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Mention: Mathématiques

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de l'Université de Picardie Jules Verne

par

Ruxi SHI

pour obtenir le grade de Docteur de l'Université de Picardie Jules Verne

Etude sur la conjecture de Fuglede et les suites oscillantes

Soutenue le 26/06/2018, après avis des rapporteurs, devant le jury d'examen :

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献给我亲爱的父母

“Der Vogel kämpft sich aus dem Ei. Das Ei ist die Welt. Wer geboren werden will, muß eine Welt zerstören.”

——Hermann Hesse, *Demian. Die Geschichte von Emil Sinclairs Jugend*

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Résumé

Dans cette thèse, nous résolvons la conjecture de Fuglede sur le corps des nombres p -adiques, et étudions certaines propriétés aléatoires des suites liées à la conjecture de Sarnak, ainsi que leurs propriétés oscillantes.

Dans la première partie, nous prouvons d'abord la conjecture de Fuglede pour des ensembles ouverts compacts dans le corps \mathbb{Q}_p des nombres p -adiques. Celle-ci indique qu'un ensemble ouvert compact dans \mathbb{Q}_p est un ensemble spectral si et seulement s'il pavé \mathbb{Q}_p par translation. Il est également prouvé qu'un ensemble ouvert compact est un ensemble spectral (ou un pavage) si et seulement s'il est p -homogène. Nous caractérisons les ensembles spectraux dans $\mathbb{Z}/p^n\mathbb{Z}$ ($p \geq 2$ premier, $n \geq 1$ entier) par la propriété de pavage et aussi par leur homogénéité. Finalement, nous montrons la conjecture de Fuglede dans \mathbb{Q}_p sans la restriction d'être ouvert compact en montrant que tout ensemble spectral ou tout pavage doivent être ouvert et compact à un ensemble de mesure nulle près.

Dans la seconde partie, nous donnons d'abord plusieurs définitions équivalentes d'une suite oscillante en termes de disjonction de différents systèmes dynamiques sur des tores. Ensuite, nous définissons la propriété de Chowla et la propriété de Sarnak pour des suites numériques prenant des valeurs 0 ou des nombres complexes de module 1. Nous prouvons que la propriété de Chowla implique la propriété de Sarnak. Il est également prouvé que pour Lebesgue presque tout $\beta > 1$, la suite $(e^{2\pi\beta^n})_{n \in \mathbb{N}}$ partage la propriété de Chowla et est par conséquent orthogonale à tout système dynamique topologique d'entropie nulle. Nous discutons également si les échantillons d'une suite aléatoire donnée ont presque sûrement la propriété de Chowla. Nous construisons certaines suites aléatoires dépendantes ayant presque sûrement la propriété de Chowla.

Mots clés: corps de nombres p -adiques, ouvert compact, ensemble spectral, pavage, conjecture de Fuglede, suites oscillantes, fonctions affines sur les tores, quasi-unipotente, distribution uniforme.

Abstract

In this thesis, we solve Fuglede's conjecture on the field of p -adic numbers, and study some randomness and the oscillating properties of sequences related to Sarnak's conjecture.

In the first part, we first prove Fuglede's conjecture for compact open sets in the field \mathbb{Q}_p of p -adic numbers which states that a compact open set in \mathbb{Q}_p is a spectral set if and only if it tiles \mathbb{Q}_p by translation. It is also proved that a compact open set is a spectral set (or a tile) if and only if it is p -homogeneous which is easy to check. We characterize spectral sets in $\mathbb{Z}/p^n\mathbb{Z}$ ($p \geq 2$ prime, $n \geq 1$ integer) by tiling property and also by homogeneity. Finally, we prove Fuglede's conjecture in \mathbb{Q}_p without the assumption of compact open sets and also show that the spectral sets (or tiles) are the sets which differ by null sets from compact open sets.

In the second part, we first give several equivalent definitions of oscillating sequences in terms of their disjointness from different dynamical systems on tori. Then we define Chowla property and Sarnak property for numerical sequences taking values 0 or complex numbers of modulus 1. We prove that Chowla property implies Sarnak property. It is also proved that for Lebesgue almost every $\beta > 1$, the sequence $(e^{2\pi\beta^n})_{n \in \mathbb{N}}$ shares Chowla property and consequently is orthogonal to all topological dynamical systems of zero entropy. We also discuss whether the samples of a given random sequence have almost surely Chowla property. Some dependent random sequences having almost surely Chowla property are constructed.

Key words: p -adic field, compact open, spectral set, tiling, Fuglede's conjecture, oscillating sequence, affine map on tori, quasi-unipotent, uniform distribution.

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Résumé des travaux de thèse

Cette thèse est consacrée à la résolution de la conjecture de Fuglede sur le corps des nombres p -adiques et à l'étude des propriétés oscillantes des suites liées à la conjecture de Sarnak.

0.1 Un aperçu de la thèse

Rappelons d'abord la conjecture de Fuglede. Soit $\Omega \subset \mathbb{R}^d$ un sous ensemble borélien de mesure de Lebesgue positive et finie. On dit que l'ensemble Ω *pave* ('tile' en anglais) \mathbb{R}^d par translation s'il existe un ensemble (dénombrable) $T \subset \mathbb{R}^d$ tel que $\{\Omega + t : t \in T\}$ est une partition de \mathbb{R}^d presque sûrement. On dit aussi que Ω est un *pavage* de \mathbb{R}^d . Dans ce cas, on dit que T est un *complément de pavage* ('tiling complement' en anglais) de Ω et le couple (Ω, T) est une *paire de pavage* ('tiling pair' en anglais). On dit que Ω est un *ensemble spectral* s'il existe un ensemble Λ de caractères continus de \mathbb{R}^d qui forment une base de Hilbert de $L^2(\Omega)$. Un tel ensemble Λ est appelé un *spectre* de Ω et le couple (Ω, Λ) est appelé une *paire spectrale*.

Voici la conjecture d'ensemble spectral due à Fuglede [43](1974):

Conjecture 0.1. *Un ensemble borélien de mesure de Lebesgue positive et finie dans \mathbb{R}^d est spectral si et seulement s'il pave \mathbb{R}^d .*

L'origine de la conjecture de Fuglede est un problème de Segal (1958). Il s'agit de caractériser les domaines $\Omega \subset \mathbb{R}^d$ tels que les opérateurs différentiels $-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_d}$, définis sur leur domaine de définition commun $C_c^\infty(\Omega)$, admettent des extensions auto-adjointes commutatives dans $L^2(\Omega)$. Fuglede a montré que c'est le cas si et seulement si Ω est un ensemble spectral, d'où le problème de caractériser les ensembles spectraux sur \mathbb{R}^d . Fuglede a également montré que c'est effectivement le cas si Ω pave \mathbb{R}^d par un réseau $A\mathbb{Z}^d$ (A étant une matrice non-singulière).

La conjecture de Fuglede a attiré une attention considérable pendant les dernières décennies. Beaucoup de résultats positifs sont obtenus dans des cas particuliers sous différentes restrictions [59, 60, 62, 65, 79, 81, 95, 96] avant que Tao [121] (2004) n'a prouvé

que l'implication "spectral \Rightarrow pavage" est fautive dans le cas \mathbb{R}^d avec $d \geq 5$, puis $d = 4$ (Matolcsi [100], Farkas et Gy [29]) et $d = 3$ (Kolountzakis et Matolcsi [84], Farkas, Matolcsi et Móra [39]). La validité de la conjecture pour les cas de $d = 1, 2$ reste inconnue. Mais Iosevich, Katz and Tao [59] ont montré que la conjecture est vraie pour les ensembles convexes dans \mathbb{R}^2 .

La conjecture 0.1 peut être formulée pour tout groupe abélien localement compact sur lequel les notions d'ensembles spectraux et de pavages sont généralisées en remplaçant la mesure de Lebesgue par la mesure de Haar sur le groupe.

La conjecture générale est loin d'être prouvée et comme nous l'avons mentionné, elle n'est pas vraie pour $\mathbb{R}^d, d \geq 3$. Le problème est d'étudier sur quels groupes abéliens localement compact la conjecture de Fuglede est vraie. En fait, il n'y a que peu de résultats connus. Nous savons que la conjecture de Fuglede est vraie sur $\mathbb{Z}/p^n\mathbb{Z}$ (Laba [89]), $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (Iosevich, Mayeli et Pakianathan [61]) et récemment $\mathbb{Z}/p^nq\mathbb{Z}$ avec $n \geq 1$ et p, q nombres premiers distincts (Malikiosis et Kolountzakis [101]).

Dans la première partie de cette thèse, nous étudions la conjecture de Fuglede sur le corps des nombres p -adiques. Un de nos théorèmes principaux affirme que la conjecture de Fuglede est vraie sur le corps des nombres p -adiques, c'est-à-dire, un ensemble borélien de mesure de Lebesgue positive et finie dans \mathbb{Q}_p est spectral si et seulement s'il pave \mathbb{Q}_p . De plus, nous prouvons que les ensembles spectraux (resp. les pavages) sont ouverts et compacts à un ensemble de mesure nulle près.

Un autre de nos théorèmes principaux déclare qu'un ensemble ouvert compact est un ensemble spectral (ou un pavage) si et seulement si son arbre associé est p -homogène. En fait, ce résultat donne une illustration géométrique des pavages et des ensembles spectraux dans le corps des nombres p -adiques. Ces résultats sont décrits en détail dans la section 1.2.

La deuxième partie de cette thèse étudie certaines propriétés aléatoires et oscillantes de suites complexes, qui sont partagées par la fonction de Möbius. La fonction de Möbius est la fonction arithmétique définie par la formule

$$\mu(n) = \begin{cases} 1 & \text{si } n = 1, \\ (-1)^k & \text{si } n \text{ est un produit de } k \text{ nombres premiers distincts,} \\ 0 & \text{sinon.} \end{cases}$$

Il est bien connu que la fonction de Möbius joue un rôle important dans la théorie des nombres. Par exemple, le théorème des nombres premiers est équivalent à l'énoncé suivant (voir [7], p.91)

$$\sum_{n=1}^N \mu(n) = o(N);$$

l'hypothèse de Riemann est équivalente à la formule suivante (voir [123])

$$\sum_{n=1}^N \mu(n) = O_{\epsilon}(N^{\frac{1}{2}+\epsilon}), \text{ pour tout } \epsilon > 0.$$

L'heuristique pseudoaléatoire de Möbius (ou le principe d'aléa de Möbius) suggère que les sous-mots finis dans la fonction Möbius μ se comportent de manière si aléatoire (ou pseudo-aléatoire) qu'une quantité substantielle de nombreux termes de la somme s'annulent. Il y a un certain nombre de façons de rendre cette heuristique précise. Comme nous l'avons déjà mentionné, l'hypothèse de Riemann peut être considérée comme une telle manifestation de l'heuristique. Une autre manifestation est la conjecture de Chowla [21] sur les corrélations de la fonction de Möbius.

Conjecture 0.2 (Chowla). *Pour chaque choix de $0 \leq a_1 < \dots < a_r, r \geq 0$, et $i_s \in \{1, 2\}$ qui ne sont pas tous égaux à 2,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu^{i_0}(n+a_1) \cdot \mu^{i_1}(n+a_2) \cdot \dots \cdot \mu^{i_r}(n+a_r) = 0.$$

Une formulation plus récente de l'heuristique aléatoire de Möbius est la conjecture suivante de Sarnak. Soit T une transformation continue d'un espace métrique compact X . D'après Sarnak [112], nous disons qu'une suite $(a(n))_{n \in \mathbb{N}}$ de nombres complexes est orthogonale au système dynamique topologique (X, T) , si

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) a(n) = 0, \tag{0.1.1}$$

pour tout $x \in X$ et pour toute $f \in C(X)$ où $C(X)$ est l'espace des fonctions continues sur X .

En 2010, Sarnak [112] a formulé la conjecture suivante.

Conjecture 0.3 (Sarnak). *La fonction de Möbius est orthogonale à tous les systèmes dynamiques topologiques d'entropie nulle.*

Sarnak [112] a également prouvé que la conjecture de Chowla implique la conjecture de Sarnak.

Récemment, El Abdalaoui, Kulaga-Przymus, Lemanczyk et de la Rue [4] ont étudié ces conjectures du point de vue de la théorie ergodique. Soit z une suite arbitraire prenant des valeurs dans $\{-1, 0, 1\}$ comme la fonction de Möbius. Nous disons que la suite z satisfait la condition (Ch) si elle satisfait la condition dans la conjecture de Chowla; de même, elle satisfait la condition (S_0) si elle satisfait la condition dans la conjecture de Sarnak. El Abdalaoui et al. ont prouvé que la condition (Ch) implique la condition (S_0) pour toutes

les suites prenant des valeurs dans $\{-1, 0, 1\}$. Ils ont aussi donné des exemples non-triviaux de suites satisfaisant la condition (Ch) et la condition (S_0) respectivement.

Nous considérons une classe plus large de suites, prenant des valeurs 0 ou des nombres complexes de module 1. Nous définissons la propriété de Chowla pour cette classe de suites et nous prouverons que de telles suites de type Chowla sont orthogonales à tous les systèmes dynamiques topologiques d'entropie nulle. Il est prouvé que pour presque tout $\beta > 1$, la suite $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ est une suite de type Chowla. Nous discutons également si les échantillons d'une suite aléatoire ont la propriété de Chowla presque sûrement. Nous construisons des suites de variables aléatoires dépendantes ayant la propriété de Chowla presque sûrement.

Nous étudions l'orthogonalité entre deux suites. Deux suites de nombres complexes $(a(n))_{n \in \mathbb{N}}$ et $(b(n))_{n \in \mathbb{N}}$ sont dites orthogonales si

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n)b(n) = 0. \quad (0.1.2)$$

Davenport [24] a prouvé que pour chaque $A > 0$,

$$\max_{\alpha \in \mathbb{R}} \left| \sum_{n=1}^N \mu(n) e^{2\pi i n \alpha} \right| = O_A \left(\frac{N}{\log^A N} \right), \text{ pour tout } N > 1. \quad (0.1.3)$$

Ceci signifie que la fonction de Möbius est orthogonale aux suites $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ pour tout $\alpha \in \mathbb{R}$. Une suite ayant une telle propriété est appelée *une suite oscillante* par Fan et Jiang dans [38] où la conjecture de Sarnak s'est avérée vraie pour une classe de dynamiques, même lorsque la fonction de Möbius est remplacée par une suite oscillante. En général, les suites oscillantes d'ordre supérieur et les suites pleinement oscillantes ont été définies par Fan dans [31] où il a prouvé que les suites pleinement oscillantes sont orthogonales à toutes les réalisations ergodiques multiples de la dynamique topologique ayant spectre quasi-discret dans le sens de Hahn–Parry. Rappelons qu'une suite $(c_n)_{n \in \mathbb{N}}$ est dite *oscillante d'ordre d* si

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i P(n)} = 0 \quad (0.1.4)$$

pour tout $P \in \mathbb{R}_d[t]$, où $\mathbb{R}_d[t]$ est l'espace des polynômes réels de degré inférieur ou égal à d . Elle est pleinement oscillante si elle est oscillante d'ordre d pour tout nombre entier $d \geq 1$.

Nous nous intéressons à la description dynamique des suites oscillantes. Nous démontrons plusieurs définitions équivalentes de suites oscillantes en termes d'orthogonalité à différents systèmes dynamiques sur des tores.

0.2 La conjecture de Fuglede

Dans cette section, nous présentons nos résultats sur la conjecture de Fuglede dans le corps des nombres p -adiques. Nous commençons par quelques remarques historiques.

0.2.1 Remarques historiques et questions ouvertes

Comme nous l'avons mentionné précédemment, Fuglede ([43]) a formulé sa conjecture concernant l'équivalence entre les pavages et les ensembles spectraux dans les espaces euclidiens.

Cette conjecture est fautive sur \mathbb{R}^d , $d \geq 3$. Cependant, elle est toujours ouverte dans les dimensions $d = 1, 2$. En fait, elle semble être difficile en dimension 1. Néanmoins, Laba [88] a prouvé que la conjecture de Fuglede est vraie pour une union de deux intervalles. Pour une union de trois intervalles, seule l'implication "Pavage \Rightarrow Spectral" a été prouvée par Bose, Kumar, Krishnan et Madan [14]. Si on regarde juste une union d'intervalles disjoints de même taille, c'est-à-dire un ensemble du type

$$\Omega = A + [0, 1),$$

où A est un ensemble fini dans \mathbb{Z} , alors dans ce cas, la conjecture est équivalente à un problème de pure théorie des nombres (voir [80]). La conjecture est toujours ouverte pour cette classe d'ensembles. Laba [89] a prouvé que la conjecture est vraie pour $\sharp A = p^n$ ($n \in \mathbb{N}$ et p un nombre premier). Elle a aussi prouvé que si Ω est un pavage alors il est aussi spectral pour $\sharp A = p^n q^m$ ($n, m \in \mathbb{N}$ et p, q nombres premiers distincts). Malikiosis et Kolountzakis [101] ont prouvé que la conjecture est vraie pour $\sharp A = p^n q$ ($n \in \mathbb{N}$ et p, q nombres premiers distincts).

En dimension 2, il y a peu de résultat. Nous savons seulement que Iosevich, Katz et Tao [59] ont prouvé la conjecture pour les domaines planaires convexes.

Comme il a déjà été mentionné, la conjecture générale de Fuglede sur un groupe abélien localement compact est formulée comme suit:

Conjecture 0.4. *Un ensemble borélien de mesure de Haar positive et finie dans un groupe abélien localement compact G est spectral si et seulement s'il pave G .*

Dans sa généralité, cette conjecture générale n'est pas vraie, par exemple, elle est fautive sur \mathbb{R}^d , $d \geq 3$. Nous pourrions plutôt demander pour quels groupes elle est vraie. En ce sens, la conjecture générale est largement ouverte. La structure algébrique des différents groupes fournirait différentes informations. Bien qu'il y ait une analyse de Fourier générale sur les groupes abéliens localement compacts (voir [108]), chaque groupe a sa propre analyse de Fourier qui est légèrement différente l'une de l'autre. Ainsi, les méthodes pour traiter la conjecture diffèrent d'un groupe à l'autre. Cela peut offrir une chance de voir les connexions entre les pavages et les ensembles spectraux dans un groupe si l'on connaît bien la structure algébrique et l'analyse de Fourier sur ce groupe. Par conséquent, il y a de nombreuses

questions à envisager. La conjecture reste ouverte même pour les groupes abéliens finis. En fait, il semble être plutôt difficile dans les groupes abéliens finis. Les “plus grands” groupes abéliens finis se sont avérés vérifier pour la conjecture de Fuglede: $\mathbb{Z}/p^n q \mathbb{Z}$ avec $n \geq 1$ et p, q étant des nombres premiers distincts par Malikiosisa et Kolountzakis [101]. On ne sait pas encore si la conjecture est vraie sur $\mathbb{Z}/p^n q^m \mathbb{Z}$ avec $n \geq 1, m > 1$ et p, q étant des nombres premiers distincts.

D’autre part, au lieu de considérer la conjecture elle-même, il est également très intéressant d’étudier la structure des pavages et des ensembles spectraux dans des groupes abéliens localement compacts. Même si la conjecture est fautive sur certains groupes, il est intéressant d’étudier la différence entre les pavages et les ensembles spectraux. Ensuite, il y a de nombreuses questions à considérer. On pourrait étudier cette question même pour des groupes cycliques finis, c’est-à-dire pouvons-nous classer des pavages et des ensembles spectraux dans des groupes cycliques finis? Cette question est encore largement ouverte. Coven et Meyerowitz [22] ont conjecturé que deux conditions algébriques ensemble (voir Section 2.6) sont suffisantes et nécessaires pour les pavages et les ensembles spectraux respectivement dans les groupes cycliques. Cependant, on sait seulement que ces deux conditions algébriques sont suffisantes pour les pavages et les ensembles spectraux dans $\mathbb{Z}/p^n q^m \mathbb{Z}$ avec $n \geq 1, m > 1$ et p, q étant nombres premiers distincts (Łaba [88]).

De plus, nous pourrions aussi poser les questions ci-dessus pour les groupes compacts (non-commutatifs), par exemple, $SO(3)$. Puisque les groupes compacts portent une mesure de Haar, les pavages sont naturellement généralisées aux groupes compacts. Pour généraliser la notion d’ensembles spectraux dans des groupes compacts, nous avons besoin de plus de la théorie de la représentation de groupe. Nous considérons d’abord un groupe fini G . Soit $\phi : G \rightarrow GL_d(\mathbb{C})$ une représentation. Nous écrivons $\phi = (\phi_{ij})_{1 \leq i, j \leq d}$. Soit R_d l’union des classes d’isomorphismes de dimension d , des représentations irréductibles de G . En vertu des relations d’orthogonalité de Schur (voir par exemple [119], p.31), l’ensemble

$$\tilde{G} := \{\sqrt{d}\phi_{ij} : \phi \in R_d, d \geq 1, 1 \leq i, j \leq d\}$$

est une base de Hilbert de $L^2(G)$. Un sous-ensemble non vide $\Omega \subset G$ est appelé *un ensemble spectral* si on peut trouver $\Lambda \subset \tilde{G}$ tel que Λ est une base de Hilbert de $L^2(\Omega)$. Puisque les relations d’orthogonalité de Schur admettent une généralisation au cas des groupes compacts en général, les notions d’ensembles spectraux sont naturellement généralisées aux groupes compacts. Ainsi nous pourrions naturellement formuler la conjecture de Fuglede sur des groupes compacts: *un ensemble borélien de mesure de Haar positive et finie dans un groupe compact est spectral si et seulement s’il pave le groupe*. Nous pourrions également poser la question de savoir à quoi ressemblent les ensembles spectraux (resp. les pavages) sur les groupes compacts. Les questions concernant les groupes compacts peuvent être étudiées dans de futures recherches, nous ne discuterons pas de ces questions dans cette thèse.

Nous présentons maintenant quelques résultats connus sur la conjecture générale sur les groupes abéliens localement compacts. Empruntant la notation de Dutkay et Lai [28],

nous écrivons $S\text{-}T(G)$, respectivement $T\text{-}S(G)$, si l'implication "Spectral \Rightarrow Pavage", respectivement "Pavage \Rightarrow Spectral", est vraie sur G . Dutkay et Lai ont montré les relations suivantes:

$$T\text{-}S(\mathbb{R}) \Leftrightarrow T\text{-}S(\mathbb{Z}) \Leftrightarrow T\text{-}S(\mathbb{Z}/N\mathbb{Z}) \text{ pour tout } N,$$

et

$$S\text{-}T(\mathbb{R}) \Rightarrow S\text{-}T(\mathbb{Z}) \Rightarrow S\text{-}T(\mathbb{Z}/N\mathbb{Z}) \text{ pour tout } N.$$

Ce résultat montre que les groupes cycliques jouent un rôle important dans l'étude de la conjecture de Fuglede sur \mathbb{R} . En fait, nous montrons que des résultats similaires se vérifient lorsque nous étudions la conjecture de Fuglede sur le corps des nombres p -adiques.

Si au lieu de groupes abéliens localement compacts nous regardons les corps locaux, alors la structure algébrique supplémentaire du corps pourrait nous donner plus d'informations sur les pavages et les ensembles spectraux. Rappelons que tout corps local est isomorphe (comme corps topologique) à l'un des suivants:

- les corps locaux archimédiens (de caractéristique zéro): le corps \mathbb{R} de nombres réels, et le corps \mathbb{C} des nombres complexes.
- les corps locaux non-archimédiens de caractéristique zéro: toutes les extensions finies du corps des nombres p -adiques \mathbb{Q}_p (où p est un nombre premier).
- les corps locaux non-archimédiens de caractéristique p (où p est un nombre premier donné): toutes les extensions finies du corps des séries formelles de Laurent $\mathbb{F}_q((T))$ sur un corps fini \mathbb{F}_q (où q est une puissance de p).

Comme nous l'avons déjà noté, la conjecture est ouverte et semble plutôt difficile dans \mathbb{R} et \mathbb{C} (isomorphe à \mathbb{R}^2). Cependant, elle est facile à réfuter sur toute extension finie du corps des séries formelles de Laurent: certains contre-exemples sont construits par une analogie des contre-exemples dans les groupes abéliens finis. L'idée principale est de considérer le corps des séries formelles de Laurent comme un \mathbb{F}_q -espace vectoriel de dimension infinie (voir Chapitre 2). Le dernier cas, le corps des nombres p -adiques, est sur une position intermédiaire, qui peut être résolue probablement. C'est une de nos motivations à étudier la conjecture de Fuglede sur le corps des nombres p -adiques. Nous discuterons de cela dans la prochain sous section.

0.2.2 La conjecture de Fuglede sur le corps des nombres p -adiques

Dans cette sous section, nous étudions la conjecture de Fuglede dans le corps des nombres p -adiques. Pour cela, nous étudions la structure des pavages compacts ouverts et des ensembles spectraux ouverts compacts.

Après une mise à l'échelle, nous supposons que les ensembles ouverts compacts sont contenus dans \mathbb{Z}_p . Clairement, un ensemble ouvert compact dans \mathbb{Z}_p a la forme

$$\bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

où $\gamma \geq 1$ est un entier et C est un sous-ensemble de $\{0, 1, \dots, p^\gamma - 1\}$ qui est identifié à $\mathbb{Z}/p^\gamma\mathbb{Z}$. Par développement p -adique, il y a une correspondance un-à-un entre $\mathbb{Z}/p^\gamma\mathbb{Z}$ et l'espace de produit X^γ avec $\sharp X = p$. Comme il est bien connu que l'espace de produit X^γ est identifié à un arbre fini $\mathcal{T}^{(\gamma)}$, $\mathbb{Z}/p^\gamma\mathbb{Z}$ peut aussi être vu comme l'arbre fini $\mathcal{T}^{(\gamma)}$. Le n -ième niveau de $\mathcal{T}^{(\gamma)}$ est constitué des chiffres du développement p -adique à la n -ième position. De plus, nous identifions chaque élément de $\mathbb{Z}/p^\gamma\mathbb{Z}$ à une branche de cet arbre fini $\mathcal{T}^{(\gamma)}$, donnée par les chiffres de son développement p -adique (voir la figure 1). Il s'ensuit que tout sous-ensemble C dans $\mathbb{Z}/p^\gamma\mathbb{Z}$ détermine un sous-arbre \mathcal{T}_C de $\mathcal{T}^{(\gamma)}$ en identifiant les éléments de C à les branches dans $\mathcal{T}^{(\gamma)}$. De plus, le sous-arbre \mathcal{T}_C est dit *homogène* si le nombre de descendants de chaque sommet ne dépend que de son niveau. Si ce nombre est 1 ou p , nous appelons \mathcal{T}_C un arbre p -homogène. Dans ce cas, l'ensemble C est dit p -homogène.

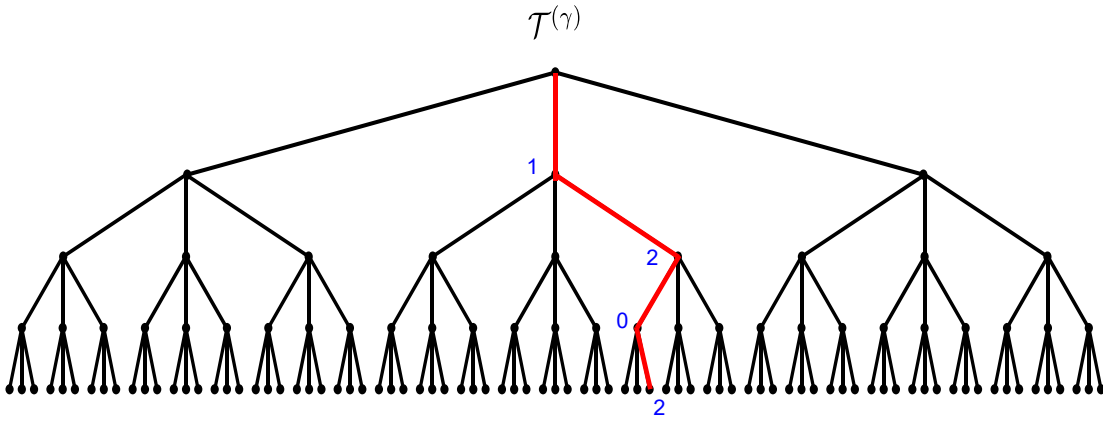


Figure 1: Pour $p = 3$, la branche de 61 peut être référencée par les chiffres (1, 2, 0, 2)

Nous démontrerons que les ensembles spectraux ouverts compacts, les pavages ouverts compacts et les ensembles p -homogènes sont cohérents.

Théorème 0.5 (Chapitre 3, Théorème 3.1). *Soit $\Omega = \sqcup_{c \in C} (c + p^\gamma\mathbb{Z}_p)$ un ensemble ouvert compact dans \mathbb{Z}_p . Les énoncés suivants sont équivalents.*

- (1) Ω est un ensemble spectral dans \mathbb{Q}_p .
- (2) \mathcal{T}_C est un arbre p -homogène.
- (3) Ω pave \mathbb{Q}_p .

Pour chaque $0 \leq n \leq \gamma$, on note

$$C_{\text{mod } p^n} := \{x \in \{0, 1, \dots, p^n - 1\} : \exists y \in C, \text{ tel que } x \equiv y \pmod{p^n}\}$$

le sous-ensemble de $\mathbb{Z}/p^n\mathbb{Z}$ déterminé par C modulo p^n . Nous obtenons également la caractérisation suivante des ensembles spectraux dans le groupe fini $\mathbb{Z}/p^\gamma\mathbb{Z}$.

Théorème 0.6 (Chapitre 3, Théorème 3.3). *Soit $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$. Les affirmations suivantes sont équivalentes.*

- (a) *C est un ensemble spectral dans $\mathbb{Z}/p^\gamma\mathbb{Z}$.*
- (b) *C pave $\mathbb{Z}/p^\gamma\mathbb{Z}$.*
- (c) *Pour chaque $n = 1, 2, \dots, \gamma - 1$, $\sharp(C_{\bmod p^n}) = p^{k_n}$ pour un nombre entier $k_n \in \mathbb{N}$, où $\sharp(C_{\bmod p^n})$ est le cardinal de l'ensemble fini $C_{\bmod p^n}$.*

Comme Tao l'a noté à la page 3 de [121], l'équivalence (a) \Leftrightarrow (b) n'est pas nouvelle, mais découle de Laba [89], qui à son tour s'appuie sur le travail de Coven-Meyerowitz [22]. Cependant, nous le prouvons en montrant que (a) et (b) sont équivalents à (c) ce qui implique la p -homogénéité de C . La p -homogénéité est facilement vérifiable. Nous pouvons l'utiliser pour décrire des ensembles spectraux finis dans $\mathbb{Z}/p^\gamma\mathbb{Z}$.

Afin de prouver la conjecture de Fuglede sur le corps des nombres p -adiques, nous devons utiliser les distributions de Bruhat-Schwartz sur \mathbb{Q}_p . La théorie des distributions p -adiques (distributions de Bruhat-Schwartz sur \mathbb{Q}_p) joue un rôle important dans la résolution des problèmes mathématiques de l'analyse et des applications p -adiques [3, 129]. L'espace des fonctions de test est constitué de toutes les fonctions uniformément localement constantes à support compact.

On note $\mu_E = \sum_{e \in E} \delta_e$ pour un ensemble discret $E \subset \mathbb{Q}_p$. La proposition suivante est valable non seulement pour \mathbb{Q}_p mais aussi pour le groupe abélien localement compact général G (voir le théorème 2.18 et le théorème 2.19). Le cas où $G = \mathbb{R}^d$ a été prouvé par Jørgensen et Pedersen [63] et le cas où G est un corps local a été prouvé par Fan [30].

Proposition 0.7 (Chapitre 3, Proposition 3.8). *Soit Ω un ensemble borélien de mesure de Haar positive et finie dans \mathbb{Q}_p . Alors nous avons les propriétés suivantes.*

- *L'ensemble Ω est un pavage avec complément de pavage T si et seulement si*

$$\mu_T * 1_\Omega(x) = 1, \quad \text{pour } \mu\text{-p.p. } x \in \mathbb{Q}_p.$$

- *L'ensemble Ω est un ensemble spectral avec spectre Λ si et seulement si*

$$\mu_\Lambda * |\widehat{1_\Omega}|^2(\xi) = \mu(\Omega)^2, \quad \forall \xi \in \widehat{\mathbb{Q}_p}.$$

Par transformation de Fourier, nous obtenons les conditions équivalentes des pavages et des ensembles spectraux dans \mathbb{Q}_p :

un ensemble borélien Ω est un pavage avec complément de pavage T dans \mathbb{Q}_p si et seulement si on a $\widehat{\mu_T} \cdot \widehat{1_\Omega} = \delta_0$,

et

un ensemble borélien Ω est un ensemble spectral avec spectre Λ dans \mathbb{Q}_p si et seulement si on a $\widehat{\mu_\Lambda} \cdot (\widehat{\frac{1}{m(\Omega)^2} |\widehat{1_\Omega}|^2}) = \delta_0$.

Puisque la fonction δ_0 a de nombreux zéros, cela nous amène à étudier les zéros de la distribution $\widehat{\mu}_E$ et la fonction \widehat{f} dans l'équation fonctionnelle suivante:

$$\widehat{\mu}_E \cdot \widehat{f} = \delta_0,$$

où $0 \leq f \in L^1(\mathbb{Q}_p)$.

Indépendamment du résultat du théorème 1.5, nous obtenons le théorème suivant.

Théorème 0.8 (Chapitre 4, Théorème 4.25). *Soit $\Omega \subset \mathbb{Q}_p$ un ensemble borélien de mesure de Haar positive et fini. Alors Ω est un ensemble spectral si et seulement s'il pave \mathbb{Q}_p . De plus, les pavages ou les ensembles spectraux dans \mathbb{Q}_p sont des ensembles ouverts compacts à un ensemble de mesure nulle près.*

Ce résultat confirme la conjecture de Fuglede sur le corps des nombres p -adiques. En combinant Théorème 1.5 et Théorème 1.8, nous obtenons que les pavages et les ensembles spectraux de \mathbb{Q}_p sont caractérisés par une propriété géométrique simple: la p -homogénéité.

0.3 Suites de type Chowla et suites oscillantes

Dans cette section, nous étudions les suites de type Chowla et les suites oscillantes. Un des outils pour étudier ces suites est la disjonction initiée par Furstenberg [44]. Nous commençons par quelques remarques historiques sur l'étude de la disjonction.

0.3.1 Disjonction de Furstenberg

Comme Furstenberg [44], nous appellerons un groupe de transformation préservant la mesure sur un espace de mesure un *processus*. Par analogie avec les entiers, Furstenberg [44] a donné deux façons suivantes pour définir que : “deux processus sont relativement premiers”, ce qui a initié la théorie de la disjonction.

Condition 1. *Deux processus T et S n'ont pas de facteur commun sauf le processus trivial réduit à un point.*

Condition 2. *Si les deux processus T et S apparaissent comme des facteurs dans un processus, alors leur produit direct $T \times S$ apparaît également comme un facteur.*

La condition 2 a été appelée par Furstenberg la disjonction de T et S . De plus, Furstenberg [44] a prouvé que la condition 2 implique la condition 1. En fait, les motivations initiales de Furstenberg pour la disjonction viennent de la classification des systèmes dynamiques liée à une question originaire en approximation diophantienne, passant par un problème de filtrage en théorie des probabilités:

Si $(\xi_n)_{n \in \mathbb{N}}$ et $(\zeta_n)_{n \in \mathbb{N}}$ sont deux processus, pouvons-nous récupérer $(\xi_n)_{n \in \mathbb{N}}$ de leur somme $(\xi_n + \zeta_n)_{n \in \mathbb{N}}$?

Beaucoup d'applications de disjonction ont déjà été montrées dans [44] par Furstenberg. Par exemple, les processus disjoints de tous les processus de Bernoulli sont caractérisés par des processus déterministes.

Furstenberg [44] a également introduit la notion de couplages, qui est devenue un outil puissant dans la théorie ergodique. L'étude de la disjonction s'avère très intéressante, car elle traite de

toutes les façons possibles de voir deux processus cohabitant à l'intérieur d'un troisième.

En commençant par les travaux pionniers de Furstenberg, la notion de couplages a été exploitée régulièrement ([109, 67, 47, 105, 45, 127, 68, 56, 77, 46, 125, 111, 76, 66, 98]). Pour une étude plus complète de la théorie ergodique par couplages, nous renvoyons au livre de Glasner [49] et à l'article de de la Rue [26].

0.3.2 Vers la conjecture de Sarnak

Comme nous l'avons déjà mentionné, la conjecture de Sarnak s'intéresse à la disjonction des systèmes dynamiques d'entropie nulle. Nous résumons ici la progression vers la conjecture de Sarnak. Nous savons maintenant que la conjecture est vraie pour les systèmes de Kronecker (Vinogradov [128]), les nilsystèmes (Green et Tao [51]), le système de Thue-Morse (Mauduit et Rivat [102]), les flots horocycliques (Bourgain, Sarnak et Ziegler [17]), Certaines classes de transformations d'échange d'intervalles (Ferenczi et Mauduit [41]) etc. Pour certains d'entre eux, le critère de KBSZ (Kàtai [73], Bourgain, Sarnak et Ziegler [17]) est utilisé pour obtenir l'orthogonalité. Pour un résumé complet de la progression de la conjecture de Sarnak, voir l'étude [40].

Comme nous l'avons mentionné, les conditions (Ch) et (S_0) pour une suite $z \in \{-1, 0, 1\}^{\mathbb{N}}$ sont obtenues à partir des conjectures de Chowla et de Sarnak respectivement en remplaçant la fonction Möbius par la suite z . El Abdalaoui et al. [4] ont prouvé que la condition (Ch) implique la condition (S_0) . Soit (X, T) un système dynamique topologique. Rappelons que $x \in X$ est dit *complètement déterministe* si pour tout point d'accumulation ν de $\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x}$ où δ_x est la mesure de Dirac en x , le système (X, T, ν) est d'entropie nulle. El Abdalaoui et al. ont aussi défini la condition (S), formellement plus forte que (S_0) en exigeant

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) z(n) = 0$$

pour tout système dynamique topologique (X, T) , tout $f \in C(X)$ et tout point complètement déterministe $x \in X$. Ils ont prouvé l'équivalence entre la condition (S) et la condition (S_0) .

0.3.3 Propriété de Chowla

Dans ce paragraphe, nous présentons notre définition de la propriété de Chowla pour les suites à valeurs 0 ou des nombres complexes de module 1, généralisant la condition (Ch)

pour les suites prenant des valeurs dans $\{-1, 0, 1\}$.

Pour définir une telle propriété de Chowla, nous devons en plus définir l'indice d'une suite dans $(S^1 \cup \{0\})^{\mathbb{N}}$ et définir les fonctions $x \mapsto x^{(n)}$ pour tout $x \in S^1 \cup \{0\}$ et pour tout $n \in \mathbb{Z}$ qui sont des analogues des puissances sur S^1 .

Soit $(z(n))_{n \in \mathbb{N}} \in (S^1 \cup \{0\})^{\mathbb{N}}$. Nous notons $U(m) = \{e^{2\pi i \frac{k}{m}} : 0 \leq k \leq m-1\}$ l'ensemble constitué de toutes les racines m -ièmes de l'unité pour $m \in \mathbb{N}_{\geq 2}$ où $\mathbb{N}_{\geq 2}$ désigne l'ensemble des entiers positifs supérieurs ou égaux à 2. L'indice de la suite $(z(n))_{n \in \mathbb{N}}$ est défini par le plus petit nombre $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ tel que le ensemble

$$\{n : z(n) \notin U(m) \cup \{0\}\}$$

est de densité nulle dans \mathbb{N} . Pour un point $x \in S^1 \cup \{0\}$ et $n \in \mathbb{Z}$, nous définissons

$$x^{(n)} := \begin{cases} x^n & \text{si } x \in S^1, \\ 0 & \text{si } x = 0. \end{cases}$$

Soit

$$I(m) := \begin{cases} \{0, 1, \dots, m-1\}, & \text{si } m \in \mathbb{N}_{\geq 2}; \\ \mathbb{Z}, & \text{si } m = \infty. \end{cases}$$

Nous définissons la propriété de Chowla pour les suites dans $S^1 \cup \{0\}$ comme suit.

Définition 0.9. Soit $z \in (S^1 \cup \{0\})^{\mathbb{N}}$ une suite d'indice $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Nous disons que la suite z a la propriété de Chowla si

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} z^{(i_0)}(n+a_1) \cdot z^{(i_1)}(n+a_2) \cdots z^{(i_r)}(n+a_r) = 0, \quad (0.3.1)$$

pour chaque choix de $0 \leq a_1 < \dots < a_r, r \geq 0$, et $i_s \in I(m)$ qui ne sont pas tous égaux à 0.

Il n'est pas difficile de voir que les suites prenant des valeurs dans $\{-1, 0, 1\}$ ont un indice égal à 2. Évidemment, la propriété de Chowla pour ces suites est exactement la condition (Ch).

De même que le résultat de la condition (Ch) impliquant la condition (S), nous montrons que les suites de type Chowla sont orthogonales à tous les systèmes dynamiques topologiques d'entropie nulle. L'outil principal de la preuve est la théorie des couplages.

Théorème 0.10 (Chapitre 7, Théorème 7.13). Si une suite dans $(S^1 \cup \{0\})^{\mathbb{N}}$ a la propriété de Chowla, alors elle est orthogonale à tous les systèmes dynamiques topologiques d'entropie nulle.

Nous étudions également à quelle condition les échantillons d'une suite aléatoire ont presque sûrement la propriété de Chowla. Nous considérons d'abord les suites aléatoires stationnaires. Nous avons la proposition suivante.

Proposition 0.11 (Chapitre 7, Proposition 7.24). *Soit $\mathcal{Z} = (Z_n)_{n \in \mathbb{N}}$ une suite de variables aléatoires stationnaires prenant des valeurs dans $S^1 \cup \{0\}$. Supposons que la suite aléatoire \mathcal{Z} ait un indice constant fini ou infini presque sûrement. Alors presque sûrement la suite aléatoire \mathcal{Z} a la propriété de Chowla si et seulement si*

$$\mathbb{E}[Z_{a_1}^{(i_1)} \cdot Z_{a_2}^{(i_2)} \cdots Z_{a_r}^{(i_r)}] = 0, \quad (0.3.2)$$

pour chaque choix de $1 \leq a_1 < \cdots < a_r, r \geq 0$, et $i_s \in I(m)$ qui ne sont pas tous égaux à 0.

En utilisant la proposition 0.11, on peut caractériser les suites aléatoires dans la proposition 0.11 quand elles sont des suites de variables indépendantes.

Corollaire 0.12 (Chapitre 7, Corollaire 7.28). *Soit (X_n) une suite de variables aléatoires stationnaires prenant des valeurs dans $S^1 \cup \{0\}$. Soit $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Supposons que presque sûrement la suite $(X_n)_{n \in \mathbb{N}}$ ait l'indice m et possède la propriété de Chowla. On a les propriétés suivantes.*

(1) *Si m est fini, alors la suite $(X_n)_{n \in \mathbb{N}}$ est indépendante si et seulement si*

$$\mathbb{E}[|X_{a_1} X_{a_2} \cdots X_{a_r}|] = \mathbb{E}[|X_{a_1}|] \mathbb{E}[|X_{a_2}|] \cdots \mathbb{E}[|X_{a_r}|] \quad (0.3.3)$$

pour chaque choix de $1 \leq a_1 < \cdots < a_r, r \geq 0$.

(2) *Si m est infini, alors la suite $(X_n)_{n \in \mathbb{N}}$ est une suite de variables indépendantes.*

C'est une question naturelle de savoir s'il y a des suites aléatoires dépendantes de la propriété de Chowla presque sûrement. Pour répondre à cette question, nous allons associer une suite aléatoire dans un système de Bernoulli avec un arbre coloré.

Soit \mathcal{A} un alphabet fini. Soit μ la mesure Bernoulli-uniforme sur $\mathcal{A}^{\mathbb{N}}$. Soit $G : \mathcal{A}^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$ une fonction continue. Notons $Z_{G,n} := G \circ S^n$. Nous expliquons comment associer le processus $(Z_{G,n})_{n \in \mathbb{N}}$ avec l'arbre coloré \mathcal{T}_G dans Chapitre 7. Nous introduisons également la notion d'homogénéité pour un arbre coloré dans le même chapitre.

Nous avons la proposition suivante.

Proposition 0.13 (Chapitre 7, Proposition 7.33). *La suite aléatoire $(Z_{G,n})_{n \in \mathbb{N}}$ a la propriété de Chowla presque sûrement si et seulement si l'arbre associé \mathcal{T}_G est homogène.*

Avec l'aide du corollaire 0.12 et de la proposition 0.13, nous pouvons construire des suites aléatoires dépendantes ayant la propriété de Chowla presque sûrement. Nous montrons juste un exemple ici pour illustrer l'existence de telles suites. Nous définissons $g : \{-1, 0, 1\}^2 \rightarrow \{-1, 1\}$ par

$$g(a, b) = \begin{cases} a & \text{si } a \in \{-1, 1\} \text{ et } b = 0; \\ 0 & \text{sinon.} \end{cases}$$

Définissons $G : \{-1, 0, 1\}^{\mathbb{N}} \rightarrow \{-1, 1\}$ par $(x_n)_{n \in \mathbb{N}} \mapsto g(x_0, x_1)$ et $Z_n = G \circ S^n$, où S est le décalage à gauche sur $\{-1, 0, 1\}^{\mathbb{N}}$. Il est facile de vérifier que la suite $(Z_n)_{n \in \mathbb{N}}$ ne satisfait pas le (0.3.3) du Corollaire 0.12, ce qui implique que $(Z_n)_{n \in \mathbb{N}}$ est une suite de variables dépendantes. D'autre part, on peut vérifier que son arbre associé \mathcal{T}_G est homogène, ce qui signifie qu'il a la propriété de Chowla presque sûrement.

0.3.4 Suites oscillantes

Dans cette sous-section, nous considérons les propriétés d'oscillation des suites complexes.

Rappelons que les suites oscillantes ont été définies dans la section 0.1. Récemment, l'orthogonalité (à certains systèmes dynamiques) de suites oscillantes a été étudiée. Par exemple, Fan et Jiang ont prouvé que les suites oscillantes d'ordre 1 sont orthogonales à tous les systèmes dynamiques dits minimalement moyen-attractifs ('minimally mean attractable' en anglais) et minimalement L-moyen-stables ('minimally mean-L-stable' en anglais) [38]; Fan a prouvé que les suites pleinement oscillantes sont orthogonales à toutes les réalisations ergodiques multiples de la dynamique topologique ayant le spectre quasi-discret au sens de Hahn-Parry [31] et à toutes les fonctions affines d'entropie nulle sur tout groupe abélien compact [32].

Comme nous l'avons mentionné, la fonction de Möbius est pleinement oscillante. De plus, il existe plusieurs façons de construire d'autres suites oscillantes: par exemple, Fan [33, 34] a prouvé que certaines suites sur nil-variétés et certaines suites q -multiplicatives de type Gelfond sont des suites pleinement oscillantes. Akiyama et Jiang [5] ont prouvé que pour presque tout $\beta > 1$, la suite $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ est pleinement oscillante.

Nous prouvons plusieurs propriétés de la suite oscillante.

Lemme 0.14 (Chapitre 6, Lemme 6.2). *Soit $d \geq 1$. Une suite de nombres complexes $(c_n)_{n \in \mathbb{N}}$ est oscillante d'ordre d si et seulement si toutes les sous-suites arithmétiques $(c_{an+b})_{n \in \mathbb{N}}$ sont oscillantes d'ordre d pour tous les entiers $a \in \mathbb{N}^*$ et $b \in \mathbb{N}$.*

Notre preuve du lemme 0.14 est basée sur un fait prouvé dans [38]. Nous remarquons qu'une forme plus forte du lemme 0.14 a été prouvée dans [32].

La proposition suivante nous dit que la définition des suites oscillantes reste valable s'il y a une perturbation pour le polynôme P par un polynôme à coefficients rationnels.

Proposition 0.15 (Chapitre 6, Proposition 6.3). *Une suite de nombres complexes $(c_n)_{n \in \mathbb{N}}$ est oscillante d'ordre d si et seulement si*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i(P(n)+Q(n))} = 0$$

pour tout $P \in \mathbb{R}_d[t]$ et tout $Q \in \mathbb{Q}[t]$.

En fait, le lemme 1.14 et la proposition 1.15 sont également utiles pour prouver le théorème 1.16.

Nous nous intéressons à la signification dynamique des suites oscillantes. Nous donnerons plusieurs descriptions équivalentes de telles suites en termes d'orthogonalité à différents systèmes dynamiques sur des tores.

Nous rappelons maintenant quelques notions concernant les groupes abéliens compacts. L'application affine T sur un groupe abélien compact X est une fonction de la forme

$$Tx = Ax + b,$$

où $b \in X$ et A est un automorphisme sur X . L'automorphisme A est aussi appelé la *section d'un automorphisme* de l'application affine T .

Comme observé dans [99] par Liu et Sarnak, un automorphisme d'entropie nulle sur $\mathbb{T}^d \times F$, où \mathbb{T}^d est un tore d -dimensionnel et F est un groupe abélien fini, est de la forme

$$T(x, y) = (Ax + By, Cy)$$

avec A un automorphisme sur \mathbb{T}^d d'entropie nulle, B un morphisme de F dans \mathbb{T}^d et C un automorphisme sur F . Un tel automorphisme est dit *spécial* si le morphisme $N := A - I$ est nilpotent d'ordre d (c'est à dire, $N^d = 0$ mais $N^{d-1} \neq 0$) et dans ce cas, l'application affine T est aussi appelée *spéciale*.

Nous avons le théorème suivant, dont le point clé est que les automorphismes d'entropie nulle sur des groupes abéliens compacts sont quasi-unipotents (voir [23, 99]).

Théorème 0.16 (Chapitre 6, Théorème 6.5). *Une suite de nombres complexes est oscillante d'ordre d si et seulement si elle est orthogonale à l'un des systèmes dynamiques suivants ou à l'une des classes suivantes de systèmes dynamiques.*

- (1) *Tous les automorphismes d'entropie nulle sur $\mathbb{T}^{d+1} \times F$ pour n'importe quel groupe abélien fini F .*
- (2) *Un automorphisme spécial d'entropie nulle sur $\mathbb{T}^{d+1} \times F$, où F est un groupe abélien fini fixé.*
- (3) *Toutes les applications affines d'entropie nulle sur $\mathbb{T}^d \times F$ pour n'importe quel groupe abélien fini F .*
- (4) *Toutes les applications affines spéciales d'entropie nulle sur $\mathbb{T}^d \times F$, où F est un groupe abélien fini fixé, qui partage la même section qu'un automorphisme.*

Nous remarquons que le groupe abélien fini F dans le théorème 0.16 peut être le groupe trivial de sorte que $\mathbb{T}^d \times F$ est exactement le tore \mathbb{T}^d .

Pour les suites pleinement oscillantes, nous avons la description équivalente suivante.

Théorème 0.17 (Chapitre 6, Théorème 6.6). *Une suite de nombres complexes est pleinement oscillante si et seulement si elle est orthogonale à toutes les applications affines d'entropie nulle sur un groupe abélien compact.*

L'implication directe du théorème 0.17 est une conséquence directe du théorème 0.16. D'autre part, l'implication réciproque est essentiellement prouvée dans [99], où Liu et Sarnak ont observé que toute orbite de toute application affine d'entropie nulle sur un

groupe abélien compact pouvait être réalisée par une orbite d'une fonction affine d'entropie nulle sur $\mathbb{T}^{d+1} \times F$ pour un groupe abélien fini F .

Une conséquence directe du théorème 0.17 est que les suites satisfaisant la condition (S_0) sont toujours pleinement oscillantes.

0.3.5 Suites $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$, $\beta > 1$

Nous présentons notre travail sur la propriété de Chowla des suites $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ pour $\beta > 1$ dans ce paragraphe. L'outil principal est la théorie de la distribution uniforme modulo 1. Nous commençons par une brève introduction à la notion de distribution uniforme modulo 1.

Historiquement parlant, de 1901 à 1910, trois articles de Bohl [13], Sierpiński [117] et Weyl [132] se consacrent en partie à la distribution de suites réelles. Quelques années plus tard, en développant les idées dans ces articles, Weyl [133, 134] a initié l'étude de la distribution uniforme modulo 1 et a énoncé divers critères pour qu'une suite donnée soit uniformément distribuée modulo 1.

Une suite $(x(n))_{n \in \mathbb{N}}$ de nombres réels est dite *uniformément distribuée modulo 1* si, pour tous nombres réels u, v avec $0 \leq u < v \leq 1$, on a

$$\lim_{N \rightarrow \infty} \frac{\#\{n : 1 \leq n \leq N, u \leq \{x_n\} \leq v\}}{N} = v - u,$$

où $\{x\}$ est la partie fractionnaire du nombre réel x .

Un des résultats de Weyl [133, 134], que l'on appelle souvent le critère de Weyl, est le théorème suivant.

Théorème 0.18 ([133, 134]). *Une suite $(x(n))_{n \in \mathbb{N}}$ de nombres réels est uniformément distribuée modulo 1 si et seulement si*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h x(n)} = 0,$$

pour tout entier non nul h .

Pour un nombre réel donné $\beta > 1$, peu de résultats sont connus pour savoir si la suite $(\beta^n)_{n \in \mathbb{N}}$ est uniformément distribuée modulo 1 et comment elle est distribuée. Par exemple, nous ne savons toujours pas si 0 est un point limite de $(\{e^n\})_{n \in \mathbb{N}}$, ni de $(\{(\frac{3}{2})^n\})_{n \in \mathbb{N}}$. Cependant, plusieurs propriétés métriques ont été établies. La première a été obtenue par Koksma [78], qui a prouvé que pour presque tout $\beta > 1$ la suite $(\beta^n)_{n \in \mathbb{N}}$ est uniformément distribuée modulo 1. En fait, Koksma [78] a prouvé le théorème général suivant.

Théorème 0.19 (Théorème de Koksma). *Soit $(f_n)_{n \in \mathbb{N}}$ une suite de fonctions réelles C^1 définies sur un intervalle $[a, b]$. On suppose*

(1) $f'_m - f'_n$ est monotone sur $[a, b]$ pour tous $m \neq n$;

(2) $\inf_{m \neq n} \min_{x \in [a, b]} |f'_m(x) - f'_n(x)| > 0$.

Alors pour presque tout $x \in [a, b]$, la suite $(f_n(x))_{n \in \mathbb{N}}$ est uniformément distribuée modulo 1.

Pour plus d'informations sur la distribution uniforme modulo 1, nous pouvons nous référer aux études [19, 87].

Comme nous l'avons mentionné, Akiyama et Jiang [5] ont prouvé que pour presque tout $\beta > 1$, la suite $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ est pleinement oscillante. Plus précisément, ils ont prouvé le théorème suivant.

Théorème 0.20 ([5], Corollaire 2). *Soit α un nombre réel non nul et $g \in C^2_+(]1, +\infty[)$ une fonction, où $C^2_+(]1, +\infty[)$ est l'espace des fonctions positives et 2 fois continuellement différentiables sur $]1, +\infty[$ dont la dérivée i -ième est non négative pour $i = 1, 2$. Alors pour presque tout nombre réel $\beta > 1$ et pour tout polynôme réel P , la suite*

$$(e^{2\pi i(\alpha \beta^n g(\beta) + P(n))})_{n \in \mathbb{N}}$$

est pleinement oscillante.

L'étape principale de leur démonstration est de montrer la distribution uniforme de la suite $(\alpha \beta^n g(\beta))_{n \in \mathbb{N}}$ pour un nombre réel α donné, une fonction $g \in C^2_+((1, \infty))$ et pour presque tout nombre réel $\beta > 1$ en utilisant le théorème de Koksma.

Comme l'indiqué dans le théorème 0.17, les suites pleinement oscillantes sont orthogonales à toutes les applications affines d'entropie nulle sur des groupes abéliens compacts. En combinant ce résultat avec le théorème 0.20, nous voyons que pour presque tout nombre réel $\beta > 1$, la suite $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ est orthogonal à cette classe de systèmes dynamiques topologiques d'entropie nulle. On peut se demander à quels systèmes dynamiques topologiques (d'entropie nulle) cette suite est orthogonale. Nous prouvons que pour presque tout nombre réel $\beta > 1$, la suite $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ est orthogonale à tous les systèmes dynamiques d'entropie nulle. Plus précisément, nous pouvons le théorème suivant.

Théorème 0.21 (Chapitre 7, Théorème 7.12). *Soit $g \in C^2_+(]1, +\infty[)$ ayant un nombre dénombrable de zéros sur $]1, +\infty[$. Alors pour presque tout nombre réel $\beta > 1$, la suite*

$$(e^{2\pi i \beta^n g(\beta)})_{n \in \mathbb{N}} \tag{0.3.4}$$

a la propriété de Chowla.

En posant $g \equiv 1$ dans le théorème 0.21, on voit qu'un corollaire direct des théorèmes 0.10 et 0.21 est que, pour presque tout $\beta > 1$, la suite $(\beta^n)_{n \in \mathbb{N}}$ est orthogonale à tous les systèmes dynamiques d'entropie nulle.

Nous désignons par $C^2_{count}(]1, +\infty[)$ l'espace de toutes les fonctions C^2 qui ont au plus dénombrable de zéros sur $]1, +\infty[$. L'étape clé de la démonstration du théorème 0.21

est d'utiliser le théorème de Koksma pour prouver la distribution uniforme de la suite $(\beta^n g(\beta))_{n \in \mathbb{N}}$ pour une fonction $g \in C_{count}^2(]1, +\infty[)$ et pour presque tout $\beta > 1$. Nous remarquons que la méthode pour prouver la distribution uniforme de telles suites est différente de celle utilisée pour démontrer le théorème 0.20 par Akiyama et Jiang. On remarque aussi que l'espace $C_{count}^2(]1, +\infty[)$ est plus grand que l'espace $C_+^2(]1, +\infty[)$. Par exemple, les fonctions $f(x) = x^2 - 5x + 6$ et $h(x) = \sin(x)$ sont dans $C_{count}^2(]1, +\infty[)$ mais pas dans $C_+^2(]1, +\infty[)$.

Chapter 1

Introduction

This thesis is devoted to solving Fuglede's conjecture on the field of p -adic numbers and to studying the oscillating properties of sequences related to Sarnak's conjecture. The thesis is thus divided into two parts. The first part consists of Chapters 2, 3 and 4, where we are interested in Fuglede's conjecture. The second part consists of Chapters 5, 6 and 7, where we are concerned with some randomness and oscillating properties of complex sequences.

1.1 An overview of the thesis

Let us first recall Fuglede's conjecture. Let $\Omega \subset \mathbb{R}^d$ be a Borel set of positive finite Lebesgue measure. The set Ω is called a *tile* of \mathbb{R}^d by translations, if there exists a (countable) set $T \subset \mathbb{R}^d$ such that $\{\Omega + t : t \in T\}$ is a partition of \mathbb{R}^d almost surely. We call T a *tiling complement* of Ω and the couple (Ω, T) a *tiling pair*. The set Ω is said to be a *spectral set* if there exists a (countable) set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda) = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$ is an orthogonal basis of the Hilbert space $L^2(\Omega)$ of square-integrable functions on Ω . Such a set Λ is called a *spectrum* of Ω and the couple (Ω, Λ) is called a *spectral pair*.

In 1974, Fuglede [43] formulated the following conjecture.

Conjecture 1.1. *A Borel set of finite positive Lebesgue measure in \mathbb{R}^d is a spectral set if and only if it is a tile.*

The motivation for Fuglede to study spectral sets is to answer a question raised by Segal about commutative self-adjoint extensions in $L^2(\Omega)$ of the partial differential operators in $C^2(\Omega)$ for suitable region Ω in \mathbb{R}^d . Fuglede [43] proved that the spectral sets are such suitable regions. This initiated the study of the spectral sets. Fuglede [43] also proved that Conjecture 1.1 is true under the extra assumption that the tiling complement or the spectrum is a full-rank lattice in \mathbb{R}^d . Many positive results were obtained [59, 60, 62, 65,

79, 81, 95, 96] before Tao [121] who gave a counterexample for Fuglede's conjecture on \mathbb{R}^5 . Now it is known that the conjecture is false in both directions for $d \geq 3$ [100, 83, 84, 29, 39]. However, the conjecture is still open in lower dimensions ($d = 1, 2$).

Conjecture 1.1 can be formulated for any locally compact abelian group on which the notions of spectral sets and tiles are generalized with the Lebesgue measure replaced by the Haar measure on the group.

The general conjecture is far from being proved and it is not true for \mathbb{R}^d , $d \geq 3$ as we have mentioned. The problem is to investigate on which locally compact abelian groups Fuglede's conjecture holds. Actually, there are only few known results. We know that Fuglede's conjecture holds on $\mathbb{Z}/p^n\mathbb{Z}$ (Łaba [89]), $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ (Iosevich, Mayeli and Pakianathan [61]) and $\mathbb{Z}/p^nq\mathbb{Z}$ with $n \geq 1$ and p, q being distinct primes (Malikiosis and Kolountzakis [101]).

In the first part of this thesis, we study Fuglede's conjecture on the field of p -adic numbers. One of our main theorems asserts that Fuglede's conjecture holds on the field of p -adic numbers, that is, a Borel set in the field of p -adic numbers is a spectral set if and only if it is a tile. Moreover, we prove that the tiles and the spectral sets are compact open up to a zero-measure set.

Another one of our main theorems states that a compact open set is a tile (or spectral set) if and only if its associated tree is homogeneous. Actually, this result yields a geometrical illustration of tiles and spectral sets in the field of p -adic numbers. These results will be described in details in Section 1.2.

The second parts of this thesis studies some randomness and oscillating properties of complex sequences, which are shared by the Möbius function. The Möbius function is an arithmetic function defined by the formula

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the Möbius function plays an important role in number theory. For example, the Prime Number Theorem is equivalent to the following statement (cf. [7], p.91)

$$\sum_{n=1}^N \mu(n) = o(N);$$

Riemann's hypothesis is equivalent to the following formula (see [123])

$$\sum_{n=1}^N \mu(n) = O_\epsilon(N^{\frac{1}{2}+\epsilon}), \text{ for each } \epsilon > 0.$$

The so-called Möbius pseudorandomness heuristic suggests that the patterns (or finite subwords) in the Möbius function μ behave so randomly (or pseudorandomly) that one

should expect a substantial amount of cancellation in sums that involve the sign fluctuation of the Möbius function in a nontrivial way, with the amount of cancellation present comparable to the amount that an analogous random sum would provide. There are a number of ways to make this heuristic precise. As already mentioned, Riemann's hypothesis can be considered as one such manifestation of the heuristic. Another manifestation is the following Chowla's conjecture [21] on the correlations of the Möbius function.

Conjecture 1.2. *For each choice of $0 \leq a_1 < \dots < a_r, r \geq 0$, and $i_s \in \{1, 2\}$ which are not all equal to 2,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu^{i_0}(n+a_1) \cdot \mu^{i_1}(n+a_2) \cdot \dots \cdot \mu^{i_r}(n+a_r) = 0.$$

A more recent formulation of the Möbius randomness heuristic is the following conjecture of Sarnak. Let T be a continuous transformation of a compact metric space X . Following Sarnak [112], we say that a sequence $(a(n))_{n \in \mathbb{N}}$ of complex numbers is orthogonal to the topological dynamical system (X, T) , if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) a(n) = 0, \quad (1.1.1)$$

for all $x \in X$ and for all $f \in C(X)$ where $C(X)$ is the space of continuous functions on X . In 2010, Sarnak [112] formulated the following conjecture.

Conjecture 1.3. *The Möbius function is orthogonal to all topological dynamical systems of zero entropy.*

Sarnak [112] also proved that Chowla's conjecture implies Sarnak's conjecture.

Recently, El Abdalaoui, Kulaga-Przymus, Lemanczyk and de la Rue [4] studied these conjectures from ergodic theory point of view. Let z be an arbitrary sequence taking values in $\{-1, 0, 1\}$ as the Möbius function. Following El Abdalaoui et al., we say that z satisfies the condition (Ch) if it satisfies the condition in Chowla's conjecture; similarly, it satisfies the condition (S_0) if it satisfies the condition in Sarnak's conjecture. El Abdalaoui et al. proved that the condition (Ch) implies the condition (S_0) for all sequences taking values in $\{-1, 0, 1\}$. They also provided some non-trivial examples of sequences satisfying the condition (Ch) and the condition (S_0) respectively.

We consider a more general class of sequences, taking values 0 or complex numbers of modulus 1. We will define Chowla property for this class of sequences and we will prove that such Chowla sequences are orthogonal to all topological dynamical systems of zero entropy. We are interested in finding Chowla sequences. It will be proved that for almost every $\beta > 1$, the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ has Chowla property. It will also be discussed whether the samples of a given random sequence have Chowla property almost surely. We will construct sequences of dependent random variables having Chowla property almost surely.

Now let us consider the orthogonality between two sequences. Two sequences of complex numbers $(a(n))_{n \in \mathbb{N}}$ and $(b(n))_{n \in \mathbb{N}}$ are said to be *orthogonal* if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a(n)b(n) = 0. \quad (1.1.2)$$

Davenport [24] proved that for each $A > 0$,

$$\max_{\alpha \in \mathbb{R}} \left| \sum_{n=1}^N \mu(n)e^{2\pi i n \alpha} \right| = O_A \left(\frac{N}{\log^A N} \right), \text{ for all } N > 1. \quad (1.1.3)$$

This means that the Möbius function is orthogonal to the sequences $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ for all $\alpha \in \mathbb{R}$. A sequence having such orthogonality is called *an oscillating sequence* by Fan and Jiang in [38] where Sarnak's conjecture was proved true for a class of dynamics even when the Möbius function is replaced by an oscillating sequence. In general, oscillating sequences of higher orders and fully oscillating sequences were defined by Fan in [31] where it was proved that the fully oscillating sequences are orthogonal to all multiple ergodic realizations of topological dynamics of quasi-discrete spectrum in the sense of Hahn–Parry. Recall that a sequence of complex numbers $(c_n)_{n \in \mathbb{N}}$ is said to be *oscillating of order d* ($d \geq 1$ being an integer) if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i P(n)} = 0 \quad (1.1.4)$$

for any $P \in \mathbb{R}_d[t]$ where $\mathbb{R}_d[t]$ is the space of real polynomials of degree smaller than or equal to d ; it is said to be *fully oscillating* if it is oscillating of order d for every integer $d \geq 1$.

We are interested in dynamical description of oscillating sequences. We will prove several equivalent definitions of oscillating sequences in terms of their orthogonality to different dynamical systems on tori.

In the following we will detail the above presentation.

1.2 Fuglede's conjecture

In this section, we present our results on Fuglede's conjecture on the field of p -adic numbers. We start with some historical remarks on Fuglede's conjecture.

1.2.1 Historical remarks and open questions

As we have mentioned before, Fuglede ([43]) formulated his conjecture concerned with the equivalence between tiles and spectral sets in Euclidean spaces. There were a lot of positive evidences towards the veracity of this conjecture under different extra assumptions

[60, 96, 81, 58, 59] before the work [121] where Tao gave a counterexample: there exists a spectral subset of \mathbb{R}^d with $d \geq 5$ which is not a tile. After that, Matolcsi [100], Matolcsi and Kolountzakis [83, 84], Farkas and Gy [29], Farkas, Matolcsi and Móra [39] gave a series of counterexamples which show that both directions of Fuglede's conjecture fail on $\mathbb{R}^d (d \geq 3)$.

However, the conjecture is still open in lower dimensions $d = 1, 2$. We will see that some special case of Fuglede's conjecture in dimension 1 is equivalent to a pure number theory problem. Actually, it appears to be rather hard in dimension 1. Nevertheless, Łaba [88] proved that Fuglede's conjecture is true for a union of two intervals. For a union of three intervals, only the "Tiling implies Spectral" part was proved by Bose, Kumar, Krishnan and Madan [14]. If one looks just at a union of disjoint intervals of same size, that is, a set of the form

$$\Omega = A + [0, 1),$$

where A is a finite set in \mathbb{Z} , then in this case, the conjecture is equivalent to a pure number theory problem (see [80]). The conjecture is still open for this class of sets. Łaba [89] proved that the conjecture is true for $\sharp A = p^n$ ($n \in \mathbb{N}$ and p prime). She also proved that if Ω is a tile then it is also spectral for $\sharp A = p^n q^m$ ($n, m \in \mathbb{N}$ and p, q distinct primes). Malikiosis and Kolountzakis [101] proved that the conjecture holds for $\sharp A = p^n q$ ($n \in \mathbb{N}$ and p, q distinct primes).

In dimension 2, there is few result. We only know that Iosevich, Katz and Tao [59] proved the conjecture for convex planar domains.

As it has already been mentioned in Section 1.1, the general Fuglede's conjecture on locally compact abelian groups is formulated as follows:

Conjecture 1.4. *A Borel set of positive and finite Haar measure in a locally compact abelian group is a spectral set if and only if it is a tile.*

In its generality, this general conjecture is not true, for instance, it fails on $\mathbb{R}^d, d \geq 3$. We could rather ask on which groups it holds. In this sense, the general conjecture is widely open. The different algebraic structures of different groups would provide diverse information. Though there is a general Fourier analysis on locally compact abelian groups (see [108]), each group has its own Fourier analysis which is slight different from each other. Thus the methods to deal with the conjecture differ from group to group. This may offer a chance to see the connections between tiles and spectral sets in some group if one knows well the algebraic structure and Fourier analysis on this group. Therefore, there are abundant questions to be thought. The question arises even for finite abelian groups. Actually, it appears to be rather hard in finite abelian groups. The "biggest" finite abelian groups proved to be true for Fuglede's conjecture is $\mathbb{Z}/p^n q \mathbb{Z}$ with $n \geq 1$ and p, q being distinct primes by Malikiosis and Kolountzakis [101]. It is not yet known whether the conjecture holds on $\mathbb{Z}/p^n q^m \mathbb{Z}$ with $n \geq 1, m > 1$ and p, q being distinct primes.

On the other hand, instead of considering the conjecture itself, it is also very interesting to study the structure of tiles and respectively spectral sets in locally compact abelian

groups. Even though the conjecture fails on certain groups, it is worth pondering the difference between the tiles and spectral sets. From this says, there are plentiful questions under consideration. We could ask this question even for finite cyclic groups, that is, can we classify tiles and spectral sets in finite cyclic groups? It appears to be far from being achieved for this question. Coven and Meyerowitz [22] conjectured that two algebraic conditions together (see Section 2.6) are sufficient and necessary for tiles and spectral sets respectively in cyclic groups. However, it is only known that these two algebraic conditions are sufficient for tiles and spectral sets in $\mathbb{Z}/p^n q^m \mathbb{Z}$ with $n \geq 1, m > 1$ and p, q being distinct primes (Laba [88]).

Moreover, we could also ask the above questions for non-commutative compact groups, for instance, $SO(3)$. Since compact groups all carry a Haar measure, the tiles are naturally generalized to compact groups. To generalize the notion of spectral sets in compact groups, we need additionally the group representation theory. We first consider a finite group G . Let $\phi : G \rightarrow GL_d(\mathbb{C})$ be a representation. We write $\phi = (\phi_{ij})_{1 \leq i, j \leq d}$. Due to Schur orthogonality relations (see for example [119], p.31), the set

$$\tilde{G} := \{\sqrt{d}\phi_{ij}\}$$

taking the union of these over all the isomorphism classes of d -dimensional irreducible representations of G for all $d \geq 1$ is a Hilbert basis of $L^2(G)$. A non-empty subset $\Omega \subset G$ is called a *spectral set* if we can find $\Lambda \subset \tilde{G}$ such that Λ is a Hilbert basis of $L^2(\Omega)$. Since Schur orthogonality relations admit a generalization to the case of compact groups in general, the notion of spectral sets are naturally generalized to compact groups. Thus we could naturally formulate Fuglede's conjecture on compact groups: *a Borel set of positive and finite Haar measure in a compact group is a spectral set if and only if it is a tile*. We could also ask the question what tiles and spectral sets look like on compact groups. The questions on compact groups may be studied in future research, but we will not discuss these questions in this thesis.

Now we present some works towards the general conjecture on locally compact abelian groups. Borrowing the notation from Dutkay and Lai [28], we write $S-T(G)$, respectively $T-S(G)$, if the direction "Spectral \Rightarrow Tiling", respectively "Tiling \Rightarrow Spectral", holds in G . Dutkay and Lai [28] showed the following relations:

$$T-S(\mathbb{R}) \Leftrightarrow T-S(\mathbb{Z}) \Leftrightarrow T-S(\mathbb{Z}/N\mathbb{Z}) \text{ for all } N,$$

and

$$S-T(\mathbb{R}) \Rightarrow S-T(\mathbb{Z}) \Rightarrow S-T(\mathbb{Z}/N\mathbb{Z}) \text{ for all } N.$$

This result shows that the cyclic groups play important roles in the study of Fuglede's conjecture in \mathbb{R} . In fact, we will show later that some similar results hold when we study Fuglede's conjecture on the field of p -adic numbers.

If instead of general locally compact abelian groups we look at local fields, then the additional algebraical structure of the field might give us more information on tiles and

spectral sets. Recall that every local field is isomorphic (as a topological field) to one of the following:

- archimedean local fields (of characteristic zero): the field \mathbb{R} of real numbers, and the field \mathbb{C} of complex numbers,
- non-Archimedean local fields of characteristic zero: finite extensions of the field of p -adic numbers \mathbb{Q}_p (where p is any prime number),
- non-Archimedean local fields of characteristic p (where p is any given prime number): finite extensions of the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field \mathbb{F}_q (where q is a power of p).

As we have already noticed, the conjecture is open and seems rather hard in \mathbb{R} and \mathbb{C} (isomorphic to \mathbb{R}^2). However, the conjecture is easy to disprove on any finite extensions of the field of formal Laurent series: some counterexamples are constructed by an analogy of the counterexamples in finite abelian groups. The main idea is to view the field of formal Laurent series as a \mathbb{F}_q vector space of infinite dimension (see Chapter 2). The last case, the field of p -adic numbers, is on an intermediate position, which can be hopefully solved. This is one of our motivations to study Fuglede's conjecture on the field of p -adic numbers. We will discuss this in the next subsection.

1.2.2 Fuglede's conjecture on the field of p -adic numbers

In this subsection, we study Fuglede's conjecture on the field of p -adic numbers. To this end, we study the structure of compact open tiles and compact open spectral sets.

Up to a scaling, we assume that the compact open sets are contained in \mathbb{Z}_p . Clearly, a compact open set in \mathbb{Z}_p has the form

$$\sqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where $\gamma \geq 1$ is an integer and C is a subset of $\{0, 1, \dots, p^\gamma - 1\}$ identified with $\mathbb{Z}/p^\gamma \mathbb{Z}$. By p -adic expansion, there is a one-to-one correspondence between $\mathbb{Z}/p^\gamma \mathbb{Z}$ and a product space X^γ with $\sharp X = p$. Since it is well known that the product space X^γ is identified with a finite tree $\mathcal{T}^{(\gamma)}$, $\mathbb{Z}/p^\gamma \mathbb{Z}$ can also be viewed as the finite tree $\mathcal{T}^{(\gamma)}$. The n -th level of $\mathcal{T}^{(\gamma)}$ consists of the digits of p -adic expansion at the n -th position. Moreover, we identify each element in $\mathbb{Z}/p^\gamma \mathbb{Z}$ with a branch in this finite tree $\mathcal{T}^{(\gamma)}$ decided by the digital set of its p -adic expansion (see Figure 1.1). It follows that any subset C in $\mathbb{Z}/p^\gamma \mathbb{Z}$ determines a subtree \mathcal{T}_C of $\mathcal{T}^{(\gamma)}$ by identifying the elements in C with the branches in $\mathcal{T}^{(\gamma)}$. Furthermore, the subtree \mathcal{T}_C is said to be *homogeneous* if the number of descendants of each vertex depends only on its level. If this number is either 1 or p , we call \mathcal{T}_C a *p -homogeneous tree*.

We will prove that the compact open spectral sets, the compact open tiles and p -homogeneous sets are coherent.

Theorem 1.5 (Chapter 3, Theorem 3.1). *Let $\Omega = \sqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p)$ be a compact open set in \mathbb{Z}_p . The following are equivalent.*

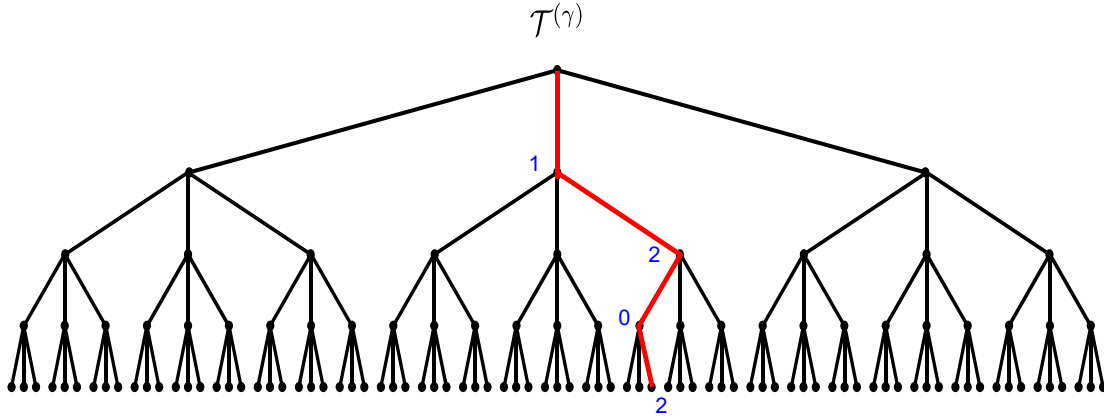


Figure 1.1: For $p = 3$, the branch of 61 can be indexed by its digits $(1, 2, 0, 2)$.

- (1) Ω is a spectral set in \mathbb{Q}_p .
- (2) \mathcal{T}_C is a p -homogenous tree.
- (3) Ω tiles \mathbb{Q}_p .

For each $0 \leq n \leq \gamma$, denote by

$$C_{\text{mod } p^n} := \{x \in \{0, 1, \dots, p^n - 1\} : \exists y \in C, \text{ such that } x \equiv y \pmod{p^n}\}$$

the subset of $\mathbb{Z}/p^n\mathbb{Z}$ determined by C modulo p^n . We also obtain the following characterization of spectral sets in the finite group $\mathbb{Z}/p^\gamma\mathbb{Z}$.

Theorem 1.6 (Chapter 3, Theorem 3.3). *Let $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$. The following statements are equivalent:*

- (a) C is a spectral set in $\mathbb{Z}/p^\gamma\mathbb{Z}$;
- (b) C is a tile of $\mathbb{Z}/p^\gamma\mathbb{Z}$;
- (c) for any $n = 1, 2, \dots, \gamma - 1$, $\#(C_{\text{mod } p^n}) = p^{k_n}$ for some integer $k_n \in \mathbb{N}$, where $\#(C_{\text{mod } p^n})$ is the cardinality of the finite set $C_{\text{mod } p^n}$.

As Tao noted on page 3 of [121], the equivalence (a) \Leftrightarrow (b) is not new, but follows from Łaba [89], which in turn builds on the work of Coven-Meyerowitz [22]. However, we prove it by showing that both (a) and (b) are equivalent to (c) which implies the p -homogeneity of C . The p -homogeneity is practically checkable. We can use it to describe finite spectral sets in $\mathbb{Z}/p^\gamma\mathbb{Z}$.

In order to prove Fuglede’s conjecture on the field of p -adic numbers, we need to use the Bruhat-Schwartz distributions on \mathbb{Q}_p . The theory of p -adic distributions (Bruhat-Schwartz distributions on \mathbb{Q}_p) plays an important role in solving mathematical problems of p -adic analysis and applications [3, 129]. The space of test functions consists of all uniformly locally constant functions of compact support.

We denote by $\mu_E = \sum_{e \in E} \delta_e$ for a discrete set $E \subset \mathbb{Q}_p$. The following proposition holds not only for \mathbb{Q}_p but also for general locally compact abelian group G (see Theorem 2.18 and Theorem 2.19). The case when $G = \mathbb{R}^d$ was proved by Jørgensen and Pedersen [63] and the case when G is a local field was proved by Fan [30].

Proposition 1.7 (Chapter 3, Proposition 3.8). *Let Ω be a Borel set of positive finite Haar measure in \mathbb{Q}_p . Then we have the followings.*

- *The set Ω is a tile with tiling complement T if and only if*

$$\mu_T * 1_\Omega(x) = 1, \text{ for } \mu\text{-a.e. } x \in \mathbb{Q}_p.$$

- *The set Ω is a spectral set with spectrum Λ if and only if*

$$\mu_\Lambda * |\widehat{1_\Omega}|^2(\xi) = \mu(\Omega)^2, \quad \forall \xi \in \widehat{\mathbb{Q}_p}.$$

By Fourier transformation, we obtain the equivalent conditions of tiles and spectral sets in \mathbb{Q}_p :

$$\text{a Borel set } \Omega \text{ is a tile with tiling complement } T \text{ in } \mathbb{Q}_p \iff \widehat{\mu_T} \cdot \widehat{1_\Omega} = \delta_0,$$

and

$$\text{a Borel set } \Omega \text{ is a spectral set with spectrum } \Lambda \text{ in } \mathbb{Q}_p \iff \widehat{\mu_\Lambda} \cdot \left(\frac{1}{\mu(\Omega)^2} |\widehat{1_\Omega}|^2 \right) = \delta_0.$$

Since the function δ_0 has abundant zeros, this leads us to study the zeros of the distribution $\widehat{\mu_E}$ and the function \widehat{f} in the following functional equation:

$$\widehat{\mu_E} \cdot \widehat{f} = \delta_0,$$

where $0 \leq f \in L^1(\mathbb{Q}_p)$.

Independent with the result of Theorem 1.5, we obtain the following theorem.

Theorem 1.8 (Chapter 4, Theorem 4.25). *Let $\Omega \subset \mathbb{Q}_p$ be a Borel set of positive finite Haar measure. Then Ω is a spectral set if and only if it is a tile in \mathbb{Q}_p . Furthermore, the tiles or spectral sets in \mathbb{Q}_p are almost compact open sets.*

This result confirms Fuglede's conjecture on the field of p -adic numbers. Combining Theorem 1.5 and Theorem 1.8, we obtain that the tiles and spectral sets in \mathbb{Q}_p are characterized by a simple geometric property: p -homogeneity.

1.3 Chowla sequences and oscillating sequences

In this section, we study the Chowla sequences and oscillating sequences. One of the tools to study such sequences is the disjointness initiated by Furstenberg [44]. We begin with some historical remarks on the study of the disjointness.

1.3.1 Furstenberg's disjointness

Following Furstenberg [44], we will call a measure-preserving transformation group on a measure space a *process*. As an analogy with the integers, Furstenberg [44] gave the following two ways to define “two processes are relatively prime”.

Condition 3. *Two processes T and S have no common factor except the trivial process reduced to one point.*

Condition 4. *Once the two processes T and S appear as factors in some process, then their direct product $T \times S$ also appears as a factor.*

Condition 2 has been called by Furstenberg the disjointness of T and S . Moreover, Furstenberg [44] proved that Condition 2 implies Condition 1. Actually, Furstenberg's initial motivations for disjointness ranged from the classification of dynamical systems to an originated question in Diophantine approximation, passing through a filtering problem in probability theory:

If $(\xi_n)_{n \in \mathbb{N}}$ and $(\zeta_n)_{n \in \mathbb{N}}$ are two processes, can we recuperate $(\xi_n)_{n \in \mathbb{N}}$ from their sum $(\xi_n + \zeta_n)_{n \in \mathbb{N}}$?

A lot of applications of disjointness have already been shown in [44] by Furstenberg. For example, the processes disjoint from all Bernoulli processes are characterized by deterministic processes.

Furstenberg [44] also introduced the notion of joinings, which has become a powerful tool in ergodic theory. The study of disjointness turns out to be quite enlightening, as it deals with

all the possible ways that two processes can be seen living together inside a third one.

Beginning with the pioneering works of Furstenberg, the conception of joinings was exploited persistently ([109, 67, 47, 105, 45, 127, 68, 56, 77, 46, 125, 111, 76, 66, 98]). For a more complete treatment of ergodic theory via joinings, we refer to Glasner's book [49] and de la Rue's survey paper [26].

1.3.2 Towards Sarnak's conjecture

As we have mentioned before, Sarnak's conjecture is concerned with the Möbius disjointness from the dynamical systems of entropy zero. Here, we summarize the progress toward Sarnak's conjecture. We now know that the conjecture is true for Kronecker systems (Vinogradov [128]), nilsystems (Green and Tao [51]), the Thue-Morse system (Mauduit and Rivat [102]), horocycles flows (Bourgain, Sarnak, and Ziegler [17]), certain class of interval exchanges (Ferenczi and Mauduit [41]) etc. For some of them, the KBSZ criterion (Kátai [73], Bourgain, Sarnak and Ziegler [17]) is used to achieve the orthogonality. For a complete summary of the progress of Sarnak's conjecture, see the survey paper [40].

As we have mentioned, the conditions (Ch) and (S_0) for a sequence $z \in \{-1, 0, 1\}^{\mathbb{N}}$ are obtained from Chowla's conjecture and Sarnak's conjecture by replacing the Möbius

function with the sequence z , respectively. El Abdalaoui et al. [4] proved that the condition (Ch) implies the condition (S_0) . Recall that a point $x \in X$ is said to be *completely deterministic* if for any accumulation point ν of $\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x}$ where δ_x is the Dirac measure supported on x , the system (X, T, ν) is of zero entropy. El Abdalaoui et al. also defined the condition (S), formally stronger than (S_0) by requiring

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) z(n) = 0$$

for any topological dynamical system (X, T) , any $f \in C(X)$ and any completely deterministic point $x \in X$. They proved the equivalence between the condition (S) and the condition (S_0) .

1.3.3 Chowla property

In this subsection, we present our definition of Chowla property for sequences taking values 0 or complex numbers of modulus 1, which generalizes the condition (Ch) for sequences taking values in $\{-1, 0, 1\}$.

To define such Chowla property, we need additionally to define the index of a sequence in $(S^1 \cup \{0\})^{\mathbb{N}}$ and define the functions $x \mapsto x^{(n)}$ for all $x \in S^1 \cup \{0\}$ and for all $n \in \mathbb{Z}$ which are analogies of the exponentiations on S^1 .

Let $(z(n))_{n \in \mathbb{N}} \in (S^1 \cup \{0\})^{\mathbb{N}}$. We denote by $U(m) = \{e^{2\pi i \frac{k}{m}} : 0 \leq k \leq m-1\}$ the set consisting of all m -th roots of the unity for $m \in \mathbb{N}_{\geq 2}$ where $\mathbb{N}_{\geq 2}$ denotes the set of positive integers larger than or equal to 2. Let $U(\infty) = S^1$. The *index* of the sequence $(z(n))_{n \in \mathbb{N}}$ is defined by the smallest number $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ such that the set

$$\{n : z(n) \notin U(m) \cup \{0\}\}$$

has zero density in \mathbb{N} . For a point $x \in S^1 \cup \{0\}$ and $n \in \mathbb{Z}$, we define

$$x^{(n)} := \begin{cases} x^n & \text{if } x \in S^1, \\ 0 & \text{if } x = 0. \end{cases}$$

Let

$$I(m) := \begin{cases} \{0, 1, \dots, m-1\}, & \text{if } m \in \mathbb{N}_{\geq 2}; \\ \mathbb{Z}, & \text{if } m = \infty. \end{cases}$$

We define Chowla property for sequences taking values in $S^1 \cup \{0\}$ as follows.

Defintion 1.9. *Let $z \in (S^1 \cup \{0\})^{\mathbb{N}}$ be a sequence of index $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. We say that the sequence z has Chowla property if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} z^{(i_0)}(n+a_1) \cdot z^{(i_1)}(n+a_2) \cdots z^{(i_r)}(n+a_r) = 0, \quad (1.3.1)$$

for each choice of $0 \leq a_1 < \dots < a_r, r \geq 0$, and $i_s \in I(m)$ which are not all equal to 0.

It is not hard to see that the sequences taking values in $\{-1, 0, 1\}$ have the index 2. Obviously, Chowla property for these sequences is exactly the condition (Ch).

Similarly to the result of the condition (Ch) implying the condition (S), we prove that Chowla sequences are orthogonal to all topological dynamical systems of zero entropy. The main tool of the proof is the theory of joining.

Theorem 1.10 (Chapter 7, Theorem 7.13). *If a sequence in $(S^1 \cup \{0\})^{\mathbb{N}}$ has Chowla property, then it is orthogonal to all topological dynamical systems of zero entropy.*

We will also study when the samples of a given random sequence have Chowla property almost surely. We first consider stationary random sequences. We have the following proposition.

Proposition 1.11 (Chapter 7, Proposition 7.24). *Let $\mathcal{Z} = (Z_n)_{n \in \mathbb{N}}$ be a sequence of stationary random variables taking values in $S^1 \cup \{0\}$. Suppose that the random process \mathcal{Z} has finite or infinite constant index almost surely. Then almost surely the process \mathcal{Z} has Chowla property if and only if*

$$\mathbb{E}[Z_{a_1}^{(i_1)} \cdot Z_{a_2}^{(i_2)} \cdot \dots \cdot Z_{a_r}^{(i_r)}] = 0, \quad (1.3.2)$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0$, and $i_s \in I(m)$ which are not all equal to 0.

Using Proposition 1.11, we can characterize the random sequences in Proposition 1.11 when they are sequences of independent variables.

Corollary 1.12 (Chapter 7, Corollary 7.28). *Let (X_n) be a sequence of stationary random variables taking values in $S^1 \cup \{0\}$. Let $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Suppose that almost surely the sequence (X_n) has the index m and has Chowla property. Then*

(1) *if m is finite, then the sequence $(X_n)_{n \in \mathbb{N}}$ is independent if and only if*

$$\mathbb{E}[|X_{a_1} X_{a_2} \cdots X_{a_r}|] = \mathbb{E}[|X_{a_1}|] \mathbb{E}[|X_{a_2}|] \cdots \mathbb{E}[|X_{a_r}|] \quad (1.3.3)$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0$;

(2) *if m is infinite, then the sequence $(X_n)_{n \in \mathbb{N}}$ is a sequence of independent variables.*

It's a natural question whether there are dependent random sequences having Chowla property almost surely. To answer this question, we will associate a random sequence in a Bernoulli system with a colored tree.

Let \mathcal{A} be a finite alphabet. Let μ be the uniform-Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. Let $G : \mathcal{A}^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$ be a continuous function. Denote $Z_{G,n} := G \circ S^n$. We will explain how to associate the process $(Z_{G,n})_{n \in \mathbb{N}}$ with the colored tree \mathcal{T}_G in Chapter 7. We will also introduce the notion of homogeneity for a colored tree in the same chapter.

We have the following proposition.

Proposition 1.13 (Chapter 7, Proposition 7.33). *The random sequence $(Z_{G,n})_{n \in \mathbb{N}}$ has Chowla property almost surely if and only if the associated tree \mathcal{T}_G is homogeneous.*

With the help of Corollary 1.12 and Proposition 1.13, we will construct dependent random sequences having Chowla property almost surely. We just show an example here to illustrate the existence of such sequences. Define $g : \{-1, 0, 1\}^2 \rightarrow \{-1, 1\}$ by

$$g(a, b) = \begin{cases} a & \text{if } a \in \{-1, 1\} \text{ and } b = 0; \\ 0 & \text{otherwise.} \end{cases}$$

We define $G : \{-1, 0, 1\}^{\mathbb{N}} \rightarrow \{-1, 1\}$ by $(x_n)_{n \in \mathbb{N}} \mapsto g(x_0, x_1)$ and $Z_n = G \circ S^n$ where S is the left shift on $\{-1, 0, 1\}^{\mathbb{N}}$. It is easy to check that the sequence $(Z_n)_{n \in \mathbb{N}}$ does not satisfy the equation (1.3.3) in Corollary 1.12, which implies that $(Z_n)_{n \in \mathbb{N}}$ is a sequence of dependent variables. On the other hand, one can check that its associated tree \mathcal{T}_G is homogeneous, which means that it has Chowla property almost surely.

1.3.4 Oscillating sequences

In this subsection, we consider the oscillating properties of complex sequences.

Recall that the oscillating sequences has been defined in Section 1.1. Recently, the orthogonal properties (to certain dynamical systems) of oscillating sequences have been studied. For example, Fan and Jiang [38] proved that the oscillating sequences of order 1 are orthogonal to all so-called minimally mean attractable and minimally mean-L-stable dynamical systems; Fan proved that fully oscillating sequences are orthogonal to all multiple ergodic realizations of topological dynamics of quasi-discrete spectrum in the sense of Hahn-Parry [31] and to all affine maps of zero entropy on any compact abelian group [32].

As we have mentioned, the Möbius function is fully oscillating. Moreover, there are several ways to construct other oscillating sequences. For instance, Fan [33, 34] proved that some sequences on nil-manifold and some q -multiplicative Gelfond type are fully oscillating sequences. Akiyama and Jiang [5] proved that for almost every $\beta > 1$, the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ is fully oscillating.

We give several properties of oscillating sequence.

Lemma 1.14 (Chapter 6, Lemma 6.2). *Let $d \geq 1$. A sequence of complex numbers $(c_n)_{n \in \mathbb{N}}$ is oscillating of order d if and only if all arithmetic subsequences $(c_{an+b})_{n \in \mathbb{N}}$ are oscillating of order d for all integers $a \in \mathbb{N}^*$ and $b \in \mathbb{N}$.*

Our proof of the above lemma is based on a fact proved in [38]. We remark that a stronger form of the above lemma was proved in [32].

The following proposition tells us that in the definition of oscillating sequences, it remains the same if there is a disturbance for the polynomial P by a polynomial with rational coefficients.

Proposition 1.15 (Chapter 6, Proposition 6.3). *A sequence of complex numbers $(c_n)_{n \in \mathbb{N}}$ is oscillating of order d if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i(P(n)+Q(n))} = 0$$

for any $P \in \mathbb{R}_d[t]$ and any $Q \in \mathbb{Q}[t]$.

Actually, Lemma 1.14 and Proposition 1.15 are also useful for proving Theorem 1.16.

We are interested in the dynamical meaning of oscillating sequences. We will give several equivalent descriptions of such sequences in terms of their orthogonality to different dynamical systems on tori.

Now we recall some notions concerning compact abelian groups. An *affine map* T on a compact abelian group X is a function of the form

$$Tx = Ax + b,$$

where $b \in X$ and A is an automorphism on X . The automorphism A is also called a *section of an automorphism* of the affine T .

As observed in [99] by Liu and Sarnak, an automorphism of zero entropy on $\mathbb{T}^d \times F$, where \mathbb{T}^d is a d -dimensional torus and F is a finite abelian group, has the form

$$T(x, y) = (Ax + By, Cy)$$

with A an automorphism on \mathbb{T}^d of zero entropy, B a morphism from F to \mathbb{T}^d and C an automorphism on F . Such automorphism T is said to be *special* if the morphism $N := A - I$ is nilpotent of order d (i.e. $N^d = 0$ but $N^{d-1} \neq 0$). Meanwhile the affine map T is also called *special*.

We have the following theorem. The key point is that the automorphisms of zero entropy on compact abelian groups are quasi-unipotent (see [23, 99]).

Theorem 1.16 (Chapter 6, Theorem 6.5). *A sequence of complex numbers is oscillating of order d if and only if it is orthogonal to one of the following dynamical systems or one of the following classes of dynamical systems.*

- (1) *All automorphisms of zero entropy on $\mathbb{T}^{d+1} \times F$ for any finite abelian group F .*
- (2) *A special automorphism of zero entropy on $\mathbb{T}^{d+1} \times F$, where F is any fixed finite abelian group F .*
- (3) *All affine maps of zero entropy on $\mathbb{T}^d \times F$ for any finite abelian group F .*
- (4) *All special affine maps of zero entropy on $\mathbb{T}^d \times F$, which share the same section of an automorphism, where F is any fixed finite abelian group.*

We remark that the finite abelian group F in Theorem 1.16 can be the trivial group so that $\mathbb{T}^d \times F$ is nothing but the torus \mathbb{T}^d .

For full oscillating sequences, we have the following equivalent description.

Theorem 1.17 (Chapter 6, Theorem 6.6). *A sequence of complex number is fully oscillating if and only if it is orthogonal to all affine maps of zero entropy on any compact abelian group.*

The necessity in Theorem 1.17 has been essentially proved in [99], where Liu and Sarnak observed that every orbit of any affine map on any compact abelian group of zero entropy can be realized by an orbit of an affine map of zero entropy on $\mathbb{T}^{d+1} \times F$ for some finite abelian group F . On the other hand, the sufficiency is a direct consequence of Theorem 1.16.

A direct consequence of Theorem 1.17 is that the sequences satisfying the condition (S_0) are always fully oscillating.

1.3.5 Sequences $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$, $\beta > 1$

We present our work on Chowla property of the sequences $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ for $\beta > 1$ in this subsection. The main tool is the theory of uniform distribution modulo one. We start with a brief introduction to the notion of uniform distribution modulo one.

Historically speaking, during 1901-1910, three papers by Bohl [13], Sierpiński [117] and Weyl [132] were devoted partly to the distribution of real sequences. A few years later, by developing the ideas in these papers, Weyl [133, 134] initiated the study of uniform distribution modulo one and stated various criteria for a given sequence to be uniformly distributed modulo one.

A sequence $(x(n))_{n \in \mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo one* if, for every couple of real numbers u, v with $0 \leq u < v \leq 1$, we have

$$\lim_{N \rightarrow \infty} \frac{\#\{n : 1 \leq n \leq N, u \leq \{x_n\} \leq v\}}{N} = v - u,$$

where $\{x\}$ denotes the fractional part of the real number x .

One of Weyl's results [133, 134] which is often referred to as Weyl's criterion, is the following theorem.

Theorem 1.18 ([133, 134]). *A sequence $(x(n))_{n \in \mathbb{N}}$ of real numbers is uniformly distributed modulo one if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h x(n)} = 0,$$

for all non-zero integers h .

For a given real number $\beta > 1$, only few results are known on whether the sequence $(\beta^n)_{n \in \mathbb{N}}$ is uniformly distributed modulo one and how it is distributed. For example, we still do not know whether 0 is a limit point of $(\{e^n\})_{n \in \mathbb{N}}$, nor of $(\{(\frac{3}{2})^n\})_{n \in \mathbb{N}}$. However, several metric statements have been established. The first one was obtained by Koksma [78], who proved that for almost every $\beta > 1$ the sequence $(\beta^n)_{n \in \mathbb{N}}$ is uniformly distributed modulo one. Actually, Koksma [78] proved the following general theorem.

Theorem 1.19 (Koksma's theorem). *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued C^1 functions defined on an interval $[a, b]$. Suppose that*

- (1) $f'_m - f'_n$ is monotone on $[a, b]$ for all $m \neq n$;
- (2) $\inf_{m \neq n} \min_{x \in [a, b]} |f'_m(x) - f'_n(x)| > 0$.

Then for almost every $x \in [a, b]$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is uniformly distributed modulo one.

For further information about uniform distribution modulo one, we may refer to the surveys [19, 87].

As we have mentioned, Akiyama and Jiang [5] proved that for almost every $\beta > 1$ the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ is fully oscillating. Actually, they proved the following theorem.

Theorem 1.20 ([5], Corollary 2). *Given any real number $\alpha \neq 0$ and any $g \in C_+^2((1, \infty))$ where $C_+^2((1, +\infty))$ is the space of all positive 2-times continuously differentiable functions on $(1, +\infty)$ whose i -th derivative is non-negative for $i = 1, 2$. Then for almost every real number $\beta > 1$ and for every real polynomial P , the sequence*

$$(e^{2\pi i(\alpha \beta^n g(\beta) + P(n))})_{n \in \mathbb{N}}$$

is fully oscillating.

The main step of their proof is to show the uniform distribution of the sequence $(\alpha \beta^n g(\beta))_{n \in \mathbb{N}}$ for a given α , a given function $g \in C_+^2((1, \infty))$ and for almost every real number $\beta > 1$ by using Koksma's theorem.

As noticed in Theorem 1.17, fully oscillating sequences are orthogonal to all affine maps on compact abelian groups of zero entropy. By Theorem 1.20, we see that for almost every real number $\beta > 1$, the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ is orthogonal to this class of topological dynamical systems of zero entropy. One may wonder to which topological dynamical systems (of zero entropy) such sequence is orthogonal. We give an answer that for almost every real number $\beta > 1$, the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ is orthogonal to all dynamical systems of zero entropy. In fact, we will prove the following theorem.

Theorem 1.21 (Chapter 7, Theorem 7.12). *Let $g \in C^2((1, +\infty))$. Suppose that the function g has at most countably many zeros on $(1, +\infty)$. Then for almost all real numbers $\beta > 1$, the sequences*

$$(e^{2\pi i \beta^n g(\beta)})_{n \in \mathbb{N}} \tag{1.3.4}$$

have Chowla property.

Putting $g \equiv 1$ in Theorem 1.21, we obtain a direct corollary of Theorem 1.10 and Theorem 1.21: for almost every $\beta > 1$, the sequence $(\beta^n)_{n \in \mathbb{N}}$ is orthogonal to all dynamical systems of entropy zero.

Denote by $C_{count}^2((1, +\infty))$ the space of all C^2 functions which have at most countably many zeros on $(1, +\infty)$. The key step of proving Theorem 1.21 is to use Koksma's theorem to prove the uniform distribution of the sequence $(\beta^n g(\beta))_{n \in \mathbb{N}}$ for a given $g \in C_{count}^2((1, +\infty))$ and for almost all real numbers $\beta > 1$. We remark that the method to prove the uniform distribution of such sequences is different from the one used to prove Theorem 1.20 by Akiyama and Jiang. It is also noticed that the space $C_{count}^2((1, +\infty))$ is larger than the space $C_+^2((1, \infty))$. For example, the functions $f(x) = x^2 - 5x + 6$ and $h(x) = \sin(x)$ are in $C_{count}^2((1, +\infty))$ but not in $C_+^2((1, \infty))$.

Chapter 2

Preliminaries to the study of Fuglede's conjecture

In this chapter, we provide some preliminary knowledge for the next two chapters where we study Fuglede's conjecture on the field of p -adic numbers. We also discuss tiles and spectral sets in general locally compact abelian groups in this chapter. The counterexamples of Fuglede's conjecture on high dimensional p -adic spaces and on the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field \mathbb{F}_q for $q = p^n$ with p prime and n positive integer are constructed. The chapter is organized as follows. In Section 2.1, we give a brief review of the study of tiles and spectral sets in \mathbb{R}^d . In Section 2.2, we study Segal's question which is a motivation for Fuglede to formulate his conjecture on tiles and spectral sets. In Section 2.3, the notions of tiles and spectral sets are generalized to general locally compact abelian groups. Section 2.4 is devoted to presenting the idea of construction of counterexamples of Fuglede's conjecture on finite abelian groups and on $\mathbb{R}^d, d \geq 3$. The original idea is from Tao [121] where he disproved Fuglede's conjecture on $\mathbb{R}^d, d \geq 5$. In Section 2.5, we give a criterion for tiles and spectral sets respectively in locally compact abelian groups. The case when $G = \mathbb{R}^d$ was proved by Jørgensen and Pedersen [63] and the case when G is a local field was proved by Fan [30]. The proof presented here follows the ideas in [63] and [30]. In Section 2.6, the proof of Schoenberg's lemma is presented. We rewrite and simplify the original proof of Schoenberg [114]. Schoenberg's lemma is a key tool to prove Fuglede's conjecture on \mathbb{Q}_p in Chapter 3 and Chapter 4. In Section 2.8, we give counterexamples of Fuglede's conjecture on high dimensional p -adic spaces which are followed from a strong connection between tiles (resp. spectral sets) in \mathbb{Q}_p^d and in $(\mathbb{Z}/p^n\mathbb{Z})^d$ for all $n \geq 0$. In Section 2.9, we disprove Fuglede's conjecture on the field of formal Laurent series by constructing counterexamples.

2.1 Introduction

The study of spectral sets was initiated by Fuglede [43], who considered a question that for which open set Ω there is a commutative extension of all the partial differential operators from $C^\infty(\Omega)$ to $L^2(\Omega)$. Fuglede found that the spectral sets are sufficient for this purpose and converted the question to classify the spectral sets. Fuglede [43] proved

that any fundamental domain given by a full-rank lattice is spectral with its dual lattice as its spectrum. But he also found that there are spectral sets which are not fundamental domains, for instance, the union of two intervals $[0, 1] \cup [2, 3]$. In the other direction, he proved that neither triangles nor circles in \mathbb{R}^2 are spectral sets.

On the other hand, we could also consider spectral measures whose support is of zero Lebesgue measure. Jørgensen and Pedersen [65] first studied the spectral structure of singular measures. They showed that the standard Cantor measure is a spectral measure if the ratio of contraction is $1/2n$. Nevertheless, there are more examples of self-similar or self-affine spectral measures in [90, 27].

Another intriguing class of sets in euclidean space is the class of tiles. Fitting individual sets (which is called tiles) together with no gaps or overlaps to fill a space gives us a tiling. Such simple illustration suggests us that the art of tiling must have originated very early in the history of civilization. By contract, the mathematical aspect of tilings is comparatively recent. The study of tiling may be initiated by Grünbaum and Shephard [52]. By now there is a wealth of literature in the subject of tiling, with topics including operator K-theory, dynamical systems, and non-commutative geometry.

Similar to spectral sets, the self-affine tile is also an intriguing subject. The study of self-affine tiles was initiated by Thurston [126] and Kenyon [75]. Following this initiation, Lagarias and Wang proved some foundational properties of self-affine tiles in a series of papers [91, 92, 93, 94]. The subject of self-affine tiles also has connections to many topics including the tiling sets, the geometric and fractal structures, the topological properties, the classification problems and application to wavelet theory.

In this thesis, we only consider the spectral sets of positive finite Haar measure and the tiles by translation.

This chapter is organized as follows. In Section 2.2, we review the motivation of Fuglede's conjecture. In Section 2.3, we recall the basic notions and properties in the theory of Fourier analysis on locally compact abelian groups; we discuss the construction of the counterexamples of Fuglede's conjecture on \mathbb{R}^d for $d \geq 3$; we prove equivalent conditions of spectral sets and tiles respectively in terms of functional equations. In Section 2.6, we are concerned with Schoenberg's lemma which deals with certain \mathbb{Z} -module generated by N -th roots of the unity. In Section 2.7, we give a brief introduction to the field of p -adic numbers and high dimensional p -adic spaces. In Section 2.8, we prove the equivalence between Fuglede's conjecture on $(\mathbb{Z}/p^n\mathbb{Z})^d$ for all $d \geq 1$ and the Fuglede's conjecture for compact open sets in \mathbb{Q}_p^d . In Section 2.8.2 we prove that Fuglede's conjecture fails on \mathbb{Q}_p^d in high dimensions. In Section 2.9, we disprove Fuglede's conjecture on the field of formal Laurent series.

2.2 Segal's question

In order to study the Fuglede's conjecture, we understand the motivation of this conjecture.

2.2.1 Nikodym sets

We present the definition of Nikodym set in this subsection. To this end, we briefly recall some notions in Schwartz distribution in \mathbb{R}^d .

Let $\Omega \subset \mathbb{R}^d$ be an open subset. Denote by $\mathcal{D}(\Omega)$ the set of all functions in $C^\infty(\Omega)$ with compact support where $C^\infty(\Omega)$ is the space of all functions in Ω that are differentiable for all degrees of differentiation. Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^d$ and $f \in \mathcal{D}(\Omega)$, we denote by $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f$. The topology on $\mathcal{D}(\Omega)$ is defined by the convergence of a sequence of elements in $\mathcal{D}(\Omega)$: a sequence $(\phi_k)_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ if the following two conditions hold:

- there is a compact set $K \subset \Omega$ such that

$$\bigcup_{k \in \mathbb{N}} \text{supp}(\phi_k) \subset K,$$

where $\text{supp}(\phi_k)$ is the support of the function ϕ_k for each $k \in \mathbb{N}$,

- for each $\alpha \in \mathbb{Z}^d$, the function $\partial^\alpha \phi_k$ tends uniformly to $\partial^\alpha \phi$ whenever k tends to infinity.

The functions in $\mathcal{D}(\Omega)$ are called *test functions*. Denote by $\mathcal{D}'(\Omega)$ the space of all continuous linear functionals on $\mathcal{D}(\Omega)$. The space $\mathcal{D}'(\Omega)$ is equipped with the weak topology and the functions in $\mathcal{D}'(\Omega)$ are called *distributions*. For any $u \in \mathcal{D}'(\Omega)$ and any $\phi \in \mathcal{D}(\Omega)$, we instead write $\langle u, \phi \rangle$ for the value of u acting on a test function ϕ . A simple observation shows that $L^2(\Omega) \subset \mathcal{D}'(\Omega)$. For $u \in \mathcal{D}'(\Omega)$ and $1 \leq i \leq n$, we define the partial derivatives $\partial_i u$ by the formulas

$$\langle \partial_i u, \phi \rangle = \langle u, \partial_i \phi \rangle,$$

for all $\phi \in \mathcal{D}(\Omega)$.

Let $\mathcal{E}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \partial_i f \in L^2(\Omega), \forall 1 \leq i \leq n\}$. The Sobolev space $H_1(\Omega)$ is defined by

$$H_1(\Omega) = \{u \in L^2(\Omega) : \partial_i u \in L^2(\Omega), \forall 1 \leq i \leq n\}.$$

It is clear that $H_1(\Omega) \subset \mathcal{E}(\Omega)$. A bounded region $\Omega \subset \mathbb{R}^d$ is said to be a *Nikodym set* if $H_1(\Omega) = \mathcal{E}(\Omega)$.

2.2.2 Fuglede's answer

It is well known that the partial derivatives $\partial_1, \partial_2, \dots, \partial_n$ are commuting operators on $C^\infty(\Omega)$ for any region Ω . In other words, if $\Omega \subset \mathbb{R}^d$ is a region and f is a function such that $\partial_i f$ and $\partial_j f$ exist and continuous for all $1 \leq i, j \leq n$, then $\partial_i \partial_j f = \partial_j \partial_i f$. Segal asked the following question:

Question 2.1. *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded region. Can we extend the partial derivatives $\partial_1, \partial_2, \dots, \partial_n$ on $C^\infty(\Omega)$ to be commutative self-adjoint operators on $L^2(\Omega)$?*

That is to say, does there exist commuting self-adjoint operators H_1, H_2, \dots, H_n on $L^2(\Omega)$ whose restriction on $C^\infty(\Omega)$ are $\partial_1, \partial_2, \dots, \partial_n$? In order to answer Segal's question, Fuglede [43] proved the following theorem.

Theorem 2.2. *Let Ω be a Nikodym set. The Segal's question has an affirmative answer if and only if Ω is a spectral set.*

Thus Segal's question is transferred to the question how to classify the spectral sets.

Recall that a (full-rank) *lattice* in \mathbb{R}^d is a subgroup of \mathbb{R}^d which is isomorphic to \mathbb{Z}^d . A *dual lattice* of a lattice T in \mathbb{R}^d is defined by the formula

$$T^* = \{\lambda : \langle \lambda, t \rangle \in \mathbb{Z}, \forall t \in T\}.$$

Fuglede [43] proved the following theorem on the classification of the spectral sets:

Theorem 2.3. *Let Ω be a subset of \mathbb{R}^d . Let T be a lattice in \mathbb{R}^d . Then the followings are equivalent.*

- (Ω, T) is a tile pair.
- (Ω, T^*) is a spectral pair.

Moreover, Fuglede [43] formulated the following conjecture.

Conjecture 2.4. *A Borel set $\Omega \subset \mathbb{R}^d$ of positive and finite Lebesgue measure is a spectral set if and only if it is a tile.*

2.3 Tiles and Spectral sets

In this section, we generalize the notions of tiles and spectral sets from Euclidean spaces to locally compact abelian groups. This leads us to formulate general Fuglede's conjecture to locally compact abelian groups. After that, the connections between Fuglede's conjectures in different locally compact abelian groups are presented.

2.3.1 Fourier analysis on locally compact abelian groups

In this subsection, we recall basic notions and theorems concerned with locally compact abelian groups. The content of this subsection forms the core of our subject and is used throughout the whole chapter. We refer the reader to Rudin's book [108] for more information.

Let G be a locally compact abelian group. The group G is then equipped with a *Haar measure* μ , which is the unique (up to a multiplicative positive constant) non-negative regular measure on G which is not identically 0 and is invariant by translation. We recall that the invariance by translation means that

$$\mu(\Omega + x) = \mu(\Omega),$$

for any $x \in G$ and any Borel subset $\Omega \in G$.

For two any Borel functions f and g on G , the convolution $f * g$ is defined by the formula

$$(f * g)(x) = \int_G f(x - y)g(y)dy,$$

provided that

$$\int_G |f(x - y)g(y)|dy < \infty. \quad (2.3.1)$$

We have the following properties of convolution as the follows.

Proposition 2.5 ([108], Page 4). (1) If (2.3.1) holds for some $x \in G$, then $(f * g)(x) = (g * f)(x)$.

(2) If $f \in L^1(G)$ and $g \in L^\infty(G)$, then $f * g$ is bounded and uniformly continuous.

(3) If $f \in L^1(G)$ and $g \in L^1(G)$, then (2.3.1) holds for almost all $x \in G$ and $f * g \in L^1(G)$ whose L^1 norm is bounded by

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

(4) If $f, g, h \in L^1(G)$, then $(f * g) * h = f * (g * h)$.

A *group character*, or simply a *character*, of G is a group homomorphism from G to the unit circle S^1 . The set of all continuous characters of G forms an abelian group, denoted by \widehat{G} , the *dual group* of G , whose addition is defined by the formula

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x),$$

for any $x \in G$ and any $\gamma_1, \gamma_2 \in \widehat{G}$.

For any $f \in L^1(G)$, the *Fourier transform* of f is defined by

$$\widehat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}dx, \text{ for all } \gamma \in \widehat{G}.$$

We will endow \widehat{G} with a topology with respect to which \widehat{G} will itself be a locally compact abelian group. Let $A(\widehat{G})$ be the set of all functions \widehat{f} with $f \in L^1(G)$. Given \widehat{G} the weakest topology such that all the functions in $A(\widehat{G})$ are continuous, the group \widehat{G} then becomes a locally compact abelian group. Repeating the same reasoning, the group \widehat{G} also has a dual group, say $\widehat{\widehat{G}}$. The Pontryagin duality theorem says that $\widehat{\widehat{G}}$ is isomorphism to G , that is

Theorem 2.6 (Pontryagin duality theorem). *Every locally compact abelian group is the dual group of its dual group.*

2.3.2 Definitions of tiles and spectral sets

Let G be a locally compact abelian group and \widehat{G} be its dual group. Let μ be the Haar measure on G . Let Ω be a Borel measurable subset in G with positive finite Haar measure. We define two kinds of sets as the generalization of tiles and spectral sets in Euclidean spaces (see Section 1.1).

Defintion 2.7. *The set Ω is a spectral set if there exists a set $\Lambda \subset \widehat{G}$ which is an orthogonal basis of the Hilbert space $L^2(\Omega)$. Such a set Λ is called a spectrum of Ω and the couple (Ω, Λ) is called a spectral pair.*

The other genre of sets under study is that of *tiles*:

Defintion 2.8. *The set Ω is a tile by translation of G if there exists a set $T \subset G$ of translates such that $\cup_{t \in T} (\Omega + t) = G$ and $\mu((t_1 + \Omega) \cap (t_2 + \Omega)) = 0$ for any $t_1 \neq t_2 \in T$. Such a set T is called a tiling complement of Ω and (Ω, T) is called a tiling pair.*

For any locally compact abelian group G , it is natural to formulate the following conjecture, called Fuglede's conjecture (or spectral set conjecture) on G :

Conjecture 2.9. *A Borel set $\Omega \subset G$ of positive and finite Haar measure is a spectral set if and only if it is a tile.*

As we have mentioned in Chapter 1, the general conjecture is far from being proved and it is usually not true as Tao [121] showed. The question becomes for which group G , Fuglede's conjecture holds.

As it has been mentioned in Chapter 1, we have the relations:

$$T\text{-}S(\mathbb{R}) \Leftrightarrow T\text{-}S(\mathbb{Z}) \Leftrightarrow T\text{-}S(\mathbb{Z}/N\mathbb{Z}) \text{ for all } N,$$

and

$$S\text{-}T(\mathbb{R}) \Rightarrow S\text{-}T(\mathbb{Z}) \Rightarrow S\text{-}T(\mathbb{Z}/N\mathbb{Z}) \text{ for all } N.$$

According to the above relations, a counterexample in a finite abelian group can be lifted to a counterexample in \mathbb{R} ; on the other hand, if the conjecture were true for every finite abelian group or \mathbb{Z} , this would hold the meaning for the original conjecture that the direction "Tiling \Rightarrow Spectral" holds in \mathbb{R} . But to obtain the other direction "Spectral \Rightarrow Tiling" in \mathbb{R} , we need additionally the statement that every spectral set in \mathbb{R} has a rational spectrum, which is far from being proved.

2.4 Counterexamples

As mentioned before, Fuglede's conjecture fails on several groups. In this subsection, we present the idea of constructing counterexamples of Fuglede's conjecture in different groups. It is also discussed that counterexamples on different groups are strongly related. Roughly speaking, a counterexample constructed in finite abelian groups gives a counterexample in Euclidean spaces. The general approach to find counterexample of Conjecture 2.4, first developed by Tao [121], is to find a counterexample in a finite abelian group and then transfer the example to the lattice \mathbb{Z}^d and finally, to \mathbb{R}^d .

2.4.1 Construction of non-spectral tiles

In this subsection, we show how to construct tiles which are not spectral. Some connection among tiles and spectral sets in different groups is also discussed here. To this end, we need the notion of the dimension of a finite abelian group. The *dimension* of a finite abelian group is defined to be the smallest d such that it is a subgroup of \mathbb{Z}^d . The connection between spectral sets in \mathbb{Z}^d and \mathbb{R}^d is shown in the following theorem.

Theorem 2.10 ([83]). *Let $\Omega \subset \mathbb{Z}^d$ and $U = (0, 1)^d$ be the unit cube. Then Ω is a spectral set in \mathbb{Z}^d if and only if $\Omega + U$ is a spectral set in \mathbb{R}^d .*

It is clear that if Ω is a tile of \mathbb{Z}^d then $\Omega + U$ is a tile of \mathbb{R}^d . It follows that a tile without spectrum in \mathbb{Z}^d will give a tile without spectrum in \mathbb{R}^d .

The connection between spectral sets in finite abelian groups and \mathbb{R}^d is shown in the following theorem.

Theorem 2.11 ([83]). *Let $\Omega \subset \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_d\mathbb{Z}$ where $n_i \in \mathbb{N}$ for all $1 \leq i \leq d$. For the set*

$$T(k) = \{0, n_1, 2n_1, \dots, (k-1)n_1\} \times \cdots \times \{0, n_d, 2n_d, \dots, (k-1)n_d\},$$

define $\Omega(k) = \Omega + T(k)$. Then, for large enough values of k , the set $\Omega(k) \subset \mathbb{Z}^d$ is spectral in \mathbb{Z}^d if and only if Ω is spectral in $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_d\mathbb{Z}$.

It is easy to check that if Ω is a tile of $\mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_d\mathbb{Z}$, then $\Omega(k)$ is a tile of \mathbb{Z}^d for all values of k . This means that a tile without spectrum in a finite abelian group of dimension d will lead to a tile without spectrum in \mathbb{Z}^d . Therefore, a non-spectral tile in some d -dimensional finite group will give a non-spectral tile in \mathbb{R}^d . In order to construct such non-spectral tile, we need additionally the notion of universal spectrum:

Definition 2.12. *Let G be a locally compact abelian group. A Borel subset $\Gamma \subset \widehat{G}$ is said to be a universal spectrum of a Borel set $\Omega \subset G$ if Γ is a spectrum of all tiling complements of Ω in G .*

This notion was originally introduced by Lagarias and Wang [95] who dealt with the case $G = \mathbb{R}^d$. Lagarias and Wang [95] also stated the following universal spectrum conjecture:

Conjecture 2.13. *All tiles in any finite abelian group possess a universal spectrum.*

Farka, Matolsci and Móra [39] proved the equivalence between the universal spectrum conjecture and the “Tiling \Rightarrow Spectral” direction for all d -dimensional finite groups.

Proposition 2.14 ([39]). *For any positive integer d , the universal spectrum conjecture is valid for all d -dimensional finite group if and only if all tiles are spectral sets in all d -dimensional finite groups.*

Hence a counterexample of the universal spectrum conjecture for a d -dimensional finite group will automatically lead to a non-spectral tile in a d -dimensional finite group. The conjecture 2.13 was first disproved in $(\mathbb{Z}/6\mathbb{Z})^5$ (Kolountzakis and Matolcsi [83]), later in $(\mathbb{Z}/6\mathbb{Z})^4$ (Farkas and Gy[29]) and $(\mathbb{Z}/24\mathbb{Z})^3$ (Farka, Matolcsi and Móra [39]). Therefore, Fuglede's conjecture fails in \mathbb{R}^d for $d \geq 3$. For completeness, we provide the counterexample in $(\mathbb{Z}/24\mathbb{Z})^3$ [39] as followings.

Proposition 2.15. [39] *Let*

$$H = \begin{pmatrix} 0 & 10 & 20 & 1 & 21 & 14 \\ 0 & 22 & 3 & 20 & 2 & 7 \\ 0 & 22 & 23 & 18 & 4 & 11 \end{pmatrix}$$

The set consisting of columns in H is a tile in $(\mathbb{Z}/24\mathbb{Z})^3$ which does not have a universal spectrum.

2.4.2 Construction of non-tile spectral sets

Now we construct some counterexamples of the direction "Spectral \Rightarrow Tiling" in Fuglede's conjecture.

A set $\Omega \subset \mathbb{Z}^d$ is called m -spectral set (resp. a m -tile) for some positive integer m if it admits a spectrum Γ (resp. tiling complement T) with all entries being multiples of $1/m$, that is to say, Ω is spectral (resp. a tile) in the group $(\mathbb{Z}/m\mathbb{Z})^d$. The connection between m -spectral sets (resp. m -tile) in finite abelian groups and \mathbb{R} is presented in the following two propositions.

Proposition 2.16 ([100]). *Let $\Omega_1 \subset \mathbb{Z}^d$ be m -spectral and $\Omega_2 \subset \mathbb{Z}^d$ be l -spectral. Then the set $\Omega = \Omega_1 + m\Omega_2$ is ml -spectral.*

Proposition 2.17 ([100]). *Suppose that the set $\Omega \subset [0, m]^d$ is not a m -tile. Then for large k , the set $\Omega_k = \Omega + (m \cdot [0, k])^d$ is not a tile in \mathbb{R}^d .*

If Ω is m -spectral, then $\Omega + m\{0, 1, \dots, k-1\}$ is a mk -spectral by Proposition 2.16. Hence $\Omega_k = \Omega + (m \cdot [0, k])^d = \Omega + m\{0, 1, \dots, k-1\} + U$ with $U = (0, 1)^d$ is spectral in \mathbb{R}^d by Theorem 2.10. It follows that a spectral set which is not a tile in $(\mathbb{Z}/m\mathbb{Z})^d$ for some integer m will give a spectral set which is not a tile in \mathbb{R}^d .

We will show an example of non-tile spectral sets in $(\mathbb{Z}/8\mathbb{Z})^3$ constructed by Kolountzakis and Matolcsi [84] using the notion of Hadamard matrices. We recall that a $k \times k$ complex matrix is called a (complex) *Hadamard matrix* if all entries of H have absolute value one and $HH^* = kI$, where I denotes the identity matrix and H^* is the Hermitian transpose of H . That is to say, the rows (and also the columns) of H form an orthogonal basis of \mathbb{C}^k . A $k \times k$ real matrix $B = (b_{lj})_{1 \leq l, j \leq k}$ is called a *log-Hadamard matrix* if $(e^{2\pi i b_{lj}})_{1 \leq l, j \leq k}$ is a Hadamard matrix.

In fact, there is a strong connection between log-Hadamard matrices and spectral sets in finite abelian groups. Let $G = (\mathbb{Z}/N\mathbb{Z})^d$ be a finite abelian group. We identify the elements

of G and \widehat{G} with d -dimensional column- and row-vectors, respectively. Let $A, B \subset G$ with $\#A = \#B = k$. We regard A and B as a $d \times k$ matrices with integer coefficients. It is not hard to see that (A, B) is a spectral pair in G if and only if $\frac{1}{N}A \cdot B^T$ is log-Hadamard, where the matrix multiplication can be taken mod N .

Let

$$H = \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 & 6 & 2 \\ 0 & 2 & 4 & 1 & 5 & 6 \\ 0 & 6 & 3 & 4 & 2 & 7 \\ 0 & 6 & 7 & 2 & 4 & 3 \\ 0 & 2 & 6 & 5 & 1 & 4 \end{pmatrix}.$$

It is easy check that H is a log-Hadamard matrix and the “mod 8 rank” of $8H$ is 3. It follows that there is a decomposition $8H = LR \pmod{8}$ where L is a 6×3 integer matrix and R is a 3×6 integer matrix. For example,

$$R = \begin{pmatrix} 0 & 2 & 4 & 1 & 5 & 6 \\ 0 & 6 & 3 & 4 & 2 & 7 \\ 0 & 6 & 7 & 2 & 4 & 3 \end{pmatrix} \text{ and } L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & 1 & 1 \end{pmatrix}.$$

This implies that the set Ω consisting of the columns of R is a spectral in $(\mathbb{Z}/8\mathbb{Z})^3$ (with spectrum consisting of the rows of L). However, since $\#\Omega = 6$ which does not divide $\#(\mathbb{Z}/8\mathbb{Z})^3 = 8^3$, the set Ω is not a tile in $(\mathbb{Z}/8\mathbb{Z})^3$.

2.5 Criteria of spectral sets and tiles

In this subsection, we prove the equivalent conditions (in terms of functional equations) of spectral sets and tiles in locally compact abelian groups.

2.5.1 Criterion of tiles

We first study the tiles in locally compact abelian groups. We have the following theorem.

Theorem 2.18. *Let Ω be a Borel subset in a locally compact abelian group G . Let μ be the Haar measure on G . Then the set Ω is a tile with tiling complement T if and only if*

$$\sum_{t \in T} 1_{\Omega}(t - x) = 1, \text{ for } \mu - \text{a.e. } x \in G. \quad (2.5.1)$$

Proof. Observe that $1_{\Omega}(t - x) = 1_{-(\Omega - t)}(x)$ which is the indicator function of the set $-(\Omega - t)$. The equation (2.5.1) means that $\{-(\Omega - t) : t \in T\}$ is a partition of G almost surely. It is clear that Ω being a tile is equivalent to (2.5.1). \square

2.5.2 Criterion of spectral sets

Now we consider the spectral sets in locally compact abelian groups. We have the following Theorem 2.19. We remark that some special cases of Theorem 2.19 were proved: the case when $G = \mathbb{R}^d$ was due to Jørgensen and Pedersen [63] and the case when G is a local field was due to Fan [30]. The proof of Theorem 2.19 is similar to the proofs in [63, 30].

Theorem 2.19. *Let Ω be a Borel subset in a locally compact abelian group G . Then the set Ω is a spectral set with spectrum Λ if and only if*

$$\sum_{\lambda \in \Lambda} |\widehat{1}_\Omega(\lambda - \xi)|^2 = \mu(\Omega)^2, \forall \xi \in \widehat{G}, \quad (2.5.2)$$

where μ is the Haar measure on G .

Proof. The inner product in $L^2(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$. For any $\lambda, \xi \in \widehat{G}$, we have

$$\langle \lambda, \xi \rangle = \int_{\Omega} \lambda \bar{\xi} d\mu = \widehat{1}_\Omega(\lambda - \xi).$$

It follows that λ and ξ are orthogonal in $L^2(\Omega)$ if and only if $\widehat{1}_\Omega(\lambda - \xi) = 0$.

The “only if” part is immediate from Parseval's equality and the fact that $\{\widehat{1}_\Omega(\lambda - \xi)\}_{\lambda \in \Lambda}$ are Fourier coefficients of ξ under the Hilbert basis $\{\lambda\}_{\lambda \in \Lambda}$.

Now we consider the “if” part. Fix any $\lambda' \in \Lambda$. Taking $\xi = \lambda'$ in (2.5.2), we obtain that

$$\mu(\Omega)^2 + \sum_{\lambda \in \Lambda \setminus \{\lambda'\}} |\widehat{1}_\Omega(\lambda - \lambda')|^2 = \mu(\Omega)^2,$$

which implies that $\widehat{1}_\Omega(\lambda - \lambda') = 0$ for all $\lambda \in \Lambda \setminus \{\lambda'\}$. That is to say, the elements in the set $\{\lambda\}_{\lambda \in \Lambda}$ are orthogonal to each other in $L^2(\Omega)$. It remains to show that $\{\lambda\}_{\lambda \in \Lambda}$ is total. The equation (2.5.2) means that

$$\forall \xi \in \widehat{G}, \xi = \sum_{\lambda \in \Lambda} \langle \xi, \lambda \rangle \lambda.$$

If $f \in L^2(\Omega)$ is orthogonal to all $\lambda \in \Lambda$, then for any $\xi \in \widehat{G}$ we have

$$\langle f, \xi \rangle = \sum_{\lambda \in \Lambda} \langle \xi, \lambda \rangle \langle f, \lambda \rangle = 0.$$

It follows that f vanishes μ -almost everywhere. We conclude that $\{\lambda\}_{\lambda \in \Lambda}$ is total. \square

Observe that both (2.5.2) and (2.5.1) are of the form

$$\mu_E * f = w, \quad (2.5.3)$$

where f is a non-negative integrable function and $w > 0$ is a positive number. The above convolution equation (2.5.3) not only has its own interest but also is our main object of study when we consider Fuglede's conjecture on the field of p -adic numbers in Chapter 5.

2.5.3 Orthonormal sets

We finish this section by a function concerning with orthonormal sets of measures. Let ν be a Borel probability measure with compact support. Let $\Lambda \subset \mathbb{R}$ be a countable set. We say that Λ is an *orthonormal set* of ν if $\widehat{\nu}(\lambda - \lambda') = 0$ for any $\lambda \neq \lambda' \in \Lambda$. It is easy to see that the spectrum of a measure is always an orthonormal set of the same measure.

We define a function with respect to ν and Λ :

$$Q_{\nu, \Lambda}(\xi) = \sum_{\lambda \in \Lambda} |\widehat{\nu}(\xi - \lambda)|^2,$$

This function characterizes whether Λ is an orthonormal set of μ . The case $G = \mathbb{R}^d$ in the following proposition was proved by Jørgensen and Pedersen [64]. The proof of the general case is contained in the proof of Theorem 2.19.

Proposition 2.20. *Let ν be a Borel probability measure with compact support in a locally compact abelian group G . Let $\Lambda \subset \widehat{G}$ be a countable set. Then*

- (i) Λ is an orthonormal set of ν if and only if $Q_{\nu, \Lambda}(\xi) \leq 1$ for all $\xi \in G$,
- (ii) Λ is a spectrum of ν if and only if $Q_{\nu, \Lambda}(\xi) = 1$ for all $\xi \in G$,
- (iii) if Λ is an orthonormal set of ν then $Q_{\nu, \Lambda}(\xi)$ is an entire function.

2.6 Schoenberg's lemma

In this section, we state Schoenberg's lemma which is a powerful tool in Chapter 3 and Chapter 4.

The Fourier condition of a spectral set is tightly related to the fact that certain sums of roots of unity must be zero. Let $N \geq 2$ be an integer and let $\omega_N = e^{2\pi i/N}$, which is a primitive N -th root of unity. Denote by \mathcal{M}_N the set of integral points $(a_0, a_1, \dots, a_{N-1}) \in \mathbb{Z}^N$ such that

$$\sum_{j=0}^{N-1} a_j \omega_N^j = 0,$$

which form a \mathbb{Z} -module. The fact that the degree over \mathbb{Q} of the extension field $\mathbb{Q}(\omega_N)$ is equal to $\phi(N)$, where ϕ is Euler's totient function, implies that the rank of \mathcal{M}_N is equal to $N - \phi(N)$. Schoenberg ([114], Theorem 1) found a set of generators. See also de Bruijn [18] and Rédei [106, 107] (actually, Rédei [106] was the first to formulate this, his incomplete proof was completed in [18] and in [107], and later independently in [114]).

We will present this in detail.

2.6.1 Cyclotomic polynomials

We begin with recalling the notion of cyclotomic polynomials. Then we show some applications of cyclotomic polynomials.

The N -th *cyclotomic polynomial* Φ_N is the unique irreducible polynomial with integer coefficients, which is a divisor of $x^N - 1$ and is not a divisor of $x^k - 1$ for any $k < N$. In other words, the N -th cyclotomic polynomial Φ_N is the minimal polynomial of ω_N . In particular, if p is a prime number, then

$$\Phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1}.$$

For general N , the Möbius inversion formula allows us to see that

$$\Phi_N(x) = \prod_{d|N} (x^{N/d} - 1)^{\mu(d)}.$$

We state some properties of cyclotomic polynomials. The proof is simple and thus omitted here. For the detailed proof, we refer to Lemma 1.1 in [22].

Lemma 2.21. *Let p be prime. Then we have*

- (1) $\prod_{d|N} \Phi_d(x) = x^N - 1$;
- (2) $\Phi_{p^{n+1}}(x) = \Phi_p(x^{p^n})$;
- (3) $\Phi_s(x^p) = \begin{cases} \Phi_{ps}(x) & \text{if } p \text{ is a factor of } s, \\ \Phi_s(x)\Phi_{ps}(x) & \text{if } p \text{ is not a factor of } s. \end{cases}$

Let A be a multi-set in $\mathbb{Z}/N\mathbb{Z}$. Recall that the *mask polynomial* of A is defined to be

$$A(X) = \sum_{a \in A} m_a X^a,$$

where m_a is the multiplicity of a in A . It is clear that the degree of $A(X)$ is at most $N - 1$. Actually, any polynomial of degree at most $N - 1$ with non-negative coefficients is a mask polynomial of some multi-set in $\mathbb{Z}/N\mathbb{Z}$. Observe that A is a subset of $\mathbb{Z}/N\mathbb{Z}$ if and only if the mask polynomial $A(X)$ has coefficients 0 and 1. Let S be the set of prime powers dividing N . Define

$$S_A = \{s \in S : \Phi_s(X) | A(X)\}.$$

Following Coven and Meyerowitz [22], we say that A satisfies the condition (T1) if

$$\sharp A = \prod_{s \in S_A} \Phi_s(1),$$

and that A satisfies the condition (T2) if $\Phi_{s_1 s_2 \dots s_m}(X)$ divides $A(X)$ for every powers of distinct primes $s_1, s_2, \dots, s_m \in S_A$. Coven and Meyerowitz [22] proved the following theorem.

Theorem 2.22 ([22], Theorem A). *If $A \subset \mathbb{Z}/N\mathbb{Z}$ satisfies the conditions (T1) and (T2), then A tiles $\mathbb{Z}/N\mathbb{Z}$ by translation.*

Coven and Meyerowitz conjectured that the conditions (T1) and (T2) are also necessary conditions for a set to be a tile in $\mathbb{Z}/N\mathbb{Z}$. Laba [89] proved that the sets satisfy the conditions (T1) and (T2) are spectral. From these developments, we see that the conditions (T1) and (T2) are positioned between the delicate relationship of tiles and spectral sets in finite abelian groups.

Let $(\mathbb{Z}/N\mathbb{Z})^*$ denote the group of all invertible elements in $\mathbb{Z}/N\mathbb{Z}$. The Galois group $\text{Gal}(\mathbb{Q}(\omega_N)/\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^*$. Then we have a natural action of $\text{Gal}(\mathbb{Q}(\omega_N))$ on the values of the mask polynomial of A , defined by

$$\sigma(A(X)) = A(X^g),$$

for $\sigma \in \text{Gal}(\mathbb{Q}(\omega_N)/\mathbb{Q})$ which is determined by $\sigma(\omega_N) = \omega_N^g$.

The following lemma allows us to see the connection between the zeros of $A(\omega^d)$ and the divisibility of $\Phi_{N/d}(X)$.

Lemma 2.23. *Let A be a subset of $\mathbb{Z}/N\mathbb{Z}$. Then we have the followings.*

(1) $A(\omega^a) = 0$ if and only if $A(\omega^{ag}) = 0$ for any $g \in (\mathbb{Z}/N\mathbb{Z})^*$.

(2) For $d|N$, we have the equivalence

$$A(\omega^d) = 0 \Leftrightarrow \Phi_{N/d}(X) | A(X) \pmod{X^N - 1}.$$

(3) Suppose that $\#\{d : A(\omega^{N/p^d}) = 0\} = k$. Then p^k divides $\#A$.

Proof. By the above action of $\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ on $A(X)$, we deduce (1). Since the cyclotomic polynomial of order N/d is the monic minimal polynomial of ω_N^d , we obtain (2). By definition, $\prod_{s \in S_A} \Phi_s(X)$ divides $A(X)$ in $\mathbb{Z}[X]$. Putting $X = 1$, we get (3). \square

2.6.2 \mathbb{Z} -module generated by the N -th roots of unity

In this subsection, we state two forms of Schoenberg's lemma.

Let

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

be the standard decomposition into prime powers. Define

$$P_{i,j} = \{(a_0, a_1, \dots, a_{N-1}) \in \mathbb{Z}^N : \text{all } a_k = 0 \text{ except } a_j = a_{j+N/p_i} = \cdots = a_{j+(p_i-1)N/p_i} = 1\},$$

for $1 \leq i \leq k$ and $0 \leq j \leq N/p_i - 1$. Schoenberg proved the following theorem.

Theorem 2.24 ([114], Theorem 1). *The set $\{P_{i,j}\}_{1 \leq i \leq k, 0 \leq j \leq N/p_i - 1}$ is a set of generators of \mathcal{M}_N .*

Denote by

$$A(X) = \sum_{j=0}^{N-1} a_j X^j.$$

Using the cyclotomic polynomials, Theorem 2.24 is reformulated as the follows by Lemma 2.23.

Theorem 2.25. *If $A(\omega_N) = 0$, then*

$$A(X) \equiv \sum_{i=1}^k P_i(X) \Phi_{p_i}(X^{N/p_i}) \pmod{X^N - 1},$$

where $P_i \in \mathbb{Z}[X]$ for all $1 \leq i \leq k$.

In fact, Theorem 2.25 follows immediately from the following property of the cyclotomic polynomial and the fact that the N -th cyclotomic polynomial Φ_N is the minimal polynomial of ω_N . The proof presented here is based on the original proof in [114].

Theorem 2.26 ([114], Theorem 2). *Let N be a positive integer. Then there exist $P_p \in \mathbb{Z}[X]$ for all p prime with $p \mid N$ such that*

$$\Phi_N(x) = \sum_{p \mid N} P_p(x) \Phi_p(x^{N/p}). \quad (2.6.1)$$

Proof. We decompose the proof into three steps.

(a) We first establish (2.6.1) for the case when $n = pq$ with p, q two distinct primes. To this end, we look at two polynomials

$$F_1(x) = x^{p-1} + x^{p-2} + \cdots + 1 \text{ and } F_2(x) = x^{q-1} + x^{q-2} + \cdots + 1.$$

Since $(p, q) = 1$, by Euclid's algorithm, there exist $P, Q \in \mathbb{Z}[X]$ such that

$$1 = P(x)F_1(x) + Q(x)F_2(x). \quad (2.6.2)$$

By multiplying $\frac{(x^{pq}-1)(x-1)}{(x^p-1)(x^q-1)}$ in both sides of (2.6.2), we obtain

$$\Phi_{pq}(x) = P(x)\Phi_p(x^q) + Q(x)\Phi_q(x^p), \quad (2.6.3)$$

which completes the proof.

(b) Now we prove the theorem for the case when N is square free. We write $N = p_1 p_2 \cdots p_k$ with p_1, p_2, \dots, p_k distinct primes. We induct on k . The claim is vacuous for $k = 1$. So suppose that $k \geq 2$ and that the claim has already been proven for $k = s \geq 1$, that is, there exist $P_i \in \mathbb{Z}[X]$ for all $1 \leq i \leq s$ such that

$$\Phi_{N/p_{s+1}}(x) = \sum_{1 \leq i \leq s} P_i(x) \Phi_{p_i}(x^{N/p_i p_{s+1}}), \quad (2.6.4)$$

where $N = p_1 p_2 \dots p_k p_{k+1}$. By Lemma 2.21 (3), we have

$$\Phi_N(x) = \frac{\Phi_{N/p_{s+1}}(x^{p_{s+1}})}{\Phi_{N/p_{s+1}}(x)}. \quad (2.6.5)$$

Applying (2.6.4) in (2.6.5), we obtain

$$\Phi_N(x) = \sum_{1 \leq i \leq s} \frac{P_i(x) \Phi_{p_i}(x^{N/p_i})}{\Phi_{N/p_{s+1}}(x)}. \quad (2.6.6)$$

Using Lemma 2.21 (3) twice, for all $1 \leq i \leq s$, we have

$$\frac{\Phi_{p_i}(x^{N/p_i})}{\Phi_{N/p_{s+1}}(x)} = \frac{\Phi_{p_i}(x^{N/p_i}) \Phi_{p_i}(x)}{\Phi_{p_i}(x^{N/p_i p_{s+1}})} = \Phi_{p_i p_{s+1}}(x^{N/p_i p_{s+1}}) \Phi_{p_i}(x^{N/p_i p_{s+1}}). \quad (2.6.7)$$

Using (2.6.3) by $x^{N/p_i p_{s+1}}$ replacing x , we deduce that

$$\Phi_{p_i p_{s+1}}(x^{N/p_i p_{s+1}}) = P(x^{N/p_i p_{s+1}}) \Phi_{p_i}(x^{N/p_i}) + Q(x^{N/p_i p_{s+1}}) \Phi_{p_{s+1}}(x^{N/p_{s+1}}), \quad (2.6.8)$$

for all $1 \leq i \leq s$. Combining (2.6.5), (2.6.7) and (2.6.8), we conclude that

$$\Phi_N(x) = \sum_{1 \leq i \leq s} P_i(x) P(x^{N/p_i p_{s+1}}) \Phi_{p_i}(x^{N/p_i}) + \sum_{1 \leq i \leq s} P_i(x) Q(x^{N/p_i p_{s+1}}) \Phi_{p_{s+1}}(x^{N/p_{s+1}}),$$

where $P_i(x) P(x^{N/p_i p_{s+1}})$ and $\sum_{1 \leq i \leq s} P_i(x) Q(x^{N/p_i p_{s+1}})$ are polynomials with integral coefficients.

(c) Now we prove the theorem. Let

$$N = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} \text{ and } m = p_1 p_2 \dots p_k.$$

Let $v = N/m$. By the Möbius inversion formula and the fact that $\mu(n) = 0$ for any n not square-free, we have

$$\Phi_N(x) = \prod_{d|N} (x^{N/d} - 1)^{\mu(d)} = \prod_{d|m} (x^{N/d} - 1)^{\mu(d)} = \Phi_m(x^v). \quad (2.6.9)$$

Since the theorem is already established for square-free m , there exist $P_i \in \mathbb{Z}[X]$ for $1 \leq i \leq k$ such that

$$\Phi_m(x^v) = \sum_{1 \leq i \leq k} P_i(x^v) \Phi_{p_i}(x^{mv/p_i}). \quad (2.6.10)$$

Combining (2.6.9) and (2.6.10), we obtain

$$\Phi_N(x) = \sum_{1 \leq i \leq k} P_i(x^v) \Phi_{p_i}(x^{N/p_i}).$$

□

2.7 Some basic notions in p -adic spaces

In this section, we give a brief introduction to the field of p -adic numbers and its finite extensions.

2.7.1 Field of p -adic numbers

Consider the field \mathbb{Q} of rational numbers and a prime $p \geq 2$. Any non-zero number $r \in \mathbb{Q}$ can be written as $r = p^v \cdot \frac{a}{b}$ where $v, a, b \in \mathbb{Z}$ and $(p, a) = 1$ and $(p, b) = 1$, where (x, y) denotes the greatest common divisor of the two integers x and y . We define $|r|_p = p^{-v_p(r)}$ for $r \neq 0$ and $|0|_p = 0$. Then $|\cdot|_p$ is a non-Archimedean absolute value. That means

- (i) $|r|_p \geq 0$ with equality only when $r = 0$;
- (ii) $|rs|_p = |r|_p |s|_p$;
- (iii) $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.

The field \mathbb{Q}_p of p -adic numbers is the completion of \mathbb{Q} under $|\cdot|_p$. A typical element x of \mathbb{Q}_p is of the form

$$x = \sum_{n=v}^{\infty} a_n p^n \quad (v \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \text{ and } a_v \neq 0). \quad (2.7.1)$$

Here, $v_p(x) := v$ is called the p -valuation of x .

The ring \mathbb{Z}_p of p -adic integers is the set of p -adic numbers with absolute value ≤ 1 , which is a compact set. Hence for every $x \in \mathbb{Q}_p$, the set $x + \mathbb{Z}_p$ is a compact neighborhood of x , which implies that the field \mathbb{Q}_p is locally compact. Thus the field \mathbb{Q}_p is a locally compact abelian group with respect to addition. We normalize the Haar measure dx (sometimes we use μ) on \mathbb{Q}_p by the equality

$$\int_{\mathbb{Z}_p} dx = 1.$$

On the other hand, the ring \mathbb{Z}_p is the set of p -adic numbers with absolute value $< p$, showing that it is open. Thus the ring \mathbb{Z}_p is a compact open set. This means that the field \mathbb{Q}_p is a totally disconnected space.

A non-trivial additive character on \mathbb{Q}_p is defined by

$$\chi(x) = e^{2\pi i \{x\}}$$

where $\{x\} = \sum_{n=v_p(x)}^{-1} a_n p^n$ is the fractional part of x in (2.7.1). From this character we can get all characters χ_ξ of \mathbb{Q}_p , by defining $\chi_\xi(x) = \chi(\xi x)$. We remark that

$$\chi(x) = e^{2\pi i k/p^d}, \quad \text{if } x \in \frac{k}{p^d} + \mathbb{Z}_p \quad (k, d \in \mathbb{Z}), \quad (2.7.2)$$

and

$$\int_{p^{-n}\mathbb{Z}_p} \chi(x) dx = 0 \quad \text{for all } n \geq 1. \quad (2.7.3)$$

The map $\xi \mapsto \chi_\xi$ from \mathbb{Q}_p to its dual group $\widehat{\mathbb{Q}}_p$ is an isomorphism. We write $\widehat{\mathbb{Q}}_p \simeq \mathbb{Q}_p$ and identify a point $\xi \in \mathbb{Q}_p$ with the point $\chi_\xi \in \widehat{\mathbb{Q}}_p$. For more information on \mathbb{Q}_p and $\widehat{\mathbb{Q}}_p$, the reader is referred to the book [129].

Now we consider the d -dimensional p -adic vector space \mathbb{Q}_p^d . For $x = (x_1, x_2, \dots, x_d) \in \mathbb{Q}_p^d$, we denote by $\{x\} = (\{x_1\}, \{x_2\}, \dots, \{x_d\})$ the fractional part of x . All continuous characters of \mathbb{Q}_p^d are

$$\chi_\xi(x) = e^{2\pi i\{\xi \cdot x\}}, \forall \xi \in \mathbb{Q}_p^d,$$

where \cdot is the scalar product in \mathbb{Q}_p^d . The map $\xi \mapsto \chi_\xi$ from \mathbb{Q}_p^d to its dual group $\widehat{\mathbb{Q}}_p^d$ is an isomorphism. For $x = (x_1, x_2, \dots, x_d) \in \mathbb{Q}_p^d$, we denote by $|x|_p = \max_{1 \leq i \leq d} \{|x_i|_p\}$.

In what follows, the following notation will be used.

Notation:

- $\mathbb{Z}_p^\times := \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$, the group of units of \mathbb{Z}_p .
- $\mathbb{L} := \{\{x\} : x \in \mathbb{Q}_p\}$, a complete set of representatives of the cosets of the additive subgroup \mathbb{Z}_p .
- $\mathbb{L}_n := p^{-n}\mathbb{L}$.
- $B(0, p^n) := p^{-n}\mathbb{Z}_p^d$, the (closed) ball centered at 0 of radius p^d in \mathbb{Q}_p^d .
- $B(x, p^n) := x + B(0, p^n)$.
- $S(x, p^n) := B(x, p^n) \setminus B(x, p^{n-1})$, a “sphere”.

The Lebesgue density theorem holds in \mathbb{Q}_p^d for the Haar measure.

Proposition 2.27 ([104]). *Let $\Omega \subset \mathbb{Q}_p^d$ be a bounded Borel set such that $\mu(\Omega) > 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{\mu(B(x, p^{-n}) \cap \Omega)}{\mu(B(x, p^{-n}))} = 1, \quad \mu\text{-a.e. } x \in \Omega.$$

2.7.2 Two simple results in \mathbb{Q}_p^d

In this subsection, we study Fourier analysis in \mathbb{Q}_p^d . We will see that the analysis in the field of p -adic numbers essentially provides some interesting properties which do not exist in Euclidean spaces.

The following lemma shows that the Fourier transform of the indicator function of a ball centered at 0 is a function of the same type and the Fourier transform of the indicator function of a compact open set is also supported by a ball, and on the ball it is the restriction of a trigonometric polynomial.

Lemma 2.28. *Let $\gamma \in \mathbb{Z}$ be an integer.*

- (a) *We have $\widehat{1_{B(0,p^\gamma)}}(\xi) = p^{\gamma d} 1_{B(0,p^{-\gamma})}(\xi)$ for all $\xi \in \mathbb{Q}_p^d$.*
- (b) *If $\Omega = \bigsqcup_j B(c_j, p^\gamma)$ is a finite union of disjoint balls of the same size, then*

$$\widehat{1_\Omega}(\xi) = p^{\gamma d} 1_{B(0,p^{-\gamma})}(\xi) \sum_j \chi(-c_j \cdot \xi). \quad (2.7.4)$$

Proof. (a) By the scaling property of the Haar measure, we have only to prove the result in the case $\gamma = 0$. Recall that

$$\widehat{1_{B(0,1)}}(\xi) = \int_{B(0,1)} \chi(-\xi \cdot x) dx.$$

When $|\xi|_p \leq 1$, the integrand is equal to 1, so $\widehat{1_{B(0,1)}}(\xi) = 1$. When $|\xi|_p > 1$, making a change of variable $x = y - z$ with $z \in B(0,1)$ chosen such that $\chi(\xi \cdot z) \neq 1$, we get

$$\widehat{1_{B(0,1)}}(\xi) = \chi(\xi \cdot z) \widehat{1_{B(0,1)}}(\xi).$$

It follows that $\widehat{1_{B(0,1)}}(\xi) = 0$ for $|\xi|_p > 1$.

- (b) is a direct consequence of (a) and of the fact

$$\widehat{1_{B(c,p^\gamma)}}(\xi) = \chi(-c \cdot \xi) \widehat{1_{B(0,p^\gamma)}}(\xi).$$

□

A complex-valued function ϕ defined on an open set $\Omega \subset \mathbb{Q}_p^d$ is said to be *locally constant* if for any $x \in \Omega$, there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\phi(x + y) = \phi(x), \forall y \in B(0, p^{l(x)}). \quad (2.7.5)$$

A direct consequence of Lemma 2.28 (a) is that

$$\widehat{1_{B(0,p^\gamma)}}(\xi + x) = \widehat{1_{B(0,p^\gamma)}}(\xi), \forall \xi \in \mathbb{Q}_p^d, \forall x \in B(0, p^{-\gamma}). \quad (2.7.6)$$

This means that the function $\widehat{1_{B(0,p^\gamma)}}$ is locally constant.

The following proposition shows that for an integrable function f , having compact support and being uniformly locally constant are dual properties for f and its Fourier transform.

Proposition 2.29. *Let $f \in L^1(\mathbb{Q}_p^d)$ be a complex-value integrable function.*

- (1) *If f has compact support, then \widehat{f} is uniformly locally constant.*
- (2) *If f is uniformly locally constant, then \widehat{f} has compact support.*

Proof. (1) Suppose that f is supported by $B(0, p^n)$. For any $x \in \mathbb{Q}_p^d$ and any $u \in B(0, p^{-n})$, we have

$$\widehat{f}(\xi + u) - \widehat{f}(\xi) = \int_{B(0, p^n)} f(y) \overline{\chi(\xi y)} (\overline{\chi(uy)} - 1) dy.$$

Notice that if $y \in B(0, p^n)$, we have $|uy|_p \leq 1$. So by (2.7.2),

$$\chi(uy) - 1 = 0.$$

Therefore, $\widehat{f}(\xi + u) - \widehat{f}(\xi) = 0$ for all $u \in B(0, p^{-n})$. Thus \widehat{f} is uniformly locally constant.

(2) Suppose that $f(x + u) = f(x)$ for any $x \in \mathbb{Q}_p^d$ and any $u \in B(0, p^n)$. Observing

$$\mathbb{Q}_p^d = \mathbb{L}_n^d + B(0, p^n) = \bigcup_{z \in \mathbb{L}_n^d} B(z, p^n),$$

we deduce

$$\widehat{f}(\xi) = \sum_{z \in \mathbb{L}_n^d} \int_{B(z, p^n)} f(y) \overline{\chi(\xi y)} dy = \sum_{z \in \mathbb{L}_n^d} f(z) \int_{B(z, p^n)} \overline{\chi(\xi y)} dy.$$

By (2.7.3), we have $\int_{B(z, p^n)} \chi(\xi y) dy = 0$ for $|\xi|_p > p^{-n}$. Therefore, $\widehat{f}(\xi) = 0$ for $|\xi|_p > p^{-n}$. \square

The above proposition not only has its own interests but also is the key which will allow us to prove that a tile Ω in \mathbb{Q}_p is an almost compact open set by showing that the support of $\widehat{1_\Omega}$ is compact (see Proposition 4.22).

2.8 Counterexamples in p -adic spaces

We have already seen the strong connection among the tiles (resp. spectral sets) in \mathbb{R}^d , \mathbb{Z}^d and finite abelian groups of dimension d . In this section, we prove a strong connection between the tiles (resp. spectral sets) in \mathbb{Q}_p^d and in finite abelian groups of dimension d . Then using this connection, we construct counterexamples of Fuglede's conjecture on high dimensional p -adic spaces.

2.8.1 Fuglede's conjecture on \mathbb{Q}_p^d and on $(\mathbb{Z}/\mathbb{Z}_{p^n})^d$

In this section, we are concerned with the relation between Fuglede's conjecture on \mathbb{Q}_p^d and on finite abelian groups $(\mathbb{Z}/\mathbb{Z}_{p^n})^d$ for all positive integers n . The discussion in this section has its own interest and is seen as a supplementary of Chapter 3 where Fuglede's conjecture is proved for all compact open sets in \mathbb{Q}_p .

We prove the following theorem in this subsection.

Theorem 2.30. *Fuglede's conjecture holds for all compact open sets in \mathbb{Q}_p^d if and only if Fuglede's conjecture holds on $(\mathbb{Z}/\mathbb{Z}_{p^n})^d$ for all n .*

We first prove that a set C is a tile in $(\mathbb{Z}/\mathbb{Z}_{p^n})^d$ for some n if and only if $C + B(0, p^{-n})$ is a tile in \mathbb{Q}_p^d . Recall that $\{x\}$ denotes the fractional part of $x \in \mathbb{Q}_p^d$ and the set \mathbb{L}^d is a complete set of representatives of the cosets of the additive subgroup \mathbb{Z}_p^d . It follows that \mathbb{L}^d identified with $((\mathbb{Q}_p/\mathbb{Z}_p)^d, +)$ has a structure of group with the addition defined by

$$\{x\} + \{y\} := \{x + y\}, \quad \forall x, y \in \mathbb{Q}_p^d.$$

But we remark that \mathbb{L}^d is not a subgroup of \mathbb{Q}_p^d .

Let A, B and C be three subsets of some Abelian group. We say that A is the *direct sum* of B and C if for each $a \in A$, there exists a unique pair $(b, c) \in B \times C$ such that $a = b + c$. In this case, we write $A = B \oplus C$. It is not hard to check that A being the direct sum of B and C implies (B, C) being spectral pair in A .

It is obvious that $\mathbb{Q}_p^d = \mathbb{Z}_p^d \oplus \mathbb{L}^d$, which implies that \mathbb{L}^d is a tiling complement of \mathbb{Z}_p^d in \mathbb{Q}_p^d . For each integer γ , recall that $\mathbb{L}_\gamma = p^{-\gamma}\mathbb{L}$. Notice that

$$\mathbb{Q}_p^d = p^{-\gamma}\mathbb{Z}_p^d \oplus p^{-\gamma}\mathbb{L}^d = B(0, p^\gamma) \oplus \mathbb{L}_\gamma^d.$$

So \mathbb{L}_γ^d is a tiling complement of $B(0, p^\gamma)$.

Let C be a subset of $(\mathbb{Z}/p^\gamma\mathbb{Z})^d \simeq \{0, 1, \dots, p^\gamma - 1\}^d$ for some positive integer γ . We have the following lemma, which characterizes whether a compact open set which tiles \mathbb{Z}_p^d .

Lemma 2.31. *The set $C + B(0, p^{-\gamma})$ tiles \mathbb{Z}_p^d if and only if C tiles $(\mathbb{Z}/p^\gamma\mathbb{Z})^d$.*

Proof. First assume that C tiles $(\mathbb{Z}/p^\gamma\mathbb{Z})^d$. That is to say, we can find $T \subset (\mathbb{Z}/p^\gamma\mathbb{Z})^d$ such that $(\mathbb{Z}/p^\gamma\mathbb{Z})^d = C \oplus T$. One can check that $\mathbb{Z}_p^d = \Omega \oplus T$, which implies that $C + B(0, p^{-\gamma})$ tiles \mathbb{Z}_p^d with tiling complement T .

Now assume that the set $C + B(0, p^{-\gamma})$ tiles \mathbb{Z}_p^d with tiling complement T . Let $T^* \subset (\mathbb{Z}/p^\gamma\mathbb{Z})^d$ with $T^* = T \pmod{p^\gamma}$. Here we identify $(\mathbb{Z}/p^\gamma\mathbb{Z})^d$ with a subset $\{0, 1, \dots, p^\gamma - 1\}^d$ of \mathbb{Q}_p^d . It is not hard to see that $(\mathbb{Z}/p^\gamma\mathbb{Z})^d = C \oplus T^*$. So C tiles $(\mathbb{Z}/p^\gamma\mathbb{Z})^d$ with tiling complement T^* . \square

Notice that for each $a \in \mathbb{Q}_p^d$, either $C + B(0, p^{-\gamma}) + a \subset \mathbb{Z}_p^d$ or $(C + B(0, p^{-\gamma}) + a) \cap \mathbb{Z}_p^d = \emptyset$. Then the set $C + B(0, p^{-\gamma})$ tiles \mathbb{Z}_p^d if and only if it tiles \mathbb{Q}_p^d . So we immediately have the following corollary.

Corollary 2.32. *The set C tiles $(\mathbb{Z}/p^\gamma\mathbb{Z})^d$ if and only if $C + B(0, p^{-\gamma})$ tiles \mathbb{Q}_p^d .*

Now we consider the spectral sets. We have the following lemma.

Lemma 2.33. *Let $n \in \mathbb{N}$. Suppose that C is a spectral set in $(\mathbb{Z}/p^n\mathbb{Z})^d$. Then $C + B(0, p^{-n})$ is a spectral set in \mathbb{Q}_p^d .*

Proof. Let Γ be a spectrum of C in $(\mathbb{Z}/p^n\mathbb{Z})^d$. We will prove that $\Gamma + p^{-n}\mathbb{L}^d$ is a spectrum of $C + B(0, p^{-n})$.

Recall that μ is the Haar measure on \mathbb{Q}_p^d . For a Borel set $\Omega \subset \mathbb{Q}_p^d$, we denote by μ_Ω the restriction of the Haar measure in Ω , that is, $\mu_\Omega = 1_\Omega \cdot \mu$. Let $m = p^{nd} \cdot \mu|_{B(0, p^{-n})}$ be a

probability measure on \mathbb{Q}_p^d . Let $\delta_C = \frac{1}{\#C} \sum_{c \in C}$ be a probability measure on $(\mathbb{Z}/p^n\mathbb{Z})^d$ (and meanwhile on \mathbb{Q}_p^d). Since (C, Γ) and $(B(0, p^{-n}), p^{-n}\mathbb{L}^d)$ are spectral pairs in $(\mathbb{Z}/p^n\mathbb{Z})^d$ and \mathbb{Q}_p^d respectively, we have

$$\sum_{\lambda \in \Gamma} |\widehat{\delta}_C(\lambda - x)|^2 = 1, \forall x \in (\mathbb{Z}/p^n\mathbb{Z})^d, \quad (2.8.1)$$

and

$$\sum_{\lambda \in p^{-n}\mathbb{L}} |\widehat{m}(\lambda - y)|^2 = 1, \forall y \in \mathbb{Q}_p^d. \quad (2.8.2)$$

We rewrite the equation (2.8.1) by regarding δ_C as the measure on \mathbb{Q}_p^d :

$$\sum_{\lambda \in \Gamma'} |\widehat{\delta}_C(\lambda - x)|^2 = 1, \forall x \in \mathbb{Q}_p^d, \quad (2.8.3)$$

where $\Gamma' = p^{-n}\Gamma \subset \mathbb{Q}_p^d$. Denote by $\Omega = C + B(0, p^{-n})$. Observe that $m * \delta_C = \frac{1}{\mu(\Omega)}\mu|_{\Omega}$. Thus

$$\frac{1}{\mu(\Omega)^2} \sum_{\lambda \in \Gamma' + p^{-n}\mathbb{L}^d} |\widehat{\mu}|_{\Omega}(\lambda - x)|^2 = \sum_{\lambda \in \Gamma' + p^{-n}\mathbb{L}^d} |\widehat{m}(\lambda - x)\widehat{\delta}_C(\lambda - x)|^2.$$

By (2.7.6), we have $\widehat{m}(z+x) = \widehat{m}(x)$ for any $z \in B(0, p^n)$ and $x \in \mathbb{Q}_p^d$. Since $\Gamma' \subset B(0, p^n)$, we have

$$\begin{aligned} \sum_{\lambda \in \Gamma' + p^{-n}\mathbb{L}^d} |\widehat{m}(\lambda - x)\widehat{\delta}_C(\lambda - x)|^2 &= \sum_{\lambda \in \Gamma', l \in p^{-n}\mathbb{L}^d} |\widehat{m}(\lambda + l - x)\widehat{\delta}_C(\lambda + l - x)|^2 \\ &= \sum_{\lambda \in \Gamma', l \in p^{-n}\mathbb{L}^d} |\widehat{m}(l - x)|^2 |\widehat{\delta}_C(\lambda + l - x)|^2. \end{aligned}$$

Combining (2.8.2) and (2.8.3), we have

$$\begin{aligned} \sum_{\lambda \in \Gamma', l \in p^{-n}\mathbb{L}^d} |\widehat{m}(l - x)|^2 |\widehat{\delta}_C(\lambda + l - x)|^2 &= \sum_{l \in p^{-n}\mathbb{L}^d} |\widehat{m}(l - x)|^2 \sum_{\lambda \in \Gamma'} |\widehat{\delta}_C(\lambda + l - x)|^2 \\ &= \sum_{l \in p^{-n}\mathbb{L}^d} |\widehat{m}(l - x)|^2 = 1. \end{aligned}$$

That is,

$$\sum_{\lambda \in \Gamma' + p^{-n}\mathbb{L}^d} |\widehat{\mu}|_{\Omega}(\lambda - x)|^2 = \mu(\Omega)^2, \forall x \in \mathbb{Q}_p^d,$$

which implies that Ω is a spectral set in \mathbb{Q}_p^d with spectrum $\Gamma' + p^{-n}\mathbb{L}^d$. \square

The following lemma establishes the fact that given a compact open spectral set in \mathbb{Z}_p^d consisting of small balls of radius $p^{-\gamma}$ ($\gamma > 0$), any spectrum of the set is uniformly distributed in the sense that any ball of radius $p^{-\gamma}$ contains exactly as many points as the number of small balls of radius $p^{-\gamma}$ in the spectral set.

Let Ω be a compact open subset of \mathbb{Z}_p^d . Assume that Ω is of the form $\Omega = \bigsqcup_{c \in C} c + p^\gamma \mathbb{Z}_p^d$, where γ is a nonnegative integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$.

Lemma 2.34. *Suppose that (Ω, Λ) is a spectral pair. Then every closed ball of radius p^γ contains $\#C$ spectrum points in Λ . That is,*

$$\#(B(a, p^\gamma) \cap \Lambda) = \#C$$

for every $a \in \mathbb{Q}_p^d$.

Proof. By Lemma 2.28, we have

$$\widehat{1_\Omega}(\lambda - \xi) = p^{-\gamma d} 1_{B(0, p^\gamma)}(\lambda - \xi) \sum_{c \in C} \chi(-(\lambda - \xi) \cdot c).$$

Then a simple computation leads to

$$\sum_{\lambda \in \Lambda} |\widehat{1_\Omega}(\lambda - \xi)|^2 = \sum_{\lambda \in \Lambda} p^{-2\gamma d} 1_{B(0, p^\gamma)}(\lambda - \xi) \left(\#C + \sum_{\substack{c, c' \in C \\ c \neq c'}} \chi((c - c') \cdot (\lambda - \xi)) \right).$$

Consider the equality in Lemma 2.19. By integrating the both sides of this equality on the ball $B(a, p^\gamma)$, we get

$$|\Omega|^2 p^{\gamma d} = p^{-2\gamma d} \sum_{\substack{\lambda \in \Lambda \\ |\lambda - a| \leq p^\gamma}} \left(\#C p^{\gamma d} + \sum_{\substack{c, c' \in C \\ c \neq c'}} \int_{B(a, p^\gamma)} \chi((c - c') \cdot (\lambda - \xi)) d\xi \right). \quad (2.8.4)$$

Here we have used the fact that two balls of the same size are either identical or disjoint. Observe that

$$\int_{B(a, p^\gamma)} \chi((c - c') \cdot (\lambda - \xi)) d\xi = \chi((c - c') \cdot \lambda) \cdot \widehat{1_{B(a, p^\gamma)}}(c - c') \quad (2.8.5)$$

and $\widehat{1_{B(a, p^\gamma)}}(c - c') = \chi(-(c - c') \cdot a) \cdot 1_{B(0, p^{-\gamma})}(c - c')$. However, by the assumption, $|c - c'|_p > p^{-\gamma}$ for $c, c' \in C$ with $c \neq c'$. Hence we have

$$\widehat{1_{B(a, p^\gamma)}}(c - c') = 0. \quad (2.8.6)$$

Applying the equalities (2.8.5) and (2.8.6) to the equality (2.8.4), we obtain

$$|\Omega|^2 \cdot p^{\gamma d} = \#C \cdot p^{-\gamma d} \cdot \#(\Lambda \cap B(a, p^\gamma)) \quad (2.8.7)$$

Since $|\Omega| = \#C \cdot p^{-\gamma d}$, we finally get $\#(\Lambda \cap B(a, p^\gamma)) = \#C$. □

The restriction that Ω is contained in \mathbb{Z}_p^d is not necessary, because scaling and translation preserves the spectral property.

Lemma 2.35. *Let $\Omega = C + B(0, p^{-n})$ be a compact open set in \mathbb{Q}_p^d where $C \subset \{0, 1, \dots, p^n - 1\}$. Suppose that Ω is a spectral set. Then C is a spectral set in $(\mathbb{Z}/p^n\mathbb{Z})^d$.*

Proof. Assume that C is not a spectral set in $(\mathbb{Z}/p^n\mathbb{Z})^d$. Let $\delta_C = \frac{1}{\sharp C} \sum_{c \in C}$ be a probability measure on $(\mathbb{Z}/p^n\mathbb{Z})^d$ (and meanwhile on \mathbb{Q}_p^d). Then by Proposition 2.20, for any set $\Lambda \subset (\mathbb{Z}/p^n\mathbb{Z})^d$ with $\sharp\Lambda = \sharp C$, there exists $x \in (\mathbb{Z}/p^n\mathbb{Z})^d$ such that

$$\sum_{\lambda \in \Lambda} |\widehat{\delta}_C(\lambda - x)|^2 > 1. \quad (2.8.8)$$

That is, for any set $\Lambda \subset B(0, p^n)$ with $\sharp\Lambda = \sharp C$, there exists $x \in \mathbb{Q}_p^d$ such that

$$\sum_{\lambda \in \Lambda} |\widehat{\delta}_C(\lambda - x)|^2 > 1. \quad (2.8.9)$$

Let $m = p^n \cdot \mu_{|B(0, p^{-n})}$ be a probability measure on \mathbb{Q}_p^d . Observe that $m * \delta_C = \frac{1}{\mu(\Omega)} \mu_{|\Omega}$. Let Λ be a spectrum of Ω . Then

$$\frac{1}{\mu(\Omega)^2} \sum_{\lambda \in \Lambda} |\widehat{\mu}_{|\Omega}(\lambda - x)|^2 = \sum_{\lambda \in \Lambda} |\widehat{m}(\lambda - x) \widehat{\delta}_C(\lambda - x)|^2.$$

We have the decomposition $\Lambda = \sqcup_{a \in p^n\mathbb{L}^d} \Lambda_a$ where $\Lambda_a = \Lambda \cap B(a, p^n)$. Then we have

$$\sum_{\lambda \in \Lambda} |\widehat{m}(\lambda - x) \widehat{\delta}_C(\lambda - x)|^2 = \sum_{a \in p^n\mathbb{L}^d} \sum_{\lambda \in \Lambda_a} |\widehat{m}(\lambda - x)|^2 |\widehat{\delta}_C(\lambda - x)|^2.$$

By (2.7.6), we have $\widehat{m}(z + x) = \widehat{m}(z)$ for any $z \in B(0, p^n)$ and $x \in \mathbb{Q}_p^d$. Since $\Lambda_a - a \subset B(0, p^n)$ for all $a \in p^n\mathbb{L}^d$, we have

$$\begin{aligned} \sum_{a \in p^n\mathbb{L}^d} \sum_{\lambda \in \Lambda_a} |\widehat{m}(\lambda - x)|^2 |\widehat{\delta}_C(\lambda - x)|^2 &= \sum_{a \in p^n\mathbb{L}^d} \sum_{\lambda \in \Lambda_a} |\widehat{m}(a - x)|^2 |\widehat{\delta}_C(\lambda - x)|^2 \\ &= \sum_{a \in p^n\mathbb{L}^d} |\widehat{m}(a - x)|^2 \sum_{\lambda \in \Lambda_a} |\widehat{\delta}_C(\lambda - x)|^2. \end{aligned}$$

Due to Lemma 2.34, the number $\sharp\Lambda_a$ is independent of a , which is equal to $\sharp C$. By Proposition 2.20,

$$\sum_{\lambda \in \Lambda_a} |\widehat{\delta}_C(\lambda - x)|^2 \geq 1, \forall a \in p^n\mathbb{L}^d.$$

Together with (2.8.8), we have

$$\sum_{a \in p^n\mathbb{L}^d} |\widehat{m}(a - x)|^2 \sum_{\lambda \in \Lambda_a} |\widehat{\delta}_C(\lambda - x)|^2 > \sum_{a \in p^n\mathbb{L}^d} |\widehat{m}(a - x)|^2 = 1. \quad (2.8.10)$$

The last equality is due to the fact that m is a spectral measure with spectrum $p^n\mathbb{L}^d$. Therefore, we obtain

$$\sum_{\lambda \in \Lambda} |\widehat{\mu}_{|\Omega}(\lambda - x)|^2 > \mu(\Omega)^2,$$

a contradiction to that the measure $\mu_{|\Omega}$ is spectral. \square

2.8.2 Fuglede's conjecture fails on high dimensional p -adic space

As an application of Theorem 2.30, similarly to the Euclidean space, we show in this subsection that Fuglede's conjecture fails on high dimensional p -adic spaces. The content in this section is treated as a additional remark of Chapter 4 where Fuglede's conjecture is proved in the field of p -adic numbers.

When p is an odd prime, an example of a $2p \times 2p$ Hadamard matrix \mathcal{L}_p whose entries were p -th roots of unity was constructed by Butson [20]. In [8], Aten et al. calculated the rank of these Hadamard matrices.

Proposition 2.36 ([8], Corollary 8.3., Corollary 8.4). *When $p \equiv 1 \pmod{4}$, the rank of \mathcal{L}_p is five. When $p \equiv 3 \pmod{4}$, the rank of \mathcal{L}_p is four.*

According to this proposition, it is proved in [8] that there exist spectral sets which are not tiles in $(\mathbb{Z}/p\mathbb{Z})^5$ for all odd primes p and in $(\mathbb{Z}/p\mathbb{Z})^4$ for all odd primes p such that $p \equiv 3 \pmod{4}$. By Theorem 2.30, this implies that there exist compact open spectral sets which are not tiles in \mathbb{Q}_p^5 for all odd primes p and in \mathbb{Q}_p^4 for all odd primes p such that $p \equiv 3 \pmod{4}$.

For $d = 3$, there exists a spectral set in the finite group $(\mathbb{Z}/8\mathbb{Z})^3$ which is not a tile [84]. This implies that there exists a compact open spectral set which is not a tile in \mathbb{Q}_2^3 by Theorem 2.30.

2.9 Fuglede's conjecture fails on the field of formal Laurent series

As mentioned before, every local field is isomorphic to \mathbb{R}, \mathbb{C} , finite extensions of \mathbb{Q}_p or finite extensions of the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field \mathbb{F}_q . We have seen that Fuglede's conjecture is far from being proved on \mathbb{R} and \mathbb{C} . On the other hand, it has been shown in Section 2.8.2 that Fuglede's conjecture fails in high dimensional p -adic space; we will prove that Fuglede's conjecture on \mathbb{Q}_p . The last case of local field –the field of formal Laurent series– is dealt with in this section. We disprove Fuglede's conjecture holds on the field of formal Laurent series $\mathbb{F}_q((T))$ over a finite field \mathbb{F}_q . Clearly, it follows that Fuglede's conjecture fails on all finite extensions of such field.

A formal Laurent series over a ring R is a series of the form

$$f = \sum_{n \in \mathbb{Z}} a_n T^n, a_n \in R,$$

where $a_n = 0$ for all but finitely many negative indices n . Given two formal Laurent series

$$f = \sum_{n \in \mathbb{Z}} a_n T^n \text{ and } g = \sum_{n \in \mathbb{Z}} b_n T^n,$$

multiplication of such series is defined by the formula

$$f \cdot g = \sum_{n \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} a_i b_{n-i} T^n;$$

addition of such series is defined by the formula

$$f + g = \sum_{n \in \mathbb{Z}} (a_n + b_n) T^n.$$

For a non-zero formal Laurent series f , the minimal integer n such that $a_n \neq 0$ is called the order of f , denoted $\text{ord}(f)$. The order of the zero series is $+\infty$. The formal Laurent series form the *ring of formal Laurent series* over R , denoted by $R((X))$. The topology of the ring of formal Laurent series over R is induced by the metric $d(f, g) = 2^{-\text{ord}(f-g)}$. Moreover, if R is a field, then $R((T))$ is in fact a field.

Now we concentrate on the field of formal Laurent series over \mathbb{F}_q where q is a power of p and \mathbb{F}_q is the finite field of order q . We denote by $\mathbb{F}_q[[T]]$ the ring of integers, consisting of all elements in $\mathbb{F}_q((T))$ whose orders are non-negative. It is not hard to see that $\mathbb{F}_q[[T]]$ is compact open. Similarly to \mathbb{Q}_p , up to a scaling, we assume that the compact open sets are contained in $\mathbb{F}_q[[T]]$. Clearly, a compact open set in $\mathbb{F}_q[[T]]$ has the form

$$\sqcup_{c \in C} (c + 2^{-\gamma} \mathbb{F}_q[[T]]),$$

where $\gamma \geq 1$ is an integer and C is a subset of $(\mathbb{F}_q)_\gamma[T]$ where $(\mathbb{F}_q)_\gamma[T]$ is the space of all polynomials over \mathbb{F}_q having degree smaller than or equal to γ . Similarly to Theorem 2.30, we have the following theorem. The proof which is almost the same, is omitted here.

Theorem 2.37. *Fuglede's conjecture holds for all compact open sets in $\mathbb{F}_q((T))$ if and only if Fuglede's conjecture holds on \mathbb{F}_q^n for all $n \in \mathbb{N}$.*

It remains to disprove Fuglede's conjecture on \mathbb{F}_q^n for some $n \in \mathbb{N}$. Observe that the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is identified with a subset of \mathbb{F}_q . We have the following lemma.

Lemma 2.38. *Let Ω be a spectral set in $(\mathbb{Z}/p\mathbb{Z})^n$. Then Ω is also a spectral set in \mathbb{F}_q^n .*

Proof. Let $\Lambda \subset (\mathbb{Z}/p\mathbb{Z})^n$ be a spectrum of Ω in $(\mathbb{Z}/p\mathbb{Z})^n$. Since Λ is also a subset set of \mathbb{F}_q^n and the dual group of \mathbb{F}_q^n is itself, the couple (Ω, Λ) is a spectral pair in \mathbb{F}_q^n . \square

Obviously, a set Ω where $\sharp\Omega$ has another prime factor different from p is not a tile in \mathbb{F}_q^n for all n . Thus the spectral sets constructed in $(\mathbb{Z}/p\mathbb{Z})^5$ for p odd prime [8] and in $(\mathbb{Z}/p\mathbb{Z})^{12}$ for $p = 2$ [121] whose cardinalities have prime factors different from p are the non-tile spectral sets in \mathbb{F}_q^n . Therefore, Fuglede's conjecture fails on $\mathbb{F}_q((T))$ for all q .

Chapter 3

Compact open spectral sets in \mathbb{Q}_p

Abstract. *In this article, we prove that a compact open set in the field \mathbb{Q}_p of p -adic numbers is a spectral set if and only if it tiles \mathbb{Q}_p by translation, and also if and only if it is p -homogeneous which is easy to check. We also characterize spectral sets in $\mathbb{Z}/p^n\mathbb{Z}$ ($p \geq 2$ prime, $n \geq 1$ integer) by tiling property and also by homogeneity. Moreover, we construct a class of singular spectral measures in \mathbb{Q}_p , some of which are self-similar measures.*

3.1 Introduction

The problem that we consider is generally rised for all locally compact Abelian groups and the results that we obtain concern only the field \mathbb{Q}_p of p -adic numbers ($p \geq 2$ being a prime). Let us first state the problem. Let G be a locally compact Abelian group and $\Omega \subset G$ be a Borel set of positive and finite Haar measure. The set Ω is said to be *spectral* if there exists a set $\Lambda \subset \widehat{G}$ of continuous characters of G which forms a Hilbert basis of the space $L^2(\Omega)$ of square Haar-integrable functions. Such a set Λ is called a *spectrum* of Ω and (Ω, Λ) is called a *spectral pair*. We say that the set Ω *tiles* G by translation if there exists a set $T \subset G$ of translates such that $\sum_{t \in T} 1_\Omega(x - t) = 1$ for almost all $x \in G$, where 1_A denotes the indicator function of a set A . Such a set T is called a *tiling complement* of Ω and (Ω, T) is called a *tiling pair*. The so-called *spectral set conjecture* states that Ω is a spectral set if and only if Ω tiles G .

This conjecture in the case $G = \mathbb{R}^d$ is the famous Fuglede spectral set conjecture [43]. Both the original Fuglede conjecture and the generalized conjecture stated above have attracted considerable attention over the last decades. For the case of \mathbb{R}^d , many positive results were obtained [59, 60, 62, 65, 79, 81, 95, 96] before Tao [121] disproved it by showing that the direction “Spectral \Rightarrow Tiling” does not hold when $d \geq 5$. Now it is known that the conjecture is false in both directions for $d \geq 3$ [100, 83, 84, 29, 39]. However, the conjecture is still open in lower dimensions ($d = 1, 2$). On the other hand, Iosevich, Katz and Tao [59] proved that Fuglede’s conjecture is true for convex planar sets. The non-convex case is

This is the paper [37] *Compact open spectral sets in \mathbb{Q}_p* in collaboration with A. H. Fan and S. L. Fan published in *Journal of Functional Analysis*, 2016, 271(12): 3628-3661.

considerably more complicated, and is not understood even in dimension 1. Lagarias and Wang [92, 95] proved that all tilings of \mathbb{R} by a bounded region must be periodic, and that the corresponding translation sets are rational up to affine transformations. This in turn leads to a structure theorem for bounded tiles, which would be crucial for the direction “Tiling \Rightarrow spectral”. Assume that $\Omega \subset \mathbb{R}$ is a finite union of intervals. The conjecture holds when Ω is a union of two intervals [88]. If Ω is a union of three intervals, it is known that “Tiling \Rightarrow spectral”; and “Spectral \Rightarrow Tiling” holds with “an additional hypothesis” [14, 15, 16].

The problem for local fields was considered by the first author of the present paper in [30] where among others, is proved the basic Landau theorem concerning the Beurling density of spectrum. In this paper, we consider the conjecture restricted for compact open sets in the field \mathbb{Q}_p of p -adic numbers.

We shall give a geometric characterization of compact open spectral sets and prove that a compact open set is a spectral set if and only if it tiles \mathbb{Q}_p . The spectra and the tiling complements of compact open spectral sets are also investigated. Subject to an isometric transformation of \mathbb{Q}_p , the spectra and tiling complements are unique and determined by the set of possible distances of different points in the compact open spectral set.

Actually, in [36], we prove that the conjecture holds in \mathbb{Q}_p without the compact open restriction. Moreover, any spectral set is proved to be a compact open set up to a Haar-null set.

Let us recall some notions and notation (we refer to [129]). The ring of p -adic integers is denoted by \mathbb{Z}_p and the Haar measure on \mathbb{Q}_p is denoted by \mathbf{m} or dx . We assume that the Haar measure is normalized so that $\mathbf{m}(\mathbb{Z}_p) = 1$. The dual group $\widehat{\mathbb{Q}_p}$ of \mathbb{Q}_p is isomorphic to \mathbb{Q}_p . Any $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{n=v_p(x)}^{\infty} a_n p^n \quad (v_p(x) \in \mathbb{Z}, a_n \in \{0, 1, 2, \dots, p-1\} \text{ and } a_{v_p(x)} \neq 0).$$

Here, the integer $v_p(x)$ is called the p -valuation of x . The fractional part $\{x\}$ of x is defined to be $\sum_{n=v_p(x)}^{-1} a_n p^n$. We fix the following character $\chi \in \widehat{\mathbb{Q}_p}$:

$$\chi(x) = e^{2\pi i \{x\}}.$$

Notice that χ is equal to 1 on \mathbb{Z}_p but is non-constant on $p^{-1}\mathbb{Z}_p$. For any $y \in \mathbb{Q}_p$, we define

$$\chi_y(x) = \chi(yx).$$

Then the map $y \mapsto \chi_y$ from \mathbb{Q}_p onto $\widehat{\mathbb{Q}_p}$ is an isomorphism.

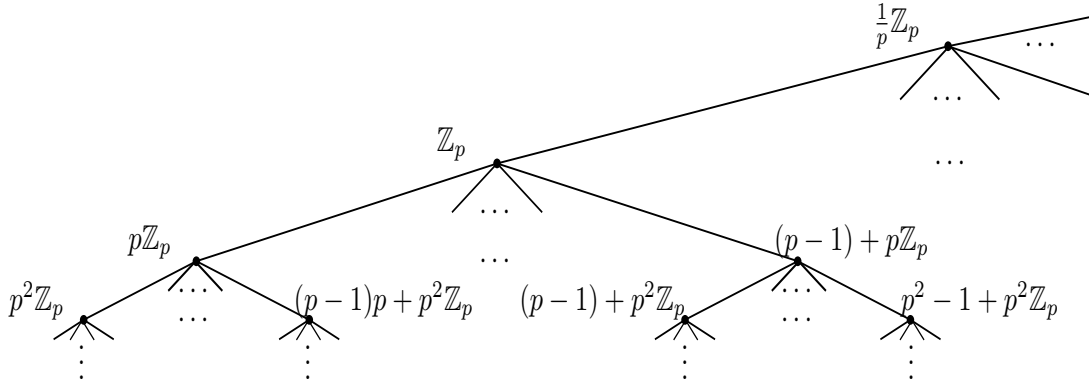


Figure 3.1: Consider \mathbb{Q}_p as an infinite tree.

In order to state our main result, we consider the field \mathbb{Q}_p as an infinite tree $(\mathcal{T}, \mathcal{E})$. The set of vertices \mathcal{T} is the set of all balls in \mathbb{Q}_p . The set of edges \mathcal{E} is the subset of $\mathcal{T} \times \mathcal{T}$ consisting of pairs $(B', B) \in \mathcal{T} \times \mathcal{T}$ such that

$$B' \subset B, \quad |B| = p|B'|,$$

where $|B|$ denotes the Haar measure of the ball B . This fact will be denoted by $B' \prec B$. We call B' a *descendant* of B , and B the *parent* of B' .

Any bounded open set O of \mathbb{Q}_p can be described by a subtree $(\mathcal{T}_O, \mathcal{E}_O)$ of $(\mathcal{T}, \mathcal{E})$. In fact, let B^* be the smallest ball containing O , which will be the root of the tree. For any given ball B contained in O , there is a unique sequence of balls B_0, B_1, \dots, B_r such that

$$B = B_0 \prec B_1 \prec B_2 \prec \dots \prec B_r = B^*.$$

We assume that the set of vertices \mathcal{T}_O is composed of all such balls B_0, B_1, \dots, B_r for all possible balls B contained in O . The set of edges \mathcal{E}_O is composed of all edges $B_i \prec B_{i+1}$ as above.

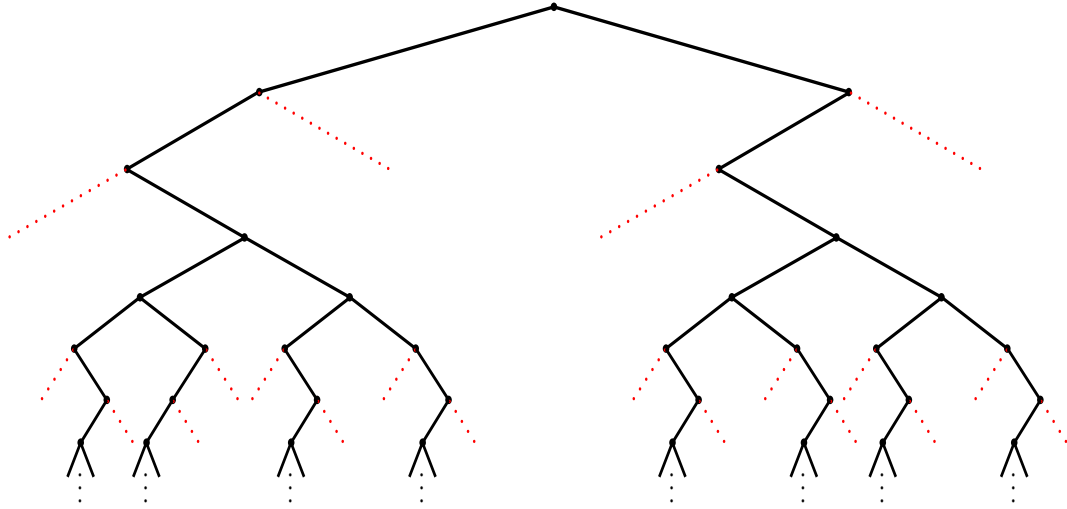


Figure 3.2: For $p = 2$, a p -homogeneous tree.

A subtree $(\mathcal{T}', \mathcal{E}')$ is said to be *homogeneous* if the number of descendants of $B \in \mathcal{T}'$ depends only on $|B|$. If this number is either 1 or p , we call $(\mathcal{T}', \mathcal{E}')$ a p -homogeneous tree.

A bounded open set is said to be *homogeneous* (resp. p -homogeneous) if the corresponding tree is homogeneous (resp. p -homogeneous).

Any compact open set can be described by a finite tree, because a compact open set is a disjoint finite union of balls of same size. In this case, as in the above construction of subtree we only consider these balls of same size as B .

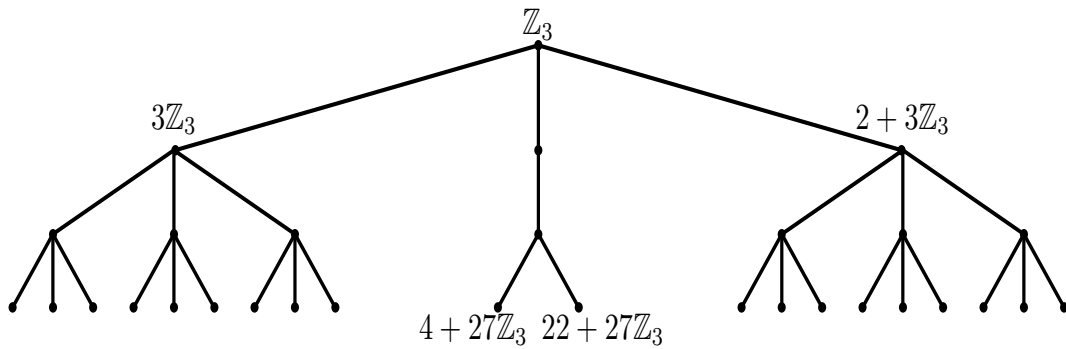


Figure 3.3: Consider the compact open set $O = 3\mathbb{Z}_3 \sqcup 3\mathbb{Z}_3 \sqcup 4 + 27\mathbb{Z}_3 \sqcup 22 + 27\mathbb{Z}_3$ as a finite tree.

We shall prove that the Fuglede conjecture holds in \mathbb{Q}_p among compact open sets and that spectral sets are characterized by their p -homogeneity.

Notice that an open compact set Ω can be written as $\bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p)$ for some finite set $C \subset \mathbb{Q}_p$ and some integer $\gamma \in \mathbb{Z}$. As we shall see in Section 3.3.2, for such a set Ω to

be spectral with Λ as spectrum if

$$\forall \lambda, \lambda' \in \Lambda, \lambda \neq \lambda', \sum_{c \in C} \chi(-c(\lambda - \lambda')) = 0 \text{ and } \sharp(\Lambda \cap B(0, p^\lambda)) = \sharp C.$$

So we are led to study the trigonometric polynomial $\sum_{c \in C} \chi(ct)$.

Theorem 3.1. *Let Ω be a compact open set in \mathbb{Q}_p . The following statements are equivalent:*

- (a) Ω is a spectral set;
- (b) Ω is p -homogeneous;
- (c) Ω tiles \mathbb{Q}_p by translation.

For any subset $\Omega \subset \mathbb{Q}_p$, the set of admissible p -orders of Ω is defined by

$$I_\Omega := \{i \in \mathbb{Z} : \exists x, y \in \Omega \text{ such that } v_p(x - y) = i\}.$$

Remark that p^{-I_Ω} is the set of possible distances of different points in Ω .

Assume that Ω is a p -homogeneous compact open set. By the definition of I_Ω , an integer $i \in I_\Omega$ if and only if the balls of radius p^{-i} in the tree \mathcal{T}_Ω has p descendants. And there is an integer γ such that $i \in I_\Omega$ if $i \geq \gamma$. This is the reason why could a compact open set be described by a finite tree.

On the other hand, it is of interest to investigate the structures of the spectra and the tiling complements of Ω . We obtain that the spectra and the tiling complements of Ω are uniquely determined by the set I_Ω , but subject to an isometric transformation of \mathbb{Q}_p .

Set $\mathbb{Z}/p\mathbb{Z} \cdot p^i := \{ap^i : 0 \leq a \leq p-1, a \in \mathbb{N}\} \subset \mathbb{Q}_p$. Recall that the addition of two subsets A and B in \mathbb{Q}_p is defined by

$$A + B := \{a + b : a \in A, b \in B\}.$$

Let $\{A_i : i \in I\}$ be a family of subsets in \mathbb{Q}_p such that all A_i contain 0. We define

$$\sum_{i \in I} A_i := \left\{ \sum_{i \in J} a_i : J \subset I \text{ finite and } a_i \in A_i \right\}.$$

Theorem 3.2. *Let Ω be a p -homogeneous compact open set in \mathbb{Q}_p with admissible p -order set I_Ω .*

- (a) *Subject to an isometric bijection of \mathbb{Q}_p ,*

$$\Lambda = \sum_{i \in I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^{-i-1}$$

is the unique spectrum of Ω .

- (b) *Subject to an isometric bijection of \mathbb{Q}_p ,*

$$T = \sum_{i \notin I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^i$$

is the unique tiling complement of Ω .

It is clear that if Ω is a spectral set with Λ as spectrum, then so are its translates $\Omega + a$ ($a \in \mathbb{Q}_p$) with spectrum Λ and its dilations $b\Omega$ ($b \in \mathbb{Q}_p^*$) with spectrum $b^{-1}\Lambda$. It is also true that the translation and the dilation don't change the tiling property and the homogeneity. Since Ω is compact open, by scaling and translation, we may assume that $\Omega \subset \mathbb{Z}_p$ and $0 \in \Omega$. So it can be represented as a disjoint union of balls of same size

$$\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where γ is a nonnegative integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$.

For each $0 \leq n \leq \gamma$, denote by

$$C_{\text{mod } p^n} := \{x \in \{0, 1, \dots, p^n - 1\} : \exists y \in C, \text{ such that } x \equiv y \pmod{p^n}\}$$

the subset of $\mathbb{Z}/p^n\mathbb{Z}$ determined by C modulo p^n .

We also obtained the following characterization of spectral sets in the finite group $\mathbb{Z}/p^\gamma\mathbb{Z}$.

Theorem 3.3. *Let $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$. The following statements are equivalent:*

- (a) C is a spectral set in $\mathbb{Z}/p^\gamma\mathbb{Z}$;
- (b) C is a tile of $\mathbb{Z}/p^\gamma\mathbb{Z}$;
- (c) For any $n = 1, 2, \dots, \gamma - 1$, $\#(C_{\text{mod } p^n}) = p^{k_n}$ for some integer $k_n \in \mathbb{N}$, where $\#(C_{\text{mod } p^n})$ is the cardinality of the finite set $C_{\text{mod } p^n}$.

As Terence Tao noted on page 3 of [121], the equivalence (a) \Leftrightarrow (b) is not new, but follows from Laba [89], which in turn builds on the work of Coven-Meyerowitz [22]. However, we prove it by showing that both (a) and (b) are equivalent to (c) which implies p -homogeneity of C . The p -homogeneity is practically checkable. We can use it to describe finite spectral sets (more precisely, the probability spectral measures uniformly distributed on finite sets) in \mathbb{Q}_p . A probability Borel measure μ on \mathbb{Q}_p is called a *spectral measure* if there exists a set $\Lambda \subset \widehat{\mathbb{Q}_p}$ such that $\{\chi_\lambda\}_{\lambda \in \Lambda}$ forms an orthonormal basis (i.e. a Hilbert basis) of the $L^2(\mu)$. Let F be a finite subset of \mathbb{Q}_p . Consider the uniform probability measure on F defined by

$$\delta_F := \frac{1}{\#F} \sum_{c \in F} \delta_c,$$

where δ_c is the dirac measure concentrated at the point c . Let

$$\gamma_F = \max_{\substack{c, c' \in F \\ c \neq c'}} v_p(c - c'),$$

where $v_p(x)$ denotes the p -valuation of $x \in \mathbb{Q}_p$. Then $p^{-\gamma_F}$ is the minimal distance between different points in F .

Theorem 3.4. *The measure δ_F is a spectral measure if and only if for each integer $\gamma > \gamma_F$, the compact open set $\Omega_\gamma := \bigsqcup_{c \in F} B(c, p^{-\gamma})$ is a spectral set.*

The above theorem provides a criterion of finite spectral set, combining with Theorem 3.1.

Corollary 3.5. *The measure δ_F is a spectral measure if and only if*

$$\sharp F = p^{\sharp I_F}.$$

Moreover, we are interested in finding more spectral measures. And we provide a class of Cantor spectral measures.

Theorem 3.6. *There exists a class of singular spectral measures in \mathbb{Q}_p .*

The article is organized as follows. In Section 3.2, we introduce some basic definitions and preliminaries on the field \mathbb{Q}_p of p -adic numbers, Fourier analysis on \mathbb{Q}_p and \mathbb{Z} -module generated by the p^n -th roots of unity. In Section 3.3, we prove Theorem 3.1. In Section 3.4, we characterize spectral sets and tiles in the finite group $\mathbb{Z}/p^\gamma\mathbb{Z}$. Theorem 3.3 is proved there. Section 3.5 is devoted to the characterization of spectra and tiling complements. Theorem 3.2 is proved at the end of this section. In Section 3.6, we characterize finite spectral sets in \mathbb{Q}_p (Theorem 3.4). In Section 3.7, we shall construct a class of singular spectral measures (Theorem 3.6) and present two concrete examples.

3.2 Preliminaries

In this section, we present some preliminaries. Some of them have their interests, like characterization of spectral measures using Fourier transform, \mathbb{Z} -module generated by the p^n -th roots of unity, uniform distribution of spectrum etc. We start with recalling p -adic numbers and related notation, and the computation of the Fourier transform of the indicator function of a compact open set.

3.2.1 The field of p -adic numbers

Consider the field \mathbb{Q} of rational numbers and a prime $p \geq 2$. Any nonzero rational number $r \in \mathbb{Q}$ can be written as $r = p^{\frac{v}{b}}$ where $v, a, b \in \mathbb{Z}$ and $(p, a) = 1$ and $(p, b) = 1$ (here (x, y) denotes the greatest common divisor of two integers x and y). We define $v_p(r) = v$ and $|r|_p = p^{-v_p(r)}$ for $r \neq 0$ and $|0|_p = 0$. Then $|\cdot|_p$ is a non-Archimedean absolute value on \mathbb{Q} . That means

- (i) $|r|_p \geq 0$ with equality only for $r = 0$;
- (ii) $|rs|_p = |r|_p |s|_p$;
- (iii) $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.

The field of p -adic numbers \mathbb{Q}_p is the completion of \mathbb{Q} under the absolute value $|\cdot|_p$. Actually a typical element of \mathbb{Q}_p is of the form of a convergent series

$$\sum_{n=N}^{\infty} a_n p^n \quad (N \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\}, a_N \neq 0).$$

Consider \mathbb{Q}_p as an additive group. Then a non-trivial group character is the following function

$$\chi(x) = e^{2\pi i\{x\}},$$

where $\{x\} = \sum_{n=N}^{-1} a_n p^n$ is the fractional part of $x = \sum_{n=N}^{\infty} a_n p^n$. From this character we can obtain all characters χ_y of \mathbb{Q}_p , which are defined by $\chi_y(x) = \chi(yx)$. Remark that each $\chi_y(\cdot)$ is uniformly locally constant, i.e.

$$\chi_y(x) = \chi_y(x'), \text{ if } |x - x'|_p \leq \frac{1}{|y|_p}.$$

The interested readers are referred to [120, 129] for further information about characters of \mathbb{Q}_p .

Notation:

$\mathbb{Z}_p^\times := \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$. It is the group of units of \mathbb{Z}_p .

$B(0, p^n) := p^{-n}\mathbb{Z}_p$. It is the (closed) ball centered at 0 of radius p^n .

$B(x, p^n) := x + B(0, p^n)$.

1_A : the indicator function of a set A .

3.2.2 Fourier transformation

Let μ be a finite Borel measure on \mathbb{Q}_p . The *Fourier transform* of μ is classically defined to be

$$\widehat{\mu}(y) = \int_{\mathbb{Q}_p} \overline{\chi}_y(x) d\mu(x) \quad (y \in \widehat{\mathbb{Q}}_p \simeq \mathbb{Q}_p).$$

The Fourier transform \widehat{f} of $f \in L^1(\mathbb{Q}_p)$ is that of μ_f where μ_f is the measure defined by $d\mu_f = f d\mathbf{m}$.

The following lemma shows that the Fourier transform of the indicator function of a ball centered at 0 is a function of the same type and the Fourier transform of the indicator function of a compact open set is also supported by a ball, and on the ball it is the restriction of a trigonometric polynomial.

Lemma 3.7. *Let $\gamma \in \mathbb{Z}$ be an integer.*

(a) *We have $\widehat{1_{B(0, p^\gamma)}}(\xi) = p^\gamma 1_{B(0, p^{-\gamma})}(\xi)$ for all $\xi \in \mathbb{Q}_p$.*

(b) *If $\Omega = \bigsqcup_j B(c_j, p^\gamma)$ is a finite union of disjoint balls of the same size, then*

$$\widehat{1_\Omega}(\xi) = p^\gamma 1_{B(0, p^{-\gamma})}(\xi) \sum_j \chi(-c_j \xi). \tag{3.2.1}$$

Proof. (a) By the scaling property of the Haar measure, we have only to prove the result in the case $\gamma = 0$. Recall that

$$\widehat{1_{B(0,1)}}(\xi) = \int_{B(0,1)} \chi(-\xi x) dx.$$

When $|\xi|_p \leq 1$, the integrand is equal to 1, so $\widehat{1_{B(0,1)}}(\xi) = 1$. When $|\xi|_p > 1$, making a change of variable $x = y - z$ with $z \in B(0, 1)$ chosen such that $\chi(\xi \cdot z) \neq 1$, we get

$$\widehat{1_{B(0,1)}}(\xi) = \chi(\xi z) \widehat{1_{B(0,1)}}(\xi).$$

It follows that $\widehat{1_{B(0,1)}}(\xi) = 0$ for $|\xi|_p > 1$.

(b) is a direct consequence of (a) and of the fact

$$\widehat{1_{B(c,p^r)}}(\xi) = \chi(-c\xi) \widehat{1_{B(0,p^r)}}(\xi).$$

□

3.2.3 A criterion of spectral measure

Let μ be a probability Borel measure on \mathbb{Q}_p . We say that μ is a *spectral measure* if there exists a set $\Lambda \subset \widehat{\mathbb{Q}_p}$ such that $\{\chi_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal basis (i.e. a Hilbert basis) of $L^2(\mu)$. Then Λ is called a *spectrum* of μ and we call (μ, Λ) a *spectral pair*. Assume that Ω is a set in \mathbb{Q}_p of positive and finite Haar measure. That Ω is a *spectral set* means the restricted measure $\frac{1}{\mathfrak{m}(\Omega)} \mathfrak{m}|_\Omega$ is a spectral measure. In this case, instead of saying $(\frac{1}{\mathfrak{m}(\Omega)} \mathfrak{m}|_\Omega, \Lambda)$ is a spectral pair, we say that (Ω, Λ) is a *spectral pair*.

Here is a criterion for a probability measure μ to be a spectral measure. The result in the case \mathbb{R}^d is due to Jørgensen and Pedersen [64]. It holds on any local field (see [30]). The proof is the same as in the Euclidean space. We reproduce the proof here for completeness.

Lemma 3.8.¹ *A Borel probability measure on \mathbb{Q}_p is a spectral measure with $\Lambda \subset \widehat{\mathbb{Q}_p}$ as its spectrum iff*

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \sum_{\lambda \in \Lambda} |\widehat{\mu}(\lambda - \xi)|^2 = 1. \quad (3.2.2)$$

In particular, a Borel set Ω such that $0 < |\Omega| < \infty$ is a spectral set with Λ as spectrum iff

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \sum_{\lambda \in \Lambda} |\widehat{1_\Omega}(\lambda - \xi)|^2 = |\Omega|^2. \quad (3.2.3)$$

Proof. Recall that $\langle f, g \rangle_\mu$ denotes the inner product in $L^2(\mu)$:

$$\langle f, g \rangle_\mu = \int f \bar{g} d\mu, \quad \forall f, g \in L^2(\mu).$$

Remark that

$$\langle \chi_\xi, \chi_\lambda \rangle_\mu = \int \chi_\xi \bar{\chi}_\lambda d\mu = \widehat{\mu}(\lambda - \xi).$$

It follows that χ_λ and χ_ξ are orthogonal in $L^2(\mu)$ iff $\widehat{\mu}(\lambda - \xi) = 0$.

¹The equivalent condition of spectral sets in locally compact abelian groups can be found in Theorem 2.19

Assume that (μ, Λ) is a spectral pair. Then (3.2.2) holds because of the Parseval equality and of the fact that $\{\widehat{\mu}(\lambda - \xi)\}_{\lambda \in \Lambda}$ are Fourier coefficients of χ_ξ under the Hilbert basis $\{\chi_\lambda\}_{\lambda \in \Lambda}$.

Now assume (3.2.2) holds. Fix any $\lambda' \in \Lambda$ and take $\xi = \lambda'$ in (3.2.2). We get

$$1 + \sum_{\lambda \in \Lambda, \lambda \neq \lambda'} |\widehat{\mu}(\lambda - \lambda')|^2 = 1,$$

which implies $\widehat{\mu}(\lambda - \lambda') = 0$ for all $\lambda \in \Lambda \setminus \{\lambda'\}$. Thus we have proved the orthogonality of $\{\chi_\lambda\}_{\lambda \in \Lambda}$. It remains to prove that $\{\chi_\lambda\}_{\lambda \in \Lambda}$ is complete. By the Hahn-Banach Theorem, what we have to prove is the implication

$$f \in L^2(\mu), \forall \lambda \in \Lambda, \langle f, \chi_\lambda \rangle_\mu = 0 \Rightarrow f = 0.$$

The condition (3.2.2) implies that

$$\forall \xi \in \widehat{\mathbb{Q}}_p, \quad \chi_\xi = \sum_{\lambda \in \Lambda} \langle \chi_\xi, \chi_\lambda \rangle_\mu \chi_\lambda.$$

This implies that χ_ξ is in the closure of the space spanned by $\{\chi_\lambda\}_{\lambda \in \Lambda}$. As f is orthogonal to χ_λ for all $\lambda \in \Lambda$. So, f is orthogonal to χ_ξ . Thus we have proved that

$$\forall \xi \in \widehat{\mathbb{Q}}_p, \quad \int \overline{\chi_\xi} f d\mu = \langle f, \chi_\xi \rangle_\mu = 0.$$

That is, the Fourier coefficients of the measure $f d\mu$ are all zero. Finally $f = 0$ μ -almost everywhere. \square

3.2.4 \mathbb{Z} -module generated by p^n -th roots of unity

The Fourier condition of a spectral set is tightly related to the fact that certain sums of roots of unity must be zero. Let $m \geq 2$ be an integer and let $\omega_m = e^{2\pi i/m}$, which is a primitive m -th root of unity. Denote by \mathcal{M}_m the set of integral points $(a_0, a_1, \dots, a_{m-1}) \in \mathbb{Z}^m$ such that

$$\sum_{j=0}^{m-1} a_j \omega_m^j = 0,$$

which form a \mathbb{Z} -module. The fact that the degree over \mathbb{Q} of the extension field $\mathbb{Q}(\omega_m)$ is equal to $\phi(m)$, where ϕ is Euler's totient function, implies that the rank of \mathcal{M}_m is equal to $m - \phi(m)$. Schoenberg ([114], Theorem 1) found a set of generators. See also de Bruijn [18] and Rédei [106, 107] (actually, Rédei [106] was the first to formulate this, his incomplete proof was completed in [18] and in [107], and later independently in [114]). Lagarias and Wang ([91], Lemma 4.1) observed that this set of generators is actually a base when m is a prime power. Let p be a prime and n be a positive integer.

Lemma 3.9 ([91, 114]). *Let $(a_0, a_1, \dots, a_{p^n-1}) \in \mathcal{M}_{p^n}$. Then for any integer $0 \leq i \leq p^{n-1} - 1$, we have $a_i = a_{i+jp^{n-1}}$ for all $j = 0, 1, \dots, p - 1$.*

We shall use Lemma 3.9 in the following two particular forms. The first one is an immediate consequence.

Lemma 3.10. *Let $(b_0, b_1, \dots, b_{p-1}) \in \mathbb{Z}^p$. If $\sum_{j=0}^{p-1} e^{2\pi i b_j/p^n} = 0$, then subject to a permutation of (b_0, \dots, b_{p-1}) , there exists $0 \leq r \leq p^{n-1} - 1$ such that*

$$b_j \equiv r + jp^{n-1} \pmod{p^n}$$

for all $j = 0, 1, \dots, p-1$.

Lemma 3.11. *Let C be a finite subset of \mathbb{Z} . If $\sum_{c \in C} e^{2\pi i c/p^n} = 0$, then $p \mid \#C$ and C can be decomposed into $\#C/p$ disjoint subsets $C_1, C_2, \dots, C_{\#C/p}$, such that each C_j consists of p points and*

$$\sum_{c \in C_j} e^{2\pi i c/p^n} = 0.$$

Proof. Observe that $e^{2\pi i c/p^n} = e^{2\pi i c'/p^n}$ if and only if $c \equiv c' \pmod{p^n}$. Fix a point $c_0 \in C$. By Lemma 3.9, there are other $p-1$ points $c_1, c_2, \dots, c_{p-1} \in C$ such that $c_j \equiv c_0 + jp^{n-1} \pmod{p^n}$ for all $1 \leq j \leq p-1$. Thus we have

$$\sum_{0 \leq i \leq p-1} e^{2\pi i c_i/p^n} = 0.$$

Set $C_1 = \{c_0, c_1, \dots, c_{p-1}\}$. So, the hypothesis is reduced to

$$\sum_{c \in C \setminus C_1} e^{2\pi i c/p^n} = 0.$$

We can complete the proof by induction. □

The following lemma states that the property $\sum_{j=0}^{m-1} \chi(\xi_j) = 0$ of the set of points $(\xi_0, \xi_1, \dots, \xi_{m-1}) \in \mathbb{Q}_p^m$ is invariant under ‘rotation’.

Lemma 3.12. *Let $\xi_0, \xi_1, \dots, \xi_{m-1}$ be m points in \mathbb{Q}_p . If $\sum_{j=0}^{m-1} \chi(\xi_j) = 0$, then $p \mid m$ and*

$$\sum_{j=0}^{m-1} \chi(x\xi_j) = 0$$

for all $x \in \mathbb{Z}_p^\times$.

Proof. By Lemma 3.11, we get $p \mid m$ and moreover, $\{\xi_0, \xi_1, \dots, \xi_{m-1}\}$ consists of m/p subsets $C_1, C_2, \dots, C_{m/p}$ such that each $C_j, 1 \leq j \leq m/p$, contains p elements and $\sum_{\xi \in C_j} \chi(\xi) = 0$.

Without loss of generality, we assume that $m = p$. By Lemma 3.10, subject to a permutation of $(\xi_0, \dots, \xi_{p-1})$, there exists $r \in \mathbb{Q}_p$ such that

$$\xi_j \equiv r + j/p \pmod{\mathbb{Z}_p}$$

for all $j = 0, 1, \dots, p-1$. Now, for any given $x \in \mathbb{Z}_p^\times$, we have

$$x\xi_j \equiv xr + \frac{x_0 j}{p} \pmod{\mathbb{Z}_p}$$

where $x_0 \in \{1, \dots, p-1\}$ is the first digit of the p -adic expansion of x . Observe that the multiplication by x_0 induces a permutation on $\{0, 1, \dots, p-1\}$. So we have

$$\sum_{j=0}^{p-1} \chi(x\xi_j) = \sum_{k=0}^{p-1} e^{2\pi i \{xr + \frac{k}{p}\}} = 0.$$

□

3.2.5 Uniform distribution of spectrum

The following lemma establishes the fact that given a compact open spectral set in \mathbb{Z}_p consisting of small balls of radius $p^{-\gamma}$ ($\gamma > 0$), any spectrum of the set is uniformly distributed in the sense that any ball of radius $p^{-\gamma}$ contains exactly as many points as the number of small balls of radius $p^{-\gamma}$ in the spectral set. This fact will contribute to proving “spectral property implies homogeneity” of Theorem 3.1.

Let Ω be a compact open subset of \mathbb{Z}_p . Assume that Ω is of the form $\Omega = \bigsqcup_{c \in C} c + p^\gamma \mathbb{Z}_p$, where γ is a positive integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$.

Lemma 3.13. *Suppose that (Ω, Λ) is a spectral pair. Then every closed ball of radius p^γ contains $\#C$ spectrum points in Λ . That is,*

$$\#(B(a, p^\gamma) \cap \Lambda) = \#C$$

for every $a \in \mathbb{Q}_p$.

Proof. By Lemma 3.7, we have

$$\widehat{1}_\Omega(\lambda - \xi) = p^{-\gamma} 1_{B(0, p^\gamma)}(\lambda - \xi) \sum_{c \in C} \chi(-(\lambda - \xi)c).$$

Then a simple computation leads to

$$\sum_{\lambda \in \Lambda} |\widehat{1}_\Omega(\lambda - \xi)|^2 = \sum_{\lambda \in \Lambda} p^{-2\gamma} 1_{B(0, p^\gamma)}(\lambda - \xi) \left(\#C + \sum_{\substack{c, c' \in C \\ c \neq c'}} \chi((c - c')(\lambda - \xi)) \right).$$

Consider the equality (3.2.3) in Lemma 3.8. By integrating both sides of this equality on the ball $B(a, p^\gamma)$, we get

$$|\Omega|^2 p^\gamma = p^{-2\gamma} \sum_{\substack{\lambda \in \Lambda \\ |\lambda - a| \leq p^\gamma}} \left(\#C p^\gamma + \sum_{\substack{c, c' \in C \\ c \neq c'}} \int_{B(a, p^\gamma)} \chi((c - c')(\lambda - \xi)) d\xi \right). \quad (3.2.4)$$

Here we have used the fact that two balls of same size are either identical or disjoint. Observe that

$$\int_{B(a,p^\gamma)} \chi((c-c')(\lambda-\xi))d\xi = \chi((c-c')\lambda) \cdot \widehat{1_{B(a,p^\gamma)}}(c-c') \quad (3.2.5)$$

and $\widehat{1_{B(a,p^\gamma)}}(c-c') = \chi(-(c-c')a) \cdot p^\gamma 1_{B(0,p^{-\gamma})}(c-c')$. However, by the assumption, $|c-c'|_p > p^{-\gamma}$ for $c, c' \in C$ with $c \neq c'$. Hence we have

$$\widehat{1_{B(a,p^\gamma)}}(c-c') = 0. \quad (3.2.6)$$

Applying the equalities (3.2.5) and (3.2.6) to the equality (3.2.4), we obtain

$$|\Omega|^2 \cdot p^\gamma = \#C \cdot p^{-\gamma} \cdot \#(\Lambda \cap B(a, p^\gamma)) \quad (3.2.7)$$

Since $|\Omega| = \#C \cdot p^{-\gamma}$, we finally get $\#(\Lambda \cap B(a, p^\gamma)) = \#C$.

□

The restriction that Ω is contained in \mathbb{Z}_p is not necessary, because scaling and translation preserves the spectral property.

3.2.6 Finite p -homogeneous trees

Let γ be a positive integer. To any $t_0 t_1 \cdots t_{\gamma-1} \in \{0, 1, 2, \dots, p-1\}^\gamma$, we associate an integer

$$c = c(t_0 t_1 \cdots t_{\gamma-1}) = \sum_{i=0}^{\gamma-1} t_i p^i \in \{0, 1, 2, \dots, p^\gamma - 1\}.$$

So $\mathbb{Z}/p^\gamma\mathbb{Z} \simeq \{0, 1, \dots, p^\gamma - 1\}$ is identified with $\{0, 1, 2, \dots, p-1\}^\gamma$ which is considered as a finite tree, denoted by $\mathcal{T}^{(\gamma)}$, see FIGURE 3.4 for an example. The set of vertices $\mathcal{T}^{(\gamma)}$ consists of the disjoint union of the sets $\mathbb{Z}/p^n\mathbb{Z}$, $0 \leq n \leq \gamma$. Each vertex, except the root of the tree, is identified with a sequence $t_0 t_1 \cdots t_{n-1}$ with $1 \leq n \leq \gamma$ and $t_i \in \{0, 1, \dots, p-1\}$. The set of edges consists of pairs $(x, y) \in \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^{n+1}\mathbb{Z}$, such that $x \equiv y \pmod{p^n}$, where $0 \leq n \leq \gamma-1$. For example, each point c of $\mathbb{Z}/p^\gamma\mathbb{Z}$ is identified with $\sum_{i=0}^{\gamma-1} t_i p^i \in \{0, 1, \dots, p^\gamma - 1\}$, which is called a *boundary point* of the tree.

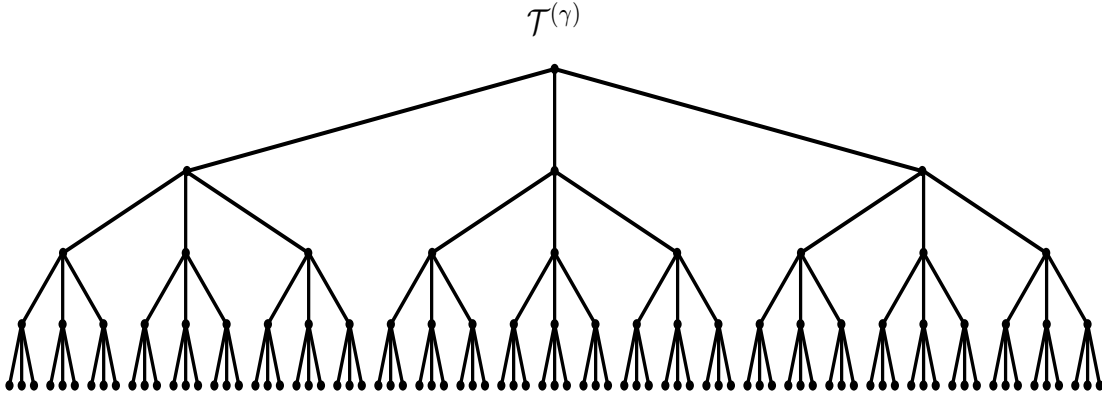


Figure 3.4: The set $\mathbb{Z}/3^4\mathbb{Z} = \{0, 1, 2, \dots, 80\}$ is considered as a tree $\mathcal{T}^{(4)}$.

Each subset $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ will determine a subtree of $\mathcal{T}^{(\gamma)}$, denoted by \mathcal{T}_C , which consists of the paths from the root to the boundary points in C . The set of vertices \mathcal{T}_C consists of the disjoint union of the sets $C_{\text{mod } p^n}, 0 \leq n \leq \gamma$. The set of edges consists of pairs $(x, y) \in C_{\text{mod } p^n} \times C_{\text{mod } p^{n+1}}$, such that $x \equiv y \pmod{p^n}$, where $0 \leq n \leq \gamma - 1$.

Now we are going to construct a class of subtrees of $\mathcal{T}^{(\gamma)}$. Let I be a subset of $\{0, 1, 2, \dots, \gamma - 1\}$ and let J be the complement of I in $\{0, 1, 2, \dots, \gamma - 1\}$. Thus I and J form a partition of $\{0, 1, 2, \dots, \gamma - 1\}$. It is allowed that I or J is empty. We say a subtree of $\mathcal{T}^{(\gamma)}$ is of $\mathcal{T}_{I,J}$ -form if its boundary points $t_0 t_1 \dots t_{\gamma-1}$ of $\mathcal{T}_{I,J}$ are those of $\mathcal{T}^{(\gamma)}$ satisfying the following conditions:

- (i) if $i \in I$, t_i can take any value of $\{0, 1, \dots, p - 1\}$;
- (ii) if $i \in J$, for any $t_0 t_1 \dots t_{i-1}$, we fix one value of $\{0, 1, \dots, p - 1\}$ which is the only value taken by t_i . In other words, t_i takes only one value from $\{0, 1, \dots, p - 1\}$ which depends on $t_0 t_1 \dots t_{i-1}$.

Remark that such a subtree depends not only on I and J but also on the values taken by t_i 's with $i \in J$. A $\mathcal{T}_{I,J}$ -form tree is called a finite p -homogeneous tree. A special subtree $\mathcal{T}_{I,J}$ is shown in FIGURE 3.5.

A set $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ is said to be p -homogeneous if the corresponding tree \mathcal{T}_C is p -homogeneous. If $C \subset \{0, 1, 2, \dots, p^\gamma - 1\}$ is considered as a subset of \mathbb{Z}_p , the tree \mathcal{T}_C could be identified with the finite tree determined by the compact open set $\Omega = \bigsqcup_{c \in C} c + p^\gamma\mathbb{Z}_p$. By definition, we immediately have the following lemma.

Lemma 3.14. *The above compact set Ω is p -homogeneous in \mathbb{Q}_p if and only if the finite set $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ is p -homogeneous.*

An algebraic criterion for the p -homogeneity of a set $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ is presented in the following theorem.

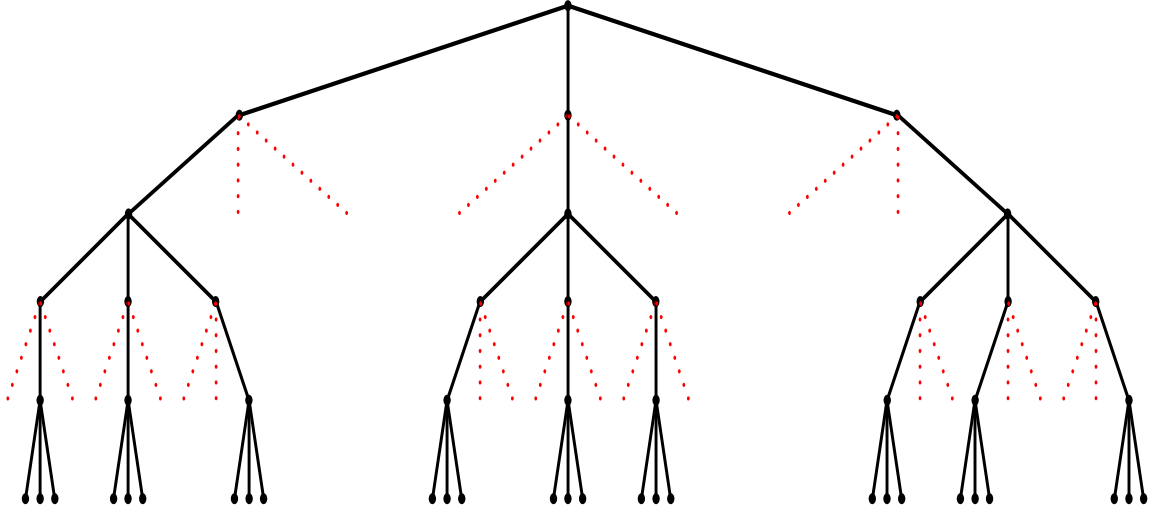


Figure 3.5: For $p = 3$, a $\mathcal{T}_{I,J}$ -form tree with $\gamma = 5, I = \{0, 2, 4\}, J = \{1, 3\}$.

Theorem 3.15. *Let γ be a positive integer and let $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$. Suppose (i) $\#C \leq p^n$ for some integer $1 \leq n \leq \gamma$; (ii) there exist n integers $1 \leq i_1 < i_2 < \cdots < i_n \leq \gamma$ such that*

$$\sum_{c \in C} e^{2\pi i c p^{-i_k}} = 0 \text{ for all } 1 \leq k \leq n. \quad (3.2.8)$$

Then $\#C = p^n$ and C is p -homogeneous. Moreover, \mathcal{T}_C is a $\mathcal{T}_{I,J}$ -form tree with $I = \{i_1 - 1, i_2 - 1, \dots, i_n - 1\}$ and $J = \{0, 1, \dots, \gamma - 1\} \setminus I$.

Proof. For simplicity, let $m = \#C$. By Lemma 3.11 and the equality (4.2.6) with $k = n$, $p \mid m$ and C can be decomposed into m/p subsets $C_1, C_2, \dots, C_{m/p}$ such that each C_j consists of p points and

$$\sum_{c \in C_j} e^{2\pi i c p^{-i_n}} = 0.$$

Then, by Lemma 3.10, we have that

$$c \equiv c' + r p^{i_n - 1} \pmod{p^{i_n}} \quad \text{for some } r \in \{0, 1, \dots, p - 1\}, \quad (3.2.9)$$

if c and c' lie in the same C_j .

Now we consider the equality (4.2.6) when $k = n - 1$. Since $i_{n-1} < i_n$, the equality (4.2.7) implies the function

$$c \mapsto e^{2\pi i c p^{-i_{n-1}}}$$

is constant on each C_j . For each $c \in C$, denote by \tilde{c} the point in $\{0, 1, 2, \dots, p^{i_{n-1}} - 1\}$ such that

$$\tilde{c} \equiv c \pmod{p^{i_{n-1}}}.$$

Observe that $e^{2\pi i c p^{-i_{n-1}}} = e^{2\pi i \tilde{c} p^{-i_{n-1}}}$ and that $\tilde{c} = \tilde{c}'$ if c and c' lie in the same C_j . Let $\tilde{C} = C_{\text{mod } p^{i_{n-1}}}$ be the set of all these \tilde{c} . So the quality (4.2.6) with $k = n - 1$ is equivalent to

$$\sum_{\tilde{c} \in \tilde{C}} e^{2\pi i \tilde{c} p^{-i_{n-1}}} = 0.$$

This equivalence follows from the facts that each C_j contains the same number of elements.

Similarly, by Lemma 3.11, we have $p \mid \frac{m}{p}$ (i.e. $p^2 \mid m$) and \tilde{C} can be decomposed into m/p^2 subsets $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_{m/p^2}$ such that each subset consists of p elements and

$$\sum_{\tilde{c} \in \tilde{C}_i} e^{2\pi i \tilde{c} p^{-i_{n-1}}} = 0.$$

By Lemma 3.10, we get that

$$\tilde{c} \equiv \tilde{c}' + r p^{i_{n-1}-1} \pmod{p^{i_{n-1}}} \quad \text{for some } r \in \{0, 1, \dots, p-1\}, \quad (3.2.10)$$

if \tilde{c} and \tilde{c}' lie in same \tilde{C}_j .

By induction, we get $p^n \mid m$. By the hypotheses $m \leq p^n$, we finally get $p^n = m$.

Furthermore, the above argument implies that \mathcal{T}_C is a p -homogeneous tree of $\mathcal{T}_{I,J}$ -form with $I = \{i_1 - 1, i_2 - 1, \dots, i_n - 1\}$ and $J = \{0, 1, \dots, \gamma - 1\} \setminus I$. \square

3.2.7 Compact open tiles in \mathbb{Q}_p

Recall that $\{x\}$ denotes the fractional part of $x \in \mathbb{Q}_p$. Let

$$\mathbb{L} := \{\{x\}, x \in \mathbb{Q}_p\},$$

which is a complete set of representatives of the cosets of the additive subgroup \mathbb{Z}_p . Then \mathbb{L} identified with $(\mathbb{Q}_p/\mathbb{Z}_p, +)$ has a structure of group with the addition defined by

$$\{x\} + \{y\} := \{x + y\}, \quad \forall x, y \in \mathbb{Q}_p.$$

Notice that \mathbb{L} is not a subgroup of \mathbb{Q}_p . Notice that \mathbb{L} is the set of p -adic rational numbers

$$\sum_{i=-n}^{-1} a_i p^i \quad (n \geq 1; 0 \leq a_i \leq p-1).$$

Let A, B and C be three subsets of some Abelian group. We say that A is the *direct sum* of B and C if for each $a \in A$, there exist a unique pair $(b, c) \in B \times C$ such that $a = b + c$. Then we write $A = B \oplus C$.

It is obvious that $\mathbb{Q}_p = \mathbb{Z}_p \oplus \mathbb{L}$, which implies that \mathbb{L} is a tiling complement of \mathbb{Z}_p in \mathbb{Q}_p . For each integer γ , let

$$\mathbb{L}_\gamma := p^{-\gamma} \mathbb{L}.$$

Notice that

$$\mathbb{Q}_p = p^{-\gamma}\mathbb{Z}_p \oplus p^{-\gamma}\mathbb{L} = B(0, p^\gamma) \oplus \mathbb{L}_\gamma.$$

So \mathbb{L}_γ is a tiling complement of $B(0, p^\gamma)$.

For a positive integer γ , let C be a subset of $\mathbb{Z}/p^\gamma\mathbb{Z} \simeq \{0, 1, \dots, p^\gamma - 1\}$. Let $\Omega = \bigsqcup_{c \in C} c + p^\gamma\mathbb{Z}_p$, where C is considered as a subset of \mathbb{Z}_p . The following lemma characterizes a finite union of balls which tiles \mathbb{Z}_p .

Lemma 3.16. *The above set Ω tiles \mathbb{Z}_p if and only if C tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$.*

Proof. Assume that C tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$, i.e. $\mathbb{Z}/p^\gamma\mathbb{Z} = C \oplus T$ for some $T \subset \mathbb{Z}/p^\gamma\mathbb{Z}$. One can check that $\mathbb{Z}_p = \Omega \oplus T$, which implies that Ω tiles \mathbb{Z}_p with tile complement T .

Assume that Ω tiles \mathbb{Z}_p with tiling complement T . Set $T^* = T \bmod p^\gamma$. One can check that $\mathbb{Z}/p^\gamma\mathbb{Z} = C \oplus T^*$. So C tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$ with tiling complement T^* . \square

Notice that for each $a \in \mathbb{Q}_p$, either $\Omega + a \subset \mathbb{Z}_p$ or $(\Omega + a) \cap \mathbb{Z}_p = \emptyset$. Then Ω tiles \mathbb{Z}_p if and only if it tiles \mathbb{Q}_p . So we immediately have the following corollary.

Corollary 3.17. *The set C tiles $\mathbb{Z}/p^\gamma\mathbb{Z}$ if and only if Ω tiles \mathbb{Q}_p .*

3.2.8 p -homogeneous discrete set in \mathbb{Q}_p

Let E be a discrete subset in \mathbb{Q}_p . Recall that

$$I_E = \{i \in \mathbb{Z} : \exists x, y \in E \text{ such that } v_p(x - y) = i\}.$$

The following lemma gives the relation between the number of elements and possible distances in a finite subset of \mathbb{Q}_p .

Lemma 3.18. *Let Λ be finite subsets of \mathbb{Q}_p . Then*

$$\#\Lambda \leq p^{\#I_\Lambda}.$$

Proof. Assume that $\#\Lambda = n$ and $I_\Lambda = \{i_1, i_2, \dots, i_n\}$ with $i_1 < i_2 < \dots < i_n$. By assumption, Λ is contained in a ball of radius p^{-i_1} . Each ball of radius p^{-i_1} consists of p balls of radius p^{-i_1-1} . So we can decompose Λ into at most p subsets E_0, E_1, \dots, E_{p-1} such that

$$|\lambda - \lambda'|_p \begin{cases} = p^{-i_1}, & \text{if } \lambda \text{ and } \lambda' \text{ lie in different } E_i, E_j. \\ < p^{-i_1}, & \text{if } \lambda \text{ and } \lambda' \text{ lie in the same } E_i. \end{cases}$$

By assumption, for each E_i , we have $I_{E_i} \subset \{i_2, i_3, \dots, i_n\}$. We apply the above argument again, with Λ replaced by each E_i . By induction, it suffices to prove the conclusion when $\#\Lambda = 1$. Obviously, $\#\Lambda \leq p$ if $\#\Lambda = 1$, which completes the proof. \square

Remark that a subset E of \mathbb{Q}_p is uniformly discrete if I_E is bounded from above. Denote γ_E by the maximum of I_E . For each integer n , set $I_E^{\geq n} := \{i \in I_E : i \geq n\}$. By Lemma 3.18, for each ball $B(a, p^{-n})$ with $n \leq \gamma_E$, we have

$$\#(E \cap B(a, p^{-n})) \leq p^{\#I_E^{\geq n}}.$$

We say a discrete set E is p -homogeneous if

$$\#(E \cap B(a, p^{-n})) = p^{\#I_E^{\geq n}} \text{ or } 0,$$

for all integers n and all $a \in \mathbb{Q}_p$. By definition, the following lemma is immediately obtained.

Lemma 3.19. *A finite set $\Lambda \subset \mathbb{Q}_p$ is p -homogeneous if and only if $\#\Lambda = p^{\#I_\Lambda}$.*

The following lemma shows that the p -homogeneous discrete sets, under isometric transformations, admit canonical forms.

Lemma 3.20. *Let E be a p -homogeneous discrete subset of \mathbb{Q}_p . Then there exists an isometric transformation $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$, such that*

$$f : E \rightarrow \widehat{E} := \left\{ \sum_{i \in I_E} \beta_i p^i \in \mathbb{Q}_p : \beta_i \in \{0, 1, 2, \dots, p-1\} \right\}.$$

Proof. Without loss of generality, we assume that E contains 0. Otherwise, we take a translate $f_a(x) = x - a$ with some $a \in E$. So $f_a(E)$ contains 0.

Recall that I_E is bounded from above and γ_E is the maximum of I_E . For integers $n > \gamma_E$, 0 is the unique point of E which lies in the balls $p^n \mathbb{Z}_p$. Now we are going to construct an isometric transformation on \mathbb{Q}_p by induction.

Step I: Let $n_0 = \gamma_E$. Then the set $E \cap p^{n_0} \mathbb{Z}_p$ consists of p points $x_0 = 0, x_1, x_2, \dots, x_{p-1}$ such that $x_j \in jp^{n_0} + p^{n_0+1} \mathbb{Z}_p$ for $0 \leq j \leq p-1$. Define

$$f_{n_0}(x) := x - x_j + jp^{n_0} \text{ if } x \in jp^{n_0} + p^{n_0+1} \mathbb{Z}_p.$$

So we obtain an isometric map $f_{n_0} : p^{n_0} \mathbb{Z}_p \rightarrow p^{n_0} \mathbb{Z}_p$ such that

$$f_{n_0}(x_j) = jp^{n_0} \text{ for all } j \in \{0, 1, \dots, p-1\}.$$

Step II: Let $n_1 = \gamma_E - 1$. We distinguish two cases:

$$n_1 \in I_E \text{ or } n_1 \notin I_E.$$

If $n_1 \in I_E$, we decompose $p^{n_1} \mathbb{Z}_p$ as

$$p^{n_1} \mathbb{Z}_p = \bigsqcup_{j=0}^{p-1} jp^{n_1} + p^{n_0} \mathbb{Z}_p.$$

Applying the similar argument as the *Step I* to each $jp^{n_1} + p^{n_0}\mathbb{Z}_p, 0 \leq j \leq p-1$, we obtain an isometric transformation g_j on $jp^{n_1} + p^{n_0}\mathbb{Z}_p$ such that $g_j(E \cap (jp^{n_1} + p^{n_0}\mathbb{Z}_p)) = \widehat{E} \cap (jp^{n_1} + p^{n_0}\mathbb{Z}_p)$. So we obtain an isometric transformation f_{n_1} on $p^{n_1}\mathbb{Z}_p$ such that

$$f_{n_1}(E \cap p^{n_1}\mathbb{Z}_p) = \widehat{E} \cap p^{n_1}\mathbb{Z}_p.$$

If $n_1 \notin I_E$, we define

$$f_{n_1}(x) = \begin{cases} f_{n_0}(x), & \text{if } x \in p^{n_0}\mathbb{Z}_p \\ x, & \text{if } x \in p^{n_1}\mathbb{Z}_p \setminus p^{n_0}\mathbb{Z}_p \end{cases}.$$

So f_{n_1} is an isometric transformation on $p^{n_1}\mathbb{Z}_p$ such that

$$f_{n_1}(E \cap p^{n_1}\mathbb{Z}_p) = \widehat{E} \cap p^{n_1}\mathbb{Z}_p.$$

By induction, we obtain an isometric transformation $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ such that $f(E) = \widehat{E}$. \square

Proposition 3.21. *Let E and E' be two p -homogeneous discrete sets in \mathbb{Q}_p . Then $I_E = I_{E'}$ if and only if there exists an isometric transformation $f : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ such that $f(E) = E'$.*

Proof. The ‘if’ part of the statement is obvious.

We are going to prove the ‘only if’ part. We claim that the isometric transformation constructed in Lemma 3.20 is a bijection. Actually, any isometric transformation of \mathbb{Q}_p is surjective, which can be deduced from the fact that isometric transformations on compact metric spaces are surjective and $\mathbb{Q}_p = \bigcup_{n \geq 0} p^{-n}\mathbb{Z}_p$. Thus, by Lemma 3.20, we have two isometric bijections $f_1, f_2 : \mathbb{Q}_p \rightarrow \mathbb{Q}_p$ such that

$$f_1(E) = f_2(E') = \widehat{E} = \left\{ \sum_{i \in I_E} \beta_i p^i \in \mathbb{Q}_p : \beta_i \in \{0, 1, 2, \dots, p-1\} \right\},$$

since $I_E = I_{E'}$. Therefore, $f_2^{-1} \circ f_1$ is an isometric transformation of \mathbb{Q}_p , which maps E onto E' . \square

3.3 Compact open spectral sets in \mathbb{Q}_p

This section is devoted to the proof of Theorem 1.1.

Let Ω be a compact open set in \mathbb{Q}_p . Therefore, without loss of generality, we assume that Ω is contained in \mathbb{Z}_p and $0 \in \Omega$. Let Ω be of the form

$$\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$.

3.3.1 Homogeneity implies spectral property

Assume that Ω is a p -homogeneous compact open set contained in \mathbb{Z}_p and containing 0. We are going to show that Ω is a spectral set by constructing a spectrum for Ω . Let I_Ω be the structure set of Ω . Then I_Ω determines a finite p -homogeneous tree of type $\mathcal{T}_{I,J}$ with $I = I_\Omega \cap \{0, 1, \dots, \gamma - 1\}$ and $J = \{0, 1, \dots, \gamma - 1\} \setminus I$.

Define

$$\Lambda = \left(\sum_{i \in I} \mathbb{Z}/p\mathbb{Z} \cdot p^{-i-1} \right) + \mathbb{L}_\gamma.$$

We claim that (Ω, Λ) is a spectral pair. To prove the claim, it suffices to check the equality (3.2.3) in Lemma 3.8. The term on the left hand side of the equality (3.2.3) is equal to

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\widehat{1}_\Omega(\lambda - \xi)|^2 &= \frac{1}{p^{2\gamma}} \sum_{\lambda \in \Lambda} 1_{B(\xi, p^\gamma)}(\lambda) \sum_{c, c' \in C} \chi((c - c')(\lambda - \xi)) \\ &= \frac{1}{p^{2\gamma}} \sum_{\lambda \in \Lambda \cap B(\xi, p^\gamma)} \sum_{c, c' \in C} \chi((c - c')(\lambda - \xi)). \end{aligned} \quad (3.3.1)$$

Let $\xi = \sum_{i=v_p(\xi)}^{\infty} \xi_i p^i \in \mathbb{Q}_p$. We set $\xi_i = 0$ if $i < v_p(\xi)$, so that $\xi = \sum_{i=-\infty}^{\infty} \xi_i p^i$. Let

$$\xi_\star = \sum_{i=-\gamma}^{-1} \xi_i p^i, \quad \xi' = \sum_{j=v_p(\xi)}^{-\gamma-1} \xi_j p^j.$$

Remark that ξ' is 0 when $v_p(\xi) > -\gamma - 1$. Then we have $\{\xi\} = \xi_\star + \xi'$ and $|\xi - \xi'|_p \leq p^\gamma$ which implies $B(\xi, p^\gamma) = B(\xi', p^\gamma)$. For $\lambda = \sum_{i=0}^n a_i p^{-i-1} \in \Lambda$, observe that $|\lambda - \xi|_p \leq p^\gamma$ if and only if $a_i = \xi_{-i-1}$ for all $i \geq \gamma$. So we get

$$\Lambda \cap B(\xi, p^\gamma) = \xi' + \sum_{i \in I} \mathbb{Z}/p\mathbb{Z} \cdot p^{-i-1},$$

which consists of $p^{\#I}$ elements. Using this last fact, the fact $|\Omega|^2 = p^{-2(\gamma - \#I)}$ and the equality (3.3.1), to prove the equality (3.2.3), we have only to prove that

$$\sum_{\lambda \in \Lambda \cap B(\xi, p^\gamma)} \chi((c - c')(\lambda - \xi)) = 0 \quad \text{for } c \neq c'. \quad (3.3.2)$$

The possible distances between c and c' are of the form p^{-i} with $i \in I$. Fix two different c and c' in C . Write

$$c - c' = p^{i_0} s,$$

for some $i_0 \in I$ and some $s \in \mathbb{Z}_p^\times$. Set $I_{i_0} = I \cap [i_0, \gamma - 1]$. For any $\lambda = \sum_{i \in I} a_i p^{-i-1} + \xi' \in \Lambda \cap B(\xi, p^\gamma)$, we have

$$\begin{aligned} (c - c')(\lambda - \xi) &\equiv (c - c') \left(\sum_{i \in I} a_i p^{-i-1} - \xi_\star \right) \pmod{\mathbb{Z}_p} \\ &\equiv -\xi_\star (c - c') + \frac{s \sum_{i \in I_{i_0}} a_i p^{\gamma-i-1}}{p^{\gamma-i_0}} \pmod{\mathbb{Z}_p} \end{aligned}$$

so that

$$\chi((c - c')(\lambda - \xi)) = \chi(-\xi_*(c - c')) \prod_{i \in I_{i_0}} \chi\left(\frac{sa_i}{p^{i-i_0+1}}\right).$$

From this, we observe that as function of λ , $\chi((c - c')(\lambda - \xi))$ only depends on the coordinates a_i of λ with $i \in I_{i_0}$. Then, by the definition of Λ , for each $\lambda = \sum_{i \in I_0} a_i p^{-i-1} + \xi' \in \Lambda \cap B(\xi, p^\gamma)$, there are $p^{\sharp(I \setminus I_{i_0})}$ points $\lambda' \in \Lambda \cap B(\xi, p^\gamma)$ such that $\chi((c - c')(\lambda - \xi)) = \chi((c - c')(\lambda' - \xi))$. So we get

$$\sum_{\lambda \in \Lambda \cap B(\xi, p^\gamma)} \chi((c - c')(\lambda - \xi)) = p^{\sharp(I \setminus I_{i_0})} \chi(-\xi_*(c - c')) \prod_{i \in I_{i_0}} \sum_{a_i=0}^{p-1} \chi\left(\frac{sa_i}{p^{i-i_0+1}}\right).$$

Therefore, we shall prove (3.3.2) if we prove that the factor corresponding to $i = i_0$ on the right hand side of the last equality is zero, i.e.

$$\sum_{a_{i_0}=0}^{p-1} \chi\left(\frac{sa_{i_0}}{p}\right) = 0. \quad (3.3.3)$$

This is really true because of Lemma 3.12 and

$$\sum_{a_{i_0}=0}^{p-1} \chi\left(\frac{a_{i_0}}{p}\right) = 0.$$

Thus we have proved that Ω is a spectral set.

3.3.2 Spectral property implies homogeneity

Assume that Λ is a spectrum of Ω . We are going to show that Ω is p -homogeneous.

By Lemma 3.13, we have $\sharp(B(0, p^\gamma) \cap \Lambda) = \sharp C$. For simplicity, let $\sharp C = m$. Set

$$D = \{|\lambda - \lambda'|_p : \lambda, \lambda' \in B(0, p^\gamma) \cap \Lambda \text{ and } \lambda \neq \lambda'\}$$

be the set of possible distances of different spectrum points in the ball $B(0, p^\gamma)$. Notice that $\log_p(D) \subset \{1, 2, \dots, \gamma\}$. Assume that $\sharp D = n$ and

$$\log_p(D) = \{i_1, i_2, \dots, i_n\} \text{ with } 1 \leq i_1 < i_2 < \dots < i_n \leq \gamma.$$

Observe that

$$\langle \chi_\lambda, \chi_{\lambda'} \rangle = \frac{1}{p^\gamma} \mathbf{1}_{B(0, p^\gamma)}(\lambda - \lambda') \sum_{c \in C} \chi(-c(\lambda - \lambda')).$$

So, the orthogonality of $\{\chi_\lambda\}_{\lambda \in \Lambda}$ implies

$$\sum_{c \in C} \chi(-c(\lambda - \lambda')) = 0 \quad (\forall \lambda, \lambda' \in \Lambda, 0 < |\lambda - \lambda'|_p \leq p^\gamma). \quad (3.3.4)$$

By (3.3.4) and Lemma 3.12, it deduces that C satisfies the conditions in Theorem 3.15. Therefore, C is a p -homogeneous tree.

On the other hand, $\sharp(B(0, p^\gamma) \cap \Lambda) = \sharp C = p^n$. Thus, by Lemma 3.19, the discrete set $B(0, p^\gamma) \cap \Lambda$ is p -homogeneous with $I_{B(0, p^\gamma) \cap \Lambda} = -\log_p(D)$.

3.3.3 Equivalence between homogeneity and tiling

Due to Lemma 3.14 and Corollary 3.17, it is sufficient to prove that C is a tile of $\mathbb{Z}/p^\gamma\mathbb{Z}$ if and only if \mathcal{T}_C is a p -homogeneous tree. We shall finish the proof when we have proved the equivalence between the tiling property and the p -homogeneity of a set in $\mathbb{Z}/p^\gamma\mathbb{Z}$. This will be done in the next section.

3.4 Spectral sets and tiles in $\mathbb{Z}/p^\gamma\mathbb{Z}$

In this section, we characterize spectral sets and tiles in the finite group $\mathbb{Z}/p^\gamma\mathbb{Z}$. Spectral sets and tiles in this group are the same which are characterized by a simple geometric property that we qualify as p -homogeneity. They can also be characterized by their Fourier transforms.

Recall that the characters of $\mathbb{Z}/p^\gamma\mathbb{Z}$ are the functions

$$x \mapsto e^{\frac{2\pi i k x}{p^\gamma}}, \quad k \in \mathbb{Z}/p^\gamma\mathbb{Z}.$$

We identify $\mathbb{Z}/p^\gamma\mathbb{Z}$ to $\{0, 1, \dots, p^\gamma - 1\}$ which can be viewed as a subset of \mathbb{Q}_p . The restriction of the characters $\chi_{\frac{k}{p^\gamma}}, k = 0, 1, 2, \dots, p^\gamma - 1$ of \mathbb{Q}_p on $\mathbb{Z}/p^\gamma\mathbb{Z}$ are exactly the characters of $\mathbb{Z}/p^\gamma\mathbb{Z}$.

For a subset C of $\mathbb{Z}/p^\gamma\mathbb{Z}$ which is viewed as a subset of \mathbb{Q}_p , let δ_C be the uniform probability measure in \mathbb{Q}_p . By definition, we immediately have the following lemma.

Lemma 3.22. *Let $C, \Lambda \subset \{0, 1, 2, \dots, p^\gamma - 1\}$. Then (C, Λ) is a spectral pair in $\mathbb{Z}/p^\gamma\mathbb{Z}$ if and only if $(\delta_C, \frac{1}{p^\gamma}\Lambda)$ is a spectral pair in \mathbb{Q}_p .*

The Fourier transform of a function f defined on $\mathbb{Z}/p^\gamma\mathbb{Z}$ is defined as follows

$$\widehat{f}(k) = \sum_{x \in \mathbb{Z}/p^\gamma\mathbb{Z}} f(x) e^{-\frac{2\pi i k x}{p^\gamma}}, \quad (\forall k \in \mathbb{Z}/p^\gamma\mathbb{Z}).$$

Theorem 3.23. *Let $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ and \mathcal{T}_C be the associated tree. The following are equivalent.*

- (1) \mathcal{T}_C is a p -homogeneous tree.
- (2) For any $1 \leq i \leq \gamma$, $\sharp(C_{\text{mod } p^i}) = p^{k_i}$, for some $k_i \in \mathbb{N}$.
- (3) There exists a subset $I \subset \mathbb{N}$ such that $\sharp I = \log_p(\sharp C)$ and $\widehat{1}_C(p^\ell) = 0$ for $\ell \in I$.
- (4) There exists a subset $I \subset \mathbb{N}$ such that $\sharp I \geq \log_p(\sharp C)$ and $\widehat{1}_C(p^\ell) = 0$ for $\ell \in I$.
- (5) C is a tile of $\mathbb{Z}/p^\gamma\mathbb{Z}$.
- (6) C is a spectral set in $\mathbb{Z}/p^\gamma\mathbb{Z}$.

Proof. (1) \Rightarrow (2): It follows from the definition of p -homogeneous subtree.

(2) \Rightarrow (3): From $\#C = p^{k_\gamma}$ we get $\log_p(\#C) = k_\gamma$. For simplicity, denote by $C_j = C_{\bmod p^j}$ for $1 \leq j \leq \gamma$.

Define

$$I := \{\gamma - j : \#C_{j-1} < \#C_j\} \subset \{0, 1, \dots, \gamma - 1\}, 1 \leq j \leq \gamma.$$

Then $\#I = k_\gamma$. For any j such that $\gamma - j \in I$, we have $\#C_j = p\#C_{j-1}$. More precisely,

$$C_j = C_{j-1} + \{0, 1, 2, \dots, p-1\}p^{j-1}.$$

Thus

$$\begin{aligned} \widehat{1}_C(p^{\gamma-j}) &= \sum_{t \in C} e^{-\frac{2\pi i}{p^j}t} = p^{k_\gamma - k_j} \sum_{t \in C_j} e^{-\frac{2\pi i}{p^j}t} \\ &= p^{k_\gamma - k_j} \sum_{t \in C_{j-1}} \sum_{l=0}^{p-1} e^{-\frac{2\pi i}{p^j}(t+lp^{j-1})} \\ &= p^{k_\gamma - k_j} \sum_{t \in C_{j-1}} e^{-\frac{2\pi i}{p^j}t} \sum_{l=0}^{p-1} e^{-\frac{2\pi i}{p}l} = 0, \end{aligned}$$

i.e. $\widehat{1}_C(p^\ell) = 0$ for $\ell \in I$.

(3) \Rightarrow (4): Obviously.

(4) \Rightarrow (1): Observe that $\widehat{1}_C(p^\ell) = 0$ means

$$\sum_{t \in C} e^{-\frac{2\pi i t}{p^{\gamma-\ell}}} = 0,$$

which is exactly the condition in Theorem 3.15. Therefore we can prove that $\#I = \log_p(\#C)$ and \mathcal{T}_C is a p -homogeneous tree.

(1) \Rightarrow (5): Assume that \mathcal{T}_C is a p -homogeneous tree $\mathcal{T}_{I,J}$. It is obvious that C has the tiling property $C \oplus S = \mathbb{Z}/p^\gamma\mathbb{Z}$ with the tiling complement

$$S = \left\{ \sum_{i \in J} a_i p^i : a_i \in \{0, 1, \dots, p-1\} \right\}.$$

(5) \Rightarrow (4): Assume that C is a tile of $\mathbb{Z}/p^\gamma\mathbb{Z}$. That is to say, there exists a set $S \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ such that $C \oplus S = \mathbb{Z}/p^\gamma\mathbb{Z}$. Since $\#(C \oplus S) = \#C \cdot \#S$, $\#C$ divides $\#(\mathbb{Z}/p^\gamma\mathbb{Z}) = p^\gamma$. The equality $C \oplus S = \mathbb{Z}/p^\gamma\mathbb{Z}$ can be rewritten as

$$\forall x \in \mathbb{Z}/p^\gamma\mathbb{Z}, \quad \sum_{y \in \mathbb{Z}/p^\gamma\mathbb{Z}} 1_C(y)1_S(x-y) = 1.$$

In other words, $1_C * 1_S = 1$, where the convolution is that in group $\mathbb{Z}/p^\gamma\mathbb{Z}$. Then we have

$$\widehat{1}_C \cdot \widehat{1}_S = p^\gamma \delta_0,$$

where δ_0 is the Dirac measure concentrated at 0. Consequently

$$Z(\widehat{1}_C) \cup Z(\widehat{1}_S) = \mathbb{Z}/p^\gamma \mathbb{Z} \setminus \{0\},$$

where $Z(\widehat{f}) := \{x : \widehat{f}(x) = 0\}$ is the set of zeros of \widehat{f} . In particular, the powers p^ℓ with $\ell = 0, 1, 2, \dots, \gamma - 1$ are zeroes of either $\widehat{1}_C$ or $\widehat{1}_S$. Let

$$C_z = \left\{ l \in \{0, 1, 2, \dots, \gamma - 1\} : \widehat{1}_C(p^\ell) = 0 \right\},$$

$$S_z = \left\{ l \in \{0, 1, 2, \dots, \gamma - 1\} : \widehat{1}_S(p^\ell) = 0 \right\}.$$

Since $C_z \cup S_z = \{0, 1, 2, \dots, \gamma - 1\}$, we have $\#C_z + \#S_z \geq \gamma$. On the other hand, we have $\log_p \#C + \log_p \#S = \gamma$. It follows that we have

$$\#C_z \geq \log_p \#C \quad \text{or} \quad \#S_z \geq \log_p \#S.$$

If $\#C_z \geq \log_p \#C$, we are done. If $\#S_z \geq \log_p \#S$, the arguments used in the proof (4) \Rightarrow (1) leads to $\#S_z = \log_p \#S$. So we have $\#C_z \geq \log_p \#C$.

(1) \Leftrightarrow (6): In Sections 3.3.1 and 3.3.2, we have proved the equivalence between (1) and that $\Omega = \bigsqcup_{c \in C} c + p^\gamma \mathbb{Z}_p$ is a spectral set in \mathbb{Q}_p . By Lemma 3.22, we have that (6) is equivalent to that δ_C is a spectral measure in \mathbb{Q}_p . Then what we have to prove is the following equivalence:

$$\Omega \text{ is a spectral set in } \mathbb{Q}_p \Leftrightarrow \delta_C \text{ is a spectral measure in } \mathbb{Q}_p.$$

Recall that $\mathbb{L}_\gamma = p^{-\gamma} \mathbb{L}$. It suffices to prove that

$$(\Omega, \Lambda_C + \mathbb{L}_\gamma) \text{ is a spectral pair} \Leftrightarrow (\delta_C, \Lambda_C) \text{ is a spectral pair in } \mathbb{Q}_p$$

where $\Lambda_C \subset B(0, p^\gamma)$ is some finite set, because it is known from Sections 3.3.1 and 3.3.2, that Ω has a spectrum of the form $\Lambda_C + \mathbb{L}_\gamma$ if it is a spectral set. By Lemma 3.8, (δ_C, Λ_C) is a spectral pair in \mathbb{Q}_p if and only if

$$\forall \xi \in \mathbb{Q}_p, \quad \sum_{\lambda \in \Lambda_C} \left| \frac{1}{\#C} \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2 = 1. \quad (3.4.1)$$

Recall that

$$\widehat{1}_\Omega(\lambda - \xi) = p^\gamma 1_{B(0, p^\gamma)}(\lambda - \xi) \sum_{c \in C} \chi(-c(\lambda - \xi)).$$

The equality (3.4.1) is then equivalent to

$$\begin{aligned} \forall \xi \in \mathbb{Q}_p, \quad & \sum_{\lambda \in \Lambda_C + \mathbb{L}_\gamma} |\widehat{1}_\Omega(\lambda - \xi)|^2 \\ &= p^{2\gamma} \sum_{\lambda \in \Lambda_C + \mathbb{L}_\gamma} 1_{B(0, p^\gamma)}(\lambda - \xi) \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2 \\ &= p^{2\gamma} \sum_{\lambda \in \Lambda_C + \xi'} \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2 \\ &= p^{2\gamma} \sum_{\lambda \in \Lambda_C} \left| \sum_{c \in C} \chi(-c\lambda) \right|^2 = (\#C)^2 p^{2\gamma} = |\Omega|^2, \end{aligned}$$

which means, by Lemma 3.8, that $(\Omega, \Lambda_C + \mathbb{L}_\gamma)$ is a spectral pair.

□

3.5 Uniqueness of spectra and tiling complements

In this section, we shall investigate the structure of the spectra and tiling complements of a p -homogeneous compact set. Without loss of generality, we assume that Ω is of the form

$$\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p),$$

where $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$. We immediately get that

$$I_\Omega \subset \mathbb{N} \text{ and } n \in I_\Omega \text{ if } n \geq \gamma.$$

Assume that Λ is a spectrum of Ω and T is a tiling complement of Ω . Notice that Λ and T are discrete subsets of \mathbb{Q}_p such that

$$|\lambda - \lambda'|_p > 1, \quad \text{if } \lambda, \lambda' \in \Lambda \text{ and } \lambda \neq \lambda'$$

and

$$|\tau - \tau'|_p > p^{-\gamma}, \quad \text{if } \tau, \tau' \in T \text{ and } \tau \neq \tau'.$$

Now we are going to characterize of the spectra and tiling components.

Theorem 3.24. *Let $\Omega \subset \mathbb{Q}_p$ be a p -homogeneous compact open set with the admissible p -order set I_Ω .*

(a) *The set Λ is a spectrum of Ω if and only if it is p -homogeneous discrete set with admissible p -order set $I_\Lambda = -(I_\Omega + 1)$.*

(b) *The set T is a tiling complement of Ω if and only if it is a p -homogeneous discrete set with admissible p -order set $I_T = \mathbb{Z} \setminus I_\Omega$.*

Proof. Without loss of generality, assume $\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p)$, where $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$. For an integer n , let

$$I_\Omega^{\leq n} = \{i \in I_\Omega, i \leq n\}.$$

(a) In Section 3.3.2, we have proved that $\Lambda \cap B(0, p^\gamma)$ is a p -homogeneous discrete set with admissible p -order set $I_{\Lambda \cap B(0, p^\gamma)} = -(I_\Omega^{\leq \gamma-1} + 1)$. Note that, any integer $n \geq \gamma$, the set Ω can be written as

$$\Omega = \bigsqcup_{c \in C_n} (c + p^n \mathbb{Z}_p),$$

where $C_n \subset \{0, 1, \dots, p^n - 1\}$. The same argument implies that the finite set $\Lambda \cap B(0, p^n)$ is p -homogeneous with $I_{\Lambda \cap B(0, p^n)} = -(I_\Omega^{\leq n-1} + 1)$. By Lemma 3.19 and the definition of

p -homogeneity, Λ is a p -homogeneous discrete set with the admissible p -order set $I_\Lambda = -(I_\Omega + 1)$.

In fact, it is routine to check that the equation (3.2.2) holds for any p -homogeneous discrete set Λ with $I_\Lambda = -(I_\Omega + 1)$. So the p -homogeneity of Λ and the equality $I_\Lambda = -(I_\Omega + 1)$ is sufficient for Λ being a spectrum of Ω .

(b) By Corollary 3.17 and Theorem 3.23, $T \cap \mathbb{Z}_p$ is a p -homogeneous discrete set with admissible p -order set

$$I_{T \cap \mathbb{Z}_p} = \{0, \dots, \gamma - 1\} \setminus I_\Omega^{\leq \gamma - 1}.$$

Similarly, for any $a \in \mathbb{Q}_p$, $T \cap (a + \mathbb{Z}_p)$ is a p -homogeneous discrete set with $I_{T \cap (a + \mathbb{Z}_p)} = I_{T \cap \mathbb{Z}_p}$. Since two balls of same size are either identical or disjoint, $T \cap (p^{-1}\mathbb{Z}_p)$ is a p -homogeneous discrete set with $I_{T \cap (p^{-1}\mathbb{Z}_p)} = I_{T \cap \mathbb{Z}_p} \cup \{-1\}$.

An argument by induction shows that $T \cap (p^{-n}\mathbb{Z}_p)$ is p -homogeneous with

$$I_{T \cap (p^{-n}\mathbb{Z}_p)} = I_{T \cap \mathbb{Z}_p} \cup \{-1, -2, \dots, -n\}.$$

As in (a), we get that T is a p -homogeneous discrete set with admissible p -order set $I_T = \mathbb{Z} \setminus I_\Omega$.

On the other hand, one can check that any p -homogenous discrete set T with $I_T = \mathbb{Z} \setminus I_\Omega$ is a tiling complement of Ω . \square

Proof of Theorem 3.2. In the proof of Theorem 3.1, we have constructed a spectrum $\Lambda = \sum_{i \in I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^{-i-1}$ for Ω and a tiling complement $T = \sum_{i \notin I_\Omega} \mathbb{Z}/p\mathbb{Z} \cdot p^i$ for Ω . Therefore, this theorem is an immediate consequence of Theorem 3.24, Lemma 3.20 and Proposition 3.21. \square

Let us finish this section by geometrically presenting the canonical spectrum and the canonical tiling complement of a compact open spectral set. Assume that $\Omega = \bigsqcup_{c \in C} (c + p^\gamma \mathbb{Z}_p)$ with $\gamma \geq 1$ is an integer and $C \subset \{0, 1, \dots, p^\gamma - 1\}$. Notice that $n \in I_\Omega$ if $n \geq \gamma$. Set $\Lambda_\gamma = \Lambda \cap B(0, p^\gamma)$ and $T_0 = T \cap B(0, 1)$. Then $\Lambda = \Lambda_\gamma \oplus \mathbb{L}_\gamma$ and $T = T_0 \oplus \mathbb{L}$. The sets Λ_γ and T_0 are p -homogeneous. If we consider $p^\gamma \Lambda_\gamma$ and T_0 as subsets of $\mathbb{Z}/p^\gamma \mathbb{Z}$, they will determine two subtrees of $\mathcal{T}^{(\gamma)}$. The following example show the relations among Ω , Λ_γ and T_0 .

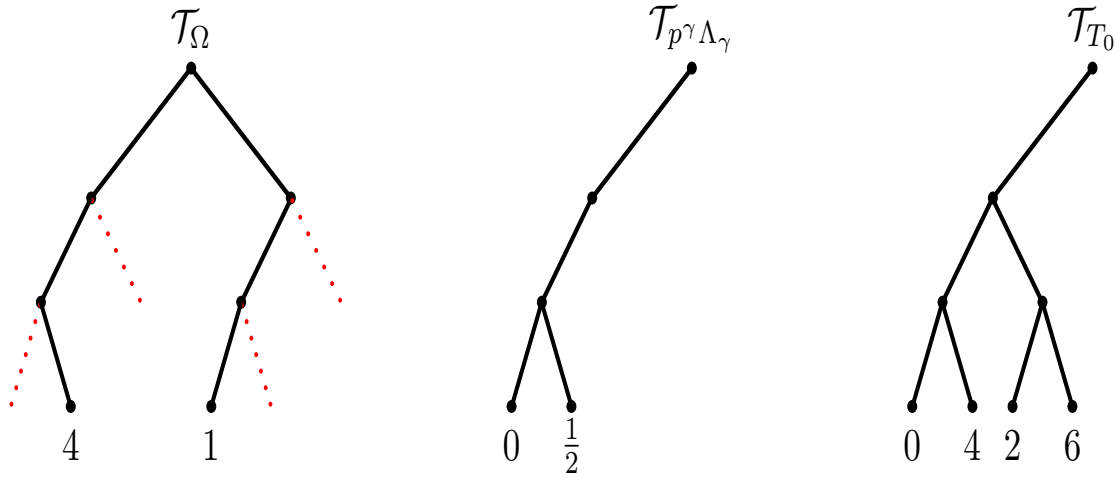


Figure 3.6: The left is the tree determined by $\Omega = (1 + 8\mathbb{Z}_2) \cup (4 + 8\mathbb{Z}_2)$; the middle is the tree determined by $\Lambda_\gamma = \{0, 1/2\}$; the right is the tree determined by $T_0 = \{0, 2, 4, 6\}$.

3.6 Finite spectral sets in \mathbb{Q}_p

The following theorem characterizes the uniform probability measures supported on some finite sets $C \subset \mathbb{Q}_p$ and it gives more information than Theorem 3.4. As we shall see, the measure δ_C is a spectral measure if and only if C is represented by an infinite p -homogeneous tree for which, from some level on, each parent has only one son. Recall that

$$\gamma_C = \max_{\substack{c, c' \in C \\ c \neq c'}} v_p(c - c').$$

Theorem 3.25. *The following are equivalent.*

- (1) *The measure δ_C is a spectral measure.*
- (2) *For each integer $\gamma > \gamma_C$, $\Omega_\gamma := \bigsqcup_{c \in C} B(c, p^{-\gamma})$ is a spectral set.*
- (3) *For some integer $\gamma_0 > \gamma_C$, $\Omega_{\gamma_0} := \bigsqcup_{c \in C} B(c, p^{-\gamma_0})$ is a spectral set.*
- (4) *For any integer $\gamma \in \mathbb{Z}$, $\Omega_\gamma := \bigcup_{c \in C} B(c, p^{-\gamma})$ is a spectral set.*

Proof. Without loss of generality, we assume that $C \subset \mathbb{Z}_p$, so that $\gamma_C \geq 0$. Recall that for any integer $\gamma \in \mathbb{Z}$, \mathbb{L}_γ denotes a complete set of representatives of the cosets of the subgroup $B(0, p^\gamma)$ of \mathbb{Q}_p .

(1) \Rightarrow (2): Fix $\gamma > \gamma_C$. Assume that Λ_C is a spectrum of δ_C , which means by Lemma 2.2 that

$$\forall \xi \in \mathbb{Q}_p, \quad \sum_{\lambda \in \Lambda_C} \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2 = (\#C)^2. \quad (3.6.1)$$

Observe that we can assume $\Lambda_C \subset B(0, p^\gamma)$. We assume $0 \in \Lambda_C$. Let λ be an arbitrary point in Λ_C , different from 0. The orthogonality of χ_0 and χ_λ is nothing but

$$\sum_{c \in C} \chi(\lambda c) = 0.$$

Apply Lemma 3.9, we get that $|\lambda|_p \leq p^{\gamma_C+1}$. We conclude that $\Lambda_C \subset B(0, p^{\gamma_C+1}) \subset B(0, p^\gamma)$ for all $\gamma > \gamma_C$.

Now we check that $(\Omega_\gamma, \Lambda_C + \mathbb{L}_\gamma)$ is a spectral pair. Recall that

$$\forall \zeta \in \mathbb{Q}_p, \quad \widehat{1_{\Omega_\gamma}}(\zeta) = p^{-2\gamma} 1_{B(0, p^\gamma)}(\zeta) \sum_{c \in C} \chi(-c\zeta). \quad (3.6.2)$$

Fix $\xi \in \mathbb{Q}_p$. By (3.6.2), we have

$$\sum_{\lambda \in \Lambda_C + \mathbb{L}_\gamma} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = p^{-2\gamma} \sum_{\lambda \in \Lambda_C + \mathbb{L}_\gamma} 1_{B(0, p^\gamma)}(\lambda - \xi) \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2$$

Since $\Lambda_C \subset B(0, p^\gamma)$, we have $B(\xi, p^\gamma) \cap (\Lambda_C + \mathbb{L}_\gamma) = \Lambda_C + \ell_\xi$ where ℓ_ξ is the unique point contained in $B(\xi, p^\gamma) \cap \mathbb{L}_\gamma$. Thus

$$\begin{aligned} \sum_{\lambda \in \Lambda_C + \mathbb{L}_\gamma} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 &= p^{-2\gamma} \sum_{\lambda \in \Lambda_C + \ell_\xi} \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2 \\ &= p^{-2\gamma} (\#C)^2 = |\Omega_\gamma|^2 \end{aligned}$$

where the second equality is a consequence of the criterion(3.6.1) and of the fact that $(\delta_C, \Lambda_C + \ell_\xi)$ is also a spectral pair. This means, by Lemma 2.2, that $(\Omega_\gamma, \Lambda_C + \mathbb{L}_\gamma)$ is a spectral pair.

(2) \Rightarrow (3): Obviously.

(3) \Rightarrow (4): Without loss of generality, we assume that $C \subset \mathbb{Z}_p$. If $\gamma \leq 0$, Ω_γ is equal to $p^{-\gamma}\mathbb{Z}_p$ which is spectral. If $1 \leq \gamma \leq \gamma_0$, Ω_γ is spectral directly by the hypothesis and Theorem 1.1. Observe that $\#(C_{\text{mod } p^\gamma}) = \#C$ for $\gamma > \gamma_C$. Therefore, if $\gamma > \gamma_0 > \gamma_C$, $C_{\text{mod } p^\gamma}$ is p -homogeneous, so that Ω_γ is a spectral set.

(4) \Rightarrow (1): For any $\xi \in \mathbb{Q}_p$, there exists an integer $\gamma > \gamma_C$ such that $\xi \in B(0, p^\gamma)$. Fix this γ depending on ξ . By the hypothesis, Ω_γ is a spectral set. Assume that Λ_γ is a spectrum of Ω_γ . That is to say

$$\forall \zeta \in \mathbb{Q}_p, \quad \sum_{\lambda \in \Lambda_\gamma} |\widehat{1_{\Omega_\gamma}}(\lambda - \zeta)|^2 = |\Omega_\gamma|^2.$$

We can assume that Λ_γ has the form $\Lambda_C + \mathbb{L}_\gamma$ (see Theorem 3.2), where $\Lambda_C \subset B(0, p^{\gamma_C+1})$.

By (3.6.2), we have

$$\begin{aligned} |\Omega_\gamma|^2 &= \sum_{\lambda \in \Lambda_\gamma} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 \\ &= p^{-2\gamma} \sum_{\lambda \in \Lambda_C + \mathbb{L}_\gamma} 1_{B(0, p^{-\gamma})}(\lambda - \xi) \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2 \\ &= p^{-2\gamma} \sum_{\lambda \in \Lambda_C} \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2. \end{aligned}$$

Since $|\Omega_\gamma|^2 = (\#C)^2 p^{-2\gamma}$, we get

$$\forall \xi \in \mathbb{Q}_p, \quad (\#C)^2 = \sum_{\lambda \in \Lambda_C} \left| \sum_{c \in C} \chi(-c(\lambda - \xi)) \right|^2.$$

This means that the measure δ_C is a spectral measure by Lemma 2.2. \square

3.7 Singular Spectral Measures

In this section, we shall construct a class of singular spectral measures. Let I, J be two disjoint infinite subsets of \mathbb{N} such that

$$I \sqcup J = \mathbb{N}.$$

For any non-negative integer γ , let $I_\gamma = I \cap \{0, 1, \dots, \gamma-1\}$ and $J_\gamma = J \cap \{0, 1, \dots, \gamma-1\}$. Let $C_{I_\gamma, J_\gamma} \subset \mathbb{Z}/p^\gamma \mathbb{Z}$ be p -homogeneous subsets corresponding to a $\mathcal{T}_{I_\gamma, J_\gamma}$ form tree as described in Section 3.2.6. Considering C_{I_γ, J_γ} as a subset of \mathbb{Z}_p , let

$$\Omega_\gamma = \bigsqcup_{c \in C_{I_\gamma, J_\gamma}} (c + p^\gamma \mathbb{Z}_p), \quad \gamma = 0, 1, 2, \dots$$

be a nested sequence of compact open sets, i.e. $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \dots$. It is obvious that the measures $\frac{1}{|\Omega_\gamma|} \mathbf{m}|_{\Omega_\gamma}$ weakly converge to a singular measure $\mu_{I, J}$ as $\gamma \rightarrow \infty$. The measure $\mu_{I, J}$ is supported on a p -homogeneous, Cantor-like set of measure 0, and the measure of an open ball with respect to $\mu_{I, J}$ is just the ‘‘proportion’’ of this support inside the ball. All this could be well defined; the proofs are simple exercises of measure theory, in particular, applications of the Portmanteau Theorem. We should remark that $\mu_{I, J}$ depends not only on I and J but also on the choice of C_{I_γ, J_γ} . Actually, the choice of C_{I_γ, J_γ} implies that the average Dirac measures $\delta_{C_{I_\gamma, J_\gamma}}$ also converge to $\mu_{I, J}$ as $\gamma \rightarrow \infty$.

Theorem 3.26. *Under the above assumption, $\mu_{I, J}$ is a spectral measure with the following set*

$$\Lambda = \left\{ \sum_{i \in I} b_i p^{-i-1} : b_i \in \{0, 1, \dots, p-1\} \right\}$$

as a spectrum.

Proof. What we have to prove is the equality (3.2.2) for the pair $(\mu_{I,J}, \Lambda)$. Since $\mu_{I,J}$ is the weak limit of $\frac{1}{|\Omega_\gamma|} \mathbf{m}|_{\Omega_\gamma}$ as $\gamma \rightarrow \infty$, we have

$$\forall \xi \in \mathbb{Q}_p, \quad \widehat{\mu_{I,J}}(\xi) = \lim_{\gamma \rightarrow \infty} \frac{1}{|\Omega_\gamma|} \cdot \widehat{1_{\Omega_\gamma}}(\xi).$$

For each integer $\gamma \geq 0$, let $\Lambda_\gamma = \Lambda \cap B(0, p^\gamma)$. Note that Ω_γ is a spectral set with spectrum $\Lambda_\gamma + \mathbb{L}_\gamma$ by Theorem 3.2. Then (3.2.3) gives the following equality

$$\forall \xi \in \mathbb{Q}_p, \quad \sum_{\lambda \in \Lambda} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = \sum_{\lambda \in \Lambda_\gamma} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = |\Omega_\gamma|^2.$$

By Fatou's Lemma, we get that

$$\sum_{\lambda \in \Lambda} |\widehat{\mu_{I,J}}(\lambda - \xi)|^2 \leq \lim_{\gamma \rightarrow \infty} \frac{1}{|\Omega_\gamma|^2} \sum_{\lambda \in \Lambda} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = 1 \quad (3.7.1)$$

for all $\xi \in \mathbb{Q}_p$. Now, for any positive integer γ_0 , we shall show that

$$\sum_{\lambda \in \Lambda} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 \geq 1_{B(0, p^{\gamma_0})}(\xi), \quad \text{for all integers } \gamma \geq \gamma_0.$$

Recall that

$$|\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = p^{-2\gamma} 1_{B(0, p^\gamma)}(\lambda - \xi) \sum_{c, c' \in C_{I,J}} \chi((c - c')(\lambda - \xi)).$$

For any $\xi \in B(0, p^{\gamma_0})$, observe that

$$\forall \lambda \in \Lambda_{\gamma_0}, \quad \chi((c - c')(\lambda - \xi)) = 1 \text{ if } |c - c'|_p \leq p^{-\gamma_0}$$

and

$$\sum_{\lambda \in \Lambda_{\gamma_0}} \chi((c - c')(\lambda - \xi)) = 0 \text{ if } |c - c'|_p > p^{-\gamma_0}.$$

For integer $\gamma \geq \gamma_0$, by calculation, there are $p^{2\sharp(I_\gamma \setminus I_{\gamma_0})} p^{\sharp I_{\gamma_0}}$ pairs $(c, c') \in C_{I_\gamma, J_\gamma} \times C_{I_\gamma, J_\gamma}$ with $|c - c'|_p \leq p^{-\gamma_0}$. So we get that

$$\sum_{\lambda \in \Lambda_{\gamma_0}} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = p^{-2(\gamma - \sharp I_\gamma)} = |\Omega_\gamma|^2 \quad \forall \xi \in B(0, p^{\gamma_0}).$$

Thus, we have

$$\lim_{\gamma \rightarrow \infty} \frac{1}{|\Omega_\gamma|^2} \sum_{\lambda \in \Lambda_{\gamma_0}} |\widehat{1_{\Omega_\gamma}}(\lambda - \xi)|^2 = 1, \quad \forall \xi \in B(0, p^{\gamma_0}),$$

or in other words,

$$\sum_{\lambda \in \Lambda_0} |\widehat{\mu_{I,J}}(\lambda - \xi)|^2 = 1, \quad \forall \xi \in B(0, p^{\gamma_0}).$$

Note that

$$1 \geq \sum_{\lambda \in \Lambda} |\widehat{\mu_{I,J}}(\lambda - \xi)|^2 \geq \sum_{\lambda \in \Lambda_0} |\widehat{\mu_{I,J}}(\lambda - \xi)|^2$$

by (3.7.1). Since γ_0 could be arbitrarily large, we have

$$\forall \xi \in \mathbb{Q}_p, \quad \sum_{\lambda \in \Lambda} |\widehat{\mu_{I,J}}(\lambda - \xi)|^2 = 1.$$

□

Assume I_γ and J_γ form a partition of $\{0, 1, \dots, \gamma - 1\}$ ($\gamma \geq 1$) such that C_{I_γ, J_γ} is a p -homogeneous tree. Then let

$$I = \bigcup_{n=0}^{\infty} (n\gamma + I_\gamma), \quad J = \bigcup_{n=0}^{\infty} (n\gamma + J_\gamma).$$

The measure constructed above in this special case is a self-similar measure generated by the following iterated function system:

$$f_c(x) = p^\gamma x + c \quad (c \in C := C_{I_\gamma, J_\gamma}).$$

Let us consider two concrete examples.

Example 1. Let $p = 2$, $\gamma = 3$ and $C = \{0, 3, 4, 7\}$. Then

$$f_0(x) = 8x, \quad f_3(x) = 8x + 3, \quad f_4(x) = 8x + 4, \quad f_7(x) = 8x + 7.$$

Observe that the tree structure of $\{0, 3, 4, 7\}$ is shown as follows

$$\begin{aligned} 0 &= 0 \cdot 1 + 0 \cdot 2 + 0 \cdot 2^2, & 3 &= 1 \cdot 1 + 1 \cdot 2 + 0 \cdot 2^2 \\ 4 &= 0 \cdot 1 + 0 \cdot 2 + 1 \cdot 2^2, & 7 &= 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 2^2. \end{aligned}$$

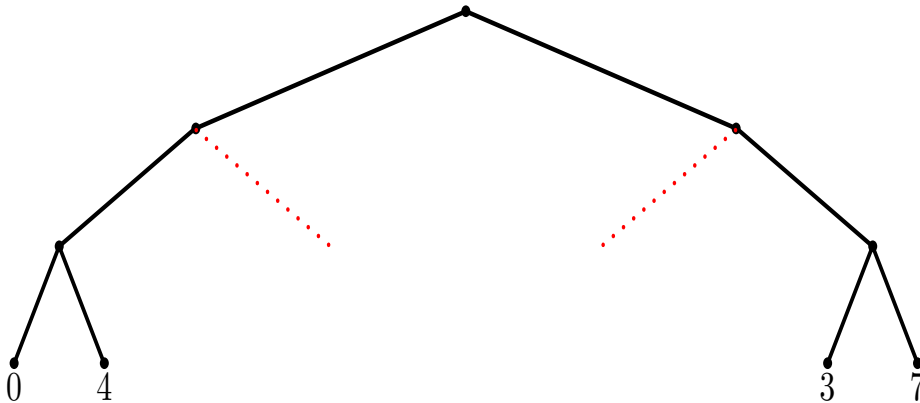


Figure 3.7: Consider $\{0, 3, 4, 7\}$ as a p -homogeneous tree.

Example 2. Let $p = 3$, $\gamma = 3$ and $C = \{0, 4, 8, 9, 13, 17, 18, 22, 26\}$. We have

$$\begin{aligned} 0 &= 0 \cdot 1 + 0 \cdot 3 + 0 \cdot 3^2, & 4 &= 1 \cdot 1 + 1 \cdot 3 + 0 \cdot 3^2, & 8 &= 2 \cdot 1 + 2 \cdot 3 + 0 \cdot 3^2 \\ 9 &= 0 \cdot 1 + 0 \cdot 3 + 1 \cdot 3^2, & 13 &= 1 \cdot 1 + 1 \cdot 3 + 1 \cdot 3^2, & 17 &= 2 \cdot 1 + 2 \cdot 3 + 1 \cdot 3^2 \\ 18 &= 0 \cdot 1 + 0 \cdot 3 + 2 \cdot 3^2, & 22 &= 1 \cdot 1 + 1 \cdot 3 + 2 \cdot 3^2, & 26 &= 2 \cdot 1 + 2 \cdot 3 + 2 \cdot 3^2. \end{aligned}$$

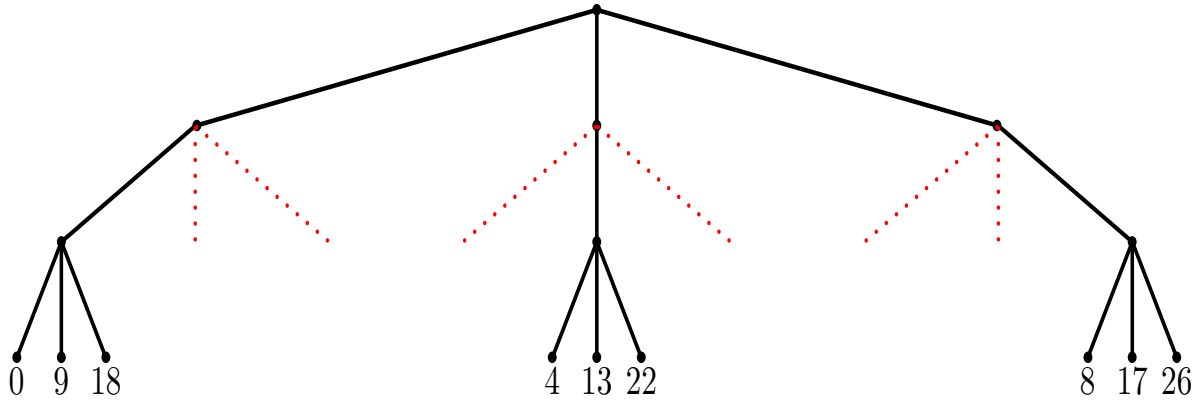


Figure 3.8: Consider $\{0, 4, 8, 9, 13, 17, 18, 22, 26\}$ as a p -homogeneous tree.

Chapter 4

Fuglede's conjecture holds in \mathbb{Q}_p

Abstract. *We prove Fuglede's conjecture in \mathbb{Q}_p which states that a Borel set of positive and finite Haar measure in \mathbb{Q}_p is a spectral set if and only if it tiles \mathbb{Q}_p by translation.*

4.1 Introduction

Let G be a locally compact abelian group and \widehat{G} its dual group. Denote by \mathbf{m} (sometimes by dx) the Haar measure on G . Consider a Borel measurable subset Ω in G with $0 < \mathbf{m}(\Omega) < \infty$. We say that Ω is a *tile* of G by translation if there exists a set $T \subset G$ of translates such that $\sum_{t \in T} 1_\Omega(x - t) = 1$ for almost all $x \in G$, where 1_A denotes the indicator function of a set A . The set T is then called a *tiling complement* of Ω and (Ω, T) is called a *tiling pair*.

In the case of $G = \mathbb{R}$ considered as an additive group, compact sets of positive measure that tile \mathbb{R} by translation were extensively studied [1, 97, 92]. The simplest case concerns compact sets consisting of finite number of unit intervals all of whose endpoints are integers. This tiling problem can be reformulated in terms of finite subsets of \mathbb{Z} which tile the group \mathbb{Z} . See [122] for references on the study of tiling problem in the group \mathbb{Z} . There are also investigations on the existence of tiles having infinitely many connected components. A large class of such tiles arises from self-similar constructions by Bandt [9], Gröchenig and Haas [50], Kenyon [74], Lagarias and Wang [91] et al. A structure theorem for bounded tiles in \mathbb{R} is obtained by Lagarias and Wang in [92] where it is proved that all tilings of \mathbb{R} by a bounded region (compact set with zero boundary measure) must be periodic, and that the corresponding tiling complements are rational up to affine transformations.

As we shall see, the tiles in the field \mathbb{Q}_p of p -adic numbers have relatively simple structures. By an *almost compact open set* we mean a Borel set $\Omega \subset \mathbb{Q}_p$ such that there exists a compact open set Ω' satisfying

$$\mathbf{m}(\Omega \setminus \Omega') = \mathbf{m}(\Omega' \setminus \Omega) = 0.$$

This is the paper [36] *Fuglede's conjecture holds in \mathbb{Q}_p* in collaboration with A. H. Fan, S. L. Fan and L. M. Liao. The arXiv version can be found in <https://arxiv.org/abs/1512.08904>.

One of our main results, stated below, concerns with the structure of tiles in \mathbb{Q}_p .

Theorem 4.1. *Assume that $\Omega \subset \mathbb{Q}_p$ is a Borel set of positive and finite Haar measure. If Ω tiles \mathbb{Q}_p by translation, then it is an almost compact open set.*

In [35], it is proved that any *bounded* tile of \mathbb{Q}_p is an almost compact open set. The proof in [35] is different from the one in the present paper. Without using the theory of distribution, that proof is more direct and easily understandable. But the boundedness is assumed as an extra condition.

We say that a Borel set $\Omega \subset G$ of positive and finite Haar measure is a *spectral set* if there exists a set $\Lambda \subset \widehat{G}$ which is an orthonormal basis of the Hilbert space $L^2(\Omega)$. Such a set Λ is called a *spectrum* of Ω and (Ω, Λ) is called a *spectral pair*.

When $G = \mathbb{R}^n$, Fuglede [31] formulated the following conjecture: *A Borel set $\Omega \subset \mathbb{R}^n$ of positive and finite Lebesgue measure is a spectral set if and only if it is a tile.* Fuglede [31] proved the conjecture in \mathbb{R}^d under the extra assumption that the spectrum or the tiling complement is a lattice of \mathbb{R}^d . There are many positive results under different extra assumptions before the work [121] where Tao gave a counterexample: there exists a spectral subset of \mathbb{R}^d with $d \geq 5$ which is not a tile. After that, Matolcsi [?], Matolcsi and Kolountzakis [83, 84], Farkas and Gy [29], Farkas, Matolcsi and Móra [39] gave a series of counterexamples which show that both directions of Fuglede's conjecture fail in $\mathbb{R}^d (d \geq 3)$. However, the conjecture is still open in low dimensions $d = 1, 2$.

The Fuglede's conjecture can be generalized to locally compact abelian groups: *A Borel set $\Omega \subset G$ of positive and finite Haar measure is a spectral set if and only if it is a tile.* In its generality, this generalized conjecture is not true. We could rather ask for which groups it holds. The question arises even for finite groups. The counterexamples in $\mathbb{R}^d, d \geq 3$ are actually constructed, basing on counterexamples in finite groups.

Let $p \geq 2$ be prime and \mathbb{Q}_p be the field of p -adic numbers. In the present paper, we will prove that Fuglede's conjecture in \mathbb{Q}_p holds.

Theorem 4.2. *Assume that $\Omega \subset \mathbb{Q}_p$ is a Borel set of positive and finite Haar measure. Then Ω is a spectral set if and only if it is a tile of \mathbb{Q}_p .*

In [37], the assertion in Theorem 4.2 was proved under the additional assumption that Ω is a compact open set in \mathbb{Q}_p , and furthermore, the compact open spectral sets were characterized by their p -homogeneity (see the definition of p -homogeneity in Section 1 of [37]).

Theorem 4.3 ([37], Theorem 1.1). *Let Ω be a compact open set in \mathbb{Q}_p . The following statements are equivalent:*

- (1) Ω is a spectral set;
- (2) Ω tiles \mathbb{Q}_p ;
- (3) Ω is p -homogenous.

Theorem 4.1 and 4.3 are two main steps towards the proof of Fuglede's conjecture in \mathbb{Q}_p . Actually, by Theorem 4.1 and 4.3, to prove Theorem 4.2, it suffices to prove “ Ω is a spectral set $\implies \Omega$ is a tile.”.

As pointed out in Proposition 3.1 of [30], which follows from [64], that (Ω, Λ) is a spectral pair in \mathbb{Q}_p is equivalent to

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \sum_{\lambda \in \Lambda} |\widehat{1_\Omega}|^2(\xi - \lambda) = \mathbf{m}(\Omega)^2,^1 \quad (4.1.1)$$

where $\widehat{1_\Omega}$ is the Fourier transform of 1_Ω . By definition, (Ω, T) is a tiling pair of \mathbb{Q}_p means that

$$\sum_{t \in T} 1_\Omega(x - t) = 1, \quad \mathbf{m}\text{-a.e. } x \in \mathbb{Q}_p \quad (4.1.2)$$

A subset E of \mathbb{Q}_p is said to be *uniformly discrete* if E is countable and $\inf_{x, y \in E} |x - y|_p > 0$, where $|\cdot|_p$ denotes the p -adic absolute value on \mathbb{Q}_p . Remark that if E is uniformly discrete, then $\text{Card}(E \cap K) < \infty$ for any compact subset K of \mathbb{Q}_p so that

$$\mu_E = \sum_{\lambda \in E} \delta_\lambda \quad (4.1.3)$$

defines a discrete Radon measure. Observe that both (4.1.1) and (4.1.2) are of the form

$$\mu_E * f = w, \quad (4.1.4)$$

where f is a non-negative integrable function and $w > 0$ is a positive number. Actually, both spectrum Λ and tiling complement T in \mathbb{Q}_p are uniformly discrete (see Proposition 4.6). The above convolution equation (4.1.4) will be our main concern.

Our proof of Theorem 4.1 and 4.2 will be based on the analysis of the set of zeros of the Fourier transforms $\widehat{\mu_\Lambda}$ and $\widehat{\mu_T}$, where both μ_Λ and μ_T are considered as Bruhat-Schwartz distributions and their Fourier transforms $\widehat{\mu_\Lambda}$ and $\widehat{\mu_T}$ are also Bruhat-Schwartz distributions.

The article is organized as follows. In Section 2, we present preliminaries on the field \mathbb{Q}_p of p -adic numbers and on the \mathbb{Z} -module generated by the p^n -th roots of unity. Some useful facts from the theory of Bruhat-Schwartz distributions are also presented. It is proved that a particular structure is shared by the set of zeros of $\widehat{\mu_E}$ for all uniform discrete sets E . In Section 3, we study the functional equation $\mu_E * f = 1$ under the assumption that E is uniformly discrete and f is a non null and non-negative integrable function. Such a relation has strong constraints on f and on E . For example, we prove that \widehat{f} has compact support. Section 4 is devoted to the proof that tiles in \mathbb{Q}_p are all almost compact open and spectral sets in \mathbb{Q}_p are all almost bounded. It is then proved that tiles in \mathbb{Q}_p are spectral sets. The proof that spectral sets in \mathbb{Q}_p are tiles is given in Section 5. In the last section, we prove the dual Fuglede's conjecture in \mathbb{Q}_p and give some remarks on the case of higher dimensions.

¹The equivalent condition of spectral sets in locally compact abelian groups can be found in Theorem 2.19.

4.2 Preliminaries

Our study on the Fuglede's conjecture on \mathbb{Q}_p is strongly related to the following functional equation

$$\mu_E * f = 1 \tag{4.2.1}$$

where $E \subset \mathbb{Q}_p$ is a uniformly discrete set and $f \in L^1(\mathbb{Q}_p)$ is a non-negative integrable function such that $\int_{\mathbb{Q}_p} f d\mathbf{m} > 0$.

The convolution in (4.2.1) is understood as a convolution of Bruhat-Schwartz distributions and even as a convolution in the Colombeau algebra of generalized functions. One of reasons is that the Fourier transform of the infinite Radon measure μ_E is not defined for the measure μ_E but for the distribution μ_E . Both spectral set and tile are characterized by special cases of the equation (4.2.1). In fact, (4.1.1) means that a spectral pair (Ω, Λ) is characterized by (4.2.1) with $f = \mathbf{m}(\Omega)^{-2} |\widehat{1_\Omega}|^2$ and $E = \Lambda$; (4.1.2) means that a tiling pair (Ω, T) is characterized by (4.2.1) with $f = 1_\Omega$ and $E = T$.

In this section, after having presented some basic facts like the Lebesgue density theorem, the uniform discreteness of spectrum and of tiling complement etc, we will recall some result from [114] on the \mathbb{Z} -module generated by p^n -th roots of unity, which is a key for our study. Then we will present some useful facts from the theory of Bruhat-Schwartz distributions and from the theory of the Colombeau algebra of generalized functions. At the end, we will investigate the set of zeros of the Fourier transform $\widehat{\mu_E}$.

4.2.1 The field of p -adic number \mathbb{Q}_p

We start with a quick recall of p -adic numbers. Consider the field \mathbb{Q} of rational numbers and a prime $p \geq 2$. Any nonzero number $r \in \mathbb{Q}$ can be written as $r = p^v \frac{a}{b}$ where $v, a, b \in \mathbb{Z}$ and $(p, a) = 1$ and $(p, b) = 1$ (here (x, y) denotes the greatest common divisor of the two integers x and y). We define $|r|_p = p^{-vp(r)}$ for $r \neq 0$ and $|0|_p = 0$. Then $|\cdot|_p$ is a non-Archimedean absolute value. That means

- (i) $|r|_p \geq 0$ with equality only when $r = 0$;
- (ii) $|rs|_p = |r|_p |s|_p$;
- (iii) $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.

The field \mathbb{Q}_p of p -adic numbers is the completion of \mathbb{Q} under $|\cdot|_p$. The ring \mathbb{Z}_p of p -adic integers is the set of p -adic numbers with absolute value ≤ 1 . A typical element x of \mathbb{Q}_p is of the form

$$x = \sum_{n=v}^{\infty} a_n p^n \quad (v \in \mathbb{Z}, a_n \in \{0, 1, \dots, p-1\} \text{ and } a_v \neq 0). \tag{4.2.2}$$

Here, $v_p(x) := v$ is called the p -valuation of x .

A non-trivial additive character on \mathbb{Q}_p is defined by

$$\chi(x) = e^{2\pi i \{x\}}$$

where $\{x\} = \sum_{n=v_p(x)}^{-1} a_n p^n$ is the fractional part of x in (4.2.2). From this character we can get all characters χ_ξ of \mathbb{Q}_p , by defining $\chi_\xi(x) = \chi(\xi x)$. We remark that

$$\chi(x) = e^{2\pi i k/p^n}, \quad \text{if } x \in \frac{k}{p^n} + \mathbb{Z}_p \quad (k, n \in \mathbb{Z}), \quad (4.2.3)$$

and

$$\int_{p^{-n}\mathbb{Z}_p} \chi(x) dx = 0 \quad \text{for all } n \geq 1. \quad (4.2.4)$$

The map $\xi \mapsto \chi_\xi$ from \mathbb{Q}_p to $\widehat{\mathbb{Q}}_p$ is an isomorphism. We write $\widehat{\mathbb{Q}}_p \simeq \mathbb{Q}_p$ and identify a point $\xi \in \mathbb{Q}_p$ with the point $\chi_\xi \in \widehat{\mathbb{Q}}_p$. For more information on \mathbb{Q}_p and $\widehat{\mathbb{Q}}_p$, the reader is referred to the book [129].

The following notation will be used in the whole paper.

Notation:

- $\mathbb{Z}_p^\times := \mathbb{Z}_p \setminus p\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p = 1\}$, the group of units of \mathbb{Z}_p .
- $B(0, p^n) := p^{-n}\mathbb{Z}_p$, the (closed) ball centered at 0 of radius p^n .
- $B(x, p^n) := x + B(0, p^n)$.
- $S(x, p^n) := B(x, p^n) \setminus B(x, p^{n-1})$, a ‘‘sphere’’.
- $\mathbb{L} := \{\{x\} : x \in \mathbb{Q}_p\}$, a complete set of representatives of the cosets of the additive subgroup \mathbb{Z}_p .
- $\mathbb{L}_n := p^{-n}\mathbb{L}$.

The Lebesgue density theorem holds in \mathbb{Q}_p for the Haar measure.

Proposition 4.4 ([104]). *Let $\Omega \subset \mathbb{Q}_p$ be a bounded Borel set such that $\mathfrak{m}(\Omega) > 0$. Then*

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{m}(B(x, p^{-n}) \cap \Omega)}{\mathfrak{m}(B(x, p^{-n}))} = 1, \quad \mathfrak{m}\text{-a.e. } x \in \Omega.$$

4.2.2 Fourier Transformation

The Fourier transformation of $f \in L^1(\mathbb{Q}_p)$ is defined to be

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p} f(x) \overline{\chi_\xi(x)} dx \quad (\forall \xi \in \widehat{\mathbb{Q}}_p \simeq \mathbb{Q}_p).$$

A complex function f defined on \mathbb{Q}_p is called *uniformly locally constant* if there exists $n \in \mathbb{Z}$ such that

$$f(x+u) = f(x) \quad \forall x \in \mathbb{Q}_p, \forall u \in B(0, p^n).$$

The following proposition shows that for an integrable function f , having compact support and being uniformly locally constant are dual properties for f and its Fourier transform.

Proposition 4.5. *Let $f \in L^1(\mathbb{Q}_p)$ be a complex-value integrable function.*

- (1) *If f has compact support, then \widehat{f} is uniformly locally constant.*
- (2) *If f is uniformly locally constant, then \widehat{f} has compact support.*

Proof. (1) Suppose that f is supported by $B(0, p^n)$. For any $x \in \mathbb{Q}_p$ and any $u \in B(0, p^{-n})$, we have

$$\widehat{f}(\xi + u) - \widehat{f}(\xi) = \int_{B(0, p^n)} f(y) \overline{\chi(\xi y)} (\chi(uy) - 1) dy.$$

Notice that if $y \in B(0, p^n)$, we have $|uy|_p \leq 1$. So by (4.2.3),

$$\chi(uy) - 1 = 0.$$

Therefore, $\widehat{f}(\xi + u) - \widehat{f}(\xi) = 0$ for all $u \in B(0, p^{-n})$. Thus \widehat{f} is uniformly locally constant.

- (2) Suppose that $f(x + u) = f(x)$ for any $x \in \mathbb{Q}_p$ and any $u \in B(0, p^n)$. Observing

$$\mathbb{Q}_p = \mathbb{L}_n + B(0, p^n) = \bigcup_{z \in \mathbb{L}_n} B(z, p^n),$$

we deduce

$$\widehat{f}(\xi) = \sum_{z \in \mathbb{L}_n} \int_{B(z, p^n)} f(y) \overline{\chi(\xi y)} dy = \sum_{z \in \mathbb{L}_n} f(z) \int_{B(z, p^n)} \overline{\chi(\xi y)} dy.$$

By (4.2.4), we have $\int_{B(z, p^n)} \chi(\xi y) dy = 0$ for $|\xi|_p > p^{-n}$. Therefore, $\widehat{f}(\xi) = 0$ for $|\xi|_p > p^{-n}$. \square

The above proposition is the key which will allow us to prove that a tile Ω in \mathbb{Q}_p is an almost compact open set by showing that the support of $\widehat{1_\Omega}$ is compact (see Proposition 4.22). Consequently, a tile is a spectral set by Theorem 4.3.

Now we prove that the spectra and tiling complements are always uniformly discrete.

Proposition 4.6. *Let $\Omega \subset \mathbb{Q}_p$ be a Borel set of positive and finite Haar measure.*

- (1) *If (Ω, Λ) is a spectral pair, then Λ is uniformly discrete.*
- (2) *If (Ω, T) is a tiling pair, then T is uniformly discrete.*

Proof. (1) By the fact $\widehat{1_\Omega}(0) = \mathbf{m}(\Omega) > 0$ and the continuity of the function $\widehat{1_\Omega}$, there exists an integer n_0 such that $\widehat{1_\Omega}(x) \neq 0$ for all $x \in B(0, p^{n_0})$. This, together with the orthogonality

$$\widehat{1_\Omega}(\lambda - \lambda') = 0 \quad \forall \lambda, \lambda' \in \Lambda \text{ distinct},$$

implies that $|\lambda - \lambda'|_p > p^{n_0}$ for different $\lambda, \lambda' \in \Lambda$.

- (2) Consider the continuous function on \mathbb{Q}_p defined by

$$f(x) := 1_\Omega * 1_{-\Omega}(x) = \int_{\mathbb{Q}_p} 1_\Omega(y) 1_\Omega(y - x) dy = \mathbf{m}(\Omega \cap (\Omega + x)).$$

The fact $f(0) = \mathbf{m}(\Omega) > 0$ and the continuity of f imply that there exists an integer n_0 such that

$$\mathbf{m}(\Omega \cap (\Omega + t)) > 0, \quad \text{for } t \in B(0, p^{n_0}). \quad (4.2.5)$$

For different $t \in T$ and $t' \in T$, the tiling property implies $\mathbf{m}((\Omega + t) \cap (\Omega + t')) = 0$. So

$$\mathbf{m}(\Omega \cap (\Omega + t - t')) = \mathbf{m}((\Omega + t) \cap (\Omega + t')) = 0$$

by the translation invariance of \mathbf{m} . Thus we must have $|t - t'|_p > p^{n_0}$, by (4.2.5). \square

4.2.3 \mathbb{Z} -module generated by p^n -th roots of unity

Let $m \geq 2$ be an integer and let $\omega_m = e^{2\pi i/m}$, which is a primitive m -th root of unity. Denote by \mathcal{M}_m the set of integral points $(a_0, a_1, \dots, a_{m-1}) \in \mathbb{Z}^m$ such that

$$\sum_{j=0}^{m-1} a_j \omega_m^j = 0.$$

The set \mathcal{M}_m is clearly a \mathbb{Z} -module. In the following we assume that $m = p^n$ is a power of a prime number.

Lemma 4.7 ([114], Theorem 1). *If $(a_0, a_1, \dots, a_{p^n-1}) \in \mathcal{M}_{p^n}$, then for any integer $0 \leq i \leq p^{n-1} - 1$ we have $a_i = a_{i+jp^{n-1}}$ for all $j = 0, 1, \dots, p-1$.*

Lemma 4.7 has the following two special forms. The first one is an immediate consequence.

Lemma 4.8. *Let $(b_0, b_1, \dots, b_{p-1}) \in \mathbb{Z}^p$. If $\sum_{j=0}^{p-1} e^{2\pi i b_j / p^n} = 0$, then subject to a permutation of (b_0, \dots, b_{p-1}) , there exist $0 \leq r \leq p^{n-1} - 1$ such that*

$$b_j \equiv r + jp^{n-1} \pmod{p^n}$$

for all $j = 0, 1, \dots, p-1$.

Lemma 4.9 ([37], Lemma 2.5). *Let C be a finite subset of \mathbb{Z} . If $\sum_{c \in C} e^{2\pi i c / p^n} = 0$, then $p \mid \text{Card}(C)$ and C is decomposed into $\text{Card}(C)/p$ disjoint subsets $C_1, C_2, \dots, C_{\text{Card}(C)/p}$, such that each C_j consists of p points and*

$$\sum_{c \in C_j} e^{2\pi i c / p^n} = 0.$$

Now applying Lemmas 4.8 and 4.9, we have the following lemma which will be useful in the paper.

Lemma 4.10. *Let $C \subset \mathbb{Q}_p$ be a finite set.*

(1) *If $\sum_{c \in C} \chi(c) = 0$, then $p \mid \text{Card}(C)$ and $\sum_{c \in C} \chi(xc) = 0$ for any $x \in \mathbb{Z}_p^\times$ (i.e. $|x|_p = 1$).*

(2) *If there exists $\xi \in \mathbb{Q}_p$ such that $\sum_{c \in C} \chi(\xi c) = 0$, then for any $c \in C$, there exists $c' \in C$ such that $|c - c'|_p = p/|\xi|_p$.*

(3) *If there exists a finite set $\mathbb{I} \subset \mathbb{Z}$ such that*

$$\sum_{c \in C} \chi(p^i c) = 0 \quad \text{for all } i \in \mathbb{I}, \quad (4.2.6)$$

then $p^{\text{Card}(\mathbb{I})} \mid \#(C)$.

Proof. (1) It is a direct consequence of Lemmas 3.10 and 3.11. See [37, Lemma 2.6] for details.

(2) Let $C = \{c_1, c_2, \dots, c_m\}$. Recall that $\chi(\xi c) = e^{2\pi i \{\xi c\}}$. There exist an integer n and a subset $\{n_1, n_2, \dots, n_m\}$ of \mathbb{Z} such that

$$\chi(\xi c_k) = e^{2\pi i n_k / p^n}, \quad k = 1, 2, \dots, m.$$

By Lemma 3.11, $p \mid \#(C)$ and C is decomposed into $\#(C)/p$ disjoint subsets $C_0, C_1, \dots, C_{\#(C)/p}$, such that each C_j consists of p points and

$$\sum_{c \in C_j} \chi(\xi c) = 0.$$

Without loss of generality, assume that $c \in C_0$. By Lemma 3.10, we have

$$|c - c'|_p = \frac{p}{|\xi|_p}, \quad \text{if } c' \in C_0 \setminus \{c\}.$$

(3) Assume that $\mathbb{I} = \{i_1, i_2, \dots, i_n\}$ with $i_1 < i_2 < \dots < i_n$. By Lemma 3.11 and the equality (4.2.6) with $i = i_1$, we have $p \mid \#(C)$ and C can be decomposed into $\#(C)/p$ subsets $C_1, C_2, \dots, C_{\#(C)/p}$ such that each C_j consists of p points and

$$\sum_{c \in C_j} \chi(p^{i_1} c) = 0.$$

By Lemma 3.10, if c and c' lie in the same C_j , we have

$$|c - c'|_p \leq p^{1+i_1}. \quad (4.2.7)$$

Now we consider the equality (4.2.6) when $i = i_2$. Since $i_1 < i_2$, (4.2.7) and (4.2.3) imply that the function

$$c \mapsto \chi(p^{i_2} c)$$

is constant on each C_j . From each C_j , take one element \tilde{c}_j . Let \tilde{C} be the set consisting of these \tilde{c}_j .

Since each C_j contains p elements, the equality (4.2.6) with $i = i_2$ is equivalent to

$$\sum_{\tilde{c} \in \tilde{C}} \chi(p^{i_2} c) = 0.$$

By Lemma 3.11, $p \mid \sharp(\tilde{C})$, which implies $p^2 \mid \sharp(C)$.

By induction, we get $p^n \mid \sharp(C)$. □

4.2.4 Bruhat-Schwartz distributions in \mathbb{Q}_p

Here we give a brief description of the theory following [3, 120, 129]. Let \mathcal{E} denote the space of the uniformly locally constant functions. The space \mathcal{D} of *Bruhat-Schwartz test functions* is, by definition, constituted of uniformly locally constant functions of compact support. Such a test function $f \in \mathcal{D}$ is a finite linear combination of indicator functions of the form $1_{B(x, p^k)}(\cdot)$, where $k \in \mathbb{Z}$ and $x \in \mathbb{Q}_p$. The largest of such numbers k is denoted by $\ell := \ell(f)$ and is called the *parameter of constancy* of f . Since $f \in \mathcal{D}$ has compact support, the minimal number $\ell' := \ell'(f)$ such that the support of f is contained in $B(0, p^{\ell'})$ exists and will be called the *parameter of compactness* of f .

Clearly, $\mathcal{D} \subset \mathcal{E}$. The space \mathcal{D} is provided with a topology of topological vector space as follows: a sequence $\{\phi_n\} \subset \mathcal{D}$ is called a *null sequence* if there is a fixed pair of $l, l' \in \mathbb{Z}$ such that each ϕ_n is constant on every ball of radius p^l and is supported by the ball $B(0, p^{l'})$ and the sequence ϕ_n tends uniformly to zero.

A *Bruhat-Schwartz distribution* f on \mathbb{Q}_p is by definition a continuous linear functional on \mathcal{D} . The value of f at $\phi \in \mathcal{D}$ will be denoted by $\langle f, \phi \rangle$. Note that linear functionals on \mathcal{D} are automatically continuous. This property allows us to easily construct distributions. Denote by \mathcal{D}' the space of Bruhat-Schwartz distributions. The space \mathcal{D}' is equipped with the weak topology induced by \mathcal{D} .

A locally integrable function f is considered as a distribution: for any $\phi \in \mathcal{D}$,

$$\langle f, \phi \rangle = \int_{\mathbb{Q}_p} f \phi d\mathbf{m}.$$

The discrete measure μ_E defined by (4.1.3) is also a distribution: for any $\phi \in \mathcal{D}$,

$$\langle \mu_E, \phi \rangle = \sum_{\lambda \in E} \phi(\lambda).$$

Here for each ϕ , the sum is finite because E is uniformly discrete and each ball contains at most a finite number of points in E . Since the test functions in \mathcal{D} are uniformly locally constant and have compact support, the following proposition is a direct consequence of Proposition 4.6 or of the fact (see also [30, Lemma 4]) that

$$\widehat{1_{B(c, p^k)}}(\xi) = \chi(-c\xi) p^k 1_{B(0, p^{-k})}(\xi). \quad (4.2.8)$$

Proposition 4.11 ([120], Chapter II 3). *The Fourier transformation $f \mapsto \widehat{f}$ is a homeomorphism from \mathcal{D} onto \mathcal{D} .*

The Fourier transform of a distribution $f \in \mathcal{D}'$ is a new distribution $\widehat{f} \in \mathcal{D}'$ defined by the duality

$$\langle \widehat{f}, \phi \rangle = \langle f, \widehat{\phi} \rangle, \quad \forall \phi \in \mathcal{D}.$$

The Fourier transformation $f \mapsto \widehat{f}$ is a homeomorphism of \mathcal{D}' onto \mathcal{D}' under the weak topology [120, Chapter II 3].

4.2.5 Zeros of the Fourier transform of a discrete measure

Let $f \in \mathcal{D}'$ be a distribution in \mathbb{Q}_p . A point $x \in \mathbb{Q}_p$ is called a *zero* of f if there exists an integer n_0 such that

$$\langle f, 1_{B(y, p^n)} \rangle = 0, \quad \text{for all } y \in B(x, p^{n_0}) \text{ and all integers } n \leq n_0.$$

Denote by \mathcal{Z}_f the set of all zeros of f . Remark that \mathcal{Z}_f is the maximal open set O on which f vanishes, i.e. $\langle f, \phi \rangle = 0$ for all $\phi \in \mathcal{D}$ such that the support of ϕ is contained in O . The *support* of a distribution f is defined as the complementary set of \mathcal{Z}_f and is denoted by $\text{supp}(f)$.

Let E be a uniformly discrete set in \mathbb{Q}_p . The following proposition characterizes the structure of $\mathcal{Z}_{\widehat{\mu_E}}$, the set of zeros of the Fourier transform of the discrete measure μ_E . It is bounded and is a union of spheres centered at 0.

Proposition 4.12. *Let E be a uniformly discrete set in \mathbb{Q}_p .*

- (1) *If $\xi \in \mathcal{Z}_{\widehat{\mu_E}}$, then $S(0, |\xi|_p) \subset \mathcal{Z}_{\widehat{\mu_E}}$.*
- (2) *The set $\mathcal{Z}_{\widehat{\mu_E}}$ is bounded.*

Proof. First remark that by using (4.2.8) we get

$$\langle \widehat{\mu_E}, 1_{B(\xi, p^{-n})} \rangle = \langle \mu_E, \widehat{1_{B(\xi, p^{-n})}} \rangle = p^{-n} \sum_{\lambda \in E \cap B(0, p^n)} \overline{\chi(\xi \lambda)}. \quad (4.2.9)$$

This expression will be used several times.

- (1) By definition, $\xi \in \mathcal{Z}_{\widehat{\mu_E}}$ implies that there exists an integer n_0 such that

$$\langle \widehat{\mu_E}, 1_{B(\xi, p^{-n})} \rangle = 0, \quad \forall n \geq n_0.$$

By (4.2.9), this is equivalent to

$$\sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi \lambda) = 0, \quad \forall n \geq n_0. \quad (4.2.10)$$

For any $\xi' \in S(0, |\xi|_p)$, we have $\xi' = u\xi$ for some $u \in \mathbb{Z}_p^\times$. By Lemma 4.10 (1) and the equality (4.2.10), we obtain

$$\sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi' \lambda) = \sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi \lambda) = 0, \quad \forall n \geq n_0.$$

Thus, again by (4.2.9), $\langle \widehat{\mu}_E, 1_{B(\xi', p^{-n})} \rangle = 0$ for $n \geq n_0$. We have thus proved $S(0, |\xi|_p) \subset \mathcal{Z}_{\widehat{\mu}_E}$.

(2) Fix $\lambda_0 \in E$. By the discreteness of E , there exists an integer n_0 such that

$$\forall \lambda \in E \setminus \{\lambda_0\}, \quad |\lambda - \lambda_0|_p \geq p^{-n_0}.$$

We are going to show that $\mathcal{Z}_{\widehat{\mu}_E} \subset B(0, p^{n_0+1})$, which is equivalent to that $\xi \notin \mathcal{Z}_{\widehat{\mu}_E}$ when $|\xi|_p \geq p^{n_0+2}$. To this end, we will prove that for all integer n large enough such that $\lambda_0 \in B(0, p^n)$, we have

$$\langle \widehat{\mu}_E, 1_{B(\xi, p^{-n})} \rangle \neq 0.$$

In fact, if this is not the case, then by (4.2.9),

$$\sum_{\lambda \in E \cap B(0, p^n)} \chi(\xi \lambda) = 0.$$

Thus, by Lemma 4.10 (2), for the given λ_0 , we can find $\lambda \in E \cap B(0, p^n)$, such that $|\lambda - \lambda_0|_p = p/|\xi|_p < p^{-n_0}$, a contradiction. □

Remark that n_0 in the above proof depends only on the structure of E . Define

$$n_E := \max_{\substack{\lambda, \lambda' \in E \\ \lambda \neq \lambda'}} v_p(\lambda - \lambda'). \quad (4.2.11)$$

According to the above proof of the second assertion of Proposition 4.12, we immediately get

$$\mathcal{Z}_{\widehat{\mu}_E} \subset B(0, p^{n_E+1}). \quad (4.2.12)$$

4.2.6 Convolution and multiplication of distributions

Denote

$$\Delta_k := 1_{B(0, p^k)}, \quad \theta_k := \widehat{\Delta}_k = p^k \cdot 1_{B(0, p^{-k})}.$$

Let $f, g \in \mathcal{D}'$ be two distributions. We define the *convolution* of f and g by

$$\langle f * g, \phi \rangle = \lim_{k \rightarrow \infty} \langle f(x), \langle g(\cdot), \Delta_k(x) \phi(x + \cdot) \rangle \rangle,$$

if the limit exists for all $\phi \in \mathcal{D}$.

Proposition 4.13 ([3], Proposition 4.7.3). *If $f \in \mathcal{D}'$, then $f * \theta_k \in \mathcal{E}$ with the parameter of constancy at least $-k$.*

We define the *multiplication* of f and g by

$$\langle f \cdot g, \phi \rangle = \lim_{k \rightarrow \infty} \langle g, (f * \theta_k) \phi \rangle,$$

if the limit exists for all $\phi \in \mathcal{D}$. The above definition of convolution is compatible with the usual convolution of two integrable functions and the definition of multiplication is compatible with the usual multiplication of two locally integrable functions.

The following proposition shows that both the convolution and the multiplication are commutative when they are well defined and the convolution of two distributions is well defined if and only if the multiplication of their Fourier transforms is well defined.

Proposition 4.14 ([129], Sections 7.1 and 7.5). *Let $f, g \in \mathcal{D}'$ be two distributions. Then*

- (1) *If $f * g$ is well defined, so is $g * f$ and $f * g = g * f$.*
- (2) *If $f \cdot g$ is well defined, so is $g \cdot f$ and $f \cdot g = g \cdot f$.*
- (3) *$f * g$ is well defined if and only if $\widehat{f \cdot g}$ is well defined. In this case, we have $\widehat{f * g} = \widehat{f \cdot g}$ and $\widehat{f \cdot g} = \widehat{f} * \widehat{g}$.*

The following proposition justifies an intuition.

Proposition 4.15. *Let $f, g \in \mathcal{D}'$ be two distributions. If $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, then $f \cdot g$ is well defined and $f \cdot g = 0$.*

Proof. Let $\phi \in \mathcal{D}$ with parameter of constancy ℓ and parameter of compactness ℓ' . By the assumption $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, the distance between the two compact sets $\text{supp}(f) \cap B(0, p^{\ell'})$ and $\text{supp}(g) \cap B(0, p^{\ell'})$ is strictly positive. So there exists an integer $n \leq \ell'$ such that for any $x \in \text{supp}(f) \cap B(0, p^{\ell'})$, the ball $B(x, p^n)$ is contained in $\mathcal{Z}_g \cap B(0, p^{\ell'})$. Therefore, the compact open set

$$X := \bigcup_{x \in \text{supp}(f) \cap B(0, p^{\ell'})} B(x, p^{-n})$$

satisfies $\text{supp}(f) \cap B(0, p^{\ell'}) \subset X \subset \mathcal{Z}_g \cap B(0, p^{\ell'})$.

Define $\phi_1(x) := \phi(x) \cdot 1_X(x)$. It follows that

$$\text{supp}(\phi - \phi_1) \subset B(0, p^{\ell'}) \setminus X \subset \mathcal{Z}_f \cap B(0, p^{\ell'}).$$

Thus, we have

$$\begin{aligned} \langle f \cdot g, \phi \rangle &= \langle f \cdot g, (\phi - \phi_1 + \phi_1) \rangle = \langle f \cdot g, \phi - \phi_1 \rangle + \langle f \cdot g, \phi_1 \rangle \\ &= \lim_{k \rightarrow \infty} \langle f, (g * \theta_k)(\phi - \phi_1) \rangle + \lim_{k \rightarrow \infty} \langle g, (f * \theta_k)\phi_1 \rangle \\ &= 0, \end{aligned}$$

where the existences of the last two limits are due to

$$\text{supp}((g * \theta_k)(\phi - \phi_1)) \subset \text{supp}(\phi - \phi_1) \subset \mathcal{Z}_f,$$

and

$$\text{supp}((f * \theta_k)\phi_1) \subset \text{supp} \phi_1 \subset \mathcal{Z}_g.$$

□

The multiplication of some special distributions has a simple form. That is the case for the multiplication of a uniformly locally constant function and a distribution.

Proposition 4.16 ([129] Section 7.5, Example 2). *Let $f \in \mathcal{E}$ and let $G \in \mathcal{D}'$. Then for any $\phi \in \mathcal{D}$, we have $\langle f \cdot G, \phi \rangle = \langle G, f\phi \rangle$.*

For a distribution $f \in \mathcal{D}'$, we define its *regularization* by the sequence of test functions ([3, Proposition 4.7.4])

$$\Delta_k \cdot (f * \theta_k) \in \mathcal{D}.$$

The regularization of a distribution converges to the distribution with respect to the weak topology.

Proposition 4.17 ([3] Lemma 14.3.1). *Let f be a distribution in \mathcal{D}' . Then $\Delta_k \cdot (f * \theta_k) \rightarrow f$ in \mathcal{D}' as $k \rightarrow \infty$. Moreover, for any test function $\phi \in \mathcal{D}$ we have*

$$\langle \Delta_k \cdot (f * \theta_k), \phi \rangle = \langle f, \phi \rangle, \quad \forall k \geq \max\{-\ell, \ell'\},$$

where ℓ and ℓ' are the parameter of constancy and the parameter of compactness of the function ϕ defined in Subsection 4.2.4.

This approximation of distribution by test functions allows us to construct a space which is bigger than the space of distributions. This larger space is the Colombeau algebra, which will be presented below. Recall that in the space of Bruhat-Schwartz distributions, the convolution and the multiplication are not well defined for all couples of distributions. But in the Colombeau algebra, the convolution and the multiplication are well defined and these two operations are associative.

4.2.7 Colombeau algebra of generalized functions

Consider the set $\mathcal{P} := \mathcal{D}^{\mathbb{N}}$ of all sequences $\{f_k\}_{k \in \mathbb{N}}$ of test functions. We introduce an algebra structure on \mathcal{P} , defining the operations component-wisely

$$\{f_k\} + \{g_k\} = \{f_k + g_k\},$$

$$\{f_k\} \cdot \{g_k\} = \{f_k \cdot g_k\},$$

where $\{f_k\}, \{g_k\} \in \mathcal{P}$.

Let \mathcal{N} be the sub-algebra of elements $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{P}$ such that for any compact set $K \subset \mathbb{Q}_p$ there exists $N \in \mathbb{N}$ such that $f_k(x) = 0$ for all $k \geq N, x \in K$. Clearly, \mathcal{N} is an ideal in the algebra \mathcal{P} .

Then we introduce the *Colombeau-type algebra*

$$\mathcal{G} = \mathcal{P}/\mathcal{N}.$$

The equivalence class of sequences which defines an element in \mathcal{G} will be denoted by $\mathbf{f} = [f_k]$, called a generalized function.

For any $\mathbf{f} = [f_k], \mathbf{g} = [g_k] \in \mathcal{G}$, the addition and multiplication are defined as

$$\mathbf{f} + \mathbf{g} = [f_k + g_k], \quad \mathbf{f} \cdot \mathbf{g} = [f_k \cdot g_k].$$

Obviously, $(\mathcal{G}, +, \cdot)$ is an associative and commutative algebra.

Theorem 4.18 ([3] Theorem 14.3.3). *The map $f \mapsto \mathbf{f} = [\Delta_k(f * \theta_k)]$ from \mathcal{D}' to \mathcal{G} is a linear embedding.*

Each distribution $f \in \mathcal{D}'$ is embedded into \mathcal{G} by the mapping which associates f to the generalized function determined by the of f . Thus we obtain that the multiplication defined on the \mathcal{D}' is associative in the following sense.

Proposition 4.19. *Let $f, g, h \in \mathcal{D}'$. If $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are well defined as multiplications of distributions, we have*

$$(f \cdot g) \cdot h = f \cdot (g \cdot h).$$

4.3 Study on $\mu_E * f = 1$

In this section, we will study the following equation

$$\mu_E * f = 1 \tag{4.3.1}$$

where $0 \leq f \in L^1(\mathbb{Q}_p)$ with $\int_{\mathbb{Q}_p} f d\mathbf{m} > 0$, and E is a uniformly discrete subset of \mathbb{Q}_p . Suppose that (E, f) is a solution of (4.3.1). We will investigate the density of E and even the distribution of E , and the supports of the Fourier transforms $\widehat{\mu}_E$ and \widehat{f} .

We say that a uniformly discrete set E has a *bounded density* if the following limit exists for some $x_0 \in \mathbb{Q}_p$

$$D(E) := \lim_{k \rightarrow \infty} \frac{\text{Card}(B(x_0, p^k) \cap E)}{\mathbf{m}(B(x_0, p^k))},$$

which is called *the density* of E . Actually, if the limit exists for some $x_0 \in \mathbb{Q}_p$, then it exists for all $x \in \mathbb{Q}_p$ and the limit is independent of x . In fact, for any $x_0, x_1 \in \mathbb{Q}_p$, when k is large enough such that $|x_0 - x_1|_p < p^k$, we have $B(x_0, p^k) = B(x_1, p^k)$.

For a function $g : \mathbb{Q}_p \rightarrow \mathbb{R}$, denote

$$\mathcal{N}_g := \{x \in \mathbb{Q}_p : g(x) = 0\}.$$

If $g \in C(\mathbb{Q}_p)$ is a continuous function, then \mathcal{N}_g is a closed set and \mathcal{Z}_g is the set of interior points of \mathcal{N}_g . But, the support of g as a continuous function is equal to the support of g as a distribution.

The following theorem gets together some properties of the solution (E, f) of the equation (4.3.1), which will be proved in this section.

Theorem 4.20. *Let $0 \leq f \in L^1(\mathbb{Q}_p)$ with $\int_{\mathbb{Q}_p} f d\mathbf{m} > 0$, and E be a uniformly discrete subset of \mathbb{Q}_p . Suppose that the equation (4.3.1) is satisfied by f and E . Then*

- (1) *The support of \widehat{f} is compact.*
- (2) *The set $\mathcal{Z}_{\widehat{\mu}_E}$ is bounded and it is the union of the punctured ball $B(0, p^{-n_f}) \setminus \{0\}$ and some spheres.*
- (3) *The density $D(E)$ exists and equals to $1/\int_{\mathbb{Q}_p} f d\mathbf{m}$. Furthermore, there exists an integer $n_f \in \mathbb{Z}$ such that for all integers $n \geq n_f$ we have*

$$\forall \xi \in \mathbb{Q}_p, \quad \text{Card}(E \cap B(\xi, p^n)) = p^n D(E).$$

Theorem 4.20 (1) and (2) will be proved in (S)3.1, the distribution of E will be discussed in (S)3.2 and the equality $D(E) = 1/\int_{\mathbb{Q}_p} f d\mathbf{m}$ will be proved in (S)3.3.

4.3.1 Compactness of $\text{supp}(\widehat{f})$ and structure of $\mathcal{Z}_{\widehat{\mu}_E}$

Our discussion is based on $\widehat{f} \cdot \widehat{\mu}_E = \delta_0$, which is implied by $f * \mu_E = 1$ (see Proposition 4.14).

Proposition 4.21. *Let $g \in C(\mathbb{Q}_p)$ be a continuous function and let $G \in \mathcal{D}'$ be a distribution. Suppose that the product $H = g \cdot G$ is well defined. Then*

$$\mathcal{Z}_H \subset \mathcal{N}_g \cup \mathcal{Z}_G.$$

Consequently, if $f * \mu_E = 1$ with $f \in L^1(\mathbb{Q}_p)$, then $\mathbb{Q}_p \setminus \{0\} \subset \mathcal{N}_{\widehat{f}} \cup \mathcal{Z}_{\widehat{\mu}_E}$, which is equivalent to

$$\{\xi \in \mathbb{Q}_p : \widehat{f}(\xi) \neq 0\} \setminus \{0\} \subset \mathcal{Z}_{\widehat{\mu}_E}. \quad (4.3.2)$$

Proof. The second assertion follows directly from the first assertion because $\widehat{f} \cdot \widehat{\mu}_E = \delta_0$ and $\mathcal{Z}_{\delta_0} = \mathbb{Q}_p \setminus \{0\}$.

We now prove the first assertion. It suffices to prove $\mathcal{Z}_H \setminus \mathcal{N}_g \subset \mathcal{Z}_G$. Take an arbitrary test function $\phi \in \mathcal{D}$ such that

$$\text{supp}(\phi) \subset \mathcal{Z}_H \setminus \mathcal{N}_g.$$

Since $\text{supp}(\phi) \subset \{x : g(x) \neq 0\}$, we can define the function

$$h(x) = \begin{cases} \frac{1}{g(x)}, & \text{for } x \in \text{supp } \phi, \\ 0, & \text{elsewhere.} \end{cases}$$

Since g is continuous, it is bounded away from 0 on $\text{supp } \phi$. So, the function h is bounded and compactly supported. Hence it belongs to $L^1(\mathbb{Q}_p)$. On the other hand, we have $h(x)g(x) = 1_{\text{supp}(\phi)}(x) \in \mathcal{D}$. Thus

$$\langle G, \phi \rangle = \langle G, 1_{\text{supp}(\phi)} \cdot \phi \rangle = \langle 1_{\text{supp}(\phi)} \cdot G, \phi \rangle = \langle (h \cdot g) \cdot G, \phi \rangle$$

where we have used Proposition 4.16 for the second equality. Notice that $\text{supp}(h) \subset \mathcal{Z}_H$. By Proposition 4.15, $h \cdot H$ is well defined and $h \cdot H = 0$. By the associativity of the multiplication (see Proposition 4.19), we get

$$\langle G, \phi \rangle = \langle h \cdot H, \phi \rangle = 0.$$

Thus we have proved $\mathcal{Z}_H \setminus \mathcal{N}_g \subset \mathcal{Z}_G$. □

Notice that $\widehat{f}(0) = \int_{\mathbb{Q}_p} f d\mathbf{m} > 0$ and \widehat{f} is a continuous function. It follows that there exists a small ball where \widehat{f} is non-vanish. Let

$$n_f := \min\{n \in \mathbb{Z} : \widehat{f}(x) \neq 0, \text{ if } x \in B(0, p^{-n})\}. \quad (4.3.3)$$

Proposition 4.22. *The Fourier transform \widehat{f} has compact support. The set $\mathcal{Z}_{\widehat{\mu}_E}$ is bounded and*

$$B(0, p^{-n_f}) \setminus \{0\} \subset \mathcal{Z}_{\widehat{\mu}_E}. \quad (4.3.4)$$

Proof. The boundedness of $\mathcal{Z}_{\widehat{\mu}_E}$ is already proved. See Proposition 4.12 (2). This together with (4.3.2) implies the compactness of $\text{supp}(\widehat{f})$. We get (4.3.4) immediately from (4.3.2). □

4.3.2 Distribution of E

The uniformly discrete set E involved in the equation (4.3.1) shares the following uniform distribution property.

Proposition 4.23. *The cardinality of $E \cap B(\xi, p^{n_f})$ is independent of $\xi \in \mathbb{Q}_p$. Consequently, the set E admits a bounded density $D(E)$. Moreover, for all integers $n \geq n_f$, we have*

$$\forall \xi \in \mathbb{Q}_p, \quad \text{Card}(E \cap B(\xi, p^n)) = p^n D(E). \quad (4.3.5)$$

Proof. For simplicity, we denote

$$E_n^\xi := E \cap B(\xi, p^n),$$

and write $E_n := E \cap B(0, p^n)$ when $\xi = 0$. It suffices to prove

$$\forall \xi \in \mathbb{Q}_p, \quad \#(E_{n_f}) = \#(E_{n_f}^\xi).$$

For any given $\xi \in \mathbb{Q}_p$, let $k = -v_p(\xi)$. If $k \leq n_f$, then $E_{n_f} = E_{n_f}^\xi$. So obviously $\#(E_{n_f}) = \#(E_{n_f}^\xi)$.

Now we suppose $k > n_f$. Then for any $\eta \in B(0, p^k) \setminus B(0, p^{n_f})$,

$$B(\eta, p^{-k}) \subset B(0, p^{-n_f}) \setminus \{0\}.$$

By (4.3.4) in Proposition 4.22, we have $\langle \widehat{\mu}_E, 1_{B(\eta, p^{-k})} \rangle = 0$, which by (4.2.9), is equivalent to

$$\sum_{\lambda \in E_k} \chi(\eta\lambda) = 0. \quad (4.3.6)$$

Taking $\eta = p^{k-1}$ in (4.3.6), we have $\sum_{\lambda \in E_k} \chi(p^{k-1}\lambda) = 0$. Observe that

$$B(0, p^k) = \bigsqcup_{i=0}^{p-1} B(ip^{-k}, p^{k-1})$$

and that the function $\chi(p^{k-1}\cdot)$ is locally constant on each ball of radius p^{k-1} . So we have

$$0 = \sum_{\lambda \in E_k} \chi(p^{k-1}\lambda) = \sum_{i=0}^{p-1} \chi\left(\frac{i}{p}\right) \text{Card}\left(E_{k-1}^{\frac{i}{p^k}}\right).$$

Applying Lemma 4.7, we obtain

$$\text{Card}\left(E_{k-1}^{\frac{i}{p^k}}\right) = \text{Card}\left(E_{k-1}^{\frac{j}{p^k}}\right), \quad \forall 0 \leq i, j \leq p-1. \quad (4.3.7)$$

Similarly, taking $\eta = p^{k-2}$ in (4.3.6), we have

$$0 = \sum_{0 \leq i, j \leq p-1} \chi\left(\frac{i}{p^2} + \frac{j}{p}\right) \text{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right).$$

Again, Lemma 4.7 implies

$$\text{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right) = \text{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{m}{p^{k-1}}}\right) \quad \forall 0 \leq i, j, m \leq p-1.$$

Since

$$\sum_{j=0}^{p-1} \text{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right) = \text{Card}\left(E_{k-1}^{\frac{i}{p^k}}\right),$$

by (4.3.7), we get

$$\text{Card}\left(E_{k-2}^{\frac{i}{p^k} + \frac{j}{p^{k-1}}}\right) = \text{Card}\left(E_{k-2}^{\frac{l}{p^k} + \frac{m}{p^{k-1}}}\right), \quad \forall 0 \leq i, j, l, m \leq p-1.$$

We continue these arguments for all $\eta = p^{k-1}, \dots, p^{n_f}$. By induction, we have

$$\text{Card}(E_{n_f}^{\xi_1}) = \text{Card}(E_{n_f}^{\xi_2}), \quad \forall \xi_1, \xi_2 \in \mathbb{L}_{n_f} \cap B(0, p^k). \quad (4.3.8)$$

Since $|\xi|_p = p^k$, there exists $\xi' \in \mathbb{L}_{n_f} \cap B(0, p^k)$, such that $E_{n_f}^{\xi} = E_{n_f}^{\xi'}$. Thus by (4.3.8),

$$\text{Card}(E_{n_f}^{\xi}) = \text{Card}(E_{n_f}^{\xi'}) = \text{Card}(E_{n_f}).$$

The formula (4.3.5) follows immediately because each ball of radius p^n with $n \geq n_f$ is a disjoint union of p^{n-n_f} balls of radius p^{n_f} so that

$$\text{Card}(E_n) = p^{n-n_f} \text{Card}(E_{n_f}).$$

□

4.3.3 Equality $D(E) = 1/\int_{\mathbb{Q}_p} f d\mathbf{m}$

Proposition 4.24. *The density $D(E)$ of E satisfies*

$$D(E) = \frac{1}{\int_{\mathbb{Q}_p} f(x) dx}.$$

Proof. By the integrability of f , the quantity

$$\epsilon_n := \int_{\mathbb{Q}_p \setminus B(0, p^n)} f(x) dx$$

tends to zero as the integer $n \rightarrow \infty$. Integrating the equality (4.3.1) over the ball $B(0, p^n)$, we have

$$\mathbf{m}(B(0, p^n)) = \sum_{\lambda \in E} \int_{B(0, p^n)} f(x - \lambda) dx. \quad (4.3.9)$$

Now we split the sum in (4.3.9) into two parts, according to $\lambda \in E \cap B(0, p^n)$ or $\lambda \in E \setminus B(0, p^n)$. Denote $I := \int_{\mathbb{Q}_p} f d\mathbf{m}$. For $\lambda \in E \cap B(0, p^n)$, we have

$$\int_{B(0, p^n)} f(x - \lambda) dx = \int_{B(0, p^n)} f(x) dx = I - \epsilon_n.$$

It follows that

$$\sum_{\lambda \in E \cap B(0, p^n)} \int_{B(0, p^n)} f(x - \lambda) dx = \text{Card}(E_n) \cdot (I - \epsilon_n). \quad (4.3.10)$$

Notice that

$$\int_{B(0, p^n)} f(x - \lambda) dx = \int_{B(-\lambda, p^n)} f(x) dx. \quad (4.3.11)$$

For $\lambda \in E \setminus B(0, p^n)$, the ball $B(-\lambda, p^n)$ is contained in $\mathbb{Q}_p \setminus B(0, p^n)$. We partition the uniformly discrete set $E \setminus B(0, p^n)$ into P_j 's such that each P_j is contained in a ball of radius p^n in $\mathbb{Q}_p \setminus B(0, p^n)$. Thus the integrals in (4.3.11) for the λ 's in the same P_j are equal. Let λ_j be a representative of P_j . Then we have

$$\sum_{\lambda \in E \setminus B(0, p^n)} \int_{B(0, p^n)} f(x - \lambda) dx = \sum_j \text{Card}(P_j) \cdot \int_{B(-\lambda_j, p^n)} f(x) dx.$$

However, by (4.3.5) in Proposition 4.23, $\text{Card}(P_j) = D(E)\mathbf{m}(B(0, p^n))$ if $n \geq n_f$. Thus, for each integer $n \geq n_f$,

$$\begin{aligned} \sum_{e \in E \setminus B(0, p^n)} \int_{B(0, p^n)} f(x - \lambda) dx &= D(E)\mathbf{m}(B(0, p^n)) \sum_j \int_{B(-\lambda_j, p^n)} f(x) dx \\ &= D(E)\mathbf{m}(B(0, p^n)) \int_{\mathbb{Q}_p \setminus B(0, p^n)} f(x) dx \\ &= D(E)\mathbf{m}(B(0, p^n))\epsilon_n. \end{aligned} \quad (4.3.12)$$

Thus from (4.3.9), (4.3.10) and (4.3.12), we finally get

$$|\mathbf{m}(B(0, p^n)) - \text{Card}(E \cap B(0, p^n)) \cdot I| \leq 2D(E)\mathbf{m}(B(0, p^n))\epsilon_n.$$

We conclude by dividing $\mathbf{m}(B(0, p^n))$ and then letting $n \rightarrow \infty$. \square

4.4 Tiles are spectral sets

The key for the proof of the following theorem is the fact that the equation $\mu_E * f = 1$ imposes that the support of the Fourier transform \widehat{f} is compact (Proposition 4.22). We also use the fact that the Fourier transform of an integrable function with compact support is uniformly locally constant (Proposition 4.5). The Lebesgue density theorem (Proposition 4.4) is used too. We recall that a point x in a Borel set Ω satisfying the equality in Proposition 4.4 is called a *density point* of Ω .

Theorem 4.25. *Without distinguishing sets which are equal modulo a set of zero Haar measure, we have the following assertions :*

- (1) *If Ω is a spectral set in \mathbb{Q}_p , then it is bounded.*
- (2) *If Ω is a tile in \mathbb{Q}_p , then it is compact and open.*
- (3) *Tiles in \mathbb{Q}_p are spectral sets.*

Proof. Since we do not distinguish sets which are equal modulo a set of zero Haar measure, we can assume that all the points in Ω are density points of Ω , according to the Lebesgue density theorem (Proposition 4.4).

(1) Assume that (Ω, Λ) is a spectral pair in \mathbb{Q}_p , which is equivalent to

$$\forall \xi \in \widehat{\mathbb{Q}_p}, \quad \mu_\Lambda * |\widehat{1_\Omega}|^2(\xi) = \mathbf{m}(\Omega)^2.$$

By Proposition 4.22, $|\widehat{1_\Omega}|^2$ has compact support. Observe that

$$|\widehat{1_\Omega}|^2(\xi) = 1_\Omega * 1_{-\Omega}(\xi) = \int_{\mathbb{Q}_p} 1_\Omega(x)1_\Omega(x - \xi)dx = \mathbf{m}(\Omega \cap (\Omega + \xi)). \quad (4.4.1)$$

We claim

$$\text{supp}(|\widehat{1_\Omega}|^2) = \overline{\Omega - \Omega}.$$

Since the inclusion $\text{supp}(|\widehat{1_\Omega}|^2) \subset \overline{\Omega - \Omega}$ is obvious, we need only to show

$$\overline{\Omega - \Omega} \subset \text{supp}(|\widehat{1_\Omega}|^2). \quad (4.4.2)$$

In fact, let $\xi \in \overline{\Omega - \Omega}$. Write $\xi = z_1 - z_2$ with $z_1, z_2 \in \Omega$. By (4.4.1), we have

$$|\widehat{1_\Omega}|^2(\xi) = \mathbf{m}(\Omega \cap (\Omega + z_1 - z_2)) = \mathbf{m}((\Omega - z_1) \cap (\Omega - z_2)).$$

Since z_1, z_2 are density points of Ω , 0 is a density point of both $\Omega - z_1$ and $\Omega - z_2$. This fact implies

$$\mathbf{m}((\Omega - z_1) \cap (\Omega - z_2)) > 0.$$

Thus we have proved $\Omega - \Omega \subset \text{supp}(\widehat{|\mathbf{1}_\Omega|^2})$. Then (4.4.2) follows. Since $\text{supp}(\widehat{|\mathbf{1}_\Omega|^2})$ is compact, the set Ω is bounded.

(2) The main argument is the same as in (1). Assume that (Ω, T) is a tiling pair in \mathbb{Q}_p , which means

$$\mu_T * \mathbf{1}_\Omega(x) = 1, \quad \mathbf{m}\text{-a.e. } x \in \mathbb{Q}_p.$$

By Proposition 4.22, $\widehat{\mathbf{1}_\Omega}$ has compact support. Then by the first assertion of Proposition 4.5, $\mathbf{1}_\Omega$ is almost uniformly locally constant, i.e. Ω is, up to a zero measure set, a union of balls with the same radius. Since Ω is of finite measure, the number of these balls is finite. So, Ω is almost compact open.

(3) It is an immediate consequence of (2) and Theorem 4.3. \square

The above proof of the fact “tiles are spectral sets” is partially based on Theorem 4.3 and partially on “tiles are compact and open” (Theorem 4.25 (2)). The proof of “spectral sets are tiles” will be not based on Theorem 4.3. That means, we are not going to show that a spectral set is a compact open set up to a set of zero measure. But the boundedness of a spectral set (Theorem 4.25 (1)) will be used.

4.5 Spectral sets are tiles

In this section, we prove that a spectral set in \mathbb{Q}_p is a tile. Assume that (Ω, Λ) is a spectral pair. The proof will be based on our knowledge on the set $\mathcal{Z}_{\widehat{\mu}_\Lambda}$. As we will see in the proof, a tiling complement can be easily constructed.

Theorem 4.26. *If Ω is a spectral set in \mathbb{Q}_p , then it is a tile.*

Proof. Suppose that (Ω, Λ) is a spectral pair. That means (see (4.1.1))

$$\sum_{\lambda \in \Lambda} |\widehat{\mathbf{1}_\Omega}|^2(x - \lambda) = \mathbf{m}(\Omega)^2, \quad \mathbf{m}\text{-a.e. } x \in \mathbb{Q}_p.$$

In other words, $\mu_\Lambda * f = 1$ where

$$f = |\widehat{\mathbf{1}_\Omega}|^2 / \mathbf{m}(\Omega)^2.$$

Let n_f be the integer defined by (4.3.3). Recall that $B(0, p^{-n_f})$ is the biggest ball centered at 0 over which \widehat{f} does not vanish. We first remark that

$$\forall n \geq n_f, \quad \text{Card}(\Lambda_n) = p^n \mathbf{m}(\Omega) \tag{4.5.1}$$

where

$$\Lambda_n := \Lambda \cap B(0, p^n).$$

In fact, Plancherel Theorem implies $\int_{\mathbb{Q}_p} f d\mathbf{m} = 1/\mathbf{m}(\Omega)$. So, by Proposition 4.24, the set Λ is of bounded density with density

$$D(\Lambda) = \frac{1}{\int_{\mathbb{Q}_p} f d\mathbf{m}} = \mathbf{m}(\Omega).$$

Thus (4.5.1) follows from (4.3.5) in Proposition 4.23.

By Theorem 4.25, the spectral set Ω is bounded, up to a Haar null measure set. Without loss of generality, we assume that $\Omega \subset \mathbb{Z}_p$.

Recall that $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ has the following properties: every sphere $S(0, p^{-n})$ either is contained in $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ or does not intersect $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ (Lemma 4.12); all spheres $S(0, p^{-n})$ with $n \geq n_f$ are contained in $\mathcal{Z}_{\widehat{\mu_\Lambda}}$ (see (4.3.4)). Let

$$\begin{aligned} \mathbb{I} &:= \{0 \leq n < n_f : S(0, p^{-n}) \subset \mathcal{Z}_{\widehat{\mu_\Lambda}}\}, \\ \mathbb{J} &:= \{0 \leq n < n_f : S(0, p^{-n}) \cap \mathcal{Z}_{\widehat{\mu_\Lambda}} = \emptyset\}. \end{aligned}$$

For $\mathcal{N}_{\widehat{f}}$, we claim

$$\bigcup_{j \in \mathbb{J}} S(0, p^{-j}) \subset \mathcal{N}_{\widehat{f}}. \quad (4.5.2)$$

In fact, otherwise for some $\xi \in S(0, p^{-j})$ with $j \in \mathbb{J}$ we have $\widehat{f}(\xi) \neq 0$. By (4.3.2) and Proposition 4.12 (1), $S(0, p^{-j}) \subset \mathcal{Z}_{\widehat{\mu_\Lambda}}$. This contradicts the fact $j \in \mathbb{J}$.

Observe that $\widehat{f}(\xi) = \mathbf{m}(\Omega \cap (\Omega + \xi))$ (see (4.4.1)) and that

$$(U - U) \setminus \{0\} \subset \bigcup_{j \in \mathbb{J}} S(0, p^{-j})$$

where

$$U := \left\{ \sum_{j \in \mathbb{J}} \alpha_j p^j, \alpha_j \in \{0, 1, \dots, p-1\} \right\}.$$

Then from (4.5.2), we get

$$(U - U) \setminus \{0\} \subset \{\xi \in \mathbb{Q}_p : \mathbf{m}(\Omega \cap (\Omega + \xi)) = 0\}. \quad (4.5.3)$$

We claim that Ω is a tile of \mathbb{Z}_p with tiling complement U , so that Ω is a tile of \mathbb{Q}_p with tiling complement $U + \mathbb{L}$.

In fact, the disjointness (up to a set of zero measure) of $\Omega + \xi_1$ and $\Omega + \xi_2$ for any two distinct $\xi_1, \xi_2 \in U$ follows directly from (4.5.3) and the invariance of the Haar measure.

So to show that Ω is a tile of \mathbb{Z}_p , it suffices to prove that the total measure of all $\Omega + \xi$ with $\xi \in U$ is equal to

$$\mathbf{m}(\Omega + U) = \text{Card}(U) \cdot \mathbf{m}(\Omega) = 1. \quad (4.5.4)$$

For $i \in \mathbb{I}$, we have $B(p^{-i}, p^{-n_f}) \subset S(0, p^{-i})$ so that $B(p^{-i}, p^{-n_f}) \subset \mathcal{Z}_{\widehat{\mu}_\Lambda}$, which implies

$$0 = \langle \widehat{\mu}_\Lambda, 1_{B(p^{-i}, p^{-N})} \rangle = \langle \mu_\Lambda, 1_{\widehat{B(p^{-i}, p^{-n_f})}} \rangle = p^{-n_f} \sum_{\lambda \in \Lambda_{n_f}} \overline{\chi(p^{-i}\lambda)}.$$

By Lemma 4.10 (3),

$$p^{\text{Card}(\mathbb{I})} \leq \text{Card}(\Lambda_{n_f}).$$

Then by the fact $\text{Card}(U) = p^{\text{Card}(\mathbb{J})}$ and (4.5.1), we have

$$\text{Card}(U) \cdot \mathbf{m}(\Omega) = p^{\text{Card}(\mathbb{J})} \cdot \frac{\text{Card}(\Lambda_{n_f})}{p^{n_f}} \geq p^{\text{Card}(\mathbb{J})} \cdot \frac{p^{\text{Card}(\mathbb{I})}}{p^{n_f}} = 1.$$

We can now conclude (4.5.4) because $\Omega + U$ is contained in \mathbb{Z}_p so that $\mathbf{m}(\Omega + U) \leq 1$. \square

4.6 Some remarks

4.6.1 Dual Fuglede's conjecture

Theorem 4.2 asserts that all spectral sets (equivalently tiles) in \mathbb{Q}_p are almost compact open. What is the topological structure of the corresponding spectra and tiling complements? This question has been answered by Theorem 1.2 of [37] for the compact open spectral sets in \mathbb{Q}_p . Recall that the spectra and tiling complements of compact open spectral sets are infinite p -homogeneous discrete sets Λ (see [37, Section 2.8]) such that there exists an integer N such that

$$(\infty, N] \cap \mathbb{Z} \subset I_\Lambda, \tag{4.6.1}$$

where $I_\Lambda := \{v_p(x - y) : x, y \in \Lambda \text{ distinct}\}$, called the set of admissible p -orders of Ω . On the other hand, for any given infinite p -homogeneous discrete sets Λ satisfying (4.6.1), we can construct two compact open sets Ω_1 and Ω_2 such that (Ω_1, Λ) is a spectral pair and (Ω_2, Λ) is a tiling pair. For details, we refer the readers to [37, Theorems 1.2 and 5.1]. So the dual Fuglede's conjecture holds in \mathbb{Q}_p (see [65] for the dual Fuglede's conjecture in \mathbb{R}^d): *a subset of \mathbb{Q}_p is a spectrum if and only if it is a tiling complement.*

4.6.2 The density of E satisfying (4.2.1) in \mathbb{Q}_p^d

Assume that E is a discrete subset in \mathbb{Q}_p^d such that $\text{Card}(E \cap K) < \infty$ for any compact subset K of \mathbb{Q}_p^d . Then $\mu_E = \sum_{\lambda \in E} \delta_\lambda$ defines a discrete Radon measure in \mathbb{Q}_p^d . Remark that E is not necessarily assumed uniformly discrete. Suppose that E satisfies the equation (4.2.1) with $f \in L^1(\mathbb{Q}_p^d)$ which is a non-negative integrable function such that $\int_{\mathbb{Q}_p^d} f d\mathbf{m} > 0$. Then for each $x_0 \in \mathbb{Q}_p^d$, the limit

$$D(E) = \lim_{k \rightarrow \infty} \frac{\text{Card}(B(x_0, p^k) \cap E)}{\mathbf{m}(B(x_0, p^k))}$$

exists and

$$D(E) = \frac{1}{\int_{\mathbb{Q}_p^d} f d\mathbf{m}}.$$

This is similar to the real case, see [80, Lemma 2].

4.6.3 Tiles and spectral sets in \mathbb{Q}_p^2 which are not compact open.

For a uniformly discrete set E in the higher dimensional space \mathbb{Q}_p^d with $d \geq 2$, the zero set of the Fourier transform of the measure μ_E is not necessarily bounded. In other words, Proposition 4.12 does not hold in \mathbb{Q}_p^d with $d \geq 2$. For example, let $E = \{(0, 0), (0, 1), \dots, (0, p-1)\}$ which is a finite subset of \mathbb{Q}_p^2 . One can check that

$$\widehat{\mathcal{Z}_{\mu_E}} = \mathbb{Q}_p \times p^{-1}\mathbb{Z}_p^\times.$$

Bounded tiles of \mathbb{Q}_p^2 are not necessarily almost compact open. Let us construct such a bounded tile of \mathbb{Q}_p^2 . We partition \mathbb{Z}_p into p Borel sets of same Haar measure, denoted A_i ($i = 0, 1, \dots, p-1$). We assume that one of A_i is not almost compact open. For example, set $S = \bigcup_{n=1}^{\infty} B(p^n, p^{-n-1})$, which is a union of countable disjoint balls. Let

$$\begin{aligned} A_0 &= S \cup (B(1, p^{-1}) \setminus (1 + S)), \\ A_1 &= (B(0, p^{-1}) \cup B(1, p^{-1})) \setminus A_0, \\ A_i &= B(i, p^{-1}) \text{ for } 2 \leq i \leq p-1. \end{aligned}$$

Then define

$$\Omega := \bigcup_{i=0}^{p-1} A_i \times B(i, p^{-1}) \subset \mathbb{Z}_p \times \mathbb{Z}_p.$$

The set Ω is not almost compact open, because any small ball centered at $(0, 0)$ meets both Ω and $(\mathbb{Z}_p \times \mathbb{Z}_p \setminus \Omega)$ with positive measure. But it is a tile of \mathbb{Q}_p^2 with tiling complement

$$T = \mathbb{L} \times \mathbb{L}_{-1}.$$

Just like on \mathbb{R}^d (see [31]), it can be proved that $(\Omega, \mathbb{L} \times \mathbb{L}_{-1})$ is a tiling pair if and only if $(\Omega, \mathbb{L} \times \mathbb{L}_1)$ is a spectral pair (see [69]).

4.6.4 Higher dimensional cases

However, the situation changes in higher dimension. In fact, it is proved in [8] that there exist spectral sets which are not tiles in \mathbb{F}_p^5 for all odd primes p and in \mathbb{F}_p^4 for all odd primes p such that $p \equiv 3 \pmod{4}$, where \mathbb{F}_p is the prime field with p elements. This implies that there exist compact open spectral sets which are not tiles in \mathbb{Q}_p^5 for all odd primes p and in \mathbb{Q}_p^4 for all odd primes p such that $p \equiv 3 \pmod{4}$.

For $d = 3$, there exists a spectral set in the finite group $(\mathbb{Z}/8\mathbb{Z})^3$ which is not a tile [84]. This implies that there exists a compact open spectral set which is not a tile in \mathbb{Q}_2^3 .

Fuglede's conjecture in \mathbb{Q}_p^2 is open, even under the compact open assumption.

Chapter 5

Preliminaries to the study of oscillating sequences

This chapter provides some preliminary knowledge for the next two chapters. Some properties of oscillating sequences are also discussed. The chapter is organized as follows. In Section 5.1, we revisit the definition of the sequences orthogonal to dynamical systems and to other sequences. In Section 5.2, we show Liu and Sarnak's observation as a complement of Theorem 6.6. Section 5.3 is devoted to proving that oscillating sequences of order 1 are orthogonal to all one-dimension dynamical systems of entropy zero. The idea of the proof follows from Karagulyan [72] where was proved the same result for the Möbius function. In Section 5.4, we provide a proof of Koksma's theorem as a complement. Koksma's theorem will be used in Chapter 7.

5.1 Introduction

As we have mentioned before, following [112], we say that a sequence $(c_n)_{n \in \mathbb{N}}$ is *orthogonal* to a dynamical system (X, T) if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n f(T^n x) = 0, \quad (5.1.1)$$

for any $x \in X$ and any $f \in C(X)$. The sequence $(f(T^n x))_{n \in \mathbb{N}}$ is sometimes called an *observable sequence*, or a *realization* of the dynamical system (X, T) . Usually, (5.1.1) is referred to the orthogonality between the sequence $(c_n)_{n \in \mathbb{N}}$ and the sequence $(f(T^n x))_{n \in \mathbb{N}}$.

We illustrate the notion of orthogonality by some simple examples.

Examples 5.1. (i) *The periodic sequences of zero mean are always orthogonal to the sequence of a constant number.*

(ii) *For every $\alpha \in \mathbb{R}$, the sequence $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ is orthogonal to the sequence of a constant number.*

(iii) *For all $P \in \mathbb{R}[t]$, the sequences $(e^{2\pi i P(n)})_{n \in \mathbb{N}}$ are orthogonal to the sequence of a constant number (see [133, 134]).*

(iv) The Möbius function is orthogonal to the sequences $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ for all $\alpha \in \mathbb{R}$ (see [24]).

(iv) The Möbius function is orthogonal to the sequences $(e^{2\pi i P(n)})_{n \in \mathbb{N}}$ for all $P \in \mathbb{R}[t]$ (see [57]).

It should be noted that we may deduce the orthogonal property to certain dynamical systems by the orthogonal property to some class of sequences. For example, it is not hard to see that if a sequence $(c_n)_{n \in \mathbb{N}}$ of complex numbers is orthogonal to a sequence of a constant number, then the sequence $(c_n)_{n \in \mathbb{N}}$ is orthogonal to the trivial system reduced to one point. Another interesting example is that if a sequence $(c_n)_{n \in \mathbb{N}}$ of complex numbers is orthogonal to all the sequences $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ for all $\alpha \in \mathbb{R}$, in other words, it is oscillating of order 1, then the sequence $(c_n)_{n \in \mathbb{N}}$ is orthogonal to all the rotations on the unit circle, that is, all the systems $(\mathbb{R}/\mathbb{Z}, R_\beta)$ for all $\beta \in \mathbb{R}$, where $R_\beta(x) = x + \beta \pmod{1}$. Moreover, Liu and Sarnak proved that if a sequence is orthogonal to all sequences $(e^{2\pi i P(n)})_{n \in \mathbb{N}}$ for all $P \in \mathbb{R}[X]$, in other words, it is fully oscillating, then it is orthogonal to all affine maps on compact abelian groups of entropy zero. This is presented in the next section.

5.2 Liu and Sarnak's observation

In this section, we present Liu and Sarnak's observation in [99]: every orbit of any affine map on any compact abelian group of zero entropy could be realized by an orbit of an affine map on $\mathbb{T}^{d+1} \times F$ for some finite abelian group F . The content of this section is rewritten from Section 2 of [99].

Let X be a compact abelian group. Let $T : x \mapsto Ax + b$ be an affine map on X where A is an automorphism and $b \in X$. Denote by \widehat{X} the dual group of X . The action of A on \widehat{X} is induced by the action of A on X , that is to say, $Ah(x) = h(Ax)$ for all $h \in \widehat{X}$ and for all $x \in X$. Fix $\phi \in \widehat{X}$. Let C_ϕ be the smallest A -invariant closed subgroup of \widehat{X} which contains ϕ . Let

$$C_\phi^\perp = \{x \in X \mid h(x) = 1, \forall h \in C_\phi\}$$

be the annihilator of C_ϕ . Let $X_\phi = X/C_\phi^\perp$ be the quotient group of X . It follows that the dual group \widehat{X}_ϕ is isomorphic to C_ϕ . It is not hard to see that

$$T(x + y) = Tx \pmod{C_\phi^\perp},$$

for any $x \in X$ and $y \in C_\phi^\perp$. It follows directly that T induces an affine map T_ϕ on the quotient space X_ϕ . Moreover the dynamical system (X_ϕ, T_ϕ) as a factor of the system (X, T) also has zero entropy.

On the other hand, by Aoki's theorem ([6], Page 13), the abelian group C_ϕ is finitely generated. The group X_ϕ as the dual group of C_ϕ is then isomorphic to $\mathbb{T}^d \times F$ with F a finite abelian group and $d \geq 0$ an integer.

For an arbitrary $x \in X$, now we look at the observable sequence $(\phi(T^n x))_{n \in \mathbb{N}}$. It is clear that $\phi(T^n x) = \widetilde{\phi}(T_\phi^n \tilde{x})$ where $\tilde{x} \in X_\phi$ is the projection of $x \in X$ on its quotient space

X_ϕ and $\tilde{\phi}$ is the character in $\widehat{X_\phi}$ induced by the character ϕ . It follows that the observable sequence $(\phi(T^n x))_{n \in \mathbb{N}}$ is the same with the observable sequence $(\tilde{\phi}(T_\phi^n \tilde{x}))_{n \in \mathbb{N}}$ realized by the affine map T_ϕ on $\mathbb{T}^d \times F$ of zero entropy. This completes the proof.

5.3 Oscillating sequences are orthogonal to one-dimension dynamics with zero entropy

In this section, we prove that oscillating sequences of order 1 are orthogonal to all one-dimension dynamical systems of entropy zero. This illustrates the statement which we have mentioned in Section 5.1 that the orthogonal property to certain dynamical systems is strongly influenced by the orthogonal property to some class of sequences. The idea of the proof follows from Karagulyan [72] where was proved the same result for the Möbius function. We also remark that the result that oscillating sequences of order 1 are orthogonal to all orientation-preserving circle homeomorphisms have already been proved by Fan and Jiang [38] but the proof which we show here is different from theirs. We begin with studying some properties of oscillating sequences of order 1.

5.3.1 Properties of oscillating sequences

Recall that an oscillating sequence $(c_n)_{n \in \mathbb{N}}$ of order 1 satisfies the *growth condition* if there exists $\lambda > 1$ such that

$$\sum_{n=1}^N |c_n|^\lambda = O(N). \tag{5.3.1}$$

For example, the Möbius function is oscillating sequences of order 1 satisfying the growth condition (5.3.1). The λ -*seminorm* of a sequence $(c_n)_{n \in \mathbb{N}}$ of complex numbers is defined by the formula

$$\|c\|_\lambda := \limsup_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{i=1}^N |c_i|^\lambda \right)^{\frac{1}{\lambda}}.$$

We use the symbol \mathcal{C}^λ to denote the space of complex sequences whose λ -seminorm are finite.

The following proposition states that the arithmetic subsequences of an oscillating sequence of order 1 are also oscillating sequences of order 1. For a similar result of oscillating sequences of higher order, we refer to Lemma 1.14 and Theorem 4 in [32].

Proposition 5.2 ([38], Proposition 4). *Let $(c_n)_{n \in \mathbb{N}}$ be an oscillating sequence. Then for any $q \geq 2$ and any $r = 1, 2, \dots, q$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{1 \leq n \leq N \\ n \equiv r \pmod q}} c_n e^{2\pi i t n} = 0, \quad \forall t \in [0, 1].$$

Let (X, T) be a topological dynamical system. Let $x \in X$. Recall that the *orbit* of x is the set $O(x) := \{T^n(x) | n \in \mathbb{N}\}$ and the *ω -limit set* $\omega(x)$ is the set of all limit points of $O(x)$.

Now we prove a sufficient condition for an oscillating sequence of order 1 to be orthogonal to the dynamical system on the unit interval.

Lemma 5.3. *Let T be a continuous transformation from the unit interval $[0, 1]$ to itself. Let $(c_n)_{n \in \mathbb{N}}$ be an oscillating sequence of order 1 satisfying the growth condition (5.3.1). Suppose that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_n 1_J(T^n(x)) = 0, \tag{5.3.2}$$

for all intervals J in $[0, 1]$ and for all $x \in [0, 1]$. Then the sequence $(c_n)_{n \in \mathbb{N}}$ is orthogonal to the dynamical system $([0, 1], T)$

Proof. Fix a continuous function $f : [0, 1] \rightarrow \mathbb{C}$. Fix $\epsilon > 0$. The uniform continuity of f allows us to divide $[0, 1]$ into disjoint intervals I_i ($0 \leq i \leq n-1$) such that $|f(x) - f(y)| < \epsilon$ for any $x, y \in I_i$. We define the step function

$$f_\epsilon(x) := \sum_{i=0}^{n-1} f(\xi_i) 1_{I_i}(x),$$

where ξ_i is some fixed point in each I_i ($0 \leq i \leq n-1$). It is clear that

$$\|f - f_\epsilon\|_\infty < \epsilon,$$

where $\|\cdot\|_\infty$ denotes the uniform norm (or sup norm) on the space of continuous functions on $[0, 1]$. It follows that

$$\left| \frac{1}{N} \sum_{n=1}^N c_n f(T^n(x)) - \frac{1}{N} \sum_{n=1}^N c_n f_\epsilon(T^n(x)) \right| \leq \|c\|_\lambda \|f - f_\epsilon\|_\infty.$$

Noticing that ϵ is arbitrary small, we complete the proof by (4). □

The principal significance of this lemma is to allow one to consider the indicator function of intervals instead of continuous functions when we need to prove the orthogonality.

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. For all $l \in \mathbb{Z}$, the *l -shift* of the sequence $(z_n)_{n \in \mathbb{N}}$ is defined by $(z_n^{(l)})_{n \in \mathbb{N}} := (z_{n+l})_{n \in \mathbb{N}}$ with $z_k = 0$ for $k \leq 0$.

Lemma 5.4. *Given $\epsilon > 0$ and $\lambda > 1$. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. Let $(c_n)_{n \in \mathbb{N}}$ be an oscillating sequence of order 1. Suppose $(c_n)_{n \in \mathbb{N}} \in \mathcal{C}^\lambda$. If there exists a shift $(z_n^{(m)})_{n \in \mathbb{N}}$ of the sequence $(z_n)_{n \in \mathbb{N}}$ for some $m \in \mathbb{N}$ such that*

$$\limsup_{N \rightarrow \infty} \frac{1}{\|c\|_\lambda} \left| \frac{1}{N} \sum_{n=1}^N c_n z_n^{(m)} \right| < \epsilon, \tag{5.3.3}$$

then the superior limit (5.3.3) holds for all shift $(z_n^{(k)})_{n \in \mathbb{N}}$ for all $k \in \mathbb{N}$.

Proof. It is not hard to see that if $(c_n)_{n \in \mathbb{N}}$ is an oscillating sequence of order 1 in \mathcal{C}^λ then each of its shifts $(c_n^{(k)})_{n \in \mathbb{N}}$ is also an oscillating sequence of order 1 in \mathcal{C}^λ . Notice that the sum of finite terms counts zero in the superior limit (5.3.3). This finishes the proof. \square

5.3.2 Interval maps of zero entropy

In [38], Fan and Jiang proved that all oscillating sequences are orthogonal to Feigenbaum zero topological entropy flows. In this section, we prove that all oscillating sequences are orthogonal to all continuous interval maps of zero entropy. The idea of our proof is from Section 3 of [72]. We have the following theorem.

Theorem 5.5. *All oscillating sequences of order 1 satisfying the growth condition (5.3.1) are orthogonal to all continuous interval maps of zero entropy.*

Proof. Let $(c_n)_{n \in \mathbb{N}}$ be an oscillating sequence of order 1 satisfying the growth condition (5.3.1). Let T be a continuous map from $[0, 1]$ to $[0, 1]$ of zero entropy. Fix a point $x \in [0, 1]$. Let us consider the ω -limit set $\omega(x)$. If $\omega(x)$ is finite, then by Proposition 5.2, we get the desired result that the sequence $(c_n)_{n \in \mathbb{N}}$ is orthogonal to $([0, 1], T)$. It remains to consider the case when the ω -limit set $\omega(x)$ is infinite. According to Proposition 5.22 in [110], for all $m \in \mathbb{N}$, there exist closed subintervals $\{L_m^k\}_{0 \leq k \leq 2^m - 1}$ of the interval $[0, 1]$ such that

- (i) L_m^k are pairwise disjoint for $0 \leq k \leq 2^m - 1$;
- (ii) $T(L_m^k) = L_m^{k+1 \pmod{2^m}}$;
- (iii) $\omega(x) \subset \mathcal{L}_m := \cup_{0 \leq k \leq 2^m - 1} L_m^k$.

The finiteness of $\partial \mathcal{L}_m$ implies $\omega(x) \cap \overset{\circ}{\mathcal{L}}_m \neq \emptyset$ where $\partial \mathcal{L}_m$ is the boundary of \mathcal{L}_m and $\overset{\circ}{\mathcal{L}}_m$ is the set of inner points of \mathcal{L}_m . It follows that $O(x) \cap \overset{\circ}{\mathcal{L}}_m \neq \emptyset$.

It is sufficient to prove (5.3.2) for all interval J . Then by Lemma 5.3, we prove the theorem. Now fix a subinterval J of $[0, 1]$ and an integer $m \in \mathbb{N}$. Set the end points of J as a and b (from left to right). We claim that the superior limit (5.3.3) holds when z_n and ϵ are replaced by $1_J(T^n(x))$ and $2 \cdot (\frac{1}{2^m})^{\frac{1}{\lambda'}}$ respectively where λ' is the dual of λ , i.e. $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$. By Lemma 5.4, we only need to show that there is $y \in O(x)$ such that the superior limit (5.3.3) holds when z_n and ϵ are replaced by $1_J(T^n(y))$ and $2 \cdot (\frac{1}{2^m})^{\frac{1}{\lambda'}}$ respectively. When m tends to infinity, we get that $2 \cdot (\frac{1}{2^m})^{\frac{1}{\lambda'}}$ tends to zero. Then the theorem follows from our claim. Thus it remains to prove the claim. According to the position between J and \mathcal{L}_m , we decompose the proof of the claim into four cases.

Case (1): $a \notin \mathcal{L}_m$ and $b \notin \mathcal{L}_m$. Then there exists a subset $K \subset \{0, 1, \dots, 2^m - 1\}$ such that

$$\bigcup_{k \in K} L_m^k \subset I \text{ and } \left(\bigcup_{i \in K^c} L_m^i \right) \cap I = \emptyset.$$

It follows that for any point $y \in O(x) \cap \mathring{\mathcal{L}}_m$, the sequence $(1_J(T^n(y)))_{n \in \mathbb{N}}$ is periodic. Therefore, by Proposition 5.2, the superior limit (5.3.3) when z_n is replaced by $1_J(T^n(y))$ is zero.

Case (2): $a \in \mathcal{L}_m$ and $b \notin \mathcal{L}_m$. It is observed that $[0, 1] \setminus \mathcal{L}_m$ is a union of disjoint intervals. According to the position of a , we pick an arbitrary point c in the first right-side interval of $[0, 1] \setminus \mathcal{L}_m$. This deduces a decomposition $J = J_1 \cup J_2$ where $J_1 = [a, c]$ and $J_2 = [c, b]$. It follows that J_1 intersects one of $\{L_m^k\}$ and J_2 is the interval of Case (1). Thus instead of $(1_J T^n(x))_{n \in \mathbb{N}}$, what is left is to consider the sequence $(1_{J_1} T^n(x))_{n \in \mathbb{N}}$. Notice the fact that for any point $y \in O(x) \cap \mathring{\mathcal{L}}_m$, if there exist $t_1, t_2 \in \mathbb{N}$ such that $T^{t_1}(y), T^{t_2}(y) \in J_1$ then $|t_1 - t_2| \geq 2^m$. Therefore, we have

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1_{J_1}(T^n(y)) \leq \frac{1}{2^m}.$$

By Hölder's inequality, we have

$$\limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{i=1}^N c_i 1_{J_1}(T^i(y)) \right| \leq \|c\|_\lambda \left(\frac{1}{2^m}\right)^{\frac{1}{\lambda}},$$

which completes our claim.

Case (3): $a \notin \mathcal{L}_m$ and $b \in \mathcal{L}_m$. It is the same calculation with Case (2) by picking an arbitrary point d in the first left-side interval of $[0, 1] \setminus \mathcal{L}_m$ according to the position of b .

Case (4): $a \in \mathcal{L}_m$ and $b \in \mathcal{L}_m$. We divide J into three intervals J_1, J_2 and J_3 by picking two points c, d where c is chosen as Case (2) and d is chosen as Case (3). We then do the same calculation with Case (2).

□

5.3.3 Circle Homeomorphism

In [38], Fan and Jiang proved that oscillating sequences of order 1 are orthogonal to all orientation-preserving circle homeomorphisms. In this section, we use a different method to prove the same result. The idea of our proof is partially from Section 4 of [72].

We write R_ρ for the rigid rotation with rotation number ρ on the unit circle. By the definition of oscillating sequences of order 1, it is clear that oscillating sequences are orthogonal to all rigid rotations on the circle.

Denote by $P : \mathbb{R} \rightarrow S^1, P(x) = e^{2\pi i x}$. Recall that every map $\phi : S^1 \rightarrow S^1$ has a unique lift $\tilde{\phi} : \mathbb{R} \rightarrow \mathbb{R}$ such that $P \circ \tilde{\phi} = \phi \circ P$ and $\tilde{\phi}(0) = 0$. Moreover, the map ϕ on the unit circle is said to be *monotone* if its lift $\tilde{\phi}$ is monotone. The *degree* of ϕ is defined by the difference $\tilde{\phi}(1) - \tilde{\phi}(0)$.

Lemma 5.6. *Let $(c_n)_{n \in \mathbb{N}}$ be an oscillating sequence satisfying the condition (5.3.1). For any interval J in the unit circle and for any irrational number ρ , we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N c_n h(R_\rho^n(x)) = 0, \quad \forall x \in S^1, \quad (5.3.4)$$

where $h = 1_J$. Moreover, let g be a continuous function $S^1 \rightarrow \mathbb{C}$ and π be a monotone map: $S^1 \rightarrow S^1$. Then the limit (5.3.4) also holds for $h = g \circ \pi$.

Proof. Since oscillating sequences of order 1 are orthogonal to all rigid rotations on the unit circle, the limit (5.3.4) holds for all $h \in C(S^1)$. Set the end points of J by a and b (from left to right). We approximate 1_J by the continuous functions f_ϵ ($\epsilon > 0$, small) which is equal to 1 on $S^1 \setminus J$, equal to 0 outside $(a - \epsilon, b + \epsilon)$ and linear otherwise. According to the construction of f_ϵ , we have

$$|1_J(x) - f_\epsilon(x)| \leq 1_{(a-\epsilon, a) \cup (b, b+\epsilon)}(x), \text{ for all } x \in S^1.$$

Since the sequence $(c_n)_{n \in \mathbb{N}}$ satisfies the condition (5.3.1), denoting by λ' the dual of λ , i.e. $\frac{1}{\lambda} + \frac{1}{\lambda'} = 1$, by Hölder's inequality, we have

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N c_n 1_J(R_\rho^n(x)) - \frac{1}{N} \sum_{n=1}^N c_n f_\epsilon(R_\rho^n(x)) \right| \\ & \leq \|c\|_\lambda \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N 1_{(a-\epsilon, a) \cup (b, b+\epsilon)}(R_\rho^n(x)) \right|^{\frac{1}{\lambda'}} \\ & \leq \|c\|_\lambda (2\epsilon)^{\frac{1}{\lambda'}}. \end{aligned}$$

The last inequation follows from the uniform distribution of $(R_\rho(x)^n)_{n \in \mathbb{N}}$ for any irrational number ρ . We can take ϵ arbitrary small in the above equation, which establishes the limit (5.3.4) for $h = 1_J$.

On the other hand, for any $\epsilon' > 0$, the uniform continuity of g allows us to divide S^1 into disjoint intervals I_i ($0 \leq i \leq n - 1$) such that for any $x, y \in I_i$ ($0 \leq i \leq n - 1$), we have $|g(x) - g(y)| < \epsilon'$. Since π is monotone, the sets $J_i := \pi^{-1}(I_i)$ ($0 \leq i \leq n - 1$) are intervals and form a partition of S^1 . We approximate f by the step function

$$F_{\epsilon'}(u) := \sum_{i=0}^{n-1} f(\xi_i) 1_{J_i}(u),$$

where ξ_i is some fixed point in J_i for $0 \leq i \leq n - 1$. It follows from the definition of $F_{\epsilon'}$ that

$$\|f - F_{\epsilon'}\|_\infty < \epsilon'.$$

Since the sequence $(c_n)_{n \in \mathbb{N}}$ satisfies the condition (5.3.1), we have

$$\left| \frac{1}{N} \sum_{n=1}^N c_n f(R_\rho^n(x)) - \frac{1}{N} \sum_{n=1}^N c_n F_{\epsilon'}(R_\rho^n(x)) \right| \leq \|c\|_\lambda \|f - F_{\epsilon'}\|_\infty, \forall x \in S^1.$$

Since ϵ' can be arbitrary small, the limit (5.3.4) holds for $h = g \circ \pi$. □

The lemma gains interest that the first part of this lemma is the inverse of Lemma 5.3 when the dynamical system $([0, 1], T)$ is replaced by (S^1, R_ρ) . The second part of this lemma is useful in the proof of Theorem 5.8.

Now we prove a lemma concerned with orientation-preserving circle homeomorphisms.

Lemma 5.7. *Let T be an orientation-preserving circle homeomorphism which is semi-conjugate to an irrational rotation R_ρ by a non-invertable continuous monotone map π of degree one. Then for any $x \in S^1$ there exists a monotone map $\pi_x : S^1 \rightarrow S^1$ such that*

(i) $\pi_x \circ \pi|_{O(x)} = id_{O(x)}$;

(ii) the diagram

$$\begin{array}{ccc} \pi(O(x)) & \xrightarrow{R_\rho} & \pi(O(x)) \\ \pi_x \downarrow & & \downarrow \pi_x \\ O(x) & \xrightarrow{T} & O(x) \end{array}$$

commutes.

Proof. Fix a point $x \in S^1$. For any $n, m \in \mathbb{N}$, if $\pi(T^n(x)) = \pi(T^m(x))$, then $R_\rho^n(\pi(x)) = R_\rho^m(\pi(x))$ by semi-conjugation. Thus $n = m$ by the fact that ρ is irrational. It follows that $\pi|_{O(x)}$ is injective.

The lift of π is denoted by $\tilde{\pi} : \mathbb{R} \rightarrow \mathbb{R}$ and the pre-image of $O(x)$ via P is denoted by $\widetilde{O(x)}$. It follows that $\tilde{\pi} : \widetilde{O(x)} \rightarrow \tilde{\pi}(\widetilde{O(x)})$ is bijective and monotone. Thus the inverse of $\tilde{\pi}$ on $\tilde{\pi}(\widetilde{O(x)})$ exists and is also monotone. We denote this inverse by $\tilde{\pi}_x$. We now extend the domain of definition of the map $\tilde{\pi}_x$ from $\tilde{\pi}(\widetilde{O(x)})$ to \mathbb{R} maintaining the monotonicity. In fact, there are several ways to do this. For example, we define the extension $\tilde{\tilde{\pi}}_x$ of $\tilde{\pi}_x$ as follows:

$$\tilde{\tilde{\pi}}_x = \inf_{y \leq z} \{ \tilde{\pi}_x(y) \mid y \in \tilde{\pi}(\widetilde{O(x)}) \}.$$

In what follows, we use the same symbol $\tilde{\tilde{\pi}}_x$ to denote its extension $\tilde{\tilde{\pi}}_x$ on \mathbb{R} . Since the degree of π is one, the projection of $\tilde{\tilde{\pi}}_x$ to S^1 , denoted by π_x , is well defined. It is easy to check that $\pi_x \circ \pi|_{O(x)} = id_{O(x)}$. In addition, by $R_\rho \circ \pi = \pi \circ T$, we have

$$\pi_x \circ (R_\rho \circ \pi)(x) = (\pi_x \circ \pi) \circ T(x) = T(x) = T \circ (\pi_x \circ \pi)(x),$$

that is, for all $y \in \pi(O(x))$,

$$\pi_x \circ R_\rho(y) = T \circ \pi_x(y).$$

It follows that the map π_x satisfies the conditions (i) and (ii), which is our desired conclusion. □

The interest of Lemma 5.7 is that if the system (S^1, T) is semi-conjugate to the rigid rotation (S^1, R_ρ) for ρ irrational, then every orbit of the system (S^1, T) can be realized by an orbit of the system (S^1, R_ρ) .

Now we have the following theorem.

Theorem 5.8. *All oscillating sequences of order 1 satisfying the condition (5.3.1) are orthogonal to all orientation-preserving circle homeomorphisms.*

Proof. Let $(c_n)_{n \in \mathbb{N}}$ be an oscillating sequence of order 1 satisfying the growth condition (5.3.1). Let T be an orientation-preserving circle homeomorphism. If the rotation number ρ of T is rational, then every orbit of T converges to a periodic orbit (see for instance [70, Proposition 11.2.2]). By Theorem 1 in [38], the oscillating sequence $(c_n)_{n \in \mathbb{N}}$ is orthogonal to the system (S^1, T) .

It remains to prove the case when the rotation number ρ of T is irrational. By Poincaré's classification theorem, T is either conjugate to the irrational rotation R_ρ or semi-conjugate to the irrational rotation R_ρ by a non-invertible continuous monotone map π of degree one. The first case follows directly from the definition of the oscillating sequence. It is sufficient to prove the second case.

Fix a point $x \in S^1$. There exists π_x satisfying the condition (i) and (ii) in Lemma 5.7. Let $y = \pi(x)$. For any $f \in C(S^1)$, the observed sequence $(f \circ T^n(x))_{n \in \mathbb{N}}$ can be rewritten by the formula

$$f \circ T^n(x) = f \circ T^n \circ \pi_x(y) = f \circ \pi_x \circ R_\rho^n(y).$$

It is not hard to see that $f \circ \pi_x$ satisfies the condition in Lemma 5.6. It follows that the limit (5.3.4) holds for $f \circ \pi_x$, which completes the proof. \square

5.4 Koksma's theorem

In this section, we give the proof of Koksma's theorem. We start with presenting some metrical results about the uniform distribution modulo one.

Let us first recall Weyl's criterion.

Theorem 5.9 ([133, 134]). *A sequence $(x(n))_{n \in \mathbb{N}}$ of real numbers is uniformly distributed modulo one if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h x(n)} = 0, \quad (5.4.1)$$

for all non-zero integers h .

Proof. By the definition of the Riemann integral, the sequence $(x(n))_{n \in \mathbb{N}}$ is uniformly distributed modulo one if and only if for every complex-valued, 1-periodic continuous function f ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x(n)) = \int_0^1 f(x) d(x). \quad (5.4.2)$$

The statement that (5.4.1) \Leftrightarrow (5.4.2) follows directly from the Stone-Weierstrass theorem stating that, for the sup norm, finite linear combinations of the functions $x \mapsto e^{2\pi i h x}$, $h \in \mathbb{Z}$ with complex coefficients, are dense in the space of complex-valued 1-periodic continuous functions. \square

An immediate application of Theorem 5.9 is that for any irrational real number α , the sequence $(n\alpha)_{n \in \mathbb{N}}$ is uniformly distributed modulo one.

Next we display an auxiliary lemma of Davenport, Erdős and LeVeque [25].

Lemma 5.10. *Let μ be a measure on a measurable space X . Let $(X_n)_{n \in \mathbb{N}}$ be a bounded sequence of measurable functions defined on X . Suppose that the series*

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_X \left| \frac{1}{N} \sum_{n=1}^N X_n \right|^2 d\mu < +\infty.$$

Then almost surely

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=1}^N X_n = 0.$$

Proof. See [25] or page 6 of [19]. \square

Now we state Koksma's theorem. The proof that we present here is based on the proof of Theorem 1.10 in [19] where it is proved that given $\xi \neq 0$, for almost every $\alpha > 1$, the sequence $(\xi \alpha^n)_{n \in \mathbb{N}}$ is uniformly distributed modulo one.

Theorem 5.11 (Koksma's theorem). *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued C^1 functions defined on an interval $[a, b] \subset \mathbb{R}$. Suppose that*

(i) $f'_m - f'_n$ is monotone on $[a, b]$ for all $m \neq n$;

(ii) $\inf_{m \neq n} \min_{x \in [a, b]} |f'_m(x) - f'_n(x)| > 0$.

Then for almost every $x \in [a, b]$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is uniformly distributed modulo one on the interval $[a, b]$.

Proof. Let N be a positive integers. Rearranging f_1, f_2, \dots, f_N , we assume that $f'_s(x) - f'_t(x) > 0$ for all $1 \leq t < s \leq N$ (by (ii)). Let m, n be integers with $1 \leq n < m \leq N$. Let h be a non-zero integer and set

$$I_{h,m,n} := \int_a^b e^{2\pi i h (f_m(x) - f_n(x))} dx.$$

Denote by $\Phi_{m,n}(x) = f_m(x) - f_n(x)$ which is strictly monotone on $[a, b]$ by (i) and (ii). Let $\Psi_{m,n}$ be the reciprocal function of $\Phi_{m,n}$ which is also strictly monotone. We observe that

$$I_{h,m,n} = \int_{\Phi_{m,n}(a)}^{\Phi_{m,n}(b)} e^{2\pi i h y} \Psi'_{m,n}(y) dy.$$

By reciprocal rule, we have

$$\Psi'_{m,n}(y) = \frac{1}{\Phi'_{m,n}(\Psi_{m,n}(y))} = \frac{1}{f'_m(\Psi_{m,n}(y)) - f'_n(\Psi_{m,n}(y))}.$$

Let

$$\epsilon := \inf_{s \neq t} \min_{x \in [a,b]} |f'_s(x) - f'_t(x)| > 0.$$

A simple computation shows that

$$\begin{aligned} & |f'_m(\Psi_{m,n}(y)) - f'_n(\Psi_{m,n}(y))| \\ &= |f'_m(\Psi_{m,n}(y)) - f'_{m-1}(\Psi_{m,n}(y))| + |f'_{m-1}(\Psi_{m,n}(y)) - f'_{m-2}(\Psi_{m,n}(y))| + \dots \\ &\quad + |f'_{n+1}(\Psi_{m,n}(y)) - f'_n(\Psi_{m,n}(y))| \\ &\geq (m-n)\epsilon. \end{aligned}$$

It follows that

$$|\Psi'_{m,n}(y)| \leq \frac{1}{(m-n)\epsilon}.$$

By the second mean value theorem, there exists c in the interval with end points $\Phi_{m,n}(a)$ and $\Phi_{m,n}(b)$ such that

$$I_{h,m,n} = \Psi'_{m,n}(\Phi_{m,n}(a)) \int_{\Phi_{m,n}(a)}^c e^{2\pi i h y} dy + \Psi'_{m,n}(\Phi_{m,n}(b)) \int_c^{\Phi_{m,n}(b)} e^{2\pi i h y} dy.$$

Thus

$$|I_{h,m,n}| \leq \frac{1}{\pi h} (|\Psi'_{m,n}(\Phi_{m,n}(a))| + |\Psi'_{m,n}(\Phi_{m,n}(b))|) \leq \frac{2}{\pi h \epsilon (m-n)}.$$

Then we have

$$\begin{aligned} I_N &:= \int_a^b \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i h f_n(x)} \right|^2 dx \leq \frac{b-a}{N} + \frac{2}{N^2} \left| \sum_{1 \leq n < m \leq N} I_{h,m,n} \right| \\ &\leq \frac{b-a}{N} + \frac{2}{N^2} \sum_{1 \leq n < m \leq N} \frac{2}{\pi h \epsilon (m-n)} \\ &\leq \frac{b-a}{N} + \frac{8}{\pi h} \cdot \frac{\log N}{N}. \end{aligned}$$

Therefore, the sum of I_N/N over the positive integers N converges. It follows from Lemma 5.10 that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i h f_n(x)} = 0,$$

for almost all $x \in [a, b]$. We deduce the result from Weyl's criterion (Theorem 5.9). \square

Consequently, for instance, the sequence (β^n) is uniformly distributed modulo one for almost all $\beta > 1$.

Chapter 6

Equivalent definitions of oscillating sequences of higher orders

Abstract. An oscillating sequence of order d is defined by the linearly disjointness from all $\{e^{2\pi iP(n)}\}_{n=1}^{\infty}$ for all real polynomials P of degree smaller or equal to d . A fully oscillating sequence is defined to be an oscillating sequence of all orders. In this paper, we give several equivalent definitions of such sequences in terms of their disjointness from different dynamical systems on tori.

Let us first recall that a sequence of complex numbers (c_n) is said to be *oscillating of order d* ($d \geq 1$ being an integer) if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi iP(n)} = 0$$

for any $P \in \mathbb{R}_d[t]$ where $\mathbb{R}_d[t]$ is the space of real polynomials of degree smaller than or equal to d ; it is said to be *fully oscillating* if it is oscillating of order d for every integer $d \geq 1$.

Sequences of the form $f_1(T^n x) f_2(T^{2n} x) \cdots f_\ell(T^{\ell n} x)$ with $f_1, \dots, f_\ell \in C(X)$ and $x \in X$ are called *multiple ergodic realizations*. Their orthogonality to fully oscillating sequences was studied in [31, 32].

The following lemma is evident by the definition of oscillating sequences because the constant term of the polynomial does not play any role.

Lemma 6.1. *Let $d \geq 1$. A sequence of complex numbers (c_n) is oscillating of order d if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i n Q(n)} = 0$$

for any $Q \in \mathbb{R}_{d-1}[t]$.

The following lemma has its own interest. The proof given here is based on a fact proved in [38].

Lemma 6.2. *Let $d \geq 1$. A sequence of complex numbers (c_n) is oscillating of order d if and only if all arithmetic subsequences (c_{an+b}) are oscillating of order d for all integers $a \in \mathbb{N}^*$ and $b \in \mathbb{N}$.*

Proof. The sufficiency being trivial, we just prove the necessity. Assume that (c_n) is oscillating of order d . By definition, it is easy to see that for any $b \in \mathbb{N}$, the translated sequence (c_{n+b}) is also oscillating of order d . Thus we only need to prove that (c_{an}) is oscillating of order d for any $a \in \mathbb{N}^*$. We will prove this by induction on d based on Proposition 4 in [38], which states that the case $d = 1$ is true.

Suppose that the assertion is proved for $d - 1$. Take any polynomial $P \in \mathbb{R}_d[t]$ and write $P(t) = wt^d + Q(t)$ with $Q \in \mathbb{R}_{d-1}[t]$. Since (c_n) is oscillating of order d , the sequence $\tilde{c}_n := c_n e^{2\pi i \frac{w}{a^d} n^d}$ is also oscillating of order d , a fortiori, \tilde{c}_n is oscillating of order $d - 1$. Then by the hypothesis of induction, \tilde{c}_{an} is also oscillating of order $d - 1$, in particular,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \tilde{c}_{an} e^{2\pi i Q(n)} = 0.$$

Now it suffices to observe

$$\forall n \in \mathbb{N}^*, \quad \tilde{c}_{an} e^{2\pi i Q(n)} = c_{an} e^{2\pi i P(n)}$$

to conclude that (c_{an}) is oscillating of order d . □

The following proposition will be used to prove Theorem 6.5.

Proposition 6.3. *A sequence of complex numbers (c_n) is oscillating of order d if and only if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i (P(n)+Q(n))} = 0$$

for any $P \in \mathbb{R}_d[t]$ and any $Q \in \mathbb{Q}[t]$.

Proof. The sufficiency being trivial, we just prove the necessity. We first remark that for any $Q \in \mathbb{Q}[t]$, the sequence $(e^{2\pi i Q(n)})$ is periodic. Indeed, on one hand, $(e^{2\pi i \frac{p}{q} n^k})$ is q -periodic for any integer k and any rational number $\frac{p}{q}$; on the other hand, the product of two periodic sequences is still periodic and the l.c.m of the two periods is a period of the product.

Let ℓ be a period of the sequence $(e^{2\pi i Q(n)})$, then

$$\begin{aligned} \sum_{n=0}^{N-1} c_n e^{2\pi i(P(n)+Q(n))} &= \sum_{j=0}^{\ell-1} \sum_{\substack{0 \leq n \leq N-1 \\ n \equiv j \pmod{\ell}}} c_n e^{2\pi i(P(n)+Q(n))} \\ &= \sum_{j=0}^{\ell-1} e^{2\pi i Q(j)} \sum_{\substack{0 \leq n \leq N-1 \\ n \equiv j \pmod{\ell}}} c_n e^{2\pi i P(n)}. \end{aligned}$$

Thus we can conclude by Lemma 6.2. \square

We finish this section by pointing out that the notion of oscillation of order $d(\geq 2)$ is stronger than the notion of weak oscillation of order $d(\geq 2)$ introduced in [38]. A sequence (c_n) is called *weakly oscillating of order d* if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i n^k t} = 0$$

for any $0 \leq k \leq d$ and any $t \in [0, 1)$. A sequence is said to be *fully weakly oscillating* if it is weakly oscillating of order d for all integers $d \geq 1$. By Lemma 6.1, it is evident that weakly oscillating sequences of order 1 are exactly oscillating sequences of order 1.

But it is not the case when $d \geq 2$.

Proposition 6.4. *Let $d \geq 2$. There exist weakly oscillating sequences of order d which are not oscillating of order d .*

Proof. The following sequence defined by

$$c_n := e^{2\pi i(n^d \alpha + n^{d-1} \beta)} \quad \text{where } \alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$$

is weakly oscillating of order d but not oscillating of order d . \square

A direct corollary of Proposition 6.4 is that a fully weakly oscillating sequence is not necessarily a fully oscillating sequence.

6.1 Statement of the results

Our main result is as follows.

Theorem 6.5. *A sequence (c_n) of complex numbers is oscillating of order d if and only if it is orthogonal to one of the following dynamical systems or one of the following classes of dynamical systems.*

- (1) *All automorphisms of zero entropy on $\mathbb{T}^{d+1} \times F$ for any finite abelian group F .*

- (2) A special automorphism of zero entropy on $\mathbb{T}^{d+1} \times F$, where F is any fixed finite abelian group F .
- (3) All affine maps of zero entropy on $\mathbb{T}^d \times F$ for any finite abelian group F .
- (4) All special affine maps of zero entropy on $\mathbb{T}^d \times F$, which share the same section of an automorphism, where F is any fixed finite abelian group.

We remark that the finite abelian group F in Theorem 6.5 can be the trivial group so that $\mathbb{T}^d \times F$ is nothing but the torus \mathbb{T}^d .

For full oscillating sequences, we have the following theorem.

Theorem 6.6. *A sequence of complex number (c_n) is full oscillating if and only if it is orthogonal to all affine maps of zero entropy on any compact abelian group.*

The necessity in Theorem 6.6 is essentially proved in [99], where the authors observe that every orbit of any affine map of zero entropy on any compact abelian group could be realized by an orbit of an affine map of zero entropy on $\mathbb{T}^{d+1} \times F$ for some finite abelian group F . On the other hand, the sufficiency is a direct consequence of Theorem 6.5.

6.2 Quasi-unipotent automorphisms

In [99], the authors proved the Möbius disjointness from affine maps on compact abelian group of zero entropy by using the fact that the automorphisms on torus of zero entropy are quasi-unipotent ([23]). In this section, we examine the orbits of quasi-unipotent automorphisms on compact abelian groups. Recall that an automorphism A on an abelian group is said to be *quasi-unipotent of type (m, l)* if $A^m = I + N$ with I the identity and $N^l = 0$, where $m \geq 1$ and $l \geq 1$ are integers.

The following fact is well known. We give a proof of it for the sake of completeness.

Lemma 6.7. *Let A be an automorphism on \mathbb{T}^d of zero entropy. Then A is quasi-unipotent of type (d, d) .*

Proof. The automorphism A can be lifted to a linear automorphism \tilde{A} of \mathbb{R}^d which preserves \mathbb{Z}^d with $\det \tilde{A} = \pm 1$. Since the dynamics (\mathbb{T}^d, T) has zero entropy, all the eigenvalues of \tilde{A} must have absolute value 1 [118]. According to Kronecker's theorem, all the eigenvalues of \tilde{A} are d -th unit roots. Thus all of eigenvalues of \tilde{A}^d are 1. By Jordan's theorem, there exists a matrix $P \in SL(d, \mathbb{C})$ such that $P\tilde{A}^dP^{-1}$ is a triangular matrix with 1 on its diagonal. Consequently we have $(P\tilde{A}^dP^{-1} - I)^d = 0$ so that $(\tilde{A}^d - I)^d = 0$. Therefore, we can write $A^d = I + N$ with $N^d = 0$. \square

Assume that the affine map $Tx = Ax + b$ has its automorphism A which is quasi-unipotent of type (m, l) . Then we say that T is quasi-unipotent of type (m, l) . Any orbits $(T^n x)$ of such an affine map has a nice behavior as shown in the following lemma.

Lemma 6.8. *Let X be a compact abelian group. Let T be an affine map on X defined by $Tx = Ax + b$ where $b \in X$ and A is a quasi-unipotent automorphism of type (m, l) . Then for any $n = qm + p \geq l$ where $1 \leq p \leq m$ and $q \in \mathbb{N}$, we have*

$$T^n x = \sum_{k=0}^{l-1} (C_q^k N^k A^p x + C_{q+1}^{k+1} N^k b^* + C_q^k N^k b_p),$$

where $b^* = \sum_{k=0}^{m-1} A^k b$ and $b_p = \sum_{k=1}^p A^k b$.

Proof. By the definition of T , we have

$$T^n x = A^n x + \sum_{j=0}^{n-1} A^j b. \quad (6.2.1)$$

Let us write

$$\sum_{j=0}^{n-1} A^j = \sum_{j=0}^{q-1} \sum_{k=1}^m A^{mj+k} + \sum_{k=1}^p A^{qm+k}.$$

Using the fact $A^m = I + N$, we get

$$\sum_{j=0}^{n-1} A^j = \sum_{j=0}^{q-1} (I + N)^j \sum_{k=0}^{m-1} A^k + (I + N)^q \sum_{k=1}^p A^k. \quad (6.2.2)$$

Recall that $N^l = 0$. Now using the interchange of summations, we have

$$\sum_{j=0}^{q-1} (I + N)^j = \sum_{j=0}^{q-1} \sum_{k=0}^j C_j^k N^k = \sum_{k=0}^{l-1} \sum_{j=k}^q C_j^k N^k = \sum_{k=0}^{l-1} N^k \sum_{j=k}^q C_j^k.$$

By the identity $\sum_{j=k}^n C_j^k = C_{n+1}^{k+1}$, we have

$$\sum_{j=0}^{q-1} (I + N)^j = \sum_{k=0}^{l-1} C_{q+1}^{k+1} N^k. \quad (6.2.3)$$

Combining (6.2.1), (6.2.2), (6.2.3) and the equation $(I + N)^q = \sum_{k=0}^{l-1} C_q^k N^k$ gives

$$\begin{aligned} T^n x &= \sum_{k=0}^{l-1} C_q^k N^k A^p x + \sum_{k=0}^{l-1} C_{q+1}^{k+1} N^k b^* + \sum_{k=0}^{l-1} C_q^k N^k b_p \\ &= \sum_{k=0}^{l-1} (C_q^k N^k A^p x + C_{q+1}^{k+1} N^k b^* + C_q^k N^k b_p). \end{aligned}$$

□

Lemma 6.8 allows us to see that the realization $(\phi(T^n x))$ for a group character $\phi \in \widehat{X}$ is related to exponentials of some real polynomials, where \widehat{X} is the dual group of X .

Lemma 6.9. *We keep the same assumption as in Lemma 6.8. Let $\phi \in \widehat{X}$, $x \in X$ and $n \geq l$. If we write $n = qm + p$ with $1 \leq p \leq m$ and $q \in \mathbb{N}$, we have*

$$\phi(T^n x) = e^{2\pi i P(q)}$$

where $P \in \mathbb{R}_l[t]$ is a polynomial depending on p . Moreover, if b is of finite order, then $P(t) = P_1(t) + P_2(t)$ with $P_1 \in \mathbb{R}_{l-1}[t]$ and $P_2 \in \mathbb{Q}[t]$.

Proof. By Lemma 6.8, we have

$$\begin{aligned} \phi(T^n x) &= \prod_{k=0}^{l-1} \phi(N^k A^p x)^{C_q^k} \phi(N^k b^*)^{C_{q+1}^{k+1}} \phi(N^k b_p)^{C_q^k} \\ &= \prod_{k=0}^{l-1} e^{2\pi i (C_q^k \beta_{k,p} + C_{q+1}^{k+1} \alpha_k + C_q^k \theta_{k,p})} = e^{2\pi i (P_1(q) + P_2(q))}, \end{aligned}$$

where

$$P_1(q) = \sum_{k=0}^{l-1} C_q^k \beta_{k,p}, \quad P_2(q) = \sum_{k=0}^{l-1} (C_{q+1}^{k+1} \alpha_k + C_q^k \theta_{k,p})$$

where $\beta_{k,p}$, α_k and $\theta_{k,p}$ are arguments such that

$$e^{2\pi i \beta_{k,p}} = \phi(N^k A^p x), \quad e^{2\pi i \alpha_k} = \phi(N^k b^*), \quad e^{2\pi i \theta_{k,p}} = \phi(N^k b_p).$$

The polynomial P_1 has degree at most $l - 1$ and the polynomial P_2 has degree at most l .

If b is of finite order so are b^* and b_p for all $1 \leq p \leq m$. Since N is homomorphic, $N^k b^*$ and $N^k b_p$ are all of finite order. This implies that α_k and $\theta_{k,p}$ are rationals so that $P_2 \in \mathbb{Q}[t]$. \square

Now we are ready to state and prove the following result on disjointness.

Proposition 6.10. *Let T be an affine map on a compact abelian group X defined by $Tx = Ax + b$. Suppose that A is quasi-unipotent of type (m, l) and that (c_n) is an oscillating sequence of order d . Then (c_n) is orthogonal to the dynamical system (X, T) if one of the following conditions is satisfied*

- (1) $d \geq l$,
- (2) $d \geq l - 1$ and b is of finite order.

Proof. Since the linear combinations of characters of X are dense in $C(X)$, we have only to check (5.1.1) for $f = \phi \in \widehat{X}$. Therefore we can conclude by combining Lemma 6.2, Proposition 6.3, Lemma 6.8 and Lemma 6.9. \square

We finish this section by two formulas concerning the powers of a unipotent map N on \mathbb{T}^d which is unipotent of order d . By the classification theorem of nilpotent matrices ([10], p. 312), under a suitable basis, N has the form

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For $n \geq d - 1$, we have

$$(I + N)^n = \sum_{k=0}^{d-1} C_n^k N^k = \begin{pmatrix} C_n^0 & C_n^1 & C_n^2 & \cdots & C_n^{d-1} \\ 0 & C_n^0 & C_n^1 & \cdots & C_n^{d-2} \\ 0 & 0 & C_n^0 & \cdots & C_n^{d-3} \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & C_n^0 \end{pmatrix}. \quad (6.2.4)$$

By the identity $\sum_{j=k}^n C_j^k = C_{n+1}^{k+1}$ and $N^d = 0$, we get

$$\sum_{k=0}^{n-1} (I + N)^k = \sum_{k=0}^{n-1} \sum_{j=0}^k C_k^j N^j = \sum_{k=0}^{d-1} C_n^{k+1} N^k,$$

that is to say,

$$\sum_{k=0}^{n-1} (I + N)^k = \begin{pmatrix} C_n^1 & C_n^2 & C_n^3 & \cdots & C_n^d \\ 0 & C_n^1 & C_n^2 & \cdots & C_n^{d-1} \\ 0 & 0 & C_n^1 & \cdots & C_n^{d-2} \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & C_n^1 \end{pmatrix}. \quad (6.2.5)$$

These two formulas (6.2.4) and (6.2.5) will be used in the proof of Theorem 6.5.

6.3 Proof of Theorem 6.5

We first observe that the implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are trivial. Let us denote by (0) the assertion “ (c_n) is oscillating of order d ”.

6.3.1 Proofs of (0) \Rightarrow (1) and (0) \Rightarrow (3)

The proofs of both implications share the same computation, which will give us an explicit expression for the orbit of an affine map of zero entropy. Let T be an affine map on $\mathbb{T}^d \times F$ of zero entropy. Then T has the form

$$T(x, y) = (Ax + By + a, Cy + b) \quad (6.3.1)$$

with $a \in \mathbb{T}^d$, $b \in F$, A an automorphism on \mathbb{T}^d of zero entropy, C an automorphism of F and B a morphism from F to \mathbb{T}^d . This was observed in [99].

For any $j \in \mathbb{N}^*$, let $(x_j, y_j) := T^j(x, y)$. It is easy to check by induction that

$$x_j = A^j x + B_j y + a_j, \quad y_j = C^j y + b_j,$$

for some $a_j \in \mathbb{T}^d$, $b_j \in F$ and B_j a morphism from F to \mathbb{T}^d . More precisely, a_j, b_j and B_j satisfy the recursive relations

$$\begin{cases} a_{j+1} = Aa_j + a + Bb_j \\ b_{j+1} = b_j + b \\ B_{j+1} = AB_j + BC^j. \end{cases}$$

Let $q = \sharp F! \cdot d$ and $M = A^q - I$ where $\sharp F!$ denotes the factorial of $\sharp F$. This choice of q makes that

(i) $C^q = I$;

(ii) A^q quasi-unipotent of type $(1, d)$, i.e. $A^q = I + M$ with $M^d = 0$.

In fact, since C is a permutation of F , the order of C divides $\sharp F!$. Then $C^{\sharp F!} = I$, a fortiori $C^q = I$. This is (i). Prove now (ii). Since A is quasi-unipotent of order (d, d) (by Lemma 6.7), we can write $A^d = I + N$ with $N^d = 0$. Since d divides q , we can write

$$M := A^q - I = (A^d - I)Q = NQ$$

where Q commute with N . So, $M^d = 0$.

Since $C^q = I$, we have $y_q = y + b_q$. Let

$$F_y(x) = A^q x + (B_q y + a_q), \quad G(y) = y + b_q.$$

Then

$$T^q(x, y) = (F_y(x), G(y)).$$

Note that F_y is an affine map on \mathbb{T}^d whose automorphism A^q is quasi-unipotent of type $(1, d)$, and G is an affine map on F whose automorphism is the identity which is a quasi-unipotent automorphism of type $(1, 1)$. It can be checked, by induction on n , that for $n \geq 1$,

$$T^{nq}(x, y) = (F_y^n(x) + H_n, G^n(y)),$$

where $H_1 = 0$ and

$$H_n = \sum_{i=1}^{n-1} i A^{q(n-i-1)} B_q b_q \quad (\forall n \geq 2) \tag{6.3.2}$$

are independent of x and y . Then for any $n \geq 2$ and $0 \leq j \leq q-1$, we have $T^{nq+j}(x, y) = T^{nq}(x_j, y_j)$ so that

$$T^{nq+j}(x, y) = (F_{y_j}^n(x_j) + H_n, G^n(y_j)). \tag{6.3.3}$$

Now let us prove (0) \Rightarrow (1). Assume that T is an automorphism of zero entropy on $\mathbb{T}^{d+1} \times F$. Then T has the form (6.3.1) with $a = 0$, $b = 0$ and A a zero entropy

automorphism on \mathbb{T}^{d+1} . Hence $a_q = 0$ and $b_q = 0$ by the above recursive relation. So (6.3.3) takes the form

$$T^{nq+j}(x, y) = (F_{y_j}^n(x_j), G^n(y_j)), \quad \text{where } F_y(x) = A^q x + B_q y.$$

The automorphism of F_y is quasi-unipotent of type $(d+1, d+1)$ and the automorphism of G is quasi-unipotent of type $(1, 1)$. Therefore the affine map $(F_y(\cdot), G(\cdot))$ on $\mathbb{T}^{d+1} \times F$ is quasi-unipotent of type $(d+1, d+1)$. On the other hand, $B_q y$ is of finite order for any $y \in F$ because F is finite and B_q is a morphism. Thus by Proposition 6.10 (2) and Lemma 6.2, for any continuous function f on $\mathbb{T}^{d+1} \times F$, we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_{nq+j} f(T^{nq+j}(x, y)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_{nq+j} f(F_{y_j}^n(x_j), G^n(y_j)) = 0. \end{aligned}$$

Taking average over $0 \leq j < q$, we finish the proof of the implication (0) \Rightarrow (1).

It remains to prove (0) \Rightarrow (3). Assume that T is an affine map of zero entropy on $\mathbb{T}^d \times F$. Then T has the form (6.3.1) and T^{nq+j} takes the form (6.3.3). The automorphism of G is quasi-unipotent of type $(1, 1)$. The automorphism of F_y is quasi-unipotent of type (d, d) but the term a_q may not be of finite order (this is the crucial difference from the implication (0) \Rightarrow (1)). We first show that $n \mapsto H_n$ can be expressed as a polynomial of n , namely

$$H_n = \sum_{k=0}^{d-1} C_n^{k+2} M^k B_q b_q. \quad (6.3.4)$$

In fact, by the identity $\sum_{j=k}^{n-2} (n-j-1) C_j^k = C_n^{k+2}$ and the fact $A^q = I + M$ with $M^d = 0$, we have

$$\begin{aligned} \sum_{j=0}^{n-2} (n-j-1) A^{qj} &= \sum_{j=0}^{n-2} (n-j-1) (I + M)^j \\ &= \sum_{j=0}^{n-2} \sum_{k=0}^j (n-j-1) C_j^k M^k \\ &= \sum_{k=0}^{d-1} \sum_{j=k}^{n-2} (n-j-1) C_j^k M^k \\ &= \sum_{k=0}^{d-1} C_n^{k+2} M^k. \end{aligned}$$

Thus (6.3.4) follows immediately from (6.3.2). Since F is finite and B_q is a morphism, $M^k B_q b_q$ are of finite order for all $0 \leq k \leq d-1$. Take an arbitrary character $\phi = \phi_1 \phi_2$

of $\mathbb{T}^d \times F$ where ϕ_1 is a character of \mathbb{T}^d and ϕ_2 is a character of F . Assume $\phi_1(x) = e^{2\pi i(\lambda_1 x_1 + \dots + \lambda_d x_d)}$ where $(\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$. By the above discussion, we have $\phi_1(H_n) = e^{2\pi i P(n)}$ where

$$P(t) = \sum_{k=0}^{d-1} C_t^{k+2} \alpha_k \in \mathbb{Q}[t]$$

with $\alpha_k \in \mathbb{Q}$ defined by $e^{2\pi i \alpha_k} = \phi_1(M^k B_q b_q)$. Then, by (6.3.3), we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_{nq+j} \phi(T^{nq+j}(x, y)) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_{nq+j} e^{2\pi i P(n)} \phi_1(F_{y_j}^n(x_j)) \phi_2(G^n(y_j)) = 0. \end{aligned}$$

The last equality follows from Lemma 6.2, Proposition 6.3 and Proposition 6.10 (1) which is applied to the affine map $(x, y) \mapsto (F_{y_j}(x), G(y))$. By taking average over $0 \leq j < q$, we complete the proof of the implication (0) \Rightarrow (3).

6.3.2 Proof of (2) \Rightarrow (0)

Let T be a special automorphism of zero entropy on $\mathbb{T}^{d+1} \times F$ where F is a finite abelian group. It has the form

$$T(x, y) = (Ax + By, Cy)$$

with $A = I + N$ an automorphism on \mathbb{T}^{d+1} of zero entropy and N of the nilpotent index $d + 1$, C an automorphism of F and B a morphism from F to \mathbb{T}^{d+1} . What we will prove is that any polynomial exponential sequence $(e^{2\pi i P(n)})$ with $P \in \mathbb{R}_d[t]$ is an observable sequence of the system $(\mathbb{T}^{d+1} \times F, T)$.

Observe that for any $n \in \mathbb{N}^*$

$$T^n(x, 0) = ((I + N)^n x, 0).$$

Let $P \in \mathbb{R}_d[t]$. Since the binomial polynomials $C_t^0, C_t^1, \dots, C_t^d$ form a basis of $\mathbb{R}_d[t]$, there exists $x := (x_0, x_1, \dots, x_d) \in \mathbb{R}^{d+1}$ such that

$$P(t) = x_0 C_t^0 + x_1 C_t^1 + \dots + x_d C_t^d.$$

Let $G : \mathbb{T}^{d+1} \rightarrow \mathbb{T}$ be the projection on the first coordinate. By the formula (6.2.4), we have

$$e^{2\pi i P(n)} = e^{2\pi i G(T^n x)}$$

where x also denotes the projection of $x \in \mathbb{R}^{d+1}$ onto \mathbb{T}^{d+1} . So, $(e^{2\pi i P(n)})$ is an observable sequence of the system $(\mathbb{T}^{d+1} \times F, T)$. This completes the proof (2) \Rightarrow (0).

6.3.3 Proof of (4) \Rightarrow (0)

Let T_b be a special affine map of zero entropy on $\mathbb{T}^d \times F$ for a finite abelian group F which has the form

$$T_b(x, y) = (Ax + By + b, Cy)$$

with $b \in \mathbb{T}^d$, $A = I + N$ an automorphism on \mathbb{T}^d of zero entropy and N of the nilpotent index d , C an automorphism of F and B a morphism from F to \mathbb{T}^d . We will prove that any polynomial exponential sequence $(e^{2\pi i P(n)})$ with $P \in \mathbb{R}_d[t]$ is an observable sequence of the system $(\mathbb{T}^d \times F, T_b)$ along the orbit of $(0, 0) \in \mathbb{T}^d \times F$.

Observe that

$$T_b^n(0, 0) = \left(\sum_{k=0}^{n-1} (I + N)^k b, 0 \right).$$

Let $Q \in \mathbb{R}_{d-1}[t]$. Since $1, C_t^1, \dots, C_t^d$ form a basis of $\mathbb{R}_d[t]$, there exists $b = (b_1, b_2, \dots, b_d) \in \mathbb{R}^d$ and $b_0 \in \mathbb{R}$ such that

$$tQ(t) = b_0 + b_1 C_t^1 + \dots + b_d C_t^d.$$

Let $G : \mathbb{T}^d \times F \rightarrow \mathbb{T}$ be the projection on the first coordinate of \mathbb{T}^d . By the formula (6.2.5), we get

$$e^{2\pi i n Q(n)} = e^{2\pi i (G(T_b^n(0,0)) + b_0)}$$

which is an observable sequence of the system $(\mathbb{T}^d \times F, T_b)$. By Lemma 6.1, this completes the proof (4) \Rightarrow (0).

Chapter 7

Construction of some Chowla sequences

For numerical sequences taking values 0 or complex numbers of modulus 1, we define Chowla property and Sarnak property. We prove that Chowla property implies Sarnak property. We also prove that for Lebesgue almost every $\beta > 1$, the sequence $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ shares Chowla property and consequently is orthogonal to all topological dynamical systems of zero entropy. It is also discussed whether the samples of a given random sequence have Chowla property almost surely. Some dependent random sequences having almost surely Chowla property are constructed.

7.1 Introduction

Recall that the Möbius function is an arithmetic function defined by the formula

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the Möbius function plays an important role in number theory: the statement

$$\sum_{n=1}^N \mu(n) = O_\epsilon(N^{\frac{1}{2}+\epsilon}), \text{ for each } \epsilon > 0,$$

is equivalent to Riemann's hypothesis (see [123]).

Chowla [21] formulated a conjecture on the higher orders correlations of the Möbius function:

Conjecture 7.1. *For each choice of $0 \leq a_1 < \dots < a_r, r \geq 0, i_s \in \{1, 2\}$ not all equal to*

This is the paper [116] *Construction of some Chowla sequences*. The arXiv version can be found in <https://arxiv.org/abs/1804.03851>.

2,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu^{i_0}(n+a_1) \cdot \mu^{i_1}(n+a_2) \cdots \mu^{i_r}(n+a_r) = 0.$$

Let T be a continuous transformation on a compact metric space X . Following Sarnak [112], we will say a sequence $(a(n))_{n \in \mathbb{N}}$ of complex numbers is orthogonal to the topological dynamical system (X, T) , if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) a(n) = 0, \tag{7.1.1}$$

for all $x \in X$ and for all $f \in C(X)$ where $C(X)$ is the space of continuous functions on X .

In 2010, Sarnak [112] formulated the following conjecture.

Conjecture 7.2. *The Möbius function is orthogonal to all the topological dynamical systems of zero entropy.*

Sarnak [112] also proved that Chowla’s conjecture implies his conjecture.

Recently, El Abdalaoui, Kulaga-Przymus, Lemanczyk and de la Rue [4] studied these conjectures from ergodic theory point of view. Let z be an arbitrary sequence taking values in $\{-1, 0, 1\}$ as the Möbius function. Following El Abdalaoui et al., we say that z satisfies the condition (Ch) if it satisfies the condition in Chowla’s conjecture; similarly, it satisfies the condition (S_0) if it satisfies the condition in Sarnak’s conjecture. They proved that the condition (Ch) implies the condition (S_0) . El Abdalaoui et al. also provided some non-trivial examples of sequences satisfying the condition (Ch) and the condition (S_0) respectively.

Let (X, T) be a topological dynamical system. Recall that a point $x \in X$ is said to be *completely deterministic* if for any accumulation point ν of $\frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x}$ where δ_x is the Dirac measure supported on x , the system (X, T, ν) is of zero entropy. El Abdalaoui et al. also defined the condition (S), formally stronger than (S_0) by requiring

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) z(n) = 0$$

for any topological dynamical system (X, T) , any $f \in C(X)$ and any completely deterministic point $x \in X$. They proved the equivalence between the condition (S) and the condition (S_0) .

Fan [31] defined that a bounded sequence $(c(n))_{n \in \mathbb{N}}$ is *oscillating of order d* ($d \geq 1$ being an integer) if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c(n) e^{2\pi i P(n)} = 0$$

for any $P \in \mathbb{R}_d[t]$ where $\mathbb{R}_d[t]$ is the space of real polynomials of degree smaller than or equal to d ; it is said to be *fully oscillating* if it is oscillating of order d for every integer

$d \geq 1$. It is clear that an oscillating sequence of order d is always oscillating of order $d - 1$ for any $d \geq 2$. In [115], we gave several equivalent definitions of oscillating sequences, one of which states that a sequence is fully oscillating if and only if it is orthogonal to all affine maps on all compact abelian groups of zero entropy. This result yields that sequences satisfying the condition (S) are fully oscillating.

In what follows, a sequence taking values in $\{-1, 0, 1\}$ is said to have *Chowla property* (resp. *Sarnak property*) if it satisfies the condition (Ch) (resp. the condition (S)). We consider a more general class of sequences, taking values 0 or complex numbers of modulus 1. We define Chowla property and Sarnak property for this class of sequences. We prove that Chowla property implies Sarnak property. A sequence having Chowla property is called a *Chowla sequence*. We are interested in finding Chowla sequences. We prove that for almost every $\beta > 1$, the sequence $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ is a Chowla sequence and consequently is orthogonal to all topological dynamical systems of zero entropy. This extends a result of Akiyama and Jiang [5] who proved that for almost every $\beta > 1$, the sequence $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ is fully oscillating. On the other hand, from a probability point of view, we discuss whether the samples of a given random sequence have almost surely Chowla property. We also construct a sequence of dependent random variables which is a Chowla sequence almost surely.

The paper is organized as follows. In Section 7.2, we will recall some basic notions in topological and measurable dynamical systems. In Section 7.3, we will introduce the definition of Chowla property for sequences taking values 0 or complex numbers of modulus 1 and the definition of Sarnak property for sequences of complex numbers. In Section 7.4, it will be proved that for almost every $\beta > 1$, the sequence $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ is a Chowla sequence. In Section 7.5, we will prove that Chowla property implies Sarnak property. In Section 7.6, we will study random sequences: we will give an equivalent condition for stationary random sequences to be Chowla sequences almost surely; we will construct sequences of dependent random variables which are Chowla sequences almost surely; we will also prove that Sarnak property does not imply Chowla property.

Through this paper, we denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_{\geq 2} = \mathbb{N} \setminus \{0, 1\}$.

7.2 Preliminary

In this section, we recall some basic notions in dynamical systems.

7.2.1 Quasi-generic measure

Let (X, T) be a topological dynamical system. Denote by $\mathcal{P}_T(X)$ the set of all T -invariant probability measures on X equipped with the weak topology. Under this weak topology, it is well known that the space $\mathcal{P}_T(X)$ is compact, convex and not empty. Let $x \in X$. Set

$$\delta_{N,T,x} := \frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x}.$$

Sometimes, we write simply $\delta_{N,x}$ if the system is fixed. Denote the set of all quasi-generic measures associated to x by

$$Q\text{-gen}(x) := \{\nu \in \mathcal{P}_T(X) : \delta_{N_k,x} \xrightarrow{k \rightarrow \infty} \nu, (N_k)_{k \in \mathbb{N}} \subset \mathbb{N}\}.$$

By compactness of $\mathcal{P}_T(X)$, the set $Q\text{-gen}(x)$ is not empty. Furthermore, the point x is said to be *completely deterministic* if for any measure $\nu \in Q\text{-gen}(x)$, the entropy $h(T, \nu)$ is zero. On the other hand, we say that the point x is *quasi-generic* for a measure $\nu \in \mathcal{P}_T(X)$ along $(N_k)_{k \in \mathbb{N}}$ if $\delta_{N_k,x} \rightarrow \nu$ as $k \rightarrow \infty$. Moreover, x is said to be *generic* for ν if the set $Q\text{-gen}(x)$ contains only one element ν .

7.2.2 Joinings

The notion of joinings was first introduced by Furstenberg [44]. It is a very powerful tool in ergodic theory. In this subsection, we will recall some basic concepts about joinings.

By a measurable dynamical system (Y, \mathcal{B}, S, μ) , we mean that (Y, \mathcal{B}) is a measurable space, S is a measurable map from Y to itself and μ is an S -invariant measure, that is to say, $\mu(S^{-1}B) = \mu(B)$ for any $B \in \mathcal{B}$.

Let (X, \mathcal{A}, T, μ) and (Y, \mathcal{B}, S, ν) be two measurable dynamical systems. We say that (Y, \mathcal{B}, S, ν) is a *factor* of (X, \mathcal{A}, T, μ) (or (X, \mathcal{A}, T, μ) is an *extension* of (Y, \mathcal{B}, S, ν)) if there exists a measurable map $\pi : X \rightarrow Y$ such that $\pi \circ S = T \circ \pi$ and the pushforward of μ via π is ν , that is, $\pi_*\mu = \nu$. We call the map π the *factor map*. In this case, it is classical that we can identify \mathcal{B} with the σ -algebra $\pi^{-1}(\mathcal{B}) \subset \mathcal{A}$, which is T -invariant. On the other hand, any T -invariant sub- σ -algebra $\mathcal{C} \subset \mathcal{A}$ will be identified with the corresponding factor $(X/\mathcal{C}, \mathcal{C}, \mu|_{\mathcal{C}}, T)$.

Let $\pi_i : (X, \mathcal{A}, T, \mu) \rightarrow (Y_i, \mathcal{B}_i, S_i, \nu_i)$ be two factor maps for $i = 1, 2$. We denote by $(Y_1, \mathcal{B}_1, S_1, \nu_1) \vee (Y_2, \mathcal{B}_2, S_2, \nu_2)$ the smallest factor of (X, T, \mathcal{A}, μ) containing both $\pi_1^{-1}(\mathcal{B}_1)$ and $\pi_2^{-1}(\mathcal{B}_2)$. It is easy to see that $(Y_1, \mathcal{B}_1, S_1, \nu_1) \vee (Y_2, \mathcal{B}_2, S_2, \nu_2)$ is a common extension of $(Y_1, \mathcal{B}_1, S_1, \nu_1)$ and $(Y_2, \mathcal{B}_2, S_2, \nu_2)$.

Recall that a measurable dynamical system is said to be a *Kolmogorov system* if any non-trivial factor of it has positive entropy. Let $\pi : (X, \mathcal{A}, T, \mu) \rightarrow (Y, \mathcal{B}, S, \nu)$ be a factor map. The *relative entropy* of the factor map π is defined by the quantity $h(T, \mu) - h(S, \nu)$. We say that the factor map π is *relatively Kolmogorov* if π is non trivial and if for any intermediate factor $(Z, \mathcal{C}, U, \rho)$ with factor map $\pi' : (Z, \mathcal{C}, U, \rho) \rightarrow (Y, \mathcal{B}, S, \nu)$, the relative entropy of π' is positive unless π' is an isomorphism.

Given measurable dynamical systems $(X_i, \mathcal{B}_i, T_i, \mu_i)$ for $i = 1, 2, \dots, k$, we recall that a *joining* ρ is a $T_1 \times \dots \times T_k$ -invariant probability measure on $(X_1 \times \dots \times X_k, \mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_k)$ such that $(\pi_i)_*(\rho) = \mu_i$, where π_i is the projection from $X_1 \times \dots \times X_k$ to X_i for $1 \leq i \leq k$. The set consisting of all joinings is denoted by $J(T_1, T_2, \dots, T_k)$. Following Furstenberg [44], $(X_1, \mathcal{B}_1, T_1, \mu_1)$ and $(X_2, \mathcal{B}_2, T_2, \mu_2)$ are said to be *disjoint* if their joining is unique, which is $\mu_1 \otimes \mu_2$.

Let $\pi_i : (X_i, \mathcal{A}_i, T_i, \mu_i) \rightarrow (Y, \mathcal{B}, S, \nu)$ be two factor maps for $i = 1, 2$. Given $\rho \in J(S, S)$,

the *relatively independent extension* of ρ is defined by $\widehat{\rho} \in J(T_1, T_2)$ such that

$$\widehat{\rho}(A_1 \times A_2) := \int_{Y \times Y} \mathbb{E}[1_{A_1} | \pi_1^{-1}(\mathcal{B})](x) \mathbb{E}[1_{A_2} | \pi_2^{-1}(\mathcal{B})](y) d\rho(x, y),$$

for each $A_i \in \mathcal{A}_i, i = 1, 2$. We say that the factor maps π_1 and π_2 are *relatively independent* over their common factor if $J(T_1, T_2)$ only consists of $\widehat{\Delta}$ where $\Delta \in J(S, S)$ is given by $\Delta(B_1 \times B_2) := \nu(B_1 \cap B_2)$ for any $B_1, B_2 \in \mathcal{B}$.

The following lemma is classical.

Lemma 7.3 ([124], Lemma 3). *Let $\pi_i : (X_i, \mathcal{A}_i, T_i, \mu_i) \rightarrow (Y, \mathcal{B}, S, \nu)$ be two factor maps for $i = 1, 2$. Suppose that π_1 is of relative zero entropy and π_2 is relatively Kolmogorov. Then π_1 and π_2 are relatively independent.*

7.2.3 Shift system

Let X be a nonempty compact metric space. The product space $X^{\mathbb{N}}$ endowed with the product topology is a compact metric space. Coordinates of $x \in X^{\mathbb{N}}$ will be denoted by $x(n)$ for $n \in \mathbb{N}$.

On the space $X^{\mathbb{N}}$ there is a natural continuous action by the *shift* $S : X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ defined by $(x(n))_{n \in \mathbb{N}} \mapsto (x(n+1))_{n \in \mathbb{N}}$. We call the topological dynamical system $(X^{\mathbb{N}}, S)$ a *(topological) shift system*. Through this paper, we always denote by S the shift on the shift system.

We recall that the subsets of $X^{\mathbb{N}}$ of the form

$$[C_0, C_1, \dots, C_k] := \{x \in X^{\mathbb{N}} : x(j) \in C_j, \forall 0 \leq j \leq k\}$$

where $k \geq 0$ and C_0, \dots, C_k are open subsets in X , are called *cylinders* and they form a basis of the product topology of $X^{\mathbb{N}}$. It follows that any Borel measure on $X^{\mathbb{N}}$ is uniquely determined by the values on these cylinders. Through this paper, we always denote \mathcal{B} the Borel σ -algebra on the shift system. For any Borel measure μ on $X^{\mathbb{N}}$, we sometimes write $(X^{\mathbb{N}}, S, \mu)$ for the measurable shift system $(X^{\mathbb{N}}, \mathcal{B}, S, \mu)$.

In what follows, we denote by $F : X^{\mathbb{N}} \rightarrow X$ the projection on the first coordinate, i.e. $(x(n))_{n \in \mathbb{N}} \mapsto x(0)$. It is easy to see that the projection is continuous.

7.3 Definitions of Chowla property

In this section, we will show how to generalize Chowla property from $\{-1, 0, 1\}^{\mathbb{N}}$ to $(S^1 \cup \{0\})^{\mathbb{N}}$ where S^1 is the unit circle. Before we really get into the generalization of Chowla property, we first generalize Sarnak property from the sequences in $\{-1, 0, 1\}^{\mathbb{N}}$ to ones in $\mathbb{C}^{\mathbb{N}}$. To this end, we define the condition (GS_0) for sequences of complex numbers as follows.

Defintion 7.4. We say that a sequence $(z(n))_{n \in \mathbb{N}}$ of complex numbers satisfies the condition (GS_0) if for each homeomorphism T of a compact metric space X with topological entropy $h(T) = 0$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) z(n) = 0,$$

for all $f \in C(X)$ and for all $x \in X$.

It is not hard to see that the the condition (GS_0) is a generalization of the condition (S_0) . Similarly, we generalize the condition (S) from $\{-1, 0, 1\}^{\mathbb{N}}$ to $\mathbb{C}^{\mathbb{N}}$.

Defintion 7.5. We say that a sequence $(z(n))_{n \in \mathbb{N}}$ of complex numbers satisfies the condition (GS) if for each homeomorphism T of a compact metric space X ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) z(n) = 0,$$

for all $f \in C(X)$ and for all completely deterministic points $x \in X$.

It is observed that the condition (GS) implies the condition (GS_0) . Similarly to the proof of $(S_0) \Rightarrow (S)$ in [4], the condition (GS_0) implying the condition (GS) follows from the characterization of completely deterministic points by Weiss [131]:

A sequence u is completely deterministic if and only if, for any $\epsilon > 0$ there exists K such that, after removing from u a subset of density less than ϵ , what is left can be covered by collection C of K -blocks such that $|C| < 2^{\epsilon K}$.

Therefore, we obtain that the condition (GS_0) is equivalent to the condition (GS) . In what follows, we say that a sequence has *Sarnak property* if it satisfies the condition (GS_0) (or equivalently the condition (GS)). Such sequence is called a *Sarnak sequence*.

Now we begin to show how to generalize Chowla property. Let $(z(n))_{n \in \mathbb{N}} \in (S^1 \cup \{0\})^{\mathbb{N}}$. For $m \in \mathbb{N}_{\geq 2}$, we denote by

$$U(m) = \{e^{2\pi i \frac{k}{m}} : 0 \leq k \leq m - 1\}$$

the set consisting of all m -th roots of the unity. We define supplementarily $U(\infty) = S^1$, which is considered as the set of points whose infinity norm is 1. The *index* of the sequence $(z(n))_{n \in \mathbb{N}}$ is defined by the smallest number $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ such that the set

$$\{n \in \mathbb{N} : z(n) \notin U(m) \cup \{0\}\}$$

has zero density in \mathbb{N} , that is,

$$\lim_{N \rightarrow +\infty} \frac{\#\{0 \leq n \leq N - 1 : z(n) \notin U(m) \cup \{0\}\}}{N} = 0.$$

We will show some examples to illustrate this definition.

Examples 7.6. (1) Every sequence $(z(n))_{n \in \mathbb{N}} \in \{-1, 0, 1\}^{\mathbb{N}}$ has the index 2.

(2) For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the sequence $(e^{2\pi i n \alpha})_{n \in \mathbb{N}}$ has the index ∞ . In general, every sequence $(z(n))_{n \in \mathbb{N}}$ taking values in $S^1 \setminus e(\mathbb{Q})$ has the index ∞ where $e(\mathbb{Q})$ denotes the set $\{e^{2\pi i a} : a \in \mathbb{Q}\}$.

(3) For any non-integer number β , the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ has the index ∞ . In fact, if β is an irrational number, then the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ taking values in $S^1 \setminus e(\mathbb{Q})$ and thus has the index ∞ ; if $\beta \in \mathbb{Q} \setminus \mathbb{Z}$, say $\beta = p/q$ with $q > 1$ and $\gcd(p, q) = 1$, then $e^{2\pi i \beta^n} \in U(q^n)$ and $e^{2\pi i \beta^n} \notin U(m)$ for any $m < q^n$ which implies that the index of the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ is ∞ .

The sequences $(S^1 \cup \{0\})^{\mathbb{N}}$ are divided into two situations according to the cases when their indice are finite or infinite. It is natural that Chowla property is thus defined for these two cases separately.

We first define the Chowla property for sequences of finite index.

Defintion 7.7. Suppose that a sequence $z \in (S^1 \cup \{0\})^{\mathbb{N}}$ has the index $m \in \mathbb{N}_{\geq 2}$. We say that the sequence z has Chowla property if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} z^{i_0}(n+a_1) \cdot z^{i_1}(n+a_2) \cdots z^{i_r}(n+a_r) = 0, \quad (7.3.1)$$

for each choice of $0 \leq a_1 < \cdots < a_r, r \geq 0, i_s \in \{1, 2, \dots, m\}$ not all equal to m .

As it has already been noticed in Example 7.6 (1), a sequence taking values in $\{-1, 0, 1\}$ has the index 2. It follows easily that a sequence taking values in $\{-1, 0, 1\}$ has Chowla property if and only if it satisfies the condition (Ch) defined in [4].

To define Chowla property for sequences of infinite index, we will need additionally the following. For a point $x \in S^1 \cup \{0\}$ and $n \in \mathbb{Z}$, we define

$$x^{(n)} := \begin{cases} x^n & \text{if } x \in S^1, \\ 0 & \text{if } x = 0, \end{cases}$$

which is viewed as an extension of power functions on S^1 . It is easy to see that $(xy)^{(n)} = x^{(n)}y^{(n)} = y^{(n)}x^{(n)}$ for all $x, y \in S^1 \cup \{0\}$ and for all $n \in \mathbb{Z}$. Now we define the Chowla property for sequences of infinite index.

Defintion 7.8. Suppose that a sequence $z \in (S^1 \cup \{0\})^{\mathbb{N}}$ has the index infinite. We say that the sequence z has Chowla property if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} z^{(i_0)}(n+a_1) \cdot z^{(i_1)}(n+a_2) \cdots z^{(i_r)}(n+a_r) = 0, \quad (7.3.2)$$

for each choice of $0 \leq a_1 < \cdots < a_r, r \geq 0, i_s \in \mathbb{Z}$ not all equal to 0.

We revisit the functions $x \mapsto x^{(n)}$ for $n \in \mathbb{Z}$. For a fixed $m \in \mathbb{N}_{\geq 2}$, it is easy to check that

$$x^{(i)} = x^{(j)} \text{ for all } x \in U(m) \cup \{0\} \iff i \equiv j \pmod{m}.$$

We define

$$I(m) := \begin{cases} \{0, 1, \dots, m-1\}, & \text{if } m \in \mathbb{N}_{\geq 2}; \\ \mathbb{Z}, & \text{if } m = \infty, \end{cases}$$

which identifies the set of functions having the form $x \mapsto x^{(n)}$ for $n \in \mathbb{Z}$. This leads us to combine Definition 7.7 and Definition 7.8 together.

Defintion 7.9. *Suppose that a sequence $z \in (S^1 \cup \{0\})^{\mathbb{N}}$ has an index $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. We say that the sequence z has Chowla property if*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} z^{(i_0)}(n+a_1) \cdot z^{(i_1)}(n+a_2) \cdots z^{(i_r)}(n+a_r) = 0, \quad (7.3.3)$$

for each choice of $0 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0.

It is not hard to check that Definition 7.9 is an unification of Definition 7.7 and Definition 7.8. A sequence will be called a *Chowla sequence* if it has Chowla property.

7.4 Sequences $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ for $\beta > 1$

In this section, we prove that the squences $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ are Chowla sequences for almost all $\beta > 1$. Actually, we prove that a more general class of sequences are Chowla sequences. To this end, we study the distribution of a given sequence $(e^{2\pi i x(n)})_{n \in \mathbb{N}}$.

In fact, the problem about the distribution of a sequence $(e^{2\pi i x(n)})_{n \in \mathbb{N}}$ is actually the one about the distribution of the sequence $(\{x(n)\})_{n \in \mathbb{N}}$ on the unite interval $[0, 1)$ where $\{x\}$ denotes the fractional part of the real number x . Recall that a sequence $(x(n))_{n \in \mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo one* if, for every real numbers u, v with $0 \leq u < v \leq 1$, we have

$$\lim_{N \rightarrow \infty} \frac{\#\{n : 1 \leq n \leq N, u \leq \{x_n\} \leq v\}}{N} = v - u.$$

For a given real number $\beta > 1$, only few results are known on the distribution of the sequence $(\beta^n)_{n \in \mathbb{N}}$. For example, we still do not know whether 0 is a limit point of $(\{e^n\})_{n \in \mathbb{N}}$, nor of $(\{(\frac{3}{2})^n\})_{n \in \mathbb{N}}$. However, several metric statements have been established. The first one was obtained by Koksma [78], who proved that for almost every $\beta > 1$ the sequence $(\beta^n)_{n \in \mathbb{N}}$ is uniformly distributed modulo one on the unit interval $[0, 1)$. Actually, Koksma [78] proved the following theorem.

Theorem 7.10 (Koksma's theorem). *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real-valued C^1 functions defined on an interval $[a, b] \subset \mathbb{R}$. Suppose that*

(1) $f'_m - f'_n$ is monotone on $[a, b]$ for all $m \neq n$;

(2) $\inf_{m \neq n} \min_{x \in [a, b]} |f'_m(x) - f'_n(x)| > 0$.

Then for almost every $x \in [a, b]$, the sequence $(f_n(x))_{n \in \mathbb{N}}$ is uniformly distributed modulo one on the unit interval $[0, 1)$.

Akiyama and Jiang [5] proved that for almost every $\beta > 1$ the sequence $(e^{2\pi i \beta^n})_{n \in \mathbb{N}}$ is fully oscillating. Actually, they proved the following theorem.

Theorem 7.11 ([5], Corollary 2). *Given any real number $\alpha \neq 0$ and any $g \in C_+^2((1, \infty))$ where $C_+^2((1, +\infty))$ is the space of all positive 2-times continuously differentiable functions on $(1, +\infty)$ whose i -th derivative is non-negative for $i = 1, 2$. Then for almost every real number $\beta > 1$ and for every real polynomial P , the sequence*

$$(e^{2\pi i(\alpha \beta^n g(\beta) + P(n))})_{n \in \mathbb{N}}$$

is fully oscillating.

The main step of their proof is to show the uniform distribution modulo one of the sequence $(\alpha \beta^n g(\beta))_{n \in \mathbb{N}}$ for a given α and a function $g \in C_+^2((1, \infty))$, for almost every real number $\beta > 1$ by using Koksma's theorem.

Now we state our main theorem in this section.

Theorem 7.12. *Given any $g \in C_{count}^2$ where the space C_{count}^2 consists of all C^2 functions which have at most countable zeros on $(1, +\infty)$. Then for almost all real numbers $\beta > 1$, the sequences*

$$(e^{2\pi i \beta^n g(\beta)})_{n \in \mathbb{N}} \tag{7.4.1}$$

are Chowla sequences.

Proof. Since the function g has at most countable zeros, for any $\epsilon > 0$ there is a union of closed intervals $\sqcup_{i \in I} J_i^\epsilon \subset (1, +\infty)$ such that

- (1) the index set I is at most countable,
- (2) J_i^ϵ is a closed interval where g does not vanish,
- (3) $m((1, +\infty) \setminus \sqcup_{i \in I} J_i^\epsilon) < \epsilon$ where m is the Lebesgue measure on the real line.

Note that if we have proved that for any $\epsilon > 0$, the sequence (7.4.1) has Chowla property for almost every $\beta \in J_i^\epsilon$, then it has Chowla property for almost every $\beta \in (1, +\infty)$. Thus it suffices to prove that for any closed interval $J \subset (1, +\infty)$ where g does not vanish, the sequence (7.4.1) has Chowla property for almost every $\beta \in J$.

Let $c_\beta(n) = e^{2\pi i \beta^n g(\beta)}$. For each choice of $0 \leq a_1 < \dots < a_r, r \geq 0, i_s \in \mathbb{Z}$ not all equal to 0, we have that

$$c_\beta^{(i_0)}(n + a_1) \cdot c_\beta^{(i_1)}(n + a_2) \cdot \dots \cdot c_\beta^{(i_r)}(n + a_r) = e^{2\pi i \beta^n h(\beta)},$$

where $h(\beta) = g(\beta) \sum_{j=1}^r i_j \beta^{a_j}$. Note that $\sum_{j=1}^r i_j x^{a_j}$ is a non-trivial real polynomial because not all i_s are equal to zero for $1 \leq s \leq r$. It follows that the function h is also in the space C_{count}^2 . Thus it is sufficient to prove that for any $g \in C_{count}^2$, for any closed interval $J \subset (1, +\infty)$ where g does not vanish, for almost every $\beta \in J$, and for large enough M , the sequence $(\beta^{n+M} g(\beta))_{n \in \mathbb{N}}$ is uniformly distributed modulo 1.

Denote by $f_n(x) = x^n g(x)$. A simple calculation allows us to see that

$$(f_m(x) - f_n(x))' = ((mx^{m-1} - nx^{n-1})g(x) + (x^m - x^n)g'(x)), \quad (7.4.2)$$

for all $m, n \in \mathbb{N}$. For given $m > n$, we deduce that for all $x > 1$,

$$\begin{aligned} \left| \frac{(f_m(x) - f_n(x))'}{x^m g(x)} \right| &= \left| \frac{m}{x} - \frac{n}{x^{m-n+1}} + \left(1 - \frac{1}{x^{m-n}}\right) \frac{g'(x)}{g(x)} \right| \\ &\geq \left(\frac{m}{x} - \frac{n}{x^{m-n+1}} - \left(1 - \frac{1}{x^{m-n}}\right) \left| \frac{g'(x)}{g(x)} \right| \right) \\ &\geq \left(\frac{m(x-1)}{x^2} - \left| \frac{g'(x)}{g(x)} \right| \right). \end{aligned} \quad (7.4.3)$$

Since $J \subset (1, +\infty)$ is a closed interval, the function $\left| \frac{g'(x)}{g(x)} \right|$ is bounded and the function $\frac{(x-1)}{x^2}$ has a positive lower bound on J . Hence the right-hand side of (7.4.3) tends to $+\infty$ when m tends to $+\infty$. It follows that there is $M_1 > 0$ such that

$$\inf_{m \neq n \geq M_1} \min_{x \in J_i} |(f_m(x) - f_n(x))'| > 0. \quad (7.4.4)$$

On the other hand, we have

$$\begin{aligned} (f'_m(x) - f'_n(x))' &= ((m(m-1)x^{m-2} - n(n-1)x^{n-2})g(x) \\ &\quad + 2(mx^{m-1} - nx^{n-1})g'(x) + (x^m - x^n)g''(x)). \end{aligned} \quad (7.4.5)$$

Similar to the calculation (7.4.3), we have

$$\begin{aligned} &\left| \frac{(f'_m(x) - f'_n(x))'}{x^m g(x)} \right| \\ &\geq \left(\frac{m(m-1)(x-1)}{x^3} - \frac{2m}{x} \left| \frac{g'(x)}{g(x)} \right| - \left| \frac{g''(x)}{g(x)} \right| \right). \end{aligned} \quad (7.4.6)$$

Since the functions $\frac{1}{x} \left| \frac{g'(x)}{g(x)} \right|$ and $\left| \frac{g''(x)}{g(x)} \right|$ are bounded on J and the function $\frac{x-1}{x^3}$ has a positive lower bound on J , the right-hand side of (7.4.6) tends to $+\infty$ when m tends to $+\infty$. It follows that there is $M_2 > 0$ such that

$$\inf_{m \neq n \geq M_2} \min_{x \in J_i} |(f'_m(x) - f'_n(x))'| > 0. \quad (7.4.7)$$

This implies that $f'_m(x) - f'_n(x)$ is monotone on J for $m \neq n \geq M_2$. By Koksma's theorem, for $M = \max\{M_1, M_2\}$, the sequence $(\beta^{n+M} g(\beta))_{n \in \mathbb{N}}$ is uniformly distributed modulo one for almost every $\beta \in J$. This completes the proof. \square

Putting $g \equiv 1$ in Theorem 7.12, we show that the sequences $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ are Chowla sequences for almost all $\beta > 1$. Moreover, by Theorem 7.13 in section 7.5, the sequences $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ are orthogonal to all dynamical systems of zero entropy for almost all $\beta > 1$.

We remark that if β is an algebraic number, then the sequence $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ is not a Chowla sequence. In fact, since β is an algebraic number, there exist r different non-negative integers a_1, a_2, \dots, a_r and r integers i_1, i_2, \dots, i_r not all equal to 0 such that

$$i_1\beta^{a_1} + i_2\beta^{a_2} + \dots + i_r\beta^{a_r} = 0. \quad (7.4.8)$$

Denote by $z(n) = e^{2\pi i\beta^n}$. Due to (7.4.8), we obtain

$$z(n + a_1)^{i_1} z(n + a_2)^{i_2} \dots z(n + a_r)^{i_r} = e^{2\pi i\beta^n (i_1\beta^{a_1} + i_2\beta^{a_2} + \dots + i_r\beta^{a_r})} = 1.$$

It follows that

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{n=0}^{N-1} z(n + a_1)^{i_1} z(n + a_2)^{i_2} \dots z(n + a_r)^{i_r} = 1. \quad (7.4.9)$$

But to have Chowla property requires the limit (7.4.9) equal to 0. Therefore, the sequence $(e^{2\pi i\beta^n})_{n \in \mathbb{N}}$ is not a Chowla sequence for any algebraic number β .

7.5 Chowla property implies Sarnak property

In this section, we show that Chowla property implies Sarnak property. The main theorem of this section is the following.

Theorem 7.13. *Chowla sequences are always orthogonal to all topological dynamical systems of zero entropy.*

Before we really get into the proof, let us remark that the idea of the proof of Theorem 7.13 is from the proof of Theorem 4.10 in [4]: the condition (Ch) implies the condition (S). Our main observation is the following lemma, which follows directly from the Stone-Weierstrass theorem.

Lemma 7.14. *The linear subspace generated by the constants and the family*

$$\mathcal{F} := \{F^{(i_1)} \circ S^{a_1} \dots F^{(i_r)} \circ S^{a_r} : 0 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)\}$$

of continuous functions is a complex algebra (closed under taking products and conjugation) of functions separating points, hence it is dense in $C((U(m) \cup \{0\})^{\mathbb{N}})$ for every $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$.

The family \mathcal{F} is an analogue of the family of \mathcal{A} in Lemma 4.6 of [4]. The difference is that \mathcal{F} is a complex algebra but \mathcal{A} is real.

Now we provide a detailed proof of Theorem 7.13. It is well known that for every $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, the system $(U(m)^{\mathbb{N}}, \mathcal{B}, S, \mu_m^{\otimes \mathbb{N}})$ is a Bernoulli system for every $m \in \mathbb{N}$, where μ_m is the equidistributed probability measure on $U(m)$ and $\mu_m^{\otimes \mathbb{N}}$ is the product measure. Since a Bernoulli system is a Kolmogorov system, the following is classical.

Proposition 7.15. *For any $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, the system $(U(m)^\mathbb{N}, S, \mu_m^{\otimes \mathbb{N}})$ is a Kolmogorov system.*

For any $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ and for any S -invariant measure ν on $(\{0, 1\}^\mathbb{N}, S)$, we define the diagram

$$\begin{array}{ccc}
 (U(m)^\mathbb{N} \times \{0, 1\}^\mathbb{N}, S, \mu_m^{\otimes \mathbb{N}} \otimes \nu) & & (7.5.1) \\
 \rho \downarrow & \searrow & \\
 ((U(m) \cup \{0\})^\mathbb{N}, S, \widehat{\nu}) & & P_2 \\
 \pi \downarrow & \swarrow & \\
 (\{0, 1\}^\mathbb{N}, S, \nu) & &
 \end{array}$$

where $\rho : (x_n, y_n)_{n \in \mathbb{N}} \mapsto (x_n y_n)_{n \in \mathbb{N}}$, $\pi : (x_n)_{n \in \mathbb{N}} \mapsto (x_n^{(0)})_{n \in \mathbb{N}}$, $P_2 : (x_n, y_n)_{n \in \mathbb{N}} \mapsto (y_n)_{n \in \mathbb{N}}$ and $\widehat{\nu} = \rho_*(\mu_m^{\otimes \mathbb{N}} \otimes \nu)$. We study this diagram in the following proposition.

Proposition 7.16. *The diagram (7.5.1) satisfies the following properties.*

- (1) *The diagram (7.5.1) commutes.*
- (2) *The factor map π is either trivial or relatively Kolmogorov.*
- (3) *The measure $\widehat{\nu}$ has the form:*

$$\widehat{\nu}(B) = \nu(\pi(B)) \prod_{i=0, b_i \neq 0}^{k-1} \mu_m(b_i),$$

for any cylinder $B = [b_0, b_1, \dots, b_{k-1}]$ in $(U(m) \cup \{0\})^\mathbb{N}$.

Proof. To prove that the diagram (7.5.1) commutes, it suffices to prove $P_2 = \pi \circ \rho$. In fact, we have

$$\pi \circ \rho((x(n), y(n))_{n \in \mathbb{N}}) = \pi((x(n)y(n))_{n \in \mathbb{N}}) = (x(n)^{(0)}y(n)^{(0)})_{n \in \mathbb{N}} = (y(n))_{n \in \mathbb{N}},$$

because of the facts that $x^{(0)} = 1$ for all $x \in U(m)$ and $y^{(0)} = y$ for all $y \in \{0, 1\}$. This completes the proof of (1).

By Proposition 7.15, the factor P_2 is relatively Kolmogorov. Since the factor map π is an intermediate factor by (1), it is either trivial or relatively Kolmogorov. This implies (2).

It remains to show the form of $\widehat{\nu}$. For any cylinder $B = [b_0, b_1, \dots, b_{k-1}]$ in $(U(m) \cup \{0\})^\mathbb{N}$, we observe that $\rho^{-1}(B)$ is the cylinder

$$[B_0, B_1, \dots, B_{k-1}] \times [b_0^{(0)}, b_1^{(0)}, \dots, b_{k-1}^{(0)}],$$

where $B_i = \{b_i\}$ if $b_i \neq 0$; and $B_i = U(m)$ if $b_i = 0$. Since the support of the probability measure μ_m is $U(m)$, we obtain

$$\begin{aligned}\widehat{\nu}(B) &= \rho_*(\mu_m^{\otimes \mathbb{N}} \otimes \nu)(B) = \mu_m^{\otimes \mathbb{N}} \otimes \nu(\rho^{-1}(B)) \\ &= \nu(\pi(B)) \prod_{i=0, b_i \neq 0}^{k-1} \mu_m(b_i),\end{aligned}$$

which completes the proof of (3). □

The following lemma follows directly from the orthogonality of continuous group characters on the group $U(m)$.

Lemma 7.17. *Let $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. For $k \in I(m)$, we have the integral*

$$\int_{U(m)} x^k d\mu_m(x) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{else.} \end{cases}$$

Now we do some calculations in the following two lemmas.

Lemma 7.18. *Let $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Let $0 \leq a_1 < \dots < a_r, r \geq 0$ and $i_s \in I(m)$ for $1 \leq s \leq r$. Then the integral*

$$\int_{(U(m) \cup \{0\})^{\mathbb{N}}} F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \dots F^{(i_r)} \circ S^{a_r} d\widehat{\nu} \quad (7.5.2)$$

equals 0 if not all i_s are zero, and equals to

$$\int_{\{0,1\}^{\mathbb{N}}} F \circ S^{a_1} \cdot F \circ S^{a_2} \dots F \circ S^{a_r} d\nu,$$

if all i_s are zero.

Proof. The second case follows directly from the facts that $\pi_*(\widehat{\nu}) = \nu$ by Proposition 7.16 and $F^{(0)} \circ S^{a_s} = F \circ S^{a_s} \circ \pi$. Now we study the first case. Fix a choice of $0 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0. A straightforward computation shows that

$$\begin{aligned}& \int_{(U(m) \cup \{0\})^{\mathbb{N}}} F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \dots F^{(i_r)} \circ S^{a_r} d\widehat{\nu} \\ &= \int_{(U(m) \cup \{0\})^{\mathbb{N}}} x(a_1)^{(i_1)} x(a_2)^{(i_2)} \dots x(a_r)^{(i_r)} d\widehat{\nu}(x),\end{aligned} \quad (7.5.3)$$

where we denote $x = (x(n))_{n \in \mathbb{N}}$. We write the right-hand side of (7.5.3) a sum of two parts:

$$\int_B x(a_1)^{(i_1)} x(a_2)^{(i_2)} \dots x(a_r)^{(i_r)} d\widehat{\nu}(x) \quad (7.5.4)$$

and

$$\int_{(U(m) \cup \{0\})^{\mathbb{N}} \setminus B} x(a_1)^{(i_1)} x(a_2)^{(i_2)} \cdots x(a_r)^{(i_r)} d\widehat{\nu}(x), \quad (7.5.5)$$

where B denotes the cylinder

$$\{x \in (U(m) \cup \{0\})^{\mathbb{N}} : x(a_s) \neq 0, \forall 1 \leq s \leq r\}.$$

A simple observation shows that the integral (7.5.5) equals to 0 because at least one of $x(a_s)$ are equal to 0 for all $1 \leq s \leq r$. On the other hand, due to the fact that $y^{(n)} = y^n$ for all $y \in U(m)$ and for all $n \in I(m)$, the integral (7.5.4) equals to

$$\int_B x(a_1)^{i_1} \cdots x(a_r)^{i_r} d\widehat{\nu}(x) = \nu(B) \prod_{j=1}^r \int_{U(m)} x(a_j)^{i_j} d\mu_m(x(a_j)). \quad (7.5.6)$$

By Lemma 7.17, the equation (7.5.6) equals to 0. Therefore, we conclude that in the first case the integral (7.5.2) equals to 0. \square

Lemma 7.19. $\mathbb{E}^{\widehat{\nu}}[F|\pi^{-1}(\mathcal{B})] = 0$.

Proof. It suffices to show $\mathbb{E}^{\widehat{\nu}}[F1_{\pi^{-1}(B)}] = 0$ for any cylinder B in $\{0, 1\}^{\mathbb{N}}$. Let $B = [b_0, b_1, \dots, b_k]$ be a cylinder in $\{0, 1\}^{\mathbb{N}}$. Then $\pi^{-1}(B) = [C_0, C_1, \dots, C_k]$ where $C_i = U(m)$ if $b_i = 1$; $C_i = \{0\}$ if $b_i = 0$ for $0 \leq i \leq k$.

If $b_0 = 0$, then

$$\mathbb{E}^{\widehat{\nu}}[F1_{\pi^{-1}(B)}] = \mathbb{E}^{\widehat{\nu}}[0 \cdot 1_{\pi^{-1}(B)}] = 0.$$

If $b_0 \neq 0$, then $C_0 = U(m)$. A simple calculation shows that

$$\mathbb{E}^{\widehat{\nu}}[F1_{\pi^{-1}(B)}] = \int_{\pi^{-1}(B)} x(0) d\widehat{\nu}(x) = \nu(B) \int_{U(m)} x(0) d\mu_m(x(0)) = 0.$$

The last equality is due to Lemma 7.17. This completes the proof. \square

Now we prove some criteria of Chowla property.

Proposition 7.20. *Let $z \in (S^1 \cup \{0\})^{\mathbb{N}}$. Then the followings are equivalent.*

- (1) *The sequence z has Chowla property.*
- (2) $Q\text{-gen}(z) = \{\widehat{\nu} : \nu \in Q\text{-gen}(\pi(z))\}$.
- (3) *As $k \rightarrow \infty$, $\delta_{N_k, \pi(z)} \rightarrow \nu$ if and only if $\delta_{N_k, z} \rightarrow \widehat{\nu}$.*

Actually, Proposition 7.20 follows directly from the following lemma.

Lemma 7.21. *Let $z \in (S^1 \cup \{0\})^{\mathbb{N}}$. Suppose that the sequence z has an index $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Suppose that $\pi(z)$ is quasi-generic for ν along $(N_k)_{k \in \mathbb{N}}$. Then*

$$\delta_{N_k, z} \rightarrow \widehat{\nu}, \quad \text{as } k \rightarrow \infty$$

if and only if

$$\lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} z^{(i_1)}(n+a_1) \cdot z^{(i_2)}(n+a_2) \cdots z^{(i_r)}(n+a_r) = 0, \quad (7.5.7)$$

for each choice of $0 \leq a_1 < \cdots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0.

Proof. “ \Rightarrow ” Fix a choice of $0 \leq a_1 < \cdots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0. Observe that

$$\begin{aligned} & \frac{1}{N_k} \sum_{n=0}^{N_k-1} z^{(i_1)}(n+a_1) \cdot z^{(i_2)}(n+a_2) \cdots z^{(i_r)}(n+a_r) \\ &= \frac{1}{N_k} \sum_{n=0}^{N_k-1} (F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \cdots F^{(i_r)} \circ S^{a_r}) \circ S^n(z), \end{aligned} \quad (7.5.8)$$

for all $k \in \mathbb{N}$. Since the measure $\widehat{\nu}$ is quasi-generic for z along $(N_k)_{k \in \mathbb{N}}$, we obtain

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} (F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \cdots F^{(i_r)} \circ S^{a_r}) \circ S^n(z) \\ &= \int_{(U(m) \cup \{0\})^{\mathbb{N}}} F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \cdots F^{(i_r)} \circ S^{a_r} d\widehat{\nu}. \end{aligned} \quad (7.5.9)$$

According to Lemma 7.18, the right-hand side of the above equation equals to 0, which completes the proof.

“ \Leftarrow ” Without loss of generality, we may assume that

$$\delta_{N_k, z} \rightarrow \theta, \quad \text{as } k \rightarrow \infty,$$

for some S -invariant probability measure θ on $(U(m) \cup \{0\})^{\mathbb{N}}$. Now we prove that the measure θ is exactly the measure $\widehat{\nu}$. Similarly to (7.5.8) and (7.5.9), we have

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} z^{(i_1)}(n+a_1) \cdot z^{(i_2)}(n+a_2) \cdots z^{(i_r)}(n+a_r) \\ &= \int_{(U(m) \cup \{0\})^{\mathbb{N}}} F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \cdots F^{(i_r)} \circ S^{a_r} d\theta, \end{aligned}$$

for each choice of $0 \leq a_1 < \cdots < a_r, r \geq 0, i_s \in I(m)$. According to the hypothesis (7.5.7), we have

$$\int_{(U(m) \cup \{0\})^{\mathbb{N}}} F^{(i_1)} \circ S^{a_1} \cdot F^{(i_2)} \circ S^{a_2} \cdots F^{(i_r)} \circ S^{a_r} d\theta = 0,$$

for each choice of $0 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0. Repeating the same argument in Lemma 7.18, we have

$$\begin{aligned} & \int_{(U(m) \cup \{0\})^{\mathbb{N}}} F^{(0)} \circ S^{a_1} \cdot F^{(0)} \circ S^{a_2} \dots F^{(0)} \circ S^{a_r} d\theta \\ &= \int_{\{0,1\}^{\mathbb{N}}} F \circ S^{a_1} \cdot F \circ S^{a_2} \dots F \circ S^{a_r} d\nu. \end{aligned}$$

Hence

$$\int_{(U(m) \cup \{0\})^{\mathbb{N}}} f d\theta = \int_{(U(m) \cup \{0\})^{\mathbb{N}}} f d\widehat{\nu},$$

for all f belonging to the family \mathcal{F} defined in Lemma 7.14. Since the family \mathcal{F} is dense in $C(X)$, by Lemma 7.14, we conclude that $\theta = \widehat{\nu}$. \square

If additionally we assume that the sequence does not take the value 0 in Proposition 7.20, then we obtain the following criterion of Chowla property for sequences in $(S^1)^{\mathbb{N}}$.

Corollary 7.22. *Let $z \in (S^1)^{\mathbb{N}}$. Suppose that z has an index $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Then the sequence z has Chowla property if and only if z is generic for $\mu_m^{\otimes \mathbb{N}}$.*

Proof. Observe that $\pi(z) = (1, 1, \dots)$. It follows that $\text{Q-gen}(\pi(z)) = \{\delta_{(1,1,\dots)}\}$, that is, $\pi(z)$ is generic for the measure $\delta_{(1,1,\dots)}$. A straightforward computation shows that the measure $\widehat{\delta}_{(1,1,\dots)}$ is exactly $\mu_m^{\otimes \mathbb{N}}$. Due to the equivalence between (1) and (2) in Proposition 7.20, we conclude that the sequence z has Chowla property if and only if z is generic for $\mu_m^{\otimes \mathbb{N}}$. \square

Let (X, T) be a topological dynamical system and $x \in X$ be a completely deterministic point. Let ν be a S -invariant measure on $\{0, 1\}^{\mathbb{N}}$. Let $z \in (U(m) \cup \{0\})^{\mathbb{N}}$ be quasi-generic for $\widehat{\nu}$ along $(N_k)_{k \in \mathbb{N}}$. Suppose that (x, z) is quasi-generic for some $T \times S$ -invariant measure ρ on $X \times (U(m) \cup \{0\})^{\mathbb{N}}$ along $(N_k)_{k \in \mathbb{N}}$. We define the diagram

$$\begin{array}{ccc} (X \times (U(m) \cup \{0\})^{\mathbb{N}}, T \times S, \rho) & \xrightarrow{P_2} & ((U(m) \cup \{0\})^{\mathbb{N}}, S, \widehat{\nu}) \\ \gamma \downarrow & & \pi \downarrow \\ (X, T, (P_1)_*(\rho)) \vee (\{0, 1\}^{\mathbb{N}}, S, \nu) & \xrightarrow{\sigma} & (\{0, 1\}^{\mathbb{N}}, S, \nu, \mathcal{B}) \end{array} \quad (7.5.10)$$

where the factor map π is already defined in diagram (7.5.1), P_i is the projection on the i -th coordinate for $i = 1, 2$ and the factor maps γ and σ are induced by the intermediate factor $(X, T, (P_1)_*(\rho)) \vee (\{0, 1\}^{\mathbb{N}}, S, \nu)$.

We have the following lemma.

Lemma 7.23. *The factor maps σ and π in digram (7.5.10) are relatively independent.*

Proof. Since P_1 is the projection on X and the point x is completely deterministic in X , we have the entropy $h(T, (P_1)_*(\rho)) = 0$. Due to Pinsker's formula (c.f. [103], Theorem 6.3), the factor map σ is of relatively entropy zero. On the other hand, by Proposition 7.16, the factor map π is relatively Kolmogorov. According to Lemma 7.3, we conclude that the factor maps σ and π are relatively independent. \square

Now we prove the main theorem in this section by using the diagram (7.5.10).

Proof of Theorem 7.13. Let (X, T) be a topological dynamical system. Let $x \in X$ be a completely deterministic point. Suppose that (x, z) is quasi-generic for some measure ρ along $(N_k)_{k \in \mathbb{N}}$. For any $f \in C(X)$, since $z(n) = F(S^n z)$, we have

$$\lim_{k \rightarrow \infty} \frac{1}{N_k} \sum_{n=0}^{N_k-1} f(T^n x) z(n) = \mathbb{E}^\rho[f \otimes F]. \quad (7.5.11)$$

On the other hand, by the criteria of Chowla property in Proposition 7.20, without loss of generality, we may assume that z is quasi-generic for some measure $\widehat{\nu}$ along $(N_k)_{k \in \mathbb{N}}$ where ν is a S -invariant probability measure on $\{0, 1\}^{\mathbb{N}}$. Observing $(P_2)_*(\rho) = \widehat{\nu}$, we establish the diagram (7.5.10). By Lemma 7.23, the factor maps σ and π are relatively independent. Let $P := (\sigma \circ \gamma, \pi \circ P_2)$. A simple computation shows that

$$\mathbb{E}^\rho[f \otimes F | P^{-1}(\mathcal{B} \otimes \mathcal{B})] = \mathbb{E}^{\gamma^*(\rho)}[f | \sigma^{-1}(\mathcal{B})] \mathbb{E}^{\widehat{\nu}}[F | \pi^{-1}(\mathcal{B})].$$

Since Lemma 7.19 tells us that $\mathbb{E}^{\widehat{\nu}}[F | \pi^{-1}(\mathcal{B})] = 0$, we obtain

$$\mathbb{E}^\rho[f \otimes F | P^{-1}(\mathcal{B} \otimes \mathcal{B})] = 0.$$

Hence we have

$$\mathbb{E}^\rho[f \otimes F] = \mathbb{E}^\rho[\mathbb{E}^\rho[f \otimes F | P^{-1}(\mathcal{B} \otimes \mathcal{B})]] = 0. \quad (7.5.12)$$

Combining (7.5.11) and (7.5.12), we complete the proof. \square

7.6 Chowla property for random sequences

In this section, we study sequences of random variables having Chowla property almost surely. We give a sufficient and necessary condition for a stationary random sequence to be a Chowla sequence almost surely. We also construct some dependent random sequences having Chowla property almost surely.

7.6.1 Chowla property for stationary random sequences

Here, we study sequences of stationary random variables.

Proposition 7.24. *Let $\mathcal{Z} = (Z_n)_{n \in \mathbb{N}}$ be a sequence of stationary random variables taking values in $S^1 \cup \{0\}$. Suppose that the random sequence \mathcal{Z} has an index $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ almost surely. Then almost surely the random sequence \mathcal{Z} is a Chowla sequence if and only if*

$$\mathbb{E}[Z_{a_1}^{(i_1)} \cdot Z_{a_2}^{(i_2)} \cdot \dots \cdot Z_{a_r}^{(i_r)}] = 0, \quad (7.6.1)$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0.

Proof. “ \Rightarrow ” Since the random sequence \mathcal{Z} is stationary, it can be viewed as a sequence $(f(T^n))_{n \in \mathbb{N}}$ for some \mathcal{C} -measurable complex function f and \mathcal{C} -measurable transformation $T : X \rightarrow X$ where X is a compact metric space and \mathcal{C} is a σ -algebra on X . By Birkhoff’s ergodic Theory, for any $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0, almost surely we have

$$\begin{aligned} & \mathbb{E}[Z_{a_1}^{(i_1)} \cdot Z_{a_2}^{(i_2)} \cdot \dots \cdot Z_{a_r}^{(i_r)} | \mathcal{I}] \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} Z_{n+a_1}^{(i_1)} \cdot Z_{n+a_2}^{(i_2)} \cdot \dots \cdot Z_{n+a_r}^{(i_r)} = 0, \end{aligned}$$

where $\mathbb{E}[f | \mathcal{I}]$ is the expectation of f conditional on the σ -algebra \mathcal{I} consisting of invariant sets of T . Thus we conclude that

$$\mathbb{E}[Z_{a_1}^{(i_1)} \cdot Z_{a_2}^{(i_2)} \cdot \dots \cdot Z_{a_r}^{(i_r)}] = \mathbb{E}[\mathbb{E}[Z_{a_1}^{(i_1)} \cdot Z_{a_2}^{(i_2)} \cdot \dots \cdot Z_{a_r}^{(i_r)} | \mathcal{I}]] = 0,$$

for any $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0.

“ \Leftarrow ” Fix a choice of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0. Let $s = \min\{1 \leq j \leq r : i_j \neq 0\}$. Define the random variables

$$Y_n := Z_{a_1+n}^{(i_1)} \cdot Z_{a_2+n}^{(i_2)} \cdot \dots \cdot Z_{a_r+n}^{(i_r)}, \quad \text{for all } n \in \mathbb{N}.$$

Then for $k > 1$ and $n \geq 1$, the product $Y_n \cdot Y_{n+k}$ has the form

$$Z_{a_1+n}^{(i'_1)} \cdot Z_{a_2+n}^{(i'_2)} \cdot \dots \cdot Z_{a_r+n+k}^{(i'_{r+k-1})} \tag{7.6.2}$$

where $i'_j \in I(m)$ and $i'_j = i_j + i_{j-k+1} \pmod{m}^1$ for $1 \leq j \leq r+k-1$ with $i_l := 0$ for $l \leq 0$ or $l \geq r+1$. It is not hard to see that $i'_s \neq 0$. It follows that the expectation of (7.6.2) vanishes by the hypothesis (7.6.1), that is,

$$\mathbb{E}[Y_n Y_{n+k}] = 0, \quad \text{for all } n, k \geq 1.$$

This means that the random sequence $(Y_n)_{n \in \mathbb{N}}$ is orthogonal². Therefore almost surely we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} Y_n = 0,$$

i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} Z_{n+a_1}^{(i_1)} \cdot Z_{n+a_2}^{(i_2)} \cdot \dots \cdot Z_{n+a_r}^{(i_r)} = 0.$$

Since there are only countably many choices of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0, we deduce that almost surely

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} Z_{n+a_1}^{(i_1)} \cdot Z_{n+a_2}^{(i_2)} \cdot \dots \cdot Z_{n+a_r}^{(i_r)} = 0,$$

for all choices of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0. □

¹In particular, if $m = \infty$, then $i'_j = i_j + i_{j-k+1}$ for $1 \leq j \leq r+k-1$.

²Recall that a random sequence $(Y_n)_{n \in \mathbb{N}}$ is *orthogonal*, if $\mathbb{E}[Y_n Y_m] = 0$ for any $n \neq m$.

We remark that if we drop the assumption that the random sequence $(Z_n)_{n \in \mathbb{N}}$ is stationary, then the proof of the sufficiency in Proposition 7.24 is still valid.

If additionally we suppose the random variables are independent and identically distributed, then we have the following equivalent condition of Chowla property for random sequences.

Corollary 7.25. *Suppose that $\mathcal{Z} = (Z_n)_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables taking values in $S^1 \cup \{0\}$. Let $m \in \mathbb{N}_{\geq 2} \cup \{0\}$. Suppose that \mathcal{Z} has the index m almost surely. Then \mathcal{Z} is a Chowla sequence almost surely if and only if $\mathbb{E}[Z_1^{(i)}] = 0$ for all $i \in I(m) \setminus \{0\}$.*

Proof. Since \mathcal{Z} is independent and identically distributed, we observe that the equivalent condition (7.6.1) of Chowla property in Proposition 7.24 becomes

$$\mathbb{E}[Z_{a_1}^{(i_1)}] \cdot \mathbb{E}[Z_{a_2}^{(i_2)}] \cdot \dots \cdot \mathbb{E}[Z_{a_r}^{(i_r)}] = 0,$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in I(m)$ not all equal to 0. This is equivalent to say that $\mathbb{E}[Z_1^{(i)}] = 0$ for all $i \in I(m) \setminus \{0\}$. \square

We show an example of sequences of independent and identically distributed random variables having Chowla property almost surely.

Example 7.26. *For any $m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, let $((U(m) \cup \{0\})^{\mathbb{N}}, S, \mu_m^{\otimes \mathbb{N}})$ be a measurable dynamical system. Recall that $F : (U(m) \cup \{0\})^{\mathbb{N}} \rightarrow (U(m) \cup \{0\})$ is the projection on the first coordinate. Let $Z_n := F \circ S^n$. Then $(Z_n)_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables, which has the index m almost surely. It is easy to check that $\mathbb{E}[Z_1^{(i)}] = 0$ for all $i \in I(m) \setminus \{0\}$ by Lemma 7.17. Therefore, almost surely the sequence $(Z_n)_{n \in \mathbb{N}}$ is a Chowla sequence.*

A natural question is raised whether there are sequences of dependent random variables having Chowla property almost surely. We will prove later (in Section 7.6.3) that such sequences exist. For this propose, we prove additionally the following lemma.

Lemma 7.27. *Let X_1, X_2, \dots, X_k be bounded random variables. The random variables X_1, \dots, X_k are independent if and only if*

$$\mathbb{E}[X_1^{n_1} X_2^{n_2} \dots X_k^{n_k}] = \mathbb{E}[X_1^{n_1}] \mathbb{E}[X_2^{n_2}] \dots \mathbb{E}[X_k^{n_k}] \quad (7.6.3)$$

for all positive integers n_1, \dots, n_k .

Proof. The necessity being trivial, we just prove the sufficiency. A simple computation allows us to see that if

$$\mathbb{E}[f_{1,i}(X_1) f_{2,i}(X_2) \dots f_{k,i}(X_k)] = \mathbb{E}[f_{1,i}(X_1)] \mathbb{E}[f_{2,i}(X_2)] \dots \mathbb{E}[f_{k,i}(X_k)]$$

holds for functions $f_{j,i}$ for all $1 \leq j \leq k$ and for all $1 \leq i \leq l$, then

$$\begin{aligned} & \mathbb{E}\left[\sum_{i=1}^l f_{1,i}(X_1) \sum_{i=1}^l f_{2,i}(X_2) \cdots \sum_{i=1}^l f_{k,i}(X_k)\right] \\ &= \mathbb{E}\left[\sum_{i=1}^l f_{1,i}(X_1)\right] \mathbb{E}\left[\sum_{i=1}^l f_{2,i}(X_2)\right] \cdots \mathbb{E}\left[\sum_{i=1}^l f_{k,i}(X_k)\right]. \end{aligned} \tag{7.6.4}$$

Combining (7.6.3) and (7.6.4), we have

$$\mathbb{E}[P_1(X_1) \cdots P_k(X_k)] = \mathbb{E}[P_1(X_1)] \mathbb{E} \cdots \mathbb{E}[P_k(X_k)], \tag{7.6.5}$$

for all P_1, \dots, P_k complex polynomials. It is well known that the random variables X_1, \dots, X_k are independent if and only if for any continuous functions f_1, f_2, \dots, f_k ,

$$\mathbb{E}[f_1(X_1) \cdots f_k(X_k)] = \mathbb{E}[f_1(X_1)] \cdots \mathbb{E}[f_k(X_k)]. \tag{7.6.6}$$

It is noticed that if additionally the random variable X_1, \dots, X_k are bounded, then ‘‘continuous functions’’ can be replaced by ‘‘bounded continuous functions’’ in the above statement. It is classical that any continuous bounded function is approximated by polynomials under uniform norm. Due to (7.6.5) and the approximation by polynomials, we see that (7.6.6) holds for any bounded continuous functions f_1, f_2, \dots, f_k , which implies that the random variables X_1, \dots, X_k are independent. \square

Now we show a sufficient and necessary condition for a sequence of stationary random variables having Chowla property almost surely to be independent.

Corollary 7.28. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of stationary random variables taking values in $S^1 \cup \{0\}$. Let $m \in \mathbb{N}_{\geq 2} \cup \{0\}$. Suppose that almost surely the sequence $(X_n)_{n \in \mathbb{N}}$ has the index m and has Chowla property. Then*

(1) *if m is finite, then the sequence $(X_n)_{n \in \mathbb{N}}$ is independent if and only if*

$$\mathbb{E}[|X_{a_1} X_{a_2} \cdots X_{a_r}|] = \mathbb{E}[|X_{a_1}|] \mathbb{E}[|X_{a_2}|] \cdots \mathbb{E}[|X_{a_r}|]$$

for each choice of $1 \leq a_1 < \cdots < a_r, r \geq 0$;

(2) *if m is infinite, then the sequence $(X_n)_{n \in \mathbb{N}}$ is always independent.*

Proof. We first consider the case when the index m is finite. Since the sequence $(X_n)_{n \in \mathbb{N}}$ has Chowla property almost surely, by Proposition 7.24, we obtain that for each choice of $1 \leq a_1 < \cdots < a_r, r \geq 0, i_s \in \mathbb{N} \setminus \{0\}$ not all equal to the multiples of m ,

$$\mathbb{E}[Z_{a_1}^{i_1} Z_{a_2}^{i_2} \cdots Z_{a_r}^{i_r}] = 0, \tag{7.6.7}$$

and at least one of $\mathbb{E}[Z_{a_1}^{i_1}], \mathbb{E}[Z_{a_2}^{i_2}], \dots, \mathbb{E}[Z_{a_r}^{i_r}]$ equal to zero. It follows that

$$\mathbb{E}[Z_{a_1}^{i_1}] \cdot \mathbb{E}[Z_{a_2}^{i_2}] \cdots \mathbb{E}[Z_{a_r}^{i_r}] = 0 = \mathbb{E}[Z_{a_1}^{i_1} Z_{a_2}^{i_2} \cdots Z_{a_r}^{i_r}], \tag{7.6.8}$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in \mathbb{N} \setminus \{0\}$ not all equal to the multiples of m . Due to Lemma 7.27 and the equation (7.6.8), the sequence $(X_n)_{n \in \mathbb{N}}$ is independent if and only if

$$\mathbb{E}[Z_{a_1}^{k_1 m}] \cdot \mathbb{E}[Z_{a_2}^{k_2 m}] \cdot \dots \cdot \mathbb{E}[Z_{a_r}^{k_r m}] = \mathbb{E}[Z_{a_1}^{k_1 m}] \mathbb{E}[Z_{a_2}^{k_2 m}] \cdot \dots \cdot \mathbb{E}[Z_{a_r}^{k_r m}], \quad (7.6.9)$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0, k_s \in \mathbb{N} \setminus \{0\}$. It is not hard to see that $X_n^{km} = |X_n|$ for every $n \in \mathbb{N}$ and for every positive integer k . It follows that the sequence $(X_n)_{n \in \mathbb{N}}$ is independent if and only if

$$\mathbb{E}[|X_{a_1} X_{a_2} \dots X_{a_r}|] = \mathbb{E}[|X_{a_1}|] \mathbb{E}[|X_{a_2}|] \cdot \dots \cdot \mathbb{E}[|X_{a_r}|],$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0$. This completes the proof of (1).

Now we consider the case when the index m is infinite. Repeating the same reasoning with (7.6.8), we obtain

$$\mathbb{E}[Z_{a_1}^{i_1}] \cdot \mathbb{E}[Z_{a_2}^{i_2}] \cdot \dots \cdot \mathbb{E}[Z_{a_r}^{i_r}] = \mathbb{E}[Z_{a_1}^{i_1} Z_{a_2}^{i_2} \dots Z_{a_r}^{i_r}] = 0, \quad (7.6.10)$$

for each choice of $1 \leq a_1 < \dots < a_r, r \geq 0, i_s \in \mathbb{N} \setminus \{0\}$. According to Lemma 7.27 and the equation (7.6.10), the sequence $(X_n)_{n \in \mathbb{N}}$ is independent. \square

7.6.2 Chowla property for independent random sequences

Now we study Chowla property for independent random sequences. A sequence taking values in $\cup_{m \in \mathbb{N}_{\geq 2}} U(m)$, having index ∞ and having Chowla property almost surely is also constructed in this subsection.

As it has already been noticed just after Proposition 7.24, the proof of the sufficiency in Proposition 7.24 is still valid without the assumption of stationarity. If additionally assuming that the random sequence is independent, we obtain the following proposition which is a direct consequence of the sufficiency in Proposition 7.24.

Proposition 7.29. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent random variables which take values in $S^1 \cup \{0\}$. Let $m \in \mathbb{N}_{\geq 2} \cup \{0\}$. Suppose that almost surely the sequence $(X_n)_{n \in \mathbb{N}}$ has the index m . Suppose that for any $k \in I(m) \setminus \{0\}$, there exists $N > 0$ such that for any $n > N$, the expectation $\mathbb{E}[X_n^k] = 0$. Then almost surely the sequence $(X_n)_{n \in \mathbb{N}}$ is a Chowla sequence.*

We remark that if the index m is finite, then the value N in the sufficient condition in Proposition 7.29 is uniform for all $k \in I(m) \setminus \{0\}$.

We show the existence of sequences taking values in $\cup_{m \in \mathbb{N}_{\geq 2}} U(m)$, having index ∞ and having Chowla property almost surely in the following example.

Example 7.30. *Let $(Z_n)_{n \geq 2}$ be a sequence of independent random variables on $(S^1 \cup \{0\})^{\mathbb{N}}$ such that Z_n has the distribution μ_n for $n \geq 2$. It easily follows that almost surely the index of $(Z_n)_{n \geq 2}$ is infinite. It is not difficult to see that*

$$\mathbb{E}[Z_n^k] = 0, \text{ for all } n \geq 2 \text{ and for all } k \in I(n) \setminus \{0\}.$$

Thus the sequence $(Z_n)_{n \geq 2}$ satisfies the condition in Proposition 7.29. Therefore, almost surely the sequence $(Z_n)_{n \geq 2}$ is a Chowla sequence.

7.6.3 Chowla property for dependent random sequences

In this subsection, we construct sequences of dependent random variables which have Chowla property almost surely. As it has been proved in Corollary 7.28 that the sequences of stationary random variables having the index ∞ and having Chowla property almost surely are always independent, we will concentrate on the situation where the index is a finite number. In what follows, we construct sequences of dependent stationary random variables which have the index 2 and have Chowla property almost surely. The sequences of dependent stationary random variables which have the finite index $m > 2$ and have Chowla property almost surely can be constructed in a similar way.

Let \mathcal{A} be a finite alphabet. Now we study the symbolic dynamics $(\mathcal{A}^{\mathbb{N}}, S, \mu)$ where μ is the uniform-Bernoulli measure. Let $G : \mathcal{A}^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$ be a continuous function. Denote by $Z_{G,n} := G \circ S^n$. We are concerned with the random sequence $(Z_{G,n})_{n \in \mathbb{N}}$. It is easy to check that the sequence $(Z_{G,n})_{n \in \mathbb{N}}$ has the index 2 almost surely.

Obviously, $G^{-1}(t)$ is a clopen subset of $\mathcal{A}^{\mathbb{N}}$ for each $t \in \{-1, 0, 1\}$. It follows that the function G is decided by finite words, i.e., there exist $l \in \mathbb{N} \setminus \{0\}$ and a function g on \mathcal{A}^l such that for all $x = (x_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$,

$$G(x) = g(x_0, x_1, \dots, x_{l-1}).$$

We associate a rooted 3-color tree \mathcal{T}_G with the sequence $(Z_{G,n})_{n \in \mathbb{N}}$.

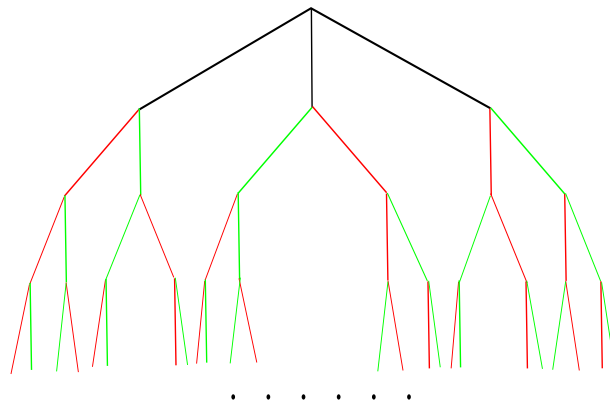
We first draw a root as the 0-th level. We then draw $\sharp\mathcal{A}$ descendants from the vertex in 0-th level as the 1-th level. The vertices at the first level are entitled by the elements in \mathcal{A} . Draw black the edges between the vertices of the 0-th level and the vertices of the 1-th level. Repeat this until we get the $(l-1)$ -th level and thus obtain a finite tree with only black edges and the property that each vertex has $\sharp\mathcal{A}$ descendants. Observe that the branch from the root to a vertex in the $(l-1)$ -th level can be viewed as a word of length $(l-1)$. Now we show how to draw the l -th level. A vertex identified by $a_1 a_2 \dots a_{l-1}$ at $(l-1)$ -th level has a descendant entitled $a \in \mathcal{A}$ if $g(a_1, a_2, \dots, a_{l-1}, a) \neq 0$. Moreover, the edge between them is red if $g(a_1, a_2, \dots, a_{l-1}, a) = 1$ and is green if $g(a_1, a_2, \dots, a_{l-1}, a) = -1$. In general, for $k \geq l-1$, a vertex identified by $a_1 a_2 \dots a_k$ in k -th level has a descendant entitled $a \in \mathcal{A}$ and only if $g(a_{k-l+2}, a_{k-l+3}, \dots, a_k, a) \neq 0$ and the color of the edge between them is red if $g(a_{k-l+2}, a_{k-l+3}, \dots, a_k, a) = 1$ and green if it has the value -1 . There edges are called the edges of the k -th level. Therefore, the rooted 3-color tree that we obtained is named by \mathcal{T}_G .

We illustrate this by the following examples.

Example 7.31. (1) Let $\mathcal{A} = \{0, 1, 2\}$. Define $g_1 : \mathcal{A}^2 \rightarrow \{-1, 0, 1\}$ by

$$\begin{aligned} (0, 0) &\mapsto 1, (1, 2) \mapsto 1, (2, 1) \mapsto 1, \\ (0, 1) &\mapsto -1, (1, 0) \mapsto -1, (2, 2) \mapsto -1, \\ (2, 0) &\mapsto 0, (0, 2) \mapsto 0, (1, 1) \mapsto 0. \end{aligned}$$

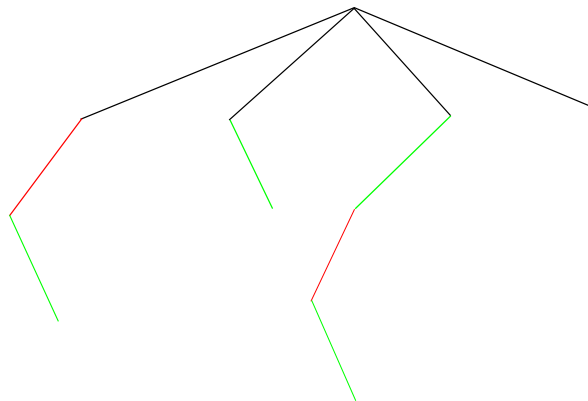
Let $G_1 : \mathcal{A}^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$, $G_1(x) = g_1(x_0, x_1)$. The associated tree \mathcal{T}_{G_1} is an infinite tree as shown in Figure 7.1.

Figure 7.1: The infinite tree \mathcal{T}_{G_1} .

(2) Let $\mathcal{A} = \{0, 1, 2, 3\}$. Define $g_1 : \mathcal{A}^2 \rightarrow \{-1, 0, 1\}$ by

$$(0, 1) \mapsto 1, (1, 3) \mapsto -1, (2, 0) \mapsto -1 \text{ and otherwise } (a, b) \mapsto 0.$$

Let $G_2 : \mathcal{A}^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$, $G_2(x) = g_2(x_0, x_1)$. The associated tree \mathcal{T}_{G_2} is a finite tree as shown in Figure 7.2.

Figure 7.2: The finite tree \mathcal{T}_{G_2}

Two branches from the root to the vertices of the same level are said to be *of different type* if they have different colors in some edges of the same level. We define the homogeneity for the tree as follows. Two branches are *of same type* if they are not of different type.

Definition 7.32. Suppose the function $G : \mathcal{A}^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$ is decided by the words of length l . We say that the associated tree \mathcal{T}_G is homogeneous if for each $k \geq l$, the numbers of branches of same type from the root to the vertices at the k -th level are constant.

It is not hard to see that Example 7.31 (1) is homogeneous but (2) is not.

Actually, Definition 7.32 means that

$$\text{Card}\{x \in \mathcal{A}^{\mathbb{N}} : G(x) = c_0, \dots, G(S^k x) = c_k\}$$

is a constant which is independent of the choice of $c = (c_n)_{n \in \mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$. Thus the following proposition is a direct consequence of Proposition 7.24.

Proposition 7.33. *Let A be a finite set. Let $G : A^{\mathbb{N}} \rightarrow \{-1, 0, 1\}$ be a continuous map. Then the random sequence $(Z_{G,n})_{n \in \mathbb{N}}$ is a Chowla sequence almost surely if and only if the associated tree \mathcal{T}_G is homogeneous.*

Now we show the existence of sequences of dependent random variables having Chowla property almost surely in the following example.

Example 7.34. *Let $\mathcal{A} = \{-1, 0, 1, \star\}$. Denote by $\tilde{\mu}_2 = \frac{1}{2}\mu_2 + \frac{1}{2}\delta_{\star}$. Then $(\mathcal{A}^{\mathbb{N}}, S, \tilde{\mu}_2^{\mathbb{N}})$ is a measurable dynamical system. Define $g : \mathcal{A}^2 \rightarrow U(m)$ by*

$$g(a, b) = \begin{cases} a & \text{if } a \neq \star \text{ and } b = \star; \\ 0 & \text{otherwise.} \end{cases}$$

We define $G : \mathcal{A}^{\mathbb{N}} \rightarrow U(m)$ by $(x_n)_{n \in \mathbb{N}} \mapsto g(x_0 x_1)$. It is not hard to see that the associated tree \mathcal{T}_G is a finite homogeneous tree (see Figure 7.3). It follows directly from Proposition 7.33 that almost surely $(Z_{G,n})_{n \in \mathbb{N}}$ is a Chowla sequence.

On the other hand, since $\mathbb{E}[|Z_1|] = \frac{1}{2}$, we obtain that

$$\mathbb{E}[|Z_1 Z_2|] = 0 \quad \text{but} \quad \mathbb{E}[|Z_1|]\mathbb{E}[|Z_2|] = \frac{1}{4},$$

that is,

$$\mathbb{E}[|Z_1 Z_2|] \neq \mathbb{E}[|Z_1|]\mathbb{E}[|Z_2|].$$

By Corollary 7.28, the sequence of random variables (Z_n) is not independent.

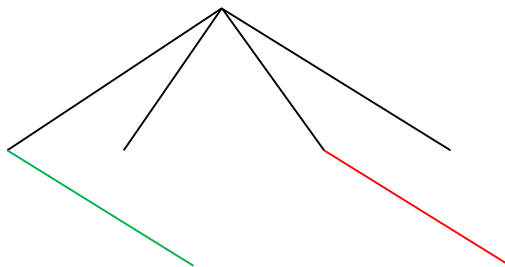


Figure 7.3: The associated tree \mathcal{T}_G

Let \mathcal{A} be a finite alphabet. Let $(\mathcal{A}^{\mathbb{N}}, S, \mu)$ be a symbolic dynamics where μ is the uniform-Bernoulli measure. Let $m \in \mathbb{N}_{\geq 2}$. Let $G : \mathcal{A}^{\mathbb{N}} \rightarrow U(m) \cup \{0\}$ be a continuous function with $G^{-1}(t)$ not empty for each $t \in U(m) \cup \{0\}$. Denote by $Z_{G,n} := G \circ S^n$. Similarly to the case when $m = 2$, we associate a rooted $m + 1$ -color tree \mathcal{T}_G with the sequence $(Z_{G,n})_{n \in \mathbb{N}}$. Similarly to Definition 7.32 and Proposition 7.33, we define the homogeneity of the associated tree \mathcal{T}_G and have the following proposition.

Proposition 7.35. *Let A be a finite set and $m \geq 2$ be an integer. Let $G : A^{\mathbb{N}} \rightarrow U(m) \cup \{0\}$ be a continuous map. Then the random sequence $(Z_{G,n})_{n \in \mathbb{N}}$ is a Chowla sequence almost surely if and only if the associated tree \mathcal{T}_G is homogeneous.*

For $m \in \mathbb{N}_{\geq 2}$, we give examples of sequences of dependent random variables having the index m and having Chowla property almost surely.

Example 7.36. *Let $m \in \mathbb{N}_{\geq 2}$. Let $\mathcal{A} = U(m) \cup \{0, \star\}$. Denote by $\tilde{\mu}_m = \frac{1}{2}\mu_m + \frac{1}{2}\delta_{\star}$. Then $(\mathcal{A}^{\mathbb{N}}, S, \tilde{\mu}_m^{\mathbb{N}})$ is a measurable dynamical system. Define $g : \mathcal{A}^2 \rightarrow U(m)$ by*

$$g(ab) = \begin{cases} a & \text{if } a \in U(m) \text{ and } b = \star; \\ 0 & \text{otherwise.} \end{cases}$$

We define $G : \mathcal{A}^{\mathbb{N}} \rightarrow U(m)$ by $(x_n)_{n \in \mathbb{N}} \mapsto g(x_0 x_1)$. It is not hard to see that the associated tree \mathcal{T}_G is a finite homogeneous tree. It follows directly from Proposition 7.35 that almost surely $(Z_{G,n})_{n \in \mathbb{N}}$ is a Chowla sequence.

On the other hand, since $\mathbb{E}[|Z_1|] = \frac{1}{2}$, we obtain that

$$\mathbb{E}[|Z_1 Z_2|] = 0 \text{ but } \mathbb{E}[|Z_1|] \mathbb{E}[|Z_2|] = \frac{1}{4},$$

that is,

$$\mathbb{E}[|Z_1 Z_2|] \neq \mathbb{E}[|Z_1|] \mathbb{E}[|Z_2|].$$

By Corollary 7.28, the sequence of random variables (Z_n) is not independent.

7.6.4 Sarnak property does not imply Chowla property

In this subsection, for every $m \in \mathbb{N}_{\geq 2} \cup \{0\}$, we prove that Sarnak property does not imply Chowla property for sequences of index m . We first prove a sufficient condition for sequences to be Sarnak sequences.

Lemma 7.37. *Let (X, T, μ) be a Kolmogorov system. Let $f : X \rightarrow \mathbb{C}$ be a measurable function. Suppose that $\mathbb{E}[f] = 0$. Then for every generic point $x \in X$ of μ , the sequence $(f \circ T^n(x))_{n \in \mathbb{N}}$ is a Sarnak sequence.*

Proof. Let (Y, U) be an arbitrary topological dynamical system. Let $y \in Y$ be a completely deterministic point. Let x be a generic point for μ . Let ρ be a quasi-generic measure for (y, x) along $(N_k)_{k \in \mathbb{N}}$ in the product system $(Y \times X, U \times T)$. It follows that for any $g \in C(Y)$,

$$\lim_{k \rightarrow +\infty} \sum_{n=0}^{N_k-1} g \circ U^n(y) \cdot f \circ T^n(x) = \mathbb{E}^{\rho}[g \otimes f]. \quad (7.6.11)$$

Since the system $(Y, U, \rho|_Y)$ is of zero entropy and (X, T, μ) is a Kolmogorov system, (X, T, μ) and $(Y, U, \rho|_Y)$ are independent. It follows that for any $g \in C(Y)$,

$$\mathbb{E}^\rho[g \otimes f] = \mathbb{E}^{\rho|_Y}[g] \cdot \mathbb{E}^\mu[f] = 0. \quad (7.6.12)$$

Combining (7.6.11) and (7.6.12), we deduce that for any $g \in C(Y)$,

$$\lim_{k \rightarrow +\infty} \sum_{n=0}^{N_k-1} g \circ U^n(y) \cdot f \circ T^n(x) = 0.$$

This completes the proof. \square

Comparing the condition in Lemma 7.37 and the necessary and sufficient condition for sequences of stationary random variables to be Chowla sequences almost surely in Proposition 7.24, we prove the existence of the sequences which have Sarnak property but not Chowla property. We illustrate this in the following examples.

Example 7.38. Recall that a Bernoulli system is always a Kolmogorov system. Let $(\{0, 1, 2\}^\mathbb{N}, S, B_3)$ be a Bernoulli system where B_3 denotes the uniform-Bernoulli measure. Let $g : \{0, 1, 2\}^2 \rightarrow \{-1, 0, 1\}$ be a function defined by

$$g(0, 1) = 1, g(1, 2) = -1 \text{ and } g(a, b) = 0 \text{ otherwise.}$$

Define $G : \{0, 1, 2\}^\mathbb{N} \rightarrow \{-1, 0, 1\}$ by $(x(n))_{n \in \mathbb{N}} \mapsto g(x(0), x(1))$. Let $Z_n := G \circ S^n$. It is easy to check that

$$\mathbb{E}[Z_1] = 0 \quad \text{but} \quad \mathbb{E}[Z_1 Z_2] = -1.$$

By Lemma 7.37, it follows that almost surely $(Z_n)_{n \in \mathbb{N}}$ is a Sarnak sequence. On the other hand, by Proposition 7.24, almost surely the sequence $(Z_n)_{n \in \mathbb{N}}$ is not a Chowla sequence.

Example 7.39. Define $h : [0, 1) \rightarrow [0, 1)$ by

$$h(t) = \begin{cases} \frac{3}{2}t & \text{if } 0 \leq t < \frac{1}{4}, \\ \frac{1}{2}t + \frac{1}{4} & \text{if } \frac{1}{4} \leq t < \frac{1}{2}, \\ \frac{3}{2}t - \frac{1}{4} & \text{if } \frac{1}{2} \leq t < \frac{3}{4}, \\ \frac{1}{2}t + \frac{1}{2} & \text{if } \frac{3}{4} \leq t < 1. \end{cases}$$

We define the function $H : S^1 \rightarrow S^1$ by the formula $H(e^{2\pi it}) = e^{2\pi i h(t)}$. Let $((S^1)^\mathbb{N}, S, \text{Leb}^{\otimes \mathbb{N}})$ be a Kolmogorov system where Leb is the Lebesgue measure on S^1 . Let $Z_n := H \circ F \circ S^n$. It is not hard to see that almost surely the index of the sequence $(Z_n)_{n \in \mathbb{N}}$ is ∞ . A simple computation allows us to see that

$$\mathbb{E}[Z_1] = 0 \quad \text{but} \quad \mathbb{E}[Z_1^2] \neq 0.$$

By Lemma 7.37 and Proposition 7.24, we conclude that almost surely $(Z_n)_{n \in \mathbb{N}}$ is a Sarnak sequence but not a Chowla sequence.

Bibliography

- [1] A. Adler and F. C. Holroyd, *Some results on one-dimensional tilings*. Geometriae Dedicata 10, 49-58 (1981)
- [2] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc., 114 (1965), 309-319.
- [3] S. Albeverio, A. Khrennikov, V. Shelkovich, *Theory of p-adic distributions: linear and nonlinear models*. Oxford University Press, Oxford, 2010.
- [4] E. H. el Abdalaoui, J. Kulaga-Przymus, M. Lemanczyk and T. de la Rue, *The Chowla and the Sarnak's conjectures from ergodic theory point of view*, Discrete and Continuous Dynamical Systems-Series A. 37, no.6, 2899-2944, 2017.
- [5] S. Akiyama, Y. Jiang. *Higher Order Oscillation and Uniform Distribution*, arXiv:1612.08376, 2016.
- [6] N. Aoki, *Topological entropy of distal affine transformations on compact abelian groups*, J. Math. Soc. Japan 23(1971), 11-17.
- [7] T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York-Heidelberg, 1976, Undergraduate Texts in Mathematics.
- [8] C. Aten, B. Ayachi, E. Bau, D. FitzPatrick, A. Iosevich, H. Liu, A. Lott, I. MacKinnon, S. Maimon, S. Nan, J. Pakianathan, G. Petridis, C. Rojas Mena, A. Sheikh, T. Tribone, J. Weill and C. Yu , *Tiling sets and spectral sets over finite fields*, arXiv:1509.01090, 2017.
- [9] C. Bandt, *Self-similar sets 5: Integer matrices and fractal tilings of \mathbb{R}^n* , Proc. Am. Math. Soc. 112 (1991), 549-562.
- [10] R. Beauregard and J. Fraleigh, *A First Course In Linear Algebra: with Optional Introduction to Groups, Rings, and Fields*, Houghton Mifflin Harcourt (HMH), (1973).
- [11] G. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. U. S. A., 17 (1931), 656-660.

- [12] N. N. Bogoliubov and N. M. Krylov, *La théorie généralisée de la mesure dans son application à l'étude de systèmes dynamiques de la mécanique non-linéaire*, Ann. Math. II (in French). Annals of Mathematics. 38 (1): 65-113, 1937.
- [13] P. Bohl, *Über ein in der Theorie der säkularen Störungen vorkommendes Problem*, J. Reine Angew. Math. 135 (1909), 189-283.
- [14] D. Bose, C. P. A. Kumar, R. Krishnan and S. Madan, *On Fuglede's conjecture for three intervals*, Online J. Anal. Comb. No. 5 (2010), 24 pp.
- [15] D. Bose and S. Madan, *Spectrum is periodic for n -intervals*, J. Funct. Anal. 260 (2011), no. 1, 308-325.
- [16] D. Bose and S. Madan, *"Spectral implies tiling" for three intervals revisited*, Forum Math. 26 (2014), no. 4, 1247-1260.
- [17] J. Bourgain, P. Sarnak, and T. Ziegler, *Disjointness of Möbius from horocycle flows*, From Fourier analysis and number theory to Radon transforms and geometry, Dev. Math., vol. 28, Springer, New York, 2013, pp. 67-83.
- [18] N. G. de Bruijn, *On the factorization of cyclic groups*, Indag. Math. Kon. Ned. Akad. Wretch. 15 (1953), 370-377.
- [19] Y. Bugeaud, *Distribution modulo one and Diophantine approximation*, Cambridge Tracts in Mathematics 193, Cambridge, 2012.
- [20] A. T. Butson, *Generalized Hadamard Matrices*, Proc. Amer. Math. Soc., 13 (6) (1962), 894-898.
- [21] S. Chowla, *The Riemann hypothesis and Hilbert's tenth problem*, *Mathematics and Its Applications, Vol. 4*, Gordon and Breach Science Publishers, New York, 1965.
- [22] E. M. Coven and A. Meyerowitz, *Tiling the integers with translates of one finite set*, J. Algebra 212 (1999), no. 1, 161-174
- [23] S. G. Dani, *Dynamical systems on homogeneous spaces*, in: *Dynamical Systems, Ergodic Theory and Applications*, Encyclopedia of Mathematical Sciences, Vol. 100, Springer.
- [24] H. Davenport, *On some infinite series involving arithmetical functions, II*, Quart. J. Math. Oxford 8 (1937), 313-320.
- [25] H. Davenport, P. Erdős and W. J. LeVeque, *On Weyl's criterion for uniform distribution*, Michigan Math. J. 10 (1963), 311-314.
- [26] T. de la Rue, *An introduction to joinings in ergodic theory*, arXiv: math/0507429, 2005.

- [27] D. Dutkay and P. Jørgensen, *Fourier frequencies in affine iterated function systems*, Journal of Functional Analysis, 247 (2007), 110-137.
- [28] D. E. Dutkay and C-K Lai, *Some reductions of the spectral set conjecture to integers*, Math. Proc. Cambridge Philos. Soc. 156 (2014), no. 1, 123-135.
- [29] B. Farkas, R. S. Gy, *Tiles with no spectra in dimension 4*, Mathematica Scandinavica, 98(2006), 44C52.
- [30] A. H. Fan, *Spectral measures on local fields*, Difference Equations, Discrete Dynamical Systems and Applications. Springer, Cham, 2015: 15-35.
- [31] A. H. Fan, *Oscillating sequences of higher orders and topological systems of quasi-discrete spectrum*, arXiv:1802.05204, 2018.
- [32] A. H. Fan, *Fully oscillating sequences and weighted multiple ergodic limit*, Comptes Rendus de l'académie des Sciences, Series I Mathematics, 2017, 355(8): 866-870.
- [33] A. H. Fan, *Topological Wiener-Wintner ergodic theorem with polynomial weights*, arXiv:1708.06093, 2017.
- [34] A. H. Fan, *Weighted Birkhoff ergodic theorem with oscillating weights*, Erg. Th. Dynam. Syst., to appear.
- [35] A. H. Fan and S. L. Fan, *Bounded tiles in \mathbb{Q}_p are compact open sets*, arXiv:1511.06404
- [36] A. H. Fan, S. L. Fan, L. M. Liao et R. X. Shi, *Fuglede's conjecture holds in \mathbb{Q}_p* , arXiv:1512.08904.
- [37] A. H. Fan, S. L. Fan et R. X. Shi, *Compact open spectral sets in \mathbb{Q}_p* , Journal of Functional Analysis, 2016, 271(12): 3628-3661.
- [38] A. H. Fan, Y. P. Jiang, *Oscillating sequences, minimal mean attractability and minimal mean-Lyapunov-stability*, Erg. Th. Dynam. Syst., to appear.
- [39] B. Farkas, M. Matolcsi, P. Mora, *On Fuglede's conjecture and the existence of universal spectra*, J. Fourier Anal. Appl. 12 (2006), no. 5, 483-494.
- [40] S. Ferenczi, J. Kulaga-Przymus, M. Lemańczyk, *Sarnak's Conjecture—what's new*, arXiv: 1710.04039, 2017.
- [41] S. Ferenczi, C. Mauduit, *On Sarnak's conjecture and Veech's question for interval exchanges*, Journal d'Analyse Mathématique, 2018, 134(2): 545-573.
- [42] J. Fourier, *Théorie analytique de la chaleur*, Paris: Firmin Didot Père et Fils, 1822.

- [43] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal., 16 (1974), 101-121.
- [44] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1-49.
- [45] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. d'Analyse Math. 31, (1977), 204-256.
- [46] E. Glasner, B. Host and D. Rudolph, *Simple systems and their higher order self-joinings*, Israel J. of Math. 78, (1992), 131-142.
- [47] E. Glasner and B. Weiss, *Minimal transformations with no common factor need not be disjoint*, Israel J. of Math. 45, (1983), 1-8.
- [48] A. O. Gel'fond, *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. 13 (1967/1968), 259-265.
- [49] E. Glasner, *Ergodic theory via joinings*, American Mathematical Soc., 2003.
- [50] K. H. Gröchenig and A. Haas, *Self-similar lattice tilings*, J. Fourier Anal. 1 (1994), 131-170.
- [51] B. Green and T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. (2) 175 (2012), no. 2, 541-566.
- [52] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, New York: W. H. Freeman, 1986.
- [53] A. Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen*, Annals of Mathematics, 1933(2), 34 (1), 147-169.
- [54] G. A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical system*, Math. Systems Theory. 3 (1969), 320-375.
- [55] K. Hensel, *Über eine neue Begründung der Theorie der algebraischen Zahlen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 1897, 6 (3): 83-88
- [56] B. Host, *Mixing of all orders and independent joinings of systems with singular spectrum*, Israel J. Math. 76, (1991), 289-298.
- [57] L. K. Hua, *Additive theory of prime numbers*, AMS Translations of Mathematical Monographs, Vol. 13, Providence, R.I. 1965
- [58] A. Iosevich, N. Katz and T. Tao, *Convex bodies with a point of curvature do not have Fourier bases*, Amer. J. Math. 123 (2001), 1, 115-120.

- [59] A. Iosevich, N. Katz, T. Tao, *The Fuglede spectral conjecture holds for convex planar domains*, Mathematical Research Letters, 10(2003), 559-570.
- [60] A. Iosevich and S. Pedersen, *Spectral and Tiling Properties of the Unit Cube*, Inter. Math. Res. Notices 16 (1998), 819-828.
- [61] A. Iosevich, A. Mayeli, J. Pakianathan, *The Fuglede's conjecture holds in $\mathbb{Z}_p \times \mathbb{Z}_p$* , arxiv:1505.00883, 2015.
- [62] P. Jørgensen and S. Pedersen, *Spectral theory for Borel sets in R^n of finite measure*, J. Funct. Anal. 107 (1992), 72-104.
- [63] P. Jørgensen and S. Pedersen, *Harmonic analysis of fractal measures*, Const. Approx., 12(1996), 1-30.
- [64] P. Jørgensen and S. Pedersen, *Dense analytic subspaces in fractal L^2 spaces*, J. Anal. Math. 75(1998),185-228.
- [65] P. Jørgensen, S. Pedersen, *Spectral pairs in Cartesian coordinates*, J. Fourier Anal. Appl. 5 (1999), 285-302.
- [66] A. del Junco, M. Lemanczyk and M. K. Mentzen, *Semisimplicity, joinings and group extensions*, Studia Math. 112, (1995), 141-164.
- [67] A. del Junco, M. Rahe and L. Swanson, *Chacon's automorphism has minimal self-joinings*, J. d'Analyse Math. 37, (1980),276-284.
- [68] A. del Junco and D.J. Rudolph, *On ergodic actions whose self-joinings are graphs*, Ergod. Th. Dynam. Sys. 7, (1987), 531-557.
- [69] M. Kadir, *Spectral sets and tiles on vector space over local fields*. J. Math. Anal. Appl. 440 (2016), no. 1, 240-249.
- [70] A. Katok, and B. Haselblatt, *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.
- [71] J.W. Kammeyer, D. J. Rudolph, *Restricted orbit equivalence for actions of discrete amenable groups*, Cambridge University Press, 2002.
- [72] D. Karagulyan, *On Möbius orthogonality for interval maps of zero entropy and orientation-preserving circle homeomorphisms*, Arkiv för matematik, 2015, 53(2): 317-327.
- [73] I. Kàtai, *A remark on a theorem of H. Daboussi*, Acta Math. Hungar. 47 (1986), no. 1-2, 223-225.
- [74] R. Kenyon, *Self-Similar Tilings*. Ph.D. thesis, Princeton University, 1990.

- [75] R. Kenyon, Self-replicating tilings, Symbolic Dynamics and its applications, (ed. P.Walters) Amer. Math. Soc., Providence, RI, 1992, 239-264.
- [76] K. H. Kim, F. W. Roush, *On the automorphism groups of subshifts*, Pure Math. Appl. Ser. B 1 (1990), no. 4, 203-230.
- [77] J. King, *Ergodic properties where order 4 implies infinite order*, Israel J. of Math. 80, (1992), 65-86.
- [78] J. F. Koksma, *Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins*, Compositio Math. 2 (1935), 250-258.
- [79] M. N. Kolountzakis, *Non-symmetric convex domains have no basis of exponentials*, Illinois Journal of Mathematics, 44.3 (2000), 542-550.
- [80] M. N. Kolountzakis, *The study of translational tiling with Fourier analysis*, Fourier analysis and convexity, 2004: 131-187.
- [81] M.N. Kolountzakis, *Packing, Tiling, Orthogonality and Completeness*, Bull. L.M.S. 32 (2000), 5, 589-599.
- [82] M. N. Kolountzakis, J. C. Lagarias, *Structure of tilings of the line by a function*, Duke Math. J, 1996, 82(3): 653-678.
- [83] M. N. Kolountzakis, M. Matolcsi, *Tiles with no spectra*, Forum Mathematicum, 18 (2006), 519-528.
- [84] M. N. Kolountzakis, M. Matolcsi, *Complex Hadamard matrices and the spectral set conjecture*, Collectanea Mathematica, 57(2006), 281-291.
- [85] M. N. Kolountzakis, M. Papadimitrakis, *A class of non-convex polytopes that admit no orthonormal basis of exponentials*, Illinois Journal of Mathematics, 2002, 46(4): 1227-1232.
- [86] A. N. Kolmogorov, *A new metric invariant of transient dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Akad. Sci. SSSR., 119 (1958), 861-864 (Russian)
- [87] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Courier Corporation, 2012.
- [88] I. Łaba, *Fuglede's conjecture for a union of two intervals*, Proc. Amer. Math. Soc. 129 (2001), no. 10, 2965-2972
- [89] I. Łaba, *The spectral set conjecture and multiplicative properties of roots of polynomials*, J. London Math. Soc. (2) 65 (2002), no. 3, 661-671.

- [90] I. Łaba and Y. Wang, *On spectral Cantor measures*, J. Funct. Anal., 193 (2002), 409-420.
- [91] J. C. Lagarias and Y. Wang, *Integral self-affine tiles in R^n I. Standard and non-standard digits sets*, J. London Math. Soc., 54 (1996), 161-179.
- [92] J. C. Lagarias and Y. Wang, *Tiling the line with translates of one tile*, Invent. Math. 124(1996), 341-365.
- [93] J. Lagarias and Y. Wang, *Self-Affine tiles in R^n* , Adv. in Math., 121 (1996), 21-49.
- [94] J. Lagarias and Y. Wang, *Integral self-affine tiles, Part II. Lattice tilings*, J. Fourier Anal. Appl. 3 (1997), 84-102.
- [95] J. C. Lagarias and Y. Wang, *Spectral sets and factorizations of finite abelian groups*, J. Funct. Anal. 145 (1997), 73-98.
- [96] J. C. Lagarias, J. A. Reed and Y. Wang, *Orthonormal bases of exponentials for the n -cube*, Duke Math. J. 103 (2000), 25-37.
- [97] K. S. Lau and H. Rao, *On one-dimensional self-similar tilings and the pq -tilings*, Trans. Amer. Math. Soc. 355, 1401-1414 (2003)
- [98] M. Lemanczyk, F. Parreau and J.-P. Thouvenot, *Gaussian automorphisms whose ergodic self-joinings are Gaussian*, Fund. Math. 164, (2000), 253-293.
- [99] J. Liu and P. Sarnak, *The Möbius function and distal flows*. *Duke Mathematical Journal*, 2015, 164, 1353-1399(7).
- [100] M. Matolcsi, *Fuglede's conjecture fails in dimension 4*, Proceedings of the American Mathematical Society, 133(2005), 3021-3026.
- [101] R-D. Malikiosis, M. N. Kolountzakis, *Fuglede's conjecture on cyclic groups of order $p^d q$* , arXiv:1612.01328, 2016.
- [102] C. Mauduit and J. Rivat, *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Ann. of Math. (2) 171 (2010), no. 3, 1591-1646.
- [103] W. Parry, *Entropy and generators in ergodic theory*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.
- [104] J. Popken, H. Turkstra, *A P -adic analogue of a theorem of Lebesgue in the theory of measure*, Nederl. Akad. Wetensch., Proc. 49, (1946) 802-814 = Indagationes Math. 8, 505-517 (1946).
- [105] M. Ratner, *Horocycle flows, joinings and rigidity of products*, Anna. of Math. 118, (1983), 277-313.

- [106] L. Rédei, *Ein Beitrag zum Problem der Faktorisierung von endlichen Abelschen Gruppen*, Acta Math. Acad. Sci. Hungar., 1:197-207, 1950.
- [107] L. Rédei, *Über das Kreisteilungspolynom*, Acta Math. Acad. Sci. Hungar., 5:27-28, 1954.
- [108] W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962, MR 27 (1969): 2808.
- [109] D. J. Rudolph, *An example of a measure-preserving transformation with minimal self-joinings and applications*, J. d'Analyse Math. 35, (1979),97-122.
- [110] S. Ruelle, *Chaos for continuous interval maps: a survey of relationship between the various sorts of chaos*, preprint, <http://www.math.u-psud.fr/~ruette/publications>, 2003.
- [111] V.V. Ryzhikov, *Joinings, intertwining operators, factors and mixing properties of dynamical systems*, Russian Acad. Izv. Math. 42, (1994), 91-114.
- [112] P. Sarnak, *Three lectures on the Möbius function, randomness and dynamics*, <https://publications.ias.edu/sites/default/files/MobiusFunctionsLectures.pdf>.
- [113] P. Sarnak, *Möbius randomness and dynamics*, Not. S. Afr. Math. Soc. 43 (2012), no. 2, 89-97.
- [114] I. J. Schoenberg, *A note on the cyclotomic polynomial*, Mathematika, 1964, 11(02): 131-136.
- [115] R. X. Shi, *Equivalent definitions of oscillating sequences of higher orders*, arXiv:1705.07827, 2017.
- [116] R. X. Shi, *Construction of some Chowla sequences*, arXiv:1804.03851, 2018
- [117] M. W. Sierpiński, *Sur la valeur asymptotique d'une certaine somme*, Krakau Anz. A (1910), 9-11.
- [118] Y. G. Sinai, *On the notion of entropy of a dynamical system*, Doklady of Russian Academy of Sciences, 124, 768-771(1959).
- [119] B. Steinberg, *Representation theory of finite groups: an introductory approach*, Springer Science & Business Media, 2011.
- [120] M. H. Taibleson, *Fourier analysis on local fields*. Princeton university press, 1975.
- [121] T. Tao, *Fuglede's conjecture is false in 5 and higher dimensions*, Math. Research Letters, 11 (2004), 251-258.

- [122] R. Tijdeman, *Decomposition of the integers as a direct sum of two subsets*, David, S. (ed) Number Theory Seminar Paris 1993 Cambridge: Cambridge University Press 1995, 261-276.
- [123] E. C. Titchmarsh, *The theory of the Riemann zeta-function, second ed.*, The Clarendon Press Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown.
- [124] J. -P. Thouvenot, *Une classe de systèmes pour lesquels la conjecture de Pinsky est vraie*, Israel Journal of Mathematics, 1975, 21(2): 208-214.
- [125] J.-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, Proceedings of the 1993 Alexandria conference, Ergodic theory and its connections with harmonic analysis, Editors: K. E. Petersen and I. A. Salama, LMS Lecture note Series 205, Cambridge University Press, 1995, 207-238.
- [126] W. Thurston, *Groups, tilings and finite state automata*, in: AMS Colloquium Lecture Notes, 1989.
- [127] W. A. Veech, *A criterion for a process to be prime*, Monatsh. Math. 94, (1982), 335-341.
- [128] I. M. Vinogradov, *Some theorems concerning the theory of primes*, Rec. Math. Moscou 2 (1937), no. 44, 179-195.
- [129] V. S. Vladimirov, I. V. Volovich and E. I. Zelenov, *P-adic analysis and mathematical physics*, World Scientific Publishing.
- [130] J. Von Neumann, *Proof of the quasi-ergodic hypothesis*, Proc. Nat. Acad. Sci. U. S. A., 18 (1932), 70-82.
- [131] B. Weiss, *Single orbit dynamics*, CBMS Regional Conference Series in Mathematics, vol. 95, American Mathematical Society, Providence, RI, 2000.
- [132] H. Weyl, *Über die Gibbsche Erscheinung und verwandte Konvergenzphänomene*, Rend. Circ. Mat. Palermo 30 (1910), 377-407.
- [133] H. Weyl, *Über ein Problem aus dem Gebiete der diophantischen Approximationen*, Gött. Nachr. (1914), 234-244.
- [134] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann. 77 (1916), 313-352.