$

where \mathbf{n}^μ is the outward normal vector of the free surface $\varepsilon\zeta$. It depends on time and space.

\mathcal{G}^μ is linear in ψ and nonlinear in ζ . See [Lan13b] for a thorough study of this operator.

We also recall the definition of a Fourier multiplier.

Definition 3.1.5. Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be a tempered distribution, let \hat{u} be its Fourier transform. Let $F \in C^\infty(\mathbb{R})$ be a smooth function such that there exists $m \in \mathbb{Z}$ for which $\forall \beta \geq 0$, $|\partial_\xi^\beta F(\xi)| \leq \tilde{C}_\beta(1 + |\xi|)^{m-|\beta|}$, where $\tilde{C}_\beta > 0$ is a constant depending on β . Then the Fourier multiplier associated with F is denoted $F(D)$ (denoted F when no confusion is possible) and defined by the formula :

$$\widehat{F(D)[u]}(\xi) = F(\xi)\hat{u}(\xi).$$

The first method to rigorously justify the Whitham equations from the water waves equations gives the two following results.

Proposition 3.1.6. *Let $\mu_{\max} > 0$. Let F_μ be the Fourier multiplier defined by*

$$F_\mu = \sqrt{\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}}.$$

There exists $n \in \mathbb{N}^$ such that for any $(\mu, \varepsilon) \in \mathcal{A}$ and $\zeta \in L_t^\infty H_x^{\alpha+n}(\mathbb{R})$ and $\psi \in L_t^\infty \dot{H}_x^{\alpha+n}(\mathbb{R})$ solutions of the water waves equations (3.1) satisfying (3.3), there exists $R_1, R_2 \in H^\alpha(\mathbb{R})$ uniformly bounded in (μ, ε) such that the quantities*

$$u^+ = \frac{\sqrt{h}-1}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v], \quad u^- = -\frac{\sqrt{h}-1}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v], \quad (3.4)$$

where $v = F_\mu^2[\partial_x \psi]$, satisfy the equations

$$\begin{cases} \partial_t u^+ + (\varepsilon \frac{3u^+ + u^-}{2} + 1)F_\mu[\partial_x u^+] = \mu \varepsilon R_1, \\ \partial_t u^- + (\varepsilon \frac{u^+ + 3u^-}{2} - 1)F_\mu[\partial_x u^-] = \mu \varepsilon R_2. \end{cases}$$

Remark 3.1.7. *The quantities u^+ and u^- can almost be seen as Riemann invariants of the Whitham-Boussinesq equations*

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon \partial_x(\zeta)v + \varepsilon \zeta \partial_x v = 0, \\ \partial_t v + F_\mu^2[\partial_x \zeta] + \varepsilon v \partial_x v = 0, \end{cases} \quad (3.5)$$

where $v = F_\mu^2[\partial_x \psi] = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}[\partial_x \psi]$, for which we know the rigorous derivation from the water waves equations (see [Eme20]). This is the main idea to prove this result.

Using Theorem 4.16 and Theorem 4.18 of [Lan13b] (the Rayleigh-Taylor condition is always satisfied when the bottom is flat, see Subsection 4.3.5 in [Lan13b]) we get the following corollary.

Corollary 3.1.8. *For any $\alpha \geq 0$, there exists $n \in \mathbb{N}^*$ such that for any $(\mu, \varepsilon) \in \mathcal{A}$, we have what follows. Consider the Cauchy problem for the water wave problem (3.1) with initial conditions $(\zeta_0, \psi_0) \in H^{\alpha+n}(\mathbb{R}) \times \dot{H}^{\alpha+n+1}(\mathbb{R})$ satisfying the non-cavitation assumption (3.3), and denote by (ζ, ψ) the corresponding solution. Let $(u_{e,0}^+, u_{e,0}^-)$ be defined by the formulas (3.4) applied to (ζ_0, ψ_0) . There exists a unique solution (u_e^+, u_e^-) of the exact diagonalized*

system

$$\begin{cases} \partial_t u_e^+ + (\varepsilon \frac{3u_e^+ + u_e^-}{2} + 1) F_\mu [\partial_x u_e^+] = 0, \\ \partial_t u_e^- + (\varepsilon \frac{u_e^+ + 3u_e^-}{2} - 1) F_\mu [\partial_x u_e^-] = 0, \end{cases} \quad (3.6)$$

with initial conditions $(u_{e,0}^+, u_{e,0}^-)$, which satisfy the following property : denote by (ζ_c, ψ_c) the following quantities

$$\zeta_c = \frac{1}{\varepsilon} \left[\left(\frac{\varepsilon}{2} (u_e^+ - u_e^-) + 1 \right)^2 - 1 \right] \quad \text{and} \quad \psi_c = \int_0^x F_\mu^{-1} [u_e^+ + u_e^-] dx, \quad (3.7)$$

then for all times $t \in [0, \frac{T}{\varepsilon}]$, one has

$$|(\zeta - \zeta_c, \psi - \psi_c)|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq \mu \varepsilon C t,$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$.

Remark 3.1.9. The existence and uniqueness of the solution (u_e^+, u_e^-) of the exact diagonalized system (3.6) with initial conditions $(u_{e,0}^+, u_{e,0}^-)$ on the required time scale is given by Proposition 3.2.9.

To write the next result we need another notation.

Notation 3.1.10. Let $k \in \mathbb{N}$ and $l \in \mathbb{N}$. A function R is said to be of order $O(\mu^k \varepsilon^l)$, denoted $R = O(\mu^k \varepsilon^l)$, if divided by $\mu^k \varepsilon^l$ this function is uniformly bounded with respect to $(\mu, \varepsilon) \in \mathcal{A}$ in the Sobolev norms $|\cdot|_{H^\alpha}$, $\alpha \geq 0$.

Proposition 3.1.11. With the same hypotheses and notations as in Proposition 3.1.6, if $u^-(0) = O(\mu)$, then there exists $T_1 > 0$ such that for all times $t \in [0, \frac{T_1}{\varepsilon}]$, $u^-(t) = O(\mu)$. Moreover for these times, u^+ satisfies the Whitham equation up to a remainder term of order $O(\mu\varepsilon)$, i.e.

$$\partial_t u^+ + F_\mu [\partial_x u^+] + \frac{3\varepsilon}{2} u^+ \partial_x u^+ = O(\mu\varepsilon).$$

If instead $u^+(0) = O(\mu)$, then there exists $T_2 > 0$ such that for all times $t \in [0, \frac{T_2}{\varepsilon}]$, $u^+(t) = O(\mu)$. Moreover for these times, u^- satisfies the counter propagating Whitham equation up to remainder term of order $O(\mu\varepsilon)$, i.e.

$$\partial_t u^- - F_\mu [\partial_x u^-] + \frac{3\varepsilon}{2} u^- \partial_x u^- = O(\mu\varepsilon). \quad (3.8)$$

Proposition 3.1.12. *In Corollary 3.1.8, if $u_e^-(0) = O(\mu)$, then one can replace ζ_c and ψ_c by*

$$\zeta_{\text{Wh},+} = \frac{1}{\varepsilon} \left[\left(\frac{\varepsilon}{2} u^+ + 1 \right)^2 - 1 \right] \quad \text{and} \quad \psi_{\text{Wh},+} = \int_0^x F_\mu^{-1}[u^+] \, dx,$$

where $u^+ \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}))$ solves the exact Whitham equation

$$\partial_t u^+ + F_\mu[\partial_x u^+] + \frac{3\varepsilon}{2} u^+ \partial_x u^+ = 0,$$

and get for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$|(\zeta - \zeta_{\text{Wh},+}, \psi - \psi_{\text{Wh},+})|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(|u_e^-(0)|_{H^{\alpha+1}} + \mu \varepsilon t),$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$.

If instead $u_e^+(0) = O(\mu)$, then one can replace ζ_c and ψ_c by

$$\zeta_{\text{Wh},-} = \frac{1}{\varepsilon} \left[\left(\frac{\varepsilon}{2} u^- - 1 \right)^2 - 1 \right] \quad \text{and} \quad \psi_{\text{Wh},-} = \int_0^x F_\mu^{-1}[u^-] \, dx,$$

where $u^- \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}))$ solves the exact counter-propagating Whitham equation

$$\partial_t u^- - F_\mu[\partial_x u^-] + \frac{3\varepsilon}{2} u^- \partial_x u^- = 0,$$

and get for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$|(\zeta - \zeta_{\text{Wh},-}, \psi - \psi_{\text{Wh},-})|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(|u_e^+(0)|_{H^{\alpha+1}} + \mu \varepsilon t),$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$.

The second method is based on a generalisation of Birkhoff's normal form algorithm developed by Bambusi in [Bam20]. Its application to our case leads us to define some operator and transformations which we need to write our main results.

Definition 3.1.13. — *Let $\alpha \geq 0$. We define $\mathcal{T}_1 : H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R}) \rightarrow H^\alpha(\mathbb{R}) \times \dot{H}^{\alpha+1}(\mathbb{R})$ by the formula*

$$\mathcal{T}_1(\zeta, v) = \begin{pmatrix} \zeta \\ \int_0^x v(y) \, dy \end{pmatrix}.$$

— Let $\alpha \geq 0$. We define $\mathcal{T}_D : H^{\alpha+1}(\mathbb{R}) \times H^{\alpha+1}(\mathbb{R}) \rightarrow H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R})$ by the formula

$$\mathcal{T}_D(r, s) = \begin{pmatrix} r + s \\ \mathbb{F}_\mu^{-1}[r - s] \end{pmatrix}.$$

— We define $\partial^{-1} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ by the formula

$$\partial^{-1}u(y) = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(y - y_1)u(y_1) dy_1.$$

— Let $\alpha \geq 0$ be a positive integer. We define $\mathcal{T}_B : W^{\alpha+1,1}(\mathbb{R}) \times W^{\alpha+1,1}(\mathbb{R}) \rightarrow W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})$ by the formula

$$\mathcal{T}_B(r, s) = \begin{pmatrix} r + \frac{\varepsilon}{4}\partial_x(r)\partial^{-1}(s) + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}s^2 \\ s + \frac{\varepsilon}{4}\partial_x(s)\partial^{-1}(r) + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}r^2 \end{pmatrix}.$$

Theorem 3.1.14. *Let $\alpha \geq 0$ be a positive integer and $(\mu, \varepsilon) \in \mathcal{A}$. Let $r, s \in \mathcal{C}^1([0, \frac{T}{\varepsilon}], W^{\alpha+7,1}(\mathbb{R}))$ be solutions of the Whitham equations*

$$\begin{cases} \partial_t r + \mathbb{F}_\mu[\partial_x r] + \frac{3\varepsilon}{2}r\partial_x r = 0, \\ \partial_t s - \mathbb{F}_\mu[\partial_x s] - \frac{3\varepsilon}{2}s\partial_x s = 0. \end{cases} \quad (3.9)$$

Let also H_{WW} be the Hamiltonian of the water waves equations (3.2) and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the Poisson tensor associated with. Then the quantities

$$\begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} := \mathcal{T}_I(\mathcal{T}_D(\mathcal{T}_B(r, s))), \quad (3.10)$$

satisfy

$$\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} = J\nabla(H_{\text{WW}})(\zeta_{\text{Wh}}, \psi_{\text{Wh}}) + (\mu\varepsilon + \varepsilon^2)R, \quad \forall t \in [0, \frac{T}{\varepsilon}],$$

where $|R|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(\mu_{\max}, |r|_{W^{\alpha+7,1}}, |s|_{W^{\alpha+7,1}})$.

Remark 3.1.15. — The existence of $r, s \in \mathcal{C}^1([0, \frac{T}{\varepsilon}], W^{\alpha+6,1}(\mathbb{R}))$ solutions of system (3.9) is given by Lemma 3.3.23.

— The remainder R is uniformly bounded in (μ, ε) for times $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$ in

$H^\alpha(\mathbb{R}) \times \dot{H}^{\alpha+1}(\mathbb{R})$. The particular time interval $[0, \frac{T}{\max(\mu, \varepsilon)}]$ comes from the estimates of r and s in $W^{\alpha+7,1}(\mathbb{R})$ given by Lemma 3.3.23. Indeed, due to the dispersive effects one has a loss of decrease of order $O((1 + \mu t)^\theta)$ for any $\theta > 1/2$, in the Sobolev spaces $W^{\beta,1}(\mathbb{R})$ where $\beta \geq 0$.

- The second equation of (3.9) is different from (3.8). One has to replace the definition of u^- in Proposition 3.1.6 by $u^- = \frac{\sqrt{h-1}}{\varepsilon} - \frac{1}{2}F_\mu^{-1}[v]$ to get

$$\partial_t u^- - F_\mu[\partial_x u^-] - \frac{3\varepsilon}{2}u^- \partial_x u^- = \mu \varepsilon R,$$

when setting $u^+(0) = O(\mu)$.

- In Theorem 3.1.14 we express the loss of regularity needed to pass rigorously from the Whitham equations (3.9) to the water waves equations (3.1).

Using again Theorem 4.16 and Theorem 4.18 of [Lan13b], we get the following corollary.

Corollary 3.1.16. *There exists $n \in \mathbb{N}^*$ such that for any $\alpha \geq 0$ and any $(\mu, \varepsilon) \in \mathcal{A}$, we have what follows. Consider the Cauchy problem for the water wave problem (3.1) with initial conditions $(\zeta_0, \psi_0) \in W^{\alpha+n,1}(\mathbb{R}) \times \dot{W}^{\alpha+n+1,1}(\mathbb{R})$ satisfying the non-cavitation assumption (3.3), and denote by (ζ, ψ) the corresponding solution. There exists a solution (r, s) of the Hamiltonian system (3.9) which satisfies the following property : denote by $(\zeta_{\text{Wh}}, \psi_{\text{Wh}})$ the solution defined by (3.10), then for all times $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$, one has*

$$|(\zeta - \zeta_{\text{Wh}}, \psi - \psi_{\text{Wh}})|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(\varepsilon^2 + (\mu\varepsilon + \varepsilon^2)t),$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{W^{\alpha+n}}, |\psi_0|_{\dot{W}^{\alpha+n+1}})$. See (3.51) for the initial conditions (r_0, s_0) associated with (r, s) .

3.1.3 Outline

In Section 3.2 we prove Proposition 3.1.6 and Proposition 3.1.11. In Subection 3.2.1 we focus on the proof of Proposition 3.1.6 using symbolic calculus to diagonalize system (3.5) (see remark 3.1.7). In Subection 3.2.2 we prove Proposition 3.1.11 using a local well-posedness result (see Proposition 3.2.9) and a stability result (see Proposition 3.2.13) on the diagonalized system (3.6). Then we prove Proposition 3.1.12.

The Section 3.3 is dedicated to the proof of Theorem 3.1.14. In Subection 3.3.1 we do a first approximation of the water waves' Hamiltonian by the one of a specific

Whitham-Boussinesq similar to (3.5) (see Proposition 3.3.2). Then we do a simple change of unknowns separating the waves into a right and left front (see Proposition 3.3.5). In Subection 3.3.2 we apply the generalised Birkhoff’s algorithm for almost smooth functions to the Hamiltonian obtained in the previous subsection. It gives an explicit transformation (3.41) allowing to pass from the latter Hamiltonian to the one of a system composed of two decoupled Whitham equations (see Property 3.3.20 and remark 3.3.22) at the desired order of precision. In Subection 3.3.3 we prove the existence of solutions of the two decoupled Whitham equations in the suitable Sobolev spaces $W^{\alpha,1}(\mathbb{R})$ for every integer $\alpha \geq 0$ (see lemma 3.3.23). Then we prove an intermediate theorem (see Theorem 3.3.24) which describes the action of the transformation resulting from the generalised Birkhoff’s algorithm on the two decoupled Whitham equations. At the end, we prove Theorem 3.1.14 and corollary 3.1.16.

3.2 Derivation of the Whitham equations from the Riemann invariants of a Whitham-Boussinesq system

The goal of this section is to prove the Propositions 3.1.6 and 3.1.11.

We consider the following Whitham-Boussinesq equations.

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon \partial_x(\zeta)v + \varepsilon \zeta \partial_x v = 0, \\ \partial_t v + F_\mu^2[\partial_x \zeta] + \varepsilon v \partial_x v = 0, \end{cases} \quad (3.11)$$

where $v = F_\mu^2[\partial_x \psi] = \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}[\partial_x \psi]$. Using Proposition 1.15 of [Eme20] we easily get the following proposition.

Proposition 3.2.1. *Let $\mu_{\max} > 0$. There exists $n \in \mathbb{N}^*$ and $T > 0$ such that for all $\alpha \geq 0$ and $p = (\mu, \varepsilon) \in \mathcal{A}$ (see Definition 3.1.2), and for every solution $(\zeta, \psi) \in C([0, \frac{T}{\varepsilon}]; H^{\alpha+n}(\mathbb{R}) \times \dot{H}^{\alpha+n+1}(\mathbb{R}))$ to the water waves equations (3.1) one has*

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon \partial_x(\zeta)v = \mu \varepsilon R_1, \\ \partial_t v + F_\mu^2[\partial_x \zeta] + \varepsilon v \partial_x v = \mu \varepsilon R_2, \end{cases} \quad (3.12)$$

with $|R_1|_{H^\alpha}, |R_2|_{H^\alpha} \leq C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{\alpha+n}}, |\psi|_{\dot{H}^{\alpha+n+1}})$.

We say that the water waves equations are consistent with the Whitham-Boussinesq equations (3.11) at precision order $O(\mu\varepsilon)$.

Remark 3.2.2. We only chose a Whitham-Boussinesq system fitted for the computations. There exists a whole class of these systems. For example, one can add a regularizing Fourier multiplier such as F_μ^2 on the nonlinear terms of system (3.11) and still be precise at order $O(\mu\varepsilon)$. The system thus obtained is

$$\begin{cases} \partial_t \zeta + \partial_x v + \varepsilon F_\mu^2 \partial_x [\zeta v] = 0, \\ \partial_t v + F_\mu^2 [\partial_x \zeta] + \frac{\varepsilon}{2} F_\mu^2 [v \partial_x v] = 0. \end{cases} \quad (3.13)$$

In [DST19] the authors proved the local well-posedness of (3.13). The reasoning of this section is the same for any Whitham-Boussinesq system such as the latter.

The system (3.11) is the starting point of our reasoning. We will adapt the method used to derive the inviscid Burgers equations from the Nonlinear Shallow Water equations which use the Riemann invariants of the latter.

Throughout this section we will use Notation 3.1.10.

3.2.1 Diagonalization of the Whitham-Boussinesq equations

In this subsection we formally diagonalize the system (3.11) to get Proposition 3.1.6. Because of the consistency of the water waves equations with the system (3.11), we can take ζ and v solutions of the latter instead of taking solutions of the water waves equations. For our purpose, we just need to prove the following proposition.

Proposition 3.2.3. *Let $\mu_{\max} > 0$, there exists $n \in \mathbb{N}$ such that for any $\alpha \geq 0$ and $(\zeta, v) \in \mathcal{C}([0, \frac{T}{\varepsilon}]; H^{\alpha+n}(\mathbb{R}) \times H^{\alpha+n}(\mathbb{R}))$ solutions of system (3.11) satisfying the non-cavitation hypothesis (3.3), the quantities*

$$u^+ = 2 \frac{\sqrt{h} - 1}{\varepsilon} + F_\mu^{-1}[v], \quad u^- = -2 \frac{\sqrt{h} - 1}{\varepsilon} + F_\mu^{-1}[v], \quad (3.14)$$

where $h = 1 + \varepsilon \zeta$, satisfy

$$\begin{cases} \partial_t u^+ + (\varepsilon \frac{3u^+ + u^-}{4} + 1) F_\mu [\partial_x u^+] = O(\mu\varepsilon), \\ \partial_t u^- + (\varepsilon \frac{u^+ + 3u^-}{4} - 1) F_\mu [\partial_x u^-] = O(\mu\varepsilon). \end{cases} \quad (3.15)$$

where $F_\mu = \sqrt{\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}}$.

Before proving this result, we recall some definition and classical results we need.

Definition 3.2.4. Let $f(x) \in C^n(\mathbb{R})$ with $n \in \mathbb{N}^*$ be such that for any $0 \leq \beta \leq n$, $|\partial_x^\beta f(x)|_{L^\infty} \leq C_\beta$, where $C_\beta > 0$ is a constant depending on β . Let also $u(\xi) \in C^\infty(\mathbb{R})$ be a smooth function such that there exists $m \in \mathbb{Z}$ for which $\forall \beta \geq 0$, $|\partial_\xi^\beta u(\xi)| \leq \tilde{C}_\beta(1 + |\xi|)^{m-|\beta|}$, where $\tilde{C}_\beta > 0$ is a constant depending on β .

Then we define the operator $Op(fu)$ as follow : for any sufficiently regular functions v we have

$$Op(fu)v(x) := f(x)u(D)[v](x),$$

where $u(D)$ is the Fourier multiplier associated with the function u .

Remark 3.2.5. The function $F_\mu(\xi) = \sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}$ defining the Fourier multiplier F_μ satisfies the condition in the previous Definition 3.2.4. Idem for its inverse or its square.

Lemma 3.2.6. Let $\mu_{\max} > 0$. Let $(\varepsilon, \mu) \in \mathcal{A} = \{(\varepsilon, \mu), 0 \leq \varepsilon \leq 1, 0 \leq \mu \leq \mu_{\max}\}$. Let G_μ be a Fourier multiplier for which there exists $n \in \mathbb{N}^*$ such that for any $\beta \geq 0$ and $u \in H^{\beta+n}(\mathbb{R})$, $|(G_\mu - 1)[u]|_{H^\beta} \lesssim \mu|u|_{H^{\beta+n}}$.

— Let $\alpha \geq 0$. Let also f be a function in $H^{\alpha+n}(\mathbb{R})$. Then for any $u \in H^{\alpha+n}(\mathbb{R})$, the commutator $[G_\mu, f]u := G_\mu[fu] - fG_\mu[u]$ satisfies

$$|[G_\mu, f]u|_{H^\alpha} = |[G_\mu - 1, f]u|_{H^\alpha} \lesssim \mu|f|_{H^{\alpha+n}}|u|_{H^{\alpha+n}}.$$

— Let $\alpha \geq 0$. Let also g be a function such that $\frac{g-1}{\varepsilon}$ is uniformly bounded in ε in $H^{\alpha+n}(\mathbb{R})$. Then for any function $u \in H^{\alpha+n}(\mathbb{R})$,

$$|[G_\mu, g]u|_{H^\alpha} = |[G_\mu - 1, g - 1]u|_{H^\alpha} \lesssim \mu\varepsilon \left| \frac{g-1}{\varepsilon} \right|_{H^{\alpha+n}} |u|_{H^{\alpha+n}}.$$

Démonstration. Use product estimates 3.4.1. □

Property 3.2.7. The Fourier multiplier F_μ satisfies the assumption of Lemma 3.2.6 with $n = 2$. Idem for its inverse or its square. The functions h , \sqrt{h} and its inverse satisfy the assumption on g of the same lemma.

Démonstration. We first prove the property on F_μ , its inverse and its square. Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be either $x \mapsto \sqrt{\frac{\tanh(x)}{x}}$, $x \mapsto \sqrt{\frac{x}{\tanh(x)}}$ or $x \mapsto \frac{\tanh(x)}{x}$. We have $F(0) = 1$ and

$F'(0) = 0$. Moreover F'' is bounded in \mathbb{R}^+ . So that using the Taylor-Lagrange inequality, we get for any $x \in \mathbb{R}^+$,

$$|F(x) - 1| \leq |F''|_{\infty} x^2. \quad (3.16)$$

Let F be either F_{μ} , its inverse or its square. Let $\beta \geq 0$ and $u \in H^{\beta+2}(\mathbb{R})$. Let also $\Lambda^{\beta} := (1 - \Delta)^{\beta/2}$, it commutes with $F - 1$. Using (3.16), we have

$$|(F - 1)[u]|_{H^{\beta}} = |\Lambda^{\beta}(F - 1)[u]|_2 \leq |(F(\sqrt{\mu}|\xi|) - 1)\widehat{\Lambda^{\beta}u}|_2 \lesssim \mu|\xi|^2|\widehat{\Lambda^{\beta}u}|_2 \lesssim \mu|u|_{H^{\beta+2}},$$

which is the result.

The fact that the function h satisfies the assumption on g is obvious.

To prove the point on \sqrt{h} , we just need to use composition estimates 3.4.2 with $G(x) = \sqrt{1+x} - 1$.

We now deal with $\frac{1}{\sqrt{h}}$. We remark that $\frac{1}{\sqrt{h}} - 1 = \frac{1-\sqrt{h}}{1+\sqrt{h}-1}$. Then we use quotient estimates 3.4.3 ($\sqrt{h} \geq \sqrt{h_{\min}}$ because ζ satisfies the non-cavitation hypothesis (3.3)). We get

$$\left| \frac{1}{\sqrt{h}} - 1 \right|_{H^{\alpha}} \leq C \left(\frac{1}{\sqrt{h_{\min}}}, |\sqrt{h} - 1|_{H^{\max(t_0, \alpha)}} \right) |\sqrt{h} - 1|_{H^{\alpha}}.$$

□

We now prove Proposition 3.2.3.

Démonstration. We can write system (3.11) under the form

$$\partial_t U + A(U) \partial_x U = 0,$$

where $U := \begin{pmatrix} \zeta \\ v \end{pmatrix}$ and $A(U) := \begin{pmatrix} \varepsilon v & h \\ F_{\mu}^2[\circ] & \varepsilon v \end{pmatrix}$, with $h := 1 + \varepsilon \zeta$.

The symbol of operator $A(U)$ is $A(U, \xi) = \begin{pmatrix} \varepsilon v & h \\ F_{\mu}^2(\xi) & \varepsilon v \end{pmatrix}$, $\xi \in \mathbb{R}$ being the frequency variable. So that for any smooth functions $W = (w_1, w_2)^T$, for all $x \in \mathbb{R}$ we have

$$A(U)W(t, x) = \text{Op}(A(U, \xi))W(t, x) = \begin{pmatrix} \varepsilon v(t, x)w_1(t, x) + h(t, x)w_2(t, x) \\ F_{\mu}^2[w_1](t, x) + \varepsilon v(t, x)w_2(t, x) \end{pmatrix}.$$

Now, we diagonalize $A(U, \xi)$.

$$\det(A(U, \xi) - \lambda I_2) = (\varepsilon v - \lambda)^2 - F_\mu^2(\xi)h,$$

so that

$$A(U, \xi) = PDP^{-1},$$

with

$$P = \begin{pmatrix} \frac{\sqrt{h}}{F_\mu(\xi)} & -\frac{\sqrt{h}}{F_\mu(\xi)} \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{F_\mu(\xi)}{2\sqrt{h}} & \frac{1}{2} \\ -\frac{F_\mu(\xi)}{2\sqrt{h}} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \varepsilon v + F_\mu(\xi)\sqrt{h} & 0 \\ 0 & \varepsilon v - F_\mu(\xi)\sqrt{h} \end{pmatrix}.$$

Based on the symbolic calculus we apply $F_\mu^{-1} \text{Op}(P^{-1})$ to the system :

$$\begin{aligned} F_\mu^{-1} \text{Op}(P^{-1}) \partial_t U + F_\mu^{-1} \text{Op}(P^{-1}) \text{Op}(A(U, \xi)) \text{Op}(PP^{-1}) \partial_x U &= 0, \\ \iff F_\mu^{-1} \text{Op}(P^{-1}) \partial_t U + F_\mu^{-1} \text{Op}(D) F_\mu F_\mu^{-1} \text{Op}(P^{-1}) \partial_x U &= R_1, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} R_1 &= F_\mu^{-1} (\text{Op}(P^{-1} A(U, \xi)) - \text{Op}(P^{-1}) \text{Op}(A(U, \xi))) \partial_x U \\ &\quad + F_\mu^{-1} \text{Op}(P^{-1} A(U, \xi)) (\text{Op}(P) \text{Op}(P^{-1}) - \text{Op}(PP^{-1})) \partial_x U \\ &\quad + F_\mu^{-1} (\text{Op}(D) - \text{Op}(P^{-1} A(U, \xi)) \text{Op}(P)) \text{Op}(P^{-1}) \partial_x U. \end{aligned}$$

We remark that for each operators F_μ^{-1} , ∂_x , $\text{Op}(P^{-1} A(U, \xi))$ and $\text{Op}(P^{-1})$ there exists $n \in \mathbb{N}$ such that for any $\beta \geq 0$, the operator is bounded from $H^{\beta+n}(\mathbb{R})$ to $H^\beta(\mathbb{R})$.

Lemma 3.2.8. R_1 is of order $O(\mu\varepsilon)$.

Démonstration. There are three terms in R_1 . We deal with each of those separately.

— First term : for any $W = (w_1, w_2) \in H^\beta(\mathbb{R}) \times H^\beta(\mathbb{R})$ with $\beta \geq 0$,

$$(\text{Op}(P^{-1} A(U, \xi)) - \text{Op}(P^{-1}) \text{Op}(A(U, \xi))) W = \frac{1}{2\sqrt{h}} \begin{pmatrix} -[F_\mu, \varepsilon v] w_1 - [F_\mu, h] w_2 \\ [F_\mu, \varepsilon v] w_1 + [F_\mu, h] w_2 \end{pmatrix}.$$

Using Lemma 3.2.6 and $h \geq h_{\min}$, we have

$$\frac{1}{2\sqrt{h}}[\mathbb{F}_\mu, \varepsilon v]w_1 = O(\mu\varepsilon) \quad \text{and} \quad \frac{1}{2\sqrt{h}}[\mathbb{F}_\mu, h]w_2 = O(\mu\varepsilon).$$

— Second term : for any $W = (w_1, w_2) \in H^\beta(\mathbb{R}) \times H^\beta(\mathbb{R})$ with $\beta \geq 0$,

$$(\text{Op}(P) \text{Op}(P^{-1}) - I_d)W = \begin{pmatrix} \sqrt{h}[\mathbb{F}_\mu^{-1}, \frac{1}{\sqrt{h}}]\mathbb{F}_\mu[w_1] \\ w_2 \end{pmatrix}.$$

Using Lemma 3.2.6 and the boundedness of \mathbb{F}_μ and \sqrt{h} we have

$$\sqrt{h}[\mathbb{F}_\mu^{-1}, \frac{1}{\sqrt{h}}]\mathbb{F}_\mu[w_1] = O(\mu\varepsilon).$$

— Third term : for any $W = (w_1, w_2) \in H^\beta(\mathbb{R}) \times H^\beta(\mathbb{R})$ with $\beta \geq 0$,

$$\begin{aligned} & (\text{Op}(D) - \text{Op}(P^{-1}A(U, \xi)) \text{Op}(P))W \\ &= \begin{pmatrix} -\frac{\varepsilon v}{2\sqrt{h}}[\mathbb{F}_\mu - 1, \sqrt{h}]\mathbb{F}_\mu^{-1}[w_1 - w_2] - \frac{1}{2}[\mathbb{F}_\mu^2, \sqrt{h}]\mathbb{F}_\mu^{-1}[w_1 - w_2] \\ \frac{\varepsilon v}{2\sqrt{h}}[\mathbb{F}_\mu - 1, \sqrt{h}]\mathbb{F}_\mu^{-1}[w_1 - w_2] - \frac{1}{2}[\mathbb{F}_\mu^2, \sqrt{h}]\mathbb{F}_\mu^{-1}[w_1 - w_2] \end{pmatrix} \end{aligned}$$

Using Lemma 3.2.6, $h \geq h_{\min}$ and the boundedness of \mathbb{F}_μ^{-1} , we have

$$\begin{cases} \frac{\varepsilon v}{2\sqrt{h}}[\mathbb{F}_\mu - 1, \sqrt{h}]\mathbb{F}_\mu^{-1}[w_1 - w_2] = O(\mu\varepsilon), \\ \frac{1}{2}[\mathbb{F}_\mu^2, \sqrt{h}]\mathbb{F}_\mu^{-1}[w_1 - w_2] = O(\mu\varepsilon). \end{cases}$$

□

We continue the proof of Proposition 3.2.3. We compute $\mathbb{F}_\mu^{-1} \text{Op}(P^{-1})\partial_t U$:

$$\mathbb{F}_\mu^{-1} \text{Op}(P^{-1})\partial_t U = \mathbb{F}_\mu^{-1} \begin{pmatrix} \frac{1}{2\sqrt{h}}\mathbb{F}_\mu[\partial_t \zeta] + \frac{\partial_t v}{2} \\ -\frac{1}{2\sqrt{h}}\mathbb{F}_\mu[\partial_t \zeta] + \frac{\partial_t v}{2} \end{pmatrix} = \begin{pmatrix} \partial_t(\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}\mathbb{F}_\mu^{-1}[v]) \\ \partial_t(-\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}\mathbb{F}_\mu^{-1}[v]) \end{pmatrix} + R_2,$$

where $R_2 = \mathbb{F}_\mu^{-1} \begin{pmatrix} -[\mathbb{F}_\mu, \frac{1}{2\sqrt{h}}]\partial_t \zeta \\ [\mathbb{F}_\mu, \frac{1}{2\sqrt{h}}]\partial_t \zeta \end{pmatrix}$. Here, to prove that R_2 is of order $O(\mu\varepsilon)$, in addition of Lemma 3.2.6 we need a control of $|\partial_t \zeta|_{H^\beta}$ for any $\beta \geq 0$. We get it using the first equation of (3.11) and product estimates 3.4.1.

The same computations for $F_\mu^{-1} \text{Op}(P^{-1})\partial_x U$ gives

$$F_\mu^{-1} \text{Op}(P^{-1})\partial_x U = \begin{pmatrix} \partial_x(\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) \\ \partial_x(-\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) \end{pmatrix} + R_3,$$

where $R_3 = F_\mu^{-1} \begin{pmatrix} -[F_\mu, \frac{1}{2\sqrt{h}}]\partial_x \zeta \\ [F_\mu, \frac{1}{2\sqrt{h}}]\partial_x \zeta \end{pmatrix}$ is of order $O(\mu\varepsilon)$.

So that

$$\begin{aligned} & F_\mu^{-1} \text{Op}(D)F_\mu F_\mu^{-1} \text{Op}(P^{-1})\partial_x U \\ &= \begin{pmatrix} \varepsilon F_\mu^{-1}[vF_\mu[\circ]] + F_\mu^{-1}[\sqrt{h}F_\mu^2[\circ]] & 0 \\ 0 & \varepsilon F_\mu^{-1}[vF_\mu[\circ]] - F_\mu^{-1}[\sqrt{h}F_\mu^2[\circ]] \end{pmatrix} \\ & \quad \times \begin{pmatrix} \partial_x(\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) \\ \partial_x(-\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) \end{pmatrix} + R_4, \end{aligned} \quad (3.18)$$

where $R_4 = F_\mu^{-1} \text{Op}(D)F_\mu[R_3] = O(\mu\varepsilon)$ because of the boundedness of the operators F_μ^{-1} , $\text{Op}(D)$ and F_μ .

Hence, we get

$$\begin{cases} \partial_t(\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) + (\varepsilon F_\mu^{-1}[v\circ] + F_\mu^{-1}[\sqrt{h}F_\mu[\circ]])F_\mu \partial_x(\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) = R_5, \\ \partial_t(-\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) + (\varepsilon F_\mu^{-1}[v\circ] - F_\mu^{-1}[\sqrt{h}F_\mu[\circ]])F_\mu \partial_x(-\frac{1}{\varepsilon}\sqrt{h} + \frac{1}{2}F_\mu^{-1}[v]) = R_6, \end{cases}$$

where R_5 and R_6 are combinations of R_1, R_2, R_3, R_4 . They are of order $O(\mu\varepsilon)$. See also that $\frac{1}{\varepsilon}\partial_x\sqrt{h}$ and $\frac{1}{\varepsilon}\partial_x\sqrt{h}$ are of order $O(1)$

But by Lemma 3.2.6, we know that for any $w \in H^\beta(\mathbb{R})$ with $\beta \geq 0$, we have

$$[F_\mu, \sqrt{h}]w = O(\mu\varepsilon) \quad \text{and} \quad [F_\mu^{-1}, v]w = O(\mu).$$

So

$$\begin{cases} \partial_t(\frac{\sqrt{h-1}}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v]) + (\varepsilon F_\mu^{-1}[v] + \sqrt{h})F_\mu \partial_x(\frac{\sqrt{h-1}}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v]) = R_7, \\ \partial_t(-\frac{\sqrt{h-1}}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v]) + (\varepsilon F_\mu^{-1}[v] - \sqrt{h})F_\mu \partial_x(-\frac{\sqrt{h-1}}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v]) = R_8, \end{cases}$$

where R_7 and R_8 are of order $O(\mu\varepsilon)$.

We now denote

$$u^+ = \frac{\sqrt{h} - 1}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v], \quad u^- = -\frac{\sqrt{h} - 1}{\varepsilon} + \frac{1}{2}F_\mu^{-1}[v].$$

We get from (3.17) and the above :

$$\begin{cases} \partial_t u^+ + (\varepsilon \frac{3u^+ + u^-}{2} + 1)F_\mu[\partial_x u^+] = O(\mu\varepsilon), \\ \partial_t u^- + (\varepsilon \frac{u^+ + 3u^-}{2} - 1)F_\mu[\partial_x u^-] = O(\mu\varepsilon). \end{cases}$$

□

3.2.2 From the diagonalized Whitham-Boussinesq equations to the Whitham equations

In this subsection we prove Proposition 3.1.11.

To prove the first part of the proposition, i.e. there exists a time $T > 0$ such that for all times $t \in [0, \frac{T}{\varepsilon}]$, $u^-(t, \cdot)$ and $u^+(t, \cdot)$ are of order $O(\mu)$, we need local well-posedness and stability results on the exact diagonalized system

$$\begin{cases} \partial_t u_e^+ + (\varepsilon \frac{3u_e^+ + u_e^-}{2} + 1)F_\mu[\partial_x u_e^+] = 0, \\ \partial_t u_e^- + (\varepsilon \frac{u_e^+ + 3u_e^-}{2} - 1)F_\mu[\partial_x u_e^-] = 0. \end{cases} \quad (3.19)$$

We begin by proving the local well-posedness of system (3.19).

Proposition 3.2.9. *(local existence) Let $(\mu, \varepsilon) \in \mathcal{A}$ (see Definition 3.1.2). Let $1/2 < t_0 \leq 1$, $\alpha \geq t_0 + 1$ and $u_{0,e}^+, u_{0,e}^- \in H^\alpha(\mathbb{R})$. Then there exists a time $T > 0$ such that system (3.19) admits a unique solution $(u_e^+, u_e^-) \in \mathcal{C}([0, \frac{T}{\varepsilon}]; H^\alpha(\mathbb{R})^2)$ with initial conditions $(u_{0,e}^+, u_{0,e}^-)$.*

Moreover $\frac{1}{T}$ and $\sup_{0 \leq t \leq \frac{T}{\varepsilon}} |u_e^+|_{H^\alpha} + |u_e^-|_{H^\alpha}$ can be estimated by $|u_{0,e}^+|_{H^\alpha}, |u_{0,e}^-|_{H^\alpha}, t_0, \mu_{\max}$ (and are independent of the parameters $(\mu, \varepsilon) \in \mathcal{A}$).

Démonstration. System (3.19) is very similar to symmetric quasilinear hyperbolic systems. The only difference is the presence of the non-local operator F_μ . The well-posedness of such systems relies on the energy estimates (a priori estimates). We will establish them here. For the rest of the proof, one can follow the one in [Mét08] where the author uses a method of regularisation suitable for our system .

First, remark that if we define $U_\varepsilon(t) := \begin{pmatrix} u_\varepsilon^+(t) \\ u_\varepsilon^-(t) \end{pmatrix}$ and

$$A(U_\varepsilon) := \begin{pmatrix} (\varepsilon \frac{3u_\varepsilon^+ + u_\varepsilon^-}{2} + 1) & 0 \\ 0 & (\varepsilon \frac{u_\varepsilon^+ + 3u_\varepsilon^-}{2} - 1) \end{pmatrix}. \quad (3.20)$$

Then system (3.19) can be written under the form

$$\partial_t U_\varepsilon + A(U_\varepsilon) F_\mu[\partial_x U_\varepsilon] = 0. \quad (3.21)$$

It justifies the following framework for energy estimates. Let $\beta \geq 0$, $R \in L^\infty([0, \frac{T}{\varepsilon}]; H^\beta(\mathbb{R}^2))$ and $\underline{U} \in L^\infty([0, \frac{T}{\varepsilon}]; H^{\max(\beta, t_0+1)}(\mathbb{R}^2))$. Let $U \in \mathcal{C}^0([0, \frac{T}{\varepsilon}]; H^\beta(\mathbb{R}^2)) \cap \mathcal{C}^1([0, \frac{T}{\varepsilon}]; H^{\beta-1}(\mathbb{R}^2))$ solves the equation

$$\partial_t U + A(\underline{U}) F_\mu[\partial_x U] = \varepsilon R. \quad (3.22)$$

Lemma 3.2.10. (*Energy estimates of order 0*) For all times $t \in [0, \frac{T}{\varepsilon}]$,

$$|U(t)|_2 \leq e^{\varepsilon \delta_0 t} |U(0)|_2 + \varepsilon \gamma_0 \int_0^t |R(t')|_2 dt',$$

where $\delta_0, \gamma_0 = C(T, |\underline{U}|_{L^\infty([0, \frac{T}{\varepsilon}]; H^{t_0+1})})$

Démonstration. In what follows we will write $(\cdot, \cdot)_2$ the scalar product in $L^2(\mathbb{R})^2$. Using the fact that U solves equation (3.22) we get

$$\frac{1}{2} \frac{d}{dt} |U|_2^2 = (\partial_t U, U)_2 = -(A(\underline{U}) F_\mu[\partial_x U], U)_2 + \varepsilon (R, U)_2.$$

But $A(\underline{U})$ is symmetric, so

$$\begin{aligned} (A(\underline{U}) F_\mu[\partial_x U], U)_2 &= (\partial_x F_\mu[U], A(\underline{U}) U)_2 = -(F_\mu[U], (\partial_x A(\underline{U})) U)_2 - (U, F_\mu[A(\underline{U}) \partial_x U])_2 \\ &= -(F_\mu[U], (\partial_x A(\underline{U})) U)_2 - (A(\underline{U}) F_\mu[\partial_x U], U)_2 - ([F_\mu, A(\underline{U})] \partial_x U, U)_2, \end{aligned}$$

where $\partial_x A(\underline{U}) = \begin{pmatrix} \varepsilon \frac{3\partial_x u_1 + \partial_x u_2}{2} & 0 \\ 0 & \varepsilon \frac{\partial_x u_1 + 3\partial_x u_2}{2} \end{pmatrix}$ and

$$[\mathbf{F}_\mu, A(\underline{U})] = \begin{pmatrix} [\mathbf{F}_\mu, \varepsilon \frac{3u_1 + u_2}{2} + 1] & 0 \\ 0 & [\mathbf{F}_\mu, \varepsilon \frac{u_1 + 3u_2}{2} - 1] \end{pmatrix} = \varepsilon \begin{pmatrix} [\mathbf{F}_\mu, \frac{3u_1 + u_2}{2}] & 0 \\ 0 & [\mathbf{F}_\mu, \frac{u_1 + 3u_2}{2}] \end{pmatrix}.$$

Hence

$$\frac{1}{2} \frac{d}{dt} |U|_2^2 = \frac{1}{2} (\mathbf{F}_\mu[U], (\partial_x A(\underline{U}))U)_2 + \frac{1}{2} ([\mathbf{F}_\mu, A(\underline{U})] \partial_x U, U)_2 + \varepsilon (R, U)_2. \quad (3.23)$$

Then using Cauchy-Schwarz inequality we get

$$\begin{aligned} |U|_2 \frac{d}{dt} |U|_2 &\leq \frac{1}{2} |\partial_x(A(\underline{U}))U|_2 |U|_2 + \frac{1}{2} |[\mathbf{F}_\mu, A(\underline{U})] \partial_x U|_2 |U|_2 + \varepsilon |R|_2 |U|_2 \\ \iff \frac{d}{dt} |U|_2 &\leq \frac{1}{2} |\partial_x(A(\underline{U}))U|_2 + \frac{1}{2} |[\mathbf{F}_\mu, A(\underline{U})] \partial_x U|_2 + \varepsilon |R|_2. \end{aligned}$$

But by product estimates 3.4.1 we have

$$|\partial_x(A(\underline{U}))U|_2 \leq \varepsilon C(|\underline{U}|_{L_t^\infty H_x^{t_0+1}}) |U|_2. \quad (3.24)$$

Moreover by commutator estimates 3.4.5 with \mathbf{F}_μ of order 0 and $\mathcal{N}^0(\mathbf{F}_\mu) \lesssim 1$ (see Definition 3.4.4), we also have

$$|[\mathbf{F}_\mu, A(\underline{U})] \partial_x U|_2 \leq \varepsilon C(|\underline{U}|_{L_t^\infty H_x^{t_0+1}}) |U|_2. \quad (3.25)$$

So that

$$\frac{d}{dt} |U|_2 \leq \varepsilon C(|\underline{U}|_{L_t^\infty H_x^{t_0+1}}) |U|_2 + \varepsilon |R|_2 \leq \varepsilon \delta_0 |U|_2 + \varepsilon |R|_2,$$

where $\delta_0 = C(|\underline{U}|_{L_t^\infty H_x^{t_0+1}})$.

We integrate this inequality in time

$$\begin{aligned} |U|_2 &\leq e^{\varepsilon \delta_0 t} |U(0)|_2 + \varepsilon \int_0^t e^{\varepsilon \delta_0 (t-t')} |R|_2 dt' \\ &\leq e^{\varepsilon \delta_0 t} |U(0)|_2 + \varepsilon e^{\delta_0 T} \int_0^t |R|_2 dt'. \end{aligned}$$

It is the thesis with $\gamma_0 = e^{\delta_0 T}$.

□

Using this lemma, we get the energy estimates of order β .

Lemma 3.2.11. (Energy estimates of order β) For all times $t \in [0, \frac{T}{\varepsilon}]$,

$$|U(t)|_{H^\beta} \leq e^{\varepsilon\delta_\beta t} |U(0)|_{H^\beta} + \varepsilon\gamma_\beta \int_0^t |R(t')|_{H^\beta} dt',$$

where $\delta_\beta, \gamma_\beta = C(T, |\underline{U}|_{L^\infty([0, \frac{T}{\varepsilon}]; H^{\max(\beta, t_0+1)})})$.

Démonstration. Let $\Lambda^\beta := (1 - \partial_x^2)^{\beta/2}$. Then $|\Lambda^\beta U|_2$ is equivalent to $|U|_{H^\beta}$. Applying Λ^β to (3.22) we have

$$\begin{aligned} \Lambda^\beta \partial_t U + \Lambda^\beta [A(\underline{U})F_\mu[\partial_x U]] &= \varepsilon\Lambda^\beta R \\ \iff \partial_t \Lambda^\beta U + A(\underline{U})F_\mu[\partial_x \Lambda^\beta U] &= \varepsilon\Lambda^\beta R + \varepsilon B_\beta(\underline{U})F_\mu[\partial_x U], \end{aligned}$$

where

$$B_\beta(\underline{U}) := \frac{1}{\varepsilon} \begin{pmatrix} [\Lambda^\beta, \varepsilon \frac{3u_1+u_2}{2} + 1] & 0 \\ 0 & [\Lambda^\beta, \varepsilon \frac{u_1+3u_2}{2} - 1] \end{pmatrix} = \begin{pmatrix} [\Lambda^\beta, \frac{3u_1+u_2}{2}] & 0 \\ 0 & [\Lambda^\beta, \frac{u_1+3u_2}{2}] \end{pmatrix}.$$

Using commutator estimates 3.4.5 and the boundedness of F_μ we get

$$|B_\beta(\underline{U})F_\mu[\partial_x U]|_2 \lesssim |\underline{U}|_{L_t^\infty H_x^{\max(t_0+1, \beta)}} |U|_{H^\beta}.$$

Then we use the energy estimates of order 0 (see Lemma 3.2.10) to get

$$\begin{aligned} |U|_{H^\beta} &\leq e^{\varepsilon\delta_0 t} |U(0)|_{H^\beta} + \varepsilon\gamma_0 \int_0^t |R|_{H^\beta} + \varepsilon\gamma_0 \int_0^t |B_\beta(\underline{U})\partial_x U|_2 dt' \\ &\lesssim e^{\varepsilon\delta_0 t} |U(0)|_{H^\beta} + \varepsilon C(T, |\underline{U}|_{L_t^\infty H_x^{\max(t_0+1, \beta)}}) \int_0^t |U|_{H^\beta} dt' + \varepsilon\gamma_0 \int_0^t |R|_{H^\beta} dt'. \end{aligned}$$

It only remains to use Grönwall's lemma to get the result. □

This concludes the proof of Proposition 3.2.9. □

Corollary 3.2.12. Let $t_0 > 1/2$ and $\alpha \geq t_0 + 1$. Let $U_e = \begin{pmatrix} u_e^+ \\ u_e^- \end{pmatrix} \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R})^2)$ solves the exact diagonalized system (3.19) with $u_e^-(0) = O(\mu)$, then for all times $[0, \frac{T}{\varepsilon}]$, $u_e^- = O(\mu)$. If instead we take $u_e^+(0) = O(\mu)$ then for all times $[0, \frac{T}{\varepsilon}]$, $u_e^+ = O(\mu)$.

Démonstration. Using energy estimates of Lemma 3.2.11 on the equation on u_e^- of system (3.19), we get

$$|u_e^-|_{H^\alpha} \leq C(T, |U_e|_{L_t^\infty H_x^\alpha}) |u_e^-(0)|_{H^\alpha}.$$

It gives the result on u_e^- . We do the same for u_e^+ with the other equation of (3.19). \square

Proposition 3.2.13. (*Stability*) Let $t_0 > 1/2$ and $\alpha \geq t_0 + 1$. Let $U_e = \begin{pmatrix} u_e^+ \\ u_e^- \end{pmatrix} \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}^2))$ solve the exact diagonalized system (3.19) and $U = \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}^2))$, defined from sufficiently regular solutions of the water waves equations satisfying the non-cavitation hypothesis (3.3) through the formulas (3.14). By Proposition 3.2.3, we have

$$\begin{cases} \partial_t U_e + A(U_e) F_\mu [\partial_x U_e] = 0, \\ \partial_t U + A(U) F_\mu [\partial_x U] = \mu \varepsilon R, \end{cases} \quad (3.26)$$

where A is defined by (3.20) and $R \in L^\infty([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}^2))$ is uniformly bounded in $(\mu, \varepsilon) \in \mathcal{A}$.

Then for all times $t \in [0, \frac{T}{\varepsilon}]$, we have

$$|U_e - U|_{H^\alpha} \leq c_\alpha (|U_e(0) - U(0)|_{H^\alpha} + \mu \varepsilon t |R|_{L_t^\infty H_x^\alpha}), \quad (3.27)$$

where $c_\alpha = C(T, |U_e|_{L_t^\infty H_x^\alpha}, |U|_{L_t^\infty H_x^{\alpha+1}})$.

In particular, if we take $U_e(0) = U(0)$ and $u^-(0) = O(\mu)$, then for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$u^- = O(\mu).$$

If instead, we take $u^+(0) = O(\mu)$, then for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$u^+ = O(\mu).$$

Démonstration. First we subtract the two equations of (3.26), we get

$$\partial_t(U_e - U) + A(U_e) F_\mu [\partial_x(U_e - U)] = -\varepsilon(\mu R + \frac{1}{\varepsilon}(A(U_e) - A(U)) F_\mu [\partial_x U]).$$

We use the estimates of order α from Lemma 3.2.11

$$\begin{aligned} |U_e - U|_{H^\alpha} &\leq e^{\varepsilon\delta_\alpha t} |U_e(0) - U(0)|_{H^\alpha} + \varepsilon\gamma_\alpha \int_0^t |\mu R(t')|_{H^\alpha} dt' \\ &\quad + \varepsilon\gamma_\alpha \int_0^t \frac{1}{\varepsilon} |(A(U_e) - A(U))F_\mu[\partial_x U]|_{H^\alpha} dt'. \end{aligned}$$

But using product estimates 3.4.1, the algebra properties of $H^\alpha(\mathbb{R})$ and the boundedness of F_μ , we have

$$|(A(U_e) - A(U))F_\mu[\partial_x U]|_{H^\alpha} \lesssim \varepsilon |U_e - U|_{H^\alpha} |U|_{H^{\alpha+1}}.$$

So that using Grönwall's lemma we get

$$\begin{aligned} |U_e - U|_{H^\alpha} &\leq \left[e^{\varepsilon\delta_\alpha t} |U_e(0) - U(0)|_{H^\alpha} + \varepsilon\gamma_\alpha \int_0^t |\mu R(t')|_{H^\alpha} dt' \right] e^{\varepsilon\gamma_\alpha \int_0^t |U|_{H^{\alpha+1}} dt'} \\ &\leq C(T, |U_e|_{L_t^\infty H_x^\alpha}) (|U_e(0) - U(0)|_{H^\alpha} + \mu\varepsilon t |R|_{L_t^\infty H_x^\alpha}) e^{\varepsilon\gamma_\alpha t |U|_{L_t^\infty H_x^{\alpha+1}}}, \end{aligned}$$

which is the first part of the result.

Now if we suppose $U_e(0) = U(0)$ and $u^-(0) = O(\mu)$, then for all times $t \in [0, \frac{T}{\varepsilon}]$, we have

$$|u^-|_{H^\alpha} \leq |u_e^- - u^-|_{H^\alpha} + |u_e^-|_{H^\alpha} \leq \mu c_\alpha T |R|_{L_t^\infty H_x^\alpha} + |u_e^-|_{H^\alpha}.$$

Using Corollary 3.2.12 we get the result. We do the same if $u^+(0) = O(\mu)$. \square

We now have all the elements to prove Proposition 3.1.11.

Démonstration. Supposing $u^-(0) = O(\mu)$, we have for all times $t \in [0, \frac{T}{\varepsilon}]$, $\varepsilon u^-(t) = O(\mu\varepsilon)$ (see Proposition 3.2.13). But u^- and u^+ solve the first equation of (3.15). So

$$\partial_t u^+ + F_\mu[\partial_x u^+] + \frac{3\varepsilon}{2} u^+ \partial_x u^+ = O(\mu\varepsilon). \quad (3.28)$$

The last equivalence coming from product estimates 3.4.1 and $|(F_\mu - 1)[u]|_{H^\beta} \lesssim \mu |u|_{H^{\beta+2}}$ for any $\beta \geq 0$ and u in $H^{\beta+2}(\mathbb{R})$.

Supposing instead $u^+(0) = O(\mu)$, we have for all times $t \in [0, \frac{T}{\varepsilon}]$, $\varepsilon u^+(t) = O(\mu\varepsilon)$.

Then, the second equation of (3.15) gives

$$\partial_t u^- - F_\mu[\partial_x u^-] + \frac{3\varepsilon}{2} u^- \partial_x u^- = O(\mu\varepsilon).$$

□

It only remains to prove Proposition 3.1.12.

Démonstration. Let $U_e = \begin{pmatrix} u_e^+ \\ u_e^- \end{pmatrix} \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R})^2)$ solve the system (3.19), with the initial data $u_{e,0}^- = O(\mu)$ and $u_{e,0}^+$, defined by (ζ_0, ψ_0) through the formulas (3.4). By the energy estimates of Lemma 3.2.11 applied on the second equation of system (3.19), we get for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$|u_e^-|_{H^\alpha} \leq C|u_e^-(0)|_{H^\alpha},$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$. So that $u^+ \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}))$, solution of the Whitham equation (which is well-posed in the Sobolev space $H^\alpha(\mathbb{R})$ [EEP15])

$$\partial_t u^+ + F_\mu[\partial_x u^+] + \frac{3\varepsilon}{2} u^+ \partial_x u^+ = 0,$$

satisfies (see (3.28))

$$\partial_t u^+ + (\varepsilon \frac{3u^+ + u_e^-}{2} + 1) F_\mu[\partial_x u^+] = \mu\varepsilon R,$$

where R is uniformly bounded in (μ, ε) in $H^\alpha(\mathbb{R})$. Using the stability estimates (3.27), with $U = \begin{pmatrix} u^+ \\ u_e^- \end{pmatrix}$ and $U_e(0) = U(0)$, we have for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$|u_e^+ - u^+|_{H^\alpha} \leq C\mu\varepsilon t,$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$.

Thus by Corollary 3.1.8 we have for all times $t \in [0, \frac{T}{\varepsilon}]$,

$$\begin{aligned} |(\zeta - \zeta_{\text{Wh},+}, \psi - \psi_{\text{Wh},+})|_{H^\alpha \times \dot{H}^{\alpha+1}} &\leq |(\zeta - \zeta_c, \psi - \psi_c)|_{H^\alpha \times \dot{H}^{\alpha+1}} \\ &\quad + |(\zeta_c - \zeta_{\text{Wh},+}, \psi_c - \psi_{\text{Wh},+})|_{H^\alpha \times \dot{H}^{\alpha+1}} \\ &\leq C(|u_e^-|_{H^{\alpha+1}} + \mu \varepsilon t) \\ &\leq C(|u_{e,0}^-|_{H^{\alpha+1}} + \mu \varepsilon t), \end{aligned}$$

where $C, T^{-1} = C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta_0|_{H^{\alpha+n}}, |\psi_0|_{\dot{H}^{\alpha+n+1}})$.

The reasoning is the same if instead $u_e^+(0) = O(\mu)$. □

3.3 Decoupling the water waves equations into two Whitham equations

The goal of this section is to prove Theorem 3.1.14. In the previous section, we proved the consistency (see Proposition 3.2.1 for the definition) of the water waves equations with the Whitham equations at a precision order $O(\mu\varepsilon)$ in the shallow water regime $\mathcal{A} := \{0 \leq \mu \leq \mu_{\max}, 0 \leq \varepsilon \leq 1\}$ (see Proposition 3.2.3). For that purpose we made a restrictive hypothesis on the initial conditions. We supposed either $u^-(0) = O(\mu)$ or $u^+(0) = O(\mu)$ (see Notation 3.1.10). In this section we use a generalisation of Birkhoff's normal form algorithm for almost smooth Hamiltonians, introduced by Bambusi in [Bam20], on the water waves Hamiltonian. It will allow us to decouple the water waves equations into two Whitham equations satisfied by the two front waves, at a precision order $O(\mu\varepsilon + \varepsilon^2)$ in the shallow water regime, without any assumption of smallness on the initial conditions.

We will only recall the elements of Bambusi's theory which are useful to understand our reasoning. For more details see [Bam20].

In this section we always suppose the parameters (μ, ε) in \mathcal{A} . Moreover every α will be a positive integer.

3.3.1 From the water waves equations to a Whitham-Boussinesq system

The starting point is the Hamiltonian of the water waves system

$$H_{\text{WW}} = \frac{1}{2} \int_{\mathbb{R}} \zeta^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} \psi \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi \, dx. \quad (3.29)$$

We use the first shallow water estimates of Proposition 1.8 in [Eme20], which we recall here for the sake of clarity.

Proposition 3.3.1. *Let $\alpha \geq 0$, and $\zeta \in H^{\alpha+4}(\mathbb{R})$ be such that (3.3) is satisfied. Let $\psi \in \dot{H}^{\alpha+3}(\mathbb{R})$, then*

$$\left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi + \partial_x (h F_\mu^2[\nabla \psi]) \right|_{H^\alpha} \leq \mu \varepsilon M(\alpha + 4) |\psi|_{\dot{H}^{\alpha+4}},$$

where $h = 1 + \varepsilon \zeta$ and for any $\beta \geq 2$, $M(\beta) := C(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^\beta})$.

Démonstration. See the proof of Proposition 1.8 in [Eme20]. □

Proposition 3.3.2. *Let H_{WW} be the Hamiltonian of the water waves equations (3.29). Then for any $\alpha \geq 0$ and any $(\zeta, \psi) \in H^{\alpha+4}(\mathbb{R}) \times \dot{H}^{\alpha+4}(\mathbb{R})$ we have*

$$|J \nabla H_{\text{WW}} - J \nabla (H_0 + \varepsilon H_1)|_{H^\alpha \times \dot{H}^{\alpha+1}} < \mu \varepsilon M(\alpha + 4) |\psi|_{\dot{H}^{\alpha+4}}, \quad (3.30)$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $H_0 := \frac{1}{2} \int_{\mathbb{R}} \zeta^2 \, dx + \frac{1}{2} \int_{\mathbb{R}} (F_\mu[\partial_x \psi])^2 \, dx$ and $H_1 := \frac{1}{2} \int_{\mathbb{R}} \zeta (F_\mu[\partial_x \psi])^2 \, dx$.

Démonstration. We defined H_{WW} as the Hamiltonian of the water waves equations, so

$$J \nabla H_{\text{WW}} = \begin{pmatrix} \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi \\ -\zeta - \frac{\varepsilon}{2} (\partial_x \psi)^2 + \frac{\mu \varepsilon}{2} \frac{(\frac{1}{\mu} \mathcal{G}^\mu[\varepsilon \zeta] \psi + \varepsilon \partial_x \zeta \partial_x \psi)^2}{1 + \varepsilon^2 \mu (\partial_x \zeta)^2} \end{pmatrix}. \quad (3.31)$$

We can easily compute $J \nabla (H_0 + \varepsilon H_1)$, we get

$$J \nabla (H_0 + \varepsilon H_1) = \begin{pmatrix} -F_\mu^2[\partial_x^2 \psi] - F_\mu[\partial_x (\varepsilon \zeta F_\mu[\partial_x \psi])] \\ -\zeta - \frac{\varepsilon}{2} (F_\mu[\partial_x \psi])^2 \end{pmatrix}. \quad (3.32)$$

We compare the first term of (3.31) and (3.32).

$$\begin{aligned} & \left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + F_\mu^2[\partial_x^2\psi] + F_\mu[\partial_x(\varepsilon\zeta F_\mu[\partial_x\psi])] \right|_{H^\alpha} \\ & \leq \left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + \partial_x(hF_\mu^2[\partial_x\psi]) \right|_{H^\alpha} + |(F_\mu - 1)[\partial_x(\varepsilon\zeta F_\mu[\partial_x\psi])]|_{H^\alpha} + |\partial_x(\varepsilon\zeta(F_\mu - 1)F_\mu[\partial_x\psi])|_{H^\alpha}. \end{aligned}$$

Hence, using Proposition 3.3.1, the fact that for any $u \in H^{\alpha+2}(\mathbb{R})$, $|(F_\mu - 1)[u]|_{H^\alpha} \lesssim \mu|u|_{H^{\alpha+2}}$ and product estimates 3.4.1, we get

$$\left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + F_\mu^2[\partial_x^2\psi] + F_\mu[\partial_x(\varepsilon\zeta F_\mu[\partial_x\psi])] \right|_{H^\alpha} \lesssim \mu\varepsilon M(\alpha + 4)|\psi|_{\dot{H}^{\alpha+4}}.$$

Then we compare the second term of (3.31) and (3.32). We first remark that there is a part of (3.31) which is clearly of order $O(\mu\varepsilon)$, we start by estimate it using quotient estimates 3.4.3, product estimates 3.4.1 and Proposition 3.4.6

$$\begin{aligned} \left| \frac{(\frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + \varepsilon\partial_x\zeta\partial_x\psi)^2}{1 + \varepsilon^2\mu(\partial_x\zeta)^2} \right|_{\dot{H}^{\alpha+1}} & \leq \left| \frac{(\frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + \varepsilon\partial_x\zeta\partial_x\psi)^2}{1 + \varepsilon^2\mu(\partial_x\zeta)^2} \right|_{H^{\alpha+1}} \\ & \leq C(\mu_{\max}, |(\partial_x\zeta)^2|_{H^{\alpha+1}}) \left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + \varepsilon\partial_x\zeta\partial_x\psi \right|_{H^{\alpha+1}}^2 \\ & \leq C(\mu_{\max}, |\zeta|_{H^{\alpha+2}}) \left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi + \varepsilon\partial_x\zeta\partial_x\psi \right|_{H^{\alpha+1}}^2 \\ & \leq M(\alpha + 3)|\psi|_{\dot{H}^{\alpha+3}}. \end{aligned}$$

Then, it only remains to see that using product estimates 3.4.1 and $|(F_\mu - 1)[u]|_{H^\alpha} \lesssim \mu|u|_{H^{\alpha+2}}$ for any $u \in H^{\alpha+2}(\mathbb{R})$, we have

$$\begin{aligned} |(\partial_x\psi)^2 - (F_\mu[\partial_x\psi])^2|_{\dot{H}^{\alpha+1}} & \leq |(\partial_x\psi)^2 - (F_\mu[\partial_x\psi])^2|_{H^{\alpha+1}} \\ & \leq |\partial_x\psi - F_\mu[\partial_x\psi]|_{H^{\alpha+1}} |\partial_x\psi + F_\mu[\partial_x\psi]|_{H^{\alpha+1}} \lesssim \mu|\psi|_{\dot{H}^{\alpha+4}} |\psi|_{\dot{H}^{\alpha+2}}. \end{aligned}$$

This ends the proof. \square

Remark 3.3.3. — *When deriving the Hamilton's equations associated with $H_0 + \varepsilon H_1$ we get the following Whitham-Boussinesq system*

$$\begin{cases} \partial_t\zeta + F_\mu^2[\partial_x^2\psi] + \varepsilon F_\mu\partial_x[\zeta F_\mu[\partial_x\psi]] = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2}(F_\mu[\partial_x\psi])^2 = 0. \end{cases}$$

It is equivalent to the system (3.11) at a precision order $O(\mu\varepsilon)$.

- We could approximate the Hamiltonian H_{WW} with any other perturbation of $H_0 + \varepsilon H_1$ of order $O(\mu\varepsilon)$. It wouldn't change the reasoning. Our choice is made to simplify the computations.

Notations 3.3.4. We will denote by $\widetilde{H}_0 + \varepsilon\widetilde{H}_1$ the Hamiltonian $H_0 + \varepsilon H_1$ rewritten in the variables (ζ, v) where $v = \partial_x \psi$. It is associated with the Poisson tensor $\widetilde{J} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$. We recall that a change of unknowns $\mathcal{T}(\eta, \omega)$ turns the initial Poisson tensor, denoted J , into $\left(\frac{\partial F}{\partial(\eta, \omega)}\right) J \left(\frac{\partial F}{\partial(\eta, \omega)}\right)^*$, where $*$ is the adjoint in $L^2(\mathbb{R})^2$.

Before applying Birkhoff's algorithm, we change the unknowns of the approximated Hamiltonian $H_0 + \varepsilon H_1$ with those usually used to diagonalize the linear part of the water waves equations.

Proposition 3.3.5. Let r and s be defined as

$$r := \frac{\zeta + F_\mu[\partial_x \psi]}{2}, \quad s := \frac{\zeta - F_\mu[\partial_x \psi]}{2}. \quad (3.33)$$

We have

$$(\widetilde{H}_0 + \varepsilon\widetilde{H}_1)(\zeta, v) = (H_0 + \varepsilon H_1)(\zeta, \psi) = \int_{\mathbb{R}} (r^2 + s^2) dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^3 + s^3) dx - \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^2 s + r s^2) dx. \quad (3.34)$$

We will denote the later Hamiltonian $H_{\text{BW}}(r, s)$. The Poisson tensor associated with is

$$J_\mu = \begin{pmatrix} -\frac{F_\mu \partial_x}{2} & 0 \\ 0 & \frac{F_\mu \partial_x}{2} \end{pmatrix}.$$

Démonstration. Easy computations. □

For later purposes we also define the inverse of the transformation (3.33).

Property 3.3.6. Let $\alpha \geq 0$ and $\mathcal{T}_D : H^{\alpha+1}(\mathbb{R}) \times H^{\alpha+1}(\mathbb{R}) \rightarrow H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R})$ be defined by

$$\mathcal{T}_D(r, s) := \begin{pmatrix} r + s \\ F_\mu^{-1}[r - s] \end{pmatrix}. \quad (3.35)$$

Then $(\widetilde{H}_0 + \varepsilon\widetilde{H}_1)(\mathcal{T}_D(r, s)) = H_{\text{BW}}(r, s)$.

Démonstration. For any $\alpha \geq 0$ and any $u \in H^{\alpha+1}(\mathbb{R})$, we have $|\mathbb{F}_\mu^{-1}[u]|_{H^\alpha} \lesssim |u|_{H^{\alpha+1}}$. \square

To separate each important part of the Hamiltonian H_{BW} we set some notations.

Notations 3.3.7. *Let*

$$\begin{cases} L := \int_{\mathbb{R}} (r^2 + s^2) \, dx, \\ Z := \frac{1}{2} \int_{\mathbb{R}} (r^3 + s^3) \, dx, \\ W := -\frac{1}{2} \int_{\mathbb{R}} (r^2 s + r s^2) \, dx. \end{cases}$$

We have

$$H_{\text{BW}} = L + \varepsilon Z + \varepsilon W.$$

Remark 3.3.8. *Our goal is to decouple the equations associated with the Hamiltonian H_{BW} . We remark that only W gives coupled terms in the Hamilton's equations.*

3.3.2 Application of Birkhoff's algorithm

We suppose first that all our objects of study are smooth. If $G(r, s)$ is a smooth function then it's corresponding Hamilton's equations are

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} = J_\mu \nabla G.$$

We will denote by Φ_G^t the corresponding flow.

If $F(r, s)$ is another smooth function, we denote the Lie derivative of F with respect to G by

$$\{F, G\}_\mu := (\nabla F, J_\mu \nabla G),$$

where (\cdot, \cdot) denotes the scalar product in $L^2(\mathbb{R})^2$.

Property 3.3.9. *Let $G(r, s)$ be a smooth function and Φ_G^t its flow associated with the Poisson tensor J_μ . Let also $F(r, s)$ be another smooth function. Then*

$$F \circ \Phi_G^\varepsilon = F + \varepsilon \{F, G\}_\mu + O(\varepsilon^2).$$

Démonstration. See Subsection 4.1 in [Bam20]. □

Suppose for now that in our case the Hamiltonian H_{BW} is smooth. Then for any smooth function $G(r, s)$ we get

$$H_{\text{BW}} \circ \Phi_G^\varepsilon = L + \varepsilon\{L, G\}_\mu + \varepsilon Z + \varepsilon W + O(\varepsilon^2).$$

Remark 3.3.10. Here it seems that to get a normal form at the order of precision $O(\varepsilon^2)$, we need G to solve the homological equation

$$\{L, G\}_\mu + W = 0. \tag{3.36}$$

In [Bam20], the author gave the solution of such equations. If

$$\lim_{\tau \rightarrow +\infty} W(\Phi_L^\tau) + W(\Phi_L^{-\tau}) = 0, \tag{3.37}$$

(in our case, we can prove this assumption using a Littlewood-Paley decomposition) then the solution of (3.36) is

$$G(r, s) = -\frac{1}{2} \int_{\mathbb{R}} \text{sgn}(\tau) W(\Phi_L^\tau(r, s)) d\tau, \tag{3.38}$$

at the additional condition that the above function is well-defined.

However the latter condition is not easy to verify in general. We expose here a naive attempt to do so in our context.

The flow Φ_L^τ satisfy

$$\frac{d}{d\tau} \Phi_L^\tau = J_\mu \nabla L(\Phi_L^\tau).$$

So that

$$\Phi_L^\tau(r, s) = \begin{pmatrix} e^{-iDF_\mu\tau} r \\ e^{iDF_\mu\tau} s \end{pmatrix},$$

and

$$W(\Phi_L^\tau(r, s)) = -\frac{1}{2} \int_{\mathbb{R}} (e^{-iDF_\mu\tau} r)^2 e^{iDF_\mu\tau} s \, dx - \frac{1}{2} \int_{\mathbb{R}} (e^{iDF_\mu\tau} s)^2 e^{-iDF_\mu\tau} r \, dx.$$

So

$$\begin{aligned} |W(\Phi_L^\tau(r, s))| &\leq \int_{\mathbb{R}} |e^{-iDF_\mu\tau} r|^2 |e^{iDF_\mu\tau} s| \, dx + \int_{\mathbb{R}} |e^{iDF_\mu\tau} s|^2 |e^{-iDF_\mu\tau} r| \, dx \\ &\leq |e^{iDF_\mu\tau} s|_\infty |e^{-iDF_\mu\tau} r|_2^2 + |e^{-iDF_\mu\tau} r|_\infty |e^{iDF_\mu\tau} s|_2^2. \end{aligned}$$

But $|e^{-iDF_\mu\tau} r|_2 = |r|_2$, and the dispersive estimates only give a decay in time of $|e^{iDF_\mu\tau} s|_\infty$ and $|e^{-iDF_\mu\tau} r|_\infty$ of order $1/\tau^{-1/2}$ [Bul16].

The idea to overcome this issue is to only solve an approximation of the homological equation at order $O(\mu\varepsilon)$.

We define another Lie derivative, associated with the Poisson tensor $J_{\text{simp}} := \begin{pmatrix} -\partial_x/2 & 0 \\ 0 & \partial_x/2 \end{pmatrix}$:

$$\{L, G\}_{\text{simp}} := (\nabla L, J_{\text{simp}} \nabla G)$$

Property 3.3.11. *We have*

$$\{L, G\}_\mu = \{L, G\}_{\text{simp}} + O(\mu)$$

Démonstration. By definition $\{L, G\}_\mu = (\nabla L, J_\mu \nabla G)$, with $J_\mu = \begin{pmatrix} -\frac{F_\mu \partial_x}{2} & 0 \\ 0 & \frac{F_\mu \partial_x}{2} \end{pmatrix}$. So

$$\{L, G\}_\mu - \{L, G\}_{\text{simp}} = ((\nabla L, (J_\mu - J_{\text{simp}}) \nabla G),$$

with $J_\mu - J_{\text{simp}} = \begin{pmatrix} -\frac{F_\mu - 1}{2} & 0 \\ 0 & \frac{F_\mu - 1}{2} \end{pmatrix}$. But for any $\alpha \geq 0$ and any $u \in H^{\alpha+2}(\mathbb{R})$, we have $|(F_\mu - 1)[u]|_{H^\alpha} \lesssim \mu |u|_{H^{\alpha+2}}$. \square

The Lie derivative associated with the Poisson tensor J_{simp} is the same one as in Bambusi's article. To write an explicit expression of the solution of the simplified homological equation associated with

$$\{L, G\}_{\text{simp}} + W = 0, \tag{3.39}$$

we need the definition of a classical primitive operator and some properties on it.

Definition/Property 3.3.12. Let $\alpha \geq 0$. We define the operator $\partial^{-1} : W^{\alpha,1}(\mathbb{R}) \rightarrow W^{\alpha,\infty}(\mathbb{R})$ by the formula

$$\partial^{-1}u(y) = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(y - y_1)u(y_1) dy_1. \quad (3.40)$$

Here are some properties on it :

- It is continuous.
- It is skew-adjoint for the scalar product in $L^2(\mathbb{R})$.
- If $\lim_{x \rightarrow +\infty} u(x) + u(-x) = 0$ (it is the case for any function $u \in W^{1,1}(\mathbb{R})$, see Corollary 8.9 in [Bré11]), then $\partial^{-1}(\partial_x u) = u$.
- $\partial_x(\partial^{-1}u) = u$.

We can now write explicitly the solution of the simplified homological equation.

Proposition 3.3.13. Let $W(r, s) = -\frac{1}{2} \int_{\mathbb{R}} (r^2 s + r s^2) dx$, and $L(r, s) = \int_{\mathbb{R}} (r^2 + s^2) dx$ for any $(r, s) \in W^{1,1}(\mathbb{R})$. Then the solution of the simplified homological equation (3.39) is

$$G(r, s) = \frac{1}{4} \int_{\mathbb{R}} [\partial^{-1}(r^2)s + \partial^{-1}(r)s^2] dx.$$

Démonstration. See Lemma 5.2 in [Bam20]. □

Here G is clearly well-defined for r and s in $W^{\alpha,1}(\mathbb{R})$ with $\alpha \geq 1$ large enough. However there is another problem, G does not generate a flow. For that reason we use the generalization of Birkhoff's algorithm in the case of "almost smooth" functions depending on small parameters (see [Bam20]). We recall here what is an almost smooth function.

Definition 3.3.14. Let $\{\mathcal{B}^\alpha\}_{\alpha \geq 0}$ be a Banach scale. A map $F(r, s, \mu, \varepsilon)$ is said to be almost smooth if $\forall \beta, \gamma \geq 0$, there exists δ and an open neighbourhood of the origin $\mathcal{U} \subset \mathcal{B}^\delta \times \mathbb{R} \times \mathbb{R}$ (where $\mathbb{R} \times \mathbb{R}$ is the domain of (μ, ε)), such that

$$F \in \mathcal{C}^\beta(\mathcal{U}, \mathcal{B}^\gamma \times \mathbb{R} \times \mathbb{R}).$$

Due to the definition of ∂^{-1} , for the rest of the article, we will work with the Banach scale $\{W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})\}_{\alpha \geq 0}$. If

- for any $\alpha \geq 0$, there exists $\alpha' \geq 0$ such that the Poisson tensor J_μ is bounded from $W^{\alpha',1}(\mathbb{R}) \times W^{\alpha',1}(\mathbb{R})$ to $W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})$,
- the function $G(r, s)$ defined in Proposition 3.38 has an almost smooth vector field, i.e. $J_\mu \nabla G$ is almost smooth, same for $\{L, G\}_{\text{simp}}$,

then the theory developed in [Bam20] gives the existence of a transformation \mathcal{T}_B such that

$$\begin{aligned} H_{\text{BW}} \circ \mathcal{T}_B &= L + \varepsilon\{L, G\}_\mu + \varepsilon Z + \varepsilon W + O(\varepsilon^2) \\ &= L + \varepsilon\{L, G\}_{\text{simp}} + \varepsilon Z + \varepsilon W + O(\mu\varepsilon + \varepsilon^2) \\ &= L + \varepsilon Z + O(\mu\varepsilon + \varepsilon^2), \end{aligned}$$

where $A = B + O(\mu^k \varepsilon^l)$ now means $\frac{A-B}{\mu^k \varepsilon^l}$ has an almost smooth vector field.

We verify the latter two conditions.

Property 3.3.15. *For any $\alpha \geq 0$ the Fourier multiplier F_μ is bounded from $W^{\alpha+1,1}(\mathbb{R})$ to $W^{\alpha,1}(\mathbb{R})$. In particular, the Poisson tensor J_μ is bounded from $W^{\alpha+2,1}(\mathbb{R}) \times W^{\alpha+2,1}(\mathbb{R})$ to $W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})$.*

The inverse Fourier multiplier of F_μ , denoted $F_\mu^{-1} := \sqrt{\frac{\sqrt{\mu}|D|}{\tanh(\sqrt{\mu}|D|)}}$, is bounded from $W^{\alpha+2,1}(\mathbb{R})$ to $W^{\alpha,1}(\mathbb{R})$.

Démonstration. First, we remark that we only need to prove the boundedness of F_μ in $W^{1,1}(\mathbb{R}) \rightarrow L^1(\mathbb{R})$. To do so, we consider an even cutoff function $\chi \in \mathcal{C}^\infty((-2, 2))$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$. Let f be in $W^{1,1}(\mathbb{R})$, we can decompose $F_\mu[f]$ into

$$\begin{aligned} F_\mu[f](x) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(y-x)\xi} F_\mu(\xi) \chi(\xi) f(y) \, dy \, d\xi + \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(y-x)\xi} F_\mu(\xi) (1 - \chi(\xi)) f(y) \, dy \, d\xi \\ &:= I_1 + I_2. \end{aligned}$$

We deal with each term separately.

— We begin with I_1 . We can write it using the inverse Fourier transform, denoted here \mathcal{F}^{-1} :

$$I_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(y-x)\xi} F_\mu(\xi) \chi(\xi) f(y) \, dy \, d\xi = \mathcal{F}^{-1}[F_\mu(\xi) \chi(\xi)] * f.$$

Using Young's inequality, we get

$$|\mathcal{F}^{-1}[F_\mu(\xi) \chi(\xi)] * f|_{L^1} \leq |\mathcal{F}^{-1}[F_\mu(\xi) \chi(\xi)]|_{L^1} |f|_{L^1}.$$

But

$$|\mathcal{F}^{-1}[F_\mu(\xi) \chi(\xi)]|_{L^1} \lesssim |F_\mu(\xi) \chi(\xi)|_{H^1}.$$

And $\forall \xi \in \mathbb{R}$ $|F_\mu(\xi)|, |F'_\mu(\xi)| \leq 1$, so $\mathcal{F}^{-1}[F_\mu(\xi)\chi(\xi)]$ is in $L^1(\mathbb{R})$. It finishes the case of I_1 .

— Now we deal with I_2 . Doing an integration by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(y-x)\xi} F_\mu(\xi)(1 - \chi(\xi))f(y) \, dy \, d\xi &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(y-x)\xi} \frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi)) \partial_y f(y) \, dy \, d\xi \\ &= \mathcal{F}^{-1}\left[\frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi))\right] * \partial_y f \end{aligned}$$

Using again Young's inequality, we get

$$\left| \mathcal{F}^{-1}\left[\frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi))\right] * \partial_y f \right|_{L^1} \leq \left| \mathcal{F}^{-1}\left[\frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi))\right] \right|_{L^1} |\partial_y f|_{L^1}.$$

But

$$\left| \mathcal{F}^{-1}\left[\frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi))\right] \right|_{L^1} \lesssim \left| \frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi)) \right|_{H^1}.$$

So $\mathcal{F}^{-1}\left[\frac{F_\mu(\xi)}{i\xi} (1 - \chi(\xi))\right] \in L^1(\mathbb{R})$. It finishes the case of I_2 .

At the end, we have $|F_\mu[f]|_{L^1} \leq |I_1|_{L^1} + |I_2|_{L^1} \lesssim |f|_{W^{1,1}}$.

With the same process we obtain the result on F_μ^{-1} (we need a second integration by parts in the high frequencies to get $\mathcal{F}^{-1}\left[\frac{F_\mu^{-1}(\xi)}{(i\xi)^2} (1 - \chi(\xi))\right]$ in L^1). \square

Property 3.3.16. *The function $G(r, s)$ defined in Proposition 3.3.13 and $\{L, G\}_{\text{simp}}$ have an almost smooth vector field.*

Démonstration. First we have $2\partial_r G = -r\partial^{-1}(s) - \frac{1}{2}\partial^{-1}(s^2)$, where we used the skew-adjoint property of ∂^{-1} . So $-2F_\mu[\partial_x \partial_r G] = F_\mu[\partial_x(r)\partial^{-1}(s)] + F_\mu[rs] + \frac{1}{2}F_\mu[s^2]$. But for any $\alpha \geq 0$, F_μ is bounded from $W^{\alpha+1,1}(\mathbb{R})$ to $W^{\alpha,1}(\mathbb{R})$, and $W^{\alpha+1,1}(\mathbb{R})$ is an algebra. We also know that ∂^{-1} is a linear operator bounded from $W^{\alpha+1,1}(\mathbb{R})$ to $W^{\alpha+1,\infty}(\mathbb{R})$, and the product of a $W^{\alpha+1,\infty}(\mathbb{R})$ with a $W^{\alpha+1,1}(\mathbb{R})$ function is still in $W^{\alpha+1,1}(\mathbb{R})$. Combining all these elements, we get the result for G .

The same arguments give the result for $\{L, G\}_{\text{simp}}$. \square

We have an explicit formulation of the transformation \mathcal{T}_B given by (4.25) in [Bam20], taking $X = J_{\text{simp}} \nabla G$, on which we can do precise estimates.

Proposition 3.3.17. *Let $\alpha \geq 0$ and $\mathcal{T}_B : W^{\alpha+1,1}(\mathbb{R}) \times W^{\alpha+1,1}(\mathbb{R}) \rightarrow W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})$ be defined by*

$$\mathcal{T}_B(r, s) = \begin{pmatrix} r \\ s \end{pmatrix} + \varepsilon J_{\text{simp}} \nabla G(r, s) = \begin{pmatrix} r + \frac{\varepsilon}{4} \partial_x(r) \partial^{-1}(s) + \frac{\varepsilon}{4} r s + \frac{\varepsilon}{8} s^2 \\ s + \frac{\varepsilon}{4} \partial_x(s) \partial^{-1}(r) + \frac{\varepsilon}{4} r s + \frac{\varepsilon}{8} r^2 \end{pmatrix}. \quad (3.41)$$

Then

$$H_{\text{BW}} \circ \mathcal{T}_B = L + \varepsilon Z + O(\mu\varepsilon + \varepsilon^2)$$

Démonstration. Let $r, s \in W^{\alpha+1,1}(\mathbb{R})$. The latter Sobolev space is an algebra, so the only terms we need to look at are $\partial_x r \partial^{-1} s$ and $\partial_x s \partial^{-1} r$. We have

$$|\partial_x(r) \partial^{-1} s|_{W^{\alpha,1}} \leq |\partial^{-1} s|_{W^{\alpha,\infty}} |\partial_x r|_{W^{\alpha,1}}.$$

Here using the continuity of ∂^{-1} from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ for any $\beta \geq 0$ and the third property of Definition/Property 3.3.12, we get

$$\begin{cases} |\partial^{-1} s|_{W^{\alpha,\infty}} \lesssim |s|_{L^1} & \text{if } \alpha = 0, \\ |\partial^{-1} s|_{W^{\alpha,\infty}} \lesssim |s|_{L^1} + \sum_{\beta \leq \alpha} |\partial_x^\beta \partial^{-1} s|_{L^\infty} = |s|_{L^1} + \sum_{\beta \leq \alpha} |\partial^{-1} \partial_x^\beta s|_{L^\infty} \lesssim |s|_{W^{\alpha,1}} & \text{if } \alpha \geq 1. \end{cases} \quad (3.42)$$

Idem for the other term. Thus $\mathcal{T}_B : W^{\alpha+1,1}(\mathbb{R}) \times W^{\alpha+1,1}(\mathbb{R}) \rightarrow W^{\alpha,1}(\mathbb{R}) \times W^{\alpha,1}(\mathbb{R})$.

From the definition of H_{BW} (see (3.34)) and (3.41) we have

$$\begin{aligned} H_{\text{BW}}(\mathcal{T}_B(r, s)) &= \int_{\mathbb{R}} \left(r^2 + \frac{\varepsilon}{2} r \partial_x(r) \partial^{-1} s + \frac{\varepsilon}{2} r^2 s + \frac{\varepsilon}{4} r s^2 \right) dx \\ &\quad + \int_{\mathbb{R}} \left(s^2 + \frac{\varepsilon}{2} s \partial_x(s) \partial^{-1} r + \frac{\varepsilon}{2} r s^2 + \frac{\varepsilon}{4} s r^2 \right) dx \\ &\quad - \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^2 s + r s^2) dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^3 + s^3) dx + O(\varepsilon^2). \end{aligned}$$

But using the skew-adjointness of ∂^{-1} we get

$$\begin{cases} \int_{\mathbb{R}} r \partial_x r \partial^{-1} s dx = \frac{1}{2} \int_{\mathbb{R}} \partial_x(r^2) \partial^{-1} s dx = -\frac{1}{2} \int_{\mathbb{R}} \partial^{-1}(\partial_x r^2) s dx = -\frac{1}{2} \int_{\mathbb{R}} r^2 s dx, \\ \int_{\mathbb{R}} s \partial_x s \partial^{-1} r dx = \frac{1}{2} \int_{\mathbb{R}} \partial_x(s^2) \partial^{-1} r dx = -\frac{1}{2} \int_{\mathbb{R}} \partial^{-1}(\partial_x s^2) r dx = -\frac{1}{2} \int_{\mathbb{R}} s^2 r dx. \end{cases}$$

Thus

$$\begin{aligned} H_{\text{BW}}(\mathcal{T}_{\text{B}}(r, s)) &= \int_{\mathbb{R}} (r^2 + s^2) \, dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} (r^3 + s^3) \, dx + O(\varepsilon^2) \\ &= L(r, s) + \varepsilon Z(r, s) + O(\varepsilon^2). \end{aligned}$$

□

Remark 3.3.18. *The theory developed in [Bam20] tells us that the transformation \mathcal{T}_{B} should be $\mathcal{T}_{\text{B}}(r, s) = \begin{pmatrix} r \\ s \end{pmatrix} + \varepsilon J_{\mu} \nabla G(r, s)$. But the precision we want is $O(\mu\varepsilon + \varepsilon^2)$, and $\varepsilon J_{\mu} = \varepsilon J_{\text{simp}} + O(\mu\varepsilon)$.*

The transformation (3.41) preserves the Hamiltonian structure associated with the Poisson tensor J_{μ} at the order of precision $O(\mu\varepsilon + \varepsilon^2)$, see the following proposition.

Proposition 3.3.19. *Let $\alpha \geq 0$. Let $r, s \in W^{\alpha+4,1}(\mathbb{R})$. Let $\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)}$ be the Jacobian matrix of the transformation T . We have*

$$\left(\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)} \right) J_{\mu} \left(\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)} \right)^* = J_{\mu} + (\mu\varepsilon + \varepsilon^2)R,$$

where $*$ is the adjoint in $L^2(\mathbb{R})^2$ and R is linear operator such that for any $U \in H^{\alpha+4}(\mathbb{R}) \times H^{\alpha+4}(\mathbb{R})$, $|RU|_{H^{\alpha} \times H^{\alpha}} \leq C(\mu_{\text{max}}, |r|_{W^{\alpha+4,1}}, |s|_{W^{\alpha+4,1}})|U|_{H^{\alpha+4} \times H^{\alpha+4}}$.

Démonstration. We easily compute the Jacobian matrix of \mathcal{T}_{B} :

$$\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)} = \begin{pmatrix} 1 + \frac{\varepsilon}{4}s + \frac{\varepsilon}{4}\partial^{-1}(s)\partial_x(\circ) & \frac{\varepsilon}{4}\partial_x(r)\partial^{-1}(\circ) + \frac{\varepsilon}{4}(r+s) \\ \frac{\varepsilon}{4}\partial_x(s)\partial^{-1}(\circ) + \frac{\varepsilon}{4}(r+s) & 1 + \frac{\varepsilon}{4}r + \frac{\varepsilon}{4}\partial^{-1}(r)\partial_x(\circ) \end{pmatrix}.$$

Its L^2 adjoint is

$$\left(\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)} \right)^* = \begin{pmatrix} 1 + \frac{\varepsilon}{4}s - \frac{\varepsilon}{4}\partial_x(\partial^{-1}(s)\circ) & \frac{\varepsilon}{4}(r+s) - \frac{\varepsilon}{4}\partial^{-1}(\partial_x(s)\circ) \\ \frac{\varepsilon}{4}(r+s) - \frac{\varepsilon}{4}\partial^{-1}(\partial_x(r)\circ) & 1 + \frac{\varepsilon}{4}r - \frac{\varepsilon}{4}\partial_x(\partial^{-1}(r)\circ) \end{pmatrix}.$$

Using the fact that both $\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)}$ and $\left(\frac{\partial \mathcal{T}_{\text{B}}}{\partial(r,s)} \right)^*$ can be written under the form $I_d + O(\varepsilon)$, we

write

$$\begin{aligned} \left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right) J_\mu \left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* &= \left[\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right) + I_d\right] J_\mu \left[\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right] + I_d \\ &= \left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right) J_\mu \left(\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right) \\ &\quad + \left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right) J_\mu + J_\mu \left(\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right) + J_\mu. \end{aligned}$$

The first term is of order $O(\varepsilon^2)$, we will estimate it later. The second and third terms are of order $O(\varepsilon)$. We prove now that their addition is of order $O(\mu\varepsilon)$. We easily compute both terms :

$$\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right) J_\mu = \frac{\varepsilon}{8} \begin{pmatrix} -sF_\mu[\partial_x(\circ)] - \partial^{-1}(s)F_\mu[\partial_x^2(\circ)] & (r+s)F_\mu[\partial_x(\circ)] + \partial_x(r)F_\mu[\circ] \\ -(r+s)F_\mu[\partial_x(\circ)] - \partial_x(s)F_\mu[\circ] & rF_\mu[\partial_x(\circ)] + \partial^{-1}(r)F_\mu[\partial_x^2(\circ)] \end{pmatrix},$$

and

$$J_\mu \left(\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right) = \frac{\varepsilon}{8} \begin{pmatrix} F_\mu[\partial_x(\partial^{-1}(s)\partial_x(\circ))] & -F_\mu[\partial_x((r+s)\circ)] + F_\mu[\partial_x(s)\circ] \\ F_\mu[\partial_x((r+s)\circ)] - F_\mu[\partial_x(r)\circ] & -F_\mu[\partial_x(\partial^{-1}(r)\partial_x(\circ))] \end{pmatrix}.$$

For now, we look at the first term of both matrices. For any $u \in H^{\alpha+4}(\mathbb{R})$

$$\begin{aligned} &|F_\mu[\partial_x(\partial^{-1}(s)\partial_x u)] - sF_\mu[\partial_x u] - \partial^{-1}(s)F_\mu[\partial_x^2 u]|_{H^\alpha} \\ &\leq |[F_\mu, s]\partial_x u|_{H^\alpha} + |[F_\mu, \partial^{-1}s]\partial_x^2 u|_{H^\alpha} \\ &= |[F_\mu - 1; s]\partial_x u|_{H^\alpha} + |[F_\mu - 1, \partial^{-1}s]\partial_x^2 u|_{H^\alpha}. \end{aligned}$$

But using Remark 3.2.7 and (3.42) we have

$$\begin{aligned} |[F_\mu - 1, \partial^{-1}s]\partial_x^2 u|_{H^\alpha} &\lesssim \mu|\partial^{-1}(s)\partial_x^2 u|_{H^{\alpha+2}} + |\partial^{-1}s|_{W^{\alpha+2,\infty}}|(F_\mu - 1)[\partial_x^2 u]|_{H^\alpha} \\ &\lesssim \mu|\partial^{-1}s|_{W^{\alpha+2,\infty}}|\partial_x^2 u|_{H^{\alpha+2}} \\ &\lesssim \mu|s|_{W^{\alpha+2,1}}|u|_{H^{\alpha+4}}. \end{aligned}$$

Using algebra properties of $H^{\alpha+2}(\mathbb{R})$ and the Sobolev embedding $W^{\alpha+3,1}(\mathbb{R}) \subset H^{\alpha+2}(\mathbb{R})$, we also have

$$|[F_\mu - 1, s]\partial_x u|_{H^\alpha} \lesssim \mu|s|_{H^{\alpha+2}}|u|_{H^{\alpha+3}} \lesssim \mu|s|_{W^{\alpha+3,1}}|u|_{H^{\alpha+3}}.$$

So that

$$|\mathbf{F}_\mu[\partial_x(\partial^{-1}(s)\partial_x u)] - s\mathbf{F}_\mu[\partial_x u] - \partial^{-1}(s)\mathbf{F}_\mu[\partial_x^2 u]|_{H^\alpha} \lesssim \mu|s|_{W^{\alpha+3,1}}|u|_{H^{\alpha+4}}.$$

The same can be done for the addition of the fourth term of both matrices.

Now we look at the addition of the second term of both matrices. For any $u \in H^{\alpha+3}(\mathbb{R})$ we have

$$\begin{aligned} (r+s)\mathbf{F}_\mu[\partial_x u] + \partial_x(r)\mathbf{F}_\mu[u] &= \partial_x((r+s)\mathbf{F}_\mu[u]) - \partial_x(r+s)\mathbf{F}_\mu[u] \\ &= \partial_x((r+s)\mathbf{F}_\mu[u]) - \partial_x(s)\mathbf{F}_\mu[u]. \end{aligned}$$

Using Lemma 3.2.6 and the Sobolev embedding $W^{\alpha+4,1}(\mathbb{R}) \subset H^{\alpha+3}(\mathbb{R})$, we get

$$\begin{aligned} &|\partial_x((r+s)\mathbf{F}_\mu[u]) - \partial_x(s)\mathbf{F}_\mu[u] - \mathbf{F}_\mu[\partial_x((r+s)u)] + \mathbf{F}_\mu[\partial_x(s)u]|_{H^\alpha} \\ &\leq |\partial_x([\mathbf{F}_\mu, r+s]u)|_{H^\alpha} + |[\mathbf{F}_\mu, \partial_x s]u|_{H^\alpha} \\ &\leq |[\mathbf{F}_\mu - 1, r+s]u|_{H^{\alpha+1}} + |[\mathbf{F}_\mu - 1, \partial_x s]u|_{H^\alpha} \\ &\lesssim \mu(|r|_{H^{\alpha+3}} + |s|_{H^{\alpha+3}})|u|_{H^{\alpha+3}} \\ &\lesssim \mu(|r|_{W^{\alpha+4,1}} + |s|_{W^{\alpha+4}})|u|_{H^{\alpha+3}}. \end{aligned}$$

The same can be done for the addition of the third term of both matrices. Thus we get for any $U \in H^{\alpha+4}(\mathbb{R}) \times H^{\alpha+4}(\mathbb{R})$

$$|\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right)J_\mu U + J_\mu\left(\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right)U|_{H^\alpha} \lesssim \mu\varepsilon(|r|_{W^{\alpha+4,1}} + |s|_{W^{\alpha+4,1}})|U|_{H^{\alpha+4}}$$

It remains to estimate $\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right)J_\mu\left(\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right)U$. Using the boundedness of \mathbf{F}_μ and the same tools as before, we get for any $U \in H^{\alpha+3}(\mathbb{R}) \times H^{\alpha+3}(\mathbb{R})$

$$|\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} - I_d\right)J_\mu\left(\left(\frac{\partial \mathcal{T}_B}{\partial(r,s)}\right)^* - I_d\right)U|_{H^\alpha} \lesssim \varepsilon^2 C(|r|_{W^{\alpha+4,1}}, |r|_{W^{\alpha+4,1}})|U|_{H^{\alpha+3}}.$$

□

3.3.3 From the Hamiltonian of the Whitham-Boussinesq system under normal form to two decoupled Whitham equations

In the previous subsections, we proved the existence of a transformation \mathcal{T}_B such that

$$\begin{aligned} H_{\text{WW}} \circ \mathcal{T}_B &= L + \varepsilon Z + O(\mu\varepsilon + \varepsilon^2) \\ &= \int_{\mathbb{R}} (r^2 + s^2) dx + \varepsilon \frac{1}{2} \int_{\mathbb{R}} (r^3 + s^3) dx + O(\mu\varepsilon + \varepsilon^2). \end{aligned}$$

The equations associated with the Hamiltonian $L + \varepsilon Z$ and the Poisson tensor J_μ are easily computed.

Property 3.3.20. *The Hamilton's equations associated with the normal form $L + \varepsilon Z$ are*

$$\begin{cases} \partial_t r + F_\mu[\partial_x r] + \frac{3\varepsilon}{2} F_\mu[r\partial_x r] = 0, \\ \partial_t s - F_\mu[\partial_x s] - \frac{3\varepsilon}{2} F_\mu[s\partial_x s] = 0. \end{cases} \quad (3.43)$$

Démonstration. The Hamilton's equations are

$$\partial_t \begin{pmatrix} r \\ s \end{pmatrix} = J_\mu \nabla(L + \varepsilon Z).$$

□

Notation 3.3.21. *We write $H_{\text{Wh}}(r, s) := L(r, s) + \varepsilon Z(r, s)$.*

Remark 3.3.22. *The two equations (3.43) are almost two Whitham equations. One just need to use that for any $\alpha \geq 0$ and any $u \in H^{\alpha+2}(\mathbb{R})$ one has $|(F_\mu - 1)[u]|_{H^\alpha} \lesssim \mu|u|_{H^{\alpha+2}}$ to get the existence of a remainder (R_1, R_2) controllable in Sobolev spaces H^α such that*

$$\begin{cases} \partial_t r + F_\mu[\partial_x r] + \frac{3\varepsilon}{2} r\partial_x r = \mu\varepsilon R_1, \\ \partial_t s - F_\mu[\partial_x s] - \frac{3\varepsilon}{2} s\partial_x s = \mu\varepsilon R_2. \end{cases}$$

Lemma 3.3.23. *Let $\theta > 1/2$ and $\alpha \geq 0$. Then for any $r_0, s_0 \in W^{\alpha+5,1}(\mathbb{R})$ there exists a time $T > 0$ such that both equations (3.43) admit a unique solution $r, s \in \mathcal{C}([0, \frac{T}{\varepsilon}], H^{\alpha+4}(\mathbb{R}))$,*

with initial conditions r_0 and s_0 , which satisfy for any times $t \in [0, \frac{T}{\varepsilon}]$

$$\begin{cases} |r(t)|_{W^{\alpha,1}} \lesssim (1 + \mu t)^\theta C(\mu_{\max}, T, |r_0|_{W^{\alpha+5,1}}), \\ |s(t)|_{W^{\alpha,1}} \lesssim (1 + \mu t)^\theta C(\mu_{\max}, T, |s_0|_{W^{\alpha+5,1}}). \end{cases}$$

In particular $r, s \in \mathcal{C}([0, \frac{T}{\varepsilon}], W^{\alpha,1}(\mathbb{R}))$ are uniformly bounded in (μ, ε) for times $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$ in $W^{\alpha,1}(\mathbb{R})$.

Moreover, for any $r_0, s_0 \in W^{\alpha+7,1}(\mathbb{R})$, both equations (3.43) admit a unique solution $r, s \in \mathcal{C}^1([0, \frac{T}{\max(\mu, \varepsilon)}], W^{\alpha,1}(\mathbb{R})) \cap \mathcal{C}([0, \frac{T}{\max(\mu, \varepsilon)}], W^{\alpha+2,1}(\mathbb{R}))$ with initial conditions r_0 and s_0 .

Démonstration. We have the Sobolev embeddings $W^{\alpha+5,1}(\mathbb{R}) \subset H^{\alpha+4}(\mathbb{R})$ and the equations (3.43) are well-posed in these Sobolev spaces (see [EEP15]). So there exists $T > 0$ such that the latter equations admit a unique solution in $\mathcal{C}([0, \frac{T}{\varepsilon}], H^{\alpha+2}(\mathbb{R}))$.

We do the rest of the reasoning for r . We use the Duhamel formula to say that

$$r(t) = e^{-itDF_\mu} r_0 + \frac{3\varepsilon}{2} \int_0^t e^{-i(t-t')DF_\mu} F_\mu[r\partial_x r] dt'.$$

We prove first that $|e^{-itDF_\mu} r_0|_{W^{\alpha,1}} \lesssim (1 + \mu t)^\theta |r_0|_{W^{\alpha+2,1}}$.

We begin by doing the same reasoning as in the proof of Property 3.3.15. We remark that we only need to prove the previous estimate in $L^1(\mathbb{R})$. Let $\chi \in \mathcal{C}^\infty((-2, 2))$ be a cutoff function such that $\chi(\xi) = 1$ for $|\xi| \leq 1$. We decompose $e^{-itDF_\mu} r_0 = e^{-itD(F_\mu-1)} e^{-itD} r_0$ into

$$\begin{aligned} e^{-itD(F_\mu-1)} e^{-itD} r_0(x) &= \mathcal{F}^{-1}[e^{-it\xi(F_\mu(\xi)-1)} \chi(\xi)] * (e^{-itD} r_0) \\ &\quad + \mathcal{F}^{-1}[e^{it\xi(F_\mu(\xi)-1)} \frac{1 - \chi(\xi)}{(i\xi)^2}] * \partial_y^2 e^{-itD} r_0 := I_1 + I_2. \end{aligned}$$

— We prove here that $|I_1|_{L^1} \lesssim (1 + \mu t)^{\frac{1}{2}+} |r_0|_{L^1}$. Using Young's inequality, we get

$$|I_1|_{L^1} \leq |\mathcal{F}^{-1}[e^{-it\xi(F_\mu(\xi)-1)} \chi(\xi)]|_{L^1} |e^{-itD} r_0|_{L^1} \leq |\mathcal{F}^{-1}[e^{-it\xi(F_\mu(\xi)-1)} \chi(\xi)]|_{L^1} |r_0|_{L^1}.$$

Now we pose $G(x) = \mathcal{F}^{-1}[e^{-it\xi(F_\mu(\xi)-1)} \chi(\xi)]$. Let $\beta \in \mathcal{C}^\infty((-2, 2))$ be a cutoff

function such that $\beta(x) = 1$ for $|x| \leq 1$. We have for any $\theta > 1/2$

$$\begin{aligned} |G|_{L^1} &\leq |\beta G|_{L^1} + |(1 - \beta)G|_{L^1} \\ &\lesssim 1 + \left| \frac{1 - \beta(x)}{x^\theta} \right|_{L^2} |x^\theta G|_{L^2}. \end{aligned}$$

We will prove the inequality $|xG|_{L^2} \lesssim 1 + \mu t$ and interpolate it with the obvious one $|G|_{L^2} \lesssim 1$.

We remark that

$$|xG|_{L^2} = |\partial_\xi \widehat{G}|_{L^2} = |\partial_\xi (e^{-it\xi(F_\mu(\xi)-1)} \chi(\xi))|_{L^2}.$$

But we have the two following estimates on F_μ : for any $\xi \in \mathbb{R}$

$$|F_\mu(\xi) - 1| \lesssim \mu \xi^2, \quad |F'_\mu(\xi)| \lesssim \mu \xi^2. \quad (3.44)$$

So

$$|\partial_\xi (e^{-it\xi(F_\mu(\xi)-1)} \chi(\xi))|_{L^2} \leq t|(F_\mu(\xi) - 1)\chi(\xi)|_{L^2} + t|\xi F'_\mu(\xi)\chi(\xi)|_{L^2} + |\chi'(\xi)|_{L^2} \lesssim 1 + \mu t.$$

Then Hölder's inequality gives us

$$|x^\theta G|_{L^2} = |(xG)^\theta G^{1-\theta}|_{L^2} \leq |(xG)^\theta|_{L^p} |G^{1-\theta}|_{L^q},$$

with $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$. It remains to take $p = \frac{2}{\theta}$ and $q = \frac{2}{1-\theta}$ to get

$$|x^\theta G|_{L^2} \leq |xG|_{L^2}^\theta |G|_{L^2}^{1-\theta} \lesssim (1 + \mu t)^\theta.$$

— We prove here that $|I_2|_{L^1} \lesssim (1 + \mu t)^\theta |r_0|_{W^{2,1}}$. Again we use Young's inequality to get

$$|I_2|_{L^1} \leq |\mathcal{F}^{-1}[e^{it\xi(F_\mu(\xi)-1)} \frac{1 - \chi(\xi)}{(i\xi)^2}]|_{L^1} |r_0|_{W^{2,1}}.$$

Then the rest of the proof is the same, with a little exception : here, using the

estimates (3.44), we have

$$\begin{aligned} & \left| \partial_\xi \left(e^{-it\xi(F_\mu(\xi)-1)} \frac{1-\chi(\xi)}{\xi^2} \right) \right|_{L^2} \\ & \leq \left| \frac{t(F_\mu(\xi)-1)(1-\chi(\xi))\xi^2 + t\xi F'_\mu(\xi)(1-\chi(\xi))\xi^2 - \chi'(\xi)\xi^2 - 2(1-\chi(\xi))\xi}{\xi^4} \right|_{L^2} \\ & \lesssim 1 + \mu t. \end{aligned}$$

Now we prove $\left| \int_0^t e^{-i(t-t')DF_\mu} F_\mu[r\partial_x r] dt' \right|_{W^{\alpha,1}} \lesssim (1+\mu t)^\theta \int_0^t |r|_{H^{\alpha+4}}^2 dt'$, so that for all times $t \in [0, \frac{T}{\varepsilon}]$ one has

$$\begin{aligned} \left| \frac{3\varepsilon}{2} \int_0^t e^{-i(t-t')DF_\mu} F_\mu[r\partial_x r] dt' \right|_{W^{\alpha,1}} & \leq (1+\mu t)^\theta C(\mu_{\max}, T, |r_0|_{H^{\alpha+4}}) \\ & \leq (1+\mu t)^\theta C(\mu_{\max}, T, |r_0|_{W^{\alpha+5,1}}). \end{aligned}$$

Using what we did before, we have

$$\begin{aligned} \left| \int_0^t e^{-i(t-t')DF_\mu} F_\mu[r\partial_x r] dt' \right|_{W^{\alpha,1}} & \leq \int_0^t |e^{-i(t-t')DF_\mu} F_\mu[r\partial_x r]|_{W^{\alpha,1}} dt' \\ & \lesssim \int_0^t (1+\mu(t-t'))^\theta |F_\mu[r\partial_x r]|_{W^{\alpha+2,1}} dt'. \end{aligned}$$

Using the continuity of F_μ from $W^{\alpha+3,1}(\mathbb{R})$ to $W^{\alpha+2,1}(\mathbb{R})$ (see Property 3.3.15), we get

$$\begin{aligned} \left| \int_0^t e^{-i(t-t')DF_\mu} F_\mu[r\partial_x r] dt' \right|_{W^{\alpha,1}} & \lesssim (1+\mu t)^\theta \int_0^t |r\partial_x r|_{W^{\alpha+3,1}} dt' \\ & \lesssim (1+\mu t)^\theta \int_0^t |r|_{H^{\alpha+4}}^2 dt'. \end{aligned}$$

It remains to prove that taking r_0 and s_0 in $W^{\alpha+7}(\mathbb{R})$, $\partial_t r$ and $\partial_t s$ are in $W^{\alpha,1}(\mathbb{R})$. We do the reasoning for r . From what we proved above, r is in $W^{\alpha+2,1}(\mathbb{R})$. It is also solution of the equation

$$\partial_t r + F_\mu[\partial_x r] + \frac{3\varepsilon}{2} F_\mu[r\partial_x r] = 0.$$

So that

$$\begin{aligned} |\partial_t r|_{W^{\alpha,1}} & \lesssim |F_\mu[\partial_x r]|_{W^{\alpha,1}} + |F_\mu[r\partial_x r]|_{W^{\alpha,1}} \\ & \leq |r|_{W^{\alpha+2,1}} + |r\partial_x r|_{W^{\alpha+1,1}} \lesssim |r|_{W^{\alpha+2,1}}^2. \end{aligned}$$

□

Theorem 3.3.24. *Let $\alpha \geq 0$. Let $r, s \in \mathcal{C}^1([0, \frac{T}{\varepsilon}], W^{\alpha+6,1}(\mathbb{R}))$ be solutions of the equations (3.43). Let also \mathcal{T}_B be the transformation (3.41), and define :*

$$\begin{pmatrix} r_c \\ s_c \end{pmatrix} := \mathcal{T}_B(r, s). \quad (3.45)$$

Then, there exists $R \in \mathcal{C}^1([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R}))$, such that one has :

$$\partial_t \begin{pmatrix} r_c \\ s_c \end{pmatrix} = J_\mu \nabla(H_{\text{BW}})(r_c, s_c) + (\mu\varepsilon + \varepsilon^2)R, \quad \forall t \in [0, \frac{T}{\varepsilon}],$$

where H_{BW} is the Hamiltonian defined in (3.34) and $|R|_{H^\alpha \times H^\alpha} \leq C(\mu_{\max}, |r|_{W^{\alpha+6,1}}, |s|_{W^{\alpha+6,1}})$. In particular, R is uniformly bounded in (μ, ε) for times $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$ in $H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R})$.

Démonstration. The existence of the solutions (r, s) is given by the previous lemma 3.3.23.

We write $\mathcal{T}_B(r, s) = \begin{pmatrix} T_1(r, s) \\ T_2(r, s) \end{pmatrix} = \begin{pmatrix} r + \frac{\varepsilon}{4}\partial_x(r)\partial^{-1}s + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}s^2 \\ s + \frac{\varepsilon}{4}\partial_x(s)\partial^{-1}r + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}r^2 \end{pmatrix}$. By definition of H_{BW} (see (3.34)) and H_{Wh} (see Notation 3.3.21), and the computations in the proof of Proposition 3.3.17 we have

$$\begin{aligned} H_{\text{BW}}(\mathcal{T}_B(r, s)) &= \int_{\mathbb{R}} (T_1(r, s)^2 + T_2(r, s)^2) dx + \frac{\varepsilon}{2} \int_{\mathbb{R}} (T_1(r, s)^3 + T_2(r, s)^3) dx \\ &\quad - \frac{\varepsilon}{2} \int_{\mathbb{R}} (T_1(r, s)^2 T_2(r, s) + T_1(r, s) T_2(r, s)^2) dx \\ &= H_{\text{Wh}}(r, s) \\ &\quad + \int_{\mathbb{R}} [(\frac{\varepsilon}{4}\partial_x(r)\partial^{-1}(s) + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}s^2)^2 + (\frac{\varepsilon}{4}\partial_x(s)\partial^{-1}(r) + \frac{\varepsilon}{4}rs + \frac{\varepsilon}{8}r^2)^2] dx \\ &\quad + \frac{\varepsilon}{2} \int_{\mathbb{R}} (T_1(r, s)^3 - r^3 + T_2(r, s)^3 - s^3) dx \\ &\quad - \frac{\varepsilon}{2} \int_{\mathbb{R}} (T_1(r, s)^2 T_2(r, s) - r^2 s + T_1(r, s) T_2(r, s)^2 - r s^2) dx \end{aligned}$$

So $H_{\text{BW}}(\mathcal{T}_B(r, s)) = H_{\text{Wh}}(r, s) + \varepsilon^2 \int_{\mathbb{R}} p(r, s, \partial_x(r)\partial^{-1}s, \partial_x(s)\partial^{-1}r) dx$, where $p : \mathbb{R}^4 \rightarrow \mathbb{R}$ is polynomial.

But $(r_c, s_c) = \mathcal{T}_B(r, s)$. So

$$\partial_t \begin{pmatrix} r_c \\ s_c \end{pmatrix} = \frac{\partial \mathcal{T}_B}{\partial (r, s)} \partial_t \begin{pmatrix} r \\ s \end{pmatrix},$$

where $\frac{\partial \mathcal{T}_B}{\partial(r,s)}$ is the Jacobian matrix of $\mathcal{T}_B(r,s)$ computed in the proof of Proposition 3.3.19. Since r and s are solutions of the Hamilton's equations associated with H_{Wh} , we have

$$\begin{aligned} \partial_t \begin{pmatrix} r_c \\ s_c \end{pmatrix} &= \frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \nabla(H_{\text{Wh}})(r,s) \\ &= \frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \nabla(H_{\text{BW}}(\mathcal{T}_B(r,s))) - \varepsilon^2 \frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \nabla \int_{\mathbb{R}} p(r,s, \partial_x(r) \partial^{-1} s, \partial_x(s) \partial^{-1} r) dx \\ &= \frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} \right)^* \nabla(H_{\text{BW}})(r_c, s_c) - \varepsilon^2 \frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \nabla \int_{\mathbb{R}} p(r,s, \partial_x(r) \partial^{-1} s, \partial_x(s) \partial^{-1} r) dx. \end{aligned}$$

Moreover by Proposition 3.3.19, we know that $\frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \left(\frac{\partial \mathcal{T}_B}{\partial(r,s)} \right)^* = J_\mu + (\mu\varepsilon + \varepsilon^2)R$, with $|RU|_{H^\alpha \times H^\alpha} \leq C(\mu_{\max}, |r|_{W^{\alpha+4,1}}, |s|_{W^{\alpha+4,1}}) |U|_{H^{\alpha+4}}$ for any $U \in H^{\alpha+4}(\mathbb{R}) \times H^{\alpha+4}(\mathbb{R})$. So, here, we need to estimate $\nabla(H_{\text{BW}})(r_c, s_c)$ in $H^{\alpha+4}(\mathbb{R}) \times H^{\alpha+4}(\mathbb{R})$.

$$\nabla(H_{\text{BW}})(\mathcal{T}_B(r,s)) = \begin{pmatrix} 2T_1(r,s) + \frac{3\varepsilon}{2}T_1(r,s)^2 - \varepsilon T_1(r,s)T_2(r,s) - \frac{\varepsilon}{2}T_2(r,s)^2 \\ 2T_2(r,s) + \frac{3\varepsilon}{2}T_2(r,s)^2 - \varepsilon T_1(r,s)T_2(r,s) - \frac{\varepsilon}{2}T_1(r,s)^2 \end{pmatrix}.$$

Combining the algebra properties of $H^{\alpha+4}(\mathbb{R})$ with the estimates

$$\begin{cases} |\partial_x(r) \partial^{-1}(s)|_{H^{\alpha+4}} \lesssim |\partial^{-1}s|_{W^{\alpha+4,\infty}} |\partial_x r|_{H^{\alpha+4}} \lesssim |s|_{W^{\alpha+4,1}} |r|_{H^{\alpha+5}} \lesssim |s|_{W^{\alpha+4,1}} |r|_{W^{\alpha+6,1}}, \\ |\partial_x(s) \partial^{-1}(r)|_{H^{\alpha+4}} \lesssim |r|_{W^{\alpha+4,1}} |s|_{W^{\alpha+6,1}} \end{cases}$$

(see (3.42)), we have

$$|\nabla(H_{\text{BW}})(\mathcal{T}_B(r,s))|_{H^{\alpha+4}} \leq C(|r|_{W^{\alpha+6,1}}, |s|_{W^{\alpha+6,1}}).$$

It remains to prove that $\frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \nabla \int_{\mathbb{R}} p(r,s, \partial_x(r) \partial^{-1} s, \partial_x(s) \partial^{-1} r) dx$ is in $\mathcal{C}^1([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R}))$.

The polynomial p is a linear combination of terms of the form

$$r^{n_1} s^{n_2} (\partial_x(r) \partial^{-1} s)^{n_3} (\partial_x(s) \partial^{-1} r)^{n_4},$$

where n_1, n_2, n_3 and n_4 are non-negative integers such that $n_1 + n_2 + n_3 + n_4 \geq 2$. Moreover

$$J_\mu \nabla \int_{\mathbb{R}} r^{n_1} s^{n_2} (\partial_x(r) \partial^{-1} s)^{n_3} (\partial_x(s) \partial^{-1} r)^{n_4} dx = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

where

$$\left\{ \begin{array}{l} M_1 = -\frac{1}{2}F_\mu \left[\partial_x \left(n_1 r^{n_1-1} s^{n_2} (\partial_x(r) \partial^{-1} s)^{n_3} (\partial_x(s) \partial^{-1} r)^{n_4} \right) \right] \\ \quad + \frac{1}{2}F_\mu \left[\partial_x^2 \left(r^{n_1} s^{n_2} n_3 (\partial_x r)^{n_3-1} (\partial^{-1} s)^{n_3} (\partial_x(s) \partial^{-1}(r))^{n_4} \right) \right] \\ \quad + \frac{1}{2}F_\mu \left[r^{n_1} s^{n_2} (\partial_x(r) \partial^{-1} s)^{n_3} (\partial_x s)^{n_4} n_4 (\partial^{-1} r)^{n_4-1} \right], \\ M_2 = \frac{1}{2}F_\mu \left[\partial_x \left(r^{n_1} n_2 s^{n_2-1} (\partial_x(r) \partial^{-1} s)^{n_3} (\partial_x(s) \partial^{-1} r)^{n_4} \right) \right] \\ \quad - \frac{1}{2}F_\mu \left[r^{n_1} s^{n_2} (\partial_x r)^{n_3} n_3 (\partial^{-1} s)^{n_3-1} (\partial_x(s) \partial^{-1}(r))^{n_4} \right] \\ \quad - \frac{1}{2}F_\mu \left[\partial_x^2 \left(r^{n_1} s^{n_2} (\partial_x(r) \partial^{-1} s)^{n_3} n_4 (\partial_x s)^{n_4-1} (\partial^{-1} r)^{n_4} \right) \right], \end{array} \right.$$

where we used $\partial_x \partial^{-1} u = u$ when $u \in L^1(\mathbb{R})$ (see Definition/Property 3.3.12). The property $n_1 + n_2 + n_3 + n_4 \geq 2$ implies that the terms containing $\partial^{-1} s$ or $\partial^{-1} r$ are always multiplied by a function in a Sobolev space $W^{\beta,1}(\mathbb{R})$ with $\beta \geq 0$.

We recall now the expression of $\frac{\partial \mathcal{T}_B}{\partial(r,s)}$:

$$\frac{\partial \mathcal{T}_B}{\partial(r,s)} = \begin{pmatrix} 1 + \frac{\varepsilon}{4}s + \frac{\varepsilon}{4}\partial^{-1}(s)\partial_x(\circ) & \frac{\varepsilon}{4}\partial_x(r)\partial^{-1}(\circ) + \frac{\varepsilon}{4}(r+s) \\ \frac{\varepsilon}{4}\partial_x(s)\partial^{-1}(\circ) + \frac{\varepsilon}{4}(r+s) & 1 + \frac{\varepsilon}{4}r + \frac{\varepsilon}{4}\partial^{-1}(r)\partial_x(\circ) \end{pmatrix}.$$

From the above computations we get

$$\frac{\partial \mathcal{T}_B}{\partial(r,s)} J_\mu \nabla \int_{\mathbb{R}} p(r,s, \partial_x(r) \partial^{-1}(s), \partial_x(s) \partial^{-1}(r)) dx \quad (3.46)$$

$$= \begin{pmatrix} M_1 + \frac{\varepsilon}{4}sM_1 + \frac{\varepsilon}{4}\partial^{-1}(s)\partial_x(M_1) + \frac{\varepsilon}{4}\partial_x(r)\partial^{-1}(M_2) + \frac{\varepsilon}{4}(r+s)M_2 \\ \frac{\varepsilon}{4}\partial_x(s)\partial^{-1}(M_1) + \frac{\varepsilon}{4}(r+s)M_1 + M_2 + \frac{\varepsilon}{4}rM_2 + \frac{\varepsilon}{4}\partial^{-1}(r)\partial_x M_2 \end{pmatrix}, \quad (3.47)$$

We estimate the terms thus obtained. Using product estimates 3.4.1 and (3.42) we have

$$\left\{ \begin{array}{l} |sM_1|_{H^\alpha} \lesssim |s|_{H^{\alpha+1}} |M_1|_{H^\alpha}, \\ |\partial^{-1}(s)\partial_x M_1|_{H^\alpha} \lesssim |\partial^{-1}s|_{W^{\alpha,\infty}} |M_1|_{H^{\alpha+1}} \lesssim |s|_{W^{\alpha,1}} |M_1|_{H^{\alpha+1}}, \\ |\partial_x(r)\partial^{-1}M_1|_{H^\alpha} \lesssim |r|_{H^{\alpha+1}} |\partial^{-1}M_1|_{W^{\alpha,\infty}} \lesssim |r|_{W^{\alpha+2,1}} |M_1|_{W^{\alpha,1}}. \end{array} \right.$$

Idem for the terms with M_2 . So that it only remains to estimate M_1 and M_2 . We do it for M_1 . Using product estimates 3.4.1 and $n_1 + n_2 + n_3 + n_4 \geq 2$, we get

$$|M_1|_{H^\alpha} \lesssim |\partial^{-1}r|_{W^{\alpha+2,\infty}}^{N_1} |\partial^{-1}s|_{W^{\alpha+2,\infty}}^{N_2} |r|_{H^{\alpha+3}}^{N_3} |s|_{H^{\alpha+3}}^{N_4},$$

where N_1, N_2, N_3 and N_4 are non negative integers such that $(N_3, N_4) \neq (0, 0)$. Then, using (3.42) and Sobolev embeddings, we have

$$|M_1|_{H^\alpha} \lesssim |r|_{W^{\alpha+4,1}}^{N_1+N_3} |s|_{W^{\alpha+4,1}}^{N_2+N_4}.$$

Moreover, using the continuity of F_μ from $W^{\alpha+1,1}(\mathbb{R})$ to $W^{\alpha,1}(\mathbb{R})$ and the algebra properties of $W^{\beta,1}(\mathbb{R})$ for any $\beta \geq 1$, we get

$$|M_1|_{W^{\alpha,1}} \lesssim |\partial^{-1}r|_{W^{\alpha+3,\infty}}^{N_1} |\partial^{-1}s|_{W^{\alpha+3,\infty}}^{N_2} |r|_{W^{\alpha+4,1}}^{N_3} |s|_{W^{\alpha+4,1}}^{N_4} \lesssim |r|_{W^{\alpha+4,1}}^{N_1+N_3} |s|_{W^{\alpha+4,1}}^{N_2+N_4}.$$

□

Combining Proposition 3.3.2 and Theorem 3.3.24 we get Theorem 3.1.14 which we recall here for the sake of clarity.

Theorem 3.3.25. *Let $\alpha \geq 0$. Let $r, s \in \mathcal{C}^1([0, \frac{T}{\varepsilon}], W^{\alpha+7,1}(\mathbb{R}))$ be solutions of the equations (3.43). Let also $\mathcal{T}_I, \mathcal{T}_D$ and \mathcal{T}_B be the transformations of Definition 3.1.13 and define :*

$$\begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} := \mathcal{T}_I(\mathcal{T}_D(\mathcal{T}_B(r, s))). \quad (3.48)$$

Then, there exists $R \in \mathcal{C}^1([0, \frac{T}{\varepsilon}], H^\alpha(\mathbb{R}) \times H^\alpha(\mathbb{R}))$, such that one has :

$$\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} = J\nabla(H_{\text{WW}})(\zeta_{\text{Wh}}, \psi_{\text{Wh}}) + (\mu\varepsilon + \varepsilon^2)R, \quad \forall t \in [0, \frac{T}{\varepsilon}],$$

where H_{WW} is the Hamiltonian defined in (3.2) and $|R|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(\mu_{\max}, |r|_{W^{\alpha+7,1}}, |s|_{W^{\alpha+7,1}})$. In particular, R is uniformly bounded in (μ, ε) for times $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$ in $H^\alpha(\mathbb{R}) \times \dot{H}^{\alpha+1}(\mathbb{R})$.

Démonstration. We start by defining the quantities

$$\begin{pmatrix} \zeta_{\text{Wh}} \\ v_{\text{Wh}} \end{pmatrix} := \mathcal{T}_D(\mathcal{T}_B(r, s)). \quad (3.49)$$

Differentiating (3.49) in time and using Theorem 3.3.24 we get

$$\begin{aligned}\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ v_{\text{Wh}} \end{pmatrix} &= \frac{\partial \mathcal{T}_D}{\partial(r, s)}(\mathcal{T}_B(r, s)) \frac{\partial \mathcal{T}_B}{\partial(r, s)} \partial_t \begin{pmatrix} r \\ s \end{pmatrix} \\ &= \frac{\partial \mathcal{T}_D}{\partial(r, s)}(\mathcal{T}_B(r, s)) J_\mu \nabla(H_{\text{BW}})(\mathcal{T}_B(r, s)) + (\mu\varepsilon + \varepsilon^2) \frac{\partial \mathcal{T}_D}{\partial(r, s)}(\mathcal{T}_B(r, s)) R_1,\end{aligned}$$

where $\frac{\partial \mathcal{T}_D}{\partial(r, s)}(\mathcal{T}_B(r, s)) = \begin{pmatrix} 1 & 1 \\ \mathbb{F}_\mu^{-1}[\circ] & -\mathbb{F}_\mu^{-1}[\circ] \end{pmatrix}$ and R_1 is the rest in Theorem 3.3.24. Using now Proposition 3.3.6 we get

$$\begin{aligned}\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ v_{\text{Wh}} \end{pmatrix} &= \tilde{J} \nabla(\tilde{H}_0 + \varepsilon \tilde{H}_1)(\mathcal{T}_D(\mathcal{T}_B(r, s))) + (\mu\varepsilon + \varepsilon^2) R_2 \\ &= \tilde{J} \nabla(\tilde{H}_0 + \varepsilon \tilde{H}_1)(\zeta_{\text{Wh}}, v_{\text{Wh}}) + (\mu\varepsilon + \varepsilon^2) R_2,\end{aligned}$$

where $\tilde{J} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$ and $|R_2|_{H^\alpha \times H^\alpha} \leq C(|r|_{W^{\alpha+7,1}}, |s|_{W^{\alpha+7,1}})$ (because for any $\beta \geq 0$ and any $u \in W^{\beta+1,1}(\mathbb{R})$, $|\mathbb{F}_\mu^{-1}[u]|_{H^\beta} \lesssim |u|_{H^{\beta+1/2}} \lesssim |u|_{W^{\beta+1}}$). Changing the unknowns into $(\zeta_{\text{Wh}}, \psi_{\text{Wh}})$ we obtain

$$\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} = J \nabla(H_0 + \varepsilon H_1)(\zeta_{\text{Wh}}, \psi_{\text{Wh}}) + (\mu\varepsilon + \varepsilon^2) R_3,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $|R_3|_{H^\alpha \times \dot{H}^{\alpha+1}} \leq C(|r|_{W^{\alpha+7,1}}, |s|_{W^{\alpha+7,1}})$. So that

$$\begin{aligned}\partial_t \begin{pmatrix} \zeta_{\text{Wh}} \\ \psi_{\text{Wh}} \end{pmatrix} &= J \nabla(H_{\text{WW}})(\zeta_{\text{Wh}}, \psi_{\text{Wh}}) + (J \nabla(H_0 + \varepsilon H_1)(\zeta_{\text{Wh}}, \psi_{\text{Wh}}) - J \nabla(H_{\text{WW}})(\zeta_{\text{Wh}}, \psi_{\text{Wh}})) \\ &\quad + (\mu\varepsilon + \varepsilon^2) R_2.\end{aligned}$$

Using Proposition 3.3.2 we get the result. The uniform boundedness in (μ, ε) for times $t \in [0, \frac{T}{\max(\mu, \varepsilon)}]$ of the remainder comes from the lemma 3.3.23. \square

It only remains to prove Corollary 3.1.16.

Démonstration. We construct (r_0, s_0) such that

$$|\mathcal{T}_I(\mathcal{T}_D(\mathcal{T}_B(r_0, s_0))) - (\zeta_0, \psi_0)|_{H^\alpha \times \dot{H}^{\alpha+1}} \lesssim \varepsilon^2. \quad (3.50)$$

We remark first that the transformations \mathcal{T}_I and \mathcal{T}_D are invertible with, for any $\beta \geq 0$, $\mathcal{T}_I^{-1} : W^{\beta,1}(\mathbb{R}) \times \dot{W}^{\beta+1,1}(\mathbb{R}) \rightarrow W^{\beta,1}(\mathbb{R}) \times W^{\beta,1}(\mathbb{R})$ and $\mathcal{T}_D^{-1} : W^{\beta+1,1}(\mathbb{R}) \times W^{\beta+1,1}(\mathbb{R}) \rightarrow W^{\beta,1}(\mathbb{R}) \times W^{\beta,1}(\mathbb{R})$ due to the continuity of F_μ from $W^{\beta+1,1}(\mathbb{R})$ to $W^{\beta,1}(\mathbb{R})$. Moreover, by definition (see (3.41)), we can write \mathcal{T}_B under the form

$$\mathcal{T}_B = \text{Id} + \varepsilon \widetilde{\mathcal{T}}_B,$$

where Id is the identity and for any $\beta \geq 0$, $\widetilde{\mathcal{T}}_B : W^{\beta+1,1}(\mathbb{R}) \times W^{\beta+1,1}(\mathbb{R}) \rightarrow W^{\beta,1}(\mathbb{R}) \times W^{\beta,1}(\mathbb{R})$. We define the transformation $\mathcal{T}_B^{\text{inv}} : W^{\beta+1,1}(\mathbb{R}) \times W^{\beta+1,1}(\mathbb{R}) \rightarrow W^{\beta,1}(\mathbb{R}) \times W^{\beta,1}(\mathbb{R})$ by

$$\mathcal{T}_B^{\text{inv}} = \text{Id} - \varepsilon \widetilde{\mathcal{T}}_B.$$

One has for any $(\eta, w) \in W^{\beta+2,1}(\mathbb{R}) \times W^{\beta+2,1}(\mathbb{R})$, $\mathcal{T}_B(\mathcal{T}_B^{\text{inv}}(\eta, w)) = \begin{pmatrix} \eta \\ w \end{pmatrix} + \varepsilon^2 R$ where $|R|_{W^{\beta,1} \times W^{\beta,1}} \leq C(|\eta|_{W^{\beta+2,1}}, |w|_{W^{\beta+2,1}})$. So that

$$(r_0, s_0) = \mathcal{T}_B^{\text{inv}}(\mathcal{T}_D^{-1}(\mathcal{T}_I^{-1}(\zeta_0, \psi_0))) \quad (3.51)$$

satisfy (3.50). □

3.4 Technical Tools

Proposition 3.4.1. (*Product estimates*)

1. Let $t_0 > 1/2$, $s \geq -t_0$ and $f \in H^\alpha \cap H^{t_0}(\mathbb{R})$, $g \in H^\alpha(\mathbb{R})$. Then $fg \in H^\alpha(\mathbb{R})$ and

$$|fg|_{H^\alpha} \lesssim |f|_{H^{\max(t_0, \alpha)}} |g|_{H^\alpha}$$

2. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that $\alpha_1 + \alpha_2 \geq 0$. Then for all $\alpha \leq \alpha_j$ ($j = 1, 2$) and

$\alpha < \alpha_1 + \alpha_2 - 1/2$, and all $f \in H^{\alpha_1}(\mathbb{R}), g \in H^{\alpha_2}(\mathbb{R})$, one has $fg \in H^\alpha(\mathbb{R})$ and

$$|fg|_{H^\alpha} \lesssim |f|_{H^{\alpha_1}} |g|_{H^{\alpha_2}}$$

Démonstration. See Appendix B.1 in [Lan13b]. □

Proposition 3.4.2. (*Composition estimates*) Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $G(0) = 0$. Also, let $t_0 > 1/2$, $\alpha \geq 0$ and $f \in H^{\max(t_0, \alpha)}(\mathbb{R})$. Then $G(f) \in H^\alpha(\mathbb{R})$ and

$$|G(f)|_{H^\alpha} \leq C(|f|_{H^{\max(t_0, \alpha)}})|f|_{H^\alpha}.$$

Démonstration. See Appendix B.1 in [Lan13b]. □

Proposition 3.4.3. (*Quotient estimates*) Let $t_0 > 1/2$, $\alpha \geq -t_0$ and $c_0 > 0$. Also let $f \in H^\alpha(\mathbb{R})$ and $g \in H^\alpha \cap H^{t_0}(\mathbb{R})$ be such that for all $x \in \mathbb{R}$, one has $1 + g(X) \geq c_0$. Then $\frac{f}{1+g}$ belongs to $H^\alpha(\mathbb{R})$ and

$$\left| \frac{f}{1+g} \right|_{H^\alpha} \leq C\left(\frac{1}{c_0}, |g|_{H^{\max(t_0, \alpha)}}\right) |f|_{H^\alpha}$$

Démonstration. See Appendix B.1 in [Lan13b]. □

Definition 3.4.4. We say that a Fourier multiplier $F(D)$ is of order α ($\alpha \in \mathbb{R}$) and write $F \in \mathcal{S}^\alpha$ if $\xi \in \mathbb{R} \mapsto F(\xi) \in \mathbb{C}$ is smooth and satisfies

$$\forall \xi \in \mathbb{R}, \forall \beta \in \mathbb{N}, \sup_{\xi \in \mathbb{R}} \langle \xi \rangle^{\beta - \alpha} |\partial^\beta F(\xi)| < \infty.$$

We also introduce the seminorm

$$\mathcal{N}^\alpha(F) = \sup_{\beta \in \mathbb{N}, \beta \leq 4} \sup_{\xi \in \mathbb{R}} \langle \xi \rangle^{\beta - \alpha} |\partial^\beta F(\xi)|.$$

Proposition 3.4.5. Let $t_0 > 1/2$, $\alpha \geq 0$ and $F \in \mathcal{S}^\alpha$. If $f \in H^\alpha \cap H^{t_0+1}$ then, for all $g \in H^{\alpha-1}$,

$$|[F(D), f]g|_2 \leq \mathcal{N}^\alpha(F) |f|_{H^{\max(t_0+1, \alpha)}} |g|_{H^{\alpha-1}}.$$

Démonstration. See Appendix B.2 in [Lan13b] for a proof of this proposition. □

Proposition 3.4.6. *Let $\alpha \geq 2$. Let $\zeta \in H^{\alpha+2}(\mathbb{R})$ be such that (3.3) is satisfied and $\psi \in \dot{H}^{\alpha+2}(\mathbb{R})$. Then one has*

$$\left| \frac{1}{\mu} \mathcal{G}^\mu[\varepsilon\zeta]\psi \right|_{H^\alpha} \leq M(s+2) |\psi|_{\dot{H}^{\alpha+2}}.$$

Démonstration. This is a direct consequence of Theorem 3.15 in [Lan13b]. □

Chapitre 4

Comparison of models for the simulation of landslide generated tsunamis

Article issu d'une collaboration avec E. Audusse¹, J.G Caldas Steinstraesser², P. Heinrich³, A. Paris⁴ et M. Parisot⁵, lors du CEMRACS 2019. Le résultat de ce travail a été publié dans le journal ESAIM : PROCEEDINGS AND SURVEYS en 2021, [Aud+21].

Abstract: In this paper, we analyze the relevance of the use of the shallow water model and the Boussinesq model to simulate tsunamis generated by a landslide. In a first part, we determine if the two models are able to reproduce waves generated by a landslide. Each model has drawbacks but it seems that it is possible to use them together to improve the simulations. In a second part we try to recover the landslide displacement from the generated wave. This problem is formulated as a minimization problem and we limit the number of parameters to determine assuming that the bottom can be well described by an empirical law.

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4.1 Introduction

This work is devoted to the numerical comparison of three different fluid mechanics models in simulating landslide tsunamis : the Navier-Stokes model, the Boussinesq model and the shallow water model. The former is considered here as a reference since the two latter are simplifications of it. However, its high computational cost and potential numerical difficulties, such as an excessive numerical diffusion, the definition of boundaries conditions or reflections at the boundaries, motivate the use of simplified models and make the computed solution not always clearly closer to the reality for some applications such as the simulation of tsunamis because of the large space scale involved. The simplified models are based on a depth-averaging. For these reasons simulations are much less expensive than with those solving the Navier-Stokes model. The shallow water model neglects the dispersive effects by assuming a hydrostatic pressure. The Boussinesq model takes into account the dispersive effects assuming small perturbation from a main water depth, but it is based on a non-conservative form that can leads to wrong solutions in the case of strongly variations of the free surface. A deeper description of the models is done in 4.2.2 as well as a description of the numerical strategy used to simulate them.

The depth-averaging models are largely used in the simulation of tsunamis [HP00 ; Héb+02 ; Le +03 ; Rod+13 ; Pou+17 ; Par+9a ; Par+9b ; LMP12]. In particular, the shallow water model is able to predict with a good accuracy the arrival time on the coast of the tsunami. However the amplitude of the wave is not well predict by the shallow water model, in particular second and third waves which are sometimes more destructive than the first despite they are smaller. The Boussinesq model leads to a better description of both the wave slope and the run-up area.

We address the question of the numerical strategy to recover the tsunami waves generated by landslides. This question is not trivial at the scale of the application and several strategies exist in the literature [Wat+03 ; GW05]. We propose to explore the possibility to recover the bottom displacement from the free surface elevation using a inverse problem of a simplified model. This strategy assumes two technical steps that have to be first validated and this is the aim of this proceeding. In a first part 4.3, we investigate the ability of the simplified models to reproduce the wave obtained by the reference model (Navier-Stokes simulated by OpenFOAM). The important question lies on the strategy used to represent the landslide and in the current work we propose to recover the bottom displacement from the result of the reference model. The drawback of this strategy is that

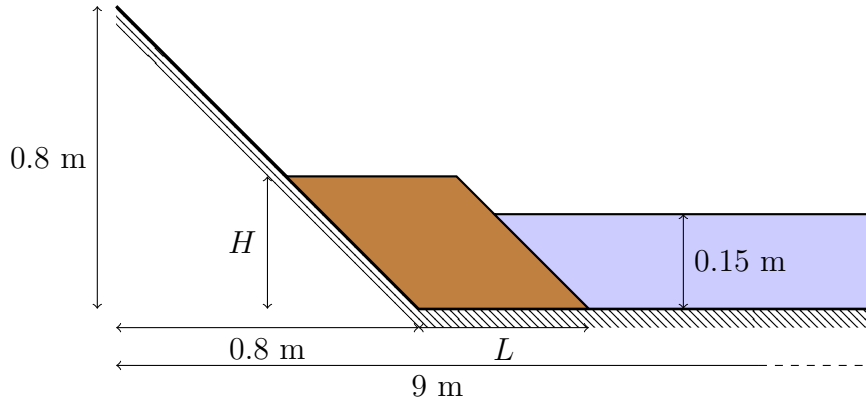


FIGURE 4.1 – Initial state of the test case. The landslide is in brown and the water is in blue.

in practice, we do not want to simulate the reference model at the scale of the application. In the second part 4.4, we investigate the possibility of recovering the bottom displacement from the free surface elevation at given positions. Inverse problems of models that develop discontinuous solutions are generally complex, see [FCD19]. Nevertheless, this preliminary study tries to identify the main practical difficulties.

4.2 Description of the study

4.2.1 The test case

The test case we based our study on is inspired from the literature, in particular [VSK14]. It is a 2D test case, where $x \in \mathbb{R}$ is the horizontal coordinate and $z \in \mathbb{R}$ is the vertical coordinate, on an idealized bathymetry, called hereafter the substratum, defined by

$$B(x) = \max(0, D - x) - D \quad (4.1)$$

with the offshore depth $D = 0.15$ m. The time coordinate will be denoted by $t \in \mathbb{R}^+$. At the initial state, the landslide is at rest and has the shape of a parallelogram of vertices $(x_0, -D)$, $(x_0 + L, -D)$, $(x_0 + L - H, H - D)$ and $(x_0 - H, H - D)$. Several values of the height H and the length L will be explored in the study. The water is initially at rest with a flat free surface at $z = 0$ m. The initial state is illustrated on 4.1.

The boundary conditions depend on the model and are described in 4.2.2.

4.2.2 The models and their numerical resolution

The numerical simulations of the landslide-generated waves were performed considering three models : the Navier-Stokes model, the Boussinesq model and the shallow water model. Because of their different mathematical properties, they are solved using different numerical tools presented in what follows. Note that the numerical strategies presented for each model are not the only ones existing in the literature. They have advantages as well as drawbacks that we try to summarize in this subsection.

For the sake of simplicity, we do not consider the second horizontal direction y since our test case does not depend on it. Note however that the models as well as the numerical tools used in this work can take it into account, (*i.e.* 3D (x, y, z) simulations for the Navier-Stokes model and 2D (x, y) simulations for the shallow water model and the Boussinesq model).

4.2.2.1 OpenFOAM solver of the free surface incompressible Navier-Stokes model

The fluid dynamics for macroscopic applications are classically described by the incompressible Navier-Stokes equations. However, the modeling of tsunamis generated by landslide also requires the description of the water surface, also called free surface, and the description of the dynamics of the landslide. In this work we propose to adopt the VOF (Volume Of Fluid) strategy, *i.e.* globally describes the flow solving the incompressible Navier-Stokes equations in the whole computational domain with physical parameters such as density and viscosity depending on the fluid locally present. More precisely, the fluid is identified using the color function $\phi(t, x, z) \in [0, 2]$ satisfying the following advection equation

$$\partial_t \phi + u \cdot \nabla \phi = 0 \quad (4.2)$$

where $u(t, x, z) \in \mathbb{R}^2$ is the velocity of the fluid. When $\phi = 0$, $\phi = 1$ and $\phi = 2$ we consider that the fluid is respectively the air, the water and the landslide. The velocity satisfies the Navier-Stokes equations

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p + \nabla \cdot (\mu \nabla u) + \rho g e_z \quad (4.3)$$

where e_z is the unit downward vector, g is the gravitational acceleration, $\rho(\phi) \in \mathbb{R}^+$ is the density of the fluid and $\mu(\phi) \in \mathbb{R}^+$ its dynamic viscosity. These two parameters are

piecewise linear reconstructions depending on the color function, *i.e.*

$$\kappa(\phi) = \begin{cases} (1 - \phi)\kappa_0 + \phi\kappa_1 & \text{if } 0 \leq \phi \leq 1 \\ (2 - \phi)\kappa_1 + (\phi - 1)\kappa_2 & \text{if } 1 < \phi \leq 2. \end{cases} \quad (4.4)$$

where κ stands for ρ and μ and the parameter ρ_0 (resp. μ_0) stands for the density (resp. viscosity) of the air, ρ_1 (resp. μ_1) stands for the density (resp. viscosity) of the water, and ρ_2 (resp. μ_2) stands for the density (resp. viscosity) of the landslide. In this study we set these parameters close to the physical values, *i.e.* $\rho_0 = 1 \text{ kg}\cdot\text{m}^3$, $\rho_1 = 1000 \text{ kg}\cdot\text{m}^3$, $\rho_2 = 1500 \text{ kg}\cdot\text{m}^3$, $\mu_0 = 1.48 \cdot 10^{-5} \text{ Pa}\cdot\text{s}$, $\mu_1 = 10^{-3} \text{ Pa}\cdot\text{s}$, $\mu_2 = 2.222 \text{ Pa}\cdot\text{s}$. The pressure $p(t, x)$ is such that the flow satisfies the divergence free condition, *i.e.*

$$\nabla \cdot u = 0. \quad (4.5)$$

The set of equations is completed by an initial condition and the boundary conditions described in the test case, see 4.2.1.

The numerical approximation of the Navier-Stokes model is realized using the module *multiphaseInterFoam* of the C++ open source software OpenFOAM (Open-source Field Operation And Manipulation) [04].

We assume a slip boundary condition at the substratum $z = B(x)$, *i.e.* $u \cdot \nabla B = 0$. The friction at the substratum is neglected to limit the difference between the models. However, it is not always clear how to translate a no-slip boundary condition or a Navier Boundary condition from the Navier-Stokes model to the shallow water model or the Boussinesq model. For computational reasons, we need to introduce a top boundary condition, set at $z = 0.65 \text{ m}$, and a left and right boundary conditions, set respectively at $x = -0.65 \text{ m}$ and $x = 8.35 \text{ m}$. All of them are defined as open boundary conditions. The space step is set in the two directions to $5 \cdot 10^{-3} \text{ m}$ in order to limit the numerical diffusion. But it makes the computation time fast enough to allow several sizes of landslides. The time step used for the computations is set to 10^{-3} s whereas the frequency of the output is set to $2 \cdot 10^{-2} \text{ s}$.

4.2.2.2 Hydra solver of the shallow water model

The shallow water model is one of the simplest for hydrology. Its well understood mathematical structure allowed very efficient and robust numerical computation. It can be

derived from the multiphase Navier-Stokes model (4.2)–(4.5) integrating only the water phase, *i.e.* $\phi = (1, 0)$, for an irrotational flow and under a long-wave assumption [Lan13a]. The classical form of the model reads

$$\begin{cases} \partial_t \bar{h} + \partial_x(\bar{h}\bar{u}) = 0, \\ \partial_t(\bar{h}\bar{u}) + \partial_x(\bar{h}\bar{u}^2 + \frac{g}{2}\bar{h}^2) = g\bar{h}\partial_x d. \end{cases} \quad (4.6)$$

where $\bar{h}(t, x)$ is the water depth and $\bar{u}(t, x)$ is the main horizontal velocity. The bottom elevation $d(t, x)$ is an input of the model and we discuss further 4.3.1.1 the strategy to obtain it.

The shallow water model is solved using the numerical tools Hydra that is based on a finite volume Godunov-type solver with MUSCL second order reconstruction and a second order Runge-Kutta time scheme [Bou04]. The bathymetry source term is treated using the hydrostatic reconstruction [Aud+04].

The left and right boundaries are set respectively at $x = -0.65$ m and $x = 8.35$ m and open boundary conditions are used. Note that the open boundary condition is not treated in the same way by the different models. The horizontal space domain was discretized with an homogeneous grid of space step $5 \cdot 10^{-3}$ m. We used an adaptive time step based on the CFL condition with the CFL parameter set to 0.9. However, the outputs are separated in time by 0.1 s.

4.2.2.3 AVALANCHE solver of the Boussinesq model

The Boussinesq model is largely used for the propagation of waves in ocean modeling. It can be derived from the multiphase Navier-Stokes model (4.2)–(4.5) by integrating only the water phase, *i.e.* $\phi = (1, 0)$, for an irrotational flow, weakly nonlinear and weakly dispersive [Lan13a]. The Boussinesq model can be written under the form [LPG08]

$$\begin{cases} \partial_t \underline{\eta} + \partial_x(\underline{h}\underline{u}) = -\partial_t d, \\ \partial_t \underline{u} + \underline{u}\partial_x \underline{u} = -g\partial_x \underline{\eta} + \frac{d^3}{3}\partial_x^2(\partial_t \underline{u}) + \frac{d}{2}\partial_t \underline{u}\partial_x^2 \underline{u} + d\partial_x d\partial_x(\partial_t \underline{u}) \end{cases} \quad (4.7)$$

where $\underline{\eta}(t, x)$ is the free surface elevation, $\underline{h}(t, x) = \underline{\eta} + d$ is the water depth, $\underline{u}(t, x)$ is the main horizontal velocity. As for the shallow water model, the bottom elevation $d(t, x)$ is an input of the model.

The Boussinesq model is solved using the numerical tools AVALANCHE developed at the CEA to simulate tsunamis (second order finite difference methods with Crank-Nicolson time scheme). It was successfully used to study subaerial or submarine landslide tsunamis [HP00; Héb+02; Le +03; Rod+13; Pou+17; Par+9a; Par+9b].

As well as in the case of the shallow water model, the friction is neglected and open boundary conditions are considered at the left ($x = -0.65$ m) and right ($x = 8.35$ m) boundaries. We use the same space step of $5 \cdot 10^{-3}$ m, an adaptive time step based on the CFL condition with the CFL parameter set to 0.9 and the outputs are separated in time by 0.1 s.

4.3 Comparison of models

4.3.1 The methodology

From the description of the models 4.2.2, an important difference can be noticed by the fact that the simplified models (4.6) and (4.7) do not describe the dynamic of the landslide but only the dynamic of the water. Several strategies exist in the literature to model landslide using the same kind of simplified model [Zec+09; Fer+17]. However the difference of the dynamic of the landslide obtained by this strategy compared to the result of the multiphase Navier-Stokes model (4.2)–(4.5) will make the comparison of the free surface unclear. Since our study focus on the waves generation, we adopt another strategy where the efficient bottom of the simplified model is not the substratum given in (4.1) but is obtained by post-treatment of the OpenFOAM Navier-Stokes results, see 4.3.1.1.

We would like to point out another major difference to keep in mind in the comparison of the models. The friction in the simplified models is neglected. This assumption is consistent with the slip condition at the substratum in the simulation of the Navier-Stokes model. However the bottom elevation is located at the interface between the landslide and the water for the simplified models. Because of the viscosity of the landslide, a friction should be considered when the landslide thickness does not vanish. Nevertheless, this value is very difficult to identify and we assumed it small enough to be neglected.

4.3.1.1 Strategy of computation of the bottom displacement

In the current section we detail the computational strategy used to define the bottom elevation in the simplified models (4.6) and (4.7). It is illustrated in 4.2.

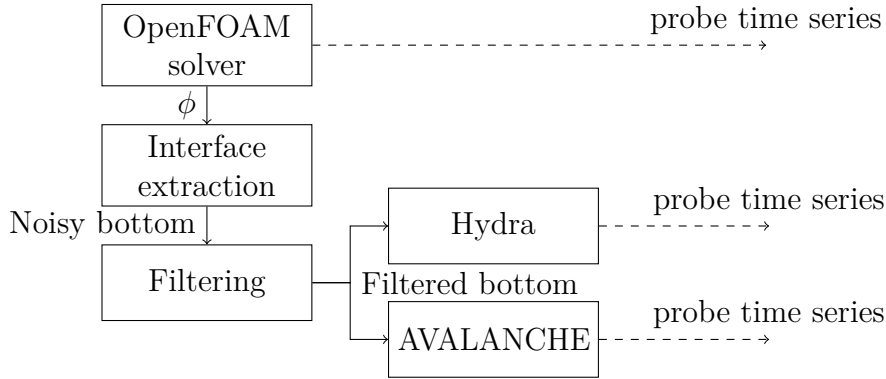


FIGURE 4.2 – Strategy of computation.

First of all, the multiphase Navier-Stokes model (4.2)–(4.5) is solved using **OpenFOAM**.

For each output of this computation, we extracted the interface between the landslide and the water, defined here as the surface satisfying $\phi = 1.5$, to be used as the bottom elevation for the simplified model (4.6) and (4.7). The **interface extraction** was realized using the software ParaView that allowed to obtain series of points on each surface where it crosses the mesh of the computation, so not regularly spaced. These series presented some oscillations, probably coming from the VOF strategy, and were polluted by some points at the substratum $z = B(x)$, in particular when the landslide becomes thin.

To clean these files we applied a **filter** that first removed the points too close to the substratum, *i.e.* $z < B(x) + 5 \cdot 10^{-3}$, then we applied a Laplacian filter to smooth the angular regions. Eventually we interpolated (linear interpolation with the left and right closer points) the series on the mesh of the simplified models. We obtained a description of the bottom elevation on the mesh of the simplified models at each output time of the OpenFOAM simulation, see 4.3. However, because of the adaptive time step strategy, we needed to interpolate these files during the computation to obtain the bottom elevation at each time step of **Hydra** and **AVALANCHE**. Once again, a linear interpolation between the closest past and future filtered bottom was used.

4.3.1.2 Simulated data based and treatment

Several simulations were performed using the strategy described in 4.3.1.1 by varying the shape of the initial landslide block using the height H (between 0.2 and 0.5 m every 0.05

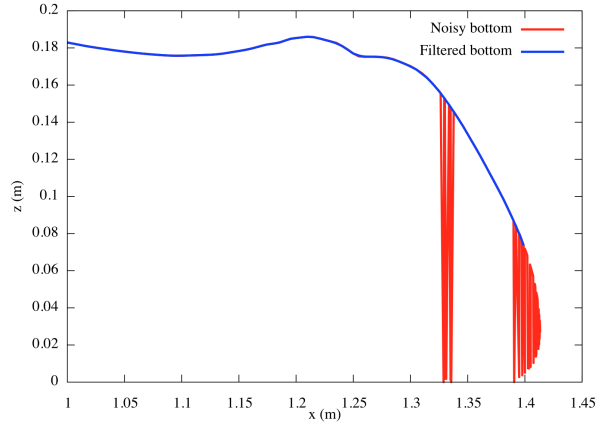


FIGURE 4.3 – Illustration of the filtering process. Red line is the bottom after the extraction step. Blue line is the bottom after the filtering step.

m) and the length L (between 0.1 and 0.6 m every 0.05 m), see 4.1. For each simulation, we compared the time series of the free surface at some gauges. This strategy follows the experimental process in which the free surface elevation is usually recovered only by some buoys at given positions.

In a first time, we compare qualitatively the results of the different models. Our first observation is that there are two distinct regions. In the area where the landslide is moving, the predominant physical phenomenon is the generation of the wave due to the variation of the bottom. Hereafter, we call this region the wave generation zone. The comparisons in this zone are described in 4.3.2.1. Far from the landslide, the dynamic of the wave is essentially a propagation phenomenon. Consequently, we call this region the wave propagation zone. The comparisons in this zone are described in 4.3.2.2.

Obviously the most important region is at the transition between the two zones defined above, around the place that the landslide finally reaches. In order to realize statistical analysis, we would like to compare the flow at a unique position, ideally in the transition area. However, it is not obvious how to clearly determine this point and it seems relevant to not change the position of the probe depending on the landslide. Hopefully, for a large majority of the simulated landslide, the right bound of the wave generation zone is close to $x = 1.15$ m. So the probe $x = 1.35$ m seems very relevant because it is located just after the place where the landslide stops for almost all the simulations. More precisely, among the 77 simulations we ran, 61 were used to identify quantities of interest to characterize the generated waves, see 4.3.2.3. In the other simulations, the landslide reaches the probe $x = 1.35$ m, completely changing the results. We also highlight the regime of validity of

the simplified models.

Eventually, in 4.3.2.4, we propose a numerical strategy trying to combine the best of both simplified models by using one or another depending on the position : we use the shallow water model (4.6) in the wave generation zone and the Boussinesq model (4.7) in the wave propagation zone. This domain decomposition strategy is well known and already used in the case of breaking waves, where the Boussinesq model is solved in the propagation zone and the shallow water model in the surf zone [FKR16].

4.3.2 The numerical results

In the two first parts, we focus on four selected cases :

- i) $(H, L) = (0.25 \text{ m}, 0.2 \text{ m})$,
- ii) $(H, L) = (0.3 \text{ m}, 0.4 \text{ m})$,
- iii) $(H, L) = (0.2 \text{ m}, 0.6 \text{ m})$,
- iv) $(H, L) = (0.45 \text{ m}, 0.4 \text{ m})$.

4.3.2.1 Comparison in the wave generation zone

4.4 represents the time series of the water surface elevation for the four test cases at gauge located at the border of their wave generation zone, *i.e.* case i) $x = 0.55 \text{ m}$, case ii) $x = 0.75 \text{ m}$, case iii) $x = 0.95 \text{ m}$ and case iv) $x = 1.15 \text{ m}$. In the same graphs we plot the landslide thickness (red dashed line) when the landslide reaches the position of the probe.

We first observe that both simplified models, for which the dynamic of the landslide was imposed as a given bathymetry, are very sensitive to the variation of the bathymetry. Indeed, in the beginning of the red dashed line in each graph in 4.4, we can observe the formation of peaks for the shallow water and the Boussinesq simulations. These peaks seem artificial, due to our strategy of imposing the bottom displacement and the fact that we do not have a proper energy closure. Studies in the literature that impose the landslide displacement using the same kind of simplified model (for example[Zec+09; Fer+17]) do not present such kind of peaks. However the dynamic of the landslide is significantly different.

We also observe that the shallow water model (4.6) presents qualitatively better results than the Boussinesq model (4.7) in the wave generation zone. This behavior was observed for all the test cases performed. Our explanation of this observation is that the Boussinesq model is weakly nonlinear. It means the water elevation in the dispersive part of the model

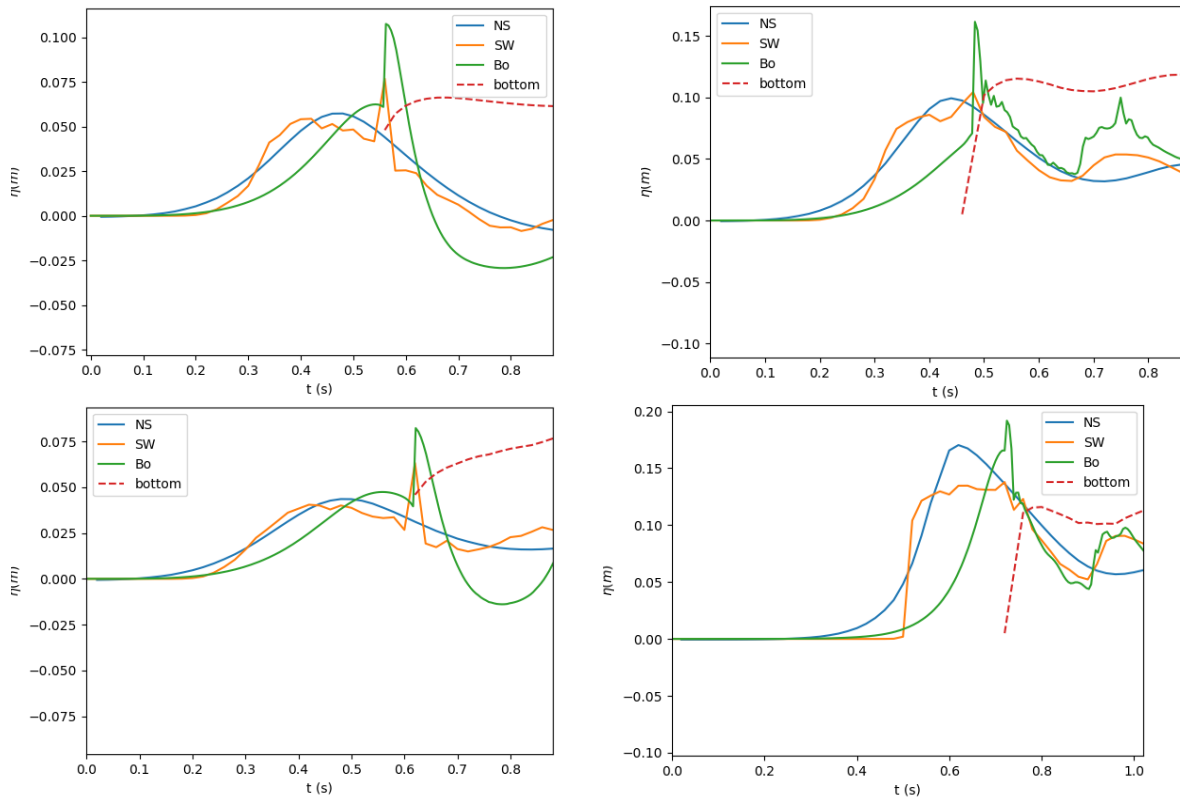


FIGURE 4.4 – Comparison between the results of the three models for four selected test cases, in the wave generation zone. The top left corresponds to the case i) at the probe $x = 0.55$ m. The top right corresponds to the case ii) at the probe $x = 0.75$ m. The bottom left corresponds to the case iii) at the probe $x = 0.95$ m. The bottom right corresponds to the case iv) at the probe $x = 1.15$ m. Blue, orange and green lines represent respectively the Navier-Stokes, the shallow water and the Boussinesq simulations. Red dashed line represents the evolution of the bottom.

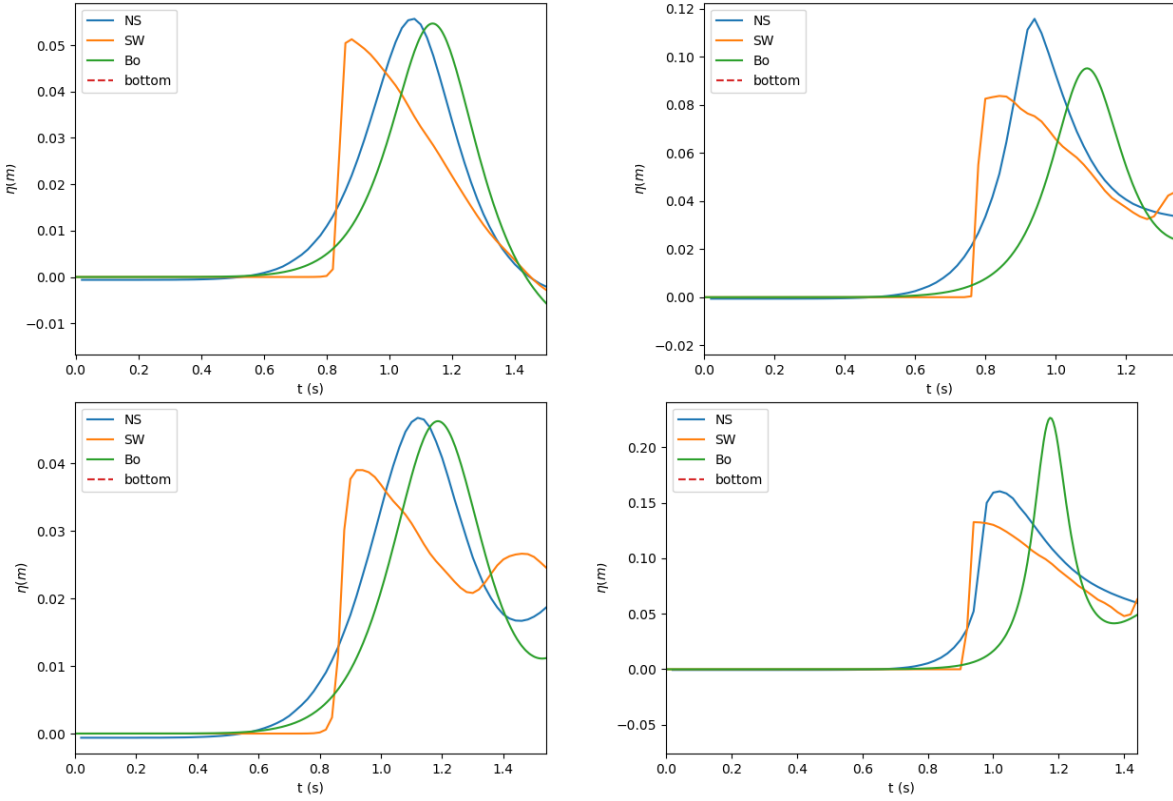


FIGURE 4.5 – Comparison between the results of the three models for four selected test cases, in the wave generation zone. The top left corresponds to the case i) at the probe $x = 1.35$ m. The top right corresponds to the case ii) at the probe $x = 1.55$ m. The bottom left corresponds to the case iii) at the probe $x = 1.75$ m. The bottom right corresponds to the case iv) at the probe $x = 1.95$ m. Blue, orange and green lines represent respectively the Navier-Stokes, the shallow water and the Boussinesq simulations.

(RHS of the momentum equation) was replaced by the mean depth d . However in the derivation of the model, the mean depth should not change in time, or at least much slower than the free surface, which is not the case in our test case. In addition, we can remark that the advection term of the Boussinesq model (LHS of the momentum equation) is written in the non-conservative form. It is well known that this form is not suitable for discontinuous solutions, at least in the case of the shallow water model, because it leads to wrong weak solutions. It is generally not a problem for the Boussinesq model, which does not develop discontinuous solutions. However, in our test case, because of the discontinuous bottom, discontinuous solutions appear.

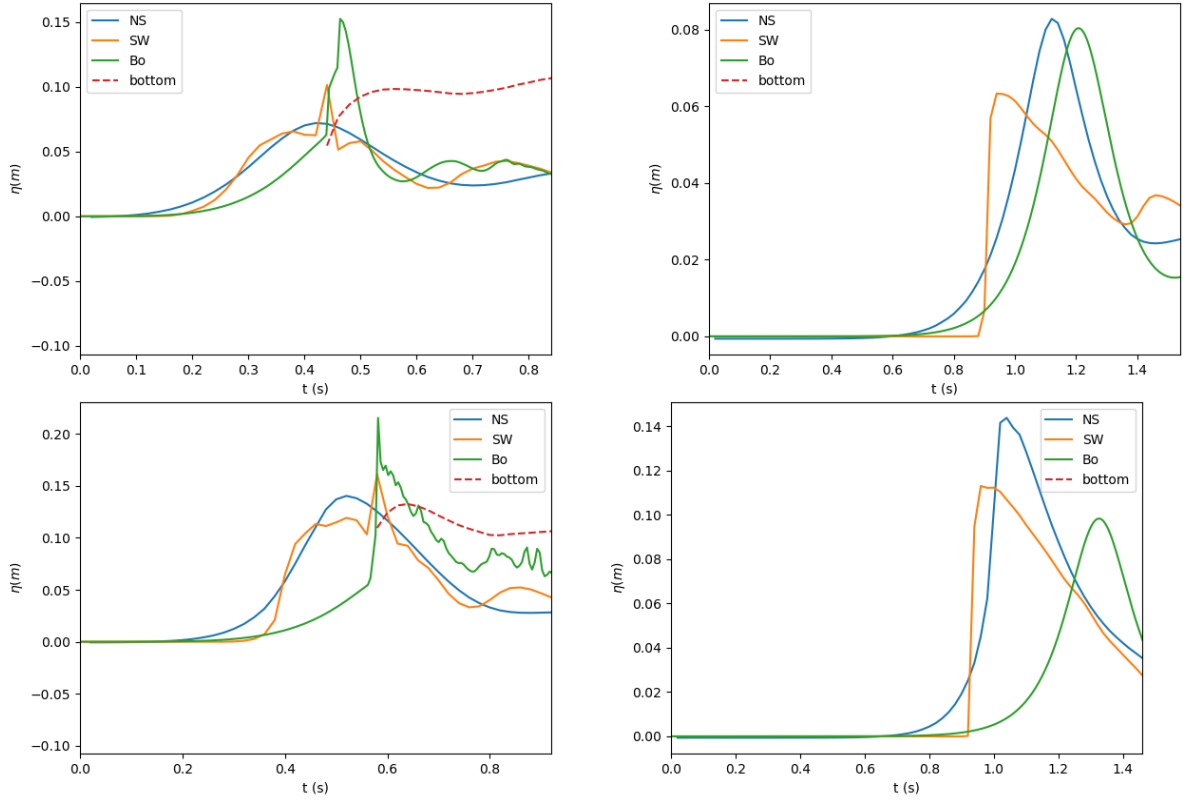


FIGURE 4.6 – Comparison between the results of the three models with the same landslide size. The top line corresponds to the parameters $(H, L) = (0.25 \text{ m}, 0.45 \text{ m})$ while the bottom line corresponds to the parameters $(H, L) = (0.45 \text{ m}, 0.25 \text{ m})$. The left column corresponds to a probe at $x = 0.75 \text{ m}$ (wave generation zone) while the right column corresponds to a probe at $x = 1.75 \text{ m}$ (wave propagation zone).

4.3.2.2 Comparison in the wave propagation zone

Here we focus on the wave propagation zone for the same test cases as in 4.3.2.1. The time series of the water surface elevation at the gauges case i) $x = 1.35 \text{ m}$, case ii) $x = 1.55 \text{ m}$, case iii) $x = 1.75 \text{ m}$ and case iv) $x = 1.95 \text{ m}$ are represented in 4.5. By definition of the wave propagation zone, the bottom does not reach the gauges.

We remark that in the wave propagation zone, the shallow water is not always the best model in recovering the results of the Navier-Stokes model. Moreover the results differ strongly from one test case to another. While for the case i) and case iii) the Boussinesq model is better, for the case iv) it is the shallow water model. Finally, for the case ii) both models seem quite far from the Navier-Stokes solution.

In the simulations realized in this work, the Boussinesq model behaves better in the

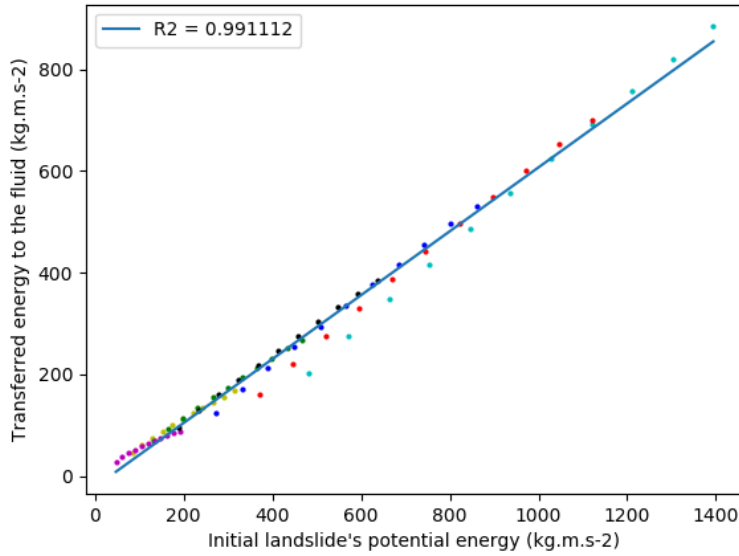


FIGURE 4.7 – Correlation between the initial energy in the landslide and the energy transferred to the water. Each point corresponds to a parameter couple (H, L) while the same color corresponds to the same H .

wave propagation zone for small landslides, while the shallow water model behaves better for big landslides. One can explain this by the fact that the Boussinesq model is a better model for the wave propagation since it takes into account the dispersive effects. However, for big landslides and since the interactions with the moving bathymetry is worse than in the case of the shallow water model, the error created during the wave generation cannot be compensated in the propagation zone. Anyway this observation suggests that the quality of the approximations given by the simplified models is related to the volume of the landslide. However the results of case ii) and case iii) are too different in spite of the same volume of the landslide. In 4.6, we show two simulations with the same volume of landslide leading to different conclusions. The graphs on the top, for parameters $(H, L) = (0.25 \text{ m}, 0.45 \text{ m})$, show better approximations with the shallow water model in the wave generation zone but worse than the Boussinesq model in the propagation zone. The graphs on the bottom, for parameters $(H, L) = (0.45 \text{ m}, 0.25 \text{ m})$, show better approximations with the shallow water model in both zones, generation and propagation. In the next section, we investigate a better criterion than the volume of the landslide to characterize the qualitative properties of the two simplified models in matching the Navier-Stokes solutions.

For now, we may conclude that the volume of the landslide is not the most determining criterium for the behavior of the shallow water model and the Boussinesq model. Instead,

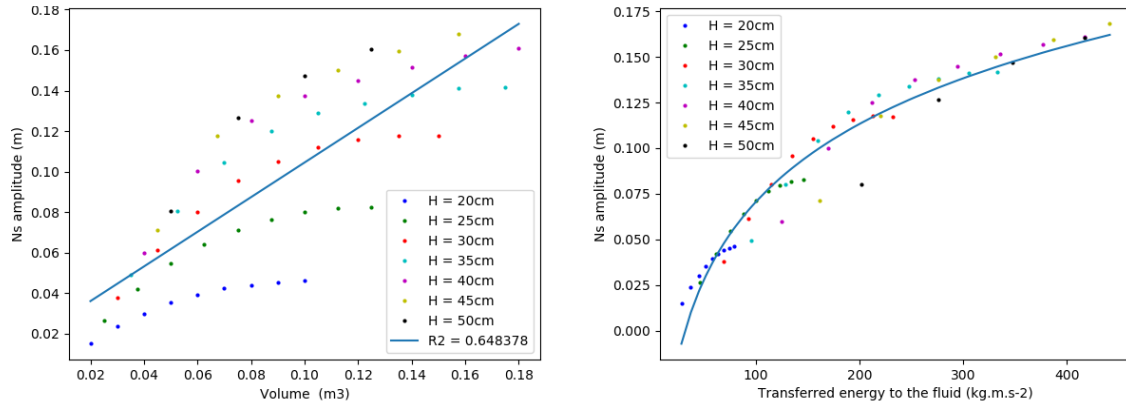


FIGURE 4.8 – Correlation between the amplitude of the wave and respectively the volume of the landslide (left) and the energy transferred to the water (right).

we can suppose that there exists a more direct link with the energy transferred from the landslide to the water. However, in practice it is not easy to measure *a priori* the energy transferred. We first show in 4.7 that the energy transferred from the landslide to the water is strongly related to the initial energy of the landslide. More precisely, for all the simulations we performed, between 50% and 60% of the initial energy was transferred to the water. Let us detail how we compute the energies. Since the flow is initially at rest, the initial energy of the landslide can be computed from its initial configuration, *i.e.* $\mathcal{P}^0 = \mathcal{P}(t = 0)$ where \mathcal{P} is the potential energy given by

$$\mathcal{P}(t) = \frac{g}{2} \int_x (\beta(t, x) - B(x))^2 dx$$

where $\beta(t, x)$ is the top interface of the landslide, *i.e.* $\phi = 1.5$. This integral is computed numerically from the output of OpenFOAM. Instead of the energy transferred, we can easily compute the final potential energy, *i.e.* $\mathcal{P}(t = 2)$. At time $t = 2$ s the landslide is at rest so all its energy is potential energy. By subtracting the two potential energies we obtain the total energy lost by the landslide during its motion. In fact, it is not exactly the energy transferred to the water (even if we neglect numerical artifices) because of the landslide's viscosity. However, once again we assume that the energy lost by viscosity effects is negligible.

4.3.2.3 Quantities of interest and correlations between the models

The correlations between some physical quantities were investigated in the probe located at the beginning of the wave propagation zone, *i.e.* $x = 1.35$ m. In case of linear

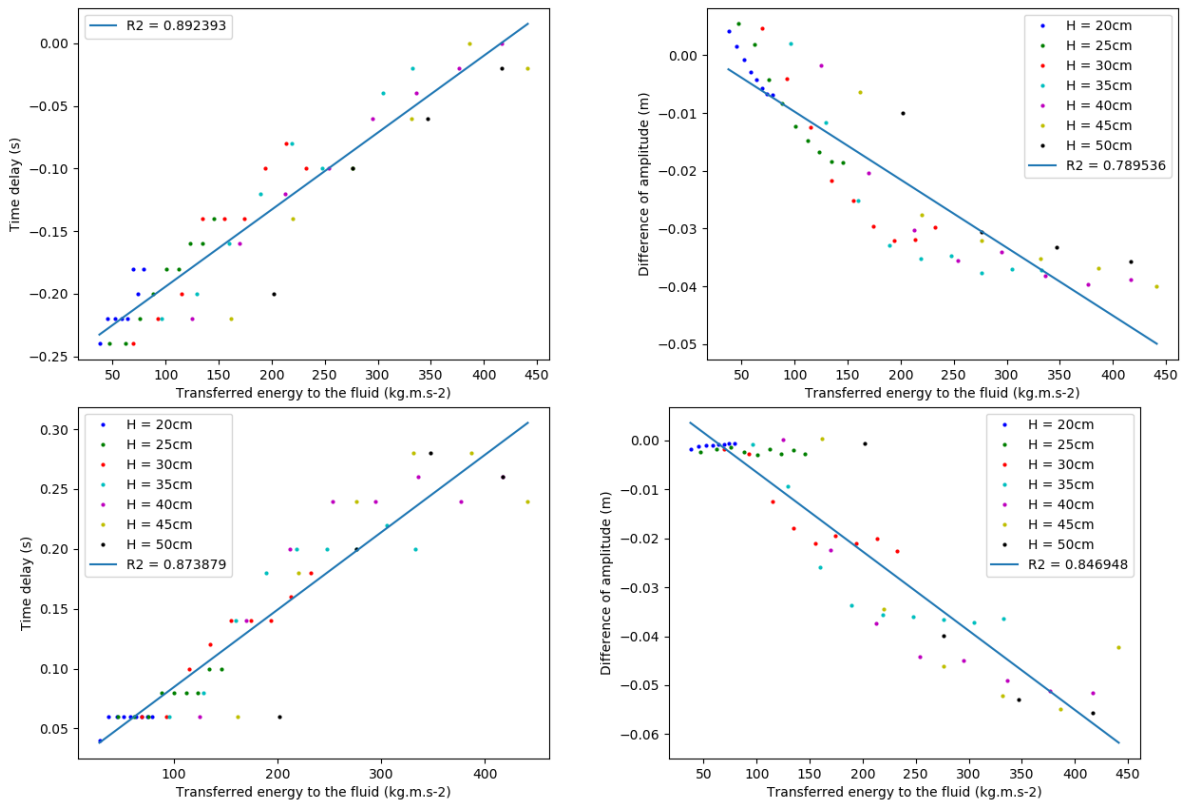


FIGURE 4.9 – Deviation of the first wave at the chosen probe in term of time delay (left column) and in term of amplitude (right column) of the shallow water model (first line) and the Boussinesq model (second line).

correlation, we indicate on the graphs the coefficient of correlation R^2 . As the extent of the wave generation zone is not the same for all the tests, the ones corresponding to the biggest amounts of landslide were not considered when computing the correlation curves presented here, since the selected gauges are in their wave generation zone, affecting the quantities considered. Therefore, among the 77 test cases performed, 61 were retained for the correlations.

First, we confirm our observation that the volume of the landslide does not characterize well the wave generated. In 4.8, the left graph shows the correlation between the amplitude of the wave generated by the landslide and the volume of the landslide. It is clear that an information is missing. The right graph illustrates the correlation between the amplitude and the energy transferred to the water. We observe that almost all the simulations can be fitted by a logarithm law.

Now we compare the models in function of the energy transferred to the water. 4.9

shows the deviation of the first wave at the chosen probe in terms of time delay (left column) and in terms of amplitude (right column) of the shallow water model (first line) and the Boussinesq model (second line). This result confirms the first observations, that the shallow water model seems to give better approximation for waves generated by important landslides (in terms of energy) while the Boussinesq model leads to better approximations for small landslides (see 4.3.2.1 and 4.3.2.2). In addition, we observe that the shallow water model is always late compared to the reference Navier-Stokes solution while the Boussinesq model is ahead. The results in terms of difference of amplitude are not as clear. For the shallow water model, a logarithm deviation seems to appear. However, the deviation of the Boussinesq model appears as a succession of logarithm that seems to indicate that some informations are still missing to characterize the flow.

4.3.2.4 A shallow water/Boussinesq mixed model

Based on the observed behaviors of the simplified models described above, we propose a simple mixed model for trying to get results closer to the Navier-Stokes reference solution. Obviously one solution would be to use a fully nonlinear weakly dispersive model, the so called Serre/Green-Naghdi model [Lan13a]. In the current state of art of numerical methods, this model is very hard to simulate, in particular close to dry areas. Even with a very robust, entropy-satisfying method [Par19] we were not able to run the test case.

We propose another strategy using the well validated numerical tools. Since the shallow water model gives a better description of the wave generation process, we propose to use it in the beginning of the simulation, before switching to the Boussinesq model after a certain time. In our case where the water is initially at rest, it leads to use the shallow water model during the generation process and the Boussinesq model during the propagation one. However, in a more complex context, as in the case of breaking waves, a space decomposition seems required [FKR16].

It remains the question of the switch time between models. This time (or localization for space decomposition) can be based on the motion of the bottom. However, to test the idea in this preliminary study, we simply fit it to reproduce as well as possible the reference Navier-Stokes solution. A visual inspection of the simulations points out the approximate instant $t_{\text{switch}} = 0.75$ s as a good choice.

In 4.10, we add the mixed model, *i.e.* the model where the numerical simulation switches from the shallow water model to the Boussinesq model, to the three fluid mechanics models already presented at the probe $x = 2.25$ m (far in the wave propagation zone). Two

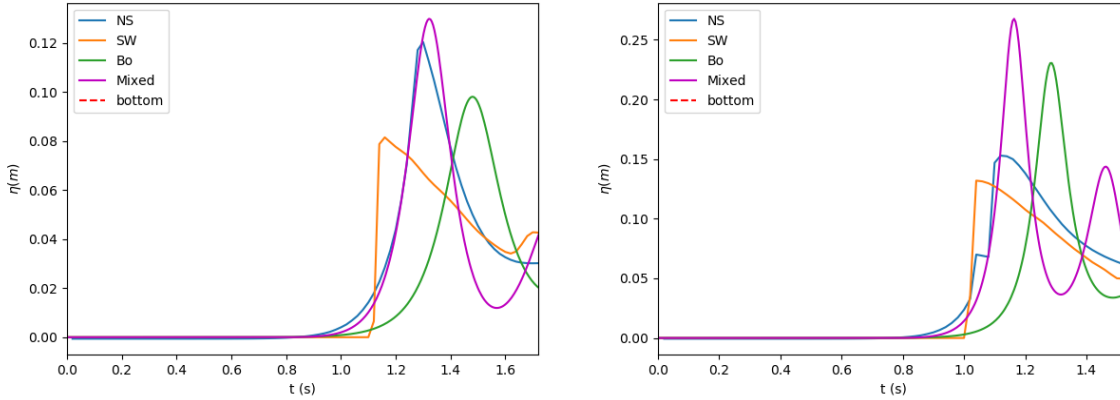


FIGURE 4.10 – Comparison between the three fluid mechanics models and the proposed mixed shallow water/Boussinesq model, for two test cases at the probe $x = 2.25$ m.

simulations are presented : in the left graph the ones with the parameters $(H, L) = (0.3 \text{ m}, 0.4 \text{ m})$ corresponding to an intermediary amount of energy, and in the right graph, the ones with the parameters $(H, L) = (0.45 \text{ m}, 0.4 \text{ m})$ corresponding to large amount of energy.

Note that we do not adapt the switch time to the simulations. For both cases, the mixed model leads to a very good agreement with the Navier-Stokes solution in term of time delay of the first wave. The amplitude of the wave with the mixed model is significantly larger when the transferred energy is too big. As far as we understand, this difference can be explained by the fact that for this case the wave energy is large enough for the wave to break in the Navier-Stokes simulation. This wave breaking phenomenon is not present in the Boussinesq model. Some breaking wave strategies existing in the literature can probably improve this result.

4.4 Inverse problem

The goal of this section is to see if we are able to recover the dynamics of a landslide knowing only the generated waves at some fixed points. This strategy is not trivial for several reasons. First the form of the landslide does not follow a general shape, so we look a priori for a function in an infinite space. Then as we see in the previous section 4.3, even with “exact” landslide shape, the difference between the simplified models and the Navier-Stokes model is significant. Eventually, the solutions of hydrodynamic models can be discontinuous. This kind of solutions are known to be not suitable for inverse problem

such as recovering a forcing leading to them [FCD19].

Anyway this preliminary study is performed to observe the result of the strategy in practice. More precisely, the first two points are technical problems that, we hope, could be solved with enough work, while the last one is a theoretical problem that can not be overcome, at least without introducing theoretical tools as Dirac measure that are complex to use in application contexts. However, It happens that computational strategies not confirmed by theory lead to good results.

4.4.1 The methodology

The problem can be summarized by the minimization problem $d = \operatorname{argmin}_{\tilde{d}}(J(\tilde{d}))$ associated to the functional

$$J(\tilde{d}) := \sum_{i=1}^N \int_0^T (\eta(x_i, t; \tilde{d}) - \eta_i^*(t))^2 dt \quad (4.8)$$

where T is the time of observation, $N > 0$ is the number of gauges, x_i are their position where the observations of the free surface $\eta_i^*(t)$ are realized. These observations can be obtained from experiments or from a reference model such as the Navier-Stokes model. The free surface of one of the simplified models $\eta(x, t; d)$ ($\bar{\eta} = \bar{h}(t, x) - d(t, x)$ for the shallow water model and $\underline{\eta}$ for the Boussinesq model) depends obviously on the bottom elevation. However, since this dependency is not explicit, the minimization problem has to be solved using numerical tools.

4.4.1.1 Modeling assumptions

As already mentioned, the bottom elevation $d(t, x)$ is a function in an infinite dimensional space (a priori in L^1 with $\int d(t, x) dx = \int d(0, x) dx + BC$ thanks to the mass conservation). The minimization problems in infinite dimensional space are much complex and the numerical resolution is less efficient and less robust than the same ones in finite dimension. To improve the robustness of the strategy, we assume that the bottom can be described by a sum of Gaussian functions, leading to a finite number of degrees of freedom. Each Gaussian is characterized by its amplitude $A(t)$, its position $X(t)$ and its standard deviation $\sigma(t) > 0$. More precisely we set

$$\tilde{d}(t, x) = -(B(x) + \sum_{i=1}^M A_i(t) \exp(-(\frac{x - X_i(t)}{\sigma_i(t)})^2))$$

where $M > 0$ is the number of Gaussians in our approximation, a priori fixed. Assuming the mass conservation for each Gaussian, its standard deviation is linked to its amplitude. More precisely, in one dimensional framework

$$0 = \partial_t \left(\int_{\mathbb{R}} A_i(t) \exp\left(-\left(\frac{x - X_i(t)}{\sigma_i(t)}\right)^2\right) dx \right) = \sqrt{\pi} \partial_t (A_i(t) \sigma_i(t))$$

$$\implies \sigma_i(t) = \frac{A_i(0)}{A_i(t)} \sigma_i(0).$$

It remains to give some assumptions on the form of the time evolution of the Gaussian's amplitude and position. For simplicity of the analysis, we assume a linear evolution :

$$A_i(t) = A_i^0 - a_i t \quad \text{and} \quad X_i(t) = X_i^0 - c_i t.$$

The initial amplitude A_i^0 and position X_i^0 are given by the initial condition, so the only parameters to determine are the decay of amplitude $a_i \in \mathbb{R}$ and the velocity of the displacement $c_i \in \mathbb{R}$. Using the same strategy, in the case of M Gaussians, we obtain $2M$ parameters to determine.

4.4.1.2 Numerical strategy

The minimization problem (4.8) in a finite dimensional space is solved with a very simple gradient descent method [NW06]. In order to limit the number of direct resolutions, we decouple the resolution of each parameter. More precisely, given a vector of unknown parameters C (in our case $C = (a_i, c_i)_{1 \leq i \leq M}$) and an initial guess C_0 , we iterate

$$C_{n+1} = C_n - \alpha_n \nabla_C^\delta J(C_n) \tag{4.9}$$

where C_n is the n^{th} approximation of the vector of unknown parameters and α^n is the descent step used for the convergence of the method. $\nabla_C^\delta J$ is an approximation of the gradient $\nabla_C J$ of the functional (4.8). Here we use the finite difference approximation, *i.e.*

$$\nabla_C^\delta J \cdot e_i = \frac{J(C_n + \delta e_i) - J(C_n - \delta e_i)}{2\delta}$$

where δ is the step of the finite difference approximation and e_i is the unit vector with all the component vanish except the i^{th} one. The descent step is α_n is chosen to assure that the sequence of approximations applied to the functional is decreasing, *i.e.* $J(C_{n+1}) < J(C_n)$.

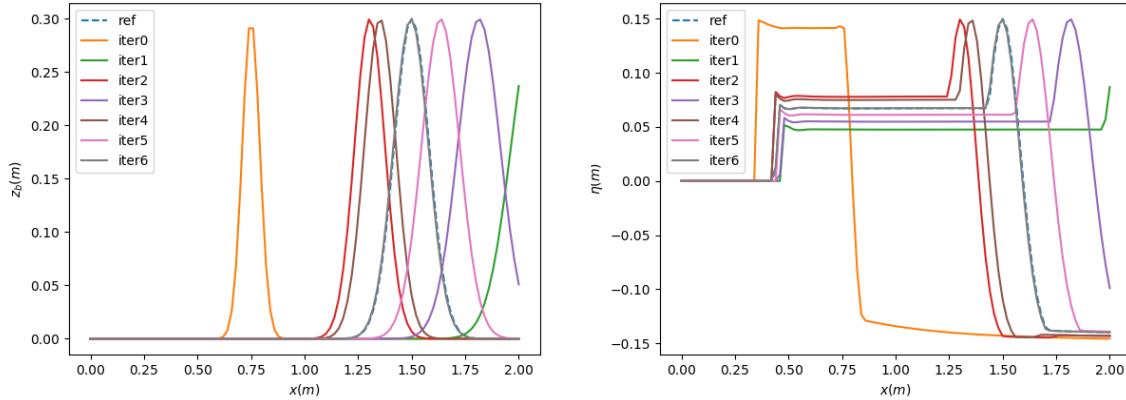


FIGURE 4.11 – Evolution of the bathymetry (left graph) and the generated surface elevation (right graph) along time and iterations for the minimization problem of a Gaussian in translation.

In practice, we consider the sequence defined by

$$\alpha_n^{q+1} = \begin{cases} \omega \alpha_n^q & \text{if } J(C_n - \alpha_n^q \nabla_C^\delta J(C_n)) > J(C_n) \\ \alpha_n^q & \text{else.} \end{cases} \quad (4.10)$$

with $0 < \omega < 1$. It is clear that the sequence converge and we set $\alpha_n = \lim_{q \rightarrow \infty} \alpha_n^q$.

Note that the estimation of the functional requires one direct computation of a simplified model (4.6) or (4.7), and two more for each parameter to estimate the discrete gradient. More efficient methods based on the alternating direction method and the secant approximation can be considered.

4.4.2 The numerical results

In this section, we illustrate the result of the minimization problem for two simple cases. For both simulations, the simplified model used is the shallow water model. The time of observation is set to $t = 2$ s, and the gauges are located every 0.2 m from $x = 1.4$ m to $x = 2.0$ m. The observations η_i^* are obtained by computing the model with the exact solution of the minimization problem (the moving bottom we look for) before forgetting it. The numerical parameters are set by $\delta = 10^{-2}$ and $\omega = 0.5$ for both simulations.

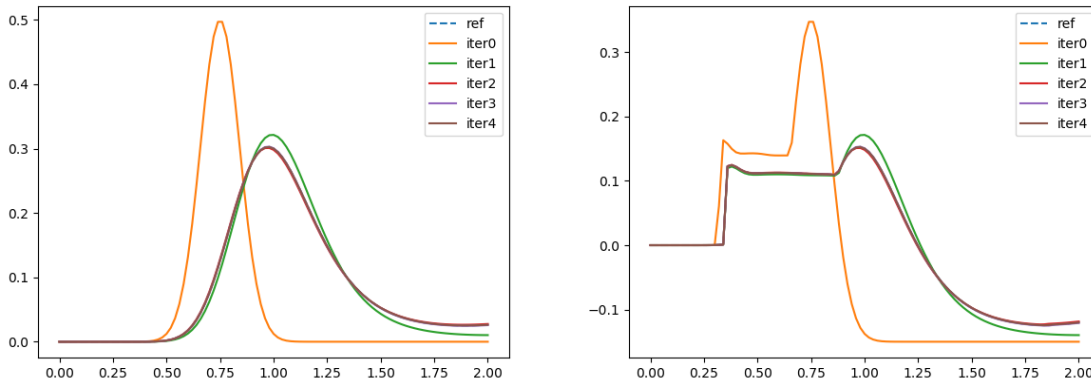


FIGURE 4.12 – Evolution of the bathymetry (left graph) and the generated surface elevation (right graph) along time and iterations for the minimization problem of a Gaussian in translation with amplitude decay.

4.4.2.1 Gaussian in translation

For this test case, we consider only one Gaussian $M = 1$ that does not decay in amplitude during the propagation, *i.e.* $a_1 = 0$. The only parameter to determine is c_1 and it is set to 0.5 to realize the observations.

In 4.11, the reference solution and the solution obtained along the iterations of the gradient descent method are plotted. We recover the velocity of the Gaussian with few iterations.

4.4.2.2 Gaussian in translation with decay

For this test case, we still consider only one Gaussian $M = 1$ but with a linear decay in amplitude. Two parameters have to be determined, *i.e.* (a_1, c_1) and it is set to $(-0.2, 0.75)$ to realize the observations.

In 4.12, the reference solution and the solution obtained along the iterations of the gradient descent method are plotted. We recover the velocity of the Gaussian with few iterations.

In both cases the minimization strategy converges in few iterations to the solution we look for. However, other numerical tests must be performed before concluding that the strategy can be used in an application context.

4.5 Conclusion

In this paper, we analysed the relevance of the use of the shallow water model or the Boussinesq model to simulate tsunamis generated by a landslide. Two questions were considered.

First we compared the two models to a reference solution obtained using the Navier-Stokes model. This comparison highlighted the fact that the Boussinesq model is not able to represent the wave generated in the wave generation zone. For large enough landslide, even in the wave propagation zone (at least not too far from the wave generation zone), the shallow water model is more efficient. We propose a strategy based on a switch from the shallow water model during the generation of the wave to the Boussinesq model during the propagation of the wave. This strategy seems relevant to improve the results of the simulations. Another solution, more satisfying in theory, is the application of more sophisticated models, as the Serre/Green-Naghdi model. However, it seems that more robust schemes, in particular for moving bathymetry or stiff solution, are required.

In addition, we highlighted the fact that the landslide is not well characterized by the volume of the landslide but can be better characterized by the energy transferred to the water. This energy is well correlated to the initial potential energy of the landslide that can be determined by the initial condition.

Finally, we addressed the question of the possibility of determining the landslide from the generated wave. It was formulated as an minimization problem, in which we searched the time-dependent bottom that minimizes the difference between a reference surface elevation and the computed surface elevation obtained by the bottom to be determined. The question was brought back to a limiting number of unknowns to determine by assuming that the bottom can be well represented by a finite (small) number of Gaussians with parameters following some empirical laws. Even if our computational analysis was too limited to conclude, the numerical strategy was validated in simple cases. Deeper computational analysis is however required, in particular a crucial point will be to determine if the sum of Gaussians is able to well approximate a realistic landslide by the inverse problem.

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Titre : Modèles de type dispersion complète en océanographie côtière

Mot clés : EDP non-locales, dispersives et non-linéaires, océanographie, justification de modèles.

Résumé : Cette thèse porte sur la justification rigoureuse de modèles de type dispersion complète en océanographie côtière. Le premier modèle de ce type est celui de G. Whitham, introduit en 1967 afin d'étudier les vagues de Stokes d'amplitude maximale et le phénomène de vagues surplombantes. Il consiste en une modification des équations de Korteweg-de Vries ayant la même relation de dispersion que celle des équations des vagues. Par la suite, de nombreux modèles de type dispersion complète, unidirectionnels comme bidirectionnels, ont été introduits dans la littérature. Dans une première partie, nous utilisons des techniques classiques d'approximations de solutions d'équations elliptiques afin de dériver des modèles de

Whitham-Boussinesq et de Whitham-Green-Naghdi. Ensuite, nous justifions complètement une classe de modèles de Whitham-Boussinesq présentant une structure d'équations quasi-linéaires non-locales. Dans une partie suivante, nous justifions le modèle de Whitham en utilisant deux méthodes différentes. L'une est adaptée à la propagation de vagues unidirectionnelles, et utilise le calcul pseudo-différentiel. L'autre est adaptée à la propagation de vagues bidirectionnelles, et est basée sur une généralisation de l'algorithme de la forme normale de Birkhoff. Dans la dernière partie, nous étudions numériquement la validité des modèles de Boussinesq et de Saint-Venant pour la propagation de tsunamis générés par des glissements de terrain.

Title: Full dispersion models in coastal oceanography

Keywords: Non-local, non-linear and dispersive PDEs, oceanography, justification of models

Abstract: This thesis is on the rigorous justification of full dispersion models in coastal oceanography. The first full dispersion model was introduced by G. Whitham in 1967 to study the Stokes waves of maximal amplitude and the wavebreaking phenomenon. It is a full dispersion modification of the Korteweg-de Vries equations which have the same dispersion relation as the general water-waves model. Afterwards, numerous unidirectional and bidirectional full dispersion models were introduced in the literature. In the first part of the thesis, we use classical techniques on free surface elliptic equations to derive rigorously some Whitham-Boussinesq

and Whitham-Green-Naghdi models. In the second part, we justify rigorously a class of non-local quasi-linear Whitham-Boussinesq systems. In the next part, we justify rigorously Whitham's model using two different methods. One is adapted to the propagation of unidirectional waves and use pseudo-differential calculus. The other is adapted to the propagation of bidirectional waves. It is based on a generalisation of Birkhoff's normal form algorithm. In the last part, we study numerically the validity of the Saint-Venant and Boussinesq models for the propagation of tsunamis generated by landslides.