About the elephant random walk

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SOUS LA DIRECTION DE **BERNARD BERCU**

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Autour de la marche aléatoire de l'éléphant

Résumé : Cette thèse porte sur l'étude de la marche aléatoire de l'éléphant et des processus qui en découlent. Cette marche aléatoire s'appelle ainsi car elle possède un paramètre de mémoire et il est bien connu que les éléphants ont une excellente mémoire et se souviennent de tous les endroits qu'ils ont visités. On va établir des résultats probabilistes de types lois des grands nombres et normalité asymptotique, mais aussi des lois du logarithme itéré et des lois fortes quadratiques à l'aide de martingales. On commence par généraliser la marche de l'éléphant pour toute dimension en utilisant des processus de comptage des pas dans chaque direction de la dimension. On s'intéresse ensuite au comportement asymptotique du centre de masse de la marche aléatoire de l'éléphant. On est amené à introduire deux martingales de sorte que leur étude simultanée permet d'obtenir des résultats analogues à ceux de la marche de l'éléphant. On réutilise ensuite cette approche afin d'étudier la marche de l'éléphant avec mémoire renforcée linéairement et la marche aléatoire de l'éléphant avec une amnésie progressive. On propose aussi une étude statistique explicite de l'estimation de la mémoire. Enfin, on présente une approche martingale pour l'étude des urnes de Pólya à deux couleurs.

Mots clés : marche aléatoire, martingale, estimation, urnes de Pólya

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About the elephant random walk

Abstract : This thesis focuses on the study of the elephant random walk and the processes related to it, using martingales. It is a stochastic process with a memory parameter introduced at the beginning of the 2000s and which induces three regimes of behavior. We aim to obtain probabilistic results such as laws of large numbers and asymptotic normality, as well as laws of iterated logarithm and quadratic strong laws. We start by generalizing the elephant random walk to dimensions greater than 2 using counting processes of the steps in each direction of the dimension. Then, we are interested in the center of mass of the random walk of the elephant random walk. We introduce two martingales such that studying them simultaneously makes it possible to obtain results analogous to those of the elephant random walk. This approach is used again to study the random walk with linearly reinforced memory or smooth amnesia. We also propose an explicit statistical analysis to estimate the memory. Finally, we present a martingale approach for the study of two-color Pólya urns.

Keywords : random walk, martingale, estimation, Pólya urns

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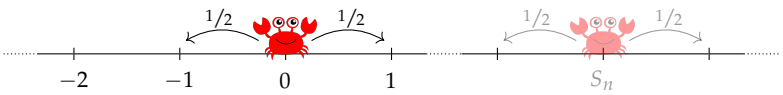
Aperçu de la thèse

Introduction

Marche aléatoire

Une **marche aléatoire** est un processus stochastique consistant en une suite de pas faits au hasard, indépendamment ou non des pas précédents. Le modèle le plus simple est celui de la **marche symétrique**. Considérons un crabe[1] qui ne peut se déplacer que de gauche à droite sur une plage de taille infinie. Le crabe démarre son voyage depuis un certain point qu'on appelle origine, puis il se déplace d'un pas vers la droite avec probabilité $1/2$ ou d'un pas vers la gauche avec probabilité $1/2$. Il répète ensuite cette action à chaque instant et de manière totalement indépendante de ce qu'il a fait avant. On représente la plage par la droite des entiers relatifs $\mathbb{Z}$.



Pour tout $n \geq 0$, la position S_{n+1} du crabe à l'instant $n+1$ est donnée par la relation de récurrence

$$S_{n+1} = S_n + X_{n+1}$$

où X_{n+1} est la variable aléatoire qui représente le $n+1$-ème pas. La famille (X_n) des pas est une suite de variables aléatoires **indépendantes et identiquement distribuées (i.i.d.)** de loi de Rademacher de paramètre $1/2$ notée $\mathcal{R}(1/2)$, c'est-à-dire que

$$X_{n+1} = \begin{cases} +1 & \text{avec probabilité } 1/2, \\ -1 & \text{avec probabilité } 1/2. \end{cases}$$

La **loi des grands nombres (LGN)** et le **théorème central limite (TCL)** pour les suites de variables i.i.d. nous indiquent immédiatement que

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{p.s.}} \mathbb{E}[X_1] = 0 \quad \text{et} \quad \frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

[1] Modélisation fortement inspirée de : *La marche du crabe (trilogie)*, Arthur de Pins (2010).

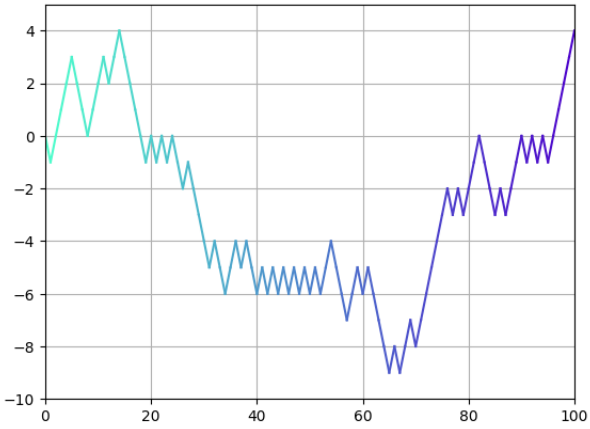
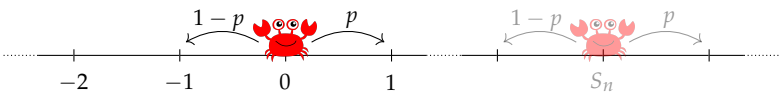


Figure 1: Marche aléatoire symétrique ($p = 0.5$).

Une première généralisation consiste à considérer des pas de loi $\mathcal{R}(p)$, où le paramètre $0 < p < 1$, de sorte que

$$X_{n+1} = \begin{cases} +1 & \text{avec probabilité} \quad p, \\ -1 & \text{avec probabilité} \quad 1-p. \end{cases}$$



Dans ce cas, la loi des grands nombres et le théorème central limite assurent que

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{p.s.}} \mathbb{E}[X_1] = 2p - 1 \quad \text{et} \quad \frac{S_n - (2p-1)n}{\sqrt{4p(1-p)n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

En particulier, la loi de la position S_{n+1} à l'instant $n+1$ n'est influencée que par la position S_n à l'instant n et le pas X_{n+1}. Plus précisement, pour $x, y \in \mathbb{Z}$, on a

$$\mathbb{P}(S_{n+1} = x \mid S_n = y) = \begin{cases} p & \text{si} \quad x = y+1, \\ 1-p & \text{si} \quad x = y-1, \\ 0 & \text{si} \quad x \neq y \pm 1. \end{cases}$$

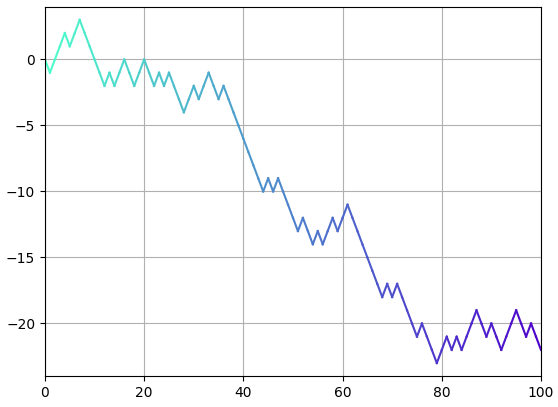


Figure 2: Marche aléatoire pour $p = 0.4$.

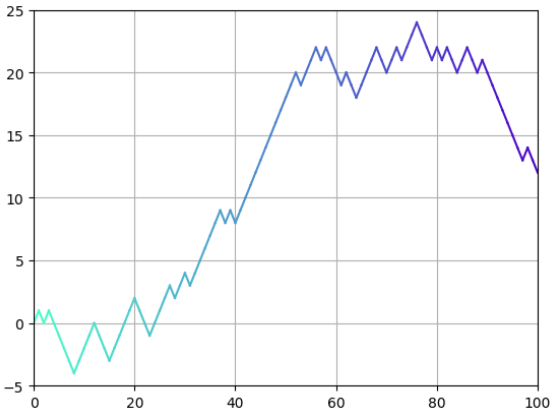
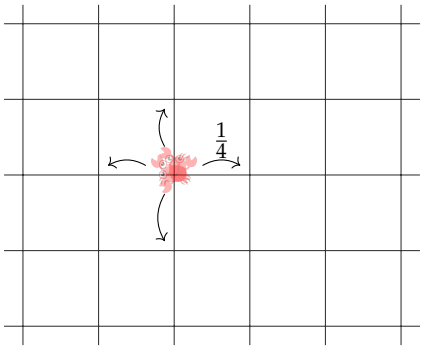


Figure 3: Marche aléatoire pour $p = 0.6$.

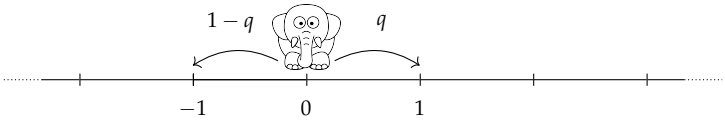
Bien qu'il ne se déplace que sur une ligne, la plage où évolue le crabe est un plan quadrillé. Imaginons à présent que le crabe rencontre un autre crabe qui peut se déplacer de haut en bas uniquement. Les crabes réalisent qu'à eux deux, ils pourront parcourir toute la plage ! On obtient alors la marche symétrique sur le réseau du plan $\mathbb{Z}^2$, c'est-à-dire en dimension 2.



On peut généraliser de la même manière pour toute dimension entière d avec $d \geq 1$.

La marche aléatoire de l'éléphant

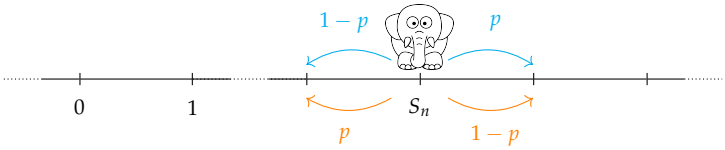
La marche aléatoire de l'éléphant (ERW) s'appelle ainsi car il est bien connu que les éléphants ont une très bonne mémoire et se souviennent de tous les endroits par où ils sont passés. Elle est définie de la manière suivante. À l'instant $n = 0$, l'éléphant se trouve à l'origine $S_0 = 0$. À l'instant $n = 1$, l'éléphant fait un pas vers la droite avec probabilité q ou un pas vers la gauche avec probabilité $1 - q$ où q est un nombre réel entre 0 et 1. La position de l'éléphant à l'instant $n = 1$ est donnée par la variable aléatoire $S_1 = X_1$ où X_1 suit la loi de Rademacher $\mathcal{R}(q)$.



Ensuite, pour chaque instant $n \geq 1$, l'éléphant choisit uniformément au hasard un instant k parmi les instants précédents $1, \ldots, n$ et on définit

$$X_{n+1} = \begin{cases} +X_k & \text{avec probabilité} \quad p, \\ -X_k & \text{avec probabilité} \quad 1-p, \end{cases}$$

où le paramètre $p \in [0, 1]$ est la mémoire de l'éléphant.



Ainsi, selon si l'éléphant se souvient d'un pas vers la droite ou vers la gauche, les probabilités sont échangées. La position de l'éléphant est donnée par la relation de récurrence

$$S_{n+1} = S_n + X_{n+1}.$$

Le cas particulier $p = 1/2$ correspond exactement à la marche aléatoire symétrique.

Contrairement à la marche aléatoire classique (symétrique ou non), le comportement de la marche aléatoire de l'éléphant change selon que sa mémoire $p < 3/4$ (régime diffusif), $p = 3/4$ (régime critique) ou $p > 3/4$ (régime superdiffusif). Les différents résultats de type loi des grands nombres et théorème central limite sont présentés ci-dessous, où L est une variable aléatoire (non dégénérée).

	Diffusif	**Critique**	**Superdiffusif**
LGN	$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{p.s.}} 0$	$\frac{S_n}{\sqrt{n}\log n} \xrightarrow[n\to\infty]{\text{p.s.}} 0$	$\frac{S_n}{n^{2p-1}} \xrightarrow[n\to\infty]{\text{p.s. / } \mathbb{L}^4} L$
TCL	$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{3-4p}\right)$	$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$	$\frac{S_n - n^{2p-1}L}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{4p-3}\right)$

Pour étudier l'ERW, les deux approches principales sont la **théorie des martingales** et les **urnes de Pólya**. L'utilisation des martingales permet d'obtenir la LGN et le TCL, mais aussi la loi du logarithme itéré (LLI) et la loi forte quadratique (LFQ). La connexion avec les urnes de Pólya permet également d'obtenir la LGN et le TCL, mais aussi le TCL fonctionnel (convergence vers un processus gaussien).

Contributions

Les contributions suivantes sont présentées dans l'ordre chronologique d'étude.

[9] BERCU, B., AND LAULIN, L. On the multi-dimensional elephant random walk. *Journal of Statistical Physics 175*, 6 (2019), 1146–1163.

[11] BERCU, B., AND LAULIN, L. On the center of mass of the elephant random walk. *Stochastic Processes and their Applications 133* (2021), 111 – 128.

[56] LAULIN, L. A martingale approach for Pólya urn processes. *Electronic Communications in Probabilities 25* (2020), 13 pp.

[57] LAULIN, L. New insights on the reinforced elephant random walk using a martingale approach. *Journal of Statistical Physics 186* (2022).

[10] BERCU, B., AND LAULIN, L. How to estimate the memory of the elephant random walk. *(Submitted) arXiv:2112.10405* (2021).

[58] LAULIN, L. Introducing smooth amnesia to the memory of the elephant random walk. *arXiv:2204.10542* (2022).

Chapitre 1 – La marche aléatoire de l'éléphant en dimension supérieure

Le but de ce chapitre est d'étudier le comportement asymptotique de la marche de l'éléphant multi-dimensionnelle (MERW). On y étend les résultats connus pour l'ERW en dimension 1 à la dimension $d \geq 2$. Pour cela, on introduit une martingale multi-dimensionnelle et on utilise la théorie des martingales.
Dans les régimes diffusif et critique, on montre la loi forte des grands nombres (LGN), la loi du logarithme itéré (LLI) et la loi forte quadratique (LFQ) pour la MERW. La normalité asymptotique (TCL) avec une bonne renormalisation est aussi obtenue. Dans le régime superdiffusif, on montre la convergence presque sûre et la convergence en moyenne quadratique vers un vecteur aléatoire de $\mathbb{R}^d$ non dégénéré.

Chapitre 2 – Le centre de masse de la marche aléatoire de l'éléphant

Le but de ce chapitre est d'étudier le comportement asymptotique du centre de masse (ou barycentre) de la marche de l'éléphant (CMERW) en dimension d. On obtient le même type de résultats connus pour la MERW. Toute l'étude repose sur l'utilisation et l'étude simultanée de deux martingales multi-dimensionnelles avec normalisation matricielle.
Dans les régimes diffusif et critique, on montre la loi forte des grands nombres (LGN), la loi du logarithme itéré (LLI) et la loi forte quadratique (LFQ) pour le CMERW. La normalité asymptotique (TCL) avec une bonne renormalisation est aussi obtenue. Dans le régime superdiffusif, on montre la convergence presque sûre et la convergence en moyenne quadratique vers un vecteur aléatoire de $\mathbb{R}^d$ non dégénéré, liée au vecteur aléatoire limite de la MERW.

Chapitre 3 – La marche aléatoire de l'éléphant renforcée

Le but de ce chapitre est d'étudier le comportement asymptotique de la marche aléatoire de l'éléphant linéairement renforcée (RERW). Le renforcement agit sur la mémoire de l'éléphant, de façon qu'un instant dont l'éléphant s'est déjà souvenu a plus de chance d'être à nouveau choisi, proportionellement au nombre de fois que l'instant a été choisi.
On obtient le même type de résultats connus pour l'ERW. Toute la stratégie repose sur l'utilisation et l'étude simultanée de deux martingales avec normalisation matricielle.
Dans les régimes diffusif et critique, on montre la loi forte des grands nombres (LGN), la loi du logarithme itéré (LLI) et la loi forte quadratique (LFQ) pour la RERW. La convergence vers un processus gaussien est aussi obtenue (TCLF).

Dans le régime superdiffusif, on montre la convergence presque sûre et la convergence en moyenne quadratique vers une variable aléatoire non dégénérée.

Chapitre 4 – La marche aléatoire de l'éléphant amnésique

Ce chapitre est une adaptation du chapitre précédent. Le renforcement de type "amnésique" agit encore une fois sur la mémoire. Ainsi, la probabilité de choisir un instant récent est beaucoup plus forte que celle de choisir un instant loin dans le passé.
On obtient à nouveau des résultats de type LGN, LLI, LFQ et TCLF, analogues à ceux du Chapitre 3.

Chapitre 5 – Estimation statistique du paramètre de mémoire

Dans ce chapitre, on propose une solution pour l'estimation de la mémoire de l'éléphant. L'estimateur est basé sur l'approximation de Taylor d'ordre 2 de la log-vraissemblance. L'étude repose à nouveau sur la théorie des martingales et les variations quadratiques qui y sont associées.
On prouve que l'estimateur est fortement consistant dans les trois régimes. On montre de plus des propriétés d'efficacité asymptotique et de normalité asymtptotique locale. Enfin on propose des intervalles de confiances exacts obtenus via des inégalités de concentration pour les martingales, des intervalles de confiance asymptotiques ainsi que des tests statistiques.

Chapitre 6 – Une approche martingale pour les urnes de Pólya

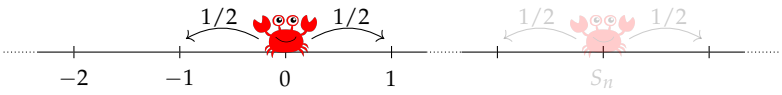
Ce chapitre présente une approche martingale pour l'étude des processus d'urnes de Pólya généralisées. Le comportement de l'urne change selon si le ratio des valeurs propres de la matrice moyenne de remplacement est plus petit que, égal à, ou plus grand que $1/2$.
On retrouve les résultats connus de LGN peu importe le type d'urnes et de TCL dans le cas des petites urnes. On montre de nouveaux résultats comme la LLI et la LFQ pour les petites urnes.

Chapitre 7 – Conclusion et perspectives

Introduction

I.1 – Random walks

A **random walk** is a mathematical process which consists of a sequence of steps performed at random, independently or not of the previous steps. The simplest model is the one of the **symmetric random walk** on the integers $\mathbb{Z}$ with steps $+1$ or -1. Consider a crab[2] that can only move to the left or to the right over a beach of infinite size. The crab starts its journey from a certain point called the origin, then it moves one step to the right with probability $1/2$ or one step to the left with probability $1/2$. Then, it repeats this action at each instant, and completely independently of what it has done before. The beach is represented by the line of the integers $\mathbb{Z}$.



For any $n \geq 0$, the position S_{n+1} of the crab at instant $n+1$ is given by the relation

$$S_{n+1} = S_n + X_{n+1}$$

where X_{n+1} is the random variable that represents the $(n+1)$-th step. The sequence (X_n) of the steps is a sequence of random variables **independent and identically distributed (i.i.d.)** with Rademacher distribution $\mathcal{R}(1/2)$, which means that

$$X_{n+1} = \begin{cases} +1 & \text{with probability} \quad 1/2, \\ -1 & \text{with probability} \quad 1/2. \end{cases}$$

The **(strong) law of large numbers (LLN)** and the **central limit theorem (CLT)** for sequences of i.i.d. random variables ensure that

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{E}[X_1] = 0 \quad \text{and} \quad \frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

[2] This model is strongly inspired by : *La marche du crabe (trilogie)*, Arthur de Pins (2010).

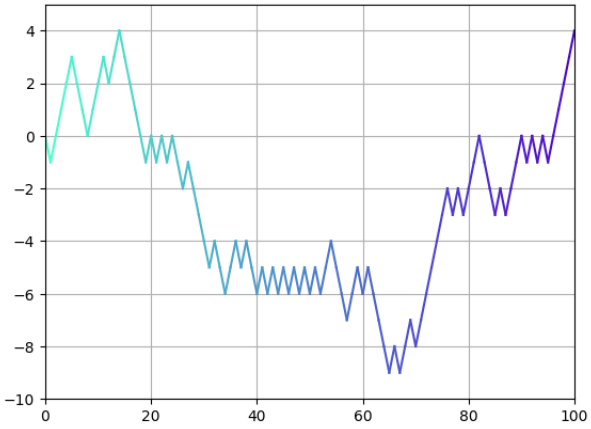


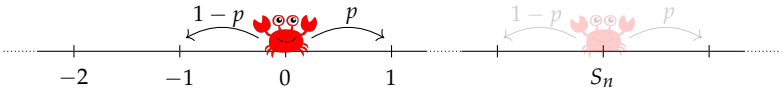
Figure I.4: A path of symmetric random walk.

More precisely, the **law of iterated logarithm** explains how big the fluctuations of $S_n/\sqrt{n}$ can be,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = -1 \quad \text{a.s.}$$

A first generalization consists in considering steps of distribution $\mathcal{R}(p)$, where $0 < p < 1$, in the way that

$$X_{n+1} = \begin{cases} +1 & \text{with probability} \quad p, \\ -1 & \text{with probability} \quad 1-p. \end{cases}$$



In that case, the law of large numbers and the central limit theorem ensure that

$$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} \mathbb{E}[X_1] = 2p-1 \quad \text{and} \quad \frac{S_n - (2p-1)n}{\sqrt{4p(1-p)n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

In particular, the law of the position S_{n+1} at the instant $n+1$ is only influenced by the position S_n at the instant n and the step X_{n+1}. More precisely, for $x, y \in \mathbb{Z}$,

$$\mathbb{P}(S_{n+1} = x \mid S_n = y) = \begin{cases} p & \text{if} \quad x = y+1, \\ 1-p & \text{if} \quad x = y-1, \\ 0 & \text{if} \quad x \neq y \pm 1. \end{cases}$$

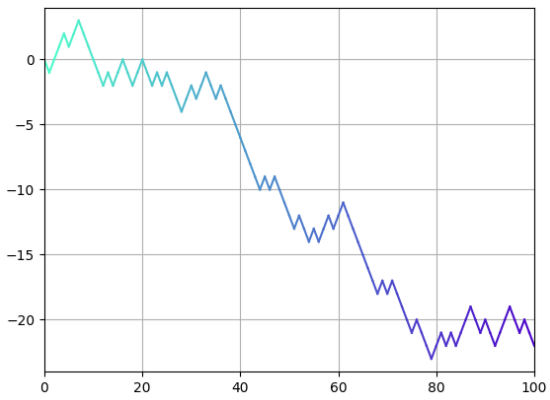


Figure I.5: A path of random walk when $p = 0.4$.

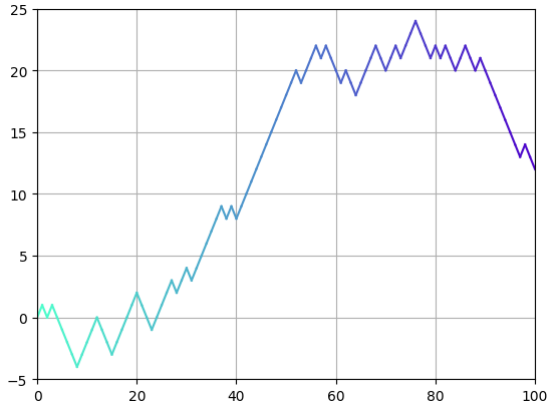
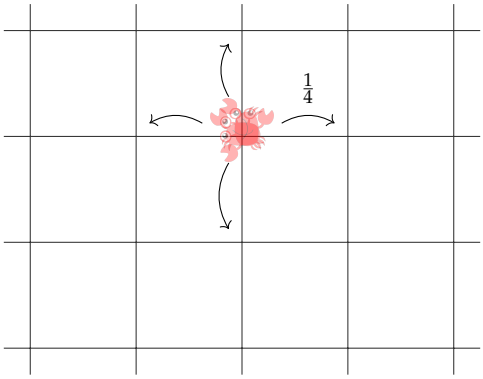


Figure I.6: A path of random walk when $p = 0.6$.

Although it only moves on the integers, the beach where the crab moves is a grid plan. Now, imagine that the crab encounters another crab which can move up and down only. The crabs realize that the two of them will be able to cover the entire beach! We then obtain the simple random walk on the lattice $\mathbb{Z}^2$, i.e. in dimension 2.



We can generalize in the same way for any integer dimension d, with $d \geq 1$. In that case, if $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ is the standard basis of $\mathbb{Z}^d$ (or $\mathbb{R}^d$), then the symmetric random walk on $\mathbb{Z}^d$ is defined by

$$\mathbf{S}_0 = 0_d, \quad \mathbf{S}_{n+1} = \mathbf{S}_n + \mathbf{X}_{n+1}$$

and, for any $1 \leq i \leq d$,

$$\mathbb{P}(\mathbf{X}_{n+1} = +\mathbf{e}_i) = \mathbb{P}(\mathbf{X}_{n+1} = -\mathbf{e}_i) = \frac{1}{2d}.$$

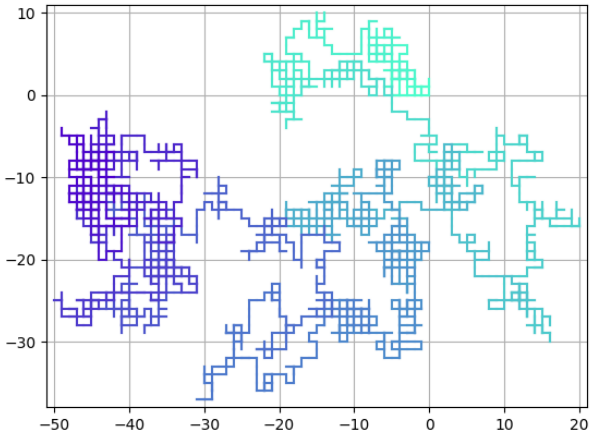


Figure I.7: A path of symmetric random walk in dimension 2.

Finally, there exists a "continuous time version" of the process. More precisely, Donsker's Theorem ensures that, in the symmetric case where $p = 0.5$,

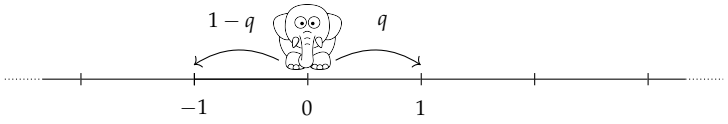
$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\right) \Longrightarrow \left(B_t,\ t \geq 0\right)$$

The notation "$\Longrightarrow$" stands for convergence with respect to the Skorokhod space. See Theorem 4.20 in [50] for one example of this version of the Theorem.
Many other questions regarding random walks deserve our attention, for example, the number of times a random walk returns to the origin, or the time it took to do so, depending on the dimension d. Those types of questions will not be explicited here.

I.2 – The Elephant Random Walk

Random walks with long-memory arose naturally in applied mathematics, theoretical physics, computer sciences and econometrics. One of them is the so-called **elephant random walk (ERW)**. It is a one-dimensional discrete-time random walk on integers, which has a complete memory of its whole history. It was introduced in 2004 by Schütz and Trimper [69] in order to investigate the long-term memory effects in non-Markovian random walks. It was referred to as the ERW in allusion to the famous saying that elephants can remember where they have been. It appears to be a time-inhomogeneous Markov chain.
A wide range of literature is now available on the ERW in dimension $d = 1$ and its extensions [3, 20, 25, 28, 27, 29, 54, 55]. One of the natural questions regarding the ERW concerns the influence of the memory parameter p on the asymptotic behavior of the ERW. Depending on the value of p with respect to $3/4$, the behavior of the ERW is quite different and we observe three regimes. More precisely, a strong law of large numbers and a central limit theorem for the position S_n, properly normalized, were established in the diffusive regime $p < 3/4$ and the critical regime $p = 3/4$, see [3, 25, 26, 69] and the more recent contributions [7, 24, 32, 36, 64, 75].

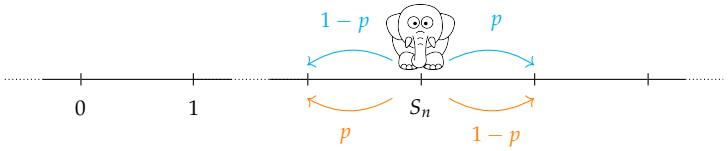
The main change between the two regimes is the normalization needed to prove the convergences.
The superdiffusive regime $p > 3/4$ turns out to be harder to deal with. Both Coletti et al. [25] and Bercu [5] proved that the limit of the position of the ERW is not Gaussian. After that, Kubota and Takei [53] showed that the fluctuation of the ERW around its limit in the superdiffusive regime is Gaussian. Finally, Bercu and Laulin in [9] extended all the results of [5] to the multi-dimensional ERW (MERW) where $d \geq 1$ and to its center of mass [11]. Moreover, functional central limit theorems were also provided via a connection to Pólya-type urns, see Baur and Bertoin [3] for the ERW, Baur [2] for a particular class of random walks with reinforced memory such as the ERW and the Shark Random Swim introduced by Businger [21], and more recently Bertenghi [13] for the MERW.
The one-dimensional ERW is defined as follows. The random walk starts at the origin at time zero, $S_0 = 0$. At time $n = 1$, the elephant moves to the right with probability q and to the left with probability $1 - q$ where q lies between zero and one. Hence, the position of the elephant at time $n = 1$ is given by $S_1 = X_1$ where X_1 has a Rademacher $\mathcal{R}(q)$ distribution.



Afterwards, at any time $n \geq 1$, the elephant chooses uniformly at random an integer k among the previous times $1, \ldots, n$, and we define

$$X_{n+1} = \begin{cases} +X_k & \text{with probability} \quad p, \\ -X_k & \text{with probability} \quad 1 - p, \end{cases}$$

where the parameter $p \in [0, 1]$ is the memory of the ERW.



Then, the position of the ERW is given by

$$S_{n+1} = S_n + X_{n+1}. \tag{I.1}$$

There are multiple ways to study the asymptotical behavior of the ERW. Baur and Bertoin [3] extensively used the connection to Pólya-type urns [44] as well as two functional limit theorems for multitype branching processes due to Janson [48], see also [23]. Bercu [5] and Coletti et al. [25] used martingales to obtain the almost sure convergences and asymptotic normality, among other results. Kürsten [55] and Businger [21] used the construction of random trees with Bernoulli percolation, which ensures that one remembers all of the past information. The first two methods are presented in the following subsections.

I.2.1 – The martingale approach

The first use of martingales was done by Coletti et al. [25] in order to obtain the law of large numbers and the central limit theorem. Afterwards, Bercu [5] used a more general martingale to obtain the law of iterated logarithm and the quadratic strong law in the diffusive and critical regimes, as well as the convergence in $\mathbb{L}^4$ in the superdiffusive regime, and also retrieved the previous results.
In order to understand well how the elephant moves, it is straightforward to see that for any time $n \geq 1$,

$$X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}} \tag{I.2}$$

where α_{n+1} and β_{n+1} are two independent discrete random variables such that α_{n+1} has a Rademacher $\mathcal{R}(p)$ distribution while β_{n+1} is uniformly distributed over the integers $\{1, \cdots, n\}$. Moreover, α_{n+1} is independent of $X_1, \ldots, X_n$.
Let $(\mathcal{F}_n)$ be the increasing sequence of σ-algebras, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. For any time $n \geq 1$, we clearly have

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = \mathbb{E}[\alpha_{n+1}] \times \mathbb{E}[X_{\beta_{n+1}}|\mathcal{F}_n] = (2p-1)\frac{S_n}{n} \quad \text{a.s.} \tag{I.3}$$

which implies that

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \gamma_n S_n \quad \text{where} \quad \gamma_n = \Big(\frac{n+2p-1}{n}\Big). \tag{I.4}$$

Moreover,

$$\prod_{k=1}^{n} \gamma_k = \frac{\Gamma(n+2p)}{\Gamma(n+1)\Gamma(2p)}$$

where Γ stands for the Euler gamma function. Therefore, let (M_n) be the sequence of random variables defined, for all $n \geq 0$, by $M_n = a_n S_n$ where $a_1 = 1$ and, for all $n \geq 2$,

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n)\Gamma(2p)}{\Gamma(n+2p-1)}. \tag{I.5}$$

Since $a_n = \gamma_n a_{n+1}$, we clearly deduce from (I.4) that for any time $n \geq 1$,

$$\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n \quad \text{a.s.}$$

In other words, the sequence (M_n) is a martingale. Obviously, for any $n \geq 1$, X_n is a binary random variable taking values in $\{+1, -1\}$. Consequently, $|S_n| \leq n$, which implies that (M_n) is locally square integrable. The martingale (M_n) can be rewritten in the additive form

$$M_n = \sum_{k=1}^{n} a_k \varepsilon_k \tag{I.6}$$

where

$$\varepsilon_n = S_n - \gamma_{n-1} S_{n-1} \tag{I.7}$$

since its increments $\Delta M_n = M_n - M_{n-1}$ satisfy $\Delta M_n = a_n S_n - a_{n-1} S_{n-1} = a_n \varepsilon_n$. The predictable quadratic variation [31] associated with (M_n) is given by $\langle M \rangle_0 = 0$ and, for all $n \geq 1$,

$$\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}[\Delta M_k^2 \mid \mathcal{F}_{k-1}]. \tag{I.8}$$

We immediately obtain from (I.4) that $\mathbb{E}[\varepsilon_{n+1}|\mathcal{F}_n] = 0$. Moreover, it follows from (I.1) together with (I.3) that

$$\mathbb{E}[S_{n+1}^2|\mathcal{F}_n] = \mathbb{E}[S_n^2 + 2S_n X_{n+1} + 1|\mathcal{F}_n] = 1 + (2\gamma_n - 1)S_n^2 \quad \text{a.s.} \tag{I.9}$$

Consequently, as $\mathbb{E}[\varepsilon_{n+1}^2|\mathcal{F}_n] = \mathbb{E}[S_{n+1}^2|\mathcal{F}_n] - \gamma_n^2 S_n^2$, we deduce from (I.9) that, for all $n \geq 1$,

$$\begin{aligned} \mathbb{E}[\varepsilon_{n+1}^2|\mathcal{F}_n] &= 1 + (2\gamma_n - 1)S_n^2 - \gamma_n^2 S_n^2 = 1 - (\gamma_n - 1)^2 S_n^2 \quad \text{a.s.} \\ &= 1 - (2p-1)^2 \Big(\frac{S_n}{n}\Big)^2 \quad \text{a.s.} \end{aligned} \tag{I.10}$$

By the same token,

$$\begin{aligned} \mathbb{E}[\varepsilon_{n+1}^4|\mathcal{F}_n] &= 1 - 3(\gamma_n - 1)^4 S_n^4 + 2(\gamma_n - 1)^2 S_n^2 \quad \text{a.s.} \\ &= 1 - 3(2p-1)^4 \Big(\frac{S_n}{n}\Big)^4 + 2(2p-1)^2 \Big(\frac{S_n}{n}\Big)^2 \quad \text{a.s.} \end{aligned} \tag{I.11}$$

Equation (I.10) is necessary to compute the quadratric variations of (M_n) while (I.11) is useful to obtain bound on the $4-th$ order moment of (ε_n). On the one hand, if $p = 1/2$, $\mathbb{E}[\varepsilon_{n+1}^2|\mathcal{F}_n] = 1$ and $\mathbb{E}[\varepsilon_{n+1}^4|\mathcal{F}_n] = 1$ a.s. On the other hand, we obtain from (I.10) and (I.11) the almost sure upper bounds

$$\sup_{n \geq 0} \mathbb{E}[\varepsilon_{n+1}^2|\mathcal{F}_n] \leq 1 \quad \text{and} \quad \sup_{n \geq 0} \mathbb{E}[\varepsilon_{n+1}^4|\mathcal{F}_n] \leq \frac{4}{3}. \tag{I.12}$$

Hereafter, we deduce from (I.6), (I.8) and (I.10) that

$$\langle M \rangle_n = \sum_{k=1}^{n} a_k^2 - (2p-1)^2 \zeta_n \quad \text{where} \quad \zeta_n = \sum_{k=1}^{n-1} a_{k+1}^2 \Big(\frac{S_k}{k}\Big)^2. \tag{I.13}$$

The asymptotic behavior of the martingale (M_n) is closely related to the one of

$$v_n = \sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} \Big(\frac{\Gamma(k)\Gamma(2p)}{\Gamma(k+2p-1)}\Big)^2.$$

We introduce the parameter $a = 2p - 1$. Via standard results on the asymptotic behavior of the Euler gamma function, we have three regimes. In the **diffusive regime** where $0 \leq p < 3/4$ or $a < 1/2$,

$$\lim_{n \to \infty} \frac{v_n}{n^{1-2p}} = \ell \quad \text{where} \quad \ell = \frac{(\Gamma(2p))^2}{1-2p}. \tag{I.14}$$

In the **critical regime** where $p = 3/4$ or $a = 1/2$,

$$\lim_{n \to \infty} \frac{v_n}{\log n} = \frac{\pi}{4}. \tag{I.15}$$

In the **superdiffusive regime** where $3/4 < p \leq 1$ or $1/2 < a \leq 1$, v_n converges to the finite value

$$\lim_{n\to\infty} v_n = \sum_{k=0}^{\infty} \Big(\frac{\Gamma(k+1)\Gamma(2p)}{\Gamma(k+2p)} \Big)^2 = \sum_{k=0}^{\infty} \frac{(1)_k\,(1)_k\,(1)_k}{(2p)_k\,(2p)_k\,k!} = {}_3F_2\Big(\begin{matrix} 1,1,1 \\ 2p,2p \end{matrix} \Big| 1 \Big) \tag{I.16}$$

where, for any $x \in \mathbb{R}$, $(x)_k = x(x+1)\cdots(x+k-1)$ for $k \geq 1$, $(x)_0 = 1$ stands for the Pochhammer symbol and ${}_3F_2$ is the generalized hypergeometric function defined by

$${}_3F_2\Big(\begin{matrix} a,b,c \\ d,e \end{matrix} \Big| z \Big) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k\,(c)_k}{(d)_k\,(e)_k\,k!} z^k.$$

The strategy here to obtain asymptotical results for the ERW relies on the theory of martingales. To be more precise, they are obtained by making use of the strong law of large numbers and the central limit theorem for martingales [31, 42] as well as the law of iterated logarithm for martingales [71, 72] and the quadratic strong law for martingales [4].

I.2.2 – The Pólya-type urns approach

This approach was first introduced by Baur and Bertoin [3] in order to obtain functional convergences for the elephant random walk. It was later generalized by Bertenghi [13] for the multidimensional ERW. The idea comes from the work of Janson [48]. The method uses a connection to Pólya-type urns that was already known before in the literature (see the survey of Pemantle [66]). A bit more precisely, given what is known from the theory of urns, it implies that the asymptotic behavior of such models is determined by the spectral decomposition of the (mean) replacement matrix of the corresponding urn.

Let (U_n) be discrete-time urn with balls of two colors, red and blue. The composition of the urn at time $n \in \mathbb{N}$ is given by a vector $U_n = (R_n, B_n)$ where R_n stands for the number of red balls and B_n for the number of blue balls at time n. The starting composition of the urn is $(1,0)$ with probability q or $(0,1)$ with probability $1-q$. Then, the urn is implemented as follows. At any time $n \geq 2$ a ball is drawn uniformly at random, its color observed, then it is returned to the urn together with a ball of the same color with probability p, or with a ball of the other color with probability $1-p$. The connection to the ERW model is straightforward. Let (S_n) denotes the ERW started from $S_0 = 0$ and such that $S_1 = R_1 - B_1$, then for every $n \geq 1$

$$S_n \overset{\mathcal{L}}{=} R_n - B_n \tag{I.17}$$

where $\overset{\mathcal{L}}{=}$ refers to equality in law. In other words, the difference between the number of red and blue balls in the urn behaves like an ERW with first step equals to $R_1 - B_1$.

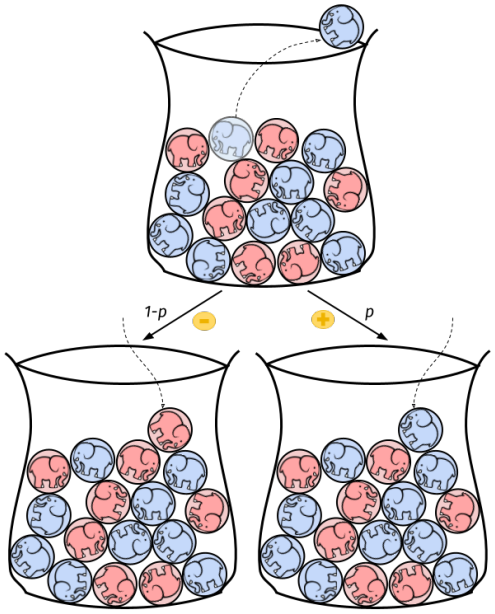


Figure I.8: The corresponding urn adapted to the ERW.

To study this process, we are interested in the spectral decomposition of the mean replacement matrix A, given by

$$A = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}. \tag{I.18}$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2p - 1 = a$ and the corresponding unit vectors in $\mathbb{L}^1$ are

$$v_1^T = \frac{1}{2}(1,\ 1), \quad v_2^T = \frac{1}{2}(1,\ -1).$$

It is well-known, see [23, 33, 34, 48], that the asymptotics of the urn depends on the ratio λ_2/λ_1 with respect to $1/2$. This is coherent and yet another good explanation to why the transition between the regimes for the ERW occurs at $a = 1/2$ which, as expected, is equivalent to $p = 3/4$.

I.2.3 – Main results

I.2.3.1 – The diffusive regime

The following results concerns the asymptotic behavior of the ERW when $0 \leq p < 3/4$, which is equivalent to $-1 < a < 1/2$. The law of large numbers is due to Coletti et al. [25], but can also be obtained using the approach of Baur and Bertoin [3].

Theorem I.1. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s. \tag{I.19}$$

The almost sure rates of convergence of the ERW were obtained by Bercu [5]. Independently, Coletti et. al obtained the law of iterated logarithm [26].

Theorem I.2. *We have the quadratic strong law*

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{S_k}{k}\Big)^2 = \frac{1}{1-2a} \quad a.s. \tag{I.20}$$

In addition, we also have the law of iterated logarithm

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2n\log\log n}\Big)^{1/2} S_n &= -\liminf_{n\to\infty} \Big(\frac{1}{2n\log\log n}\Big)^{1/2} S_n \\ &= \frac{1}{\sqrt{1-2a}} \quad a.s. \end{aligned} \tag{I.21}$$

In particular,

$$\limsup_{n\to\infty} \frac{S_n^2}{2n\log\log n} = \frac{1}{1-2a} \quad a.s. \tag{I.22}$$

The next result is devoted to the functional convergence and the asymptotic normality of the ERW in the diffusive regime $0 \leq p < 3/4$. The distributional convergence holds in the Skorokhod space $D([0,\infty[)$ of right-continuous functions with left-hand limits. See [19, Chapter 3] for more details on the definition of the distributional convergence in the Skorokhod Space $D([0,\infty[)$.
The functional convergence was obtained by Baur and Bertoin [3]. The asymptotic normality can be deduced from the distributional convergence, but it can also be obtained using martingales [25, 5].

Theorem I.3. *We have the following convergence in $D(0,\infty)$*

$$\Big(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big)$$

where $\big(W_t,\ t \geq 0\big)$ is a real-valued mean-zero Gaussian process starting from the origin and

$$\mathbb{E}[W_s W_t] = \frac{1}{1-2a} s \Big(\frac{t}{s}\Big)^a.$$

In particular, we have the asymptotic normality

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{1-2a}\Big). \tag{I.23}$$

Remark I.4. *In the particular case $p = 1/2$, one find again the central limit theorem for the simple random walk*

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

Remark I.5. *In 2017, Coletti et al. [26] proved a strong invariance principle for the ERW in the three regimes.*

Finally, we consider the counting process of zeros,

$$Z_n = \mathrm{card}\big\{1 \leq i \leq n,\ S_j = 0\big\}, \quad n \geq 1.$$

Coletti and Papageorgiou [24] pointed the first result in the next theorem. After that, Bertoin [16] explicited the asymptotic more precisely.

Theorem I.6. *The ERW is recurrent in the diffuse regime, which is equivalent to*

$$\lim_{n\to\infty} Z_n = +\infty \quad a.s.$$

In particular,

$$\frac{Z_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} V \tag{I.24}$$

where V is some random variable.

I.2.3.2 – The critical regime

Hereafter, we investigate the critical regime where $a = 1/2$ and the memory parameter is $p = 3/4$. This law of large numbers can be found in Bercu [5].

Theorem I.7. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} = 0 \quad a.s. \tag{I.25}$$

Once again, the almost sure rates of convergence of the ERW were obtained by Bercu [5]. Independently, Coletti et. al obtained the law of iterated logarithm [26].

Theorem I.8. *We have the quadratic strong law*

$$\limsup_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \Big(\frac{S_k}{k\log k}\Big)^2 = 1 \quad a.s. \tag{I.26}$$

In addition, we also have the law of iterated logarithm

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} S_n &= -\liminf_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} S_n \\ &= 1 \quad a.s. \end{aligned} \tag{I.27}$$

In particular,

$$\limsup_{n\to\infty} \frac{S_n^2}{2n\log n\log\log\log n} = 1 \quad a.s. \tag{I.28}$$

One can observe a very unusual rate of convergence in the law of iterated logarithm. The next result deals with the functional convergence [3] and the asymptotic normality [5, 25]of the ERW in the critical regime $p = 3/4$.

Theorem I.9. *We have the following convergence in $D(0,\infty)$*

$$\Big(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n^t\log n}},\ t \geq 0\Big) \Longrightarrow \big(B_t,\ t \geq 0\big)$$

where $(B_t,\ t \geq 0)$ is a one-dimensional standard Brownian motion. In particular, we have the asymptotic normality

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1). \tag{I.29}$$

I.2.3.3 – The superdiffusive regime

Finally, we are interested in the more complicated superdiffusive regime where $1/2 < a \leq 1$ and $3/4 < p \leq 1$. The law of large numbers was proved in the works of Baur and Bertoin [3] and [5]. The convergence in $\mathbb{L}^2$ is one of the first things known about the ERW [69, 29]. After that, Bercu [5] proved the convergence in $\mathbb{L}^4$.

Theorem I.10. *We have the almost sure convergence*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{n^a},\ t \geq 0\Big) \Longrightarrow (\Lambda_t,\ t \geq 0) \tag{I.30}$$

where the limiting $\Lambda_t = t^a L$ and L is some non-degenerate random variable. In particular, we have

$$\lim_{n\to\infty} \frac{S_n}{n^a} = L \quad a.s. \tag{I.31}$$

Theorem I.11. *The convergence also holds in $\mathbb{L}^4$, which means that*

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{S_n}{n^a} - L\Big|^4\Big] = 0. \tag{I.32}$$

The first three moments of S_n were previously calculated in [29] in the special case $q = 1$. After that, Bercu [5] gave the computation of the first four moments.

Theorem I.12. *The first four moments of L are given by*

$$\mathbb{E}[L] = \frac{2q-1}{\Gamma(2p)}, \tag{I.33}$$

$$\mathbb{E}[L^2] = \frac{1}{(4p-3)\Gamma(2(2p-1))}, \tag{I.34}$$

$$\mathbb{E}[L^3] = \frac{2p(2q-1)}{(2p-1)(4p-3)\Gamma(3(2p-1))}, \tag{I.35}$$

$$\mathbb{E}[L^4] = \frac{6(8p^2-4p-1)}{(8p-5)(4p-3)^2\Gamma(4(2p-1))}. \tag{I.36}$$

It appears that the random variable L is not Gaussian [29, 3, 5].

Remark I.13. *Bercu et al. [7] used the computations of the moments of L to obtain new results on hypergeometric functions.*

Finally, Kubota and Takei [53] showed that, even though L is not Gaussian, the fluctuation of the ERW aroud its limit L are Gaussian. They used martingale theory and Hall and Heyde results [42].

Theorem I.14. *We have the asymptotic normality*

$$\frac{S_n - n^a L}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{2a-1}\Big). \tag{I.37}$$

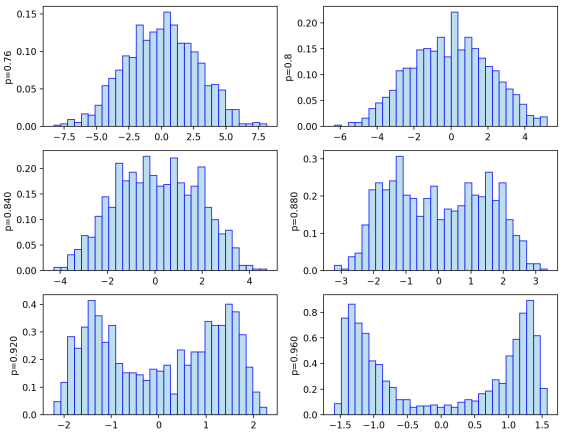


Figure I.9: Histogram of L values when $q = 0.5$ depending on the value of p.

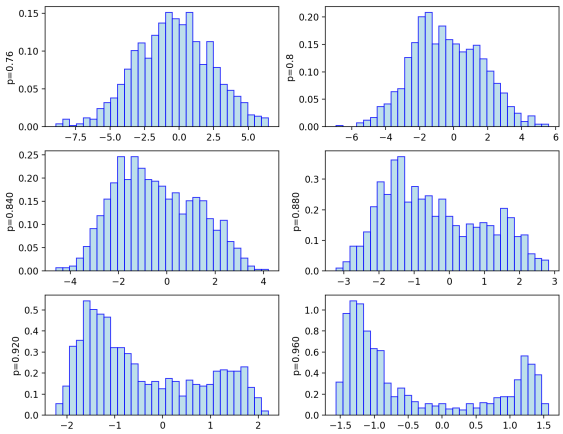


Figure I.10: Histogram of L values when $q = 0.3$ depending on the value of p.

I.3 – ERW with general steps distribution

In the recent work [18], Bertoin introduced the noise reinforced Brownian motion $\big(\hat{B}_t,\ t \geq 0\big)$ as the universal limit of random walks with some steps reinforcement. More precisely, let (X_n) be a sequence of i.i.d. real random variables and (ε_n) a sequence of i.i.d. Bernoulli random variables with parameter $\hat{p}$. Then, set $\hat{X}_1 = X_1$ and, for $n \geq 1$,

$$\hat{X}_{n+1} = \begin{cases} X_{n+1} & \text{if } \varepsilon_{n+1} = 0, \\ \hat{X}_{\mathcal{U}(n)} & \text{if } \varepsilon_{n+1} = 1, \end{cases}$$

where $\mathcal{U}(n)$ stands for the uniform distribution on $\{1, \ldots, n\}$. The sequence

$$\hat{S}_n = \hat{X}_1 + \ldots + \hat{X}_n$$

is referred to as the (positively) step-reinforced random walk (pSRRW). In his work, Bertoin showed that, if $\hat{p} \in (0, 1/2)$, $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = 1$

$$\Big(\frac{\hat{S}_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(\hat{B}_t,\ t \geq 0\big)$$

where $\big(\hat{B}_t,\ t \geq 0\big)$ has the same distribution as

$$\Big(\frac{t^{\hat{p}}}{\sqrt{1-2\hat{p}}} B_{t^{1-2\hat{p}}},\ t \geq 0\Big).$$

In the case where X_1 has the standard Rademacher distribution $\mathcal{R}(1/2)$, Kürsten [55] explained that $\hat{S}$ is the elephant random walk with memory parameter

$$p = \frac{\hat{p}+1}{2}$$

(hence, such that $a = \hat{p}$). It implies that some of the results on the ERW when $a > 0$ can be retrieved by studying the pSRRW, and that the process W in Theorem I.3 is exactly $\hat{B}$. When X_1 has a symmetric stable distribution, $\hat{S}$ is the so-called shark random swim which has been studied by Businger [21].

By the same token, it is possible to define the (negatively or counterbalanced) step reinforced random walk (nSRRW) process S, the main change being that $\check{X}_n = -\check{X}_{\mathcal{U}(n-1)}$ if $\varepsilon_n = 1$. In that case, when X_1 has the standard Rademacher distribution $\mathcal{R}(1/2)$, $\check{S}$ is simply the elephant random walk with memory parameter

$$p = \frac{1-\check{p}}{2}$$

(hence, such that $a = -\check{p}$).
Consequently, the ERW can be understood using both the pSRRW and the nSRRW. We would also like to notify the reader that this approach of the ERW using the pSRRW is related to generalized (or correlated) Bernoulli processes, as studied by Heyde [47]. In this case and with the corresponding notations, the parameters are $\theta = \hat{p}$ (or $\theta = a$) and $p = 1/2$. The behavior of the pSSRW $\hat{S}_n$ is equal is distribition to the one of $S_n - n$, where S_n is the generalized binomial, see Drezner and Farnum [30]. This approach only works for $a > 0$.
The asymptotic results regarding the SRRW have been obtained by using martingale theory and embeding the SRRW in a branching process via the introduction of a Yule process. The following have been established by Bertoin [18, 17, 15] and Bertenghi [12].

Theorem I.15. (Step reinforced random walk) *Let $X_1 \in \mathbb{L}^2$ and $\hat{p} \in]1/2, 1]$. Then, we have the law of large numbers*

$$\lim_{n\to\infty} \frac{\hat{S}_n - n\mathbb{E}[X_1]}{n^{\hat{p}}} \xrightarrow[n\to\infty]{\mathbb{P}} \hat{W}$$

where $\hat{W}$ is some non-degenerate random variable. Moreover, we also have the asymptotic normality

$$\frac{\hat{S}_n - n\mathbb{E}[X_1] - n^{\hat{p}}\hat{W}}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{\mathbb{V}[X_1]}{2\hat{p}-1}\right).$$

Theorem I.16. (Counterbalanced random walk) *Let $X_1 \in \mathbb{L}^1$ and $\hat{p} \in [0, 1]$. Then, we have the law of large numbers*

$$\frac{\check{S}_n}{n} \xrightarrow[n\to\infty]{\mathbb{L}^2} \frac{\hat{p}}{2-\hat{p}}\mathbb{E}[X_1].$$

Moreover, if $X_1 \in \mathbb{L}^2$, we also have the asymptotic normality

$$\frac{\check{S}_n - n\frac{\hat{p}}{2-\hat{p}}\mathbb{E}[X_1]}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{\mathbb{E}[X_1^2] - \left(\frac{\hat{p}}{2-\hat{p}}\mathbb{E}[X_1]\right)^2}{3-2\hat{p}}\right).$$

Finally, very recently, Bertenghi and Rosales-Ortiz [14] established the following results which give exactly the relation between the usual random walk and the reinforced or counterbalanced random walks associated.

Theorem I.17. (Joint invariance principles) *Fix $\hat{p} \in [0, 1/2[$ and consider the triplet $(S_n, \hat{S}_n, \check{S}_n)$ consisting of the random walk (S_n) with its positively and negatively reinforced versions of parameter $\hat{p}$. Assume further that X_1 is centered, $\mathbb{E}[X_1] = 0$, with variance $\mathbb{V}[X_1] = 1$. Then, the following weak convergence holds in the sense of Skorokhod,*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}, \frac{\hat{S}_{\lfloor nt \rfloor}}{\sqrt{n}}, \frac{\check{S}_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(B_t, \hat{B}_t, \check{B}_t\ t \geq 0\big),$$

where the processes B, $\hat{B}$, $\check{B}$ denote respectively a standard BM, a positively noise reinforced BM and a counterbalanced BM with covariances,

$$\mathbb{E}[B_s \check{B}_t] = t^{-\hat{p}}(t \wedge s)^{1+\hat{p}} \frac{1-\hat{p}}{1+\hat{p}},\ \mathbb{E}[B_s \hat{B}_t] = t^{\hat{p}}(t \wedge s)^{1-\hat{p}},\ \mathbb{E}[\hat{B}_s \check{B}_t] = t^{\hat{p}} s^{-\hat{p}}(t \wedge s) \frac{1-\hat{p}}{1+\hat{p}}.$$

I.4 – Outline of the thesis

Contributions

The following contributions are sorted chronogically.

[9] BERCU, B., AND LAULIN, L. On the multi-dimensional elephant random walk. *Journal of Statistical Physics 175*, 6 (2019), 1146–1163.

[11] BERCU, B., AND LAULIN, L. On the center of mass of the elephant random walk. *Stochastic Processes and their Applications 133* (2021), 111 – 128.

[56] LAULIN, L. A martingale approach for Pólya urn processes. *Electronic Communications in Probabilities 25* (2020), 13 pp.

[57] LAULIN, L. New insights on the reinforced elephant random walk using a martingale approach. *Journal of Statistical Physics 186* (2022).

[10] BERCU, B., AND LAULIN, L. How to estimate the memory of the elephant random walk. *(Submitted) arXiv:2112.10405* (2021).

[58] LAULIN, L. Introducing smooth amnesia to the memory of the elephant random walk. *arXiv:2204.10542* (2022).

Chapter 1 – The multidimensional elephant random walk

The goal of this chapter is to extend the results on the one-dimensional ERW to the mutidimensional elephant random walk (MERW) in $\mathbb{R}^d$, using a martigale approach.

In the diffusive and critical regimes, we establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for the MERW. The asymptotic normality of the MERW, properly normalized, is also provided. In the superdiffusive regime, we prove the almost sure convergence as well as the mean square convergence of the MERW to a non degenerate random vector of $\mathbb{R}^d$.

Chapter 2 – The center of masse of the elephant random walk

The goal of this chapter is to investigate the asymptotic behavior of the center of mass of the elephant random walk (CMERW) in $\mathbb{R}^d$. The entire analysis relies on asymptotic results for multi-dimensional martingales.
In the diffusive and critical regimes, we establish the almost sure convergence, the law of iterated logarithm and the quadratric strong law for CMERW. The asymptotic normality of the center of mass, properly normalized, is also provided. Finally, we prove a strong limit theorem for the center of mass in the superdiffusive regime.

Chapter 3 – The lineary reinforced elephant random walk

This chapter is devoted to a direct martingale approach for the linearly reinforced elephant random walk (RERW). All the analysis relies on asymptotic results for multi-dimensional martingales with matrix normalization.
We establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for the RERW in the diffusive and critical regimes. The distributional convergences of the RERW to some Gaussian processes are also provided. In the superdiffusive regime, we prove the distributional convergence as well as the mean square convergence of the RERW.

Chapter 4 – The amnesic elephant random walk

This chapter is an adaptation of Chapter 3 and gives a direct martingale approach for some type of amnesic change in the memory of the ERW (AERW). Once again, all the analysis relies on asymptotic results for multi-dimensional martingales with matrix normalization.

Chapter 5 – How to estimate the memory parameter

In this chapter, we introduce an original way to estimate the memory parameter of the elephant random walk. Our estimator is nothing more than a quasi-maximum likelihood estimator, based on a second order Taylor approximation of the log-likelihood function. The analysis relies on asymptotic results for martingales and the quadratic variations associated.
We show the almost sure convergence of our estimate in the diffusive, critical and superdiffusive regimes. The local asymptotic normality of our statistical procedure is established in the diffusive regime, while the local asymptotic mixed normality is proven in the superdiffusive regime. Asymptotic and exact confidence intervals as well as statistical tests are also provided.

Chapter 6 – A martingale approach for Pólya urn processes

This chapter is devoted to a direct martingale approach for Pólya urn models. A Pólya process is said to be small when the ratio of its replacement matrix eigenvalues is less than or equal to $1/2$, otherwise it is called large.
We find again some well-known results on the asymptotic behavior for small and large urn processes. We also provide new almost sure properties for small urn processes.

Chapter 7 – Conclusion and perspectives

1 The Multi-dimensional Elephant Random Walk

This chapter presents the results of [9] :

BERCU, B., AND LAULIN, L. On the Multi-dimensional Elephant Random Walk. *J. Stat. Phys. 175*, 6 (2019), 1146–1163.

1.1 – Introduction

Over the last decade, the Elephant Random Walk has received considerable attention in the mathematical physics literature in the diffusive regime $p < 3/4$ and the critical regime $p = 3/4$, see e.g. [3, 5, 20, 25, 28, 27, 29, 54, 55, 65] and the references therein.

Surprisingly, to the best of our knowledge, no references were available on the multi-dimensional elephant random walk (MERW) on $\mathbb{Z}^d$, except [28, 63] in the special case $d = 2$. The goal of the paper explicited in this chapter is to fill the gap by extending the results on the one-dimensional ERW to the MERW. To be more precise, we shall study the influence of the memory parameter p on the MERW and we will show that the critical value is given by

$$p_d = \frac{2d+1}{4d}.$$

In the diffusive and critical regimes $p \leq p_d$, the reader will find the natural extension to higher dimension of the results established in [3, 5, 25, 26] on the almost

sure asymptotic behavior of the ERW as well as on its asymptotic normality. In the superdiffusive regime $p > p_d$, we will also prove some extensions of the results in [5, 28, 63].
Our strategy is to make an extensive use of the theory of martingales [31, 42], in particular the strong law of large numbers and the central limit theorem for multi-dimensional martingales [31], as well as the law of iterated logarithm [71, 72].

We strongly believe that our approach could be successfully extended to MERW with stops [28, 43], to amnesiac MERW [27], as well as to MERW with reinforced memory [3, 44].

The chapter is organized as follows. In Section 1.2, we introduce the exact MERW and the multi-dimensional martingale we will extensively make use of. The main results are given in Section 1.3. As usual, we first investigate the diffusive regime $p < p_d$ and we establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for the MERW. The asymptotic normality of the MERW, properly normalized, is also provided. Next, we prove similar results in the critical regime $p = p_d$. At last, we study the superdiffusive regime $p > p_d$ and we prove the almost sure convergence as well as the mean square convergence of the MERW to a non-degenerate random vector. Our martingale approach is described in Section 1.4, while all technical proofs are postponed to Sections 1.6 and 1.7. We also give an alternative approach using Pólya-type urns in Section 1.5.

1.2 – The multi-dimensional elephant random walk

First of all, let us introduce the MERW. It is the natural extension to higher dimension of the one-dimensional ERW defined in the pioneer work of Schütz and Trimper [69]. For a given dimension $d \geq 1$, let $(\mathbf{S}_n)$ be a random walk on $\mathbb{Z}^d$, starting at the origin at time zero, $\mathbf{S}_0 = 0$. At time $n = 1$, the elephant moves in one of the $2d$ directions with the same probability $1/2d$. Afterwards, at time $n \geq 1$, the elephant chooses uniformly at random an integer k among the previous times $1, \ldots, n$. Then, it moves exactly in the same direction as that of time k with probability p or in one of the $2d-1$ remaining directions with the same probability $(1-p)/(2d-1)$, where the parameter p stands for the memory parameter of the MERW. Denote

$$\mathbf{I}_d = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{J}_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

and let $(\mathbf{A}_{n+1})$ be a sequence of random matrices such that

$$\mathbf{A}_n = \begin{cases} +\mathbf{I}_d & \text{with probability} \quad p \\ -\mathbf{I}_d & \text{with probability} \quad \frac{1-p}{2d-1} \\ +\mathbf{J}_d & \text{with probability} \quad \frac{1-p}{2d-1} \\ -\mathbf{J}_d & \text{with probability} \quad \frac{1-p}{2d-1} \\ \vdots & \\ +\mathbf{J}_d^{d-1} & \text{with probability} \quad \frac{1-p}{2d-1} \\ -\mathbf{J}_d^{d-1} & \text{with probability} \quad \frac{1-p}{2d-1} \end{cases}.$$

One can observe that the permutation matrix $\mathbf{J}_d$ satisfies $\mathbf{J}_d^d = \mathbf{I}_d$. Therefore, the position of the elephant at time $n \geq 1$ is given by

$$\mathbf{S}_{n+1} = \mathbf{S}_n + \mathbf{X}_{n+1}. \tag{1.1}$$

It follows from our very definition of the MERW that at any $n \geq 1$, $\mathbf{X}_{n+1} = \mathbf{A}_{n+1}\mathbf{X}_{b_{n+1}}$ where $\mathbf{A}_{n+1}$ is the random $d \times d$ matrix described before while b_{n+1} is a random variable uniformly distributed on $\{1, ..., n\}$. Moreover, as $\mathbf{A}_{n+1}$ and b_{n+1} are conditionally independent, we clearly have

$$\mathbb{E}\left[\mathbf{X}_{n+1} \mid \mathcal{F}_n\right] = \mathbb{E}\left[\mathbf{A}_{n+1}\right] \mathbb{E}\left[\mathbf{X}_{b_{n+1}} \mid \mathcal{F}_n\right] \tag{1.2}$$

where $\mathcal{F}_n$ stands for the σ-algebra, $\mathcal{F}_n = \sigma(\mathbf{X}_1, \ldots, \mathbf{X}_n)$. Hence, we can deduce from the law of total probability that at any time $n \geq 1$,

$$\mathbb{E}\left[\mathbf{X}_{n+1} \mid \mathcal{F}_n\right] = \frac{1}{n}\Big(\frac{2dp-1}{2d-1}\Big)\mathbf{S}_n = \frac{a}{n}\mathbf{S}_n \quad \text{a.s.} \tag{1.3}$$

where a is the fundamental parameter of the MERW,

$$a = \frac{2dp-1}{2d-1}. \tag{1.4}$$

Consequently, we immediately obtain from (1.1) and (1.3) that for any $n \geq 1$,

$$\mathbb{E}\left[\mathbf{S}_{n+1} \mid \mathcal{F}_n\right] = \gamma_n \mathbf{S}_n \quad \text{where} \quad \gamma_n = 1 + \frac{a}{n}. \tag{1.5}$$

Furthermore,

$$\prod_{k=1}^{n} \gamma_k = \frac{\Gamma(a+1+n)}{\Gamma(a+1)\Gamma(n+1)}$$

where Γ is the standard Euler Gamma function. The critical value associated with the memory parameter p of the MERW is

$$p_d = \frac{2d+1}{4d}. \tag{1.6}$$

As a matter of fact,

$$a < \frac{1}{2} \Longleftrightarrow p < p_d, \quad a = \frac{1}{2} \Longleftrightarrow p = p_d, \quad a > \frac{1}{2} \Longleftrightarrow p > p_d.$$

All our investigation in the three regimes relies on a martingale approach. To be more precise, the asymptotic behavior of $(\mathbf{S}_n)$ is closely related to the one of the sequence $(\mathbf{M}_n)$ defined, for all $n \geq 0$, by $\mathbf{M}_n = a_n \mathbf{S}_n$ where $a_0 = 1$, $a_1 = 1$ and, for all $n \geq 2$,

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(a+1)\Gamma(n)}{\Gamma(n+a)}. \tag{1.7}$$

It follows from a well-known property of the Euler Gamma function that

$$\lim_{n \to \infty} \frac{\Gamma(n+a)}{\Gamma(n) n^a} = 1. \tag{1.8}$$

Hence, we obtain from (1.7) and (1.8) that

$$\lim_{n \to \infty} n^a a_n = \Gamma(a+1). \tag{1.9}$$

Furthermore, since $a_n = \gamma_n a_{n+1}$, we can deduce from (1.5) that for all $n \geq 1$,

$$\mathbb{E}\left[\mathbf{M}_{n+1} \mid \mathcal{F}_n\right] = \mathbf{M}_n \quad \text{a.s.}$$

It means that $(\mathbf{M}_n)$ is a multi-dimensional martingale. Our goal is to extend the results recently established in [5] to MERW.

One can observe that our approach is much more tricky than that of [5] as it requires to study the asymptotic behavior of the multi-dimensional martingale $(\mathbf{M}_n)$. More precisley, while this appears to be similar to the ERW, the main difficulty here relies on the introduction of the $N_n^X(i)$ process counting the number of times a direction has been chosen (positively or negatively). This was not necessary in dimension 1 due to the fact that $S_n = S_n^+ - S_n^-$ and $S_n^+ + S_n^- = n$. Hence, there was only one direction. Finally, the reader can note that the ERW is a time-inhomogeneous Markov chain, while in contrast the MERW in dimensions greater or equal to two is non-Markovian.

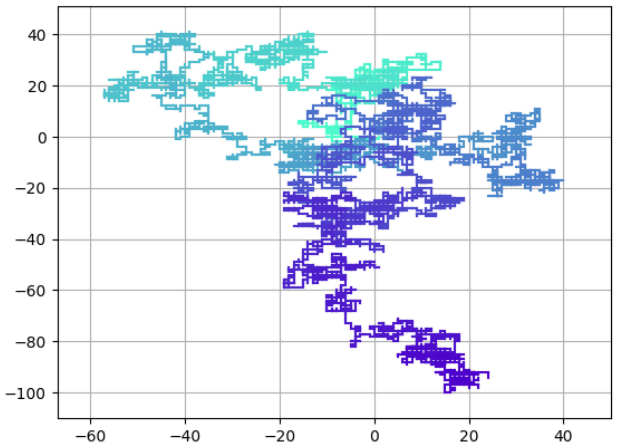


Figure 1.1: The 2-dimensional ERW when $p = 0.4$.

1.3 – Main results

1.3.1 – The diffusive regime

Our first result deals with the strong law of large numbers for the MERW in the diffusive regime where $0 \leq p < p_d$.

Theorem 1.1. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{S}_n = 0 \quad a.s. \tag{1.10}$$

Remark 1.2. *For any $\alpha > 1/2$, we have the more precise result*

$$\lim_{n\to\infty} \frac{1}{n^\alpha}\mathbf{S}_n = 0 \quad a.s.$$

Some refinements on the almost sure rates of convergence for the MERW are as follows.

Theorem 1.3. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2}\mathbf{S}_k\mathbf{S}_k^T = \frac{1}{d(1-2a)}\mathbf{I}_d \quad a.s. \tag{1.11}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\mathbf{S}_k\|^2}{k^2} = \frac{1}{(1-2a)} \quad a.s. \tag{1.12}$$

Moreover, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|\mathbf{S}_n\|^2}{2n\log\log n} = \frac{1}{(1-2a)d} \quad a.s. \tag{1.13}$$

Our next result is devoted to the asymptotic normality of the MERW in the diffusive regime $0 \leq p < p_d$.

Theorem 1.4. *We have the asymptotic normality*

$$\frac{1}{\sqrt{n}}\mathbf{S}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{(1-2a)d}\mathbf{I}_d\Big). \tag{1.14}$$

Remark 1.5. *We clearly have from* (1.4) *that*

$$\frac{1}{1-2a} = \frac{2d-1}{2d(1-2p)+1}.$$

Hence, in the special case $d = 1$, the critical value $p_d = 3/4$ and the asymptotic variance

$$\frac{1}{1-2a} = \frac{1}{3-4p}.$$

Consequently, we find again the asymptotic normality for the one-dimensional ERW in the diffusive regime $0 \leq p < 3/4$ recently established in [3, 5, 25].

1.3.2 – The critical regime

We now focus our attention on the critical regime where the memory parameter $p = p_d$.

Theorem 1.6. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n\log n}}\mathbf{S}_n = 0 \quad a.s. \tag{1.15}$$

Remark 1.7. *For any $\alpha > 1/2$, we have the more precise result*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}(\log n)^{\alpha}}\mathbf{S}_n = 0 \quad a.s.$$

We continue with some refinements on the almost sure rates of convergence for the MERW.

Theorem 1.8. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n}\sum_{k=2}^{n} \frac{1}{(k\log k)^2}\mathbf{S}_k\mathbf{S}_k^T = \frac{1}{d}\mathbf{I}_d \quad a.s \tag{1.16}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log\log n}\sum_{k=2}^{n} \frac{\|\mathbf{S}_k\|^2}{(k\log k)^2} = 1 \quad a.s. \tag{1.17}$$

Moreover, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|\mathbf{S}_n\|^2}{2n\log n\log\log\log n} = \frac{1}{d} \quad a.s. \tag{1.18}$$

Our next result concerns the asymptotic normality of the MERW in the critical regime $p = p_d$.

Theorem 1.9. *We have the asymptotic normality*

$$\frac{1}{\sqrt{n\log n}}\mathbf{S}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{d}\mathbf{I}_d\Big). \tag{1.19}$$

Remark 1.10. *As before, in the special case $d = 1$, we find again [3, 5, 25] the asymptotic normality for the one-dimensional ERW*

$$\frac{\mathbf{S}_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

1.3.3 – The superdiffusive regime

Finally, we get a handle on the more arduous superdiffusive regime where $p_d < p \leq 1$.

Theorem 1.11. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{n^a}\mathbf{S}_n = \mathbf{L}_d \quad a.s. \tag{1.20}$$

where the limiting value L is a non-degenerate random vector. We also have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big\|\frac{1}{n^a}\mathbf{S}_n - \mathbf{L}_d\Big\|^2\Big] = 0. \tag{1.21}$$

Theorem 1.12. *The expected value of L is* $\mathbb{E}[\mathbf{L}_d] = 0$*, while its covariance matrix is given by*

$$\mathbb{E}\left[\mathbf{L}_d\mathbf{L}_d^T\right] = \frac{1}{d(2a-1)\Gamma(2a)}\mathbf{I}_d. \tag{1.22}$$

In particular,

$$\mathbb{E}\left[\|\mathbf{L}_d\|^2\right] = \frac{1}{(2a-1)\Gamma(2a)}. \tag{1.23}$$

Remark 1.13. *Another possibility for the MERW is that, at time* $n = 1$*, the elephant moves in one direction, say the first direction* $\mathbf{e}_1$ *of the standard basis* $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ *of* $\mathbb{R}^d$*, with probability q or in one of the* $2d-1$ *remaining directions with the same probability* $(1-q)/(2d-1)$*, where the parameter q lies in the interval* $[0,1]$*. Afterwards, at any time* $n \geq 2$*, the elephant moves exactly as before, which means that his steps are given by* (1.2)*. Then, the results of Section 1.3 hold true except Theorem 1.12 where*

$$\mathbb{E}[\mathbf{L}_d] = \frac{1}{\Gamma(a+1)}\Big(\frac{2dq-1}{2d-1}\Big)\mathbf{e}_1$$

and

$$\mathbb{E}[\mathbf{L}_d\mathbf{L}_d^T] = \frac{1}{\Gamma(2a+1)}\Big(\frac{2dq-1}{2d-1}\Big)\Big(\mathbf{e}_1\mathbf{e}_1^T - \frac{1}{d}\mathbf{I}_d\Big) + \frac{1}{d(2a-1)\Gamma(2a)}\mathbf{I}_d,$$

which also leads to

$$\mathbb{E}\left[\|\mathbf{L}_d\|^2\right] = \frac{1}{(2a-1)\Gamma(2a)}.$$

1.4 – A multi-dimensional martingale approach

It is clear that for any time $n \geq 1$, $\|\mathbf{X}_n\| = 1$. Consequently, it follows from (1.1) that $\|\mathbf{S}_n\| \leq n$. Therefore, the sequence $(\mathbf{M}_n)$ given, for all $n \geq 0$, by $\mathbf{M}_n = a_n\mathbf{S}_n$, is a locally square-integrable multi-dimensional martingale. It can be rewritten in the additive form

$$\mathbf{M}_n = \sum_{k=1}^{n} a_k\boldsymbol{\varepsilon}_k \tag{1.24}$$

where

$$\boldsymbol{\varepsilon}_n = \mathbf{S}_n - \gamma_{n-1}\mathbf{S}_{n-1} \tag{1.25}$$

since its increments $\Delta \mathbf{M}_n = \mathbf{M}_n - \mathbf{M}_{n-1}$ satisfy $\Delta \mathbf{M}_n = a_n \mathbf{S}_n - a_{n-1}\mathbf{S}_{n-1} = a_n \varepsilon_n$. The predictable quadratic variation associated with $(\mathbf{M}_n)$ is the random square matrix of order d given, for all $n \geq 1$, by

$$\langle \mathbf{M} \rangle_n = \sum_{k=1}^{n} \mathbb{E}\left[\Delta \mathbf{M}_k (\Delta \mathbf{M}_k)^T | \mathcal{F}_{k-1} \right]. \tag{1.26}$$

We already saw from (1.5) that $\mathbb{E}\left[\varepsilon_{n+1} \mid \mathcal{F}_n\right] = 0$. Moreover, we deduce from (1.1) together with (1.3) that

$$\begin{aligned} \mathbb{E}\left[\mathbf{S}_{n+1}\mathbf{S}_{n+1}^T \mid \mathcal{F}_n \right] &= \mathbb{E}\left[\mathbf{S}_n \mathbf{S}_n^T \mid \mathcal{F}_n \right] + \frac{2a}{n}\mathbf{S}_n \mathbf{S}_n^T + \mathbb{E}\left[\mathbf{X}_{n+1}\mathbf{X}_{n+1}^T \mid \mathcal{F}_n \right] \\ &= \left(1 + \frac{2a}{n}\right) \mathbf{S}_n \mathbf{S}_n^T + \mathbb{E}\left[\mathbf{X}_{n+1}\mathbf{X}_{n+1}^T \mid \mathcal{F}_n \right] \quad \text{a.s.} \end{aligned} \tag{1.27}$$

In order to calculate the right-hand side of (1.27), one can notice that for any $n \geq 1$,

$$\mathbf{X}_n \mathbf{X}_n^T = \sum_{i=1}^{d} \mathbb{1}_{X_n^i \neq 0} \mathbf{e}_i \mathbf{e}_i^T$$

where $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ stands for the standard basis of the Euclidean space $\mathbb{R}^d$ and X_n^i is the i-th coordinate of the random vector $\mathbf{X}_n$. Moreover, it follows from (1.2) together with the law of total probability that, at any time $n \geq 1$ and for any $1 \leq i \leq d$,

$$\begin{aligned} \mathbb{P}(\mathbf{X}_{n+1}^i \neq 0 | \mathcal{F}_n) &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{P}((\mathbf{A}_n \mathbf{X}_k)^i \neq 0 | \mathcal{F}_n) \\ &= \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{\mathbf{X}_k^i \neq 0} \mathbb{P}(\mathbf{A}_n = \pm \mathbf{I}_d) + \frac{1}{n}\sum_{k=1}^{n} (1 - \mathbb{1}_{\mathbf{X}_k^i \neq 0}) \mathbb{P}(\mathbf{A}_n = \pm \mathbf{J}_d) \\ &= \frac{N_n^X(i)}{n}\Big(\mathbb{P}(\mathbf{A}_n = \mathbf{I}_d) - \mathbb{P}(\mathbf{A}_n = \mathbf{J}_d) \Big) + 2\mathbb{P}(\mathbf{A}_n = \mathbf{J}_d) \end{aligned}$$

which implies that for any $1 \leq i \leq d$,

$$\mathbb{E}\big[\mathbb{1}_{\mathbf{X}_{n+1}^i \neq 0} \mid \mathcal{F}_n \big] = \frac{a}{n} N_n^X(i) + \frac{(1-a)}{d} \quad \text{a.s.} \tag{1.28}$$

where

$$N_n^X(i) = \sum_{k=1}^{n} \mathbb{1}_{\mathbf{X}_k^i \neq 0}$$

and the parameter a is given by (1.4). Hence, we infer from (1.27) and (1.28) that

$$\mathbb{E}\left[\mathbf{X}_{n+1}\mathbf{X}_{n+1}^T \mid \mathcal{F}_n \right] = \frac{a}{n}\mathbf{\Sigma}_n + \frac{(1-a)}{d}\mathbf{I}_d \quad \text{a.s.} \tag{1.29}$$

where

$$\mathbf{\Sigma}_n = \sum_{i=1}^{d} N_n^X(i) \mathbf{e}_i \mathbf{e}_i^T. \tag{1.30}$$

One can observe the elementary fact that for all $n \geq 1$, $\mathrm{Tr}(\mathbf{\Sigma}_n) = n$ where $\mathrm{Tr}(\mathbf{\Sigma}_n)$ stands for the trace of the positive definite matrix $\mathbf{\Sigma}_n$. Therefore, we obtain from (1.27) together with (1.29) that

$$\begin{aligned}
\mathbb{E}\left[\varepsilon_{n+1}\varepsilon_{n+1}^T \mid \mathcal{F}_n\right] &= \mathbb{E}\left[\mathbf{S}_{n+1}\mathbf{S}_{n+1}^T \mid \mathcal{F}_n\right] - \gamma_n^2 \mathbf{S}_n\mathbf{S}_n^T \\
&= \Big(1 + \frac{2a}{n}\Big)\mathbf{S}_n\mathbf{S}_n^T + \frac{a}{n}\mathbf{\Sigma}_n + \frac{(1-a)}{d}\mathbf{I}_d - \gamma_n^2 \mathbf{S}_n\mathbf{S}_n^T \\
&= \frac{a}{n}\mathbf{\Sigma}_n + \frac{(1-a)}{d}\mathbf{I}_d - \Big(\frac{a}{n}\Big)^2 \mathbf{S}_n\mathbf{S}_n^T \quad \text{a.s.} \qquad (1.31)
\end{aligned}$$

which ensures that

$$\begin{aligned}
\mathbb{E}\left[\|\varepsilon_{n+1}\|^2 \mid \mathcal{F}_n\right] &= \frac{a}{n}\mathrm{Tr}(\mathbf{\Sigma}_n) + \frac{1-a}{d}\mathrm{Tr}(\mathbf{I}_d) - \Big(\frac{a}{n}\Big)^2 \|\mathbf{S}_n\|^2 \\
&= 1 - (\gamma_n - 1)^2\|\mathbf{S}_n\|^2 \quad \text{a.s.} \qquad (1.32)
\end{aligned}$$

By the same token,

$$\mathbb{E}\left[\|\varepsilon_{n+1}\|^4 \mid \mathcal{F}_n\right] = 1 - 3(\gamma_n - 1)^4\|\mathbf{S}_n\|^4 - 2(\gamma_n - 1)^2\|\mathbf{S}_n\|^2 + 4(\gamma_n - 1)^2\xi_n$$

where, thanks to (1.29),

$$\xi_n = \mathbb{E}\left[\langle \mathbf{S}_n, \mathbf{X}_{n+1}\rangle^2 | \mathcal{F}_n\right] = \frac{a}{n}\mathbf{S}_n^T\mathbf{\Sigma}_n\mathbf{S}_n + \frac{(1-a)}{d}\|\mathbf{S}_n\|^2.$$

It leads to

$$\begin{aligned}
\mathbb{E}\left[\|\varepsilon_{n+1}\|^4 \mid \mathcal{F}_n\right] &= 1 - 3(\gamma_n - 1)^4\|\mathbf{S}_n\|^4 - 2\Big(1 - \frac{2(1-a)}{d}\Big)(\gamma_n - 1)^2\|\mathbf{S}_n\|^2 \\
&\quad + \frac{4a}{n}(\gamma_n - 1)^2\mathbf{S}_n^T\mathbf{\Sigma}_n\mathbf{S}_n \quad \text{a.s.} \qquad (1.33)
\end{aligned}$$

Therefore, as $\mathbf{\Sigma}_n \leq n\mathbf{I}_d$ for the usual order of positive definite matrices, we clearly obtain from (1.33) that

$$\begin{aligned}
\mathbb{E}\left[\|\varepsilon_{n+1}\|^4 \mid \mathcal{F}_n\right] &\leq 1 - 3(\gamma_n - 1)^4\|\mathbf{S}_n\|^4 \\
&\quad + \frac{2}{d}(\gamma_n - 1)^2\Big(2a(d-1) + 2 - d\Big)\|\mathbf{S}_n\|^2 \quad \text{a.s.} \qquad (1.34)
\end{aligned}$$

Consequently, we obtain from (1.32) and (1.34) the almost sure upper bounds

$$\sup_{n\geq 0}\mathbb{E}\left[\|\varepsilon_{n+1}\|^2 \mid \mathcal{F}_n\right] \leq 1 \quad \text{and} \quad \sup_{n\geq 0}\mathbb{E}\left[\|\varepsilon_{n+1}\|^4 \mid \mathcal{F}_n\right] \leq \frac{4}{3} \quad \text{a.s.} \qquad (1.35)$$

Hereafter, we deduce from (1.26) and (1.31) that

$$\begin{aligned}
\langle \mathbf{M}\rangle_n &= a_1^2\mathbb{E}[\varepsilon_1\varepsilon_1^T] + \sum_{k=1}^{n-1} a_{k+1}^2\mathbb{E}\left[\varepsilon_{k+1}\varepsilon_{k+1}^T \mid \mathcal{F}_k\right] \\
&= \frac{1}{d}\mathbf{I}_d\sum_{k=1}^{n} a_k^2 + a\sum_{k=1}^{n-1} a_{k+1}^2\Big(\frac{1}{k}\mathbf{\Sigma}_k - \frac{1}{d}\mathbf{I}_d\Big) - \zeta_n \qquad (1.36)
\end{aligned}$$

where

$$\zeta_n = a^2\sum_{k=1}^{n-1}\Big(\frac{a_{k+1}}{k}\Big)^2\mathbf{S}_k\mathbf{S}_k^T.$$

Hence, by taking the trace on both sides of (1.36), we find that

$$\mathrm{Tr}\langle \mathbf{M} \rangle_n = \sum_{k=1}^{n} a_k^2 - a^2 \sum_{k=1}^{n-1} \Big(\frac{a_{k+1}}{k} \Big)^2 \|\mathbf{S}_k\|^2. \tag{1.37}$$

The asymptotic behavior of the multi-dimensional martingale $(\mathbf{M}_n)$ is closely related to the one of

$$v_n = \sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} \Big(\frac{\Gamma(a+1)\Gamma(k)}{\Gamma(a+k)} \Big)^2.$$

One can observe that we always have $\mathrm{Tr}\langle \mathbf{M} \rangle_n \leq v_n$. Once again, we have three regimes. In the diffusive regime where $a < 1/2$,

$$\lim_{n\to\infty} \frac{v_n}{n^{1-2a}} = \ell \quad \text{where} \quad \ell = \frac{(\Gamma(a+1))^2}{1-2a}. \tag{1.38}$$

In the critical regime where $a = 1/2$,

$$\lim_{n\to\infty} \frac{v_n}{\log n} = (\Gamma(a+1))^2 = \frac{\pi}{4}. \tag{1.39}$$

Finally, in the superdiffusive regime where $a > 1/2$, v_n converges to the finite value

$$\begin{aligned} \lim_{n\to\infty} v_n &= \sum_{k=0}^{\infty} \Big(\frac{\Gamma(a+1)\Gamma(k+1)}{\Gamma(a+k+1)} \Big)^2 = \sum_{k=0}^{\infty} \frac{(1)_k\,(1)_k\,(1)_k}{(a+1)_k\,(a+1)_k\,k!} \\ &= {}_3F_2\Big(\begin{matrix} 1,1,1 \\ a+1, a+1 \end{matrix} \Big| 1 \Big) \end{aligned} \tag{1.40}$$

where, for any $\alpha \in \mathbb{R}$, $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ for $k \geq 1$, $(\alpha)_0 = 1$ stands for the Pochhammer symbol and ${}_3F_2$ is the generalized hypergeometric function defined by

$${}_3F_2\Big(\begin{matrix} a,b,c \\ d,e \end{matrix} \Big| z \Big) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k\,(c)_k}{(d)_k\,(e)_k\,k!} z^k.$$

1.5 – Another approach using Pólya-type urns

In his work [13], Bertenghi used the approach developed by Baur and Bertoin [3] to obtain functional limit theorems for the MERW. More precisely, if you consider an urn filled with balls of $2d$ distinct colors and its composition at any time $n \in \mathbb{N}$ $U_n = \big(U_n^1, \ldots, U_n^{2d}\big)$ where each component U_n^k represents the numbers of balls of color k at time n. A time zero, there are no balls in the urn such that $U_0 = \mathbf{0}_{2d}$. Then, we set $U_1 = (1, 0, \ldots, 0)$ meaning that we start the urn with exactly one ball of the first color. Then, we add one ball at each time as follows. At any time $n \geq 2$, we draw a ball uniformly at random from the urn, observe its colour, put it back to the urn and add a ball of the same color with probability p, or add a ball of the one of the $2d-1$ other colors with probability $(1-p)/(2d-1)$ for

each color (uniformly). The connection to the MERW model is once again quite simple: If $\mathbf{S}_n$ denotes the position of the MERW started from zero at time zero, then

$$\mathbf{S}_n \overset{\mathcal{L}}{=} (X_n^1 - X_n^{d+1})\mathbf{e}_1 + \ldots + (X_n^d - X_n^{2d})\mathbf{e}_d.$$

To study this process, it is once again necessary to explicit mean replacement matrix of order $2d$, which is given in this case by

$$A = \begin{pmatrix} p & \frac{1-p}{2d-1} & \cdots & \cdots & \frac{1-p}{2d-1} \\ \frac{1-p}{2d-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \frac{1-p}{2d-1} \\ \frac{1-p}{2d-1} & \cdots & \cdots & \frac{1-p}{2d-1} & p \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = (2dp-1)/(2d-1)$with mutiplicity $2d-1$. Then, studying the ratio $\lambda_2/\lambda_1 < 1/2$ is indeed equivalent to $a < 1/2$.

Theorem 1.14. *The following convergences in $D([0,\infty[)$ hold. In the diffusive regime,*

$$\Big(\frac{\mathbf{S}_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(\mathbf{W}_t,\ t \geq 0\big) \tag{1.41}$$

where $\big(\mathbf{W}_t,\ t \geq 0\big)$ is a $\mathbb{R}^d$-valued centered Gaussian process starting from the origin with covariance

$$\mathbb{E}[\mathbf{W}_s\mathbf{W}_t] = \frac{1}{(1-2a)d} s\Big(\frac{t}{s}\Big)^a \mathbf{I}_d \tag{1.42}$$

for $0 < s \leq t$. In the critical regime,

$$\Big(\frac{\mathbf{S}_{\lfloor n^t \rfloor}}{\sqrt{n^t \log n}},\ t \geq 0\Big) \Longrightarrow \frac{1}{\sqrt{d}}\big(\mathbf{B}_t,\ t \geq 0\big) \tag{1.43}$$

where $\big(\mathbf{B}_t,\ t \geq 0\big)$ is a standard d-dimensional Brownian motion.

1.6 – Proofs of the almost sure convergence results

1.6.1 – The diffusive regime

Proof of Theorem 1.1. First of all, we focus our attention on the proof of the almost sure convergence (1.10). We already saw from (1.37) that $\mathrm{Tr}\langle \mathbf{M} \rangle_n \leq v_n$. Moreover, we obtain from (1.38) that, in the diffusive regime where $0 < a < 1/2$, v_n increases to infinity with the speed n^{1-2a}. On the one hand, it follows from the strong law of large numbers for multi-dimensional martingales, see Theorem A.3, that for any $\gamma > 0$,

$$\frac{\|\mathbf{M}_n\|^2}{\lambda_{\max}\langle \mathbf{M} \rangle_n} = o\Big(\Big(\log \mathrm{Tr}\langle \mathbf{M} \rangle_n\Big)^{1+\gamma}\Big) \quad \text{a.s} \tag{1.44}$$

where $\lambda_{\max}\langle \mathbf{M} \rangle_n$ stands for the maximal eigenvalue of the random square matrix $\langle \mathbf{M} \rangle_n$. However, as $\langle \mathbf{M} \rangle_n$ is a positive definite matrix and $\mathrm{Tr}\langle \mathbf{M} \rangle_n \leq v_n$, we clearly have

$$\lambda_{\max}\langle \mathbf{M} \rangle_n \leq \mathrm{Tr}\langle \mathbf{M} \rangle_n \leq v_n.$$

Consequently, we obtain from (1.44) that

$$\|\mathbf{M}_n\|^2 = o\big(v_n(\log v_n)^{1+\gamma}\big) \quad \text{a.s}$$

which implies that

$$\|\mathbf{M}_n\|^2 = o\big(n^{1-2a}(\log n)^{1+\gamma}\big) \quad \text{a.s.} \tag{1.45}$$

Hence, as $\mathbf{M}_n = a_n \mathbf{S}_n$, it follows from (1.9) and (1.45) that for any $\gamma > 0$,

$$\|\mathbf{S}_n\|^2 = o\big(n(\log n)^{1+\gamma}\big) \quad \text{a.s.}$$

which completes the proof of Theorem 1.1.

■

Proof of Theorem 1.3. We shall now proceed to the proof of the almost sure rates of convergence given in Theorem 1.3. First of all, we claim that

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{\Sigma}_n = \frac{1}{d}\mathbf{I}_d \quad \text{a.s.} \tag{1.46}$$

where $\mathbf{\Sigma}_n$ is the random square matrix of order d given by (1.30). As a matter of fact, in order to prove (1.46) it is only necessary to show that for any $1 \leq i \leq d$,

$$\lim_{n\to\infty} \frac{N_n^X(i)}{n} = \frac{1}{d} \quad \text{a.s.} \tag{1.47}$$

For any $1 \leq i \leq d$, denote

$$\Lambda_n(i) = \frac{N_n^X(i)}{n}.$$

One can observe that

$$\Lambda_{n+1}(i) = \frac{n}{n+1}\Lambda_n(i) + \frac{1}{n+1}\mathbb{1}_{X_{n+1}^i \neq 0}$$

which leads, via (1.28), to the recurrence relation

$$\Lambda_{n+1}(i) = \frac{n}{n+1}\gamma_n\Lambda_n(i) + \frac{(1-a)}{d(n+1)} + \frac{1}{n+1}\delta_{n+1}(i) \tag{1.48}$$

where $\delta_{n+1}(i) = \mathbb{1}_{X_{n+1}^i \neq 0} - \mathbb{E}[\mathbb{1}_{X_{n+1}^i \neq 0} \mid \mathcal{F}_n]$. After straightforward calculations, the solution of this recurrence relation is given by

$$\Lambda_n(i) = \frac{1}{na_n}\Big(\Lambda_1(i) + \frac{(1-a)}{d}\sum_{k=2}^{n} a_k + L_n(i)\Big) \tag{1.49}$$

where

$$L_n(i) = \sum_{k=2}^{n} a_k \delta_k(i).$$

However, $(L_n(i))$ is a square-integrable real martingale with predictable quadratic variation $\langle L(i) \rangle_n$ satisfying $\langle L(i) \rangle_n \leq v_n$ a.s. Then, it follows from the standard strong law of large numbers for martingales given by Theorem A.2 that $(L_n(i))^2 = O(v_n \log v_n)$ a.s. Consequently, as na_n^2 is equivalent to $(1-2a)v_n$, we obtain that for any $1 \leq i \leq d$,

$$\lim_{n \to \infty} \frac{1}{na_n} L_n(i) = 0 \quad \text{a.s.} \tag{1.50}$$

Furthermore, one can easily check from (1.9) that

$$\lim_{n \to \infty} \frac{1}{na_n} \sum_{k=1}^{n} a_k = \frac{1}{1-a}. \tag{1.51}$$

Therefore, we find from (1.49) together with (1.50) and (1.51) that for any $1 \leq i \leq d$,

$$\lim_{n \to \infty} \Lambda_n(i) = \frac{1}{d} \quad \text{a.s.} \tag{1.52}$$

which immediately leads to (1.47). Hereafter, it follows from the conjunction of (1.10), (1.31) and (1.47) that

$$\lim_{n \to \infty} \mathbb{E}\left[\varepsilon_{n+1} \varepsilon_{n+1}^T \mid \mathcal{F}_n \right] = \frac{1}{d} \mathbf{I}_d \quad \text{a.s.} \tag{1.53}$$

By the same token, we also obtain from (1.36) and Toeplitz lemma that

$$\lim_{n \to \infty} \frac{1}{v_n} \langle \mathbf{M} \rangle_n = \frac{1}{d} \mathbf{I}_d \quad \text{a.s.} \tag{1.54}$$

We are now in the position to prove the quadratic strong law (1.11). For any vector $\mathbf{u}$ of $\mathbb{R}^d$, denote $\mathbf{M}_n(\mathbf{u}) = \langle \mathbf{u}, \mathbf{M}_n \rangle$ and $\varepsilon_n(u) = \langle \mathbf{u}, \varepsilon_n \rangle$. We clearly have from (1.24)

$$\mathbf{M}_n(\mathbf{u}) = \sum_{k=1}^{n} a_k \varepsilon_k(\mathbf{u}).$$

Consequently, $(\mathbf{M}_n(\mathbf{u}))$ is a square-integrable real martingale. Moreover, it follows from (1.53) that

$$\lim_{n \to \infty} \mathbb{E}\left[|\varepsilon_{n+1}(\mathbf{u})|^2 \mid \mathcal{F}_n \right] = \frac{1}{d} \|\mathbf{u}\|^2 \quad \text{a.s.}$$

Moreover, we can deduce from (1.35) and the Cauchy-Schwarz inequality that

$$\sup_{n \geq 0} \mathbb{E}\left[|\varepsilon_{n+1}(\mathbf{u})|^4 \mid \mathcal{F}_n \right] \leq \frac{4}{3} \|\mathbf{u}\|^4 \quad \text{a.s.}$$

Furthermore, we clearly have from (1.9) and (1.38) that

$$\lim_{n \to \infty} n f_n = 1 - 2a \quad \text{where} \quad f_n = \frac{a_n^2}{v_n},$$

which of course implies that f_n converges to zero. Therefore, it follows from the quadratic strong law for real martingales, see Theorem A.5, that for any vector u of $\mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{1}{\log v_n} \sum_{k=1}^{n} f_k\Big(\frac{\mathbf{M}_k^2(u)}{v_k}\Big) = \frac{1}{d}\|\mathbf{u}\|^2 \quad \text{a.s.} \tag{1.55}$$

Consequently, we find from (1.38) and (1.55) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{a_k^2}{v_k^2}\mathbf{M}_k^2(\mathbf{u}) = \frac{(1-2a)}{d}\|u\|^2 \quad \text{a.s.} \tag{1.56}$$

Hereafter, as $\mathbf{M}_n = a_n\mathbf{S}_n$ and $n^2a_n^4$ is equivalent to $(1-2a)^2v_n^2$, we obtain from (1.56) that for any vector u of $\mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2}\mathbf{u}^T\mathbf{S}_k\mathbf{S}_k^T\mathbf{u} = \frac{1}{d(1-2a)}\|\mathbf{u}\|^2 \quad \text{a.s.} \tag{1.57}$$

By virtue of the second part of Proposition 4.2.8 in [31], we can conclude from (1.57) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2}\mathbf{S}_k\mathbf{S}_k^T = \frac{1}{d(1-2a)}\mathbf{I}_d \quad \text{a.s.} \tag{1.58}$$

which completes the proof of (1.11). By taking the trace on both sides of (1.58), we immediately obtain (1.12). Finally, we shall proceed to the proof of the law of iterated logarithm given by (1.13). We already saw that $a_n^4v_n^{-2}$ is equivalent to $(1-2a)^2n^{-2}$. It ensures that

$$\sum_{n=1}^{+\infty} \frac{a_n^4}{v_n^2} < +\infty. \tag{1.59}$$

Hence, it follows from the law of iterated logarithm for real martingales due to Stout [71, 72], see Theorem A.4, that for any vector $\mathbf{u}$ of $\mathbb{R}^d$,

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2v_n\log\log v_n}\Big)^{1/2}\mathbf{M}_n(\mathbf{u}) &= -\liminf_{n\to\infty} \Big(\frac{1}{2v_n\log\log v_n}\Big)^{1/2}\mathbf{M}_n(\mathbf{u}) \\ &= \frac{1}{\sqrt{d}}\|\mathbf{u}\| \quad \text{a.s.} \end{aligned} \tag{1.60}$$

Consequently, as $\mathbf{M}_n(u) = a_n\langle \mathbf{u}, \mathbf{S}_n\rangle$, we obtain from (1.38) together with (1.60) that

$$\begin{aligned} \limsup_{n\to\infty} \Big(\frac{1}{2n\log\log n}\Big)^{1/2}\langle \mathbf{u}, \mathbf{S}_n\rangle &= -\liminf_{n\to\infty} \Big(\frac{1}{2n\log\log n}\Big)^{1/2}\langle \mathbf{u}, \mathbf{S}_n\rangle \\ &= \frac{1}{\sqrt{d(1-2a)}}\|\mathbf{u}\| \quad \text{a.s.} \end{aligned}$$

In particular, for any vector u of $\mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{1}{2n\log\log n}\langle \mathbf{u}, \mathbf{S}_n\rangle^2 = \frac{1}{d(1-2a)}\|\mathbf{u}\|^2 \quad \text{a.s.} \tag{1.61}$$

By taking all rational points on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the bound in (1.61) holds simultaneously for all of them, which implies that

$$\limsup_{n\to\infty} \frac{\|\mathbf{S}_n\|^2}{2n\log\log n} \leq \sup_{u\in\mathbb{Q}^d\cap\mathbb{S}^{d-1}} \limsup_{n\to\infty} \frac{\|\mathbf{S}_n\|^2}{2n\log\log n} = \frac{1}{d(1-2a)} \quad \text{a.s.}$$

In addition, for any single $\mathbf{u} \in \mathbb{S}^{d-1}$, we also obtain the reverse inequality

$$\limsup_{n\to\infty} \frac{\|\mathbf{S}_n\|^2}{2n\log\log n} \geq \limsup_{n\to\infty} \frac{\langle u, \mathbf{M}_n\rangle^2}{2n\log\log n} = \frac{1}{d(1-2a)} \quad \text{a.s.}$$

It immediately leads to

$$\limsup_{n\to\infty} \frac{\|\mathbf{S}_n\|^2}{2n\log\log n} = \frac{1}{d(1-2a)} \quad \text{a.s.}$$

which achieves the proof of Theorem 1.3.

■

1.6.2 – The critical regime

Proof of Theorem 1.6. We already saw from (1.39) that in the critical regime where $a = 1/2$, v_n increases slowly to infinity with a logarithmic speed $\log n$. We obtain once again from Theorem A.3 that for any $\gamma > 0$,

$$\|\mathbf{M}_n\|^2 = o\big(v_n(\log v_n)^{1+\gamma}\big) \quad \text{a.s}$$

which leads to

$$\|\mathbf{M}_n\|^2 = o\big(\log n(\log\log n)^{1+\gamma}\big) \quad \text{a.s.} \tag{1.62}$$

However, we clearly have from (1.9) with $a = 1/2$ that

$$\lim_{n\to\infty} na_n^2 = \frac{\pi}{4}. \tag{1.63}$$

Consequently, as $\mathbf{M}_n = a_n\mathbf{S}_n$, we deduce from (1.62) and (1.63) that for any $\gamma > 0$,

$$\|\mathbf{S}_n\|^2 = o\big(n\log n(\log\log n)^{1+\gamma}\big) \quad \text{a.s.}$$

which completes the proof of Theorem 1.6.

■

Proof of Theorem 1.8. The proof of Theorem 1.8 is left to the reader as it follows the same lines as that of Theorem 1.3.

■

1.6.3 – The superdiffusive regime

Proof of Theorem 1.11. We already saw from (1.40) that in the superdiffusive regime where $1/2 < a \leq 1$, v_n converges to a finite value. As previously seen, $\mathrm{Tr}\langle \mathbf{M} \rangle_n \leq v_n$. Hence, we clearly have

$$\lim_{n\to\infty} \mathrm{Tr}\langle \mathbf{M} \rangle_n < \infty \quad \text{a.s.}$$

Therefore, if

$$\mathbf{L}_n = \frac{\mathbf{M}_n}{\Gamma(a+1)}, \tag{1.64}$$

we can deduce from Theorem A.3 that

$$\lim_{n\to\infty} \mathbf{M}_n = \mathbf{M} \quad \text{and} \quad \lim_{n\to\infty} \mathbf{L}_n = \mathbf{L}_d \quad \text{a.s.} \tag{1.65}$$

where the limiting values $\mathbf{M}$ and $\mathbf{L}_d$ are the random vectors of $\mathbb{R}^d$ given by the following series (which are not absolutely convergent)

$$\mathbf{M} = \sum_{k=1}^{\infty} a_k \boldsymbol{\varepsilon}_k \quad \text{and} \quad \mathbf{L}_d = \frac{1}{\Gamma(a+1)} \sum_{k=1}^{\infty} a_k \boldsymbol{\varepsilon}_k.$$

Consequently, as $\mathbf{M}_n = a_n \mathbf{S}_n$, (1.20) clearly follows from (1.9) and (1.65). We now focus our attention on the mean square convergence (1.21). As $\mathbf{M}_0 = 0$, we have from (1.24) and (1.26) that for all $n \geq 1$,

$$\mathbb{E}[\|\mathbf{M}_n\|^2] = \sum_{k=1}^{n} \mathbb{E}[\|\Delta \mathbf{M}_k\|^2] = \mathbb{E}[\mathrm{Tr}\langle \mathbf{M} \rangle_n] \leq v_n.$$

Hence, we obtain from (1.40) that

$$\sup_{n\geq 1} \mathbb{E}\left[\|\mathbf{M}_n\|^2\right] \leq {}_3F_2\Big(\begin{matrix}1,1,1\\ a+1, a+1\end{matrix}\Big|1\Big) < \infty,$$

which means that the martingale $(\mathbf{M}_n)$ is bounded in $\mathbb{L}^2$. Therefore, we have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\big[\|\mathbf{M}_n - \mathbf{M}\|^2\big] = 0,$$

which clearly leads to (1.21).

■

Proof of Theorem 1.12. First of all, we clearly have for all $n \geq 1$, $\mathbb{E}[\mathbf{M}_n] = 0$ which implies that $\mathbb{E}[M] = 0$ leading to $\mathbb{E}[L] = 0$. Moreover, taking expectation on both sides of (1.27) and (1.29), we obtain that for all $n \geq 1$,

$$\begin{aligned} \mathbb{E}\left[\mathbf{S}_{n+1}\mathbf{S}_{n+1}^T\right] &= \Big(1 + \frac{2a}{n}\Big)\mathbb{E}\left[\mathbf{S}_n\mathbf{S}_n^T\right] + \mathbb{E}\left[\mathbf{X}_{n+1}\mathbf{X}_{n+1}^T\right] \\ &= \Big(1 + \frac{2a}{n}\Big)\mathbb{E}\left[\mathbf{S}_n\mathbf{S}_n^T\right] + \frac{a}{n}\mathbb{E}\left[\boldsymbol{\Sigma}_n\right] + \frac{(1-a)}{d}\mathbf{I}_d. \end{aligned} \tag{1.66}$$

However, we claim that

$$\mathbb{E}\left[\mathbf{\Sigma}_n\right] = \frac{n}{d}\mathbf{I}_d. \tag{1.67}$$

As a matter of fact, taking expectation on both sides of (1.49), we find that for any $1 \leq i \leq d$,

$$\mathbb{E}[\Lambda_n(i)] = \frac{1}{na_n}\Big(\mathbb{E}[\Lambda_1(i)] + \frac{(1-a)}{d}\sum_{k=2}^{n} a_k\Big). \tag{1.68}$$

On the one hand, we clearly have

$$\mathbb{E}[\Lambda_1(i)] = \frac{1}{d}.$$

On the other hand, it follows from Lemma B.1 in [5] that

$$\begin{aligned}\sum_{k=2}^{n} a_k &= \sum_{k=2}^{n} \frac{\Gamma(a+1)\Gamma(k)}{\Gamma(k+a)} = \sum_{k=1}^{n-1} \frac{\Gamma(a+1)\Gamma(k+1)}{\Gamma(k+a+1)} \\ &= \frac{1}{(a-1)}\left(1 - \frac{\Gamma(a+1)\Gamma(n+1)}{\Gamma(a+n)}\right) = \frac{(1-na_n)}{(a-1)}.\end{aligned} \tag{1.69}$$

Consequently, we can deduce from (1.68) and (1.69) that for any $1 \leq i \leq d$,

$$\mathbb{E}[\Lambda_n(i)] = \frac{1}{na_n}\Big(\frac{1}{d} - \frac{(1-na_n)}{d}\Big) = \frac{1}{d}. \tag{1.70}$$

Therefore, we get from (1.30) and (1.70) that

$$\mathbb{E}[\mathbf{\Sigma}_n] = n\sum_{i=1}^{d} \mathbb{E}[\Lambda_n(i)]\mathbf{e}_i\mathbf{e}_i^T = \frac{n}{d}\sum_{i=1}^{d} \mathbf{e}_i\mathbf{e}_i^T = \frac{n}{d}\mathbf{I}_d.$$

Hereafter, we obtain from (1.66) and (1.67) that

$$\mathbb{E}\left[\mathbf{S}_{n+1}\mathbf{S}_{n+1}^T\right] = \Big(1 + \frac{2a}{n}\Big)\mathbb{E}\left[\mathbf{S}_n\mathbf{S}_n^T\right] + \frac{1}{d}\mathbf{I}_d. \tag{1.71}$$

It is not hard to see that the solution of this recurrence relation is given by

$$\begin{aligned}\mathbb{E}\left[\mathbf{S}_n\mathbf{S}_n^T\right] &= \frac{\Gamma(n+2a)}{\Gamma(2a+1)\Gamma(n)}\left(\mathbb{E}[\mathbf{S}_1\mathbf{S}_1^T] + \frac{1}{d}\sum_{k=1}^{n-1}\frac{\Gamma(2a+1)\Gamma(k+1)}{\Gamma(k+2a+1)}\mathbf{I}_d\right) \\ &= \frac{\Gamma(n+2a)}{\Gamma(n)}\left(\sum_{k=1}^{n}\frac{\Gamma(k)}{\Gamma(k+2a)}\right)\frac{1}{d}\mathbf{I}_d\end{aligned} \tag{1.72}$$

since

$$\mathbb{E}[\mathbf{S}_1\mathbf{S}_1^T] = \frac{1}{d}\mathbf{I}_d.$$

Therefore, it follows once again from Lemma B.1 in [5] that

$$\mathbb{E}\left[\mathbf{S}_n\mathbf{S}_n^T\right] = \frac{n}{(2a-1)}\left(\frac{\Gamma(n+2a)}{\Gamma(n+1)\Gamma(2a)} - 1\right)\frac{1}{d}\mathbf{I}_d. \tag{1.73}$$

Hence, we obtain from (1.64) together with (1.73) that

$$\begin{aligned}\mathbb{E}[L_nL_n^T] &= \frac{na_n^2}{(2a-1)(\Gamma(a+1))^2}\left(\frac{\Gamma(n+2a)}{\Gamma(n+1)\Gamma(2a)} - 1\right)\frac{1}{d}\mathbf{I}_d \\ &= \frac{n}{(2a-1)}\left(\frac{\Gamma(n)}{\Gamma(n+a)}\right)^2\left(\frac{\Gamma(n+2a)}{\Gamma(n+1)\Gamma(2a)} - 1\right)\frac{1}{d}\mathbf{I}_d.\end{aligned} \tag{1.74}$$

Finally, we find from (1.21) and (1.74) that

$$\lim_{n\to\infty} \mathbb{E}[L_n L_n^T] = \mathbb{E}[\mathbf{L}_d \mathbf{L}_d^T] = \frac{1}{d(2a-1)\Gamma(2a)} \mathbf{I}_d$$

which achieves the proof of Theorem 1.12.

■

1.7 – Proofs of the asymptotic normality results

1.7.1 – The diffusive regime

Proof of Theorem 1.4. In order to establish the asymptotic normality (1.14), we shall make use of the central limit theorem for multi-dimensional martingales, given by Theorem A.6. First of all, we already saw from (1.54) that

$$\lim_{n\to\infty} \frac{1}{v_n} \langle \mathbf{M} \rangle_n = \frac{1}{d} \mathbf{I}_d \quad \text{a.s.} \tag{1.75}$$

Consequently, it only remains to show that $(\mathbf{M}_n)$ satisfies Lindeberg's condition, in other words, for all $\varepsilon > 0$,

$$\frac{1}{v_n} \sum_{k=1}^{n} \mathbb{E}\left[\|\Delta \mathbf{M}_n\|^2 \mathbb{1}_{\{\|\Delta \mathbf{M}_n\| \geq \varepsilon \sqrt{v_n}\}} | \mathcal{F}_{k-1} \right] \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

We have from (1.35) that for all $\varepsilon > 0$

$$\begin{aligned}
\frac{1}{v_n} \sum_{k=1}^{n} \mathbb{E}\left[\|\Delta \mathbf{M}_n\|^2 \mathbb{1}_{\{\|\Delta \mathbf{M}_n\| \geq \varepsilon \sqrt{v_n}\}} | \mathcal{F}_{k-1} \right] &\leq \frac{1}{\varepsilon^2 v_n^2} \sum_{k=1}^{n} \mathbb{E}\left[\|\Delta \mathbf{M}_n\|^4 | \mathcal{F}_{k-1} \right] \\
&\leq \sup_{1\leq k\leq n} \mathbb{E}\left[\|\boldsymbol{\varepsilon}_k\|^4 \mid \mathcal{F}_{k-1} \right] \frac{1}{\varepsilon^2 v_n^2} \sum_{k=1}^{n} a_k^4 \\
&\leq \frac{4}{3\varepsilon^2 v_n^2} \sum_{k=1}^{n} a_k^4.
\end{aligned}$$

However, we already saw from (1.59) that

$$\sum_{n=1}^{+\infty} \frac{a_n^4}{v_n^2} < +\infty.$$

Hence, it follows from Kronecker's lemma that

$$\lim_{n\to\infty} \frac{1}{v_n^2} \sum_{k=1}^{n} a_k^4 = 0,$$

which ensures that Lindeberg's condition is satisfied. Therefore, we can conclude from the central limit theorem for martingales that

$$\frac{1}{\sqrt{v_n}} \mathbf{M}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{d} \mathbf{I}_d\Big). \tag{1.76}$$

As $\mathbf{M}_n = a_n \mathbf{S}_n$ and $\sqrt{n} a_n$ is equivalent to $\sqrt{v_n(1-2a)}$, we find from (1.76) that

$$\frac{1}{\sqrt{n}} \mathbf{S}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{d(1-2a)} \mathbf{I}_d\Big),$$

which completes the proof of Theorem 1.4.

■

1.7.2 – The critical regime

Proof of Theorem 1.9. Via the same lines as in the proof of (1.54), we can deduce from (1.15), (1.37) and (1.39) that in the critical regime

$$\lim_{n\to\infty} \frac{1}{v_n} \langle \mathbf{M} \rangle_n = \frac{1}{d} \mathbf{I}_d \quad \text{a.s.} \tag{1.77}$$

Moreover, it follows from (1.39) and (1.63) that $a_n^2 v_n^{-1}$ is equivalent to $(n \log n)^{-1}$. It implies that

$$\sum_{k=1}^{\infty} \frac{a_n^4}{v_n^2} < +\infty. \tag{1.78}$$

As previously seen, we infer from (1.78) that $(\mathbf{M}_n)$ satisfies Lindeberg's condition. Therefore, we can conclude from the central limit theorem for martingales that

$$\frac{1}{\sqrt{v_n}} \mathbf{M}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{d} \mathbf{I}_d\Big). \tag{1.79}$$

Finally, as $\mathbf{M}_n = a_n \mathbf{S}_n$ and $a_n \sqrt{n \log n}$ is equivalent to $\sqrt{v_n}$, we obtain from that (1.79) that

$$\frac{1}{\sqrt{n \log n}} \mathbf{S}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{d} \mathbf{I}_d\Big),$$

which achieves the proof of Theorem 1.9.

■

2 The Center of Mass of the Elephant Random Walk

This chapter presents the results of [11] :

BERCU, B., AND LAULIN, L. On the Center of Mass of the Elephant Random Walk. *Stochastic Process. Appl. 133* (2021), 111 – 128.

2.1 – Introduction

Let $(\mathbf{S}_n)$ be a standard random walk in $\mathbb{R}^d$. The center of mass $\mathbf{G}_n$ of $\mathbf{S}_n$ is defined by

$$\mathbf{G}_n = \frac{1}{n}\sum_{k=1}^{n} \mathbf{S}_k. \tag{2.1}$$

The question of the asymptotic behavior of $\mathbf{G}_n$ was first raised by Paul Erdös. Very recently, Lo and Wade [62] extended the results of Grill [38] by studying the asymptotic behavior of $(\mathbf{G}_n)$. More precisely, let $\mathbf{S}_n = \mathbf{X}_1 + \cdots + \mathbf{X}_n$ where the increments $(\mathbf{X}_n)$ are independent and identically distributed square integrable random vectors of $\mathbb{R}^d$ with mean μ and covariance matrix Γ. They proved the strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{G}_n = \frac{1}{2}\mu \quad \text{a.s.} \tag{2.2}$$

together with the asymptotic normality,

$$\frac{1}{\sqrt{n}}\Big(\mathbf{G}_n - \frac{n}{2}\mu\Big) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3}\Gamma\Big). \tag{2.3}$$

The proofs of many results on the convex hull C_n as well as on the center of mass $\mathbf{G}_n$ rely on independence and exchangeability of the increments of the walk. For example, one can observe that

$$\mathbf{G}_n = \frac{1}{n}\sum_{k=1}^{n}\mathbf{S}_k = \frac{1}{n}\sum_{k=1}^{n}(n-k+1)\mathbf{X}_k \tag{2.4}$$

shares the same distribution as

$$\mathbf{\Sigma}_n = \frac{1}{n}\sum_{k=1}^{n}k\mathbf{X}_k.$$

A natural question concerns the asymptotic behavior of $\mathbf{G}_n$ in other situations where the increments of the walk are not independent and not identically distributed. In this chapter, we investigate the asymptotic behavior of the center of mass of the multi-dimensional elephant random walk introduced in Chapter 1.
Our strategy for proving asymptotic results for the center of mass of the elephant random walk (CMERW) is as follows. On the one hand, the behavior of position $\mathbf{S}_n$ is closely related to the one of the locally square-integrable martingale adapted to the filtration $(\mathcal{F}_n)$. The sequence $(\mathbf{M}_n)$ is defined, for all $n \geq 0$, by $\mathbf{M}_n = a_n\mathbf{S}_n$ with $a_1 = 1$ and, for all $n \geq 2$,

$$a_n = \prod_{k=1}^{n-1}\Big(\frac{k}{k+a}\Big) = \frac{\Gamma(a+1)\Gamma(n)}{\Gamma(n+a)} \tag{2.5}$$

where Γ stands for the Euler Gamma function and a is the fundamental parameter of the ERW defined by

$$a = \frac{2dp-1}{2d-1}. \tag{2.6}$$

It can be rewritten, see (1.24) or [9], in the additive form

$$\mathbf{M}_n = \sum_{k=1}^{n}a_k\boldsymbol{\varepsilon}_k \tag{2.7}$$

where $\boldsymbol{\varepsilon}_1 = \mathbf{S}_1$ and, for all $n \geq 2$,

$$\boldsymbol{\varepsilon}_n = \mathbf{S}_n - \Big(\frac{a_{n-1}}{a_n}\Big)\mathbf{S}_{n-1} = \mathbf{S}_n - \Big(1+\frac{a}{n-1}\Big)\mathbf{S}_{n-1}. \tag{2.8}$$

On the other hand, an analogue of equation (2.4) is given by

$$\begin{aligned}\mathbf{G}_n &= \frac{1}{n}\sum_{k=1}^{n}\mathbf{S}_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\mathbf{M}_k = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\sum_{\ell=1}^{k}a_\ell\boldsymbol{\varepsilon}_\ell = \frac{1}{n}\sum_{k=1}^{n}a_k\boldsymbol{\varepsilon}_k\sum_{\ell=k}^{n}\frac{1}{a_\ell},\\ &= \frac{1}{n}\sum_{k=1}^{n}a_k(b_n-b_{k-1})\boldsymbol{\varepsilon}_k\end{aligned} \tag{2.9}$$

where the sequence (b_n) is given by $b_0 = 0$ and, for all $n \geq 1$,

$$b_n = \sum_{k=1}^{n}\frac{1}{a_k}. \tag{2.10}$$

Denoting

$$\mathbf{N}_n = \sum_{k=1}^{n} a_k b_{k-1} \varepsilon_k, \tag{2.11}$$

it is straightforward to see that $\mathbb{E}\left[\mathbf{N}_{n+1} \mid \mathcal{F}_n\right] = \mathbf{N}_n$ a.s. since $\mathbb{E}\left[\varepsilon_{n+1} \mid \mathcal{F}_n\right] = 0$. Hence, $(\mathbf{N}_n)$ is also a locally square-integrable martingale adapted to the filtration $(\mathcal{F}_n)$. We deduce from (2.9) that

$$\mathbf{G}_n = \frac{1}{n}(b_n \mathbf{M}_n - \mathbf{N}_n). \tag{2.12}$$

Relation (2.12) allows us to establish the asymptotic behavior of the CMERW via an extensive use of the strong law of large numbers and the central limit theorem for multi-dimensional martingales [22], [31], [42], [73].
The chapter is organized as follows. The main results are given in Section 2.2. We first investigate the diffusive regime $p < p_d$ and we establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for the CMERW. The asymptotic normality of the CMERW, properly normalized, is also provided. Next, we prove similar results in the critical regime $p = p_d$. Finally, we establish a strong limit theorem in the superdiffusive regime $p > p_d$. Our martinagle approach is described in Section 2.3 while all technical proofs are postponed to Sections 2.4 and 2.5.

2.2 – Main results

2.2.1 – The diffusive regime

Our first result deals with the strong law of large numbers for the CMERW in the diffusive regime where $0 \leq p < p_d$. The following strong law for the CMERW will be deduced as a simple consequence of the strong law for $(\mathbf{S}_n)$.

Theorem 2.1. *We have the almost sure convergence*

$$\lim_{n \to \infty} \frac{1}{n} \mathbf{G}_n = 0 \quad a.s. \tag{2.13}$$

Remark 2.2. *For any $\alpha > 1/2$, we have the more precise result*

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \mathbf{G}_n = 0 \quad a.s.$$

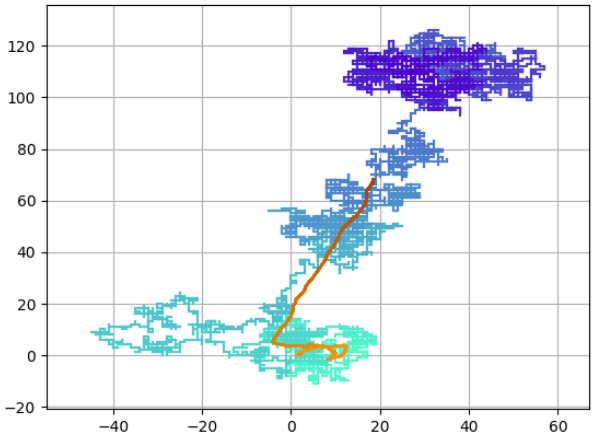


Figure 2.1: The 2-dimensional ERW in blue and the CMERW in red, for $n = 10^4$ steps and a diffusive memory parameter $p = 1/2$.

The almost sure rates of convergence for CMERW are as follows.

Theorem 2.3. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2} \mathbf{G}_k \mathbf{G}_k^T = \frac{2}{3(1-2a)(2-a)d} \mathbf{I}_d \quad a.s. \tag{2.14}$$

where $\mathbf{I}_d$ *stands for the identity matrix of order* d*. In particular,*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\mathbf{G}_k\|^2}{k^2} = \frac{2}{3(1-2a)(2-a)} \quad a.s. \tag{2.15}$$

Moreover, we have the upper-bound in the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|\mathbf{G}_n\|^2}{2n \log\log n} \leq \frac{\left(\sqrt{3}+\sqrt{1-2a}\right)^2}{3(a+1)^2(1-2a)d} \quad a.s. \tag{2.16}$$

We are now interested in the asymptotic normality of the CMERW.

Theorem 2.4. *We have the asymptotic normality*

$$\frac{1}{\sqrt{n}} \mathbf{G}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{2}{3(1-2a)(2-a)d} \mathbf{I}_d\right). \tag{2.17}$$

Remark 2.5. *It is possible to show that the following convergence in* $D([0,\infty[)$ *holds*

$$\left(\frac{\mathbf{G}_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\right) \Longrightarrow \left(\mathcal{G}_t,\ t \geq 0\right)$$

where $\left(\mathcal{G}_t,\ t \geq 0\right)$ *is a real-valued centered Gaussian process starting from the origin with covariance*

$$\mathbb{E}[\mathcal{G}_s \mathcal{G}_t] = \frac{s}{d(a+1)(2-a)} \left(\frac{1}{1-2a} \left(\frac{t}{s}\right)^a - \frac{1}{3}\left(\frac{s}{t}\right)\right) \mathbf{I}_d$$

for $0 < s \leq t$*.*

Remark 2.6. *One can observe from Theorem 3.3 in [9] that the ratio of the asymptotic variances between the CMERW and the ERW is given by*

$$R(a) = \frac{2}{3(2-a)}.$$

In the diffusive regime, this ratio lies between $2/9$ *and* $4/9$ *and it is always smaller than* 1*, as one can see in Figure 2.1. Moreover, in the special case where the elephant moves in one of the* $2d$ *directions with the same probability* $p = 1/2d < p_d$*, it follows from* (2.6) *that the fundamental parameter* $a = 0$*. Consequently, we deduce from* (2.17) *that*

$$\frac{1}{\sqrt{n}}\mathbf{G}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3d}\mathbf{I}_d\Big).$$

We find again the asymptotic normality (2.3) *where the mean value* $\mu = 0$ *and the covariance matrix* $\Gamma = \frac{1}{d}\mathbf{I}_d$*, that is the result in the simple symmetric case.*

Remark 2.7. *The convergence* (2.17) *can also be obtained using the functional result from Bertenghi [13], given here in Theorem 1.14. More precisely, we observe that*

$$\frac{\mathbf{G}_n}{\sqrt{n}} = \int_0^1 \frac{\mathbf{S}_{\lfloor nt \rfloor}}{\sqrt{n}} \mathrm{d}t.$$

Consequently, $\mathbf{G}_n/\sqrt{n}$ *is a continuous function of* $\mathbf{S}_{\lfloor nt \rfloor}/\sqrt{n}$ *in* $D([0,1])$*. Hence, the functional distribution from Theorem 1.14 gives us that*

$$\frac{\mathbf{G}_n}{\sqrt{n}} = \int_0^1 \frac{\mathbf{S}_{\lfloor nt \rfloor}}{\sqrt{n}} \mathrm{d}t \xrightarrow[n\to\infty]{\mathcal{L}} \int_0^1 \mathbf{W}_t \mathrm{d}t.$$

2.2.2 – The critical regime

Hereafter, we investigate the critical regime where the memory parameter $p = p_d$.

Theorem 2.8. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}\log n}\mathbf{G}_n = 0 \quad a.s. \tag{2.18}$$

Remark 2.9. *For any* $\alpha > 1/2$*, we have the more precise result*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}(\log n)^\alpha}\mathbf{G}_n = 0 \quad a.s.$$

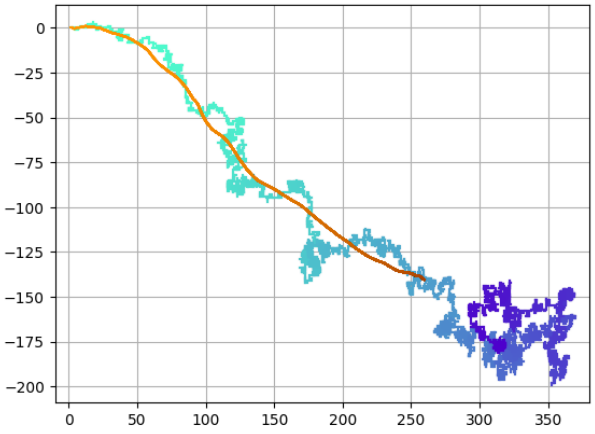


Figure 2.2: The 2-dimensional ERW in blue and the CMERW hull in red, for $n = 10^4$ steps and the critical memory parameter $p = 5/8$.

The almost sure rates of convergence for the CMERW are as follows.

Theorem 2.10. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{1}{(k\log k)^2} \mathbf{G}_k \mathbf{G}_k^T = \frac{4}{9d} \mathbf{I}_d \quad a.s. \tag{2.19}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{\|\mathbf{G}_k\|^2}{(k\log k)^2} = \frac{4}{9} \quad a.s. \tag{2.20}$$

Moreover, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|\mathbf{G}_n\|^2}{2n\log n \log\log\log n} = \frac{4}{9d} \quad a.s. \tag{2.21}$$

Our next result concerns the asymptotic normality of the CMERW.

Theorem 2.11. *We have the asymptotic normality*

$$\frac{1}{\sqrt{n\log n}} \mathbf{G}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{4}{9d}\mathbf{I}_d\right). \tag{2.22}$$

Remark 2.12. *In the critical regime, the ratio of the asymptotic variances between the CMERW and the ERW is* $4/9$.

2.2.3 – The superdiffusive regime

Finally, we focus our attention on the superdiffusive regime where $p > p_d$. The almost sure convergence of $(\mathbf{S}_n)$, properly normalized by n^a, yields the following strong limit theorem for the CMERW.

Theorem 2.13. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{1}{n^a} \mathbf{G}_n = \mathbf{G} \quad a.s. \tag{2.23}$$

where the limiting value G is a non-degenerate random vector of $\mathbb{R}^d$. We also have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\left[\left\|\frac{1}{n^a}\mathbf{G}_n - \mathbf{G}\right\|^2\right] = 0. \tag{2.24}$$

Remark 2.14. *The limiting value $\mathbf{G}$ is in fact completely related to the one of the MERW $\mathbf{L}_d$ as*

$$\mathbf{G} = \frac{1}{a+1}\mathbf{L}_d.$$

Remark 2.15. *The expected value of $\mathbf{G}$ is zero and its covariance matrix is given by*

$$\mathbb{E}\left[\mathbf{G}\mathbf{G}^T\right] = \frac{1}{d(a+1)^2(2a-1)^2\Gamma(2a-1)}\mathbf{I}_d.$$

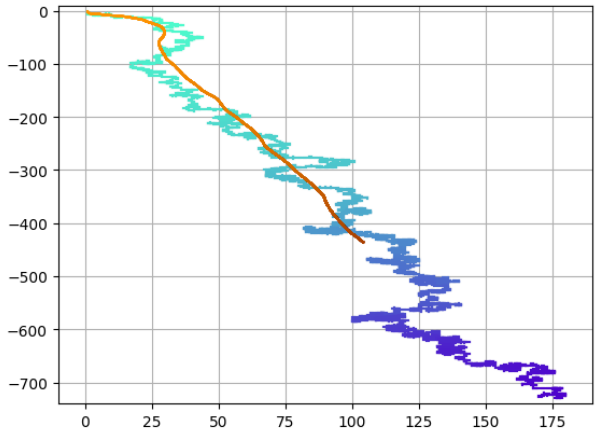


Figure 2.3: The 2-dimensional ERW in blue, the CMERW in red, for $n = 10^4$ steps and a superdiffusive memory parameter $p = 3/4$.

2.3 – A multi-dimensional martingale approach

We already saw from (2.12) that the CMERW can be rewritten as

$$\mathbf{G}_n = \frac{1}{n}(b_n\mathbf{M}_n - \mathbf{N}_n).$$

In order to investigate the asymptotic behavior of $(\mathbf{G}_n)$, we introduce the multi-dimensional martingale $(\mathcal{M}_n)$ defined by

$$\mathcal{M}_n = \begin{pmatrix} \mathbf{M}_n \\ \mathbf{N}_n \end{pmatrix} \tag{2.25}$$

where $(\mathbf{M}_n)$ and $(\mathbf{N}_n)$ are the two locally square-integrable martingales given by (2.7) and (2.11). The main difficulty we face here is that the predictable quadratic variations of $(\mathbf{M}_n)$ and $(\mathbf{N}_n)$ increase to infinity with two different speeds. A matrix normalization is necessary to establish the asymptotic behavior of the

CMERW. Let (V_n) be the sequence of positive definite diagonal matrices of order $2d$ given by

$$V_n = \frac{1}{n\sqrt{n}} \begin{pmatrix} b_n & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{I}_d \tag{2.26}$$

where $A \otimes B$ stands for the Kronecker product of the matrices A and B.

Lemma 2.16. *The sequence $(\mathcal{M}_n)$ is a locally square-integrable martingale of $\mathbb{R}^{2d}$. Its predictable quadratic variation $\langle \mathcal{M} \rangle_n$ satisfies in the diffusive regime where $a < 1/2$,*

$$\lim_{n\to\infty} V_n \langle \mathcal{M} \rangle_n V_n^T = V \quad a.s. \tag{2.27}$$

where the limiting matrix

$$V = \frac{1}{d(a+1)^2} \begin{pmatrix} \frac{1}{1-2a} & \frac{1}{2-a} \\ \frac{1}{2-a} & \frac{1}{3} \end{pmatrix} \otimes \mathbf{I}_d. \tag{2.28}$$

Remark 2.17. *Via the same lines as in the proof of Lemma 2.16, we find that in the critical regime $a = 1/2$, the sequence of normalization matrices (V_n) has to be replaced by*

$$W_n = \frac{1}{n\sqrt{n \log n}} \begin{pmatrix} b_n & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{I}_d. \tag{2.29}$$

Moreover, the limiting matrix in (2.27) *must be changed by*

$$W = \frac{4}{9d} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{I}_d. \tag{2.30}$$

Proof of Lemma 2.16. The increments of the ERW are bounded by 1, that is for any time $n \geq 1$, $\|\mathbf{X}_n\| = 1$. Hence, it follows from (1.1) that $\|\mathbf{S}_n\| \leq n$ and $\|\mathbf{G}_n\| \leq n$ which imply that $\|\mathbf{M}_n\| \leq na_n$ and $\|\mathbf{N}_n\| \leq na_nb_n + n^2$. We already saw in Section 2.1 that $(\mathcal{M}_n)$ is a locally square-integrable martingale. Denote $\Delta\mathbf{M}_n = \mathbf{M}_n - \mathbf{M}_{n-1}$, and similarly for other processes. It follows from (2.7), (2.8) and (2.11) that the predictable quadratic variation associated with $(\mathcal{M}_n)$ is the square matrix of order $2d$ given, for all $n \geq 1$, by

$$\langle \mathcal{M} \rangle_n = \sum_{k=1}^{n} \mathbb{E}\left[\begin{pmatrix} \Delta\mathbf{M}_k \\ \Delta\mathbf{N}_k \end{pmatrix} \begin{pmatrix} \Delta\mathbf{M}_k \\ \Delta\mathbf{N}_k \end{pmatrix}^T \Big| \mathcal{F}_{k-1} \right] = \sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes \mathbb{E}\left[\varepsilon_k \varepsilon_k^T \big| \mathcal{F}_{k-1} \right]. \tag{2.31}$$

Moreover, we deduce from formulas (A.7) and (B.3) in [9] that for all $n \geq 1$,

$$\mathbb{E}\left[\varepsilon_{n+1} \varepsilon_{n+1}^T \mid \mathcal{F}_n \right] = \frac{1}{d}\mathbf{I}_d + a\Big(\frac{1}{n}\mathbf{\Sigma}_n - \frac{1}{d}\mathbf{I}_d\Big) - \Big(\frac{a}{n}\Big)^2 \mathbf{S}_n \mathbf{S}_n^T \quad \text{a.s.} \tag{2.32}$$

where $\mathbf{\Sigma}_n$ is a random positive definite matrix of order d satisfying

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{\Sigma}_n = \frac{1}{d}\mathbf{I}_d \quad \text{a.s.} \tag{2.33}$$

Consequently, we obtain from (2.32) that

$$\langle \mathcal{M} \rangle_n = \frac{1}{d} \sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes \mathbf{I}_d + a \sum_{k=1}^{n-1} a_{k+1}^2 \begin{pmatrix} 1 & b_k \\ b_k & b_k^2 \end{pmatrix} \otimes \Big(\frac{1}{k} \mathbf{\Sigma}_k - \frac{1}{d} \mathbf{I}_d \Big) - \xi_n \tag{2.34}$$

where

$$\xi_n = a^2 \sum_{k=1}^{n-1} \Big(\frac{a_{k+1}}{k} \Big)^2 \begin{pmatrix} 1 & b_k \\ b_k & b_k^2 \end{pmatrix} \otimes \mathbf{S}_k \mathbf{S}_k^T.$$

According to Theorem 3.1 in [9], the remainder ξ_n plays a negligible role as

$$\lim_{n \to \infty} \frac{\mathbf{S}_n}{n} = 0 \quad \text{a.s.} \tag{2.35}$$

Hereafter, it is not hard to see that

$$V_n \Big(\sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes \mathbf{I}_d \Big) V_n^T = \frac{1}{n^3} \begin{pmatrix} b_n^2 \sum_{k=1}^{n} a_k^2 & b_n \sum_{k=1}^{n} a_k^2 b_{k-1} \\ b_n \sum_{k=1}^{n} a_k^2 b_{k-1} & \sum_{k=1}^{n} a_k^2 b_{k-1}^2 \end{pmatrix} \otimes \mathbf{I}_d.$$

Furthermore, from a well-known property of the Euler Gamma function, we have

$$\lim_{n \to \infty} \frac{\Gamma(n+a)}{\Gamma(n) n^a} = 1. \tag{2.36}$$

Hence, we obtain from (2.5), (2.10) and (2.36) that

$$\lim_{n \to \infty} n^a a_n = \Gamma(a+1) \quad \text{and} \quad \lim_{n \to \infty} \frac{b_n}{n^{a+1}} = \frac{1}{\Gamma(a+2)}. \tag{2.37}$$

Consequently, as soon as $a < 1/2$, we immediately find from (2.37) that

$$\begin{aligned} \lim_{n \to \infty} \frac{b_n^2}{n^3} \sum_{k=1}^{n} a_k^2 &= \frac{1}{(1-2a)(a+1)^2}, \\ \lim_{n \to \infty} \frac{b_n}{n^3} \sum_{k=1}^{n} a_k^2 b_{k-1} &= \frac{1}{(2-a)(a+1)^2}, \\ \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} a_k^2 b_{k-1}^2 &= \frac{1}{3(a+1)^2}. \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} V_n \Big(\sum_{k=1}^{n} a_k^2 \begin{pmatrix} 1 & b_{k-1} \\ b_{k-1} & b_{k-1}^2 \end{pmatrix} \otimes \mathbf{I}_d \Big) V_n^T = \frac{1}{(a+1)^2} \begin{pmatrix} \frac{1}{1-2a} & \frac{1}{2-a} \\ \frac{1}{2-a} & \frac{1}{3} \end{pmatrix} \otimes \mathbf{I}_d. \tag{2.38}$$

Finally, it follows from the combinaition of (2.33), (2.34), (2.35) and (2.38) that

$$\lim_{n \to \infty} V_n \langle \mathcal{M} \rangle_n V_n^T = \frac{1}{d(a+1)^2} \begin{pmatrix} \frac{1}{1-2a} & \frac{1}{2-a} \\ \frac{1}{2-a} & \frac{1}{3} \end{pmatrix} \otimes \mathbf{I}_d. \quad \text{a.s.} \tag{2.39}$$

which is exactly what we wanted to prove.

■

2.4 – Proofs of the almost sure convergence results

2.4.1 – The diffusive regime.

Proof of Theorem 2.1. We already saw from Theorem 3.1 in [9] that

$$\lim_{n\to\infty} \frac{\mathbf{S}_n}{n} = 0 \quad \text{a.s.} \tag{2.40}$$

Consequently, the almost sure convergence (2.13) immediately follows from (2.40) together with the Toeplitz lemma given e.g. by Lemma 2.2.13 in [31].

■

Proof of Theorem 2.3. Our goal is to check that all the hypotheses of Theorem A.8 are satisfied. Thanks to Lemma 2.16, hypothesis (H.1) holds almost surely. In order to verify that Lindeberg's condition (H.2) is satisfied, we have from (2.25) together with (2.7), (2.11) and V_n given by (2.26) that for all $1 \leq k \leq n$

$$V_n \Delta \mathcal{M}_k = \frac{a_k}{n\sqrt{n}} \begin{pmatrix} b_n \varepsilon_k \\ b_{k-1} \varepsilon_k \end{pmatrix},$$

which implies that

$$\|V_n \Delta \mathcal{M}_k\|^2 \leq \frac{2a_k^2 b_n^2}{n^3} \|\varepsilon_k\|^2. \tag{2.41}$$

Consequently, we obtain that for all $\varepsilon > 0$,

$$\begin{aligned} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] &\leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big], \\ &\leq \frac{4b_n^4}{\varepsilon^2 n^6} \sum_{k=1}^{n} a_k^4 \mathbb{E}\big[\|\varepsilon_k\|^4 \mid \mathcal{F}_{k-1}\big], \\ &\leq \frac{4b_n^4}{\varepsilon^2 n^6} \sup_{1\leq k\leq n} \mathbb{E}\big[\|\varepsilon_k\|^4 \mid \mathcal{F}_{k-1}\big] \sum_{k=1}^{n} a_k^4. \end{aligned} \tag{2.42}$$

However, it follows from the right-hand side of formula (4.11) in [9] that

$$\sup_{1\leq k\leq n} \mathbb{E}\big[\|\varepsilon_k\|^4 \mid \mathcal{F}_{k-1}\big] \leq \frac{4}{3} \quad \text{a.s.} \tag{2.43}$$

Therefore, we infer from (2.42) that for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] \leq \frac{16\, b_n^4}{3\, \varepsilon^2 n^6} \sum_{k=1}^{n} a_k^4 \quad \text{a.s.} \tag{2.44}$$

Moreover, we have from (2.37) that

$$b_n^4 \sum_{k=1}^{n} a_k^4 = O(n^5). \tag{2.45}$$

Consequently, (2.44) together with (2.45) ensure that Lindeberg's condition (H.2) holds almost surely, that is for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] = 0 \quad \text{a.s.} \tag{2.46}$$

We will now check that condition (H.3) is satisfied in the special case $\beta = 2$, that is

$$\sum_{n=1}^{\infty} \frac{1}{\big(\log(\det V_n^{-1})^2\big)^2} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_n\|^4 \mid \mathcal{F}_{n-1}\big] < \infty \quad \text{a.s.} \tag{2.47}$$

We have from (2.26) that

$$\det V_n^{-1} = \Big(\frac{n^{3/2}}{b_n}\Big)^d. \tag{2.48}$$

Hence, we find from (2.37) and (2.48) that

$$\lim_{n\to\infty} \frac{\log(\det V_n^{-1})^2}{\log n} = d(1-2a). \tag{2.49}$$

Consequently, we can replace $\log(\det V_n^{-1})^2$ by $\log n$ in (2.47). Hereafter, we obtain from (2.41) and (2.43) that

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{(\log n)^2} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_n\|^4 \mid \mathcal{F}_{n-1}\big] &= O\Big(\sum_{n=1}^{\infty} \frac{1}{(\log n)^2} \frac{a_n^4 b_n^4}{n^6} \mathbb{E}\big[\|\varepsilon_n\|^4 \mid \mathcal{F}_{n-1}\big]\Big), \\ &= O\Big(\sum_{n=1}^{\infty} \frac{1}{(\log n)^2} \frac{a_n^4 b_n^4}{n^6}\Big). \end{aligned} \tag{2.50}$$

However, we have from (2.37) that

$$\lim_{n\to\infty} \frac{a_n^4 b_n^4}{n^4} = \frac{1}{(a+1)^4}. \tag{2.51}$$

Therefore, (2.50) together with (2.51) immediately lead to (2.47). We are now in a position to apply the quadratic strong law given by Theorem A.8. We have from (A.11) and (2.49) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) V_k \mathcal{M}_k \mathcal{M}_k^T V_k^T = d(1-2a)V \quad \text{a.s.} \tag{2.52}$$

where the limiting matrix V is given by (2.28). However, it follows from (2.9), (2.25) and (2.26) that

$$\frac{1}{\sqrt{n}} \mathbf{G}_n = \mathbf{v}^T V_n \mathcal{M}_n \quad \text{where} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \mathbf{I}_d. \tag{2.53}$$

Consequently, we deduce from (2.52) and (2.53) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) \frac{1}{k} \mathbf{G}_k \mathbf{G}_k^T = d(1-2a)\mathbf{v}^T V \mathbf{v} \quad \text{a.s.} \tag{2.54}$$

Furthermore, we obtain from (2.48) and (2.37) that

$$\lim_{n\to\infty} n\Big(\frac{(\det V_n)^2-(\det V_{n+1})^2}{(\det V_n)^2}\Big) = d(1-2a).$$

Hence, (2.54) clearly leads to convergence (2.14),

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k^2}\mathbf{G}_k\mathbf{G}_k^T = \mathbf{v}^T V\mathbf{v} = \frac{2}{3(1-2a)(2-a)d}\mathbf{I}_d \quad \text{a.s.} \tag{2.55}$$

By taking the trace on both sides of (2.55), we also obtain that

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{\|\mathbf{G}_k\|^2}{k^2} = \frac{2}{3(1-2a)(2-a)} \quad \text{a.s.} \tag{2.56}$$

Finally, we shall proceed to the proof of the upper-bound (2.16) in the law of iterated logarithm. Denote

$$\tau_n = \sum_{k=1}^{n} a_k^2 b_{k-1}^2. \tag{2.57}$$

We already saw from (2.51) that $a_n^4 b_{n-1}^4 \tau_n^{-2}$ is equivalent to $9n^{-2}$. It implies that

$$\sum_{n=1}^{+\infty}\frac{a_n^4 b_{n-1}^4}{\tau_n^2} < \infty. \tag{2.58}$$

Moreover, we have from (2.33), (2.34), (2.35) and (2.57) that

$$\lim_{n\to\infty}\frac{1}{\tau_n}\langle N\rangle_n = \frac{1}{d}\mathbf{I}_d \quad \text{a.s.}$$

Consequently, we deduce from the law of iterated logarithm for martingales due to Stout [72], see Theorem A.4, that $(\mathbf{N}_n)$ satisfies for any vector $\mathbf{u}\in\mathbb{R}^d$,

$$\begin{aligned}\limsup_{n\to\infty}\Big(\frac{1}{2\tau_n\log\log\tau_n}\Big)^{1/2}\langle\mathbf{u},\mathbf{N}_n\rangle &= -\liminf_{n\to\infty}\Big(\frac{1}{2\tau_n\log\log\tau_n}\Big)^{1/2}\langle\mathbf{u},\mathbf{N}_n\rangle \\ &= \frac{1}{\sqrt{d}}\|\mathbf{u}\| \quad \text{a.s.}\end{aligned} \tag{2.59}$$

However, since τ_n is equivalent to $n^3/3(a+1)^2$, (2.59) immediately leads to

$$\begin{aligned}\limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle\mathbf{u},\mathbf{N}_n\rangle &= -\liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle\mathbf{u},\mathbf{N}_n\rangle \\ &= \frac{1}{\sqrt{3d}(a+1)}\|\mathbf{u}\| \quad \text{a.s.}\end{aligned} \tag{2.60}$$

Furthermore, it was already shown by formula (5.17) in [9] that for any vector $\mathbf{u}\in\mathbb{R}^d$,

$$\begin{aligned}\limsup_{n\to\infty}\Big(\frac{n^{2a}}{2n\log\log n}\Big)^{1/2}\langle\mathbf{u},\mathbf{M}_n\rangle &= -\liminf_{n\to\infty}\Big(\frac{n^{2a}}{2n\log\log n}\Big)^{1/2}\langle\mathbf{u},\mathbf{M}_n\rangle \\ &= \frac{\Gamma(a+1)}{\sqrt{d(1-2a)}}\|\mathbf{u}\| \quad \text{a.s.}\end{aligned} \tag{2.61}$$

Therefore, we deduce from (2.12) and (2.37) together with (2.60) and (2.61) that for any vector $\mathbf{u}$ of $\mathbb{R}^d$,

$$\limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\langle \mathbf{u}, \mathbf{G}_n\rangle = \limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle \mathbf{u}, b_n\mathbf{M}_n - \mathbf{N}_n\rangle$$

$$\leq \limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle \mathbf{u}, b_n\mathbf{M}_n\rangle \tag{2.62}$$

$$+ \limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle \mathbf{u}, -\mathbf{N}_n\rangle$$

$$\leq \limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle \mathbf{u}, b_n\mathbf{M}_n\rangle \tag{2.63}$$

$$- \liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle \mathbf{u}, \mathbf{N}_n\rangle$$

$$\leq \frac{\|\mathbf{u}\|}{\sqrt{d}(a+1)}\Big(\frac{1}{\sqrt{1-2a}}+\frac{1}{\sqrt{3}}\Big) \quad \text{a.s.} \tag{2.64}$$

By the same token, we also find that for any vector u of $\mathbb{R}^d$,

$$\liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\langle \mathbf{u}, \mathbf{G}_n\rangle = \liminf_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)^{1/2}\frac{1}{n}\langle \mathbf{u}, b_n\mathbf{M}_n - \mathbf{N}_n\rangle \tag{2.65}$$

$$\geq -\frac{\|\mathbf{u}\|}{\sqrt{d}(a+1)}\Big(\frac{1}{\sqrt{1-2a}}+\frac{1}{\sqrt{3}}\Big) \quad \text{a.s.} \tag{2.66}$$

Consequently, we obtain from (2.64) and (2.66) that for any vector $\mathbf{u}$ of $\mathbb{R}^d$,

$$\limsup_{n\to\infty}\Big(\frac{1}{2n\log\log n}\Big)\langle \mathbf{u}, \mathbf{G}_n\rangle^2 \leq \frac{\|\mathbf{u}\|^2}{d(a+1)^2}\Big(\frac{1}{\sqrt{1-2a}}+\frac{1}{\sqrt{3}}\Big)^2 \quad \text{a.s.} \tag{2.67}$$

One can observe that the upper-bound in (2.67) is close to the optimal bound

$$(\mathbf{vu})^T V\mathbf{vu} = \frac{2\|\mathbf{u}\|^2}{3(1-2a)(2-a)d}.$$

Finally, by taking all rational points on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, the bound in (2.67) holds simultaneously for all of them, which implies that

$$\limsup_{n\to\infty}\frac{\|\mathbf{G}_n\|^2}{2n\log\log n} \leq \sup_{\mathbf{u}\in\mathbb{Q}^d\cap\mathbb{S}^{d-1}}\limsup_{n\to\infty}\frac{\langle \mathbf{u}, \mathbf{G}_n\rangle^2}{2n\log\log n} \leq \frac{\big(\sqrt{3}+\sqrt{1-2a}\big)^2}{3(a+1)^2(1-2a)d} \quad \text{a.s.}$$

completing the proof of Theorem 2.3.

■

2.4.2 – The critical regime.

Proof of Theorem 2.8. We have from Theorem 3.4 in [9] that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}\log n}\mathbf{S}_n = 0 \quad \text{a.s.} \tag{2.68}$$

Hence, (2.18) clearly follows from (2.68) together with the Toeplitz lemma.

■

Proof of Theorem 2.10. The proof of the quadratic strong law (2.19) is left to the reader as it follows essentially the same lines as that of (2.14). The only minor change is that the matrix V_n has to be replaced by the matrix W_n defined in (2.30). We shall now proceed to the proof of the law of iterated logarithm given by (2.21). On the one hand, it follows from (2.60) with $a = 1/2$ that for any vector $\mathbf{u} \in \mathbb{R}^d$,

$$\begin{aligned}\limsup_{n\to\infty} \Big(\frac{1}{2n\log\log n}\Big)^{1/2} \frac{1}{n}\langle \mathbf{u}, \mathbf{N}_n\rangle &= -\liminf_{n\to\infty} \Big(\frac{1}{2n\log\log n}\Big)^{1/2} \frac{1}{n}\langle \mathbf{u}, \mathbf{N}_n\rangle \\ &= \frac{2}{3\sqrt{3}}\|\mathbf{u}\| \quad \text{a.s.}\end{aligned}$$

which immediately leads to

$$\limsup_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} \frac{1}{n}\langle \mathbf{u}, \mathbf{N}_n\rangle = 0 \quad \text{a.s.}$$

On the other hand, we obtain from the law of iterated logarithm for $\mathbf{S}_n$ given in Theorem 3.5 of [9] that for any vector $u \in \mathbb{R}^d$,

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2}\langle \mathbf{u}, \mathbf{G}_n\rangle \\ &\quad= \limsup_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} \frac{1}{n}\langle \mathbf{u}, b_n\mathbf{M}_n - \mathbf{N}_n\rangle \\ &\quad= \limsup_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} \frac{1}{n}\langle \mathbf{u}, b_n\mathbf{M}_n\rangle \\ &\quad= \limsup_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} \frac{1}{n}\langle \mathbf{u}, a_n b_n\mathbf{S}_n\rangle \\ &\quad= \limsup_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} \frac{2}{3}\langle \mathbf{u}, \mathbf{S}_n\rangle \\ &\quad= -\liminf_{n\to\infty} \Big(\frac{1}{2n\log n\log\log\log n}\Big)^{1/2} \frac{2}{3}\langle \mathbf{u}, \mathbf{S}_n\rangle \\ &\quad= \frac{2}{3\sqrt{d}}\|u\| \quad \text{a.s.}\end{aligned} \tag{2.69}$$

Hence, we clearly deduce from (2.69) that for any vector $u \in \mathbb{R}^d$,

$$\limsup_{n\to\infty} \frac{1}{2n\log n\log\log\log n}\langle u, \mathbf{G}_n\rangle^2 = \frac{4}{9d}\|\mathbf{u}\|^2 \quad \text{a.s.} \tag{2.70}$$

By taking all rational points on the unit sphere $\mathbb{S}_{d-1}$ in $\mathbb{R}^d$, the bound in (2.70) holds simultaneously for all of them, which implies that

$$\limsup_{n\to\infty} \frac{\|\mathbf{G}_n\|^2}{2n\log n\log\log\log n} \leq \sup_{u\in\mathbb{Q}^d\cap\mathbb{S}^{d-1}} \limsup_{n\to\infty} \frac{\langle u, \mathbf{G}_n\rangle^2}{2n\log n\log\log\log n} = \frac{4}{9} \quad \text{a.s.}$$

In addition, for any single $\mathbf{u} \in \mathbb{S}^{d-1}$, we also obtain the reverse inequality

$$\limsup_{n\to\infty} \frac{\|\mathbf{G}_n\|^2}{2n\log n\log\log\log n} \geq \limsup_{n\to\infty} \frac{\langle u, \mathbf{G}_n\rangle^2}{2n\log n\log\log\log n} = \frac{4}{9} \quad \text{a.s.}$$

It immediately leads to (2.21) which achieves the proof of Theorem 2.10.

■

2.4.3 – The superdiffusive regime.

Proof of Theorem 2.13. It follows from Theorem 3.7 in [9] that

$$\lim_{n\to\infty} \frac{1}{n^a}\mathbf{S}_n = \mathbf{L}_d \quad \text{a.s.} \tag{2.71}$$

where the limiting value L is a non-degenerate random vector of $\mathbb{R}^d$. Hence, (2.71) together with the Toeplitz lemma imply (2.23) where the limiting value

$$\mathbf{G} = \frac{1}{a+1}\mathbf{L}_d.$$

Moreover, we have from (2.12) that

$$\begin{aligned}\mathbb{E}\Big[\Big\|\frac{1}{n^a}\mathbf{G}_n - \mathbf{G}\Big\|^2\Big] &= \mathbb{E}\Big[\Big\|\frac{1}{n^{a+1}}(b_n\mathbf{M}_n - \mathbf{N}_n) - \mathbf{G}\Big\|^2\Big],\\ &\leq 2\mathbb{E}\Big[\Big\|\frac{a_nb_n}{n^{a+1}}\mathbf{S}_n - \mathbf{G}\Big\|^2\Big] + 2\mathbb{E}\Big[\Big\|\frac{1}{n^{a+1}}\mathbf{N}_n\Big\|^2\Big].\end{aligned}$$

On the one hand, we already saw from (2.37) that

$$\lim_{n\to\infty} \frac{a_nb_n}{n} = \frac{1}{a+1}.$$

Consequently, we deduce from the mean square convergence (3.12) in [9] that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big\|\frac{a_nb_n}{n^{a+1}}\mathbf{S}_n - \mathbf{G}\Big\|^2\Big] = 0. \tag{2.72}$$

On the other hand, $\mathbb{E}[\|\mathbf{N}_n\|^2] = \mathbb{E}[\mathrm{Tr}\langle\mathbf{N}_n\rangle)] \leq \tau_n$ where τ_n is given by (2.57). Since τ_n is equivalent to $n^3/3(a+1)^2$ and $a > 1/2$, it is not hard to see that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big\|\frac{1}{n^{a+1}}\mathbf{N}_n\Big\|^2\Big] = 0. \tag{2.73}$$

Finally, we obtain (2.24) from (2.72) and (2.73), completing the proof of Theorem 2.13.

■

2.5 – Proofs of the asymptotic normality results

2.5.1 – The diffusive regime.

Proof of Theorem 2.4. On the one hand, we already saw from (2.53) that

$$\frac{1}{\sqrt{n}}\mathbf{G}_n = \mathbf{v}^T V_n \mathcal{M}_n \quad \text{where} \quad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \mathbf{I}_d.$$

On the other hand, we deduce from (2.27) and (2.46) that the two conditions $(\mathrm{H}.1)$ and $(\mathrm{H}.2)$ of Theorem A.7 are satisfied. Consequently, we obtain that

$$\frac{1}{\sqrt{n}}\mathbf{G}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, \mathbf{v}^T\mathbf{v}\big)$$

where the matrix V is given by (2.28). It clearly leads to (2.17) as

$$\mathbf{v}^T V \mathbf{v} = \frac{2}{3(1-2a)(2-a)d}\mathbf{I}_d.$$

■

2.5.2 – The critical regime.

Proof of Theorem 2.11. The proof follows exactly the same lines as that of Theorem 2.4 replacing V_n by W_n. The details are left to the reader.

■

3 The linearly Reinforced Elephant Random Walk

This chapter presents the results of [57] :
LAULIN, L. New insights on the Reinforced Elephant Random Walk using a martingale approach. *Journal of Statistical Physics 186* (2021).

3.1 – Introduction

Reinforced random walks have generated much interest in the recent years with the focus being mainly on graphs, edge or vertex reinforced random walk, see for example [52] or [66] for a comprehensive and extensive overview on the subject, as well as the recent contributions [2, 15].
In this chapter, we investigate a special case of reinforced random walks in connection with the Elephant Random Walk (ERW).
The reinforcement we are interested in here acts on the memory. As it was done in [5], we can write the $(n+1)$-th increment X_{n+1} under the form

$$X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}}. \tag{3.1}$$

In the case of the ERW, we had $\alpha_{n+1} \sim \mathcal{R}(p)$ and $\beta_{n+1} \sim \mathcal{U}\{1,\dots,n\}$. The major change for the RERW is that the distribution of β_n is no longer uniform but depends on a reinforcement parameter $c \geq 0$. Whereas the original ERW (when $c = 0$) had a Markovian property, providing that one considers both the position

of the ERW and the time, the reinforcement implies that the RERW (when $c > 0$) is strongly non-Markovian : the position of the RERW at time n depends on all of the steps performed up to time n as well as on their weight. Precisely, the conditional distribution of X_{n+1} no longer depends on the position at time n, but on the "weight" of each previous instants and the "weighted" random walk associated.

Very recently, Baur [2] studied the asymptotic behavior of the RERW using a Pólya-type urns approach. He established interesting functional limit theorems thanks to the seminal work of Janson [48]. Our strategy here is different as it relies on a martingale approach. On the one hand, we prove new almost sure convergence results such as strong laws of large numbers, laws of iterated logarithms, as well as quadratic strong laws. On the other hand, we give an alternative method to obtain the functional limit theorems without making use of the results from [48]. The martingale approach we propose fulfills these two objectives. The main strength of our approach is that calculations are self-contained and rather easy to follow. It should also be noted that using the martingale theory is sufficient on its own to obtain all the results presented in this chapter. Moreover, we strongly believe that this approach could be used to study several variations of the ERW with reinforced memory or more generally reinforced random walks, as done in Chapter 4.

This chapter is organized as follows: The model of reinforced memory is presented in Section 3.2 while the main results are given in Section 3.3. We first investigate the diffusive regime and we establish the strong law of large numbers, the law of iterated logarithm and the quadratic strong law for the RERW. The functional central limit theorem is also provided. Next, we prove similar results in the critical regime. Finally, we establish a strong limit theorem in the superdiffusive regime. Our martingale approach is described in Section 3.4. Finally, all technical proofs are postponed to Sections 3.6–3.7.

3.2 – The reinforced elephant random walk

We assume in all the sequel that the memory parameter $p \neq 1/2$ since the particular case $p = 1/2$ reduces to the standard random walk. Let $\mathcal{F}_n$ be the natural σ-algebra up to time n, $\mathcal{F}_n = \sigma(X_1, \ldots, X_n, \beta_1, \ldots, \beta_n)$, and denote by $\rho_n(k)$ the weight of the instant k after n steps. The ERW is associated with the special case where $\rho_n(k) = 1$ if $k \leq n$ and 0 elsewise. Adding a reinforcement of weight c, where c is a non-negative real number, implies that the weight $\rho_n(k)$ of instant k is modified as follows:

$$\rho_n(k) = \begin{cases} 0 & \text{if } k \geq n+1, \\ 1 & \text{if } k = n, \\ \rho_{n-1}(k) + c\mathbb{1}_{\beta_n = k} & \text{if } 1 \leq k < n. \end{cases}$$

Consequently, it follows from the very definition of $\rho_n(k)$ that the conditional distribution of β_{n+1} is given by, for $1 \leq k \leq n$,

$$\mathbb{P}(\beta_{n+1} = k \mid \mathcal{F}_n) = \frac{\rho_n(k)}{\sum_{j=1}^{n} \rho_n(j)} = \frac{\rho_n(k)}{(c+1)n - c}.$$

The parameter c represents the intensity of the reinforcement. The reader can notice that the case $c = 0$ corresponds to the traditional ERW, and that in this case the distribution of β_{n+1} is only dependent of the time n.

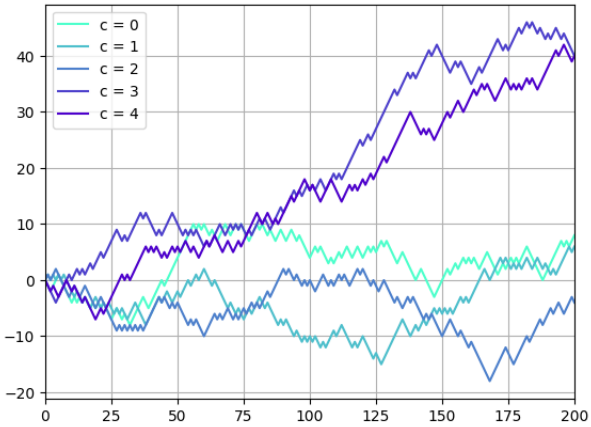


Figure 3.1: Multiple paths of the RERW depending on the reinforcement parameter c when $p = 0.35$.

Hereafter, recall that $a = 2p - 1$, such that $-1 \leq a \leq 1$. We have by the definition of X_n,

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\alpha_{n+1}]\mathbb{E}[X_{\beta_{n+1}} \mid \mathcal{F}_n] \\
&= a\mathbb{E}\Big[\sum_{k=1}^{n} X_k \mathbb{1}_{\beta_{n+1}=k} \mid \mathcal{F}_n\Big] \\
&= \frac{a}{(c+1)n - c} \sum_{k=1}^{n} X_k \rho_n(k).
\end{aligned}$$

Then, denote

$$Y_n = \sum_{k=1}^{n} X_k \rho_n(k) \tag{3.2}$$

such that $Y_n = S_n$ when $c = 0$, and

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \frac{a}{(c+1)n - c} Y_n. \tag{3.3}$$

Hence, we immediately get

$$\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = S_n + \frac{a}{(c+1)n - c} Y_n. \tag{3.4}$$

Hereafter, notice that

$$Y_{n+1} = \sum_{k=1}^{n+1} X_k \rho_{n+1}(k) = \sum_{k=1}^{n} X_k \big(\rho_n(k) + c\mathbb{1}_{\beta_{n+1}=k}\big) + X_{n+1} = Y_n + (\alpha_{n+1} + c) X_{\beta_{n+1}} \tag{3.5}$$

we obtain

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \Big(1 + \frac{a+c}{(c+1)n-c}\Big) Y_n. \tag{3.6}$$

Finally, for any $n \geq 1$ let

$$\gamma_n = 1 + \frac{a+c}{(c+1)n-c} = \frac{n+a\lambda}{n-c\lambda} \quad \text{where} \quad \lambda = \frac{1}{c+1} \tag{3.7}$$

and

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n-c\lambda)\Gamma(1+a\lambda)}{\Gamma(n+a\lambda)\Gamma(\lambda)}. \tag{3.8}$$

It follows from standard calculations on the Gamma function that

$$\lim_{n\to\infty} n^{(a+c)\lambda} a_n = \frac{\Gamma(1+a\lambda)}{\Gamma(\lambda)}. \tag{3.9}$$

Our strategy for proving asymptotic results for the reinforced elephant random walk is as follows. On the one hand, the behavior of the position S_n of the RERW is closely related to the one of the sequences (M_n) and (N_n) defined for all $n \geq 0$ by

$$M_n = a_n Y_n \quad \text{and} \quad N_n = S_n - \frac{a}{a+c} Y_n. \tag{3.10}$$

We immediately get from (3.6) and (3.8) that (M_n) is a locally square-integrable martingale adapted to $\mathcal{F}_n$. Moreover, we have from (3.3), (3.4) and (3.6) that

$$E\Big[S_{n+1} - \frac{a}{a+c} Y_{n+1} \mid \mathcal{F}_n\Big] = S_n - \frac{a}{a+c} Y_n$$

which means that (N_n) is also a locally square-integrable martingale adapted to $\mathcal{F}_n$. On the other hand, we can rewrite S_n as

$$S_n = N_n + \frac{a}{a+c} a_n^{-1} M_n \tag{3.11}$$

and equation (3.11) allows us to establish the asymptotic behavior of the RERW via an extensive use of the strong law of large numbers and the functional central limit theorem for multi-dimensional martingales [22], [31], [42], [73].

3.3 – Main results

3.3.1 – The diffusive regime

Our first result deals with the strong law of large numbers for the RERW in the diffusive regime where $p < (3-c)/4$.

Theorem 3.1. *We have the convergence*

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s. \tag{3.12}$$

The almost sure rate of convergence for RERW is as follows.

Theorem 3.2. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k^2}{k^2} = \frac{2ac+c-1}{2a+c-1} \quad a.s. \tag{3.13}$$

Remark 3.3. *In addition, we could also obtain an upper-bound for the law of iterated logarithm as it was done for the center of mass of the MERW in [11].*

Hereafter, we are interested in the distributional convergence of the RERW, which holds in the Skorokhod space $D([0,\infty[)$ of right-continuous functions with left-hand limits. The following theorem was first obtained by Baur [2, Theorem 3.2] in the case of a memory parameter equal to $(p+1)/2$.

The notation "$\Longrightarrow$" indicates convergence with respect to the Skorokhod space while "$\xrightarrow{\mathcal{L}}$" stands for convergence in distribution (or weak convergence). See [19, Chapter 3] for more details on the definition of the distributional convergence in the Skorokhod Space $D([0,\infty[)$.

Theorem 3.4. *The following convergence in $D([0,\infty[)$ holds*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big) \tag{3.14}$$

where $\big(W_t,\ t \geq 0\big)$ is a real-valued centered Gaussian process starting from the origin with covariance

$$\mathbb{E}[W_s W_t] = \frac{a(1-c^2)}{(a+c)(1-2a-c)} s\Big(\frac{t}{s}\Big)^{\lambda(a+c)} + \frac{c(a+1)}{a+c} s \tag{3.15}$$

for $0 < s \leq t$. In particular, we have

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{2ac+c-1}{2a+c-1}\right). \tag{3.16}$$

Remark 3.5. *When $c = 0$ we find again the results from [3] for the ERW*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big)$$

where $\big(W_t,\ t \geq 0\big)$ is a real-valued mean-zero Gaussian process starting from the origin and

$$\mathbb{E}[W_s W_t] = \frac{1}{1-2a} s\Big(\frac{t}{s}\Big)^{a}.$$

In particular, we also obtain the asymptotic normality from [5, 25]

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{1-2a}\right).$$

As it was done in [13], we also obtain the asymptotic normality for the center of mass of the RERW defined by

$$G_n = \frac{1}{n} \sum_{k=1}^{n} S_n.$$

Corollary 3.6. *We have the asymptotic normality*

$$\frac{G_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{2-c(c+1+3ca+3a-2a^2)}{3(2+c-a)(1-2a-c)}\right). \tag{3.17}$$

Remark 3.7. *When $c = 0$, we find again the asympotic normality established in [11, 13]*

$$\frac{G_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{2}{3(1-2a)(2-a)}\right).$$

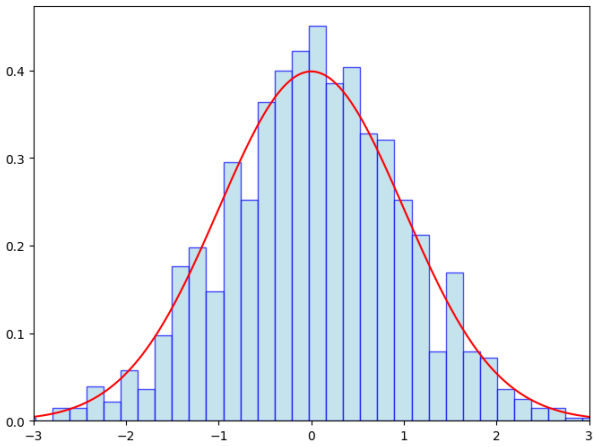


Figure 3.2: Asymptotic normality for the RERW in the diffusive regime, when $p = 0.35$ and $c = 1$.

3.3.2 – The critical regime

Hereafter, we investigate the critical regime where $p = (3-c)/4$.

Theorem 3.8. *We have the convergence*

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} = 0 \quad a.s. \tag{3.18}$$

The almost sure rates of convergence for the RERW are as follows.

Theorem 3.9. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \frac{S_k^2}{(k\log k)^2} = \frac{(c-1)^2}{c+1} \quad a.s. \tag{3.19}$$

In addition, we also have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{S_n^2}{2n\log n\log\log\log n} = \frac{(c-1)^2}{c+1} \quad a.s. \tag{3.20}$$

Once again, our next result concerns the functional convergence in distribution for the RERW. The following theorem was also first obtained by Baur [2, Theorem 3.2].

Theorem 3.10. *The following convergence in $D([0,\infty[)$ holds*

$$\left(\frac{S_{\lfloor n^t \rfloor}}{\sqrt{n^t \log n}},\ t \geq 0\right) \Longrightarrow \sqrt{\frac{(c-1)^2}{(c+1)}}\left(B_t,\ t \geq 0\right) \tag{3.21}$$

where $(B_t,\ t \geq 0)$ is a one-dimensional standard Brownian motion. In particular, we have

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{(c-1)^2}{c+1}\right). \tag{3.22}$$

Remark 3.11. *When $c = 0$, we find again the results from [3] for the ERW*

$$\left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n^t \log n}},\ t \geq 0\right) \Longrightarrow \left(B_t,\ t \geq 0\right)$$

where $(B_t,\ t \geq 0)$ is a one-dimensional standard Brownian motion. In particular, we find once again the asymptotic normality from [3, 5, 25]

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

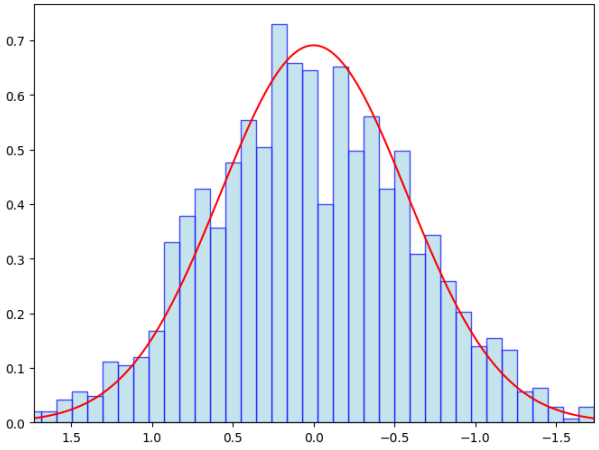


Figure 3.3: Asymptotic normality for the RERW in the critical regime, when $c = 2$ (i.e. $p = 0.25$).

3.3.3 – The superdiffusive regime

Finally, we focus our attention on the superdiffusive regime where $p > (3-c)/4$. The reader can notice that it is the only type of behavior for the RERW that still holds when $c > 3$ since $a \geq -1$. The following convergence in $D([0,\infty[)$ can also be found in [2, Theorem 3.2]. The almost sure and mean-square convergences are new.

Theorem 3.12. *We have the following distributional convergence in $D([0,\infty[)$*

$$\left(\frac{S_{\lfloor nt \rfloor}}{n^{\lambda(c+a)}},\ t \geq 0\right) \Longrightarrow \left(\Lambda_t,\ t \geq 0\right) \tag{3.23}$$

where the limiting $\Lambda_t = t^{\lambda(c+a)} L_c$, L_c *being some non-denegerate random variable. In particular, we have*

$$\lim_{n\to\infty} \frac{S_n}{n^{(a+c)\lambda}} = L_c \quad a.s. \tag{3.24}$$

We also have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{S_n}{n^{(a+c)\lambda}} - L_c\Big|^2\Big] = 0. \tag{3.25}$$

Theorem 3.13. *The expected value of* L_c *is*

$$\mathbb{E}[L_c] = \frac{a(2q-1)\Gamma(\lambda)}{(a+c)\Gamma(1+a\lambda)} \tag{3.26}$$

while its variance is given by

$$\mathbb{E}\big[L_c^2\big] = \frac{a^2(1+2ac+c^2)\Gamma(\lambda)}{(a+c)^2\lambda(2a+c-1)\Gamma((2a+c)\lambda)}. \tag{3.27}$$

Remark 3.14. *When* $c = 0$, *we find once again the moments of* L *established in [5]*

$$\mathbb{E}[L] = \frac{2q-1}{\Gamma(a+1)} \quad \textit{and} \quad \mathbb{E}\big[L^2\big] = \frac{1}{(2a-1)\Gamma(2a)}.$$

3.4 – A two-dimensional martingale approach

In order to investigate the asymptotic behavior of (S_n), we introduce the two-dimensional martingale $(\mathcal{M}_n)$ defined by

$$\mathcal{M}_n = \begin{pmatrix} N_n \\ M_n \end{pmatrix} \tag{3.28}$$

where (M_n) and (N_n) are the two locally square-integrable martingales introduced in (3.10). As for the center of mass of the ERW [11], the main difficulty we face is that the predictable quadratic variations of (M_n) and (N_n) increase to infinity with two different speeds. A matrix normalization will again be necessary to establish the asymptotic behavior of the RERW. We will study $(\mathcal{M}_n)$, instead of (M_n) or (N_n).

Let $\varepsilon_{n+1} = Y_{n+1} - \gamma_n Y_n$ and $\xi_n = (\alpha_n - a)X_{\beta_n}$. We have from equations (3.5), (3.8) and (3.10)

$$\begin{aligned}
\Delta\mathcal{M}_{n+1} &= \mathcal{M}_{n+1} - \mathcal{M}_n \\
&= \begin{pmatrix} S_{n+1} - S_n - \frac{a}{a+c}(Y_{n+1} - Y_n) \\ a_{n+1}Y_{n+1} - a_n Y_n \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{n+1}X_{\beta_{n+1}} - \frac{a}{a+c}(\alpha_{n+1}+c)X_{\beta_{n+1}} \\ a_{n+1}\varepsilon_{n+1} \end{pmatrix} \\
&= a_{n+1}\varepsilon_{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{c}{a+c}\xi_{n+1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.29}$$

We also find from (3.5) that

$$\begin{aligned}\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] &= \mathbb{E}[Y_{n+1}^2 \mid \mathcal{F}_n] - \gamma_n^2 Y_n^2 \\ &= Y_n^2 + 2(\gamma_n - 1)Y_n^2 + 1 + 2ac + c^2 - \gamma_n^2 Y_n^2 \\ &= 1 + 2ac + c^2 - (\gamma_n - 1)^2 Y_n^2. \end{aligned} \tag{3.30}$$

In addition, we obtain once again from (3.5) that

$$\mathbb{E}[\xi_{n+1}^2 \mid \mathcal{F}_n] = 1 - a^2 \tag{3.31}$$

and finally

$$\begin{aligned}\mathbb{E}[\varepsilon_{n+1}\xi_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}\big[\big((1-\gamma_n)Y_n + (\alpha_{n+1}+c)X_{\beta_{n+1}}\big)(\alpha_{n+1}-a)X_{\beta_{n+1}} \mid \mathcal{F}_n\big] \\ &= \mathbb{E}\big[\big((1-\gamma_n)(\alpha_{n+1}-a)Y_n X_{\beta_{n+1}} + (\alpha_{n+1}+c)(\alpha_{n+1}-a) \mid \mathcal{F}_n\big] \\ &= 1 - a^2. \end{aligned} \tag{3.32}$$

Hereafter, we deduce from (3.29), (3.30), (3.31) and (3.32) that

$$\begin{aligned}\mathbb{E}\big[(\Delta\mathcal{M}_{n+1})(\Delta\mathcal{M}_{n+1})^T \mid \mathcal{F}_n\big] &= a_{n+1}^2\big(1 + 2ac + c^2 - (\gamma_k - 1)^2 Y_k^2\big)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + a_{n+1}\frac{c}{a+c}(1-a^2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\quad + \Big(\frac{c}{a+c}\Big)^2(1-a^2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

We are now able to compute the quadratic variation of $\mathcal{M}_n$, that is

$$\begin{aligned}\langle \mathcal{M} \rangle_n &= \sum_{k=0}^{n-1} a_{k+1}^2\big(1 + 2ac + c^2 - (\gamma_k - 1)^2 Y_k^2\big)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \sum_{k=0}^{n-1} a_{k+1}\frac{c}{a+c}(1-a^2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\quad + n\Big(\frac{c}{a+c}\Big)^2(1-a^2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle \mathcal{M} \rangle_n &= v_n(1 + 2ac + c^2)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + w_n\frac{c}{a+c}(1-a^2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\quad + n\Big(\frac{c}{a+c}\Big)^2(1-a^2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \mathcal{R}_n\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned} \tag{3.33}$$

where

$$v_n = \sum_{k=1}^{n} a_k^2\,, \quad w_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad \mathcal{R}_n = \sum_{k=0}^{n-1} a_{k+1}^2(\gamma_k - 1)^2 Y_k^2.$$

Hereafter, we immediately deduce from (3.33) that

$$\langle M \rangle_n = (1 + 2ac + c^2) \sum_{k=1}^{n} a_k^2 - \mathcal{R}_n \tag{3.34}$$

and that

$$\langle N \rangle_n = \left(\frac{c}{a+c}\right)^2 (1 - a^2)n. \tag{3.35}$$

The asympotic behavior of M_n is closely related to the one of (v_n) as one can observe that we always have $\langle M \rangle_n \leq (1 + 2ac + c^2)v_n$ and thus $\langle M \rangle_n = O(v_n)$. Consequently to the definition of (a_n), we have three regimes of behavior for (M_n). In the diffusive regime where $p < (3-c)/4$ or $a < (1-c)/2$,

$$\lim_{n\to\infty} \frac{v_n}{n^{1-2(a+c)\lambda}} = \ell \quad \text{where} \quad \ell = \frac{1}{1 - 2(a+c)\lambda} \left(\frac{\Gamma(1+a\lambda)}{\Gamma(\lambda)}\right)^2. \tag{3.36}$$

In the critical regime where $p = (3-c)/4$ or $a = (1-c)/2$,

$$\lim_{n\to\infty} \frac{v_n}{\log n} = \left(\frac{\Gamma(\frac{c+3}{2(c+1)})}{\Gamma(\frac{1}{c+1})}\right)^2. \tag{3.37}$$

In the superdiffusive regime where $p > (3-c)/4$ or $a > (1-c)/2$,

$$\lim_{n\to\infty} v_n = \sum_{n=1}^{\infty} \left(\frac{\Gamma(n - c\lambda)\Gamma(1 + a\lambda)}{\Gamma(n + a\lambda)\Gamma(\lambda)}\right)^2 < +\infty. \tag{3.38}$$

3.5 – Another approach using Pòlya-type urns

As it was done in [2, 3, 13], it is possible to use another approach based on Pòlya-type urns and the results from [48]. This was presented in the work of Baur [2]. Here, we have to consider an urn $U_n = (G_n, B_n, R_n)^T$ for $n \in \mathbb{N}$, with balls of three different types and with mean replacement matrix given by

$$A = \begin{pmatrix} c+p & 1-p & 1-p \\ 0 & c & c \\ 1-p & p & p \end{pmatrix}. \tag{3.39}$$

The coefficient a_{ij} of the matrix A represents the mean number of balls of type i which are added to the urn if a ball of type j is drawn, observed and then returned to the urn. Here, let say we have three colors of balls that are green, blue and red. The numbers of balls of each color at instant $n \geq 1$ are given by G_n, B_n and R_n. In our configuration, the number of red balls corresponds to the number of steps towards the right direction. The number of blue balls corresponds to the additional weight of the right direction. The number of green balls corresponds to the total weight of the left direction. For example, let say a green ball is drawn, it is then returned to the urn together c (because a step to the left was remembered).

Then, with probability p one other green ball is added, meaning a step to the left is performed, and with probability $1-p$ one red ball is added, meaning a step to the right is performed. No blue balls are added because the instant remembered was a left one. Hereafter, it follows from the dynamics of the urn that the number of steps to the right of the RERW until time n is distributed as R_n. Consequently, we have for the position S_n of the reinforced ERW at time n that

$$S_n \overset{\mathcal{L}}{=} 2R_n - n. \tag{3.40}$$

Hereafter, the eigenvalues associated with the mean replacement matrix A defined in (3.39) are $\lambda_1 = c+1$, $\lambda_2 = c+a$ and $\lambda_3 = 0$ and the corresponding unit vectors in L^1 are

$$v_1^T = \frac{1}{2(c+1)}(c+1,\ c,\ 1), \quad v_2^T = \frac{1}{2(c+a)}(-(c+a),\ c,\ a), \quad v_3^T = \frac{1}{2}(0,\ -1,\ 1).$$

Then, we denote u_1, u_2 and u_3 the vectors of a corresponding dual basis where

$$\begin{aligned} u_1^T &= (1,\ 1,\ 1), \quad u_2^T = (-1,\ 1,\ 1), \\ u_3^T &= \frac{1}{(c+1)(c+a)}(c(a-1),\ -(2a+ca+c),\ c(2c+a+1)). \end{aligned}$$

The study of the process (U_n) relies on the value of the ratio λ_2/λ_1. In particular, the case $\lambda_2/\lambda_1 = 1/2$ corresponds to the case where $a = (1-c)/2$, which is coherent with the previous trichotomy. This connection allows us to retrieve the results from Theorems 3.4 and 3.10 using Theorem 3.31 from [48]. We also find again the distributional convergence (3.23) from Theorem 3.12 using once again [48], Theorem 3.24.

3.6 – Proofs of the almost sure convergence results

Lemma 3.15. *Let (V_n) be the sequence of positive definite diagonal matrices of order 2 given by*

$$V_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a}{a+c} a_n^{-1} \end{pmatrix}. \tag{3.41}$$

Then, the quadratic variation of $\langle \mathcal{M} \rangle_n$ satisfies in the diffusive regime where $a < (1-c)/2$,

$$\lim_{n\to\infty} V_n \langle \mathcal{M} \rangle_n V_n = V \quad a.s. \tag{3.42}$$

where the matrix V is given by

$$V = \frac{1}{(a+c)^2} \begin{pmatrix} c^2(1-a^2) & ac(c+1)(1+a) \\ ac(c+1)(1+a) & \dfrac{a^2(1+2ac+c^2)(c+1)}{1-c-2a} \end{pmatrix}. \tag{3.43}$$

Remark 3.16. *Following the same steps as in the proof of Lemma 3.15, we find that in the critical regime $a = (1-c)/2$, the sequence of normalization matrices (V_n) has to be replaced by*

$$W_n = \frac{1}{\sqrt{n \log n}} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a}{a+c} a_n^{-1} \end{pmatrix}. \tag{3.44}$$

The limit matrix V also needs to be replaced by

$$W = \frac{(c-1)^2}{c+1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.45}$$

Proof of Lemma 3.15. We immediately obtain from Theorem 3.1 and (3.9), (3.33), (3.36) that

$$\begin{aligned} \lim_{n\to\infty} V_n \langle \mathcal{M} \rangle_n V_n^T &= \Big(\frac{a}{a+c}\Big)^2 \frac{1}{1-2\lambda(a+c)} (1+2ac+c^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \frac{1}{1-\lambda}(1-a^2)\frac{ac}{(a+c)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\quad + (1-a^2)\Big(\frac{c}{a+c}\Big)^2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{(a+c)^2} \begin{pmatrix} c^2(1-a^2) & ac(c+1)(1+a) \\ ac(c+1)(1+a) & \dfrac{a^2(1+2ac+c^2)(c+1)}{1-c-2a} \end{pmatrix} \end{aligned}$$

which is exactly what we wanted to prove.

■

3.6.1 – The diffusive regime

Proof of Theorem 3.1. We shall make extensive use of the strong law of large numbers for martingales, see Theorem A.2. First, we have for M_n that for any $\gamma > 0$,

$$M_n^2 = O\big((\log v_n)^{1+\gamma} v_n\big) \quad \text{a.s.}$$

Then, by definition of M_n and as a_n is asymptotically equivalent to $n^{-(a+c)\lambda}$ and v_n is asymptotically equivalent to $n^{1-2(a+c)\lambda}$, it ensures

$$\frac{Y_n^2}{n^2} = O\Big((\log n)^{1+\gamma} \frac{n^{1-2(a+c)\lambda}}{n^{2(1-(a+c)\lambda)}}\Big) \quad \text{a.s.}$$

and finally

$$\frac{Y_n^2}{n^2} = O\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \quad \text{a.s.}$$

This implies that

$$\lim_{n\to\infty} \frac{Y_n}{n} = 0 \quad \text{a.s.} \tag{3.46}$$

We now focus our attention on N_n. By the same token as before, we have that for any $\gamma > 0$,

$$N_n^2 = O\big((\log n)^{1+\gamma} n\big) \quad \text{a.s.}$$

which by definition of N_n gives us

$$\frac{\big(S_n - \frac{a}{a+c} Y_n\big)^2}{n^2} = O\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \quad \text{a.s.}$$

and we conclude

$$\lim_{n\to\infty} \frac{S_n}{n} - \frac{a}{a+c}\frac{Y_n}{n} = 0 \quad \text{a.s.} \tag{3.47}$$

This achieves the proof of Theorem 3.1 as the convergences (3.46) and (3.47) hold almost surely.

■

Proof of Theorem 3.2. We need to check that all the hypotheses of Theorem A.8 are satisfied. Thanks to Lemma 3.15, hypothesis (H.1) holds almost surely. In order to verify that Lindeberg's condition (H.2) is satisfied, we have from (3.10) together with (3.28) and V_n given by (3.41) that for all $1 \leq k \leq n$

$$V_n \Delta \mathcal{M}_k = \frac{1}{(a+c)\sqrt{n}} \begin{pmatrix} c\xi_{n+1} \\ a a_n^{-1} a_k \varepsilon_k \end{pmatrix}$$

which implies that

$$\|V_n \Delta \mathcal{M}_k\|^2 = \frac{1}{(a+c)^2 n}\big(c^2 + a^2 a_n^{-2} a_k^2 \varepsilon_k^2\big) \tag{3.48}$$

and

$$\|V_n \Delta \mathcal{M}_k\|^4 = \frac{1}{(a+c)^4 n^2}\big(c^4 + 2a^2c^2 a_n^{-2} a_k^2 \varepsilon_k^2 + a_n^{-4} a_k^4 \varepsilon_k^4\big). \tag{3.49}$$

Consequently, we obtain that for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big]. \tag{3.50}$$

It follows from (3.9) that

$$a_n^{-2} \sum_{k=1}^{n} a_k^2 = O(n) \quad \text{and} \quad a_n^{-4} \sum_{k=1}^{n} a_k^4 = O(n).$$

Hence, using that the sequence (ε_n) is uniformly bounded

$$\sup_{1\leq k\leq n} |\varepsilon_k| \leq c + 2 \quad \text{a.s.} \tag{3.51}$$

we find that

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big] = O\Big(\frac{1}{n}\Big) \quad \text{a.s.}$$

which ensures that Lindeberg's condition (H.2) holds almost surely, that is for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta\mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n\Delta\mathcal{M}_k\|>\varepsilon\}} \mid \mathcal{F}_{k-1}\big] = 0 \quad \text{a.s.} \tag{3.52}$$

Hereafter, we need to verify (H.3) is satisfied in the special case $\beta = 2$ that is

$$\sum_{n=1}^{\infty} \frac{1}{\big(\log(\det V_n^{-1})^2\big)^2} \mathbb{E}\big[\|V_n\Delta\mathcal{M}_n\|^4 \mid \mathcal{F}_{n-1}\big] < \infty \quad \text{a.s.}$$

We immediately have from (3.41)

$$\det V_n^{-1} = \frac{a+c}{a}\sqrt{n}a_n. \tag{3.53}$$

Hence, we obtain from (3.9) and (3.53) that

$$\lim_{n\to\infty} \frac{\log(\det V_n^{-1})^2}{\log n} = 1 - 2(a+c)\lambda. \tag{3.54}$$

Therefore, we can replace $\log(\det V_n^{-1})^2$ by $\log n$ in (3.6.1). Hereafter, we obtain from (3.49) and (3.51) that

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^2} \mathbb{E}\big[\|V_n\Delta\mathcal{M}_n\|^4 \mid \mathcal{F}_{n-1}\big] = O\Big(\sum_{n=1}^{\infty} \frac{1}{(n\log n)^2}\Big). \tag{3.55}$$

Thus, (3.55) guarentees that (H.3) is verified. We are now going to apply the quadratic strong law given by Theorem A.2 in [11]. We get from equation (3.54) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2}\Big) V_k\mathcal{M}_k\mathcal{M}_k^T V_k^T = (1-2(a+c)\lambda)V \quad \text{a.s.} \tag{3.56}$$

However, we obtain from (3.9) and (3.53) that

$$\lim_{n\to\infty} n\Big(\frac{(\det V_n)^2 - (\det V_{n+1})^2}{(\det V_n)^2}\Big) = 1 - 2(a+c)\lambda. \tag{3.57}$$

Finally, let $u = \Big(1,1\Big)^T$ we have

$$u^T V_n \mathcal{M}_n = \frac{S_n}{\sqrt{n}} \tag{3.58}$$

and we deduce from (3.56), (3.57) and (3.58) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k^2}{k^2} = v^T V v \quad \text{a.s.} \tag{3.59}$$

which, together with

$$u^T V u = \frac{2ac + c - 1}{2a + c - 1} \tag{3.60}$$

completes the proof of Theorem 3.2.

■

3.6.2 – The critical regime

Proof of Theorem 3.8. Again, we shall make use of the strong law of large numbers for martingales, see Theorem A.2. First, we have for M_n that for any $\gamma > 0$,

$$M_n^2 = O\big((\log v_n)^{1+\gamma} v_n\big) \quad \text{a.s.}$$

which by definition of M_n and as a_n is asymptotically equivalent to $n^{-1/2}$ and v_n is asymptotically equivalent to $\log n$ ensures that

$$\frac{Y_n^2}{(\sqrt{n}\log n)^2} = O\Big((\log\log n)^{1+\gamma}\frac{\log n}{(\log n)^2}\Big) \quad \text{a.s.}$$

and finally that

$$\frac{Y_n^2}{(\sqrt{n}\log n)^2} = O\Big(\frac{(\log\log n)^{1+\gamma}}{\log n}\Big) \quad \text{a.s.}$$

This implies that

$$\lim_{n\to\infty} \frac{Y_n}{\sqrt{n}\log n} = 0 \quad \text{a.s.} \tag{3.61}$$

In addition, we still have that for any $\gamma > 0$,

$$N_n^2 = O\big((\log n)^{1+\gamma} n\big) \quad \text{a.s.}$$

which by definition of N_n gives us

$$\frac{\big(S_n - \frac{a}{a+c}Y_n\big)^2}{(\sqrt{n}\log n)^2} = O\Big((\log n)^{\gamma-1}\Big) \quad \text{a.s.}$$

Taking e.g. $\gamma = \frac{1}{2}$ we can conclude that

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} - \frac{a}{a+c}\frac{Y_n}{\sqrt{n}\log n} = 0 \quad \text{a.s.} \tag{3.62}$$

This achieves the proof of Theorem 3.8 as the convergences (3.61) and (3.62) hold almost surely.

■

Proof of Theorem 3.9. The proof of the quadratic strong law (3.19) is left to the reader as it follows essentially the same lines as that of (3.13). The only minor change is that the matrix V_n has to be replaced by the matrix W_n defined in (3.44). We shall now proceed to the proof of the law of iterated logarithm given by (3.20). On the one hand, it follows from (3.9) and (3.36) that

$$\sum_{n=1}^{+\infty} \frac{a_n^4}{v_n^2} < \infty. \tag{3.63}$$

Moreover, we have from (3.34) and (3.35) that

$$\lim_{n\to\infty} \frac{\langle M\rangle_n}{v_n} = 1 + 2ac + c^2 \quad \text{a.s.} \quad \text{and} \quad \lim_{n\to\infty} \frac{\langle N\rangle_n}{n} = \Big(\frac{c}{a+c}\Big)^2(1-a^2) \quad \text{a.s.}$$

Consequently, we deduce from the law of iterated logarithm for martingales due to Stout [72], see Theorem A.4, that (M_n) satisfies when $a = (1-c)/2$

$$\begin{aligned}\limsup_{n\to\infty} \frac{M_n}{(2v_n \log\log v_n)^{1/2}} &= -\liminf_{n\to\infty} \frac{M_n}{(2v_n \log\log v_n)^{1/2}} \\ &= \sqrt{1+c} \quad \text{a.s.}\end{aligned}$$

However, as $a_n v_n^{-1/2}$ is asymptotically equivalent to $(n\log n)^{-1/2}$, we immediately obtain from (3.37) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{Y_n}{(2n\log n\log\log\log n)^{1/2}} &= -\liminf_{n\to\infty} \frac{Y_n}{(2n\log n\log\log\log n)^{1/2}} \\ &= \sqrt{1+c} \quad \text{a.s.} \qquad (3.64)\end{aligned}$$

The law of iterated logarithm for martingales also allow us to find that (N_n) satisfies

$$\begin{aligned}\limsup_{n\to\infty} \frac{N_n}{(2n\log\log n)^{1/2}} &= -\liminf_{n\to\infty} \frac{N_n}{(2n\log\log n)^{1/2}} \\ &= \frac{2c}{c+1}\sqrt{(1-a^2)} \quad \text{a.s.}\end{aligned}$$

which ensures that

$$\limsup_{n\to\infty} \frac{N_n}{(2n\log n\log\log\log n)^{1/2}} = 0 \quad \text{a.s.}$$

Hence, we deduce from (3.11) and (3.64) that

$$\begin{aligned}\limsup_{n\to\infty} \frac{S_n}{(2n\log n\log\log\log n)^{1/2}} &= \limsup_{n\to\infty} \frac{N_n + \frac{1-c}{1+c} a_n^{-1} M_n}{(2n\log n\log\log\log n)^{1/2}} \\ &= \limsup_{n\to\infty} \frac{1-c}{1+c} \frac{Y_n}{(2n\log n\log\log\log n)^{1/2}} \\ &= -\liminf_{n\to\infty} \frac{1-c}{1+c} \frac{Y_n}{(2n\log n\log\log\log n)^{1/2}} \\ &= -\liminf_{n\to\infty} \frac{S_n}{(2n\log n\log\log\log n)^{1/2}}.\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} \frac{S_n^2}{2n\log n\log\log\log n} &= \limsup_{n\to\infty} \Big(\frac{1-c}{1+c}\Big)^2 \frac{Y_n^2}{2n\log n\log\log\log n} \\ &= \frac{(1-c)^2}{1+c}\end{aligned}$$

which immediately leads to (3.20), thus completing the proof of Theorem 3.9.

■

3.6.3 – The superdiffusive regime

Proof of Theorem 3.12. Hereafter, we shall again make extensive use of the strong law of large numbers for martingales in order to prove (3.24), see Theorem A.2. When $a > (1-c)/2$, we have from (3.38) that v_n converges. Hence, as $\langle M \rangle_n \leq (1 + 2ac + c^2)v_n$, we clealy have that $\langle M \rangle_\infty < \infty$ almost surely and we can conclude that

$$\lim_{n\to\infty} M_n = M \quad \text{a.s.} \quad \text{where} \quad M = \sum_{k=1}^{\infty} a_k \varepsilon_k$$

which by definition of M_n and as a_n is asymptotically equivalent to $\frac{\Gamma(1+a\lambda)}{\Gamma(\lambda)} n^{-(a+c)\lambda}$ ensures that

$$\lim_{n\to\infty} \frac{Y_n}{n^{(a+c)\lambda}} = Y \quad \text{a.s.} \quad \text{where} \quad Y = \frac{\Gamma(\lambda)}{\Gamma(1+a\lambda)} M. \tag{3.65}$$

Moreover, we still have that for any $\gamma > 0$,

$$N_n^2 = O\big((\log n)^{1+\gamma} n\big) \quad \text{a.s.}$$

which by definition of N_n gives us for all $t \geq 0$

$$\frac{\big(S_n - \frac{a}{a+c} Y_n\big)^2}{n^{2(a+c)\lambda}} = O\Big(\frac{(\log n)^{1+\gamma}}{n^{2(a+c)\lambda - 1}}\Big) \quad \text{a.s.}$$

As $a > (1-c)/2$ in the superdiffusive regime, we obtain thanks to (3.46) that for all $t \geq 0$

$$\lim_{n\to\infty} \frac{S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor^{(a+c)\lambda}} - \frac{a}{a+c} \frac{Y_{\lfloor nt \rfloor}}{\lfloor nt \rfloor^{(a+c)\lambda}} = 0 \quad \text{a.s.} \tag{3.66}$$

The convergences (3.65) and (3.66) hold almost surely and $\lfloor nt \rfloor$ is asymptotically equivalent to nt which implies

$$\lim_{n\to\infty} \frac{S_{\lfloor nt \rfloor}}{n^{(a+c)\lambda}} = t^{(a+c)\lambda} L_c \quad \text{a.s.} \tag{3.67}$$

Finally, the fact that (3.67) holds almost surely ensures that it also holds for the finite-dimensional distributions, and we obtain (3.23) with $\Lambda_t = t^{(a+c)\lambda} L_c$ and $L_c = \frac{a}{a+c} Y$.

We shall now proceed to the proof of the mean square convergence (3.25). On the one hand, as $M_0 = 0$ we have from (3.34) that

$$\mathbb{E}\big[M_n^2\big] = \mathbb{E}\big[\langle M \rangle_n\big] \leq (1 + 2ac + c^2)v_n.$$

Hence, we obtain from (3.38) that

$$\sup_{n\geq 1} \mathbb{E}\big[M_n^2\big] < \infty$$

which ensures that the martingale (M_n) is bounded in $\mathbb{L}^2$. Therefore, we have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\big[\ |\ M_n - M\ |^2\ \big] = 0$$

which implies that

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{Y_n}{n^{(a+c)\lambda}} - Y\Big|^2\Big] = 0. \tag{3.68}$$

On the other hand, for any $n \geq 0$, the martingale (N_n) satisfies

$$\mathbb{E}\big[N_n^2\big] = \mathbb{E}\big[\langle N\rangle_n\big] \leq (1-a^2)\Big(\frac{c}{a+c}\Big)^2 n$$

and since $(a+c)\lambda > \frac{1}{2}$ we obtain

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{N_n}{n^{(a+c)\lambda}}\Big|^2\Big] = 0. \tag{3.69}$$

Finally, we obtain the mean square convergence (3.25) from (3.68) and (3.69) and we achieve the proof of Theorem 3.12.

■

Proof of Theorem 3.13. We start by the calculation of the expectation (3.26). We immediately have from (3.6) that

$$\mathbb{E}[Y_{n+1}] = \gamma_n \mathbb{E}[Y_n] = \Big(\frac{n+a\lambda}{n-c\lambda}\Big)\mathbb{E}[Y_n]$$

which leads to

$$\mathbb{E}[Y_n] = \prod_{k=1}^{n-1}\Big(\frac{k+a\lambda}{k-c\lambda}\Big)\mathbb{E}[Y_1] = \prod_{k=1}^{n-1}\Big(\frac{k+a\lambda}{k-c\lambda}\Big)\mathbb{E}[X_1] = (2q-1)a_n^{-1}. \tag{3.70}$$

Hence, we immediately get equation (3.26) from (3.70), that is

$$\mathbb{E}[L_c] = \frac{a\Gamma(\lambda)}{(a+c)\Gamma(1+a\lambda)}\mathbb{E}[M] = \frac{a\Gamma(\lambda)}{(a+c)\Gamma(1+a\lambda)}\mathbb{E}[M_n] = \frac{a(2q-1)\Gamma(\lambda)}{(a+c)\Gamma(1+a\lambda)}.$$

Hereafter, we obtain from (3.30) by taking expectation on both sides that

$$\mathbb{E}[Y_{n+1}^2] = 1 + 2ac + c^2 + (2\gamma_n - 1)\mathbb{E}[Y_n^2] = 1 + 2ac + c^2 + \Big(\frac{n+(2a+c)\lambda}{n-c\lambda}\Big)\mathbb{E}[Y_n^2]$$

and thanks to well-known recursive relation solutions and Lemma B.1 in [5], we get

$$\begin{aligned}
\mathbb{E}[Y_n^2] &= (1+2ac+c^2)\prod_{k=0}^{n-1}\Big(\frac{k+(2a+c)\lambda}{k-c\lambda}\Big)\sum_{k=0}^{n-1}\prod_{i=0}^{k}\frac{i-c\lambda}{i+(2a+c)\lambda} \\
&= \frac{(1+2ac+c^2)\Gamma(n+(2a+c)\lambda)\Gamma(\lambda)}{\Gamma(n-c\lambda)\Gamma(1+(2a+c)\lambda)}\sum_{k=0}^{n-1}\frac{\Gamma(k+\lambda)\Gamma(1+(2a+c)\lambda)}{\Gamma(k+1+(2a+c)\lambda)\Gamma(\lambda)} \\
&= \frac{(1+2ac+c^2)\Gamma(n+(2a+c)\lambda)}{\Gamma(n-c\lambda)}\sum_{k=1}^{n}\frac{\Gamma(k+\lambda-1)}{\Gamma(k+(2a+c)\lambda)} \\
&= \frac{(1+2ac+c^2)\Gamma(n+(2a+c)\lambda)}{\lambda(2a+c-1)\Gamma(n-c\lambda)}\Big(\frac{\Gamma(\lambda)}{\Gamma((2a+c)\lambda)} - \frac{\Gamma(n+\lambda)}{\Gamma(n+(2a+c)\lambda)}\Big).
\end{aligned}$$

Hence, we obtain from (3.9), (3.10) and (3.68) that

$$\mathbb{E}[Y^2] = \lim_{n\to\infty}\frac{\mathbb{E}[Y_n^2]}{n^{2(a+c)\lambda}} = \frac{(1+2ac+c^2)\Gamma(\lambda)}{\lambda(2a+c-1)\Gamma((2a+c)\lambda)}. \tag{3.71}$$

■

3.7 – Proofs of the functional limit theorems

3.7.1 – The diffusive regime

Proof of Theorem 3.4. In order to apply Theorem A.9 in the Appendix, we must verify that (H′.1), (H′.2) and (H.4) are satisfied.

(H′.1) We have from (3.42) and the fact that $a_{\lfloor nt \rfloor}$ is asymptotically equivalent to $t^{-(a+c)\lambda} a_n$ that

$$V_n \langle \mathcal{M} \rangle_{\lfloor nt \rfloor} V_n^T \underset{n\to\infty}{\longrightarrow} V_t \quad \text{a.s.}$$

where

$$V_t = \frac{1}{(a+c)^2} \begin{pmatrix} c^2(1-a^2)t & ac(c+1)(1+a)t^{1-(a+c)\lambda} \\ ac(c+1)(1+a)t^{1-(a+c)\lambda} & \dfrac{a^2(1+2ac+c^2)(c+1)}{1-c-2a} t^{1-2(a+c)\lambda} \end{pmatrix}.$$

(H′.2) We also get that Lindeberg's condition is satisfied as we already know from (3.52) that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1} \big] = 0 \quad \text{a.s.}$$

which implies from (3.49) and the fact that $V_n V_{\lfloor nt \rfloor}^{-1}$ converges

$$\begin{aligned} \lim_{n\to\infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1} \big] &\leq \lim_{n\to\infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \big] \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|(V_n V_{\lfloor nt \rfloor}^{-1}) V_{\lfloor nt \rfloor} \Delta \mathcal{M}_k\|^4 \big] \\ &= 0 \quad \text{a.s.} \end{aligned}$$

(H.4) In this particular case, we have $V_t = tK_1 + t^{\alpha_2} K_2 + t^{\alpha_3} K_3$ where

$$\alpha_2 = 1 - (a+c)\lambda > 0 \quad \text{and} \quad \alpha_3 = 1 - 2(a+c)\lambda > 0$$

as $a \leq (1-c)/2$, and the matrices are symmetric

$$K_1 = \frac{c^2(1-a^2)}{(a+c)^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; K_2 = \frac{ac(c+1)(a+1)}{(a+c)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$K_3 = \frac{a^2(1+2ac+c^2)(c+1)}{(1-2a-c)(a+c)^2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, we obtain that

$$\big(V_n \mathcal{M}_{\lfloor nt \rfloor}, \, t \geq 0 \big) \Longrightarrow \big(\mathcal{B}_t, \, t \geq 0 \big)$$

where $\mathcal{B}$ is defined as in (A.9). Finally, using the fact that $S_{\lfloor nt \rfloor}$ is asymptotically equivalent to $N_{\lfloor nt \rfloor} + t^{(a+c)\lambda} \frac{a}{a+c} a_n^{-1} M_{\lfloor nt \rfloor}$ and multiplying $u_t = \begin{pmatrix} 1 \\ t^{(a+c)\lambda} \end{pmatrix}$ we conclude

$$\big(\frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor},\ t \geq 0\big) \Longrightarrow \big(W_t,\ t \geq 0\big) \tag{3.72}$$

where $W_t = u_t^T \mathcal{B}_t$. It only remains to compute the covariance function of W that is for $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}\big[W_s W_t\big] &= u_s^T \mathbb{E}\big[\mathcal{B}_s \mathcal{B}_t^T\big] u_t \\
&= u_s^T V_s u_t \\
&= u_s^T \big(sK_1 + s^{1-(a+c)\lambda} K_2 + s^{1-2(a+c)\lambda} K_3\big) u_t \\
&= \frac{c^2(1-a^2)}{(a+c)^2} s + \frac{ac(c+1)(a+1)}{(a+c)^2} s^{1-(a+c)\lambda} \big(s^{(a+c)\lambda} + t^{(a+c)\lambda}\big) \\
&\quad + \frac{a^2(1+2ac+c^2)(c+1)}{(1-2a-c)(a+c)^2} s^{1-2(a+c)\lambda} (st)^{(a+c)\lambda} \\
&= \Big(\frac{c^2(1-a^2)}{(a+c)^2} + \frac{ac(c+1)(a+1)}{(a+c)^2}\Big) s \\
&\quad + \Big(\frac{ac(c+1)(a+1)}{(a+c)^2} + \frac{a^2(1+2ac+c^2)(c+1)}{(1-2a-c)(a+c)^2}\Big) s \Big(\frac{t}{s}\Big)^{(a+c)\lambda} \\
&= \frac{c(a+1)}{a+c} s + \frac{a(1-c^2)}{(a+c)(1-2a-c)} s \Big(\frac{t}{s}\Big)^{(a+c)\lambda}.
\end{aligned}$$

■

Proof of Corollary 3.6. As for Corollary 4.1 from [13], we observe that

$$\frac{G_n}{\sqrt{n}} = \int_0^1 \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \mathrm{d}t.$$

Consquently, $G_n/\sqrt{n}$ is a continuous function of $S_{\lfloor nt \rfloor}/\sqrt{n}$ in $D([0,1])$. Hence, the functional distribution from Theorem 3.4 gives us that

$$\frac{G_n}{\sqrt{n}} = \int_0^1 \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \mathrm{d}t \xrightarrow[n \to \infty]{\mathcal{L}} \int_0^1 W_t \mathrm{d}t.$$

The process $\big(W_t,\ t \geq 0\big)$ is a continuous real-valued and centered Gaussian process starting from the origin, which implies that $\int_0^1 W_t \mathrm{d}t$ is also one. Its covariance

is given by

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^1 W_s\mathrm{d}s\Big)\Big(\int_0^1 W_t\mathrm{d}t\Big)\Big] &= 2\int_0^1\int_0^t \mathbb{E}\big[W_sW_t\big]\mathrm{d}s\mathrm{d}t \\ &= 2\frac{a(1-c^2)}{(a+c)(1-2a-c)}\int_0^1\int_0^t s\Big(\frac{t}{s}\Big)^{\lambda(a+c)}\mathrm{d}s\mathrm{d}t \\ &\quad + 2\frac{c(a+1)}{a+c}\int_0^1\int_0^t s\mathrm{d}s\mathrm{d}t \\ &= \frac{2a(1-c^2)(c+1)}{3(2+c-a)(a+c)(1-2a-c)} + \frac{c(a+1)}{3(a+c)} \\ &= \frac{2-c(c+1+3ca+3a-2a^2)}{3(2+c-a)(1-2a-c)}.\end{aligned}$$

■

3.7.2 – The critical regime

Proof of Theorem 3.10. First, we have from (3.35) that for all $t \geq 0$

$$\frac{\langle N\rangle_{\lfloor n^t\rfloor}}{n^t\log n} \longrightarrow 0 \quad \text{a.s.}$$

which implies from Theorem A.2 that

$$\frac{N_{\lfloor n^t\rfloor}}{n^t\log n} \longrightarrow 0 \quad \text{a.s.} \tag{3.73}$$

Hereafter, in order to apply Theorem A.9 to the one-dimensional martingale (M_n), we must once again verify that (H′.1), (H′.2) and (H.4) are satisfied.

(H′.1) Let $w_n = \sqrt{v_n^{-1}}$, we have from (3.34), Remark 3.16 and the fact that $a_{\lfloor n^t\rfloor}$ is asymptotically equivalent to $n^{-t/2}$ that

$$w_n\langle M\rangle_{\lfloor n^t\rfloor}w_n \underset{n\to\infty}{\longrightarrow} t(c+1) \quad \text{a.s.}$$

(H′.2) We also get that Lindeberg's condition is satisfied as v_n is increasing as $\log n$ and we have for all $\varepsilon > 0$

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{v_n}\sum_{k=1}^{n^t}\mathbb{E}\big[\Delta M_k^2\mathbb{1}_{\{|\Delta M_k|>\varepsilon\sqrt{v_n}\}}\mid\mathcal{F}_{k-1}\big] &\leq \lim_{n\to\infty}\frac{1}{\varepsilon^2v_n^2}\sum_{k=1}^{\lfloor n^t\rfloor}\mathbb{E}\big[\Delta M_k^4\big] \\ &\leq \lim_{n\to\infty}\Big(\frac{v_{\lfloor n^t\rfloor}}{v_n}\Big)^2\frac{1}{\varepsilon^2v_{\lfloor n^t\rfloor}^2}\sum_{k=1}^{\lfloor n^t\rfloor}\mathbb{E}\big[\Delta M_k^4\big] \\ &\leq \lim_{n\to\infty}\frac{t^2}{\varepsilon^2(\log n^t)^2}\sum_{k=1}^{\lfloor n^t\rfloor}\mathbb{E}\big[\Delta M_k^4\big].\end{aligned}$$

Moreover, we have from the very definition of M_n that

$$\sum_{k=1}^{n} \mathbb{E}\big[\Delta M_k^4\big] = O\Big(\sum_{k=1}^{n} a_k^4\Big) \quad \text{a.s.}$$

and as a_n is asymptotically equivalent to $n^{-1/2}$, we can conclude that

$$\lim_{n\to\infty} \frac{1}{v_n} \sum_{k=1}^{n^t} \mathbb{E}\big[\Delta M_k^2 \mathbb{1}_{\{|\Delta M_k| > \varepsilon\sqrt{v_n}\}} \mid \mathcal{F}_{k-1}\big] = 0 \quad \text{a.s.}$$

(H.4) In this particular case, we have $w_t = t(c+1)$. Hence, we obtain that

$$\big(w_n M_{\lfloor n^t \rfloor},\ t \geq 0\big) \Longrightarrow \big(W_t,\ t \geq 0\big)$$

where W is defined as in Theorem A.9. Moreover, when $a = (1-c)/2$ we obtain from (3.10), (3.73) and the fact that $(a_{\lfloor n^t \rfloor} v_n)^{-1}$ is asymptotically equivalent to $\sqrt{n^t \log n}^{-1}$ that

$$\Big(\frac{S_{\lfloor n^t \rfloor}}{\sqrt{n^t \log n}} - \frac{N_{\lfloor n^t \rfloor}}{\sqrt{n^t \log n}},\ t \geq 0\Big) \Longrightarrow \frac{1-c}{c+1}\big(W_t,\ t \geq 0\big).$$

Consequently, using that W is a centered Brownian motion with variance $(c+1)$, we can conclude that

$$\Big(\frac{S_{\lfloor n^t \rfloor}}{\sqrt{n^t \log n}},\ t \geq 0\Big) \Longrightarrow \sqrt{\frac{(1-c)^2}{c+1}}\big(B_t,\ t \geq 0\big)$$

and this achieves the proof of Theorem 3.10.

■

4 The Amnesic Elephant Random Walk

This chapter presents the results of [58] :

LAULIN, L. Introducing smooth amnesia to the memory of the elephant random walk. *arXiv:2204.10542* (2022).

4.1 – Introduction

In Chapter 3, we investigated how the ERW behaves if the distribution of the memory of the elephant is no longer uniform at over the previous instants, but dependent on the past and the number of times an instant is remembered.

The idea of this chapter is to use the tools developed in Chapters 2 and 3 to study how changing the memory allows us to induce amnesia to the ERW. More precisely, the new distribution of the memory β_n is such that the probability of choosing a fixed instant $k \in \mathbb{N}^*$ at time $n \geq k$ decreases approximatly with speed $(\beta+1)k^\beta/n^{\beta+1}$ for some amnesia parameter $\beta \geq 0$.

The very interesting question of amnesic elephant random walk (AERW) has not really been studied. Gut and Stadmüller [40, 41] investigated variations of the memory for the special cases of ERW with delays or gradually increasing memory. In [40] the elephant could stop and only remembered the first (and second) step it took. Consequently, it did not induce a phase transition. In [41], the elephant only remembered a portion of its past (recent or distant), this portion being fixed or depending on the time n, but was always "small".

The entire study we conduct below can be generalized when $\beta < 0$ is not an integer. This can be interpreted as cases where the elephant only remembers the

first steps it performed. When $\beta < -1$, it appears that the AERW only have one regime that is the diffusive regime. This observation is coherent with the work of Gut and Stadmüller [41].

As in the previous chapters, we can write the $(n+1)$-th increment X_{n+1} under the form

$$X_{n+1} = \alpha_{n+1} X_{\beta_{n+1}}. \tag{4.1}$$

In the case of the ERW, we had $\alpha_{n+1} \sim \mathcal{R}(p)$ and $\beta_{n+1} \sim \mathcal{U}\{1,\dots,n\}$. The major change for the AERW is that the distribution of β_n is no longer uniform but depends on the amnesia parameter $\beta \geq 0$. In this case, the elephant chooses an instant according to β_{n+1} as follows,

$$\mathbb{P}(\beta_{n+1} = k) = \frac{(\beta+1)\Gamma(k+\beta)\Gamma(n)}{\Gamma(k)\Gamma(n+\beta+1)} = \frac{(\beta+1)}{n}\frac{\mu_k}{\mu_{n+1}} \quad \text{for } 1 \leq k \leq n, \tag{4.2}$$

where

$$\mu_n = \prod_{k=1}^{n-1}\left(1+\frac{\beta}{k}\right) = \frac{\Gamma(n+\beta)}{\Gamma(n)\Gamma(\beta+1)}. \tag{4.3}$$

The case $\beta = 0$ corresponds to the traditional ERW. As β grows, the probability of choosing a recent instant gets bigger.

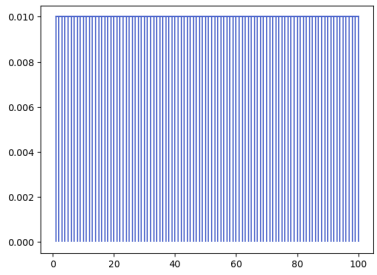


Figure 4.1: Mass function of β_{100} in the usual case $\beta = 0$.

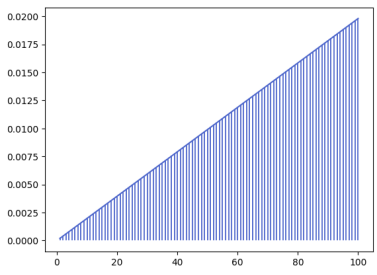


Figure 4.2: Mass function of β_{100} when $\beta = 1$.

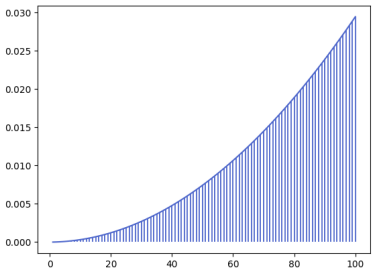


Figure 4.3: Mass function of β_{100} when $\beta = 2$.

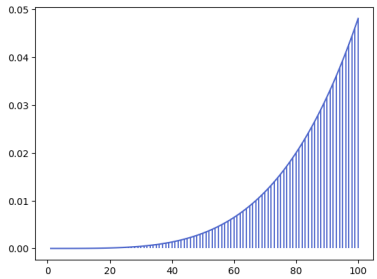


Figure 4.4: Mass function of β_{100} when $\beta = 3$.

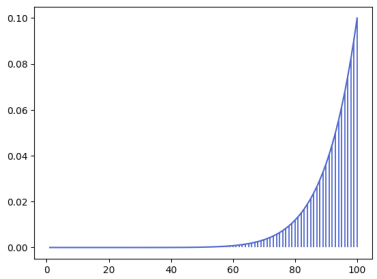


Figure 4.5: Mass function of β_{100} when $\beta = 10$.

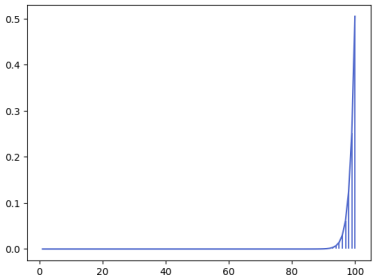


Figure 4.6: Mass function of β_{100} when $\beta = 100$.

We have by the definition of the step X_{n+1} given in (4.1) that

$$\begin{aligned}
\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\alpha_{n+1}]\mathbb{E}[X_{\beta_{n+1}} \mid \mathcal{F}_n] \\
&= (2p-1)\mathbb{E}\Big[\sum_{k=1}^{n} X_k \mathbb{1}_{\beta_{n+1}=k} \mid \mathcal{F}_n\Big] \\
&= \frac{(2p-1)(\beta+1)}{n\mu_{n+1}} \sum_{k=1}^{n} X_k \mu_k.
\end{aligned} \tag{4.4}$$

Then, denote $a = 2p - 1$ and

$$Y_n = \sum_{k=1}^{n} X_k \mu_k. \tag{4.5}$$

We deduce from (4.4) that

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \Big(1 + \frac{a(\beta+1)}{n}\Big) Y_n. \tag{4.6}$$

Hereafter, for any $n \geq 1$, let

$$\gamma_n = 1 + \frac{a(\beta+1)}{n} \tag{4.7}$$

and

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n)\Gamma(a(\beta+1)+1)}{\Gamma(n+a(\beta+1))}. \tag{4.8}$$

It follows from standard results on the Gamma function that

$$\lim_{n\to\infty} n^{a(\beta+1)} a_n = \Gamma(a(\beta+1)+1). \tag{4.9}$$

Our strategy for proving asymptotic results for the AERW is as follows. On the one hand, the behavior of the position S_n is closely related to the one of the sequences (M_n) and (N_n) defined, for all $n \geq 0$, by

$$M_n = a_n Y_n \quad \text{and} \quad N_n = S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)} \mu_n^{-1} Y_n. \tag{4.10}$$

We immediately get from (4.6) and (4.8) that (M_n) is a locally square-integrable martingale adapted to $(\mathcal{F}_n)$. Moreover, we have from (4.4) that

$$E\Big[S_{n+1} + \frac{a(\beta+1)}{\beta - a(\beta+1)} \mu_{n+1}^{-1} Y_{n+1} \mid \mathcal{F}_n\Big] = S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)} \mu_n^{-1} Y_n$$

which also means that (N_n) is also a locally square-integrable martingale adapted to $\mathcal{F}_n$. On the other hand, we can rewrite S_n as

$$S_n = N_n - \frac{a(\beta+1)}{\beta - a(\beta+1)} (\mu_n a_n)^{-1} M_n \tag{4.11}$$

and equation (4.11) allows us to establish the asymptotic behavior of the AERW via an extensive use of the same tools as in Chapters 2 and 3.

The main results of this chapter are given in Section 4.2. We first investigate the diffusive regime and we establish the strong law of large numbers, the law of iterated logarithm and the quadratic strong law for the AERW. The functional central limit theorem is also provided. Next, we prove similar results in the critical regime. Finally, we establish a strong limit theorem in the superdiffusive regime. Our martingale approach is described in Section 4.3. Finally, we give some of the technical proofs in Section 4.4.

4.2 – Main results

4.2.1 – The diffusive regime

Our first result deals with the strong law of large numbers for the AERW in the diffusive regime where $p < \frac{4\beta+3}{4(\beta+1)}$. The following strong law for the AERW will be deduced from both the strong laws for (N_n) and (M_n).

Theorem 4.1. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad a.s. \tag{4.12}$$

The almost sure rate of convergence for the AERW is as follows, for

$$\sigma_\beta^2 = \frac{2\beta + 1 - a}{(1-a)(1+2\beta-2a(\beta+1))}.$$

Theorem 4.2. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{S_k^2}{k^2} = \sigma_\beta^2 \quad a.s. \tag{4.13}$$

Hereafter, we are interested in the distributional convergence of the AERW, which holds in the Skorokhod space $D([0,\infty[)$ of right-continuous functions with left-hand limits.

Theorem 4.3. *The following convergence in distribution in $D([0,\infty[)$ holds*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big) \tag{4.14}$$

where $\big(W_t,\ t \geq 0\big)$ is a real-valued centered Gaussian process starting from the origin with covariance

$$\begin{aligned}\mathbb{E}[W_s W_t] = {} & \frac{a(1+\beta)(1-a)+a\beta}{(2(\beta+1)(1-a)-1)(a-\beta(1-a))(1-a)} s\Big(\frac{t}{s}\Big)^{a-\beta(1-a)} \\ & + \frac{\beta}{(\beta(1-a)-a)(1-a)} s\end{aligned} \tag{4.15}$$

for $0 < s \leq t$. In particular, we have

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \sigma_\beta^2\right). \tag{4.16}$$

Remark 4.4. *When $\beta = 0$ we find again the results from [3] for the ERW*

$$\Big(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}},\ t \geq 0\Big) \Longrightarrow \big(W_t,\ t \geq 0\big)$$

where $\big(W_t,\ t \geq 0\big)$ is a real-valued mean-zero Gaussian process starting from the origin and

$$\mathbb{E}[W_s W_t] = \frac{1}{1-2a} s\Big(\frac{t}{s}\Big)^a.$$

4.2.2 – The critical regime

Hereafter, we investigate the critical regime where $p = \frac{4\beta+3}{4(\beta+1)}$. It is interesting to notice that, when β is really large (or $\beta \to \infty$) the critical regime is reached for the memory parameter $p = 1$. Hence, the greater β is, the more there are values of the memory parameter p for which the AERW stays in the diffusive regime; but whatever the value of β, we still observe a phase transition.

Theorem 4.5. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}\log n} = 0 \quad a.s. \tag{4.17}$$

The almost sure rates of convergence for the AERW are as follows.

Theorem 4.6. *We have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \frac{S_k^2}{(k\log k)^2} = (2\beta+1)^2 \quad a.s. \tag{4.18}$$

In addition, we also have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{S_n^2}{2n\log n\log\log\log n} = (2\beta+1)^2 \quad a.s. \tag{4.19}$$

Once again, our next result concerns the asymptotic normality of the AERW.

Theorem 4.7. *The following convergence in distribution in $D([0,\infty[)$ holds*

$$\Big(\frac{S_{\lfloor n^t \rfloor}}{\sqrt{n^t\log n}},\ t \geq 0\Big) \Longrightarrow (2\beta+1)\big(B_t,\ t \geq 0\big) \tag{4.20}$$

where $(B_t,\ \ t \geq 0)$ is a one-dimensional standard Brownian motion. In particular, we have the asymptotic normality

$$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, (2\beta+1)^2\Big). \tag{4.21}$$

4.2.3 – The superdiffusive regime

Finally, we focus our attention on the superdiffusive regime where $p > \frac{4\beta+3}{4(\beta+1)}$.

Theorem 4.8. *We have the almost sure convergence*

$$\lim_{n\to\infty} \frac{S_n}{n^{a(\beta+1)-\beta}} = L_\beta \quad a.s. \tag{4.22}$$

where the limiting L_β is a non-degenerate random variable. We also have the mean square convergence

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{S_n}{n^{a(\beta+1)-\beta}} - L_\beta\Big|^2\Big] = 0. \tag{4.23}$$

Theorem 4.9. *The expected value of L_β is*

$$\mathbb{E}[L_\beta] = \frac{a(\beta+1)(2q-1)\Gamma(\beta+1)}{\big(a(\beta+1)-\beta\big)\Gamma\big(a(\beta+1)+1\big)} \tag{4.24}$$

while its second order moment is given by

$$\mathbb{E}\big[L_\beta^2\big] = \frac{a^2(\beta+1)^2\Gamma(\beta+1)^2\Gamma\big(2(a-1)(\beta+1)+1\big)}{\big(a(\beta+1)-\beta\big)^2\Gamma\big((2a-1)(\beta+1)+1\big)^2}. \tag{4.25}$$

Remark 4.10. *When $\beta = 0$ we find again the expected values for the ERW from [5]*

$$\mathbb{E}[L] = \frac{2q-1}{\Gamma(a+1)} \quad \textit{and} \quad \mathbb{E}[L^2] = \frac{1}{(2a-1)\Gamma(2a)}.$$

4.3 – A two-dimensional martingale approach

In order to investigate the asymptotic behavior of (S_n), we introduce the two-dimensional martingale $(\mathcal{M}_n)$ defined by

$$\mathcal{M}_n = \begin{pmatrix} N_n \\ M_n \end{pmatrix} \tag{4.26}$$

where (M_n) and (N_n) are the two locally square-integrable martingales introduced in (4.10). As for the CMERW and the RERW, the main difficulty we face is that the predictable quadratic variations of (M_n) and (N_n) increase to infinity with two different speeds. A matrix normalization will again be necessary to establish the asymptotic behavior of the AERW. We will alternatively study $(\mathcal{M}_n)$, (M_n) or (N_n). Denote the martingale increment $\varepsilon_{n+1} = Y_{n+1} - \gamma_n Y_n$. We obtain that

$$\begin{aligned}
\Delta\mathcal{M}_{n+1} &= \mathcal{M}_{n+1} - \mathcal{M}_n \\
&= \begin{pmatrix} S_{n+1} - S_n + \frac{a(\beta+1)}{\beta - a(\beta+1)}\big(\frac{Y_{n+1}}{\mu_{n+1}} - \frac{Y_n}{\mu_n}\big) \\ a_{n+1}Y_{n+1} - a_n Y_n \end{pmatrix} \\
&= \begin{pmatrix} \big(1 + \frac{a(\beta+1)}{\beta - a(\beta+1)}\big)X_{n+1} - \frac{a(\beta+1)}{(\beta - a(\beta+1))\mu_{n+1}}\frac{\beta}{n}Y_n \\ a_{n+1}\varepsilon_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\beta}{(\beta - a(\beta+1))\mu_{n+1}}\big(Y_n + X_{n+1}\mu_{n+1} - (\gamma_n - 1)Y_n\big) \\ a_{n+1}\varepsilon_{n+1}. \end{pmatrix}
\end{aligned}$$

Consequently

$$\Delta\mathcal{M}_{n+1} = \begin{pmatrix} \frac{\beta}{(\beta - a(\beta+1))\mu_{n+1}} \\ a_{n+1} \end{pmatrix}\varepsilon_{n+1}.$$

We also obtain that

$$\begin{aligned}
\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] &= \mathbb{E}[Y_{n+1}^2 \mid \mathcal{F}_n] - \gamma_n^2 Y_n^2 \\
&= Y_n^2 + 2(\gamma_n - 1)Y_n^2 + \mu_{n+1}^2 - \gamma_n^2 Y_n^2 \\
&= \mu_{n+1}^2 - (\gamma_n - 1)^2 Y_n^2.
\end{aligned} \tag{4.27}$$

Therefore, we deduce that

$$\mathbb{E}\big[(\Delta\mathcal{M}_{n+1})(\Delta\mathcal{M}_{n+1})^T \mid \mathcal{F}_n\big] = (\mu_{n+1}^2 - (\gamma_n - 1)^2\Upsilon_n^2) \begin{pmatrix} \big(\frac{\beta}{(\beta - a(\beta+1))\mu_{n+1}}\big)^2 & \frac{\beta a_{n+1}}{(\beta - a(\beta+1))\mu_{n+1}} \\ \frac{\beta a_{n+1}}{(\beta - a(\beta+1))\mu_{n+1}} & a_{n+1}^2 \end{pmatrix}.$$

We are now able to compute the quadratic variation of $\mathcal{M}_n$

$$\langle \mathcal{M} \rangle_n = \sum_{k=0}^{n-1} \begin{pmatrix} \big(\frac{\beta}{\beta - a(\beta+1)}\big)^2 & \frac{\beta a_{k+1}\mu_{k+1}}{\beta - a(\beta+1)} \\ \frac{\beta a_{k+1}\mu_{k+1}}{\beta - a(\beta+1)} & (a_{k+1}\mu_{k+1})^2 \end{pmatrix} - \xi_n \tag{4.28}$$

where

$$\xi_n = \sum_{k=0}^{n-1} (\gamma_k - 1)^2 \Upsilon_k^2 \begin{pmatrix} \big(\frac{\beta}{(\beta - a(\beta+1))}\big)^2 & \frac{\beta a_{k+1}\mu_{k+1}}{(\beta - a(\beta+1))} \\ \frac{\beta a_{k+1}\mu_{k+1}}{(\beta - a(\beta+1))} & (a_{k+1}\mu_{k+1})^2 \end{pmatrix}.$$

Hereafter, we immediately deduce from (4.28) that

$$\langle M \rangle_n = \sum_{k=1}^{n} (a_k \mu_k)^2 - \zeta_n \quad \text{where} \quad \zeta_n = \sum_{k=1}^{n} a_k^2 (\gamma_k - 1)^2 \Upsilon_k^2 \tag{4.29}$$

and

$$\langle N \rangle_n = \Big(\frac{\beta}{\beta - a(\beta+1)}\Big)^2 n. \tag{4.30}$$

The asymptotic behavior of M_n is closely related to the one of

$$w_n = \sum_{k=1}^{n} (a_k \mu_k)^2 \tag{4.31}$$

as one can observe that we always have $\langle M \rangle_n \leq w_n$ and that ζ_n is negligible when compared to w_n. Consequently, it follows from the definitions of (a_n) and (μ_n) that we have three regimes of behavior for (M_n). In the diffusive regime where is $p < \frac{4\beta+3}{4(\beta+1)}$ or $a < 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n\to\infty} \frac{w_n}{n^{1-2(a(\beta+1)-\beta)}} = \ell \quad \text{where} \quad \ell = \frac{1}{1 + 2(\beta - a(\beta+1))} \Big(\frac{\Gamma(a(\beta+1)+1)}{\Gamma(\beta+1)}\Big)^2. \tag{4.32}$$

In the critical regime where $p = \frac{4\beta+3}{4(\beta+1)}$ or $a = 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n\to\infty} \frac{w_n}{\log n} = \Big(\frac{\Gamma(\beta + 1 + \frac{1}{2})}{\Gamma(\beta+1)}\Big)^2. \tag{4.33}$$

In the superdiffusive regime where $p > \frac{4\beta+3}{4(\beta+1)}$ or $a > 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n\to\infty} w_n = \sum_{k=1}^{\infty} \Big(\frac{\Gamma(a(\beta+1)+1)\Gamma(k+\beta)}{\Gamma(k + a(\beta+1))\Gamma(\beta+1)}\Big)^2 < +\infty. \tag{4.34}$$

4.4 – Preview of the proofs

Lemma 4.11. *Let (V_n) be the sequence of positive definite diagonal matrices of order 2 given by*

$$V_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{a(\beta+1)}{\beta - a(\beta+1)}(a_n \mu_n)^{-1} \end{pmatrix}. \tag{4.35}$$

Let $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *such that*

$$v^T V_n \mathcal{M}_n = \frac{S_n}{\sqrt{n}}. \tag{4.36}$$

The quadratic variation of $\langle \mathcal{M} \rangle_n$ satisfies in the diffusive regime where is $a < 1 - \frac{1}{2(\beta+1)}$,

$$\lim_{n \to \infty} V_n \langle \mathcal{M} \rangle_n V_n^T = V \quad a.s. \tag{4.37}$$

where the matrix V is given by

$$V = \frac{1}{(\beta - a(\beta+1))^2} \begin{pmatrix} \beta^2 & \frac{a\beta}{1-a} \\ \frac{a\beta}{1-a} & \frac{a^2(\beta+1)^2}{1+2\beta-2a(\beta+1)} \end{pmatrix}. \tag{4.38}$$

Remark 4.12. *Following the same steps as in the proof of Lemma 4.11, we find that in the critical regime $a = 1 - \frac{1}{2(\beta+1)}$, the sequence of normalization matrices (V_n) has to be replaced by*

$$W_n = \frac{1}{\sqrt{n \log n}} \begin{pmatrix} 1 & 0 \\ 0 & 2a(\beta+1)(a_n \mu_n)^{-1} \end{pmatrix}. \tag{4.39}$$

The limit matrix V also needs to be replaced by

$$W = (2\beta+1)^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.40}$$

Proof of Lemma 4.11. We obtain from Theorem 4.1, equations (4.9) and (4.32) that

$$\begin{aligned} &\lim_{n \to \infty} V_n \langle \mathcal{M} \rangle_n V_n^T \\ &\quad = \lim_{n \to \infty} \frac{1}{n} \begin{pmatrix} \sum_{k=0}^{n-1} \left(\frac{\beta}{(\beta - a(\beta+1))} \right)^2 & \frac{a(\beta+1)\beta}{(\beta - a(\beta+1))^2 a_n \mu_n} \sum_{k=0}^{n-1} a_{k+1} \mu_{k+1} \\ \frac{a(\beta+1)\beta}{(\beta - a(\beta+1))^2 a_n \mu_n} \sum_{k=0}^{n-1} a_{k+1} \mu_{k+1} & \left(\frac{a(\beta+1)}{(\beta - a(\beta+1)) a_n \mu_n} \right)^2 \sum_{k=0}^{n-1} (a_{k+1} \mu_{k+1})^2 \end{pmatrix} \\ &\quad = \frac{1}{(\beta - a(\beta+1))^2} \begin{pmatrix} \beta^2 & \frac{a(\beta+1)\beta}{\beta+1-a(\beta+1)} \\ \frac{a(\beta+1)\beta}{\beta+1-a(\beta+1)} & \frac{a(\beta+1)^2}{2(\beta - a(\beta+1))+1} \end{pmatrix} \end{aligned}$$

which is exactly what we wanted to prove.

■

The proofs of all the results in Section 4.2 follow essentially the same lines as the ones explicited in Sections 3.6 and 3.7. We give two examples, both in the diffusive regime, in order to convince the reader.

Proof of Theorem 4.1. Again, we shall make extensive use of the strong law of large numbers for martingales, see Theorem A.2. First, we have for (M_n) that for any $\gamma > 0$,

$$M_n^2 = O\big((\log w_n)^{1+\gamma} w_n\big) \quad \text{a.s.}$$

which by definition of M_n and as a_n is asymptotically equivalent to $n^{-a(\beta+1)}$ and w_n is asymptotically equivalent to $n^{1+2(\beta-a(\beta+1))}$ ensures that

$$\frac{Y_n^2}{n^2} = O\Big((\log n)^{1+\gamma} \frac{n^{1+2(\beta-a(\beta+1))}}{n^{2(1-a(\beta+1))}}\Big) \quad \text{a.s.}$$

Finally as μ_n is asymptotically equivalent to n^β, we obtain that

$$\frac{Y_n^2}{(\mu_n n)^2} = O\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \quad \text{a.s.}$$

which reduces to

$$\lim_{n\to\infty} \frac{Y_n}{\mu_n n} = 0 \quad \text{a.s.} \tag{4.41}$$

We now focus our attention on (N_n). By the same token as before, we have that for any $\gamma > 0$,

$$N_n^2 = O\big((\log n)^{1+\gamma} n\big) \quad \text{a.s.}$$

which by definition of (N_n) gives us

$$\frac{\big(S_n - \frac{a(\beta+1)}{\beta-a(\beta+1)} \mu_n^{-1} Y_n\big)^2}{n^2} = O\Big(\frac{(\log n)^{1+\gamma}}{n}\Big) \quad \text{a.s.}$$

and we conclude that

$$\lim_{n\to\infty} \frac{S_n}{n} - \frac{a(\beta+1)}{\beta - a(\beta+1)} \frac{Y_n}{\mu_n n} = 0 \quad \text{a.s.} \tag{4.42}$$

This achieves the proof of Theorem 4.1 as the convergences (4.41) and (4.42) hold almost surely.

■

Proof of Theorem 4.3. In order to apply Theorem A.9 in the Appendix, we must verify that (H'.1), (H'.2) and (H.4) are satisfied.

(H'.1) We have from (4.37) and the fact that $a_{\lfloor nt \rfloor}$ is asymptotically equivalent to $t^{-a(\beta+1)} a_n$ that

$$V_n \langle \mathcal{M} \rangle_{\lfloor nt \rfloor} V_n^T \xrightarrow[n\to\infty]{} V_t \quad \text{a.s.}$$

where

$$V_t = \frac{1}{(\beta - a(\beta+1))^2} \begin{pmatrix} \beta^2 t & \frac{a\beta}{1-a} t^{1+\beta-a(\beta+1)} \\ \frac{a\beta}{1-a} t^{1+\beta-a(\beta+1)} & \frac{a^2(\beta+1)^2}{1+2\beta-2a(\beta+1)} t^{1+2\beta-2a(\beta+1)} \end{pmatrix}.$$

(H'.2) In order to verify that Lindeberg's condition is satisfied, we start by deducing from (4.10) together with (4.26) and V_n given by (4.35) that for all $1 \leq k \leq n$

$$V_n \Delta \mathcal{M}_n = \frac{1}{(\beta - a(\beta+1))\sqrt{n}\mu_n} \binom{\beta}{a} \varepsilon_n$$

which implies that

$$\|V_n \Delta \mathcal{M}_k\|^2 = \frac{1}{(\beta - a(\beta+1))^2 n \mu_n^2} (\beta^2 + a^2) \varepsilon_n^2. \tag{4.43}$$

Consequently, we obtain that for all $\varepsilon > 0$,

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big]. \tag{4.44}$$

It follows from (4.9) that

$$a_n^{-2} \sum_{k=1}^{n} a_k^2 = O(n) \quad \text{and} \quad a_n^{-4} \sum_{k=1}^{n} a_k^4 = O(n).$$

Hence, using that the sequence (ε_n) is bounded

$$\sup_{1 \leq k \leq n} |\varepsilon_k| \leq (\beta+2)\mu_n \quad \text{a.s.} \tag{4.45}$$

we find that

$$\sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4 \mid \mathcal{F}_{k-1}\big] = O\Big(\frac{1}{n}\Big) \quad \text{a.s.}$$

which ensures that Lindeberg's condition (H.2) holds almost surely, that is for all $\varepsilon > 0$,

$$\lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] = 0 \quad \text{a.s.} \tag{4.46}$$

Since $V_n V_{\lfloor nt \rfloor}^{-1}$ converges, we immediately obtain that

$$\begin{aligned}
\lim_{n \to \infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \mid \mathcal{F}_{k-1}\big] &\leq \lim_{n \to \infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^4\big] \\
&\leq \lim_{n \to \infty} \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big[\|(V_n V_{\lfloor nt \rfloor}^{-1}) V_{\lfloor nt \rfloor} \Delta \mathcal{M}_k\|^4\big] \\
&= 0 \quad \text{a.s.}
\end{aligned}$$

(H.4) In this particular case, we have $V_t = tK_1 + t^{\alpha_2} K_2 + t^{\alpha_3} K_3$ where

$$\alpha_2 = 1 - a(\beta+1) > 0 \quad \text{and} \quad \alpha_3 = 1 - 2a(\beta+1) > 0$$

as $a < 1 - \frac{1}{2(\beta+1)}$, and the matrices are symmetric

$$K_1 = \frac{\beta^2}{(\beta - a(\beta+1))^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; K_2 = \frac{a\beta}{(1-a)(\beta - a(\beta+1))^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$K_3 = \frac{a^2(\beta+1)^2}{(1+2\beta-2a(\beta+1))(\beta-a(\beta+1))^2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consequently, we obtain that

$$\big(V_n\mathcal{M}_{\lfloor nt \rfloor},\, t \geq 0\big) \Longrightarrow \big(\mathcal{B}_t,\, t \geq 0\big)$$

where $\mathcal{B}$ is defined as in (A.9). Finally, using the fact that $S_{\lfloor nt \rfloor}$ is asymptotically equivalent to $N_{\lfloor nt \rfloor} + t^{\beta-a(\beta+1)}\frac{a(\beta+1)}{\beta-a(\beta+1)}(\mu_n a_n)^{-1}M_{\lfloor nt \rfloor}$, and multiplying by $u_t = \begin{pmatrix} 1 \\ t^{a(\beta+1)-\beta} \end{pmatrix}$, we conclude

$$\big(\frac{1}{\sqrt{n}}S_{\lfloor nt \rfloor},\, t \geq 0\big) \Longrightarrow \big(W_t,\, t \geq 0\big) \tag{4.47}$$

where $W_t = u_t^T\mathcal{B}_t$. It only remains to compute the covariance function of (W_t) that is for $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}\big[W_sW_t\big] &= u_s^T\mathbb{E}\big[\mathcal{B}_s\mathcal{B}_t^T\big]u_t \\
&= u_s^TV_su_t \\
&= u_s^T\big(sK_1 + s^{1+\beta-a(\beta+1)}K_2 + s^{1+2\beta-2a(\beta+1)}K_3\big)u_t \\
&= \frac{\beta^2}{(\beta-a(\beta+1))^2}s + \frac{a\beta s^{1+\beta-a(\beta+1)}}{(1-a)(\beta-a(\beta+1))^2}\big(s^{a(\beta+1)-\beta} + t^{a(\beta+1)-\beta}\big) \\
&\quad + \frac{a^2(\beta+1)^2}{(1+2\beta-2a(\beta+1))(\beta-a(\beta+1))^2}s^{1+2\beta-2a(\beta+1)}(st)^{a(\beta+1)-\beta} \\
&= \frac{a(1+\beta)(1-a)+a\beta}{(2(\beta+1)(1-a)-1)(a-\beta(1-a))(1-a)}s\Big(\frac{t}{s}\Big)^{a-\beta(1-a)} \\
&\quad + \frac{\beta}{(\beta(1-a)-a)(1-a)}s.
\end{aligned}$$

■

5 Statistical estimation of the memory parameter

This chapter presents the results of [10] :

BERCU, B. AND LAULIN, L. How to estimate the memory of the Elephant Random Walk. *arXiv:2112.10405* (2021).

5.1 – Introduction

In this chapter, we aim to provide a statistical analysis of the ERW. We start by recalling to the reader that $X_1 \sim \mathcal{R}(q)$ and that, for all $n \geq 1$, the $(n+1)$-th step is performed by choosing at random an integer k among the previous times $1, \ldots, n$ and the elephant moves according to

$$X_{n+1} = \begin{cases} +X_k & \text{with probability} \quad p, \\ -X_k & \text{with probability} \quad 1-p. \end{cases} \tag{5.1}$$

The position of the ERW at time $n+1$ is given by

$$S_{n+1} = S_n + X_{n+1}. \tag{5.2}$$

We have already seen that the asymptotic behavior of the ERW is closely related to the value of the memory parameter p. Whatever the value of p in $[0,1]$, it has been shown that

$$\lim_{n\to\infty} \frac{S_n}{n} = 0 \quad \text{a.s.} \tag{5.3}$$

Moreover, it has been proven in the diffusive regime $0 \leq p < 3/4$ that

$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3-4p}\Big), \tag{5.4}$$

while in the critical regime $p = 3/4$ that

$$\frac{S_n}{\sqrt{n \log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1). \tag{5.5}$$

We refer the reader to [3, 25, 26, 5] and to the recent contributions [17, 24, 32, 64, 75]. In the superdiffusive regime $3/4 < p \leq 1$, it has been established by three different approaches [3, 5, 25] that

$$\lim_{n\to\infty} \frac{S_n}{n^{2p-1}} = L \quad \text{a.s.} \tag{5.6}$$

where L is a non-degenerate random variable which is not Gaussian [5]. However, the fluctuation of the ERW around its limit L is Gaussian [53] since, on the event $\{L^2 > 0\}$,

$$\sqrt{n^{4p-3}}\Big(\frac{S_n}{n^{2p-1}} - L\Big) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{4p-3}\Big). \tag{5.7}$$

In this chapter, we shall focus our attention on the parametric estimation of the memory parameter p. To the best of our knowledge, no one has tackled this statistical analysis. It has been mentioned by Heyde [47] that, in the case of generalized Bernoulli processes, it is possible to consistently estimate θ by maximum likelihood but that an explicit expression is not available for the estimator. Heyde added that the Fisher information for a sample of size n only increases proportionally to $\log n$ if $\theta < 1/2$, $(\log n)^2$ if $\theta = 1/2$ and $n^{2\theta-1}$ if $\theta > 1/2$, but gave no more details. We also find those speeds in our analysis.

Our estimator is explicitly given, for all $n \geq 2$, by

$$\widehat{p}_n = \frac{\displaystyle\sum_{k=1}^{n-1} \frac{S_k}{k}\Big(X_{k+1} + \frac{S_k}{k}\Big)}{\displaystyle 2\sum_{k=1}^{n-1}\Big(\frac{S_k}{k}\Big)^2}. \tag{5.8}$$

The chapter is organized as follows. In Section 5.2, we explain in detail how we are led to introduce the estimator $\widehat{p}_n$ via a quasi-maximum likelihood approach. Section 5.3 is devoted to the main results of the chapter. We show the almost sure convergence of $\widehat{p}_n$ to p whatever the value of the memory parameter. This preliminary estimation allows us to say whether the ERW is in the diffusive, critical or superdiffusive regimes. The local asymptotic normality of our statistical procedure is established in the diffusive regime, while the local asymptotic mixed normality is proven in the superdiffusive regime. In both regimes, asymptotic and exact confidence intervals as well as statistical tests are also provided. Our martingale approach is described in Section 5.4, while all technical proofs are postponed to Section 5.5.

5.2 – Quasi-maximum likelihood estimation

Denote by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ the σ-algebra of events occurring up to time n. It follows from (5.1) that for all $n \geq 1$,

$$\begin{aligned}\mathbb{P}(X_{n+1} = 1 \mid \mathcal{F}_n) &= \frac{p}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_k=1\}} + \frac{(1-p)}{n}\sum_{k=1}^{n} \mathbb{1}_{\{X_k=-1\}}, \\ &= \frac{p}{2n}\Big(n + S_n\Big) + \frac{(1-p)}{2n}\Big(n - S_n\Big), \\ &= \frac{1}{2}\Big(1 + (2p-1)\frac{S_n}{n}\Big).\end{aligned}$$

It clearly means that the conditional distribution of X_{n+1} given $\mathcal{F}_n$ is a Rademacher $\mathcal{R}(p_n)$ distribution where

$$p_n = \frac{1}{2}\Big(1 + a\frac{S_n}{n}\Big) \quad \text{and} \quad a = 2p - 1. \tag{5.9}$$

Therefore, we obtain that for $x_{n+1} \in \{-1, 1\}$

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid \mathcal{F}_n) = p_n^{(1+x_{n+1})/2}(1-p_n)^{(1-x_{n+1})/2}. \tag{5.10}$$

For all $n \geq 1$ and $x \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$, let $\mathbb{P}_p(x) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)$. We clearly deduce from (5.10) that for all $n \geq 2$,

$$\begin{aligned}\mathbb{P}_p(x) &= \prod_{k=1}^{n-1} \mathbb{P}(X_{k+1} = x_{k+1} \mid X_1 = x_1, \ldots, X_k = x_k)\mathbb{P}(X_1 = x_1), \\ &= \prod_{k=1}^{n-1} \Big(p_k^{(1+x_{k+1})/2}(1-p_k)^{(1-x_{k+1})/2}\Big) q^{(1+x_1)/2}(1-q)^{(1-x_1)/2}\end{aligned}$$

where, for all $1 \leq k \leq n$, S_k is replaced by $s_k = x_1 + \cdots + x_k$ in the definition of p_k. Consequently, the likelihood function associated with $(X_1, \ldots, X_n)$ is given by

$$L_n(p) = \prod_{k=1}^{n-1} \Big(p_k^{(1+X_{k+1})/2}(1-p_k)^{(1-X_{k+1})/2}\Big) q^{(1+X_1)/2}(1-q)^{(1-X_1)/2}. \tag{5.11}$$

It is easier to work with the log-likelihood function $\ell_n(p) = \log(L_n(p))$. We have from (5.11) that

$$\begin{aligned}\ell_n(p) = \sum_{k=1}^{n-1} \Big(\frac{1+X_{k+1}}{2}\Big)\log p_k + \Big(\frac{1-X_{k+1}}{2}\Big)\log(1-p_k) \\ + \Big(\frac{1+X_1}{2}\Big)\log q + \Big(\frac{1-X_1}{2}\Big)\log(1-q).\end{aligned} \tag{5.12}$$

Hence, if $\overline{X}_n$ stands for the empirical mean of $(X_1, \ldots, X_n)$, it follows from (5.9) and (5.12) that

$$\begin{aligned}\ell_n'(p) &= \sum_{k=1}^{n-1} (1 + X_{k+1}) \Big(\frac{\overline{X}_k}{1 + a\overline{X}_k} \Big) - (1 - X_{k+1}) \Big(\frac{\overline{X}_k}{1 - a\overline{X}_k} \Big), \\ &= \sum_{k=1}^{n-1} \frac{2\overline{X}_k (X_{k+1} - a\overline{X}_k)}{1 - a^2 \overline{X}_k^2}, \\ &= \sum_{k=1}^{n-1} \frac{2X_{k+1}\overline{X}_k}{1 + aX_{k+1}\overline{X}_k}. \end{aligned} \tag{5.13}$$

It is well-known that the process $(\ell_n'(p))$ is a locally square integrable martingale [45]. Its predictable quadratic variation is nothing else than the conditional Fisher information $I_n(p)$ associated with $(X_1, \ldots, X_n)$. We shall see that

$$I_n(p) = \sum_{k=1}^{n-1} \frac{\overline{X}_k^2}{p_k(1 - p_k)}. \tag{5.14}$$

It is not possible to find an explicit solution of the equation $\ell_n'(p) = 0$. However, we already saw from (5.3) that whatever the value of p in $[0, 1]$, $\overline{X}_n$ goes to zero almost surely. Consequently, it makes sense to replace $\ell_n(p)$ by its second order Taylor approximation

$$\begin{aligned}\lambda_n(p) = \sum_{k=1}^{n-1} a\overline{X}_k \Big(X_{k+1} - \frac{a}{2}\overline{X}_k \Big) - (n-1)\log 2 \\ + \Big(\frac{1 + X_1}{2} \Big) \log q + \Big(\frac{1 - X_1}{2} \Big) \log(1 - q). \end{aligned} \tag{5.15}$$

Since $a = 2p - 1$, (5.15) clearly implies that

$$\lambda_n'(p) = \sum_{k=1}^{n-1} 2X_{k+1}\overline{X}_k (1 - aX_{k+1}\overline{X}_k) \quad \text{and} \quad \lambda_n''(p) = -4 \sum_{k=1}^{n-1} \overline{X}_k^2.$$

Therefore, λ_n is a strictly concave function reaching its maximum at the value where its first derivative is equal to zero, which leads to

$$\widehat{p}_n = \frac{\displaystyle\sum_{k=1}^{n-1} \frac{S_k}{k} \Big(X_{k+1} + \frac{S_k}{k} \Big)}{\displaystyle 2 \sum_{k=1}^{n-1} \Big(\frac{S_k}{k} \Big)^2}.$$

It appears that our statistical approach is the most efficient strategy as it satisfies the local asymptotic normality (LAN) property in the diffusive regime and the local asymptotic mixed normality (LAMN) property in the superdiffusive regime [74].

5.3 – Main results

Our first result deals with the almost sure convergence of $\widehat{p}_n$ to p.

Theorem 5.1. *Whatever the value of the memory parameter p in $[0,1]$, $\widehat{p}_n$ is a strongly consistent estimator of p,*

$$\lim_{n\to\infty} \widehat{p}_n = p \qquad a.s. \tag{5.16}$$

5.3.1 – The diffusive regime

Our next result is devoted to the asymptotic normality of the estimator $\widehat{p}_n$ in the diffusive regime where $0 \leq p < 3/4$. Denote by $I(p)$ the asymptotic Fisher information

$$I(p) = \frac{4}{3-4p}. \tag{5.17}$$

Theorem 5.2. *We have the asymptotic normality*

$$\sqrt{\log n}(\widehat{p}_n - p) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{3-4p}{4}\Big). \tag{5.18}$$

It means that $\widehat{p}_n$ is an asymptotically efficient estimator of p. In particular,

$$2\sqrt{\log n}\frac{(\widehat{p}_n - p)}{\sqrt{3-4\widehat{p}_n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1). \tag{5.19}$$

We now focus our attention on the LAN property in the diffusive regime.

Theorem 5.3. *The sequence of experiments $(\mathbb{P}_n(p), p \in [0,3/4[)$ is locally asymptotically normal. More precisely, there exists a sequence of real random variables $(\Delta_n(p))$ such that*

$$\Delta_n(p) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, I(p)\big)$$

and for any sequence of real numbers (h_n) converging to h, the log-likelihood ratio satisfies

$$\log\Big(\frac{L_n(p + (\log n)^{-1/2}h_n)}{L_n(p)}\Big) = h\Delta_n(p) - \frac{h^2}{2}I(p) + o(1) \quad a.s. \tag{5.20}$$

Our next result concerns an asymptotic confidence interval for the memory parameter p.

Theorem 5.4. *In the diffusive regime and for any $0 < \alpha < 1$, we have the asymptotic confidence interval for p with confidence level $1-\alpha$,*

$$\mathcal{I}(p) = \Big[\widehat{p}_n - \frac{\sqrt{3-4\widehat{p}_n}}{2\sqrt{\log n}}t_{1-\alpha/2},\ \widehat{p}_n + \frac{\sqrt{3-4\widehat{p}_n}}{2\sqrt{\log n}}t_{1-\alpha/2}\Big] \tag{5.21}$$

where $t_{1-\alpha/2}$ stands for the $(1-\alpha/2)$-quantile of the standard $\mathcal{N}(0,1)$ distribution.

5.3.2 – The critical regime

We now focus our attention on the more complicated critical regime where $p = 3/4$. Denote by V_n a suitable approximation of the conditional Fisher information $I_n(p)$ given by (5.14),

$$V_n = 4 \sum_{k=1}^{n-1} \Big(\frac{S_k}{k} \Big)^2. \tag{5.22}$$

Theorem 5.5. *We have the convergence in distribution*

$$\frac{1}{(\log n)^2} V_n \xrightarrow[n\to\infty]{\mathcal{L}} 4\Lambda \tag{5.23}$$

where Λ stands for the integral of the squared standard Brownian motion

$$\Lambda = \int_0^1 B_t^2 \mathrm{d}t. \tag{5.24}$$

Remark 5.6. *It is impossible to prove the almost sure convergence as well as the convergence in probability in* (5.23)*. By the sharp analysis of Li [60, 61] concerning the L_2-norm of the Brownian motion, we can only show that*

$$\liminf_{n\to\infty} \Big(\frac{\log\log\log n}{(\log n)^2} \Big) V_n = \frac{1}{2} \quad a.s.$$

while

$$\limsup_{n\to\infty} \Big(\frac{1}{(\log n)^2 \log\log\log n} \Big) V_n = \frac{32}{\pi^2} \quad a.s.$$

This is the reason why we cannot establish the asymptotic normality of our estimator $\widehat{p}_n$ in the critical regime.

Remark 5.7. *It follows from the Karhunen-Loève expansion of the Brownian motion that*

$$\Lambda = \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2 \pi^2} \xi_n^2 \tag{5.25}$$

where (ξ_n) is a sequence of independent and identically distributed random variables with $\mathcal{N}(0,1)$ distribution, see e.g. Lemma 4 in [60]. Formula (5.25) *allows the numerical computation of the α-quantiles of Λ, see [51].*

5.3.3 – The superdiffusive regime

Our next result deals with the asymptotic normality of $\widehat{p}_n$ in the superdiffusive regime where $3/4 < p \leq 1$. We recall here that L is the limiting non-degenerate random variable given in (5.6). The conditional hypothesis $\{L^2 > 0\}$ we are working under is believed to be satisfied almost surely. However, it has not yet been proved as the study of the distribution of L appears to be a really hard problem.

Theorem 5.8. *Conditionally on the event* $\{L^2 > 0\}$, *we have the asymptotic normality*

$$\sqrt{V_n}(\widehat{p}_n - p) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1). \tag{5.26}$$

Figures 1 and 2 show the asymptotic normality of our estimator $\widehat{p}_n$ in the diffusive and superdiffusive regimes with $p = 0.4$ and $p = 0.9$, respectively. The density function of the standard normal distribution is in red and the bins represent $N = 3000$ different values of $\sqrt{V_n}(\widehat{p}_n - p)$ for $n = 1000$. We have used equation (5.26) to obtain both of the figures, as Theorem 5.8 is also true in the diffusive regime. In fact, using directly the approximation of V_n made in Theorem 5.2 can not provide such good convergence results by simulations in the diffusive regime since V_n increases almost surely to $4/(3-4p)$ with the slow speed $\log n$.

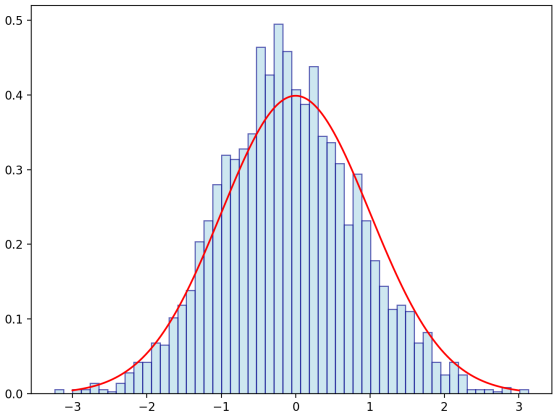


Figure 5.1: Asymptotic normality for $p = 0.4$.

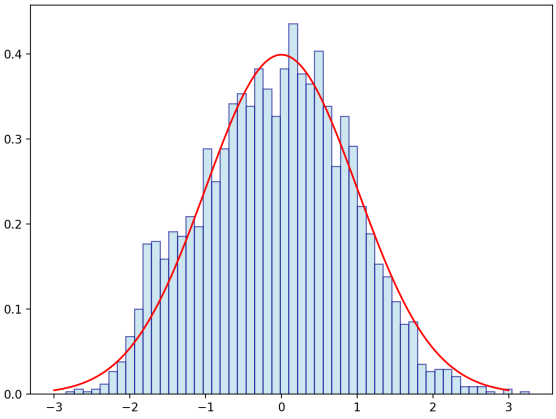


Figure 5.2: Asymptotic normality for $p = 0.9$.

The LAMN property in the superdiffusive regime is as follows.

Theorem 5.9. *Conditionally on the event* $\{L^2 > 0\}$, *the sequence of experiments* $(\mathbb{P}_n(p), p \in]3/4, 1])$ *is locally asymptotically mixed normal. More precisely, there exists two sequences of real random variables* $(\Delta_n(p))$ *and* $(J_n(p))$ *such that*

$$\big(\Delta_n(p), J_n(p)\big) \xrightarrow[n\to\infty]{\mathcal{L}} \big(\Delta(p), J(p)\big)$$

and that the conditional distribution of $\Delta(p)$ *given* $J(p) = J$ *is a standard* $\mathcal{N}(0, J)$ *distribution, and for any sequence of real numbers* (h_n) *converging to* h, *the log-likelihood ratio satisfies*

$$\log\left(\frac{L_n(p + (n^{4p-3})^{-1/2}h_n)}{L_n(p)}\right) = h\Delta_n(p) - \frac{h^2}{2}J_n(p) + o(1) \quad a.s. \tag{5.27}$$

We also propose an asymptotic confidence interval for the memory parameter p.

Theorem 5.10. *In the superdiffusive regime and for any* $0 < \alpha < 1$, *we have conditionally on the event* $\{L^2 > 0\}$, *the asymptotic confidence interval for* p *with confidence level* $1 - \alpha$,

$$\mathcal{I}(p) = \left[\widehat{p}_n - \frac{1}{\sqrt{V_n}}t_{1-\alpha/2},\ \widehat{p}_n + \frac{1}{\sqrt{V_n}}t_{1-\alpha/2}\right] \tag{5.28}$$

where $t_{1-\alpha/2}$ *stands for the* $(1-\alpha/2)$*-quantile of the standard* $\mathcal{N}(0,1)$ *distribution.*

5.3.4 – Exact confidence intervals

Our purpose is now to provide an exact confidence interval for the memory parameter p whatever its value in $[0,1]$.

Theorem 5.11. *For any $0 < \alpha < 1$, an exact confidence interval for p with confidence level $1-\alpha$ is given, for all $n \geq 1$, by*

$$\mathcal{J}(p) = \left[\widehat{p}_n - \frac{2\sqrt{3n\log(2/\alpha)}}{V_n}, \ \widehat{p}_n + \frac{2\sqrt{3n\log(2/\alpha)}}{V_n} \right]. \tag{5.29}$$

Moreover, in the diffusive regime with $1/4 \leq p < 3/4$, the exact confidence interval $\mathcal{J}(p)$ can be slightly improved by

$$\mathcal{K}(p) = \left[\widehat{p}_n - \frac{\sqrt{29n\log(2/\alpha)}}{\sqrt{3}V_n}, \ \widehat{p}_n + \frac{\sqrt{29n\log(2/\alpha)}}{\sqrt{3}V_n} \right]. \tag{5.30}$$

Remark 5.12. *Our confidence interval is better than the one obtained using Azuma-Hoeffding inequality which is given, for all $n \geq 3$, by*

$$\mathcal{A}(p) = \left[\widehat{p}_n - \frac{2\sqrt{8n\log(2/\alpha)}}{V_n}, \ \widehat{p}_n + \frac{2\sqrt{8n\log(2/\alpha)}}{V_n} \right]. \tag{5.31}$$

Figure 3 shows the three confidence intervals $\mathcal{I}(p)$, $\mathcal{J}(p)$ and $\mathcal{A}(p)$ in the superdiffusive regime with $p = 0.9$, for n varying from 1 to 100. As expected, the asymptotic confidence interval $\mathcal{I}(p)$ is always more accurate than $\mathcal{J}(p)$ and $\mathcal{A}(p)$, providing that the Gaussian approximation is justified. One can also observe that $\mathcal{J}(p)$ and $\mathcal{A}(p)$ are always true whatever the value of n and that $\mathcal{J}(p)$ is more accurate than $\mathcal{A}(p)$.

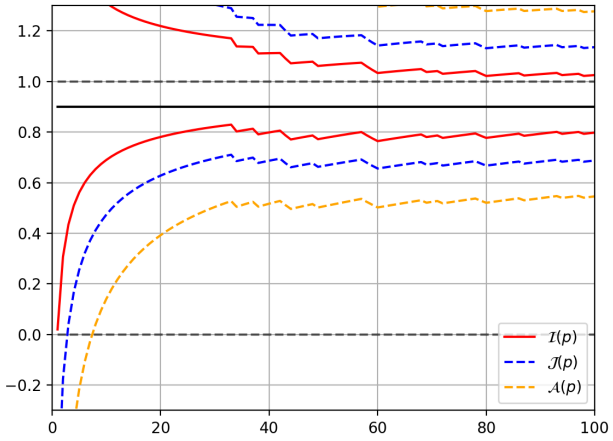


Figure 5.3: Confidence intervals for $p = 0.9$ and $\alpha = 0.05$.

5.3.5 – Statistical tests

We are now in position to propose a bilateral statistical test built on our statistic $\widehat{p}_n$. We start by fixing some memory value $0 < p_0 < 1$ such that $p_0 \neq 3/4$. Our goal is to test

$$\mathcal{H}_0 \ : \ \text{“}p = p_0\text{”} \quad \text{against} \quad \mathcal{H}_1 \ : \ \text{“}p \neq p_0\text{”}.$$

Theorem 5.13. *Under the null hypothesis* $\mathcal{H}_0$: "$p = p_0$",

$$V_n(\widehat{p}_n - p_0)^2 \xrightarrow[n\to\infty]{\mathcal{L}} \chi^2 \tag{5.32}$$

where χ^2 *has a Chi-square distribution with one degree of freedom. Moreover, under the alternative hypothesis* $\mathcal{H}_1$: "$p \neq p_0$",

$$\lim_{n\to\infty} V_n(\widehat{p}_n - p_0)^2 = +\infty \quad a.s. \tag{5.33}$$

For a significance level α where $0 < \alpha < 1$, the acceptance and rejection regions are given by $\mathcal{A} = [0, z_\alpha]$ and $\mathcal{R} =]z_\alpha, +\infty[$ where z_α stands for the $(1-\alpha)$-quantile of the Chi-square distribution with one degree of freedom. The null hypothesis $\mathcal{H}_0$ will not be rejected if the empirical value

$$V_n(\widehat{p}_n - p_0)^2 \leq z_\alpha$$

and will be rejected otherwise.

The purpose of our second test is to find out if the ERW is in the critical or the diffusive regime. Concretely, we wish to test

$$\mathcal{H}_0 : "p = 3/4" \quad \text{against} \quad \mathcal{H}_1 : "p < 3/4".$$

We immediately obtain Theorem 5.14, whose proof directly follows from (5.23).

Theorem 5.14. *Under the null hypothesis* $\mathcal{H}_0$: "$p = 3/4$",

$$\frac{1}{(\log n)^2} V_n \xrightarrow[n\to\infty]{\mathcal{L}} 4\Lambda \tag{5.34}$$

where Λ *is the integral of the squared Brownian motion given by* (5.24). *Moreover, under the alternative hypothesis* $\mathcal{H}_1$: "$p < 3/4$",

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} V_n = 0 \quad a.s. \tag{5.35}$$

For a significance level α where $0 < \alpha < 1$, the acceptance and rejection regions are given by $\mathcal{A} = [\lambda_{1-\alpha}, +\infty[$ and $\mathcal{R} = [0, \lambda_{1-\alpha}[$ where $\lambda_{1-\alpha}$ stands for the α-quantile of the random variable Λ which can be found in [51]. For example, $\lambda_{0.05} = 1.656$ and $\lambda_{0.10} = 1.196$. The null hypothesis $\mathcal{H}_0$ will not be rejected if the empirical value

$$\frac{1}{4(\log n)^2} V_n \geq \lambda_{1-\alpha}$$

and will be rejected otherwise. The goal of our third is to find out if the ERW is in the critical or superdiffusive regime. More precisely, we wish to test

$$\mathcal{H}_0 : "p = 3/4" \quad \text{against} \quad \mathcal{H}_1 : "p > 3/4".$$

Theorem 5.15. *Under the null hypothesis $\mathcal{H}_0$: "$p = 3/4$",*

$$\frac{1}{(\log n)^2} V_n \xrightarrow[n\to\infty]{\mathcal{L}} 4\Lambda \tag{5.36}$$

where Λ is given by (5.24)*. Moreover, under the alternative hypothesis $\mathcal{H}_1$: "$p > 3/4$" and conditionally to $\{L^2 > 0\}$,*

$$\lim_{n\to\infty} \frac{1}{(\log n)^2} V_n = +\infty \quad a.s. \tag{5.37}$$

For a significance level α where $0 < \alpha < 1$, the acceptance and rejection regions are given by $\mathcal{A} = [0, \lambda_\alpha]$ and $\mathcal{R} =]\lambda_\alpha, +\infty[$ where λ_α stands for the $(1-\alpha)$-quantile of Λ, see [51]. The null hypothesis $\mathcal{H}_0$ will not be rejected if the empirical value

$$\frac{1}{4(\log n)^2} V_n \leq \lambda_\alpha$$

and will be rejected otherwise.

5.4 – The martingale approach

We already saw at the beginning of Section 5.2 that for all $n \geq 1$,

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = a\Big(\frac{S_n}{n}\Big) \tag{5.38}$$

where $a = 2p - 1$. For all $n \geq 1$, let

$$\varepsilon_{n+1} = X_{n+1} - a\Big(\frac{S_n}{n}\Big)$$

with the initial value $\varepsilon_1 = X_1$. Since (X_n) is a binary sequence of random variables taking values in $\{+1, -1\}$, it clearly follows from (5.38) that (ε_n) is a martingale difference sequence such that for all $n \geq 1$,

$$\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] = 1 - a^2\Big(\frac{S_n}{n}\Big)^2. \tag{5.39}$$

Equation (5.39) immediately implies that

$$\sup_{n\geq 1} \mathbb{E}[\varepsilon_n^2] \leq 1.$$

Denote for all $n \geq 2$,

$$M_n = \sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)\varepsilon_{k+1} \tag{5.40}$$

with $M_1 = 0$. As $|S_n| \leq n$, (M_n) is a locally square integrable martingale. Its predictable quadratic variation is given by $\langle M\rangle_1 = 0$ and for all $n \geq 2$,

$$\langle M\rangle_n = \sum_{k=1}^{n-1} \mathbb{E}[\Delta M_{k+1}^2 \mid \mathcal{F}_k] = \sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2 \mathbb{E}[\varepsilon_{k+1}^2 \mid \mathcal{F}_k].$$

We obtain from (5.39) that

$$\langle M \rangle_n = \sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2 \Big(1 - a^2 \Big(\frac{S_k}{k}\Big)^2\Big) = \sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2 - a^2 \sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^4. \tag{5.41}$$

Consequently, we deduce from (5.3) and (5.22) that

$$\lim_{n\to\infty} \frac{V_n}{\langle M \rangle_n} = 4 \quad \text{a.s.} \tag{5.42}$$

which means that the asymptotic behavior of the martingale (M_n) is closely related to the one of the conditional Fisher information $I_n(p)$ and its approximation V_n. Moreover, we have from (5.8) that

$$\widehat{p}_n = \frac{\displaystyle\sum_{k=1}^{n-1} \frac{S_k}{k}\Big(X_{k+1} + \frac{S_k}{k}\Big)}{\displaystyle 2\sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2} = \frac{\displaystyle\sum_{k=1}^{n-1} \frac{S_k}{k}\Big(X_{k+1} - a\frac{S_k}{k}\Big) + (a+1)\sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2}{\displaystyle 2\sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2}$$

which reduces, via (5.40), to

$$\widehat{p}_n - p = \frac{2M_n}{V_n}. \tag{5.43}$$

It ensures that the study of the asymptotic behavior of $\widehat{p}_n$ can be achieved through convergence results for the martingale (M_n).

5.5 – Proofs of the main results

Proof of Theorem 5.1. In the diffusive regime $0 \leq p < 3/4$, we have from the quadratic strong law given by Theorem 3.2 in [5] that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \Big(\frac{S_k}{k}\Big)^2 = \frac{1}{3-4p} \quad \text{a.s.} \tag{5.44}$$

which implies that

$$\lim_{n\to\infty} \frac{V_n}{\log n} = \frac{4}{3-4p} \quad \text{a.s.} \tag{5.45}$$

In the critical regime $p = 3/4$, it follows once again from the quadratic strong law given by Theorem 3.5 in [5] that

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \Big(\frac{S_k}{k\log k}\Big)^2 = 1 \quad \text{a.s.}$$

leading to

$$\lim_{n\to\infty} V_n = +\infty \quad \text{a.s.}$$

In the superdiffusive regime $3/4 < p \leq 1$, we deduce from (5.3) together with Toeplitz's lemma that

$$\lim_{n\to\infty} \frac{1}{n^{4p-3}} \sum_{k=1}^{n} \Big(\frac{S_k}{k}\Big)^2 = \frac{L^2}{4p-3} \quad \text{a.s.} \tag{5.46}$$

which ensures that

$$\lim_{n\to\infty} \frac{V_n}{n^{4p-3}} = \frac{4L^2}{4p-3} \quad \text{a.s.} \tag{5.47}$$

where L is a non-degenerate random variable. Consequently, whatever the value of the memory parameter p in $[0,1]$, we obtain that V_n is increasing to infinity almost surely. Hence, we get from (5.42) that $\langle M \rangle_n$ also goes to infinty almost surely in the three regimes. Therefore, we can conclude from the strong law of large numbers for martingales, see Theorem A.1, that

$$\lim_{n\to\infty} \frac{M_n}{V_n} = 0 \quad \text{a.s.} \tag{5.48}$$

Finally, (5.43) together with (5.48) immediately lead to (5.16).

■

5.5.1 – The diffusive regime

Proof of Theorem 5.2. In the diffusive regime $0 \leq p < 3/4$, we already saw from (5.44) that

$$\lim_{n\to\infty} \frac{\langle M \rangle_n}{\log n} = \frac{1}{3-4p} \quad \text{a.s.}$$

Moreover, (M_n) satisfies the conditional Lindeberg condition, that is for all $\varepsilon > 0$,

$$\frac{1}{\log n} \sum_{k=1}^{n-1} \mathbb{E}\Big[|\Delta M_{k+1}|^2 \mathbb{1}_{|\Delta M_{k+1}| > \varepsilon \sqrt{\log n}} \mid \mathcal{F}_k \Big] \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

where, for all $n \geq 1$,

$$\Delta M_{n+1} = M_{n+1} - M_n = \Big(\frac{S_n}{n}\Big)\varepsilon_{n+1}.$$

As a matter of fact, as $|S_n| \leq n$, we clearly have $|\varepsilon_{n+1}| \leq 2$ and $|\Delta M_{n+1}| \leq 2$. Hence, we obtain that for all $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{\log n} \sum_{k=1}^{n-1} \mathbb{E}\Big[(\Delta M_{k+1})^2 \mathbb{1}_{|\Delta M_{k+1}| > \varepsilon \sqrt{\log n}} \mid \mathcal{F}_k \Big] &\leq \frac{1}{\varepsilon^2 (\log n)^2} \sum_{k=1}^{n-1} \mathbb{E}\Big[(\Delta M_{k+1})^4 \mid \mathcal{F}_k \Big], \\
&\leq \frac{4}{\varepsilon^2 \, (\log n)^2} \sum_{k=1}^{n-1} \Big(\frac{S_k}{k}\Big)^2, \\
&\leq \frac{V_n}{\varepsilon^2 (\log n)^2}.
\end{aligned}$$

Therefore, we clearly deduce from (5.45) that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n-1} \mathbb{E}\Big[|\Delta M_{k+1}|^2 \mathbb{1}_{|\Delta M_{k+1}| > \varepsilon \sqrt{\log n}} \mid \mathcal{F}_k \Big] = 0 \quad \text{a.s.}$$

which means that the conditional Lindeberg condition is satisfied. Hence, we can conclude from Corollary 3.1 in [42] that

$$\frac{M_n}{\sqrt{\langle M \rangle_n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1). \tag{5.49}$$

Finally, we obtain from (5.42) and (5.43) together with Slutsky's Lemma that

$$\sqrt{V_n}(\widehat{p}_n - p) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

leading via (5.45) to

$$\sqrt{\log n}(\widehat{p}_n - p) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{3-4p}{4}\Big).$$

One can observe that the asymptotic variance is the inverse of the Fisher information given by (5.17), which completes the proof of Theorem 5.2.

■

Proof of Theorem 5.3. As in the proof of Theorem 7.2 in [74] devoted to the Taylor expansion of the log-likelihood ratio, let

$$\log(1+x) = x - \frac{x^2}{2} + x^2 R(x)$$

where the function $R(x)$ tends to zero as x goes to zero. For any sequence of real numbers (h_n) converging to h, we have from (5.12) that

$$\begin{aligned}\ell_n(p + (\log n)^{-1/2}h_n) - \ell_n(p) = &\sum_{k=1}^{n-1}\Big(\frac{1+X_{k+1}}{2}\Big)\log\Big(1 + \frac{2(\log n)^{-1/2}h_n\overline{X}_k}{1+a\overline{X}_k}\Big)\\ &+ \sum_{k=1}^{n-1}\Big(\frac{1-X_{k+1}}{2}\Big)\log\Big(1 - \frac{2(\log n)^{-1/2}h_n\overline{X}_k}{1-a\overline{X}_k}\Big)\end{aligned}$$

Consequently, we obtain the Taylor expansion

$$\begin{aligned}\ell_n(p + (\log n)^{-1/2}h_n) - \ell_n(p) = &\sum_{k=1}^{n-1}(1+X_{k+1})\Big(\frac{(\log n)^{-1/2}h_n\overline{X}_k}{1+a\overline{X}_k} - \frac{(\log n)^{-1}h_n^2\overline{X}_k^2}{(1+a\overline{X}_k)^2}\Big)\\ &+ 2\sum_{k=1}^{n-1}(1+X_{k+1})\frac{(\log n)^{-1}h_n^2\overline{X}_k^2}{(1+a\overline{X}_k)^2}R\Big(\frac{2(\log n)^{-1/2}h_n\overline{X}_k}{1+a\overline{X}_k}\Big)\\ &- \sum_{k=1}^{n-1}(1-X_{k+1})\Big(\frac{(\log n)^{-1/2}h_n\overline{X}_k}{1-a\overline{X}_k} + \frac{(\log n)^{-1}h_n^2\overline{X}_k^2}{(1-a\overline{X}_k)^2}\Big)\\ &+ 2\sum_{k=1}^{n-1}(1-X_{k+1})\frac{(\log n)^{-1}h_n^2\overline{X}_k^2}{(1-a\overline{X}_k)^2}R\Big(\frac{2(\log n)^{-1/2}h_n\overline{X}_k}{1-a\overline{X}_k}\Big).\end{aligned}$$

From now on, we are going to make repeated use that (X_n) is a binary sequence of random variables taking values in $\{+1,-1\}$. We can split the log-likelihood ratio into three terms,

$$\ell_n(p + (\log n)^{-1/2}h_n) - \ell_n(p) = \frac{2h_n}{\sqrt{\log n}}P_n - \frac{2h_n^2}{\log n}Q_n + \frac{2h_n^2}{\log n}R_n \qquad (5.50)$$

where

$$P_n = \frac{1}{2}\sum_{k=1}^{n-1}\Big(\frac{(1+X_{k+1})\overline{X}_k}{1+a\overline{X}_k} - \frac{(1-X_{k+1})\overline{X}_k}{1-a\overline{X}_k}\Big) = \frac{1}{2}\sum_{k=1}^{n-1}\frac{2(X_{k+1}-a\overline{X}_k)\overline{X}_k}{1-(a\overline{X}_k)^2}$$
$$= \sum_{k=1}^{n-1}\frac{(X_{k+1}-a\overline{X}_k)\overline{X}_k}{X_{k+1}^2-(a\overline{X}_k)^2} = \sum_{k=1}^{n-1}\frac{\overline{X}_k}{X_{k+1}+a\overline{X}_k} = \sum_{k=1}^{n-1}\frac{X_{k+1}\overline{X}_k}{1+aX_{k+1}\overline{X}_k},$$

$$Q_n = \frac{1}{2}\sum_{k=1}^{n-1}\Big(\frac{(1+X_{k+1})\overline{X}_k^2}{(1+a\overline{X}_k)^2} + \frac{(1-X_{k+1})\overline{X}_k^2}{(1-a\overline{X}_k)^2}\Big) = \frac{1}{2}\sum_{k=1}^{n-1}\frac{2(1-2a\overline{X}_kX_{k+1}+a^2\overline{X}_k^2)\overline{X}_k^2}{(1-(a\overline{X}_k)^2)^2}$$
$$= \sum_{k=1}^{n-1}\frac{(X_{k+1}^2-2a\overline{X}_kX_{k+1}+a^2\overline{X}_k^2)\overline{X}_k^2}{(X_{k+1}^2-(a\overline{X}_k)^2)^2} = \sum_{k=1}^{n-1}\frac{(X_{k+1}-a\overline{X}_k)^2\overline{X}_k^2}{(X_{k+1}+a\overline{X}_k)^2(X_{k+1}-a\overline{X}_k)^2}$$
$$= \sum_{k=1}^{n-1}\frac{\overline{X}_k^2}{(X_{k+1}+a\overline{X}_k)^2} = \sum_{k=1}^{n-1}\frac{\overline{X}_k^2}{(1+aX_{k+1}\overline{X}_k)^2},$$

and

$$R_n = \sum_{k=1}^{n-1}\Big(\frac{(1+X_{k+1})\overline{X}_k^2}{(1+a\overline{X}_k)^2}R\Big(\frac{2(\log n)^{-1/2}h_n\overline{X}_k}{1+a\overline{X}_k}\Big) + \frac{(1-X_{k+1})\overline{X}_k^2}{(1-a\overline{X}_k)^2}R\Big(\frac{2(\log n)^{-1/2}h_n\overline{X}_k}{1-a\overline{X}_k}\Big)\Big).$$

On the one hand, we have

$$P_n = \sum_{k=1}^{n-1}\frac{X_{k+1}\overline{X}_k}{1+aX_{k+1}\overline{X}_k} = \sum_{k=1}^{n-1}\frac{X_{k+1}\overline{X}_k\big(1-aX_{k+1}\overline{X}_k\big)}{\big(1+aX_{k+1}\overline{X}_k\big)\big(1-aX_{k+1}\overline{X}_k\big)}$$
$$= \sum_{k=1}^{n-1}\frac{\overline{X}_k\big(X_{k+1}-a\overline{X}_k\big)}{1-a^2\overline{X}_k^2} = \sum_{k=1}^{n-1}\frac{\overline{X}_k\,\varepsilon_{k+1}}{1-a^2\overline{X}_k^2}.$$

It clearly means that the sequence (P_n) is a square integrable martingale. We obtain from (5.39) that the predictable quadratic variation associated with (P_n) is given by

$$\langle P\rangle_n = \sum_{k=1}^{n-1}\frac{\overline{X}_k^2}{\big(1-a^2\overline{X}_k^2\big)^2}\mathbb{E}[\varepsilon_{k+1}^2 \mid \mathcal{F}_k] = \sum_{k=1}^{n-1}\frac{\overline{X}_k^2}{1-a^2\overline{X}_k^2}.$$

Hence, we immediately deduce from (5.3) and (5.44) that

$$\lim_{n\to\infty}\frac{\langle P\rangle_n}{\log n} = \frac{1}{3-4p} \quad \text{a.s.}$$

Moreover, as it was previously done for the martingale (M_n), one can easily check that (P_n) satisfies the conditional Lindeberg condition. Consequently, it follows from Corollary 3.1 in [42] that

$$\frac{P_n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3-4p}\Big). \tag{5.51}$$

On the other hand, we also have from (5.3) and (5.22) that

$$Q_n = \sum_{k=1}^{n-1}\frac{\overline{X}_k^2}{(1+aX_{k+1}\overline{X}_k)^2} = \frac{1}{4}V_n + o(V_n) \quad \text{a.s.} \tag{5.52}$$

In the same way,

$$|R_n| = o\left(\sum_{k=1}^{n-1} \frac{\overline{X}_k^2}{(1+a\overline{X}_k)^2}\right) + o\left(\sum_{k=1}^{n-1} \frac{\overline{X}_k^2}{(1-a\overline{X}_k)^2}\right) = o(V_n) \quad \text{a.s.} \tag{5.53}$$

Finally, we obtain from the conjunction of (5.50), (5.51), (5.52) and (5.53) that

$$\ell_n(p + (\log n)^{-1/2} h_n) - \ell_n(p) = h_n \Delta_n(p) - \frac{h_n^2}{2} \frac{V_n}{\log n} + o(1) \quad \text{a.s.} \tag{5.54}$$

where

$$\Delta_n(p) = \frac{2P_n}{\sqrt{\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{4}{3-4p}\Big),$$

which is exactly what we wanted to prove.

■

Proof of Theorem 5.4. The proof directly follows from Theorem 5.2. Indeed, we obtain from the asymptotic normality (5.19) that for any $0 < \alpha < 1$,

$$\lim_{n\to\infty} \mathbb{P}\Big(2\sqrt{\log n} \frac{|\widehat{p}_n - p|}{\sqrt{3-4\widehat{p}_n}} \leq t_{1-\alpha/2}\Big) = 1 - \alpha$$

where $t_{1-\alpha/2}$ stands for the $(1-\alpha/2)$-quantile of the standard $\mathcal{N}(0,1)$ distribution. Moreover, one can easily see that

$$\mathbb{P}\Big(2\sqrt{\log n} \frac{|\widehat{p}_n - p|}{\sqrt{3-4\widehat{p}_n}} \leq t_{1-\alpha/2}\Big) = \mathbb{P}\Big(\widehat{p}_n - \frac{\sqrt{3-4\widehat{p}_n}}{2\sqrt{\log n}} t_{1-\alpha/2} \leq p \leq \widehat{p}_n + \frac{\sqrt{3-4\widehat{p}_n}}{2\sqrt{\log n}} t_{1-\alpha/2}\Big).$$

It implies that

$$\lim_{n\to\infty} \mathbb{P}\Big(p \in \Big[\widehat{p}_n - \frac{\sqrt{3-4\widehat{p}_n}}{2\sqrt{\log n}} t_{1-\alpha/2},\ \widehat{p}_n + \frac{\sqrt{3-4\widehat{p}_n}}{2\sqrt{\log n}} t_{1-\alpha/2}\Big]\Big) = 1 - \alpha,$$

which completes the proof of Theorem 5.4.

■

5.5.2 – The critical regime

Proof of Theorem 5.5. It follows from (5.2) and (5.38) with $a = 1/2$ that for all $n \geq 1$,

$$S_{n+1} = \Big(1 + \frac{1}{2n}\Big) S_n + \varepsilon_{n+1}.$$

It clearly implies that for all $n \geq 1$,

$$\overline{X}_{n+1} = \Big(1 - \frac{1}{2(n+1)}\Big) \overline{X}_n + \frac{1}{n+1} \varepsilon_{n+1}. \tag{5.55}$$

Consequently, we obtain from (5.55) that for all $n \geq 2$,

$$\overline{X}_n = \prod_{k=2}^{n} \Big(1 - \frac{1}{2k}\Big) X_1 + \sum_{k=2}^{n} \prod_{i=k+1}^{n} \Big(1 - \frac{1}{2i}\Big) \frac{1}{k} \varepsilon_k$$

leading to

$$\overline{X}_n = \frac{\Gamma(n+1/2)}{\Gamma(n+1)}\Big(\frac{2}{\sqrt{\pi}}X_1 + \mathcal{M}_n\Big) \tag{5.56}$$

where

$$\mathcal{M}_n = \sum_{k=2}^{n} \frac{\Gamma(k)}{\Gamma(k+1/2)}\varepsilon_k. \tag{5.57}$$

We already saw that (ε_n) is a martingale difference sequence satisfying (5.39). Hence, $(\mathcal{M}_n)$ is a locally square integrable martingale with predictable quadratic variation given, for all $n \geq 2$,

$$\langle \mathcal{M} \rangle_n = \sum_{k=2}^{n} \Big(\frac{\Gamma(k)}{\Gamma(k+1/2)}\Big)^2 - \frac{1}{4}\sum_{k=2}^{n} \Big(\frac{\Gamma(k)}{\Gamma(k+1/2)}\Big)^2 \overline{X}_k^2. \tag{5.58}$$

Moreover, one can easily see that $\mathbb{E}[S_n^2] = nH_n$ where H_n stands for the harmonic number

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Therefore, we obtain from (5.58) that

$$\langle \mathcal{M} \rangle_n - \mathbb{E}[\langle \mathcal{M} \rangle_n] = -\frac{1}{4}\sum_{k=2}^{n} \Big(\frac{\Gamma(k)}{\Gamma(k+1/2)}\Big)^2 \Big(\overline{X}_k^2 - \frac{H_k}{k}\Big). \tag{5.59}$$

On the one hand, we have for all $n \geq 1$,

$$n + \frac{1}{4} < \Big(\frac{\Gamma(n+1)}{\Gamma(n+1/2)}\Big)^2 < n + \frac{1}{2},$$

which is equivalent to

$$\frac{1}{n} + \frac{1}{4n^2} < \Big(\frac{\Gamma(n)}{\Gamma(n+1/2)}\Big)^2 < \frac{1}{n} + \frac{1}{2n^2}.$$

It ensures that

$$\sum_{n=2}^{\infty} \Big(\frac{\Gamma(n)}{\Gamma(n+1/2)}\Big)^2 \frac{H_n}{n} < \sum_{n=2}^{\infty} \frac{2H_n}{n^2} < \infty. \tag{5.60}$$

On the other hand, it follows from the quadratic strong law for the ERW given in Theorem 3.5 of [5] that

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{\overline{X}_k^2}{(\log k)^2} = 1 \quad \text{a.s.} \tag{5.61}$$

Hence, we get from (5.61) together with Toeplitz lemma [31] that

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \frac{\overline{X}_k^2}{k} = 0 \quad \text{a.s.}$$

which leads to

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=2}^{n} \Big(\frac{\Gamma(k)}{\Gamma(k+1/2)}\Big)^2 \overline{X}_k^2 = 0 \quad \text{a.s.} \tag{5.62}$$

Thus, we obtain from (5.59), (5.60) and (5.62) that

$$\langle \mathcal{M} \rangle_n - \mathbb{E}[\langle \mathcal{M} \rangle_n] = o(\log\log n) \quad \text{a.s.}$$

Therefore, we deduce from the strong invariance principle for martingales given in Theorem 2.1 of [70] with $a_n = \log\log n$ and $b_n = \log n$ that

$$\mathcal{M}_n - B_{\log n} = o(\log\log n) \quad \text{a.s.} \tag{5.63}$$

Consequently, we obtain from (5.56) and (5.63) the decomposition

$$\overline{X}_n = \frac{\Gamma(n+1/2)}{\Gamma(n+1)} B_{\log n} + R_n \quad \text{a.s.} \tag{5.64}$$

where the remainder R_n satisfies

$$\sum_{k=1}^{n} R_k^2 = o\big(\log n (\log\log n)^2\big) \quad \text{a.s.} \tag{5.65}$$

In order to prove (5.23), it only remains to show that

$$\frac{1}{(\log n)^2} \sum_{k=1}^{n} \Big(\frac{\Gamma(k+1/2)}{\Gamma(k+1)}\Big)^2 B_{\log k}^2 \xrightarrow[n\to\infty]{\mathcal{L}} \Lambda \tag{5.66}$$

where Λ is the integral of the squared standard Brownian motion

$$\Lambda = \int_0^1 B_t^2 \mathrm{d}t.$$

We have for all $n \geq 1$,

$$\frac{1}{n+1} < \Big(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\Big)^2 < \frac{1}{n}.$$

Consequently, the left-hand side in (5.66) shares the same asymptotic behavior as

$$\frac{1}{(\log n)^2} \sum_{k=1}^{n} \frac{1}{k} B_{\log k}^2.$$

Moreover, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k} B_{\log k}^2 &= \int_1^n \frac{1}{t} B_{\log t}^2 \mathrm{d}t + o(\log n) \quad \text{a.s.} \\ &= \int_0^{\log n} B_s^2 \mathrm{d}s + o(\log n) \quad \text{a.s.} \end{aligned} \tag{5.67}$$

using the change of variables $s = \log t$. Hereafter, it follows from the self-similarity of the Brownian motion that

$$\int_0^{\log n} B_s^2 \mathrm{d}s \overset{\mathcal{L}}{=} (\log n)^2 \int_0^1 B_t^2 \mathrm{d}t = (\log n)^2 \Lambda. \tag{5.68}$$

Finally, we deduce from (5.64), (5.65), (5.67) and (5.68) that

$$\frac{1}{(\log n)^2} \sum_{k=1}^{n} \overline{X}_k^2 \xrightarrow[n\to\infty]{\mathcal{L}} \Lambda$$

which completes the proof of Theorem 5.5.

■

5.5.3 – The superdiffusive regime

Proof of Theorem 5.8. In the superdiffusive regime $3/4 < p \leq 1$, we already saw from (5.46) that

$$\lim_{n\to\infty} \frac{\langle M \rangle_n}{n^{4p-3}} = \frac{L^2}{4p-3} \quad \text{a.s.}$$

■

Moreover, as it was previously done in the diffusive regime, it is not hard to see that (M_n) satisfies the conditional Lindeberg condition. Hence, it follows from (5.43) together with Corollary 3.2 in [42] that, conditionally on the event $\{L^2 > 0\}$, we have the asymptotic normality

$$\sqrt{V_n}(\widehat{p}_n - p) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

Proof of Theorem 5.9. As it was previously done in the proof of Theorem 5.3, we can split the log-likelihood ratio into three terms,

$$\ell_n(p + (n^{4p-3})^{-1/2} h_n) - \ell_n(p) = \frac{2h_n}{\sqrt{n^{4p-3}}} P_n - \frac{2h_n^2}{n^{4p-3}} Q_n + \frac{2h_n^2}{n^{4p-3}} R_n \quad \text{a.s.} \tag{5.69}$$

where the random variables P_n and Q_n are exactly the same, while the speed $\log n$ is replaced by n^{4p-3} in the expression of R_n. We immediately deduce from (5.6) and (5.46) that

$$\lim_{n\to\infty} \frac{\langle P \rangle_n}{n^{4p-3}} = \frac{L^2}{4p-3} \quad \text{a.s.}$$

Once again, (P_n) satisfies the conditional Lindeberg condition in the superdiffusive regime. Hence, it follows from Corollary 3.2 in [42] that, conditionally on the event $\{L^2 > 0\}$,

$$\frac{P_n}{\sqrt{n^{4p-3}}} \xrightarrow[n\to\infty]{\mathcal{L}} L \times \mathcal{N}\Big(0, \frac{1}{4p-3}\Big). \tag{5.70}$$

Hereafter, we obtain from equations (5.69), (5.70), (5.52) and (5.53) that

$$\ell_n(p + (n^{4p-3})^{-1/2} h_n) - \ell_n(p) = h_n \Delta_n(p) - \frac{h_n^2}{2} J_n(p) + o(1) \quad \text{a.s.} \tag{5.71}$$

where

$$\Delta_n(p) = \frac{2P_n}{\sqrt{n^{4p-3}}} \quad \text{and} \quad J_n(p) = \frac{V_n}{n^{4p-3}}.$$

Finally, as the convergence in (5.70) is stable [42], we immediately have that

$$\big(\Delta_n(p), J_n(p)\big) \xrightarrow[n\to\infty]{\mathcal{L}} \big(\Delta(p), J(p)\big)$$

where

$$J(p) = \frac{4L^2}{4p-3}.$$

In addition, conditionally on the event $\{J(p) = J\}$,

$$\Delta_n(p) = \frac{2P_n}{\sqrt{n^{4p-3}}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, J),$$

which completes the proof of Theorem 5.9.

■

5.5.4 – Exact confidence intervals

Proof of Theorem 5.11. In order to prove Theorem 5.11, we shall make use of concentration inequalities for martingales [8]. First of all, one can observe that (M_n) is a bounded difference martingale as equation (5.40) implies that for all $n \geq 2$,

$$|\Delta M_n| = \left|\varepsilon_n \overline{X}_{n-1}\right| \leq (1 + |a|)\left|\overline{X}_{n-1}\right| \quad \text{a.s.}$$

Inspired by the Azuma-Hoeffding inequality for bounded difference martingales, denote

$$B_n = (1 + |a|)^2 \sum_{k=1}^{n-1} \overline{X}_k^2.$$

Since $|a| \leq 1$, we clearly have from (5.41) that

$$5\langle M \rangle_n + B_n \leq \frac{1}{4}\big(5 + (1 + |a|)^2\big) V_n \leq 9n.$$

Hence, Theorem 3.4 in [8] ensures again that for any $x > 0$,

$$\mathbb{P}(|M_n| \geq x) \leq 2\exp\Big(-\frac{x^2}{3n}\Big). \tag{5.72}$$

Consequently, it follows from (5.43) and (5.72) that for any $x > 0$,

$$\mathbb{P}(V_n|\widehat{p}_n - p| \geq 2nx) \leq 2\exp\Big(-\frac{nx^2}{3}\Big). \tag{5.73}$$

Hereafter, denote

$$\alpha = 2\exp\Big(-\frac{nx^2}{3}\Big).$$

As soon as $nx^2 > 3\log(2)$, the value $0 < \alpha < 1$. Therefore, we deduce from (5.73) that an exact confidence interval for p, with confidence level $1 - \alpha$, is given by

$$\mathcal{J}(p) = \left[\widehat{p}_n - \frac{2\sqrt{3n\log(2/\alpha)}}{V_n},\ \widehat{p}_n + \frac{2\sqrt{3n\log(2/\alpha)}}{V_n}\right].$$

In the diffusive regime with $1/4 \leq p < 3/4$, we have $|a| \leq 1/2$ which implies that

$$5\langle M \rangle_n + B_n \leq \frac{1}{4}\big(5 + (1 + |a|)^2\big) V_n \leq \frac{29}{4}n.$$

Hence, proceeding as in the previous calculation, we obtain the exact confidence interval for p, with confidence level $1 - \alpha$,

$$\mathcal{K}(p) = \left[\widehat{p}_n - \frac{\sqrt{29n\log(2/\alpha)}}{\sqrt{3}V_n},\ \widehat{p}_n + \frac{\sqrt{29n\log(2/\alpha)}}{\sqrt{3}V_n}\right].$$

■

5.5.5 – Statistical tests

Proof of Theorem 5.13. The proof is quite straightforward. As a matter of fact, we already know from (5.18) or (5.26) that under the null hypothesis $\mathcal{H}_0$,

$$\sqrt{V_n}(\widehat{p}_n - p_0) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$$

which immediately implies (5.32). It only remains to show that under the alternative hypothesis $\mathcal{H}_1$, our test's statistic goes to infinity. Under $\mathcal{H}_1$, we obtain from Theorem 5.1 that

$$\lim_{n\to\infty} \widehat{p}_n - p_0 = p - p_0 \quad \text{a.s.}$$

and this limit is not zero. Consequently,

$$\lim_{n\to\infty} V_n(\widehat{p}_n - p_0)^2 = +\infty \quad \text{a.s.}$$

as we already saw that whatever the value of the memory parameter p in $[0,1]$, V_n is increasing to infinity almost surely, completing the proof of Theorem 5.13.

■

Proof of Theorem 5.14. The first part of Theorem 5.14 immediately follows from (5.23) under the null hypothesis. It only remains to show that under the alternative hypothesis $\mathcal{H}_1$, our test's statistic goes to 0. Under $\mathcal{H}_1$, we have from equation (5.45) that

$$\lim_{n\to\infty} \frac{V_n}{\log n} = \frac{4}{3-4p} \quad \text{a.s.}$$

which clearly implies that

$$\lim_{n\to\infty} \frac{V_n}{(\log n)^2} = 0 \quad \text{a.s.}$$

which is exactly what we wanted to prove.

■

Proof of Theorem 5.15. The first part of Theorem 5.15 follows once again from (5.23) under the null hypothesis. It only remains to show that under the alternative hypothesis $\mathcal{H}_1$, our test's statistic goes to infinity. Under $\mathcal{H}_1$, we have from equation (5.47) that

$$\lim_{n\to\infty} \frac{V_n}{n^{4p-3}} = \frac{4L^2}{4p-3} \quad \text{a.s.}$$

which implies that, conditionally on $\{L^2 > 0\}$,

$$\lim_{n\to\infty} \frac{V_n}{(\log n)^2} = +\infty \quad \text{a.s.}$$

completing the proof of Theorem 5.15.

■

6 A martingale approach for Pòlya urn processes

This chapter presents the results of [56] :

LAULIN, L. A martingale approach for Pólya urn processes. *Electron. Commun. Probab. 25* (2020), 13 pp.

6.1 – Introduction

At the inital time $n = 0$, an urn is filled with $\alpha \geq 0$ red balls and $\beta \geq 0$ white balls. Then, at any time $n \geq 1$ one ball is drawn randomly from the urn and its color observed. If it is red it is then returned to the urn together with a additional red balls and $b \geq 0$ white ones. If it is white it is then returned to the urn together with $c \geq 0$ additional red balls and d white ones. The model corresponding replacement matrix is given, for $a, b, c, d \in \mathbb{N}$, by

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The urn process is said to be *balanced* if the total number of balls added at each step is a constant, $S = a + b = c + d \geq 1$. Thanks to the balance assumption, S is the maximum eigenvalue of R^T. In fact, S is the Perron–Frobenius eigenvalue so it is simple. Moreover, the second eigenvalue of R^T is given by $m = a - c = d - b$. Throughout the rest of this chapter, our processes will be balanced and we shall denote

$$\sigma = m/S < 1$$

the ratio of the two eigenvalues. It is straightforward that the respective eigenvectors of R^T are given by

$$v_1 = \frac{S}{b+c}\begin{pmatrix} c \\ b \end{pmatrix} \quad \text{and} \quad v_2 = \frac{S}{b+c}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can rewrite R^T under the following form

$$R^T = PDP^{-1} = \frac{1}{b+c}\begin{pmatrix} c & 1 \\ b & -1 \end{pmatrix}\begin{pmatrix} S & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 & 1 \\ b & -c \end{pmatrix}.$$

Hereafter, let us define the process (U_n), the composition of the urn at time n, by

$$U_n = \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \quad \text{and} \quad U_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where X_n is the number of red balls and Y_n is the number of white ones. Then, let $\tau = \alpha + \beta \geq 1$ and $\tau_n = \tau + nS$ be the number of balls inside the urn at time n. In particular, one can observe that $X_n + Y_n = \tau_n$ is a deterministic quantity.
The traditional Pólya urn model corresponds to the case where the replacement matrix R is diagonal, while the generalized Pólya urn model corresponds to the case where the replacement matrix R is not diagonal.

The questions about the asymptotic behavior of (U_n) have been extensively studied, firstly by Freedman [35] and by many after, see for example [23, 33, 34, 48, 67, 49]. We also refer the reader to Pouyanne's CIMPA summer school lectures 2014 [68] for a very comprehensive survey on Pólya urn processes that has been a great source of inspiration. The reader may notice that this study is related to Bercu [5] on the elephant random walk. This is due to the paper of Baur and Bertoin [3] on the connection between elephant random walks and Pólya-type urns.

Our strategy is to use the martingale theory [31, 42] in order to propose a direct proof of the asymptotic normality associated with (U_n). We also establish new refinements on the almost sure convergence of (U_n). The chapter is organized as follows. In Section 6.2, we briefly present the traditional Pólya urn model, as well as the martingale related to this case. We establish the almost sure convergence and the asymptotic normality for this martingale. In Section 6.3, we present the generalized Pólya urn model with again the martingale related to this case, and we also give the main results for this model. Hence, we first investigate small urn regime where $\sigma \leq 1/2$ and we establish the almost sure convergence, the law of iterated logarithm and the quadratic strong law for (U_n). The asymptotic normality of the urn composition is also provided. We finally study the large urn where $\sigma > 1/2$ and we prove the almost sure convergence as well as the mean square convergence of (U_n) to a non-degenerate random vector whose moments are given. The proofs are postponed to Sections 6.4 and 6.5.

6.2 – Traditional Pólya urn model

This model corresponds to the case where the replacement matrix is diagonal

$$R = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}.$$

It means that at any time $n \geq 1$, one ball is drawn randomly from the urn, its color observed and it is then returned to the urn together with $S \geq 1$ additional balls of the same color. Let us define the process (M_n) by

$$M_n = \frac{X_n}{\tau_n}$$

and write

$$X_n = \alpha + S \sum_{k=1}^{n} \varepsilon_k$$

where the conditional distribution of ε_{n+1} given the past up to time n is $\mathcal{L}(\varepsilon_{n+1} \mid \mathcal{F}_n) = \mathcal{B}(M_n)$ and $\mathcal{B}$ denotes the Bernoulli distribution. We clearly have

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$$

which means that (M_n) is a martingale. We have $\Delta M_{n+1} = \frac{S}{\tau_{n+1}}\big(\varepsilon_{n+1} - M_n\big)$. Hence,

$$\mathbb{E}\big[\Delta M_{n+1}^2 | \mathcal{F}_n\big] = \frac{S^2}{\tau_{n+1}^2}\Big(\mathbb{E}\big[\varepsilon_{n+1}^2 | \mathcal{F}_n\big] - M_n^2\Big) = \frac{S^2 M_n(1 - M_n)}{\tau_{n+1}^2}.$$

We now focus our attention on the asymptotic behavior of (M_n).

Theorem 6.1. *The process (M_n) converges to a random variable M_∞ almost surely and in any $\mathbb{L}^p$ for $p \geq 1$. The limit M_∞ has a beta distribution, with parameters $\frac{\alpha}{S}$ and $\frac{\beta}{S}$.*

Remark 6.2. *This result was first proved by Freedman, Theorem 2.2 in [35].*

Our first new result on the Gaussian fluctuation of (M_n) is as follows.

Theorem 6.3. *We have the following convergence in distribution*

$$\sqrt{n}\frac{M_\infty - M_n}{\sqrt{M_n(1 - M_n)}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, 1\big). \tag{6.1}$$

6.3 – Generalized Pólya urn model

This model corresponds to the case where the replacement matrix is not diagonal,

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let us rewrite

$$X_n = \alpha + a\sum_{k=1}^{n}\varepsilon_k + c\sum_{k=1}^{n}(1-\varepsilon_k)$$

where the conditional distribution of ε_{n+1} given the past up to time n is $\mathcal{L}(\varepsilon_{n+1} \mid \mathcal{F}_n) = \mathcal{B}(\tau_n^{-1}X_n)$. We have

$$U_{n+1} = U_n + R^T\begin{pmatrix}\varepsilon_{n+1}\\ 1-\varepsilon_{n+1}\end{pmatrix}$$

and

$$U_n - \mathbb{E}[U_n] = \begin{pmatrix}X_n - \mathbb{E}[X_n]\\ Y_n - \mathbb{E}[Y_n]\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}\big(X_n - \mathbb{E}[X_n]\big) = \frac{b+c}{S}\big(X_n - \mathbb{E}[X_n]\big)v_2.$$

Hence, we obtain that

$$\begin{aligned}
\mathbb{E}\big[U_{n+1} - \mathbb{E}[U_{n+1}] \mid \mathcal{F}_n\big] &= U_n - \mathbb{E}[U_n] + R^T\mathbb{E}\Big[\begin{pmatrix}\varepsilon_{n+1}\\ 1-\varepsilon_{n+1}\end{pmatrix} - \mathbb{E}\big[\begin{pmatrix}\varepsilon_{n+1}\\ 1-\varepsilon_{n+1}\end{pmatrix}\big] \mid \mathcal{F}_n\Big]\\
&= \big(I_2 + \tau_n^{-1}R^T\big)\Big(U_n - \mathbb{E}[U_n]\Big)\\
&= \big(I_2 + \tau_n^{-1}R^T\big)\begin{pmatrix}1\\ -1\end{pmatrix}\big(X_n - E[X_n]\big)\\
&= \big(1 + \tau_n^{-1}m\big)\begin{pmatrix}1\\ -1\end{pmatrix}\big(X_n - E[X_n]\big)\\
&= \big(1 + \tau_n^{-1}m\big)\big(U_n - \mathbb{E}[U_n]\big). \qquad (6.2)
\end{aligned}$$

Finally, denote

$$\sigma_n = \prod_{k=0}^{n-1}\big(1 + \tau_k^{-1}m\big)^{-1} = \frac{\Gamma(n + \frac{\tau}{S})\Gamma(\frac{\tau}{S} + \sigma)}{\Gamma(\frac{\tau}{S})\Gamma(n + \frac{\tau}{S} + \sigma)}. \qquad (6.3)$$

One can observe that

$$\lim_{n\to\infty} n^{\sigma}\sigma_n = \lambda \quad \text{where} \quad \lambda = \frac{\Gamma(\frac{\tau}{S} + \sigma)}{\Gamma(\frac{\tau}{S})}. \qquad (6.4)$$

Hereafter, we define the process (M_n) by

$$M_n = \sigma_n\big(U_n - \mathbb{E}[U_n]\big). \qquad (6.5)$$

Thanks to equation (6.2), we immediately get that

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n.$$

Hence, the sequence (M_n) is a locally bounded and square integrable martingale. We are now allowed to compute the quadratic variation of (M_n). First of all

$$\Delta M_{n+1} = m\sigma_{n+1}\big(\varepsilon_{n+1} - \mathbb{E}[\varepsilon_{n+1} \mid \mathcal{F}_n]\big)\begin{pmatrix}1\\ -1\end{pmatrix} = m\sigma_{n+1}\big(\varepsilon_{n+1} - \tau_n^{-1}X_n\big)\begin{pmatrix}1\\ -1\end{pmatrix}. \qquad (6.6)$$

Moreover,

$$\mathbb{E}\big[\big(\varepsilon_{n+1} - \tau_n^{-1}X_n\big)^2\big|\mathcal{F}_n\big] = \tau_n^{-1}X_n\big(1 - \tau_n^{-1}X_n\big). \tag{6.7}$$

Consequently, we obtain from (6.6) and (6.7) that

$$\mathbb{E}\big[\Delta M_{n+1}\Delta M_{n+1}^T\big|\mathcal{F}_n\big] = m^2\sigma_{n+1}^2\tau_n^{-1}X_n\big(1 - \tau_n^{-1}X_n\big)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{6.8}$$

Therefore

$$\begin{aligned} \langle M\rangle_n &= \sum_{k=0}^{n-1}\mathbb{E}\big[\Delta M_{k+1}\Delta M_{k+1}^T\big|\mathcal{F}_k\big] \\ &= m^2\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\sum_{k=0}^{n-1}\sigma_{k+1}^2\tau_k^{-1}X_k\big(1 - \tau_k^{-1}X_k\big). \end{aligned} \tag{6.9}$$

As $\tau_k^{-1}X_k\big(1 - \tau_k^{-1}X_k\big) \leq \frac{1}{4}$, it is not hard to see that

$$\mathrm{Tr}\langle M\rangle_n \leq m^2 w_n \quad \text{where} \quad w_n = \sum_{k=1}^{n}\sigma_k^2. \tag{6.10}$$

The asymptotic behavior of (M_n) is closely related to the one of (w_n) with the following trichotomy:

- The diffusive regime where $\sigma < 1/2$: The urn is said to be small and we have
$$\lim_{n\to\infty}\frac{w_n}{n^{1-2\sigma}} = \frac{\lambda^2}{1-2\sigma}.$$
- The critical regime where $\sigma = 1/2$: The urn is said to be critically small and we have
$$\lim_{n\to\infty}\frac{w_n}{\log n} = \frac{\Gamma(\frac{\tau}{S} + \frac{1}{2})}{\Gamma(\frac{\tau}{S})}.$$
- The superdiffusive regime where $\sigma > 1/2$: The urn is said to be large and we have
$$\lim_{n\to\infty} w_n = \sum_{k=0}^{\infty}\Big(\frac{\Gamma(k + \frac{\tau}{S})\Gamma(\frac{\tau}{S} + \sigma)}{\Gamma(\frac{\tau}{S})\Gamma(k + \frac{\tau}{S} + \sigma)}\Big)^2 < +\infty.$$

Proposition 6.4. *We have for small and large urns*

$$\mathbb{E}[U_n] = nv_1 + \sigma_n^{-1}\Big(\frac{b\alpha - c\beta}{S}\Big)v_2 + \frac{\tau}{S}v_1. \tag{6.11}$$

Proof of Proposition 6.4. First of all, denote $\Lambda_n = I_2 + \tau_n^{-1}R^T = P\big(I_2 + \tau_n^{-1}D\big)P^{-1}$ where I_2 is the identity matrix of order 2, and $T_n = \prod_{k=0}^{n-1}\Lambda_k$. For any $n \in \mathbb{N}$, T_n is diagonalisable and

$$T_n = PD_nP^{-1} = \frac{1}{b+c}\begin{pmatrix} c & 1 \\ b & -1 \end{pmatrix}\begin{pmatrix} \tau_n/\tau & 0 \\ 0 & \sigma_n^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ b & -c \end{pmatrix}.$$

Since $E[U_{n+1} \mid \mathcal{F}_n] = \Lambda_n U_n$ we easily get that $\mathbb{E}[U_n] = T_n U_0$, which leads to

$$\begin{aligned}\mathbb{E}[U_n] &= \frac{1}{b+c}\Big(\frac{\tau_n}{\tau}\begin{pmatrix} c & c \\ b & b \end{pmatrix} + \sigma_n^{-1}\begin{pmatrix} b & -c \\ -b & c \end{pmatrix}\Big)U_0 \\ &= nv_1 + \frac{\tau}{S}v_1 + \sigma_n^{-1}\frac{b\alpha - c\beta}{S}v_2.\end{aligned}$$

■

6.3.1 – Small urns

The almost sure convergence of (U_n) for small urns is due to Janson, Theorem 3.16 in [48].

Theorem 6.5. *When the urn is small, $\sigma < 1/2$, we have the following convergence*

$$\lim_{n\to\infty} \frac{U_n}{n} = v_1 \tag{6.12}$$

almost surely and in any $\mathbb{L}^p$, $p \geq 1$.

Our new refinements on the almost sure rates of convergence are as follows.

Theorem 6.6. *When the urn is small and $bc \neq 0$, we have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k^2}(U_k - kv_1)(U_k - kv_1)^T = \frac{1}{1-2\sigma}\frac{bcm^2}{(b+c)^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad a.s. \tag{6.13}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log n}\sum_{k=1}^{n}\frac{\|U_k - kv_1\|^2}{k^2} = \frac{2}{1-2\sigma}\frac{bcm^2}{(b+c)^2} \quad a.s. \tag{6.14}$$

Moreover, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|U_n - nv_1\|^2}{2n\log\log n} = \frac{2}{1-2\sigma}\frac{bcm^2}{(b+c)^2} \quad a.s. \tag{6.15}$$

Remark 6.7. *The law of iterated logarithm for (X_n) was previously established by Bai, Hu and Zhang via a strong approximation argument, see Corollary 2.1 in [1].*

Theorem 6.8. *When the urn is small and $bc \neq 0$, we have the following asymptotic normality*

$$\frac{U_n - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, K) \tag{6.16}$$

where $K = \dfrac{1}{1-2\sigma}\dfrac{bcm^2}{(b+c)^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Remark 6.9. *An invariance principle for (X_n) was proved by Gouet, see Proposition 2.1 in [37].*

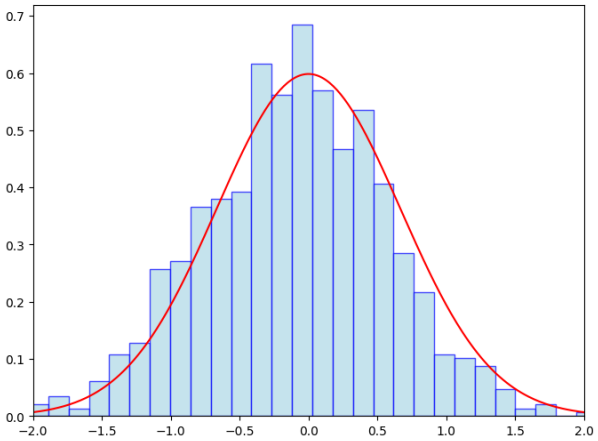


Figure 6.1: Asymptotic normality for $\alpha = 1$, $\beta = 2$ and $R = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

6.3.2 – Critically small urns

The almost sure convergence of (U_n) for critically small urns is again due to Janson, Theorem 3.16 in [48].

Theorem 6.10. *When the urn is critically small, $\sigma = 1/2$, we have the following convergence*

$$\lim_{n\to\infty} \frac{U_n}{n} = v_1 \tag{6.17}$$

almost surely and in any $\mathbb{L}^p$, $p \geq 1$.

Once again, we have some refinements on the almost sure rates of convergence.

Theorem 6.11. *When the urn is critically small and $bc \neq 0$, we have the quadratic strong law*

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \frac{1}{(k\log k)^2} (U_k - kv_1)(U_k - kv_1)^T = bc \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad a.s. \tag{6.18}$$

In particular,

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \frac{\|U_k - kv_1\|^2}{(k\log k)^2} = 2bc \quad a.s. \tag{6.19}$$

Moreover, we have the law of iterated logarithm

$$\limsup_{n\to\infty} \frac{\|U_n - nv_1\|^2}{2\log n \log\log\log n} = 2bc \quad a.s. \tag{6.20}$$

Remark 6.12. *The law of iterated logarithm for (X_n) was also established by Bai, Hu and Zhang via a strong approximation argument, see Corollary 2.2 in [1].*

Theorem 6.13. *When the urn is critically small and $bc \neq 0$, we have the following asymptotic normality*

$$\frac{U_n - nv_1}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, K) \tag{6.21}$$

where $K = bc \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Remark 6.14. *An invariance principle for* (X_n) *was also proved by Gouet, see Proposition 2.1 in [37].*

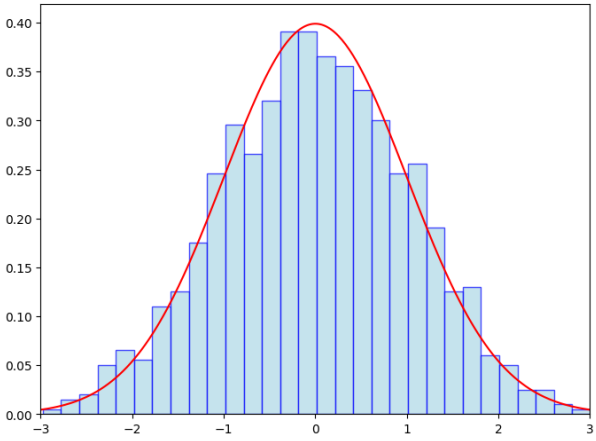


Figure 6.2: Asymptotic normality in the case of critically small urns for $\alpha = \beta = 1$ and $R = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

6.3.3 – Large urns

The convergences of $n^{-\sigma}(U_n - nv_1)$ to Wv_2 first appeared in Pouyanne [67], Theorem 3.5. The almost sure convergence of (U_n) for large urns is again due to Janson, Theorem 3.16 in [48]. The explicit calculations of the moments of W are new.

Theorem 6.15. *When the urn is large,* $\sigma > 1/2$*, we have the following convergence*

$$\lim_{n\to\infty} \frac{U_n}{n} = v_1 \tag{6.22}$$

almost surely and in any $\mathbb{L}^p$*,* $p \geq 1$*. Moreover, we have*

$$\lim_{n\to\infty} \frac{U_n - nv_1}{n^\sigma} = Wv_2 \tag{6.23}$$

almost surely and in $\mathbb{L}^2$*, where* W *is a real-valued random variable and*

$$\mathbb{E}[W] = \frac{\Gamma(\frac{\tau}{S})}{\Gamma(\frac{\tau}{S}+\sigma)} \frac{b\alpha - c\beta}{S}, \tag{6.24}$$

$$\mathbb{E}[W^2] = \sigma^2 \frac{\Gamma(\frac{\tau}{S})}{\Gamma(\frac{\tau}{S}+2\sigma)} \Big(\frac{bc}{2\sigma - 1} \frac{\tau}{S} + (b-c) \frac{b\alpha - c\beta}{\sigma S} + \frac{(b\alpha - c\beta)^2}{\sigma^2 S^2} \Big). \tag{6.25}$$

6.4 – Proofs of the almost sure convergence results

6.4.1 – Generalized urn model – small urns

Proof of Theorem 6.5. We denote the maximum eigenvalue of $\langle M\rangle_n$ by $\lambda_{max}\langle M\rangle_n$. We make use of the strong law of large numbers for martingales given by Theorem A.3, that is for any $\gamma > 0$,

$$\frac{\|M_n\|^2}{\lambda_{max}\langle M\rangle_n} = o\big((\log \mathrm{Tr}\langle M\rangle_n)^{1+\gamma}\big) \quad \text{a.s.}$$

It follows from (6.10) that

$$\|M_n\|^2 = o\big(w_n(\log w_n)^{1+\gamma}\big) \quad \text{a.s.}$$

which implies

$$\|M_n\|^2 = o\big(n^{1-2\sigma}(\log n)^{1+\gamma}\big) \quad \text{a.s.}$$

Hence, we deduce from (6.4) and (6.5) that

$$\|U_n - \mathbb{E}[U_n]\|^2 = o\big(n(\log n)^{1+\gamma}\big) \quad \text{a.s.}$$

which completes the proof for the almost sure convergence. The convergence in any $\mathbb{L}^p$ for $p \geq 1$ holds since $n^{-1}\|U_n - \mathbb{E}[U_n]\|$ is uniformly bounded by $2\sqrt{2}(\tau + S)$.

■

Proof of Theorem 6.6. We shall make use of Theorem A.5. For any $u \in \mathbb{R}^2$ let $M_n(u) = \langle u, M_n\rangle$ and denote $f_n = \dfrac{\sigma_n^2}{w_n}$. We have from (6.4) and (6.10) that f_n is asymptotically equivalent to $(1-2\sigma)n^{-1}$ and converges to 0. Moreover, we obtain from equations (6.9), (6.12) and the Toeplitz lemma that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{w_n}\langle M\rangle_n &= \lim_{n\to\infty}\frac{m^2}{w_n}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\sum_{k=0}^{n-1}\sigma_{k+1}^2\tau_k^{-1}X_k\big(1-\tau_k^{-1}X_k\big)\\ &= \frac{bcm^2}{(b+c)^2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \quad \text{a.s.}\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{w_n}\langle M\rangle_n = (1-2\sigma)K \quad \text{a.s.} \tag{6.26}$$

where K is the covariance matrix from Theorem 6.8.. Therefore, we get from (6.26) that

$$\lim_{n\to\infty}\frac{1}{\log w_n}\sum_{k=1}^{n} f_k\Big(\frac{M_k(u)^2}{w_k}\Big) = (1-2\sigma)u^TKu \quad \text{a.s.}$$

which leads to

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n} f_k^2 u^T(U_k - E[U_k])(U_n - E[U_k])^T u = (1-2\sigma)^2 u^TKu \quad \text{a.s.}$$

Furthermore, we have from (6.11) that $\mathbb{E}[U_n]$ is asymptotically equivalent to nv_1. Consequently, we obtain that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^2} (U_k - kv_1)(U_k - kv_1)^T = K \quad \text{a.s.}$$

We now focus our attention on the law of iterated logarithm. We already saw that

$$\sum_{n=1}^{\infty} \frac{\sigma_n^4}{w_n^2} < \infty.$$

Hence, it follows from the law of iterated logarithm for real martingales that first appeared in Stout [71, 72], see Theorem A.4, that for any $u \in \mathbb{R}^d$,

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{\sqrt{2w_n \log\log w_n}} M_n(u) &= -\liminf_{n\to\infty} \frac{1}{\sqrt{2w_n \log\log w_n}} M_n(u) \\ &= \sqrt{(1-2\sigma) u^T K u} \quad \text{a.s.} \end{aligned}$$

Consequently, as $M_n(u) = \sigma_n \langle u, U_n - \mathbb{E}[U_n] \rangle$, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \langle u, U_n - \mathbb{E}[U_n] \rangle &= -\liminf_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \langle u, U_n - \mathbb{E}[U_n] \rangle \\ &= \sqrt{u^T K u} \quad \text{a.s.} \end{aligned}$$

In particular, for any vector $u \in \mathbb{R}^2$

$$\limsup_{n\to\infty} \frac{1}{2n \log\log n} u^T (U_n - \mathbb{E}[U_n])(U_n - \mathbb{E}[U_n])^T u = u^T K u \quad \text{a.s.}$$

Finally, we deduce from $U_n - \mathbb{E}[U_n] = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (X_n - \mathbb{E}[X_n])$ and for $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{2n \log\log n} \|U_n - \mathbb{E}[U_n]\|^2 &= \limsup_{n\to\infty} \frac{2}{2n \log\log n} (X_n - \mathbb{E}[X_n])^2 \\ &= \frac{2}{1-2\sigma} \frac{bcm^2}{(b+c)^2} \quad \text{a.s.} \end{aligned}$$

which together with (6.11) completes the proof of Theorem 6.6.

■

6.4.2 – Generalized urn model – critically small urns

Proof of Theorem 6.10. Again, we make use of the strong law of large numbers for martingales given e.g. by Theorem A.3, that is for any $\gamma > 0$,

$$\frac{\|M_n\|^2}{\lambda_{max} \langle M \rangle_n} = o\big((\log \mathrm{Tr} \langle M \rangle_n)^{1+\gamma} \big) \quad \text{a.s.}$$

Since $\mathrm{Tr}\langle M\rangle_n \leq m^2 w_n$ and the quadratic version of M_n is a semi-definite positive matrix we have $\lambda_{max}\langle M\rangle_n \leq m^2 w_n$ so that

$$\|M_n\|^2 = o\big(w_n(\log w_n)^{1+\gamma}\big) \quad \text{a.s.}$$

which implies

$$\|M_n\|^2 = o\big(\log n(\log\log n)^{1+\gamma}\big) \quad \text{a.s.}$$

Moreover, by definition of M_n and using σ_n equivalent we get

$$\|U_n - \mathbb{E}[U_n]\|^2 = o\big(\sqrt{n}\log n(\log\log n)^{1+\gamma}\big) \quad \text{a.s.}$$

which completes the proof for the almost sure convergence. The convergence in any $\mathbb{L}^p$ for $p \geq 1$ holds by the same arguments as in the proof of Theorem 6.5.

■

Proof of Theorem 6.11. We shall once again make use of Theorem A.5. For any $u \in \mathbb{R}^2$ let $M_n(u) = \langle u, M_n\rangle$ and denote $f_n = \dfrac{\sigma_n^2}{w_n}$. We have from (6.4) that f_n is equivalent to $(n\log n)-1$ and converges to 0. When $\sigma = 1/2$ we have $b + c = m$. Moreover, we obtain from equations (6.9), (6.17) and Toeplitz lemma that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{w_n}\langle M\rangle_n &= \lim_{n\to\infty}\frac{m^2}{w_n}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\sum_{k=0}^{n-1}\sigma_{k+1}^2\tau_k^{-1}X_k\big(1-\tau_k^{-1}X_k\big)\\ &= bc\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \quad \text{a.s.}\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\frac{1}{w_n}\langle M\rangle_n = K \quad \text{a.s.} \tag{6.27}$$

Therefore, we get from (6.26) that

$$\lim_{n\to\infty}\frac{1}{\log w_n}\sum_{k=1}^{n} f_k\Big(\frac{M_k(u)^2}{w_k}\Big) = u^T K u \quad \text{a.s.}$$

which leads to

$$\lim_{n\to\infty}\frac{1}{\log\log n}\sum_{k=1}^{n} f_k^2 u^T(U_k - E[U_k])(U_n - E[U_k])^T u = u^T K u \quad \text{a.s.}$$

Consequently, we obtain from (6.11) that

$$\lim_{n\to\infty}\frac{1}{\log\log n}\sum_{k=1}^{n}\frac{1}{(k\log k)^2}(U_k - kv_1)(U_k - kv_1)^T = K \quad \text{a.s.}$$

We now focus our attention on the law of iterated logarithm. It is not hard to see that

$$\sum_{n=1}^{\infty}\frac{\sigma_n^4}{w_n^2} < \infty.$$

Hence, it follows once again from the law of iterated logarithm for real martingales, see Theorem A.4, that for any $u \in \mathbb{R}^d$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{\sqrt{2w_n \log\log w_n}} M_n(u) &= -\liminf_{n\to\infty} \frac{1}{\sqrt{2w_n \log\log w_n}} M_n(u) \\ &= \sqrt{u^T K u} \quad \text{a.s.}\end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} &\frac{1}{\sqrt{2\log n \log\log\log n}} \langle u, U_n - \mathbb{E}[U_n]\rangle \\ &= -\liminf_{n\to\infty} \frac{1}{\sqrt{2\log n \log\log\log n}} \langle u, U_n - \mathbb{E}[U_n]\rangle \\ &= \sqrt{u^T K u} \quad \text{a.s.}\end{aligned}$$

In particular, for any vector $u \in \mathbb{R}^2$

$$\limsup_{n\to\infty} \frac{1}{2\log n \log\log\log n} u^T (U_n - \mathbb{E}[U_n])(U_n - \mathbb{E}[U_n])u = u^T K u \quad \text{a.s.}$$

Finally, we deduce once again from $U_n - \mathbb{E}[U_n] = \begin{pmatrix} 1 \\ -1 \end{pmatrix} (X_n - \mathbb{E}[X_n])$ and for $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ that

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{2\log n \log\log\log n} \|U_n - \mathbb{E}[U_n]\|^2 &= \limsup_{n\to\infty} \frac{2}{2\log n \log\log\log n} (X_n - \mathbb{E}[X_n])^2 \\ &= 2bc \quad \text{a.s.}\end{aligned}$$

which together with (6.11) completes the proof of Theorem 6.11.

■

6.4.3 – Generalized urn model – large urns

Proof of Theorem 6.15. First, as $\mathrm{Tr}\langle M\rangle_n \leq m^2 w_n < \infty$, we have that (M_n) converges almost surely to a random vector Mv_2, where M is a real-valued random variable and

$$\lim_{n\to\infty} \sigma_n \big(X_n - \mathbb{E}[X_n]\big) = \frac{S}{b+c} M = \frac{1}{1-\sigma} M \quad \text{a.s.}$$

Hence, it follows from (6.5) that

$$\lim_{n\to\infty} \sigma_n (U_n - \mathbb{E}[U_n]) = Mv_2 \quad \text{a.s.} \tag{6.28}$$

which implies via (6.4) that

$$\lim_{n\to\infty} \sigma_n \|U_n - \mathbb{E}[U_n]\| = \lim_{n\to\infty} \frac{\lambda}{n^\sigma} \|U_n - \mathbb{E}[U_n]\| = \|Mv_2\| \quad \text{a.s.}$$

Therefore, we obtain that

$$\lim_{n\to\infty} \frac{\|U_n - \mathbb{E}[U_n]\|}{n} = 0 \quad \text{a.s.} \tag{6.29}$$

Hence, we deduce (6.22) from (6.11), (6.28) and (6.29). The convergence in any $\mathbb{L}^p$ for $p \geq 1$ holds again by the same arguments as before. We now focus our attention on equation (6.23). We have from (6.11) and (6.28) that

$$\lim_{n\to\infty} \sigma_n\big(U_n - \mathbb{E}[U_n]\big) = \lim_{n\to\infty} \sigma_n\big(U_n - nv_1\big) - \Big(\frac{b\alpha - c\beta}{S}\Big)v_2 = Mv_2 \quad \text{a.s.}$$

Consequently,

$$\lim_{n\to\infty} \frac{U_n - nv_1}{n^\sigma} = Wv_2 \quad \text{a.s.}$$

where the random variable W is given by

$$W = \frac{1}{\lambda}\big(M + \frac{b\alpha - c\beta}{S}\big). \tag{6.30}$$

Using the fact that

$$\mathbb{E}\big[\|M_n\|^2\big] = \mathbb{E}\big[\mathrm{Tr}\langle M\rangle_n\big] \leq m^2 w_n,$$

we get

$$\sup_{n\geq 1} \mathbb{E}\big[\|M_n\|^2\big] < \infty$$

which means that (M_n) is a martingale bounded in $\mathbb{L}^2$, thus converging in $\mathbb{L}^2$. Finally, as $\mathbb{E}[M_n] = 0$ and (M_n) converges in $\mathbb{L}^1$ to M, $\mathbb{E}[M] = 0$. Hence, we find from (6.4.3) that

$$\mathbb{E}[W] = \frac{\Gamma(\frac{\tau}{S})}{\Gamma(\frac{\tau}{S} + \sigma)} \frac{b\alpha - c\beta}{S}.$$

We shall now proceed to the computation of $\mathbb{E}[W^2]$. It is not hard to see that

$$\mathbb{E}\big[(X_{n+1} - \mathbb{E}[X_{n+1}])^2\big] = (1 + 2m\tau_n^{-1})\mathbb{E}\big[(X_n - \mathbb{E}[X_n])^2\big] + m^2\tau_n^{-1}\mathbb{E}[X_n]\big(1 - \tau_n^{-1}\mathbb{E}[X_n]\big)$$

which leads to

$$\begin{aligned}
\mathbb{E}\big[(X_n - \mathbb{E}[X_n])^2\big] &= m^2 \frac{\Gamma(n + \frac{\tau}{S} + 2\sigma)}{\Gamma(n + \frac{\tau}{S})} \sum_{k=0}^{n-1} \frac{\Gamma(k + 1 + \frac{\tau}{S})}{\Gamma(k + 1 + \frac{\tau}{S} + 2\sigma)} \tau_k^{-1}\mathbb{E}[X_k]\big(1 - \tau_k^{-1}\mathbb{E}[X_k]\big) \\
&= \frac{\sigma^2}{(1-\sigma)^2} \frac{\Gamma(n + \frac{\tau}{S} + 2\sigma)}{\Gamma(n + \frac{\tau}{S})} S_n.
\end{aligned}$$

It follows from (6.11) that

$$S_n = bcA_n + (b - c)\frac{b\alpha - c\beta}{S} \frac{\Gamma(\frac{\tau}{S})}{\Gamma(\frac{\tau}{S} + \sigma)} B_n - \frac{(b\alpha - c\beta)^2}{S^2} \frac{\Gamma(\frac{\tau}{S})^2}{\Gamma(\frac{\tau}{S} + \sigma)^2} C_n$$

where A_n, B_n and C_n are as follows, and we obtain from lemma B.1 in [5] that

$$A_n = \sum_{k=1}^{n} \frac{\Gamma(k + \frac{\tau}{S})}{\Gamma(k + \frac{\tau}{S} + 2\sigma)} = \frac{1}{2\sigma - 1}\big(\frac{\Gamma(\frac{\tau}{S} + 1)}{\Gamma(\frac{\tau}{S} + 2\sigma)} - \frac{\Gamma(n + \frac{\tau}{S} + 1)}{\Gamma(n + \frac{\tau}{S} + 2\sigma)}\big),$$

$$B_n = \sum_{k=1}^{n} \frac{\Gamma(k-1+\frac{\tau}{S}+\sigma)}{\Gamma(k+\frac{\tau}{S}+2\sigma)} = \frac{1}{\sigma}\Big(\frac{\Gamma(\frac{\tau}{S}+\sigma)}{\Gamma(\frac{\tau}{S}+2\sigma)} - \frac{\Gamma(n+\frac{\tau}{S}+\sigma)}{\Gamma(n+\frac{\tau}{S}+2\sigma)}\Big),$$

$$C_n = \sum_{k=1}^{n} \frac{\Gamma(k-1+\frac{\tau}{S}+\sigma)^2}{\Gamma(k+\frac{\tau}{S})\Gamma(k+\frac{\tau}{S}+2\sigma)} = \frac{1}{\sigma^2}\Big(\frac{\Gamma(n+\frac{\tau}{S}+\sigma)^2}{\Gamma(n+\frac{\tau}{S})\Gamma(n+\frac{\tau}{S}+2\sigma)} - \frac{\Gamma(\frac{\tau}{S}+\sigma)^2}{\Gamma(\frac{\tau}{S})\Gamma(\frac{\tau}{S}+2\sigma)}\Big).$$

Consequently, we have

$$\mathbb{E}[M^2] = \frac{\sigma^2\lambda^2\Gamma(\frac{\tau}{S})}{\Gamma(\frac{\tau}{S}+2\sigma)}\Big(\frac{bc}{2\sigma-1}\frac{\tau}{S} + (b-c)\frac{b\alpha-c\beta}{\sigma S} + \frac{(b\alpha-c\beta)^2}{\sigma^2 S^2}\Big) - \frac{(b\alpha-c\beta)^2}{S^2} \tag{6.31}$$

which via (6.30) and (6.31) achieves the proof of Theorem 6.15.

■

6.5 – Proofs of the asymptotic normality results

6.5.1 – Traditional urn model

Proof of Theorem 6.3. We shall make use of part (b) of Theorem 1 and Corollaries 1 and 2 from [46]. Let

$$s_n^2 = \sum_{k=n}^{\infty} \mathbb{E}[\Delta M_k^2].$$

It is not hard to see that

$$\lim_{n\to\infty} s_n^2 = 0$$

since

$$\sum_{n=1}^{\infty} \mathbb{E}[\Delta M_n^2] \leq \frac{S^2}{4}\sum_{n=1}^{\infty}\frac{1}{\tau_n^2} < +\infty.$$

Moreover, using the convergence of (M_n) in $\mathbb{L}^2$ and the moments of a beta distribution with parameters $\frac{\alpha}{S}$ and $\frac{\beta}{S}$, we get that

$$\lim_{n\to\infty}\Big(\sum_{k=n}^{\infty}\frac{1}{\tau_{k+1}^2}\Big)^{-1} s_n^2 = \frac{\alpha\beta S^2}{(\alpha+\beta)(\alpha+\beta+S)},$$

leading to

$$\lim_{n\to\infty} n s_n^2 = \ell \quad \text{where} \quad \ell = \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+S)}.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{s_n^2}\sum_{k=n}^{\infty}\mathbb{E}\big[\Delta M_{k+1}^2|\mathcal{F}_k\big] &= \lim_{n\to\infty}\frac{1}{s_n^2}\sum_{k=n}^{\infty}\frac{c^2 M_k(1-M_k)}{\tau_{k+1}^2} \quad \text{a.s.} \\
&= \lim_{n\to\infty}\frac{1}{\ell S^2}\Big(\sum_{k=n}^{\infty}\frac{1}{\tau_{k+1}^2}\Big)^{-1}\sum_{k=n}^{\infty}\frac{S^2 M_k(1-M_k)}{\tau_{k+1}^2} \quad \text{a.s.} \\
&= \frac{M_\infty(1-M_\infty)}{\ell} \quad \text{a.s.}
\end{aligned}$$

Consequently, the first condition of part (b) of Corollary 1 in [46] is satisfied with $\eta^2 = \ell^{-1} M_\infty(1 - M_\infty)$. Let us now focus on the second condition of Corollary 1 in [46] and let $\varepsilon > 0$. On the one hand, we get that for all $\varepsilon > 0$

$$\frac{1}{s_n^2} \sum_{k=n}^{\infty} \mathbb{E}\big[\Delta M_{k+1}^2 \mathbb{1}_{|\Delta M_{k+1}| > \varepsilon s_n}\big] \leq \frac{1}{\varepsilon^2 s_n^4} \sum_{k=n}^{\infty} \mathbb{E}\big[\Delta M_{k+1}^4\big] \leq \frac{7S^4}{\varepsilon^2 s_n^4} \sum_{k=n}^{\infty} \frac{1}{\tau_k^4} \leq \frac{7}{\varepsilon^2 s_n^4} \sum_{k=n}^{\infty} \frac{1}{k^4}.$$

On the other and, using that s_n^4 increases at speed n^2 and that

$$\lim_{n\to\infty} 3n^3 \sum_{k=n}^{\infty} \frac{1}{k^4} = 1,$$

we can conclude that

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=n}^{\infty} \mathbb{E}\big[\Delta M_k^2 \mathbb{1}_{|\Delta M_k| > \varepsilon s_n}\big] = 0 \quad \text{a.s.}$$

Hereafter, we easily get that

$$\sum_{k=1}^{\infty} \frac{1}{s_k^4} \mathbb{E}\big[\Delta M_k^4 | \mathcal{F}_{k-1}\big] \leq 7 \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty. \tag{6.32}$$

Noting that

$$\sum_{k=1}^{n} \frac{1}{s_k^2} \big(|\Delta M_k|^2 - \mathbb{E}\big[|\Delta M_k|^2 \mid \mathcal{F}_{k-1}\big]\big)$$

is a martingale, the equation (6.32) proves that its bracket is convergent, wich implies that the martingale is also convergent. This gives us

$$\sum_{k=1}^{\infty} \frac{1}{s_k^2} \big(|\Delta M_k|^2 - \mathbb{E}\big[|\Delta M_k|^2 \mid \mathcal{F}_{k-1}\big]\big) < +\infty \quad \text{a.s.}$$

Hence, the second condition of Corollary 1 in [46] is satisfied. Therefore we obtain that

$$\frac{M_\infty - M_n}{\sqrt{\langle M \rangle_\infty - \langle M \rangle_n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, 1\big). \tag{6.33}$$

Moreover, since

$$\lim_{n\to\infty} \sqrt{\frac{M_n(1 - M_n)}{n(\langle M \rangle_\infty - \langle M \rangle_n)}} = 1 \quad \text{a.s.}$$

we finally obtain from Slutsky's Lemma that

$$\sqrt{n} \frac{M_\infty - M_n}{\sqrt{M_n(1 - M_n)}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, 1\big). \tag{6.34}$$

which achieves the proof of Theorem 6.3.

■

6.5.2 – Generalized urn model – small urns

Proof of Theorem 6.8. We shall make use of the central limit theorem for multivariate martingales, see Theorem A.6. First of all, we already saw from (6.26) that

$$\lim_{n\to\infty} \frac{1}{w_n}\langle M\rangle_n = (1-2\sigma)\Gamma \quad \text{a.s.}$$

It only remains to show that Lindeberg's condition is satisfied, that is for all $\varepsilon > 0$,

$$\frac{1}{w_n}\sum_{k=0}^{n-1} \mathbb{E}\big[\|\Delta M_{k+1}\|^2 \mathbb{1}_{\|\Delta M_{k+1}\|\geq \varepsilon\sqrt{w_n}} |\mathcal{F}_k\big] \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

We clearly have

$$\frac{1}{w_n}\sum_{k=0}^{n-1} \mathbb{E}\big[\|\Delta M_{k+1}\|^2 \mathbb{1}_{\|\Delta M_{k+1}\|\geq \varepsilon\sqrt{w_n}} |\mathcal{F}_k\big] \leq \frac{1}{\varepsilon w_n^2}\sum_{k=0}^{n-1} \mathbb{E}\big[\|\Delta M_{k+1}\|^4\big] \leq \frac{m^2}{\varepsilon w_n^2}\sum_{k=0}^{n-1} \sigma_k^4 \quad \text{a.s.}$$

However, it is not hard to see that

$$\lim_{n\to\infty} \frac{1}{w_n^2}\sum_{k=0}^{n-1} \sigma_k^4 = 0$$

which ensures Lindeberg's condition is satisfied. Consequently, we can conclude that

$$\frac{M_n}{\sqrt{w_n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, (1-2\sigma)\Gamma\big).$$

As $M_n = \sigma_n\big(U_n - \mathbb{E}[U_n]\big)$ and $\sqrt{n}\sigma_n$ is asymptotically equivalent to $\sqrt{(1-2\sigma)w_n}$, together with (6.11), we obtain that

$$\frac{U_n - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, \Gamma\big).$$

■

6.5.3 – Generalized urn model – critically small urns

Proof of Theorem 6.13. We shall also make use of the central limit thoerem for multivariate martingales. We already saw from (6.27) that

$$\lim_{n\to\infty} \frac{1}{w_n}\langle M\rangle_n = bc \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Once again, it only remains to show that Lindeberg's condition is satisfied, that is for all $\varepsilon > 0$,

$$\frac{1}{w_n}\sum_{k=0}^{n-1} \mathbb{E}\big[\|\Delta M_{k+1}\|^2 \mathbb{1}_{\|\Delta M_{k+1}\|\geq \varepsilon\sqrt{w_n}} |\mathcal{F}_k\big] \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

As in the proof of Theorem (6.8), we have

$$\frac{1}{w_n}\sum_{k=0}^{n-1} \mathbb{E}\big[\|\Delta M_{k+1}\|^2 \mathbb{1}_{\|\Delta M_{k+1}\|\geq \varepsilon\sqrt{w_n}} |\mathcal{F}_k\big] \leq \frac{1}{\varepsilon w_n^2}\sum_{k=0}^{n-1} \mathbb{E}\big[\|\Delta M_{k+1}\|^4\big] \leq \frac{m^2}{2\varepsilon w_n^2}\sum_{k=0}^{n-1} \sigma_k^4. \quad \text{a.s.}$$

It is not hard to see that once again

$$\lim_{n\to\infty} \frac{1}{w_n^2} \sum_{k=0}^{n-1} \sigma_k^4 = 0.$$

Hence, Lindeberg's condition is satisfied and we find that

$$\frac{M_n}{\sqrt{w_n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, \Gamma\big).$$

As $M_n = \sigma_n\big(U_n - \mathbb{E}[U_n]\big)$ and $\sigma_n\sqrt{n\log n}$ is asymptotically equivalent to $\sqrt{w_n}$, together with (6.11), we can conclude that

$$\frac{U_n - nv_1}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0, \Gamma\big).$$

■

7 Conclusion and perspectives

In this thesis, we studied the elephant random walk (ERW) and processes related to it. Most of the processes we studied in this work are non-Markovian. The martingale theory turned out to be an excellent solution to overcome this and to solve the problems studied in this thesis.
At first, we generalized the elephant random walk to any dimension. To do so, we used the counting processes of the steps in each direction of the dimension and auxillary martingales deduced from those processes. We showed probabilistic results such as laws of large numbers and asymptotic normality, as well as laws of iterated logarithm and quadratic strong laws in the three regimes of behavior, depending on the dimension. Afterwards, we focused our attention on the center of mass of the random walk. We explained why it was not possible to find a martingale from this process only. To overcome this issue, we proposed a suitable approach that consists in finding two martingales such that studying them simultaneously made possible to obtain results analogous to those of the ERW. Then, we used this method again to study the elephant random walk with linearly reinforced memory or smooth amnesia. Next, we gave an explicit estimator of the memory of the ERW, using a quasi-likelihood estimate. Finally, because of the link between the ERW and Pólya-type urns, our last work constisted in giving a martingale approach to study the two-color case.

Here are some research perspectives related to the works presented above.

Who (or what) is L ?

The question of the distribution of L is one of the trickiest question regarding the ERW. It seems hard to obtain explicitly the law of L based on the methods previously introduced. For example, the ERW is related to Pólya urns with random replacement but the distributions of the limiting random variables for large urns have only been explicited in the case of two-color urns with deterministic

replacement.
It might be possible to show the absolute continuity of its distribution using the connection with random recursive trees and Bernoulli percolation [55, 21], and the more general framework introduced in [12, 17].
Another approach could come from the last work of Bercu [6] on the ERW with stops. The calculation of the Pochhammer moments together with a nice identity that links the Pochhammer and the classical moments ensure that the random variable couting the numbers of non-zero steps, properly normalized, converges almost surely to a Mittag-Leffler distribution. We can only conjecture here that the distribution of L is closely related to a mixture of Mittag-Leffler distributions.

ERW with general steps distribution

There are various ways to modify the behavior of the ERW. A first question is about the effects of step-reinforced random walks with independent but not identically distributed steps, as a generalization of [12]. The interesting parts here are the processes that could appear when looking at the convergence in $D(0,\infty)$ and the use of martingale theory.
Another variation deals with the distribution of the memory. This distribution is usually uniform [9, 17, 21], but it can be modified in an appropriate way, see [2] and Chapters 3 and 4. It would be interesting to study how other changes in the distribution of the memory affect the behavior of the ERW, and more generally of step-reinforced RW. Again, being able to establish the universality of the limiting process, as it was done by Bertoin [18] for the noise reinforced Brownian motion (nBM), would be a nice achievement.

Local limit theorems for Pólya urns processes

The question of the recurrence/transience property of a random walk is important. The ERW is recurrent in the diffusive and critical regimes, and transient in the superdiffusive regime. More precisely, it was recently proved by Bertoin [17] that the ERW is positive recurrent in the diffusive regime (only when $p < 1/4$). However, no results are available in the multi-dimensional setting. A first idea would be to prove local limit theorems in the case of generalized Pólya urns with deterministic replacement. This result would make it possible to conclude on the the recurrent cases for a fixed number of colors $m \geq 3$. If that is done, it would be interesting to try to generalize this to random replacement (possibly with some stronger hypothesis) in order to conclude on the (multi-dimensional) ERW.

Estimation for Póya urns processes

In 2014, Le Goff and Soulier [59] explained how to estimate one particular parameter in the special case of two-color urn model when only one ball is added

at each instant. They considered more general law for the replacement, possibly nonlinear, and based their estimation on multiple realisations of the process. Surprisingly, no other statistical study have been led on the estimation of parameters for generalized two-color Pólya urn models. Hence, a good question is to determine explicit estimators of the parameters in the balanced case with deterministic replacement, which reduces to two parameters. Then, it may be possible to adapt the method to the random replacement case which could offer a new way to estimate the memory parameter of the ERW.

Estimation for variations of the ERW

Very few results are devoted to statistical inference for elephant random walks. In [10] we addressed this problem using the well-known approach of the quasi-maximum likelihood estimator. This method could also be adapted in the case of ERW with general steps distribution, providing the law of the steps have a second order moment. It should also be possible to propose a statistical estimation of the parameters in the case of ERW with stops [6, 39], or with memory changes [57, 58] as well as the minimal random walk [64]. The difficulty here is that there are at least two parameters to estimate simultaneously.

A Martingale Theory

In this appendix, we give results regarding martingale theory that are extensively used in the presented work. Most of the following results were obtained thanks to the impressive work of Hall and Heyde on the subject [42, 46, 45]. We refer the reader to the book of Duflo [31] for a very complete survey of martingales theory. This book will be given as a reference for many of the following Theorems.

A.1 – "Classical" results

The following results can be found in [31], see Theorems 1.3.15, 1.3.24 and 4.3.15.

Theorem A.1. (Law of Large Numbers for Martingales) *Let (M_n) be a real square integrable martingale adapted to $\mathcal{F}$ with quadratic variations $\langle M\rangle_n$, and set $\langle M\rangle_\infty = \lim\langle M\rangle_n$.*

1. *Suppose that $\langle M\rangle_\infty < \infty$, then $M_n \xrightarrow[n\to\infty]{a.s.} M_\infty$ where M_∞ is a finite random variable.*
2. *Suppose that $\langle M\rangle_\infty = \infty$, then*

$$\frac{M_n}{\langle M\rangle_n} \xrightarrow[n\to\infty]{a.s.} 0$$

and more precisely, for all $\gamma > 0$, we have

$$\frac{M_n}{\langle M\rangle_n} = o\Big(\big(\log\langle M\rangle_n\big)^{1+\gamma}\Big) \quad a.s. \tag{A.1}$$

Theorem A.2. (Law of Large Numbers for regressive series) *Let (ε_n) be a square-integrable sequence of d-dimensional random variables adapted to $\mathcal{F}$ such that*

$$\mathbb{E}[\varepsilon_{n+1} \mid \mathcal{F}_n] = 0, \quad \sup_n \mathbb{E}[\|\varepsilon_{n+1}\|^2 \mid \mathcal{F}_n] \leq C$$

for C a finite random variable. Suppose that (Φ_n) is another sequence of d-dimensional random variables adapted to $\mathcal{F}$. Then, set $s_n = \sum_{k=0}^n \|\Phi_k\|^2$ and let $M_n = \sum_{k=1}^n \langle \Phi_{k-1}, \varepsilon_k\rangle$ be a martingale adapted to $\mathcal{F}$.

1. *Suppose that* $s_\infty < \infty$, *then* $M_n \xrightarrow[n\to\infty]{a.s.} M_\infty$ *where* M_∞ *is a finite random variable.*
2. *Suppose that* $s_\infty = \infty$, *then*

$$\frac{M_n}{s_{n-1}} \xrightarrow[n\to\infty]{a.s.} 0 \tag{A.2}$$

and more precisely, for all $\gamma > 0$, *we have*

$$M_n^2 = O\big((\log s_{n-1})^{1+\gamma} s_{n-1}\big) \quad a.s. \tag{A.3}$$

Moreover, if C *is a constant and if for some* $a > 2$, $\sup_n \mathbb{E}[\|\varepsilon_{n+1}\|^a \mid \mathcal{F}_n] < \infty$, *we also have*

$$M_n^2 = O\big((\log s_{n-1}) s_{n-1}\big) \quad a.s. \tag{A.4}$$

Theorem A.3. (Law of Large Numbers for vector Martingales) *Suppose that* (M_n) *is a square-integrable vector martingale adapted to* $\mathcal{F}$ *with quadratic variations* $\langle M\rangle_n$.

1. *Suppose that* $Tr(\langle M\rangle_\infty) < \infty$, *then* $M_n \xrightarrow[n\to\infty]{a.s.} M_\infty$ *where* M_∞ *is a finite random variable.*
2. *Suppose that* $Tr(\langle M\rangle_\infty) = \infty$, *then*

$$\frac{M_n}{\lambda_{max}\langle M\rangle_n} \xrightarrow[n\to\infty]{a.s.} 0 \tag{A.5}$$

and more precisely,

$$\frac{\|M_n\|^2}{\lambda_{max}\langle M\rangle_n} = o\Big(\Big(\log Tr\langle M\rangle_n\Big)^{1+\gamma}\Big) \quad a.s \tag{A.6}$$

We give the law of iterated logarithm for real martingales which is due to Stout [71, 72], see also Corollary 6.4.25 in [31].

Theorem A.4. (Law of iterated logarithm) *Let* $M_n = \sum \Phi_{k-1}\varepsilon_k$ *be a real martingale defined as in Theorem A.2, with the same notations. Assume that* $\mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] \le \sigma^2$ *and that for some* $1 < a \le 2$, $\sup_n \mathbb{E}[|\varepsilon_{n+1}|^{2a} \mid \mathcal{F}_n] < \infty$. *Let* (T_n) *be a sequence adapted to* $\mathcal{F}$ *such that* $|\Phi_n| \le T_n$, *and set* $\tau_n = \sum_{k=0}^n T_k^2$. *Then, if* $\tau_\infty = \infty$ *and* $\sum T_k^{2a}/\tau_k^a < \infty$, *we have*

$$\limsup_{n\to\infty} \Big(\frac{1}{2\tau_n \log\log \tau_n}\Big)^{1/2} M_n = -\liminf_{n\to\infty} \Big(\frac{1}{2\tau_n \log\log \tau_n}\Big)^{1/2} M_n = \sigma \quad a.s. \tag{A.7}$$

The following result is an adaptation of the one in [4], see Theorem 3.

Theorem A.5. (Quadratic strong law) *Let* $M_n = \sum \Phi_{k-1}\varepsilon_k$ *be a real martingale defined as in Theorem A.2, with the same notations. Moreover, assume that* $\lim_n \mathbb{E}[\varepsilon_{n+1}^2 \mid \mathcal{F}_n] = \sigma^2$ *and that for some* $a > 2$, $\sup_n \mathbb{E}[|\varepsilon_{n+1}|^a \mid \mathcal{F}_n] < \infty$. *Let the explosion coefficient associated with* (Φ_n) *be defined by* $f_n = \Phi_n/s_n$ *and assume that* $f_n \xrightarrow[n\to\infty]{a.s.} 0$. *Then,*

$$\lim_{n\to\infty} \frac{1}{\log s_n} \sum_{k=1}^n f_k\Big(\frac{M_k^2}{s_{k-1}}\Big) = \sigma^2 \quad a.s. \tag{A.8}$$

The following result is the usual central limit theorem for vector martingales, it can be found for example in [31], see Corollary 2.1.10.

Theorem A.6. (Central Limit Theorem for vector Martingales) *Let (M_n) be a square integrable vector martingale adapted to $\mathcal{F}$ with quadratic variations $\langle M\rangle_n$. Let $\Delta M_n = M_n - M_{n-1}$ and suppose that for (s_n) a real deterministic sequence, such that s_n is increasing to $+\infty$, the following assumptions hold :*

1. $\dfrac{1}{s_n}\langle M\rangle_n \xrightarrow[n\to\infty]{\mathbb{P}} \Gamma$;
2. *Lindeberg's condition is satisfied, that is, for all $\varepsilon > 0$*

$$\frac{1}{s_n}\sum_{k=1}^{n}\mathbb{E}\left[\|\Delta M_n\|^2 \mathbb{1}_{\{\|\Delta M_n\|\geq \varepsilon\sqrt{s_n}\}}|\mathcal{F}_{k-1}\right] \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Then, we have

$$\frac{1}{s_n}M_n \xrightarrow[n\to\infty]{a.s.} 0 \quad \text{and} \quad \frac{1}{\sqrt{s_n}}M_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0,\Gamma\big). \tag{A.9}$$

A.2 – Non-standard results on martingales

The proofs of our results rely on two non-standard central limit theorem and quadratic strong law for multi-dimensional martingales. A simplified version of Theorem 1 of Touati [73] is as follows.

Theorem A.7. (Central limit for matrix normalisation) *Let $(\mathcal{M}_n)$ be a locally square-integrable martingale of $\mathbb{R}^\delta$ adapted to a filtration $(\mathcal{F}_n)$, with predictable quadratic variation $\langle\mathcal{M}\rangle_n$. Let (V_n) be a sequence of non-random square matrices of order δ such that $\|V_n\|$ decreases to 0 as n goes to infinity. Assume that there exists a symmetric and positive semi-definite matrix V such that*

$$V_n\langle\mathcal{M}\rangle_n V_n^T \xrightarrow[n\to\infty]{\mathbb{P}} V. \tag{H.1}$$

Moreover, assume that Lindeberg's condition is satisfied, that is for all $\varepsilon > 0$,

$$\sum_{k=1}^{n}\mathbb{E}\big[\|V_n\Delta\mathcal{M}_k\|^2\mathbb{1}_{\{\|V_n\Delta\mathcal{M}_k\|>\varepsilon\}}\big|\mathcal{F}_{k-1}\big] \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{H.2}$$

where $\Delta\mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$. Then, we have the asymptotic normality

$$V_n\mathcal{M}_n \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\big(0,V\big). \tag{A.10}$$

The quadratic strong law requires more restrictive assumptions. The following result is a simplified version of Theorem 2.1 of Chaabane and Maaouia [22] where the normalization matrices (V_n) are diagonal.

Theorem A.8. (Quadratic strong law for matrix normalisation) *Let $(\mathcal{M}_n)$ be a locally square-integrable martingale of $\mathbb{R}^\Delta$ adapted to a filtration $(\mathcal{F}_n)$, with predictable quadratic*

variation $\langle \mathcal{M} \rangle_n$. *Let* (V_n) *be a sequence of non-random positive definite diagonal matrices of order* Δ *such that its diagonal terms decrease to zero at polynomial rates. Assume that* (H.1) *and* (H.2) *hold almost surely. Moreover, suppose that there exists* $\beta \in]1, 2]$ *such that*

$$\text{(H.3)} \qquad \sum_{n=1}^{\infty} \frac{1}{\big(\log(\det V_n^{-1})^2\big)^{\beta}} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_n\|^{2\beta} \big| \mathcal{F}_{n-1}\big] < \infty \quad a.s.$$

Then, we have the quadratic strong law

$$\lim_{n\to\infty} \frac{1}{\log(\det V_n^{-1})^2} \sum_{k=1}^{n} \Big(\frac{(\det V_k)^2 - (\det V_{k+1})^2}{(\det V_k)^2} \Big) V_k \mathcal{M}_k \mathcal{M}_k^T V_k^T = V \quad a.s. \tag{A.11}$$

The proofs of our functional results rely on the non-standard functional central limit theorem, which is a simplified version of Theorem 1 part 2) of Touati [73].

Theorem A.9. (Functional convergence for matrix normalisation) *Let* $(\mathcal{M}_n)$ *be a locally square-integrable martingale of* $\mathbb{R}^\delta$ *adapted to a filtration* $(\mathcal{F}_n)$, *with predictable quadratic variation* $\langle \mathcal{M} \rangle_n$. *Let* (V_n) *be a sequence of non-random square matrices of order* δ *such that* $\|V_n\|$ *decreases to* 0 *as* n *goes to infinity. Moreover let* $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-decreasing function going to infinity at infinity. Assume that there exists a symmetric and positive semi-definite matrix* V_t *that is deterministic and such that for all* $t \geq 0$

$$\text{(H'.1)} \qquad V_n \langle \mathcal{M} \rangle_{\tau(nt)} V_n^T \xrightarrow[n\to\infty]{\mathbb{P}} V_t.$$

Moreover, assume that Lindeberg's condition is satisfied, that is for all $t \geq 0$ *and* $\varepsilon > 0$,

$$\text{(H'.2)} \qquad \sum_{k=1}^{\tau(nt)} \mathbb{E}\big[\|V_n \Delta \mathcal{M}_k\|^2 \mathbb{1}_{\{\|V_n \Delta \mathcal{M}_k\| > \varepsilon\}} \big| \mathcal{F}_{k-1}\big] \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

where $\Delta \mathcal{M}_n = \mathcal{M}_n - \mathcal{M}_{n-1}$. *Finally, assume that*

$$\text{(H.4)} \qquad V_t = \sum_{j=1}^{q} t^{\alpha_j} K_j$$

where $\alpha_j > 0$ *and* K_j *is a symmetric matrix, for some* $q \in \mathbb{N}^*$. *Then, we have the distributional convergence in the Skorokhod space* $D([0, \infty[)$ *of right-continuous functions with left-hand limits,*

$$\big(V_n \mathcal{M}_{\tau(nt)}, t \geq 0\big) \Longrightarrow \big(\mathcal{W}_t,\ t \geq 0\big) \tag{A.12}$$

where $\mathcal{W} = \big(\mathcal{W}_t,\ t \geq 0\big)$ *is a continuous* $\mathbb{R}^d$*-valued centered Gaussian process starting at 0 with covariance, for* $0 \leq s \leq t$,

$$\mathbb{E}[\mathcal{W}_s \mathcal{W}_t^T] = V_s. \tag{A.13}$$

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