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Edouard Dossetto

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# Macro-dynamics of the energy transition: robustness, money velocity and public policies

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## Composition du jury

- M. Jean-Bernard Châtelain, Professeur des Universités, Paris 1 Panthéon-Sorbonne, PSE
- M. Matheus R. Grasselli, Professeur des Universités, McMaster University *Rapporteur*
- M. Stefano Demichelis, Professeur des Universités, Università Di Pavia *Rapporteur*
- M. Adrien Nguyen-Huu, Maître de conférence, Université de Montpellier
- M. Joseph E. Stiglitz, Professeur des Universités, Columbia University

**“L’Université Paris 1 Panthéon-Sorbonne n’entend donner aucune approbation, ni improbation aux opinions émises dans cette thèse ; elles doivent être considérées comme propres à leur auteur”**

*À mes grands-parents*

# Acknowledgments

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# Résumé Substantiel

Cette thèse prend son origine dans le discours fondateur de Mark Carney (gouverneur de la Banque d'Angleterre) sur les risques financiers induits par le climat.

La probabilité d'un "*climate-driven Minsky moment*" [Carney, 2019] ou moment de Minsky (induit par le climat) - terme utilisé pour désigner un effondrement soudain du prix des actifs et pour dépeindre les crises financières survenues en 2001 et 2007-2009 - combinée à l'incapacité d'une grande majorité des modèles de l'époque à prévoir de telles crises nous invite à nous interroger sur la fiabilité des modèles actuels pour l'orientation des décideurs. Si l'on ne peut ignorer l'ampleur des impacts socio-économiques de la récente crise sanitaire et la remise en question de nos politiques économiques qu'elle a engendrée, cette thèse se concentre sur un autre défi majeur de notre époque: la nécessaire transition de nos systèmes économiques pour assurer la résilience de nos sociétés face à cette autre menace fondamentale qu'est le changement climatique.

Le changement climatique constitue en effet une menace considérable pour nos sociétés en raison de son ampleur et de son impact dans tous les secteurs et toutes les régions du monde,<sup>1</sup> son caractère prévisible,<sup>2</sup> son irréversibilité<sup>3</sup> et sa dépendance vis-à-vis des actions à court terme.<sup>4</sup> Avec le sommet de Rio, le protocole de Kyoto et maintenant les règles de mise en œuvre de l'accord de Paris (finalisé en 2021), les États se sont positionnés en pierres angulaires de la lutte contre le changement climatique. Lors de la COP21, la sphère politique mondiale a confirmé sa volonté de faciliter la transition vers une société bas carbone.<sup>5</sup>

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<sup>1</sup>Le rapport du *Network for Greening the Financial System* (NGFS) indique que le changement climatique affectera tous les agents économiques (ménages, entreprises, gouvernements). Les risques seront probablement corrélés à des points de basculement et potentiellement aggravés par ceux-ci, de manière non linéaire. Cela signifie que les impacts pourraient être beaucoup plus importants, plus étendus et plus variés que ceux d'autres changements structurels.

<sup>2</sup>Bien que les résultats exacts, l'horizon temporel et la trajectoire future soient incertains, il existe un degré élevé de certitude pour qu'une certaine combinaison de risques physiques et de transition se matérialise à l'avenir. [Carney, 2019]

<sup>3</sup>Comme le montrent les travaux du GIEC, la concentration des émissions de gaz à effet de serre (GES) dans l'atmosphère détermine l'impact du changement climatique et aucune technologie mature n'a encore une portée suffisante pour inverser le processus. Au-delà d'un certain seuil, elles présentent un fort potentiel pour imposer des conséquences irréversibles sur une multitude d'écosystèmes, même si l'horizon temporel et le degré de gravité restent incertains [IPCC, 2014]. De nombreux changements dus aux émissions passées et futures de gaz à effet de serre seront irréversibles, en particulier vis-à-vis des modifications des océans, des calottes glaciaires et du niveau mondial des mers. [IPCC, 2021]

<sup>4</sup>Les actions menées aujourd'hui détermineront la nature et l'ampleur des catastrophes liées au climat. Elles exigent donc des stratégies fiables et globales de long terme. Par exemple : des réductions fortes, rapides et durables des émissions de CH<sub>4</sub> limiteraient également l'effet de réchauffement résultant de la diminution de la pollution par les aérosols et amélioreraient la qualité de l'air. [IPCC, 2021]

<sup>5</sup>Par la suite, la loi française sur la transition énergétique, adoptée en juillet 2015, exige des entreprises et des institutions financières qu'elles rendent compte des risques climatiques et qu'elles limitent la hausse mondiale des températures à moins de 2 degrés Celsius, et de préférence aussi près que possible de 1,5 degré Celsius. Le 'Green

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La demande d’actions immédiates par le Groupe d’experts intergouvernemental sur l’évolution du climat ([IPCC, 2014], [IPCC, 2018] et plus récemment dans le rapport de son premier groupe de travail [IPCC, 2021]) afin d’atténuer les impacts à court terme et l’irréversibilité à grande échelle du changement climatique, nécessite la construction de modèles fiables pour garantir la transition vers une économie à faible émission de carbone (ou même une économie neutre du point de vue climatique). Le développement de mon travail vise à étendre un modèle de transition macro-économique existant, stock-flux cohérent, avec une dynamique non linéaire et à en évaluer la fiabilité. La dynamique de ce type de modèle conduit en effet à des équilibres multiples incorporant la possibilité d’un effondrement. L’utilisation d’un tel modèle fournit donc un cadre dans lequel les politiques publiques capables d’éviter ce que Marc Carney appelle un “*climate-driven Minsky moment*” [Carney, 2019], peuvent être analysées.

Ce terme fait référence à l’hypothèse d’instabilité financière (FIH) de Minsky, qui a analysé dans les années 1970 la probabilité d’une instabilité intrinsèque du système financier et la manière dont elle peut impliquer une nouvelle crise comparable en terme d’ampleur à celle de 1929. Comme expliqué dans [McIsaac, 2016], le principal avantage de ce cadre est sa capacité à reproduire une crise financière, telle que la crise des subprimes, de manière endogène, alors que la grande majorité des *Dynamic Stochastic General Equilibrium* (DSGE), très étudiés dans la littérature, ne peuvent guère considérer ces situations autrement que comme des “*cygnes noirs*” ou des risques importants ayant une très faible probabilité de se matérialiser.

Il existe donc, d’une part, un besoin évident de disposer de modèles de transition vers une société neutre en carbone, capable de prédire le risque d’un “*moment de Minsky induit par le climat*”, afin d’orienter les décisions de politique publique ; et d’autre part, la conscience que la production d’un tel modèle présente de grandes difficultés analytiques, comme le souligne [Carney, 2019], aggravées par l’incapacité des modèles macro-économiques standard à prévoir les crises financières des années 2000.

Cet effet cocktail motive une analyse de ces modèles à la frontière de la modélisation mathématique, des hypothèses économiques (ou financières), et des objectifs et contraintes écologiques. Cette analyse en introduction de mon travail se fonde sur le rôle ambigu des modèles, dû à l’incertitude et à la nature non-bijective de la modélisation (Section 1.2), et la crise des modèles macro-économiques, dont certaines caractéristiques conduisent aux incapacités indiquées ci-dessus (Section 1.3).

### **L’ambiguïté des modèles, source d’incertitudes pour l’orientation des décideurs.**

Il est en effet essentiel de tenir compte de la polyvalence et de l’ambiguïté des modèles pour orienter au mieux nos décisions publiques. Les modèles mathématiques et les simulations informatiques sont utilisés dans une grande variété de domaines et d’institutions. Les écologistes les développent par exemple pour étudier les écosystèmes et la dynamique des populations. Les institutions du monde entier les utilisent pour étudier les économies mondiales et nationales. En ce qui concerne le changement climatique, deux grands types de modèles sont utilisés : a) les modèles physiques qui évaluent l’évolution du climat et b) les modèles macro-économiques qui évaluent l’impact économique ou

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Deal” de l’UE et son paquet “Fit-for-55” comprennent une législation contraignante pour garantir que l’UE atteigne ses objectifs climatiques et énergétiques de 2030. Lors de la COP26, la présidence britannique a réussi à placer la neutralité carbone au centre des discussions.

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financier du changement climatique. Dans certains cas, les modèles physiques permettent de corroborer ou de falsifier des hypothèses sur les systèmes écologiques, et les modèles macro-économiques permettent d’orienter les politiques publiques. Par exemple, en France, le Commissariat général du développement durable (du Ministère de la transition écologique) évalue l’impact de la stratégie nationale bas carbone avec le modèle Three-ME2. Ses résultats prévoient une augmentation du PIB de 1,6 d’ici 2035, alors que le ministère français des Finances prévoit un effet de contraction, avec son propre modèle *Mésange* (voir [Grandjean and Giraud, 2017]).

*Les modèles physiques* (modèles climatiques ou météorologiques) utilisés pour prédire l’évolution de l’environnement peuvent être évalués par leur capacité à simuler les conditions climatiques actuelles et passées, et à reproduire les variations climatiques à chaque échelle de temps. Ces modèles ont donc une capacité prédictive qui permet d’identifier une politique publique efficace avec fiabilité (robustesse) [Grandjean and Giraud, 2017].

Pour les *modèles macro-économiques*, atteindre des résultats robustes est un sujet beaucoup plus sensible en raison a) des échecs d’une grande majorité de modèles à prédire les crises des années 2000,<sup>6</sup> et b) de la sensibilité (manque de robustesse) des modèles actuels.<sup>7</sup> Il est clair aujourd’hui que cette sensibilité a conduit à de grandes sous-estimations de l’impact économique du climat.<sup>8</sup> C’est pourquoi une analyse de sensibilité globale des modèles macro-économiques est nécessaire pour éclairer les discours sur la fiabilité de leurs résultats, et ainsi contribuer à tempérer la crise qu’ils traversent.

La capacité du modèle à représenter fidèlement la réalité dépend en effet, à la fois d’études empiriques et de considérations théoriques. Par exemple, les fonctions comportementales (comme la courbe de Phillips ou la fonction d’investissement en économie) sont construites pour s’adapter à des données qui impliquent un grand nombre de calibrations ou d’estimations (basées sur des études empiriques [Grandjean and Giraud, 2017]), alors que la fonction de production (comme [Cobb and Douglas, 1928], [Leontief, 1951], CES [Solow, 1956] ou Putty-clay au sens de [Akerlof and Stiglitz, 1969]) est un concept clé de la théorie de la croissance économique, et peut donc faire apparaître des divergences entre les lois de production réelles et leur représentation fonctionnelle (voir [Stiglitz, 2018]).

La variété des entrées d’un modèle souffre, elle aussi, d’incertitudes sur ses valeurs réelles. Toute prévision doit donc tenir compte de ces décalages possibles afin de fournir des résultats fiables (robustes). Cette considération critique est l’objet de l’analyse d’incertitude (UA) et de l’analyse de sensibilité (SA). En particulier dans les sciences sociales, ou dans des environnements économiques complexes, où ces calibrations et estimations de paramètres conduisent à des résultats très sensibles aux paramètres, l’impact des incertitudes des entrées ne peut être considéré comme négligeable. Ainsi, la remise en question de la robustesse des modèles macro-économiques est de plus en plus considérée comme essentielle pour mener des politiques publiques dans ces domaines [European Commission, 2009].

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<sup>6</sup>Voir Section 1.3.2.

<sup>7</sup>Voir une revue de certains de ces modèles dans la Section 2.4.

<sup>8</sup>Comme exposé dans le livre de [Pottier, 2014].



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Les modèles macro-économiques écologiques tentent de relier les environnements climatiques et économiques. Certains d'entre eux visent à évaluer l'impact d'une politique publique,<sup>9</sup> avec une analyse coûts-bénéfices ou coût-efficacité, à l'échelle nationale, régionale ou mondiale. Inspirés de [Laffargue, 2012] et [Grandjean and Giraud, 2017], ces modèles macro-économiques sont classés comme suit :

- Modèles néo-keynésiens : avec des ajustements de court terme, ces modèles sont pilotés par la demande globale (par exemple, Mésange). Des déséquilibres temporaires sont possibles en raison de l'inertie des ajustements de quantités ;
- Modèles d'équilibre général (CGE) : basés sur une représentation walrasienne de l'économie, où les prix sont parfaitement flexibles et où les préférences d'un agent *rationnel* optimisant une fonction d'utilité sont explicitement décrites. Divers types de modèles existent dans cette catégorie (tels que les modèles d'évaluation intégrée - Integrated assessment model comme celui de [Nordhaus, 2008]). En général, les institutions utilisent à la fois un modèle DSGE estimé (Global Integrated Monetary and Fiscal Model - GIMF, New Area-Wide Model of The Euro area - NAWM, ...) et un modèle néo-keynésien pour leurs prévisions ;
- Modèles de déséquilibre (comme ThreeMe, un modèle post-keynésien) : avec une dynamique des prix spécifique, indépendante de l'ajustement de l'offre et de la demande, une production multisectorielle et une dynamique des salaires à travers une nouvelle courbe de Phillips à court terme. Le modèle GEMMES (General Monetary and Multisectoral Macrodynamics for the Ecological Shift) est basé sur les travaux de [Grasselli and Costa Lima, 2012] avec une dynamique non linéaire hors équilibre inspirée des travaux de [Goodwin, 1967], [Akerlof and Stiglitz, 1969], [Keen, 1995]. Ces équations sont basées sur la dynamique proie-prédateur de Lotka-Volterra, qui prend en compte la dette privée et/ou publique. Ces types de modèles sont capables de représenter l'hypothèse d'instabilité financière de Minsky (FIH) et l'éventualité de krachs. La dynamique stochastique permet l'émergence d'indicateurs agrégés, mais ils sont fortement critiqués en raison de leur forte dépendance aux paramètres, de leur caractère myope (sans anticipations) et de leur croissance exogène. Cependant, ils semblent être les plus à même de représenter les trajectoires que nous voulons explorer, si nous pouvons compenser ces critiques. Une version estimée du modèle existe avec une boucle de rétroaction climatique développée dans [Bovari et al., 2018] et une première approche dédiée à l'économie brésilienne a été mise en œuvre par [Bastidas et al., 2017].

Pour faire face au risque d'un moment de Minsky lié au climat, en utilisant un modèle robuste pour orienter les politiques publiques afin de garantir une transition vers une société neutre en carbone, ces modèles de déséquilibre semblent les plus convaincants car capables de surmonter certaines des défaillances des autres modèles macro-économiques.<sup>10</sup>

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<sup>9</sup>Mésange (Bercy), Three-Me (OFCE / ADEME), Imaclim (CIRED), Nemesis (Erasmus), Gemini (Lausanne), GEM E3 (Commission européenne), GEMMES (AFD), etc.

<sup>10</sup>Un aperçu plus large de ces critiques est disponible dans les chapitres de la thèse, mais nous pouvons en afficher certaines ici, par exemple :

- La plupart des nouveaux modèles classiques n'ont pas d'énergie ou de matière dans leur fonction de production (exception : lorsqu'on utilise la fonction de production KLEM avec Capital (K), Travail (L), Énergie (E), Matière (M), par exemple, [Stiglitz, 1974a]). Ces modèles considèrent généralement que, l'élasticité du PIB à l'énergie est égale à la quantité nominale d'énergie dans le PIB, ce qui est négligeable. En fait, [Giraud and Kahraman, 2014] a montré que cette part en volume représente en moyenne 60% (6 fois plus que ce que les nouveaux économistes classiques admettent habituellement).
- La monnaie est toujours considérée comme neutre lorsqu'elle est prise en compte, alors qu'elle est démontrée

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Selon l'analyse de [Grandjean and Giraud, 2017], on peut en effet s'interroger sur la capacité des modèles macro-économiques à prédire des trajectoires fiables, notamment en raison de la définition de leurs entrées,<sup>11</sup> l'ambiguïté de leurs indicateurs de production (PIB<sup>12</sup> ; Coût des politiques publiques ; Taux de chômage<sup>13</sup>), leur manque de back-testing, leur sensibilité potentiellement élevée aux paramètres calibrés, leur incapacité à représenter les trajectoires hors équilibre à équilibres multiples, notamment en économie du changement climatique.

Dans le cas spécifique de l'économie du changement climatique, les modélisateurs sont en effet confrontés à deux domaines principaux : l'économie et l'environnement. Dans le domaine de l'environnement, les chercheurs modélisent des interactions complexes entre des composants dépendant d'observables (éventuellement probabilistes). En général, ces comportements peuvent être décrits par des lois et des équations physiques. Ces équations fondamentales, telles que la conservation de l'énergie, le transfert de chaleur, etc., ont déjà été établies et testées empiriquement. La résolution de ces équations, par le biais d'une méthode numérique et d'un échantillonnage choisis, peut introduire certaines approximations. Ces équations font intervenir divers paramètres, parmi lesquels certaines constantes fondamentales ne sont pas discutables (comme la constante de Planck ou de Boltzman). Les incertitudes des modèles ne proviennent donc pas principalement des équations elles-mêmes, mais plutôt de la manière dont elles sont mises en œuvre et résolues afin de représenter le système réel.

En macroéconomie en revanche, les modélisateurs doivent représenter des interactions complexes entre des unités de décision complexes (les humains), dont la description comportementale par des équations mathématiques n'est pas indiscutable [Grandjean and Giraud, 2017]. Outre les doutes légitimes que l'on peut avoir sur la pertinence de certains indicateurs et les biais possibles dans la méthodologie de leur construction, les données sous-jacentes de ces modèles macro-économiques sont difficiles d'accès en raison de leur nature 'socialement construite'. Leur construction dépend donc d'institutions disposant de techniques et de capacités différentes pour évaluer finement ces données d'entrée. D'autres paramètres d'entrée sont par ailleurs supposés constants, comme le *taux de dépréciation du capital*, ce qui est tout sauf certain.

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comme endogène dans le livre *Illusion Financière* de [Giraud, 2014].

- Ils n'admettent pas de dynamique globale hors équilibre (voir par exemple les modèles CGE) ;
- Ils sont principalement linéaires dans la dynamique présumée, même s'ils représentent des phénomènes très complexes ;
- Ils ne prennent pas en compte la dette privée (comme le montre les travaux de [Keen, 2013] sur le lien dette-déflation. Si un endettement excessif est si susceptible de générer une déflation, il est pour le moins surprenant que le ratio d'endettement ne fasse pas partie d'un tel modèle — mentionnons une exception notable, [Krugman and Eggertsson, 2013], qui est cependant une approche sans monnaie...) ni le secteur bancaire (uniquement considéré comme des intermédiaires financiers). Voir par exemple *Debt-Deflation versus the Liquidity Trap : the Dilemma of Nonconventional Monetary Policy* de [Giraud and Pottier, 2016] et *Illusion Financière*, [Giraud, 2014].
- En raison des hypothèses anticipations rationnelles et de demande globale, ils ne peuvent pas prédire un équilibre avec un sous-emploi important (du moins lorsque le marché du travail est parfaitement flexible) et peuvent à peine considérer les crises financières comme des *cygnes noirs*.

<sup>11</sup>Par exemple, la controverse sur le capital de Cambridge à la fin des années 1950 exprime l'écart entre la définition de la nature et du rôle des biens d'équipement. [Stiglitz, 1974b].

<sup>12</sup>Voir une revue des critiques dans [Jany-Catrice and Méda, 2015] ou la proposition de [Piketty, 2009].

<sup>13</sup>Les experts de la BCE estiment le taux d'atonie du marché du travail à environ 18% dans la zone euro, soit presque le double du taux officiel d'Eurostat qui se situe autour de 9,5%. Cela montre le paradoxe actuel concernant le calcul de cet indicateur [European Central Bank, 2017]

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Les incertitudes de ces entrées et la difficulté d'accès aux données nuisent à la fiabilité des résultats ou à la qualité des recommandations en termes de politique publique (surtout lorsque le résultat lui-même est un indicateur ambigu). Les modèles macro-économiques sont en effet susceptibles de dépendre de nombreux paramètres. Si certains d'entre eux sont estimés, un grand nombre de paramètres ne sont que calibrés. Les méthodes d'estimation peuvent conduire à des décalages entre le résultat et la réalité. Ces types d'inadéquation peuvent être réduits en testant rétrospectivement (ce que l'on nomme *back-testing*) le modèle sur des données historiques.

Ces estimations et calibrations des paramètres peuvent avoir des impacts importants sur les résultats. La finalité de l'analyse de sensibilité est de décrire le lien entre l'impact sur les sorties et les incertitudes sur les entrées. Ce type d'analyse est détaillé dans l'annexe A.2. Dans un système très complexe, tel que les sciences de l'environnement ou la macroéconomie, avec de nombreuses non-linéarités et paramètres, nous pouvons reconnaître le fait qu'une incertitude mineure peut entraîner une grande variation de la sortie.

Parmi les différentes analyses de sensibilité, la méthode bayésienne est la plus fréquemment utilisée pour calibrer les modèles DSGE, mais cette méthode comporte de nombreux biais *a priori*. L'inférence bayésienne utilisée pour construire l'analyse de sensibilité des modèles DSGE est donc fortement dépendante des choix *a priori* des modélisateurs. Dans un domaine où les faits et les *valeurs* sont difficiles à distinguer [Pottier, 2014], on peut comprendre comment une méthode bayésienne pour calibrer un modèle DSGE pourrait potentiellement confirmer les biais originaux des modélisateurs. D'autres analyses de sensibilité (dites locales) impliquent la sélection d'un point nominal autour duquel tester la sensibilité du modèle.<sup>14</sup> Le point nominal sélectionné dans l'espace d'entrée (quelle que soit la méthode choisie) est critique pour l'analyse de sensibilité, car le modélisateur l'explore dans un voisinage proche de cette valeur particulière.

Dans un domaine où les incertitudes sur les paramètres sont importantes, en raison de données difficilement accessibles et socialement construites, on peut comprendre la nécessité de "back-tester" le modèle et d'éprouver sa robustesse à ces incertitudes par l'analyse de sensibilité, afin de fournir le résultat le plus fiable, même si la sortie reste discutable.

### **Sélection d'une famille de modèle capable de décrire les risques induits par un moment de Minsky dû au climat**

Le modèle historique de Solow est un modèle dynamique, mais souvent utilisé pour analyser un état stationnaire à long terme. Cette habitude est tout sauf scientifique, car elle présuppose que notre économie réelle a atteint son équilibre. Les modèles d'équilibre général tentent de calculer de nombreux équilibres statiques à chaque pas de temps, en tenant compte des éventuels chocs exogènes pour représenter la grande volatilité des indicateurs économiques réels. Cependant, ces modèles considèrent la cinématique du système à travers le prisme d'une succession d'équilibres. Les modèles DSGE quant à eux, supposent que l'équilibre de Solow est atteint, mais que des chocs exogènes en éloignent le système. Leurs analyses consistent à calculer le retour à l'état d'équilibre de long terme, quelle que soit la nature du choc. Par essence, ces modèles ne paraissent donc pas en mesure de prédire les trajectoires vers un effondrement ou un *moment de Minsky induit par le climat*. La dynamique hors équilibre d'un DSGE est par construction toujours transitoire.

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<sup>14</sup>Ces méthodes sont également présentées en annexe A.2.4.

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Pour respecter un objectif climatique, nous devons intégrer les contraintes climatiques dans le modèle macro-économique afin de décrire les trajectoires de transition (comme cela a été remarqué dans [Canfin et al., 2016]). L’objectif est de garantir une meilleure cohérence entre l’analyse à court terme et les objectifs de décarbonation à long terme. Pour cette raison, selon [Grandjean and Giraud, 2017], les modèles macro-économiques doivent :

- Être “Stock-Flow Consistent” (une condition de base pour garantir la fiabilité du modèle en termes de comptabilité) ;
- Être “Physic-Consistent” (et en particulier avec les lois physiques, telles que la 2<sup>nd</sup> loi de la thermodynamique) ;
- Décrire explicitement la circulation de la monnaie, et ne pas la supposer neutre *a priori* ;
- Être “back-testé” et analysé en termes de sensibilité aux paramètres ;
- Éviter les variables cachées ou infalsifiables.

Une typologie de modèles plus intéressante que les autres pour notre étude sont ainsi les modèles de déséquilibre mentionnés précédemment (tels que Three-Me et son extension IMACLIM-3ME [Gherzi, 2020], [Akerlof and Stiglitz, 1969] ou [Grasselli and Costa Lima, 2012]). Ces modèles évaluent par construction les trajectoires vers des équilibres multiples et sont capables de décrire la dynamique globale hors équilibre.

Contrairement aux modèles macro-économiques habituels, le modèle [Grasselli and Costa Lima, 2012] est capable d’éviter certains de leurs défauts et de capturer les caractéristiques spécifiques dont nous avons besoin pour prédire un éventuel effondrement (provoqué par exemple, par une quantité explosive de dettes). Ce modèle est basé sur les travaux de Goodwin et introduit la dette [Goodwin, 1967] et [Keen, 1995]. La nature et la stabilité des équilibres de dette ont été analysées dans [Grasselli and Costa Lima, 2012] et [McIsaac, 2016] pour les extensions de Van Der Ploeg.<sup>15</sup> Le modèle est basé sur la logique prédateur-proie de Lotka-Volterra, avec une équation classique d’accumulation du capital et deux fonctions comportementales (sur, d’une part, l’investissement dépendant des profits, et, d’autre part, le lien entre le taux d’emploi et la part des salaires, à travers une courbe de Phillips à court terme), la variation de la dette privée (comme la différence entre l’investissement et les profits) et un taux de croissance exogène de la population et de la productivité du travail. Les défauts du modèle résident dans son modèle de croissance exogène, l’utilisation de la loi de Say (qui peut être assouplie par l’introduction d’inventaires [Grasselli and Nguyen-Huu, 2018]), sa nature myope (sans anticipations), et sa forte dépendance aux paramètres.<sup>16</sup>

Dans le cadre de notre travail, trois axes de recherche sont explorés pour évaluer la capacité de ce modèle à répondre à l’objectif de notre thèse : la construction d’un modèle de transition robuste qui doit être en mesure de fournir des informations sur les politiques publiques permettant d’éviter soit un équilibre d’attraction avec du capital brun uniquement, soit un effondrement dû à un “*moment de Minsky induit par le climat*”. Cette construction est basée sur une structure stock-flux cohérente avec une dynamique non linéaire conduisant à des équilibres multiples, qui peut intégrer ce moment de Minsky. Ces axes sont les suivants :

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<sup>15</sup>Introduisant une fonction de production CES au lieu de la fonction de production Leontief [Van der Ploeg, 1985].

<sup>16</sup>Voir aussi l’étude de [Pottier and Nguyen-Huu, 2017] sur la dépendance de la dynamique du modèle à la fonction d’investissement choisie.

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- Quantifier la robustesse d’un modèle et comparer sa sensibilité à celle d’autres modèles macro-économiques ;
  - Étendre un modèle macro-économique existant, [Grasselli and Costa Lima, 2012], en introduisant les inventaires, la vitesse de la monnaie, l’intervention du gouvernement afin d’aider l’orientation des politiques publiques en conjonction avec les modèles actuels appliqués ;
  - Évaluer l’impact des politiques publiques afin de garantir une transition entre les différentes natures de capital dans ce modèle.

Pour répondre aux critiques formulées à l’encontre du modèle [Grasselli and Costa Lima, 2012] concernant sa forte dépendance vis-à-vis des paramètres, le chapitre 2 explore la robustesse du modèle avec deux fonctions de production (Leontief et CES), à travers une analyse de sensibilité globale avec des indices de Sobol’ estimés par la méthode [Saltelli, 2002].<sup>17</sup> Le chapitre passe également en revue certaines analyses sur des modèles DSGE, de la théorie des cycle réels (RBC) et des modèles d’évaluation intégrée (IAM). Le chapitre 3 vise à améliorer notre modèle en introduisant la vélocité de la monnaie (selon la théorie quantitative de la monnaie) dans un cadre assouplissant la loi de Say par l’introduction des inventaires (inspirée par les travaux de [Grasselli and Nguyen-Huu, 2018]). Finalement, le modèle de transition est construit avec deux natures de capital et peut être généralisé dans un cadre “Putty-Clay” (voir [Akerlof and Stiglitz, 1969], [Cass and Stiglitz, 1969]). Le modèle de transition le plus simple possible, avec seulement deux natures de capital, fournit déjà un critère de stabilité et des leviers possibles par l’intervention du gouvernement (au sens de [Costa Lima and Grasselli, 2014]). Ce chapitre 4 étudie les équilibres multiples à long terme et les critères de stabilité du modèle de transition avec différents types d’intervention gouvernementale.

## Quantifier la robustesse d’un modèle et comparer sa sensibilité à celle d’autres modèles macro-économiques

Dans le chapitre 2, l’objectif est de tester la robustesse du modèle [Grasselli and Costa Lima, 2012] en termes de sensibilité aux paramètres. Ce travail a été mené avec le soutien du programme Alliance (Alliance Doctoral Mobility Grant) et la Columbia University Graduate School of Arts and Sciences, sur l’invitation du Prof. Joseph E. Stiglitz.

Les modèles macro-économiques actuellement utilisés sont confrontés à diverses défaillances (voir [Stiglitz, 2018]), notamment dans un environnement économique complexe impliquant un grand nombre de paramètres calibrés et estimés, ce qui conduit à des résultats très sensibles aux paramètres (voir [Grandjean and Giraud, 2017]). Par conséquent, la construction d’un modèle, notamment en sciences sociales ou en macroéconomie, nécessite de considérer trois types d’incertitudes :

- Incertitude sur la précision entre le modèle et la réalité (*Sensibilité aux représentations*) ;
- Incertitudes numériques correspondant aux choix d’échantillonnage ;
- Incertitudes liées aux paramètres (*Sensibilité aux paramètres*).

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<sup>17</sup>Le choix de la méthode est basé sur un état de l’art dans ce domaine (voir annexe A.2). Notons que la dépendance aux valeurs initiales a été explorée à travers l’étude du bassin d’attraction par [McIsaac, 2016]. Par conséquent, nous pouvons concentrer notre étude sur les paramètres uniquement.

Après une étude des différentes méthodes d'analyse de sensibilité (voir l'annexe A.2.5), une analyse de sensibilité globale (GSA) avec des indices de Sobol estimés par la méthode [Saltelli, 2002] a été choisie en fonction des caractéristiques de notre modèle : présence de non-linéarités, petit nombre d'entrées, et coûts de calcul modestes.

Inspiré par les travaux du chimiste Cukier, I.M. Sobol' a en effet proposé une analyse de sensibilité globale basée sur la décomposition de la variance par le biais d'un échantillonnage de Monte-Carlo. Étant donné le modèle suivant  $Y = f(\mathbf{X})$ , cette analyse de la variance (ANOVA) vise à définir la part de variabilité de la sortie due à ses entrées, ce que l'on appelle les "indices de Sobol'".

Dans le cadre de notre étude, nous nous intéressons à une hiérarchisation des facteurs, afin de qualifier l'influence relative de chaque facteur d'entrée sur la sortie. Le principe d'application de l'ANOVA avec indices de Sobol est décrit ci-dessous, mais l'estimation de ces indices reste difficile (en particulier pour l'effet de Sobol de second ordre, représentant les effets conjoints entre les sorties) et est développée en annexe A.2.6. Plusieurs études fructueuses ont déjà été menées en économie sur les modèles DSGE (par exemple, [Ratto, 2008]).

Dans son article fondateur [Sobol, 1967], Sobol a proposé d'exprimer  $f \in L^1([0, 1]^p, dx)$  en sommes de dimension croissante:

$$f(x_1, \dots, x_p) = f_0 + \sum_{i=1}^p f_i(x_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(x_i, x_j) + \dots + f_{1,\dots,p}(x_1, \dots, x_p). \quad (1)$$

En imposant que  $\forall s \in \{1, \dots, p\}, \forall i_1 < i_2 < \dots < i_s \in \{1, \dots, p\}^s$  et  $\forall k \in \{i_1, \dots, i_s\}$ ,

$$\int_0^1 f_{i_1, \dots, i_s}(x_{i_1}, \dots, x_{i_s}) dx_k = 0$$

ce développement de  $f$  en  $2^n$  sommes de différentes dimensions existe et est unique, et les termes associés sont orthogonaux deux à deux dans  $L^1([0, 1]^p, dx)$ .

Supposons maintenant que l'on puisse exprimer une variable aléatoire de carré intégrable,  $Y$ , sous la forme  $Y = f(\mathbf{X})$  où  $\mathbf{X} = (X_1, \dots, X_p)$  est un vecteur de  $p$  variables aléatoires indépendantes uniformément distribuées sur  $[0, 1]$ , nous avons <sup>18</sup>

$$Y = f_0 + \sum_{i=1}^p f_i(X_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(X_i, X_j) + \dots + f_{1,\dots,p}(X_1, \dots, X_p). \quad (2)$$

Par conséquent, en utilisant l'orthogonalité par paire dans la décomposition de Sobol', nous pouvons obtenir facilement une expression récursive de chaque terme en utilisant les espérances condition-

<sup>18</sup>Ici l'indépendance des variables aléatoires  $(X_1, \dots, X_p)$  est une hypothèse forte nécessaire pour obtenir la décomposition de la variance sous-jacente alors que nous pouvons évidemment relâcher l'hypothèse sur la distribution en nous rappelant que pour une variable aléatoire réelle  $Z$ ,  $F_Z^-(U)$  et  $Z$  sont équidistribués lorsque  $U$  est une variable aléatoire uniforme sur  $[0, 1]$  et  $F_Z^-$  le pseudo-inverse de la fonction de distribution de  $Z$ . Dans le cas de variables dépendantes, on peut s'opposer à la difficile généralisation des expressions récursives ci-dessus de chaque terme en utilisant les espérances conditionnelles. Voir la note de bas de page 23 pour plus d'explications.

nelles

$$\begin{aligned}
f_0 &= E(Y) \\
f_i(X_i) &= E(Y|X_i) - E(Y) \\
f_{i,j}(X_i, X_j) &= E(Y|X_i, X_j) - f_i(X_i) - f_j(X_j) - E(Y) \\
&\vdots
\end{aligned} \tag{3}$$

et déduire la décomposition dite ANOVA-HDMR pour la variance de  $Y$ .

$$\begin{aligned}
V(Y) &= \sum_{i=1}^p V_i + \sum_{i=1}^p \sum_{j>i}^p V_{i,j} + \dots + V_{1,\dots,p} \\
&\Downarrow \\
1 &= \sum_{i=1}^p \frac{V_i}{V(Y)} + \sum_{i=1}^p \sum_{j>i}^p \frac{V_{i,j}}{V(Y)} + \dots + \frac{V_{1,\dots,p}}{V(Y)}
\end{aligned} \tag{4}$$

où

$$\begin{aligned}
V_i &= V(E(Y|X_i)) \\
V_{i,j} &= V(f_{i,j}(X_i, X_j)) = V[E(Y|X_i, X_j)] - V[E(Y|X_i)] - V[E(Y|X_j)] \\
&\vdots
\end{aligned} \tag{5}$$

Maintenant nous pouvons définir les indices de Sobol :  $\forall s \in \{1, \dots, p\}, \forall i_1 < i_2 < \dots < i_s \in \{1, \dots, p\}^s$ .

$$S_{i_1, \dots, i_s} = \frac{V_{i_1, \dots, i_s}}{V(Y)} \tag{6}$$

où  $S_i$  est une mesure de l'impact de  $X_i$  sur la variance totale,  $S_{i,j}$  est une mesure de l'impact conjoint de  $(X_i, X_j)$  sur la variance totale, etc....

Dans cet esprit, pour mesurer l'impact total (somme des effets uniques et des effets conjoints liés) de  $X_i$  sur la variance de  $Y$  nous définissons<sup>19</sup>

$$S_{T_i} = \frac{E(V(Y|X_{\bar{i}}))}{V(Y)} = 1 - \frac{V(E(Y|X_{\bar{i}}))}{V(Y)} \tag{7}$$

où  $X_{\bar{i}} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_p)$ . Par exemple pour un modèle à 3 entrées, nous avons

$$S_{T_1} = S_1 + S_{1,2} + S_{1,3} + S_{1,2,3} = 1 - S_{2,3}. \tag{8}$$

<sup>19</sup>Les propriétés des indices de Sobol' sont bien connues dans la littérature, et peuvent être résumées comme suit :

1.  $\sum_{s=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq p} S_{i_1, \dots, i_s} = 1$
2.  $\forall i \in \{1, \dots, p\}, S_i, S_{T_i} \in [0, 1]$ .

Ces deux propriétés correspondent à l'interprétation des indices de Sobol' comme contributions relatives à la variance de la sortie  $V(Y)$ .



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Le processus pour estimer les indices est basé sur la méthode [Saltelli, 2002] : deux matrices d'échantillonnage initiales sont générées par des *Latin Hypercube Sampling (LHS)*. Une revue de la littérature montre que des GSA ont également été réalisées sur des modèles DSGE, RBC et le modèle DICE [Nordhaus, 2008].

Avec la méthode des indices de Sobol' estimés selon [Saltelli, 2002], la sensibilité d'une sortie du modèle [Grasselli and Costa Lima, 2012] à ses entrées est plus équilibrée que celle d'un modèle DSGE, RBC ou IAM (DICE), dépendant principalement d'un unique paramètre souvent très difficile à estimer. Notre étude montre aussi qu'il est possible d'améliorer légèrement la robustesse du modèle en endogénéisant les paramètres principalement responsables de cette sensibilité. Ainsi, le modèle [Grasselli and Costa Lima, 2012] avec une fonction de production CES est légèrement plus robuste qu'avec une fonction de production Leontief.

Nous avons également étudié le modèle original [Goodwin, 1967] (voir annexe A.5) et nous avons remarqué que la sensibilité du modèle original dépend des mêmes paramètres que son extension et que ses performances en termes de robustesse sont légèrement meilleures que celles du modèle [Grasselli and Costa Lima, 2012], ce qui pourrait être dû au plus petit nombre de non-linéarités. Ces résultats sont assez rassurants quant au fait que l'extension du modèle ne semble pas affecter la nature profonde de sa sensibilité. Ils nous amènent à nous interroger sur un éventuel compromis entre la robustesse et le réalisme d'un modèle pour trouver un équilibre entre un modèle très robuste incapable de s'approcher de la réalité (tel que le modèle [Goodwin, 1967]), et un modèle très fin et très sensible (comme le DSGE de [Ratto, 2008]). De ce point de vue, le modèle [Grasselli and Costa Lima, 2012] semble être un compromis intéressant, avec une description plus fine de l'économie et une sensibilité équilibrée entre plusieurs entrées.

### **Étendre un modèle macro-économique existant, [Grasselli and Costa Lima, 2012], en introduisant les inventaires et la vitesse de la monnaie**

Jusqu'à présent, l'analyse de la monnaie se heurte à un certain nombre de paradoxes et d'impossibilités qui affecte notre compréhension de l'interaction entre la monnaie, sa vélocité et la macro-dynamique. Dans le chapitre 3, nous proposons une dynamique stock-flux cohérente avec concurrence imparfaite où ces paradoxes peuvent être levés.

Une version préliminaire, sans monnaie, de notre dynamique non linéaire et stock-flux cohérente est réalisée dans un premier temps. En plus de fixer la notation pour le modèle complet, nous nous écartons de la littérature précédente en nous concentrant sur une économie avec une fonction de production CES, une tarification oligopolistique et une dynamique des inventaires. Ce modèle présente quatre équilibres de long terme, économiquement significatifs, et localement stables pour cette dynamique, dont l'un est caractérisé par un ratio d'endettement fini et un emploi strictement positif — l'équilibre solovien — et un autre par un endettement infini et un emploi nul.

Dans un second temps, nous introduisons la monnaie à travers la célèbre 'équation d'échange', où  $T$  est le volume des transactions. Dans ce cadre, s'il y a une offre excédentaire,  $Y \geq Y_d$ , le volume effectif de transaction est  $Y_d$ . Si, au contraire, la demande excède l'offre,  $Y \leq Y_d$ , les entreprises vendront des stocks pour répondre à la demande, et par conséquent,  $T$  est toujours égal à  $Y_d$ , tant que le stock d'inventaire reste non négatif. Si ce stock devient nul, le secteur de la production ne



peut plus répondre à la demande chaque fois que  $Y \leq Y_d$  : un certain rationnement doit avoir lieu et le volume de transaction doit être égal à la production :  $T = Y$ . Nous laissons cette dernière situation pour une étude plus approfondie,<sup>20</sup> et nous concentrerons donc notre analyse sur les solutions “intérieures” du système différentiel représentant le modèle avec monnaie où  $V \geq 0$  tout au long du processus. Par conséquent, l’équation d’échange se lit comme suit

$$M\mathbf{v} = pY_d, \quad (9)$$

où  $M$  doit être compris comme la somme des comptes bancaires détenus par des organismes non gouvernementaux (ménages et entreprises). Il s’agit d’un stock endogène de *monnaie intérieure*, généré en reconnaissance de dette.<sup>21</sup> La variable,  $\mathbf{v}$ , définie par (9), désigne la vitesse de la monnaie, à savoir la vitesse à laquelle les transactions réelles se produisent dans l’économie.

Mais d’où vient cet argent ? Dans notre économie de crédit, il ne peut être créé que par un crédit bancaire supplémentaire. Inversement, lorsqu’une dette est remboursée (que ce soit une dette des firmes,  $D_f$ , ou des ménages,  $D_h$ ), la quantité de monnaie correspondante retourne au bilan du système bancaire, et est donc ‘détruite’. En conséquence,

$$\dot{M} := \dot{D}_f + \dot{D}_h. \quad (10)$$

Remarquez que, chaque fois que  $\dot{D}_f + \dot{D}_h = 0$ , aucune monnaie supplémentaire ne circule dans l’économie : l’un des deux secteurs prête à l’autre l’incrément de dette nécessaire. Avec les équations dynamiques de la dette, l’équation (10) conduit à la dynamique endogène suivante de la monnaie :

$$\dot{M} = -c\dot{V} + p\delta K. \quad (11)$$

où  $V$  est le volume d’inventaires, valorisé au coût unitaire de production,  $c$  tandis que  $\delta$  est le taux de dépréciation du capital,  $K$ , en termes réels, valorisé au prix  $p$ .

Malgré sa simplicité, l’équation (11) fournit quelques informations utiles. En effet, si l’on maintenait la ‘loi’ de Say et que l’on négligeait le coût irrécupérable induit par la dépréciation du capital, la quantité de monnaie  $M$  serait nécessairement constante, et deviendrait donc, là encore, non pertinente. On déduirait également de (11) que  $\dot{D}_f = -\dot{D}_h$ . Le fameux problème de la poule et de l’œuf — qui cause l’autre, l’épargne ou le crédit ? — devient à nouveau inextricable. Cela signifie que, pour résoudre cette énigme et pouvoir observer la création endogène de crédit dans un modèle macro-économique, l’abandon de la “loi” de Say est une condition nécessaire.

L’introduction de la monnaie permet de résoudre les premiers paradoxes auxquels nous venons de faire allusion : il n’est pas nécessaire de s’appuyer sur l’hypothèse d’anticipations rationnelles pour que l’économie atteigne un équilibre de long terme avec un taux d’inflation stable. Nous montrons aussi que la monnaie, telle qu’elle est ici introduite, s’avère ne pas être neutre à la fois à court et à long terme. Sa création est ici déterminée de manière endogène par le crédit bancaire. La cohérence stock-flux permet de suivre la création monétaire et son déplacement le long du circuit de l’économie agrégée.

<sup>20</sup>Considérant que ce type de “problème limite” conduit à une dynamique non différentiable, qui sera étudiée dans un travail ultérieur.

<sup>21</sup>C’est-à-dire que nous négligeons ici la monnaie extérieure qui serait libre de dette, voir [Gurley and Shaw, 1960].

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Dans ce modèle, l'évolution de sa vitesse de circulation peut être observée. En particulier, la baisse de la vitesse de circulation de la monnaie observée dans plusieurs pays à la suite d'une crise financière peut être analysée. Notre modèle apporte aussi un éclairage nouveau sur les liens entre la vitesse de circulation de la monnaie et la trajectoire de la dette. La faible inflation, une faible croissance du revenu réel, une augmentation de la dette privée et une diminution de la vélocité des échanges s'avèrent être les caractéristiques des trajectoires macro-économiques menant à un endettement de longue durée. Nous fournissons les conditions pour lesquelles, cependant, une politique monétaire à faible taux d'intérêt peut stimuler l'activité économique afin d'échapper à la trappe à liquidité.

Par exemple l'intervention de la banque centrale peut avoir un effet positif évident pour prévenir une crise caractérisée par l'effondrement des taux d'emploi. La politique monétaire prévient ce résultat en conditionnant le taux d'intérêt nominal à court terme au ratio dette privée/résultat : plus ce dernier est élevé, plus le coût de l'argent doit être faible. Les résultats en matière de persistance sont beaucoup plus forts que, par exemple, l'éventuelle instabilité locale et asymptotique d'états stationnaires indésirables pour des valeurs de paramètres typiques du modèle : l'intervention de la banque centrale, sous la forme d'une politique monétaire suffisamment réactive, empêche l'économie de rester en permanence à des niveaux d'emploi arbitrairement bas, quelles que soient les conditions initiales du système. Pour faire écho à [Costa Lima and Grasselli, 2014], il se peut que la stabilisation d'une économie monétaire instable soit un ordre trop élevé pour la banque centrale, mais que la déstabilisation d'une crise stable de dette-déflation soit possible.

Il y a, bien sûr, un certain nombre d'aspects institutionnels que nous avons négligés dans ce chapitre : réserves obligatoires, exigences en matière d'adéquation des fonds propres (comme le cadre de Bâle III/CRR), réglementation prudentielle... [Bovari et al., 2020] ont introduit quelques raffinements dans la modélisation du système bancaire. dans une économie "réelle". Ajoutés au cadre actuel, ces raffinements pourraient nous permettre d'aborder des questions importantes autour d'une réglementation des banques basée sur l'effet de levier contracyclique.

Peut-être est-il tout aussi important de considérer les économies ouvertes et l'interaction entre la création de crédit non neutre à partir de rien, les taux de change et les emprunts internationaux. La thèse principale défendue par [Werner, 2014b] devrait alors être examinée afin de vérifier dans quelles conditions les pays en développement auraient dû s'appuyer sur la création monétaire nationale plutôt que d'emprunter de l'argent à l'étranger, au risque de se retrouver "englués dans une spirale de dettes en devises étrangères, alors qu'en réalité les banques étrangères ont simplement créé l'argent à partir de rien, ce que le pays en développement aurait pu arranger par le biais de leurs propres banques nationales" ([Werner, 2016]). Comme l'indique clairement cet auteur, le fait de considérer la création monétaire endogène "a également des implications sur la question de savoir qui doit payer pour le sauvetage des banques, en déplaçant le pendule des contribuables vers les banquiers centraux". Nous laissons aux recherches futures le soin d'étudier ces questions fascinantes.

### **Évaluer des leviers de politiques publiques pour garantir une transition entre différentes natures de capital dans ce modèle.**

Après avoir étudié la robustesse du modèle et sa capacité à intégrer la vitesse de circulation de la monnaie et à relâcher la loi de Say, l'objectif de ce chapitre 4 est de proposer un modèle simple de transition énergétique des combustibles fossiles aux énergies renouvelables.

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Nous considérons donc deux types de capital, un capital “brun” (ou sale) qui a besoin d’énergie fossile pour fonctionner, et un capital “vert” (ou propre), qui est moins productif mais ne dépend pas de l’énergie fossile. Le prix de l’énergie fossile doit être compris comme le prix relatif des énergies fossiles et renouvelables. L’investissement privé est orienté vers le type de capital le plus rentable, en fonction du prix (exogène) de l’énergie fossile. Les deux types de capital sont de type “putty-clay” ([Johansen, 1959]) par rapport à l’énergie : une fois qu’un bien d’équipement est créé, son rapport de consommation de capital à l’énergie est fixé une fois pour toutes, et les deux types de capital ne sont pas substituables. Cette structure de production s’inscrit dans la même macrodynamique proie-prédateur que précédemment, stock-flux cohérente [Goodwin, 1967] où l’investissement privé peut être financé par la dette [Grasselli and Costa Lima, 2012]. La transition vers les énergies renouvelables a lieu si l’économie fonctionne asymptotiquement avec du capital vert uniquement. Deux facteurs principaux sont mis en évidence dans ce chapitre 4 : le ratio production/capital du capital brun par rapport au capital vert et le prix relatif des énergies fossiles par rapport aux énergies renouvelables.

Dans ce travail, l’investissement est orienté vers le type de capital le plus rentable, en fonction de sa productivité et du prix relatif de l’énergie fossile, mais ne s’ajuste pas instantanément (bien que plus l’écart entre les taux de profit est élevé, plus l’ajustement est rapide). Cette inertie de l’investissement peut être considérée comme reflétant les effets de complémentarité entre les deux types de capital (qui sont par ailleurs de parfaits substituts en tant que facteurs de production), la spécialisation de la main-d’œuvre ou les externalités liées aux infrastructures. Elle s’écarte du passage instantané du capital brun au capital vert, souvent postulé dans la littérature.<sup>22</sup>

Suite à [Grasselli and Nguyen-Huu, 2018], et comme dans le chapitre précédent, la loi de Say est abandonnée ici et la politique publique est introduite par le biais des dépenses publiques et des taxes d’une manière similaire à [Costa Lima and Grasselli, 2014]. L’interaction entre la dynamique des inventaires et la politique publique conduit à une dynamique similaire à celle bidimensionnelle de type Aubry-Mather sur le tore avec un petit cycle (lié à la dynamique proie-prédateur sous-jacente entre l’emploi et les salaires) entrelacé avec un plus grand cycle induit par la réponse publique contracyclique. Avec la dynamique de la dette privée, de la part des salaires et du sous-emploi, cela conduit à un système dynamique non linéaire à 10 dimensions. Nous montrons que ce système peut être réduit à un système à 5 dimensions. Ce système n’est cependant pas autonome car il dépend du prix de l’énergie fossile, donné de manière exogène, qui détermine l’allocation des investissements.

Pour cette étude, nous considérons donc une économie fermée à quatre secteurs comprenant des ménages, un secteur bancaire, un secteur public et des entreprises privées produisant un seul bien de consommation à capital brun ou vert. Par souci de simplicité, nous nous limitons au cas extrême de deux types de capital : le capital brun d’une part, qui a besoin d’un flux constant d’énergie fossile pour produire ; et le capital vert d’autre part, qui fonctionne sans énergie fossile mais est moins productif que le capital sale.

La production,  $Y$ , peut être produite de deux manières : soit par rapport au stock,  $K_1$ , de capital brun avec un flux,  $E$ , d’énergie fossile et de travail,  $L_1$ , soit en exploitant un stock de capital propre,

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<sup>22</sup>Voir, par exemple, le modèle DICE de Nordhaus [Nordhaus, 2014] ; pour un modèle de changement climatique partageant notre cadre théorique, voir [Bovari et al., 2018].

$K_2$ , avec un flux,  $L_2$ , de travail seul.<sup>23</sup> Pour chaque type  $k = 1, 2$ , le ratio capital/production  $\nu_k$  est constant. Les biens d'équipement installés, quel que soit leur type, ne peuvent pas être convertis les uns dans les autres : si les investisseurs veulent passer d'un type à un autre, ils doivent investir dans de nouveaux biens d'équipement. En ce qui concerne la production de biens, les deux types de biens,  $Y_1$  et  $Y_2$ , sont de parfaits substituts :

$$Y = Y_1 + Y_2 \text{ avec } Y_1 = \frac{K_1}{\nu_1} = \frac{E}{\nu_1} = a_1 L_1 \text{ et } Y_2 = \frac{K_2}{\nu_2} = a_2 L_2, \quad (12)$$

où  $\nu_1, \nu_2, a_1, a_2 > 0$  avec  $a_k$  le taux de productivité du travail pour chaque type de capital  $k = 1, 2$ . Le plein emploi du capital est supposé tout au long de l'équation. Toutes les quantités sont exprimées en termes réels et sont des quantités nettes, ce qui signifie que les recettes et les dépenses intermédiaires sont déduites de la production annuelle finale.

On montre que le système n'admet que deux états stationnaires correspondant respectivement à un équilibre fonctionnant et investissant uniquement en capital vert et un autre avec uniquement du capital brun. On peut montrer que, sous des conditions tout à fait raisonnables, détaillées dans l'annexe C.1.2, dans chaque cas, huit points d'équilibre émergent. Par conséquent, le modèle présente 16 états stables à long terme.

Comme dans la littérature consacrée à ce type de macrodynamique (voir [Grasselli and Costa Lima, 2012] et [Giraud and Grasselli, 2019]), nous montrons que, indépendamment de la nature du capital prévalant à long terme, il existe un état stationnaire souhaitable correspondant à une part de salaire et un taux d'emploi positifs, et un niveau fini de ratios d'endettement, de subventions et de taxes. Suivant une remarque de [Giraud and Grasselli, 2019], c'est l'analogie d'un équilibre *solovien* parce que le taux de croissance réel, à cet état d'équilibre, est égal à la croissance de la productivité du travail (progrès technique) plus la croissance de la population. Si nous utilisions une fonction de production CES, le modèle de Solow deviendrait un cas particulier du présent modèle.

Le deuxième type d'équilibre à long terme est un état stationnaire déflationniste associé à une montée en flèche des ratios d'endettement tandis que les salaires et l'emploi tendent vers zéro. Sous des spécifications raisonnables de la courbe de Phillips à court terme, ainsi que des fonctions d'investissement et de consommation, nous montrons que ces équilibres déflationnistes par la dette, qu'ils soient sales ou propres, sont localement instables. Ceci contraste fortement avec la littérature déjà mentionnée. La force qui déstabilise les équilibres déflationnistes s'avère être l'introduction de l'énergie dans le processus de production et l'inertie avec laquelle l'investissement agrégé passe d'un type de capital à un autre. Cela soulève une question intéressante pour les recherches futures : quelles sont les forces qui poussent une économie vers la déflation par la dette dans un monde où plusieurs types de capitaux coexistent et où l'investissement présente une certaine inertie ? [Costa Lima and Grasselli, 2014] avaient fourni un mécanisme complètement différent qui déstabilise la crise dette-déflation, basé sur l'intervention publique.

Deux autres équilibres correspondent à des états déflationnistes avec un niveau de dette fini : l'un avec un salaire nul, l'autre avec un salaire positif. Leur stabilité asymptotique locale ne peut être jugée a priori et doit être vérifiée au cas par cas. Il en va de même pour l'équilibre dit de

<sup>23</sup>Pour simplifier les notations, nous n'introduisons pas explicitement les énergies renouvelables, qui pourraient être utilisées par les deux types de capital. un tel ajout pourrait facilement être fait sans améliorer significativement nos résultats, ni leur clarté.

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l'esclavage (sans salaire mais avec emploi). Tous, cependant, sont structurellement instables et donc non pertinents dans notre étude.

Le principal moteur de la transition vers la durabilité de l'économie s'avère être la différence entre le prix relatif de l'énergie fossile et l'écart de productivité entre les deux types de capital. Si le prix de l'énergie fossile reste trop faible par rapport à l'écart de productivité entre les deux types de capital, l'économie fonctionne asymptotiquement exclusivement avec du capital brun. Le coût de l'énergie fossile est compensé à long terme par une diminution de la part des salaires, grâce à la productivité plus élevée du capital. Inversement, lorsque le prix de l'énergie fossile est suffisamment dissuasif, l'investissement se concentre finalement sur le capital propre. Des simulations numériques montrent les effets de changements dans les paramètres et dans le prix relatif de l'énergie fossile.

Afin de satisfaire ce critère de transition que notre étude a mis en lumière,  $\chi_E - \Delta\nu > 0$ , nous pouvons introduire une taxe carbone,  $\chi_c$  (selon, par exemple, le corridor de valeurs établi par [Stiglitz et al., 2017]), afin d'augmenter le prix des énergies fossiles,  $\chi_E$  ou réduire le ratio de capital vert par rapport à la production,  $\nu_2$ . En effet, comme  $\Delta\nu = \frac{1}{\nu_1} - \frac{1}{\nu_2}$ , le critère de transition sera plus facilement satisfait. La première option risque de réduire les profits privés et donc de favoriser l'endettement privé. La deuxième option présente le risque d'augmenter la dette publique. Nous identifions les conditions sous lesquelles la transition vers le capital vert se fait sans conduire à un surendettement, qu'il soit privé ou public.

Bien que ces deux outils politiques aient été largement préconisés dans la littérature, leur efficacité à long terme n'a pas été clairement démontrée jusqu'à présent. Lorsque seul l'outil de la taxe carbone est utilisé, nous montrons que les critères de transition et de stabilité locale sont identiques :  $\chi_E + \chi_c - \Delta\nu > 0$ . Lorsqu'elle est suffisamment élevée, une taxe sur le carbone incite fortement à passer des combustibles fossiles aux énergies renouvelables, mais au prix d'une pénalisation des entreprises. L'état stationnaire vert de type solovien est localement asymptotiquement stable, et si l'économie converge, elle doit devenir asymptotiquement verte car les équilibres sales s'avèrent asymptotiquement instables. Inversement, si la taxe carbone reste trop faible, l'économie ne peut s'empêcher, si elle converge, de le faire vers l'un des nombreux équilibres à forte intensité de combustibles fossiles, car les états d'équilibre verts deviennent instables.

La deuxième option est réalisée en introduisant des subventions publiques à l'innovation verte, dans l'espoir de pouvoir garantir que la transition aura lieu, quel que soit le point de départ de l'économie. Inspirés par l'équation classique d'accumulation du capital, nous proposons une dynamique similaire pour le stock de connaissances qui a un impact sur la productivité du travail et donc sur le ratio capital/production,  $\nu_2$ , du capital vert, qui influence à son tour la productivité du travail associée au capital vert.

Sous les mêmes conditions raisonnables que précédemment, le modèle présente les mêmes types d'états stationnaires - qu'ils soient bruns ou verts. Le principal écart par rapport à la section précédente provient toutefois de l'introduction de la variable  $\nu_2$  : de nouveaux équilibres apparaissent, moins manichéens que précédemment, puisqu'ils sont compatibles avec la coexistence du capital vert et du capital brun à long terme. Mais ils soulèvent, bien sûr, de nouvelles questions : dans quelles conditions l'économie convergera-t-elle vers un état durable à zéro carbone, par opposition à une situation à faible carbone où, même à long terme, le capital sale continuerait à fonctionner, bien que de manière modérée ?

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Pour chacune des variantes propres de ces équilibres, nous avons toujours des équilibres localement stables sous les hypothèses standard : la situation souhaitable avec des salaires et des emplois positifs, et un niveau fini de ratios d'endettement, des subventions et des taxes. Comme précédemment, l'état d'équilibre déflationniste associé à une montée en flèche des ratios d'endettement alors que les salaires et les emplois se réduisent à zéro est localement instable. Ce qui est nouveau, cependant, c'est que, le régime permanent de type solovien brun s'avère également localement instable sous réserve que le rapport du stock de connaissance sur le capital vert est tel que sa dérivée par rapport au temps reste bornée supérieurement chaque fois que le volume de capital vert tend vers zéro. Cette restriction revient à dire que, chaque fois que le capital propre se réduit à zéro (en raison du manque d'investissements verts et de la dépréciation exponentielle), le savoir-faire écologique encapsulé dans  $R$  se réduit encore plus rapidement. Cela peut être réalisé, par exemple, en imposant que le taux de décroissance des connaissances soit supérieur au taux de dépréciation du capital vert.

Dans ce chapitre, nous avons aussi présenté une paramétrisation particulière conduisant à une transition avec un faible prix de l'énergie fossile. Dans cet exemple, le financement public et privé de la R&D verte permet d'assurer la transition, même si l'économie suivait initialement une voie carbonée. L'introduction d'une taxe carbone accélère la transition mais ne la garantit pas. Comme dans [Acemoglu et al., 2012], tout retard dans la mise en œuvre des politiques publiques de transition peut être très coûteux en temps et en argent. Cependant, contrairement au résultat obtenu dans cet article, ici, aucune transition ne peut se produire en cas de laisser-faire.

De plus, nous avons montré qu'en général, de nouveaux types d'états stables à long terme émergent, où le capital brun et le capital vert coexistaient. Dans notre exemple spécifique, cela est dû au faible niveau d'investissements publics et privés dans la R&D verte. Bien que moins simplistes que nos précédents équilibres manichéens, ces états stables hybrides suggèrent que, non seulement le laissez-faire est insuffisant, mais qu'une intervention publique trop timide peut échouer à conduire l'économie vers une situation de neutralité carbone à long terme.

Plusieurs améliorations de cet article peuvent être envisagées. La dynamique des prix devrait évidemment intégrer le coût de l'énergie et être au moins partiellement orientée par la demande. Cela permettrait aux prix de la consommation de refléter l'augmentation du coût des combustibles fossiles chaque fois qu'une taxe sur le carbone est mise en pratique. L'impact sur la demande effective serait vraisemblablement non négligeable. La principale question serait alors de savoir si une demande déprimée serait capable d'empêcher le secteur privé d'investir suffisamment dans les infrastructures vertes pour atteindre un équilibre propre de type solovien. La même question serait probablement renforcée si les ménages consommaient également de l'énergie (et pas seulement le secteur de la production) et supportaient donc directement une partie du coût supplémentaire induit par une taxe carbone.

D'autre part, nous nous sommes exclusivement concentrés, ici, sur l'analyse à long terme. Une analyse à court terme permettrait également de comprendre les avantages et les inconvénients des leviers politiques considérés dans ce papier et qui ont été très débattus aux États-Unis dans la première partie du mandat Biden. En particulier, l'instabilité asymptotique de l'état stationnaire de type solovien "sale" devrait être mise en perspective avec l'urgence temporelle induite par le changement climatique.

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Une autre amélioration de ce papier est laissée pour un travail ultérieur, qui consiste à endogénéiser le prix,  $\chi_E$ , de l'énergie fossile. Enfin, en se basant sur le présent travail, on pourrait introduire une variété de différents types de capitaux dans un secteur de production “putty-clay” similaire à celui introduit par [Akerlof and Stiglitz, 1969] afin d'obtenir un cadre plus réaliste que le cas à deux capitaux envisagé ici.

Pour conclure, ces cinq années de doctorat ont permis de tester la robustesse d'un modèle de déséquilibre, capable de simuler un moment de Minsky. Les extensions de ce modèle ne semblent pas fondamentalement altérer sa robustesse, des améliorations lui ont été apportées pour répondre aux principales critiques faites à cette famille de modèles. Ces améliorations concernent notamment le relâchement de la “loi” de Say par l'introduction des inventaires, qui se révèlent indispensable pour pouvoir étudier la création monétaire. La vélocité de la monnaie et la création monétaire par les banques peut être suivies dans cette version étendue du modèle d'origine [Grasselli and Costa Lima, 2012]. Enfin, notre travail a consisté à construire un modèle de transition simple, avec seulement deux types de capitaux, basé sur [Grasselli and Costa Lima, 2012] et à en étudier les multiples équilibres et les leviers possibles pour garantir la transition de l'un vers l'autre capital.



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# List of Abbreviations

ANOVA	ANalysis Of the VAriance
BP	Beyond Petroleum (formerly British Petroleum)
CES	Constant Elasticity of Substitution
CGE	Computable General Equilibrium
COP	Conference Of the Parties
CPE	Chaos Polynomial Expansion
CPI	Consumer Price Index
DICE	Dynamic Integrated model of Climate and the Economy
DoE	Design of Experiment
DSGE	Dynamic Stochastic General Equilibrium
DSGM	Derivative-based Global Sensitivity Measure
E3	Economy-Energy-Environment
EC	European Commission
ECB	European Central Bank
EU	European Union
FAST	Fourier Amplitude Sensitivity Test
FF	Full or Fractional Factorial
FIH	Financial Instability Hypothesis
FIPECO	FInance Publique et ECOnomie
GAMS	General Algebraic Modeling System
GDP	Gross Domestic Product
GEMMES	General Monetary and Multisectoral Macrodynamics for the Ecological Shift
GSA	Global Sensitivity Analysis
HANDY	Human And Nature DYnamics
HDMR	High Dimensional Model Representation
IAM	Integrated Assessment Model
IES	Intertemporal Elasticity of Substitution

## CONTENTS

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IMF	International Monetary Fund
IPCC	Intergovernmental Panel on Climate Change
LED	Light Emitting Diode
LHS	Latin Hypercube Sampling
LSA	Local Sensitivity Analysis
MC	Monte-Carlo
MCA	Monte-Carlo Analysis
NFGS	Network for Greening the Financial System
OAT	One-at-A-Time
OECD	Organisation for Economic Co-operation and Development
OLH	Optimal Latin Hypercube
OLHS	Optimal Latin Hypercube Sampling
PCC	Partial Correlation Coefficient
PI	Proportional-Integral
PPP	Purchasing Power Parity
PRRC	Partial Rank Correlation Coefficient
QoI	Quantity of Interest
RBC	Real Business-Cycle
R&D	Research & Development
RE	Renewable Energies
SA	Sensitivity Analysis
SCC	Social Cost of Carbon
SDR	State Dependent Regression
SRC	Standardized Regression Coefficient
SRRC	Standardized Regression Rank Coefficient
TFP	Total Factor Productivity
UA	Uncertainties Analysis
UN	United Nations
VAR	Vector AutoRegression
WB	World Bank

# Chapter 1

## Introduction

The origin of this thesis lies in Mark Carney's (Governor of the Bank of England) foundational speech about climate-induced financial risks. In 2015, he stated that the climate change challenge creates three types of potential risks for financial institutions: a) *Physical climate risks* associated with the physical damages to assets,<sup>1</sup> b) *liability risks* associated with the impacts that could arise tomorrow if parties who have suffered loss or damage from the effects of climate change seek compensation from those they hold responsible,<sup>2</sup> and c) *Carbon risks* associated with the financial viability of a necessary transition to a low-carbon economy [Carney, 2015].

Three years later, in their October 2018 progress report, the members of the Network for Greening the Financial System (NGFS) recognised that the challenges we face are unprecedented, urgent and analytically difficult. Indeed, while dealing with climate change, we are at the same time facing a deep crisis in financial and macro-economic models.

The likelihood of a *climate-driven* Minsky moment [Carney, 2019] - the term used to refer to a sudden collapse in asset prices and to depict the financial crises that occurred in 2001 and 2007-2009 - combined with the inability of a large majority of models at that time to predict such crises invites us to question the reliability of today's models for the orientation of decision makers. While we cannot ignore the magnitude of the socio-economic impacts of the recent health crisis and the questioning of our economic policies that it has engendered, this thesis focuses on another major challenge of our time: the necessary transition of our economic systems to ensure the resilience of our societies in the face of this other fundamental threat that is climate change.

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<sup>1</sup>These changes in turn may translate into adaptation costs and economic loss of value.

<sup>2</sup>At the time, one might have thought that it would be decades before such claims were made, and that carbon extractors and emitters - and, if covered by liability insurance, their insurers - would be the hardest hit [Carney, 2015]. But in 2021, at COP26, loss and damages were at the centre of the discussion and result in a cover decision [on Climate Change 1/CP.26, 2021]

The demand for immediate actions by the Intergovernmental Panel on Climate Change, [IPCC, 2014], [IPCC, 2018] and more recently in the report of its first Working Group [IPCC, 2021], in order to mitigate the short-term impacts and far-reaching irreversibility of climate change, requires the building of reliable models to guarantee transition to a low-carbon economy (or even a climate neutral economy). The development of my work aims at evaluating such reliability and offering extensions to an existing stock-flow consistent macro-economic transition model with a non-linear dynamic. This dynamic leads to multiple equilibria incorporating the possibility of a collapse. The use of such a model provides a framework where public policies capable of avoiding it can be analysed.

## 1.1 “*The world is on an unsustainable path*” [BP, 2019]

Climate change is undoubtedly a tremendous threat to our societies due to its magnitude and far-reaching impact across all sectors and geographies,<sup>3</sup> its foreseeable nature,<sup>4</sup> its irreversibility<sup>5</sup> and its dependency on short-term actions.<sup>6</sup> With the Rio Summit, the Kyoto protocol and now the Paris Agreement Rulebook (finalized in 2021), States positioned themselves as cornerstones in the fight against climate change. During the COP21, the global political sphere reconfirmed its willingness to facilitate the transition to a low-carbon society. France’s Energy Transition Law, adopted in July of 2015, requires companies and financial institutions to report on climate risk and limit the global rise of temperatures to less than 2°C, and preferably as close to 1.5°C as possible. The EU’s Green Deal and its “Fit-for-55” package include binding legislation to ensure that the EU meets its 2030 climate and energy targets. At COP26, the UK Presidency succeeded in putting carbon neutrality at the centre of discussions.

### 1.1.1 A long-time known trajectory

The extreme effects of climate change are already easily observed around the world. From oppressive heatwaves in North America to typhoons in south-east Asia and severe droughts in Africa and Australia, climate-related catastrophes are damaging infrastructure, degrading public health, decreasing productivity, and diminishing our collective wellbeing [Carney, 2019]. As exemplified by the foreign electronic or automotive industry after flooding in Thailand,<sup>7</sup> and the five-fold increase in insurance losses over the past three decades, these repercussions are increasingly severe and global in reach. In the energy sector, data compiled in the BP Statistical Review of World Energy suggest that:

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<sup>3</sup>NFGS report states: climate change will affect all agents in the economy (households, businesses, governments). The risks will likely be correlated with and potentially aggravated by tipping points, in a non-linear fashion. This means the impacts could be much larger, and more widespread and diverse than those of other structural changes.

<sup>4</sup>While the exact outcomes, time-horizon and future pathway are uncertain, there is a high degree of certainty that some combination of physical and transition risks will materialise in the future [Carney, 2019].

<sup>5</sup>As shown in the work reported by the IPCC group, the concentration of greenhouse gas (GHG) emissions in the atmosphere determines the impact of climate change and no mature technology is yet far-reaching enough to reverse the process. Above a certain threshold, they demonstrate great potential to impose irreversible consequences on a multitude of ecosystems, even if the time-horizon and degree of severity remain uncertain [IPCC, 2014]. Many changes due to past and future greenhouse gas emissions are irreversible for centuries to millennia, especially changes in the ocean, ice sheets and global sea level. [IPCC, 2021]

<sup>6</sup>Actions today will determine the nature and magnitude of climate-related disasters. Thus, they demand reliable and comprehensive long-term strategies. E.g.: strong, rapid and sustained reductions in CH<sub>4</sub> emissions would also limit the warming effect resulting from declining aerosol pollution and would improve air quality. [IPCC, 2021]

<sup>7</sup>In 2011, Western Digital and Nissan suffered losses of 235M and 67M, respectively (see [Chongvilaivan, 2012]).

*“In 2018, global energy demand and carbon emissions from energy use grew at their fastest rate since 2010/11, moving even further away from the accelerated transition envisaged by the Paris climate goals.”*

According to the authors, the growth in energy consumption in 2019 is clearly climate-related because families and businesses around the world increased their demand for cooling and heating in response to an unusually large number of hot and cold days. This rise in energy consumption led to an acceleration in carbon emissions. Despite the many lock-downs around the world and the shutdown of many economic activities leading to an unprecedented drop of 5.4 per cent in 2020, global carbon dioxide emissions bounced back to pre-COVID levels, and concentrations of GHGs in the atmosphere continue to rise.<sup>8</sup> More generally, experts conclude that the global pace of progress is “inconsistent with the Paris climate goals”. They stated:

*“The world is on an unsustainable path: the longer carbon emissions continue to rise, the harder and more costly will be the eventual adjustment to net-zero carbon emissions.”*  
[BP, 2019]

In their highly influential book *The limits to growth*<sup>9</sup> and 1974 study about economic growth and resource depletion, Meadows and colleagues accurately illustrated the societal risks associated with surpassing sustainable thresholds<sup>10</sup> nearly fifty years ago [Meadows et al., 1972] and [Meadows et al., 1974] for our societies. Most of the scenarios the researchers depicted ultimately led to global economic collapse. Back-testing the model forty years later [Turner, 2008], [Turner, 2012] and [Turner, 2014] updated a version of the model demonstrating that the collapse is even more probable.<sup>11</sup> However, the researchers’ conclusions have not been taken into serious consideration due to their basic core economic model.

In [McIsaac, 2016] and [Bovari et al., 2018], the authors used a much finer economic model and the United Nations (UN) population growth series to explore the question of whether or not global warming could induce a similar breakdown of the world economy. They are reaching the same conclusions as their counterparts, with even more precision about the process of climate change on economic fundamentals.<sup>12</sup>

Motesharrei and colleagues develop the *Human And Nature Dynamics* (HANDY) model to illustrate the factors which led to the collapse or sustainability of advanced, complex, and powerful societies throughout history [Motesharrei et al., 2014]. The authors identified two important factors

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<sup>8</sup>The Emissions Gap Report 2021 shows that new national climate pledges combined with other mitigation measures put the world on track for a global temperature rise of 2.7°C by the end of the century. [UNEP, 2021]

<sup>9</sup>Updated in [Meadows and Randers, 2012].

<sup>10</sup>Also warned in [Barnosky et al., 2012].

<sup>11</sup>Turner’s model forecasts a collapse in living standards due to resource scarcity and carbon saturation in the twenty-first century.

<sup>12</sup>Their findings are threefold: 1) the +2°C target is already out of reach, absent negative emissions. A result that is confirmed by the IPCC report WG I [IPCC, 2021] and the UNFCCC synthesis report in Oct 2021 of nationally determined contributions which assesses the temperature rise at 2.7°C. [UNFCCC, 2021]; 2) the long-run (resp. short-run) consequences of the process of climate change on economic fundamentals may lead to severe economic consequences absent (resp. too rapid) proactive climate policies. Global warming (resp. too fast transition) forces the private sector to leverage in order to compensate for output and capital losses (resp. to lower carbon emissions), thus endangering financial stability; 3) Implementing an adequate carbon price trajectory, as well as increasing the wage share, fostering employment, and reducing private debt make it easier to avoid unintended degrowth and to reach a +2.7°C target.

which appeared frequently throughout the study: 1) the stretching of resources due to the strain placed on the ecological carrying capacity,<sup>13</sup> and 2) the economic stratification of society into Elites and Masses (or “Commoners”).<sup>14</sup> They conclude that the model produces two different scenarios of collapse, with one being due to a scarcity of labor (which follows an inequality-induced famine) and the other to a scarcity of Nature (caused by a depletion of natural resources).

*“In both of these scenarios, the Elites, due to their wealth, do not suffer the detrimental effects of the environmental collapse until much later than the Commoners. This buffer of wealth allows Elites to continue “business as usual” despite the impending catastrophe. It is likely that this is an important mechanism that would help explain how historical collapses were allowed to occur by elites who appear to be oblivious to the catastrophic trajectory”.*

### 1.1.2 A “climate-driven Minsky moment” [Carney, 2019]

Although the trajectories to a resource-driven collapse are well-modeled for a long time, the last decade has witnessed the emergence of threat due to the *financialization*<sup>15</sup> of societies combined with these climate-related risks. Indeed, as detailed in the work of the 2°Investing Initiative group<sup>16</sup> [Cleveland et al., 2015], a systemic risk to financial stability associated with the transition to a low-carbon economy has not been carefully investigated to date. Such an assessment of systemic risk would require a better understanding of climate and carbon risks in the realm of financial institutions. Whereas no such analysis exists yet, the topic is now gaining recognition and priority on the agendas of macro-prudential authorities.

To avoid being “oblivious to the catastrophic trajectory”<sup>17</sup> [Motesharrei et al., 2014], thirty-four central banks and supervisors across five continents, which account for half of all global greenhouse gas emissions, created the Network for Greening the Financial System (NFGS) in 2017. The coalition’s first report delivers recommendations to ensure a smooth transition to a low-carbon economy, which include integrating the monitoring of climate-related financial risks into day-to-day supervisory work, adding sustainability criterion into portfolio management, collaborating to share knowledge about climate-risks data, and developing the capacity for in-house climate-risk assessments. However the report recognizes the analytically difficult character of this approach.

At the same time, the shift towards greater electrification seems more and more necessary with the increasing demand for power. Consequently, the necessary energy transition can only take place if accompanied by a corresponding decarbonization program [BP, 2019]. For the financial sector, this multifaceted transition demands new resolutions and actions by operators (disclosure) and investors (divestment and engagement) [Cleveland et al., 2015] to mitigate the risk of what Mark Carney calls a “climate-driven Minsky moment” [Carney, 2019].

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<sup>13</sup> [Abel, 2013], [Catton, 1982], [Kammen et al., 1994], [Ladurie, 1987], [Ponting, 1993], [Redman, 2004], [Wood, 1998] and [Wright, 2004].

<sup>14</sup> [Brenner, 1976], [Turchin and Nefedov, 2009], [Diamond and Van Dijk, 2008] and [Goldstone, 1991].

<sup>15</sup>Financialization is defined as “the growing importance of finance in daily activities, both at the national and global levels”. At the heart of the problem of financialization is a growing inequality of income and wealth, as Nobel Laureate Joseph E. Stiglitz shrewdly observed [Sheng, 2013].

<sup>16</sup>A multi-stakeholder think-tank working to align the finance sector with 2°C climate goals.

<sup>17</sup>Also written in [Carney, 2019] as “the obvious risks before our eyes”.



This term refers to the Minsky’s Financial Instability Hypothesis (FIH), which analyzed in the 1970s the likelihood of an intrinsic instability of the financial system and how it can imply a new financial crisis comparable in terms of magnitude to that of 1929. As explained in [McIsaac, 2016], the primary advantage of this framework is its ability to reproduce a financial crisis, such as the subprime mortgage crisis, endogenously, while the large majority of *Dynamic Stochastic General Equilibrium* (DSGE) models, which are well developed in literature, can hardly consider these situations other than as “*black swans*” or large risks having a very small probability to materialize.

On the one hand, there is a clear need for a transition model to a carbon-neutral society, that can predict the risk of a “*climate-driven Minsky moment*”, in order to orientate public policy decisions. On the other hand, there is great analytical difficulty in producing such a model, as pointed out by [Carney, 2019], which is compounded by the inability of standard macro-economic models to predict the financial crises of the 2000s.

This cocktail effect motivates an analysis of the nature of these models at the intersection of mathematical modeling, economic (or financial) assumptions, and ecological objectives and constraints. This analysis is the object of the following sections about: the ambiguous role of models, due to uncertainty and the non-bijective nature of modeling (Section 1.2), and the crisis of macro-economic models, of which some features leads to this inability indicated above (Section 1.3).

## 1.2 The ambiguous role of models

Considering the versatility and ambiguity of models is critical. For example, mathematical models and computer simulations are used in a wide variety of fields and institutions. Ecologists at all levels of experience and scope develop them to study ecosystems and population dynamics. Institutions around the world use them to study global and national economies. In regard to climate change, two main types of models are used: a) *physical models* which assess the evolution of climate and b) macro-economic models which assess the economic or financial impact of climate change. In some instances, physical models help to corroborate or falsify hypotheses about the ecological systems, and macro-economic models help to guide public policies.<sup>18</sup>

As noted by [Cariboni et al., 2007], models are often utilized in order to answer specific questions about the current or future behavior of the system under analysis (e.g., is extinction within a fixed time horizon a concrete threat?) or to investigate interactions between model components [Zaldivar et al., 1998]. More specifically, such system and interactions may be represented by mathematical models and computer simulations such as differential equations, interacting particles, or algebraic equations. The models used to design structures, or to help policy decisions, influence in turn the development of new behaviors in many spheres of society.

However, models themselves are built under uncertainties in their representation of reality, with samplings of continuous phenomena, and the possible errors in the values of their own factors (e.g., a population growth rate) what is called the parameterization of the model. These uncertain factors

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<sup>18</sup>E.g., in France, the Commissioner-General for Sustainable Development (of the Ministry of Environment) assesses the impact of the National Low-Carbon Strategy with the Three-ME2 model. Their results forecast a GDP increase of 1.6 by 2035, whereas the French Ministry of Finance predicts a contractionary effect, with its own *Mésange* model (see [Grandjean and Giraud, 2017]).

come in a variety of forms.<sup>19</sup> While seeking scientific insight, designing the model, or making decisions, the user of these models is thus limited by such uncertainties surrounding their input variables and parameters (both are called factors). The reliability of the results produced by the model, called robustness, is defined as its ability to retain its qualitative properties independently of the errors that may arise from these uncertainties. A sensitivity analysis of the model’s input variables and parameters allows for a quantification of its robustness of uncertainty.

### 1.2.1 Uncertainty: the very essence of a model

In their major book about modeling, [Saltelli et al., 2000] illustrate the role of models in a scientific process in the sense of [Rosen, 1991], summarized in Figure 1.1. The non-bijective nature of modeling generates different ambiguities surrounding the role of models in various fields. This nature is due to the variety of uncertainties one may face in the building of a model.

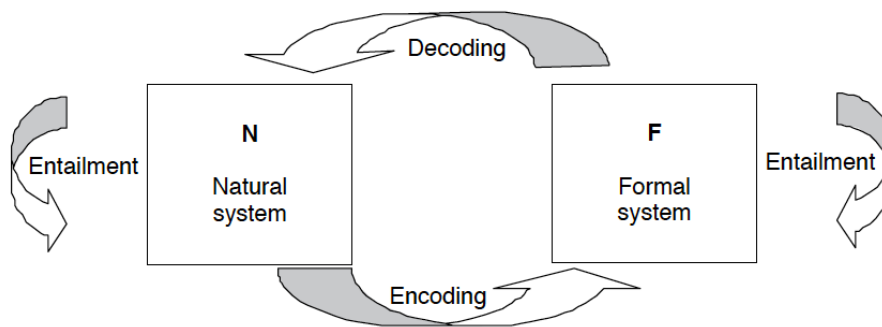


Figure 1.1: Modeling after [Rosen, 1991]

The “world”, or “natural system” on the left of the Rosen’s diagram, is the object of this investigation. The model operates on the assumption that there are rules governing its behavior, be it natural (as in the case of the diagram) or artificial. The object of the model is to represent such rules. To this end, we “encode” the world into a system of formal relationships such as hypotheses, differential equations, etc. The decoding of this set of structures allows us to predict or understand the “world’s” behavior, because both the “world” system and model are “entailed” with these rules. The Rosen’s intuition postulates that there are no rules dictating how to encode or decode either the “world” or the model, which would imply that the relationship between the “world” and its representation is not a bijective function.<sup>20</sup> In his words:

*“While the world and the model are each internally “entailed”, nothing entails the world with the model”.*

As explained by [Saltelli et al., 2008a], among the reasons for this paradox is the fact that the portion of the world captured by the model is an arbitrary “enclosure” of an otherwise open,

<sup>19</sup>E.g., chemical, material, or mechanical properties, electrical capacitance or resistance, or atmospheric conditions, behavioral economic function, etc.

<sup>20</sup>In mathematics, a bijection, bijective function, or one-to-one correspondence is a function between the elements of two sets, where each element of one set is paired with exactly one element of the other set, and each element of the other set is paired with exactly one element of the first set. There are no unpaired elements. [Nikulin, 2001].

interconnected system.<sup>21</sup> This idea might be quite disturbing for a student accustomed to the beauty and apparent self-evidence of physical laws, but, as the authors say:

*“Practitioners of modeling have come to live with the rather unpleasant reality that more than one model may be compatible with the same set of data or evidence. Some have gone so far as to coin a word for this paradox: equifinality<sup>22</sup> [Beven, 1993], [Beven and Beven, 2001] and [Saltelli et al., 2004] meaning that different models can lead to the same end.”*

Besides this partial “enclosure” of the model, which leads to *equifinality*, another ambiguity of modeling resides in the “non-injective” nature of modeling, by which various realities can be modeled by the same formal system. The scientific inquiry and modeling strategy might consequently be rather arduous as a result of the ambiguity between model and reality. For instance, in macroeconomics, Pottier’s criticism of the [Acemoglu et al., 2012] model illustrates this “second layer” of uncertainty in modeling, in which the model’s mathematical formalism could represent either innovation, or nudges to managers, or the learning-by-doing abilities of the system - all with the same modeling prediction (decoding) of very different public policies [Pottier, 2014].

Moreover, when modeling continuous phenomena with differential equations, mathematical resolutions or computing simulations involve sampling techniques that lead to mismatches between the continuous reality and its discrete representation.<sup>23</sup> Thus, when we define a model, we must eventually define its inputs and outputs according to its nature and purpose. As summarized in [Saltelli et al., 2008a], a model can be:

- Leading to **diagnostic or prognostic**: a distinction between models used to understand a law or to predict the behavior of a system ;
- **Data-driven or law-driven**: a law-driven model tries to put together accepted laws which have been attributed to the system, in order to predict its behaviour. For example, we use Darcy’s and Ficks’ laws to understand the motion of a solute in water flowing through a porous medium. A data-driven model tries to treat the solute as a signal and to derive its properties statistically.<sup>24</sup>

In every case, the model’s ability to faithfully represent the reality depends both on empirical studies and theoretical considerations. For example, behavioral functions (such as the Phillips curve or the investment behavioral function in economics) are built to fit data that involve a large number of calibrations or estimations (based on empirical studies [Grandjean and Giraud, 2017]), whereas the production function (such as [Cobb and Douglas, 1928], [Leontief, 1951], CES [Solow, 1956] or Putty-clay in the sense of [Akerlof and Stiglitz, 1969]) is a key concept of economic growth theory, and thus can present discrepancies between the actual production laws and their functional representation (see [Stiglitz, 2018]).

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<sup>21</sup>Especially when dealing with the study of nonobservable parts of a system. e.g.,when astrophysicists introduce dark matter into equations to explain the expansion of the Universe.

<sup>22</sup>Also called model indeterminacy.

<sup>23</sup>In the field of digital signal processing, the Nyquist-Shannon signal sampling theorem gives a criterion about the minimal sampling frequency according to the maximal signal frequency to avoid distortion of the information [Shannon, 1949].

<sup>24</sup>Many other categorizations of models are possible: [Bell, 1988] distinguishes between formal (axiomatic), descriptive and normative models (rules an agent should follow to reach a target).

A model's variety of inputs suffer the uncertainty of their actual values. Any forecast should thus take into account these possible mismatches in order to provide reliable (robust) results. This critical consideration is the object of the uncertainty (UA) and sensitivity analysis (SA). Especially in the social sciences, or in complex economic environments, where these calibrations and estimations of parameters lead to highly parameter-sensitive results, the impact of inputs' uncertainties cannot be considered negligible. Thus, questioning the robustness of macro-economic models comes to be more and more seen as essential to conduct public policies in these fields [European Commission, 2009].

### 1.2.2 The sensitivity analysis to build reliable models

A possible definition of sensitivity analysis is given by [Saltelli et al., 2004]: *“The study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model inputs”*.

While building a model or interpreting numerical simulations, Sensitivity Analysis (SA) methods are thus powerful tools, providing insight on the uncertainty in a model's output [Iooss and Lemaître, 2015] and its relative dependence on different sources of uncertainty in the model's input [Saltelli et al., 2000] in order to quantify the relative importance of these input variables on the output. More specifically, the purpose of SA is to answer the questions:

- *“Which of the uncertain input factors are most influential in determining the variability affecting the inference?”*
- *“If the uncertainty of one of the inputs could be eliminated, which one should be chosen in order to reduce to the minimum the variance of the output of interest?”*
- *“Are there factors whose effect on the output is so low that they can be confidently fixed anywhere in their ranges of variation without affecting the results?”*

An overview of SA methodologies can be found in Chapter 2 and its appendix. Some relevant applications of SA techniques to ecological and environmental science include, atmospheric chemistry [Campolongo et al., 1999], transport emissions [Kioutsioukis et al., 2005], geographic information systems [Crosetto and Tarantola, 2001], environmental management [EPA, 2003] and population dynamics [Zaldivar et al., 2000].

According to Saltelli and colleagues,<sup>25</sup> the role of UA and SA is crucial in ecological risk assessment, which utilizes analytical models to estimate the impact of human actions on natural resources and interpret the significance of that impact in light of the uncertainties identified in each component of the evaluation process.

Some effort has been put into understanding the role of SA from an environmental regulatory point of view. Both the report on Good Practice Guidance and Uncertainty Management in National Greenhouse Gas Inventories [EPA, 2003] and the Draft Guidance on the Development, Evaluation and Application of Regulatory Environmental Models [IPCC, 2014] provide information about model use and model science in environmental settings. The EPA's (Environmental Protection Agency) report also contains recommendations on good practices for UA and SA [EPA,

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<sup>25</sup> [Saltelli et al., 2000], [Saltelli et al., 2004] and [Saltelli et al., 2008a].

2003]. In Europe, sensitivity analysis is mentioned in the guidelines for impact assessment [European Commission, 2009]. Among them, the Global Sensitivity Analysis (GSA) appears to fulfill the macro-prudential authorities' need for robust models.

According to the model's goals, nature and properties, types of uncertain inputs, and outputs (also called quantities of interest – QoI), various GSA can be used. In [Iooss and Lemaître, 2015], the authors give many application examples ; for instance, [Makowski et al., 2006] analyze, for a crop model prediction, the contribution of 13 genetic parameters on the variance of two outputs. Additionally, Lefebvre and colleagues use GSA to determine the most influential input among a large number of inputs (around thirty) [Lefebvre et al., 2010].

According to [Iooss and Lemaître, 2015], the objectives of GSA may include:

- identify and prioritize the most influential inputs,
- identify non-influential inputs in order to fix them to nominal values,
- map the output behavior in function of the inputs by focusing on a specific domain of inputs if necessary,
- calibrate some model inputs using some available information (real output observations, constraints, etc.).

When the model is non-linear and non-monotonic, with a small number of parameters and a short running time for each simulation, the De Rocquigny's decision diagram suggests using the GSA method with the so-called *Sobol' indices* [De Rocquigny, 2008]. The method, developed by the Russian mathematician Sobol in [Sobol, 1967], is based on the decomposition of the output variance: for a given function  $f \in L^1([0, 1]^p, dx)$  Sobol considers a decomposition into terms of increasing dimensions

$$f(x_1, \dots, x_p) = f_0 + \sum_{i=1}^p f_i(x_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(x_i, x_j) + \dots + f_{1,\dots,p}(x_1, \dots, x_p) \quad (1)$$

and imposes that  $\forall s \in \{1, \dots, p\}, \forall i_1 < i_2 < \dots < i_s \in \{1, \dots, p\}^s$  and  $\forall k \in \{i_1, \dots, i_s\}$ ,

$$\int_0^1 f_{i_1, \dots, i_s}(x_{i_1}, \dots, x_{i_s}) dx_k = 0 .$$

Consequently this expansion of  $f$  into  $2^p$  summands of different dimensions exists, is unique and the related terms are pairwise orthogonal in  $L^1([0, 1]^p, dx)$ . Assuming now a square integrable random variable  $Y$  as  $Y = f(\mathbf{X})$  where  $\mathbf{X} = (X_1, \dots, X_p)$  is a vector of  $p$  independent random variables uniformly distributed on  $[0, 1]$ , we have:<sup>26</sup>

$$Y = f_0 + \sum_{i=1}^p f_i(X_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(X_i, X_j) + \dots + f_{1,\dots,p}(X_1, \dots, X_p). \quad (2)$$

---

<sup>26</sup>Here one can see that the variance decomposition strongly depends on the independence of the random variables  $(X_1, \dots, X_p)$ .

which implies<sup>27</sup>

$$V(Y) = \sum_{i=1}^p V_i + \sum_{i=1}^p \sum_{j>i}^p V_{i,j} + \dots + V_{1,\dots,p} \Leftrightarrow 1 = \sum_{i=1}^p \frac{V_i}{V(Y)} + \sum_{i=1}^p \sum_{j>i}^p \frac{V_{i,j}}{V(Y)} + \dots + \frac{V_{1,\dots,p}}{V(Y)} \quad (4)$$

where  $S_{i_1,\dots,i_s} = \frac{V_{i_1,\dots,i_s}}{V(Y)}$ .  $S_i$  is a measure of the single impact of  $X_i$  on the total variance,  $S_{i,j}$  is a measure of the joint impact of  $(X_i, X_j)$  on the total variance, etc... In this spirit, to quantify the total impact (sum of single and related joint effects) of  $X_i$  on the variance of  $Y$  we define

$$S_{T_i} = \frac{E(V(Y|X_{\bar{i}}))}{V(Y)} = 1 - \frac{V(E(Y|X_{\bar{i}}))}{V(Y)} \quad (5)$$

where  $X_{\bar{i}} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_p)$ .

For example, in a model with 3 inputs we have

$$S_{T_1} = S_1 + S_{1,2} + S_{1,3} + S_{1,2,3} = 1 - S_{2,3}. \quad (6)$$

Monte Carlo sampling-based methods have been developed to estimate these Sobol' indices (see [Saltelli, 2002]), although McKay, Iman and colleagues demonstrate the efficiency of other types of stratified sampling with a pick freeze method (such as Latin Hypercube Sampling - see [McKay et al., 2000] and [Iman, 1999]). These methods (GSA with Sobol' indices estimated by the [Saltelli, 2002] method) are specifically designed for non-linear models with interactions between parameters. In general, Global Sensitivity Analysis can be conducted with variance-based, frequency-based or meta-models methods. Their results are quite similar [Saltelli and Bolado, 1998], but latter two methods require the *a priori* choice of an upper bound (in the degree of the polynomial or the harmonic), which can introduce biases in the study to assess actual values of the sensitivity indices.

*Physical models* (climate or meteorological models) used to predict the evolution of the environment can be evaluated through their ability to: a) simulate present and past climate conditions and b) reproduce climate variations at each time-scale. These models thus have a predictive ability that allows for the identification of an effective public policy with reliability (robustness) [Grandjean and Giraud, 2017].

For *macro-economic models*, the challenge of achieving this reliability is a much more sensitive subject because of a) the failures of a large majority of models to predict the 2000s crises,<sup>28</sup> and b) the sensitivity (lack of robustness) of today's models.<sup>29</sup> It is clear today that this sensitivity has

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<sup>27</sup>Using the pairwise orthogonality in the Sobol' decomposition, we can obtain easily a recursive expression of each term using conditional expectations

$$\begin{aligned} f_0 &= E(Y) \\ f_i(X_i) &= E(Y|X_i) - E(Y) \\ f_{i,j}(X_i, X_j) &= E(Y|X_i, X_j) - f_i(X_i) - f_j(X_j) - E(Y) \\ &\vdots \end{aligned} \quad (3)$$

and deduce the so-called ANOVA-HDMR decomposition for the variance of  $Y$ .

<sup>28</sup>See Section 1.3.2 and 1.3.3.

<sup>29</sup>See a review of some of these models in Section 2.4.

led to large underestimations of the economic impact of climate.<sup>30</sup> Therefore, a global sensitivity analysis of macro-economic models is necessary to inform discourses around the reliability of their results, and thus to help make up for the crisis they are facing.

### 1.3 The crisis of macro-economic models

Macroeconomics has historically been a controversial sphere, despite the existence of a mainstream theory. The mainstream theory failed to predict the crises of the 2000's in spite of the multitude of contemporary models, vivifying the old fatalistic diagnosis of the crisis of macro-economic models. We should interpret this diagnosis as a demand for the selection of a model that is able to accommodate our scientific expectations in climate change economics.

#### 1.3.1 An historical typology of macro-economic models

Various types of macro-economics models can be described:

- Statistical models, such as Vector AutoRegression (VAR) models, link linear statistical relationships to macro-economic variables of multiple time series. They are a generalization of the univariate autoregressive model. According to [Bjornland, 2000], even though VAR models may not satisfy Lucas' criteria for policy intervention, they are still useful for anticipating the impact of policy actions that fall within the realm of historical experience. In particular, shifts in policy rules can be somewhat subsumed under stable policy rules ;
- Theoretical models: since the publication of *The General Theory of Employment, Interest and Money* [Keynes, 1936], economists have been divided into micro and macroeconomics. Micro-economic models are interested in individuals' and firms' behavior and solving utility or profit maximization problems under certain constraints. Macro-economic models study aggregated indicators which consider the whole economy. The theory is based on a mathematical interpretation of the Keynes' and Hicks' work about the Great Depression [McIsaac, 2016]. In the 1960s, Lucas and his contemporaries of the new classical macroeconomics developed micro-foundations (criticized by the Sonnenschein-Mantel-Debreu Theorem) and a complex mathematical basis known as *Dynamic Stochastic General Equilibrium* (DSGE). Under the assumption of an absence of rigidities in market mechanisms,<sup>31</sup> the DSGE models add imperfections and exogenous shocks that alter the equilibrium and impinging economic growth. These models overpowered macroeconomics and are widespread among national and international institutions.<sup>32</sup> Generally, they are founded on the following assumptions [Annicchiarico,

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<sup>30</sup>Exposed in the book of [Pottier, 2014].

<sup>31</sup>First models called *Real Business Cycle* models postulated a representative consumer who operates in perfectly competitive markets.

<sup>32</sup>Example of DSGE models across the world [Grandjean and Giraud, 2017]:

- Europe: Germany: [Pytlarczyk, 2005] ; France: Mésange [Bardaji et al., 2017] ; UK: BEQM [Harrison et al., 2005] ;
- Euro Zone: BCE, NAWM [Christiano et al., 2010] ; European Commission: QUEST III [Ratto, 2008] and DYNARE [Adjemian et al., 2011] ;
- Americas: Brasil: SAMBA ; Canada: ToTEM [Fenton and Murchison, 2006] ; United States: SIGMA, [Erceg et al., 2006] ;
- International Institutions, e.g., IMF: GIMF [Laxton et al., 2010] ;
- World Bank: Integrated macroeconomic model for poverty analysis (IMMPA) is a dynamic CGE.



2010]:

- Perfect competition in all markets ;
- All prices adjust instantaneously ;
- Rational expectations ;
- No asymmetric information ;
- The competitive equilibrium is Pareto optimal ;
- Firms are identical and price takers ;
- Infinitely lived identical price-taking households.

The models' general equilibrium nature is presumed to capture the interaction between policy actions and agents' behavior, while the models specify assumptions about the stochastic shocks that give rise to economic fluctuations. Hence, the models are presumed to “*trace more clearly the shocks' transmission to the economy*” [Sbordone et al., 2010].

An other classification of these models can be done by way of their different purposes:

- Cyclical forecasts models (e.g., Mésange, Egée and Opale for the French Ministry of Economy) where the future values of the main indicators are computed with very short-tem horizon (around one month up to two years) ;
- Integrated Assessment Models (IAM) which aim to assess economic impacts of climate change. They are subdivided into a) Process-based IAM [Sathaye and Shukla, 2013], also called E3 (Economy-Energy-Environment), usually with a cost-effectiveness analysis such as Imaclim-R (CIRED), and b) those based on a cost-benefit analysis (such as the famous DICE [Nordhaus, 2008], PAGES, FUND, etc.). IAM are usually based on a Computable General Equilibrium model (CGE) with a climate feedback loop (e.g. a damage function) ;
- Theoretical macro-economic models which aim to explore a particular phenomenon without necessarily an empirical basis, to identify *stylized facts* both corroborated by the internal logic of the model and the data. However, as detailed in [Pottier, 2014], this juxtaposition of the model and the data, without an actual confrontation, might lead to poor analyses called *just-so-stories*.

Ecological macro-economic models attempt to link climate and economic environments. Some of them aim to assess the impact of a public policy,<sup>33</sup> with cost-benefits or cost-effectiveness analysis, at a national, regional or global scale. Inspired by [Laffargue, 2012] and [Grandjean and Giraud, 2017], these macro-economic models are classified as:

- Neo-Keynesian based models: with quantity adjustment in the short-run, these models are driven by aggregate demand (e.g., Mésange). Temporary disequilibria are possible due to the inertia of quantities adjustments ;
- Computable General Equilibrium models: based on a Walrasian representation of the economy, where prices are perfectly flexible and preferences of a *rational* agent optimizing a utility function are explicitly described. Various types of models exist in this category (such as

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<sup>33</sup>Mésange (Bercy), Three-Me (OFCE / ADEME), Imaclim (CIRED), Nemesis (Erasmus), Gemini (Lausanne), GEM E3 (European Commission), GEMMES (AFD), etc.



Integrated Assessment Models). Generally, institutions use both an estimated DSGE (Global Integrated Monetary and Fiscal model - GIMF, New Area-Wide Model of The Euro area - NAWM, ...) and a neo-keynesian model for their forecast ;

- Disequilibrium models (such as ThreeMe, a post-Keynesian model): with a specific prices' dynamic, independent of the adjustment in supply / demand, a multisectoral production and a wages' dynamic through a new short-term Phillips curve. GEMMES is based on the work of [Grasselli and Costa Lima, 2012] with a non-linear out-of-equilibrium dynamic inspired by the work of [Goodwin, 1967], [Akerlof and Stiglitz, 1969], [Keen, 1995]. These equations are based on the Predator-Prey Logic of Lotka-Volterra, which takes into account private and/or public debt. These types of models are able to represent the Minsky's Financial Instability Hypothesis (FIH) and predict crashes. The stochastic dynamic allows the emergence in aggregated indicators, but they are heavily criticized due to their high-dependency on parameters, their myopic nature (without expectations) and exogenous growth. However, they seem to be the most able to represent the trajectories we want to explore, if we can compensate for these criticisms. An estimated version of the model exists with a Climate Feedback Loop developed in [Bovari et al., 2018] and a first approach dedicated to the Brazilian economy was implemented by [Bastidas et al., 2017].

For the purpose of addressing the risk of a “*climate-driven Minsky moment*”, using a robust model to orientate public policies towards a smooth transition to a carbon-neutral society, these disequilibrium models seem indeed the most compelling because they are able to overcome some of the failures of other macro-economic models.

### 1.3.2 A largely documented diagnostic of a crisis

The crisis of economics and economic models is an ancient topic that was reignited by the inability of major macro-economic models to predict the financial crises of the 2000s. This inability reveals a failure both in the profession, as expressed in [Galbraith, 2009], [Boyer, 2012] and [Guesnerie, 2013], and in the core theory of the model at that time. Prof. Blanchard<sup>34</sup> who claimed in his 2009 paper that “the state of macro is good” [Blanchard, 2009],<sup>35</sup> stated in 2016 [Blanchard, 2016]:

*“I see the current DSGE model as seriously flawed [...] They are based on unappealing assumptions. Not just simplifying assumptions, as any model must, but assumptions profoundly at odds with what we know about consumers and firms.”*

In [Christiaens, 2018], the author wrote the Kocherlakota's<sup>36</sup> words:

*“...we do not have a settled successful theory of the macro-economy. The choice made 25-40 years ago - made then for a number of excellent reasons - should not be treated as written in stone or even in a pen”*

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<sup>34</sup>Chief economist at the International Monetary Fund between 2008 - 2015.

<sup>35</sup>While Robert Lucas, in his presidential address to the American Economic Association at the University of Chicago in 2003, was declaring that “central problem of depression-prevention has been solved”.

<sup>36</sup>President of the Federal Reserve Bank of Minneapolis 2009.

More recently, the Nobel Laureate, Prof. Joseph E. Stiglitz,<sup>37</sup> wrote that DSGE “*failed to incorporate key aspects of economic behavior, e.g., incorporating insights from information economics and behavioral economics. Inadequate modelling of the financial sector meant they were ill-suited for predicting or responding to a financial crisis*” in [Stiglitz, 2018], while Prof. Paul Romer,<sup>38</sup> speaking about DSGE, declared also in *The Trouble with Macroeconomics* [Romer, 2016]: “*For more than three decades, macroeconomics has gone backwards [...] The trouble is not so much that macroeconomists say things that are inconsistent with the facts. The real trouble is that other economists do not care that the macroeconomists do not care about the facts.*”

In France, the FIPECO<sup>39</sup> presented a comparison between the forecast from the French Ministry of Economy and reality, and concluded by: “*In economic matters, the observed reality is never compatible with predictions. Statistically, they noticed that the modeling does not perform better than a simple persistence model. These criticisms about DSGE are not new. Indeed in his book, *Debunking Economics: The Naked Emperor Dethroned?*, [Keen, 2011] developed a series of theoretical criticisms of mainstream macroeconomics. A broader overview of these criticisms is available in the next section, but we can have some of them displayed here, such as:*

- Most of new classical models have no energy or matter in their production function (exception: when using KLEM production function with Capital (K), Labor (L), Energy (E), Matter (M), e.g., [Stiglitz, 1974a]);<sup>40</sup>
- Money is always considered neutral when it is taken into account;<sup>41</sup>
- They do not admit global out-of-equilibrium dynamics (see e.g., CGE models);
- They are mainly linear in the presumed dynamics, even if they represent very complex phenomenon;
- They do not take into account private debt<sup>42</sup> (exception: [Krugman and Eggertsson, 2013] but no money...) nor the banking sector (only considered as financial intermediaries);<sup>43</sup>
- Due to rational expectations and aggregate demande assumptions, they cannot predict an equilibrium with large underemployment (at least when the labor market is perfectly flexible) and can barely consider financial crises as *black swans*.

Perhaps the most compelling criticism of classical macroeconomic models is that they fail to consider private debt, which generates an inability to predict trajectories to collapse other than *black swans*. For example, the oil-debt link is well-illustrated by Prof. Joseph E. Stiglitz in his book *The Great Divide* [Stiglitz, 2015], in which he illustrates how the Federal Reserve contributes

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<sup>37</sup>Nobel Laureate in 2001 and World Bank Chief Economist between 1997-2000.

<sup>38</sup>Nobel Laureate in 2018 and World Bank Chief Economist between 2016 - 2018.

<sup>39</sup>Finance Publique et Economie, independent and free-access association analyzing public finance and french economy. Source <http://www.fipeco.fr/pdf/0.48968300%201464608610.pdf>

<sup>40</sup>These models generally consider that, according to the cost-share theorem, the GDP-elasticity to energy equals the nominal amount of energy in GDP, which is negligible. In fact, [Giraud and Kahraman, 2014] showed that this share in volume accounts in average for 60% (6 times greater than what new classical economists usually admit).

<sup>41</sup>Whereas it is demonstrated as endogenous in the book *Illusion Financière* of [Giraud, 2014].

<sup>42</sup>See the work of [Keen, 2013] about the link debt-deflation. If excessive debt is so likely to generate a deflation, it is at least surprising the debt ratio was not part of such a model.

<sup>43</sup>See *Debt-Deflation versus the Liquidity Trap: the Dilemma of Nonconventional Monetary Policy* of [Giraud and Pottier, 2016] and *Illusion Financière*, [Giraud, 2014].

to the aggravating factors of the financial crisis by misunderstanding the attenuating power of debt to limit oil price inflation:

*“The war in Iraq made matters worse, because it led to soaring oil prices. With America so dependent on oil imports, we had to spend several hundred billion more to purchase oil [...] Normally this would have led to an economic slowdown as it had in the 1970s. But the Fed met the challenge in the most myopic way imaginable. The flood of liquidity made money readily available in mortgage markets even to those who would normally not be able to borrow. And, yes, this succeeded in forestalling an economic downturn ; America’s household saving rate plummeted to zero. But it should have been clear that we were living on borrowed money and borrowed time.”*

Indeed, the very low interest rates of the Fed (from 0.98% to 5.3 % between 2002 and 2006) allow every American to live on credit, thus circumventing inflation in oil prices [McIsaac, 2016]. The book’s analysis illustrates how the financial crisis is, above all, a crisis of private debt. It does so by explaining that DSGE models at the time were incapable of considering such a crisis probable. This incapability motivates a revival of Minsky’s idea of the Financial Instability Hypothesis (FIH) about the intrinsic instability of capitalism.<sup>44</sup>

According to the analysis of [Grandjean and Giraud, 2017], one can also question the ability of macro-economic models to predict reliable trajectories, especially because of their input definition,<sup>45</sup> the ambiguity of their output indicators (GDP<sup>46</sup> ; Cost of public policy ; Unemployment rate<sup>47</sup>), their lack of back-testing, their potentially high sensitivity to calibrated parameters, their inability to represent out-of-equilibrium trajectories to multiple equilibria, especially in climate change economics.

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<sup>44</sup>The Minsky’s idea was to consider a time in which economic stability incites investors to take risks (with optimism). This risk-taking reflects innovations transforming production and society. However, they also perturb the economic environment. With its sheep-like behavior, the capitalist system switches from a time of constant growth to a time of *euphoria* where expectations about the future become increasingly optimistic. Financial levers are easier. Investments are encouraged by risk-taking, and they begin to exceed profits, leading to an increase in private debt. The financing demand skyrockets. Subsequently, interest rates increase so much that financial viability may not be satisfied. At the peak of this euphoric period, when financial-market operators feel a too big mismatch between valuation and actual values, they sell their assets, igniting a credit collapse. Because of the increasing debt and high interest rates, profits are fewer and fewer, which will make investments subsequently plummet. The euphoric period thus leads to a recession in which profit rates come back to more moderate values. This story is at the heart of the naturalist novel *L’Argent* by Emile Zola, in which the author depicts money and debt as living entities and describes the premisses of the collapse of the Napoleonic 2nd Empire. In [Krugman and Eggertsson, 2013], the authors illustrate this phenomenon with the coyote and the road runner cartoons, in which the coyote continues to run even when he is mid-air and about to fall from a cliff. When optimism comes back, the same scheme will appear, but this time, the debt amount at the beginning of the euphoric period will be higher. For this reason, [Krugman and Eggertsson, 2013] try to put the debt at the heart of new classical DSGE models, but without money. Despite their attempts, the large majority of new-classical models still does not take into account private debt in their dynamic.

<sup>45</sup>E.g., the Cambridge capital controversy in the late 1950s expresses the gap between definition of the nature and role of capital goods. [Stiglitz, 1974b].

<sup>46</sup>See a review of the criticisms in [Jany-Catrice and Méda, 2015] or the proposition of [Piketty, 2009].

<sup>47</sup>The ECB experts estimate the labour market slack around 18% in the Euro Zone, almost twice the official rate from Eurostat around 9.5%. This shows the current paradox about the calculation of this indicator [European Central Bank, 2017]

### 1.3.3 Macro-economic modeling: map of the problematic

In climate change economics, modelers are facing two main fields: Economics and Environment. In environmental science, researchers are modeling complex interactions between components depending on (possibly probabilistic) observables. However these behaviors can usually be described through physical laws and equations. These fundamental equations, such as the conservation of energy, heat transfer, etc., have already been established and empirically tested. Solving these equations, through a selected numerical method and sampling, can introduce some approximations. These equations involve various parameters, among which some fundamental constants are not questionable (such as the Planck's or Boltzman's constant). The uncertainties in the models are thus not primarily coming from the equations themselves, but rather from the way they are implemented and solved in order to represent the actual system.

In Macroeconomics, modelers have to represent complex interactions between complex decision units (humans), of which the behavioral description through mathematical equations is not something unquestionable [Grandjean and Giraud, 2017]. Furthermore, whereas all environmental models are governed by the laws of thermodynamics,<sup>48</sup> a majority of macro-economic models still ignore them, despite the words of Sir [Eddington, 2012]:<sup>49</sup>

*“The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations, then so much the worse for Maxwell's equations. If it is found to be contradicted by observation, well, these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation.”*

Some works of economists, such as [Georgescu-Roegen, 1993] or [Ayres and Warr, 2005], are pioneers in taking into account physical laws and constraints into macroeconomic theory, but this kind of link is still rare in the field. Moreover, an important number of models are facing other structural problems in the definition of their inputs / outputs (indicators), the back-testing of their results, and their high-dependency on parameters, sampling choices, and representations.

#### The ambiguity of indicators based on hard-to-access data

Usual macro-economic models deliver predictive trajectories about some indicators (or outputs, also-called Quantity of Interest in Sensitivity Analysis). The relevance of these indicators is questionable because of their ambiguous definition, their difficult methodology of construction and the hard-to-access data they are based on.

For instance, the main indicator of these models, Gross Domestic Product (GDP), is criticized heavily through a large literature (see [Jany-Catrice and Méda, 2015] for a survey, or the [Piketty, 2009] proposition to switch the indicator to Net National Income). According to the [Kaya, 2014]'s

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<sup>48</sup>More specifically, the second law of thermodynamics which states that the total entropy of an isolated system can never decrease over time. It means that a natural process runs only in one sense, and is not reversible. For example, heat always flows spontaneously from hotter to colder bodies, and never the reverse, unless external work is performed on the system. This implies the impossibility of a perpetual motion machine, or the Carnot's theorem that limits the maximum efficiency for any possible engine.

<sup>49</sup>British astrophysicist and philosopher of science.

equation, about the link between GDP growth and CO<sub>2</sub> emissions growth, it seems indeed dubious to consider it as a relevant indicator to the task of building a climate policy. [Giraud and Kahraman, 2014] stated the link between energy consumption growth and GDP growth, evidenced in the [BP, 2019] report, questioning the integrity of pursuing GDP growth if that growth is so strongly linked to increases in our (mostly fossil) energy consumption.

According to [OECD, 1996], the unemployment rate is also an ambiguous indicator, because: (1) many classifications of unemployment may lead to various possible results due to accounting manipulations; (2) in France, the methodology differs according to the institutions and thus deliver different results; and (3) the indicator is based on a strong conventional component that can be caused by many sources (discouraged workers; unregistered; ...).

The (expected) cost of a public policy is often used to orientate a policy maker's decision. This is the case, for example, in Integrated Assessment Models (IAM) such as DICE, which relies on a cost-benefit analysis. This analysis is generally conducted through the maximization of the aggregated utility function.<sup>50</sup> The aggregation methodology is debatable, as detailed in the works of many economists.<sup>51</sup> One of the best example is the Sonnenschein-Mantel-Debreu Theorem which proves that microeconomic rationality assumptions have no equivalent macroeconomic implications, leading to the conclusion that aggregate demand can be any polynomial [Rizvi, 2006]. Furthermore, the utility's maximization is generally based on consumption, without consideration for a possible limit to this consumption.

Besides the dubious relevance of these indicators, and the possible biases in their building methodology, the underlying inputs or data of these macro-economic models are hard-to-access due to their *socially-built* nature. Their construction thus depends on institutions with different techniques and abilities to finely assess the input. Other input parameters are assumed to be constant, such as the *depreciation rate of capital*, which is anything but certain.

### **A need for back-testing and assessing sensitivity of models**

These inputs' uncertainties and the difficulty to access to data impinge the reliability of the results or the recommendation in terms of public policy (especially when the result itself is an ambiguous indicator). The macro-economic models are indeed likely dependent on many parameters. While some of them are estimated, a large number of parameters are only calibrated. The estimation methods can lead to mismatches between the result and the reality. These types of mismatches can be reduced through back-testing the model on historical data.

These estimations and calibrations of parameters can have important impacts on the outputs. This is the end of Sensitivity Analysis: to describe the link between impact on the outputs and uncertainties on the inputs. This type of analysis is detailed in appendix A.2. In a very complex system, such as environmental science, or macroeconomics, with numerous nonlinearities and parameters, we can acknowledge the fact that a minor uncertainty might lead large variation of the output.

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<sup>50</sup>Inspired from the Ramsey-Cass-Koopmans model, maximizing levels of consumption over successive generations [Collard, 2012].

<sup>51</sup>E.g., [Arrow, 1999], [Arrow et al., 2004].

Among the various sensitivity analyses, the bayesian method is the most frequently used to calibrate DSGE models, but this method involves many *a priori* biases. The bayesian inference used to build the sensitivity analysis of the DSGE models is consequently highly dependent on the *a priori* choices of the modelers. In a field where facts and *values* are hard to distinguish [Pottier, 2014], one can understand how a bayesian method to calibrate a DSGE model could potentially confirm the original biases of the modelers. Other SA (so-called local) implies the selection of a nominal point around which testing the sensitivity of the model.<sup>52</sup> The selected nominal point in the input's space (whatever the method selection is) is critical for the sensitivity analysis, because the modeler is exploring it in a close neighborhood of this particular value.

In a field with important uncertainties on parameters, due to hard-to-access and socially-built data, one can understand the need for back-testing the model and experiencing its robustness to these uncertainties through sensitivity analysis, in order to deliver the most reliable result, even if the output remains questionable.

### The questionable validity of functional representations and dynamics

Among the different representations of the production function in Macroeconomics, one of the most famous is the Cobb-Douglas production function, which is still used in various models and was at the center of the Cambridge Capital Controversy.<sup>53</sup> The theory itself of a production function is empirically hard to describe (even at a micro-level) and covers many theoretical challenges about the possible substitution between factors, the elasticity and inertia of these substitutions, etc.

Many other functional representations in economics<sup>54</sup> are unfalsifiable in the sense of [Popper, 1973],<sup>55</sup> who proposed that “*the criterion of the scientific status of a theory is its falsifiability, or refutability, or testability*” [Popper, 1963]. This argument helps us to understand that we need to take into account the limitation of these models and not dismiss all functional representations, following the aphorism of [Box, 1976]: “*All models are wrong, some are useful.*”

In addition to these functional representations, the very different dynamics between the different types of models can be questioned. For this reason, major institutions are generally using several models with different dynamics to compute a range of scenarios covering these theoretical divergences in the functional representations.

The historical Solow's model is dynamic but often used to analyse a steady-state in the long-run. This habit is anything but scientific, because it presupposes that our real economy has reached its equilibrium, and nothing could be less certain. Computable General Equilibrium models try to calculate many static equilibrium at each time step, taking into account possible exogenous shocks to represent the high volatility of actual economic indicators. However these models view the kinematics of the system through the lens of a succession of equilibria. DSGE models assume the Solow's equilibrium is reached, but exogenous shocks move the system away. Their analyses

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<sup>52</sup>These methods are also presented in appendix A.2.4.

<sup>53</sup>See the work of [Samuelson, 1966], and the more recent book of [Felipe and McCombie, 2013], which is summarized in [Guerrien and Gun, 2014].

<sup>54</sup>E.g., the utility function, which can not be observed [Hodgson, 2012].

<sup>55</sup>A statement, hypothesis, or theory has falsifiability (or is falsifiable) if it is contradicted by a basic statement, which, in an eventual successful or failed falsification, must respectively correspond to a true or hypothetical observation.



consist in computing the return to the long-run steady-state, whatever the nature of the shock is. By essence, these models are not able to predict trajectories to collapse or a *climate-driven Minsky moment*. The out-of-equilibrium dynamic of a DSGE is by construction always transitional.

A more interesting typology of models for our study are the **disequilibrium** models mentioned above (such as Three-Me and its extension IMACLIM-3ME [Gherzi, 2020], [Akerlof and Stiglitz, 1969] or [Grasselli and Costa Lima, 2012]). These models assess by construction the trajectories to multiple equilibria and are able to describe global out-of-equilibrium dynamics. The GEMMES model<sup>56</sup> is an application of this type of model about various geography [Bastidas et al., 2017] and production functions [McIsaac, 2016], with climate feedback loop [Bovari et al., 2018] and various features (Government intervention [Costa Lima and Grasselli, 2014], Inventories [Grasselli and Nguyen Huu, 2016], Inequalities [Giraud and Grasselli, 2017], etc.).

One conclusion drawn from this line of thought is that we cannot use DSGE models to represent the trajectories we seek to explore in order to avoid a “*climate-driven Minsky moment*”. For this reason, we used a disequilibrium model, as an alternative to the previously described defaults. As detailed in appendix A.1.1, the primarily necessary features we aim to offer are a non-linear, global out-of-equilibrium dynamic, and a consideration of money and private debt, in which mass unemployment and financial crashes can occur and the climate feedback loop can be analyzed.

The selected model is based on the work of Goodwin and introduces debt [Goodwin, 1967] and [Keen, 1995]. The nature and stability of the debt equilibria were analyzed in [Grasselli and Costa Lima, 2012] and [McIsaac, 2016] for the Van Der Ploeg’s Extensions.<sup>57</sup> The model is based on the Lotka-Volterra Predator-Prey Logic, with a classical capital’s accumulation equation and two behavioral functions (about, on the one hand, investment depending on profits, and, on the other hand, the link between employment rate and wage share, through a short-term Phillips curve), private debt variation (as the difference between investment and profits) and exogenous population and labor productivity growth rate. The flaws of the model lie in its exogenous growth model, the use of the Say’s law (which can be relaxed by the introduction of inventories [Grasselli and Nguyen-Huu, 2018]), its myopic nature (without expectations), and its high parameter-dependency<sup>58</sup> and its criticism stems from these flaws.

In order to contribute effectively to the improvements of financial and macro-economic models built to assess public policies for preventing climate-driven collapse, a robust transition model have to be provided in order to display transition trajectories, while bearing in mind the obstacles of the climate change economics.

### 1.3.4 A mix of science and ethical values in Climate Change Economics

The standard evaluation of the economic impacts of climate change highlights specific issues, be they technical (about the right indicator) or more philosophical. This evaluation is indeed contained within a big-picture portrayal of human history as a relentless and inevitable trend “*toward greater levels of social complexity, political organization, and economic specialization, with the development of more complex and capable technologies supporting ever-growing population, all sustained by the*

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<sup>56</sup>Developed at the French Development Agency under the direction of Gaël Giraud.

<sup>57</sup>Introducing a CES production function instead of the Leontief production function [Van der Ploeg, 1985].

<sup>58</sup>See also the study of [Pottier and Nguyen-Huu, 2017] about the dependency of the model dynamics to the selected investment function.

*mobilization of ever-increasing quantities of material, energy, and information*” [Motesharrei et al., 2014] on the one hand.

In this story, human society is well-satisfied with its achievements. Wealth has never been higher, a new liberalism leads us to a capitalism without any complexes, and there is no problem that innovation, or a right price, cannot solve, because our knowledge is in a constant state of progression. Any problems remaining in our society do not really matter because they can be solved by way of political will, or more preferably, by letting the market mechanisms guide us to the harmony. From an economic point of view, the ecological problem is rather simple. Ecological damages are externalities. The obvious solution consists of putting a price on the externality so that it can be used optimally. The price adjustment would make up the cost that the externality may impose upon society.<sup>59</sup>

However, on the other hand, in the same field (climate change economics), a very different vision exists, as described, for example, in [Pottier, 2014]. In this story, climate threats are not mere accidents, but rather consequences of a specific way of life that liberalism and globalisation fostered around the world. This lifestyle is reliant on the consumption and exploitation of natural resources, and an “out of sight, out of mind” attitude toward waste, which ultimately ends up in natural habitats. This lifestyle inevitably leads to a mass depletion of resources and biodiversity. The widespread combustion of fossil fuels releases greenhouse gases into the atmosphere at unprecedented levels and provokes climate disruption. Overfishing and ocean acidification destroy marine species. This ecological crisis is the result of a dysfunctional society.

These antinomic visions co-exist in the same discipline, with one being inspired by biology and the other by economics. The gap between the two visions can be attributed to a deficiency in our tools for understanding climate change from the economic perspective, and also to an epistemological mixing of values, as is shown in [Pottier, 2014]. Generally speaking, three main tools can be utilised to reframe the ecological crisis according to its economic impacts: the discount rate, a damage function of the cost-benefit analysis, and the cost-effectiveness analysis.

- Discount rate: in most macro-economic models addressing climate, the discount rate represents the *preference for the present*, covers a wide range of possible values,<sup>60</sup> and is a very influential parameter on the output (see the controversy about the [Stiglitz et al., 2017] report in [Godard, 2007]). Hidden behind the technical debate around the actual values of the discount is an ethical question: should we value people living today more than future generations [Beckerman et al., 2007]? For example, [Mertens and Rubinchik, 2012] state that discount rate must equal the growth rate of real per capita income whereas [Roemer, 2011] shows that “*a very small discount rate (large discount factor) should be used*”;
- Damage functions: they aim at assessing the economic impact of climate change through a function depending usually on increases in temperature, as they impact production [Nordhaus, 2008] or labor productivity [Burke et al., 2015], but very large uncertainties remain about

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<sup>59</sup>This position is defended by important economists claiming that ecological issues can be addressed through a relevant price. See, for example, the op-ed article of Tirole and Gollier who advocates for a global carbon price [Tirole and Gollier, 2015], or the famous reports of [Stiglitz et al., 2017] or [Canfin et al., 2016].

<sup>60</sup>As noticed by [Grandjean and Giraud, 2017], the Ramsey’s equation states that the discount rate must equal the sum of the time-preference rate and the product between the growth rate of the economy by the risk-averse rate. But the debate is still open about the values of these elements.



the parameter and the form of these functions, which leads to important variations between scenarios such as those from [Nordhaus, 2008], [Weitzman, 2012] or [Dietz and Stern, 2015];

- Cost-effectiveness analysis: according to [Pindyck, 2017], cost-benefit analyses, and especially damage functions, “*create a perception of knowledge and precision, but that perception is illusory and misleading*”. For this reason, numerous models are now involved in cost-effectiveness analysis through a disaggregation of economic sectors in order to avoid criticism on aggregated production, and model in a finer way the relationship between production and climate.

Beyond the technical debate, [Pottier, 2014] shows how the canonical epistemology (mainly inspired by Milton Friedman) of economic analysis built a positive and normative economics on the separation between facts and values. [Ferraton, 2008] contests this separation, and shows that economists act as if they are reaching objective, positive results and conclusions when, in fact, they are always tainted with philosophical elements. With the absence of axiological neutrality [Ferraton, 2011], inspired from the Weberian ideal of research motivation, one must be very careful with their personal values, and not just their quantitative ones, when conducting research.

To follow these guidelines and contribute to research in economics in the fairest way, I will try to be explicit, at each step of my work, about my assumptions and the values which underlie them. My own perspective on climate change economics is that natural habitats are degraded because of human activity, on the one hand. On the other hand, the society’s financialization introduces strong biases into our understanding of our own system, making its reactions highly non-linear. From the perspective of an economist, I agree with Minsky’s idea of an intrinsic instability of capitalism. Moreover, I do believe, as the experiment showed us, that our mainstream models are not able to capture the magnitude of the problem - both economically and ecologically. As a former student of mechanical engineering, I also consider that our macro-economic models suffer from their parameter-sensitivity, and thus that their results are highly questionable.

Consequently, I will direct my research to a family of models able to take into account a “*climate-driven Minsky moment*”: disequilibrium models. Furthermore, I will try to assess and improve their robustness. I am deeply convinced of our need for robust macro-economic models not only to assess the economic impacts of climate change, but more importantly, to identify the public policies best able to redirect our system’s trajectory to a more sustainable path.

## 1.4 Research questions and executive summary

### 1.4.1 Research questions and axes

This work seeks to address three research questions with the objective of building a robust transition model that can take into account a “*climate-driven Minsky moment*”. These questions are: what are the primary necessary features of a model to assess the macrodynamics of the transition? How should we evaluate or model these features quantitatively (from an economic viewpoint)? Which public policies are likely to guarantee a transition in this model with these features?

In this framework, these questions correspond to three research axes, each of them leading to the publication of an article (working papers). These axes are the following:

- Quantifying the robustness of a model and comparing its sensitivity with those of other macroeconomic models ;

- Extending an existing macro-economic model, [Grasselli and Costa Lima, 2012], by introducing inventories, money velocity, government intervention in order to help the orientation of public policies in conjunction with applied current models ;
- Assessing the impact of public policies in order to guarantee a transition between different natures of capital in this model.

### 1.4.2 Thesis organization

This robust transition model must be able to provide information about public policies avoiding either an attracting equilibria with *brown* capital only, or a collapse due to a “*climate-driven Minsky moment*”. This building is based on a stock-flow consistent model with a non-linear dynamic leading to multiple equilibria, that can integrate this Minsky moment.

To comply with the climate objective, we have to integrate climatic constraints into the macro-economic model in order to describe transition trajectories (as noticed in [Canfin et al., 2016]). The aim is to guarantee the best consistency between short-run analysis and long-term decarbonation objectives. For this reason, according to [Grandjean and Giraud, 2017], macro-economic models have to:

- Be Stock-Flow Consistent (a basic condition to guarantee the reliability of the model in terms of accounting);
- Be Physic-Consistent (and especially with the physical laws, such as the 2<sup>nd</sup> law of thermodynamics);
- Explicitly describe the money circulation, and not assume *a priori* a neutral money;
- Be back-tested and analyzed in terms of parameter-sensitivity;
- Avoid hidden or unfalsifiable variables.

In contrast with usual macro-economic models, the [Grasselli and Costa Lima, 2012] model is able to avoid some of their multiple failures and to capture specific features we need in order to predict possible collapse driven (e.g., by an explosive amount of debt). This [Grasselli and Costa Lima, 2012] model has been backtested and extended through e.g. the work of [McIsaac, 2016] and [Bovari et al., 2018] with a climate feedback loop.

But one can highlight some criticisms to the model such as its exogenous growth model, its assumptions about the Say’s Law,<sup>61</sup> or its myopic nature (no expectations are introduced in the model) or its input-dependency (parameter and initial values of state variables): all stability criterion are satisfied under certain conditions on parameters.

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<sup>61</sup>Which equals output and demand, and consequently drives the expression of the investment behavioral function according to the adage: “*all profits are reinvested and all wages are consumed*”. There is no inventories or buffer between output and demand.

To respond to the criticism made against the original [Grasselli and Costa Lima, 2012] model about its high parameter-dependency, Chapter 2 explores the robustness of the model with two production functions (Leontief and CES), through a GSA with Sobol’ indices estimated by the [Saltelli, 2002] method.<sup>62</sup> It also reviews some global SA about Dynamic Stochastic General Equilibrium (DSGE), Real Business Cycle (RBC) models and Integrated Assessment Models (IAM). Chapter 3 extends the model by introducing money velocity (according to the Quantitative Theory of Money) in a framework relaxing the Say’s law by the introduction of inventories (inspired by the work of [Grasselli and Nguyen Huu, 2016]). Eventually, the transition model is built with two natures of capital and can be generalized through the Putty-Clay framework (see [Akerlof and Stiglitz, 1969], [Cass and Stiglitz, 1969]). The simplest transition model, with two natures of capital only, delivers yet stability criterion and possible levers through government intervention (in the sense of [Costa Lima and Grasselli, 2014]). This Chapter 4 studies the long-run multiple equilibria and stability criteria of the transition model with different types of government intervention.

## **Chapter 2: *Building a global sensitivity analysis to quantify the robustness of macro-economic models***

In Chapter 2, the robustness of the [Grasselli and Costa Lima, 2012] model was tested in terms of sensitivity to parameters. After a study of the various sensitivity analysis methods (see also the appendix of Chapter 2), a Global Sensitivity Analysis (GSA) with Sobol’ indices estimated by the [Saltelli, 2002] method was selected according to the features of the models: nonlinearities, a small number of inputs, and modest computational costs. The process is based on two initial sampling matrices generated by Latin Hypercube Sampling (LHS). A review of literature shows that GSA has been also conducted on the DSGE, RBC, and DICE models [Nordhaus, 2008]. This work was done with the support of the Alliance Program (Alliance Doctoral Mobility Grant) in Columbia University Graduate School of Arts and Sciences, under the invitation of Prof. Joseph E. Stiglitz.

With the Sobol’ indices estimated by the [Saltelli, 2002] method, the sensitivity of an output in the [Grasselli and Costa Lima, 2012] model to its inputs is more balanced than the one in a DSGE, RBC or IAM (DICE)<sup>63</sup> model – mainly depending on a single parameter hard-to-assess. One can slightly improve the model’s robustness by endogeneizing the main influential parameter as in the case of a CES production function.

Eventually we also studied the original [Goodwin, 1967] model (see appendix A.5) and noticed that the sensitivity of the original model depends on the same parameters as its extension. We showed that its performance in terms of robustness are slightly better than the [Grasselli and Costa Lima, 2012] model, which might be due to the smaller number of nonlinearities.

These results are quite reassuring that refining the model does not seem to affect the deep nature of its sensitivity. They might make us wonder about a possible compromise between the robustness and the *realism* of a model to find an equilibrium between a very robust model unable to approximate

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<sup>62</sup>The selection of the method is based on a State of Art in this field (see appendix A.2). Note that the dependency to initial values has been explored through the study of basin of attraction by [McIsaac, 2016]. Consequently we can focus our study on parameters only.

<sup>63</sup>See respectively [Ratto, 2008], [Harenberg et al., 2017], [Nordhaus, 2008], [Miftakhova, 2019], [Butler et al., 2014] and [Anderson et al., 2014].

reality (such as the [Goodwin, 1967] model), and a very fitted and very sensitive model (such as the DSGE of [Ratto, 2008]). From this point of view, the [Grasselli and Costa Lima, 2012] model seems to be an interesting compromise, with a finer description of the economy and a more robust structure.

### **Chapter 3: *Money Velocity in an Imperfectly Competitive, Stock-Flow Consistent Dynamics***

So far, the analytics of money has been cluttered by a number of paradoxes and impossibilities obstructing access to a proper understanding of the interaction between money, its velocity and macro-dynamics. We propose a stock-flow consistent dynamics in continuous time with imperfect competition where these paradoxes can be solved. Money turns out to be non-neutral both in the short- and the long-run. Its creation by credit banking allows its quantity circulating in the economy, as well as the speed at which it circulates, to be endogenously determined by the need to finance investment and consumption. In particular, the decline in money velocity observed in several countries in the aftermath of the Great financial crisis can be analyzed.

Our set-up sheds new light on the links between money velocity and the debt-deflationary path on which several countries seem to have been progressing despite unconventional monetary policies. Low inflation, low real income growth, increasing private debt and a declining velocity of trades turn out to be the hallmark of the macro-economic trajectories leading to a debt-deflationary long-run crisis. We provide conditions under which, however, a low-interest rate monetary policy can stimulate economic activity so as to escape from the liquidity trap.

### **Chapter 4: *Directed Technical Change in a Stock-flow Consistent Dynamics with Government: from Brown to Green Capital***

Now we explore the robustness of the model and its ability to integer money velocity and to relax Say's law, the aim of this chapter is to propose a simple model of energy transition from fossil fuels to renewable energies. We consider here the extreme case of two natures of capital, one brown (that needs fossil energy to operate), the other, green (free of fossil energy), which, however, is less productive than the dirty one. This dual capital structure is embedded into a stock-flow consistent macrodynamics where investment can be financed by debt and the productive sector adjusts shifts in demand thanks to inventories. The transition is said to take place if the economy does asymptotically operate with green capital only.

Absent any public intervention, the economy converges to a purely brown production sector: unsurprisingly, renewable energies are kicked out as long as green capital remains less productive than the brown one. The inertia affecting the transition of aggregate investment from one type of capital to the other, however, turns out to destabilize the debt-deflationary equilibria of the dynamics. Finally, we pinpoint two main levers for a government intervention: increasing the energy price *via* a carbon tax or the output-to-capital ratio of green capital *via* publicly subsidized directed technical innovation. The first option runs the risk of reducing private profits, hence fostering private debts. The second runs the dual risk of boosting public debt. We identify conditions under which the transition to green capital takes place without leading to an overhang of debt, be it private or public. Numerical simulations illustrate the properties of each equilibrium.

## Chapter 2

# Building a global sensitivity analysis to quantify the robustness of macro-economic models

Currently used macro-economic models are facing various defaults (see [Stiglitz, 2018]), especially in a complex economic environment involving a large number of calibrated / estimated parameters which leads to highly parameter-sensitive results (see [Grandjean and Giraud, 2017]). Moreover, in social sciences, where most data are approximations, the impact of inputs' uncertainties cannot be considered negligible, and questioning the robustness of our macro-economic models becomes more and more essential to conduct public policies in these fields [European Commission, 2009]. Consequently, building a model, especially in social sciences or macroeconomics, requires considering three types of uncertainties:

- Uncertainty on the accuracy between model and reality (*Sensitivity to representations*);
- Numerical uncertainties corresponding to sampling choices;
- Parameter uncertainties (*Sensitivity to parameters*).

In the frame of the sensitivity to parameters, inputs' uncertainties are usually modeled by probability density functions. We estimate the probability density of the input and try to quantify its propagation through the model. Unlike local sensitivity analysis (LSA), global sensitivity analysis (GSA) methods evaluate the effect of this propagation while all other factors are varied as well and thus they account for interactions between variables and do not depend on the stipulation of a benchmark. Among the large number of available approaches<sup>1</sup>, the method of global sensitivity indices developed by Sobol' (1993, 2001) is based on the ANOVA decomposition of a high dimensional model representation [Kucherenko and Shah, 2007].

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<sup>1</sup>Such that, for example:

- Momentum Methods, where the output's average and variance are compared to the probability density of inputs (hard in highly non-linear model) ;
- Sampling Methods (which can be conducted either with a Monte-Carlo or a Latin Hyper Cube sampling). These methods are very time-consuming and can miss some non-linearity phenomena ;
- Response Surface Methods (also called Emulators or Meta-Models), which are less time consuming and ensure to capture combined parameters' phenomenon.

This Sobol’ GSA is superior to other SA methods, such as those based on joint effects or regression coefficients, because it doesn’t require linearity assumptions. Moreover, it captures both individual and interaction effects of the input parameters on the output variance, overcoming drawbacks such as the “curse of dimensionality” [Bellman, 1957]. Various efficient practical implementations of these indices have been proposed in the literature (see [Saltelli, 2002] for a survey). In macroeconomics, a number of SA already exists for DSGE models (see [Ratto, 2008], and [Harenberg et al., 2017] for a survey of the methods or the SA made by Nordhaus about the canonical model [Nordhaus, 2008]).

Our sensitivity analysis is applied to a specific type of macro-economic models such as the [Grasselli and Costa Lima, 2012] model, where investment can be financed by debt. The main features of this approach when compared to competing macro-economic models are its stock-flow consistent methodology leading to multiple equilibria and its 3D non-linear dynamics describing the time evolution of the wage share, employment rate, and private debt of the system.

This type of models, also called *disequilibrium models* (such as Three-Me, or [Akerlof and Stiglitz, 1969], or [Grasselli and Costa Lima, 2012] based on a Lotka-Volterra approach between wages and employment rate<sup>2</sup>), can answer to some of the criticisms of a large majority of usual models because of their global out-of-equilibrium non-linear dynamics. For a more detailed motivation of the selection of this family of models, we refer the reader to the introduction.<sup>3</sup>

With this family of models, we can indeed study trajectories towards various equilibria, be they “good” or “bad” (e.g. in the case of a crisis), their basin of attractions [McIsaac, 2016] and manage public policies to stay in it [Costa Lima and Grasselli, 2014]. Many extensions are available to cope with fundamental economic aspects: money velocity can be introduced to overcome neutral or exogenous money, we can consider different trajectories for private and public debts, integrate climate feedback loops [Bovari et al., 2018] and associated transition mechanism or observe the non-linear inventory dynamics of the model without Say’s Law [Grasselli and Nguyen-Huu, 2018].

However, the output values of the model (wage share, employment rate and debt ratio of the system), the stability criterion of the considered equilibria and the dimensions of their basins of attraction are obviously dependent on certain parameters of the system. This motivates a dedicated sensitivity analysis to understand the possibility for a decision-maker to influence certain inputs in order to drastically transform the trajectory of the system.

The rest of this study is organized as follows. Section 2.1 is dedicated to a brief review of the features of the model [Grasselli and Costa Lima, 2012]. Then an overview of the various SA methods is conducted in Section 2.2 in order to motivate the selected method - a global sensitivity analysis (GSA) with Sobol’ indices - according to the characteristics of the model. Section 2.3 contains our application of the GSA to the model and concludes about its macroeconomic consequences and the possible improvements we should make, i.e. endogenizing growth or relaxing the assumption of constant capital-to-output ratio. In Section 2.4, we propose a qualitative comparison with other GSA conducted on various types of macro-economic models such as Dynamic Stochastic General Equilibrium (DSGE) model [Ratto, 2008], Real Business Cycle (RBC) model [Harenberg et al., 2017] or Integrated Assessment Model: DICE [Nordhaus, 2008].

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<sup>2</sup>Inspired from [Goodwin, 1967] and developed by [Keen, 1995].

<sup>3</sup>See also Appendix A.1 for the justification of our assumption on the model.



## 2.1 General structure of the [Grasselli and Costa Lima, 2012] model

The model is a 3D non-linear dynamical system describing the time evolution of the wage share, employment rate, and private debt of the system. It proposes a stock-flow consistent approach leading to multiple equilibria where investments can be financed by private debt. Two behavioral functions are assumed between wages and employment on the one hand, and, on the other hand, about investment depending on the net profit. A Leontief's production function is assumed involving a fixed capital-to-output ratio. Consumption, here, is only an adjustment variable. We refer the reader to see appendix A.1.1 and the above-mentioned article for comprehensive information about the properties and construction of the model. In this section, we only present a summary of the essential elements for the continuation of our study.

### 2.1.1 Some classical assumptions

First, a classical capital's accumulation equation is assumed, i.e. that  $\dot{K}$ , the change in capital stock  $K$ , is depending on the investment  $I$  and a fixed depreciation rate  $\delta$ :

$$\dot{K} = I - \delta K . \quad (1)$$

Assuming the Say's Law, total sales demand  $Y_d$  is equal to the total yearly output  $Y$  (so consumption  $C$  acts only as an adjustment variable):

$$C + I = Y_d = Y , \quad (2)$$

where  $\Pi$  is the net profit for firms after paying wages,  $W$ .

As in [Grasselli and Costa Lima, 2012], we consider the following Leontief production function:

$$Y = \min \left\{ \frac{K}{\nu} ; aL \right\} , \quad (3)$$

which means, when assuming full capital utilization:  $Y = \frac{K}{\nu} = aL$ , where  $\nu$  is a constant capital-to-output ratio,  $L$  is the number of employed workers and  $a$  is the labour productivity, that is to say, the number of units of output per worker per year. All quantities are assumed to be quoted in real rather nominal terms, thereby already incorporating the effects of inflation, and are net quantities, meaning that intermediate revenues and expenditures are deducted from the final yearly output. Let the labor productivity be given by  $a(t) = a_0 e^{\alpha t}$  and the total labour force by:  $N(t) = N_0 e^{\beta t}$ , we then define the employment rate:  $\lambda = \frac{L(t)}{N(t)}$

### 2.1.2 Stock-flow consistency

### 2.1.3 Aggregate behavioural rules

The rate of change in real wages  $w$  is an increasing function of the employment rate  $\lambda$ :

$$\dot{w} = \phi(\lambda)w \quad (4)$$

where  $\phi(\cdot)$  is known as an increasing short-run Philips curve (see, e.g., [Gordon, 2011] for a historical survey, [Gordon, 2013], [Mankiw, 2001] and [Mankiw, 2016], as well as [Grasselli and Nguyen Huu, 2016]). Moreover, we define the wage share:  $\omega := \frac{W}{Y} = \frac{wL}{aL} = \frac{w}{a}$ .

Inspired from [Minsky, 1982], and formalized by [Keen, 1995], the aggregate investment is expressed as a function,  $\kappa(\cdot)$  of the net profit share:

$$I = \kappa(\pi)Y, \quad (5)$$

where the net profit share,  $\pi := \frac{\Pi}{Y}$ , with  $\Pi$  the net profit after paying wages and interests on debt (i.e.  $\Pi := Y - W - rD$ , where  $D$  is the amount of debt in real term and  $r$  a constant interest rate). This implies that the **growth rate** for the economy in this model,  $\hat{Y}$ , is:

$$\hat{Y} := \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{I - \delta K}{K} = \frac{\kappa(\pi)}{\nu} - \delta. \quad (6)$$

As in Minsky's and Kindleberger's theories, current cash-flows validate past liabilities and form the basis for investment decisions.<sup>4</sup> The change in real debt,  $D$ , is thus :

$$\dot{D} := I - \Pi. \quad (7)$$

The dynamical variables in this model are so the wage share  $\omega$ , the employment rate  $\lambda$  and the debt ratio  $d := \frac{D}{Y}$ . Thus the differential system is:

$$\begin{cases} \frac{\dot{\omega}}{\omega} = \frac{\dot{w}}{w} - \frac{\dot{a}}{a} = \Phi(\lambda) - \alpha \Leftrightarrow \dot{\omega} = \omega (\Phi(\lambda) - \alpha) \\ \frac{\dot{\lambda}}{\lambda} = \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} = \frac{\kappa(\pi)}{\nu} - \delta - \alpha - \beta \Leftrightarrow \dot{\lambda} = \lambda \left( \frac{\kappa(\pi)}{\nu} - \delta - \alpha - \beta \right) \\ \frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(\pi)Y - \pi Y}{D} - \left( \frac{\kappa(\pi)}{\nu} - \delta \right) \Leftrightarrow \dot{d} = \kappa(\pi) - \pi - d \left( \frac{\kappa(\pi)}{\nu} - \delta \right) \end{cases} \quad (8)$$

Under the quite reasonable conditions presented in [Grasselli and Costa Lima, 2012], the dynamical system admits only three equilibria:

- a so-called *Good* Equilibrium – corresponding to a desirable situation with positive equilibrium final values of wages and employment, and a finite level of debt ratio,  $(\omega_1; \lambda_1; d_1)$ , – which is locally stable under certain conditions on the set of parameters ;
- a *Slavery* Equilibrium with employment but no wages, which is structurally unstable, and so will not be studied further ;
- an Explosive-debt Equilibrium (also called the *Bad* Equilibrium) associated with a skyrocketing level of debt ratios while wages and employment shrink to zero. This long-term steady state is also locally stable for a broad family of parameters.

Figures 2.1a and 2.1b are the trajectories of the two locally stable equilibria of the system considering a short-run Phillips curve of the form,  $\phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0$ , and the following investment function:  $\kappa(\pi) = k_0 + k_1 e^{k_2 \pi e}$ , as in [Grasselli and Costa Lima, 2012] and with parameters specified below (see Table 2.1).<sup>5</sup>

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<sup>4</sup>For simplicity again, the model assumes that non-financial firms injects all their profits,  $\Pi$ , for investment purposes (no redistribution of their expected profit to their shareholders).

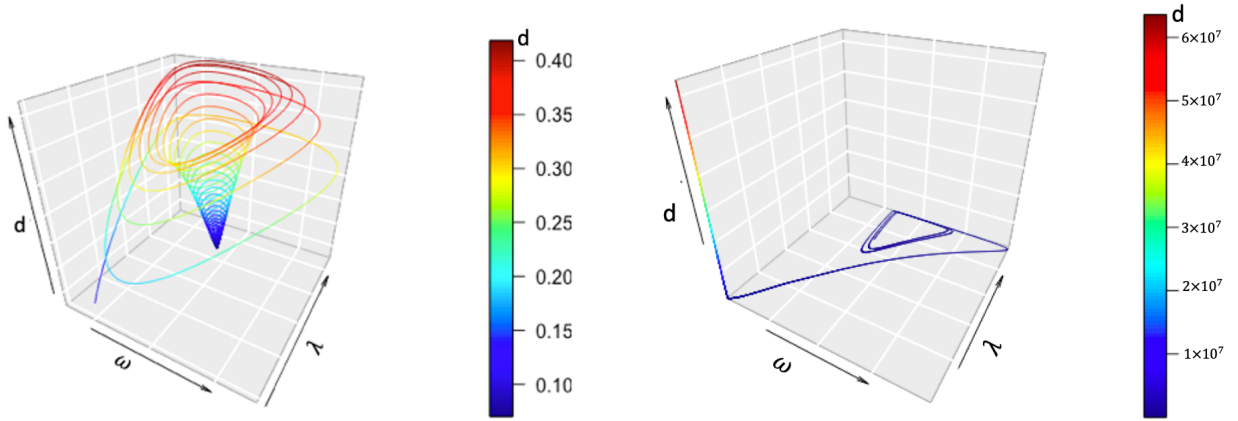
<sup>5</sup>As exposed in [McIsaac, 2016] based on World Bank Data, Penn World Table and BIS Statistics Explorer.



CHAPTER 2. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

Symbol & Definition		Value	Symbol & Definition		Value
$\alpha$	labor productivity growth rate	0.025	$\phi_0$	Phillips curve $\phi(\lambda)$ parameter	$\frac{0.04}{1-0.04^2}$
$\beta$	population growth rate	0.02	$\phi_1$	Phillips curve $\phi(\lambda)$ parameter	$\frac{0.04^3}{1-0.04}$
$\delta$	depreciation rate	0.01	$k_0$	Investment function $\kappa(\pi)$ constant	-0.0065
$\nu$	capital-to-output ratio	3	$k_1$	Investment function $\kappa(\pi)$ parameter	$e^{-5}$
$r$	short-run nominal interest rate	3%	$k_2$	$\kappa(\pi)$ exponential parameter	20

Table 2.1: Canonical parameterization of [Grasselli and Costa Lima, 2012].



(a) Trajectory from initial values:  $\omega_0 = 0.8$ ,  $\lambda_0 = 0.9$  and  $d_0 = 0.1$  to the *Good* Equilibrium with final values:  $(\omega_1, \lambda_1, d_1) = (0.8361, 0.9686, 0.0702)$

(b) Trajectory from initial values:  $\omega_0 = 0.7$ ,  $\lambda_0 = 0.7$  and  $d_0 = 0.1$  to the *Explosive-debt* Equilibrium with final values  $(0, 0, 9 \cdot 10^{101})$

Figure 2.1: Trajectories leading to the locally stable equilibria of system of the [Grasselli and Costa Lima, 2012] model with the set of parameters defined in Table 2.1.

[McIsaac, 2016] studied the basins of attraction<sup>6</sup> of these equilibria. The dimension of the basins are depending on the set of parameters, as they are modifying the condition of stability (Routh-Hurwitz criterion). Figure 2.2 represents the basin of attraction of the *Good* Equilibrium<sup>7,8</sup>.

<sup>6</sup>I.e. all initial values of input's variables leading to this equilibrium according to our choice of parameters.

<sup>7</sup>Figure displayed with the .R code of [Augier, 2018]

<sup>8</sup>One can notice that the initial value of  $(\omega, \lambda)$  leading to the *Good* Equilibrium (green points of the Figure 2.2) are in a neighborhood of this equilibrium (the equilibrium values are detailed in Fig. 2.1a:  $(\omega_1, \lambda_1) = (0.8361, 0.9686)$ ). Note that, for the debt ratio the range of possible initial values leading to the *Good* Equilibrium is quite large compared with the one of  $\omega$  or  $\lambda$ .

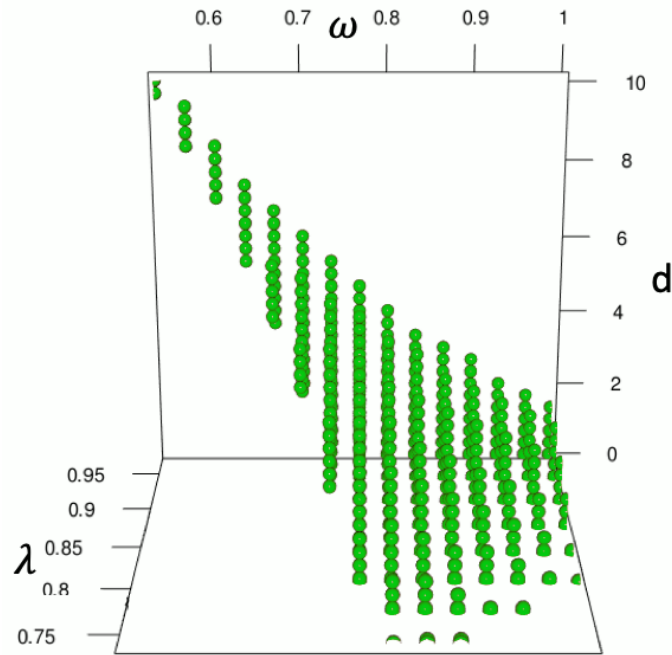


Figure 2.2: Basin of attraction of the *Good* Equilibrium for the set of parameters defined in Table 2.1 ; each green point is a set of initial values for the inputs ( $\omega$ ;  $\lambda$ ;  $d$ ) leading to the *Good* Equilibrium ; the rest of the space leads to the *Explosive Debt* Equilibrium

## 2.2 Motivation of our uncertainties analysis (UA) and sensitivity analysis (SA) methods

Numerical models are now widely used to study complex systems involving large numbers of parameters, be it in engineering, computation science, environmental science, macroeconomics or finance. Their ability to faithfully represent the reality depends both on empirical studies and theoretical considerations.

As an example, behavioral functions<sup>9</sup> are built to fit data, involving a large number of calibration or estimation (based on empirical studies) which leads to highly parameter-sensitive results [Grandjean and Giraud, 2017], whereas production function<sup>10</sup> is a key concept of the economic growth theory, and thus can present discrepancies between the actual production laws and their functional representation (see [Stiglitz, 2018]).

These different types of model's inputs suffer from various nature of uncertainties and any prevision should be undertaken into a probabilistic frame to take them into account.<sup>11</sup> This is the

<sup>9</sup>Such as the Philips' curve or the investment behavioral function in [Grasselli and Costa Lima, 2012].

<sup>10</sup>Such as [Cobb and Douglas, 1928], Putty-clay in the sense of [Akerlof and Stiglitz, 1969], CES [Solow, 1956] or [Leontief, 1951] as used in [Grasselli and Costa Lima, 2012].

<sup>11</sup>To provide high-quality controls of model-based previsions, SA has been confirmed by [European Commission, 2009] in a working document (*The EC handbook for extended impact assessment* which states that *a good sensitivity analysis should conduct analyses over the full range of plausible values of key parameters and their interactions, to assess how impacts change in response to changes in key parameters.*) and the journal *Science Online* reviewed by [Saltelli et al., 2008a] highlights the importance of SA and UA in falsifying or corroborating a simulation model.

principle of an uncertainty and sensitivity analysis.<sup>12</sup>

Uncertainties analysis (UA) studies the propagation of uncertainties in the model by assessing the uncertainty on the outputs due to uncertainties on the inputs, whereas sensitivity analysis is designed to assess (qualitatively or quantitatively) the respective contribution of each inputs to the output uncertainties.<sup>13</sup>

We refer the interested reader to appendix A.2.1 and A.2.2 for more details on UA et SA, including a survey of classical tools and a comprehensive justification of the approach used in this chapter to analyse 3D non-linear dynamical systems.

### 2.2.1 Definition and selection of uncertainty and sensitivity analysis

When building a simulation model, many sources of uncertainty can indeed arise. To improve the reliability of the process, it is necessary to understand these risks. According to [Macdonald et al., 1999] and [De Wit, 2001], the sources of uncertainty can be classified as follows:

- **Uncertainty on the accuracy between model and reality:** This uncertainty appears when building the mathematical representation of the real economic phenomenon. Mathematics are an approximation of the economic reality based on restrictive assumptions, such as, for example, rational agent, Say’s Law, or the production function used in our macro-economic models.
- **Numerical uncertainty corresponding to sampling choices:** In macroeconomics, as in several other fields, mathematical models are often described by partial derivative equations, without explicit closed-form solutions. Numerical resolution of these equations involves an approximation of the continuous model by finite difference schemes. With smaller discretizations,<sup>14</sup> one can reduce the impact of this uncertainty, but we can not totally discard it.
- **Parameter uncertainty:** macro-economic models are usually built with a high number of calibrated or estimated parameters, involving uncertainties about their actual values. The knowledge of these parameter uncertainties and the quantification of their impact on the outputs of a model is at the core of the so-called “propagation of uncertainties” [Saltelli et al., 2008a] and [Bouleau, 2022].<sup>15</sup>

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<sup>12</sup>See appendix A.2 for a definition and appendix A.2.1 and A.2.2 for a review of these various techniques

<sup>13</sup>See e.g. [Lam and Hui, 1995], [Macdonald et al., 1999], [Helton et al., 2006], [Saltelli et al., 2008a]. The difference between the definition of uncertainty analysis (UA) and sensitivity analysis (SA) is stated by [Helton et al., 2006] as that: “*Specially, uncertainty analysis refers to the determination of the uncertainty in analysis results that derives from uncertainty in analysis inputs, and sensitivity analysis refers to the determination of the contributions of individual uncertain inputs to the uncertainty in analysis results.*” UA and SA are most often run together, where the SA is used to rank the uncertainty sources (identified by UA), according to their influences on outputs [Saltelli et al., 2008a]. Another usual definition of a SA method is the following: *The study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input* [Saltelli et al., 2008b].

<sup>14</sup>In the field of digital signal processing, the Nyquist-Shannon signal sampling theorem gives a criterion about the minimal sampling frequency according to the maximal signal frequency to avoid distortion of the information [Shannon, 1949].

<sup>15</sup>See also [Bouleau, 2008] and [Bouleau, 2019].

To assess the impact of uncertainties on outputs, we have to propagate the uncertainties through the model. We so choose a technique to explore the numerical model. These inputs' uncertainties are usually modeled by a probability density functions, used widely since the 70s [Sobol, 1967] [McKay et al., 2000].

Among the various methods to conduct an uncertainty analysis (UA),<sup>16</sup> we selected a stratified sampling with a Latin Hypercube Sampling (LHS) method in order to maximize the inputs space coverage<sup>17</sup> and because of its features when dealing with very non-linear model, without linear approximations, and its ability to be optimized - as detailed in [McKay et al., 2000] or [Saltelli, 2002].

For sensitivity analysis, a wide range of techniques is also available, but these techniques are commonly grouped into local and global methods [Lomas and Eppel, 1992], which distinctions are listed by [Hoes and De Vann, 2005].<sup>18</sup> Local SA are considered to be computationally faster and easier, but less accurate when compared to the sophisticated global SA methods. According to the literature, it also has been stated that the global SA should be used, when input variables are in a non-linear model and from different magnitudes of uncertainty sources [Cukier et al., 1973].<sup>19</sup> To alleviate the lacks of local SA, when studying very non-linear models, or exploring large inputs spaces, one can conduct a global sensitivity analysis.<sup>20</sup>

Global sensitivity analysis (GSA) are indeed used for many different purposes, such as identifying critical regions in the space of inputs, establishing importance of inputs factors in regard with their impact on outputs, simplifying models, etc. Most of these GSA in literature are based on derivatives  $\frac{\partial Y_i}{\partial X_i}$  of an output  $Y_i$  versus an input  $X_i$ , which can be thought as a mathematical definition of the sensitivity of  $Y_i$  versus  $X_i$ . The derivative based approach is very efficient in computer time, but are unwarranted when model inputs are uncertain or when the model is of unknown linearity.

In order to help the practitioner to choose the most appropriate method for its problem and its model, some authors have proposed decision trees. Figure 2.3 reproduces the flowchart of [De Rocquigny, 2008]. Although useful to fix some ideas, such diagrams are rather simple and should be used with caution.

Among the various GSA methods<sup>21</sup> and according to the decision tree above (Figure 2.3), we selected a Sobol's indices method (with the [Saltelli, 2002] estimation method<sup>22</sup>) in order to study the [Grasselli and Costa Lima, 2012] macroeconomic (*a priori* non monotonic) model, with a non-linear dynamics, partial derivative equations and around 10 inputs (parameters of the model),

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<sup>16</sup>See appendix A.2.1 for a description of these methods.

<sup>17</sup>LHS are indeed generally used to improve the inputs space coverage, especially when processing is expensive due to a high number of parameters or wide ranges of uncertainty.

<sup>18</sup>See also in Table A.1 of appendix A.2.2.

<sup>19</sup>LSA indeed focuses on the local impact of input variables on model outputs. It is usually carried out by assuming the linear or monotonous model, or computing partial derivatives of the outputs, with respect to a small interval fractional variation of an input variable around its normal value [Saltelli et al., 2008a].

<sup>20</sup>See appendix A.2.2 for an overview of the different available methods.

<sup>21</sup>See appendix A.2.5 for an overview of these methods and their features.

<sup>22</sup>See appendix A.2.6 for an exhaustive description of this method and appendix A.2.6 for the motivation to use it in order to estimate Sobol' indices.

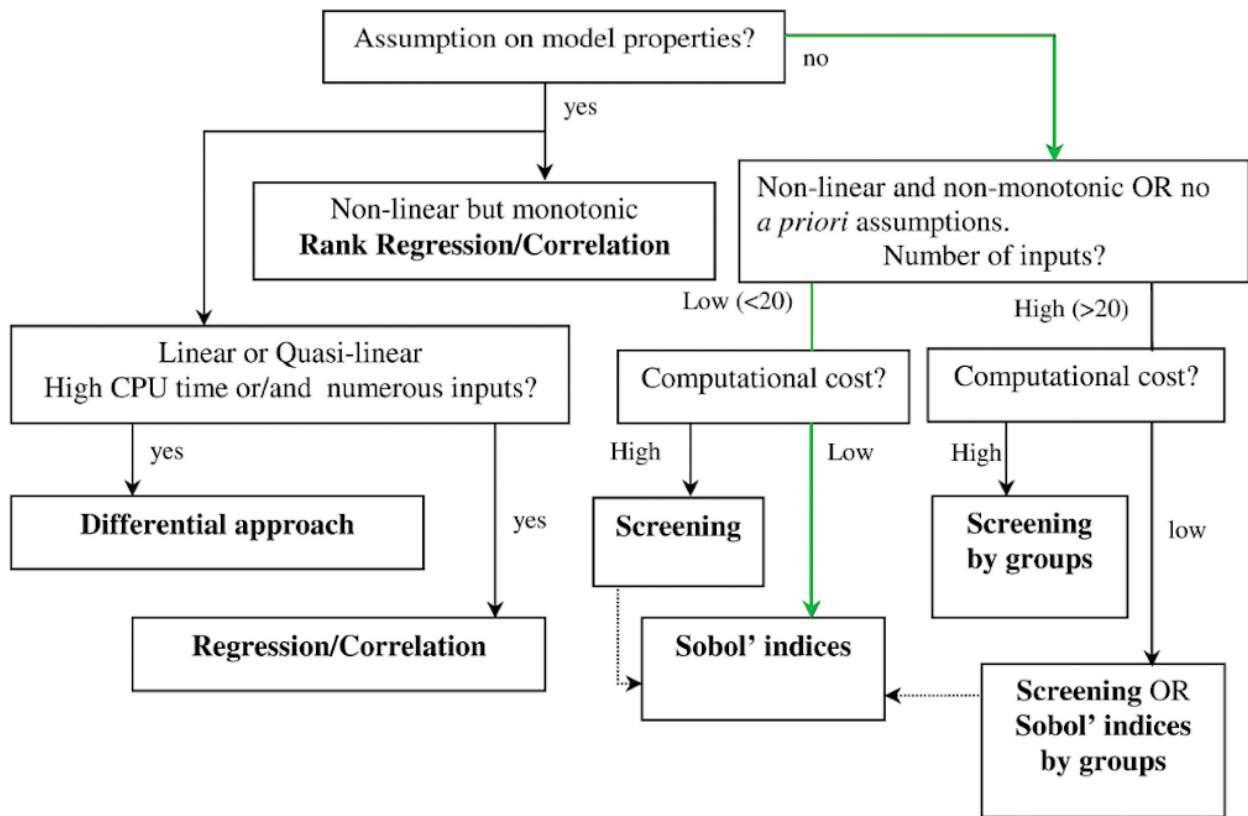


Figure 2.3: Decision diagram for the choice of a SA method (from [De Rocquigny, 2008])

### 2.2.2 A global sensitivity analysis with Sobol' indices

As recommended in [Saltelli et al., 2008b], the relative influence of the inputs variance on output variance allows better results about the global sensitivity measure of a very non-linear system. *This is in order to exploit the statistical properties of variance [...] to investigate how factors contribute to the variance. Interesting features of variance-based methods are:*

- *Model independence: the sensitivity measure is model-free ;*
- *Capacity to capture the influence of the full range of variation of each input factor ;*
- *Appreciation of interaction effects among input factors;*
- *Capacity to tackle groups of input factors: uncertain factors might pertain to different logical types, and it might be desirable to decompose the uncertainty according to these types.*

The main drawbacks of variance-based measures are their computational cost, which is not a big issue for our model (less than ten inputs) and the high number of possible methods, leading to different rankings of factors according to their objectives. To avoid this kind of confusing results we have to define a **Setting**. *A setting is a way of framing the sensitivity quest in such a way that the answer can be confidently entrusted to a well-identified measure [Saltelli et al., 2008b].*

Inspired by the work of the chemists Cukier, I.M. Sobol' proposed a Variance-based Global Sensitivity Analysis through a straightforward Monte-Carlo implementation of the concept of computing

sensitivity for arbitrary group of factors. Given the following model  $Y = f(\mathbf{X})$ : this ANalysis Of the VAriance (ANOVA) aims at defining the share of variability of the output due to its inputs, what is called Sobol' indices.

In the frame of our study, we are interested in a Factor Prioritization Setting, to qualify the relative influence of each input factors on the output. The application principle of the ANOVA with Sobol' indices is described below, but the estimation of these indices remains challenging (especially for second-order Sobol' effect, representing joint effects between output) and is developed in appendix A.2.6. However various successful studies were conducted about DSGE models (e.g. [Ratto, 2008]).

### 2.2.3 Construction of the Sobol' indices

In the case of a linear model  $Y$  with  $p \in \mathbb{N}^*$  independent inputs  $(X_1, \dots, X_p)$ , we have

$$Y = \sum_{i=1}^p a_i X_i, \quad a_i \in \mathbb{R}. \quad (9)$$

Assuming the independence of the inputs,<sup>23</sup> the variance of  $Y$  is given by

$$V(Y) = \sum_{i=1}^p a_i^2 V(X_i) \quad (10)$$

and, in this case, the sensitivity of  $Y$  to  $X_i$  is usually quantified measuring the impact of  $X_i$  on the total variance by the so-called Standardized Regression Coefficient:

$$SRC_i = a_i^2 \frac{V(X_i)}{V(Y)} \quad (11)$$

fulfilling by construction  $\sum_{i=1}^p SRC_i = 1$ .

What is more, in this particular case, we don't have interaction effects in the sense that the impact of two or more inputs is just the sum of their single effects. For a non-linear model, where we can not *a priori* make hypothesis on the structure, we need to define sensitivity indices from a well-chosen decomposition of the output variance.

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<sup>23</sup>The issue of GSA with dependent variables has been the object of intense recent research but is not considered in our study as we make the classical assumptions of *a priori* independent inputs at a first step of the study. A prospect of this work can be to check the independence of these inputs. We review here the different improvements made to take into account dependent inputs. About Sobol' indices, [Xu and Gertner, 2008] exhibits a Sobol' indices decomposition into dependent/correlated and independent/uncorrelated components for linear models. With this decomposition, [Li et al., 2010] build the Sobol' indices for a general model. An other technique is proposed by [Mara and Tarantola, 2011], using the Gram-Schmidt process to decorrelate the inputs variables. They also propose to define new indices through the Sobol' indices of the decorrelated problem. [Chastaing et al., 2013] provide a theoretical framework to generalize the ANOVA decomposition to problems with dependent variables, based on the work of [Hooker, 2007]. In contrast to the other works which focus on generalizing the ANOVA decomposition, [Kucherenko et al., 2012] develop the Sobol' indices via the law of total variance. Recent work of [Tarantola and Mara, 2017] considered estimating Sobol' indices with dependent variables using the Fourier Amplitude Sensitivity Test. In our study, inputs of the model are the exogenous parameters of [Grasselli and Costa Lima, 2012], assumed to be independent. Thus we can only consider the core method of the Sobol' indices to assess its sensitivity to its inputs. But these extensions can be useful when using e.g. a Taylor's rule to endogeneize the interest rate  $r$ .

In his seminal paper [Sobol, 1967], Sobol proposed to expand  $f \in L^1([0, 1]^p, dx)$  into summands of increasing dimensions

$$f(x_1, \dots, x_p) = f_0 + \sum_{i=1}^p f_i(x_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(x_i, x_j) + \dots + f_{1,\dots,p}(x_1, \dots, x_p). \quad (12)$$

Imposing that  $\forall s \in \{1, \dots, p\}$ ,  $\forall i_1 < i_2 < \dots < i_s \in \{1, \dots, p\}^s$  and  $\forall k \in \{i_1, \dots, i_s\}$ ,

$$\int_0^1 f_{i_1, \dots, i_s}(x_{i_1}, \dots, x_{i_s}) dx_k = 0$$

this expansion of  $f$  into  $2^p$  summands of different dimensions exists, is unique and the related terms are pairwise orthogonal in  $L^1([0, 1]^p, dx)$ .

Suppose now that we can express a square integrable random variable  $Y$  as  $Y = f(\mathbf{X})$  where  $\mathbf{X} = (X_1, \dots, X_p)$  is a vector of  $p$  independent random variables uniformly distributed on  $[0, 1]$ , we have<sup>24</sup>

$$Y = f_0 + \sum_{i=1}^p f_i(X_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(X_i, X_j) + \dots + f_{1,\dots,p}(X_1, \dots, X_p). \quad (13)$$

Consequently, using the pairwise orthogonality in the Sobol' decomposition, we can obtain easily a recursive expression of each term using conditional expectations

$$\begin{aligned} f_0 &= E(Y) \\ f_i(X_i) &= E(Y|X_i) - E(Y) \\ f_{i,j}(X_i, X_j) &= E(Y|X_i, X_j) - f_i(X_i) - f_j(X_j) - E(Y) \\ &\vdots \end{aligned} \quad (14)$$

and deduce the so-called ANOVA-HDMR decomposition for the variance of  $Y$

$$\begin{aligned} V(Y) &= \sum_{i=1}^p V_i + \sum_{i=1}^p \sum_{j>i}^p V_{i,j} + \dots + V_{1,\dots,p} \\ &\Updownarrow \\ 1 &= \sum_{i=1}^p \frac{V_i}{V(Y)} + \sum_{i=1}^p \sum_{j>i}^p \frac{V_{i,j}}{V(Y)} + \dots + \frac{V_{1,\dots,p}}{V(Y)} \end{aligned} \quad (15)$$

where

---

<sup>24</sup>Here the independence of the random variables  $(X_1, \dots, X_p)$  is a strong hypothesis necessary to obtain the underlying variance decomposition while we can obviously relax the assumption on the distribution remembering that for a real random variable  $Z$ ,  $F_Z^-(U)$  and  $Z$  are equidistributed when  $U$  is an uniform random variable on  $[0, 1]$  and  $F_Z^-$  the pseudo-inverse of the distribution function of  $Z$ . In the case of dependent variables, one can oppose the difficult generalisation of the above recursive expressions of each term using conditionnal expectations. See the footnote 23 for more explanations.



$$\begin{aligned}
 V_i &= V(E(Y|X_i)) \\
 V_{i,j} &= V(f_{i,j}(X_i, X_j)) = V[E(Y|X_i, X_j)] - V[E(Y|X_i)] - V[E(Y|X_j)] \\
 &\vdots
 \end{aligned} \tag{16}$$

Now, inspired by the linear case, we can define the Sobol indices:  $\forall s \in \{1, \dots, p\}, \forall i_1 < i_2 < \dots < i_s \in \{1, \dots, p\}^s$

$$S_{i_1, \dots, i_s} = \frac{V_{i_1, \dots, i_s}}{V(Y)} \tag{17}$$

where  $S_i$  is a measure of the single impact of  $X_i$  on the total variance,  $S_{i,j}$  is a measure of the joint impact of  $(X_i, X_j)$  on the total variance, etc...

In this spirit, to quantify the total impact (sum of single and related joint effects) of  $X_i$  on the variance of  $Y$  we define

$$S_{T_i} = \frac{E(V(Y|X_{\bar{i}}))}{V(Y)} = 1 - \frac{V(E(Y|X_{\bar{i}}))}{V(Y)} \tag{18}$$

where  $X_{\bar{i}} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_p)$ .

For example, in a model with 3 inputs we have

$$S_{T_1} = S_1 + S_{1,2} + S_{1,3} + S_{1,2,3} = 1 - S_{2,3}. \tag{19}$$

Properties of the Sobol' indices are well known in the literature, and can be summarized as:

1.  $\sum_{s=1}^p \sum_{1 \leq i_1 < i_2 < \dots < i_s \leq p} S_{i_1, \dots, i_s} = 1$
2.  $\forall i \in \{1, \dots, p\}, S_i, S_{T_i} \in [0, 1]$
3.  $\forall i \in \{1, \dots, p\}, S_i \leq S_{T_i}$

The first two properties corresponds to the interpretation of the Sobol' indices as relative contributions to the variance of the output  $V(Y)$ . The third property expresses that  $S_i$  as the contribution of  $X_i$  is contained in  $S_{T_i}$  as the contribution of  $X_i$  and all of its interactions.

**Conclusion** To summarize this section, our study will be based on a global sensitivity analysis with a Latin Hypercube Sampling method to identify the inputs that are the most contributors to the variance of the output (by comparing their total impact Sobol' indices,  $S_{T_i}$ ). We will then conduct a global sensitivity analysis (GSA) on these main influent inputs in order to quantify their contribution to the outputs uncertainty. This GSA will be conducted by comparing first-order and total-effect Sobol' indices estimated with the [Saltelli, 2002] method.



## 2.3 Global sensitivity analysis of the [Grasselli and Costa Lima, 2012] model

The aim of this section is to test the robustness of the [Grasselli and Costa Lima, 2012] model, i.e. here to analyze the global sensitivity of its outputs to uncertainties on its inputs. Consequently, we first have to state the inputs/outputs of our study. We give details about the range of the inputs, the sampling and the design of the experiment. From this point, we can estimate the Sobol's indices with a strong basis of various combinations of our inputs.

This GSA with the Sobol' indices estimated by the [Saltelli, 2002] method will be conducted with:

1. A qualitative analysis with an Optimal Latin Hypercube Sampling to identify the main influential parameters of the [Grasselli and Costa Lima, 2012] model on the outputs - i.e. the *Good* Equilibrium values of the wage share, the employment rate and the debt-to-output ratio. This step investigates all the parameters and their joint effect. The identification is done by comparing the relative total-effect Sobol' indices of each input in subsection 2.3.3.
2. The refining of our meshing about the main influential parameters in order to capture a quantitative assessment of the first-order, total-effect and second-order (joint effect) Sobol' indices of these inputs. Refining is done with 200, 300, and 500 simulations which was sufficient to guarantee a maximal standard deviation less than 0.05 (see subsection 2.3.4) .

### 2.3.1 Identification of the inputs/outputs of our study

The inputs of the [Grasselli and Costa Lima, 2012] model are both input's variables (such as wage share, employment rate or debt ratio) and ex-ante fixed parameters (such as the labor productivity growth rate or the capital-to-output ratio). As the impact of input's variables has been extensively studied<sup>25</sup> in the [McIsaac, 2016], [Bovari et al., 2018] and [Augier, 2018], especially through basins of attraction, we will not studied it further here.

The inputs of our study are consequently the parameters of the model. At a first stage, we consider the impact of all the parameters. The range of their possible values are chosen to guarantee the convergence to the *Good* Equilibrium (see Table 2.2). These ranges might seem small, but wider ranges does not guarantee a sufficient number of combinations leading to the *Good* Equilibrium, due to its stability criterion. As one can show it, when running the sensitivity analysis with wider ranges, the number of combinations avoiding the large basin of attraction of the *Explosive Debt* Equilibrium shrinks drastically and therefore is not more relevant than smaller ranges in a neighborhood of the *Good* Equilibrium we want to study.

In this work, we are only looking at the values of the *Good* Equilibrium  $(\omega_1, \lambda_1, d_1)$ <sup>26</sup> as outputs of our study and how they can be influenced by uncertainties on the inputs. One can also be interested in observing the impact of inputs on the *Explosive Debt* Equilibrium values, but as  $(\omega, \lambda, d) \rightarrow (0, 0, +\infty)$ , the examination of their variance is less meaningful, and is highly dependent on the time horizon we consider. As the *Slavery* Equilibrium is structurally unstable, we do not study it further.

---

<sup>25</sup>But these studies take as assumptions certain values of parameters that guarantee the stability of the studied equilibria. One can wonder about the uncertainties of these parameters and their potential impact on the outputs of the model.

<sup>26</sup>See equation (26) of the appendix A.1.1 for the expression of these final values at the *Good* Equilibrium.

Symbol	Definition	Min Value	Max Value	Range	Nom. Value
$\alpha$	labor productivity growth rate	0	0.03	0.03	0.025
$\beta$	population growth rate	0.015	0.03	0.015	0.02
$\delta$	depreciation rate	0.005	0.03	0.025	0.01
$\nu$	capital-to-output ratio	2	4.5	2.5	3
$\phi_0$	Philips Curve $\Phi(\lambda)$ parameter	0.038	0.042	0.004	$\frac{0.04}{1-0.04^2}$
$\phi_1$	Philips Curve $\Phi(\lambda)$ parameter	6.333 E-5	7 E-5	6.67 E-6	$\frac{0.04^3}{1-0.04}$
$k_0$	Investment function $\kappa(\pi)$ constant	-0.007	0.0004	0.074	-0.0065
$k_1$	Investment function $\kappa(\pi)$ parameter	0.0064	0.007	0.0006	$e^{-5}$
$k_2$	$\kappa(\pi)$ exponential parameter	19.8	20.4	0.6	20
$r$	nominal interest rate	1%	8%	0.07	3%

Table 2.2: Definition and Range of the inputs of our study

### 2.3.2 Sampling method: Latin Hypercube

One of the main issues of this type of study lies in the sampling choices: a too precise sampling leads to very long rendering due to the skyrocketing number of simulations to launch<sup>27</sup>; and a too wide one can miss some important phenomenon due to the nonlinearity of our model.

We compare here two types of Latin Hypercube Sampling: the so-called Random one and the Optimal Latin Hypercube Sampling. Comparison here is made in regard with a Full Factorial Sampling - when every possible combination of levels of each inputs is tested.

- **The (Random) Latin Hypercube Sampling (LHS)** is a Design of Experiment (DoE) ensuring a maximum inputs' space coverage [Iman, 1999]. It is a stratified statistical method generating a near-random sample of parameter values from a multidimensional distribution. According to [McKay et al., 2000], it confers to LHS a bigger precision than other stratified sampling methods;
- **The Optimal Latin Hypercube Sampling (OLHS)** can be implement to deliver even better coverage ensuring to capture the maximum of information with the minimum of combinations<sup>28</sup>. As a comparison with more usual methods such as One-At-A-Time samplings for every inputs, also called Fractional Factorial Sampling (or Full Factorial Sampling - FF), the OLHS is known to be a much more efficient technique. These different techniques are

<sup>27</sup>As an example, in the [Saltelli, 2002] method (see appendix A.2.6), with  $p$  inputs, and  $n$  combinations of the input values to test, we need to build first two samples of the  $n$ -combinations, then run  $n * p$  combinations for the estimation of the first-order Sobol' indices,  $n * p$  combinations for the estimation of the total-effect Sobol' indices, and  $n * \binom{p}{2}$  combinations for the estimation of the joint effect. E.g. with 10 inputs and 100 combinations, it leads to  $100 * 10 * 2 + 100 * 10 + 100 * 10 + 100 * \binom{10}{2} = 8500$  combinations. With no-parallelization, the running time of one simulation is around 10s, i.e. a total time of 23h30.

<sup>28</sup>These types of matrix are not reproducible (unless using the same random seed), because their construction is based on a randomization, followed by a stochastic optimization process which aims at spreading points as evenly as possible within the design space (according to an optimality criteria such *maxi-min distance criteria*).

detailed in [Burman and Plackett, 1946], improved by [Box and Behnken, 1960] and Taguchi methods [Karna et al., 2012].

The concept of these DoE can be illustrated by Figure 2.4 for two parameters X1, X2. With the same number of combinations, the FF explores only 3 levels (with redundant levels of a parameter), missing thus some areas of the inputs space, whereas Random LHS guarantees that no combination contains the same value for a given parameter and OHLS satisfies an evenly spreading.<sup>29</sup>

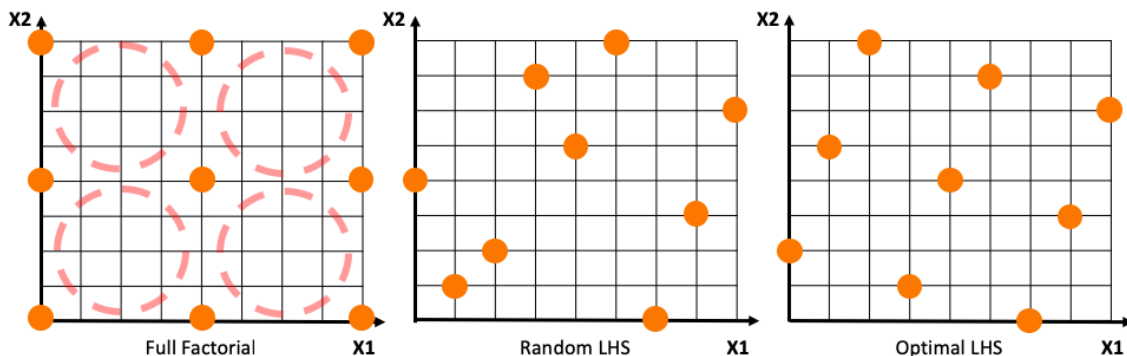


Figure 2.4: Comparaision of samplings with only three possible levels of two inputs

For our study with the OLHS, we choose a sampling of 100 levels for each parameter as it guarantees an inputs' space coverage  $\geq 95\%$  (see Figure A.4b from [Iman, 1999]) and is more easily reproducible than a Random LHS.

### 2.3.3 Sobol' indices computation for the [Grasselli and Costa Lima, 2012] inputs

In this part, we present our results regarding the relative importance of the inputs according to their first-order and total-effect Sobol' indices in order to capture joint effects. This leads to the identification of four main influential parameters: the labor productivity growth rate  $\alpha$ , the population growth rate  $\beta$ , the depreciation rate  $\delta$  and the capital-to-output ratio  $\nu$ . One can notice that the nominal interest rate  $r$  is very influential at first-order, but does not have many joint effects, and so is not a key parameter in terms of total-effect Sobol' indices.

Complying the [Saltelli, 2002] method to estimate the Sobol' indices of each input of the [Grasselli and Costa Lima, 2012] model, we conduct the calculation with the following steps:

1. We build two sample matrix  $\mathbf{X}(1)$  and  $\mathbf{X}(2)$  with the Optimal Latin Hypercube sampling (100 combinations of inputs, ensuring an inputs' space coverage  $\geq 95\%$ ) ;
2. We build the dedicated matrix for the first-order, second-order and total effect Sobol' indices with the [McKay et al., 2000] method (See appendix A.2.6).<sup>30</sup> ;

<sup>29</sup>In our study, considering, for example, (4 inputs  $\times$  7 levels =) 2041 simulations to run with FF Sampling leads to almost the same standard deviations of the Sobol' indices than 100 simulations of OLHS with ten replicates and is consequently twice slower.

<sup>30</sup>First-order Matrix contains 10 columns (10 inputs) and 100 lines (100 combinations of inputs), Total-effect Matrix contains also 10 columns and 100 lines, whereas Second-order Matrix contains 10 inputs and  $100 * \binom{10}{2} = 4500$  lines.

3. We run the [Grasselli and Costa Lima, 2012] model with every combination of parameters (each line of these matrix) and record the *Good* Equilibrium values: wage share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$  ;
4. We compute the Sobol' indices by comparing the variance of this output vectors to the expected variance of these vectors with no perturbations, and we replicate the whole process 10 times in order to display the standard deviation of our results.

Our results are gathered in Table 2.3.<sup>31</sup> We rank the parameters according to their total-effect Sobol' indices, in order to take into account the possible joint effects between them. This ranking is of course different from the one at first-order (without joint effects).<sup>32</sup> According to the total-effect Sobol' indices, the most influential parameters for all outputs are:

- The labor productivity growth rate  $\alpha$  with a mean total-effect on all outputs about 0.861,
- The population growth rate  $\beta$ : 0.788 ,
- The depreciation rate  $\delta$ : 0.894,
- And the capital-to-output ratio  $\nu$ : 0.8995.

Here it is important to note that this ranking is possible only because it is very stable regardless of the chosen output. If this were not the case no overall conclusion would be possible

One can notice that the nominal interest rate which has an important first-order Sobol' indices on  $\omega_1$ , is not very influential at a global scale because its interaction with other parameters does not lead to important variation of the outputs.

An interesting question here is to wonder if the most influential parameters we highlighted are hard to estimate or calibrate. Indeed, a non-sensitive parameter that is difficult to estimate does not represent a big problem for the robustness of the model, or a very sensitive parameter that is easy to estimate or calibrate gives confidence in the results of the model, and conversely.

As detailed in [McIsaac, 2016], demographic data from United Nations time series are available and can easily permit approximations through logistic function (but more rarely with an exponential as in the [Grasselli and Costa Lima, 2012] - or only when considering periods around 50 years - which can be the case in this family of models). This gives some confidence about the ability to estimate the population growth rate,  $\beta$  with a certain confidence.

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<sup>31</sup>Results about second-order Sobol' indices (joint effects between factors) are displayed in appendix A.3.

<sup>32</sup>Indeed, as one can see in Table 2.3, the analysis of the most influential parameters on each output **at first-order** gives dissonant information because it does not capture the joint effects between parameters which are undoubtedly important in this non-linear model. **On the wage share**,  $\omega_1$ , the most influential parameters are the capital-to-output ratio  $\nu$ , with a first-order Sobol' indices about 0.243, the nominal interest rate  $r$ : 0.215, the population growth rate  $\beta$ : 0.118, and the constant from the investment function  $k_0$ : 0.095. This result can be surprising because one can expect the labor productivity growth rate  $\alpha$  to be, one of the main first-order influential parameter. The reason why it is not the case might be due to the feedback loops of the Differential System containing proportional-integral (PI) controls. Indeed, the [Grasselli and Costa Lima, 2012] model reacts as a Closed Loop Transfer Function with proportional-integral control in its direct chain. In the field of mechanical engineering (and especially in process control) PI controls are known to constrain static perturbation on the inputs (such the ones of  $\alpha$ ,  $\phi_0$ , or  $\phi_1$ ). **On the employment rate**,  $\lambda_1$ , the most influential parameters are the labor productivity growth rate  $\alpha$ : 0.146, the investment function constant  $k_0$ : 0.139, the parameter of its exponential shape  $k_2$ : 0.125 and the depreciation rate  $\delta$ : 0.122. **On the debt ratio**,  $d_1$ , the most influential ones are again the capital-to-output ratio,  $\nu$ : 0.138, the population growth rate,  $\beta$ : 0.119, the labor productivity growth rate,  $\alpha$ : 0.106, and the investment function constant  $k_0$ : 0.094.

CHAPTER 2. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

Input	Description	First-order $S_i$			Total effect $S_i^T$			Infl. Rank
		$\omega_1$	$\lambda_1$	$d_1$	$\omega_1$	$\lambda_1$	$d_1$	#
$\alpha$	labor productivity growth rate	0.092 (0.0548)	0.146 (0.0966)	0.106 (0.0831)	0.827 (0.1498)	0.8995 (0.0938)	0.855 (0.0962)	3
$\beta$	population growth rate	0.118 (0.0533)	0.076 (0.0529)	0.119 (0.0947)	0.805 (0.1347)	0.763 (0.1640)	0.798 (0.1817)	4
$\delta$	depreciation rate	0.079 (0.0571)	0.122 (0.0592)	0.076 (0.0579)	0.900 (0.0592)	0.864 (0.1376)	0.917 (0.0424)	2
$\nu$	capital-to-output ratio	0.243 (0.0870)	0.094 (0.0745)	0.138 (0.0628)	0.925 (0.0548)	0.864 (0.1110)	0.910 (0.0617)	1
$\phi_0$	Philips Curve $\Phi(\lambda)$ parameter	0.00706 (0.0048)	0.032 (0.0440)	0.0064 (0.0064)	0.1464 (0.003)	0.156 (0.0875)	0.1458 (0.003)	10
$\phi_1$	Philips Curve $\Phi(\lambda)$ parameter	0.00707 (0.0048)	0.041 (0.0428)	0.0057 (0.0053)	0.1466 (0.003)	0.225 (0.2030)	0.1458 (0.003)	9
$k_0$	Investment function $\kappa(\pi)$ constant	0.095 (0.0699)	0.139 (0.0702)	0.094 (0.0597)	0.252 (0.3132)	0.254 (0.2733)	0.231 (0.2783)	8
$k_1$	$\kappa(\pi)$ parameter	0.063 (0.0432)	0.066 (0.0631)	0.077 (0.0581)	0.325 (0.3472)	0.347 (0.3466)	0.369 (0.3404)	6
$k_2$	$\kappa(\pi)$ exponential parameter	0.078 (0.0386)	0.125 (0.0929)	0.040 (0.0480)	0.427 (0.3952)	0.461 (0.3776)	0.428 (0.3824)	5
$r$	nominal interest rate	0.215 (0.0811)	0.102 (0.0710)	0.068 (0.0500)	0.308 (0.0493)	0.374 (0.2266)	0.146 (0.0030)	7

Table 2.3: Sobol' indices and standard deviations (with 10 bootstrap replicates) of all inputs (ranked from #1 the main influent parameter to #10 the less influent one) of the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wage share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$

As also exposed in their paper, by averaging the time series of the labour productivity growth rate (from 1990 to 2011 - World Bank and Penn World Data), a better approximation is obtained than with the OLS regression. This can give some confidence in the possibility to use a fixed parameter to estimate this parameter.

On the contrary, the capital-to-output ratio is hard to define and is undoubtedly variable (between 3 and 3.5 in the period 1990 - 2011 according the World Bank Data and Penn World Table.) and the depreciation rate,  $\delta$ , is often *calibrated as an educated guess* [Smets and Wouters, 2007]. One can imagine that such difficulty to estimate them can impede the robustness of the model.

In the next subsection 2.3.4, we will quantify more precisely the influence of these parameters and we will see in the next section 2.4, that the sensitivity of the model to these difficult-to-estimate

parameters is quite low compared to other more traditional models.

### 2.3.4 Convergence of the Sobol' indices of the model

After identifying the most influential parameters we need to refine our meshing to quantitatively assess their contribution to the variance of the outputs. Consequently we are exploring the most influential parameters,<sup>33</sup> by refining the meshing of this input's space following the method exposed in [Harenberg et al., 2017] to give a better assessment of the Sobol' indices. As in [Harenberg et al., 2017], we choose to stop this exploration when the standard deviation of every indices is less than 0.05.

This exploration is conducted with an increasing number of simulations. We need to go up to 500 simulations to ensure that the standard deviation of each indice is less than 0.05. The indices and standard deviations for 500 simulations are displayed in Table 2.4.<sup>34</sup> Figure 2.5 displays the value of each Sobol' indice and its standard deviation for each exploration of 200, 300, 400 and 500 simulations respectively.

N = 500 simulations		First-order $S_i$			Total effect $S_i^T$		
Input	Description	$\omega_1$	$\lambda_1$	$d_1$	$\omega_1$	$\lambda_1$	$d_1$
$\alpha$	labor productivity growth rate	0.088 (0.0470)	0.133 (0.0482)	0.105 (0.0344)	0.948 (0.0321)	0.933 (0.0125)	0.950 (0.0315)
$\beta$	population growth rate	0.077 (0.0421)	0.070 (0.0260)	0.090 (0.0376)	0.835 (0.0463)	0.869 (0.0454)	0.875 (0.0431)
$\delta$	depreciation rate	0.053 (0.0227)	0.100 (0.0248)	0.073 (0.0447)	0.919 (0.0421)	0.929 (0.0322)	0.938 (0.0171)
$\nu$	capital-to-output ratio	0.189 (0.0498)	0.060 (0.0457)	0.122 (0.0403)	0.935 (0.0377)	0.938 (0.0281)	0.942 (0.0304)

Table 2.4: Sobol' indices and standard deviations (with 10 bootstrap replicates and 500 simulations) of the main influent inputs of the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wage share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$

Figure 2.5a displays the convergence of the four most influential parameter on the three outputs ( $\omega_1, \lambda_1, d_1$ ). One can notice that the highest first-order Sobol' indices is obtained for the influence of the capital-to-output ratio,  $\nu$ , to the wage share,  $\omega_1$  (see also Figure 2.6) with an indice around 0.189 (with a standard deviation around 0.0498). Figure 2.5b, 2.5c and 2.5d are an extraction of Figure 2.5a and displays the results for the wage share, the employment rate and debt ratio respectively.

This result is important for the rest of the study to understand if this model is **qualitatively** more robust than others classical models (by comparing their highest first-order Sobol' indice for example), but also to see how we can improve the robustness of the [Grasselli and Costa Lima, 2012] model (by "endogenizing" the most influential parameter).

<sup>33</sup>Labor productivity growth rate  $\alpha$ ; population growth rate  $\beta$  ; depreciation rate  $\delta$  and capital-to-output ration  $\nu$ .

<sup>34</sup>See in appendix A.4 for the results of each exploration.



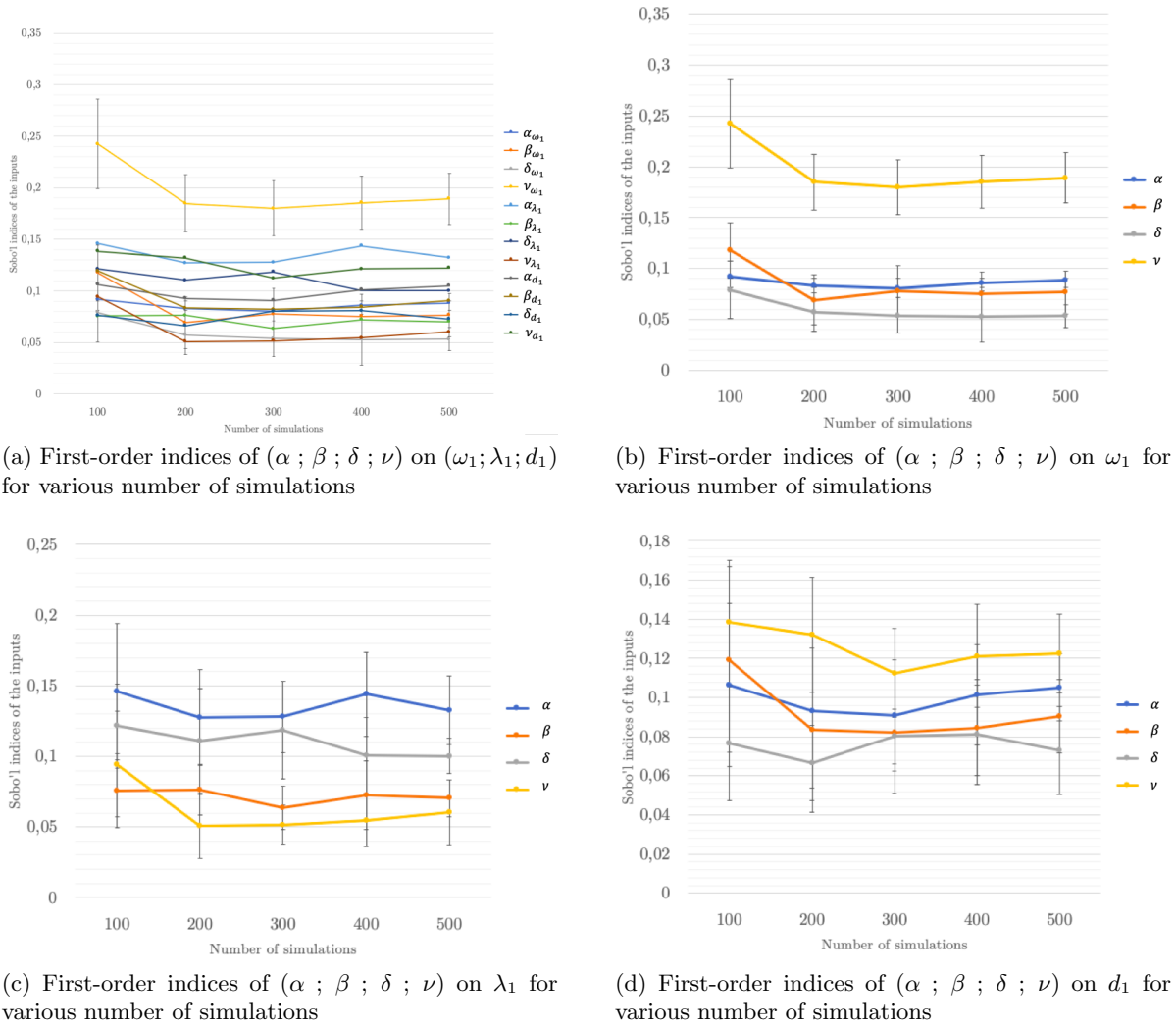


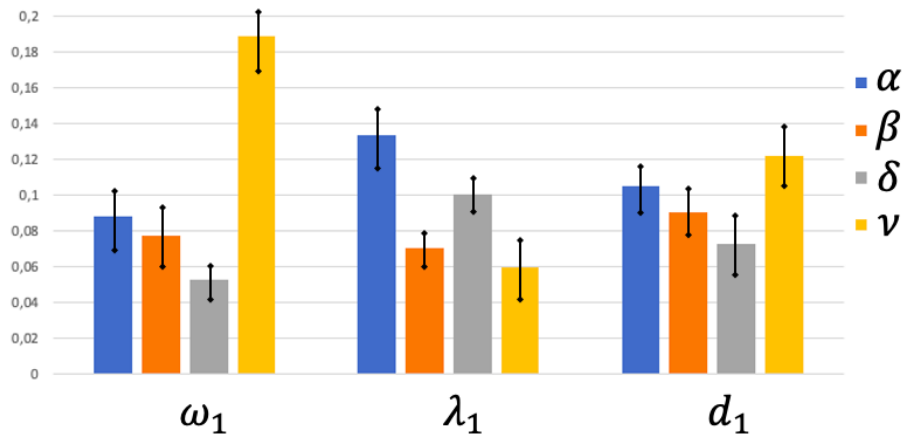
Figure 2.5: Convergence of the Sobol' indices of the main influential inputs  $(\alpha ; \beta ; \delta ; \nu)$  on  $(\omega_1; \lambda; d_1)$

Another remarkable result can be identified by studying the original [Goodwin, 1967] model:<sup>35</sup> in this simpler model, the most influential parameters are still the labor productivity,  $\alpha$ , the population growth rate,  $\beta$ , the depreciation rate,  $\delta$ , and the capital-to-output ratio,  $\nu$ . This result seems to confirm the idea that **refining this model**(e.g. by introducing a debt ratio as in [Grasselli and Costa Lima, 2012]) **does not fundamentally affect the nature of its sensitivity**.

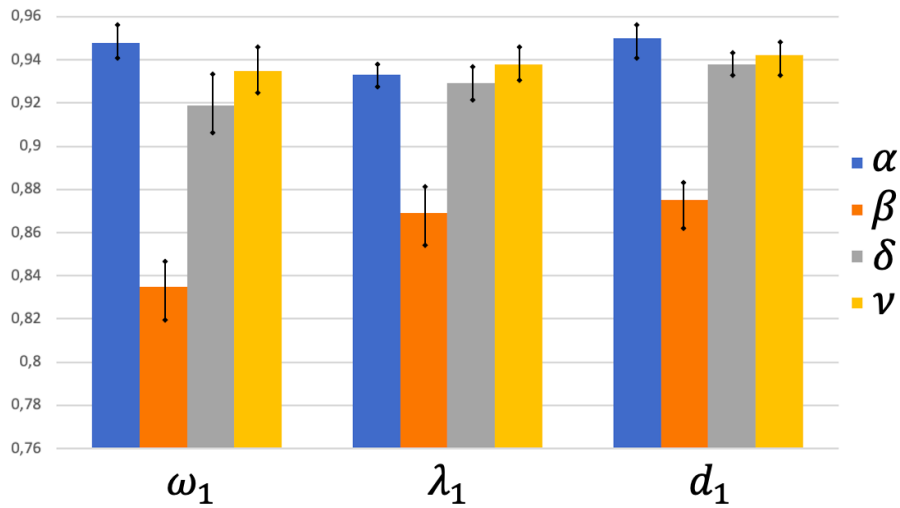
Since the maximum first-order Sobol' index of the model depends on the uncertainty on the capital-to-output ratio, one can decide to improve the robustness of the model by changing its production function from Leontief to CES in order to “endogenise” this capital-to-output ratio.<sup>36</sup> The results of this analysis shows that at first-order and for 100 simulations (10 bootstrap), the [Grasselli and Costa Lima, 2012] model with CES production is slightly more robust to its inputs than the same model with a Leontief production function. At a total-effect order, the quadruplet

<sup>35</sup>See appendix A.5 for the whole study.

<sup>36</sup>See appendix A.5.1 for an exhaustive analysis.



(a) First-order Sobol' indices and standard deviation of  $(\alpha ; \beta ; \delta ; \nu)$  on  $(\omega_1; \lambda_1; d_1)$



(b) Total-effect Sobol' indices and standard deviation of  $(\alpha ; \beta ; \delta ; \nu)$  on  $\omega_1$

Figure 2.6: Sobol' indices of [Grasselli and Costa Lima, 2012] for the most influential parameters and their standard deviation.

$(\alpha, \beta, \delta, \nu)$  is replaced by  $(\alpha, \beta, \delta, C)$  which seems coherent with the sensitivity of [Grasselli and Costa Lima, 2012] be it with a Leontief or a CES production function.



## 2.4 Comparison with other macro-economic models

In the economic literature, a number of sensitivity or uncertainties analysis are conducted with various methods (mainly Local or Bayesian approaches), but a very little number of them are Global Sensitivity Analysis (GSA). The European Commission in its [European Commission, 2009] report highlighted however its importance in order to conduct public policies. [Ratto, 2008], for the Joint Research Center of the European Commission, delivers a GSA with the DYNARE code [Adjemian et al., 2011] of a core DSGE model: the [Lubik and Schorfheide, 2007] model.

[Harenberg et al., 2017] and [Harenberg et al., 2019] are successful studies comparing different techniques to analyze sensitivity of a canonical Real Business Cycle (RBC) model, and especially through GSA. [Miftakhova, 2019] and [Miftakhova, 2021] applied the same GSA method (with Chaos Polynomial Expansion) about the most commonly employed Integrated Assessment Model (IAM), DICE [Nordhaus, 2008]. The author states that, when applied to a selection of parameters, the method produces results consistent with the conclusions of [Nordhaus, 2008]. Full analysis however changes drastically the ranking and the relative impact of the inputs with the most influential parameters being among those omitted from the aforementioned restricted setting (confirmed by [Anderson et al., 2014], [Butler et al., 2014]).

In this part, we qualitatively compare the first-order Sobol' indices of [Grasselli and Costa Lima, 2012] with these models. This comparison should be taken with caution as the quantities of interest of each model are different.

### 2.4.1 Review of the [Ratto, 2008] GSA about the [Lubik and Schorfheide, 2007] model

The [Lubik and Schorfheide, 2007] model is a a small-scale, structural general equilibrium model of a small open economy (SOE), in the variables aggregated output ( $y_t$ ), CPI (Consumer Price Index) inflation rate ( $\pi_t$ ), nominal interest rate ( $R_t$ ), terms of trade ( $q_t$ ), exogenous world output ( $y_t^*$ ), world inflation shock ( $\pi_t^*$ ) and nominal exchange rate ( $e_t$ ). It is a simplified version of the [Galián and Monacelli, 2005] model. As explained in [Lubik and Schorfheide, 2007], the model consists of a forward-looking (open economy) IS (Investment-Savings)-curve and a Phillips curve. Monetary policy is described by an interest rate rule, while the exchange rate is introduced via the definition of the CPI and under the assumption of Purchasing power parity (PPP). Specifically, the evolution of the SOE is determined by the following equations. The consumption Euler equation is rewritten as an open economy IS-curve:

$$y_t = E_t y_{t+1} - [\tau + \alpha(2 - \alpha)(1 - \tau)](R_t - E_t \pi_{t+1}) - \alpha[\tau + \alpha(2 - \alpha)(1 - \tau)]E_t \Delta q_{t+1} - \alpha(2 - \alpha) \frac{1 - \tau}{\tau} \Delta y_{t+1}^* - E_t z_{t+1} \quad (20)$$

where  $\alpha \in [0; 1]$  is the import share, and  $\tau$  the inter-temporal substitution elasticity,  $z_t$  is the growth rate of an underlying non-stationary world technology process  $A_t$ .<sup>37</sup>

<sup>37</sup>In order to guarantee the stationarity of the model, all real variables are therefore expressed in terms of percentage deviations from  $A_t$ . See [Lubik and Schorfheide, 2005] for further discussion of such a specification. Note also that optimal price setting of domestic firms leads to the open economy Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \alpha \beta E_t \Delta q_{t+1} - \alpha \Delta q_t + \frac{k}{\tau + \alpha(2 - \alpha)(1 - \tau)} (y_t - \bar{y}_t) \quad (21)$$

and  $\pi_t = \Delta e_t + (1 - \alpha) \Delta q_t + \pi_t^*$

Assuming monetary policy is described by an interest rate rule, the authors includes nominal exchange rate depreciation  $\Delta e_t$  in the policy rule:

$$R_t = \rho_R R_{t-1} + (1 - \rho_R)[\psi_1 \pi_t + \psi_2 y_t + \psi_3 \Delta e_t] + \varepsilon_t^R \quad (22)$$

where  $\psi_1, \psi_2, \psi_3 > 0$  are policy coefficients,  $\rho_R \in [0, 1]$  a smoothing persistence term and  $\varepsilon_t^R$  an exogenous policy shock.

The GSA method applied in [Ratto, 2008] is a High Dimensional Model Representation (HDMR) [M. Sobol, 1990] of the DSGE model described as:

$$E_t(g(y_{t+1}; y_t; y_{t-1}; u_t; \mathbf{X})) = 0 \quad (23)$$

where  $y_t$  is the vector of endogenous variables,  $u_t$  the vector of exogenous shocks, and  $\mathbf{X} = (X_1; \dots; X_p)$ , the array of structural parameters they study. They estimate the state dependent regression coefficients of their truncated HDMR, by a procedure of recursive filtering and smoothing estimation (Kalman filter) based on a Monte-Carlo sampling of their data.

The study shows the use of the HDMR to characterize the relationship between the reduced form coefficients of this rational expectation model and the structural coefficients. They explain that “*the polynomial parameterization of the SDR estimation of the HDMR provides the analyst with a direct, albeit approximated, analytic representation of the solution procedure of rational expectation models*”.

They analyse in particular the reduced form coefficients describing the relationships between  $\pi_t$  vs.  $R_{t-1}$ . The conclusions of their study are that their SDR estimation explains more than 99% of the total variation of the log-transformed coefficient (within sample).

The largest impact is given by  $\rho_R$  (smoothing persistence term of the nominal interest rate), with a first-order Sobol’ indices  $S_i = 0.896$ , followed by  $k$  (a slope coefficient treated as structural parameter even if it is defined as a function depending on underlying parameters capturing the degree of prices stickiness, labor supply and demand elasticities) with a  $S_i = 0.06$  and  $\psi_1$ , a policy parameter, with  $S_i = 0.04$ . All the remaining parameters have an irrelevant effect.

As a conclusion, we can underline that a model with a first-order Sobol’ indices equal to 0.9 is highly likely to be sensitive to uncertainties on the input responsible for this variation. If the alluded input is moreover a smoothing persistence parameter hard to estimate empirically, one can wonder about the global robustness of such a model and its recommendations, especially when working to orientate public policies of the European Commission. Of course, we can hardly compare our results to this study, as the quantity of interest ( $\pi_t$  vs.  $R_{t-1}$ ) is very different from ours.<sup>38</sup>

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where  $\bar{y}_t = -\alpha(2 - \alpha)(1 - \tau)/\tau y_t^*$  is potential output in the absence of nominal rigidities, and  $k > 0$  a slope coefficient, depending on underlying structural parameters, such as labor supply and demand elasticities and parameters capturing the degree of price stickiness. In order to study exchange rate policies, the authors introduce the nominal exchange rate  $e_t$  via the definition of the CPI and assumed that relative PPP holds, leading the previous second equation, where  $\pi_t^*$  is a world inflation shock, treated as an unobservable.

<sup>38</sup>Sobol’ indices of the [Grasselli and Costa Lima, 2012] model are less than 0.2 for each quantity of interest.

### 2.4.2 Review of the [Harenberg et al., 2017] GSA about a canonical RBC model

The aim of this article is to transfer the recent advances in sensitivity analysis methods from the engineering and mathematic fields to economics. It gathers and compares different methods and especially the GSA with Sobol' indices, applied to a canonical Real Business Cycle model with capital adjustment cost. The model was widely studied (see e.g. [Aruoba et al., 2006] or [Den Haan et al., 2011]) and focus, following the economic literature, on two quantities of interest: average production and the ratio of the variance of log production in the model over its empirical counterpart. The estimation of the indices relies on the chaos polynomial expansion (CPE) approximation to identify interactions and calculate statistical moments as well as the probability density of the quantities of interest. The allocation problem is described by a dynamical optimization:

$$V(k_t; a_t) = \max_{\{c_t; l_t; i_t\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t \frac{(c_t^\chi (1-l_t)^{1-\chi})^{1-\frac{1}{\tau}}}{1-\frac{1}{\tau}} \quad (24)$$

where  $c_t$  is the consumption,  $1-l_t$  is leisure in each period,  $\beta$  the discount factor,  $\tau$  the intertemporal elasticity of substitution (IES), and  $\chi$  the share parameter in the composite commodity. The decision variables are consumption  $c_t$ , labor  $l_t$ , and investment  $i_t$ . The aggregate resource constraint is given by

$$q_t = c_t + i_t + \frac{\phi}{2} k_t \left( \frac{i_t}{k_t} - \delta \right)^2 \quad (25)$$

with  $q_t$  the quantity of produced goods,  $k_t$  the capital stock, and  $\delta$  its depreciation rate. The production technology is assumed to be:  $q_t = e^{a_t} k_t^\alpha l_t^{1-\alpha}$  depending on  $a_t$  the productivity of the factors, with a technical substitution rate  $\alpha$ .<sup>39</sup>

The inputs of the study are the seven parameters of the system: the discount factor  $\beta$ , the IES  $\tau$ , the share parameter in the composite commodity  $\chi$ , the technical substitution rate  $\alpha$ , the depreciation rate  $\delta$ , the auto joint effects coefficient  $\rho$ , strength of adjustment cost  $\phi$ , the standard deviation of the normally distributed shock  $\sigma$ .

Two quantities of interest are observed as outputs: average production,  $Q_1 = \mathbb{E}[q_t]$ , and the variance of log production in the model over its empirical counterpart  $Q_2 = \frac{1}{\sigma_q^2} Var[\log(q_t)]$  with  $\sigma_q^2$  the empirical variance of log production in the data (commonly set at 0.019 in the literature). The study displays the first-order and total-effect Sobol' indices of each input in relation with the two quantities of interest (see [Harenberg et al., 2017] and the appendix of [Harenberg et al., 2019]).

In [Harenberg et al., 2017], the main influential parameter on the average production,  $\mathbb{E}[q_t]$ , is the capital share  $\alpha$ , with a first-order Sobol' indice about to 0.75 and a total-effect Sobol' indice about 0.85. Here again, it is a high sensitivity for a model to a single parameter. One can also notice that there is no big differences between total-effect and first-order Sobol' indices in this model, which means that it is not very non-linear.

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<sup>39</sup>Capital adjustment cost are modelled as in [Den Haan et al., 2011] with  $\phi$  governing the size of cost. The capital transition and stochastic productivity process are given by

$$\begin{aligned} k_{t+1} &= i_t + (1-\delta)k_t \\ a_{t+1} &= \rho a_t + \varepsilon_{t+1} \end{aligned} \quad (26)$$

where  $\rho$  is an auto joint effects coefficient and  $\varepsilon_t$  a normally distributed shock (std. deviation:  $\sigma$ ).

### 2.4.3 Review of the [Miftakhova, 2019], [Butler et al., 2014] and [Anderson et al., 2014] GSA about DICE [Nordhaus, 2008]

Parameter uncertainty in Integrated Assessment Models (IAM) is highly debated but only partially explored in the literature: expert opinions on the importance of particular inputs (let alone their probability distributions) often diverge, hampering policy assessment and the decision-making process ([Tol, 1965], [Heal and Millner, 2014], [Athanasoglou, 2014]).

In this study, the authors implement a GSA with Sobol' indices or density sensitivity indices about the most commonly employed IAM: DICE [Nordhaus, 2008]. They compared their results to the GSA developed in [Nordhaus, 2008]. The authors demonstrate the application of the method, when applied to the same selection of parameters, produces results consistent with the conclusions of [Nordhaus, 2008]. Full analysis, however, changes the ranking dramatically, with the most influential parameters being among those omitted from the aforementioned restricted setting. We also show that caution is needed when analyzing the full set of parameters because the credibility of the results relies on the independence assumption.

DICE is a neoclassical growth model that resolves a trade-off between consumption, investment, and emissions reduction. Given a discount rate  $\rho$ , the model maximizes total social welfare over its time span,

$$W = \sum_{t=1}^{T_m^{ax}} (1 + \rho)^{-t} U(C_t, L_t) \quad (27)$$

with a constant-elasticity  $\alpha$  of the utility function of consumption  $C_t$ , and labor  $L_t$ .

$$U(C_t, L_t) = \frac{(C_t/L_t)^{1-\alpha} - 1}{1-\alpha} L_t \quad (28)$$

Production function takes the standard Cobb-Douglas form with Hicks neutral total factor productivity (TFP)  $A_t$ , labor  $L_t$ , capital  $K_t$ , as input factors. Total output  $Y_t$  is reduced by the cost of mitigation, a power function of emission reduction policy  $\mu$ , and by the damage induced by the climate change,  $\Omega_t$ ,

$$Y_t = \Omega_t (1 - \theta_1 \mu^{\theta_2}) A_t L_t^{1-\gamma} K_t^\gamma \quad (29)$$

where the reduction in economic output caused by climate change is a function of the atmospheric temperature anomaly  $T_{A_t}$  (measured as the increase in the atmospheric temperature from the preindustrial level),

$$\Omega_t = \frac{1}{1 + a_1 T_{A_t} + a_2 T_{A_t}^3} \quad (30)$$

Given a depreciation rate  $\delta$ , and investment  $I_t$ , capital accumulates according to:

$$K_t = (1 - \delta)K_{t-1} + I_{t-1} \quad (31)$$

The production of goods affects the climate system by introducing anthropogenic  $CO_2$  emissions at a rate determined by the emissions intensity  $\sigma_t$  and reduced by the chosen rate of mitigation  $\mu_t$ , the total emissions levels is the sum this anthropogenic and a natural source  $E_{tree_t}$

$$\begin{aligned} E_t &= E_{CO_2t} + E_{tree_t} \\ E_{CO_2t} &= \sigma_t (1 - \mu_t) A_t L_t^{1-\gamma} K_t^\gamma \end{aligned} \quad (32)$$

The core of the climate module is its carbon cycle with a three-layer model (atmospheric, upper layer and lower ocean of  $CO_2$  concentrations) with transitions among layers:

$$\begin{aligned} M_{AT_{t+1}} &= E_t + b_{11}M_{AT_t} + b_{21}M_{U_t} \\ M_{U_{t+1}} &= b_{12}M_{AT_t} + b_{22}M_{U_t} + b_{32}M_{L_t} \\ M_{L_{t+1}} &= b_{23}M_{U_t} + b_{33}M_{L_t} \end{aligned} \tag{33}$$

The net change in atmospheric  $CO_2$  concentrations indices the net energy flow to the atmosphere (called radiative forcing,  $F_t$ ),

$$F_t = \eta \log_2 \left( \frac{M_{AT_t} + M_{AT_{t+1}}}{2M_{AT_{1750}}} \right) + F_{EX_t} \tag{34}$$

where  $F_{EX_t}$  is the exogenous radiative forcing. This energy causes a change in the atmospheric temperature,  $T_{A_t}$ , and, subsequently, the ocean temperature,  $T_{O_t}$ ,

$$T_{A_{t+1}} = T_{A_t} \tag{35}$$

[Butler et al., 2014] displayed how all inputs can be represented in a synthetic figure. Respecting the selection made by [Nordhaus, 2008], [Miftakhova, 2019] and [Miftakhova, 2021] preselects the same eight input parameters that presumably affect the inference from the model the most, and assume that they are normally distributed, as summarized in Table 2.5.

Parameter	Definition	Mean	Std
$a_2$	Damage function coefficient	0.0028	0.0013
$b_{12}$	Carbon cycle transition	0.189	0.017
$g_{A0}$	Initial growth rate of TFP	0.092	0.040
$g_0$	Initial change of decarbonization	-0.07	0.02
$CCum$	Maximum extraction of fossil fuels	6,000	1,200
$pBACK$	Cost of backstop technology	1.17	0.468
$L_{ASYM}$	Asymptotic population	8,600	1,892
$\lambda$	Equilibrium climate sensitivity	3.00	1.11

Table 2.5: Parameters selected for the analysis and their distributions (from [Nordhaus, 2008]).

For these eight parameters, the results of first-order and total-effect Sobol' indices are consistent with [Nordhaus, 2008]: the damage function coefficient,  $a_2$ , contributes to the variance of the output the most, followed by climate sensitivity,  $\lambda$ , and the initial growth rate of TFP,  $g_{A0}$ . Asymptotic population size,  $L_{ASYM}$ , plays a far less significant role, while the effect of the uncertainty in the rest of the parameters is negligible.

As explained by [Miftakhova, 2019], the inference above relies on the preliminary subjective selection of a set of parameters that potentially have a strong effect on the SCC (Social Cost of Carbon). Performing a generalized experiment by treating all parameters equally and assuming that we do not have any information on their probability distributions, leads to very different results. The computed Sobol' indices reveal a notable difference to the previous, restricted setting. The biggest effects are indeed attributed to the damage function exponent,  $a_3$  (first-order Sobol' indice about 0.3), capital elasticity,  $\gamma$  (about 0.16), and elasticity of marginal utility of consumption,  $\alpha$ , (about 0.12) all three missing in the subset selected for the previous analysis.

The resulting ranking therefore suggests that it can be highly misleading to rely on a prior subjective judgment about the importance of input parameters. To infer a model's sensitivity to the uncertainty in its input factors it is necessary to consider their complete set and discriminate among the parameters based on a full quantitative decomposition of uncertainty in the model. The results of [Miftakhova, 2019] are in agreement with those of [Anderson et al., 2014].

We can not directly compare these various studies in terms of Sobol' indices, as the outputs are not the same, but one can notice that the outputs of some of these models are highly sensitive (between 0.55 to 0.9 for the main quantities of interest of DSGE and RBC) to their inputs.<sup>40</sup>

A more precise study of the DICE model reveals that not only the ranking, but also the values of the GSA of [Nordhaus, 2008] are biased, due to a prior selection of parameters [Butler et al., 2014], [Anderson et al., 2014], [Miftakhova, 2019]. These new studies deliver values of maximal Sobol' indices about 0.3 when studying the output: Social Cost of Carbon 2015. But here again, the main influential input is arbitrary imposed (damage function exponent  $a_3$  in [Nordhaus, 2008]) which can be drastically different (see [Bovari et al., 2018] for a comparison of various damage functions).

## 2.5 Conclusion

In this study, we developed a sensitivity analysis of the parameters of a macro-economic model. We study the relative influence of uncertainties on all inputs' parameters of our model: the [Grasselli and Costa Lima, 2012] with a Leontief production function. We selected a variance-based Global Sensitivity Analysis (GSA), because this model contains a large number of nonlinearities and *a priori* independent inputs. Due to the small number of inputs of the model (and consequently its low computational cost), we chose the Sobol' indices method to conduct our GSA according to the decision tree of [De Rocquigny, 2008]. The calculation of the Sobol' indices is based on combinations of inputs values obtained from a Design of Experiment (DoE): an Optimal Latin Hypercube Sampling, according to the [Saltelli, 2002] method of estimation.

At a first step, we regarded all the possible inputs of the model and observed their impact on three outputs: the *Good* Equilibrium value of the wage share, the employment rate and the debt ratio of the economy. The most influential parameters on these outputs are the labor productivity growth rate, the population growth rate, the depreciation rate and the capital-to-output ratio. The last one has the maximal first-order Sobol' indice (less than 0.2). These values are not very dependent on the number of simulations as we show it by successively increasing the number of simulations until the satisfaction of a certain criterion about the standard deviation of our results (less than 0.05).

One can notice here that two of these most influential parameters can be assessed with a certain confidence as detailed in [McIsaac, 2016] (from demographic data from United Nations time series for the population growth rate and by averaging the time series of the labour productivity growth rate). On the contrary, the capital-to-output ratio and the depreciation rate are hard to assess that can impede the robustness of the model.

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<sup>40</sup>By the way, these inputs (discount factor  $\beta$ , capital share  $\alpha$ , auto joint effects coefficient  $\rho$  in [Harenberg et al., 2017] or smoothing persistence of the interest rate  $\rho_R$  in [Ratto, 2008]) are generally hard to estimate, what can lead consequently to questionable results about the prediction of these models.



Then we displayed various similar GSA about DSGE, RBC and IAM model, and their maximal first-order Sobol' indices. We can not directly compare these results as the outputs and inputs of each study are different, but we can question the reliability of the information delivered by these models highly sensitive on an arbitrarily fixed parameter.

Indeed, our work has identified the most influential parameters of the model [Grasselli and Costa Lima, 2012] to help modelers refine their work when building predictions with this model. On the contrary of DSGE or RBC model of [Ratto, 2008] and [Harenberg et al., 2017] (which are very dependent on a single parameter), we show that the sensitivity of the [Grasselli and Costa Lima, 2012] model is more balanced between four parameters (the capital-to-output ratio, the depreciation rate, the population growth rate and the labor productivity growth rate).

We can make no quantitative conclusions about the relative robustness of a model compared with an other one, because their respective robustness depends on the uncertainty on the most influential parameters (the smoothing persistence for [Ratto, 2008], the capital share for [Harenberg et al., 2017] and the four parameters of [Grasselli and Costa Lima, 2012]).

However, we can notice that the model observed in [Ratto, 2008] and [Harenberg et al., 2017] are highly sensitive to a single parameter that is generally hard to assess. Modelers can now choose to focus their efforts on refining the value of this parameter hard-to-assess when working with these type of models or they can decide to work with [Grasselli and Costa Lima, 2012] where large uncertainties in each of the four influential parameters in [Grasselli and Costa Lima, 2012] will be less likely and have an equal impact on the variance of the output (the maximum share of the impact is 0.19).

A prospect of the study can be to improve this balanced sensitivity of our model. One can think about endogeneizing the most influential parameters (e.g. capital-to-output ratio and labor productivity growth rate) either by using different production function, such as a Constant Elasticity Substitution (CES) function (see [McIsaac, 2016] for the use of the Van der Ploeg's extension) or a Putty-Clay production function (see [Akerlof and Stiglitz, 1969]) instead of our Leontief production function ; or by developing the model with an endogenous growth.

The study of the [Grasselli and Costa Lima, 2012] model with a CES production function is provided in appendix A.5.1. At a total-effect level (i.e. with all joint effects between inputs), the main influent parameters are the same as before. At a first-order level, the main influent parameter on the wage share is now the nominal interest rate. An improvement of these performances is possible by introducing a Taylor' rule to endogeneize it.

Eventually we also studied the original [Goodwin, 1967] model (see appendix A.5) and noticed that the sensitivity of the original model depends on the same parameters as its extension. This is quite reassuring that refining the model does not seem to affect the deep nature of its sensitivity as we can improve it by multiple features about money, inventory or different natures of capital.

We showed that the performance of the original [Goodwin, 1967] in terms of Sobol' indices are slightly better than the [Grasselli and Costa Lima, 2012] model, which might be due to the smaller number of nonlinearities. These results might make us wonder about a possible compromise between the robustness and the *realism* of a model to find an equilibrium between a very robust model unable



to approximate reality (such as the [Goodwin, 1967] model), and a very fitted and very sensitive model (such as the DSGE of [Ratto, 2008]). From this point of view, the [Grasselli and Costa Lima, 2012] model seems to be an interesting compromise, with a finer description of the economy and a more balanced sensitivity.

## Chapter 3

# Money Velocity in an Imperfectly Competitive, Stock-Flow Consistent Dynamics

### Abstract

So far, the analytics of money has been cluttered by a number of paradoxes and impossibility results obstructing access to a proper understanding of the interaction between money, its velocity and macro-dynamics.

We propose a stock-flow consistent dynamics in continuous time with imperfect competition where these paradoxes can be solved. Money turns out to be non-neutral both in the short- and the long-run. Its creation by credit banking allows its quantity circulating in the economy, as well as the speed at which it circulates, to be endogenously determined by the need to finance investment and consumption. In particular, the decline in money velocity observed in several countries in the aftermath of the Great financial crisis can be analyzed.

Our set-up sheds new light on the links between money velocity and the debt-deflationary path on which several countries seem to have been progressing despite unconventional monetary policies. Low inflation, low real income growth, increasing private debt and a declining velocity of trades turn out to be the hallmark of the macro-economic trajectories leading to a debt-deflationary long-run crisis. We provide conditions under which, however, a low-interest rate monetary policy can stimulate economic activity so as to escape from the liquidity trap.

### 3.1 Introduction

From 2008:Q1 to 2019:Q3, the  $M1$ -velocity of trades in the US economy, measured as the number of times a US Dollar has changed hand within one year through some trade in the real economy<sup>1</sup>, has shrunk from more than 10 to less than 6. This decline is well approximated by an exponential decreasing at the constant rate -1.4% (see Figure 3.1). Another example of a declining velocity is provided by the Czech Koruna over the period 2008-2018, which fell from 2.5 to 1.5, hence losing 40% (see Figure 2).

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<sup>1</sup>Income velocity is measured, here, as the ratio between nominal GDP and the monetary aggregate,  $M_1$ .

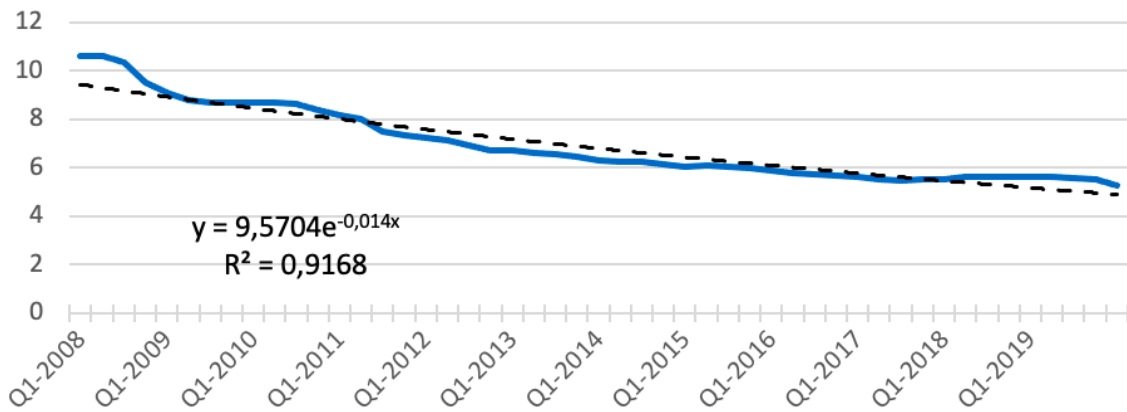


Figure 3.1: **Income velocity in US Dollars, 2008-2019.**

More generally, during the ten years following the Great Financial Crisis, the following OECD currencies have experienced an average slow-down of -1.3% in the speed at which they circulate in their respective real economy: the US Dollar, the Euro, the Australian and the Canadian Dollar, the Swiss Franc, the Swedish Krona, the Chilean Peso, the Korean Won, the Turkish Lira, the Czech Koruna and the Zloty. Significant exceptions are provided by the Iceland Krona, the Indian Rupee, the Brazilian Real, the Norwegian Krone and the Russian Ruble.

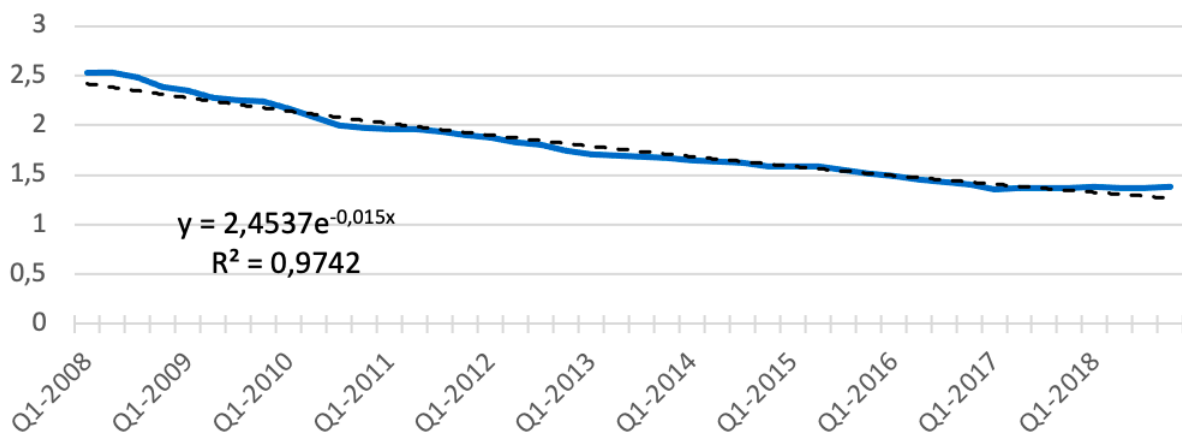


Figure 3.2: **Income velocity in Czech Koruna, 2008-2018.**

This illustrates the first of three stylized facts that seem to have characterized several OECD countries in the aftermath of the Great Financial Crisis:

1. A sharp decline in money velocity,
2. A strong growth of the ratio private debt/GDP, and
3. A stagnation of real GDP itself — or, at best, a slow real growth rate compared to the paces observed up to 2007.

Indeed, most of these countries experienced an significant increase of the private debt-to-GDP ratio: Australia, Canada, Poland, Sweden, Switzerland, Turkey, Chile and 9 Euro Zone countries (Austria,

Belgium, France, Finland, Greece, Ireland, Luxembourg, Netherlands, Slovakia) have gone through an exponential increase of this ratio at the average rate of 2.8% between Q1 2008 and Q1 2014. To give just one example: in Canada, this ratio jumped from about 175% to more than 250% within 6 years (see Figure 3).

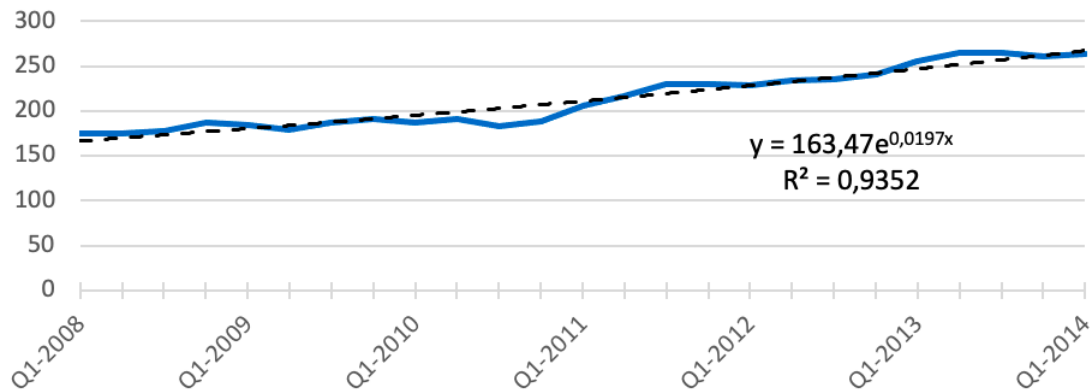


Figure 3.3: **Private debt/GDP Canada, 2008-2014.**

In order to put these magnitudes into perspective, we present a macro-dynamics which can be viewed as a monetary extension of the Solow model (see [Van der Ploeg, 1985] and [Bastidas et al., 2019]).<sup>2</sup> At variance with the celebrated workhorse, here, the phase space admits at least two long-run equilibria: the first one is but the embedding within our setting of the unique Solow equilibrium; the second steady state is a debt-deflationary one *à la* [Fisher, 1933] where an overhang of private debt leads to a sharp decline of money velocity and ultimately a fall in GDP. [?] had provided a full characterization of debt-deflation and the liquidity trap induced by an excess of private debt but their analysis remained entirely confined to a static equilibrium framework and money was not viewed as being endogenously created by banking credit. Here, by contrast, we provide a narrative of the *path* ultimately leading to a liquidity trap, along which several countries may have been traveling in the recent past — and, as we shall shortly see, money exclusively arises from endogenously determined banking credit.

A simple calibration of our model leads to evolution rates of both money velocity and debt ratios that are quite close to the ones recently observed in the countries mentioned above.<sup>3</sup> If confirmed, it would mean that these countries are converging towards a debt-deflationary equilibrium.

### 3.1.1 Paradoxes associated with money

The theory surrounding money, however, is notoriously paved with various paradoxes which, so far, have not been solved.

#### A. Banking credit creation

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<sup>2</sup>See also [Grasselli and Costa Lima, 2012], [Costa Lima and Grasselli, 2014], [Grasselli and Nguyen-Huu, 2018], [Giraud and Grasselli, 2019] and [McIsaac, 2016].

<sup>3</sup>Of course, there are exceptions e.g. Hungary, Island, Israël, Japan, Russia New Zealand and 5 Euro Zone countries (Germany, Portugal, Latvia, Slovenia, Spain) but this seems to provide a pattern.

First, as shown by [Werner, 2016],<sup>4</sup> during the past century, three different theories of banking were dominant at different times: (1) The currently prevalent financial intermediation theory of banking says that banks collect deposits and then lend these out, just like other non-bank financial intermediaries. (2) The older fractional reserve theory of banking says that each individual bank is a financial intermediary without the power to create money, but the banking system collectively is able to create money through the process of ‘multiple deposit expansion’ (the ‘money multiplier’) fueled by the money created by the central bank. (3) The credit creation theory of banking, predominant a century ago, argues that each individual bank creates credit and money newly when granting a bank loan. Since according to the dominant financial intermediation theory, banks are virtually identical with other non-bank financial intermediaries, they are usually neglected in the models used in economics or by central bankers. “Should this theory not be correct, as writes [Werner, 2016], currently prevailing economics modelling and policy-making would be without empirical foundation.” Confirming what had already been assessed by [Werner, 2014a], [Werner, 2016] proves that the financial intermediation and the fractional reserve theories of banking are rejected by the evidence.

As concluded by [Werner, 2016], “one of the implications of this study is that it does not make much sense to build economic theories of the financial sector, if these are not based on institutional (and accounting) realities. The role of accounting and law in economics should be increased, both in research and in the teaching of economics. This includes the role of national income accounting and flow of funds information [...], which have to be reconciled with those records of the banks. These are not only the ‘central settlement bureau, a kind of clearing house or bookkeeping centre for the economic system?’ ([Schumpeter and Nichol, 1934], p. 124), but also the creators and allocators of the money supply. The reflection of empirical bank reality within theories and textbooks surely must become the ‘new normal’ in finance and economics.”

This paper aims at paving the road towards such a renewal of economic modelling by endorsing the credit creation theory of banking within a stock-flow consistent setting.<sup>5</sup>

### *B. Instability of inflation*

Second, monetarist theories assume that money supply can anchor prices through the famous equation of exchange first expressed by ([Fisher, 1912]):

$$M\mathbf{v} = pT, \tag{1}$$

where  $M$  is money,  $p$  is the aggregate price index,  $T$  is real output, and  $\mathbf{v}$  stands for money velocity, measuring how rapidly the value of output  $pT$  “turns over” with respect to the money supply,  $M$ . Most monetarist models assume from the outset that  $\mathbf{v}$  is an increasing function of the inflation rate,  $i$ . At first glance, this seems all the more natural as the return to holding money is  $-i$ . However, as emphasized, e.g., by [Taylor, 2009], pp. 70 sq, this leads to a seemingly insurmountable puzzle. Indeed, differentiating (1) and rearranging, one gets

$$\frac{di}{dt} = \frac{\mathbf{v}}{d\mathbf{v}/di} (i + g - \hat{M}), \tag{2}$$

where a hat signifies growth rate. Suppose that  $T$  and  $M$  are predetermined (or determined by market forces that are independent from  $\mathbf{v}$  and  $p$ ), then imposing  $d\mathbf{v}/di > 0$  induces an unstable dynamics of inflation: a higher value of  $i$  increases  $di/dt$ , which feeds back into a further increase

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<sup>4</sup>See also [Werner, 2014a], [Werner, 2014b], [Werner, 2014c].

<sup>5</sup>See [Godley and Lavoie, 2006] for stock-flow consistent macro-modelling.

in  $i$ . In other words, apart in some degenerate situations, this viewpoint necessarily leads to the conclusion that hyperinflation is unavoidable. Models with perfect foresight therefore try to avoid this stumbling block by imposing from the outset that asset holders will rationally anticipate an inflation path along which  $\mathbf{v}$  is constant and  $di/dt = 0$ , so that (2) reduces to

$$i = \hat{M} - g.$$

This argument, however, has to face yet another paradox, namely the result provided by [?] showing that such a “perfect foresight” inflation path is not locally stable: any mistake by any asset holder in the formation of expectations along this path will lead to another inflation path from which no perfect foresight expectations can jump back to the initial trajectory. Generically, therefore, economic actors will fail to coordinate on this “perfect foresight” inflation path.<sup>6</sup> As a consequence, the economy will sooner or later leave this “rational” trajectory — and shall never return to it, being unavoidably trapped by some hyperinflation spiral, courtesy of Eq. (2).

Next, money velocity is not observed to be wildly unbounded: even in chronically inflationary Brazil, the ratio of GDP to money supply reached a level of around 65 in the mid-1990s, which may have been the upper-bound ever recorded in history. This empirical observation is yet another characteristic needing explanation. Indeed, by contrast, Eq. (1) and (2) do not provide any upper-bound on  $\mathbf{v}$ . Is there any mechanism, hidden behind these two equations, that would give an implicit cap on the speed of money circulation?

### C. Money neutrality?

Finally, monetary analysis has been plagued for decades by an ongoing debate on the alleged neutrality of money. Money is said to be neutral when a once-and-for-all change in the money supply or money demand has no real effects. Money is super-neutral when a change in the *growth rate* of its supply (or demand) has no real effect. And money is non-neutral when a change in its supply or demand does have real effects. Current mainstream wisdom claims that<sup>7</sup>

- 1/ Money is neutral in the long run.
- 2/ Money is approximately super-neutral in the long run, but not exactly so. This is especially true if there are other distortions such as taxes on nominal investment income.
- 3/ Money is strongly non-neutral in the short run, as monetary shocks affected real wages, real output, employment, real interest rates, debt defaults, and many other real variables. The short run can last for years.

Statement 3/ is confirmed by empirical investigations.<sup>8</sup> The difficulty with 1/ and 2/, of course, is that, if money is neutral in the long-run, it implies that any real short-term effects are ultimately reversed. To be convincing, a sensible monetary theory should be able to describe how this reversal happens across time. How long is the long-run? We are not aware of any theoretical framework compatible with the credit creation theory of banking that would highlight a mechanism allegedly responsible for turning short-term non-neutral money into being long-run neutral. In this paper, we show that within our framework, 3/ is confirmed but neither 1/ nor 2/ hold: money is non-neutral at every time scale — and even with perfectly flexible prices.<sup>9</sup>

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<sup>6</sup>See also [Guesnerie, 2005].

<sup>7</sup>See, e.g., for a good synthesis of this standpoint.

<sup>8</sup>For classical references, see [Friedman M., 1963], the Volcker disinflation, and [Mussa, 1986].

<sup>9</sup>More often than not, short-run money non-neutrality obtains thanks to some price stickiness or wage rigidity (see, e.g., [Mankiw and Romer, 1991] for a textbook illustration). Here, such an expedient is superfluous: both prices and wages are perfectly flexible.

Throughout the paper, we have tried to cast the discussion within the simplest possible setting. Nonetheless, among the not completely standard features of our model is the fact that we need to drop Say’s law so as to be able to consider the endogenous dynamics of inventories. The rationale for this is the following. As observed by [Pottier and Nguyen-Huu, 2017], when combined with usual accounting identities, Say’s law makes the monetary interpretation of any dynamics akin to the one studied here quite controversial: everything goes as if firms would borrow “money” out of households’ savings (or vice versa), so that “money” could equally well be replaced by a consumption good or by some numéraire. In other words, despite the formal invocation of “money”, one would still deal with the modelling of a “real” (moneyless) economy. As we shall see, decoupling effective demand from the supply forces turns out to be a necessary condition for the genuine emergence of money: at each point of time, the stock of inventories is then modified proportionately to the gap between saving and debt, which allows for money creation (or destruction).

### 3.1.2 Monetary policy and persistence

In the years following the Great Recession, the United States, Japan as well as several European countries have turned to unconventional monetary policies to stimulate economic activity. After two decades of slow growth, the Bank of Japan decided to employ a zero interest rate policy (ZIRP) to combat deflation and promote economic recovery [Kuroda, 2016]. A similar policy has been implemented by the US and UK.<sup>10</sup> Under ZIRP, however, the central bank can no longer reduce interest rates, rendering conventional monetary policy ineffective — so goes the argument. On the other hand, as seen in the Eurozone, over-extending a zero interest rate policy can also result in negative interest rates.<sup>11</sup> Thus, several authors have challenged the value of zero interest rate policies, pointing to liquidity traps amongst several other pitfalls.<sup>12</sup> Our analysis confirms the primary benefit of low interest rates to be their ability to stimulate economic activity. We identify conditions under which, near-zero interest rates lower the cost of borrowing, hence spur spending on business capital and investments. Businesses’ increased capital spending then creates jobs and consumption opportunities. Under the identified conditions, there is no need for the central bank to set an implied negative nominal interest rate, where loans actually receive interest.

To prove this result, we adopt the persistence viewpoint. Persistence theory (see [Smith and Thieme, 2011] and [Costa Lima and Grasselli, 2014]) studies the long term behaviour of dynamical systems, in particular the possibility that one or more variables remain bounded away from zero. It has been successfully applied by [] in order to assess the impact of public spending in a moneyless economy. Here, we show that a near-zero interest policy can guarantee that the employment rate does not remain indefinitely trapped at arbitrarily small values. This is in sharp contrast with what happens in the model without monetary policy intervention, where the employment rate is guaranteed to converge to zero and remain there forever if the initial conditions are in the basin of attraction of the debt-deflationary long-run steady state. Furthermore, as with any persistence result, Proposition 1 is a global one: no matter how disastrous the initial conditions are, a sufficiently responsive monetary policy can bring the economy back from a state of crisis associated with zero employment rates.

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<sup>10</sup>See Bank of England, “Monetary Policy Report” [Bank of England, 2020] and Board of Governors of the Federal Reserve System. “Open Market Operations.” [Federal Reserve, 2021b].

<sup>11</sup>European central bank. “The ECB’s Negative Interest Rate.” [European Central Bank, 2021]

<sup>12</sup>Federal Reserve Bank of St. Louis. “The Liquidity Trap: An Alternative Explanation for Today’s Low Inflation.” [Federal Reserve, 2021a].



The rest of the paper is organized as follows. We set the stage in section 2 with a preliminary, moneyless version of our non-linear, stock-flow consistent dynamics. Besides fixing the notation for the full-blown model, we depart from the previous literature by focusing on an economy with a CES production function, oligopoly pricing, and inventory dynamics. Section 3 exhibit four economically meaningful and locally stable long-run steady states for this dynamics, among which one is characterized by a finite debt ratio and strictly positive employment — the Solovian equilibrium— and another one by infinite debt and zero employment. The next section is dedicated to the introduction of money. Section 5 shows that it solves the first two paradoxes just alluded to: there is no need to rely on the rational expectation hypothesis for the economy to reach a long-run steady state with a stable inflation rate. We show money to be non-neutral both in the short- and long-term. Its creation is endogenously determined by banking credit. Stock-flow consistency then proves to enable tracking money creation and its travel along the circuit of the aggregate economy. The monetarist interpretation of the equation of exchange holds at the monetary Solovian equilibrium, but only there. Moreover, near-zero monetary policy is shown to enable avoiding the liquidity trap associated with the debt-deflationary steady-states. A last section concludes.

## 3.2 The moneyless economy

This section introduces our main notations and the underlying “real” (i.e., moneyless) economy. Let us consider a three-sector closed economy,  $\mathcal{E}$ , with no public sector, comprising households, a banking sector and private non-financial firms producing a single consumption good. Labor forces,  $N$ , grow at some deterministic rate  $\hat{N} := \dot{N}/N = n(t) \geq 0$ , possibly time-dependent. We denote the stock of capital in real terms by  $K$ , the total output by  $Y$ , and the number of employed workers by  $L$ . For the sake of simplicity, we assume that the economy is endowed with a CES production function

$$Y := A \left[ bK^{-\eta} + (1-b)e^{\alpha t} L^{-\eta} \right]^{-1/\eta},$$

where  $A > 0$  is a (constant) total factor productivity,  $b \in [0, 1]$  reflects the capital intensity of production,  $\alpha > 0$  is the (constant) growth rate of Harrod-neutral technical progress, and the short-run elasticity between capital and labor is captured by  $\zeta := \frac{1}{1+\eta}$ .<sup>13</sup> As usually, changes in capital stock are given by

$$\dot{K} := I - \delta K, \tag{3}$$

where  $\delta > 0$  is a constant depreciation rate and  $I$  stands for the real investment flow.

### 3.2.1 Stock-flow consistent inventories

Let us denote total real demand by  $Y_d$ :

$$Y_d := C + I, \tag{4}$$

---

<sup>13</sup>Following [Bastidas et al., 2019], we shall restrict ourselves to the economically more relevant case where  $\eta \geq 0$ , so that  $\zeta \in [0, 1]$ . Even though the empirics of this family of endogenous growth cycle models needs to be further developed, we chose a CES production function since, when backtested on the U.S. economy at least, it seems to outperform the Leontieff specification, [McIsaac, 2021]. On the other hand, the Leontieff approach implies that  $\nu$  be constant across time—which is obviously not the case in any country where data is available— while the Leontieff viewpoint allows for an endogenous capital-to-output ratio.

where  $C$  denotes aggregate consumption. The difference between output and demand determines changes in the level of inventories,  $V$ , held by firms:

$$\dot{V} := Y - Y_d. \quad (5)$$

Let  $p \geq 0$  refer to a price deflator. Nominal sales are given by  $pY_d = pC + pI$  while nominal output is

$$Y_n := pY_d + c\dot{V}, \quad (6)$$

where  $c := \mathbf{w}L/Y$ , is the unit cost of production, with  $\mathbf{w}$  being the unitary money wage and  $W := \mathbf{w}L$  the wage bill. Denoting  $\omega := \mathbf{w}L/pY$  the wage share, one gets  $c = p\omega$ . In other words, we follow [Godley and Lavoie, 2006] (eq. (8.24) and (8.25)), [Grasselli and Nguyen-Huu, 2018] by valuing inventory changes at cost  $c$ , consistently with current accounting practices:  $c\dot{V}$  is what is incurred by the acquisition department of a firm<sup>14</sup> in order to purchase unsold goods from the production department.<sup>15</sup>

Let  $r \geq 0$  be a constant, nominal short-run interest rate<sup>16</sup> while  $D_f$  denotes the nominal stock of firms' debts. The pre-dividend net profit of firms after paying wages, interest on debt, and accounting for depreciation (i.e., consumption of fixed capital) is given by<sup>17</sup>

$$\Pi_f := Y_n - W - rD_f - p\delta K. \quad (7)$$

### 3.2.2 Expected profits and debts

As in Minsky's and Kindleberger's theories, current cash-flows validate past liabilities and form the basis for investment decisions. At variance with these authors, however, we introduce here a small twist by letting investment depend upon expected sales,  $pY$ , and not  $Y_n$ . This enables to circumvent any circularity in the definition of profits (since  $Y_n$  itself depends upon current investments) and, quite realistically, make investment depend upon (a reduced-form of) entrepreneurs' expectations. Non-financial firms redistribute only a (constant) fraction,  $\Delta \in (0, 1)$ , of their expected profit,  $\Delta_f := \Delta\Pi_e$ , to their shareholders. The retained profits (or savings),  $\Pi_f - \Delta\Pi_e$ , are reinjected for investment purposes. It follows from firms' budget constraint that

$$\dot{D}_f := pI - S_f = pI - \Pi_f + \Delta\Pi_e. \quad (8)$$

Observe that, whenever  $Y_n < pY$ , firms' expected profit turns out to have been over-optimistic:  $\Pi_f < \Pi_e$ . The fraction  $\Delta(\Pi_e - \Pi_f)$  that is distributed to shareholders is therefore financed not with cash currently on hand, but through new debt financing, embodied in (10).<sup>18</sup>

Next, as in [Giraud and Grasselli, 2019], the household budget constraint implies that, whenever nominal household consumption exceeds their disposable income, the difference needs to be financed

<sup>14</sup>Represented in the capital account of Table 4.1, see below.

<sup>15</sup>Therefore,  $Y_n = pY$  if, and only if, either  $p = c$  or  $\dot{V} = 0$ .

<sup>16</sup>Section 5 will consider time-varying short-run interest rate policies.

<sup>17</sup>As pointed out by [Grasselli and Nguyen-Huu, 2018], even though changes in inventory add to profits, sales constitute the only way for firms to have positive gross (i.e., before interest payment and depreciation of capital) profit, since  $pC + pI + c\dot{V} - W = pY_d + c(\dot{V} - Y) = (p - c)Y_d$ .

<sup>18</sup>Debt-financed dividends, known as dividend recap, have become an almost standard practice in the U.S. in the last decade. According to Moody's Investors Service, in the first three quarters of 2012 there were over \$ 16.5 billion in debt-financed dividend payments (see the *Wall-Street journal*, <https://on.wsj.com/3jqQvb5>).

by an increase,  $\dot{D}_h$ , in debt. Conversely, if households' disposable income exceeds nominal household consumption, household saving is used to reimburse (part of) their net aggregate debt. Firms and banks are privately owned by households.<sup>19</sup> Let us denote by  $\Delta_f$  and  $\Delta_b$  the dividends received by households from firms and banks respectively. The households' disposable income is  $W + \Delta_f + \Delta_b - rD_h$ . The banks' current profit is  $\Pi_b := r(D_f + D_h)$ . For simplicity again, we assume that, facing negligible operating costs, banks redistribute all their profits,  $\Delta_b := r(D_f + D_h)$ .

On the supply side, expected profit,  $\Pi_e$ , is defined as the profit that would be accrued were aggregate demand to absorb the output,  $Y$ .<sup>20</sup>

$$\Pi_e := pY - W - rD_f - p\delta K.$$

The dynamics of households' debt can now be written

$$\begin{aligned} \dot{D}_h &:= pC - (W + \Delta_f + \Delta_b - rD_h) = pC - W - rD_f - \Delta\Pi_e \\ &= Y_n - c\dot{V} - pI - W - rD_f - \Delta\Pi_e \\ &= -\dot{D}_f - c\dot{V} + p\delta K. \end{aligned} \quad (9)$$

As will be clear shortly, the latter equation is key for the introduction of money.

### 3.2.3 Some accounting

The balance sheet, transactions, and flow of funds for this economy are described in Table [4.1]. Unless otherwise stated, all quantities are functions of time. Households' assets consist thus of firms' equity,  $E_f$ , bank equity,  $E_b$ , and current money deposits,  $M_h$ . Their liabilities consist of loans,  $L_h$ , resulting in net worth (i.e., wealth)  $X_h := E + M_h - L_h$ , with  $E := E_f + E_b$ . Firms' assets are the capital stock,  $pK$ , plus inventories,  $cV$ , and current money deposits,  $M_f$ , whereas their liabilities are their loans,  $L_f$ , and equity  $E_f := pK + cV + M_f - L_f$ .<sup>21</sup> Similarly, loans,  $L := L_h + L_f$  are the assets of the banking sector. Its liabilities are the deposits,  $M := M_h + M_f$  whereas bank equity is  $E_b = L - M$ . Hence, the net worth of firms and banks remains  $X_f = X_b = 0$ . Again for simplicity, we assume that firms do not keep positive deposits, preferring to use any balances to repay their loans instead. As already said, all their net profits minus dividends are reinvested:  $\Pi_f - \Delta_f = \Pi_f - \Delta\Pi^e = S_f$ . Savings of the banking sector are given by<sup>22</sup>

$$S_b = rL - rM - \Delta_b = r(D_f + D_h) - \Delta_b = 0. \quad (10)$$

Finally, the financial balance row on Table [4.1] obeys the following *ex post* accounting identity between total nominal savings and investment in the economy  $\mathcal{E}$ :

$$S := S_h + S_f + S_b = p(I - \delta K) + c\dot{V}. \quad (11)$$

We eventually get an expression of the total wealth,  $X$ , in  $\mathcal{E}$ :

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<sup>19</sup>For simplicity, at variance with [Giraud and Grasselli, 2019], we don't draw any distinction between wage-earners and "rentiers" earning dividends.

<sup>20</sup>Expected profit could arguably be thought of as depending upon current sales, and not supply capacity. This, however, would make the analysis more complex without impairing qualitatively our main results.

<sup>21</sup>Here, equity is treated as a balancing variable so that the net worth of non-financial firms is always  $X_f = 0$ .

<sup>22</sup>To save notations, we assume that banks have no consumption. Several alternative specifications of banks' behavior would be conceivable. Details can be found in [Giraud and Grasselli, 2019].

CHAPTER 3. MONEY VELOCITY IN AN IMPERFECTLY COMPETITIVE, STOCK-FLOW CONSISTENT DYNAMICS

	Households	Firms	Banks	Row Sum
<b>Balance sheet</b>				
Capital stock		$+pK$		$pK$
Inventory		$+cV$		$cV$
Deposits	$+M_h$	$+M_f$	$-M$	0
Loans	$-L_h$	$-L_f$	$L$	0
Equity	$E$	$-E_f$	$-E_b$	0
Column sum (Net worth)	$X_h$	$X_f = 0$	$X_b = 0$	$X$
<b>Transactions</b>				
		Current	Capital	
Consumption	$-pC$	$+pC$		0
Investment		$+pI$	$-pI$	0
Change in Inventory		$+c\dot{V}$	$-c\dot{V}$	0
Depreciation		$-p\delta K$	$+p\delta K$	0
Wages	$+wL$	$-wL$		0
Interest on loans	$-rL_h$	$-rL_f$	$+rL$	0
Interest on deposits	$+rM_h$	$+rM_f$	$-rM$	0
Pre-dividend net profits		$-\Pi_f$	$+\Pi_f$	0
Dividends	$\Delta_f + \Delta_b$		$-\Delta_f$	0
Column sum (balances)	$S_h$	0	$S_f - p(I - \delta K) - c\dot{V}$	$S_b = 0$
<b>Flows of funds</b>				
Change in capital stock		$p\dot{K}$		$p\dot{K}$
Change in inventory		$c\dot{V}$		$c\dot{V}$
Change in deposits	$+\dot{M}_h$	$+\dot{M}_f$	$-\dot{M}$	0
Change in loans	$-\dot{L}_h$	$-\dot{L}_f$	$+\dot{L}$	0
Column sum (savings)	$S_h$	$S_f$	$S_b$	$p\dot{K} + c\dot{V}$
Change in firm equity	$+\dot{E}_f$	$-(S_f + p\dot{K} + c\dot{V})$		0
Change in bank equity	$+\dot{E}_b$		$-S_b$	0
Change in net worth	$\dot{E} + S_h$			$(p\dot{K}) + (c\dot{V})$

Table 3.1: Balance sheet, transactions and flow of funds for the three-sector, closed economy with inventories and prices

$$\begin{aligned}
 X &= X_h = E_f + E_b + M_h - L_h \\
 &= pK + cV.
 \end{aligned}$$

In words, total wealth equals the wealth of the households, which in turn is equivalent to the productive assets of the corporate sector.

### 3.2.4 Aggregate behavioural rules

Aggregate behavioural rules will now add some flesh to the skeleton provided by Table [4.1]. Similarly to [Grasselli and Nguyen-Huu, 2018], the aggregate investment function of firms is given

by

$$I := \frac{\kappa(\pi_e)}{\nu(\omega)} K, \quad (12)$$

where  $\nu(\omega) := K/Y$  is the endogenous capital-to-output ratio, and  $\kappa(\cdot)$  is a continuous, increasing function of the normalized expected profit:

$$\pi_e := \frac{\Pi_e}{pY} = 1 - \omega - rd_f - \delta\nu(\omega). \quad (13)$$

The rate of change in real wages is a function of the current employment rate and inflation:

$$\frac{\dot{w}}{w} := \phi(\lambda) + \gamma \frac{\dot{p}}{p}, \quad (14)$$

where  $\phi(\cdot)$  is an increasing short-run Phillips curve, and  $\gamma \in [0, 1]$  is a measure of money illusion — with  $\gamma = 1$  corresponding to no monetary illusion.<sup>23</sup>

Aggregate consumption is given by a function,  $c_h(\cdot)$ , of disposable income (cf. (11)):

$$C := c_h(\omega + rd_f + \Delta\pi_e)Y. \quad (15)$$

Finally, the short-run price-wage dynamics is given by:<sup>24</sup>

$$i := i(\omega) := \frac{\dot{p}}{p} = \eta_p(\mu\omega - 1). \quad (16)$$

In words, prices relax at speed  $1/\eta_p > 0$  around the unit cost of production,  $c = p\omega$ , inflated by some markup,  $\mu \geq 1$ , whose magnitude depends upon the imperfection of competition on the commodity market. The markup epitomizes the gap between the average cost of production,  $wL/Y = c$ , and the long-run price.<sup>25</sup>

### 3.2.5 Oligopoly pricing

The production sector is assumed to behave like a large oligopoly which, given some stock of capital  $K$ , adjusts the quantity,  $L$ , of hired labor so as to maximise its current, nominal net profit,  $\Pi_f$ . The Marshallian first-order condition of this optimization programme is  $w\partial L/\partial Y = p$  or, equivalently,  $\partial Y/\partial L = w/p$ . However, due to imperfect competition, the ruling price,  $p$ , will usually be higher than marginal cost. More precisely, a monopoly's decision problem is usually thought of as choosing the quantity,  $Y$ , so as to maximise its profit:

$$\max_{Y \geq 0} p(Y)Y - c(Y), \quad (17)$$

where  $p(\cdot)$  stands for the inverse aggregate demand function, and  $c(\cdot)$  for the monopoly's cost function. As is well-known, under standard smoothness assumptions, a non-zero solution,  $Y^* > 0$ , to (17) is given by

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<sup>23</sup>See, e.g., [Gordon, 2011] for a historical survey, [Gordon, 2013], [Mankiw, 2001] and [Mankiw, 2016], as well as [Grasselli and Nguyen Huu, 2016]).

<sup>24</sup>See for example [Godard, 2007], Section 8.3.

<sup>25</sup>This is a cost-pushed inflation dynamics. There are good arguments showing that, in certain countries and for certain time periods, inflation can also be demand-pulled. We could have let inflation in (21) depend upon the dynamics of inventories at the cost, however, of a more complex analysis involving Hopf bifurcations leading to steady states being replaced by limit-cycles. This extension is left for further research.

$$p'(Y^*)Y^* + p(Y^*) = c'(Y^*). \quad (18)$$

Since, usually,  $p(\cdot)$  is decreasing, we get indeed that  $p(Y^*) > c'(Y^*)$ . Moreover, it is equally well-known that, in general, the average cost of production,  $c(Y^*)/Y^* = wL^*/Y^* = \omega^*$  at equilibrium is located somewhere above the marginal cost,  $c'(Y^*) = w\partial L/\partial Y$ , and below the monopoly price,  $p(Y^*)$ . In our setting, this means that there exists some  $\sigma \geq \mu \geq 1$  such that  $p = \mu\omega$  and

$$p = \sigma\mathbf{w} \frac{\partial L}{\partial Y}, \quad (19)$$

Of course, when  $\sigma = \mu = 1$ , we recover the textbook (perfectly competitive) case. On the other hand, the price dynamics (21) is precisely saying that prices are relaxing towards the markup augmented average cost. Now, some manipulations analogous to the ones performed by [Van der Ploeg, 1985] lead to the endogenous capital-to-output ratio as a function of  $\omega$  and  $\sigma$  :

$$\nu(\omega) := \frac{K}{Y} = \frac{1}{A} \left( \frac{1 - \sigma\omega}{b} \right)^{-1/\eta}, \quad (20)$$

$$g := g(\omega, \lambda, d_f) := \hat{Y} = \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta - \frac{\dot{\omega}}{(1 - \sigma\omega)\eta}. \quad (21)$$

As an immediate consequence, whenever  $\omega$  is increasing along the endogenous business cycle, real growth is a decreasing function of  $\sigma$ , hence will tend to be lower if the oligopolistic power of firms increases. Since, however,  $\mu$  should increase as well, inflation should accelerate: the resulting outcome on nominal growth remains ambiguous. On the contrary, along the declining phase of  $\omega$ , a tougher competition will depress both real and nominal growth. However, at a long-run steady state where, say,  $\dot{\omega} = 0$ ,  $g$  is unambiguously a decreasing function of the oligopolistic power of firms as long as wages are non-zero.

Labor productivity  $a := \frac{Y}{L}$  obtains in a similar way:

$$a(\omega(t)) = Ae^{\alpha t} \left( \frac{\sigma\omega(t)}{1 - b} \right)^{\frac{1}{\eta}}. \quad (22)$$

Therefore, since  $\eta > 0$  (see footnote 1), labor productivity will tend to rise if competition deteriorates on the commodity market while salary negotiations on the labour market remain untouched. This should not come as a surprise: an increase in  $\sigma$  works exactly as a decrease in the workers' real wage. More generally, (19) is equivalent to

$$\frac{\mathbf{w}}{p} = \frac{1}{\sigma} \frac{\partial Y}{\partial L}.$$

Taking  $\sigma > 1$  is a way to reflect, e.g., the productivity-pay gap ([Stansbury and Summers, 2018]).<sup>26</sup>

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<sup>26</sup>There is a deeper justification for  $\sigma > 1$ , provided by the criticism formulated in [Kucherenko and Shah, 2007] and [?] on the textbook first-order condition of profit-maximization. Once it is taken into account that the total differential of profit should intervene in the first-order characterization of profit-optimization (and not just a partial derivative) the standard equality between marginal revenue and marginal cost leads to a condition analogous to (19) — even within a Marshallian “atomistic” set-up.

### 3.3 Long-run “real” perspectives

This section provides a brief account of the phase space of our basic “real” (i.e., moneyless) economy. A numerical example illustrates the main emerging properties.

#### 3.3.1 The moneyless vector field

The employment rate dynamics can now be expressed as:

$$\hat{\lambda} = \hat{Y} - \hat{a} - \hat{N} = \frac{\kappa(\pi_e)}{\nu(\omega)} - \frac{\dot{\omega}}{\eta\omega(1-\sigma\omega)} - (\alpha + n + \delta). \quad (23)$$

The wage share dynamics also obtains as

$$\frac{\dot{\omega}}{\omega} = \frac{\eta}{\eta+1} \left( \phi(\lambda) - \alpha - (1-\gamma) i(\omega) \right). \quad (24)$$

The debt ratios are defined by  $d_f := \frac{D_f}{pY}$ , and  $d_h := \frac{D_h}{pY}$ . Writing  $c_h := C/Y$ , their dynamics follow from (11), (10) and (21):

$$\begin{aligned} \hat{d}_f &= \frac{\kappa(\pi_e) - (1-\omega)y_d + \Delta\pi_e + \delta\nu(\omega)}{d_f} + r - g - i \\ \text{and } \hat{d}_h &= \frac{c_h(\omega + rd_f + \Delta\pi_e) - \omega - rd_f - \Delta\pi_e}{d_h} - g - i. \end{aligned}$$

where the normalized effective demand is given by:

$$y_d := Y_d/Y = c_h(\omega + rd_f + \Delta\pi_e) + \kappa(\pi_e).^{27} \quad (25)$$

The overall dynamics therefore reduces to a 4-dimensional, non-linear autonomous system:<sup>28</sup>

$$\begin{cases} \dot{\omega} = \frac{\eta\omega}{\eta+1} \left[ \phi(\lambda) - \alpha - (1-\gamma)i(\omega) \right] \\ \dot{\lambda} = \lambda \left[ \frac{\kappa(\pi_e)}{\nu(\omega)} - \frac{\dot{\omega}}{\eta\omega(1-\sigma\omega)} - (\alpha + n + \delta) \right] \\ \dot{d}_f = \kappa(\pi_e) - (1-\omega)(y_d - \Delta) + (1-\Delta)\delta\nu(\omega) - d_f [g(\omega, \lambda, d_f) - (1-\Delta)r + i(\omega)] \\ \dot{d}_h = c_h(\omega + rd_f + \Delta\pi_e) - \omega - rd_f - \Delta\pi_e - d_h [g(\omega, \lambda, d_f) + i(\omega)]. \end{cases} \quad (26)$$

where expected profit,  $\pi_e$ , is defined by (13) as a function of  $\omega$  and  $d_f$ .

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<sup>27</sup>This relationship between the demand-to-output ratio,  $y_d$ , the investment behavioural function,  $\kappa(\cdot)$ , and aggregate consumption,  $c_h(\cdot)$ , comes directly from  $Y_d = C + I$ . It eases the study of (26) by reducing the number of differential equations: households’ debt turns out to be entirely determined by the other variables, as noticed in Appendix B.1.1.

<sup>28</sup>The capital-to-output ratio,  $\nu(\omega)$ , can be treated as an auxiliary variable. Indeed,  $\frac{\dot{\nu}}{\nu} = \frac{\dot{\omega}}{\eta(1-\sigma\omega)}$  so that  $\dot{\nu}(\omega) = 0$  if, and only if,  $\dot{\omega} = 0$  which is by definition true at any equilibrium point.



Normalising our state variables with  $Y_n$  (instead of  $pY$ ), one would recover the alternative share of wages, corporate debt ratio and profit rate obtained by using the expressions:

$$\frac{Y_n}{pY} = \frac{pY_d + c(Y - Y_d)}{pY} = \frac{(p - c)Y_d}{pY} + \frac{c}{p} = (1 - \omega)y_d + \omega ,$$

$$\text{so that } \begin{cases} \tilde{\omega} = \frac{W}{pY} \frac{pY}{Y_n} = \frac{\omega}{(1 - \omega)y_d + \omega} \\ \tilde{d}_f = \frac{D_f}{pY} \frac{pY}{Y_n} = \frac{d_f}{(1 - \omega)y_d + \omega} \\ \tilde{\pi}_f = 1 - \omega_n - rd_{nf} - \delta \frac{\nu(\omega)}{(1 - \omega)y_d + \omega} = \frac{(1 - \omega)y_d - rd_f - \delta\nu(\omega)}{(1 - \omega)y_d + \omega} . \end{cases} \quad (27)$$

### 3.3.2 Long-term real steady states

Under reasonable conditions detailed in Appendix C.1.2,<sup>29</sup> the vector field (26) exhibits four types of meaningful steady points, analogous to those described in [Grasselli and Nguyen Huu, 2016]. The first one is a Solovian equilibrium corresponding to a desirable situation with positive wages and employment, and a finite level of debt ratios,  $d_h$  and  $d_f$ .<sup>30</sup> Following [Giraud and Grasselli, 2019] we call it Solovian because the growth rate,  $g$ , at this steady state equals labor productivity growth  $\alpha$  (technical progress) plus population growth,  $n$ . Moreover it verifies the Golden rule. Solow's workhorse model being a particular case of the present one,<sup>31</sup> this equilibrium can be interpreted as the embedding of the unique equilibrium exhibited by the standard Solow model within our broader setting. This steady state, however, is no more globally stable as in the Solow model, but turns out to be locally stable under standard assumptions. Fig. 3.3.3 provides an illustration of the way inflation and the demand-to-output ratio,  $y_d$ , evolve in the neighborhood of such a desirable equilibrium.

The second long-run steady state of (26) is a debt-deflationary situation associated with a skyrocketing level of debt ratios while wages and employment shrink to zero.<sup>32</sup> This long-term collapse is also locally stable for a broad family of parameters. Fig. 3.3.3 shows how oscillations of inflation and the demand-to-output ratio,  $y_d$ , damp as the economy converges towards such an undesirable steady state. The next two steady states correspond to deflationary states with a finite debt level — one with zero wage, the other one with positive wages. Their local asymptotic stability cannot be judged a priori and must be checked on a case-by-case basis. Eventually, there are two other steady states corresponding respectively to a *trivial*  $(\omega, \lambda, d_f) = (0, 0, 0)$  and a *slavery*  $(\omega, \lambda, d_f) = (0, \lambda_5, d_{f_5})$  long-run equilibrium. Both, however, are structurally unstable and therefore irrelevant.

<sup>29</sup>See Appendix C.1.2 for a complete analysis of the existence and stability of steady states.

<sup>30</sup>See Figure 3.4a for the trajectory towards this steady state.

<sup>31</sup>It can be recovered by putting  $r = \gamma = 0$ ,  $\sigma = \mu = 1$ ,  $c_h(x) \equiv x$ ,  $\kappa(\pi_e) \equiv \pi_e$ , and  $\varphi \equiv 0$ .

<sup>32</sup>See Figure 3.4b for the trajectory towards this debt-deflationary equilibrium.

Symbol & Definition		Value	Symbol & Definition		Value
$\alpha$	labor productivity growth rate	0.025	$\phi_0$	Phillips curve $\phi(\lambda)$ parameter	$\frac{0.04}{1-0.04^2}$
$n$	population growth rate	0.02	$\phi_1$	Phillips curve $\phi(\lambda)$ parameter	$\frac{0.04^3}{1-0.04}$
$\delta$	depreciation rate	0.01	$k_0$	Investment function constant	-0.0065
$A$	Factor productivity	1/3	$k_1$	Investment function parameter	$e^{-5}$
$b$	Share of capital	0.135	$k_2$	$\kappa(\cdot)$ exponential parameter	20
$\eta$	Elasticity of substitution	50	$r$	Short-run nominal interest rate	3%
$\gamma$	Degree of monetary illusion	0.8	$\mu$	Mark-up	1.8
$c_-$	min. value of the consumption ratio	0.01	$c_+$	max. value of the consumption ratio	0.99
$\eta_p$	Price relaxation	0.03	$\Delta$	Share of profits saved by firms	0.1

Table 3.2: **Parameters**

### 3.3.3 Numerical Example

The baseline parameters for our simulations are provided in Table [3.2].<sup>33</sup> We rely on a short-run Phillips curve of the form:

$$\phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0, \quad (28)$$

with parameters specified below. The investment function is taken from [Grasselli and Costa Lima, 2012]:  $\kappa(\pi_e) = k_0 + k_1 e^{k_2 \pi_e}$ . Following [Giraud and Grasselli, 2019], the consumption ratio is given by:

$$c_h(\omega + rd_f + \Delta\pi_e) = \min \left\{ c_-; c_+ \tanh(\omega + rd_f + \Delta\pi_e) \right\}. \quad (29)$$

By analogy with the Goodwin cycles empirically identified for the US economy (see [Grasselli and Maheshwari, 2017], [McIsaac, 2021]), one cycle at the beginning of a trajectory corresponds to 10-15 years, so that one time unit in our discrete-time simulation must correspond to approximately one week.

Let us wrap together the various elements of this preliminary (moneyless) model. Dropping Say's law, allowing for some substitutability between capital and labour (or, equivalently, endogenizing the capital-to-output ratio), and for imperfect competition on the commodity market does not qualitatively change the phase space of a closed economy,  $\mathcal{E}$ . We recover endogenous business cycles which, depending upon the basin of attraction to which their starting point belongs, will converge to one of a finite number of long-run steady states, broadly similar to the ones already found in the above cited literature. As already suggested in [Bastidas et al., 2019], the CES production function seems to induce a smaller magnitude of endogenous cycles and a more rapid convergence to the Solovian equilibrium than, say, a Leontief technology. Notice as well that, in all the examples we have gone through, the debt ratio,  $d_f$ , never decreases along any trajectory leading to one of the two stable steady states (this need not be true with a Leontief production structure). Finally, the basin of attraction of the Solovian equilibrium shrinks as competition becomes more oligopolistic (see Figure 3.6) making it more difficult to reach this desirable steady state.

<sup>33</sup>See [Grasselli and Costa Lima, 2012], [McIsaac, 2016], [Grasselli and Nguyen Huu, 2016] and [Giraud and Grasselli, 2019] for exhaustive justifications of these values.

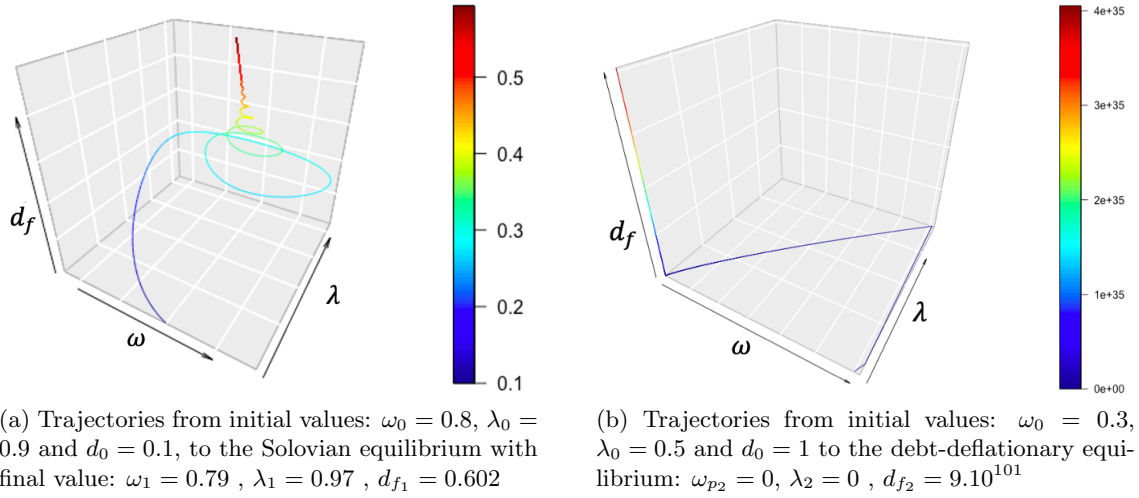


Figure 3.4: Trajectories leading to the locally stable steady states of (26).

### 3.4 Introducing money

In this section, we explicitly introduce money. So far, indeed,  $\mathcal{E}$  could be interpreted as a purely "real" (moneyless) economy. Indeed, prices given by (21) can be viewed as being only relative prices. The system (26) is 0-homogeneous with respect to  $(p, w)$ : multiplying both the consumption price index and the unitary wage by the same  $\beta > 0$  would leave the dynamics of  $\mathcal{E}$  unchanged. Similarly, the dynamics of debts (11) can be interpreted as follows: any additional credit received by either households or firms is lent to them by firms, taking either from the stock of inventories or from the sunk cost linked to capital depreciation.<sup>34</sup> In other words, there is no nominal anchor for prices and wages. As we now show, however, money can be introduced so as to provide this gauge.

Remember the celebrated "equation of exchange" (1), where  $T$  is the volume of transactions. In our setting, if there is oversupply,  $Y \geq Y_d$ , the effective volume of transaction is  $Y_d$ . If, on the contrary, demand exceeds supply,  $Y \leq Y_d$ , firms will sell inventories to meet demand, and as a consequence,  $T$  is still equal to  $Y_d$ , as long as the stock of inventories remains non-negative. If this stock becomes null, the production sector can no more meet demand whenever  $Y \leq Y_d$ : some rationing must occur and the volume of transaction should equal production:  $T = Y$ . We leave the latter situation for further investigation,<sup>35</sup> and shall therefore focus our analysis on "interior" solutions to (26) where  $V \geq 0$  throughout.<sup>36</sup> As a consequence, the equation of exchange reads

$$M\mathbf{v} = pY_d, \quad (30)$$

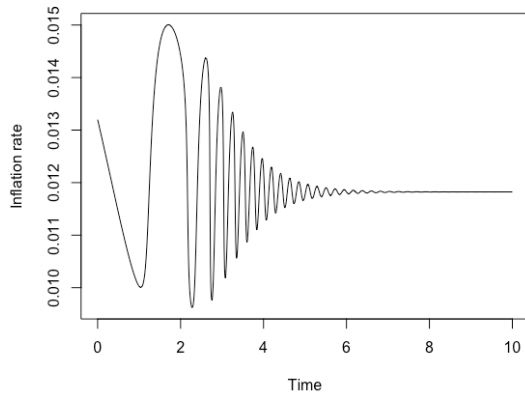
where  $M$  is to be understood as the sum of bank accounts held by non-governmental bodies (households and firms). It is an endogenous stock of *inside* money, generated as acknowledgement of debt.<sup>37</sup> The variable,  $\mathbf{v}$ , defined by (30), denotes *income velocity*, namely the speed at which real

<sup>34</sup>A similar interpretation — leading to the same conclusion that  $\mathcal{E}$  is moneyless — has been advocated by [Pottier and Nguyen-Huu, 2017], in a simpler framework.

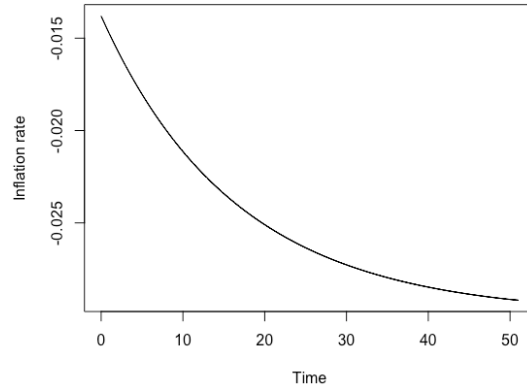
<sup>35</sup>Considering this type of "boundary problem" leads to a non-differentiable dynamics, to be studied in a subsequent work.

<sup>36</sup>This restriction was actually already at work in [Grasselli and Nguyen Huu, 2016] as well as in the previous sections of the present paper.

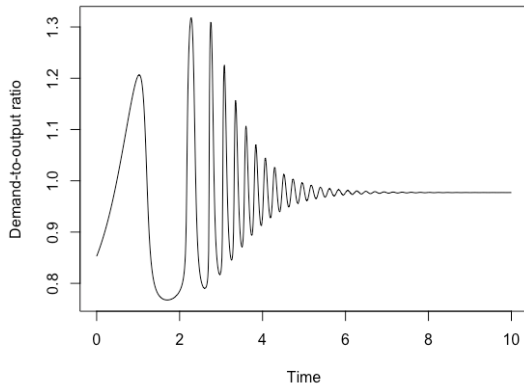
<sup>37</sup>That is, we neglect here outside money that would be free of debt, see [Gurley and Shaw, 1960].



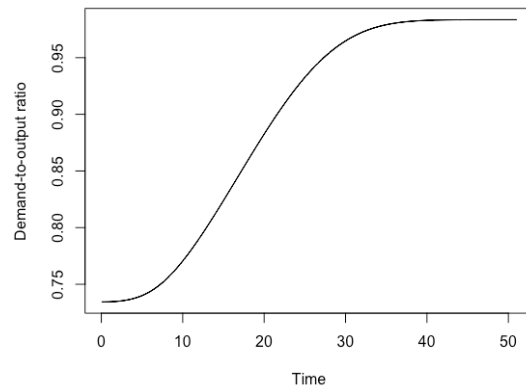
(a) Convergence of inflation,  $i$ , to its Solovian asymptotic value:  $i_1 = 1.2\%$



(b) Convergence of inflation,  $i$ , to its debt-deflationary asymptotic value  $\underline{i} = -3\%$

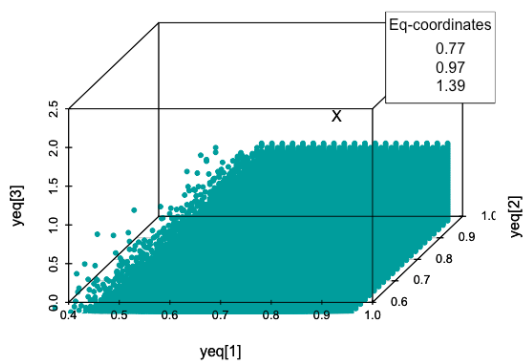


(c) Convergence of the demand-to-output ratio,  $y_d$ , along a path to the Solovian equilibrium, with final value:  $y_{d1} = 0.83$

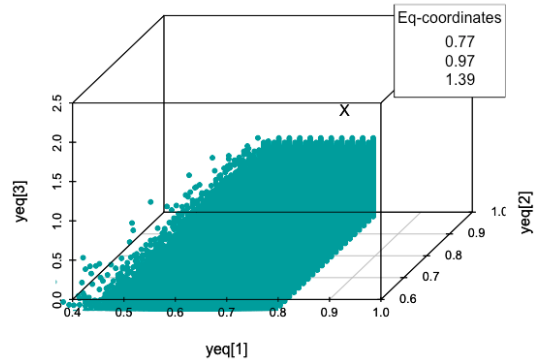


(d) Convergence of the demand-to-output ratio,  $y_d$ , to its upper-bound,  $c_+ + k_0$ , along a path to the debt-deflationary equilibrium

Figure 3.5: Inflation and demand-to-output ratio



(a) Basin of attraction of the Solovian equilibrium with perfect competition:  $\sigma = \mu = 1$ .



(b) Basin of attraction of the Solovian equilibrium with imperfect competition:  $\sigma > \mu > 1$ .

Figure 3.6: Influence of imperfect competition on the basins of attraction.

transactions occur in the economy. In a similar vein, equation (2) must be interpreted as

$$\frac{di}{dt} = \frac{\mathbf{v}}{d\mathbf{v}/di} \left( i + g_d - \hat{M} \right),$$

with  $g_d := \dot{Y}_d/Y_d$ .

Where does money come from? In our credit-economy, it can only be created through some additional banking credit. Conversely, when a debt is reimbursed, the corresponding quantity of money flows back to the balance sheet of the banking system, and is therefore “destroyed”. As a consequence,

$$\dot{M} := \dot{D}_f + \dot{D}_h. \quad (31)$$

Notice that, whenever  $\dot{D}_f + \dot{D}_h = 0$ , no additional money flows into the economy: one of either sector is lending to the other the increment of debt required. Together with the debt dynamics in (11) and (10), Eq. (31) leads to the following endogenous dynamics of money:

$$\dot{M} = -c\dot{V} + p\delta K. \quad (32)$$

Despite its simplicity, Eq. (32) delivers some useful insights. Indeed, were we to maintain Say’s “law” and to neglect the sunk cost induced by capital depreciation, the quantity of money  $M$  would necessarily be constant, and would therefore become, again, irrelevant. One would also deduce from (32) that  $\dot{D}_f = -\dot{D}_h$ . The famous chicken-and-egg problem — who causes the other, savings or loans? — would again become inextricable. This means that, in order to solve this conundrum and to be able to observe endogenous credit creation in a macro-economic model, giving up Say’s “law” is a necessary condition.

### 3.4.1 The non-neutral dynamics of money

Defining the money-to-output ratio by  $m := M/pY$ , one gets

$$\dot{m} = -\omega(1 - y_d) + \delta\nu(\omega) - m[g + i], \quad (33)$$

which we add to the dynamical system (26). The new vector field therefore involves the following state variables  $(\omega, \lambda, d_f, d_h, m)$ . Income velocity,  $\mathbf{v}$ , can be written  $\mathbf{v} = \frac{y_d}{m}$ , and is, therefore, a function of  $\omega, \lambda, d_f, d_h$ , and  $m$ . Its dynamics is given by

$$\hat{\mathbf{v}} = \hat{y}_d - \hat{m}. \quad (34)$$

Monetarist theories often assume that  $\mathbf{v}$  is an increasing function of the inflation rate: the return to holding money being equal to  $-i$ , as inflation runs faster, it erodes the real value of the money stock more rapidly so that households and firms should flee to other assets. In our framework, this property is broadly implied by (1).

From a slightly different viewpoint, one could argue that the speed at which prices adjust should also be an increasing function of  $\mathbf{v}$ : the more frequently trades take place, the more often price changes can materialize. For instance, this would certainly be true in (double) auction mechanisms.<sup>38</sup> Here, this could be captured by introducing income velocity into the differential equation that governs inflation (21) as follows:

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<sup>38</sup>Cf. e.g. [?].

$$i(\omega, d_f, d_h) = \eta_p \mathbf{v}(\mu\omega - 1). \quad (35)$$

In words, (35) means that the speed,  $\eta_p \mathbf{v}$ , at which prices relax to changes in their long-run value (given by  $\mu\omega$ ) is an increasing function of  $\mathbf{v}$ .

Whether we adopt (35) or keep (16) in order to account for price dynamics, notice that, being now dependent upon  $\dot{\omega}$  (see (21)), the real growth rate,  $g$ , depends upon  $\mathbf{v}$ , and hence  $m$ . Therefore, at least in the short-and medium-run, money is non-neutral.

Notice, moreover, that the instability problem, linked to the conventional interpretation of the equation of exchange alluded to in the Introduction, no more shows up. Indeed, (2)

$$\frac{di}{dt} = \frac{\mathbf{v}}{d\mathbf{v}/di} (i + g_d - \hat{M})$$

does not necessarily lead to unstable inflation since, from now on, both  $g$ ,  $i$  and  $\hat{m}$  depend upon  $\mathbf{v}$ . As a matter of fact, courtesy of (24), the dynamics of inflation can equivalently be written:

$$\frac{di}{dt} = \eta_p m \dot{\omega} = \frac{\eta_p m \eta \omega}{\eta + 1} [\phi(\lambda) - \alpha - (1 - \gamma)i].$$

Since  $\gamma \in (0, 1)$ , a higher value of  $i$  decreases  $di/dt$ , which now feeds back into a decrease in  $i$ . The overall dynamics boils down to the following 4-dimensional system:

$$\begin{cases} \dot{\omega} = \frac{\eta\omega}{\eta + 1} [\phi(\lambda) - \alpha - (1 - \gamma)\eta_p \mathbf{v}(\mu\omega - 1)] \\ \dot{\lambda} = \lambda \left[ \frac{\kappa(\pi_e)}{\nu(\omega)} - \frac{\dot{\omega}}{\eta\omega(1 - \sigma\omega)} - (\alpha + n + \delta) \right] \\ \dot{d}_f = \kappa(\pi_e) - (1 - \omega)(y_d - \Delta) + (1 - \Delta)\delta\nu(\omega) - d_f [g - (1 - \Delta)r + \eta_p \mathbf{v}(\mu\omega - 1)] \\ \dot{m} = -\omega(1 - y_d) + \delta\nu(\omega) - m[g + \eta_p \mathbf{v}(\mu\omega - 1)]. \end{cases} \quad (36)$$

As in the moneyless case,  $d_h$  can be treated an auxiliary variable following:

$$\dot{d}_h = c_h(\omega + rd_f + \Delta\pi_e) - \omega - rd_f - \Delta\pi_e - d_h [g + \eta_p \mathbf{v}(\mu\omega - 1)].$$

### 3.4.2 Endogeneous money creation

Let us track on Table 3.3 the path followed by money creation, that is, by an additional increment of credit,  $\dot{\ell} = \dot{\ell}_h + \dot{\ell}_f$ , distributed at time  $t$  by the banking sector to, respectively, households and firms. Suppose that households use this money injection in order to increase their current consumption while firms use this additional funding to finance some additional investment. Suppose also that, the surge in demand is fulfilled by firms by drawing on the inventories. The accounting value of the stock of unsold production will be reduced by  $\frac{\varepsilon}{p}\dot{\ell}_h = \omega\dot{\ell}_h$ . As a consequence, the aggregate profit of the corporate sector will increase by  $\omega\dot{\ell}_h$ . Suppose that a fraction  $\theta \in (0, 1)$  of this additional profit is redistributed to the firms' shareholders of firms. Therefore, households' money deposits will increase by  $\theta\dot{\ell}_h$ . On the side of corporates, they receive the initial loan arising from the banking sector,  $\dot{\ell}_f$ , which, being spent for investment, remains on the balance sheet of the productive sector. Firms' deposits, however, are also raised by the money left after dividends have been paid,  $(1 - \theta)\dot{\ell}_h$ .

	Households	Firms	Banks	Row Sum
<b>Balance sheet</b>				
Capital stock		$+pK$		$+pK$
Inventory		$+cV$		$+cV$
Deposits	$+M_h$	$+M_f$	$-M$	0
Loans	$-L_h$	$-L_f$	$L$	0
Equity	$E$	$-E_f$	$-E_b$	0
Column sum (Net worth)	$X_h$	$X_f = 0$	$X_b = 0$	$X$
<b>Transactions</b>				
		Current	Capital	
Consumption	$-pC - \dot{\ell}_h$	$+pC + \dot{\ell}_h$		0
Investment		$+pI + \dot{\ell}_f$	$-pI - \dot{\ell}_f$	0
Changes in Inventory		$+c\dot{V} - \omega\dot{\ell}_h$	$-c\dot{V} + \omega\dot{\ell}_h$	0
Accounting memo [GDP]		$[Y_n]$		
Depreciation		$-p\delta K$	$+p\delta K$	0
Wages	$+W$	$-W$		0
Interest on loans	$-rL_h$	$-rL_f$	$+rL$	0
Interest on deposits	$+rM_h$	$+rM_f$	$-rM$	0
Pre-dividend net profits		$-\Pi_f - \dot{\ell}_h$	$+\Pi_f + \dot{\ell}_h$	0
Dividends	$+\Delta_f + \theta\dot{\ell}_h + \Delta_b$		$-\Delta_f - \theta\dot{\ell}_h$	0
Column sum (balances)	$S_h - (1 - \theta)\dot{\ell}_h$	0	$S_f + (1 - \theta)\dot{\ell}_h - p\dot{K} - c\dot{V}$	$S_b = 0$
<b>Flows of funds</b>				
Change in capital stock		$+p\dot{K}$		$+p\dot{K}$
Change in inventory		$c\dot{V}$		$c\dot{V}$
Change in deposits	$+\dot{M}_h + \theta\dot{\ell}_h$	$+\dot{M}_f + (1 - \theta)\dot{\ell}_h + \dot{\ell}_f$	$-\dot{M} - \dot{\ell}$	0
Change in loans	$-\dot{L}_h - \dot{\ell}_h$	$-\dot{L}_f - \dot{\ell}_f$	$+L + \dot{\ell}$	0
Column sum (savings)	$S_h - (1 - \theta)\dot{\ell}_h$	$S_f + (1 - \theta)\dot{\ell}_h$	$S_b$	$p\dot{K} + c\dot{V}$
Change in firm equity	$+\dot{E}_f$	$-(S_f + p\dot{K} + c\dot{V})$		0
Change in bank equity	$+\dot{E}_b$		$-S_b$	0
Change in net worth	$\dot{E} + S_h$			$(p\dot{K}) + (c\dot{V})$

Table 3.3: Endogenous money creation



## 3.5 Long-run non-neutrality and monetary policy

We now turn to the long-run properties of our monetary economy.

### 3.5.1 Monetary steady states

First, observe that the equilibrium condition for the money ratio,  $\dot{m} = 0$ , implies

$$g^* := g(\omega^*, \lambda^*, d_f^*, m^*) = \frac{\delta\nu(\omega^*) - \omega^*(1 - y_d^*)}{m^*} - i(\omega^*). \quad (37)$$

In words, growth is a non-trivial function of  $m^*$ , even at equilibrium. Therefore, unless the right-hand side of (37) turns out to be independent from the path followed by  $m$ , money is non-neutral *even in the long-run*.

In order to check whether (37) is path-dependent with respect to  $m$ , let us study the long-run steady states with money. Under the same technical conditions as in section 2, (36) exhibits 7 meaningful types of equilibrium points which contain the monetary analogs of the three steady states that had already emerged in the long-run pattern of the real economy.<sup>39</sup> Only two of them, however, turn out to be locally stable under reasonable conditions: the Solovian equilibrium with money  $(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)$  and the debt-deflationary equilibrium with skyrocketing levels of debt *and* money  $(0, 0, +\infty, +\infty)$  but with zero wage and employment. See Figure 3.7a for an illustration of a trajectory towards the Solovian equilibrium, and 3.7b for the associated evolution of the money-to-output ratio. Fig. 3.8a and 3.8b illustrate convergence towards a debt-deflationary crisis.

Therefore, once again, the third equilibrium (debt-deflation but with a finite, asymptotic level of debt ratio) remains locally unstable for a large family of parameters. Consequently, in the sequel, we focus on the first two zeroes of the vector field with money, which survive as meaningful and locally stable long-run steady states.

As detailed in Appendix B.2.2, in a neighborhood of the **monetary Solovian equilibrium**  $(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)$ ,

- The expected profit rate converges to  $\pi_{e1}^* = 1 - \omega_1^* - rd_{f_1}^* + \delta\nu(\omega_1^*)$ ;
- The households' debt ratio converges to  $d_{h1}^*$ ;<sup>40</sup>
- The demand-to-output ratio,  $y_d$ , converges to  $y_{d1}^* = \nu(\omega_1^*)(\alpha + n + \delta) + c_h(\omega_1^* + rd_{f_1}^* + \Delta\pi_{e1}^*)$ , which can be either higher or lower than 1, depending on the aggregate consumption function  $c_h(\cdot)$ ;<sup>41</sup>

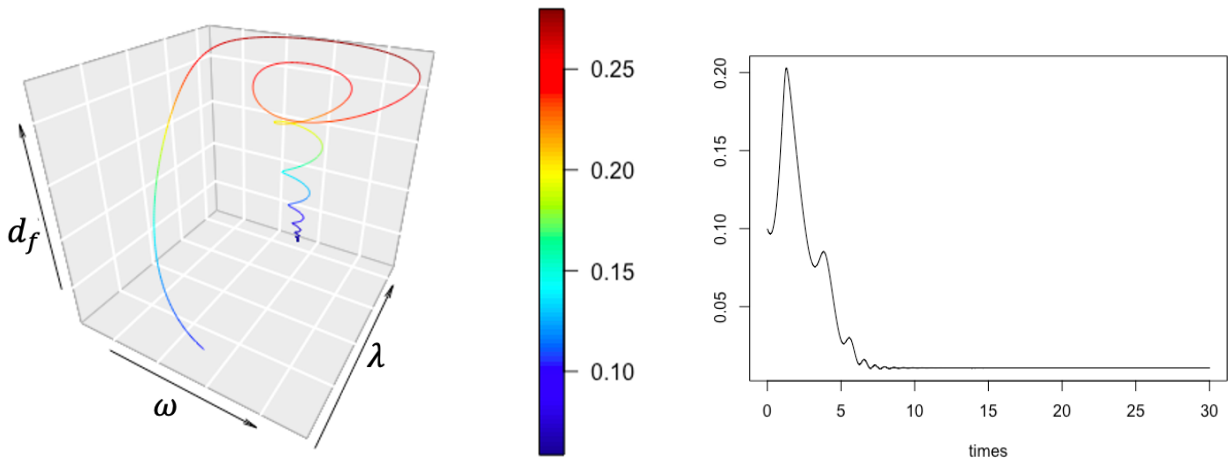
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<sup>39</sup>Other steady states are structurally unstable and will not be studied further: a *trivial* Equilibrium  $(0, 0, 0, m_4^*)$ , a *slavery* equilibrium  $(0, \lambda_5^*, d_{f_5}^*, m_5^*)$  or any steady state with  $m = 0$ , such as  $(\omega_6, \lambda_6, d_{f_6}, 0)$ . See Appendix B.2.2 for further details.

<sup>40</sup>Indeed  $d_{h1}^*$  can be deduced from others variables which converge. See Appendix C.1.4 for the demonstration, which is still true after introducing money velocity in Appendix B.2.2.

<sup>41</sup>With our leading parameterization,  $y_{d1} = 0.98$ , so the Solovian equilibrium is overproducing and consequently the stock of inventories grows to infinity. Notice that inventories,  $V$ , grow to infinity along this long-run steady state with overproduction ( $y_{d1}^* < 1$ ) but plummet if  $y_{d1}^* > 1$ . With the consumption function of our numerical example, the latter situation is quite unlikely to happen:  $y_{d1}^* > 1$  is equivalent to  $c_h(\omega_1^* + rd_{f_1}^* + \Delta\pi_{e1}^*) + \kappa(\pi_{e1}^*) > 1$ . In the neighborhood of the Solovian equilibrium,  $\kappa(\pi_{e1}^*) = \nu(\alpha + n + \delta) \sim 10^{-2}$  under reasonable specifications. The consumption function,  $c_h(\cdot)$ , is a continuously strictly increasing function which tends to  $c_1\omega_1 + c_2rd_{f_1}$ , with  $c_1, c_2 > 0$ . Consequently  $y_{d1}^* > 1$  means  $c_1(1 - \pi_{e1}^* - rd_{f_1}^* + \delta\nu(\omega_1^*)) + c_2rd_{f_1}^* + \nu(\alpha + n + \delta) > 1$ , where  $1 - c_1 \sim 10^{-1}$ ,  $c_1\pi_{e1}^* > 0$  and  $c_1\delta\nu(\omega_1^*) \sim 10^{-2}$ . Thus, in order to satisfy  $y_{d1}^* > 1$ , we must have at least  $c_2 - c_1 > 0$  which implies that the volume of wealth dedicated to consumption coming from the redistributed banks' profit be greater than the volume of income dedicated to consumption stemming from wages — which is unlikely in practice.

- The income velocity of money converges to  $\mathbf{v}_1^* = \frac{y_{d_1}^*}{m_1^*} > 0$ ;
- Inflation converges to  $i_1^*$  and the growth rate of the quantity of circulating money  $\hat{M} = \hat{m} + \hat{p} + \hat{Y}$  converges to  $\alpha + n + i_1^*$ . The standard monetarist equation with rational expectations therefore holds at the monetary Solovian equilibrium:  $\hat{M} = i + g$ .
- In our numerical example (introduced in section 4.4.2), income velocity converges towards  $\sim 100$  around the monetary Solovian steady-state. Once  $\mathbf{v}$  is introduced in the price dynamics (35), inflation is therefore asymptotically 100 times larger than with (21), while the asymptotic private debt ratio is 10 times smaller. In both cases, the asymptotic employment rate stands above 0.96, there is overproduction and households accumulate positive net savings.



(a) Trajectories from initial values:  $\omega_0 = 0.8$ ,  $\lambda_0 = 0.9$  and  $d_0 = 0.1$ ,  $m_0 = 0.1$  to final value:  $\omega_1^* = 0.79113030$ ,  $\lambda_1^* = 0.98529608$ , and  $d_{f_1}^* = 0.05865288$

(b) Evolution of the money-to-output ratio along the path to the Solovian Equilibrium towards  $m_1^* = 0.01076340$ .

Figure 3.7: Trajectories leading to the Solovian equilibrium of (27).

Along a path to the locally asymptotically stable **monetary debt-deflationary** steady state  $(0, 0, +\infty, +\infty)$ :<sup>42</sup>

- The expected profit rate,  $\pi_e$ , tends to  $-\infty$ ;
- The investment share,  $\kappa(\pi_e)$ , converges to its lower bound value,  $k_0$ ;
- The consumption share reaches its upper-bound,  $c_+$ ;<sup>43</sup>
- This drives the demand-to-output ratio to  $\bar{y}_d = k_0 + c_+$ , a positive value which can be higher or lower than 1. Because production is plummeting, the volume of demand  $Y_d = y_d Y$  tends to 0 (despite the fact that the consumption share stays positive).
- Since  $\mathbf{v} = \frac{y_d}{m}$ , income velocity vanishes in a neighborhood of the asymptotic collapse;
- In our numerical example, asymptotic inflation is either nihil or negative ( $i = -\eta_p = -0.03$ ) depending upon whether one adopts (21) or (35) for the price dynamics.

Equation (37) then implies that  $g = 0$  at this debt-deflationary equilibrium. As a consequence, money is non neutral in the long-run in as much various trajectories of  $m$  lead to different steady

<sup>42</sup>See Appendix B.2.2 for details.

<sup>43</sup>The consumption function here is driven by firms' debt which is held as a liability by the banks' shareholders. Consequently,  $c_h(\omega + rd_f + \Delta\pi_e)$  tends to its upper bound  $c_+$ , because  $d_f$  tends to  $+\infty$  and  $\Delta < 1$ .

states, with distinct real growth rates. Notice, however, that the Monetarist master equation,  $\hat{M} = g + i$ , being true at the monetary Solovian equilibrium, money *looks neutral* at *this* equilibrium. This apparent long-run neutrality, however, is an artifact from static equilibrium models which neglect the dynamical path leading to this *particular* equilibrium.

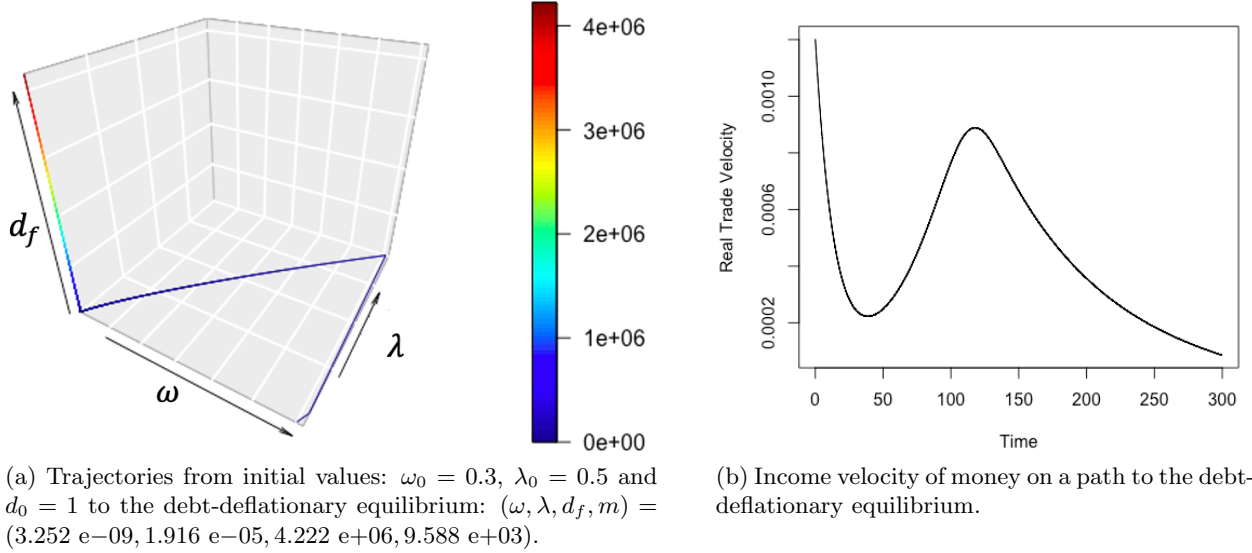


Figure 3.8: Trajectories leading to solutions of (27) to  $(0, 0, +\infty, +\infty)$ .

### 3.5.2 Monetary persistence

This section shows how a well-tuned monetary policy and banking regulation can guarantee that the economy does not reach a state of permanently zero employment. For that purpose, let us extract from Appendix A the following technical assumptions:

**Assumption A** a) (ii) *The short-run Phillips curve verifies  $\phi(0) < \alpha - (1 - \gamma)\eta_p$ .*

**Assumption B** *The investment function satisfies the following boundary condition:*

$$\kappa_0(0) > (\alpha + n + \delta)\nu(0) = (\alpha + n + \delta)\frac{b^{1/\eta}}{A}.$$

In the sequel, following [Smith and Thieme, 2011], a dynamical system is said to be  $X$ -UWP (Uniformly Weakly Persistent with respect to the functional  $t \mapsto X(t)$ ) if there exists  $\varepsilon > 0$  such that  $\overline{\lim}_{t \rightarrow +\infty} X(t) \geq \varepsilon$ . The main result of this section is the following:

**Proposition 1.** *Whenever inflation is given by (16) and under Assumptions A 1) (ii) and B, the monetary economy defined by (36) is  $\lambda$ -UWP, provided the central bank implements an interest rate policy,  $t \mapsto r(t)$ , such that*

$$r(t) < \frac{\varepsilon}{d_f(t)}, \quad (38)$$

*as soon as  $d_f(t) > 0$  and  $\lambda(t) < \varepsilon$ , for some low  $\varepsilon > 0$ .*

Before entering into the proof, let us briefly comment this result. The main takeaway is that a specific monetary policy is sufficient to drive a monetary economy out of the dangerous waters of the liquidity trap each time the employment rate becomes too low. This policy consists in lowering the short-run nominal interest rate so that (38) holds provided aggregate corporate debts are positive. This is but a stylized version of the policy put into practice in a number of countries after the Great recession (Japan, UK, US, euro zone...), except that, here, it is not framed as a function of some inflation target — as in the conventional Taylor rule ([Taylor, 1993])— but according to the debt ratio of the private sector. To the best of our knowledge, this is the first time such a criterion emerges as a condition for a successful monetary policy when an economy is threatened by a liquidity trap.

That inflation be captured by (16) means that it does not depend on income velocity but does not prevent the reverse causality to be true. Assumption **A** 1) (ii) is weak and standard in the literature devoted to this class of macro-economic dynamics. Assumption **B** is somewhat stronger. It complements **A** 2) (ii), which we use to prove the existence of both the monetary Solovian equilibrium and some economically irrelevant (and structurally non-existent) steady states (see Appendix B.2.2). **A** 2) (ii) basically says that, provided  $rd_f$  remains bounded away from zero, the investment-to-output ratio should be bounded from above by a cap given by

$$(\alpha + n + \delta) \frac{b^{1/\eta}}{A}. \quad (39)$$

On the contrary, **B** requires (normalized) aggregate investment to exceed this level whenever  $rd_f = 0$ . In words, if the central bank implements ZIRP, it should successfully encourage low-cost borrowing up to the point that investment crosses what is, otherwise, an upper-bound of theirs. Notice, however, that our main result does *not* require ZIRP but only a very low nominal interest strategy, provided the private sector's investment appetite is strong enough. According to standard calibrations (including our numerical example), the threshold (39) is close to 10-15%. Evidently, in some countries, the private sector is too weak to reach like 15% of its output — however unconventional the monetary policy might be. This suggests that, for such countries, either a public intervention is needed along the lines advocated, e.g., by [Mazzucato, 2011] or negative interest nominal short-term interest rates should be envisaged in order to escape from the liquidity trap.

**Proof** As in [Costa Lima and Grasselli, 2014], we proceed by contradiction. Suppose that, for every  $\varepsilon > 0$ , there exists a time  $t_0 > 0$  such that  $\forall t \geq t_0, \lambda(t) < \varepsilon$ . Then, given that, obviously,  $i(\omega) \geq -\eta_p$ , the differential equation followed by  $\omega$  in (36) implies that

$$\omega(t) < \omega(t_0) e^{(t-t_0) \frac{\eta}{\eta+1} [\phi(\varepsilon) - \alpha + (1-\gamma)\eta_p]}.$$

It follows from Assumption **B** a)(ii) that, whenever  $\varepsilon$  is chosen small enough so that

$$\phi(\varepsilon) - \alpha + (1-\gamma)\eta_p < 0,$$

there exists  $t_1 > t_0$  such that,  $\forall t \geq t_1, \omega(t) < \varepsilon$ . Consequently, (20) and  $\eta > 0$  imply that the capital-to-output ratio verifies

$$\nu(\omega(t)) < \frac{1}{A} \left( \frac{b}{1-\sigma\varepsilon} \right)^{1/\eta}.$$

Now, (13) and (38) imply that the normalized expected profit satisfies

$$\pi_\varepsilon(t) > 1 - 2\varepsilon - \frac{\delta}{A} \left( \frac{b}{1-\sigma\varepsilon} \right)^{1/\eta} > 0. \quad (40)$$

The last inequality arises from Assumption **B**.

On the other hand, the motion of the wage share,  $\omega$ , is such that

$$\frac{\dot{\omega}}{\eta\omega(1-\sigma\omega)} = \frac{1}{(\eta+1)(1-\sigma\omega)} \left[ \phi(\lambda) - \alpha - (1-\gamma)i \right].$$

Consequently, for  $t > t_1$ ,

$$\frac{\dot{\omega}(t)}{\eta\omega(t)(1-\sigma\omega(t))} \leq \frac{1}{(\eta+1)(1-\sigma\varepsilon)} \left[ \phi(\varepsilon) - \alpha + (1-\gamma)\eta_p \right] =: \Gamma_\varepsilon.$$

It follows, again, from **A** a)(ii) that the right-hand side,  $\Gamma_\varepsilon$ , is negative. Consequently, the motion of the employment rate,  $\lambda$ , in (36) leads to

$$\lambda(t) \geq \lambda(t_1) e^{(t-t_1) \left[ \frac{\kappa(\pi_\varepsilon)}{\nu(\varepsilon)} - \Gamma_\varepsilon - (\alpha+n+\delta) \right]} > \lambda(t_1) e^{(t-t_1) \left[ \frac{\kappa(\pi_\varepsilon)}{\nu(\varepsilon)} - (\alpha+n+\delta) \right]}.$$

Therefore, (40) guarantees that there exists  $t_2 > t_1$  such that, for every  $t \geq t_2$ ,  $\lambda(t) > \varepsilon$ . A contradiction. □

### 3.6 Conclusion

In this paper, a first attempt at modelling a monetary economy according to the lines advocated by [Werner, 2016] has been provided. We showed that it is possible to make explicit endogenous money creation through banking credit in a macroeconomic dynamics so as to circumvent the famous conundrums associated with inflation instability. No rational expectation is needed for that purpose.

In the absence of monetary policy, the dynamics is primarily driven by the investment decisions of firms based on profit levels and by the consumption decisions of households based on income: high profits lead to high investment and capital expansion financed by increasing private debt levels; high incomes lead to high effective demand financed by increasing household debts. We have shown that a key for letting money enter the picture is to drop Say's law, so that the imbalance between debts and savings must be filled with new credit banking. The dynamic analysis showed that the interplay between effective demand, production and money creation can lead either to an equilibrium with a finite private debt-to-output ratio, positive but stable inflation and positive money velocity — the monetary Solovian equilibrium — or to another steady state where debts become infinite while the wage share, the employment rate, and money velocity both collapse to zero. Inflation either collapses as well or can even become negative — a hallmark for the deflationary trap.

Moreover, money has been shown to be non-neutral both in the short- and the long-run, even though the monetarist interpretation of the exchange equation happens to be true at the monetary Solovian equilibrium — and only there. The fact that banks create credit and money out of nothing, if used appropriately, results in non-hyper-inflationary growth, provided the economy crosses the basin of attraction of the monetary Solovian equilibrium.

Eventually, we have shown that central bank intervention may have a clear positive effect in preventing a crisis characterized by collapsing employment rates. Monetary policy prevents this outcome by conditioning the short-run nominal interest rate upon the private debt-to-outcome ratio: The larger the latter, the smaller should the cost of money be. The persistence result of section 4 are much stronger than, say, the possible local, asymptotic instability of undesirable steady-states

under typical parameter values of the model: central bank intervention, in the form of a responsive enough monetary policy, prevents the economy from remaining permanently at arbitrarily low levels of employment regardless of the initial conditions of the system. To echo [Costa Lima and Grasselli, 2014], it may be that stabilizing an unstable monetary economy is too tall an order for central bank, but destabilizing a stable debt-deflationary crisis is possible.

There are, of course, a number of institutional aspects that we have neglected in this first tentative pass: reserve requirements, capital adequacy requirements (such as the Basel III/CRR framework), prudential regulation... [Bovari et al., 2020] have introduced some refinements in the modelling of the banking system. within a “real” economy. Added to the present framework, these refinements could enable us to tackle important issues around a countercyclical leverage-based regulation of banks.

Perhaps as important is to consider open economies and the interplay between non-neutral credit creation out of thin air, exchange rates, and international borrowing. The main thesis defended by [Werner, 2014b] should then be examined in order to check under which conditions developing countries should have relied on domestic money creation rather than borrowing money from abroad, at the risk of getting “ensnared in spiralling foreign currency debt, when actually the foreign banks just created the money out of nothing, something the developing country could have arranged for through its own domestic banks” ([Werner, 2016]). As made clear by this author, considering endogenous money creation “also has implications for the question of who should pay for bank bailouts, shifting the pendulum from burdening tax-payers towards central bankers”. We leave for further research the task of investigating these fascinating questions.

## Chapter 4

# Directed Technical Change in an inventory Stock-flow Consistent Dynamics with Government: from dirty to clean Capital

### Abstract

The aim of this paper is to propose a simple model of energy transition from fossil fuels to renewable energies.

We consider here the extreme case of two natures of capital, one brown (that needs fossil energy to operate), the other, green (free of fossil energy), which, however, is less productive than the dirty one. This dual capital structure is embedded into a stock-flow consistent macrodynamics where investment can be financed by debt and the productive sector adjusts shifts in demand thanks to inventories. The transition is said to take place if the economy does asymptotically operate with green capital only.

Absent any public intervention, the economy converges to a purely brown production sector: unsurprisingly, renewable energies are kicked out as long as green capital remains less productive than the brown one. The inertia affecting the transition of aggregate investment from one type of capital to the other, however, turns out to destabilize the debt-deflationary equilibria of the dynamics. Finally, we pinpoint two main levers for a government intervention: increasing the energy price *via* a carbon tax or the output-to-capital ratio of green capital *via* publicly subsidized directed technical innovation. The first option runs the risk of reducing private profits, hence fostering private debts. The second runs the dual risk of boosting public debt. We identify conditions under which the transition to green capital takes place without leading to an overhang of debt, be it private or public. Numerical simulations illustrate the properties of each equilibrium.

## 4.1 Introduction

Directing technical change towards alternative energy sources to fossil fuels is of crucial importance for the ecological transition. This paper aims at developing a simple framework for analysing the driving forces shaping the biases of capital increase that may, or may not, favour the shift towards renewable energies. We consider two types of capital, a “brown” (or dirty) capital which



needs fossil energy to operate, and a “green” (or clean) capital, which is less productive but does not rely on fossil energy.<sup>1</sup> Private investment is directed towards the most profitable capital type, depending on the (exogenous) price of fossil energy. Both types of capital are putty-clay ([Johansen, 1959]) with respect to energy: once a capital good is created, its capital to energy consumption ratio is fixed once for all, and both types of capital are not substitutable. This production structure is embedded into a stock-flow consistent prey-predatory macrodynamics [Goodwin, 1967] where private investment can be financed by debt [Grasselli and Costa Lima, 2012]. The transition towards renewable energies takes place if the economy does asymptotically operate with green capital only. Two main drivers for this to happen are highlighted in this paper: the output-to-capital ratio of *brown* versus *green* capital and the relative price of fossil energy compared with renewable energies.

Investment is directed towards the most profitable capital type, depending on its productivity and the relative price of fossil energy, but does not adjust instantly (although the higher the gap between the rates of profit, the quicker the adjustment). This inertia of investment can be thought of as capturing complementarity effects between the two types of capital (which are otherwise perfect substitutes as production factors), specialization of workforce or infrastructure externalities. It departs from the instantaneous shift from brown to green capital often postulated in the literature.<sup>2</sup>

Following [Grasselli and Nguyen-Huu, 2018], Say’s law is dropped here and public policy is introduced through public spending and taxes in a way similar to [Costa Lima and Grasselli, 2014]. The interplay between inventory dynamics and public policy leads to a rich dynamics similar to a two-dimensional Aubry-Mather-like dynamics on the torus with a small cycle (linked to the underlying prey-predatory dynamics between employment and wages) intertwined with a larger one induced by the county-cyclical public response. Together with the dynamics of private debt, wage share and underemployment, it leads to a 10-dimensional non-linear dynamical system. We show that this system can be reduced to a 5-dimensional one. That system, however, is not autonomous because it depends on the exogenously given price of fossil energy, which determines the allocation of investment.

As in the literature devoted to this type of macrodynamics (see [Grasselli and Costa Lima, 2012] and [Giraud and Grasselli, 2019]), we show that, independently of the nature of capital prevailing in the long-run, there are two types of steady states: a *Solovian-like* one, which is but the extension to our setting of the (unique) equilibrium of the standard Solow-Swan dynamics; and a *debt-deflationary* one, where an overhang of private debts leads the economy to a liquidity trap where trades, production and employment collapse.

Under reasonable specifications of the short-run Phillips curve, as well as investment and consumption functions, we show that debt-deflationary equilibria, whether dirty or clean, are locally unstable. This stands in sharp contrast with the literature already mentioned. The force that destabilizes deflationary equilibria turns out to be the inertia with which aggregate investment shifts from one type of capital to another. [Costa Lima and Grasselli, 2014] have provided a completely different mechanism that destabilizes the debt-deflationary crisis, based on public intervention. Our result provides insight on this problem by suggesting that we might need to dig further the real causes of the debt-deflation disease, hence the remedies that could help cure it.

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<sup>1</sup>To save notations, we do not make the dissipation of renewable energies explicit. Nevertheless the price of fossil energy is to be understood as the relative price of fossil to renewable energies.

<sup>2</sup>See, e.g. in Nordhaus’ DICE [Nordhaus, 2014] model; for a climate change model sharing our theoretical framework, see [Bovari et al., 2018].

The main driver of the sustainability transition of the economy turns out to be the difference between the relative price of fossil energy and the productivity gap between the two types of capital. If the fossil energy price remains too low with respect to the differential in productivity between the two types of capital, the economy does asymptotically operate exclusively with brown capital. The cost of fossil energy is compensated in the long-run by a decrease in the share of wages courtesy of the higher productivity of capital. Conversely, whenever the fossil energy price is sufficiently deterrent, the investment ultimately concentrate on the clean capital. Numerical simulations show the effects of changes in the parameters and in the relative fossil energy price.

Consequently, the main levers for a government intervention consist in increasing either the relative fossil energy price or the productivity of clean capital. Both tend to discourage investment directed at brown factors, leading to directed technological innovation towards green infrastructures. The transition is said to take place if the economy does asymptotically operate with green capital only. We show that this happens if the carbon tax sufficiently increases the relative fossil energy price while directed innovation boosts the productivity of clean capital. Moreover, when a carbon tax is adequately combined with public subsidies

The rest of the paper is organised as follows. The following section sets up the two types of capital within an inventory dynamics with government intervention. It leads to the identification of a quantitative criterion for the green transition to take place, namely that the relative fossil energy price be higher than the productivity gap between the two types of capital. In section 3, we construct the 10-dimensional dynamical system followed by this dual economy, and study the local stability of its long-run steady states. In the next section, we refine the public policy of directed technical change so as to identify conditions that ensure that the economy will ultimately shift towards clean capital. The last section concludes.

## 4.2 Inventory dynamics with government intervention

We consider a four-sector closed economy,  $\mathcal{E}$ , comprising households, a banking sector, a public sector and private firms producing a single consumption good with brown or green capital. For the sake of simplicity, we restrict ourselves to the extreme case of two types of capital: brown capital on the one hand, which needs a constant flow of fossil energy to produce output ; and green capital on the other hand, which operates without fossil energy but is less productive than dirty capital.

### 4.2.1 Dirty *vs* clean capital

Output,  $Y$ , can be produced in two ways: either courtesy of a stock,  $K_1$ , of brown capital with a flow,  $E$ , of fossil energy and labour,  $L_1$ , or operating a stock of clean capital,  $K_2$ , with a flow,  $L_2$ , of labour alone.<sup>3</sup> For each type  $k = 1, 2$ , the capital-to-output ratio  $\nu_k$  is constant. Installed capital goods of any type cannot be converted into one another: If investors want to switch from one type to another, they have to invest into new capital goods. As regards output production, both types of outputs,  $Y_1$  and  $Y_2$ , are perfect substitutes:

$$Y = Y_1 + Y_2 \text{ with } Y_1 = \frac{K_1}{\nu_1} = \frac{E}{\nu_1} = a_1 L_1 \text{ and } Y_2 = \frac{K_2}{\nu_2} = a_2 L_2, \quad (1)$$

---

<sup>3</sup>To simplify notations, we don't introduce explicitly renewable energies, which might be used by both types of capital. such an addition could easily be done without significantly improving our results, nor their clarity.

where  $\nu_1, \nu_2, a_1, a_2 > 0$ . Full employment of capital is assumed throughout.<sup>4</sup> All quantities are expressed in real terms, and are net quantities, meaning that intermediate revenues and expenditures are deducted from the final yearly output.

As usually, capital accumulation depends upon investment,  $I$ , and a constant depreciation rate,  $\delta > 0$ , for both types of capital goods:

$$\forall k = 1, 2 \quad \dot{K}_k = I_k - \delta K_k. \quad (2)$$

For the sake of our argument, the two types of capital differ only by their productivity:  $\nu_1 > \nu_2$ . Therefore, we impose that they have the same “capital-labor ratio”:<sup>5</sup>

$$\frac{K_1}{\nu_1} = a_1 \nu_1 = a_2 \nu_2 = \frac{K_2}{\nu_2} \quad (3)$$

Labor productivities,  $a_1$  and  $a_2$ , are assumed to grow exponentially:

$$\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{a_2} = \alpha, \quad (4)$$

with the same (constant) growth rate,  $\alpha > 0$ , of Harrod-neutral technical progress. Total labor forces,  $N$ , grow at some deterministic time-dependent rate  $\hat{N} := \dot{N}/N = n(t) \geq 0$ . The employment rate is defined as  $\lambda := \frac{L(t)}{N(t)}$ .

### 4.2.2 Expected profits, taxes and subsidies

Let us now introduce inventories along the lines of [Grasselli and Nguyen-Huu, 2018], as well as government intervention (as in [Costa Lima and Grasselli, 2014]). For simplicity, let us restrict government intervention to simple “base-level” taxes and subsidies targeting the net profits of private non-financial firms. Getting rid of Say’s Law is a turnkey in order to tackle the issue of “degrowth” and consumption sobriety. The household’s sector is introduced in the same way as in [Giraud and Grasselli, 2019].

Let us denote total real demand by  $Y_d$ :

$$Y_d := C + I + G_E, \quad (5)$$

where  $I = I_1 + I_2$  is the total investment volume for both types of capital goods,  $C$  denotes aggregate consumption, and  $G_E$ , government expenditures (also in real terms).

The difference between output and demand determines changes in the level of inventories,  $V$ , held by firms:

$$\dot{V} := Y - Y_d. \quad (6)$$

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<sup>4</sup>For more thought on this assumption, see the conclusion.

<sup>5</sup>Another possible assumption would be that the productivity of labor is the same across the technologies, i.e.  $a_1 = a_2$ .

Let  $p \geq 0$  denote the unit price level for both investment, government expenditure and consumption good. Nominal sales are given by  $pY_d = pC + pI + pG_E$ . Let  $\mathbf{w}$  designate the unitary money wage,  $W := \mathbf{w}L$  the wage bill,  $\omega := \mathbf{w}L/pY$  the wage share, and  $c := \mathbf{w}L/Y = p\omega$ , the unit cost of production. Nominal output is

$$Y_n := pY_d + c\dot{V}. \quad (7)$$

In other words, we follow [Godley and Lavoie, 2006] (eq. (8.24) and (8.25)), [Grasselli and Nguyen-Huu, 2018] by valuing inventory changes at cost  $c$ , consistently with current accounting practices:  $c\dot{V}$  is what is incurred by the acquisition department of a firm<sup>6</sup> in order to purchase unsold goods from the production department.<sup>7</sup>

Let  $r \geq 0$  be a constant, nominal short-run interest rate charging private debts while  $D_f$  denotes the nominal stock of firms' debts. The pre-dividend net profit of non-financial firms,  $\Pi_f$  (after paying wages, interest on debt, energy purchase, taxes,  $T$ , and receiving subventions,  $G$ ) is given by:

$$\Pi_f := Y_n - W - rD_f - p_E E + G - T. \quad (8)$$

We must distinguish here this profit from their expected profit,  $\Pi_e$ , defined as:

$$\Pi_e := pY - W - rD_f - p_E E + G - T. \quad (9)$$

On the household side, as in [Giraud and Grasselli, 2019], the budget constraint implies that, whenever nominal household consumption exceeds their disposable income, the difference needs to be financed by an increase,  $\dot{D}_h$ , in debt. Conversely, if households' disposable income exceeds nominal household consumption, households' saving is used to reimburse (part of) their net aggregate debt. As banks are privately owned by households,<sup>8</sup> let us denote by  $\Delta_b$  the dividends received by households from banks. Let also assume that public debt is directly hold by households and constitutes another source of income. The households' disposable income is  $W + \Delta_b + r_g D_g - rD_h$ . The banks' current profit is  $\Pi_b := r(D_f + D_h)$ .<sup>9</sup> For simplicity again, we assume that, facing negligible operating costs, banks redistribute all their profits,  $\Delta_b := r(D_f + D_h)$ .

Current cash-flows validate past liabilities and form the basis for investment decisions. At variance with these authors, however, we introduce here a small twist by letting investment depend upon expected sales,  $pY$ , and not  $Y_n$ .<sup>10</sup> This enables to circumvent any circularity in the definition of profits (since  $Y_n$  itself depends upon current investments) and, quite realistically, make investment depend upon (a reduced-form of) entrepreneurs' expectations. Once more, for simplicity, we assume that non-financial firms inject all their profits,  $\Pi_f$ , for investment purposes (no redistribution of their expected profit to their shareholders).<sup>11</sup> Whence, changes in corporate debts are given by

$$\dot{D}_f := pI - \Pi_f. \quad (10)$$

---

<sup>6</sup>Represented in the capital account of Table [4.1], see below.

<sup>7</sup>Therefore,  $Y_n = pY$  if, and only if, either  $p = c$  or  $\dot{V} = 0$ .

<sup>8</sup>For simplicity, at variance with [Giraud and Grasselli, 2019], we don't draw any distinction between wage-earners and "rentiers" earning dividends.

<sup>9</sup>For the sake of simplicity, we assume the same interest rates for both households and firms debt.

<sup>10</sup>As a consequence, whenever  $Y_n < pY$ , firms' expected profit turns out to have been over-optimistic:  $\Pi_f < \Pi_e$ .

<sup>11</sup>See [Dossetto and Giraud, 2020] for an explicit treatment of dividends. By contrast, banks' dividends *must* be assumed to be redistributed in order to keep the stock-flow consistency of the dynamics.

The dynamics of households' debt can now be written as the difference between consumption spending and disposable income:

$$\begin{aligned}\dot{D}_h &:= pC - (W + \Delta_b + r_g D_g - rD_h) = pC - W - rD_f - r_g D_g \\ &= Y_n - c\dot{V} - pI - pG_E - W - rD_f - r_g D_g \\ &= -\dot{D}_f - \dot{D}_g - c\dot{V}.\end{aligned}\quad (11)$$

Changes in public debt,  $D_g$ , depend on the volume of public expenditures,  $G_E$ , and the State net revenues,  $T^*$ , which take into account revenues from taxes, the public debt servicing and the cost of subsidies to firms:

$$\begin{aligned}\dot{D}_g &= pG_E - T^* \\ \text{with } T^* &:= T - r_g D_g - G,\end{aligned}\quad (12)$$

where  $r_g$  is a constant, nominal short-run interest rate charged on sovereign debt, while  $G$  and  $T$  are base-level subsidies and taxes. The dynamics of the latter primarily depends on the overall state of the economy as measured by the level of output, the employment rate and non-financial firms' profits:

$$\begin{aligned}\dot{G}(t) &= \Gamma(\lambda)Y & \text{with } \dot{\Gamma}(\lambda) < 0, \\ \dot{T}(t) &= \Theta(\pi_f)Y & \text{with } \dot{\Theta}(\pi_f) > 0.\end{aligned}\quad (13)$$

We postulate the dynamics for expenditures to take the form:  $\dot{G}_E = \Gamma_E(\omega, \lambda, \pi_f, g, \tau, G_E, Y)$ .

### 4.2.3 Some accounting

The balance sheet, transactions, and flow of funds for this economy are described in Table [4.1]. Unless otherwise stated, all quantities are functions of time. Government debt,  $D_g$ , is the only liability of the public sector.<sup>12</sup> Following the simplifying assumptions of [Costa Lima and Grasselli, 2014], public sector does not keep bank deposits or make banks loans. Households' assets consist thus of government debt, firm equity,  $E_f$ , bank equity,  $E_b$  and current money deposits,  $M_h$ . Their liabilities consist of loans,  $L_h$ , resulting in net worth (i.e., wealth)  $X_h := E_h + M_h - L_h + D_g$ , with  $E_h := E_f + E_b$ . Firms' assets are the capital stock,  $pK$ , plus inventories,  $cV$ , and current money deposits,  $M_f$ , whereas their liabilities are their loans,  $L_f$ , and equity is  $E_f := pK + cV + M_f - L_f$ .<sup>13</sup> Similarly, loans,  $L := L_h + L_f$ , are the assets of the banking sector, and its liabilities are the deposits,  $M := M_h + M_f$ , whereas bank equity is  $E_b = L - M$ . Hence, the net worth of firms and banks remains  $X_f = X_b = 0$ . Again for simplicity, we assume that firms do not keep positive deposits, preferring to use any balances to repay their loans instead. As already said, all their net profits are reinvested:  $\Pi_f = S_f$ . Savings of the banking sector are given by<sup>14</sup>

$$S_b = rL - rM - \Delta_b = r(D_f + D_h) - \Delta_b = 0.\quad (14)$$

Finally, the financial balance row on Table [4.1] obeys the following *ex post* accounting identity between total nominal savings and investment in the economy  $\mathcal{E}$ :

$$S := S_h + S_f + S_b + S_g = pI + c\dot{V} - pEE.\quad (15)$$

---

<sup>12</sup>Government debt is not hold by firms or banks.

<sup>13</sup>Here, equity is treated as a balancing variable so that the net worth of non-financial firms is always  $X_f = 0$ .

<sup>14</sup>To save notations, we assume that banks have no consumption. Several alternative specifications of banks' behavior would be conceivable. Details can be found in [Giraud and Grasselli, 2019].

We eventually get an expression for total wealth,  $X$ , in  $\mathcal{E}$ :

$$\begin{aligned} X &= X_h + X_g = E_f + E_b + M_h - L_h + D_g - D_d \\ &= pK + cV. \end{aligned}$$

In words, total wealth equals the wealth of the households and the government, which in turn is equivalent to the productive assets of the corporate sector.

Aggregate behavioural rules will now complete the structure provided by Table [4.1].

#### 4.2.4 Aggregate behavioural rules

Business decisions will depend upon the rates of profit of each type of capital:

$$\pi_{K_1} := \frac{pY_1 - W_1 - p_E E}{pK_1} = \frac{1}{\nu_1} \left( 1 - \frac{\mathbf{w}}{a_1 p} \right) - \frac{p_E}{p} \quad \text{and} \quad \pi_{K_2} := \frac{1}{\nu_2} \left( 1 - \frac{\mathbf{w}}{a_2 p} \right).$$

The flow of investment,  $I$ , is allocated between the two types of capital goods according to the following rule:

$$I = I_1 + I_2 \quad \text{where} \quad I_1 = (1 - \theta)I \quad \text{and} \quad I_2 = \theta I, \quad (16)$$

where the variable,  $\theta \in [0, 1]$ , represents the proportion of investment directed towards green capital. Its dynamics is determined by the gap between the two rates of profit  $\pi_{K_1}$  and  $\pi_{K_2}$ :

$$\dot{\theta} = \sigma\theta(1 - \theta)(\pi_{K_2} - \pi_{K_1}), \quad (17)$$

where  $\sigma\theta(1 - \theta)$  characterizes the inertia of investment.<sup>15</sup> When  $\sigma \rightarrow +\infty$ , investment adjusts instantly as would be assumed in a neo-classical equilibrium model. Then, one would have  $\theta = 1$  as soon as  $\pi_{K_2} > \pi_{K_1}$  and  $\theta = 0$  otherwise.<sup>16</sup>

The aggregate investment function of firms is given by

$$I := \kappa(\pi_e)Y, \quad (18)$$

where  $\kappa(\cdot)$  is a continuous, increasing function of the normalized expected profit,  $\pi_e := \frac{\Pi_e}{pY}$ .

The rate of change in money wages is a function of the current employment rate and inflation:

$$\frac{\dot{\mathbf{w}}}{\mathbf{w}} := \phi(\lambda) + \gamma \frac{\dot{p}}{p}, \quad (19)$$

where  $\phi(\cdot)$  is an increasing short-run Phillips curve and  $\gamma \in [0, 1]$  is a measure of money illusion, where  $\gamma = 1$  corresponds to no monetary illusion (see, e.g., [Gordon, 2011] for a historical survey, [Gordon, 2013], [Mankiw, 2001] and [Mankiw, 2016], as well as [Grasselli and Nguyen Huu, 2016]).

<sup>15</sup>This inertia of investment can be thought of as capturing complementarity effects between the two types of capital that are not formalized by the production function, specialization of workforce or infrastructure externalities. Of course, a single parameter,  $\sigma$ , is not enough to translate the complexity of these effects. In particular, infrastructure externalities and learning-by-doing suggest that  $\sigma$  should depend on  $\varepsilon$ . But this dependency need not be symmetrical and we would thus have to distinguish between an adjustment speed,  $\sigma^+$ , when the transition is going forward and another one,  $\sigma^-$ , when the economy actually increases its investment in dirty capital. These improvements are outside the scope of this paper. For now, let us just keep in mind that a small  $\sigma$  means that investment reacts very slowly to a gap between the rates of profits.

<sup>16</sup>Obviously, the speed of investment adjustment depends upon the liquidity of markets and money velocity, see [Dossetto and Giraud, 2020]. These aspects would go beyond the scope of this paper, and  $\sigma$  will be treated as a constant parameter.

CHAPTER 4. DIRECTED TECHNICAL CHANGE IN AN INVENTORY STOCK-FLOW  
CONSISTENT DYNAMICS WITH GOVERNMENT: FROM DIRTY TO CLEAN CAPITAL

	Households	Firms	Banks	Gov.	Row Sum
<b>Balance sheet</b>					
Capital stock		$+pK$			$+pK$
Inventory		$+cV$			$+cV$
Public Debt	$D_g$			$-D_g$	0
Deposits	$+M_h$	$+M_f$	$-M$		0
Loans	$-L_h$	$-L_f$	$L$		0
Equity	$E$	$-E_f$	$-E_b$		0
Column sum (Net worth)	$X_h$	$X_f = 0$	$X_b = 0$	$X_g$	$X$
<b>Transactions</b>					
		Current	Capital		
Consumption	$-pC$	$+pC$			0
Investment		$+pI$	$-pI$		0
Government expenditure		$+pG_E$		$-pG_E$	0
Changes in inventory		$+c\dot{V}$	$-c\dot{V}$		0
Accounting memo [GDP]		$[Y_n]$			
Wages	$+W$	$-W$			0
Cost of Energy		$-p_E E$	$+p_E E$		0
Interest on loans	$-rL_h$	$-rL_f$		$+rL$	0
Interest on deposits	$+rM_h$	$+rM_f$		$-rM$	0
Interest on public debt	$+r_g D_g$			$-r_g D_g$	0
Subsidies		$+G$		$-G$	0
Taxation		$-T$		$+T$	0
Dividends	$\Delta_b$			$-\Delta_b$	0
Column sum (balances)	$S_h$	$\Pi_f$	$-pI - c\dot{V} + p_E E$	$S_b = 0$	$S_g$
<b>Flows of funds</b>					
Change in capital stock		$+p\dot{K}$			$+p\dot{K}$
Change in inventory		$c\dot{V}$			$c\dot{V}$
Change in deposits	$+M_h$	$+M_f$	$-M$		0
Change in loans	$-L_h$	$-L_f$	$+L$		0
Change in public debt	$+D_g$			$-D_g$	0
Column sum (savings)	$S_h$	$S_f$	$S_b$	$S_g$	$p\dot{K} + c\dot{V}$
Change in firm equity	$+E_f$	$-(S_f + p\dot{K} + c\dot{V})$			0
Change in bank equity	$+E_b$		$-S_b$		0
Change in net worth	$\dot{E} + S_h$	0	0	$S_g$	$(p\dot{K}) + (c\dot{V})$

Table 4.1: Balance sheet, transactions and flow of funds for the four-sector, closed economy with inventories and prices



As a proxy for aggregate consumption, we adopt a function,  $c(\cdot)$ , of disposable income (11):

$$C := c(\omega + rd_f + r_g d_g)Y, \quad (20)$$

where  $d_f := \frac{D_f}{pY}$  and  $d_g := \frac{D_g}{pY}$  are the normalized levels of public and private debts held by households.

A short-run price-wage dynamics is given by:<sup>17</sup>

$$i(\omega) := \frac{\dot{p}}{p} = \eta_p(m\omega - 1). \quad (21)$$

Prices relax at speed  $1/\eta_p > 0$  around the unit cost of production,  $c = p\omega$ , inflated by some markup,  $m \geq 1$ , whose magnitude depends upon the imperfection of competition on the commodity market. The markup epitomizes the gap between the average cost of production,  $wL/Y = c$ , and the (long-run) equilibrium price,  $p$ .

### 4.3 Long-run stability study

The real growth rate of the economy,  $\mathbf{g} := \hat{Y} := \dot{Y}/Y$ , can be expressed as:

$$\begin{aligned} \mathbf{g} &= \frac{1}{Y} \frac{d}{dt} \left( \frac{K_1}{\nu_1} + \frac{K_2}{\nu_2} \right) = \frac{1}{Y} \left[ \frac{I_1 - \delta K_1}{\nu_1} + \frac{I_2 - \delta K_2}{\nu_2} \right] \\ &= \frac{\kappa(\pi_e)}{\nu_\theta} - \delta, \end{aligned} \quad (22)$$

where  $\frac{1}{\nu_\theta} = \frac{1-\theta}{\nu_1} + \frac{\theta}{\nu_2}$  is the mean productivity of current investment.

The non-financial firms' debt ratio dynamics,  $d_f = D_f/pY$ , is defined by:

$$\frac{\dot{d}_f}{d_f} = \frac{\dot{D}_f}{D_f} - \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} = \frac{\kappa(\pi_e) - \pi_f}{d_f} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right), \quad (23)$$

where  $\pi_f := \frac{\Pi_f}{pY}$  is the normalized corporate net profit.

The households' debt ratio dynamics,  $d_h = D_h/pY$ , follows:

$$\begin{aligned} \frac{\dot{d}_h}{d_h} &= \frac{\dot{D}_h}{D_h} - \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} \\ &= \frac{c(\omega + rd_f + r_g d_g) - \omega - rd_f}{d_h} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right). \end{aligned} \quad (24)$$

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<sup>17</sup>See for example [Godard, 2007], Section 8.3.

The kinetics of the ratio of government expenditure,  $g_e := \frac{G_E}{Y}$ , tax ratio,  $\tau := \frac{T}{pY}$ , and subsidies ratio,  $g := \frac{G}{pY}$  can be respectively written:

$$\begin{aligned} \dot{g}_e &= \frac{\Gamma_E(\omega, \lambda, \pi_f, g, \tau, G_E, Y)}{Y} - g_e \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta \right), \\ \dot{\tau} &= \Theta(\pi_f) - \tau \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right), \\ \dot{g} &= \Gamma(\lambda) - g \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right). \end{aligned} \quad (25)$$

Thus the public debt ratio,  $d_g = \frac{D_g}{pY}$ , follows the ordinary differential equation:

$$\begin{aligned} \frac{\dot{d}_g}{d_g} &= \frac{\dot{D}_g}{D_g} - \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} \\ \Rightarrow \dot{d}_g &= g_e + g - \tau + d_g \left( r_g - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - i \right). \end{aligned} \quad (26)$$

Notice that, courtesy of (1),  $\varepsilon = \frac{Y_1}{Y} = \frac{E}{\nu_1 Y}$ , is directly related to the brown energy intensity of production. The variable  $\varepsilon$  can therefore serve as an index of the depth of the energy shift towards low-carbon energies:  $\varepsilon \rightarrow 1$  means that the economy remains entirely fossil-based in the long-run. On the contrary,  $\varepsilon \rightarrow 0$  means that, asymptotically, no dirty capital operates any longer. Its dynamics is given by:

$$\begin{aligned} \frac{\dot{\varepsilon}}{\varepsilon} &= \frac{\dot{Y}_1}{Y_1} - \frac{\dot{Y}}{Y} = \frac{\dot{K}_1}{\nu_1 Y_1} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta \right) = \frac{(1-\theta)I - \delta K_1}{\nu_1 Y_1} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta \right) = \frac{(1-\theta)I}{\nu_1 Y} \frac{Y}{Y_1} - \frac{\kappa(\pi_e)}{\nu_\theta} \\ \Leftrightarrow \dot{\varepsilon} &= \kappa(\pi_e) \left( \frac{1-\theta}{\nu_1} - \frac{\varepsilon}{\nu_\theta} \right) \end{aligned} \quad (27)$$

Hence, denoting  $\mu_\varepsilon := \frac{1-\varepsilon}{\nu_1} + \frac{\varepsilon}{\nu_2}$  and  $\Delta\nu := \frac{1}{\nu_1} - \frac{1}{\nu_2}$ ,

$$\dot{\varepsilon} = \frac{\kappa(\pi_e)}{\Delta\nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right).$$

The new auxiliary variable,  $\mu_\varepsilon$ , captures the mismatch between the share of green investment in the economy and the productivity gap between the two types of capital.

The wage share,  $\omega = \frac{W}{pY}$ , becomes:

$$\begin{aligned} \omega &= \mathbf{w} \left( \frac{L_1 + L_2}{pY} \right) = \mathbf{w} \left( \frac{L_1}{Y_1} \frac{Y_1}{pY} + \frac{L_2}{Y_2} \frac{Y_2}{pY} \right) = \mathbf{w} \left( \frac{\varepsilon}{a_1 p} + \frac{1-\varepsilon}{a_2 p} \right) \\ &= \frac{\mathbf{w}}{a_1 p} \left( \varepsilon + (1-\varepsilon) \frac{\nu_2}{\nu_1} \right) = \mathbf{w} \frac{\nu_2 \mu_\varepsilon}{a_1 p}, \end{aligned} \quad (28)$$

and consequently, its motion can be expressed as:

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathbf{w}}}{\mathbf{w}} - \frac{\dot{a}_1}{a_1} - \frac{\dot{p}}{p} + \frac{\dot{\mu}_\varepsilon}{\mu_\varepsilon} = \phi(\lambda) - (1-\gamma)i - \alpha - \frac{\dot{\varepsilon} \Delta\nu}{\mu_\varepsilon}. \quad (29)$$

As for the employment rate dynamics, it reads:

$$\begin{aligned} \lambda &= \frac{L}{N} = \frac{L}{Y} \frac{Y}{N} = \frac{\nu_2 \mu_\varepsilon}{a_1} \frac{Y}{N} \\ \Rightarrow \frac{\dot{\lambda}}{\lambda} &= \frac{\dot{\mu}_\varepsilon}{\mu_\varepsilon} - \frac{\dot{a}_1}{a_1} + \frac{\dot{Y}}{Y} - \frac{\dot{N}}{N} = \frac{\kappa(\pi_e)}{\nu_\theta} - \delta - \alpha - n - \frac{\dot{\Delta\nu}}{\mu_\varepsilon}. \end{aligned} \quad (30)$$

The new auxiliary variable,  $\Delta\nu$ , measures the productivity gap between the two types of capital. It allows us to express the difference between rates of profit as

$$\pi_{K_2} - \pi_{K_1} = \frac{pE}{p} - \Delta\nu = \chi_E - \Delta\nu, \quad (31)$$

where  $\chi_E := \frac{pE}{p}$  is the relative price of energy.

Thus, changes in the rate,  $\theta$ , which measures the proportion of investment directed towards  $K_2$ , can be written as:<sup>18</sup>

$$\dot{\theta} = \sigma\theta(1 - \theta)(\pi_{K_2} - \pi_{K_1}) = \sigma\theta(1 - \theta)(\chi_E - \Delta\nu). \quad (32)$$

It now becomes clear that  $\Delta\nu$  plays the role of a threshold for the ‘oil price’:<sup>19</sup> if  $\chi_E > \Delta\nu$ , oil is expensive enough to make the green technology attractive albeit being less productive. The share of green investment rises and the transition towards a zero-carbon economy takes place. But if  $\chi_E < \Delta\nu$ , oil remains cheap enough to make the green investment unattractive and no exit from the fossil fuel trap can spontaneously take place.

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<sup>18</sup>Its dynamics is determined by the gap between the two rates of profit weighted by the inertia of investment

<sup>19</sup>At first glance,  $\Delta\nu$  and  $\chi_E$  should not be of the same dimension. But we normalized capital stock  $K$  in (2) so that one unit of output invested provides one unit of capital, and energy consumption  $E$  in (1) so that one unit of brown capital uses one unit of energy per unit of time. As a consequence, the productivity gap  $\Delta\nu$  and the relative price of oil  $\chi_E$  both have the dimension,  $t^{-1}$ , of the inverse of time.

The overall dynamics therefore leads to a 10-dimensional non-linear system:<sup>20</sup>

$$\left\{ \begin{array}{l} \dot{\omega} = \phi(\lambda) - (1 - \gamma)i - \alpha - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} \\ \dot{\lambda} = \frac{\kappa(\pi_e)}{\nu_\theta} - \delta - \alpha - n - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} \\ \dot{d}_f = \kappa(\pi_e) - (1 - \omega)y_d + \chi_E\nu_1\varepsilon - g + \tau + d_f \left( r - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - i \right) \\ \dot{d}_h = c(\omega + rd_f + r_gd_g) - \omega - rd_f - d_h \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\ \dot{g}_e = \frac{\Gamma_E(\omega, \lambda, \pi, g, \tau, G_E, Y)}{Y} - g_e \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta \right) \\ \dot{d}_g = g_e + g - \tau + d_g \left( r_g - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - i \right) \\ \dot{g} = \Gamma(\lambda) - g \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\ \dot{\tau} = \Theta(\pi_f) - \tau \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\ \dot{\varepsilon} = \frac{\kappa(\pi_e)}{\Delta\nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1\nu_2} \right) \\ \dot{\theta} = \sigma\theta(1 - \theta)(\chi_E - \Delta\nu), \end{array} \right. \quad (33)$$

with the following auxiliary variables:

$$\left\{ \begin{array}{l} \pi_e = 1 - \omega - rd_f - \chi_E\nu_1\varepsilon + g - \tau \\ \frac{1}{\nu_\theta} = \frac{1 - \theta}{\nu_1} + \frac{\theta}{\nu_2} \\ \mu_\varepsilon = \frac{1 - \varepsilon}{\nu_1} + \frac{\varepsilon}{\nu_2} \\ \Delta\nu = \frac{1}{\nu_1} - \frac{1}{\nu_2} \\ y_d := \frac{Y_d}{Y} = c(\omega + rd_f + r_gd_g) + \kappa(\pi_e) + g_e \\ i = \eta_p(m\omega - 1). \end{array} \right. \quad (34)$$

One readily sees that the last two differential equations of (33) admit only two steady states  $(\varepsilon, \theta) = (0, 1)$  or  $(1, 0)$ . This corresponds respectively to an equilibrium operating and investing only in green capital ( $\theta = 1$  and  $\varepsilon = 0$ ) and another one with brown capital only ( $\theta = 0$  and  $\varepsilon = 1$ ).

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<sup>20</sup>As shown in Appendix C.1.2,  $d_h$  can be considered as an auxiliary variable of the system. For the same reason, and following [Costa Lima and Grasselli, 2014], we solve the  $g_e$ -equation separately, especially because  $g_e$  does not depend here on the public debt ratio  $d_g$ . Note that, if this was the case, one could still solve separately the couple  $(g_e, d_g)$  as auxiliary variables depending on the rest of the system.

It can be shown that, under quite reasonable conditions, detailed in Appendix C.1.2,<sup>21</sup> in each case, eight equilibrium points emerge. Hence, the vector field (33) exhibits 16 long-run steady states.

The most desirable one is a “good” steady state corresponding to a positive wage share and employment rate, and a finite level of debt ratios,  $d_g$ ,  $d_h$  and  $d_f$ , subsidies,  $g$ , and taxes,  $\tau$ . Following a remark from [Giraud and Grasselli, 2019], it is the analog of the Solovian-like because the real growth rate, at this steady state, equals labor productivity growth  $\alpha$  (technical progress) plus population growth,  $n$ . Moreover it verifies the Golden rule. Were we to use a CES production function, Solow’s workhorse model would become a particular case of the present one,<sup>22</sup> this equilibrium can be interpreted as the analog of the embedding of the unique equilibrium exhibited by the standard Solow model within our broader setting. This steady state, however, is no longer globally stable as in the Solow model, but turns out to be locally stable under standard assumptions.<sup>23</sup>

The second long-run equilibrium of (33) is a debt-deflationary steady state associated with a skyrocketing level of debt ratios while wages and employment shrink to zero. This long-term steady state is locally asymptotically unstable — which sharply contrasts with the takeaway of the previous literature where, usually, both the Solovian-like and the debt-deflationary equilibria are stable under reasonable conditions. As shown in Appendix C.1.4, the main reason for the instability of debt-deflationary equilibria is the introduction of energy in the production process.

Two other equilibria correspond to deflationary states with a finite debt level – one with zero wage, the other one with positive wages. Their local asymptotic stability cannot be judged a priori and must be checked on a case-by-case basis. The same holds the so-called *slavery* equilibrium (with no wages but employment). Eventually, there are three other infinite-valued equilibria. All of them, however, are structurally unstable and therefore irrelevant.

### Leading numerical Example

The baseline parameters for our numerical example are provided in Table [4.2].<sup>24</sup> We use a short-run Phillips curve of the form:

$$\phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0, \quad (35)$$

with parameters specified below. The investment function is taken from [Grasselli and Costa Lima, 2012]:  $\kappa(\pi_e) = k_0 + k_1 e^{k_2 \pi_e}$ . The tax and subsidies function are taken from [Costa Lima and Grasselli, 2014]  $\Theta(\pi_f) = \theta_0 + \theta_1 e^{\theta_2 \pi_f}$  and  $\Gamma(\lambda) = \gamma_0(1-\lambda)$ . Following [Grasselli and Nguyen-Huu, 2018], the consumption ratio is given by  $pC = c_1 W + c_2 D_f$ . These functions satisfy the assumptions in Appendix C.1.2.

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<sup>21</sup>See Appendix C.1.2 for a complete analysis of the existence and stability of equilibria.

<sup>22</sup>It could then be recovered by putting  $r = \gamma = 0$ ,  $m = 1$ ,  $c_h(x) \equiv x$ ,  $\kappa(\pi_e) \equiv \pi_e$ , and  $\varphi \equiv 0$ .

<sup>23</sup>See Figure 4.1a for the trajectories of the brown Solovian-like equilibrium and Figure 4.1b for the green one.

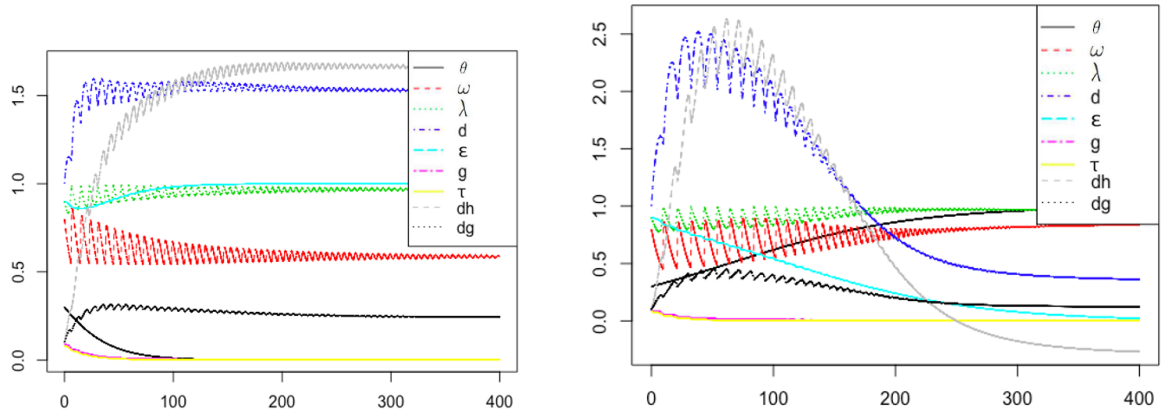
<sup>24</sup>See [Grasselli and Costa Lima, 2012], [McIsaac, 2016], [Grasselli and Nguyen-Huu, 2016] and [Giraud and Grasselli, 2019] for exhaustive justifications of these values.

CHAPTER 4. DIRECTED TECHNICAL CHANGE IN AN INVENTORY STOCK-FLOW CONSISTENT DYNAMICS WITH GOVERNMENT: FROM DIRTY TO CLEAN CAPITAL

Symbol & Definition		Value	Symbol & Definition		Value
$\alpha$	Labor productivity growth rate	0.025	$\phi_0$	Phillips curve $\phi(\lambda)$ parameter	$\frac{0.04}{1-0.04^2}$
$n$	Population growth rate	0.02	$\phi_1$	Phillips curve $\phi(\lambda)$ parameter	$\frac{0.04^3}{1-0.04}$
$\delta$	Depreciation rate	0.01	$k_0$	Investment function $\kappa(\pi_e)$ constant	-0.0065
$\nu_1$	brown capital-to-output ratio	2	$k_1$	Investment function $\kappa(\pi_e)$ parameter	$e^{-5}$
$\nu_2$	green capital-to-output ratio	3	$k_2$	$\kappa(\pi_e)$ exponential parameter	20
$r$	Nominal interest rate	3%	$r_g$	Public interest rate	0%
$\gamma$	Degree of monetary illusion	0.08	$m$	Prices mark-up	1.8
$c_1$	Fraction of income consumed	0.95	$c_2$	Fraction of wealth consumed	0.1
$\eta_p$	Speed of lagged adjustment	0.03	$\gamma_0$	Subsidies function $\Gamma(\lambda)$ constant	0.0045
$\theta_0$	Tax function $\Theta(\pi_f)$ constant	- 0.2	$\theta_1$	Tax function $\Theta(\pi_f)$ parameter	0.2
$\theta_2$	$\Theta(\pi_f)$ exponential parameter	0.01	$\theta_1$	Share of profits saved by firms	0.2
$p_E$	Cost of brown energy	0.18	$\sigma$	Inertia of green investments	1

Table 4.2: Parameters

As illustrated in Figure 4.1b, an initial positive value of  $g - \tau$  ensures the convergence of these trajectories to a green equilibrium. These values will decrease as the transition is effective ( $\varepsilon \rightarrow 0$ ). The private debt ratio is also declining in the long-run because profits are not impinged by fossil energy prices.



(a) Trajectory for  $p_E = 0.13$  from initial values:  $\omega = 0.6$ ,  $\lambda = 0.9$ ,  $d = 1$ ,  $\varepsilon = 0.9$ ,  $\theta = 0.3$ ,  $g = 0.003$ ,  $\tau = 0.08$  to *Brown Solovian-like* equilibrium with final values:  $\omega_1 = 0.55$ ,  $\lambda_1 = 0.97$ ,  $d_1 = 1.48$ ,  $\bar{\varepsilon} = 0.99$ ,  $\bar{\theta} = 1.8 \text{ E} - 07$ ,  $g_1 = 0.003$ ,  $\tau_1 = 0.006$ ,  $\pi_1 = 0.14$

(b) Trajectory for  $p_E = 0.18$  from initial values:  $\omega = 0.6$ ,  $\lambda = 0.9$ ,  $d = 1$ ,  $\varepsilon = 0.9$ ,  $\theta = 0.3$ ,  $g = 0.09$ ,  $\tau = 0.08$  to *Green Solovian-like* equilibrium with final values:  $\omega_1 = 0.82$ ,  $\lambda_1 = 0.97$ ,  $d_1 = 0.16$ ,  $\bar{\varepsilon} = 0.02$ ,  $\bar{\theta} = 0.99$ ,  $g_1 = 0.002$ ,  $\tau_1 = 0.005$ ,  $\pi_1 = 0.16$

Figure 4.1: Trajectories to the locally stable equilibria of (33)

As detailed in Appendix C.1.2, this Leading Example delivers some quantitative results in a neighborhood of the analog of the **Solovian-like equilibrium**  $(\omega_1, \lambda_1, \pi_{e_1}, \bar{\varepsilon}, \bar{\theta})$ , be it brown or green:

- The expected profit rate converges to  $\pi_{e_1}$  that is higher in the *Green Solovian-like* equilibrium

than in the *Brown Solovian-like* equilibrium where it is impinged by the price of fossil energy.

- The demand-to-output ratio,  $y_d$ , converges to  $y_{d_1} = \nu_\theta(\alpha + n + \delta) + c(\omega_1 + rd_{f_1}) + \bar{g}_e$ , which can be either higher or lower than 1, depending on the aggregate consumption function. With our Leading example,  $y_{d_1} = 0.83$  in the case of brown equilibrium whereas  $y_{d_1} = 0.99$  in the case of green equilibrium. Consequently, the *Brown Solovian-like* equilibrium is more overproducing than the *Green Solovian-like* equilibrium.<sup>25</sup> Notice that inventories,  $V$ , grow to infinity along this long-run steady state with overproduction ( $y_{d_1} < 1$ ) but plummet if  $y_{d_1} > 1$ . With the consumption function of our Leading Example, the latter situation is quite unlikely to happen.<sup>26</sup>
- The households' debt ratio converges to  $d_{h_1} = 1.65$  in the neighborhood of the *Brown Solovian-like* equilibrium and  $d_{h_1} = -0.18$  in the neighborhood of the green one.<sup>27</sup>
- Public debt ratio converges to  $d_{g_1} = 0.24$  in the neighborhood of the *Brown Solovian-like* equilibrium and  $d_{g_1} = 0.13$  in the neighborhood of the green one.
- The pre-dividend profit rate of firms,  $\pi_f$ , converge to  $\pi_{f_1} = \pi_{e_1} - (1 - \omega_1)(1 - y_{d_1}) = 0.05$  in the neighborhood of the *Brown Solovian-like* equilibrium and  $\pi_{f_1} = 0.14$  in the neighborhood of the green one.
- The inflation rate converges to  $i_1 = 1.5\%$  in the neighborhood of the *Brown Solovian-like* equilibrium and  $i_1 = 1.4\%$  in the neighborhood of the green one.

## 4.4 Introducing government levers

We are now interested in introducing governmental levers acting on the master transition criterion:

$$\chi_E - \Delta\nu > 0.$$

This can be achieved either by influencing the relative energy price,  $\chi_E$ , through a carbon tax (according to, e.g., the corridor of values established by [Stiglitz et al., 2017]) or by reducing the difference,  $\Delta\nu$ , in capital-to-output ratios courtesy of publicly subsidized directed technical change targeting the output-to-capital ratio of green capital. Although these two policy tools have been widely advocated for in the literature, their long-run efficiency has not been clearly demonstrated so far. When high enough, a carbon tax provides a strong incentive to shift from fossil fuel to renewable energies but at the cost of penalizing firms, reducing their profits, hence potentially fostering corporate debts. Conversely, public subsidies may boost sovereign indebtedness up to a non-sustainable level. Under which conditions will either of these forces prevail?

<sup>25</sup>This implies that the stock of inventories grows to infinity.

<sup>26</sup>Indeed,  $y_{d_1} > 1$  is equivalent to  $c(\omega_1 + rd_{f_1}) + \kappa(\pi_{e_1}) + \bar{g}_e > 1$ . In the neighborhood of the Solovian-like equilibrium,  $\kappa(\pi_{e_1}) = \nu_\theta(\alpha + n + \delta) \sim 10^{-2}$  under reasonable specifications. The consumption function,  $c(\cdot)$ , is a continuously strictly increasing function which tends to  $c_1\omega_1 + c_2rd_{f_1}$ , with  $c_1, c_2 > 0$ , and we can reasonably assume  $\bar{g}_e \sim 10^{-2}$  (government expenditures other than remuneration). Consequently  $y_{d_1} > 1$  means  $c_1(1 - \pi_{e_1} - rd_{f_1} - p_E\nu_1\bar{e} - \tau_1 + g_1) + c_2rd_{f_1} + \nu_\theta(\alpha + n + \delta) + \bar{g}_e > 1 \Leftrightarrow (c_2 - c_1)rd_{f_1} + \bar{g}_e > 1 - c_1 + c_1(\pi_{e_1} + p_E\nu_1\bar{e} - g_1 + \tau_1) - \nu_\theta(\alpha + n + \delta)$ , where  $1 - c_1 \sim 10^{-1}$ ,  $c_1(\pi_{e_1} + p_E\nu_1\bar{e} - g_1 + \tau_1) > 0$  and  $\nu_\theta(\alpha + n + \delta) \sim 10^{-1}$ . Thus, in order to guarantee  $y_{d_1} > 1$  for a broad range of parameter, we need  $c_2 - c_1 > 0$  which implies that the volume of wealth dedicated to consumption coming from the redistributed banks' profit be greater than the volume of income dedicated to consumption stemming from wages – which is very unlikely in practice.

<sup>27</sup>Indeed  $d_{h_1}^*$  can be deduced from others variables which converge.



#### 4.4.1 Introduction of a carbon tax

We first use the following simple form of carbon tax:

$$T_c = k_c E \quad (36)$$

where  $k_c > 0$  is a constant, so that  $T_c$  is directly proportional to the flow of fossil energy entering the economy. The stock-flow matrix with this carbon tax is displayed in Table 4.3 (see in red the changes made).

This carbon tax impacts the net profit of firms,  $\pi_f$ , as well as their expected profit,  $\pi_e$ :

$$\begin{aligned} \pi_f &= (1 - \omega)y_d - rd_f - \chi_E \nu_1 \varepsilon + g - \tau - \tau_c, \\ \pi_e &= 1 - \omega - rd_f - \chi_E \nu_1 \varepsilon + g - \tau - \tau_c, \end{aligned} \quad (37)$$

where  $\tau_c = T_c/pY = \frac{k_c}{p} \nu_1 \varepsilon = \chi_c \nu_1 \varepsilon$  with  $\chi_c := \frac{k_c}{p}$ .

Consequently, the difference between the two rates of profit becomes

$$\pi_{K_2} - \pi_{K_1} = \frac{1 - \frac{w}{a_2}}{\nu_2} - \frac{1 - \frac{w}{a_1}}{\nu_1} + \frac{pE + k_c}{p} = \chi_E + \chi_c - \Delta\nu$$

and, thus, the new dynamics of green investment,  $\theta^*$ , is:

$$\dot{\theta}^* = \sigma \theta^* (1 - \theta^*) (\chi_E + \chi_c - \Delta\nu). \quad (38)$$

Unsurprisingly, the carbon tax also reduces the public debt ratio:<sup>28</sup>

$$\dot{d}_g = g_e + g - \tau - \tau_c + d_g \left( r_g - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - i \right).$$

Eventually, expected profit in (4) is also modified by  $\chi_c$ :

$$\begin{aligned} \dot{\pi}_e &= -\omega(\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + [1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon] \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\ &\quad - r [\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)] + \dot{\varepsilon} \left( \omega \frac{\Delta\nu}{\mu_\varepsilon} - pE\nu_1 \right). \end{aligned} \quad (39)$$

Apart from these mild changes, the differential system is nearly identical to (33).

The Jacobian matrix of this new system is similar to (40) (see Appendix C.1 for the whole study) except:

$$\frac{\partial f_5}{\partial \varepsilon} = \sigma \theta^* (1 - \theta^*) \chi_c \nu_1 \quad (40)$$

For each equilibrium of the system (33),  $\theta^* \rightarrow 0$  or 1, making the above term null. Consequently the Jacobian matrix at each equilibrium is the same as before, apart of the last eigenvalue. This last eigenvalue is now  $\sigma(1 - 2\bar{\theta})(\chi_E + \chi_c - \Delta\nu)$  at *Solovian-like* and debt-deflationary equilibria, whether brown or green.

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<sup>28</sup>Despite this modification,  $d_g$  remains an auxiliary variable of the system and can therefore be removed from the stability study.

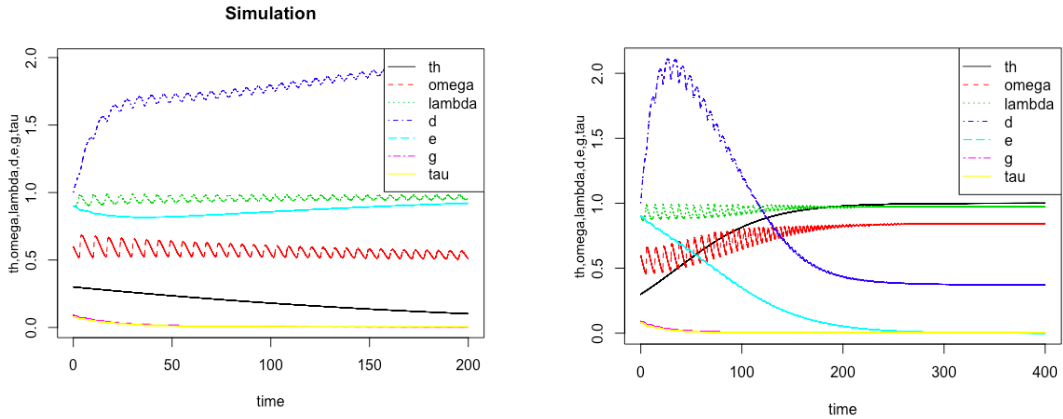
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	Households	Firms	Banks	Gov.	Row Sum
<b>Balance sheet</b>					
Capital stock		$+pK$			$+pK$
Inventory		$+cV$			$+cV$
Public Debt	$D_g$			$-D_g$	0
Deposits	$+M_h$	$+M_f$	$-M$		0
Loans	$-L_h$	$-L_f$	$L$		0
Equity	$E$	$-E_f$	$-E_b$		0
Column sum (Net worth)	$X_h$	$X_f = 0$	$X_b = 0$	$X_g$	$X$
<b>Transactions</b>					
		Current	Capital		
Consumption	$-pC$	$+pC$			0
Investment		$+pI$	$-pI$		0
Government expenditure		$+pG_E$		$-pG_E$	0
Changes in inventory		$+c\dot{V}$	$-c\dot{V}$		0
Accounting memo [GDP]		$[Y_n]$			
Wages	$+W$	$-W$			0
Cost of Energy		$-p_E E$	$+p_E E$		0
Interest on loans	$-rL_h$	$-rL_f$	$+rL$		0
Interest on deposits	$+rM_h$	$+rM_f$	$-rM$		0
Interest on public debt	$+r_g D_g$			$-r_g D_g$	0
Subsidies		$+G$		$-G$	0
Taxation		$-T - T_c$		$+T + T_c$	0
Dividends	$\Delta_b$		$-\Delta_b$		0
Column sum (balances)	$S_h$	$\Pi_f$	$-pI - c\dot{V} + p_E E$	$S_b = 0$	$S_g$
<b>Flows of funds</b>					
Change in capital stock		$+p\dot{K}$			$+p\dot{K}$
Change in inventory		$c\dot{V}$			$c\dot{V}$
Change in deposits	$+M_h$	$+M_f$	$-M$		0
Change in loans	$-L_h$	$-L_f$	$+L$		0
Change in public debt	$+D_g$			$-D_g$	0
Column sum (savings)	$S_h$	$S_f$	$S_b$	$S_g$	$p\dot{K} + c\dot{V}$
Change in firm equity	$+E_f$	$-(S_f + p\dot{K} + c\dot{V})$			0
Change in bank equity	$+E_b$		$-S_b$		0
Change in net worth	$\dot{E} + S_h$	0	0	$S_g$	$(p\dot{K}) + (c\dot{V})$

Table 4.3: Balance sheet, transactions and flow of funds for the four-sector, closed economy with inventories, prices and carbon tax

As a conclusion, the transition and the local stability criteria are identical:  $\chi_E + \chi_c - \Delta\nu > 0$ . For a sufficiently high  $\chi_c := k_c/p$ , this criterion is satisfied, the green Solovian-like steady state is locally asymptotically stable, and if the economy converges, it must become asymptotically green (which means that it started in the basin of attraction of this equilibrium) because the dirty equilibria are unstable. Conversely, if the carbon tax remains too low, the economy cannot help but, if it converges, do so towards one of the several fossil-fuel intensive equilibria because green steady-states are unstable.

With our Leading Example, we can display the following trajectories with different values of  $\chi_c$ :  $\chi_c = 0.03$ , no transition occurs or  $\chi_c = 0.06$ , the transition criterion is satisfied (See Figure 4.2).



(a) Trajectories with  $\chi_c = 0.03$ : no transition occurs, the system asymptotically converges to the Brown Solovian-like equilibrium as  $\chi_E + \chi_c - \Delta\nu < 0$

(b) Trajectories with  $\chi_c = 0.06$ : the transition criterion is satisfied,  $\chi_E + \chi_c - \Delta\nu > 0$ , and the system asymptotically converge to the Green Solovian-like equilibrium

Figure 4.2: **Trajectories along a path to the Brown Solovian-like equilibrium of (55),  $\chi_E = 0.13$ , with carbon tax only.**

#### 4.4.2 Innovation and directed technical change

In order to satisfy the transition criterion  $\chi_E - \Delta\nu > 0$ , the previous section introduced a carbon tax so as to rise the fossil energy price. Another way to guarantee that the economy goes green consists in reducing the capital-to-output ratio of green capital. As  $\Delta\nu = \frac{1}{\nu_1} - \frac{1}{\nu_2}$  will shrink, the transition criterion will be more easily satisfied at the cost of more public debt.

Inspired by the classical capital accumulation equation, we propose the following dynamics for a stock of knowledge which impacts the labor productivity and so the capital-to-output ratio,  $\nu_2$ , of the green capital. The dynamics of knowledge is assumed to follow:

$$\dot{R} = I_{RD} - \delta_{RD}R, \quad (41)$$

where  $\dot{R}$  measures the speed at which innovations emerge and  $I_{RD}$  denotes investment in R&D:

$$I_{RD} = \theta_{RD}I_2 + \sigma_{RD}\tau_c Y.$$

In words, a share,  $\theta_{RD} \in [0, 1]$ , of investment in green capital,  $I_2$ , is dedicated to R&D while a fraction,  $\sigma_{RD} \in [0, 1]$ , of the carbon tax revenues is used to fund further investments. The parameter,  $\delta_{RD} > 0$ , captures the pace at which knowledge becomes obsolete.<sup>29</sup>

Relaxing the assumption that  $\frac{\dot{a}_1}{a_1} = \frac{\dot{a}_2}{a_2}$ , and keeping  $a_1$  still equal to  $a_1(t) = A_1 e^{\alpha t}$ , we get the transmission of this knowledge to the labor productivity:

$$a_2(t) = \rho^{\sigma_\nu} A_2 e^{\alpha t} \quad (42)$$

where  $\rho := R/K_2$  is the ratio of knowledge compared to green capital, and  $\sigma_\nu$  is a coefficient representing the incomplete transmission of knowledge.

It follows from (3), that the dynamics of the green capital,  $\nu_2 = \frac{a_1 \nu_1}{a_2}$ , can be written as a function of the evolution of knowledge:

$$\frac{\dot{\nu}_2}{\nu_2} = \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} = \alpha - (\alpha + \sigma_\nu \hat{\rho}) = -\sigma_\nu \hat{\rho} \quad (43)$$

Note that the evolution of the knowledge-to-capital ratio is given by:

$$\hat{\rho} = \hat{R} - \hat{K}_2 \Rightarrow \dot{\rho} = \frac{\theta_{RD} \theta \kappa(\pi_e) + \sigma_{RD} \tau_c - \rho \theta \kappa(\pi_e)}{\nu_2 (1 - \varepsilon)}. \quad (44)$$

The stock-flow matrix with carbon tax and directed technical change capturing green R&D is displayed in Table 4.4 (see in red the changes made). Firms' assets are still the capital stock,  $pK$ , plus inventories,  $cV$ , and current money deposits,  $M_f$ , whereas their liabilities are their loans,  $L_f$ , and equity is  $E_f := pK + cV + M_f - L_f$ , but, now, the investment of firms is either capital investment,  $I = I_1 + I_2(1 - \theta_{RD})$ , or R&D investment,  $I_{RD}$ , as defined in (41).

### Dynamics with innovation and public subsidies

The real growth rate of the economy,  $\mathbf{g}$ , can now be expressed as:

$$\begin{aligned} \mathbf{g} &= \frac{1}{Y} \frac{d}{dt} \left( \frac{K_1}{\nu_1} + \frac{K_2}{\nu_2} \right) = \frac{1}{Y} \left[ \frac{I_1 - \delta K_1}{\nu_1} + \frac{I_2 - \delta K_2}{\nu_2} - \frac{\dot{\nu}_2 K_2}{\nu_2^2} \right] \\ &= \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon), \end{aligned} \quad (45)$$

where  $\pi_e$ , as before, depends upon the carbon tax:  $\pi_e = 1 - \omega - r d_f - \chi_E \nu_1 \varepsilon + g - \tau - \tau_c$ .

Consequently, the non-financial firms' debt ratio dynamics,  $d_f = D_f/pY$ , is defined by:

$$\frac{\dot{d}_f}{d_f} = \frac{\dot{D}_f}{D_f} - \mathbf{g} - \frac{\dot{p}}{p} = \frac{\kappa(\pi_e) - \pi_f}{d_f} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right). \quad (46)$$

In the same way, the households' debt ratio dynamics,  $d_h = D_h/pY$ , follows:

$$\begin{aligned} \frac{\dot{d}_h}{d_h} &= \frac{\dot{D}_h}{D_h} - \mathbf{g} - \frac{\dot{p}}{p} \\ &= \frac{c(\omega + r d_f + r_g d_g) - \omega - r d_f}{d_h} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right). \end{aligned} \quad (47)$$

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<sup>29</sup>A more realistic model would assume that the obsolescence rate of knowledge is itself a function of the speed of innovations. This type of refinement is left for further research.

CHAPTER 4. DIRECTED TECHNICAL CHANGE IN AN INVENTORY STOCK-FLOW  
CONSISTENT DYNAMICS WITH GOVERNMENT: FROM DIRTY TO CLEAN CAPITAL

	Households	Firms	Banks	Gov.	Row Sum
<b>Balance sheet</b>					
Capital stock		$+pK$			$+pK$
Inventory		$+cV$			$+cV$
Public Debt	$D_g$			$-D_g$	0
Deposits	$+M_h$	$+M_f$	$-M$		0
Loans	$-L_h$	$-L_f$	$L$		0
Equity	$E$	$-E_f$	$-E_b$		0
Column sum (Net worth)	$X_h$	$X_f = 0$	$X_b = 0$	$X_g$	$X$
<b>Transactions</b>					
Consumption	$-pC$	Current $+pC$	Capital		0
Investment		$+pI + pI_{RD}$	$-pI - p\theta_{RD}I_2$	$-p\sigma_{RD}T_c$	0
Government expenditure		$+pG_E$		$-pG_E$	0
Changes in inventory		$+c\dot{V}$	$-c\dot{V}$		0
Accounting memo [GDP]		$[Y_n]$			
Wages	$+W$	$-W$			0
Cost of Energy		$-p_E E$	$+p_E E$		0
Interest on loans	$-rL_h$	$-rL_f$	$+rL$		0
Interest on deposits	$+rM_h$	$+rM_f$	$-rM$		0
Interest on public debt	$+r_g D_g$			$-r_g D_g$	0
Subsidies		$+G$		$-G$	0
Taxation		$-T - T_c$		$+T + T_c$	0
Dividends	$\Delta_b$			$-\Delta_b$	0
Column sum (balances)	$S_h$	$\Pi_f$	$-pI - p\theta_{RD}I_2 - c\dot{V} + p_E E$	$S_b = 0$	$S_g$
<b>Flows of funds</b>					
Change in capital stock		$+p\dot{K}$			$+p\dot{K}$
Change in inventory		$+c\dot{V}$			$+c\dot{V}$
Change in deposits	$+M_h$	$+M_f$	$-M$		0
Change in loans	$-L_h$	$-L_f$	$+L$		0
Change in public debt	$+D_g$			$-D_g$	0
Column sum (savings)	$S_h$	$S_f$	$S_b$	$S_g$	$p\dot{K} + c\dot{V}$
Change in firm equity	$+E_f$	$-S_f - c\dot{V} - p\dot{K}$			0
Change in bank equity	$+E_b$		$-S_b$		0
Change in net worth	$\dot{E} + S_h$			$S_g$	$(p\dot{K}) + (c\dot{V})$

Table 4.4: Balance sheet, transactions and flow of funds for the four-sector, closed economy with inventories, prices and carbon tax

The kinetics of the government expenditure ratio,  $g_e := \frac{G_E}{Y}$ , tax ratio,  $\tau := \frac{T}{pY}$ , and subsidy ratio,  $g := \frac{G}{pY}$ , can be respectively written:

$$\begin{aligned} \dot{g}_e &= \frac{\Gamma_E(\omega, \lambda, \pi_f, g, \tau, G_E, Y)}{Y} - g_e \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) \right), \\ \dot{\tau} &= \Theta(\pi_f) - \tau \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) + i \right), \\ \dot{g} &= \Gamma(\lambda) - g \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) + i \right). \end{aligned} \quad (48)$$

Since the carbon tax is unchanged:  $\tau_c = T_c/pY = \frac{k_c}{p}\nu_1\varepsilon = \chi_c\nu_1\varepsilon$  with  $\chi_c := \frac{k_c}{p}$ , the public debt ratio,  $d_g = \frac{D_g}{pY}$ , evolves according to

$$\begin{aligned} \frac{\dot{d}_g}{d_g} &= \frac{\dot{D}_g}{D_g} - \mathbf{g} - \frac{\dot{p}}{p} \\ \Rightarrow \dot{d}_g &= g_e + g - \tau - \tau_c + d_g \left( r_g - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - \sigma_\nu \hat{\rho}(1 - \varepsilon) - i \right). \end{aligned} \quad (49)$$

The dynamics of the share of output produced out of brown capital,  $\varepsilon := Y_1/Y$ , is given by:

$$\begin{aligned} \frac{\dot{\varepsilon}}{\varepsilon} &= \frac{\dot{Y}_1}{Y_1} - \frac{\dot{Y}}{Y} = \frac{\dot{K}_1}{\nu_1 Y_1} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) \right) \\ &= \frac{(1 - \theta)I - \delta K_1}{\nu_1 Y_1} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) \right) = \frac{(1 - \theta)I}{\nu_1 Y} \frac{Y}{Y_1} - \frac{\kappa(\pi_e)}{\nu_\theta} - \sigma_\nu \hat{\rho}(1 - \varepsilon) \\ \Rightarrow \dot{\varepsilon} &= \kappa(\pi_e) \left( \frac{1 - \theta}{\nu_1} - \frac{\varepsilon}{\nu_\theta} \right) - \sigma_\nu \hat{\rho}(1 - \varepsilon)\varepsilon = \frac{\kappa(\pi_e)}{\Delta\nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) - \sigma_\nu \hat{\rho}(1 - \varepsilon)\varepsilon. \end{aligned} \quad (50)$$

The wage share,  $\omega = \frac{W}{pY}$ , still equals:

$$\omega = \mathbf{w} \left( \frac{L_1 + L_2}{pY} \right) = \mathbf{w} \left( \frac{L_1}{Y_1} \frac{Y_1}{pY} + \frac{L_2}{Y_2} \frac{Y_2}{pY} \right) = \mathbf{w} \left( \frac{\varepsilon}{a_1 p} + \frac{1 - \varepsilon}{a_2 p} \right) = \frac{\mathbf{w}}{a_1 p} \left( \varepsilon + (1 - \varepsilon) \frac{\nu_2}{\nu_1} \right) = \mathbf{w} \frac{\nu_2 \mu_\varepsilon}{a_1 p}, \quad (51)$$

where  $\mu_\varepsilon = \frac{1 - \varepsilon}{\nu_1} + \frac{\varepsilon}{\nu_2}$  and consequently  $\dot{\mu}_\varepsilon = -\Delta\nu\dot{\varepsilon} + \frac{\sigma_\nu \hat{\rho} \varepsilon}{\nu_2}$ . Consequently, the wage share dynamics can now be expressed as:

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathbf{w}}}{\mathbf{w}} + \frac{\dot{\nu}_2}{\nu_2} - \frac{\dot{a}_1}{a_1} - \frac{\dot{p}}{p} + \frac{\dot{\mu}_\varepsilon}{\mu_\varepsilon} = \phi(\lambda) - (1 - \gamma)i - \alpha - \sigma_\nu \hat{\rho} - \frac{\dot{\varepsilon} \Delta\nu}{\mu_\varepsilon} + \frac{\sigma_\nu \hat{\rho} \varepsilon}{\nu_2 \mu_\varepsilon}. \quad (52)$$

The employment rate dynamics becomes:

$$\begin{aligned} \lambda &= \frac{L}{N} = \frac{L}{Y} \frac{Y}{N} = \frac{\nu_2 \mu_\varepsilon}{a_1} \frac{Y}{N} \\ \Rightarrow \frac{\dot{\lambda}}{\lambda} &= \frac{\dot{\mu}_\varepsilon}{\mu_\varepsilon} + \frac{\dot{\nu}_2}{\nu_2} - \frac{\dot{a}_1}{a_1} + \mathbf{g} - \frac{\dot{N}}{N} = \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) - \alpha - \sigma_\nu \hat{\rho} - n - \frac{\dot{\varepsilon} \Delta\nu}{\mu_\varepsilon} + \frac{\sigma_\nu \hat{\rho} \varepsilon}{\nu_2 \mu_\varepsilon}. \end{aligned} \quad (53)$$

The gap between rates of profit remains unchanged:  $\pi_{K_2} - \pi_{K_1} = \frac{1-\frac{w}{a_2}}{\nu_2} - \frac{1-\frac{w}{a_1}}{\nu_1} + \frac{p_E+k_c}{p} = \chi_E + \chi_c - \Delta\nu$  and so is the motion of the share  $\theta^*$ , of green investment:

$$\dot{\theta}^* = \sigma\theta^*(1-\theta^*)(\chi_E + \chi_c - \Delta\nu). \quad (54)$$

The differential system followed by our economy is now 11-dimensional:

$$\left\{ \begin{array}{l} \frac{\dot{\omega}}{\omega} = \phi(\lambda) - (1-\gamma)i - \alpha - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \sigma_\nu\hat{\rho}\left(1 - \frac{\varepsilon}{\nu_2\mu_\varepsilon}\right) \\ \frac{\dot{\lambda}}{\lambda} = \frac{\kappa(\pi_e)}{\nu_\theta} - \delta - \alpha - n - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \sigma_\nu\hat{\rho}\varepsilon\left(1 - \frac{1}{\nu_2\mu_\varepsilon}\right) \\ \dot{d}_f = \kappa(\pi_e) - (1-\omega)y_d + \chi_E\nu_1\varepsilon - g + \tau + \tau_c + d_f\left(r - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - \sigma_\nu\hat{\rho}(1-\varepsilon) - i\right) \\ \dot{d}_h = c(\omega + rd_f + r_gd_g) - \omega - rd_f - d_h\left(\frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu\hat{\rho}(1-\varepsilon) + i\right) \\ \dot{g}_e = \frac{\Gamma_E(\omega, \lambda, \pi, g, \tau, \tau_c, G_E, Y)}{Y} - g_e\left(\frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu\hat{\rho}(1-\varepsilon)\right) \\ \dot{d}_g = g_e + g - \tau - \tau_c + d_g\left(r_g - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - \sigma_\nu\hat{\rho}(1-\varepsilon) - i\right) \\ \dot{g} = \Gamma(\lambda) - g\left(\frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu\hat{\rho}(1-\varepsilon) + i\right) \\ \dot{\tau} = \Theta(\pi_f) - \tau\left(\frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu\hat{\rho}(1-\varepsilon) + i\right) \\ \dot{\varepsilon} = \frac{\kappa(\pi_e)}{\Delta\nu}\left(\frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1\nu_2}\right) - \sigma_\nu\hat{\rho}(1-\varepsilon)\varepsilon \\ \dot{\theta}^* = \sigma\theta^*(1-\theta^*)(\chi_E + \chi_c - \Delta\nu) \\ \dot{\rho} = \frac{(\theta_{RD} - \rho)\theta\kappa(\pi_e) + \sigma_{RD}\tau_c}{\nu_2(1-\varepsilon)} \end{array} \right. \quad (55)$$



### Auxiliary variables

$$\left\{ \begin{array}{l} \hat{\nu}_2 = -\sigma_\nu \hat{\rho} \\ \tau_c = \chi_c \nu_1 \varepsilon \\ \pi_e = 1 - \omega - rd_f - \chi_E \nu_1 \varepsilon + g - \tau - \tau_c \\ i = \eta_p (m\omega - 1). \\ \frac{1}{\nu_\theta} = \frac{1 - \theta}{\nu_1} + \frac{\theta}{\nu_2} \\ \mu_\varepsilon = \frac{1 - \varepsilon}{\nu_1} + \frac{\varepsilon}{\nu_2} \\ \Delta\nu = \frac{1}{\nu_1} - \frac{1}{\nu_2} \\ y_d := \frac{Y_d}{Y} = c(\omega + rd_f + r_g d_g) + \kappa(\pi_e) + g_e \\ \pi_e - \pi_f = (1 - \omega)(1 - y_d) \end{array} \right. \quad (56)$$

Under the same reasonable conditions as before (detailed in Appendix C.1.2),<sup>30</sup> the vector field (55) exhibits the same types of steady states —be they brown or green depending upon to the solution of the  $(\varepsilon, \theta)$ -equations. The main departure with the previous section, however, arises from the introduction of the variable  $\nu_2$ : new equilibria appear in which  $\varepsilon$  and  $\theta$  need not equal 0 nor 1. These new steady states are less manichean than before, since they are compatible with the coexistence of green and brown capital in the long run. But they raise, of course, new questions: under which conditions will the economy converge to a zero-carbon sustainable state, as opposed to a *low*-carbon situation where, even in the long-term, dirty capital would still operate, albeit in a moderate way?

For each of the clean variants of the three nature of equilibria, we still have locally stable equilibria under standard assumptions : the desirable situation with positive wages and employment, and a finite level of debt ratios,  $d_g$ ,  $d_h$  and  $d_f$ , subsidies  $g$  and tax  $\tau$ . As before debt-deflationary steady state associated with a skyrocketing level of debt ratios while wages and employment shrink to zero are locally unstable. What is new, however, is that, under the following technical assumption, the brown Solovian-like steady state turns out to be locally unstable as well:

**Assumption C** *The ratio,  $\rho := R/K_2$ , is such that its time-derivative,  $\dot{\rho}$ , remains bounded from above whenever  $K_2 \rightarrow 0^+$ .*

This restriction amounts to saying that, whenever clean capital shrinks to zero (because of the lack of green investment and exponential depreciation), the ecological know-how encapsulated in  $R$  shrinks even more rapidly. This can be achieved, e.g., by imposing that the decay rate of knowledge,  $\delta_{RD}$ , in (41) be such that  $\delta_{RD} > \delta$ .<sup>31</sup>

### Leading numerical Example

<sup>30</sup>See Appendix C.2 for a complete analysis of the existence and stability of equilibria.

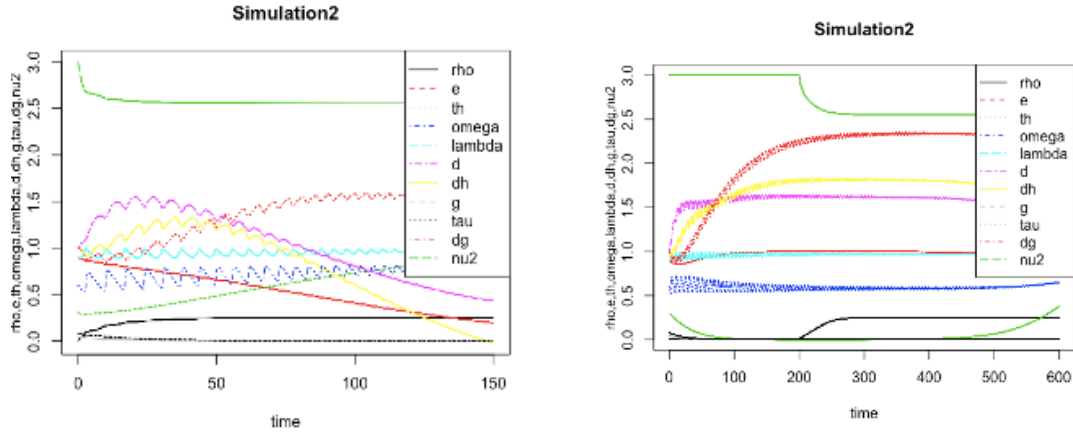
<sup>31</sup>The other equilibria also appear to be locally unstable, and will not be further considered. We let the reader verify they are still locally or structurally unstable with the new system (55).

Symbol & Definition		Value	Symbol & Definition		Value
$\chi_c$	Carbon tax ratio	0.005	$\theta_{RD}$	Share of investment in R&D	0.25
$\sigma_\nu$	Knolwedge transmission factor	0.05	$\sigma_{RD}$	Carbon tax share to public R&D	0.25
lag	Delay before intervention	0			

Table 4.5: **New Parameters**

The baseline parameters for our simulations are the same as in Table [4.2]<sup>32</sup> to which we have added the parameterization of Table [4.5]. Depending on these values, transition from a initial state leading to brown equilibria ( $\chi_e = 0.13$ ,  $\nu_1 = 2$ ,  $\nu_2 = 3$ ) occurs or not.

Assuming first, that public and private fundings for green R&D are equal, we find that transition occurs whatever is the delay of intervention (lag = 0 in Figure 4.3a or lag = 200 - which corresponds to almost a century - Figure 4.3b) and whether there is a carbon tax or not. The more the intervention is delayed, the longer the transition is (around half a century with no lag while it last almost two century with lag = 200) and consequently the volume of private and public debt is even more important.



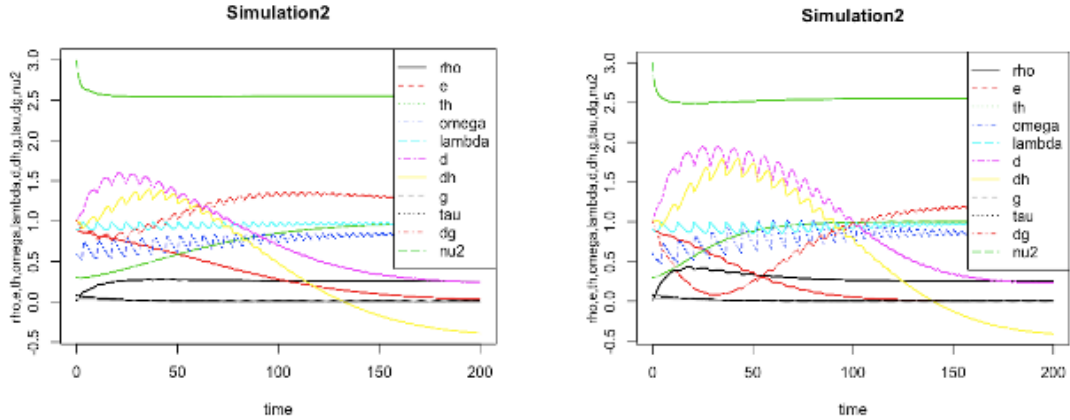
(a) Trajectories with no carbon tax  $\chi_c = 0$  and no delay in intervention lag = 0: transition occurs, the system asymptotically converges to the Green Solovian-like equilibrium because  $\nu_2$  is such that  $\chi_E - \Delta\nu$  becomes positive

(b) Trajectories with no carbon tax  $\chi_c = 0$  and an important delay lag = 200: transition occurs because the criterion will be satisfied,  $\chi_E + \chi_c - \Delta\nu > 0$ , and eventually the system asymptotically converge to the Green Solovian-like equilibrium

Figure 4.3: **Trajectories of (55) from a path leading to the Brown Solovian-like equilibrium,  $\chi_E = 0.13$  to the Green Solovian-like equilibrium, with no carbon tax**

The introduction of carbon tax reduces the public debt during the transition phase and accelerates the transition. See Figure 4.4a with a small carbon tax,  $\chi_c = 0.005$ , where the transition phase lasts 150 time steps (around 75 years) compared with Figure 4.4b with an important carbon tax,  $\chi_c = 0.03$ , where the transition phase lasts 'only' 50 time steps (around 25 years).

<sup>32</sup>See [Grasselli and Costa Lima, 2012], [McIsaac, 2016], [Grasselli and Nguyen Huu, 2016] and [Giraud and Grasselli, 2019] for exhaustive justifications of these values.



(a) Trajectories with small carbon tax  $\chi_c = 0.005$  and no delay in intervention lag = 0: transition slowly occurs

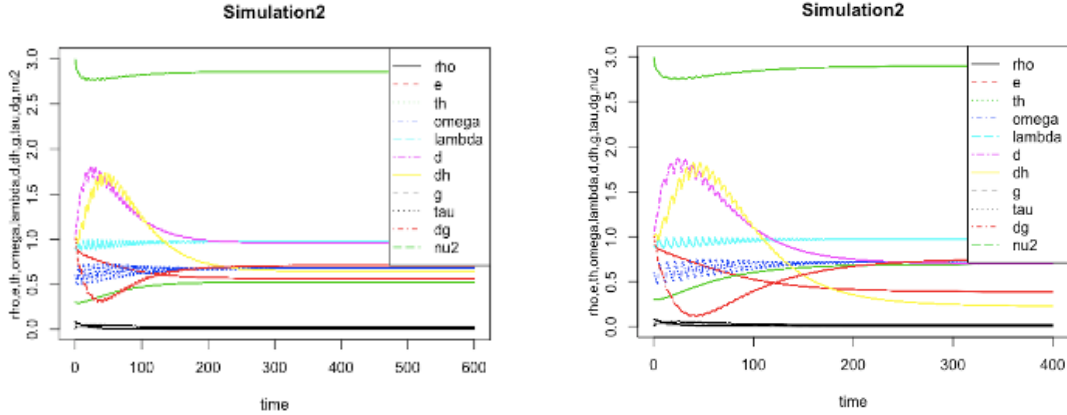
(b) Trajectories with important carbon tax  $\chi_c = 0.03$  and no delay lag = 0: transition quickly occurs

Figure 4.4: Trajectories of (55) from a path leading to the Brown Solovian-like equilibrium,  $\chi_E = 0.13$  to the Green Solovian-like equilibrium, with different carbon taxes

### Hybrid equilibria

Apart from the simplistic equilibria, all-green or all-brown, of the previous section, the dynamical system with government intervention (55) displays new long-run steady states where both brown and green capital coexist ( $\varepsilon \neq 0$  or 1 and  $\theta \neq 0$  or 1).

These hybrid equilibria are not locally stable in our numerical Example but can be turned into stable ones with alternate parameterizations. This can be achieved by assuming, for instance, that public investment in green R&D is low  $\sigma_{RD} = 0.05$  (instead of  $\sigma_{RD} = 0.25$ ) and that private investment is even lower  $\theta_{RD} = 0.01$  (instead of  $\theta_{RD} = 0.25$ , one can display Figure 4.5a with  $\chi_c = 0.02$  or Figure 4.5b with  $\chi_c = 0.025$ . Small variations of carbon tax here affect the equilibrium values of the share of output produced from brown capital,  $\varepsilon$  and the share of green investments,  $\theta$ .



(a) Trajectories with  $\chi_c = 0.02$  lead towards equilibrium values:  $\varepsilon = 0.56$  and  $\theta = 0.54$

(b) Trajectories with  $\chi_c = 0.025$  lead towards equilibrium values:  $\varepsilon = 0.4$  and  $\theta = 0.7$

Figure 4.5: Trajectories of (55) with low private and public investments,  $\theta_{RD} = 0.01$  and  $\sigma_{RD} = 0.05$  leading from the Brown Solovian-like equilibrium ( $\chi_E = 0.13$ ) to intermediate Solovian-like equilibria.

## 4.5 Concluding remarks

In this paper, we first built a simple transition model from fossil fuel-intensive to fossil-free capital in a dynamics with public and private debts, inventories and government intervention. Multiple long-run equilibria are the rule so that the main question of economic policy becomes: how to drive an economy towards the basin of attraction of some desirable long-run steady state?

We identified conditions under which the transition to green capital can be implemented without leading to an overhang of debt, be it private or public. The key parameter driving whether transition towards renewable energies takes place is the difference between the price of fossil energy and the productivity gap between the two types of capital:  $\chi_E - \Delta\nu$ . Surprisingly, because of inertia in the allocation of investment, the debt-deflationary equilibria, whether brown or green, turn out *not* to be locally stable. Each of them is crossed by a one-dimensional repulsive submanifold. This raises an interesting question for future research: what are the forces that drive an economy towards debt-deflation in a world when multiple types of capital coexist and investment exhibits some kind of inertia?

Introducing governmental levers is done in this paper both by influencing the energy price through a carbon tax (according to the corridor of values established by [Stiglitz et al., 2017]) and by reducing the capital-to-output gap between the two types of capital.

Whenever only the carbon tax tool is used, we have shown that the transition and the local stability criteria happen to be identical:  $\chi_E + \chi_c - \Delta\nu > 0$ . For a sufficiently high carbon tax, this criterion is satisfied, the green Solovian-like steady state is locally asymptotically stable, and if the economy converges, it must become asymptotically green because the dirty equilibria turn out to be asymptotically unstable. Conversely, if the carbon tax remains too low, the economy cannot help but, if it converges, do so towards one of the several fossil-fuel intensive equilibria because green steady-states become unstable.

In order to capture directed technical change, we introduced public subsidies of green innovation, with the hope to be able to ensure that the transition will take place, whatever being the starting point of the economy.

We exhibited a particular parameterization leading to a transition with a low price of fossil energy. In this example, public and private funding for green R&D ensure that the transition is achieved, even if the economy was originally following a dirty path. Introducing a carbon tax accelerates the transition but does not guarantee it. As in [Acemoglu et al., 2012], any delay in the implementation of public transition policies can be very costly in time and money. Contrary to the result in [Acemoglu et al., 2012], however, here, no transition can occur under *laissez-faire*.

Moreover, we have shown that, in general, new types of long-run steady states emerge, where brown and green capital coexist. In our specific example, this is due to the low level of public and private investments in green R& D. While less simplistic than our previous 0-1 equilibria, these hybrid steady states suggest that, not only is *laissez-faire* insufficient but a too timid public intervention may fail to drive the economy towards a zero-carbon long-run situation. A surprising result, eventually, is that, under a realistic assumption (Assumption C), the dirty Solovian-like equilibrium becomes asymptotically unstable.

Several improvements of this paper should be considered. The price dynamics should obviously incorporate the cost of energy and be at least partially demand-driven. This would allow consumption prices to reflect the increasing cost of fossil fuel whenever a carbon tax is put into practice. The impact on effective demand would presumably be non-negligible. The main question would then become whether a depressed demand would be able to prevent the private sector from investing sufficiently in green infrastructures so as to reach the clean Solovian-like equilibrium. The same question would be probably reinforced if households were to consume energy as well (and not just the production sector) and would therefore directly bear part of the additional cost induced by a carbon tax.

On the other hand, we have focused exclusively, here, on the long-run analysis. A short-run analysis would also help understand the pros and cons of the political levers considered in this paper and which have been much debated in the U.S. in the first part of the Biden mandate. In particular, the asymptotic instability of the dirty Solovian-like steady state should be put in perspective with the time urgency induced by global warming.

Another improvement of this paper is left for further work, which consists in endogenizing the price,  $p_E$ , of fossil energy. Eventually, building on the present work, one might introduce a variety of different types of capital in a full-blown Putty-Clay production sector similar to the one introduced by [Akerlof and Stiglitz, 1969] in order to get a more realistic setting than the extreme case envisaged here.

## Chapter 5

# Conclusion

The aim of our work was to develop a robust transition model that should be able to provide information on public policies to avoid either an attractive equilibrium with fossil fuel-intensive capital only or a collapse due to a climate-induced Minsky moment. The construction of such a model was based on [Grasselli and Costa Lima, 2012], which belongs to the family of disequilibrium models. This model was selected because of its consistent stock-flow structure with nonlinear dynamics leading to multiple equilibria, which can incorporate this Minsky moment.

To cope with the criticisms formulated against this family of model, especially concerning their strong dependence on parameters, we developed a sensitivity analysis of the parameters of [Grasselli and Costa Lima, 2012]. We study the relative influence of uncertainties on all inputs' parameters with a variance-based Global Sensitivity Analysis (GSA), because this model contains a large number of nonlinearities and *a priori* independent inputs. Due to its small number of inputs (and consequently its low computational cost), we chose a GSA with Sobol' indices. The calculation of the Sobol' indices was based on combinations of inputs values obtained from a Design of Experiment (DoE): an Optimal Latin Hypercube Sampling, according to the [Saltelli, 2002] method of estimation.

We noted that the most influential parameters on the model outputs are the labor productivity growth rate, the population growth rate, the depreciation rate and the capital-to-output ratio. The last one has the maximal first-order Sobol' indice (less than 0.2). These values are not very dependent on the number of simulations as we show it by successively increasing their number until the satisfaction of a certain criterion about the standard deviation of our results (less than 0.05). Two of these most influential parameters can be assessed with a certain confidence as detailed in [McIsaac, 2016] (from demographic data from United Nations time series for the population growth rate and by averaging the time series of the labour productivity growth rate). On the contrary, the capital-to-output ratio and the depreciation rate are hard to assess that can impede the robustness of the model.

Although we cannot directly compare these results to other similar GSA on DSGE, RBC and IAM models, because the outputs and inputs of each study are different, we can note that the model observed in [Ratto, 2008], [Harenberg et al., 2017] or [Nordhaus, 2008] are highly sensitive to a single parameter that is generally hard to assess while the sensitivity of the [Grasselli and Costa Lima, 2012] model with a Leontief production function is more balanced between four parameters. Moreover when studying the original [Goodwin, 1967] model, we noticed that its sensitivity was dependent

on the same parameters as its extension, that is quite reassuring that refining the model does not seem to affect the deep nature of its sensitivity,

As the extensions of this model do not seem to fundamentally alter its robustness, improvements have been made in particular concerning the relaxation of Say’s “law” through the introduction of inventories, which is essential for studying money creation. The dynamic analysis of this “extended” model showed that the interplay between effective demand, production and money creation can lead either to an equilibrium with a finite private debt-to-output ratio, positive but stable inflation and positive money velocity — the *monetary Solovian equilibrium*— or to another steady state where debts skyrocket while the wage share, the employment rate, and money velocity both collapse. Inflation either collapses as well or can even become negative — a hallmark for the deflationary trap.

Moreover, money has been shown to be non-neutral both in the short- and the long-run, even though the monetarist interpretation of the exchange equation happens to be true at the monetary Solovian equilibrium — and only there. The fact that banks create credit and money out of nothing, if used appropriately, results in non-hyper-inflationary growth, provided the economy crosses the basin of attraction of the monetary Solovian equilibrium. We also showed that central bank intervention may have a clear positive effect in preventing a crisis characterized by collapsing employment rates. According to the persistence result, central bank intervention, in the form of a responsive enough monetary policy, prevents the economy from remaining permanently at arbitrarily low levels of employment regardless of the initial conditions of the system.

Eventually a simple transition model is constructed with two natures of capital (can be generalized in a “Putty-Clay” framework in the sense of [Akerlof and Stiglitz, 1969], [Cass and Stiglitz, 1969]), from fossil fuel-intensive to fossil-free capital in a dynamics with public and private debts, inventories and government intervention in order to answer the question: how to drive an economy towards the basin of attraction of some desirable long-run steady state? We identified conditions under which the transition to green capital can be implemented without leading to an overhang of debt, be it private or public. The key parameter driving whether transition towards renewable energies takes place is the difference between the price of fossil energy and the productivity gap between the two types of capital. Surprisingly, because of inertia in the allocation of investment, the debt-deflationary equilibria, whether brown or green, turn out *not* to be locally stable.

Introducing governmental levers is done in this paper both by influencing the energy price through a carbon tax (according to the corridor of values established by [Stiglitz et al., 2017]) and by reducing the capital-to-output gap between the two types of capital. Whenever only the carbon tax tool is used, we have shown that the transition and the local stability criteria happen to be identical:  $\chi_E + \chi_c - \Delta\nu > 0$ . For a sufficiently high carbon tax, this criterion is satisfied, and if the economy converges, it must become asymptotically green because the dirty equilibria turn out to be asymptotically unstable. Conversely, if the carbon tax remains too low, the economy cannot help but, if it converges, do so towards one of the several fossil-fuel intensive equilibria because green steady-states become unstable.

In order to capture directed technical change, we introduced public subsidies of green innovation, with the hope to be able to ensure that the transition will take place, whatever being the starting point of the economy. We exhibited a particular parameterization leading to a transition with a



low price of fossil energy. In this example, public and private funding for green R&D ensure that the transition is achieved, even if the economy was originally following a dirty path. Introducing a carbon tax accelerates the transition but does not guarantee it. As in [Acemoglu et al., 2012], any delay in the implementation of public transition policies can be very costly in time and money. Contrary to the result in [Acemoglu et al., 2012], however, here, no transition can occur under *laissez-faire*.

Moreover, we have shown that, in general, new types of long-run steady states emerge, where brown and green capital coexist. In our specific example, this is due to the low level of public and private investments in green R&D. While less simplistic than our previous 0-1 equilibria, these hybrid steady states suggest that, not only is *laissez-faire* insufficient but a too timid public intervention may fail to drive the economy towards a zero-carbon long-run situation. A surprising result, eventually, is that, under a realistic assumption, the dirty Solovian-like equilibrium becomes asymptotically unstable.

Several improvements of this work should be considered. About the introduction of money, there are, of course, a number of institutional aspects that we have neglected in this first tentative pass: reserve requirements, capital adequacy requirements (such as the Basel III/CRR framework), prudential regulation... Added to the present framework, these refinements could enable us to tackle important issues around a countercyclical leverage-based regulation of banks. About transition, we focused exclusively, here, on the long-run analysis. A short-run analysis would also help understand the pros and cons of the political levers considered that should be put in perspective with the time urgency induced by global warming.

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## APPENDIX

## Appendix A

# Building a global sensitivity analysis to quantify the robustness of macro-economic models

### A.1 Overview of macro-economic modeling: Features and criticisms of the [Grasselli and Costa Lima, 2012] model

In regard to various possible classifications, [Grandjean and Giraud, 2017] displayed a comparison between climate and macro-economic models to explain the problems that modelers in Climate Change Economics are facing. As written in introduction,<sup>1</sup> they classified models between:

- New-keynesian based models (e.g., Mésange of the French Ministry of Economy) in which temporary disequilibrium is possible due to the inertia of quantities adjustments;
- Computable General Equilibrium models (such as Integrated Assessment Models - IAM, usually). Generally, institutions use both an estimated *Dynamic Stochastic General Equilibrium* or DSGE (such as Global Integrated Monetary and Fiscal model - GIMF, New Area-Wide Model of The Euro area - NAWM, ...) and a neo-keynesian model for their forecast;
- Disequilibrium models, and more specifically those able to represent the Minsky's Finance Instability Hypothesis (FIH) and predict crashes such as GEMMES (General Monetary and Multisectorial Macrodynamics for the Ecological Shift). Their stochastic dynamic allows the emergence in aggregated indicators, but they are very criticized due to the high-dependency on their parameters, their myopic nature (without expectations) and exogenous growth. However they seem the most able to represent the trajectories we want to explore, if we can address these criticisms. An estimated version of the model with a Climate Feedback Loop exists as a result of the work of [Bovari et al., 2018] and a first approach dedicated to the brazilian economy was implemented by [Bastidas et al., 2017].

In their analysis, [Grandjean and Giraud, 2017] displayed different features of these models which lead to uncertainties in regard to their ability to predict reliable trajectories. These uncertainties

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<sup>1</sup>See section 1.3.1.

are linked to their inputs definition<sup>2</sup>, the ambiguity of their output indicators (GDP<sup>3</sup>; Cost of public policy; Unemployment rate<sup>4</sup>), their lack of back-testing, their potentially large sensitivity to calibrated parameters and their inability to represent out-of-equilibrium trajectories to multiple equilibria, especially to a global collapse.

The more interesting typology of models for our study seems to be the **disequilibrium** models (such as Three-Me and its extension IMACLIM-3ME [Gherzi, 2020], or [Akerlof and Stiglitz, 1969], or [Grasselli and Costa Lima, 2012] based on a Lotka-Volterra approach between wages and employment rate<sup>5</sup>). These models assess by construction the trajectories to multiple equilibria and are able to describe global out-of-equilibrium dynamics. The GEMMES model<sup>6</sup> is an application of this type of model about various geography [Bastidas et al., 2017] and production functions [McIsaac, 2016], with climate feedback loop [Bovari et al., 2018], and features (Government intervention [Costa Lima and Grasselli, 2014], Inventories [Grasselli and Nguyen Huu, 2016], Inequalities [Giraud and Grasselli, 2017], etc.).

### A.1.1 Features and criticisms of the [Grasselli and Costa Lima, 2012] model

To address the need for macro-economic models to assess possible trajectories to a *climate driven Minsky moment*, I will focus my research on a particular family of disequilibrium models. This family, and especially the [Grasselli and Costa Lima, 2012] model, can indeed answer to some of the criticisms of a large majority of usual models such as:

- Not taking into account energy or matter in their production function;
- Neutral or exogenous money;
- Generally no global out-of-equilibrium dynamics;
- Mainly linear in the presumed dynamics;
- Without private debt or massive unemployment;
- Without banking sector;
- Using rational expectations that constrain trajectories.

The selected model is based on the work of [Goodwin, 1967], with debt introduction [Keen, 1995], of which the nature and stability of the equilibria were analyzed in [Grasselli and Costa Lima, 2012] and [McIsaac, 2016] for the Van Der Ploeg’s Extensions<sup>7</sup>. Based on the Lotka-Volterra Predator-Prey Logic, with a classical capital’s accumulation equation and two behavioral functions (about, on the one hand, investment depending on profits, and, on the other hand, the link between employment rate and wages share, through a short-term Phillips curve), the model takes into

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<sup>2</sup>E.g., the Cambridge capital controversy in the late 1950s expresses the gap between definition of the nature and role of capital goods [Stiglitz, 1974b].

<sup>3</sup>See a review of the criticisms in [Jany-Catrice and Méda, 2015] or the proposition of [Piketty, 2009].

<sup>4</sup>The ECB experts estimate the labour market slack around 18% in the Euro Zone, almost twice greater than the official rate from Eurostat around 9.5%. This shows the current paradox about the calculation of this indicator: <http://bit.ly/2q2bVBz> .

<sup>5</sup>Inspired from [Goodwin, 1967] and developed by [Keen, 1995].

<sup>6</sup>Developed at the French Development Agency under the direction of Gaël Giraud.

<sup>7</sup>Introducing a CES production function instead of the Leontief production function [Van der Ploeg, 1985].

account debt variation (as the difference between investment and profits), but with an exogenous population and labor productivity growth rate. The major criticisms of this type of model lie in its exogenous growth model, the use of the Say’s law (which can be relaxed by the introduction of inventories [Grasselli and Nguyen Huu, 2016]) its myopic nature (without expectations) and its high parameter-dependency<sup>8</sup>.

In conjunction with the current models (used by institutions such as the French Ministry of Economy or Environment), this work aims at providing a model which gives a broader view of the transition challenges by offering:

- A more robust, non-linear, global out-of-equilibrium dynamic;
- A Stock-Flow Consistency, with two natures of capital;
- An analysis of multiple equilibria and their stability criterion;
- Various possible extensions: climate feedback loop [Bovari et al., 2018], government intervention [Costa Lima and Grasselli, 2014], inventories [Grasselli and Nguyen Huu, 2016], inequalities [Giraud and Grasselli, 2017], etc., and applications (Brazil: [Bastidas et al., 2017]).

**Features and underlying assumptions of the [Grasselli and Costa Lima, 2012] model**

Based on the Lotka-Volterra Predator-Prey Logic<sup>9</sup>, the original model follows the 2D (two dimensions) dynamical system of the form:

$$\dot{x}_t = x_t (\Phi(y_t) - \alpha) , \tag{1}$$

$$\dot{y}_t = y_t (\gamma - \kappa(x_t)) , \tag{2}$$

in which the predator population dynamics,  $x_t$ , is increased by a function depending on the number of preys, and decreased by a loss rate of the predators due to either natural death or emigration, whereas preys are decreased in function of the number of predators and increased through a natural (exponential) growth rate  $\gamma$ . These equations have a periodic solution (similar to a simple harmonic motion if linearized).

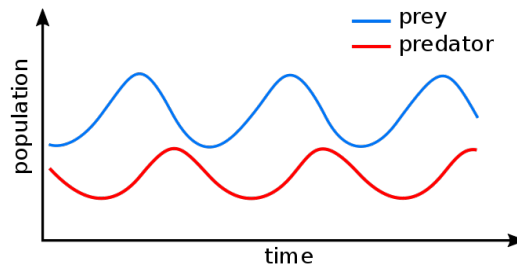


Figure A.1: Evolution of a Predator-Prey population (interpreted from [Lotka, 1920] and [Volterra, 1928])

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<sup>8</sup>See also the study of [Pottier and Nguyen-Huu, 2017] about the dependency of the model dynamic to the selected investment function.

<sup>9</sup>An example of a Kolmogorov model [Hoppensteadt, 2006] which is a more general framework that can model the dynamics of ecological systems with Predator-Prey interactions, competition, disease, and mutualism.

The general frame, built on this logic, contains a production function (originally: a Leontief production function but [Van der Ploeg, 1985] offers an extension with a Constant Elasticity of Substitution production function) such as:

$$Y = \min \left\{ \frac{K}{\nu}, aL \right\} . \quad (3)$$

As in [Grasselli and Costa Lima, 2012],  $Y$  is the total yearly output,  $K$  is the stock of capital,  $\nu$  is a constant capital-to-output ratio,  $L$  is the number of employed workers, and  $a$  is the labour productivity, that is to say, the number of units of output per worker per year. All quantities are assumed to be quoted in real rather nominal terms, thereby already incorporating the effects of inflation, and are net quantities, meaning that intermediate revenues and expenditures are deducted from the final yearly output. For the sake of simplicity in the calculation of growth rate, demography and labor productivity are canonically given by the following exponential shaped functions<sup>10</sup>:

$$N(t) = N_0 e^{\beta t} , \quad (4)$$

$$a(t) = a_0 e^{\alpha t} , \quad (5)$$

where  $N_0$  and  $a_0$  are initial values (whatever is the initial time) that would not impact the calculation of growth rate  $\frac{\dot{a}}{a} = \alpha$  or  $\frac{\dot{N}}{N} = \beta$ . These constant growth rates are tested in our sensitivity analysis. For macroeconomists, their sum recalls the *Good* balanced growth path (which equals the sum of the technological progress and population growth rate).

Defining the employment rate  $\lambda = \frac{L(t)}{N(t)}$  and the full capital utilization assumption allows us to write:

$$Y = \min \left\{ \frac{K}{\nu}, aL \right\} = \frac{K}{\nu} = aL = a\lambda N . \quad (6)$$

Two key assumptions are made in the model:

- On the one hand, about the link between wages  $w$  (or inflation rate) and employment rate  $\lambda$ . This first behavioral function assumes that the rate of change in real wages  $w$  is an increasing function of the employment rate  $\lambda$ . This relation is known as the Phillips curve  $\Phi$ :

$$\dot{w} = \Phi(\lambda)w \quad (7)$$

$$\text{where : } \Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0 \text{ with } \phi_0 = \frac{0.04}{1-0.04^2} , \phi_1 = \frac{0.04^3}{1-0.04^2} . \quad (8)$$

This form and parameterization of the Phillips Curve guarantee that  $\forall t, 0 \leq \lambda(t) \leq 1$  because (as shown in [Desai et al., 2006])  $\Phi$  is a continuously differentiable function on  $[0, 1]$  and

$$\begin{aligned} \dot{\Phi}(\lambda) &> 0 \text{ on } [0, 1] , \\ \Phi(0) &< \alpha , \end{aligned} \quad (9)$$

$$\lim_{\lambda \rightarrow 1^-} \Phi(\lambda) = +\infty .$$

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<sup>10</sup> [McIsaac, 2016] offered an extension with a population dynamic based on the recommended United Nations sigmoid.

- On the other hand, assuming the Say's Law, which can be summarized by "All wages are consumed and all profits are reinvested", total sales demand  $Y_d$  is equal to the output  $Y$  (so consumption  $C$  acts only as an adjusting variable):

$$C + I = Y_d = Y = W + \Pi , \quad (10)$$

where  $\Pi$  is the net profit for firms after paying wages  $W$ .

Assuming also a classical capital accumulation's equation:

$$\dot{K} = (Y - wL) - \delta K = (1 - \omega)Y - \delta K , \quad (11)$$

where  $\delta$  is the depreciation rate of capital (assumed to be constant), and the wage share,  $\omega := W/Y = wL/Y$ , one can deduce the following wage share (predator) and employment rate (prey) evolution:

$$\begin{aligned} \frac{\dot{\omega}}{\omega} &= \frac{\dot{w}}{w} - \frac{\dot{a}}{a} = \Phi(\lambda) - \alpha , \\ \frac{\dot{\lambda}}{\lambda} &= \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} = \frac{1 - \omega}{\nu} - \alpha - \beta - \delta , \end{aligned} \quad (12)$$

$$\Leftrightarrow \begin{cases} \dot{\omega} = \omega (\Phi(\lambda) - \alpha) \\ \dot{\lambda} = \lambda \left( \frac{1 - \omega}{\nu} - \alpha - \beta - \delta \right) . \end{cases} \quad (13)$$

This differential system (13) leads to two different equilibria:

- Trivial equilibrium:  $(\omega_0, \lambda_0) = (0, 0)$ ;
- Non trivial equilibrium:  $(\omega_1, \lambda_1) = (1 - \nu(\alpha + \beta + \delta), \Phi^{-1}(\alpha))$ .

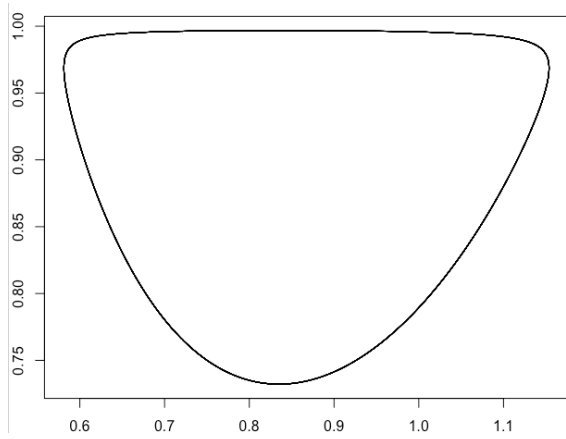
The stability of the non trivial equilibrium is studied in [Grasselli and Costa Lima, 2012] through the Lyapunov function,  $H(\cdot)$ , associated to the system (using separating variables and integrating the equation for  $d\lambda/d\omega$ ):

$$H(\omega, \lambda) = \int_{\lambda_0}^{\lambda} \frac{\phi(s)}{s} ds - \alpha \ln(\lambda) + \frac{\omega}{\nu} - \left( \frac{1}{\nu} - \alpha - \beta - \delta \right) \ln(\omega) , \quad (14)$$

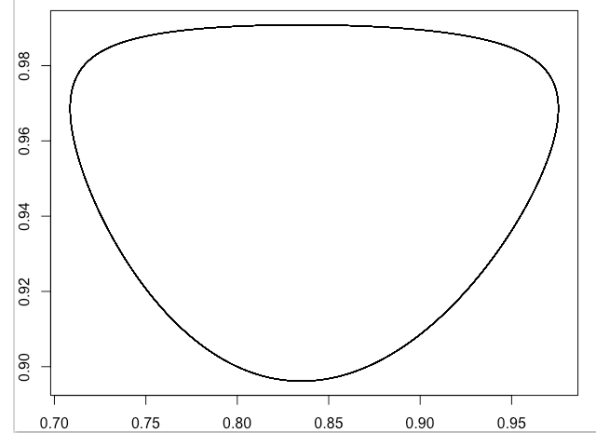
$$\text{where } \dot{H}(\omega, \lambda) = \nabla H \cdot (\dot{\omega}, \dot{\lambda}) = 0 , \quad (15)$$

which corroborates the conservative behavior of the system, observed on Figure A.2b.

With the parameterization of [Grasselli and Costa Lima, 2012], one can observe the limit circle trajectory on Figure A.2. One obvious drawback of the model is that it does not constrain the variables  $\omega$  and  $\lambda$  to remain in the unit square, as should be the case given their economic interpretation. For example, for initial values  $\omega = 0.75$  and  $\lambda = 0.75$ , we obtain Figure A.2a, whereas for initial values  $\omega = 0.8$  and  $\lambda = 0.9$ , we obtain the constrained Figure A.2b. In order to avoid  $\omega > 1$ , we can choose a normalized Phillips curve to constraint  $\omega \in [0, 1]$  (see appendix B.3 for the study of the system with the normalized Phillips curve).



(a) Evolution to the trajectories in the 2D model for initial values:  $\omega_0 = 0.75$ ,  $\lambda_0 = 0.75$ .



(b) Evolution to the trajectories in the 2D model for initial values:  $\omega_0 = 0.8$ ,  $\lambda_0 = 0.9$ .

Figure A.2: Trajectories of the locally stable equilibria of the 2D-model with the set of parameters defined in [Grasselli and Costa Lima, 2012].

### Introducing a banking sector to finance new investments

Inspired from [Minsky, 1982], and formalized by [Keen, 1995], [Grasselli and Costa Lima, 2012] modeled investment as a function of the net profit after paying wages and interests on debt  $\Pi = Y - W - rD$  (where  $D$  is the amount of debt in real term and  $r$  a constant interest rate):

$$I = \kappa(\pi)Y , \quad (16)$$

with  $\pi$ , the net profit share:  $\pi = \frac{\Pi}{Y}$ .

Investments can now exceed profits, thus generating debt. With this debt ( $D$  amount of debt in real terms), the net profit after paying wages and interest on debt is:

$$\Pi = (1 - \omega - rd)Y = \pi Y , \quad (17)$$

where  $\pi$  is the net profit share,  $r$  is a constant short-term real interest rate and  $d = D/Y$  is the debt ratio in the economy. The change in capital stock depends on the investment  $I$  and the depreciation of this capital:

$$\dot{K} = I - \delta K = \kappa(1 - \omega - rd)Y - \delta K . \quad (18)$$

To guarantee the existence of the relevant equilibria we made the following additional assumptions on  $\kappa(\cdot)$ , a continuously increasing differentiable function such that:

$$\begin{aligned} \dot{\kappa}(\pi) &> 0 \text{ on } [0, 1] , \\ \lim_{\pi \rightarrow -\infty} \kappa(\pi) = \kappa(-\infty) &\leq \nu(\max(r, \alpha + \beta) + \delta) \leq \lim_{\pi \rightarrow +\infty} \kappa(\pi) \leq 1 , \\ \lim_{\pi \rightarrow -\infty} \pi^2 \dot{\kappa}(\pi) &< +\infty . \end{aligned} \quad (19)$$



APPENDIX A. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

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In [Grasselli and Costa Lima, 2012], the authors take for instance<sup>11</sup> the following form of  $\kappa$ , complying with the previous assumptions:

$$\kappa(\pi) = k_0 + k_1 e^{k_2 \pi} . \quad (20)$$

The variation of  $D$  is consequently built on the difference between the demand in investment and profits:

$$\dot{D} = I - \Pi . \quad (21)$$

With this definition of capital accumulation, the growth rate of the economy and the employment rate dynamics become:

$$\frac{\dot{Y}}{Y} = \frac{\kappa(\pi)}{\nu} - \delta , \quad (22)$$

$$\frac{\dot{\lambda}}{\lambda} = \frac{\kappa(\pi)}{\nu} - \alpha - \beta - \delta . \quad (23)$$

Defining  $d = D/Y$ , the debt ratio of the economy, one can express its dynamics through:

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{(\kappa(\pi) - \pi)Y}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(\pi) - \pi}{d} - \left( \frac{\kappa(\pi)}{\nu} - \delta \right) . \quad (24)$$

The [Grasselli and Costa Lima, 2012] model now owns three state variables: the wage share  $\omega$ , the employment rate  $\lambda$  and the debt ratio of the economy  $d$ . The 3D differential system<sup>12</sup> becomes:

$$\begin{cases} \dot{\omega} = \omega [\Phi(\lambda) - \alpha] , \\ \dot{\lambda} = \lambda \left[ \frac{\kappa(\pi)}{\nu} - \delta - \alpha - \beta \right] , \\ \dot{d} = \kappa(\pi) - \pi - d \left[ \frac{\kappa(\pi)}{\nu} - \delta \right] . \end{cases} \quad (25)$$

The basic intuitions from this differential system are the following truisms:

- Wages share  $\omega \nearrow$  if wage rise exceeds growth in labor productivity;
- Employment rate  $\lambda \nearrow$  if economic growth  $>$  the sum of pop. + labor productivity growth;
- Debt-to-output ratio  $d \nearrow$  if rate of debt growth exceeds rate of growth of GDP.

<sup>11</sup>A more exhaustive study of the different form of  $\kappa$  is done in [Pottier and Nguyen-Huu, 2017].

<sup>12</sup>With the following notation:

$\omega$	wage share $w(t)/a(t)$	$\alpha$	labor productivity rate
$\lambda$	employment rate $L(t)/N(t)$	$\beta$	population growth rate
$d$	debt ratio $D/Y$	$\delta$	depreciation rate
$\pi$	net profit share $\Pi/Y$	$\Phi(\lambda)$	Short-term Phillips curve
$\kappa(\pi)$	Investment function	$\nu$	capital-to-output ratio

## APPENDIX A. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

The analysis of the system leads to the identification of two economically meaningful long-run equilibria:

- A *Good* equilibrium, locally stable: in which final values of employment rate and wage share are positive with finite level of debt, following a Solovian balanced growth<sup>13</sup>. One can calculate the equilibrium for each State Variable  $(\omega_1, \lambda_1, d_1)$ :

$$\begin{aligned}\lambda_1 &= \Phi^{-1}(\alpha) , \\ d_1 &= \frac{\kappa(\pi_1) - \pi_1}{\alpha + \beta} , \\ \omega_1 &= 1 - \pi_1 - rd_1 , \\ \text{with } \pi_1 &= \kappa^{-1}(\nu(\alpha + \beta + \delta)) .\end{aligned}\tag{26}$$

- A *Slavery* equilibrium (with no wages but employment), structurally unstable and economically meaningless;
- A *Explosive-Debt* equilibrium, locally stable: in which employment rate and wage share are collapsing with an infinite level of debt. It is the situation of economic crisis.

Graphical representations and mathematical proofs of these results are presented in [Grasselli and Costa Lima, 2012]. Figures A.3a and A.3b show the trajectories of these two locally stable equilibria.

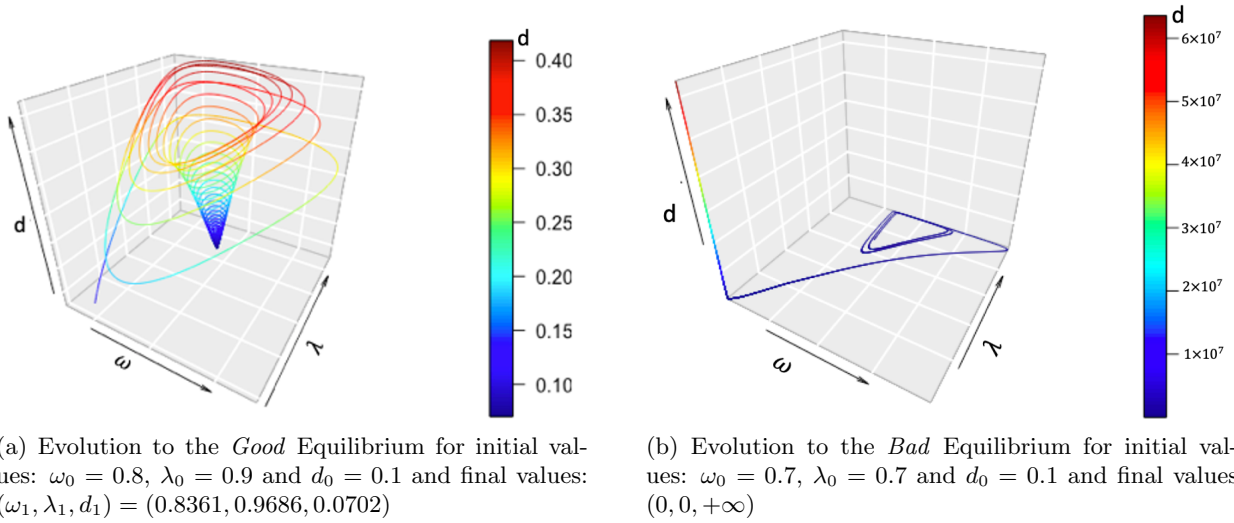


Figure A.3: Trajectories of the two locally stable equilibria of the [Grasselli and Costa Lima, 2012] model with the set of parameters defined in Table 2.1

The methodology used to study the stability of each equilibria is the following:

<sup>13</sup>The growth rate of the economy equals in the long-run to the sum of the technical progress growth and the population growth.

- Building the Jacobian Matrix of the system:

$$J = \begin{pmatrix} \Phi(\lambda) - \alpha & \omega \dot{\Phi}(\lambda) & 0 \\ -\frac{\lambda \dot{\kappa}(\pi)}{\nu} & \frac{\kappa(\pi) - \nu(\alpha + \beta + \delta)}{\nu} & -\frac{r \lambda \dot{\kappa}(\pi)}{\nu} \\ \frac{(d - \nu) \dot{\kappa}(\pi) + \nu}{\nu} & 0 & r + \delta + \frac{r(d - \nu) \dot{\kappa}(\pi) - \kappa(\pi)}{\nu} \end{pmatrix}; \quad (27)$$

- For each equilibrium, computing the Jacobian Matrix (with the change of variable  $u = 1/d$  in *Explosive Debt* case. For instance, in the *Good* Equilibrium, the Jacobian Matrix becomes:

$$J = \begin{pmatrix} 0 & K_0 & 0 \\ -K_1 & 0 & -rK_1 \\ K_2 & 0 & rK_2 - (\alpha + \beta) \end{pmatrix}, \quad (28)$$

where:

$$\begin{aligned} K_0 &= \omega \dot{\Phi}(\lambda), \\ K_1 &= \frac{\lambda \dot{\kappa}(\pi)}{\nu}, \\ K_2 &= \frac{(d - \nu) \dot{\kappa}(\pi) + \nu}{\nu}; \end{aligned} \quad (29)$$

- Extracting the characteristic polynomial  $P(X) = \det(J - XId)$ , here:

$$P(X) = X^3 + (\alpha + \beta - rK_2)X^2 + K_0K_1X + K_0K_1(\alpha + \beta); \quad (30)$$

- Computing a stability criterion (in our case: Routh-Hurwitz [Routh, 1877] and [Hurwitz, 1963]): “Necessary and sufficient condition for all roots of the characteristic polynomial to have negative real parts is”:

$$\frac{\dot{\kappa}(\pi_1)}{\nu}(\pi_1 + \nu(\alpha + \beta) - \kappa(\pi_1)) - (\alpha + \beta) > 0; \quad (31)$$

- Verifying the criterion is satisfied by the parameterization<sup>14</sup> or not.

### Basins of Attraction

[McIsaac, 2016] studied the basins of attraction of these equilibria. Their dimensions depend on the set of parameters, as they are modifying the condition of stability (Routh-Hurwitz criterion [Routh, 1877] and [Hurwitz, 1963]).<sup>15</sup>

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<sup>14</sup>See section 4.4.2 for the definition of our set of parameters in this work and [Grasselli and Costa Lima, 2012], [McIsaac, 2016], [Grasselli and Nguyen Huu, 2016] and [Giraud and Grasselli, 2019] for exhaustive justifications of these values.

<sup>15</sup>Figure 2.2 represents the basin of attraction — i.e. all initial values of input’s variables leading to this equilibrium according to our set of parameters — of the *Good* Equilibrium. The figures is displayed with the .R code of [Augier, 2018].

Many criticisms to this 3D-model can be highlighted, such as the absence of government, the assumption of the Say's law, which are relaxed with the method of [Costa Lima and Grasselli, 2014] and [Grasselli and Nguyen Huu, 2016], and the impossibility to study a transition between different natures of capital (due to the sole nature of capital here). The aim of our last chapter is to address this last point. Another usual criticism of this 3D-model is the absence of substitution between factors. This issue was addressed through the extension [Van der Ploeg, 1985], developed in [McIsaac, 2016].

### Introducing a new production function (with a Constant Elasticity of Substitution)

Implemented by [McIsaac, 2016], following the work of [Van der Ploeg, 1985], this extension incorporates substitution between labor and capital into the original model by using the CES production function:

$$Y(t) = C(bK^{-\eta} + (1-b)(a(t)L)^{-\eta})^{-\frac{1}{\eta}}, \quad (32)$$

where  $C > 0$  and  $0 < b < 1$  are constants and the elasticity of substitution is defined by:  $\sigma = \frac{1}{1+\eta}$ . Using the hypothesis  $\frac{\partial Y}{\partial L} = w$ , the author finds that the optimal capital-to-output ratio and productivity are given by:

$$\nu(t) = \frac{K(t)}{Y(t)} = \frac{1}{C} \left( \frac{1-\omega(t)}{b} \right)^{-\frac{1}{\eta}}, \quad (33)$$

$$a(t) = \frac{Y(t)}{L(t)} = a_0 e^{\alpha t} \left( \frac{\omega(t)}{1-b} \right)^{\frac{1}{\eta}}. \quad (34)$$

As previously we obtain a slightly modified wage share and employment rate dynamics:

$$\begin{aligned} \dot{w} &= \Phi(\lambda)w, \\ \omega &= \frac{w(t)}{a(t)}, \\ \frac{\dot{\omega}}{\omega} &= \frac{\dot{w}}{w} - \frac{\dot{a}}{a} = \frac{\eta}{\eta+1}(\Phi(\lambda) - \alpha), \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{\dot{\lambda}}{\lambda} &= \frac{\dot{L}}{L} - \frac{\dot{N}}{N} = \left( \frac{\dot{K}}{K} - \frac{\dot{\nu}}{\nu} - \frac{\dot{a}}{a} \right) - \frac{\dot{N}}{N}, \\ \frac{\dot{\lambda}}{\lambda} &= Cb^{-\frac{1}{\eta}}(1-\omega)^{1+\frac{1}{\eta}} - \delta - \beta - \alpha - \frac{1}{\eta} \frac{\dot{\omega}}{\omega(1-\omega)}. \end{aligned}$$

Consequently the 2D-differential system becomes:

$$\frac{\dot{\omega}}{\omega} = \frac{\eta}{\eta+1}(\Phi(\lambda) - \alpha), \quad (36)$$

$$\frac{\dot{\lambda}}{\lambda} = Cb^{-\frac{1}{\eta}}(1-\omega)^{1+\frac{1}{\eta}} - \delta - \beta - \alpha - \frac{1}{\eta} \frac{\dot{\omega}}{\omega(1-\omega)}. \quad (37)$$

To extend the basic two dimensional model, we introduce debt (D amount of debt in real terms). The net profit after paying wages and interest on debt, change in capital stock, is as before:

$$\begin{aligned} \pi &= (1 - \omega - rd)Y , \\ \dot{K} &= \kappa(\pi)Y - \delta K , \quad \text{where } \kappa(\pi) = k_0 + k_1 e^{k_2 \pi} \\ \Rightarrow \frac{\dot{K}}{K} &= \frac{\kappa(\pi)}{\nu} - \delta . \end{aligned} \tag{38}$$

Consequently, growth rates and employment rate dynamics become:

$$\frac{\dot{\lambda}}{\lambda} = \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{\frac{1}{\eta}} - \delta - \beta - \alpha - \frac{\dot{\omega}}{\eta \omega(1-\omega)} , \tag{39}$$

$$g(\omega, d) = \frac{\dot{Y}}{Y} = \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{\frac{1}{\eta}} - \delta - \frac{\dot{\omega}}{\eta(1-\omega)} . \tag{40}$$

It follows the debt ratio dynamics:

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(\pi) - \pi}{d} - \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{\frac{1}{\eta}} + \delta + \frac{\dot{\omega}}{\eta(1-\omega)} . \tag{41}$$

Thus the 3D-differential system becomes:

$$\begin{cases} \dot{\omega} = \omega \frac{\eta}{\eta+1} (\Phi(\lambda) - \alpha) \\ \dot{\lambda} = \lambda \left( \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{\frac{1}{\eta}} - \delta - \beta - \alpha - \frac{\dot{\omega}}{\eta \omega(1-\omega)} \right) \\ \dot{d} = d \left( r - \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{\frac{1}{\eta}} + \delta + \frac{\dot{\omega}}{\eta(1-\omega)} \right) + \kappa(\pi) - (1-\omega) \end{cases} . \tag{42}$$

The nature of the equilibria is the same as previously: *Good*, *Slavery* and *Explosive* Equilibrium. Denoting  $\pi = 1 - \omega - r d$ , the Jacobian for the differential system with a price wage dynamics becomes:

$$J(\omega, \lambda, d) = \begin{pmatrix} \frac{\partial f_1}{\partial \omega} & \omega \frac{\eta}{\eta+1} \dot{\Phi}(\lambda) & 0 \\ \frac{\partial f_2}{\partial \omega} & \frac{\partial f_2}{\partial \lambda} & -r \lambda \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{1/\eta} \\ \frac{\partial f_3}{\partial \omega} & 0 & \frac{\partial f_3}{\partial d} \end{pmatrix} , \tag{43}$$

where

$$\begin{aligned}
 \frac{\partial f_1}{\partial \omega} &= \frac{\eta}{\eta + 1} (\Phi(\lambda) - \alpha) , \\
 \frac{\partial f_2}{\partial \omega} &= -\lambda \left( \frac{\kappa(\pi)C}{\eta(1-\omega)} \left( \frac{1-\omega}{b} \right)^{1/\eta} + \dot{\kappa}(\pi)C \left( \frac{1-\omega}{b} \right)^{1/\eta} + \frac{\dot{\omega}\eta(1-2\omega)}{(\omega\eta(1-\omega))^2} \right) , \\
 \frac{\partial f_2}{\partial \lambda} &= \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{1/\eta} - (\alpha + \beta + \delta) + \frac{\dot{\omega}}{\omega\eta(1-\omega)} , \\
 \frac{\partial f_3}{\partial \omega} &= d \left( \frac{\kappa(\pi)C}{\eta(1-\omega)} \left( \frac{1-\omega}{b} \right)^{1/\eta} + \dot{\kappa}(\pi)C \left( \frac{1-\omega}{b} \right)^{1/\eta} - \frac{\dot{\omega}\eta(1-2\omega)}{(\omega\eta(1-\omega))^2} \right) - \dot{\kappa}(\pi) + 1 , \\
 \frac{\partial f_3}{\partial d} &= r + \delta - \kappa(\pi)C \left( \frac{1-\omega}{b} \right)^{1/\eta} + \frac{\dot{\omega}}{\eta\omega(1-\omega)} + rd\dot{\kappa}(\pi)C \left( \frac{1-\omega}{b} \right)^{1/\eta} - r\dot{\kappa}(\pi) .
 \end{aligned} \tag{44}$$

With the same methodology, we identify the stability criterion of each equilibria. The characteristic polynomial of this *Good* Equilibrium fulfills already some of the conditions of the Routh-Hurwitz criterion with the same parameterization as before. The new necessary and sufficient condition becomes (also satisfied with our parameters):

$$K_0 K_2^e (-r K_3^e) + r K_0 \lambda_{p1} \frac{\kappa(\pi_1)C}{\eta(1-\omega_{p1})} \left( \frac{1-\omega_{p1}}{b} \right)^{1/\eta} (\dot{\kappa}(\pi_1) - 1) > 0 . \tag{45}$$

The first two eigenvalues of the Jacobian Matrix of the *Explosive Debt* Equilibrium are negative. So this equilibrium is stable if and only if:  $k_0 C \left( \frac{1}{b} \right)^{1/\eta} - (r + \delta) < 0$  which is satisfied with our model parameters.

**Conclusion** The 3D base-model of [Grasselli and Costa Lima, 2012] is the core of our study because of its ability to take into account private debt and represent trajectories towards multiple equilibria with a non-linear dynamic. Two main locally stable equilibria exist: one, looking like a Solovian balanced growth path in the long-run, with a growth rate equals to the sum of the technical progress and population growth rates, and the other one, with an explosive level of debt, no wages share and no employment rate.

The introduction of different features in the model is possible. They sometimes increase the number of equilibria and impact their stability criterion. For this reason, [Costa Lima and Grasselli, 2014] tried to destabilize the *Explosive Debt* Equilibrium with a government intervention. Other possible extensions were implemented such as climate feedback loop [Bovari et al., 2018], Inventories [Grasselli and Nguyen Huu, 2016], or Inequalities [Giraud and Grasselli, 2017].

Moreover, this type of model provides information about the initial values leading to each equilibrium, once the stability criterion are satisfied. These *basins of attraction* are studied by [McIsaac, 2016] and show how little is the volume leading to the *Good* Equilibrium. These volumes of course depend on the set of parameters.

## A.2 Some reminder on sensitivity analysis (SA)

Sensitivity analysis (SA) is designed to assess (qualitatively or quantitatively) the outputs variation due to inputs variation [Lam and Hui, 1995], [Macdonald et al., 1999], [Helton et al., 2006], [Saltelli et al., 2008a]. One can thus consider the SA as an "input-output" analysis of the simulation model [Corson, 1992], leading to a classification of the inputs as "sensitive" or "robust" (or "insensitive") to the outputs. Sensitive inputs means that changes in their values have an important impact on the outputs values, while robust inputs have negligible impacts [Macdonald et al., 1999]. The influence of sensitive inputs depends on the choice of value ranges: if a sensitive input is known to within a close tolerance, the predicted outputs will not be severely disturbed by its uncertainty (the standardized regression coefficient calculated to measure its sensitivity will correspondingly become smaller) [Hamby, 1994]

SA are most often run together with an uncertainty analysis (UA), where the SA is used to rank the uncertainty sources (identified by UA), according to their influences on outputs [Saltelli et al., 2008a]. The difference between the definition of uncertainty analysis (UA) and sensitivity analysis (SA) is stated by [Helton et al., 2006] as that: *"Specially, uncertainty analysis refers to the determination of the uncertainty in analysis results that derives from uncertainty in analysis inputs, and sensitivity analysis refers to the determination of the contributions of individual uncertain inputs to the uncertainty in analysis results."*

The following subsections aim at giving definitions, categories, techniques and applications of uncertainty and sensitivity analysis in order to motivate the selected method in our work.

### A.2.1 Uncertainties Analysis (UA): exploring the space of the input factors

The analysis of the uncertainties contains two steps: first, we estimate the probability density of the input with statistical estimation techniques or through experts, and then quantify its propagation in the model through a wide range of possible methods.<sup>16</sup> Each of them belongs to the following classification, regarding the output  $Y$  of a mathematical model  $f(X)$  where  $X$  is the  $p$ -inputs vector  $X = \{X_1, X_2, \dots, X_p\}$  defined on the identity hypercube  $\Omega_p = [0, 1]^p$ .

#### Monte-Carlo (MC) Analysis

In literature, one can find deterministic and probabilistic form of the Monte-Carlo analysis, which is the most widely used method for sampling, due to its fast computation and easy understanding [Tian, 2013]. The concept is to analyse the approximate distribution of possible outputs with probabilistic variables [Lomas and Eppel, 1992] [Hopfe et al., 2007]. The total output uncertainty is assessed by the total inputs uncertainties regardless of the amount of the variables and their possible interactions. The principle is the following:

1. Generate a sampling of  $N$ -values of the  $p$ -inputs vector  $X = (X_1, \dots, X_p)$  on the basis of the density of their joint probability;
2. Calculate the deterministic output for each set of parameter;
3. Assess the statistical properties of the output: mean, variance, density of probability.

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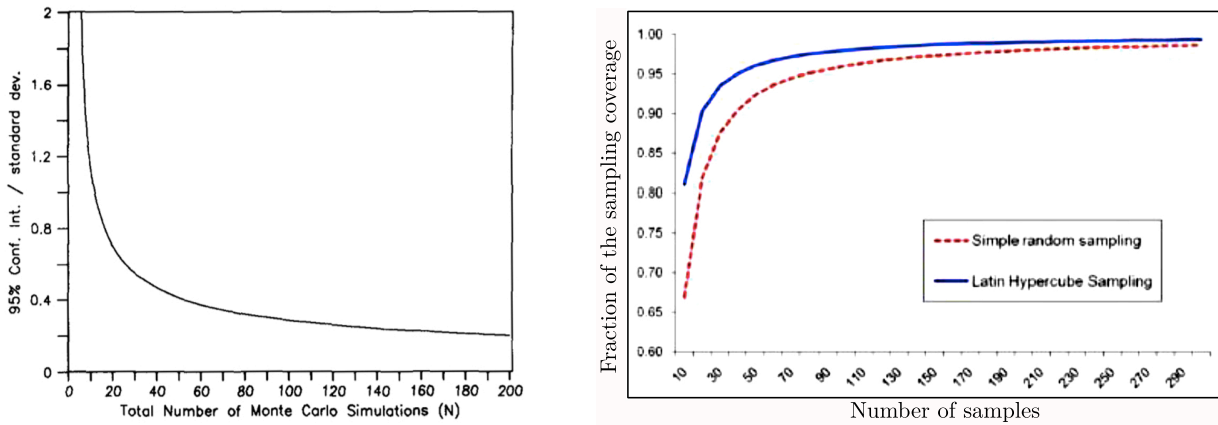
<sup>16</sup>See appendix A.2.1 for a more precise description of these methods.

Thus we assess the expectation and variance of the output such as:

$$E[Y] \approx \frac{1}{N} \sum_{i=1}^N f(X_i)$$

$$V[Y] \approx \frac{1}{N-1} \sum_{i=1}^N (f(X_i) - E(X))^2$$
(46)

[Lomas and Eppel, 1992] stated that with a sufficiently high number of repetitions, the Gaussian distribution of outputs will be exhibited, regardless of the types of variables probability distributions: the performance of MCA can be marginally improved after 60 to 80 simulations (see Figure A.4a)



(a) Relationship between normalized confidence interval and number of Monte-Carlo simulations [Lomas and Eppel, 1992]

(b) Sampling coverage for simple random sampling and LHS [Iman, 1999], showing the number of samples on the x-axis, and the fraction of the sampling coverage on the y-axis

Figure A.4: Performances of MC simulations [Lomas and Eppel, 1992], simple random sampling and LHS [Iman, 1999] according to the number of simulated samples.

To diminish the number of repetitions and improve sample coverage, three sampling techniques are widely used: simple random sampling, stratified sampling and Latin Hypercube sampling (LHS) [Wang, 2014]:

- Simple random sampling: a basic sampling technique, working through randomly generating samples and adjusting them to the target output, by means of the probability distributions of samples;
- Stratified sampling: an improvement sampling method over the simple random sampling, forcing samples to conform to the whole distribution being analysed. To achieve this, each sample value is randomly selected within each stratum, where several strata have equal probability, divided according to the probability distribution of the target output.
- Latin hypercube sampling (LHS): an evolution of the stratified sampling, dividing the input variables into strata, and then generating samples that have variable values from different stratum.

Stratified sampling are generally used to improve the inputs space coverage, especially when processing is expensive due to a high number of parameters or wide ranges of uncertainty. However,



according [Macdonald et al., 1999] and [Iman, 1999], results from different sampling methods are quite similar (at a 5% level see Figure A.4b) in increasing the robustness of MCA, after approximately 60 simulations. [Saltelli et al., 2008a] state that one can not make firm conclusions about the superiority of LHS over random sampling method. Thus, stratified sampling are used in the case of fewer simulations required.

### Latin Hypercube Sampling (LHS)

Samples generated from MC are likely grouped in an inputs region of highest probability, leading to a lack of information in others regions. LHS, as a stratified sampling, give a more uniform coverage of the inputs space. This sampling technique described by [McKay et al., 2000] is one of most widely used in litterature [Dubus et al., 2003] [McMahon et al., 2001]. The principle of the method is the following:

1. Stratify the random variables repartition in N equiprobable intervals (with N, an *a priori* setting).
2. For each input, random sampling is done on every interval, so all the possible levels of the input are explored. N values are so obtained for each input, indexed by the corresponding interval. Random permutations of these intervals are done to create a new sample
3. Obtained values for each input are then associated to give N n-uplets.

Thus, no samples can contain the same value for a given parameter. The sample is evenly spread on the range of each inputs variation and so the distribution of the output will be more precise, as shown in [McKay et al., 2000].

Optimization process can be add to the LHS to deliver even better coverage, they are called Optimal Latin Hypercube Sampling (OLHS). The construction of an LHS is based on a first step of randomization followed by a stochastic optimization process which aims at spreading points as evenly as possible within the design space (according to a optimality criteria such as the maximin distance criteria). One can notice that this type of matrix is not reproducible (unless using the same random seed). [McKay et al., 2000] **compared LHS with MC sampling and conclude that LHS gives better statistical results on the outputs.**

### Bayesian approach

Bayesian methods are now widely used for the estimation and evaluation of dynamic stochastic general equilibrium (DSGE) models [Del Negro and Schorfheide, 2008]. This approach has been discussed by many authors in the literature in the last few years (e.g. [Schorfheide, 2000], [Lubik and Schorfheide, 2007]). A comprehensive survey is provided in [An and Schorfheide, 2007]. Despite their importance, the literature has paid little attention to the systematic elicitation of priors. Prior distributions either reflect subjective opinions or summarize information derived from data not included in the estimation sample (recent exceptions in medical studies [McCandless and Gustafson, 2017], computing science [Le Gratiet et al., 2018] and macroeconomics [Jacobi et al., 2019]).

Due to the formulation of priors and the need to linearize first the model, the bayesian approaches can not be good candidates for our study, as we desire especially to study the robustness of non-linear models without linearly approximating it. Nevertheless, Bayesian approaches have multiple success and applications such as in the DYNARE Software of the European Commission (a DSGE and

overlapping generations model controlled by a sensitivity analysis [Adjemian et al., 2011]) or [Smets and Wouters, 2003], [Oakley and O’Hagan, 2004]

**Conclusion** Conducting our uncertainty analysis on a very non-linear model, without linear approximations, leads to consider a sampling method with stratification, such as Latin Hypercube Sampling, because of its features [McKay et al., 2000] and its ability to be optimized.

### A.2.2 Sensitivity Analysis (SA): from uncertainties in the input to uncertainty in the output

A possible definition of sensitivity analysis (usually conducted in parallel of an UA) is the following: *The study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input* [Saltelli et al., 2008b]. A wide range of techniques are available to conduct sensitivity analysis. This part will present a brief review of them. We develop some of these techniques in next parts.

According to [Saltelli et al., 2008a] SA can be classified into qualitative SA (such as Screening Methods) or Quantitative SA which can be carried for a single input parameter or multiple parameters. Quantitative SA are commonly grouped into local and global methods [Lomas and Eppel, 1992], which distinctions are listed by [Hoes and De Vann, 2005] in Table A.1.

The principle of SA is to regard the mathematics of simulation as a "black box". We only alter the model descriptions such as the initial conditions, the boundary constraints and solution methods and run repeatedly the simulations of each set of inputs as a separate model [Wang, 2014]. Statistical techniques are adopted to analyse the relationship between the perturbations (different sets of input variables) and outputs and so quantify the uncertainty [MacDonald, 2002].

Local SA assesses output uncertainty by increasing a small amount of each input value. For global SA, the input space is sampled and the impacts of all inputs on the output uncertainty are assessed. Most common methods in SA are Screening methods and Finite-difference approximation, Monte-Carlo Analysis, ANalysis Of VAriance (ANOVA).

Even if the methods of local SA are considered to be computational faster and easier. Less accurate is their inherent problem, compared to the sophisticated global SA methods. According to literatures, it also has been stated that the global SA should be used, when input variables are in a non-linear model and from different magnitudes of uncertainty sources [Cukier et al., 1973]. The main methods of SA are further compared in Table A.2.

### A.2.3 Screening Methods

Screening methods are presented, inter alia, by [Campolongo, 2005], they are meant to classify inputs of the model by their impact on outputs. They explore the behavior of the system in presence of a high number of inputs by using discrete sampling techniques on each input. The discretization of inputs in different values (called *levels*) make them so-called deterministic methods, because they do not use probability on inputs. This is the reason why they are usually easy to implement and faster in terms of computation time than other SA methods. These methods are usually classified as qualitative and would not be used in our quantitative study. However the principle of screening

APPENDIX A. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

Items	Local SA	Global SA
Aim	It is meant for the determination of partial derivation of the output in relation to a specific input variable.	It is meant for the determination of output uncertainty in relation to the overall input variables.
Input variables	Input variables are sampled one-at-once (remaining variables stay at their 'base-case' values).	Input variables are sampled simultaneously.
Relationship between input variables and outputs	A linear relationship is assumed between input variables and outputs.	A linear relationship is not necessarily assumed between input variables and outputs.
Distribution of input variables	There is only one assigned distribution to input variables (typically a normal distribution).	Each input variable can be assigned to different variation/distribution (typically a normal distribution).
Numbers of simulations	On average, a large number of simulations are required (but the exact numbers depend on the LSA method).	The number of simulations in local SA is less than in global SA.

Table A.1: Comparison of local and global SA methods [Hoes and De Vann, 2005] and [Hopfe et al., 2007].

Method Name	I	II	III	IV
Monte-Carlo Analysis (MCA)	Yes	Yes	Yes	No
Variant based methods (e.g. ANOVA)	Yes	Yes	Yes	Yes
Screening methods	No/Yes	Yes	Yes	Yes
Local methods (e.g. Finite-difference approximation)	No	No	No	Yes
I: Input variables	The influences of input variables (on outputs) will incorporate the effects of the range of input variation and the probability density function (particularly, the normal or uniform distribution).			
II: Variation of input variables	Different to partial derivatives, the variation of one input variable will cause the changes of others.			
III: Model independence	The performance of sensitivity techniques will not be affected by model nonlinearity, if input variables are independent.			
IV: Treat group variables as they were individuals	The influence of individual variable should be evaluated, even variables are varied as a group. It improves the agility of results interpretation.			

Table A.2: Comparison of four different methods to conduct global sensitivity analysis (European commission, 2005) [Hopfe et al., 2007].

through a Design of Experiment (DoE) with a Optimal Latin Hypercube Sampling inspired the first step of our analysis.

### Simple or High dimensional Screenings' methods

This type of techniques comes from the field of Design of Experiment (DoE) and are easy to implement as they do not need binding assumptions about the nature of the model. Among these assumptions, independence of the inputs, monotony of the output variation in relation with each input and small number of inputs impacting the system (compared to the total number of inputs) are the most famous in litterature [Iooss, 2011] leading to the following methods:

- **Fractional Factorial Screening:** in the field of DoE, we commonly have a greater number of observations than the number of inputs to study. This leads to the probabilistic estimation of the coefficients representing the influence of inputs on output. These influence are categorized in: main effects (or first-order effects), joint effect effects (second-order effects), etc., which can be analyzed with a Full Factorial DoE (where every possible combination of levels of each inputs is tested). However, in a lot of applications, a sufficient number of observations can be hard to provide or might be biased, due to a lack of information. In this context, Fractional Factorial screening (as a part of the so-called oversaturated designs) allows to determine the main effects with a smaller number of simulations. The technique is detailed by [Burman and Plackett, 1946], improved by [Box and Behnken, 1960] and Taguchi methods [Karna et al., 2012].
- **Group Screening:** More adapted to numerical experiments, it consists in the division of inputs into group to identify the more influent ones. This is meant to multiply the group composition by discarding the less influent until the most influent one in obtained. The principle is the following:
  1. Independent inputs are grouped on the apriori intuition of the researcher. Each group-input is then studied as a input with two levels:
    - Group-input is at the + level when every input in the group are at their + level;
    - Group-input is at the - level when every input in the group are at their - level;
  2. Once the most influent group are identified, non influent are discarded of the second step where each group is either divided into smaller groups with less inputs, or each input in the most influent group is one-at-a-time tested.

This method need however a prior knowledge of the change direction of the output according to the change direction of each input, which is not the case in very non-linear macro-economic models.

- **Sequential Bifurcation for Factor-Screening:** A two-groups screening without DoE, which uses regression analysis or variance-based analysis. This approach is close to dichotomic search (or binary search), where experiments are sequentially chosen by using previous results to guide the research of most influent inputs. As for group screening, the computation cost of this function depends on the number of influent inputs, the classification strategy ad the ability to gather most influent inputs in the same group.

## Morris' Method

Campolongo's Screening methods have been developed and improved by Morris, relying on the idea that a model with a high number of inputs is hard to explore, but a small number of inputs are often very influent. A qualitative approach of the Morris' method consist in identifying them first, according to their relative effets:

- negligible effects;
- linear effects without joint effect;
- non-linear effects with joint effect.

With a finite-element approximation, Morris improved the common derivative methods, and can explore bigger variation of the inputs. This lead to what he called "elementary effect", which the total impact on the output due to the variation of the inputs. By repeating the study, he computes the statistical properties of this elementary effect.

The calculation of Morris is thus a mean and variance of derivative calculations for different combinations of the inputs rather than a fine study of a particular "nominal point" (combination of the input) conducted by the common derivative methods. This technique is explored in the global sensitivity analysis subsection, as a Quantitative SA.

### A.2.4 Local Sensitivity Analysis (LSA)

LSA focuses on the local impact of input variables on model outputs. It is usually carried out by assuming the linear or monotonous model, or computing partial derivatives of the outputs, with respect to a small interval fractional variation of an input variable around its normal value [Saltelli et al., 2008a]. The interval could be the same for all of input variables, but it is not related to the knowledge degree of input variables. Since the model we want to study contains nonlinearities, we should avoid Local Sensitivity Analysis, but we review some of these methods here as they pave the way for global sensitivity analysis.

For now, assume that the parameters are all statistically independent and that any combination of parameter values  $X_r$  with  $r = 1...p$  is equally likely. The  $p$  sampled parameter values define a point in  $p$ -dimensional space, and sample points are evenly spread throughout a  $p$ -dimensional hypercube of side 1.

#### Linear model

As a first approximation, the dependency can be viewed as linear in each parameter. So we can write the model as:

$$Y = b_0 + \sum_{i=1}^p b_i X_i \quad (47)$$

As the  $p$ -inputs are assumed to be independent, the variance is calculated by:

$$V(Y) = \sum_{i=1}^p b_i^2 V(X_i) \quad (48)$$

where  $b_i^2 V(X_i)$  is the share of the variance due to the  $X_i$  input. So we can build an indice of the variance of the output due to an input through the *Standard Regression Coefficient* (SRC) indice which is always positive ( $SRC_i \in [0, 1]$ ):

$$SRC_i = \frac{b_i^2 V(X_i)}{V(Y)} \quad (49)$$

We can notice that this indice equals the square of the linear joint effect coefficient  $\rho_{X_i, Y}$ . It is indeed easy to verify that:

$$\begin{aligned} Cov(X_i, Y) &= b_i V(X_i) \\ \Rightarrow \rho_{X_i, Y} &= \frac{Cov(X_i, Y)}{\sqrt{V(X_i)V(Y)}} = b_i \sqrt{\frac{V(X_i)}{V(Y)}} \end{aligned} \quad (50)$$

However, it is sometimes hard to assess the  $SRC_i$  coefficient when the repetition of the simulation are not done on the same set of inputs values. joint effect (second-order effect) between inputs can appears and affect the share of the variance due to a particular input [Saporta, 1990]. This is the reason why we use the *Partial joint effect Coefficient* (PCC) indice:

$$PCC_i = \rho_{X_i, Y|X_{\sim i}} = \frac{Cov(X_i, Y|X_{\sim i})}{\sqrt{V(X_i|X_{\sim i})V(Y|X_{\sim i})}} \quad (51)$$

Where  $X_{\sim i}$  is the  $X$  vector without its  $i$ -th component. These indices are explored in [Saporta, 1990]. If their results are not equal, the classification they imply is the same (by taking  $PCC$  in absolute terms as it can be negative).

### One-at-a-time (OAT) sampling

When solving the matrix problem representing the linear and/or monotonous model  $Y = b_0 + \sum_{i=1}^p b_i X_i$  with  $N$  simulations

$$\begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_p \end{bmatrix} = \begin{bmatrix} 1 & x_1^1 & \dots & x_p^1 \\ 1 & x_1^2 & \dots & x_p^2 \\ \dots & \dots & \dots & \dots \\ 1 & x_1^N & \dots & x_p^N \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_p \end{bmatrix} \quad (52)$$

one can carefully select the data points in order to limit numerical problem. One way of simplifying the matrix  $X_i^j$  is to use an One-at-a-time (OAT) design where only one input is changed at each simulation. As an example with  $X_i \in \{0, 1\}$ , assume the following first set of simulations:

$$\begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_p \end{bmatrix} \quad (53)$$

This system can thus easily simplified by operating the rows into:

$$\begin{bmatrix} y_1 \\ y_2 - y_1 \\ \dots \\ y_p - y_{kp_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \dots \\ b_p \end{bmatrix} \quad (54)$$

[Saltelli et al., 2008a] explains by this equation that if there is any change in value between  $y_i$  and  $y_{i+1}$ , it can only be attributed to a change in parameter  $X_i$  (complicated by random effects if the model is stochastic). So the quantity  $\Delta y_i = y_{i+1} - y_i$  is an estimate of the effect on  $y$  by changing  $X_i$  from 0 to 1. It is applicable everywhere if the linear model is appropriate, and for some region around the current sample point otherwise.<sup>17</sup>

However OAT sampling is inefficient when the number of parameters  $p$  is large and only a few of them are influential [Saltelli et al., 2008a] (about chemical kinetics, see the comparison of [Zádor and Turányi, 2006]). As the macro-economic models we are studying is very non-linear and as we can not apriori define the number of influential input, we do not use this family of methods.

Extensions with different types of sampling, such as Monte-Carlo sampling, Fractional Factorial DoE, Stratified or Latin Hypercube Sampling exists and have been described above, and are applicable in the frame of linear and/or monotonous model. Generalization of the method exists known as the Finite differences analysis. However, to capture joint effect with other inputs or nonlinearities (which can hardly be identified with the OAT sensitivity analysis) a combination with scenario analysis, discussed next, might be necessary.

### Scenario analysis

In this very common LSA in economics, several inputs values are simultaneously changed to figure changes in economy, allowing to capture some joint effect between inputs. This intuitively appealing method remains hard to apply in order to isolate the effect of each joint effect.<sup>18</sup> As explained in [Harenberg et al., 2019], one solution consists in combining scenario analysis with

<sup>17</sup>This is the work developed with success by [Erb and Michaels, 1998] about biological models with closed time interval and small distance to a "nominal point". In our example, we only take two possible extreme value of the domain of each input. More complicated paths through the sample space would involve much smaller changes to the parameter values, which would allow the estimation of elementary effects, as previously explained in the Morris' method.

<sup>18</sup>About the canonical RBC model, evoked in section 2.4.2 [Harenberg et al., 2019], the authors explain that scenarii can not capture a particularly plausible economic environment, but rather to exemplify the approach. One drawback that becomes apparent is the high level of discretion typically involved in choosing scenarii and corresponding parameter values, which is due to the local nature of this sensitivity measure. As an example, the scenarii can be:

- "Baseline" Scenario: where all inputs are kept at their baseline values.
- "High risk and risk aversion" Scenario: where  $\sigma$ , and  $\rho$ , the standard deviation and autojoint effect of the total factor productivity (TFP), are at their upper bounds and  $\tau$ , the intertemporal elasticity of substitution (IES), is at its lower bound, so that risk aversion  $1 - \tau$  is high. All other inputs are kept their baseline values.
- "High capital utilization and frictions" Scenario: where  $\alpha$ , the capital share,  $\delta$ , the depreciation rate and  $\phi$ , capital adjustment cost, are at their upper bounds, and all other inputs are kept at their baseline values.

Displaying the observed output from these different scenarii allows us to compare the scenarii and evaluate the impact of joint parameter changes on the outputs. However, scenario analysis can not specify which parameter or which interaction between parameters is important in each case.



OAT finite differences to tease out individual parameter effects and interactions, which is known as scenario decomposition or generalized Tornado diagrams carried out with success in [Borgonovo et al., 2011] and [Borgonovo and Plischke, 2016]. However, such Tornado diagrams are rarely encountered in economic studies and are outside the scope of this study, since they also suffer from the fact that they are local and linear.<sup>19</sup>

### Partial derivatives

Computing partial derivatives of the outputs, with respect to a small interval fractional variation of an input variable around its normal value, can be used as a sensitivity measure. It delivers information on the influence of an input in the local neighborhood of a "nominal point".

Among the advantages of the method over common screening methods are that it covers the entire input space and that it is not restricted to linear influences. Partial differences at different points in the input space, so-called elementary effects by [Morris, 1991], are computed:

$$d_i(X) = \frac{|Y(x_1, \dots, x_i + \Delta_p, \dots, x_k) - Y(X)|}{\Delta_p} \quad (55)$$

where  $\{x_1, \dots, x_k\}$  is sampled on a grid of  $p$  levels,  $x_i$  is the value of the  $i$ -th component of  $X$  and  $\Delta_p$  is a predetermined multiple of  $\frac{1}{p-1}$ . The distribution of the  $d_i$  provides information on the behavior of  $X_i$ , and we can study its statistical properties: overall influence of the input is characterized by the mean of  $d_i(X)$ , so called Morris Importance Measure, standard deviation characterizes the linearity of the  $X_i$  influence on the output  $Y$ . One can use various sampling techniques (Random Values, Stratified Sampling, Monte-Carlo Sampling, LHS,...) to elaborate designs in order to calculate this distribution of  $d_i$ .

Eventually, computing the Jacobian matrix of a system at various particular points should give a global sensitivity measure. This is the idea of Derivative-based Global Sensitivity Measure (DSGM) as developed by [Sobol' and Kucherenko, 2009] and improved by [Lamboni et al., 2013].

#### A.2.5 Global Sensitivity Analysis (GSA)

To alleviate the lacks of Local Sensitivity Analysis, when studying very non-linear models, or exploring large inputs spaces, one can conduct a Global Sensitivity Analysis. A wide range of GSA methods are existing. In this part, we motivate the selected method of our study, that is a GSA with Sobol' indices, by reviewing some of the most famous methods, such that:

- Derivative-based Global Sensitivity Measures (DGSM)<sup>20</sup> [Lamboni et al., 2013]

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<sup>19</sup>However some of these techniques are still successfully used in linear macro-economic models or by approximating nonlinearities with small perturbations spaces around the equilibrium, such as e.g. [Iskrev, 2010], [Gourieroux et al., 2000] or the DSGE and overlapping generations models from the European Commission [Adjemian et al., 2011]. These techniques often used partial derivatives to conduct their sensitivity analysis.

<sup>20</sup>Defined in [Saltelli et al., 2008b] as:

*"derivatives are only informative at the base point where they are computed and do not provide for an exploration of the rest of the space of the input factors. This would matter relatively little for linear systems, in which the property at a point away from the baseline can be computed quickly by linear extrapolation using first-order point derivatives, but it would matter greatly for nonlinear ones"*, as can be macro-economic models.



- Variance-based sensitivity indices (Sobol' indices) for independent inputs and its extension for dependent inputs (Shapley effects)
- Frequency-based GSA: Fourier Amplitude Sensitivity Test (FAST) [Cukier et al., 1973]
- Other GSA in macroeconomics: Quantitative Meta-model (based on Chaos polynomial) [Harenberg et al., 2019] , [Fajraoui, 2014]

**Derivative-based Global Sensitivity Measures** The original [Morris, 1991] method aims at measuring local sensitivity through "elementary effect" with a finite element approximation. A generalization of this method to global measurement was given in [Sobol' and Kucherenko, 2009] in order to capture nonlinearities between output and inputs. This technique knew some successful studies e.g. in finance about Option pricing [Campolongo et al., 2006] or Asset Allocation [Manganeli, 2004].

Before [Sobol' and Kucherenko, 2009], this Morris' method with finite elements approximation was assumed to be reliable to assess the total indices of the inputs influence, with comparable results to the total Sobol' indices  $S_{T_i}$  (see next paragraph). But their numerical experiments shows that SA based on  $v_i$  can not allow to classify inputs by order of influence.<sup>21</sup> This is why this method are usually classified in screening methods. Even if they are less time-costly than total Sobol' indices calculation methods they can not be appropriate to our study.

**Frequency-based GSA: Fourier Amplitude Sensitivity Test (FAST)** The FAST method was first developed by [Cukier et al., 1973], [Cukier et al., 1978] and [Schaibly and Shuler, 1973]. It approximates the pluri-dimensionnal decomposition with mono-dimensionnal decompositions along a curve on the domain  $[0, 1]^p$ . This curve is defined by parametric equations:

$$x_i(s) = g_i(\sin(\omega_i s)) \quad \text{for } i = 1, \dots, p \quad (56)$$

where  $g_i$  are the function to determine in order to guarantee an uniform coverage of the identity hypercube (see example below),  $(\omega_1, \dots, \omega_p) \in \mathbb{N}^p$  are integer frequency linearly independent (no linear combinations between them). Thus, when  $s$  is varying in  $\mathbb{R}$ , the vector  $(x_1(s), \dots, x_p(s))$  is varying on the curve in  $[0, 1]^p$ . [Saltelli and Bolado, 1998] showed, with a high number of experiments that the results about FAST sensitivity indices are equivalent to those obtained with Sobol' methods.<sup>22</sup>

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<sup>21</sup>The demonstration of this result is available in [Lamboni et al., 2013]. Indeed, the total-impact indices of this derivative-based method,  $v_i$ , approximates  $v_i = \int_{\Omega^n} \left( \frac{\partial f(x)}{\partial x_i} \right)^2 dx$  and the authors showed that the total Sobol' indices,  $S_{T_i} \leq 4C_i \frac{v_i}{V}$  with  $C_i = \frac{1}{4\pi^2}$ . Consequently, if  $X_i$  is not influent, its total-impact indices,  $v_i \approx 0$ , showing that the Morris' method is useful when assessing what inputs are not influent.

<sup>22</sup> [Cukier et al., 1973] show that  $f_0 = \int_{[0,1]^p} f(x)dx = \lim_{T \rightarrow -\infty} \frac{1}{2\pi} \int_{-T}^T f(x(s))ds$ . With integer frequency, the curves are  $2\pi$ -periodic:  $f_0 = \frac{1}{2\pi} \int_{-2\pi}^{\pi} f(x(s))ds$ . These calculations are applied to compute the variance of the model  $Y = f(X_1, \dots, X_p)$ . [Saltelli et al., 1999] introduced an Extended FAST method, to compute total sensitivity indices by assessing the share of the variance due to all the inputs but  $X_i$  as the sum of the square of all Fourier coefficients linked to frequencies different from  $\omega_i$  and its harmonic.

However the estimation of the indices involves to define  $g_i$ ,  $\omega_i$ <sup>23</sup> and  $M$ , an upper bound to assess the coefficient, which must also be chosen, as infinite sums are not computable.<sup>24</sup>  $M$  is so a compromise between the quality of the indices and their computation cost.<sup>25</sup>

One of the advantages of this method is that indices can be calculated independently from each others, with the same sample, whereas Sobol' method involves at least two samples. However, the Sobol' method is stochastic, it allows to display confident interval about the estimation of the indices, which is impossible with FAST, as the estimation is deterministic once the set of frequency is chosen. [Saltelli and Bolado, 1998] compared FAST and Sobol' methods on a various number of models and concluded that FAST was computationally less complex and less time-costly. Nevertheless, FAST can be biased due to the choice of the frequency, whereas Sobol' always converge to the actual value of the sensitivity indices.

In our study, the complexity of the model due to the number of inputs and nonlinearities is not too important to motivate a FAST. To avoid biases due to the choice of frequency we will so not use a FAST to analyse the sensitivity of our model.

**Variance-based GSA: ANalysis Of the VAriance (ANOVA) with Sobol' indices** As recommended in [Saltelli et al., 2008b], the relative influence of the inputs variance on output variance allows better results about the global sensitivity measure of a very non-linear system. *This is in order to exploit the statistical properties of variance [...] to investigate how factors contribute to the variance. Interesting features of variance-based methods are:*

- *Model independence: the sensitivity measure is model-free;*
- *Capacity to capture the influence of the full range of variation of each input factor;*
- *Appreciation of interaction effects among input factors;*

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<sup>23</sup>The definition of the function  $g_i$  have been proposed with different techniques. One of the most known remains the one of [Saltelli et al., 1999], because it gives the best space coverage with a uniform distribution of the samples:

$$x_i(s) = g_i(\sin(\omega_i(s))) = \frac{1}{2} + \frac{1}{\pi} \arcsin(\sin(\omega_i(s))) \quad (57)$$

For the frequency, the method consists in selecting a high frequency for the observed input  $X_i$  and smaller frequency for the others inputs. But this kind of method tend to overestimate the influence of the inputs on the output variance [Saltelli et al., 1999]. The algorithm proposed by the authors is:

1. Choosing  $\omega_i$  frequency according to the impact on the minimum number of simulations  $N$ ;
2. Determining the maximum value of the others frequency values:  $\max(\omega_{\sim i}) = \frac{\omega_i}{2M}$ ;
3. All the other frequencies are chosen between 1 and  $\max(\omega_{\sim i})$  with two conditions:
  - (a) Interval between each frequency should be maximal;
  - (b) Number of variables  $X_i$  with the same frequency should be minimal.

[Saltelli et al., 1999] determine thus the minimal sampling dimension pour the assessment of these indices:

$$N = 2M \max(\omega_{\sim i}) + 1 \quad (58)$$

<sup>24</sup> $M$  acts as a maximum harmonic in the calculation, and is chosen according to: The bigger, the more reliable will be the indices and the smaller, the faster simulations will be.

<sup>25</sup> [Cukier et al., 1975] estimates empirically that  $M$  should be in 4 or 6, whatever is the dimension of the model.

- *Capacity to tackle groups of input factors: uncertain factors might pertain to different logical types, and it might be desirable to decompose the uncertainty according to these types.*

The main drawbacks of variance-based measures are their computational cost, which is not a big issue for our model (less than ten inputs) and the high number of possible methods, leading to different rankings of factors according to their objectives.

The application principle of the ANOVA with Sobol' indices is described in section 2.2.3. The estimation of these indices remains challenging (especially for second-order Sobol' effect, representing joint effect between output<sup>26</sup>) and is developed in the next subsection A.2.6. Various successful studies were conducted about DSGE models (e.g. [Ratto, 2008]).

Suppose that we can express a square integrable random variable  $Y$  as  $Y = f(\mathbf{X})$  where  $\mathbf{X} = (X_1, \dots, X_p)$  is a vector of  $p$  independent random variables uniformly distributed on  $[0, 1]$ . From the so-called [M. Sobol, 1990] decomposition, it is possible to expand  $f \in L^2([0, 1]^p, dx)$  into orthogonal summands of increasing dimensions

$$f(x_1, \dots, x_p) = f_0 + \sum_{i=1}^p f_i(x_i) + \sum_{i=1}^p \sum_{j>i}^p f_{i,j}(x_i, x_j) + \dots + f_{1,\dots,p}(x_1, \dots, x_p) \quad (59)$$

and to obtain the classical ANOVA decomposition

$$\begin{aligned} V(Y) &= \sum_{i=1}^p V_i + \sum_{i=1}^p \sum_{j>i}^p V_{i,j} + \dots + V_{1,\dots,p} \\ &\Updownarrow \\ 1 &= \sum_{i=1}^p \frac{V_i}{V(Y)} + \sum_{i=1}^p \sum_{j>i}^p \frac{V_{i,j}}{V(Y)} + \dots + \frac{V_{1,\dots,p}}{V(Y)} \end{aligned} \quad (60)$$

where

$$\begin{aligned} V_i &= V(E(Y|X_i)) \\ V_{i,j} &= V(f_{i,j}(X_i, X_j)) = V(E(Y|X_i, X_j)) - V(E(Y|X_i)) - V(E(Y|X_j)). \\ &\vdots \end{aligned} \quad (61)$$

---

<sup>26</sup>The issue of GSA with dependent variables has been the object of intense recent research but is not considered in our study as we make the classical assumptions of *a priori* independent inputs at a first step of the study. A prospect of this work can be to check the independence of these inputs. We review here the different improvements made to take into account dependent inputs. About Sobol' indices, [Xu and Gertner, 2008] exhibits a Sobol' indices decomposition into dependent/correlated and independent/uncorrelated components for linear models. With this decomposition, [Li et al., 2010] build the Sobol' indices for a general model. An other technique is proposed by [Mara and Tarantola, 2011], using the Gram-Schmidt process to decorrelate the inputs variables. They also propose to define new indices through the Sobol' indices of the decorrelated problem. [Chastaing et al., 2013] provide a theoretical framework to generalize the ANOVA decomposition to problems with dependent variables, based on the work of [Hooker, 2007]. In contrast to the other works which focus on generalizing the ANOVA decomposition, [Kucherenko et al., 2012] develop the Sobol' indices via the law of total variance. Recent work of [Tarantola and Mara, 2017] considered estimating Sobol' indices with dependent variables using the Fourier Amplitude Sensitivity Test. In our study, inputs of the model are the exogenous parameters of [Grasselli and Costa Lima, 2012], assumed to be independent. Thus we can only consider the core method of the Sobol' indices to assess its sensitivity to its inputs. But these extensions can be useful when using e.g. a Taylor's rule to endogeneize the interest rate  $r$ .

In this way, we can define the Sobol indices:  $\forall s \in \{1, \dots, p\}, \forall i_1 < i_2 < \dots < i_s \in \{1, \dots, p\}^s$

$$S_{i_1, \dots, i_s} = \frac{V_{i_1, \dots, i_s}}{V(Y)}. \quad (62)$$

The general principle of the estimation of these indices is usually based on a Monte-Carlo analysis<sup>27</sup> with independent inputs (also called pick-freeze method) with the scheme of [Sobol, 1993] and [Saltelli, 2002]<sup>28</sup> or using replicated array-based Latin hypercube sample [McKay et al., 2000].<sup>29</sup>

**Other GSA in macroeconomics:** Beside variance-based analysis (or frequency-based analysis) one can use emulators build on orthogonal polynomials (Hermite, Legendre, ...). Among them, Chaos Polynomial Expansion (CPE) are the most famous in the litterature due to their efficeine in convergence (see [Harenberg et al., 2019] for a recent succes about canonical real business cycles models). This development is based on the theory of [Wiener, 1938] and the works of [Cameron and Martin, 1947], with a recent interest in numerical stochastic process.<sup>30</sup> The principle is to build a meta-model through CPE, which has the features to be reversible, and the disadvantages to need truncatures (in a same way as FAST) involving an *a priori* arbitrary choice about the degree of the polynomial.

The technique involves a lot of similarity with the ANOVA decomposition since the CPE are orthogonal for a Gaussian process. The estimation of the coefficient is still challenging. There are two group of method:

- Intrusive methods [Le Maître et al., 2002] [Matthies and Keese, 2003]: they consists in introducing calculation of the polynomial coefficient in the model. They allow thus to estimate these coefficients with a single run of the simulation. But the method is vey complex to implement depending on the degree of nonlinearities of the model [Berveiller, 2005];
- Non-intrusive methods [Li and Zhang, 2007], [Blatman and Sudret, 2010]: they consider the model as a "black box", and estimate the coefficient through numerical simulation in dedicating points of the inputs spaces (such as a Latin Hypercube Sampling), but their reliability is highly dependent on the number of simulations [Fajraoui, 2014], and of the choice of the DoE [Sudret, 2008].

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<sup>27</sup>Others methods consists in estimating Sobol' first order indices with with B-spline Smoothing [Ratto and Pagano, 2010] or with Saltelli's so-called "extended-FAST" method [Saltelli et al., 1999].

<sup>28</sup>The seminal method of [Sobol, 1993] to compute the indices given by the variance decomposition up to a specified order, improved with the Saltelli's scheme [Saltelli, 2002] to compute first order, second order and total indices has many variations: inspired from the work of Jansen-Sobol's [Jansen, 1999], Mauntz-Kucherenko's scheme computes first order and total indices using improved formulas for small indices [Tarantola et al., 2007] ; the Martinez's scheme uses joint effect coefficient-based formulas associated with theoretical confidence [Martinez, 2011] ; Janon-Monod uses optimal asymptotic variance [Monod et al., 2006] [Janon et al., 2014] ; or via unique matrix permutations [Alex Mara and Rakoto Joseph, 2008] ; via 3 input independent matrices [Owen, 2013] for first-order and total-effect indices using Liu-Owen's scheme [Liu and Owen, 2006] and pick-freeze scheme [Fruth et al., 2014].

<sup>29</sup>Other sampling methods can be used for the estimation of the Sobol' first order and closed second order indices with, for example, replicated orthogonal array-based Latin hypercube sample [Tissot and Prieur, 2015], or under inequality constraints by extension of the replication procedure or with kriging-based global sensitivity analysis [Le Gratiot and Garnier, 2014] - available in the [Pujol et al., 2015] package.

<sup>30</sup>See e.g. [Ghanem and Spanos, 1991]

**Conclusion:** Since the model we study contains nonlinearities and we want to assess the relative effects of its parameters on the output, we should avoid Local Sensitivity Analysis. We can conduct a screening method at a first step but we overall must conduct a Global Sensitivity Analysis. The first screening method can be based on Latin Hypercube Sampling, according to its features [McKay et al., 2000].

The Global Sensitivity Analysis (GSA) can either be conducted with variance-based, frequency-based or meta-models methods. Their results are quite equivalent [Saltelli and Bolado, 1998], but the last two need the *apriori* choice of an upper bound (in the degree of the polynomial or the harmonic), which can introduce biases in the study to assess actual values of the sensitivity indices. Consequently we use a GSA with the Sobol' indices methods.

As the inputs are assume to be independent, we do not need the introduction of the possible extension, and we only have to study the different method to estimate the Sobol' indices. These methods are commonly based on Monte-Carlo or quasi-Monte-Carlo sampling. But [McKay et al., 2000] introduces an improvement of the space coverage with the help of Latin Hypercube Sampling. This is the object of the next subsection.

### A.2.6 Estimation of the indices

Sobol' indices can sometimes be formally calculated when the analytic form of the model is known and simple. But, in the case of our study, the model is represented as a black box because this function is either unknown or too complex. We have to approximate these indices using fast numerical tools. Let us remark that the Sobol' indices may be expressed using ratio of variances of various conditional expectations. Thus, the use of basic Monte Carlo methods to approximate them seems to be prohibitive in this case due to a quadratic time complexity.

In this subsection, we review an approach adapted to our choice of analysis: the Sobol' estimation method, improved by [Saltelli, 2002] and [McKay et al., 2000] leading to a linear time complexity. Some success-studies with this estimation methods have been conducted by [Canova, 1995] about DSGE, in finance with [Kucherenko, 2010] about GSA with Monte-Carlo pricing and [Gilquin et al., 2016] about a Land use and Transport Integrated Model.

We generate<sup>31</sup> two independent input sample matrix of size  $N$  for  $\mathbf{X} = (X_1, \dots, X_p)$ :

$$\mathbf{X}^{(1)} = \begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} & \dots & x_{1p}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_{N1}^{(1)} & x_{N2}^{(1)} & \dots & x_{Np}^{(1)} \end{pmatrix} \quad \mathbf{X}^{(2)} = \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} & \dots & x_{1p}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_{N1}^{(2)} & x_{N2}^{(2)} & \dots & x_{Np}^{(2)} \end{pmatrix} \quad (63)$$

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<sup>31</sup>Even if classical random number generators can be used, Quasi Monte-Carlo sampling can be more efficient according to [Sobol, 1998], [Bratley et al., 1992] and [Joe and Kuo, 2008], because it offers a better speed of convergence. What is more, [McKay et al., 2000] develops an improved method based on Latin Hypercube Sampling instead of Monte-Carlo sampling in order to generate sets of inputs to test with a better space coverage for a smaller number of simulations needed. The latter is implemented in this work.

For  $j \in \{1, \dots, p\}$ , the first order Sobol' indice  $S_j$  measuring the single impact of  $X_j$  on the total variance is given by  $S_j = \frac{V_j}{V(Y)} = \frac{V(E(Y|X_j))}{V(Y)}$  where

$$V_j = \underbrace{E(E(Y | X_j)^2)}_{U_j} - E(Y)^2. \quad (64)$$

Classically,  $E[Y]$  and  $V(Y)$  are respectively approximated using

$$\bar{Y} = \frac{1}{2N} \sum_{i=1}^N \left( f(x_{i1}^{(1)}, \dots, x_{ij}^{(1)}, \dots, x_{ip}^{(1)}) + f(x_{i1}^{(2)}, \dots, x_{ij}^{(2)}, \dots, x_{ip}^{(2)}) \right)$$

and

$$\hat{V} = \frac{1}{2N} \sum_{i=1}^N \left( f^2(x_{i1}^{(1)}, \dots, x_{ij}^{(1)}, \dots, x_{ip}^{(1)}) + f^2(x_{i1}^{(2)}, \dots, x_{ij}^{(2)}, \dots, x_{ip}^{(2)}) \right) - \bar{Y}^2$$

while the term  $U_j$  is estimated, using [Saltelli, 2002] fixing method (see also [Jacques, 2011]), by

$$\hat{U}_j = \frac{1}{N} \sum_{i=1}^N f(x_{i1}^{(1)}, \dots, x_{ij}^{(1)}, \dots, x_{ip}^{(1)}) \times f(x_{i1}^{(2)}, \dots, x_{i(j-1)}^{(2)}, x_{ij}^{(1)}, x_{i(j+1)}^{(2)}, \dots, x_{ip}^{(2)}).$$

Thus, only using  $N(p+2)$  runs of the model, the first order Sobol' indices may be estimated using

$$\hat{S}_j = \frac{\hat{U}_j - \bar{Y}^2}{\hat{V}}. \quad (65)$$

In the same way, for  $j < k \in \{1, \dots, p\}^2$ , the second-order Sobol' index  $S_{j,k}$  measuring the joint impact of  $(X_j, X_k)$  on the total variance (removing the first order impacts of  $X_j$  and  $X_k$ ) is given by

$$S_{j,k} = \frac{V(E(Y|X_j, X_k)) - V(E(Y|X_j)) - V(E(Y|X_k))}{V(Y)} = \frac{\overbrace{E(E(Y | X_j, X_k)^2)}^{U_{j,k}} - U_j - U_k + E(Y)^2}{V(Y)}$$

and may be approximated by

$$\hat{S}_{j,k} = \frac{\hat{U}_{j,k} - \hat{U}_j - \hat{U}_k + \bar{Y}^2}{\hat{V}} \quad (66)$$

where

$$\hat{U}_{j,k} = \frac{1}{N} \sum_{i=1}^N f(x_{i1}^{(1)}, \dots, x_{ij}^{(1)}, \dots, x_{ik}^{(1)}, \dots, x_{ip}^{(1)}) \times f(x_{i1}^{(2)}, \dots, x_{i(j-1)}^{(2)}, x_{ij}^{(1)}, x_{i(j+1)}^{(2)}, \dots, x_{i(k-1)}^{(2)}, x_{ik}^{(1)}, x_{i(k+1)}^{(2)}, \dots, x_{ip}^{(2)}). \quad (67)$$

Obviously, the approximation of these second-order indices relies on the estimation of the first-order ones. More generally, the computation of higher-order indices becomes more and more cumbersome and difficult to handle in practice when the number  $p$  of inputs is large. Nevertheless, in order

to capture the global impact (sum of single and related joint effects) of  $X_j$  on the variance of  $Y$ , remind that the total-effect Sobol' index may be expressed as

$$S_{T_j} = \frac{E(V(Y|X_{\bar{j}}))}{V(Y)} = 1 - \frac{V(E(Y|X_{\bar{j}}))}{V(Y)} = 1 - \frac{\overbrace{E(E(Y|X_{\bar{j}})^2) - E(Y)^2}^{U_j}}{V(Y)} \quad (68)$$

where  $X_{\bar{j}} = (X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_p)$ . Thus, at a reasonable computational cost (only using  $N(p+2)$  runs of the model), total-effect Sobol' indices may be estimated as follows

$$\hat{S}_{T_j} = 1 - \frac{\hat{U}_{\bar{j}} - \bar{Y}^2}{\hat{V}} \quad (69)$$

where

$$\hat{U}_{\bar{j}} = \frac{1}{N} \sum_{i=1}^N f(x_{i1}^{(1)}, \dots, x_{ij}^{(1)}, \dots, x_{ip}^{(1)}) \times f(x_{i1}^{(1)}, \dots, x_{i(j-1)}^{(1)}, x_{ij}^{(2)}, x_{i(j+1)}^{(1)}, \dots, x_{ip}^{(1)}).$$

### A.2.7 Synthesis and selection of the GSA: Sobol' indices estimated by Mc Kay's method

Even if this list of UA and SA techniques is not exhaustive, this review has illustrated the great variety of available methods, positioning in terms of assumptions and type of results. A canonical synthesis was provided in [Iooss and Lemaître, 2015] (see Figure A.5) which has several levels of reading:

- Distinction between screening methods (identification of non-influential variables among a large number) and more precise variance-based quantitative methods;
- Positioning methods based on their cost in terms of model calls number (which linearly depends in the number of inputs for most of the methods);
- Positioning methods based on their assumptions about the model complexity and regularity;
- Distinction between the type of information provided by each method;
- Identification of methods which require some a priori knowledge about the model behavior.

Based on the characteristics of the different methods, some authors [De Rocquigny, 2008] have proposed decision trees to help the practitioner to choose the most appropriate method for its problem and its model. Figure 2.3 reproduces the flowchart of [De Rocquigny, 2008]. Although useful to fix some ideas, such diagrams are rather simple and should be used with caution.

In many applications, outputs can be vector or function (temporal, spatial, spatio-temporal,...),<sup>32</sup> The case of functional inputs also receives a growing interest<sup>33</sup> but its treatment in a functional statistical framework remains to be done [Iooss, 2011].

<sup>32</sup>see [Campbell et al., 2006], [Auder et al., 2010], [Lamboni et al., 2013] who have produced first SA results on such problems.

<sup>33</sup> [Iooss and Ribatet, 2009], [Saint-Geours et al., 2011].



APPENDIX A. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

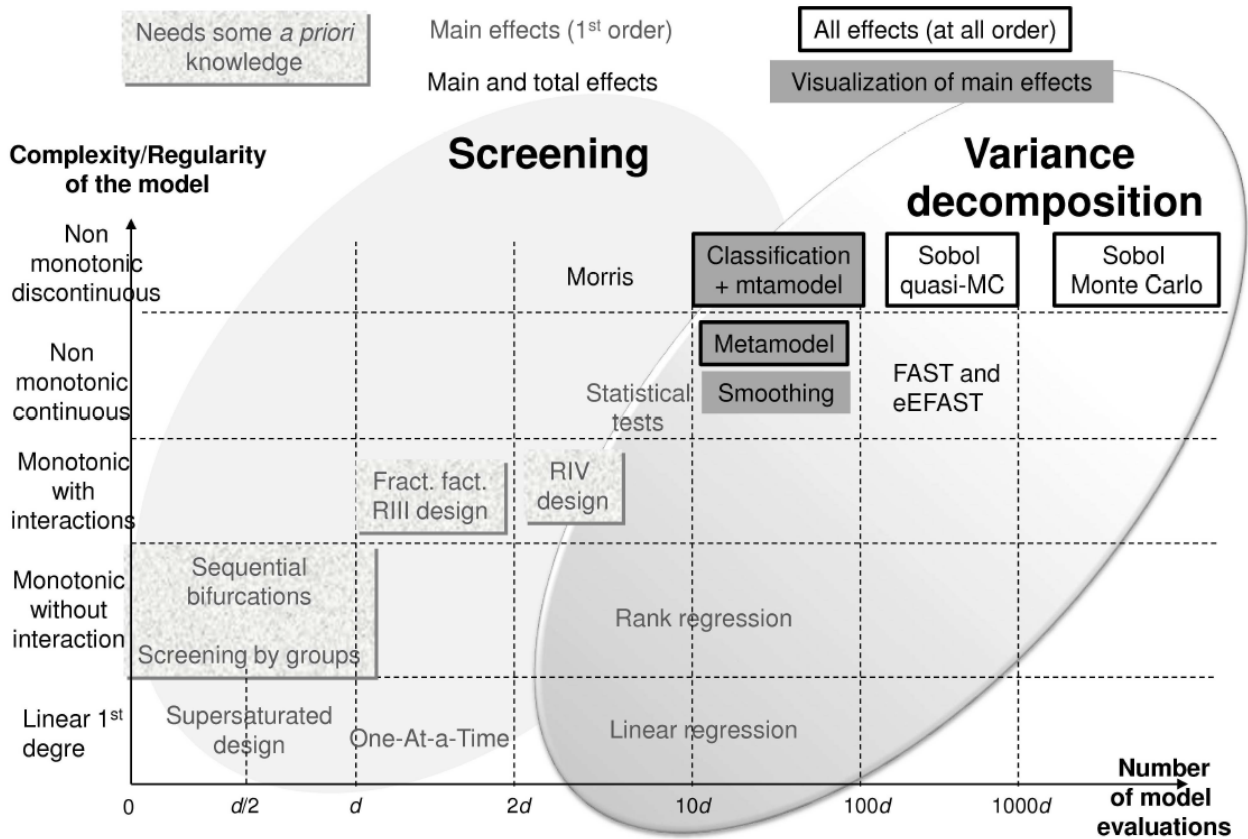


Figure A.5: SA methods graphical synthesis [Iooss and Lemaître, 2015].

To study the [Grasselli and Costa Lima, 2012] macroeconomic (*a priori* non monotonic) model, with a non-linear dynamic, partial derivative equations, and around 10 inputs (parameters of the model), we have thus:

1. (Qualitative SA and UA) To conduct first a screening method with a Latin Hypercube Sampling (according to [Iman, 1999]), in order to identify the propagation of uncertainties and the inputs with negligible effects;
2. (Quantitative SA) Then, to refine our study, by assessing the First order, Second order and Total effect Sobol' indices with the [Saltelli, 2002] estimation method.



### A.3 Second-order Sobol' indices of the [Grasselli and Costa Lima, 2012] model with 100 simulations

In this appendix, we display the second-order Sobol' indices of the [Grasselli and Costa Lima, 2012] model, computed as explained in Appendix A.2.6, with 100 simulations for the parameters described in the Table 2.3 of the Subsection 2.3.3. These results, as reported in Tables A.3 and A.4, are coherent with the properties of the Sobol' indices (see subsection 2.2.3 about the Variance-based GSA) where the total-effect Sobol' indices of one input is always greater than the sum of the contribution of the first-order Sobol' indice of this input and its joint effects (second-order Sobol' indices). Note that The main influent joint effect between inputs is due to the same four parameters highlighted at first-order:  $\alpha - \nu$  for the wage share, around 0.2346,  $\alpha - \delta$  for the employment rate, around 0.1982, and  $\beta - \nu$  for the debt ratio, around 0.1897. The results are once again relatively stable independently of the outputs.

N = 100 simulations	Second-order Sobol' indices $S_{ij}$ of the [Grasselli and Costa Lima, 2012] model						
Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$	Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$
$\alpha - \beta$	0.0797 (0.0439)	0.1769 (0.0818)	0.1237 (0.0736)	$k_0 - r$	0.0014 (0.0014)	0.0012 (0.0006)	0.0017 (0.0032)
$\beta - \delta$	0.1082 (0.1044)	0.1353 (0.0827)	0.13476 (0.1235)	$\alpha - \phi_0$	0.0005 (0.0002)	0.0029 (0.0042)	0.0017 (0.0033)
$\delta - \nu$	0.1913 (0.1142)	0.1191 (0.0793)	0.1314 (0.0852)	$\beta - \phi_1$	0.0005 (0.0002)	0.0058 (0.0121)	0.0017 (0.0034)
$\nu - \phi_0$	0.0015 (0.0016)	0.0406 (0.0705)	0.03499 (0.0274)	$\delta - k_0$	0.0007 (0.0006)	0.0018 (0.0012)	0.0024 (0.0034)
$\phi_0 - \phi_1$	0.0004 (0.0002)	0.0013 (0.0011)	0.0016 (0.0032)	$\nu - k_1$	0.0020 (0.0006)	0.0976 (0.0926)	0.1310 (0.0976)
$\phi_1 - k_0$	0.0007 (0.0005)	0.0012 (0.0008)	0.0017 (0.0030)	$\phi_0 - k_2$	0.0005 (0.0002)	0.0044 (0.0095)	0.0017 (0.0034)
$k_0 - k_1$	0.0007 (0.0006)	0.0012 (0.0008)	0.0019 (0.0031)	$\phi_1 - r$	0.0004 (0.0002)	0.0008 (0.0006)	0.0017 (0.0032)
$k_1 - k_2$	0.0432 (0.0306)	0.0120 (0.0058)	0.0455 (0.0356)	$\alpha - \phi_1$	0.0380 (0.0236)	0.0483 (0.0599)	0.0429 (0.0297)
$k_2 - r$	0.0018 (0.0012)	0.0858 (0.0576)	0.0017 (0.0034)	$\beta - k_0$	0.0911 (0.0712)	0.0014 (0.0006)	0.0021 (0.0061)
$\alpha - \delta$	0.0877 (0.0527)	0.1982 (0.1151)	0.1208 (0.0886)	$\delta - k_1$	0.0406 (0.0188)	0.0246 (0.0245)	0.0341 (0.0254)
$\beta - \nu$	0.2211 (0.1112)	0.1403 (0.0535)	0.1897 (0.0765)	$\nu - k_2$	0.0017 (0.0011)	0.0012 (0.0007)	0.0276 (0.0311)
$\delta - \phi_0$	0.0675 (0.0506)	0.0047 (0.0092)	0.0642 (0.0537)	$\phi_0 - r$	0.0005 (0.0002)	0.0012 (0.0007)	0.0017 (0.0032)

Table A.3: Part 1 / Second-order Sobol' indices and standard deviations (with 10 bootstrap replicates and 100 simulations) of all inputs in the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$  and employment rate  $\lambda_1$

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N = 100 simulations	Second-order Sobol' indices $S_{ij}$ of the [Grasselli and Costa Lima, 2012] model						
Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$	Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$
$\nu - \phi_1$	0.0004 (0.0002)	0.0077 (0.0179)	0.0017 (0.0033)	$\alpha - k_0$	0.0217 (0.0154)	0.034 (0.0510)	0.0387 (0.0261)
$\phi_0 - k_0$	0.0004 (0.0003)	0.0017 (0.0017)	0.0017 (0.0032)	$\beta - k_1$	0.0306 (0.0256)	0.0229 (0.0258)	0.0228 (0.0249)
$\phi_1 - k_1$	0.0004 (0.0003)	0.0027 (0.0049)	0.0017 (0.0033)	$\delta - k_2$	0.0993 (0.0798)	0.0387 (0.0406)	0.0631 (0.0489)
$k_0 - k_2$	0.0008 (0.0006)	0.0014 (0.0004)	0.0020 (0.0032)	$\nu - r$	0.0028 (0.0011)	0.0921 (0.0747)	0.0006 (0.0004)
$k_1 - r$	0.0012 (0.0009)	0.001 (0.006)	0.0017 (0.0032)	$\alpha - k_1$	0.0464 (0.0324)	0.0299 (0.0338)	0.0178 (0.0110)
$\alpha - \nu$	0.2346 (0.1655)	0.1602 (0.0873)	0.1842 (0.1272)	$\beta - k_2$	0.0184 (0.0213)	0.1341 (0.0321)	0.1193 (0.1152)
$\beta - \phi_0$	0.0004 (0.0002)	0.0036 (0.0702)	0.0017 (0.0034)	$\delta - r$	0.0018 (0.0009)	0.0375 (0.0619)	0.0315 (0.0401)
$\delta - \phi_1$	0.0013 (0.0015)	0.0410 (0.0718)	0.0352 (0.0273)	$\alpha - k_2$	0.0972 (0.0641)	0.0229 (0.0112)	0.0904 (0.0878)
$\nu - k_0$	0.0245 (0.0090)	0.0387 (0.0569)	0.0434 (0.0403)	$\beta - r$	0.0018 (0.0010)	0.0009 (0.0007)	0.0020 (0.0034)
$\phi_0 - k_1$	0.0005 (0.0002)	0.0022 (0.0354)	0.0017 (0.0032)	$\alpha - r$	0.002 (0.0015)	0.0012 (0.0007)	0.0017 (0.0033)
$\phi_1 - k_2$	0.0004 (0.0002)	0.0094 (0.0244)	0.00017 (0.0034)				

Table A.4: Part 2 / Second-order Sobol' indices and standard deviations (with 10 bootstrap replicates and 100 simulations) of all inputs in the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$  and employment rate  $\lambda_1$

#### A.4 Convergence of the Sobol' indices of the [Grasselli and Costa Lima, 2012] model with a Leontief production function

In this appendix, we refine the meshing of our inputs (labor productivity growth rate  $\alpha$ ; population growth rate  $\beta$ ; depreciation rate  $\delta$  and capital-to-output ration  $\nu$ ) to give a better assessment of the Sobol' indices of the [Grasselli and Costa Lima, 2012] model outputs: the *Good* Equilibrium values of the wage share  $\omega_1$ , the employment rate  $\lambda_1$  and the debt ratio of the economy  $d_1$ . Following [Harenberg et al., 2017], we choose to stop this exploration when the standard deviation of every indices is less than 0.05. These exploration with an increasing number of simulations (200, 300, 400, 500) leads to the results displayed in Tables A.5, A.6 and A.7). Once again, one can notice the stability of the results whatever is the number of simulations.

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	First-order $S_i$	N = 200 simulations			N = 300 simulations		
Input	Description	$\omega_1$	$\lambda_1$	$d_1$	$\omega_1$	$\lambda_1$	$d_1$
$\alpha$	labor productivity growth rate	0.083 (0.0406)	0.127 (0.0677)	0.093 (0.0783)	0.080 (0.0440)	0.128 (0.0509)	0.091 (0.0570)
$\beta$	population growth rate	0.069 (0.0500)	0.076 (0.0358)	0.083 (0.0839)	0.078 (0.0499)	0.063 (0.0306)	0.082 (0.0618)
$\delta$	depreciation rate	0.057 (0.0376)	0.111 (0.0746)	0.066 (0.0384)	0.054 (0.0347)	0.118 (0.0685)	0.080 (0.0282)
$\nu$	capital-to-output ratio	0.185 (0.0551)	0.051 (0.0457)	0.132 (0.0585)	0.180 (0.0533)	0.052 (0.0278)	0.113 (0.0456)

Table A.5: First-order Sobol' indices and standard deviations (10 bootstrap replicates for each 200 & 300 simulations) of the main influent inputs of the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$

	First-order $S_i$	N = 400 simulations			N = 500 simulations		
Input	Description	$\omega_1$	$\lambda_1$	$d_1$	$\omega_1$	$\lambda_1$	$d_1$
$\alpha$	labor productivity growth rate	0.086 (0.0351)	0.144 (0.0597)	0.101 (0.0511)	0.088 (0.0470)	0.133 (0.0482)	0.105 (0.0344)
$\beta$	population growth rate	0.075 (0.0430)	0.072 (0.0487)	0.085 (0.0489)	0.077 (0.0421)	0.070 (0.0260)	0.090 (0.0376)
$\delta$	depreciation rate	0.053 (0.0497)	0.101 (0.0543)	0.081 (0.0511)	0.053 (0.0227)	0.100 (0.0248)	0.073 (0.0447)
$\nu$	capital-to-output ratio	0.186 (0.0516)	0.054 (0.0363)	0.121 (0.0530)	0.189 (0.0498)	0.060 (0.0457)	0.122 (0.0403)

Table A.6: First-order Sobol' indices and standard deviations (10 bootstrap replicates of each 400 & 500 simulations) of the main influent inputs of the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$

N = 500 simulations	Second-order Sobol' indices $S_{ij}$		
Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$
$\alpha - \beta$	0.053 (0.0331)	0.068 (0.0548)	0.056 (0.0307)
$\beta - \delta$	0.070 (0.0530)	0.091 (0.0532)	0.068 (0.0603)
$\delta - \nu$	0.064 (0.0463)	0.061 (0.0352)	0.066 (0.0504)
$\alpha - \delta$	0.054 (0.0449)	0.098 (0.05997)	0.058 (0.0469)
$\beta - \nu$	0.076 (0.0505)	0.064 (0.0516)	0.079 (0.0543)
$\alpha - \nu$	0.067 (0.0491)	0.057 (0.0341)	0.073 (0.0467)

Table A.7: Second-order Sobol' indices and standard deviations (10 bootstrap replicates and 500 simulations) of the main influent inputs of the [Grasselli and Costa Lima, 2012] model on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$

## A.5 Sobol' indices of the [Goodwin, 1967] model

Inspired from the description of the [Goodwin, 1967] model by [Grasselli and Costa Lima, 2012], we undertake a GSA of the model with the same method: GSA with Sobol' indices estimated by [Saltelli, 2002] method. The outputs of our study are the *Good* Equilibrium values of the wage share  $\omega_1$  and the employment rate  $\lambda_1$ , because there is no debt in this model. The inputs of our study are all the parameters of the model with the same ranges (see Table A.8) as in the GSA of section 2.3. The results are reported in Table A.9 and show that, at first-order and for only 100 simulations, the [Goodwin, 1967] model is more robust to its inputs than the [Grasselli and Costa Lima, 2012] (the capital-to-output ratio  $\nu$  is slightly less sensitive), at a first-order and total-effect point of view. This last result is coherent with the idea that more nonlinearities and interactions between inputs are possible in the [Grasselli and Costa Lima, 2012] than in [Goodwin, 1967]. These results makes us wonder about a possible compromise between the robustness and the *realism* of a model to find an equilibrium between a very robust model unable to predict the reality (such as the [Goodwin, 1967] model), and a very fitted and very sensitive model (such as the DSGE of [Ratto, 2008]). From this point of view, the [Grasselli and Costa Lima, 2012] model seems an interesting compromise, with a finer description of the economy and a more robust structure.

Symbol	Definition	Min Value	Max Value	Range	Nom. Value
$\alpha$	labor productivity growth rate	0	0.03	0.03	0.025
$\beta$	population growth rate	0.015	0.03	0.015	0.02
$\delta$	depreciation rate	0.005	0.03	0.025	0.01
$\nu$	capital-to-output ratio	2	4.5	2.5	3
$\phi_0$	Philips Curve $\Phi(\lambda)$ parameter	0.038	0.042	0.004	$\frac{0.04}{1-0.04^2}$
$\phi_1$	Philips Curve $\Phi(\lambda)$ parameter	6.333 E-5	7 E-5	6.67 E-6	$\frac{0.04^3}{1-0.04}$

Table A.8: Definition and Range of the inputs of the GSA for the [Goodwin, 1967] model

Input	[Goodwin, 1967] - 100 simulations	First-order $S_i$		Total effect $S_i^T$	
	Description	$\omega_1$	$\lambda_1$	$\omega_1$	$\lambda_1$
$\alpha$	labor productivity growth rate	0.068 (0.0388)	0.050 (0.0448)	0.630 (0.0484)	0.955 (0.0317)
$\beta$	population growth rate	0.050 (0.0183)	0.074 (0.0693)	0.618 (0.2825)	0.726 (0.3882)
$\delta$	depreciation rate	0.084 (0.0544)	0.060 (0.0431)	0.677 (0.0474)	0.849 (0.0562)
$\nu$	capital-to-output ratio	0.240 (0.0581)	0.056 (0.0486)	0.857 (0.0428)	0.926 (0.2833)
$\phi_0$	Philips Curve $\Phi(\lambda)$ parameter	0.028 (0.0523)	0.012 (0.0263)	0.297 (0.0443)	0.115 (0.2798)
$\phi_1$	Philips Curve $\Phi(\lambda)$ parameter	0.028 (0.0523)	0.012 (0.0263)	0.297 (0.0443)	0.115 (0.0444)

Table A.9: Sobol' indices and standard deviations of all inputs of the [Goodwin, 1967] model on the outputs: the *Good* Equilibrium values of wages share,  $\omega_1$ , and employment rate,  $\lambda_1$ .

N = 100 simulations	Second-order Sobol' indices $S_{ij}$ of the [Goodwin, 1967] model	
	$\omega_1$	$\lambda_1$
$\alpha - \beta$	0.057 (0.0103)	0.076 (0.0204)
$\beta - \delta$	0.085 (0.0153)	0.102 (0.0190)
$\delta - \nu$	0.106 (0.0173)	0.066 (0.0161)
$\nu - \phi_0$	0.032 (0.0161)	0.017 (0.008)
$\phi_0 - \phi_1$	0.023 (0.0129)	0.016 (0.0259)
$\alpha - \delta$	0.098 (0.0174)	0.098 (0.0240)
$\beta - \nu$	0.087 (0.0174)	0.137 (0.0089)
$\delta - \phi_0$	0.034 (0.0168)	0.017 (0.0085)
$\nu - \phi_1$	0.029 (0.0147)	0.017 (0.0168)
$\alpha - \nu$	0.085 (0.0213)	0.097 (0.0083)
$\beta - \phi_0$	0.032 (0.0161)	0.017 (0.0089)
$\delta - \phi_1$	0.035 (0.0180)	0.017 (0.0084)
$\alpha - \phi_0$	0.020 (0.0130)	0.017 (0.0083)
$\beta - \phi_1$	0.033 (0.0166)	0.017 (0.0084)
$\alpha - \phi_1$	0.035 (0.0179)	0.017 (0.0084)

Table A.10: Second-order Sobol' indices and standard deviations of all inputs of the [Goodwin, 1967] model on the outputs.

What is remarkable here is that the most influential parameters are the same for both models, be it [Goodwin, 1967] or [Grasselli and Costa Lima, 2012]: the labor productivity,  $\alpha$ , the population growth rate,  $\beta$ , the depreciation rate,  $\delta$ , and the capital-to-output ratio,  $\nu$ . This result seems to confirm the idea that **refining this model**(e.g. by introducing a debt ratio as in [Grasselli and Costa Lima, 2012]) **does not fundamentally affect the nature of its sensitivity**.

### A.5.1 Sobol' indices of the [Grasselli and Costa Lima, 2012] model with CES production function

In order to improve the performance of the model in terms of robustness, one can decide to switch the production function from a Leontief shape to a Constant Elasticity of Substitution (CES) production function as in the work of [McIsaac, 2016] based on the Van der Ploeg's extension. This transformation endogeneizes the capital-to-output ratio  $\nu$  responsible for the main impact in terms of Sobol' indices on the output of our study.

[Van der Ploeg, 1985] indeed relaxed the assumption that capital and labor cannot be substituted with each other by endowing the economy with a CES production function:

$$Y = C[bK^{-\eta} + (1 - b)(aL)^{-\eta}]^{-1/\eta} \quad (70)$$

where  $C > 0$  is the factor productivity,  $b \in [0, 1]$  is the share of capital and the short-run elasticity of substitution between factors is  $\sigma = \frac{1}{1+\eta}$ . One can notice that when  $\eta \rightarrow +\infty$ , one can obtain a Leontief production function without substitutability (where  $C = \frac{1}{\nu}$ ), and if  $\eta \rightarrow 0$  this form leads to a Cobb-Douglas production function. The first-order condition of the profit maximization leads

to the following shape of  $\nu(t)$ :

$$\nu(t) = \frac{1}{C} \left( \frac{1 - \omega(t)}{b} \right)^{-1/\eta} \quad (71)$$

Instead of  $\nu$ , the study owns now three more parameters: the factor productivity,  $C$ , the share of capital,  $b$ , and a parameter,  $\eta$ , controlling the elasticity of substitution with the following ranges (see Table A.11). When applying the same methodology to this model, we obtain the following

Symbol	Definition	Min Value	Max Value	Range	Nom. Value
$C$	Factor productivity	1/4.5	1/2	0.2778	1/3
$b$	Share of capital	0.1	0.3	0.2	0.135
$\eta$	Controlling the elasticity of substitution	0	600	600	50

Table A.11: Definition and Range of the inputs of the GSA for the [Grasselli and Costa Lima, 2012] model with CES production function

Sobol' indices (Tables A.12, A.13 and A.14).

The results of this analysis shows that at first-order and for 100 simulations (10 bootstrap), the [Grasselli and Costa Lima, 2012] model with CES production is slightly more robust to its inputs than the same model with a Leontief production function. At a total-effect order, the quadruplet  $(\alpha, \beta, \delta, \nu)$  is replaced by  $(\alpha, \beta, \delta, C)$ <sup>34</sup> which seems coherent with the sensitivity of [Grasselli and Costa Lima, 2012] be it with a Leontief or a CES production function.

Quantitatively, at first-order the main influent parameter on the wage share is now the nominal interest rate  $r$  around 0.174 (since  $\nu$  is now endogeneized), but at a total-effect order, this input is not very influent due to its few joint effect with others inputs.

The main influent parameter on the employment rate is still the labor productivity growth rate  $\alpha$  around 0.196 (greater than with a Leontief production function) and the main influent parameter on the debt ratio is the factor productivity  $C$  around 0.127. At a total-effect level, the most impacting inputs (with all joint effects) are still the labor productivity growth rate  $\alpha$ , the population growth rate  $\beta$ , the depreciation rate  $\delta$  and now the factor productivity  $C$ .

Due to the increased number of possible joint effect, the value of the second-order Sobol' indices are smaller than with a Leontief production function (66 possible joint effect instead of 45). The main influent joint effect on the outputs are now  $\alpha - \beta$  for the wage share, around 0.0383 (instead of 0.2346 for  $\alpha - \nu$ ),  $\alpha - \delta$  for the employment rate, around 0.0397 (instead of 0.1982 for the same inputs in the model with a Leontief function),  $\alpha - \beta$  for the debt ratio, around 0.0409 (instead of 0.1897 for  $\beta - \nu$ ).

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<sup>34</sup>Note that:  $\nu(t) = \frac{1}{C} \left( \frac{1 - \omega(t)}{b} \right)^{-1/\eta}$  which means  $C$  and  $\nu$  are closely related. Note e.g. that  $\nu(t) \rightarrow 1/C$  when the elasticity of substitution,  $\eta$  tends to  $\infty$ .

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Input	Description	First-order $S_i$			Total effect $S_i^T$		
		$\omega_1$	$\lambda_1$	$d_1$	$\omega_1$	$\lambda_1$	$d_1$
$\alpha$	labor productivity growth rate	0.135 (0.0788)	0.196 (0.16477)	0.101 (0.0806)	0.822 (0.1971)	0.863 (0.1470)	0.791 (0.2102)
$\beta$	population growth rate	0.096 (0.0923)	0.107 (0.0705)	0.100 (0.0664)	0.754 (0.1876)	0.728 (0.3113)	0.755 (0.2464)
$\delta$	depreciation rate	0.127 (0.0640)	0.099 (0.0722)	0.124 (0.0786)	0.865 (0.1808)	0.792 (0.2289)	0.868 (0.1597)
$\mathcal{C}$	factor productivity	0.106 (0.0886)	0.114 (0.0902)	0.127 (0.0088)	0.876 (0.1635)	0.856 (0.1312)	0.876 (0.1784)
$b$	Share of capital	0.085 (0.0799)	0.115 (0.0773)	0.082 (0.0628)	0.254 (0.2226)	0.288 (0.2473)	0.221 (0.2662)
$\eta$	Controlling the elasticity of substitution	0.087 (0.0707)	0.122 (0.0934)	0.073 (0.0535)	0.282 (0.2978)	0.288 (0.2763)	0.312 (0.3624)
$\phi_0$	Philips Curve $\Phi(\lambda)$ parameter	0.0091 (0.0078)	0.0121 (0.0082)	0.0085 (0.0060)	0.185 (0.1385)	0.445 (0.2238)	0.224 (0.3037)
$\phi_1$	Philips Curve $\Phi(\lambda)$ parameter	0.0089 (0.0079)	0.0154 (0.0011)	0.0087 (0.0059)	0.185 (0.1386)	0.547 (0.5121)	0.272 (0.6022)
$k_0$	Investment function $\kappa(\pi)$ constant	0.083 (0.0826)	0.137 (0.0784)	0.096 (0.0589)	0.320 (0.3640)	0.320 (0.3583)	0.224 (0.3318)
$k_1$	$\kappa(\pi)$ parameter	0.099 (0.0913)	0.124 (0.0998)	0.093 (0.0644)	0.343 (0.3907)	0.383 (0.3354)	0.308 (0.4289)
$k_2$	$\kappa(\pi)$ exponential parameter	0.105 (0.0891)	0.141 (0.1035)	0.094 (0.0611)	0.429 (0.4132)	0.445 (0.4493)	0.387 (0.4506)
$r$	nominal interest rate	0.174 (0.1029)	0.128 (0.1016)	0.090 (0.0589)	0.310 (0.0544)	0.265 (0.1323)	0.182 (0.0094)

Table A.12: Sobol' indices and standard deviations (with 10 bootstrap replicates) of all inputs of the [Grasselli and Costa Lima, 2012] model with a CES production function on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$ , employment rate  $\lambda_1$  and debt ratio  $d_1$

The [Grasselli and Costa Lima, 2012] model is slightly more robust to its inputs than the same model with Leontief production function. In order to improve the first-order performance of the model in terms of robustness, one can be interested in endogeneizing the nominal interest rate  $r$  by introducing a Taylor's rule, or the labor productivity growth rate (as suggested before in the model with Leontief production function).



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N = 100 simulations	Second-order Sobol' indices $S_{ij}$ of the [Grasselli and Costa Lima, 2012] model						
Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$	Joint-effect of inputs	$\omega_1$	$\lambda_1$	$d_1$
$\alpha - \beta$	0.0383 (0.0340)	0.0221 (0.0249)	0.0409 (0.0642)	$\phi_1 - k_1$	0.0092 (0.0094)	0.0187 (0.0121)	0.0166 (0.0221)
$\beta - \delta$	0.0343 (0.0316)	0.0318 (0.0571)	0.0257 (0.0203)	$k_0 - k_2$	0.0268 (0.0285)	0.0222 (0.0162)	0.0156 (0.0081)
$\delta - C$	0.0282 (0.0307)	0.0353 (0.0559)	0.0205 (0.0176)	$k_1 - r$	0.0015 (0.0011)	0.0012 (0.0009)	0.0011 (0.0005)
$C - b$	0.0214 (0.0252)	0.0118 (0.0126)	0.0387 (0.0602)	$\alpha - C$	0.0168 (0.0178)	0.0266 (0.0351)	0.0289 (0.0316)
$b - \eta$	0.0023 (0.0027)	0.0129 (0.0122)	0.0009 (0.0006)	$\beta - b$	0.0026 (0.0021)	0.0222 (0.0236)	0.0317 (0.0435)
$\eta - \phi_0$	0.0035 (0.0005)	0.0168 (0.0136)	0.0011 (0.0667)	$\delta - \eta$	0.0187 (0.0131)	0.0149 (0.0119)	0.0198 (0.0198)
$\phi_0 - \phi_1$	0.0004 (0.0006)	0.0138 (0.009)	0.0011 (0.0006)	$C - \phi_0$	0.0123 (0.0156)	0.0138 (0.0085)	0.0010 (0.0006)
$\phi_1 - k_0$	0.0008 (0.0005)	0.0179 (0.011)	0.0010 (0.0006)	$b - \phi_1$	0.0199 (0.0236)	0.0172 (0.0599)	0.0253 (0.0297)
$k_0 - k_1$	0.0111 (0.0062)	0.0167 (0.0112)	0.0025 (0.0039)	$\eta - k_0$	0.0026 (0.0032)	0.0012 (0.0009)	0.0015 (0.0015)
$k_1 - k_2$	0.0223 (0.0282)	0.0162 (0.0107)	0.0134 (0.0052)	$\phi_0 - k_1$	0.0200 (0.0197)	0.0175 (0.0129)	0.0256 (0.0253)
$k_2 - r$	0.0012 (0.0011)	0.0046 (0.0067)	0.0011 (0.0005)	$\phi_1 - k_2$	0.0200 (0.0197)	0.0197 (0.0136)	0.0256 (0.0253)
$\alpha - \delta$	0.0307 (0.0443)	0.0397 (0.0866)	0.0246 (0.0179)	$k_0 - r$	0.0009 (0.0007)	0.0016 (0.0011)	0.0018 (0.0023)
$\beta - C$	0.0133 (0.0069)	0.0381 (0.0748)	0.0262 (0.0227)	$\alpha - b$	0.0302 (0.0374)	0.0157 (0.0132)	0.0256 (0.0277)
$\delta - b$	0.0252 (0.0273)	0.0127 (0.0131)	0.0217 (0.0192)	$\beta - \eta$	0.0139 (0.0119)	0.0150 (0.0115)	0.0205 (0.0167)
$C - \eta$	0.0215 (0.0209)	0.0131 (0.0105)	0.0182 (0.0165)	$\delta - \phi_0$	0.0190 (0.0176)	0.0145 (0.0103)	0.0256 (0.0261)
$b - \phi_0$	0.0025 (0.0035)	0.0141 (0.0131)	0.0023 (0.0025)	$C - \phi_1$	0.0174 (0.0165)	0.0209 (0.0159)	0.0242 (0.0242)
$\eta - \phi_1$	0.0016 (0.0018)	0.0154 (0.0118)	0.0016 (0.0016)	$b - k_0$	0.0223 (0.0188)	0.0187 (0.0151)	0.0029 (0.0029)
$\phi_0 - k_0$	0.0046 (0.0051)	0.0174 (0.0122)	0.0011 (0.0006)	$\eta - k_1$	0.0257 (0.0272)	0.0176 (0.0142)	0.0055 (0.0061)

Table A.13: Part 1 / Second-order Sobol' indices and standard deviations (with 10 bootstrap replicates and 100 simulations) of all inputs in the [Grasselli and Costa Lima, 2012] model with CES production function on the outputs: the *Good* Equilibrium values of wages share  $\omega_1$  and employment rate  $\lambda_1$

APPENDIX A. BUILDING A GLOBAL SENSITIVITY ANALYSIS TO QUANTIFY THE ROBUSTNESS OF MACRO-ECONOMIC MODELS

N = 100 simulations	Second-order Sobol' indices $S_{ij}$ of the [Grasselli and Costa Lima, 2012] model						
	Joint-effect of inputs			Joint-effect of inputs	Joint-effect of inputs		
	$\omega_1$	$\lambda_1$	$d_1$		$\omega_1$	$\lambda_1$	$d_1$
$\phi_0 - k_2$	0.0199 (0.0197)	0.0113 (0.0099)	0.0256 (0.0100)	$C - k_2$	0.0203 (0.0123)	0.0145 (0.0101)	0.0157 (0.0105)
$\phi_1 - r$	0.0003 (0.0002)	0.0013 (0.0010)	0.0011 (0.0006)	$b - r$	0.0008 (0.0004)	0.0154 (0.0138)	0.0011 (0.0006)
$\alpha - \eta$	0.0256 (0.0355)	0.0157 (0.0119)	0.0203 (0.0221)	$\alpha - k_0$	0.0248 (0.0283)	0.0191 (0.0125)	0.0322 (0.0312)
$\beta - \phi_0$	0.0193 (0.0188)	0.0155 (0.0081)	0.0255 (0.0249)	$\beta - k_1$	0.0324 (0.0466)	0.0157 (0.0074)	0.0308 (0.0511)
$\delta - \phi_1$	0.0190 (0.0176)	0.0173 (0.0137)	0.0258 (0.0261)	$\delta - k_2$	0.0165 (0.0108)	0.0146 (0.0111)	0.0177 (0.0174)
$C - k_0$	0.0275 (0.0381)	0.0150 (0.0122)	0.0020 (0.0019)	$C - r$	0.0446 (0.0833)	0.0012 (0.0009)	0.0023 (0.0039)
$b - k_1$	0.0186 (0.0153)	0.0154 (0.0127)	0.0019 (0.0018)	$\alpha - k_1$	0.0214 (0.012)	0.0147 (0.0150)	0.0334 (0.0545)
$\eta - k_2$	0.0199 (0.0217)	0.0197 (0.0138)	0.0114 (0.0141)	$\beta - k_2$	0.0242 (0.0224)	0.0189 (0.0115)	0.0243 (0.0335)
$\phi_0 - r$	0.0007 (0.0015)	0.0168 (0.0129)	0.0010 (0.0005)	$\delta - r$	0.0238 (0.0161)	0.0149 (0.0107)	0.0181 (0.0163)
$\alpha - \phi_0$	0.0213 (0.0215)	0.0159 (0.0123)	0.0271 (0.0265)	$\alpha - k_2$	0.0182 (0.0130)	0.0174 (0.0121)	0.0134 (0.0070)
$\beta - \phi_1$	0.0193 (0.0188)	0.0155 (0.0101)	0.0255 (0.0249)	$\beta - r$	0.0253 (0.0201)	0.0165 (0.0127)	0.0254 (0.0252)
$\delta - k_0$	0.0188 (0.0159)	0.0259 (0.0363)	0.0266 (0.0306)	$\alpha - r$	0.0314 (0.0249)	0.0176 (0.0133)	0.0192 (0.0180)
$C - k_1$	0.0323 (0.0362)	0.0129 (0.0088)	0.0192 (0.0115)	$b - k_2$	0.0247 (0.0253)	0.0187 (0.0138)	0.0053 (0.0058)
$\eta - r$	0.0012 (0.0006)	0.0186 (0.0009)	0.0010 (0.0006)	$\alpha - \phi_1$	0.0213 (0.0798)	0.0152 (0.0406)	0.0271 (0.0489)
$\beta - k_0$	0.0015 (0.0455)	0.001 (0.0115)	0.0017 (0.0411)	$\delta - k_1$	0.0223 (0.0239)	0.0240 (0.0363)	0.0286 (0.0305)

Table A.14: Part 2 / Second-order Sobol' indices and standard deviations (10 bootstrap replicates, 100 simulations) of all inputs in the CES [Grasselli and Costa Lima, 2012] model on the outputs

## Appendix B

# Money Velocity in an Imperfectly Competitive, Stock-Flow Consistent Dynamics

### B.1 Long-run properties of the moneyless model

This section analyzes the local, asymptotic stability of long-run steady states in the vector field (26).

#### B.1.1 The reduced dynamical system

Notice first that the behavior of the households' debt ratio,  $d_h$ , in (26), can be deduced from the other variables while the latter are independent from  $d_h$ . We therefore focus on the 3-dimensional system  $(\omega, \lambda, d_f)$ , together with the following auxiliary variables:

$$\begin{cases} \nu(\omega) = \frac{1}{A} \left( \frac{1 - \sigma\omega}{b} \right)^{-1/\eta} \\ i(\omega) = \eta_p(\mu\omega - 1) \\ \pi_e = 1 - \omega - rd_f - \delta\nu(\omega) \\ y_d = c_h(\omega + rd_f + \Delta\pi_e) + \kappa(\pi_e) \\ \dot{d}_h = c_h(\omega + rd_f + \Delta\pi_e) - \omega - rd_f - \Delta\pi_e - d_h \left( \frac{\kappa(\pi_e)}{\nu} - \delta - \frac{\dot{\omega}}{\eta(1 - \sigma\omega)} + i(\omega) \right). \end{cases} \quad (1)$$

The associated Jacobian matrix is:

$$J(\omega, \lambda, d_f) = \begin{pmatrix} \frac{\partial f_1}{\partial \omega} & \omega \frac{\eta}{\eta+1} \phi'(\lambda) & 0 \\ \frac{\partial f_2}{\partial \omega} & \frac{\partial f_2}{\partial \lambda} & -r\lambda \frac{\kappa'(\pi_e)}{\nu(\omega)} \\ \frac{\partial f_3}{\partial \omega} & \frac{d_f}{\eta+1} \frac{\omega}{1-\sigma\omega} \phi'(\lambda) & \frac{\partial f_3}{\partial d_f} \end{pmatrix} \quad (2)$$

where

$$\begin{aligned}
\frac{\partial f_1}{\partial \omega} &:= \frac{\eta}{\eta+1} \left[ \phi(\lambda) - \alpha - (1-\gamma)i(\omega) \right] - \frac{\eta\omega}{\eta+1} (1-\gamma)\eta_p\mu \\
\frac{\partial f_2}{\partial \omega} &:= -\lambda \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)i(\omega)}{(\eta+1)(1-\sigma\omega)^2} - \frac{(1-\gamma)\eta_p\mu}{(\eta+1)(1-\sigma\omega)} \right) \\
\frac{\partial f_2}{\partial \lambda} &:= \frac{\kappa(\pi_e)}{\nu(\omega)} - (\alpha + n + \delta) - \frac{\phi(\lambda) - \alpha - (1-\gamma)i(\omega)}{(\eta+1)(1-\sigma\omega)} - \lambda \frac{\phi'(\lambda)}{(\eta+1)(1-\sigma\omega)} \\
\frac{\partial f_3}{\partial \omega} &:= d_f \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} - \eta_p\mu + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)i(\omega)}{(\eta+1)(1-\sigma\omega)^2} - \frac{\omega(1-\gamma)\eta_p\mu}{(\eta+1)(1-\sigma\omega)} \right) \\
&\quad - \kappa'(\pi_e) \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + (y_d - \Delta) - (1-\omega) \frac{\partial y_d}{\partial \omega} + \frac{(1-\Delta)\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \\
\frac{\partial f_3}{\partial d_f} &:= (1-\Delta)r - r\kappa'(\pi_e) - (1-\omega) \frac{\partial y_d}{\partial d_f} - \left( \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta + i(\omega) - \omega \frac{\phi(\lambda) - \alpha - (1-\gamma)i(\omega)}{(\eta+1)(1-\sigma\omega)} \right) \\
&\quad + rd_f \frac{\kappa'(\pi_e)}{\nu(\omega)}
\end{aligned} \tag{3}$$

### B.1.2 Existence and stability of long-run equilibria

Let us adopt the following set of rather weak assumptions:<sup>1</sup>

#### Assumption A

1. The (short-run Phillips) function  $\phi^2$  is continuously differentiable on  $[0, 1]$  and

- (i)  $\phi'(\lambda) > 0$  on  $[0; 1]$
- (ii)  $\phi(0) < \alpha + (1-\gamma)i(0) = \alpha - (1-\gamma)\eta_p$
- (iii)  $\lim_{\lambda \rightarrow 1^-} \phi(\lambda) = +\infty$ .

2. The investment function  $\kappa(\cdot)$  is increasing, continuously differentiable on  $\mathbb{R}$ . There is some  $\rho > 0$  such that, as long as  $rd_f > \rho$ :<sup>3</sup>

- (i)  $\kappa'(\pi_e) > 0$  on  $[0, 1]$
- (ii)  $\lim_{\omega \rightarrow 0} \kappa(\omega, d_f) = \kappa_0(d_f) \leq (\alpha + n + \delta)\nu(0) = (\alpha + n + \delta) \frac{b^{1/\eta}}{A}$
- (iii)  $\lim_{\pi_e \rightarrow -\infty} \pi_e^2 \kappa'(\pi_e) < +\infty$ .

---

<sup>1</sup>Of course, if one is only concerned with existence, continuous differentiability can be replaced by local Lipschitzianity.

<sup>2</sup>Its form and parameterization guarantee that  $\forall t, 0 \leq \lambda(t) \leq 1$ , as in [Desai et al., 2006]. Observe also that (ii) implies  $\phi(0) < \alpha$  because  $i(0) = -\eta_p < 0$ .

<sup>3</sup>These restrictions are sufficient to guarantee the existence of the relevant equilibria alluded to in section 4.3, as in [Grasselli and Costa Lima, 2012]. The lower bound,  $\rho$ , will be useful when dealing with persistence in section 3.5.2.

3. The consumption function  $c_h(\cdot)$  is strictly increasing and continuously differentiable on  $\mathbb{R}^+$ , and verifies:<sup>4</sup>

$$\begin{aligned} \text{(i)} \quad & \forall x \leq 0, c_h(x) = c_- > 0 \\ \text{(ii)} \quad & \lim_{x \rightarrow +\infty} c_h(x) = c_+. \end{aligned} \tag{6}$$

4. Eventually,  $r + \delta - A b^{\frac{1}{\eta}} \kappa(0) > 0$ .<sup>5</sup>

Our numerical example 4.4.2 complies with this set of assumptions.

### The Solovian equilibrium

The Solovian equilibrium corresponds to a desirable long-run steady state with positive wages and employment, and a finite level of debt  $(\omega_1; \lambda_1; d_{f_1})$ . To check its existence, observe that  $\dot{\lambda} = 0$  in the second equation of (26) whenever

$$\kappa(\pi_1) = \frac{\alpha + n + \delta}{A} \left( \frac{1 - \sigma\omega_1}{b} \right)^{-\frac{1}{\eta}} = (\alpha + n + \delta)\nu(\omega_1). \tag{7}$$

Equation (15) can be uniquely solved for  $\pi_1$  because of condition (9), which implies that  $d_{f_1} = (1 - \pi_1 - \omega_1 - \delta\nu(\omega_1))/r$  is uniquely determined by  $\omega_1$ . Given  $\omega_1$ , we also obtain  $\lambda_1 = \phi^{-1}(\alpha + (1 - \gamma)i_1)$ , where  $i_1 = \eta_p(\mu\omega_1 - 1)$ , which belongs to  $[0; 1]$ , courtesy of (8) (ii) and provided  $i_1 > 0$ .

Because  $d_h$  can be deduced from the other state variables,  $c_h(\omega_1 + rd_{f_1} + \Delta\pi_1)$  depends only upon  $\omega_1$ . The same holds for  $y_{d_1} = \kappa(\pi_1) + c_h(\omega_1 + rd_{f_1} + \Delta\pi_1)$ . Consequently, in order to guarantee  $\dot{d} = 0$  in the third equation of (26), we need:  $\kappa(\pi_1) - (1 - \omega_1)(y_{d_1} - \Delta) + (1 - \Delta)\delta\nu(\omega_1) - d_{f_1}[\alpha + n + i_1 - (1 - \Delta)r] = 0$ , which leads to a nonlinear equation in  $\omega_1$ .

Each side of (15) is a continuous function of  $\omega \in [0; 1]$ . The right-hand side of the equation is null if  $\omega = 1$ , and equals  $(\alpha + n + \delta)\nu(0) > 0$  if  $\omega = 0$ . In the left-hand side,  $\kappa(\cdot)$  is an increasing function of  $\omega$ . Consequently, assumption (9) (ii) ensures the existence of at least one solution of (15). At this steady state, the growth rate of the economy,  $g = \frac{\dot{Y}}{Y} + \frac{\dot{p}}{p}$  with  $\frac{\dot{Y}}{Y} = \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta - \frac{\dot{\omega}}{\eta(1 - \sigma\omega)}$ , becomes:  $g = \alpha + n + i_1$ .

The Jacobian matrix becomes:

$$\begin{pmatrix} K_i & K_0 & 0 \\ -K_1 K_\omega - K_1^e & -K_\lambda & -rK_1 \\ K_2 K_\omega + K_2^e + K_{y\omega} & K_{d_f} & rK_2 + (1 - \Delta)r - K_{y_d} - (\alpha + n + \bar{i}) \end{pmatrix} \tag{8}$$

---

<sup>4</sup>Condition (i) expresses the fact that households need a minimum level of subsistence consumption even at negative income, whereas (ii) says that total normalized consumption can be higher than 1 (and consequently exceed total output, by sourcing from the stock of inventories) but must have an upper bound. See [Giraud and Grasselli, 2019] for details.

<sup>5</sup>This condition is borrowed from [Bastidas et al., 2019], and turns out to be sufficient for the *slavery* equilibrium to exist.

with

$$\begin{aligned}
K_i &= -\frac{\eta\omega_1}{\eta+1}(1-\gamma)\eta_p\mu \quad , \quad K_0 = \frac{\eta\omega_1}{\eta+1} \phi'(\lambda_1), \\
K_1 &= \lambda_1 \frac{\kappa'(\pi_1)}{\nu(\omega_1)} \quad , \quad K_\lambda = \lambda_1 \frac{\phi'(\lambda_1)}{(\eta+1)(1-\sigma\omega_1)} \\
K_1^e &= \lambda_1 \left[ \frac{\sigma\kappa(\pi_1)}{\eta(1-\sigma\omega_1)\nu(\omega_1)} - \frac{(1-\gamma)\eta_p\mu}{(\eta+1)(1-\sigma\omega_1)} \right], \\
K_2 &= \frac{d_{f_1} - \nu(\omega_1)}{\nu(\omega_1)} \kappa'(\pi_1) \quad , \quad K_\omega = \left( 1 + \frac{\delta\sigma\nu(\omega_1)}{\eta(1-\sigma\omega_1)} \right) \\
K_2^e &= d_{f_1} \left( \frac{\sigma\kappa(\pi_1)}{\eta(1-\sigma\omega_1)\nu(\omega_1)} - \eta_p\mu - \omega_1 \frac{(1-\gamma)\eta_p\mu}{(\eta+1)(1-\sigma\omega_1)} \right) \\
K_{y\omega} &= y_{d_1} - \Delta - (1-\omega_1) \frac{\partial y_d}{\partial \omega |_{(\omega_1; \lambda_1; d_{f_1})}} + \frac{(1-\Delta)\delta\sigma\nu(\omega_1)}{\eta(1-\sigma\omega_1)} \\
K_{d_f} &= \frac{d_{f_1}\omega_1\phi'(\lambda_1)}{(\eta+1)(1-\sigma\omega_1)} \quad , \quad K_{y_d} = (1-\omega_1) \frac{\partial y_d}{\partial d_{f_1} |_{(\omega_1; \lambda_1; d_{f_1})}} .
\end{aligned} \tag{9}$$

The characteristic polynomial is  $P(X) = X^3 + a_2X^2 + a_1X + a_0$ , where

$$\begin{aligned}
a_0 &= K_0[rK_1(K_2K_\omega + K_2^e + K_{y\omega}) - b_2(K_1K_\omega + K_1^e)] - K_i(rK_1K_{d_f} - b_2K_\lambda) \\
a_1 &= rK_1K_{d_f} - b_2K_\lambda + K_i(b_2 - K_\lambda) + K_0(K_1K_\omega + K_1^e) \\
a_2 &= K_\lambda - b_2 - K_i \quad \text{where } b_2 = rK_2 + (1-\Delta)r - K_{y_d} - (\alpha + n + \bar{i}).
\end{aligned} \tag{10}$$

The Routh-Hurwitz Stability criterion requires  $a_3 \geq 0$ ,  $a_2 \geq 0$ ,  $-\frac{a_3a_0 - a_2a_1}{a_2} \geq 0$ ,  $a_0 \geq 0$ , which, here, means:

$$K_\lambda - rK_2 - (1-\Delta)r + K_{y_d} + (\alpha + n + i_1) - K_i \geq 0, \tag{11}$$

$$K_0[rK_1(K_2K_\omega + K_2^e + K_{y\omega}) - b_2(K_1K_\omega + K_1^e)] - K_i(rK_1K_{d_f} - b_2K_\lambda) \geq 0, \tag{12}$$

$$a_2a_1 - a_0 \geq 0. \tag{13}$$

This condition is satisfied for a wide range of parameters and, in particular, by the parameterization of our numerical example.<sup>6</sup>

### Deflationary equilibria with finite debt

1) Another economically doubtful equilibrium exists,  $(\omega, \lambda, d_f) = (\omega_2, 0, d_{f_2})$ , provided some  $(\omega_2, d_{f_2})$  solves the nonlinear equations:

$$\begin{cases} i(\omega_2) = \frac{\phi(0) - \alpha}{1 - \gamma} \\ 0 = \kappa(\pi_2) - (1 - \omega_2)(y_{d_2} - \Delta) + (1 - \Delta)\delta\nu(\omega_2) - d_{f_2} \left[ \frac{\kappa(\pi_2)}{\nu(\omega_2)} - \delta - (1 - \Delta)r + i(\omega_2) \right] \end{cases} \tag{14}$$

with  $\pi_2 = 1 - \omega_2 - rd_{f_2} - \delta\nu(\omega_2)$  and  $y_{d_2} = \kappa(\pi_2) + c_h(\omega_2 + rd_{f_2} + \Delta\pi_2)$ . Assumption (8) (ii) implies that  $i(\omega_2) < 0$ , which motivates the name of this *deflationary* equilibrium. Positive wages

<sup>6</sup>Section 4.4.2 and the final values :  $(\omega_1, \lambda_1, d_{f_1}, \pi_1, i_1, y_{d_1}) = (0.791, 0.969, 0.592, 0.161, 0.013, 0.915)$ .

being distributed to unemployed workers make however this Cornucopian equilibrium economically hardly relevant. Its Jacobian matrix is lower triangular. The stability criterion is:

$$(1 - \Delta)r - r\kappa'(\pi_2) - (1 - \omega_2) \left( (1 - \Delta)rc'_h(\omega_2 + rd_{f_2} + \Delta\pi_2) - r\kappa'(\pi_2) \right) - \left( \frac{\kappa(\pi_2)}{\nu(\omega_2)} - \delta + i(\omega_2) \right) + rd_{f_2} \frac{\kappa'(\pi_2)}{\nu(\omega_2)} < 0, \quad (15)$$

which is equivalent to

$$(1 - \Delta)r(1 - (1 - \omega_2)c'_h(\omega_2 + rd_{f_2} + \Delta\pi_2)) + \delta - i(\omega_2) + r\kappa'(\pi_2) \frac{d_{f_2} - \omega_2\nu(\omega_2)}{\nu(\omega_2)} - \frac{\kappa(\pi_2)}{\nu(\omega_2)} < 0.$$

Given that  $\delta > 0$ ,  $\omega_2 \in [0, 1]$ ,  $c'_h \geq 0$  and  $i(\omega_2) < 0$ , the local stability cannot be ruled out *a priori*. Notice, however, that, with the peculiar investment function chosen for our leading numerical example,  $\kappa'(\pi_e) = k_1k_2e^{k_2\pi_e} = k_2(\kappa(\pi_e) - k_0)$  which implies that the stability criterion becomes:

$$(1 - \Delta)r(1 - (1 - \omega_2)c'_h(\omega_2 + rd_{f_2} + \Delta\pi_2)) + \delta - i(\omega_2) - rk_2k_0 \frac{d_{f_2} - \omega_2\nu(\omega_2)}{\nu(\omega_2)} + \kappa(\pi_2) \frac{rk_2(d_{f_2} - \omega_2\nu(\omega_2)) - 1}{\nu(\omega_2)} < 0. \quad (16)$$

Zero-profit should presumably lead to zero investment:  $\kappa(0) = k_0 + k_1 = 0$ , so that  $k_0$  should be negative. If this is so, and for a wide range of parameters  $r$  and  $k_2$ , criterion (16) is violated. This holds, in particular, for our leading parameterization.

2) An economically more relevant, deflationary steady state is given by  $(\omega, \lambda, d_f) = (0, 0, d_{f_3})$ , provided  $d_{f_3}$  solves the nonlinear equation:

$$\kappa(\pi_3) - (1 - \Delta)y_{d_3} + (1 - \Delta)\delta\nu(0) - d_{f_3} \left[ \frac{\kappa(\pi_3)}{\nu(0)} - \delta - (1 - \Delta)r + i(0) \right] = 0, \quad (17)$$

where  $\pi_3 = 1 - rd_{f_3} - \delta\nu(0)$  and  $y_{d_3} = \kappa(\pi_3) + c_h(rd_{f_3} + \Delta\pi_3)$ . Here again,  $i(0) = -\eta_p < 0$ , and the Jacobian matrix is lower triangular. Under Assumption (8) (ii), the sign of its eigenvalues depends on the chosen parameterization, so that local stability cannot be ruled out *a priori*. Indeed, the stability criterion is:<sup>7</sup>

$$(1 - \Delta)r - (1 - \Delta)rc'_h(rd_{f_3} + \Delta\pi_3) + r\kappa'(\pi_3) - \left( \frac{\kappa(\pi_3)}{\nu(0)} - \delta - \eta_p \right) + rd_{f_3} \frac{\kappa'(\pi_3)}{\nu(0)} < 0 \quad (18)$$

With our specific investment function,  $\kappa(\pi_e) = k_0 + k_1e^{k_2\pi_e}$ ,  $\kappa'(\pi_e) = k_1k_2e^{k_2\pi_e} = k_2(\kappa(\pi_e) - k_0)$ , so that the stability criterion becomes :

$$(1 - \Delta)(1 - c'_h(rd_{f_3} + \Delta\pi_3))r + \delta + \eta_p - rk_2k_0 \frac{d_{f_3} - \Delta\nu(0)}{\nu(0)} + \kappa(\pi_3) \frac{rk_2(d_{f_3} - \Delta\nu(0)) - 1}{\nu(0)} < 0. \quad (19)$$

As before, there is a broad range of parameters  $r$ ,  $k_2$ ,  $\Delta$  — including our canonical parameterization — that does not satisfy this stability condition. On the other hand, (19) does not depend upon  $\sigma$ .

### Trivial and slavery steady states

As in the literature quoted, there exist as well a trivial and a slavery equilibrium, given respectively by:

$$(\omega, \lambda, d_f) = (0, 0, 0) \quad \text{and} \quad (\omega, \lambda, d_f) = (0, \lambda_5, d_{f_5}). \quad (20)$$

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<sup>7</sup>Whenever  $\Phi(0) < \alpha + (1 - \gamma)i(0)$  is replaced with the (stronger) condition  $\Phi(0) < \alpha$ , the first eigenvalue becomes positive and so this equilibrium is locally unstable.



In our differential system (26), the presence of depreciation into the profit equation implies that  $(\omega, \lambda, d_f) = (0, 0, 0)$  leads to  $\dot{d} = 0$  if, and only if,  $\kappa(\pi_4) - (y_{d_4} - \Delta) + (1 - \Delta)\delta\nu(0) = 0$ , where  $\pi_4 = 1 - \delta\nu(0)$  and  $y_{d_4} = \kappa(\pi_4) + c_h(\Delta\pi_4)$ . This a structurally unstable condition on the model's parameters.

In a similar way, for  $(0, \lambda_5, d_{f_5})$  to be an equilibrium, one needs  $1 - rd_{f_5} - \delta\nu(0) = \kappa^{-1}(\nu(0)(\alpha + n + \delta))$  where

$$d_{f_5} = \frac{\nu(0)(\alpha + n + \delta) - (y_{d_5} - \Delta) + (1 - \Delta)\delta\nu(0)}{\alpha + n - (1 - \Delta)r + i(0)},$$

and  $y_{d_5} = \nu(0)(\alpha + n + \delta) + c_h(rd_{f_5} + \Delta\pi_5)$ .

Hence, both equilibria are structurally unstable.

### Infinite-valued debt-deflationary equilibria

Contrary to [Grasselli and Nguyen-Huu, 2018], a long-run equilibrium with an unbounded debt ratio,  $(\omega, \lambda, d_f) = (0, 0, +\infty)$ , exists as a solution of (26). Let us indeed make the change of variable  $q = 1/d_f$ , as in [Grasselli and Costa Lima, 2012], and observe that  $(\omega, \lambda, q) = (0, 0, 0)$  is a zero of the transformed system:

$$\begin{cases} \dot{\omega} = \frac{\eta\omega}{\eta+1} [\phi(\lambda) - \alpha - (1-\gamma)i(\omega)] \\ \dot{\lambda} = \lambda \left[ \frac{\kappa(\pi_e)}{\nu(\omega)} - \frac{\dot{\omega}}{\eta\omega(1-\sigma\omega)} - (\alpha + n + \delta) \right] \\ \dot{q} = q \left[ g(\omega, 1/q) - (1-\Delta)r + i(\omega) \right] - q^2 \left[ \kappa(\pi_e) - (1-\omega)(y_d - \Delta) + (1-\Delta)\delta\nu(\omega) \right] \end{cases} \quad (21)$$

The Jacobian matrix becomes:

$$\begin{pmatrix} \frac{\partial f_1}{\partial \omega} & \omega \frac{\eta}{\eta+1} \phi'(\lambda) & 0 \\ \frac{\partial f_2}{\partial \omega} & \frac{\partial f_2}{\partial \lambda} & \frac{r}{q^2} \lambda \frac{\kappa'(\pi_e)}{\nu(\omega)} \\ \frac{\partial f_3^*}{\partial \omega} & -\frac{q}{\eta+1} \frac{\omega}{1-\sigma\omega} \phi'(\lambda) & \frac{\partial f_3^*}{\partial q} \end{pmatrix} \quad (22)$$

where

$$\begin{aligned} \frac{\partial f_3^*}{\partial \omega} &= -q \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} - \eta_p\mu + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)i(\omega)}{(\eta+1)(1-\sigma\omega)^2} - \frac{\omega(1-\gamma)\eta_p\mu}{(\eta+1)(1-\sigma\omega)} \right) \\ &\quad - q^2 \left[ -\kappa'(\pi_e) \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + y_d - \Delta - (1-\omega) \frac{\partial y_d}{\partial \omega} + \frac{(1-\Delta)\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right] \\ \frac{\partial f_3^*}{\partial q} &= \left( \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta - \omega \frac{\phi(\lambda) - \alpha - (1-\gamma)i(\omega)}{(\eta+1)(1-\sigma\omega)} - (1-\Delta)r + i(\omega) \right) + \frac{r\kappa'(\pi_e)}{q\nu(\omega)} \\ &\quad - 2q[\kappa(\pi_e) - (1-\omega)(y_d - \Delta) + (1-\Delta)\delta\nu(\omega)] - r\kappa'(\pi_e) + q^2(1-\omega) \frac{\partial y_d}{\partial q} \end{aligned} \quad (23)$$

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Since  $\kappa(\pi_e) \xrightarrow{q \rightarrow 0} k_0$ ,  $\kappa'(\pi_e) \xrightarrow{q \rightarrow 0} 0$ ,  $i(\omega) \xrightarrow{\omega \rightarrow 0} -\eta_p$ , Assumption (10) (ii)  $c(x) \xrightarrow{x \rightarrow +\infty} c_+$ <sup>8</sup> and Assumption (9) (iii),  $\lim_{\pi_e \rightarrow -\infty} \pi_e^2 \kappa'(\pi_e) < +\infty$ , the Jacobian matrix becomes:

$$\begin{pmatrix} \frac{\eta}{\eta+1}(\phi(0) - \alpha + (1 - \gamma)\eta_p) & 0 & 0 \\ 0 & k_0 A \left(\frac{1}{b}\right)^{1/\eta} - (\alpha + n + \delta) - \frac{\phi(0) - \alpha + (1 - \gamma)\eta_p}{\eta + 1} & 0 \\ 0 & 0 & k_0 A \left(\frac{1}{b}\right)^{1/\eta} - \delta - (1 - \Delta)r - \eta_p \end{pmatrix} \quad (24)$$

The (explosive) debt-deflationary steady state is locally, asymptotically stable if, and only if,:

$$\begin{aligned} \phi(0) - \alpha + (1 - \gamma)\eta_p &\leq 0 \\ k_0 A \left(\frac{1}{b}\right)^{1/\eta} - (\alpha + n + \delta) - \frac{\phi(0) - \alpha + (1 - \gamma)\eta_p}{\eta + 1} &\leq 0 \\ k_0 A \left(\frac{1}{b}\right)^{1/\eta} - \delta - (1 - \Delta)r - \eta_p &\leq 0, \end{aligned} \quad (25)$$

which is satisfied by a broad range of parameterizations, including our numerical example.

Notice that there exists another debt-deflationary equilibrium,  $(\omega; \lambda; d_f) = (\omega_6; \lambda_6; +\infty)$ , where  $(\omega_6; \lambda_6)$  is solution of the following equation:<sup>9</sup>

$$\begin{cases} k_0 = \frac{\alpha + n + \delta}{A} \left(\frac{1 - \sigma\omega_6}{b}\right)^{-\frac{1}{\eta}} = (\alpha + n + \delta)\nu(\omega_6), \\ \lambda_6 = \phi^{-1}(\alpha + (1 - \gamma)i_6) \end{cases} \quad (26)$$

with  $i_6 = \eta_p(\mu\omega_6 - 1)$ , which, courtesy of (8) (ii), always exist provided  $i_6 > 0$ .

An educated guess suggests, however, that the first condition is likely to be violated:  $\alpha$ ,  $n$  and  $\delta$  should be of the magnitude  $10^{-2}$ ,  $\nu(\omega) \simeq 3$  whereas  $k_0 \leq 10^{-4}$  (and in our specific investment function,  $k_0$  is even negative). Consequently, this additional debt-deflationary equilibrium is very unlikely to exist.

Eventually, a deflationary state with an infinite debt ratio,  $(\omega, \lambda, d_f) = (\omega_2, 0, +\infty)$ <sup>10</sup> is also a zero of (21). In the neighborhood of this equilibrium,  $\pi_8 \rightarrow -\infty$ ,  $\kappa(\pi_8) \rightarrow k_0$  and  $y_{d_8} \rightarrow k_0 + c_+$ . The Jacobian matrix can be transformed into a lower triangular matrix whose eigenvalues are easy to identify. Their sign depends on the selected parameterization, so that the stability of the steady state cannot be ruled out *a priori*. With our particular parameterization, the three stability conditions above are verified. None the less, a steady state with positive wages and zero-employment is hardly economically meaningful.

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<sup>8</sup>Notice that  $\omega + rd_f + \Delta\pi_e \rightarrow +\infty$ , hence,  $y_d = \kappa(\pi) + c_h(\omega + rd_f + \Delta\pi_e) \rightarrow k_0 + c_+ > 0$ . Consequently, on a path to this explosive debt-deflationary equilibrium, the demand ratio  $y_d$  can be greater or lower than 1, depending upon the upper bound  $c_+$ . Because production is collapsing anyway, demand  $Y_d = y_d Y$  is following the same path.

<sup>9</sup>Remember that  $d_f$  tends to infinity,  $\dot{d} \rightarrow 0$ ,  $\pi_e \rightarrow -\infty$  and  $\kappa(\pi_e) \rightarrow k_0$ .

<sup>10</sup>With  $\eta_p(\mu\omega_2 - 1) = \frac{\phi(0) - \alpha}{1 - \gamma} < 0$  due to Assumption (8) (ii).

## B.2 The monetary dynamics

Let us now introduce money and Income velocity as in section 3.4.

### B.2.1 The reduced monetary dynamical system

As before, the last variable,  $d_h$ , of the system (36), can be deduced from  $(\omega, \lambda, d_f, m)$ . The same holds for Income velocity  $\mathbf{v} = y_d/m$ , while the evolution of  $(\omega, \lambda, d_f, m)$  is independent from  $d_h$  and  $\mathbf{v}$ . We therefore focus on the following 4-dimensional system with  $(\omega, \lambda, d_f, m)$  as state variables:

$$\begin{cases} \dot{\omega} = \frac{\eta\omega}{\eta+1} \left[ \phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1) \right] \\ \dot{\lambda} = \lambda \left[ \frac{\kappa(\pi_e)}{\nu(\omega)} - \frac{\dot{\omega}}{\eta\omega(1-\sigma\omega)} - (\alpha + n + \delta) \right] \\ \dot{d}_f = \kappa(\pi_e) - (1-\omega)(y_d - \Delta) + (1-\Delta)\delta\nu(\omega) - d_f [g(\omega, \lambda, d_f) - (1-\Delta)r + \eta_p \mathbf{v}(\mu\omega - 1)] \\ \dot{m} = -\omega(1-y_d) + \delta\nu(\omega) - m[g(\omega, \lambda, d_f, m) + \eta_p \mathbf{v}(\mu\omega - 1)]. \end{cases} \quad (27)$$

New auxiliary variables are:

$$\begin{cases} \nu(\omega) = \frac{1}{A} \left( \frac{1-\sigma\omega}{b} \right)^{-1/\eta} \\ \pi_e = 1 - \omega - rd_f - \delta\nu(\omega) \\ y_d = c_h(\omega + rd_f + \Delta\pi_e) + \kappa(\pi_e) \\ \dot{d}_h = c_h(\omega + rd_f + \Delta\pi_e) - \omega - rd_f - \Delta\pi_e - d_h \left( \frac{\kappa(\pi_e)}{\nu} - \delta - \frac{\dot{\omega}}{\eta(1-\sigma\omega)} + i(\omega) \right) \\ v = \frac{y_d}{m} \\ i(\omega, m) = \eta_p \mathbf{v}(\mu\omega - 1) \\ g(\omega, \lambda, d_f, m) = \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta - \frac{\dot{\omega}}{(1-\sigma\omega)\eta}. \end{cases} \quad (28)$$

The Jacobian matrix,  $J^*(\omega, \lambda, d_f, m)$ , associated with (27) is:

$$\begin{pmatrix} \frac{\partial f_1^*}{\partial \omega} & \omega \frac{\eta}{\eta+1} \phi'(\lambda) & -\frac{\eta\omega}{\eta+1} \frac{(1-\gamma)\eta_p}{m} \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) & \frac{\eta\omega(1-\gamma)}{\eta+1} \frac{\eta_p y_d}{m^2} (\mu\omega - 1) \\ \frac{\partial f_2^*}{\partial \omega} & \frac{\partial f_2^*}{\partial \lambda} & \frac{\partial f_2^*}{\partial d_f} & \lambda \frac{(1-\gamma)\eta_p y_d (\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)m^2} \\ \frac{\partial f_3^*}{\partial \omega} & \frac{d_f}{\eta+1} \frac{\omega}{1-\sigma\omega} \phi'(\lambda) & \frac{\partial f_3^*}{\partial d_f} & d_f \eta_p \frac{y_d}{m^2} (\mu\omega - 1) \left[ \frac{\omega(1-\gamma)}{(1-\sigma\omega)(\eta+1)} + 1 \right] \\ \frac{\partial f_4^*}{\partial \omega} & \frac{m}{\eta+1} \frac{\omega}{1-\omega} \phi'(\lambda) & \frac{\partial f_4^*}{\partial d_f} & \frac{\partial f_4^*}{\partial m} \end{pmatrix} \quad (29)$$

where

$$\begin{aligned}
\frac{\partial f_1^*}{\partial \omega} &= \frac{\eta}{\eta+1} \left[ \phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1) \right] - \frac{\eta\omega}{\eta+1} (1-\gamma) \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \\
\frac{\partial f_2^*}{\partial \omega} &= -\lambda \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)^2} \right. \\
&\quad \left. - \frac{(1-\gamma)\eta_p}{(\eta+1)(1-\sigma\omega)m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \right) \\
\frac{\partial f_3^*}{\partial \omega} &= d_f \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} - \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)^2} \right. \\
&\quad \left. - \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \right) - \kappa'(\pi_e) \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + y_d - \Delta - (1-\omega) \frac{\partial y_d}{\partial \omega} + \frac{(1-\Delta)\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \\
\frac{\partial f_4^*}{\partial \omega} &= -(1-y_d) + \omega \frac{\partial y_d}{\partial \omega} + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} + m \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} \right. \\
&\quad \left. + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)^2} - \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] - \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \right)
\end{aligned} \tag{30}$$

$$\begin{aligned}
\frac{\partial f_2}{\partial \lambda} &= \frac{\kappa(\pi_e)}{\nu(\omega)} - (\alpha + n + \delta) - \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)} - \lambda \frac{\phi'(\lambda)}{(\eta+1)(1-\sigma\omega)} \\
\frac{\partial f_2^*}{\partial d_f} &= -r \lambda \frac{\kappa'(\pi_e)}{\nu(\omega)} - \lambda \frac{(1-\gamma)\eta_p}{m(\eta+1)} \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) \\
\frac{\partial f_3^*}{\partial d_f} &= (1-\Delta)r - r\kappa'(\pi_e) - (1-\omega) \frac{\partial y_d}{\partial d_f} - \left( \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta + \eta_p \mathbf{v}(\mu\omega - 1) - \omega \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)} \right) \\
&\quad + d_f \left[ \frac{r\kappa'(\pi_e)}{\nu(\omega)} - \frac{\eta_p}{m} \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) - \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \frac{\eta_p}{m} \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) \right] \\
\frac{\partial f_4^*}{\partial d_f} &= \omega \frac{\partial y_d}{\partial d_f} - \eta_p \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) + m \left[ \frac{r\kappa'(\pi_e)}{\nu(\omega)} - \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \frac{\eta_p}{m} \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) \right] \\
\frac{\partial f_4^*}{\partial m} &= -g(\omega, \lambda, d_f, m) + \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \frac{\eta_p y_d}{m} (\mu\omega - 1).
\end{aligned} \tag{31}$$

## B.2.2 Long-run equilibria with money

Here is the list of steady states for (27).

### - Finite-debt ratio equilibria

- The monetary Solovian equilibrium,  $(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*) \in \mathbb{R}_{++}^4$ , where  $m_1^*$  is uniquely determined by  $\omega_1^*$ :<sup>11</sup>

$$m_1^* = \frac{1}{\alpha + n} \left[ -\omega_1^* (1 - y_{d_1}^*) + \delta\nu(\omega_1^*) - \eta_p y_{d_1}^* (\mu\omega_1^* - 1) \right]. \tag{32}$$

- A deflationary state with non-zero wage  $(\omega_2^*, 0, d_{f_2}^*, m_2^*)$  where  $(\omega_2^*, d_{f_2}^*, m_2^*)$  solves the nonlinear

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<sup>11</sup>Because  $y_{d_1}^*$  is also only depending on  $\omega_1^*$ , see Appendix C.1.4.

equations:

$$\begin{cases} i(\omega_2^*, m_2^*) = \eta_p \frac{y_{d_2}^*}{m_2^*} (\mu \omega_2^* - 1) = \frac{\phi(0) - \alpha}{1 - \gamma} < 0 \quad \text{due to technical conditions (8) (ii)} \\ 0 = \kappa(\pi_2^*) - (1 - \omega_2^*)(y_{d_2}^* - \Delta) + (1 - \Delta)\delta\nu(\omega_2^*) - d_{f_2}^* \left[ \frac{\kappa(\pi_2^*)}{\nu(\omega_2^*)} - \delta - (1 - \Delta)r + i(\omega_2^*, m_2^*) \right], \\ 0 = \omega_2^*(1 - y_{d_2}^*) - \delta\nu(\omega_2^*) + \eta_p y_{d_2}^* (\mu \omega_2 - 1) + m_2^* \left[ \frac{\kappa(\pi_2^*)}{\nu(\omega_2^*)} - \delta \right] \end{cases} \quad (33)$$

with  $\pi_2^* = 1 - \omega_2^* - r d_{f_2}^* - \delta\nu(\omega_2^*)$  and  $y_{d_2}^* = \kappa(\pi_2^*) + c_h(\omega_2^* + r d_{f_2}^* + \Delta\pi_2)$ .  
The characteristic polynomial of the Jacobian matrix at this equilibrium is

$$\left( \frac{\kappa(\pi_2^*)}{\nu(\omega_2^*)} - (\alpha + n + \delta) - X \right) P_3(X),$$

where  $P_3(X) = X^3 + a_2 X^2 + a_1 X + a_0$ , with

$$\begin{aligned} a_0 &= J_{31}(J_{13}J_{44} - J_{43}J_{14}) - J_{11}(J_{33}J_{44} - J_{43}J_{34}) - J_{41}(J_{13}J_{34} - J_{33}J_{14}) \\ a_1 &= J_{44}J_{33} - J_{34}J_{43} + J_{11}(J_{44} + J_{33}) - J_{31}J_{13} - J_{14}J_{41} \\ a_2 &= -(J_{11} + J_{44} + J_{33}) \end{aligned} \quad \text{with } J_{ij} = \frac{\partial f_i^*}{\partial x_j} \Big|_{(\omega_2^*; 0; d_{f_2}^*, m_2^*)} \quad (34)$$

According to the Routh-Hurwitz\* criterion, this equilibrium is locally stable if, and only if,  $a_3 \geq 0$ ,  $a_2 \geq 0$ ,  $a_2 a_1 \geq a_3 a_0$ ,  $a_0 \geq 0$ . Whether these conditions are satisfied depends upon the selected parameterization and, in general, local (un)stability cannot be rule out. Notice that, with the particular parameterization of our numerical example (4.4.2), the second condition,  $a_2 \geq 0$ , is not verified, so that the debt-deflationary equilibrium turns out to be unstable.

- A deflationary state with zero-employment-wage  $(0, 0, d_{f_3}^*, m_3^*)$ ,
  - An equilibrium with infinite money ratio  $(\bar{\omega}^*; \bar{\lambda}^*; \bar{d}_f^*; +\infty)$ ,
  - A *Trivial* Equilibrium  $(0, 0, 0, m_4^*)$ .
  - A *Slavery* Equilibria  $(0, \lambda_5^*, d_{f_5}^*, m_5^*)$  with  $\lambda_5^* > 0$ .
  - Any equilibrium with  $m = 0$ , such as  $(\omega_6, \lambda_6, d_{f_6}, 0)$ .
- **Steady states with infinite debt ratio**
- Debt-deflationary equilibria with finite money ratio  $(0, 0, +\infty, m_7^*)$  or  $(\omega_9^*; \lambda_9^*; +\infty, m_9^*) > 0$ ,
  - A Cornucopian deflationary state with infinite debt ratio  $(\omega_8^*, 0, +\infty, m_8^*)$ ,
  - Debt-deflationary equilibria with skyrocketing levels of money  $(\omega_{10}^*; \lambda_{10}^*; +\infty, +\infty)$  or  $(0, 0, +\infty, +\infty)$

### Monetary Solovian equilibrium

At the monetary Solovian equilibrium  $(\omega_1^*; \lambda_1^*; d_{f_1}^*, m_1^*)$  defined above, the Jacobian matrix is given by:

$$\begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ -\lambda_1^*(K_1 K_\omega + K_{21}) & -\lambda_1^* K_{22} & -\lambda_1^*(r K_1 + K_{23}) & \lambda_1^* K_{24} \\ K_2 K_\omega + d_{f_1} K_{31} + K_{y\omega} & J_{32} & r K_2 + (1 - \Delta)r - K_{y d} - (\alpha + n + i_1) - K_3 & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{pmatrix}, \quad (35)$$

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with

$$\begin{aligned}
J_{11} &= -\frac{\eta\omega_1^*}{\eta+1}(1-\gamma)\frac{\eta_p}{m_1^*}\left[y_{d_1}^*\mu + (\mu\omega_1^* - 1)\frac{\partial y_d}{\partial \omega_1^*}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)}\right], & J_{12} &= \frac{\eta\omega_1^*}{\eta+1}\phi'(\lambda_1^*), \\
J_{13} &= -\frac{\eta\omega_1^*}{\eta+1}(1-\gamma)\frac{\eta_p}{m_1^*}(\mu\omega_1^* - 1)\frac{\partial y_d}{\partial d_f}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)}, & i_1^* &= \eta_p\frac{y_{d_1}^*}{m_1^*}(\mu\omega_1^* - 1) \\
K_1 &= \frac{\kappa'(\pi_1^*)}{\nu(\omega_1^*)}, & K_{22} &= \frac{\phi'(\lambda_1^*)}{(\eta+1)(1-\sigma\omega_1^*)}, & K_{24} &= \frac{(1-\gamma)i_1}{(\eta+1)(1-\sigma\omega_1^*)m_1^*} \\
K_{21} &= \frac{\sigma\kappa(\pi_1^*)}{\eta(1-\sigma\omega_1^*)\nu(\omega_1^*)} - \frac{(1-\gamma)\eta_p}{(\eta+1)(1-\sigma\omega_1^*)m_1^*}\left[y_d\mu + (\mu\omega_1^* - 1)\frac{\partial y_d}{\partial \omega_1^*}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*)}\right], & J_{14} &= \eta\omega_1^*K_{24}(1-\sigma\omega_1^*), \\
K_{31} &= \frac{\sigma\kappa(\pi_1^*)}{\eta(1-\sigma\omega_1^*)\nu(\omega_1^*)} - \frac{\eta_p}{m_1^*}\left[y_d\mu + (\mu\omega_1^* - 1)\frac{\partial y_d}{\partial \omega_1^*}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)}\right]\left(1 + \frac{\omega_1^*(1-\gamma)}{(\eta+1)(1-\sigma\omega_1^*)}\right) \\
K_3 &= \frac{\eta_p}{m_1^*}\left(1 + \frac{\omega_1^*(1-\gamma)}{(\eta+1)(1-\sigma\omega_1^*)}\right)\frac{\partial y_d}{\partial d_f}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)}, & K_{23} &= \frac{(1-\gamma)\eta_p(\mu\omega_1^* - 1)}{(\eta+1)(1-\sigma\omega_1^*)m_1^*}\frac{\partial y_d}{\partial d_f}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)} \\
K_{y\omega} &= y_{d_1}^* - \Delta - (1-\omega_1^*)\frac{\partial y_d}{\partial \omega_1^*}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)} + \frac{(1-\Delta)\delta\sigma\nu(\omega_1^*)}{\eta(1-\sigma\omega_1^*)}, & K_{yd} &= (1-\omega_1^*)\frac{\partial y_d}{\partial d_f}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)} \\
J_{32} &= d_{f_1}^*\omega_1^*K_{22}, & J_{34} &= \frac{d_{f_1}^*i_1}{m_1^*}\left(1 + \frac{\omega_1^*(1-\gamma)}{(\eta+1)(1-\sigma\omega_1^*)}\right), & J_{42} &= m_1^*\omega_1^*K_{22} \\
J_{41} &= -1 + y_{d_1}^* + \omega_1^*\frac{\partial y_d}{\partial \omega_1^*}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)} + \frac{\delta\sigma\nu(\omega_1^*)}{\eta(1-\sigma\omega_1^*)} + m_1^*(K_1K_\omega + K_{31}), & K_2 &= d_{f_1}^*K_1 - \kappa'(\pi_1^*) \\
J_{43} &= m_1^*rK_1 + \left(\omega_1^* - \eta_p(\mu\omega_1^* - 1) - \frac{\omega_1^*(1-\gamma)\eta_p}{(\eta+1)(1-\sigma\omega_1^*)}(\mu\omega_1^* - 1)\right)\frac{\partial y_d}{\partial d_f}\Big|_{(\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*)}
\end{aligned} \tag{36}$$

$$J_{44} = -(\alpha + n) + \frac{\omega_1^*(1-\gamma)\eta_py_{d_1}^*}{(\eta+1)(1-\sigma\omega_1^*)m_1^*}(\mu\omega_1^* - 1), \quad K_\omega = \left(1 + \frac{\delta\sigma\nu(\omega_1^*)}{\eta(1-\sigma\omega_1^*)}\right) \tag{37}$$

The characteristic polynomial is  $P(X) = X^4 + a_3X^3 + a_2X^2 + a_1X + a_0$ , where

$$\begin{aligned}
a_4 &= 1 \\
a_3 &= \lambda_1^*K_{22} - J_{11} - J_{33} - J_{44} \\
a_2 &= J_{33}J_{44} - J_{43}J_{34} + J_{31}J_{13} - J_{41}J_{14} + (J_{33} + J_{44})(J_{11} - \lambda_1^*K_{22}) \\
&\quad + \lambda_1^*[J_{32}(rK_1 + K_{23}) + K_{24}J_{42} - K_{22}J_{11} + J_{12}(K_1K_\omega + K_{21})] \\
a_1 &= \lambda_1^*[(J_{13}J_{32} + J_{14}J_{42} - (J_{33} + J_{44})J_{12})(K_1K_\omega + K_{21}) + K_{22}[J_{11}(J_{33} + J_{44}) - J_{14}J_{41} - J_{13}J_{31}] \\
&\quad + K_{24}(J_{42}J_{33} - J_{32}J_{43} - J_{41}J_{12} + J_{11}J_{42}) + (J_{32}J_{44} - J_{11}J_{32} + J_{42}J_{34} + J_{31}J_{12})(rK_1 + K_{23})] \\
&\quad - (J_{11} - \lambda_1^*K_{22})(J_{33}J_{44} - J_{34}J_{43}) - J_{31}(J_{13}J_{44} - J_{43}J_{14}) + J_{41}(J_{13}J_{34} - J_{33}J_{44}) \\
a_0 &= \lambda_1^*[K_{22}(J_{31}(J_{13}J_{44} - J_{43}J_{14}) - J_{11}(J_{33}J_{44} - J_{43}J_{34}) + J_{41}(J_{33}J_{14} - J_{13}J_{34})) \\
&\quad + K_{24}(J_{11}(J_{32}J_{43} - J_{42}J_{33}) + J_{31}(J_{42}J_{13} - J_{12}J_{43}) + J_{41}(J_{12}J_{33} - J_{32}J_{13})) \\
&\quad + (K_1K_\omega + K_{21})(J_{12}(J_{33}J_{44} - J_{43}J_{34}) + J_{42}(J_{13}J_{34} - J_{33}J_{14}) - J_{32}(J_{13}J_{44} - J_{43}J_{14})) \\
&\quad + (rK_1 + K_{23})(J_{31}(J_{42}J_{14} - J_{12}J_{44}) + J_{11}(J_{32}J_{44} - J_{42}J_{34}) + J_{41}(J_{12}J_{34} - J_{32}J_{14})]
\end{aligned} \tag{38}$$

with  $J_{31} = K_2K_\omega + d_{f_1}K_{31} + K_{y\omega}$  and  $J_{33} = rK_2 + (1-\Delta)r - K_{yd} - (\alpha + n + i_1) - K_3$

The Routh-Hurwitz Stability criterion leads to the following local, asymptotic stability criterion:  $a_3 \geq 0$ ,  $a_3a_2 - a_4a_1 \geq 0$ ,  $(a_3a_2 - a_4a_1)a_1 - a_0a_3^2 > 0$  and  $a_0 \geq 0$ .

The parameterization of our numerical example (4.4.2) yields:

$$\begin{aligned} (\omega_1^*, \lambda_1^*, d_{f_1}^*, m_1^*, \pi_1^*, i_1^*, y_{d_1}^*) &= (0.791, 0.985, 0.059, 0.011, 0.177, 0.078, 0.980) , \\ \text{we obtain } (a_3, a_2, a_1, a_0) &= (5.954, 42.862, 44.882, 8.547) , \end{aligned} \quad (39)$$

which satisfy the stability criterion.

### Other finite-debt ratio equilibria

In the same way as in section C.1.4, one can exhibit several other equilibria with finite debt ratio. 1)( $\omega, \lambda, d_f, m$ ) = (0, 0,  $d_{f_3}^*, m_3^*$ ), with ( $d_{f_3}^*, m_3^*$ ) solving the nonlinear equations:

$$\begin{aligned} \kappa(\pi_3^*) - y_{d_3}^* + \Delta + (1 - \Delta)\delta\nu(0) - d_{f_3}^* \left[ \frac{\kappa(\pi_3^*)}{\nu(0)} - \delta - (1 - \Delta)r + i(0, m_3^*) \right] &= 0 \\ m_3^* \left[ \frac{\kappa(\pi_3^*)}{\nu(0)} - \delta \right] &= \delta\nu(0) + \eta_p y_{d_3}^* \end{aligned} \quad (40)$$

$$\text{where } \pi_3^* = 1 - r d_{f_3}^* - \delta\nu(0) \text{ and } y_{d_3}^* = \kappa(\pi_3^*) + c_h(r d_{f_3}^* + \Delta\pi_3^*).$$

Here again, we have  $i(0, m_3^*) = -\eta_p \frac{y_{d_3}^*}{m_3^*} < 0$ , which motivates the name of this *deflationary* equilibrium.<sup>12</sup> The characteristic polynomial of the Jacobian matrix at the equilibrium point (0, 0,  $d_{f_3}^*, m_3^*$ ) is:

$$\left( X - \frac{\eta}{\eta + 1} [\Phi(0) - \alpha + (1 - \gamma)i(0, m_3^*)] \right) \left( X - \left[ \frac{\kappa(\pi_3^*)}{\nu(0)} - (\alpha + n + \delta) - \frac{\Phi(0) - \alpha + (1 - \gamma)\eta_p \frac{y_{d_3}^*}{m_3^*}}{\eta + 1} \right] \right) P_2(X) \quad (41)$$

where  $P_2(X) = X^2 - (J_{44} + J_{33})X + J_{44}J_{33} - J_{34}J_{43}$ , with  $J_{ij} = \frac{\partial f_i^*}{\partial x_j} |_{(0;0;d_{f_3}^*, m_3^*)}$ . Under the assumptions adopted in section C.1.2, the signs of the roots of this polynomial depend on the selected parameterization of our economy, so that neither stability, nor instability can be *a priori* ruled out. With the specific parameterization of our numerical example, the two roots of  $P_2(X)$  are positive: this equilibrium is locally unstable.

2) The change of variable,  $n = 1/m$ , enables to verify that  $(\bar{\omega}^*; \bar{\lambda}^*; \bar{d}_f^*; +\infty)$  is an equilibrium for the system, with  $\bar{\lambda}^* = \Phi^{-1}(\alpha)$ ,  $\bar{\pi}_e^*$  and  $\bar{d}_f^*$  uniquely defined by  $\bar{\omega}^*$ <sup>13</sup> and where  $\bar{\omega}^*$  solves the nonlinear equation:

$$0 = \kappa(\bar{\pi}_e^*) - (1 - \bar{\omega}^*)(\bar{y}_d^* - \Delta) + (1 - \Delta)\delta\nu(\bar{\omega}^*) - \bar{d}_f^*[\alpha + n - (1 - \Delta)r].$$

Now, the fourth line of the Jacobian matrix contains the following elements:

$$\begin{aligned} \frac{\partial f_n^*}{\partial \omega} &= n \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1 - \sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1 - \sigma\omega)\nu(\omega)} + \sigma \frac{\phi(\lambda) - \alpha - (1 - \gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta + 1)(1 - \sigma\omega)^2} \right. \\ &\quad \left. - \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \left( 1 + \frac{\omega(1 - \gamma)}{(\eta + 1)(1 - \sigma\omega)} \right) \right) \\ &\quad + n^2 \left[ (1 - y_d) - \omega \frac{\partial y_d}{\partial \omega} - \frac{\delta\sigma\nu(\omega)}{\eta(1 - \sigma\omega)} + \eta_p \left( y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right) \right] \\ \frac{\partial f_n^*}{\partial \lambda} &= -n \frac{\omega\phi'(\lambda)}{(n + 1)(1 - \omega)} \\ \frac{\partial f_n^*}{\partial d_f} &= n^2 \left[ -\omega \frac{\partial y_d}{\partial d_f} + \eta_p \frac{\partial y_d}{\partial d_f} (\mu\omega - 1) \right] + n \left[ -r \frac{\kappa'(\pi_e)}{\nu(\omega)} \right] \\ \frac{\partial f_n^*}{\partial n} &= 2n[\omega(1 - y_d) - \delta\nu(\omega) + \eta_p y_d(\mu\omega - 1)] + g(\omega, \lambda, d_f, 1/n) + n \left[ -\frac{\omega(1 - \gamma)\eta_p y_d(\mu\omega - 1)}{(\eta + 1)(1 - \sigma\omega)} \right]. \end{aligned} \quad (42)$$

<sup>12</sup>The deflationary nature of this equilibrium holds as long as  $m_3^* > 0$ , i.e.  $\kappa(\pi_3) > \delta\nu(0)$ , which means a non-negative growth rate of the economy fueled by firms' debts.

<sup>13</sup>Indeed  $\kappa(\bar{\pi}_e^*) = \nu(\bar{\omega}^*)(\alpha + n + \delta)$  and  $\bar{d}_f^* = \frac{1 - \bar{\pi}_e^* - \bar{\omega}^* - \delta\nu(\bar{\omega}^*)}{r}$ .

Each element of this fourth line tends to 0 when  $(\omega; \lambda; d_f; n) \rightarrow (\bar{\omega}^*; \bar{\lambda}^*; \bar{d}_f^*; 0)$ , with the exception of the last one:  $\frac{\partial f_n^*}{\partial n} \rightarrow \alpha + n > 0$ . This last element is a positive root of the associated characteristic polynomial, which implies that this equilibrium is locally unstable.

### Trivial and Slavery steady states

The *trivial* Equilibrium is structurally unstable as before. More precisely,  $(\omega, \lambda, d_f, m) = (0, 0, 0, m_4^*)$  with  $m_4^* \neq 0$  is no longer an equilibrium due to the introduction of depreciation into the profit equation:  $\dot{d} = 0$  if, and only if,  $\kappa(\pi_4^*) - (1 - \Delta)y_{d_4}^* + (1 - \Delta)\delta\nu(0) = 0$ , with  $\pi_4^* = 1 - \delta\nu(0)$  and  $y_{d_4}^* = \kappa(\pi_4^*) + c(\Delta\pi_{f_4}^*)$ , a condition that is structurally unstable.<sup>14</sup>

Similarly, as before, the so-called *Slavery* state  $(0, \lambda_5^*, d_{f_5}^*, m_5^*)$  is a long-run equilibrium only provided a structurally unstable condition be satisfied.<sup>15</sup> Moreover, both the trivial and the slavery equilibria lack economic relevance, and will not be studied further.

More generally one can show that there is no structurally stable equilibrium with  $m = 0$ . Indeed, let us assume the existence of an equilibrium of the form  $(\omega_6^*, \lambda_6^*, d_{f_6}^*, 0)$  for the differential system which implies that:

$$\begin{aligned} \dot{m} = 0 \Leftrightarrow 0 &= -\omega_6^*(1 - y_{d_6}^*) + \delta\nu(\omega_6^*) - \eta_p y_{d_6}^* (\mu\omega_6^* - 1) \\ &\Leftrightarrow \omega_6^*(1 - y_{d_6}^*) + \eta_p y_{d_6}^* (\mu\omega_6^* - 1) = \delta\nu(\omega_6^*) \end{aligned} \quad (43)$$

which can be uniquely solved by  $\omega_6^* \neq 0$ . This condition implies that  $\lambda_6^* = \Phi^{-1}(\alpha + (1 - \gamma)\eta_p \frac{y_{d_6}^*}{m_6^*} (\mu\omega_6^* - 1))$  so that  $\dot{\omega} = 0$ . Because  $m_6^* \rightarrow 0$ , this means that  $\lambda_6^* \rightarrow 1^-$  according to hypothesis (8) (iii). Let us take:

$$\kappa(\bar{\pi}_6^*) = (\alpha + n + \delta)\nu(\omega_6^*), \quad (44)$$

so that  $\dot{\lambda} = 0$  in the second equation of (27). We know that equation (44) can be uniquely solved for  $\pi_6^*$  because of condition (9), which implies that  $d_{f_6}^* = (1 - \pi_6^* - \omega_6^* - \delta\nu(\omega_6^*))/r$  is uniquely determined by  $\omega_6^*$ . Consequently, in order to guarantee  $\dot{d} = 0$  in the third equation of (27), we need:  $0 = \kappa(\pi_6^*) - (1 - \omega_6^*)(y_{d_6}^* - \Delta) + (1 - \Delta)\delta\nu(\omega_6^*) - d_{f_6}^*[\alpha + n - (1 - \Delta)r]$ , which leads to a second nonlinear equation that  $\omega_6^*$  must satisfy — this over-determination of  $\omega_6^*$  makes very such equilibrium (if any) structurally unstable since any infinitesimal change in the coefficients of, say, the first equation will prevent  $\omega_6^*$  from solving the second, and vice versa.

### The Debt-deflationary equilibrium

To see that  $(\omega, \lambda, d_f, m) = (0, 0, +\infty, +\infty)$  is an equilibrium, let use the same change of variable  $q := 1/d_f$  as in [Grasselli and Costa Lima, 2012], and  $n := 1/m$ . Observe that  $(\omega, \lambda, q, n) = (0, 0, 0, 0)$  is an equilibrium point of the transformed system:

$$\begin{cases} \dot{\omega} = \frac{\eta\omega}{\eta + 1} \left[ \phi(\lambda) - \alpha - (1 - \gamma)\eta_p \mathbf{v}(\mu\omega - 1) \right] \\ \dot{\lambda} = \lambda \left[ \frac{\kappa(\pi_e)}{\nu(\omega)} - \frac{\dot{\omega}}{\eta\omega(1 - \sigma\omega)} - (\alpha + n + \delta) \right] \\ \dot{q} = q \left[ g(\omega, \lambda, 1/q, 1/n) - (1 - \Delta)r + \eta_p \mathbf{v}(\mu\omega - 1) \right] - q^2 \left[ \kappa(\pi_e) - (1 - \omega)(y_d - \Delta) + (1 - \Delta)\delta\nu(\omega) \right] \\ \dot{n} = n^2 \left[ \omega(1 - y_d) - \delta\nu(\omega) + \eta_p y_d (\mu\omega - 1) \right] + n \left[ g(\omega, \lambda, 1/q, 1/n) \right]. \end{cases} \quad (45)$$

The Jacobian matrix of (45) is

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<sup>14</sup>No equilibrium can satisfy  $m_4^* = 0$ . In that case, indeed,  $\dot{m} = \delta\nu(0) + \eta_p y_{d_4}^* > 0$ . The monetary ratio,  $m$ , tending to 0 would imply an inflation rate plummeting to  $-\infty$ , which lacks any economic meaning.

<sup>15</sup>Here again, the model parameters must indeed satisfy a binding condition which makes this equilibrium structurally unstable:  $1 - rd_{f_5}^* - \delta\nu(0) = \kappa^{-1}(\nu(0)(\alpha + n + \delta))$  where  $d_{f_5}^* = \frac{\nu(0)(\alpha + n + \delta) - y_{d_5}^* + \Delta + (1 - \Delta)\delta\nu(0)}{\alpha + n - (1 - \Delta)r + i(0, m_5^*)}$ ,  $m_5^* = \frac{\delta\nu(0) + \eta_p y_{d_5}^*}{\alpha + n}$  and  $y_{d_5}^* = \nu(0)(\alpha + n + \delta) + c_n(rd_{f_5}^* + \Delta\pi_5^*)$ . This implies that  $d_{f_5}^*$  must solve simultaneously two distinct equations.



$$\begin{pmatrix} \frac{\partial f_1^*}{\partial \omega} & \omega \frac{\eta}{\eta+1} \phi'(\lambda) & -\frac{\eta \omega}{\eta+1} (1-\gamma) \eta_p n \frac{\partial y_d}{\partial q} (\mu\omega - 1) & -\omega \frac{\eta(1-\gamma)}{\eta+1} \eta_p y_d (\mu\omega - 1) \\ \frac{\partial f_2^*}{\partial \omega} & \frac{\partial f_2^*}{\partial \lambda} & \lambda \left( \frac{r\kappa'(\pi_e)}{q^2 \nu(\omega)} + \frac{(1-\gamma)\eta_p n}{(\eta+1)} \frac{\partial y_d}{\partial q} \frac{(\mu\omega-1)}{(1-\sigma\omega)} \right) & -\lambda \frac{(1-\gamma)\eta_p y_d (\mu\omega-1)}{(\eta+1)(1-\sigma\omega)} \\ \frac{\partial f_q^*}{\partial \omega} & -\frac{q}{\eta+1} \frac{\omega}{1-\sigma\omega} \phi'(\lambda) & \frac{\partial f_q^*}{\partial q} & q \left[ \frac{\omega(1-\gamma)}{(1-\sigma\omega)(\eta+1)} + 1 \right] \eta_p y_d (\mu\omega - 1) \\ \frac{\partial f_n^*}{\partial \omega} & \frac{\partial f_n^*}{\partial \lambda} & \frac{\partial f_n^*}{\partial q} & \frac{\partial f_n^*}{\partial n} \end{pmatrix} \quad (46)$$

where

$$\begin{aligned} \frac{\partial f_q^*}{\partial \omega} &= -q \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} - \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \right. \\ &\quad \left. + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)^2} - \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \right) \\ &\quad - q^2 \left[ -\kappa'(\pi_e) \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + y_d - \Delta - (1-\omega) \frac{\partial y_d}{\partial \omega} + \frac{(1-\Delta)\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right] \\ \frac{\partial f_q^*}{\partial q} &= \left( \frac{\kappa(\pi_e)}{\nu(\omega)} - \delta - (1-\Delta)r + \eta_p \mathbf{v}(\mu\omega - 1) - \omega \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)} \right) + \frac{r\kappa'(\pi_e)}{q\nu(\omega)} - \eta_p n \frac{\partial y_d}{\partial q} (\mu\omega - 1) \\ &\quad - \frac{\omega(1-\gamma)\eta_p n}{(\eta+1)(1-\sigma\omega)} \frac{\partial y_d}{\partial q} (\mu\omega - 1) - 2q[\kappa(\pi_e) - (1-\omega)(y_d - \Delta) + (1-\Delta)\delta\nu(\omega)] - r\kappa'(\pi_e) + q^2(1-\omega) \frac{\partial y_d}{\partial q} \\ \frac{\partial f_n^*}{\partial \omega} &= n \left( \frac{\kappa'(\pi_e)}{\nu(\omega)} \left( 1 + \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} \right) + \frac{\sigma\kappa(\pi_e)}{\eta(1-\sigma\omega)\nu(\omega)} \right. \\ &\quad \left. + \sigma \frac{\phi(\lambda) - \alpha - (1-\gamma)\eta_p \mathbf{v}(\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)^2} - \frac{\eta_p}{m} \left[ y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right] \left( 1 + \frac{\omega(1-\gamma)}{(\eta+1)(1-\sigma\omega)} \right) \right) \\ &\quad + n^2 \left[ (1-y_d) - \omega \frac{\partial y_d}{\partial \omega} - \frac{\delta\sigma\nu(\omega)}{\eta(1-\sigma\omega)} + \eta_p \left( y_d \mu + \frac{\partial y_d}{\partial \omega} (\mu\omega - 1) \right) \right] \\ \frac{\partial f_n^*}{\partial \lambda} &= -n \frac{\omega\phi'(\lambda)}{(n+1)(1-\sigma\omega)} \\ \frac{\partial f_n^*}{\partial q} &= n^2 \left[ -\omega \frac{\partial y_d}{\partial q} + \eta_p \frac{\partial y_d}{\partial q} (\mu\omega - 1) \right] + n \left[ -\frac{r\kappa'(\pi_e)}{q^2 \nu(\omega)} - \frac{\omega(1-\gamma)\eta_p n}{(\eta+1)(1-\sigma\omega)} \frac{\partial y_d}{\partial q} (\mu\omega - 1) \right] \\ \frac{\partial f_n^*}{\partial n} &= 2n[\omega(1-y_d) - \delta\nu(\omega) + \eta_p y_d (\mu\omega - 1)] + g(\omega, \lambda, d_f, 1/n) + n \left[ -\frac{\omega(1-\gamma)\eta_p y_d (\mu\omega - 1)}{(\eta+1)(1-\sigma\omega)} \right] \end{aligned} \quad (47)$$

Observe  $(\omega, \lambda, q, n) = (0, 0, 0, 0)$  is an equilibrium point of (45), which implies that  $\kappa(\pi_e) \xrightarrow{q \rightarrow 0} k_0$ ,  $\kappa'(\pi_e) \xrightarrow{q \rightarrow 0} 0$ ,  $i(\omega) \xrightarrow{\omega \rightarrow 0} 0$ . Consequently, because of condition (10) (ii)  $c(x) \xrightarrow{x \rightarrow +\infty} c_+$ <sup>16</sup> and condition 9 (iii),  $\lim_{\pi_e \rightarrow -\infty} \pi_e^2 \kappa'(\pi_e) < +\infty$ , the matrix (46) becomes:

<sup>16</sup>One can notice here that  $y_d = \kappa(\pi) + c_h(\omega + rd_f + \Delta\pi_e) \rightarrow k_0 + c_+$  (because  $\omega + rd_f + \Delta\pi_e \rightarrow +\infty$ ), a finite positive value. Consequently, on a path to the infinite-valued equilibrium, the demand ratio  $y_d$  can be greater or lower than 1 according to the upper bound  $c_+$  and, because production is collapsing, demand  $Y_d = y_d Y$  is following the same path.

$$\begin{pmatrix} \frac{\eta}{\eta+1}(\phi(0) - \alpha) & 0 & 0 & 0 \\ 0 & \frac{k_0}{\nu(0)} - (\alpha + n + \delta) - \frac{\phi(0) - \alpha}{\eta+1} & 0 & 0 \\ 0 & 0 & \frac{k_0}{\nu(0)} - \delta - (1 - \Delta)r & 0 \\ 0 & 0 & 0 & \frac{k_0}{\nu(0)} - \delta \end{pmatrix} \quad (48)$$

Local, asymptotic stability obtains if, and only if,:

$$\begin{aligned} \phi(0) - \alpha &\leq 0, \\ \frac{k_0}{\nu(0)} - (\alpha + n + \delta) - \frac{\phi(0) - \alpha}{\eta + 1} &\leq 0, \\ \frac{k_0}{\nu(0)} - \delta - (1 - \Delta)r &\leq 0, \\ \frac{k_0}{\nu(0)} - \delta &\leq 0. \end{aligned} \quad (49)$$

which is satisfied with our parameterization.

### Other equilibria with infinite debt ratio

1) A second debt-deflationary equilibrium exists:  $(\omega, \lambda, d_f, m) = (0, 0, +\infty, m_7^*)$  with  $m_7^* \left[ \frac{\kappa(\pi_7^*)}{\nu(0)} - \delta \right] = \delta\nu(0) + \eta_p y_{d_7}^*$ . Because  $d_f \rightarrow +\infty$ , we have  $\pi_7^* \rightarrow -\infty$ . Consequently, our specification of the increasing investment function  $\kappa(\pi_7^*) \rightarrow k_0 < 0$  implies  $m_7^* < 0$  (economically meaningless). The Jacobian matrix with the change of variable  $q = 1/d_f$  is a lower triangular matrix at this equilibrium point, from which we can immediately obtain the four eigenvalues:

$$\begin{aligned} \Phi(0) - \alpha + (1 - \gamma)\eta_p \frac{y_{d_7}}{m_7^*}, \quad \frac{k_0}{\nu(0)} - (\alpha + n + \delta) - \frac{\Phi(0) - \alpha + (1 - \gamma)\eta_p \frac{y_{d_7}}{m_7^*}}{(\eta + 1)} \\ J_{3,3}^*(\omega, \lambda, q, m)|_{(0;0;0;m_7^*)}, \quad -\frac{k_0}{\nu(0)} + \delta \end{aligned} \quad (50)$$

With this specific investment function, the last eigenvalue is positive which implies that this equilibrium is locally unstable.

2) There exists also a deflationary state with infinite-debt level,  $(\omega, \lambda, d_f, m) = (\omega_8^*, 0, +\infty, m_8^*)$ , where  $(\omega_8^*, m_8^*)$  must be solution of the following system:

$$\begin{cases} i(\omega_8^*, m_8^*) = \eta_p \frac{y_{d_8}^*}{m_8^*} (\mu\omega_8^* - 1) = \frac{\phi(0) - \alpha}{1 - \gamma} < 0 \quad \text{due to technical conditions (8) (ii)} \\ 0 = \omega_8^* (1 - y_{d_8}^*) - \delta\nu(\omega_8^*) + \eta_p y_{d_8}^* (\mu\omega_8^* - 1) + m_8^* \left[ \frac{\kappa(\pi_8^*)}{\nu(\omega_8^*)} - \delta \right] \end{cases} \quad (51)$$

In the neighborhood of this equilibrium,  $\pi_8^* \rightarrow -\infty$ ,  $\kappa(\pi_8^*) \rightarrow k_0$  and  $y_{d_8}^* \rightarrow k_0 + c_+$ . As for deflationary states with finite-debt levels, one can transform the Jacobian matrix into a lower triangular matrix leading to the identification of its eigenvalues. Their sign depends on the selected parameterization and cannot be determined *a priori*. With the particular parameterization of our numerical example, the first three stability condition are still verified, but the last one is not, which means that this equilibrium is locally unstable.

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3) Due to the introduction of a CES production function, another debt-deflationary equilibrium appears, namely  $(\omega; \lambda; d_f; m) = (\omega_9^*; \lambda_9^*; +\infty, m_9^*)$  or  $(\omega_9^*; \lambda_{10}^*; +\infty, +\infty)$ , where  $(\omega_9^*; \lambda_9^*)$  solves the following equations:

$$\begin{cases} k_0 = \frac{\alpha + n + \delta}{A} \left( \frac{1 - \sigma \omega_9^*}{b} \right)^{-\frac{1}{\eta}} = (\alpha + n + \delta) \nu(\omega_9^*), \\ \lambda_9^* = \phi^{-1}(\alpha + (1 - \gamma) i_9), \end{cases} \quad (52)$$

where  $d_f \rightarrow \infty, \dot{d} \rightarrow 0, \pi_e \rightarrow -\infty$  and  $\kappa(\pi_e) \rightarrow k_0$ , and with  $i_9^* = \eta_p \frac{y_{d_9}^*}{m_9^*} (\mu \omega_9^* - 1)$ .<sup>17</sup>

As shown above, the  $\dot{m}$  - equation of the system can be solved either with

$$m_9^* = \frac{-\omega_9^*(1 - y_{d_9}^*) + \delta \nu(\omega_9^*) + \eta_p y_{d_9}^* (\mu \omega_9^* - 1)}{\alpha + n}$$

or with  $m \rightarrow +\infty$ , which implies  $i(\omega, m) \rightarrow 0$  and consequently,  $\lambda_{10}^* = \phi^{-1}(\alpha)$ . The first condition in (52), however, is likely to be empirically violated because  $\alpha, n$  and  $\delta$  are close to  $10^{-2}$ ,  $\nu(\omega) \simeq 3$  whereas  $k_0 \leq 10^{-4}$ .<sup>18</sup> Consequently, we will not further consider this equilibrium.

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<sup>17</sup>Courtesy of (8) (ii), a solution to (52) exists provided  $i_9^* \geq 0$ .

<sup>18</sup>For the specific investment function of our numerical example,  $k_0$  is even negative.

SUPPLEMENTARY MATERIALS

### B.3 Normalization of the Philips' Curve

In this section, we just give a quick taste of the impacts that an other definition of the Philips' Curve could have on the original model [Grasselli and Costa Lima, 2012]. In order to avoid  $\omega > 1$ , one can choose e.g. a normalized Phillips curve to constraint  $\omega \in [0, 1]$ .

In [Grasselli and Costa Lima, 2012], the Philips' Curve,  $\Phi(\lambda)$ , is defined by:

$$\Phi(\lambda) = -\phi_0 + \frac{\phi_1}{(1-\lambda)^2}$$

We here propose a normalization such as the differential system becomes

$$\begin{aligned} \dot{\omega} &= \omega ((1-\omega)\Phi(\lambda) - \alpha) \\ \dot{\lambda} &= \lambda \left( \frac{1-\omega}{\nu} - \alpha - \beta - \delta \right). \end{aligned} \tag{53}$$

and we obtain the following dissipative phenomenon:

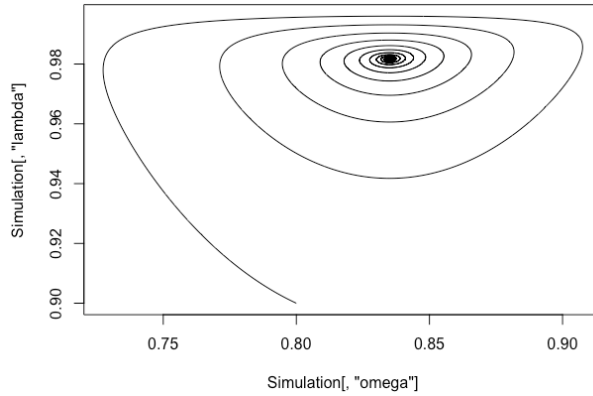


Figure B.1: Goodwin with Normalized Philips Curve  $\omega_0 = 0.8$  and  $\lambda_0 = 0.9$

#### B.3.1 Properties

Using the Lyapunov function previously associated to the system,

$$H(\omega, \lambda) = \int_{\lambda_0}^{\lambda} \frac{\phi(s)}{s} ds - \alpha \ln(\lambda) + \frac{\omega}{\nu} - \left( \frac{1}{\nu} - \alpha - \beta - \delta \right) \ln(\omega), \tag{54}$$

and calculating, with the new system  $(\dot{\omega}; \dot{\lambda})$ ,  $\frac{dH}{dt} = \nabla H \cdot (\dot{\omega}, \dot{\lambda})$  leads to:

$$\begin{aligned}
 \frac{dH}{dt} &= \frac{\partial H}{\partial \omega} \cdot \dot{\omega} + \frac{\partial H}{\partial \lambda} \cdot \dot{\lambda} \\
 &= \left( \frac{1}{\nu} - \left( \frac{1}{\nu} - (\alpha + \beta + \delta) \right) \frac{1}{\omega} \right) \cdot \omega ((1 - \omega)\Phi(\lambda) - \alpha) \\
 &\quad + \left( \frac{\Phi(\lambda)}{\lambda} - \frac{\alpha}{\lambda} \right) \cdot \lambda \left( \frac{1 - \omega}{\nu} - (\alpha + \beta + \delta) \right) \\
 &= \left( \frac{\omega}{\nu} - \left( \frac{1}{\nu} - (\alpha + \beta + \delta) \right) \right) (-\omega)\Phi(\lambda) \\
 &= \frac{\dot{\lambda}}{\lambda} \omega \Phi(\lambda) \\
 &= \left( \frac{1 - \omega}{\nu} - \alpha - \beta - \delta \right) \omega \Phi(\lambda),
 \end{aligned} \tag{55}$$

which means that the conservative behaviour of the system depends on the +/- sign of this last equation. We first study the sign of  $\Phi(\lambda)$

$$\Phi(\lambda) = -\phi_0 + \frac{\phi_1}{(1 - \lambda)^2} \tag{56}$$

with  $\phi_0 = \frac{0.04}{1 - 0.04^2}$  and  $\phi_1 = \frac{0.04^3}{1 - 0.04^2}$

Consequently, for each  $\lambda \geq 0.96$  we have  $\Phi(\lambda) \geq 0$

Assuming  $\lambda \geq 0.96$ , we study the function  $f$  (defined for  $x \in [0; 1]$ ):

$$f(x) = \left( \frac{1 - x}{\nu} - \alpha - \beta - \delta \right) x \Phi(\lambda) \tag{57}$$

which is a polynomial with two roots :

$$\begin{cases} x_1 = 0 \\ x_2 = 1 - \nu(\alpha + \beta + \delta) \cong 0.835 \end{cases} \tag{58}$$

We then study the sign of the derivative of  $f$

$$\dot{f}(x) = \left( \frac{1 - x}{\nu} - (\alpha + \beta + \delta) - \frac{x}{\nu} \right) \Phi(\lambda) \tag{59}$$

$$\Rightarrow \dot{f}(x) \geq 0 \Leftrightarrow x \leq \frac{1 - \nu(\alpha + \beta + \delta)}{2} \tag{60}$$

Thus, it implies the following evolution of  $f$  :

This means that the derivative of the potential function  $H(\omega, \lambda)$  (Lyapunov function associated to the system) is negative for all  $\omega_p \in ]x_2; 1]$  and positive otherwise, when  $\lambda \geq 0.96$ .

Assuming now  $\lambda \leq 0.96$ , which implies  $\Phi(\lambda) \leq 0$ , as a consequence :

$$\dot{f}(x) = \left( \frac{1 - x}{\nu} - (\alpha + \beta + \delta) - \frac{x}{\nu} \right) \Phi(\lambda) \tag{61}$$

$$\Rightarrow \dot{f}(x) \geq 0 \Leftrightarrow x \geq \frac{1 - \nu(\alpha + \beta + \delta)}{2} \tag{62}$$

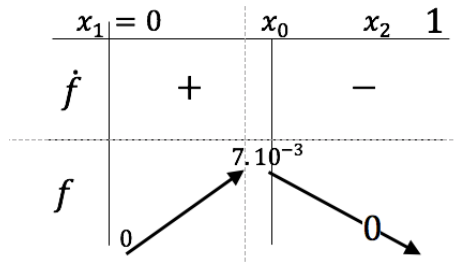


Figure B.2: Evolution of the function  $f$  for  $\lambda = 0.98$

Thus, it implies the following evolution of  $f$  :

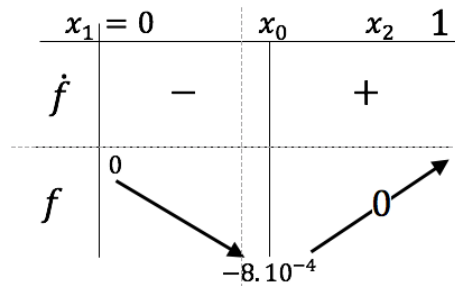


Figure B.3: Evolution of the function  $f$  for  $\lambda = 0.95$

This means that the derivative of the potential function  $H(\omega, \lambda)$  (Lyapunov function associated to the system) is negative for all  $\omega_p \in [0; x_2[$  and positive otherwise, when  $\lambda \leq 0.96$ . Thus, the system has a dissipative behavior in general, except for the value of  $\omega_p \in [x_2; 1]$  when  $\lambda \leq 0.96$  and value of  $\omega \in [0; x_1[$  when  $\lambda \geq 0.96$ , which seems strange (we do not observe expansive behavior on the graph between these values) What could be summed up in the following figure representing the different behaviors of the system according to value of  $(\omega; \lambda)$  :

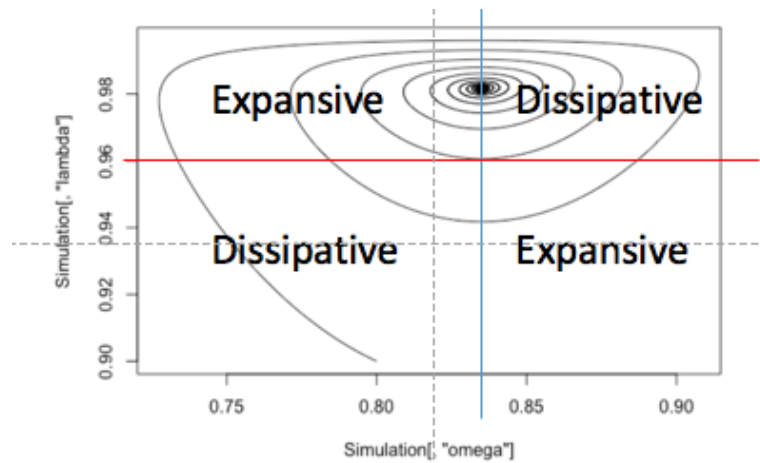


Figure B.4: Evolution of the function  $f$  for  $\lambda = 0.95$   
[Grasselli and Nguyen Huu, 2016]

## Appendix C

# Directed Technical Change in an inventory Stock-flow Consistent Dynamics with Government: from dirty to clean Capital

### C.1 Stability of equilibria in the transition model with government

This section analyzes the local, asymptotic stability of long-run steady states of the dynamical system (33).

#### C.1.1 The reduced dynamical system

Notice first that the behavior of the households' debt ratio,  $d_h$ , in (33), can be deduced from the other variables while the latter are independent from  $d_h$ .

Based on the observation of actual french government's interest rates, we consider  $r_g$  close to 0, in order to simplify the model. This implies that consumption only depends now on wage share and firm's debt ratio  $c(\omega, rd_f)$ , and that  $d_g$  becomes an auxiliary variable of the differential system, as no other variable depends on it.

As in [Costa Lima and Grasselli, 2014], “as long as the dynamics for government expenditures does not depend explicitly on the level of government debt, equation (48) can be solved separately first and then used to solve equation (49), and government debt ratio converges exponentially fast with rate ”  $\frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i - r_g$ , to the equilibrium value:

$$\bar{d}_g = \begin{cases} \frac{\bar{g}_e + \bar{g} - \bar{\tau}}{A} & \text{if } A = \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i - r_g > 0 \\ +\infty & \text{if } A < 0 \text{ or } A = 0 \text{ and } \bar{g}_e + g_1 - \tau_1 > 0 \\ 0 & \text{if } A = 0 \text{ and } \bar{g}_e + g_1 - \tau_1 < 0. \end{cases} \quad (1)$$

Moreover, with the same method as in [Costa Lima and Grasselli, 2014], using the differential system and that:  $\pi_e = 1 - \omega - rd_f + g - \tau - \chi_E \nu_1 \varepsilon$ , we can write:

$$\begin{aligned}
 \dot{\pi}_e &= -\dot{\omega} - rd_f + \dot{g} - \dot{\tau} - p_E \nu_1 \dot{\varepsilon} \\
 &= -\omega \left( \phi(\lambda) - (1 - \gamma)i - \alpha - \frac{\dot{\varepsilon} \Delta \nu}{\mu_\varepsilon} \right) \\
 &\quad - r \left[ \kappa(\pi_e) - (1 - \omega)y_d + \chi_E \nu_1 \varepsilon - g + \tau + d_f \left( r - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - i \right) \right] + \Gamma(\lambda) \\
 &\quad - g \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) - \Theta(\pi_f) + \tau \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) - p_E \nu_1 \dot{\varepsilon}
 \end{aligned} \tag{2}$$

$$\begin{aligned}
 \Rightarrow \dot{\pi}_e &= -\omega(\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + [1 - \omega - \pi_e - \chi_E \nu_1 \varepsilon] \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\
 &\quad - r [\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)] + \dot{\varepsilon} \left( \omega \frac{\Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right).
 \end{aligned} \tag{3}$$

Thus, the previous system reduces to the following five-dimensionnal system<sup>1</sup> in which we consider  $g_e$  quickly convergent to its equilibrium value  $\bar{g}_e$ :

$$\left\{ \begin{aligned}
 \frac{\dot{\omega}}{\omega} &= \phi(\lambda) - (1 - \gamma)i - \alpha - \frac{\Delta \nu}{\mu_\varepsilon} \dot{\varepsilon} \\
 \frac{\dot{\lambda}}{\lambda} &= \frac{\kappa(\pi_e)}{\nu_\theta} - \alpha - n - \delta - \frac{\Delta \nu}{\mu_\varepsilon} \dot{\varepsilon} \\
 \dot{\pi}_e &= -\omega(\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + [1 - \omega - \pi_e - \chi_E \nu_1 \varepsilon] \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\
 &\quad - r [\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)] + \dot{\varepsilon} \left( \omega \frac{\Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \\
 \dot{\varepsilon} &= \frac{\kappa(\pi_e)}{\Delta \nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) \\
 \dot{\theta} &= \sigma \theta (1 - \theta) (\chi_E - \Delta \nu),
 \end{aligned} \right. \tag{4}$$

---

<sup>1</sup>It will be enough to characterize the equilibria in which the economy either prospers or collapses. But we have to notice that when working with the reduced system, we can not recover  $d_f$ ,  $g$  and  $\tau$  separately, but rather the combination  $rd_f - g + \tau$ .



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with the following auxiliary variables:

$$\left\{ \begin{array}{l} rd_f - g + \tau = 1 - \omega - \pi_e - \chi_E \nu_1 \varepsilon \\ \frac{1}{\nu_\theta} = \frac{1 - \theta}{\nu_1} + \frac{\theta}{\nu_2} \\ \mu_\varepsilon = \frac{1 - \varepsilon}{\nu_1} + \frac{\varepsilon}{\nu_2} \\ \Delta \nu = \frac{1}{\nu_1} - \frac{1}{\nu_2} \\ y_d = c(\omega, d_f) + \kappa(\pi_e) + g_e \\ \dot{d}_h = c(\omega, d_f) - \omega - rd_f - d_h \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) \\ \dot{d}_g = g_e + g - \tau + d_g \left( r_g - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - i \right) \\ \pi_e - \pi_f = (1 - \omega)(1 - y_d), \end{array} \right. \quad (5)$$

while assuming that  $\dot{g}_e = \frac{\Gamma_E(\omega, \lambda, \pi, g, \tau, G_E, Y)}{Y} - g_e \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta \right)$  quickly converges to a finite equilibrium value  $\bar{g}_e$ .

Considering  $g_e$  at its equilibrium value, the associated Jacobian matrix is:

$$\left( \begin{array}{ccccc} \frac{\partial f_1}{\partial \omega} & \omega \dot{\phi}(\lambda) & -\omega \frac{\dot{\kappa}(\pi_e)}{\mu_\varepsilon} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & \omega \frac{\kappa(\pi_e) \Delta \nu}{\mu_\varepsilon^2 \nu_1 \nu_2} & \omega \Delta \nu \kappa(\pi_e) \\ 0 & \frac{\partial f_2}{\partial \lambda} & \lambda \frac{\dot{\kappa}(\pi_e)}{\mu_\varepsilon \nu_1 \nu_2} & \lambda \frac{\kappa(\pi_e) \Delta \nu}{\mu_\varepsilon^2 \nu_1 \nu_2} & 0 \\ \frac{\partial f_3}{\partial \omega} & \dot{\Gamma}(\lambda) - \omega \dot{\phi}(\lambda) & \frac{\partial f_3}{\partial \pi_e} & \frac{\partial f_3}{\partial \varepsilon} & \frac{\partial f_3}{\partial \theta} \\ 0 & 0 & \frac{\dot{\kappa}(\pi_e)}{\Delta \nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & -\frac{\kappa(\pi_e)}{\nu_\theta} & -\kappa(\pi_e) \mu_\varepsilon \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\theta)(\chi_E - \Delta \nu) \end{array} \right) \quad (6)$$

Where:

$$\begin{aligned}
\frac{\partial f_1}{\partial \omega} &= \phi(\lambda) - \alpha - (1 - \gamma)i - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \omega(1 - \gamma)\eta_p m \\
\frac{\partial f_2}{\partial \lambda} &= \frac{\kappa(\pi_e)}{\nu_\theta} - \alpha - n - \delta - \dot{\varepsilon}\frac{\Delta\nu}{\mu_\varepsilon} \\
\frac{\partial f_3}{\partial \omega} &= - \left( \phi(\lambda) - \alpha - (1 - \gamma)i - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} \right) + \omega\eta_p(1 - \gamma)m - \frac{\partial \pi_f}{\partial \omega} \dot{\Theta}(\pi_f) \\
&\quad + (1 - \omega - \pi_e - \chi_{E\nu_1\varepsilon})\eta_p m - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) + r \left( 1 - y_d + (1 - \omega) \frac{\partial y_d}{\partial \omega} \right) \\
\frac{\partial f_3}{\partial \pi_e} &= - \frac{\partial \pi_f}{\partial \pi_e} \dot{\Theta}(\pi_f) + (1 - \omega - \pi_e - \chi_{E\nu_1\varepsilon}) \frac{\dot{\kappa}(\pi_e)}{\nu_\theta} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) - r \left[ \dot{\kappa}(\pi_e) - 1 - (1 - \omega) \frac{\partial y_d}{\partial \pi_e} \right] \\
&\quad + \frac{\omega \dot{\kappa}(\pi_e)}{\mu_\varepsilon} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) \\
\frac{\partial f_3}{\partial \varepsilon} &= - p_{E\nu_1} \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + i \right) + \dot{\varepsilon}\omega \left( \frac{\Delta\nu}{\mu_\varepsilon} \right)^2 - \left( \frac{\omega \Delta\nu}{\mu_\varepsilon} - p_{E\nu_1} \right) \frac{\kappa(\pi_e)}{\nu_\theta} \\
\frac{\partial f_3}{\partial \theta} &= - \left( \frac{\omega \Delta\nu}{\mu_\varepsilon} - p_{E\nu_1} \right) \kappa(\pi_e) \mu_\varepsilon - (1 - \omega - \pi_e - \chi_{E\nu_1\varepsilon}) \kappa(\pi_e) \Delta\nu
\end{aligned} \tag{7}$$

### C.1.2 Existence of long-run equilibria

Let us adopt the following set of weak assumptions:

- The (Phillips) function  $\phi^2$  is continuously differentiable on  $[0, 1]$  and

$$\begin{aligned}
&\text{(i) } \phi'(\lambda) > 0 \text{ on } [0; 1] \\
&\text{(ii) } \phi(0) < \alpha + (1 - \gamma)i(0) \\
&\text{(iii) } \lim_{\lambda \rightarrow 1^-} \phi(\lambda) = +\infty,
\end{aligned} \tag{8}$$

- The investment function  $\kappa(\cdot)$  is increasing, continuously differentiable on  $\mathbb{R}$ , and satisfies:<sup>3</sup>

$$\begin{aligned}
&\text{(i) } \kappa'(\pi_e) > 0 \text{ on } [0; 1] \\
&\text{(ii) } \lim_{\pi_e \rightarrow 0} \kappa(\pi_e) = 0 \\
&\text{(iii) } \lim_{\pi_e \rightarrow -\infty} \pi_e^2 \kappa'(\pi_e) < +\infty
\end{aligned} \tag{9}$$

- The consumption function  $c(\cdot)$  is strictly increasing and continuously differentiable on  $\mathbb{R}^+$ , and verifies<sup>4</sup>:

$$\begin{aligned}
&\text{(i) } \forall x \leq 0, c(x) = c_- > 0 \\
&\text{(ii) } \lim_{x \rightarrow +\infty} c(x) = c_+,
\end{aligned} \tag{10}$$

---

<sup>2</sup>Its form and parameterization guarantee that  $\forall t, 0 \leq \lambda(t) \leq 1$ , as in [Desai et al., 2006]. Observe also that (ii) implies  $\phi(0) < \alpha$  because  $i(0) = -\eta_p < 0$ .

<sup>3</sup>These restrictions are sufficient to guarantee the existence of the relevant equilibria alluded to in section 4.3, as in [Grasselli and Costa Lima, 2012].

<sup>4</sup>Condition (i) expresses the fact that households need a minimum level of subsistence consumption even at negative income, whereas (ii) says that total normalized consumption can be higher than 1 (and consequently exceed total output, by sourcing from the stock of inventories) but must have an upper bound. See [Giraud and Grasselli, 2019] for details.

– The tax function  $\Theta(\cdot)$  is increasing, continuously differentiable on  $\mathbb{R}$  and satisfies:

$$\begin{aligned} & \text{(i) } \forall x \geq 0 \quad \Theta'(x) > 0 \\ & \text{(ii) } \lim_{x \rightarrow 0} \Theta(x) = 0 \\ & \text{(iii) } \lim_{x \rightarrow -\infty} x^2 \Theta'(x) < +\infty, \end{aligned} \tag{11}$$

– the subsidies function  $\Gamma(\cdot)$  is decreasing, continuously differentiable on  $[0; 1]$  and satisfies:

$$\begin{aligned} & \text{(i) } \forall x \geq 0 \quad \Gamma'(x) < 0 \\ & \text{(ii) } \lim_{x \rightarrow 0} \Gamma(x) = \gamma_0 > 0 \\ & \text{(iii) } \lim_{x \rightarrow 1} \Gamma(x) = 0. \end{aligned} \tag{12}$$

Our Leading Example 4.3 complies with this set of assumptions.

### C.1.3 Analysis of the different equilibria

The last equation of the system (33) admits only two solutions:  $\theta = 0$  (corresponding to a society where there is no more investment in green capital) or  $\theta = 1$  (corresponding to a society investing only in green capital).

1.  $\theta_0 = 0$ , solution for the fifth equation:  $\dot{\theta} = \sigma\theta(1 - \theta)(\chi_E - \Delta\nu)$ , implies  $\nu_\theta = \nu_1$  while  $\dot{\varepsilon} = 0 \Leftrightarrow \mu_\varepsilon = \frac{1}{\nu_2} \Rightarrow \varepsilon = 1$ . In this case, we have to study the so-called brown equilibria of the following system:

$$\begin{cases} \frac{\dot{\omega}}{\omega} = \phi(\lambda) - \alpha - (1 - \gamma)i \\ \frac{\dot{\lambda}}{\lambda} = \frac{\kappa(\pi_e)}{\nu_1} - \alpha - n - \delta \\ \dot{\pi}_e = -\omega(\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + [1 - \omega - \pi_e - p_E\nu_1] \left( \frac{\kappa(\pi_e)}{\nu_1} - \delta + i \right) \\ \quad - r[\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)]. \end{cases} \tag{13}$$

The differential system is a Goodwin-Keen system with government, inventories and prices and a net profit which is at equilibrium values  $\bar{\pi}_e = 1 - \bar{\omega} - r\bar{d}_f + \bar{g} - \bar{\tau} - p_E\nu_1$ .<sup>5</sup>

2.  $\theta_1 = 1$ , solution for the fifth equation:  $\dot{\theta} = \sigma\theta(1 - \theta)(\chi_E - \Delta\nu)$ , implies  $\nu_\theta = \nu_2$  while  $\dot{\varepsilon} = 0 \Leftrightarrow \mu_\varepsilon = \frac{1}{\nu_1} \Rightarrow \varepsilon = 0$ . In this case, the differential system is now a Goodwin-Keen model with government, inventories and prices, with  $\pi_e = 1 - \omega - rd_f + g - \tau$ . So we have to study the so-called green equilibria of the system:

$$\begin{cases} \dot{\omega} = \omega(\phi(\lambda) - \alpha - (1 - \gamma)i) \\ \dot{\lambda} = \lambda \left( \frac{\kappa(\pi_e)}{\nu_2} - \alpha - n - \delta \right) \\ \dot{\pi}_e = -\omega(\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + [1 - \omega - \pi_e] \left( \frac{\kappa(\pi_e)}{\nu_2} - \delta + i \right) \\ \quad - r[\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)]. \end{cases} \tag{14}$$

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<sup>5</sup>in that case:  $\nu_\theta = \nu_1$ .

In this case, the differential system is a Goodwin-Keen system with government, inventories and prices and a net profit which is at equilibrium values  $\bar{\pi}_e = 1 - \bar{\omega} - r\bar{d}_f + \bar{g} - \bar{\tau}$ .

Both differential system (13) and (14) admits 7 equilibria:

1. a Solovian-like equilibrium  $(\omega_1, \lambda_1, \pi_{e_1})$  which is locally stable with our Leading Example
2. a debt-deflationary equilibrium  $(0, 0, -\infty)$  which is locally unstable with our Leading Example
3. a *slavery* equilibrium  $(0, \lambda_3, \pi_{e_3})$  – structurally unstable
4. Deflationary finite-valued equilibria  $(\omega_4, 0, \pi_{e_4})$  and  $(0, 0, \pi_{e_5})$  – both structurally unstable
5. Other infinite-valued equilibria  $(0, 0, +\infty)$  and  $(\omega_4, 0, \pm\infty)$  – both structurally unstable

### C.1.4 Stability study of equilibria with brown or green capital

#### The Solovian-like equilibrium

The Solovian-like equilibrium corresponds to a desirable long-run steady state with positive wages and employment, and a finite level of debt  $(\omega_1, \lambda_1, \pi_{e_1})$ . To check its existence, observe that  $\dot{\lambda} = 0$  in the second equation of (33) whenever

$$\kappa(\pi_{e_1}) = (\alpha + n + \delta)\nu_\theta. \quad (15)$$

Equation (15) can be uniquely solved for  $\pi_{e_1}$  because of condition (9). Given  $\omega_1$ , we also obtain  $\lambda_1 = \phi^{-1}(\alpha + (1 - \gamma)i_1)$ , where  $i_1 = \eta_p(m\omega_1 - 1)$ , which belongs to  $[0; 1]$ , courtesy of (8) (ii) and provided  $i_1 > 0$ .

Because of assumption (10),  $c(\cdot)$  is uniquely determined by  $\omega$  and  $d_f$  and so does  $y_d$ , and consequently  $\pi_f$ . Assuming  $\omega_1 \neq 1$ <sup>6</sup> implies that  $d_f$  has to converge to  $\bar{d}_f$ , be it finite or not, in order to have  $\dot{\pi}_e = 0$ . This leads to a finite value of  $c(\omega_1 + r\bar{d}_f) = c_1 \in [c_-; c_+]$ . Thus  $y_{d1} = \kappa(\pi_{e_1}) + c_1 + \bar{g}_e$  and  $\pi_{f1} = \pi_{e_1} - (1 - \omega_1)(1 - y_{d1})$ .

Consequently, in order to guarantee  $\dot{\pi}_e = 0$  in the third equation of (33), we need  $\omega_1$  such that:  $\Gamma(\lambda_1) - \Theta(\pi_{f1}) + [1 - \omega_1 - \pi_{e_1} - p_e\nu_1\bar{\varepsilon}](\alpha + n + i_1) - r[\kappa(\pi_{e_1}) - \pi_{f1}] = 0$ . By the way,  $(\omega_1, \lambda_1, \pi_{e_1})$  as defined above, is an equilibrium for (33).

The Jacobian matrix becomes:

$$J_1 = \begin{pmatrix} -\omega_1(1 - \gamma)\eta_p m & K_0 & 0 & \omega_1 K_\varepsilon & \omega_1 \Delta\nu\kappa(\pi_{e_1}) \\ 0 & 0 & \lambda_1 \frac{\dot{\kappa}(\pi_{e_1})}{\mu_\varepsilon \nu_1 \nu_2} & \lambda_1 K_\varepsilon & 0 \\ K_{31} & \dot{\Gamma}(\lambda_1) - K_0 & K_{33} & K_{34} & K_{35} \\ 0 & 0 & 0 & -\frac{\kappa(\pi_{e_1})}{\nu_\theta} & -\kappa(\pi_{e_1})\mu_\varepsilon \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\bar{\theta})(\chi_E - \Delta\nu) \end{pmatrix} \quad (16)$$

---

<sup>6</sup>Notice that  $\omega_1$  can not be equal to 1, otherwise  $\pi_e \neq 0$ .

Where:

$$\begin{aligned}
K_0 &= \omega_1 \dot{\phi}(\lambda_1) \\
K_\varepsilon &= \frac{\kappa(\pi_{e1})\Delta\nu}{\mu_\varepsilon^2\nu_1\nu_2} \\
K_{31} &= \omega_1\eta_p m(1-\gamma) - \dot{\Theta}(\pi_f) \frac{\partial\pi_f}{\partial\omega} \Big|_{(\omega_1, \lambda_1, \pi_{e1})} + K_d\eta_p m - (\alpha + n + i_1) + r \left( 1 - y_{d1} + (1 - \omega_1) \frac{\partial y_d}{\partial\omega} \Big|_{(\omega_1, \lambda_1, \pi_{e1})} \right) \\
K_d &= 1 - \omega_1 - \pi_{e1} - p_E\nu_1\bar{\varepsilon} \\
K_{33} &= \dot{\Theta}(\pi_{f1}) \frac{\partial\pi_f}{\partial\pi_e} \Big|_{(\omega_1, \lambda_1, \pi_{e1})} + K_d \frac{\dot{\kappa}(\pi_{e1})}{\nu_\theta} - (\alpha + n + i_1) - r \left[ \dot{\kappa}(\pi_e) - 1 - (1 - \omega_1) \frac{\partial y_d}{\partial\pi_e} \Big|_{(\omega_1, \lambda_1, \pi_{e1})} \right] \\
K_{34} &= p_E\nu_1(\delta - i_1) - \frac{\omega_1\Delta\nu}{\mu_\varepsilon}(\alpha + n + \delta) \quad K_{35} = - \left( \frac{\omega_1\Delta\nu}{\mu_\varepsilon} - p_E\nu_1 \right) \kappa(\pi_{e1})\mu_\varepsilon - K_d\Delta\nu\kappa(\pi_{e1})
\end{aligned} \tag{17}$$

The characteristic polynomial of this matrix is:  $P(X) = (X - \sigma(1 - 2\bar{\theta})(\chi_E - \Delta\nu)) \left( X + \frac{\kappa(\pi_{e1})}{\nu_\theta} \right) P_3(X)$  where  $P_3(X) = a_3X^3 + a_2X^2 + a_1X + a_0$  with:

$$\begin{aligned}
a_3 &= 1 \\
a_2 &= -K_{33} + \eta_p m(1 - \gamma)\omega_1 \\
a_1 &= -K_{33}\omega_1\eta_p m(1 - \gamma) - \frac{\lambda_1\dot{\kappa}(\pi_{e1})}{\mu_\varepsilon\nu_1\nu_2}(\dot{\Gamma}(\lambda_1) - K_0) \\
a_0 &= -(\dot{\Gamma}(\lambda_1) - K_0)\lambda_1 \frac{\dot{\kappa}(\pi_{e1})}{\mu_\varepsilon\nu_1\nu_2} \eta_p m(1 - \gamma)\omega_1 - K_0 K_{31} \lambda_1 \frac{\dot{\kappa}(\pi_{e1})}{\mu_\varepsilon\nu_1\nu_2}.
\end{aligned} \tag{18}$$

The stability of the equilibrium is guaranteed by the Routh-Hurwitz's criterion if and only if:

$$\begin{aligned}
a_3 &\geq 0 \\
a_2 &\geq 0 \\
a_0 &\geq 0 \\
a_2a_1 - a_3a_0 &\geq 0 \\
\frac{\kappa(\pi_{e1})}{\nu_\theta} &> 0 \\
\sigma(1 - 2\bar{\theta})(\chi_E - \Delta\nu) &< 0.
\end{aligned} \tag{19}$$

These conditions are satisfied for a wide range of parameters and, in particular, by the parameterization of our Leading Example.

### Debt-deflationary Equilibrium

To study the debt-deflationary Equilibrium  $(0; 0; -\infty)$ , let us make the change of variable  $\pi_e = \frac{1}{q}$ . The new system is the same as before except for the  $\pi_e$ -equation:

$$\begin{aligned}
\dot{q} &= -q^2 \left[ -\omega(\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + \left( 1 - \omega - \frac{1}{q} - \chi_E\nu_1\varepsilon \right) \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) \right. \\
&\quad \left. - r \left( \kappa(1/q) - \frac{1}{q} + (1 - \omega)(1 - y_d) \right) + \dot{\varepsilon} \left( \omega \frac{\Delta\nu}{\mu_\varepsilon} - p_E\nu_1 \right) \right].
\end{aligned} \tag{20}$$

The Jacobian matrix associated to this new system is:

$$\begin{pmatrix} \frac{\partial f_1}{\partial \omega} & \omega \dot{\phi}(\lambda) & \frac{\omega}{-q^2} \frac{\dot{\kappa}(1/q)}{\mu_\varepsilon} \left( \frac{1}{\nu_1 \nu_2} - \frac{\mu_\varepsilon}{\nu_\theta} \right) & \omega \frac{\Delta \nu}{\mu_\varepsilon^2} \frac{\kappa(1/q)}{\nu_1 \nu_2} & \omega \Delta \nu \kappa(1/q) \\ 0 & \frac{\partial f_2}{\partial \lambda} & \frac{\lambda}{-q^2} \frac{\dot{\kappa}(1/q)}{\mu_\varepsilon \nu_1 \nu_2} & \lambda \frac{\Delta \nu}{\mu_\varepsilon^2} \frac{\kappa(1/q)}{\nu_1 \nu_2} & 0 \\ \frac{\partial f_3}{\partial \omega} & -q^2 (\dot{\Gamma}(\lambda) - \omega \dot{\phi}(\lambda)) & \frac{\partial f_3}{\partial q} & \frac{\partial f_3}{\partial \varepsilon} & \frac{\partial f_3}{\partial \theta} \\ 0 & 0 & \frac{\dot{\kappa}(1/q)}{-q^2 \Delta \nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & -\frac{\kappa(1/q)}{\nu_\theta} & -\kappa(1/q) \mu_\varepsilon \\ 0 & 0 & 0 & 0 & \sigma(1-2\theta)(\chi_E - \Delta \nu) \end{pmatrix} \quad (21)$$

Where:

$$\begin{aligned} \frac{\partial f_1}{\partial \omega} &= \phi(\lambda) - \alpha - (1-\gamma)i - \frac{\dot{\varepsilon} \Delta \nu}{\mu_\varepsilon} - \omega(1-\gamma)\eta_p m \\ \frac{\partial f_2}{\partial \lambda} &= \frac{\kappa(\pi_e)}{\nu_\theta} - \alpha - n - \delta - \dot{\varepsilon} \frac{\Delta \nu}{\mu_\varepsilon} \\ \frac{\partial f_3}{\partial \omega} &= -q^2 \left[ - \left( \phi(\lambda) - \alpha - (1-\gamma)i - \frac{\dot{\varepsilon} \Delta \nu}{\mu_\varepsilon} \right) + \omega \eta_p (1-\gamma)m - \frac{\partial \pi_f}{\partial \omega} \dot{\Theta}(\pi_f) + \left( 1 - \omega - \frac{1}{q} - \chi_E \nu_1 \varepsilon \right) \eta_p m \right. \\ &\quad \left. - \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) + r \left( 1 - y_d + (1-\omega) \frac{\partial y_d}{\partial \omega} \right) \right] \\ \frac{\partial f_3}{\partial q} &= -q^2 \left[ - \frac{\partial \pi_f}{\partial q} \dot{\Theta}(\pi_f) + \left( 1 - \omega - \frac{1}{q} - \chi_E \nu_1 \varepsilon \right) \frac{\dot{\kappa}(1/q)}{-q^2 \nu_\theta} + \frac{1}{q^2} \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) \right. \\ &\quad \left. - r \left( - \frac{\dot{\kappa}(1/q)}{q^2} + \frac{1}{q^2} - (1-\omega) \frac{\partial y_d}{\partial q} \right) \right] \\ &\quad - 2q \left[ -\omega(\phi(\lambda) - (1-\gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) + \left[ 1 - \omega - \frac{1}{q} - p_E \nu_1 \varepsilon \right] \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) \right. \\ &\quad \left. - r \left[ \kappa(1/q) - \frac{1}{q} + (1-\omega)(1-y_d) \right] + \dot{\varepsilon} \left( \omega \frac{\Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \right] \\ \frac{\partial f_3}{\partial \varepsilon} &= -q^2 \left[ -p_E \nu_1 \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) + \dot{\varepsilon} \omega \left( \frac{\Delta \nu}{\mu_\varepsilon} \right)^2 - \left( \frac{\omega \Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \frac{\kappa(1/q)}{\nu_\theta} \right] \\ \frac{\partial f_3}{\partial \theta} &= -q^2 \left[ - \left( \frac{\omega \Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \kappa(1/q) \mu_\varepsilon - \left( 1 - \omega - \frac{1}{q} - \chi_E \nu_1 \varepsilon \right) \kappa(1/q) \Delta \nu \right] \end{aligned} \quad (22)$$

Consequently, at the equilibrium  $(\omega, \lambda, q) = (0, 0, 0)$ , the Jacobian Matrix becomes:

$$\begin{pmatrix} \phi(0) - \alpha + (1 - \gamma)\eta_p & 0 & 0 & 0 & 0 \\ 0 & \frac{k_0}{\nu_\theta} - \alpha - n - \delta & 0 & 0 & 0 \\ 0 & 0 & \frac{k_0}{\nu_\theta} - \delta - \eta_p - r & 0 & 0 \\ 0 & 0 & 0 & -\frac{k_0}{\nu_\theta} & -k_0\mu_\varepsilon \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\bar{\theta})(\chi_E - \Delta\nu) \end{pmatrix} \quad (23)$$

Which is stable if and only if:

$$\begin{aligned} \phi(0) - \alpha + (1 - \gamma)\eta_p &< 0 \\ \frac{k_0}{\nu_\theta} - \alpha - n - \delta &< 0 \\ \frac{k_0}{\nu_\theta} - \delta - \eta_p - r &< 0 \\ -\frac{k_0}{\nu_\theta} &< 0 \\ \sigma(1 - 2\bar{\theta})(\chi_E - \Delta\nu) &< 0 \end{aligned} \quad (24)$$

The fourth condition,  $k_0 > 0$ , is likely to be violated under any reasonable specifications of the investment function: it is hard, indeed, to imagine a productive sector with zero profit that would still spend a non-negligible amount of money on investment. In particular with our aggregate investment function satisfying assumptions (9) (ii),  $k_0 < 0$ .<sup>7</sup>

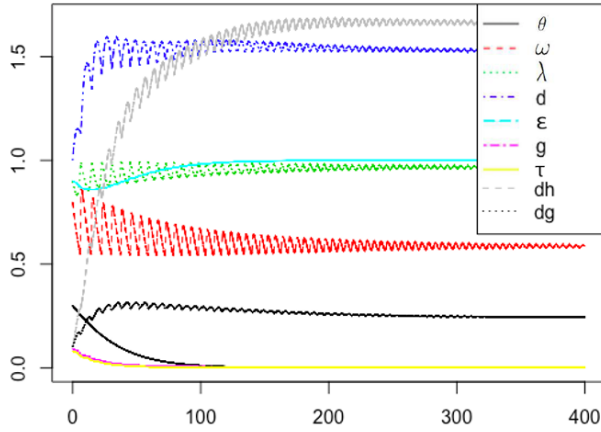
Whence, the debt-deflationary equilibrium admits a one-dimensional repulsive submanifold of the phase space, and is therefore locally unstable. Numerical simulations suggest that the inertia of aggregate investment plays a role in this lack of asymptotic stability. Consider, indeed, an economy trapped in the neighborhood of a debt-deflationary steady state: the excess of private debt induced by the cost of investing in either type of capital has a strong recessionary effect on investment, so that output is shrinking to zero, as well as employment, profits and wages. Despite the overall depression, the inertia embodied in (4.2.4) implies that output,  $Y$ , is still produced out of  $K_2$  and  $K_1$ , so that  $\varepsilon = Y_1/Y$  need not converge to 1 (resp. 0) as it should were the economy to converge towards the brown (resp. green) deflationary equilibrium.

In order to display the trajectories leading to these debt-deflationary equilibria, one can make other assumptions (e.g  $k_0 > 0$ ) corresponding to firms investing even if there is no expected profit (which is not economically meaningful) so that the debt-deflationary Equilibrium will become locally stable, as in Figure C.1b. This assumption ( $k_0 = 0.05$ ) does not impact the local stability of the other equilibria, but does not meet our general assumptions.

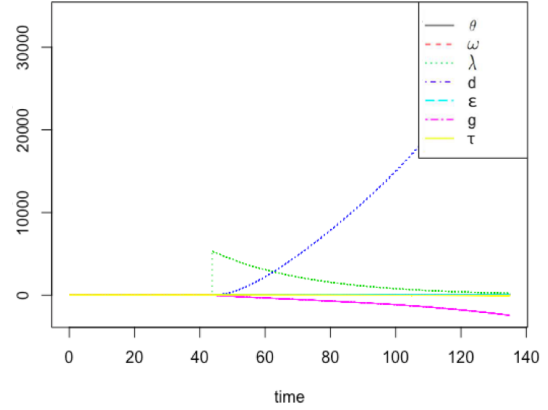
Here are some illustrations of the paths leading to these two locally stable equilibria (see Table [4.2] for the parameterization).

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<sup>7</sup>Indeed, assuming  $\lim_{\pi_\varepsilon \rightarrow 0} \kappa(\pi_\varepsilon) = 0$  implies that  $k_0 < 0$  with our specific aggregate investment function.

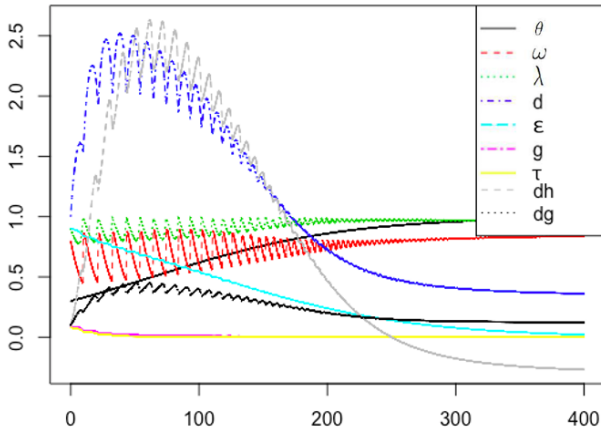


(a) Trajectory from initial values:  $\omega = 0.6$ ,  $\lambda = 0.9$ ,  $d = 1$ ,  $\varepsilon = 0.9$ ,  $\theta = 0.3$ ,  $g = 0.003$ ,  $\tau = 0.08$  to Solovian-like equilibrium with final values:  $\omega_1 = 0.5546858$ ,  $\lambda_1 = 0.9733608$ ,  $d_1 = 1.478804$ ,  $\bar{\varepsilon} = 0.9999996$ ,  $\bar{\theta} = 1.829661e - 07$ ,  $g_1 = 0.003236408$ ,  $\tau_1 = 0.006333863$ ,  $\pi_1 = 0.1378527$

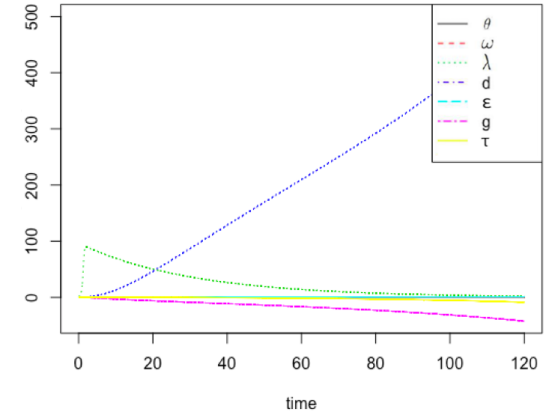


(b) Trajectory (with  $k_0 = 0.05$ ) from initial values:  $\omega = 0.53$ ,  $\lambda = 0.9$ ,  $d = 1$ ,  $\varepsilon = 0.9$ ,  $\theta = 0.3$ ,  $g = 0.09$ ,  $\tau = 0.08$  to debt-deflationary equilibrium with final values:  $\omega_2 = 0$ ,  $\lambda_2 = 0$ ,  $d_2 = 351.99$ ,  $\bar{\varepsilon} = 1$ ,  $\bar{\theta} = 0$ ,  $g_2 = -26, 99$ ,  $\tau_2 = -0.67$

Figure C.1: Trajectories to the brown locally stable equilibria of (33)



(a) Trajectory from initial values:  $\omega = 0.6$ ,  $\lambda = 0.9$ ,  $d = 1$ ,  $\varepsilon = 0.9$ ,  $\theta = 0.3$ ,  $g = 0.09$ ,  $\tau = 0.08$  to Solovian-like equilibrium with final values:  $\omega_1 = 0.8226136$ ,  $\lambda_1 = 0.9713984$ ,  $d_1 = 0.1589752$ ,  $\bar{\varepsilon} = 0.0217836$ ,  $\bar{\theta} = 0.9888603$ ,  $g_1 = 0.0021762718$ ,  $\tau_1 = 0.005458815$ ,  $\pi_1 = 0.1614925$



(b) Trajectory (with  $k_0 = 0.05$ ) from initial values:  $\omega = 0.5$ ,  $\lambda = 0.9$ ,  $d = 1$ ,  $\varepsilon = 0.9$ ,  $\theta = 0.3$ ,  $g = 0.09$ ,  $\tau = 0.08$  to debt-deflationary equilibrium with final values:  $\omega_2 = 0$ ,  $\lambda_2 = 0$ ,  $d_2 = 1.1 \text{ E}12$ ,  $\bar{\varepsilon} = 0$ ,  $\bar{\theta} = 1$ ,  $g_2 = 0.12$ ,  $\tau_2 = 6.8 \text{ E}10$

Figure C.2: Trajectories to the green locally stable equilibria of (33)

To reach the green equilibria, one can notice the initial positive value of  $g - \tau$  needed to ensure the convergence of these trajectories to green equilibria. These values will decrease as the transition is effective ( $\varepsilon \rightarrow 0$ ). The private debt is also falling in the long-run because profits are not impinged by fossil energy prices.



### Slavery Equilibrium

As in the literature quoted, there exist a slavery equilibrium  $(0; \lambda_3; \pi_{e_3}; \bar{\varepsilon}; \bar{\theta})$ , be it brown or green:

$$\pi_{e_3} = \pi_{e_1} = \kappa^{-1}(\nu_\theta(\alpha + n + \delta))$$

and 
$$\lambda_3 = \Gamma^{-1} \left[ \Theta(\pi_{f_3}) - [1 - \pi_{e_3} - p_E \nu_1 \bar{\varepsilon}] \left( \frac{\kappa(\pi_{e_3})}{\nu_\theta} - \delta - \eta_p \right) + r [\kappa(\pi_{e_3}) - \pi_{e_3} + (1 - y_{d_3})] \right]. \quad (25)$$

Consequently the Jacobian Matrix becomes:

$$\begin{pmatrix} \phi(0) - \alpha + (1 - \gamma)\eta_p & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 \frac{\dot{\kappa}(\pi_{e_3})}{\mu_\varepsilon \nu_1 \nu_2} & \lambda_3 \frac{\kappa(\pi_{e_3}) \Delta \nu}{\mu_\varepsilon^2 \nu_1 \nu_2} & 0 \\ K_{31} & \dot{\Gamma}(\lambda_3) & K_{33} & K_{34} & K_{35} \\ 0 & 0 & 0 & -\frac{\kappa(\pi_{e_3})}{\nu_\theta} & -\mu_\varepsilon \kappa(\pi_{e_3}) \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\bar{\theta})(\chi_E - \Delta \nu) \end{pmatrix} \quad (26)$$

Where:

$$\begin{aligned} K_d &= 1 - \pi_{e_3} - p_E \nu_1 \bar{\varepsilon} \\ K_{31} &= -(\phi(0) - \alpha + (1 - \gamma)\eta_p) - \dot{\Theta}(\pi_{f_3}) \frac{\partial \pi_f}{\partial \omega} \Big|_{(0, \lambda_3, \pi_{e_3})} + K_d \eta_p m - (\alpha + n - \eta_p) + r \left[ 1 - y_{d_3} + \frac{\partial y_{d_3}}{\partial \omega} \Big|_{(0, \lambda_3, \pi_{e_3})} \right] \\ K_{33} &= -\dot{\Theta}(\pi_{f_3}) \frac{\partial \pi_f}{\partial \pi_e} \Big|_{(0, \lambda_3, \pi_{e_3})} + K_d \frac{\dot{\kappa}(\pi_{e_3})}{\nu_\theta} - (\alpha + n - \eta_p) - r \left[ \dot{\kappa}(\pi_{e_3}) - 1 - \frac{\partial y_{d_3}}{\partial \pi_e} \Big|_{(0, \lambda_3, \pi_{e_3})} \right] \\ K_{34} &= p_E \nu_1 (\delta + \eta_p) \quad K_{35} = p_E \nu_1 \mu_\varepsilon \kappa(\pi_{e_3}) - K_d \Delta \nu \kappa(\pi_{e_3}) \end{aligned} \quad (27)$$

The characteristic polynomial of this matrix is:  $P(X) = (X - (\phi(0) - \alpha + (1 - \gamma)\eta_p))(X - \sigma(1 - 2\bar{\theta})(\chi_E - \Delta \nu)) \left( X + \frac{\kappa(\pi_{e_3})}{\nu_1} \right) P_2(X)$  where  $P_2(X) = a_2 X^2 + a_1 X + a_0$  with:

$$\begin{aligned} a_2 &= 1 \\ a_1 &= -K_{33} \\ a_0 &= -\lambda_3 \frac{\dot{\Gamma}(\lambda_3) \dot{\kappa}(\pi_{e_3})}{\mu_\varepsilon \nu_1 \nu_2}. \end{aligned} \quad (28)$$

The stability of the equilibrium is guaranteed by the Routh-Hurwitz's criterion if and only if:

$$\begin{aligned} a_2 &\geq 0 \\ a_1 &\geq 0 \\ a_0 &\geq 0 \\ \frac{\kappa(\pi_{e_3})}{\nu_\theta} &> 0 \\ \phi(0) - \alpha + (1 - \gamma)\eta_p &< 0 \\ \sigma(1 - 2\bar{\theta})(\chi_E - \Delta \nu) &< 0. \end{aligned} \quad (29)$$

which is not satisfied by a broad range of parameterizations,<sup>8</sup> including our Leading Example. Whence, the *slavery* equilibrium is not structurally stable and will not be further studied.

### Deflationary finite-valued equilibria

1) Another economically doubtful equilibrium  $(\omega, \lambda, \pi_e, \varepsilon, \theta) = (\omega_4, 0, \pi_{e_4}, \bar{\varepsilon}, \bar{\theta})$  exists if there exists  $(\omega_4, \pi_{e_4})$  solving the nonlinear equations:

$$\begin{cases} i(\omega_4) = \frac{\phi(0) - \alpha}{1 - \gamma} \\ 0 = \Gamma(0) - \Theta(\pi_{f_4}) + [1 - \omega_4 - \pi_{e_4} - p_E \nu_1 \bar{\varepsilon}] \left( \frac{\kappa(\pi_{e_4})}{\nu_\theta} - \delta + i(\omega_4) \right) - r [\kappa(\pi_{e_4}) - \pi_{e_4} + (1 - \omega_4)(1 - y_{d_4})]. \end{cases} \quad (30)$$

Assumption (8) (ii) implies that  $i(\omega_2) < 0$ , which motivates the name of this *deflationary* equilibrium. Positive wages being distributed to unemployed workers make however this Cornucopian equilibrium economically hardly relevant.

Consequently the Jacobian Matrix becomes:

$$\begin{pmatrix} -\omega_4(1 - \gamma)\eta_p m & K_0 & 0 & \omega_4 K_\varepsilon & \omega_4 \Delta \nu \kappa(\pi_{e_4}) \\ 0 & K_{22} & 0 & 0 & 0 \\ K_{31} & \dot{\Gamma}(0) - K_0 & K_{33} & K_{34} & K_{35} \\ 0 & 0 & 0 & -\frac{\kappa(\pi_{e_4})}{\nu_\theta} & -\kappa(\pi_{e_4})\mu_\varepsilon \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\bar{\theta})(\chi_E - \Delta \nu) \end{pmatrix} \quad (31)$$

---

<sup>8</sup>Indeed  $K_{33} = -\dot{\Theta}(\pi_{f_3}) \frac{\partial \pi_f}{\partial \pi_e} \Big|_{(0, \lambda_3, \pi_{e_3})} + K_d \frac{\dot{\kappa}(\pi_{e_3})}{\nu_\theta} - (\alpha + n - \eta_p) - r \left[ \dot{\kappa}(\pi_{e_3}) - 1 - \frac{\partial y_{d_3}}{\partial \pi_e} \Big|_{(0, \lambda_3, \pi_{e_3})} \right]$  with  $\frac{\partial \pi_f}{\partial \pi_e} = \frac{\partial y_d}{\partial \pi_e}$ .

For a broad range of parameters  $r - \dot{\Theta}(\pi_{f_3}) > 0$ . For instance, in our leading,  $r$  is around  $10^{-2}$  and  $\dot{\Theta}(\pi_{f_3})$  is around  $10^{-3}$ . Thus  $-\dot{\Theta}(\pi_{f_3}) \frac{\partial \pi_f}{\partial \pi_e} \Big|_{(0, \lambda_3, \pi_{e_3})} + r \frac{\partial y_{d_3}}{\partial \pi_e} \Big|_{(0, \lambda_3, \pi_{e_3})} > 0$ . By the way,  $\alpha + n - \eta_p - r \sim 10^{-2}$  and negative for a broad range of values (and so it is in our Leading Example). Lastly  $K_d/\nu_\theta \sim 10^{-1}$  for a broad range of values and so  $K_d/\nu_\theta > r$ . Consequently, there is a large range of parameters for which  $K_{33}$  is positive, which implies that the Routh-Hurwitz's criterion is not satisfied.

Where:

$$\begin{aligned}
K_0 &= \omega_4 \dot{\phi}(0) \\
K_\varepsilon &= \frac{\kappa(\pi_{e_4})\Delta\nu}{\mu_\varepsilon^2\nu_1\nu_2} \\
K_{31} &= \omega_4\eta_p m(1-\gamma) - \dot{\Theta}(\pi_{f_4}) \frac{\partial\pi_f}{\partial\omega|_{(\omega_4,0,\pi_{e_4})}} + K_d\eta_p m - \left( \frac{\kappa(\pi_{e_4})}{\nu_\theta} - \delta + i(\omega_4) \right) \\
&\quad + r \left( 1 - y_{d_4} + (1 - \omega_4) \frac{\partial y_d}{\partial\omega|_{(\omega_4,0,\pi_{e_4})}} \right) \\
K_d &= 1 - \omega_4 - \pi_{e_4} - p_E\nu_1\bar{\varepsilon} \\
K_{22} &= \frac{\kappa(\pi_{e_4})}{\nu_\theta} - \alpha - n - \delta \\
K_{33} &= -\dot{\Theta}(\pi_{f_4}) \frac{\partial\pi_f}{\partial\pi_e|_{(\omega_4,0,\pi_{e_4})}} + K_d \frac{\dot{\kappa}(\pi_{e_4})}{\nu_\theta} - \left( \frac{\kappa(\pi_{e_4})}{\nu_\theta} - \delta + i(\omega_4) \right) - r \left[ \dot{\kappa}(\pi_{e_4}) - 1 - (1 - \omega_4) \frac{\partial y_d}{\partial\pi_e|_{(\omega_4,0,\pi_{e_4})}} \right] \\
K_{34} &= -p_E\nu_1 \left( \frac{\kappa(\pi_{e_4})}{\nu_\theta} - \delta + i(\omega_4) \right) - \left( \frac{\omega_4\Delta\nu}{\mu_\varepsilon} - p_E\nu_1 \right) \frac{\kappa(\pi_{e_4})}{\nu_\theta} \\
K_{35} &= - \left( \frac{\omega_4\Delta\nu}{\mu_\varepsilon} - p_E\nu_1 \right) \kappa(\pi_{e_4})\mu_\varepsilon - K_d\Delta\nu\kappa(\pi_{e_4})
\end{aligned} \tag{32}$$

This matrix can be transformed into an upper triangular matrix of which stability criterion are the signs of the eigenvalues:  $K_{33}$ ,  $-\eta_p m(1-\gamma)\omega_4$ ,  $K_{22}$ ,  $-\frac{\kappa(\pi_{e_4})}{\nu_\theta}$  and  $\sigma(1-2\bar{\theta})(\chi_E - \Delta\nu)$ . According to our assumptions, all these eigenvalues have negative parts but  $K_{33}$ , which is positive for a broad range of parameterization.<sup>9</sup> Consequently this equilibrium is structurally unstable.

2) An economically more relevant, deflationary steady state is given by  $(\omega, \lambda, \pi_e, \varepsilon, \theta) = (0, 0, \pi_{e_6}, \bar{\varepsilon}, \bar{\theta})$ , provided  $\pi_{e_5}$  solving the nonlinear equation:

$$0 = \Gamma(0) - \Theta(\pi_{f_5}) + [1 - \pi_{e_6} - p_E\nu_1\bar{\varepsilon}] \left( \frac{\kappa(\pi_{e_6})}{\nu_\theta} - \delta - \eta_p \right) - r [\kappa(\pi_{e_6}) - \pi_{e_6} + (1 - y_{d_5})]. \tag{33}$$

Here again,  $i(0) = -\eta_p < 0$ , and the Jacobian matrix can be transformed into an upper triangular matrix.

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<sup>9</sup>Indeed, for the same reasons as in the *slavery* case, there exists a broad range of parameters values that makes  $K_{33} > 0$ . If  $K_{22} = \frac{\kappa(\pi_{e_4})}{\nu_\theta} - \alpha - n - \delta < 0$  is negative, there is a broad range of value that makes also  $\frac{\kappa(\pi_{e_4})}{\nu_\theta} - \eta_p - r - \delta < 0$  because  $r$ ,  $\eta_p$ ,  $\alpha$  and  $n$  are of the same order of magnitude.  $K_d/\nu_\theta$  is still greater than  $r$ . Lastly  $r(1 - \omega_4) \frac{\partial y_d}{\partial\pi_e|_{(\omega_4,0,\pi_{e_4})}}$  is still greater than  $-\Theta(\pi_{f_4}) \frac{\partial\pi_f}{\partial\pi_e|_{(\omega_4,0,\pi_{e_4})}}$  for a broad range of values.

Consequently the Jacobian Matrix becomes:

$$\begin{pmatrix} \phi(0) - \alpha + (1 - \gamma)\eta_p & 0 & 0 & 0 & 0 \\ 0 & K_{22} & 0 & 0 & 0 \\ K_{31} & \Gamma(0) & K_{33} & K_{34} & K_{35} \\ 0 & 0 & 0 & -\frac{\kappa(\pi_{e6})}{\nu_1} & \frac{\mu_\varepsilon}{\Delta\nu}\Delta\nu\kappa(\pi_{e1}) \\ 0 & 0 & 0 & 0 & \sigma(\chi_E - \Delta\nu) \end{pmatrix} \quad (34)$$

Where:

$$\begin{aligned} \Delta\nu\kappa(\pi_{e1}) &= \frac{\kappa(\pi_{e6})\Delta\nu}{\nu_1^2} & K_d &= 1 - \pi_{e6} - p_E\nu_1\bar{\varepsilon} & K_{22} &= \frac{\kappa(\pi_{e6})}{\nu_\theta} - \alpha - n - \delta \\ K_{31} &= -(\phi(0) - \alpha + (1 - \gamma)\eta_p) - \dot{\Theta}(\pi_{f5})\frac{\partial\pi_f}{\partial\pi_e|_{(0,0,\pi_{e5})}} + K_d\eta_p m - \left(\frac{\kappa(\pi_{e6})}{\nu_\theta} - \delta - \eta_p\right) \\ &\quad + r\left(1 - y_{d5} + \frac{\partial y_d}{\partial\omega|_{(0,0,\pi_{e5})}}\right) \\ K_{33} &= -\dot{\Theta}(\pi_{f5})\frac{\partial\pi_f}{\partial\pi_e|_{(0,0,\pi_{e5})}} + K_d\frac{\dot{\kappa}(\pi_{e6})}{\nu_\theta} - \left(\frac{\kappa(\pi_{e6})}{\nu_\theta} - \delta - \eta_p\right) - r\left[\dot{\kappa}(\pi_{e6}) - 1 - \frac{\partial y_d}{\partial\pi_e|_{(0,0,\pi_{e5})}}\right] \\ K_{34} &= p_E\nu_1(\delta + \eta_p) & K_{35} &= -p_E\frac{\kappa(\pi_{e6})\mu_\varepsilon}{\nu_1} + K_d\Delta\nu\kappa(\pi_{e5}) \end{aligned} \quad (35)$$

For the same reasons as before, either  $K_{22}$  or  $K_{33}$  is positive for a broad range of value which means that all eigenvalues have not negative parts. Consequently this equilibrium is structurally unstable.

### Other infinite-valued equilibria

Another infinite-valued equilibrium is given by  $(\omega, \lambda, \pi_e, \varepsilon, \theta) = (0, 0, +\infty, \bar{\varepsilon}, \bar{\theta})$ . Indeed  $\pi_e \rightarrow +\infty$  implies  $q = \frac{1}{\pi_e} \rightarrow 0$  that solves  $\dot{q}$ -equation (46) of the new system. The Jacobian matrix becomes an upper triangular matrix. One of its eigenvalues is  $\frac{\kappa(1/q)}{\nu_\theta} - \alpha - n - \delta$  which can never be negative because  $\kappa$  is assumed to be strickly increasing. So this equilibrium is structurally unstable.

Eventually, a deflationary state with an infinite profit ratio, be it positive or negative,  $(\omega, \lambda, d_f) = (\omega_4, 0, \pm\infty)$ <sup>10</sup> is also a zero of (4). In the neighborhood of this equilibrium, two cases can be studied:

- If  $\pi_e \rightarrow -\infty$ , due to an explosive level of private debt for example,  $\kappa(\pi_e) \rightarrow k_0$  and  $y_d \rightarrow k_0 + c_+ + \bar{g}_e$ . The Jacobian matrix can be transformed into an upper triangular matrix whose eigenvalues are easy to identify. Their sign depends on the selected parameterization, so that the stability of the steady state cannot be ruled out *a priori*. As in the case of the *Bad equilibrium*, the stability condition  $-\frac{k_0}{\nu_\theta} < 0$  is not verified with our particular parameterization. Other values of parameterization might satisfy this condition. Nonetheless, a steady state with positive wages, negative profits and zero-employment is hardly economically meaningful.

<sup>10</sup>With  $\eta_p(\mu\omega_4 - 1) = \frac{\phi(0) - \alpha}{1 - \gamma} < 0$  due to Assumption (8) (ii).

- If  $\pi_e \rightarrow +\infty$ , due to an explosive level of subsidies for example, the characteristic polynomial associated to the Jacobian Matrix admits  $\frac{\kappa(1/q)}{\nu_\theta} - \alpha - n - \delta$  as root, that is not negative because  $\kappa$  is assumed to be strictly increasing. So this equilibrium is structurally unstable.

## C.2 Stability of equilibria with subsidies and carbon tax

This section analyzes the local, asymptotic stability of long-run steady states of the dynamical system (55).

### C.2.1 The reduced dynamical system

Notice as before that the behavior of the households' debt ratio,  $d_h$ , in (55), can still be deduced from the other variables while the latter are independent from  $d_h$ .

Regarding again  $r_g$  close to 0, consumption only depends now on wage share and firm's debt ratio  $c(\omega, rd_f)$ , and consequently  $d_g$  is still an auxiliary variable of the differential system, as no other variable depends on it.

With the same consideration about  $g_e$  as before, we assume that its equation dynamics can be solved separately.

Eventually, using the same method as [Costa Lima and Grasselli, 2014] and because  $\pi_e = 1 - \omega - rd_f + g - \tau - \tau_c - \chi_E \nu_1 \varepsilon$ , we can write:

$$\begin{aligned}
 \dot{\pi}_e &= -\dot{\omega} - rd_f + \dot{g} - \dot{\tau} - (\chi_E + \chi_c) \nu_1 \dot{\varepsilon} \\
 &= -\omega \left( \phi(\lambda) - (1 - \gamma)i - \alpha - \frac{\dot{\varepsilon} \Delta \nu}{\mu_\varepsilon} - \sigma_\nu \hat{\rho} \left( 1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon} \right) \right) \\
 &\quad - r \left[ \kappa(\pi_e) - (1 - \omega)y_d + \chi_E \nu_1 \varepsilon - g + \tau + d_f \left( r - \frac{\kappa(\pi_e)}{\nu_\theta} + \delta - \sigma_\nu \hat{\rho}(1 - \varepsilon) - i \right) \right] + \Gamma(\lambda) \\
 &\quad - g \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) + i \right) - \Theta(\pi_f) + \tau \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) + i \right) - (\chi_E + \chi_c) \nu_1 \dot{\varepsilon}
 \end{aligned} \tag{36}$$

$$\begin{aligned}
 \Rightarrow \dot{\pi}_e &= -\omega \left( \phi(\lambda) - (1 - \gamma)i - \alpha - \sigma_\nu \hat{\rho} \left( 1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon} \right) \right) + \Gamma(\lambda) - \Theta(\pi_f) \\
 &\quad + [1 - \omega - \pi_e - (\chi_E + \chi_c) \nu_1 \varepsilon] \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) + i \right) \\
 &\quad - r [\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)] + \dot{\varepsilon} \left( \omega \frac{\Delta \nu}{\mu_\varepsilon} - (\chi_E + \chi_c) \nu_1 \right).
 \end{aligned} \tag{37}$$

Thus, the previous system reduces to the following five-dimensionnal system<sup>11</sup> in which we consider

<sup>11</sup>It will be enough to characterize the equilibria in which the economy either prospers or collapses. But we have to notice that when working with the reduced system, we can not recover  $d_f$ ,  $g$  and  $\tau$  separately, but rather the combination  $rd_f - g + \tau$ .

$g_e$  quickly convergent to its equilibrium value  $\bar{g}_e$ :

$$\left\{ \begin{array}{l} \dot{\omega} = \phi(\lambda) - (1 - \gamma)i - \alpha - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \sigma_\nu \hat{\rho} \left(1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon}\right) \\ \dot{\lambda} = \frac{\kappa(\pi_e)}{\nu_\theta} - \delta - \alpha - n - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \sigma_\nu \hat{\rho} \varepsilon \left(1 - \frac{1}{\nu_2 \mu_\varepsilon}\right) \\ \dot{\pi}_e = -\omega \left( \phi(\lambda) - (1 - \gamma)i - \alpha - \sigma_\nu \hat{\rho} \left(1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon}\right) \right) + \Gamma(\lambda) - \Theta(\pi_f) + \dot{\varepsilon} \left( \omega \frac{\Delta\nu}{\mu_\varepsilon} - (\chi_E + \chi_c)\nu_1 \right) \\ \quad + [1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon] \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho}(1 - \varepsilon) + i \right) - r [\kappa(\pi_e) - \pi_e + (1 - \omega)(1 - y_d)] \\ \dot{\varepsilon} = \frac{\kappa(\pi_e)}{\Delta\nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) - \sigma_\nu \hat{\rho}(1 - \varepsilon)\varepsilon \\ \dot{\theta}^* = \sigma\theta^*(1 - \theta^*)(\chi_E + \chi_c - \Delta\nu) \\ \dot{\rho} = \frac{(\theta_{RD} - \rho)\theta^*\kappa(\pi_e) + \sigma_{RD}\tau_c}{\nu_2(1 - \varepsilon)}. \end{array} \right. \quad (38)$$

with the following auxiliary variables:

$$\left\{ \begin{array}{l} rd_f - g + \tau + \tau_c = 1 - \omega - \pi_e - \chi_E \nu_1 \varepsilon \\ \hat{\nu}_2 = -\sigma_\nu \hat{\rho} \\ \tau_c = \chi_c \nu_1 \varepsilon \\ i = \eta_p(m\omega - 1). \\ \frac{1}{\nu_\theta} = \frac{1 - \theta}{\nu_1} + \frac{\theta}{\nu_2} \\ \mu_\varepsilon = \frac{1 - \varepsilon}{\nu_1} + \frac{\varepsilon}{\nu_2} \\ \Delta\nu = \frac{1}{\nu_1} - \frac{1}{\nu_2} \\ y_d := \frac{Y_d}{Y} = c(\omega + rd_f + r_g d_g) + \kappa(\pi_e) + g_e \\ \pi_e - \pi_f = (1 - \omega)(1 - y_d) \end{array} \right. \quad (39)$$

while assuming that  $\dot{g}_e = \frac{\Gamma_E(\omega, \lambda, \pi, g, \tau, G_E, Y)}{Y} - g_e \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta \right)$  quickly converges to a finite equilibrium value  $\bar{g}_e$ .

APPENDIX C. DIRECTED TECHNICAL CHANGE IN AN INVENTORY STOCK-FLOW  
CONSISTENT DYNAMICS WITH GOVERNMENT: FROM DIRTY TO CLEAN CAPITAL

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Considering  $g_e$  at its equilibrium value, the associated Jacobian matrix is:

$$\begin{pmatrix} \frac{\partial f_1}{\partial \omega} & \omega \dot{\phi}(\lambda) & -\omega \frac{\dot{\kappa}(\pi_e)}{\mu_\varepsilon} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & \frac{\partial f_1}{\partial \varepsilon} & \omega \Delta \nu \kappa(\pi_e) & \frac{\partial f_1}{\partial \rho} \\ 0 & \frac{\partial f_2}{\partial \lambda} & \lambda \frac{\dot{\kappa}(\pi_e)}{\mu_\varepsilon \nu_1 \nu_2} & \frac{\partial f_2}{\partial \varepsilon} & 0 & \frac{\partial f_2}{\partial \rho} \\ \frac{\partial f_3}{\partial \omega} & \dot{\Gamma}(\lambda) - \omega \dot{\phi}(\lambda) & \frac{\partial f_3}{\partial \pi_e} & \frac{\partial f_3}{\partial \varepsilon} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \rho} \\ 0 & 0 & \frac{\dot{\kappa}(\pi_e)}{\Delta \nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & \frac{\partial f_4}{\partial \varepsilon} & \frac{\partial f_4}{\partial \theta^*} & \frac{\partial f_4}{\partial \rho} \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\theta^*)(\chi_E + \chi_c - \Delta \nu) & -\sigma\theta^*(1 - \theta^*) \frac{\sigma_\nu}{\nu_2 \rho} \\ 0 & 0 & \frac{(\theta_{RD} - \rho)\theta^* \dot{\kappa}(\pi_e)}{\nu_2(1 - \varepsilon)} & \frac{\partial f_6}{\partial \varepsilon} & \frac{\partial f_6}{\partial \theta^*} & \frac{\partial f_6}{\partial \rho} \end{pmatrix} \quad (40)$$

Where:

$$\begin{aligned}
\frac{\partial f_1}{\partial \omega} &= \phi(\lambda) - \alpha - (1 - \gamma)i - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \sigma_\nu \hat{\rho} \left(1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon}\right) - \omega(1 - \gamma)\eta_p m \\
\frac{\partial f_1}{\partial \varepsilon} &= \omega \left[ \frac{\kappa(\pi_e)\Delta\nu}{\mu_\varepsilon^2 \nu_1 \nu_2} + \sigma_\nu \hat{\rho} K_{\rho_1} \right] \quad \text{where} \quad K_{\rho_1} = \frac{\Delta\nu(1 - 2\varepsilon)}{\mu_\varepsilon} + (1 - \varepsilon)\varepsilon \left(\frac{\Delta\nu}{\mu_\varepsilon}\right)^2 + \left(\frac{1}{\nu_2 \mu_\varepsilon} + \frac{\Delta\nu \varepsilon}{\nu_2 \mu_\varepsilon^2}\right) \\
\frac{\partial f_1}{\partial \rho} &= \omega \sigma_\nu \frac{\hat{\rho}}{\rho} \left(1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon}\right) + \omega \frac{\sigma_\nu^2 \hat{\rho} \varepsilon}{\rho \nu_2 \mu_\varepsilon} \left(1 - \frac{1}{\mu_\varepsilon \nu_2}\right) - \omega \frac{\Delta\nu}{\mu_\varepsilon} \frac{\partial f_4}{\partial \rho} - \omega \dot{\varepsilon} \frac{\partial}{\partial \rho} \left(\frac{\Delta\nu}{\mu_\varepsilon}\right) \\
\frac{\partial f_2}{\partial \lambda} &= \frac{\kappa(\pi_e)}{\nu_\theta} - \delta - \alpha - n - \frac{\dot{\varepsilon}\Delta\nu}{\mu_\varepsilon} - \sigma_\nu \hat{\rho} \varepsilon \left(1 - \frac{1}{\nu_2 \mu_\varepsilon}\right) \\
\frac{\partial f_2}{\partial \varepsilon} &= \lambda \left[ \frac{\kappa(\pi_e)\Delta\nu}{\mu_\varepsilon^2 \nu_1 \nu_2} + \Delta\nu \sigma_\nu \hat{\rho} \frac{\mu_\varepsilon(1 - 2\varepsilon) + \varepsilon(1 - \varepsilon)\Delta\nu}{\mu_\varepsilon^2} - \sigma_\nu \hat{\rho} \left(1 - \frac{1}{\nu_2 \mu_\varepsilon} - \frac{\varepsilon \Delta\nu}{\nu_2 \mu_\varepsilon}\right) \right] \\
\frac{\partial f_2}{\partial \rho} &= \lambda \kappa(\pi_e) \frac{\theta \sigma_\nu}{\nu_2 \rho} - \lambda \dot{\varepsilon} \frac{\partial}{\partial \rho} \left(\frac{\Delta\nu}{\mu_\varepsilon}\right) - \lambda \frac{\Delta\nu}{\mu_\varepsilon} \frac{\partial f_4}{\partial \rho} + \lambda \sigma_\nu \varepsilon \frac{\hat{\rho}}{\rho} \left(1 - \frac{1}{\nu_2 \mu_\varepsilon}\right) + \lambda \frac{\sigma_\nu^2 \hat{\rho} \varepsilon}{\mu_\varepsilon \nu_2 \rho} \left(1 - \frac{\varepsilon}{\mu_\varepsilon \nu_2}\right) \\
\frac{\partial f_3}{\partial \omega} &= - \left( \phi(\lambda) - \alpha - (1 - \gamma)i - \sigma_\nu \hat{\rho} \left(1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon}\right) \right) + \omega \eta_p (1 - \gamma)m - \dot{\Theta}(\pi_f) \left[ 1 - y_d + (1 - \omega) \frac{\partial y_d}{\partial \omega} \right] \\
&\quad + \dot{\varepsilon} \frac{\Delta\nu}{\mu_\varepsilon} + \eta_p m (1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon) - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right) + r \left( 1 - y_d + (1 - \omega) \frac{\partial y_d}{\partial \omega} \right) \\
\frac{\partial f_3}{\partial \pi_e} &= - \frac{\partial \pi_f}{\partial \pi_e} \dot{\Theta}(\pi_f) + [1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon] \frac{\dot{\kappa}(\pi_e)}{\nu_\theta} - \left( \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right) \\
&\quad - r \left[ \dot{\kappa}(\pi_e) - 1 - (1 - \omega) \frac{\partial y_d}{\partial \pi_e} \right] + \left( \frac{\omega \Delta\nu}{\mu_\varepsilon} - (\chi_e + \chi_c)\nu_1 \right) \frac{\dot{\kappa}(\pi_e)}{\Delta\nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) \\
\frac{\partial f_3}{\partial \varepsilon} &= -\omega \sigma_\nu \hat{\rho} \frac{\mu_\varepsilon + \Delta\nu \varepsilon}{\nu_2 \mu_\varepsilon^2} + \dot{\varepsilon} \omega \left(\frac{\Delta\nu}{\mu_\varepsilon}\right)^2 + \left(\frac{\omega \Delta\nu}{\mu_\varepsilon} - (\chi_E + \chi_c)\nu_1\right) \left(-\frac{\kappa(\pi_e)}{\nu_\theta} - \sigma_\nu \hat{\rho} (1 - 2\varepsilon)\right) \\
&\quad - (\chi_E + \chi_c)\nu_1 \left[ \frac{\kappa(\pi_e)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right] - \sigma_\nu \hat{\rho} [1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon] \\
\frac{\partial f_3}{\partial \theta^*} &= - \left( \frac{\omega \Delta\nu}{\mu_\varepsilon} - (\chi_E + \chi_c)\nu_1 \right) \left[ -\kappa(\pi_e)\mu_\varepsilon - \sigma_\nu (1 - \varepsilon) \frac{\varepsilon}{\rho} \frac{\partial f_6}{\partial \theta^*} \right] \\
&\quad - (1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon) \left[ -\kappa(\pi_e)\Delta\nu - \frac{\sigma_\nu (1 - \varepsilon)}{\rho} \frac{\partial f_6}{\partial \theta^*} \right] \\
\frac{\partial f_3}{\partial \rho} &= -\omega \sigma_\nu \frac{\hat{\rho}}{\rho} \left(1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon}\right) - \omega \sigma_\nu \hat{\rho} \varepsilon \frac{\partial}{\partial \rho} \left(\frac{1}{\nu_2 \mu_\varepsilon}\right) + \dot{\varepsilon} \left( \omega \Delta\nu \frac{\sigma_\nu \varepsilon}{\nu_2 \rho \mu_\varepsilon^2} - \frac{\omega \sigma_\nu}{\mu_\varepsilon \nu_2 \rho} \right) + \left[ \omega \frac{\Delta\nu}{\mu_\varepsilon} - (\chi_E + \chi_c)\nu_1 \right] \frac{\partial f_4}{\partial \rho} \\
&\quad + [1 - \omega - \pi_e - (\chi_E + \chi_c)\nu_1 \varepsilon] \left( \kappa(\pi_e) \frac{\sigma_\nu \theta}{\nu_2 \rho} - \sigma_\nu \frac{\hat{\rho}}{\rho} (1 - \varepsilon) \right) \quad ; \quad \frac{\partial f_6}{\partial \theta^*} = \frac{(\theta_{RD} - \rho)\kappa(\pi_e)}{\nu_2 (1 - \varepsilon)} \\
\frac{\partial f_4}{\partial \varepsilon} &= -\frac{\kappa(\pi_e)}{\nu_\theta} - \sigma_\nu \hat{\rho} (1 - 2\varepsilon) \quad ; \quad \frac{\partial f_4}{\partial \theta^*} = -\kappa(\pi_e)\mu_\varepsilon - \sigma_\nu (1 - \varepsilon) \frac{\varepsilon}{\rho} \frac{\partial f_6}{\partial \theta^*} \\
\frac{\partial f_4}{\partial \rho} &= \frac{\kappa(\pi_e)}{\Delta\nu} \left( \frac{1}{\nu_\theta} \frac{\partial \mu_\varepsilon}{\partial \rho} + \frac{\mu_\varepsilon \sigma_\nu \theta}{\nu_2 \rho} - \frac{\sigma_\nu}{\nu_1 \nu_2 \rho} \right) + \frac{\kappa(\pi_e)}{\Delta\nu^2} \left( \frac{-\sigma_\nu}{\nu_2 \rho} \right) \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) + \sigma_\nu (1 - \varepsilon) \varepsilon \frac{\hat{\rho}}{\rho} \\
\frac{\partial f_6}{\partial \varepsilon} &= \frac{(\theta_{RD} - \rho)\theta^* \kappa(\pi_e) + \sigma_{RD} \chi_c \nu_1}{\nu_2 (1 - \varepsilon)^2} \quad ; \quad \frac{\partial f_6}{\partial \rho} = \frac{\theta^* \kappa(\pi_e)}{\nu_2 (1 - \varepsilon)} \left[ \left( \frac{\theta_{RD}}{\rho} - 1 \right) \sigma_\nu - 1 \right] \\
\frac{\partial \mu_\varepsilon}{\partial \rho} &= \frac{\varepsilon \sigma_\nu}{\nu_2 \rho} \quad ; \quad \frac{\partial 1/\nu_\theta}{\partial \rho} = \frac{\theta \sigma_\nu}{\nu_2 \rho} \quad ; \quad \frac{\partial \nu_2}{\partial \rho} = \frac{\partial \nu_2}{\partial t} \frac{\nu_2}{\nu_2} \frac{\rho}{\rho} \frac{\partial t}{\partial \rho} = \frac{\dot{\nu}_2}{\hat{\rho}} \frac{\nu_2}{\rho} = -\sigma_\nu \frac{\nu_2}{\rho}
\end{aligned} \tag{41}$$



### C.2.2 Analysis of the different equilibria

With the same set of assumptions as before and with the parameter's value of our Leading Example, we study the stability of the two locally stable equilibria of the previous model: the Solovian-like and the Debt-deflationary Equilibrium. We let to an other study, the possibility to analyse other meaningful equilibria.

#### The Solovian-like Equilibrium

This Solovian-like equilibrium corresponds to a desirable long-run steady state with positive wages and employment, and a finite level of debt  $(\omega_1, \lambda_1, \pi_{e_1})$ . This equilibrium may be green or brown according to the asymptotic value of  $(\varepsilon, \theta)$ .

At the Green Solovian-like Equilibrium,  $\theta = 1$  and  $\varepsilon = 0$ :  $\rho$  has to be equal to  $\rho_1 := \theta_{RD}$  in order to ensure  $\dot{\rho} = 0$ . At the Brown Solovian-like Equilibrium ( $\theta = 0$  and  $\varepsilon = 1$ ), one can use the change of variable  $\rho = 1/\psi$  to show that  $\rho \rightarrow +\infty$ . We study now the stability of these two equilibria.

At the Green Solovian-like Equilibrium, the Jacobian matrix becomes is the same as the 5x5 matrix of the appendix C.1.4,  $J_1$ , with the following sixth line and column:

$$\left( \begin{array}{cccccc}
 & & & & & 0 \\
 & & & & & \lambda_1 \frac{\kappa(\pi_{e_1})\sigma_\nu}{\bar{\nu}_2\rho_1} \\
 & & & & & \\
 J_1 & & & & & (1 - \omega_1 - \pi_{e_1}) \frac{\kappa(\pi_{e_1})\sigma_\nu}{\bar{\nu}_2\rho_1} \\
 & & & & & 0 \\
 & & & & & 0 \\
 0 & 0 & 0 & \sigma_{RD}\chi_c \frac{\nu_1}{\bar{\nu}_2} & 0 & -\frac{\kappa(\pi_{e_1})}{\bar{\nu}_2}
 \end{array} \right) \tag{42}$$

which can be transformed into an upper triangular matrix.

The characteristic polynomial of this matrix admits a new negative root,  $-\frac{\kappa(\pi_{e_1})}{\bar{\nu}_2}$ , which means that the Green Solovian-like Equilibrium is still locally asymptotically stable.

At the Brown Solovian-like Equilibrium,  $(\omega, \lambda, \pi_e, \varepsilon, \theta^*, \psi) = (\omega_1, \lambda_1, \pi_{e_1}, 1, 0, 0)$ , with the change of variable  $\psi = 1/\rho$ , the last equation of the system becomes:  $\dot{\psi} = -\frac{\psi(\theta_{RD}\psi-1)\theta^*\kappa(\pi_e)+\psi^2\sigma_{RD}\tau_c}{\nu_2(1-\varepsilon)}$ .

Consequently the Jacobian matrix for the modified system becomes:

$$\left( \begin{array}{cccccc} \frac{\partial f_1^*}{\partial \omega} & \omega \dot{\phi}(\lambda) & -\omega \frac{\dot{\kappa}(\pi_e)}{\mu_\varepsilon} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & \frac{\partial f_1^*}{\partial \varepsilon} & \omega \Delta \nu \kappa(\pi_e) & \frac{\partial f_1^*}{\partial \psi} \\ 0 & \frac{\partial f_2^*}{\partial \lambda} & \lambda \frac{\dot{\kappa}(\pi_e)}{\mu_\varepsilon \nu_1 \nu_2} & \frac{\partial f_2^*}{\partial \varepsilon} & 0 & \frac{\partial f_2^*}{\partial \psi} \\ \frac{\partial f_3^*}{\partial \omega} & \dot{\Gamma}(\lambda) - \omega \dot{\phi}(\lambda) & \frac{\partial f_3^*}{\partial \pi_e} & \frac{\partial f_3^*}{\partial \varepsilon} & \frac{\partial f_3^*}{\partial \theta} & \frac{\partial f_3^*}{\partial \psi} \\ 0 & 0 & \frac{\dot{\kappa}(\pi_e)}{\Delta \nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & \frac{\partial f_4^*}{\partial \varepsilon} & \frac{\partial f_4^*}{\partial \theta^*} & \frac{\partial f_4^*}{\partial \psi} \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\theta^*)(\chi_E + \chi_c - \Delta \nu) & \sigma \theta^*(1 - \theta^*) \frac{\sigma_\nu}{\nu_2 \psi} \\ 0 & 0 & -\frac{\psi(\theta_{RD}\psi - 1)\theta^* \dot{\kappa}(\pi_e)}{\nu_2(1 - \varepsilon)} & -\frac{\psi^2 \sigma_{RD} \chi_c \nu_1}{\nu_2(1 - \varepsilon)^2} & -\frac{\psi(\theta_{RD}\psi - 1)\kappa(\pi_e)}{\nu_2(1 - \varepsilon)} & \frac{\partial f_6^*}{\partial \psi} \end{array} \right) \quad (43)$$

In order to analyse the local stability of this brown Solovian like equilibrium (where  $\rho$  tends to  $+\infty$ ), let us endorse Assumption **C**:  $\dot{\rho}$  remains bounded from above as  $t \rightarrow +\infty$ . This implies that:

$$\frac{1}{\nu_2(1 - \varepsilon)} = \frac{\dot{\rho}}{(\theta_{RD} - \rho)\theta^* \kappa(\pi_e) + \sigma_{RD} \tau_c}. \quad (44)$$

Consequently, each term of the last line of the Jacobian matrix (43) tends to 0. Indeed:<sup>12</sup>

$$\begin{aligned} -\frac{\psi(\theta_{RD}\psi - 1)\theta^* \dot{\kappa}(\pi_e)}{\nu_2(1 - \varepsilon)} &= -\frac{\psi(\theta_{RD}\psi - 1)\theta^* \dot{\kappa}(\pi_e) \dot{\rho}}{(\theta_{RD} - \rho)\theta^* \kappa(\pi_e) + \sigma_{RD} \tau_c} \rightarrow 0 \\ -\frac{\psi^2 \sigma_{RD} \chi_c \nu_1}{\nu_2(1 - \varepsilon)^2} &= -\frac{\psi^2 \sigma_{RD} \chi_c \nu_1 \dot{\rho}^2}{[(\theta_{RD} - \rho)\theta^* \kappa(\pi_e) + \sigma_{RD} \tau_c]^2} \rightarrow 0 \\ -\frac{\psi(\theta_{RD}\psi - 1)\kappa(\pi_e)}{\nu_2(1 - \varepsilon)} &= -\frac{\psi(\theta_{RD}\psi - 1)\kappa(\pi_e) \dot{\rho}}{(\theta_{RD} - \rho)\theta^* \kappa(\pi_e) + \sigma_{RD} \tau_c} \rightarrow 0 \\ \frac{\partial f_6^*}{\partial \psi} &= -\frac{[(2 - \sigma_\nu)\psi \theta_{RD} - (1 - \sigma_\nu)]\theta^* \kappa(\pi_e) + (2 - \sigma_\nu)\psi \sigma_{RD} \tau_c}{\nu_2(1 - \varepsilon)} \\ &= -\frac{[(2 - \sigma_\nu)\psi \theta_{RD} - (1 - \sigma_\nu)]\theta^* \kappa(\pi_e) + (2 - \sigma_\nu)\psi \sigma_{RD} \tau_c \dot{\rho}}{(\theta_{RD} - \rho)\theta^* \kappa(\pi_e) + \sigma_{RD} \tau_c} \rightarrow 0 \end{aligned} \quad (45)$$

This means that the Jacobian matrix (43) can be turned into an upper triangular matrix of which the characteristic polynomial admits a non-negative root. Consequently, the dirty Solovian-like equilibrium is locally asymptotically unstable. Our numerical Example illustrates how the introduction of green R&D destabilizes the brown Solovian-like steady state (see Figure 4.3b: green transition does occur — with high private debt levels and much more slowly but still).

<sup>12</sup>Note that  $\sigma_{RD} \tau_c$  tends to a constant:  $\sigma_{RD} \chi_c \nu_1$ .

### Debt-deflationary Equilibrium

As before, to study the debt-deflationary Equilibrium  $(0; 0; -\infty)$ , let us make the change of variable  $\pi_e = \frac{1}{q}$ . The new system is the same as before except for the  $\pi_e$ -equation:

$$\dot{q} = -q^2 \left[ -\omega \left( \phi(\lambda) - (1 - \gamma)i - \alpha - \sigma_\nu \hat{\rho} \left( 1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon} \right) \right) + \Gamma(\lambda) - \Theta(\pi_f) + \dot{\varepsilon} \left( \omega \frac{\Delta \nu}{\mu_\varepsilon} - (\chi_E + \chi_c) \nu_1 \right) \right. \\ \left. + \left( 1 - \omega - \frac{1}{q} - (\chi_E + \chi_c) \nu_1 \varepsilon \right) \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right) - r \left( \kappa(1/q) - \frac{1}{q} + (1 - \omega)(1 - y_d) \right) \right]. \quad (46)$$

The Jacobian matrix associated to this new system is:

$$\begin{pmatrix} \frac{\partial f_1}{\partial \omega} & \omega \dot{\phi}(\lambda) & \frac{\omega}{-q^2} \frac{\dot{\kappa}(1/q)}{\mu_\varepsilon} \left( \frac{1}{\nu_1 \nu_2} - \frac{\mu_\varepsilon}{\nu_\theta} \right) & \omega \frac{\Delta \nu}{\mu_\varepsilon^2} \frac{\kappa(1/q)}{\nu_1 \nu_2} & \omega \Delta \nu \kappa(1/q) & \frac{\partial f_1}{\partial \rho} \\ 0 & \frac{\partial f_2}{\partial \lambda} & \frac{\lambda}{-q^2} \frac{\dot{\kappa}(1/q)}{\mu_\varepsilon \nu_1 \nu_2} & \lambda \frac{\Delta \nu}{\mu_\varepsilon^2} \frac{\kappa(1/q)}{\nu_1 \nu_2} & 0 & \frac{\partial f_2}{\partial \rho} \\ \frac{\partial f_3}{\partial \omega} & -q^2 (\dot{\Gamma}(\lambda) - \omega \dot{\phi}(\lambda)) & \frac{\partial f_3}{\partial q} & \frac{\partial f_3}{\partial \varepsilon} & \frac{\partial f_3}{\partial \theta} & \frac{\partial f_3}{\partial \rho} \\ 0 & 0 & \frac{\dot{\kappa}(1/q)}{-q^2 \Delta \nu} \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1 \nu_2} \right) & -\frac{\kappa(1/q)}{\nu_\theta} & -\kappa(1/q) \mu_\varepsilon & \frac{\partial f_4}{\partial \rho} \\ 0 & 0 & 0 & 0 & \sigma(1 - 2\theta^*)(\chi_E + \chi_c - \Delta \nu) & \frac{\partial f_5}{\partial \rho} \\ 0 & 0 & \frac{(\theta_{RD} - \rho) \theta^* \dot{\kappa}(\pi_e)}{\nu_2 (1 - \varepsilon)} & \frac{\partial f_6}{\partial \varepsilon} & \frac{\partial f_6}{\partial \theta^*} & \frac{\partial f_6}{\partial \rho} \end{pmatrix} \quad (47)$$

Where:

$$\begin{aligned} \frac{\partial f_3}{\partial \omega} &= -q^2 \left[ -\left( \phi(\lambda) - \alpha - (1 - \gamma)i - \frac{\dot{\varepsilon} \Delta \nu}{\mu_\varepsilon} - \sigma_\nu \hat{\rho} \left( 1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon} \right) \right) + \omega \eta_p (1 - \gamma) m - \frac{\partial \pi_f}{\partial \omega} \dot{\Theta}(\pi_f) \right. \\ &\quad \left. + \left( 1 - \omega - \frac{1}{q} - (\chi_E + \chi_c) \nu_1 \varepsilon \right) \eta_p m - \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right) + r \left( 1 - y_d + (1 - \omega) \frac{\partial y_d}{\partial \omega} \right) \right] \\ \frac{\partial f_3}{\partial q} &= -q^2 \left[ -\frac{\partial \pi_f}{\partial q} \dot{\Theta}(\pi_f) + \left( 1 - \omega - \frac{1}{q} - (\chi_E + \chi_c) \nu_1 \varepsilon \right) \frac{\dot{\kappa}(1/q)}{-q^2 \nu_\theta} + \frac{1}{q^2} \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + \sigma_\nu \hat{\rho} (1 - \varepsilon) + i \right) \right. \\ &\quad \left. - r \left( -\frac{\dot{\kappa}(1/q)}{q^2} + \frac{1}{q^2} - (1 - \omega) \frac{\partial y_d}{\partial q} \right) \right] - 2q \left[ -\omega (\phi(\lambda) - (1 - \gamma)i - \alpha) + \Gamma(\lambda) - \Theta(\pi_f) \right. \\ &\quad \left. + \left[ 1 - \omega - \frac{1}{q} - p_E \nu_1 \varepsilon \right] \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) - r \left[ \kappa(1/q) - \frac{1}{q} + (1 - \omega)(1 - y_d) \right] + \dot{\varepsilon} \left( \omega \frac{\Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \right] \\ \frac{\partial f_3}{\partial \varepsilon} &= -q^2 \left[ -p_E \nu_1 \left( \frac{\kappa(1/q)}{\nu_\theta} - \delta + i \right) + \dot{\varepsilon} \omega \left( \frac{\Delta \nu}{\mu_\varepsilon} \right)^2 - \left( \frac{\omega \Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \frac{\kappa(1/q)}{\nu_\theta} \right] \\ \frac{\partial f_3}{\partial \theta} &= -q^2 \left[ -\left( \frac{\omega \Delta \nu}{\mu_\varepsilon} - p_E \nu_1 \right) \kappa(1/q) \mu_\varepsilon - \left( 1 - \omega - \frac{1}{q} - \chi_E \nu_1 \varepsilon \right) \kappa(1/q) \Delta \nu \right] \\ \frac{\partial f_3}{\partial \rho} &= -q^2 \left[ -\omega \sigma_\nu \frac{\hat{\rho}}{\rho} \left( 1 - \frac{\varepsilon}{\nu_2 \mu_\varepsilon} \right) - \omega \sigma_\nu \hat{\rho} \varepsilon \frac{\partial}{\partial \rho} \left( \frac{1}{\nu_2 \mu_\varepsilon} \right) + \dot{\varepsilon} \left( \omega \Delta \nu \frac{\sigma_\nu \varepsilon}{\nu_2 \rho \mu_\varepsilon^2} - \frac{\omega \sigma_\nu}{\mu_\varepsilon \nu_2 \rho} \right) + \left[ \omega \frac{\Delta \nu}{\mu_\varepsilon} - (\chi_E + \chi_c) \nu_1 \right] \frac{\partial f_4}{\partial \rho} \right. \\ &\quad \left. + \left[ 1 - \omega - \frac{1}{q} - (\chi_E + \chi_c) \nu_1 \varepsilon \right] \left( \kappa(1/q) \frac{\sigma_\nu \theta}{\nu_2 \rho} - \sigma_\nu \frac{\hat{\rho}}{\rho} (1 - \varepsilon) \right) \right] \end{aligned} \quad (48)$$

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Consequently, at the equilibrium  $(\omega, \lambda, q) = (0, 0, 0)$ , the Jacobian Matrix becomes:

$$\begin{pmatrix} \phi(0) - \alpha + (1 - \gamma)\eta_p & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{k_0}{\nu_\theta} - \alpha - n - \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_0}{\nu_\theta} - \delta - \eta_p - r & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{k_0}{\nu_\theta} & \frac{\partial f_4}{\partial \theta} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} & \frac{\partial f_4}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} \\ 0 & 0 & 0 & 0 & \frac{\partial f_5}{\partial \theta} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} & \frac{\partial f_5}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} \\ 0 & 0 & 0 & \frac{\partial f_6}{\partial \varepsilon} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} & \frac{(\theta_{RD} - \bar{\rho})k_0}{\bar{\nu}_2(1 - \bar{\varepsilon})} & \frac{\partial f_6}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} \end{pmatrix} \quad (49)$$

with

$$\begin{aligned} \frac{\partial f_4}{\partial \theta} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &= -k_0\mu_\varepsilon - \sigma_\nu(1 - \bar{\varepsilon})\frac{\bar{\varepsilon}}{\bar{\rho}}\frac{(\theta_{RD} - \bar{\rho})k_0}{\bar{\nu}_2(1 - \bar{\varepsilon})} \\ \frac{\partial f_4}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &= \frac{k_0}{\Delta\nu} \left( \frac{\bar{\varepsilon}\sigma_\nu}{\nu_\theta\bar{\nu}_2\bar{\rho}} + \frac{\mu_\varepsilon\sigma_\nu\bar{\theta}}{\bar{\nu}_2\bar{\rho}} - \frac{\sigma_\nu}{\nu_1\bar{\nu}_2\bar{\rho}} \right) + \frac{k_0}{\Delta\nu^2} \left( \frac{-\sigma_\nu}{\bar{\nu}_2\bar{\rho}} \right) \left( \frac{\mu_\varepsilon}{\nu_\theta} - \frac{1}{\nu_1\bar{\nu}_2} \right) \\ \frac{\partial f_5}{\partial \theta} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &= \sigma(1 - 2\bar{\theta})(\chi_E + \chi_c - \Delta\nu) \\ \frac{\partial f_5}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &= -\sigma\bar{\theta}^*(1 - \bar{\theta}^*)\frac{\sigma_\nu}{\bar{\nu}_2\bar{\rho}} \\ \frac{\partial f_6}{\partial \varepsilon} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &= \frac{(\theta_{RD} - \bar{\rho})\bar{\theta}^*k_0 + \sigma_{RD}\chi_c\nu_1}{\bar{\nu}_2(1 - \bar{\varepsilon})^2} \\ \frac{\partial f_6}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &= \frac{\bar{\theta}^*k_0}{\bar{\nu}_2(1 - \bar{\varepsilon})} \left[ \left( \frac{\theta_{RD}}{\bar{\rho}} - 1 \right) \sigma_\nu - 1 \right] \end{aligned} \quad (50)$$

where, whether  $\varepsilon \rightarrow 1$  and  $\theta \rightarrow 0$  (brown equilibrium), or  $\varepsilon \rightarrow 0$  and  $\theta \rightarrow 1$  (green):

$$\begin{aligned} \frac{\partial f_4}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &\rightarrow 0 \\ \frac{\partial f_5}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &\rightarrow 0 \end{aligned} \quad (51)$$

Consequently, the Jacobian Matrix of the modified system can be turned into an upper triangular

matrix which is stable if and only if:

$$\begin{aligned}
 \phi(0) - \alpha + (1 - \gamma)\eta_p &< 0 \\
 \frac{k_0}{\nu_\theta} - \alpha - n - \delta &< 0 \\
 \frac{k_0}{\nu_\theta} - \delta - \eta_p - r &< 0 \\
 -\frac{k_0}{\nu_\theta} &< 0 \\
 \sigma(1 - 2\bar{\theta})(\chi_E + \chi_c - \Delta\nu) &< 0 \\
 \frac{\partial f_6}{\partial \rho} \Big|_{(0,0,0,\bar{\varepsilon},\bar{\theta},\bar{\rho})} &< 0
 \end{aligned} \tag{52}$$

The condition  $-\frac{k_0}{\nu_1} < 0$  is not verified with our aggregate investment function satisfying assumptions (9) (ii) and thus, the debt-deflationary equilibria, be it brown or green, is still locally unstable.<sup>13</sup>

### Hybrid Equilibrium

Due to the introduction of a variable  $\nu_2$ , we can now study another type of equilibrium where  $\nu_2$  is a solution of  $\dot{\theta} = 0$ . Indeed,  $\frac{1}{\bar{\nu}_2} = \frac{1}{\nu_1} - \chi_E - \chi_c \Rightarrow \dot{\theta} = 0$  and so  $\bar{\nu}_2$  is solution of the fifth equation of (38).

Because  $\hat{\nu}_2 = -\sigma_\nu \hat{\rho}$ , if  $\nu_2$  tends to  $\bar{\nu}_2$ , consequently  $\rho$  tends to  $\rho_3$ , which has no reason to be equal to  $\rho_1 = \theta_{RD}$  (like it was the case in the Solovian-like Equilibrium).

As a consequence, if  $(\varepsilon, \theta) = (\varepsilon_3, \theta_3)$  such that:

$$\begin{cases} \frac{\mu_{\varepsilon_3}}{\nu_{\theta_3}} = \frac{1}{\nu_1 \bar{\nu}_2} \\ (\theta_{RD} - \rho_3)\theta_3 \kappa(\pi_{\varepsilon_3}) + \sigma_{RD} \chi_c \nu_1 \varepsilon_3 = 0 \end{cases} \tag{53}$$

with  $\pi_{\varepsilon_3} = \kappa^{-1}(\nu_{\theta_3}(\alpha + n + \delta))$ ,  $\lambda_3 = \phi^{-1}(\alpha + (1 - \gamma)i_3)$ , and  $\omega_3$ , solution of the third equation of (38), consequently  $(\omega_3, \lambda_3, \pi_{\varepsilon_3}, \varepsilon_3, \theta_3, \rho_3)$  is an equilibrium for (38).

The Jacobian matrix becomes:

$$J_3 = \begin{pmatrix} J_{11} & J_{12} & 0 & & \omega_3 \Delta \nu \kappa(\pi_{\varepsilon_3}) & J_{16} \\ 0 & 0 & J_{23} & J_{24} & 0 & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ 0 & 0 & 0 & J_{44} & J_{45} & J_{46} \\ 0 & 0 & 0 & 0 & 0 & J_{56} \\ 0 & 0 & J_{63} & J_{64} & J_{65} & J_{66} \end{pmatrix} \tag{54}$$

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<sup>13</sup>Indeed, assuming  $\lim_{\pi_e \rightarrow 0} \kappa(\pi_e) = 0$  implies that  $k_0 < 0$ .

Where:

$$\begin{aligned}
 J_{11} &= -\omega_3(1-\gamma)\eta_p m & J_{12} &= \omega_3 \dot{\phi}(\lambda_3) & J_{14} &= \omega_3 K_\varepsilon & \text{with } K_\varepsilon &= \frac{\kappa(\pi_{e_3})\Delta\nu}{\mu_{\varepsilon_3}^2 \nu_1 \bar{\nu}_2} & J_{16} &= -\omega_3 \frac{\Delta\nu}{\mu_{\varepsilon_3}} J_{46} \\
 J_{23} &= \lambda_3 \frac{\dot{\kappa}(\pi_{e_3})}{\mu_{\varepsilon_3} \nu_1 \bar{\nu}_2} & J_{24} &= \lambda_3 K_\varepsilon & J_{26} &= \lambda_3 \kappa(\pi_{e_3}) \frac{\theta_3 \sigma_\nu}{\bar{\nu}_2 \rho_3} - \lambda_3 \frac{\Delta\nu}{\mu_{\varepsilon_3}} J_{46} \\
 & \text{with } J_{46} &= \frac{\sigma_\nu \kappa(\pi_{e_3})}{\Delta\nu \bar{\nu}_2 \rho_3} \left[ \frac{\varepsilon_3}{\nu \theta_3} + \mu_{\varepsilon_3} \theta_3 - \frac{1}{\nu_1} \right] \\
 J_{36} &= \left[ \omega_3 \frac{\Delta\nu}{\mu_{\varepsilon_3}} - (\chi_E + \chi_c) \nu_1 \right] J_{46} + (1 - \omega_3 - \pi_{e_3} - (\chi_E + \chi_c) \nu_1 \varepsilon_3) \kappa(\pi_{e_3}) \frac{\sigma_\nu \theta_3}{\bar{\nu}_2 \rho_3} & J_{56} &= -\sigma \theta_3 (1 - \theta_3) \frac{\sigma_\nu}{\bar{\nu}_2 \rho_3} \\
 J_{31} &= \omega_3 \eta_p m (1 - \gamma) - \dot{\Theta}(\pi_{f_3}) \frac{\partial \pi_f}{\partial \omega} \Big|_{(\omega_3, \lambda_3, \pi_{e_3}, \varepsilon_3, \theta_3, \rho_3)} + K_d \eta_p m - (\alpha + n + i_3) \\
 & \quad + r \left( 1 - y_{d_3} + (1 - \omega_3) \frac{\partial y_d}{\partial \omega} \Big|_{(\omega_3, \lambda_3, \pi_{e_3}, \varepsilon_3, \theta_3, \rho_3)} \right) \\
 K_d &= 1 - \omega_3 - \pi_{e_3} - \chi_E \nu_1 \varepsilon_3 & J_{32} &= \dot{\Gamma}(\lambda_3) - J_{12} & J_{66} &= \frac{\theta_3 \kappa(\pi_{e_3})}{\bar{\nu}_2 (1 - \varepsilon_3)} \left[ \left( \frac{\theta_{RD}}{\rho_3} - 1 \right) \sigma_\nu - 1 \right] \\
 J_{33} &= \dot{\Theta}(\pi_{f_3}) \frac{\partial \pi_f}{\partial \pi_e} \Big|_{(\omega_3, \lambda_3, \pi_{e_3}, \varepsilon_3, \theta_3, \rho_3)} + K_d \frac{\dot{\kappa}(\pi_{e_3})}{\nu \theta_3} - (\alpha + n + i_3) - r \left[ \dot{\kappa}(\pi_{e_3}) - 1 - (1 - \omega_3) \frac{\partial y_d}{\partial \pi_e} \Big|_{(\omega_3, \lambda_3, \pi_{e_3}, \varepsilon_3, \theta_3, \rho_3)} \right] \\
 J_{34} &= (\chi_E + \chi_c) \nu_1 (\delta - i_3) - \left( \frac{\omega_3 \Delta\nu}{\mu_{\varepsilon_3}} \right) (\alpha + n + \delta) & J_{35} &= - \left( \frac{\omega_3 \Delta\nu}{\mu_{\varepsilon_3}} - (\chi_E + \chi_c) \nu_1 \right) J_{45} \\
 J_{44} &= -(\alpha + n + \delta) & J_{45} &= -\kappa(\pi_{e_3}) \mu_{\varepsilon_3} + \sigma_\nu (1 - \varepsilon_3) \frac{\varepsilon_3}{\rho_3} J_{65} \\
 J_{63} &= \frac{(\theta_{RD} - \rho_3) \theta_3^* \dot{\kappa}(\pi_{e_3})}{\bar{\nu}_2 (1 - \varepsilon_3)} & J_{64} &= \frac{\sigma_{RD} \chi_c \nu_1}{\bar{\nu}_2 (1 - \varepsilon_3)} & J_{65} &= \frac{(\theta_{RD} - \rho_3) \kappa(\pi_{e_3})}{\bar{\nu}_2 (1 - \varepsilon_3)}
 \end{aligned} \tag{55}$$

The characteristic polynomial of this matrix is:  $P(X) = a_6 X^6 + a_5 X^5 + a_4 X^4 + a_3 X^3 + a_2 X^2 + a_1 X + a_0$  with:

$$\begin{aligned}
 a_6 &= 1 \\
 a_5 &= -(J_{11} + J_{33} + J_{44} + J_{66}) \\
 a_4 &= J_{11}(J_{33} + J_{44} + J_{66}) + J_{33}(J_{44} + J_{66}) - J_{56}J_{65} + J_{44}J_{66} - J_{46}J_{64} - J_{23}J_{32} \\
 a_3 &= -J_{11}[J_{33}(J_{44} + J_{66}) - J_{56}J_{65} + J_{44}J_{66} - J_{46}J_{64}] + J_{44}J_{56}J_{65} - J_{64}J_{45}J_{56} - J_{31}J_{12}J_{23} - J_{16}J_{31}J_{63} \\
 & \quad + J_{33}[J_{56}J_{65} - J_{44}J_{66} + J_{64}J_{46}] - J_{32}J_{63}J_{26} + J_{32}J_{11}J_{23} + J_{32}J_{23}(J_{44} + J_{66}) \\
 a_2 &= -J_{11}[J_{44}J_{56}J_{65} - J_{64}J_{45}J_{56} + J_{33}(J_{56}J_{65} - J_{44}J_{66} + J_{64}J_{46})] - J_{33}(J_{44}J_{56}J_{65} + J_{64}J_{45}J_{56}) \\
 & \quad - J_{32}J_{11}[J_{23}(J_{44} + J_{66}) - J_{63}J_{26}] + J_{32}[J_{23}(J_{56}J_{65} - J_{66}J_{44} + J_{46}J_{64}) + J_{63}J_{26}J_{44}] \\
 & \quad + J_{31}J_{12}J_{23}(J_{44} + J_{66}) + J_{24}J_{31}J_{12}J_{63} + J_{31}J_{63}(J_{14}J_{46} + J_{44}J_{16} + \omega_3 \Delta\nu \kappa(\pi_{e_3}) J_{56}) \\
 a_1 &= J_{11}J_{33}(J_{44}J_{56}J_{65} + J_{64}J_{45}J_{56}) - J_{32}J_{11}[J_{23}(J_{56}J_{65} - J_{44}J_{66} + J_{64}J_{46}) - J_{63}J_{24}J_{46} + J_{65}J_{26}J_{44}] \\
 & \quad - J_{32}J_{23}J_{44}J_{56}J_{65} + J_{32}J_{23}J_{64}J_{45}J_{56} + J_{32}J_{63}J_{24}J_{45}J_{56} + J_{31}J_{12}J_{23}(J_{56}J_{65} - J_{44}J_{66} + J_{46}J_{64}) \\
 & \quad - J_{31}J_{12}J_{63}(J_{24}J_{46} + J_{26}J_{44}) + J_{31}J_{63}(J_{14}J_{45}J_{56} + J_{44}J_{56}\omega_3\Delta\nu\kappa(\pi_{e_3})) \\
 a_0 &= -J_{32}J_{11}[-J_{23}J_{44}J_{56}J_{65} + J_{23}J_{64}J_{45}J_{56} + J_{63}J_{24}J_{45}J_{56}] + J_{31}J_{12}J_{23}(-J_{44}J_{56}J_{65} + J_{64}J_{45}J_{56}) \\
 & \quad - J_{31}J_{12}J_{63}J_{24}J_{45}J_{56}.
 \end{aligned} \tag{56}$$

The stability of the equilibrium is guaranteed by the Routh-Hurwitz's criterion if and only if:

$$\begin{aligned}
 & a_6 \geq 0 \\
 & a_5 \geq 0 \\
 & b_1 = \frac{a_5 a_4 - a_3 a_6}{a_5} \geq 0 \\
 & c_1 = \frac{b_1 a_3 - a_5 b_2}{b_1} \geq 0 \quad \text{where } b_2 = \frac{a_3 a_2 - a_1 a_4}{a_3} \\
 & d_1 = \frac{c_1 b_2 - c_2 b_1}{c_1} > 0 \quad \text{where } c_2 = \frac{b_2 a_1 - a_0 a_3}{b_2}.
 \end{aligned} \tag{57}$$

These conditions are satisfied for a wide range of parameters and, in particular, by the parameterization of our Leading Example.

# Bibliography

- [Abel, 2013] Abel, W. (2013). *Agricultural Fluctuations in Europe: From the Thirteenth to twentieth centuries*. Routledge.
- [Acemoglu et al., 2012] Acemoglu, D., Aghion, P., Bursztyn, L., and Hemous, D. (2012). The environment and directed technical change. *American Economic Review*, 102(1):131–66.
- [Adjemian et al., 2011] Adjemian, S., Bastani, H., Juillard, M., Mihoubi, F., Perendia, G., Ratto, M., and Villemot, S. (2011). Dynare: Reference manual, version 4.
- [Akerlof and Stiglitz, 1969] Akerlof, G. and Stiglitz, J. E. (1969). Capital, Wages and Structural Unemployment. *Economic Journal*, 1969, vol. 79, issue 314, 269-81.
- [Alex Mara and Rakoto Joseph, 2008] Alex Mara, T. and Rakoto Joseph, O. (2008). Comparison of some efficient methods to evaluate the main effect of computer model factors. *Journal of Statistical Computation and Simulation*, 78(2):167–178.
- [An and Schorfheide, 2007] An, S. and Schorfheide, F. (2007). Bayesian analysis of dsge models. *Econometric Reviews*, 26(2-4):113–172.
- [Anderson et al., 2014] Anderson, B., Borgonovo, E., Galeotti, M., and Roson, R. (2014). Uncertainty in climate change modeling: Can global sensitivity analysis be of help? *Risk Analysis*, Vol. 34, No. 2, DOI: 10.1111/risa.12117.
- [Annicchiarico, 2010] Annicchiarico, B. (2010). Microfoundations of dsge models: Lecture.
- [Arrow et al., 2004] Arrow, K., Dasgupta, P., Goulder, L., Daily, G., Ehrlich, P., Heal, G., Levin, S., Mäler, K.-G., Schneider, S., Starrett, D., et al. (2004). Are we consuming too much? *Journal of Economic Perspectives*, 18(3):147–172.
- [Arrow, 1999] Arrow, K. J. (1999). Amartya k. sen’s contributions to the study of social welfare. *The Scandinavian Journal of Economics*, 101(2):163–172.
- [Aruoba et al., 2006] Aruoba, S. B. B., Fernández-Villaverde, J., and Rubio-Ramírez, J. F. (2006). Comparing solution methods for dynamic equilibrium economies. *Journal of Economic Dynamics and Control* 30 (12), 2477-2508.
- [Athanasoglou, 2014] Athanasoglou, S. (2014). Multidimensional welfare rankings under weight imprecision: a social choice perspective. *Social Choice and Welfare* 44, 719-744.
- [Auder et al., 2010] Auder, B., Iooss, B., and Marquès, M. (2010). Computer experiments with functional outputs: Global sensitivity analysis and metamodeling. *Procedia - Social and Behavioral Sciences*, 2(6):7583 – 7584.



- [Augier, 2018] Augier, S. (2018). Finding basin of attraction: Different algorithms. *Work in progress, Agence Française de développement*.
- [Ayres and Warr, 2005] Ayres, R. U. and Warr, B. (2005). Accounting for growth: the role of physical work. *Structural Change and Economic Dynamics*, 16(2):181–209.
- [Bank of England, 2020] Bank of England, . (2020). Monetary policy report. Technical report, Bank of England.
- [Bardaji et al., 2017] Bardaji, J., Campagne, B., Khder, M.-B., Lafféter, Q., Simon, O., Dufernez, A. S., Elezaar, C., Leblanc, P., Masson, E., and Partouche, H. (2017). The MESANGE macroeconometric model: re-estimation and innovations. Technical report, Institut National de la Statistique et des Etudes Economiques.
- [Barnosky et al., 2012] Barnosky, A. D., Hadly, E. A., Bascompte, J., Berlow, E. L., Brown, J. H., Fortelius, M., Getz, W. M., Harte, J., Hastings, A., Marquet, P. A., et al. (2012). Approaching a state shift in earth’s biosphere. *Nature*, 486(7401):52.
- [Bastidas et al., 2019] Bastidas, D., Fabre, A., and Mc Isaac, F. (2019). Minskyan classical growth cycles: stability analysis of a stock-flow consistent macrodynamic model. *Mathematics and Financial Economics*, 13(3):359–391.
- [Bastidas et al., 2017] Bastidas, D., Giraud, G., and McIsaac, F. (2017). Growth, underemployment, capital and international trades in brazil: a multisectorial, monetary and stock-flow consistent analysis. *Workingpaper, AFD*.
- [Beckerman et al., 2007] Beckerman, W., Hepburn, C., et al. (2007). Ethics of the discount rate in the stern review on the economics of climate change. *World Economics-Henley on Thames-*, 8(1):187.
- [Bell, 1988] Bell, D. E. (1988). *Decision making: Descriptive, normative, and prescriptive interactions*. cambridge university Press.
- [Bellman, 1957] Bellman, R. E. (1957). Dynamic programming. *Rand Corporation. Princeton University Press. ISBN 978-0-691-07951-6*.
- [Berveiller, 2005] Berveiller, M. (2005). *Stochastic finite element analysis: intrusive and non-intrusive approaches for reliability analysis*. Theses, Université Blaise Pascal - Clermont-Ferrand II.
- [Beven, 1993] Beven, K. (1993). Prophecy, reality and uncertainty in distributed hydrological modelling. *Advances in water resources*, 16(1):41–51.
- [Beven and Beven, 2001] Beven, K. J. and Beven, K. (2001). *Rainfall-runoff modelling*. Wiley Online Library.
- [Bjornland, 2000] Bjornland, H. C. (2000). Var models in macroeconomic research. *Statistics Norway*, 14:1–29.
- [Blanchard, 2009] Blanchard, O. (2009). The state of macro. *Annu. Rev. Econ.*, 1(1):209–228.
- [Blanchard, 2016] Blanchard, O. (2016). Do dsge models have a future? *Revista de Economía Institucional*, 18(35):39–46.

- [Blatman and Sudret, 2010] Blatman, G. and Sudret, B. (2010). Efficient computation of global sensitivity indices using sparse polynomial chaos expansions. *Reliability Engineering & System Safety*, 95:1216–1229.
- [Borgonovo et al., 2011] Borgonovo, E., Castaings, W., and Tarantola, S. (2011). Moment independent importance measures: New results and analytical test cases. *Risk Analysis*, 31(3):404–428.
- [Borgonovo and Plischke, 2016] Borgonovo, E. and Plischke, E. (2016). Sensitivity analysis: A review of recent advances. *European Journal of Operational Research*, 248(3):869–887.
- [Bouleau, 2008] Bouleau, N. (2008). Error calculus for finance and physics. the language of dirichlet forms. *Berlin, Boston: De Gruyter*. Retrieved 14 Jun. 2019, from <https://www.degruyter.com/view/product/36729>.
- [Bouleau, 2019] Bouleau, N. (2019). Théorie des erreurs. *Spartacus-Cassini*.
- [Bouleau, 2022] Bouleau, N. (2022). The mathematics of errors. *Springer International Publishing*.
- [Bovari et al., 2020] Bovari, E., Giraud, G., and McIsaac, F. (2020). Financial impacts of climate change mitigation policies and their macroeconomic implications: a stock-flow consistent approach. *Climate Policy*, 20(2):179–198.
- [Bovari et al., 2018] Bovari, E., McIsaac, F., and Giraud, G. (2018). Coping with Collapse : A Stock-Flow Consistent Monetary Macrodynamics of Global Warming. *Ecological Economics Volume 147*, pages 383–398.
- [Box, 1976] Box, G. E. (1976). Science and statistics. *Journal of the American Statistical Association*, 71(356):791–799.
- [Box and Behnken, 1960] Box, G. E. P. and Behnken, D. W. (1960). Some new three level designs for the study of quantitative variables. *Technometrics*, 2(4):455–475.
- [Boyer, 2012] Boyer, R. (2012). La discipline économique des années 1930 à nos jours. *Le Débat*, 2:148–166.
- [BP, 2019] BP (2019). 2019 bp statistical review of world energy. Technical Report 2019 - 68th, BP Report 2019.
- [Bratley et al., 1992] Bratley, P., Fox, B. L., and Niederreiter, H. (1992). Implementation and tests of low-discrepancy sequences. *ACM Trans. Model. Comput. Simul.*, 2(3):195–213.
- [Brenner, 1976] Brenner, R. (1976). Agrarian class structure and economic development in pre-industrial europe. *Past & present*, 70(1):30–75.
- [Burke et al., 2015] Burke, M., Hsiang, S. M., and Miguel, E. (2015). Global non-linear effect of temperature on economic production. *Nature*, 527(7577):235.
- [Burman and Plackett, 1946] Burman, J. P. and Plackett, R. L. (1946). The Design of Optimum Multifactorial Experiments. *Biometrika*, 33(4):305–325.
- [Butler et al., 2014] Butler, M. P., Reed, P. M., Fisher-Vanden, K., Keller, K., and Wagener, T. (2014). Identifying parametric controls and dependencies in integrated assessment models using global sensitivity analysis. *Environmental Modelling & Software* 59 (2014) 10–29.

## BIBLIOGRAPHY

---

- [Cameron and Martin, 1947] Cameron, R. H. and Martin, W. T. (1947). The orthogonal development of non-linear functionals in series of fourier-hermite functionals. *Annals of Mathematics*, 48(2):385–392.
- [Campbell et al., 2006] Campbell, K., McKay, M. D., and Williams, B. J. (2006). Sensitivity analysis when model outputs are functions. *Reliability Engineering and System Safety*, 91(10):1468 – 1472.
- [Campolongo, 2005] Campolongo, F. (2005). Screening methods in sensitivity analysis: The morris method and its applications. *JRC-Ispra (Joint research centre, European commission)*.
- [Campolongo et al., 2006] Campolongo, F., Cariboni, J., and Schoutens, W. (2006). The importance of jumps in pricing european options. *Reliability Engineering & System Safety*, 91:1148–1154.
- [Campolongo et al., 1999] Campolongo, F., Saltelli, A., Jensen, N., Wilson, J., and Hjorth, J. (1999). The role of multiphase chemistry in the oxidation of dimethylsulphide (dms). a latitude dependent analysis. *Journal of Atmospheric Chemistry*, 32(3):327–356.
- [Canfin et al., 2016] Canfin, P., GrandJean, A., and Mestrallet, G. (2016). Rapport de la mission, propositions pour des prix du carbone alignés avec l’accord de paris.
- [Canova, 1995] Canova, F. (1995). Sensitivity analysis and model evaluation in simulated dynamic general equilibrium economies. *International Economic Review*, 36(2):477–501.
- [Cariboni et al., 2007] Cariboni, J., Gatelli, D., Liska, R., and Saltelli, A. (2007). The role of sensitivity analysis in ecological modelling. *Ecological modelling*, 203(1-2):167–182.
- [Carney, 2015] Carney, M. (2015). Breaking the tragedy of the horizon - climate change and financial stability - speech by mark carney. <https://www.bankofengland.co.uk/speech/2015/breaking-the-tragedy-of-the-horizon-climate-change-and-financial-stability>. 2015-09-29.
- [Carney, 2019] Carney, M. (2019). A new report from the network for greening the financial system. <https://www.theguardian.com/commentisfree/2019/apr/17/the-financial-sector-must-be-at-the-heart-of-tackling-climate-change>. 2019.
- [Cass and Stiglitz, 1969] Cass, D. and Stiglitz, J. (1969). The implications of alternative saving and expectations hypotheses for choices of technique and patterns of growth. *Journal of Political Economy*.
- [Catton, 1982] Catton, W. R. (1982). *Overshoot: The ecological basis of revolutionary change*. University of Illinois Press.
- [Chastaing et al., 2013] Chastaing, G., Prieur, C., and Gamboa, F. (2013). Generalized sobol sensitivity indices for dependent variables: numerical methods. *Journal of Statistical Computation and Simulation*, 85.
- [Chongvilaivan, 2012] Chongvilaivan, A. (2012). Thailand’s 2011 flooding: Its impact on direct exports and global supply chains. Technical report, ARTNeT working paper series.
- [Christiaens, 2018] Christiaens, T. (2018). Can we avoid another financial crisis? (steve keen). *Marx & Philosophy Review of Books*.

## BIBLIOGRAPHY

---

- [Christiano et al., 2010] Christiano, L. J., Trabandt, M., and Walentin, K. (2010). Dsge models for monetary policy analysis. In *Handbook of monetary economics*, volume 3, pages 285–367. Elsevier.
- [Cleveland et al., 2015] Cleveland, S., Schuwerk, R., and Weber, C. (2015). Carbon asset risk: From rhetoric to action.
- [Cobb and Douglas, 1928] Cobb, C. W. and Douglas, P. H. (1928). A theory of production. *The American Economic Review*, Vol. 18, No. 1, Supplement, *Papers and Proceedings of the Fortieth Annual Meeting of the American Economic Association (Mar., 1928)*, pp. 139- 165.
- [Collard, 2012] Collard, D. (2012). *Generations of economists*. Routledge.
- [Corson, 1992] Corson, G. (1992). Input-output sensitivity of building energy simulations. *ASHRAE Winter Meeting*, 98(1), pp. 618-626.
- [Costa Lima and Grasselli, 2014] Costa Lima, B. and Grasselli, M. R. (2014). Destabilizing a stable crisis: employment persistence and government intervention in macroeconomics. *Structural Change and Economic Dynamics*, Volume 30, Pages 30-51.
- [Crosetto and Tarantola, 2001] Crosetto, M. and Tarantola, S. (2001). Uncertainty and sensitivity analysis: tools for gis-based model implementation. *International Journal of Geographical Information Science*, 15(5):415–437.
- [Cukier et al., 1973] Cukier, R., Fortuin, C., Shuler, K., Petschek, A., and Schaibly, J. (1973). Study of the sensitivity of coupled reaction systems to uncertainties in rate coefficients: I theory. *The Journal of chemical physics*, 59(8), pp. 3873.
- [Cukier et al., 1978] Cukier, R., Levine, H., and Shuler, K. (1978). Nonlinear sensitivity analysis of multiparameter model systems. *Journal of Computational Physics*, 26(1):1 – 42.
- [Cukier et al., 1975] Cukier, R. I., Schaibly, J. H., and Shuler, K. E. (1975). Study of the sensitivity of coupled reaction systems to uncertainties in rate coefficients. iii. analysis of the approximations. *The Journal of Chemical Physics*, 63(3):1140–1149.
- [De Rocquigny, 2008] De Rocquigny, E. (2008). *Uncertainty in Industrial Practice*. John Wiley & Sons, Ltd, 1 edition.
- [De Wit, 2001] De Wit, M. (2001). Uncertainty in predictions of thermal comfort in buildings. *Delft University, PhD thesis*.
- [Del Negro and Schofheide, 2008] Del Negro, M. and Schofheide, F. (2008). Forming priors for dsge models (and how it affects the assessment of nominal rigidities). *Journal of Monetary Economics*, 55(7):1191 – 1208.
- [Den Haan et al., 2011] Den Haan, W. J., Judd, K. L., and Juillard, M. (2011). Computational suite of models with heterogeneous agents ii: Multi-country real business cycle models. *Journal of Economic Dynamics and Control* 35 (2), 175-177.
- [Desai et al., 2006] Desai, M., Henry, B., and Mosley, A. and Pemberton, M. (2006). Clarification of the goodwin model of the growth cycle. *J. Econ. Dyn. Control* 30(12), 2661-2670.

## BIBLIOGRAPHY

---

- [Diamond and Van Dijk, 2008] Diamond, J. M. and Van Dijk, M. (2008). Collapse: How societies choose to fail or succeed. *Anthropos: International Review of Anthropology and Linguistics*, 103(2):587.
- [Dietz and Stern, 2015] Dietz, S. and Stern, N. (2015). Endogenous growth, convexity of damage and climate risk: How nordhaus' framework supports deep cuts in carbon emissions. *Economic Journal*, 0(583):574-620.
- [Dossetto and Giraud, 2020] Dossetto, E. and Giraud, G. (2020). Money velocity in an imperfectly competitive, stock-flow consistent dynamics. *forthcoming*.
- [Dubus et al., 2003] Dubus, I. G., Brown, C. D., and Beulke, S. (2003). Sensitivity analyses for four pesticide leaching models. *Pest Management Science*, 59(9):962–982.
- [Eddington, 2012] Eddington, A. (2012). *The nature of the physical world: Gifford lectures (1927)*. Cambridge University Press.
- [EPA, 2003] EPA (2003). Crem draft guidance for environmental models. council for regulatory environmental modelling. Technical report, CREM.
- [Erb and Michaels, 1998] Erb, R. S. and Michaels, G. S. (1998). Sensitivity of biological models to errors in parameter estimates. *Pacific Symposium on Biocomputing. Pacific Symposium on Biocomputing*, pages 53–64.
- [Erceg et al., 2006] Erceg, C., Guerrieri, L., and Gust, C. (2006). Sigma: A new open economy model for policy analysis. *International Journal of Central Banking*, 2(1).
- [European Central Bank, 2017] European Central Bank, . (2017). Economic bulletin issue 3. [https://www.ecb.europa.eu/pub/pdf/other/ebbox201703\\_3.en.pdf](https://www.ecb.europa.eu/pub/pdf/other/ebbox201703_3.en.pdf).
- [European Central Bank, 2021] European Central Bank, . (2021). The ECB's negative interest rate. <https://www.ecb.europa.eu/ecb/educational/explainers/tell-me-more/html/why-negative-interest-rate.en.html>.
- [European Commission, 2009] European Commission, . (2009). Global sensitivity analysis for macroeconomic models. *European Commission : Joint Research Centre - Institute for the Protection and Security of the Citizen (IPSC)*. Available at: <http://ipsc.jrc.ec.europa.eu/events.php?id=79>.
- [Fajraoui, 2014] Fajraoui, N. (2014). Analyse de sensibilité globale et polynômes du chaos pour l'estimation des paramètres : application aux transferts en milieu poreux. *PhD thesis Ecole Doctorale Sciences de la Terre et Environnement Strasbourg*.
- [Federal Reserve, 2021a] Federal Reserve, . (2021a). The liquidity trap: An alternative explanation for today's low inflation. <https://www.federalreserve.gov/aboutthefed/files/newyorkfinstmt2020.pdf>.
- [Federal Reserve, 2021b] Federal Reserve, . (2021b). Open market operations. Technical report, Board of Governors of the Federal Reserve System.
- [Felipe and McCombie, 2013] Felipe, J. and McCombie, J. S. (2013). *The Aggregate Production Function and the Measurement of Technical Change: Not Even Wrong*. Edward Elgar Publishing.

- [Fenton and Murchison, 2006] Fenton, P. and Murchison, S. (2006). Totem: The bank of canada's new projection and policy-analysis model. *Bank of Canada Review*, 2006:5–18.
- [Ferraton, 2008] Ferraton, C. (2008). *Les valeurs guident et accompagnent notre recherche, l'institutionnalisme de Myrdal*. ENS éditions.
- [Ferraton, 2011] Ferraton, C. (2011). L'objectivité en économie politique selon gunnar myrdal. *Revue de philosophie économique*, 12(2):29–52.
- [Fisher, 1912] Fisher, I. (1912). “the equation of exchange” for 1911, and forecast. *The American Economic Review*, 2(2):302–319.
- [Fisher, 1933] Fisher, I. (1933). The debt-deflation theory of great depressions. *Econometrica: Journal of the Econometric Society*, pages 337–357.
- [Friedman M., 1963] Friedman M., S. A. (1963). A monetary history of the united states.
- [Fruth et al., 2014] Fruth, J., Roustant, O., and Kuhnt, S. (2014). Total interaction index: A variance-based sensitivity index for second-order interaction screening. *Journal of Statistical Planning and Inference*, 147:212–223.
- [Galbraith, 2009] Galbraith, J. K. (2009). Pourrait-on enfin avancer sur la réforme de l'enseignement de l'économie ? *Revue du MAUSS*, pages 261–265.
- [Galiánd Monacelli, 2005] Gali, J. and Monacelli, T. (2005). Monetary policy and exchange rate volatility in a small open economy. *Review of Economic Studies* 72.
- [Georgescu-Roegen, 1993] Georgescu-Roegen, N. (1993). The entropy law and the economic problem. *Valuing the Earth: Economics, ecology, ethics*, pages 75–88.
- [Ghanem and Spanos, 1991] Ghanem, R. and Spanos, P. (1991). Stochastic finite elements : a spectral approach. *Berlin journal Springer*.
- [Ghers, 2020] Ghersi, F. (2020). *IMACLIM-3ME*. Centre international de recherche sur l'environnement et le développement.
- [Gilquin et al., 2016] Gilquin, L., Capelle, T., Arnaud, E., and Prieur, C. (2016). Sensitivity analysis and optimisation of a land use and transport integrated model. *Journal de la Société Française de Statistique hal-01291774*.
- [Giraud, 2014] Giraud, G. (2014). *Illusion financière (3e édition revue et augmentée)*. Editions de l'Atelier.
- [Giraud and Grasselli, 2017] Giraud, G. and Grasselli, M. (2017). The macrodynamics of household debt, growth, and inequality. Technical report, mimeo: Centre économie de la Sorbonne, Paris.
- [Giraud and Grasselli, 2019] Giraud, G. and Grasselli, M. (2019). Household debt: The missing link between inequality and secular stagnation. *Journal of Economic Behavior & Organization*.
- [Giraud and Kahraman, 2014] Giraud, G. and Kahraman, Z. (2014). How dependent is growth from primary energy? the dependency ratio of energy in 33 countries (1970-2011). *Documents de Travail du Centre d'Economie de la Sorbonne*.

- [Giraud and Pottier, 2016] Giraud, G. and Pottier, A. (2016). Debt-deflation versus the liquidity trap: the dilemma of nonconventional monetary policy. *Economic Theory*, 62(1-2):383–408.
- [Godard, 2007] Godard, O. (2007). Le rapport stern sur l'économie du changement climatique était-il une manipulation grossière de la méthodologie économique ? *Revue d'économie politique*, 117(4):475–506.
- [Godley and Lavoie, 2006] Godley, W. and Lavoie, M. (2006). *Monetary economics: an integrated approach to credit, money, income, production and wealth*. Springer.
- [Goldstone, 1991] Goldstone, J. A. (1991). *Revolution and rebellion in the early modern world*. Univ of California Press.
- [Goodwin, 1967] Goodwin, R. M. (1967). A growth cycle, in: Feinstein, c. h. (ed.), socialism, capitalism and economic growth . *Cambridge University Press, London, pp. 54-58*.
- [Gordon, 2011] Gordon, R. J. (2011). The history of the phillips curve: Consensus and bifurcation. *Economica*, 78(309):10–50.
- [Gordon, 2013] Gordon, R. J. (2013). The phillips curve is alive and well: Inflation and the nairu during the slow recovery. Technical report, National Bureau of Economic Research.
- [Gourieroux et al., 2000] Gourieroux, C., Laurent, J. P., and Scaillet, O. (2000). Sensitivity analysis of values at risk. *Journal of Empirical Finance*, 7(3-4):225–245.
- [Grandjean and Giraud, 2017] Grandjean, A. and Giraud, G. (2017). Comparaison des modèles météorologiques, climatiques et économiques : quelles capacités, quelles limites, quels usages ? *WP*.
- [Grasselli and Costa Lima, 2012] Grasselli, M. R. and Costa Lima, B. (2012). An Analysis of the Keen Model for Credit Expansion, Asset Price Bubbles and Financial Fragility. *Mathematical and Financial Economics - Volume 6*, pages 191–210.
- [Grasselli and Maheshwari, 2017] Grasselli, M. R. and Maheshwari, A. (2017). A comment on testing goodwin: growth cycles in ten oecd countries? *Cambridge Journal of Economics*, 41(6):1761–1766.
- [Grasselli and Nguyen Huu, 2016] Grasselli, M. R. and Nguyen Huu, A. (2016). Inventory growth cycles with debt-financed investment. *Working paper*.
- [Grasselli and Nguyen-Huu, 2018] Grasselli, M. R. and Nguyen-Huu, A. (2018). Inventory growth cycles with debt-financed investment. *Structural Change and Economic Dynamics*, 44:1–13.
- [Guerrien and Gun, 2014] Guerrien, B. and Gun, O. (2014). En finir, pour toujours, avec la fonction de production agrégée. *jesus felipe and john sl mcombie, the aggregate production function and the measurement of technical change: Not even wrong*.
- [Guesnerie, 2005] Guesnerie, R. (2005). *Assessing Rational Expectations 2: eductive stability in economics*. MIT press.
- [Guesnerie, 2013] Guesnerie, R. (2013). L'économie saisie par le doute. 2013-11-13.

- [Gurley and Shaw, 1960] Gurley, J. G. and Shaw, E. S. (1960). Money in a theory of finance (Washington, DC: Brookings Institution). *Transactions Demand for Cash. The Review of Economics and Statistics*, 38(3).
- [Hamby, 1994] Hamby, D. (1994). A review of techniques for parameter sensitivity analysis of environmental models. *Environmental Monitoring and Assessment*, 32, pp. 135-154.
- [Harenberg et al., 2017] Harenberg, D., Marelli, S., Sudret, B., and Winschel, V. (2017). Uncertainty quantification and global sensitivity analysis for economic models. *Economics Working Paper Series, No. 17/265, ETH Zurich, CER-ETH - Center of Economic Research, Zurich*, <http://dx.doi.org/10.3929/ethz-a-010820135>.
- [Harenberg et al., 2019] Harenberg, D., Marelli, S., Sudret, B., and Winschel, V. (2019). Uncertainty quantification and global sensitivity analysis for economic models. *Quantitative Economics*, 10(1):1–41.
- [Harrison et al., 2005] Harrison, R., Nikolov, K., Quinn, M., Ramsay, G., Scott, A., and Thomas, R. (2005). The Bank of England Quarterly Model. Technical report, Bank of England, Threadneedle Street, London, EC2R 8AHs.
- [Heal and Millner, 2014] Heal, G. and Millner, A. (2014). Reflections: Uncertainty and decision making in climate change economics. *Review of Environmental Economics and Policy* 8, 120-137.
- [Helton et al., 2006] Helton, J., Johnson, J., Sallaberry, C., and Storlie, C. (2006). Survey of sampling-based methods for uncertainty and sensitivity analysis. *Reliability Engineering and System Safety*, 91(10), pp. 1175-120.
- [Hodgson, 2012] Hodgson, G. M. (2012). *From pleasure machines to moral communities: an evolutionary economics without homo economicus*. University of Chicago Press.
- [Hoes and De Vann, 2005] Hoes, P. and De Vann, C. (2005). De gevoeligheidsanalyse van gebouw simulaties. *Masterproject 3. Technische Universiteit Eindhoven, Unit Building Physics and Systems (BPS)*.
- [Hooker, 2007] Hooker, G. (2007). Generalized functional anova diagnostics for high-dimensional functions of dependent variables. *Journal of Computational and Graphical Statistics*, 16(3):709–732.
- [Hopfe et al., 2007] Hopfe, C., Hensen, J., and Plokker, W. (2007). Uncertainty and sensitivity analysis for detailed design support. *Proceedings of the 10th International IBPSA Building Simulation Conference 2007*.
- [Hoppensteadt, 2006] Hoppensteadt, F. (2006). Predator-prey model. *Scholarpedia*, 1(10):1563.
- [Hurwitz, 1963] Hurwitz, A. (1963). Ueber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt. In *Mathematische Werke*, pages 533–545. Springer.
- [Iman, 1999] Iman, R. (1999). Appendix a: Latin hypercube sampling. *Encyclopedia of Statistical Sciences, Update, 3*, pp. 408-411.
- [Iooss, 2011] Iooss, B. (2011). Revue sur l'analyse de sensibilité globale de modèles numériques. *Journal de la Société Française de Statistique*, 152(1):1–23.



## BIBLIOGRAPHY

---

- [Iooss and Lemaître, 2015] Iooss, B. and Lemaître, P. (2015). A review on global sensitivity analysis methods. In Meloni, C. and Dellino, G., editors, *Uncertainty management in Simulation-Optimization of Complex Systems: Algorithms and Applications*. Springer.
- [Iooss and Ribatet, 2009] Iooss, B. and Ribatet, M. (2009). Global sensitivity analysis of computer models with functional inputs. *Reliability Engineering and System Safety*, 94(7):1194 – 1204.
- [IPCC, 2014] IPCC (2014). Climate change 2014 - working group iii contribution to the fifth assessment report of the intergovernmental panel on climate change. *IPCC*.
- [IPCC, 2018] IPCC (2018). *Global Warming of 1.5°C: An IPCC Special Report on the Impacts of Global Warming of 1.5°C Above Pre-industrial Levels and Related Global Greenhouse Gas Emission Pathways, in the Context of Strengthening the Global Response to the Threat of Climate Change, Sustainable Development, and Efforts to Eradicate Poverty*. Intergovernmental Panel on Climate Change.
- [IPCC, 2021] IPCC (2021). *AR6 Climate Change 2021: The Physical Science Basis*. Intergovernmental Panel on Climate Change - WG I.
- [Iskrev, 2010] Iskrev, N. (2010). Local identification in dsge models. *Journal of Monetary Economics*, 57(2):189 – 202.
- [Jacobi et al., 2019] Jacobi, L., Joshi, M., and Zhu, D. (2019). Automated sensitivity computations for bayesian markov chain monte carlo inference: A new approach for prior robustness and convergence analysis. *Working progress*.
- [Jacques, 2011] Jacques, J. (2011). Pratique de l’analyse de sensibilité : comment évaluer l’impact des entrées aléatoires sur la sortie d’un modèle mathématique. *PUB. IRMA, LILLE 2011, Vol. 71, No III, p 9*.
- [Janon et al., 2014] Janon, A., Klein, T., Lagnoux, A., Nodet, M., and Prieur, C. (2014). Asymptotic normality and efficiency of two sobol index estimators. *ESAIM: Probability and Statistics*, 18:342–364.
- [Jansen, 1999] Jansen, M. J. (1999). Analysis of variance designs for model output. *Computer Physics Communications*, 117(1-2):35–43.
- [Jany-Catrice and Méda, 2015] Jany-Catrice, F. and Méda, D. (2015). *Faut-il attendre la croissance ?* La Documentation française.
- [Joe and Kuo, 2008] Joe, S. and Kuo, F. (2008). Constructing sobol sequences with better two-dimensional projections. *SIAM Journal on Scientific Computing*, 30(5):2635–2654.
- [Johansen, 1959] Johansen, L. (1959). Substitution versus fixed production coefficients in the theory of economic growth : A synthesis. *Econometrica*, 27(2), 157-176. <http://www.jstor.org/stable/1909440>.
- [Kammen et al., 1994] Kammen, D. M., Smith, K. R., Rambo, A. T., and Khalil, M. (1994). Preindustrial human environmental impacts: are there lessons for global change science and policy? *Chemosphere*.
- [Karna et al., 2012] Karna, D. S., D Ph, K., and P Scholar, A. (2012). An overview on taguchi method. *International Journal of Engineering and Mathematical Sciences, Volume 1, pp.11-18*.

## BIBLIOGRAPHY

---

- [Kaya, 2014] Kaya (2014). Equation de kaya par jm jancovici. <https://jancovici.com/changement-climatique/economie/quest-ce-que-lequation-de-kaya/>.
- [Keen, 1995] Keen, S. (1995). Finance and economic breakdown: Modeling minsky’s financial instability hypothesis. *Journal of Post-Keynesian Economics*, 1995, vol. 17, issue 4, 607-635.
- [Keen, 2011] Keen, S. (2011). *Debunking economics: The naked emperor dethroned?* Zed Books Ltd.
- [Keen, 2013] Keen, S. (2013). A monetary minsky model of the great moderation and the great recession. *Journal of Economic Behavior & Organization*, 86:221–235.
- [Keynes, 1936] Keynes, J. M. (1936). *The general theory of employment, interest, and money - ed 2018*. Springer.
- [Kioutsioukis et al., 2005] Kioutsioukis, I., Melas, D., Zerefos, C., and Ziomas, I. (2005). Efficient sensitivity computations in 3d air quality models. *Computer Physics Communications*, 167(1):23–33.
- [Krugman and Eggertsson, 2013] Krugman, P. and Eggertsson, G. (2013). Debt, deleveraging, and the liquidity trap: A fisher-minsky-koo approach. *The Quarterly Journal of Economics*.
- [Kucherenko, 2010] Kucherenko, S. . (2010). The importance of being global. application of global sensitivity analysis in monte carlo option pricing. *Convergence (London, England)*.ISSN: 1354-8565. Volume: 1 Issue: 2 Page: 2.
- [Kucherenko and Shah, 2007] Kucherenko, S. and Shah, N. (2007). The importance of being global application of global sensitivity analysis in monte carlo option pricing. *Imperial College London, SW7 2AZ, UK,*.
- [Kucherenko et al., 2012] Kucherenko, S., Tarantola, S., and Annoni, P. (2012). Estimation of global sensitivity indices for models with dependent variables. *Computer Physics Communications*, 183:937–946.
- [Kuroda, 2016] Kuroda, H. (2016). The battle against deflation: The evolution of monetary policy and japan’s experience. *Speech given at Columbia University, New York, April, 13*.
- [Ladurie, 1987] Ladurie, E. L. R. (1987). *The French Peasantry, 1450-1660*. Univ of California Press.
- [Laffargue, 2012] Laffargue, J.-P. (2012). Une comparaison des modèles macro-économétriques et dsge dans l’évaluation des politiques économiques: une comparaison basée sur les modèles mésange et egée. *Economie et statistique*, 451(1):45–68.
- [Lam and Hui, 1995] Lam, J. and Hui, S. (1995). Sensitivity analysis of energy performance of office buildings. *Building and Environment*, 31(1), pp. 27-39.
- [Lamboni et al., 2013] Lamboni, M., Iooss, B., Popelin, A.-L., and Gamboa, F. (2013). Derivative-based global sensitivity measures: General links with sobol’ indices and numerical tests. *Mathematics and Computers in Simulation*.
- [Laxton et al., 2010] Laxton, D., Mursula, S., Kumhof, M., and Muir, D. (2010). The Global Integrated Monetary and Fiscal Model (GIMF): Theoretical Structure. Technical Report 10/34, International Monetary Fund.

## BIBLIOGRAPHY

---

- [Le Gratiet et al., 2018] Le Gratiet, L., Cannamela, C., and Iooss, B. (2018). A bayesian approach for global sensitivity analysis of (multifidelity) computer codes. *SIAM/ASA Journal on Uncertainty Quantification*, 2(1):336–363.
- [Le Gratiet and Garnier, 2014] Le Gratiet, L. and Garnier, J. (2014). Recursive co-kriging model for design of computer experiments with multiple levels of fidelity. *International Journal for Uncertainty Quantification*, 4(5).
- [Le Maître et al., 2002] Le Maître, O. P., Reagan, M. T., Najm, H. N., Ghanem, R. G., and Knio, O. M. (2002). A Stochastic Projection Method for Fluid Flow. II. Random Process. *Journal of Computational Physics*, 181:9–44.
- [Lefebvre et al., 2010] Lefebvre, S., Roblin, A., Varet, S., and Durand, G. (2010). A methodological approach for statistical evaluation of aircraft infrared signature. *Reliability Engineering & System Safety*, 95(5):484–493.
- [Leontief, 1951] Leontief, W. W. (1951). The structure of american economy, 1919-1939 : an empirical application of equilibrium analysis. *New York :. Oxford University Press,. 1951.. 264 p. : Il., tablas. (Es). Edición ; 2 ed.*
- [Li et al., 2010] Li, G., Rabitz, H., Yelvington, P. E., Oluwole, O. O., Bacon, F., Kolb, C. E., and Schoendorf, J. (2010). Global sensitivity analysis for systems with independent and/or correlated inputs. *The Journal of Physical Chemistry A*, 114(19):6022–6032.
- [Li and Zhang, 2007] Li, H. and Zhang, D. (2007). Probabilistic collocation method for flow in porous media: Comparisons with other stochastic methods. *Water Resources Research*, 43(9).
- [Liu and Owen, 2006] Liu, R. and Owen, A. B. (2006). Estimating mean dimensionality of analysis of variance decompositions. *Journal of the American Statistical Association*, 101(474):712–721.
- [Lomas and Eppel, 1992] Lomas, K. and Eppel, H. (1992). Sensitivity analysis techniques for building thermal simulation programs. *Energy and Buildings*, 19(1), pp. 21-44.
- [Lotka, 1920] Lotka, A. J. (1920). Analytical note on certain rhythmic relations in organic systems. *Proceedings of the National Academy of Sciences*, 6(7):410–415.
- [Lubik and Schorfheide, 2005] Lubik, T. and Schorfheide, F. (2005). A bayesian look at new open economy macroeconomics. *NBER macroeconomics annual*, 20:313–366.
- [Lubik and Schorfheide, 2007] Lubik, T. A. and Schorfheide, F. (2007). Do central banks respond to exchange rate movements? A structural investigation. *Journal of Monetary Economics*, 54(4):1069–1087.
- [M. Sobol, 1990] M. Sobol, I. (1990). Sensitivity estimates for nonlinear mathematical models. *Matematicheskoe Modelirovanie*, 2.
- [MacDonald, 2002] MacDonald, I. (2002). Quantifying the effects of uncertainty in building simulation. *Department of Mechanical Engineering. University of Strathclyde, PhD thesis.*
- [Macdonald et al., 1999] Macdonald, I., Clarke, J., and Strachan, P. (1999). Assessing uncertainty in building simulation. *Proceedings 6th Building Simulation 1999.*

- [Makowski et al., 2006] Makowski, D., Naud, C., Jeuffroy, M.-H., Barbottin, A., and Monod, H. (2006). Global sensitivity analysis for calculating the contribution of genetic parameters to the variance of crop model prediction. *Reliability Engineering & System Safety*, 91(10-11):1142–1147.
- [Manganelli, 2004] Manganelli, S. (2004). Asset allocation by variance sensitivity analysis. *Journal of Financial Econometrics*, 2:370–389.
- [Mankiw, 2001] Mankiw, N. G. (2001). The inexorable and mysterious tradeoff between inflation and unemployment. *The Economic Journal*, 111(471):45–61.
- [Mankiw, 2016] Mankiw, N. G. (2016). *Principles of economics*. Cengage Learning.
- [Mankiw and Romer, 1991] Mankiw, N. G. and Romer, D. (1991). *New Keynesian Economics-Vol. 1: Imperfect Competition and Sticky Prices*, volume 1. The MIT Press.
- [Mara and Tarantola, 2011] Mara, T. and Tarantola, S. (2011). Variance-based sensitivity indices for models with dependent inputs. *Reliability Engineering and System Safety - RELIAB ENG SYST SAFETY*, 107.
- [Martinez, 2011] Martinez, J. (2011). Analyse de sensibilité globale par décomposition de la variance. *Presentation in Journée des GdR Ondes & Mascot*, 13.
- [Matthies and Keese, 2003] Matthies, H. G. and Keese, A. (2003). Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. *Informatik-Berichte der Technischen Universität Braunschweig*, 2003-08.
- [Mazzucato, 2011] Mazzucato, M. (2011). The entrepreneurial state. *Soundings*, 49(49):131–142.
- [McCandless and Gustafson, 2017] McCandless, L. C. and Gustafson, P. (2017). A comparison of bayesian and monte carlo sensitivity analysis for unmeasured confounding. *Statistics in Medicine*, 36(18):2887–2901.
- [McIsaac, 2016] McIsaac, F. (2016). *Energy and Money in New Frameworks for Macro-dynamics*. PhD thesis, Université Paris 1, Panthéon - Sorbonne.
- [McIsaac, 2021] McIsaac, F. (2021). Testing goodwin with a stochastic differential approach? the united states (1948–2019). *Metroeconomica*, 72(4):696–730.
- [McKay et al., 2000] McKay, M. D., Beckman, R. J., and Conover, W. J. (2000). A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 42.
- [McMahon et al., 2001] McMahon, A., Heathcote, J., Carey, M., Erskine, A., and Barker, J. (2001). Guidance on assigning values to uncertain parameters in subsurface contaminant fate and transport modelling. *National Groundwater and Contaminated Land Centre report NC/99/38/3*.
- [Meadows and Randers, 2012] Meadows, D. and Randers, J. (2012). *The limits to growth: the 30-year update*. Routledge.
- [Meadows et al., 1972] Meadows, D. H., Meadows, D. H., Randers, J., and Behrens III, W. W. (1972). The limits to growth: a report to the club of rome (1972). *Google Scholar*.

## BIBLIOGRAPHY

---

- [Meadows et al., 1974] Meadows, D. L., Behrens, W. W., Meadows, D. H., Naill, R. F., Randers, J., and Zahn, E. (1974). *Dynamics of growth in a finite world*. Wright-Allen Press Cambridge, MA.
- [Mertens and Rubinchik, 2012] Mertens, J.-F. and Rubinchik, A. (2012). Intergenerational equity and the discount rate for policy analysis. *Macroeconomic Dynamics*, 16(1):61–93.
- [Miftakhova, 2019] Miftakhova, A. (2019). Global sensitivity analysis for integrated assessment modeling. *Working paper, University of Zurich*.
- [Miftakhova, 2021] Miftakhova, A. (2021). Global sensitivity analysis for optimal climate policies: Finding what truly matters. *Economic Modelling*, 105:105653.
- [Minsky, 1982] Minsky, H. P. (1982). Can “it” happen again? essays on instability and finance, me sharpe. *Inc.: Armonk, NY*.
- [Monod et al., 2006] Monod, H., Naud, C., and Makowski, D. (2006). Uncertainty and sensitivity analysis for crop models. *Working with dynamic crop models: Evaluation, analysis, parameterization, and applications*, 4:55–100.
- [Morris, 1991] Morris, M. D. (1991). Factorial sampling plans for preliminary computational experiments. *Communications in Statistics - Theory and Methods* 33 (2), 161174.
- [Motesharrei et al., 2014] Motesharrei, S., Rivas, J., and Kalnay, E. (2014). Human and nature dynamics (handy): Modeling inequality and use of resources in the collapse or sustainability of societies. *Ecological Economics*, 101:90–102.
- [Mussa, 1986] Mussa, M. (1986). Nominal exchange rate regimes and the behavior of real exchange rates: Evidence and implications. In *Carnegie-Rochester Conference series on public policy*, volume 25, pages 117–214. Elsevier.
- [Nikulin, 2001] Nikulin, M. (2001). Hazewinkel, michiel, encyclopaedia of mathematics: an updated and annotated translation of the soviet’ mathematical encyclopaedia. *Reidel Sold and distributed in the USA and Canada by Kluwer Academic Publishers*.
- [Nordhaus, 2008] Nordhaus, W. (2008). A question of balance: Weighing the options on global warming policies. *Yale University Press*.
- [Nordhaus, 2014] Nordhaus, W. (2014). Estimates of the social cost of carbon: Concepts and results from the dice-2013r model and alternative approaches. *Journal of the Association of Environmental and Resource Economists*, Vol. 1, No. 1/2 (Spring/Summer 2014), pp. 273-312.
- [Oakley and O’Hagan, 2004] Oakley, J. E. and O’Hagan, A. (2004). Probabilistic sensitivity analysis of complex models: A bayesian approach. *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, 66(3):751–769.
- [OECD, 1996] OECD (1996). *OECD employment outlook: July 1996*. OECD, Paris, France.
- [on Climate Change 1/CP.26, 2021] on Climate Change 1/CP.26, U. N. F. C. (2021). Draft cop decision proposed by the president.
- [Owen, 2013] Owen, A. B. (2013). Better estimation of small sobol’sensitivity indices. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 23(2):1–17.

## BIBLIOGRAPHY

---

- [Piketty, 2009] Piketty, T. (2009). Fini le pib, retournons au revenu national. 2009-10-06.
- [Pindyck, 2017] Pindyck, R. S. (2017). The use and misuse of models for climate policy. *Review of Environmental Economics and Policy*, 11(1):100–114.
- [Ponting, 1993] Ponting, C. (1993). *A green history of the world. The environment and the collapse of great civilizations*. New York New York Penguin Books 1993.
- [Popper, 1963] Popper, K. (1963). *Conjectures and refutations: The growth of scientific knowledge - ed 2014*. routledge.
- [Popper, 1973] Popper, K. (1973). La logique de la découverte scientifique (1934), trad. *Nicole Thyssen-Rutten et Philippe Devaux, Payot, coll.Bibliothèque scientifique, Paris*.
- [Pottier, 2014] Pottier, A. (2014). *L'économie dans l'impasse climatique: développement matériel, théorie immatérielle et utopie auto-stabilisatrice*. PhD thesis, Paris, EHESS.
- [Pottier and Nguyen-Huu, 2017] Pottier, A. and Nguyen-Huu, A. (2017). Debt and investment in the keen model: a reappraisal of modelling minsky. *Review of Keynesian Economics*, 5(4):631–647.
- [Pujol et al., 2015] Pujol, G., Iooss, B., Iooss, M. B., and DiceDesign, S. (2015). Package “sensitivity”.
- [Pytlarczyk, 2005] Pytlarczyk, E. (2005). An estimated dsge model for the german economy within the euro area. Discussion Paper Series 1: Economic Studies 2005,33, Deutsche Bundesbank.
- [Ratto, 2008] Ratto, M. (2008). Analysing DSGE Models with Global Sensitivity Analysis. *Computational Economics*, 31(2):115–139.
- [Ratto and Pagano, 2010] Ratto, M. and Pagano, A. (2010). Using recursive algorithms for the efficient identification of smoothing spline anova models. *AStA Advances in Statistical Analysis*, 94(4):367–388.
- [Redman, 2004] Redman, C. L. (2004). *The archaeology of global change: the impact of humans on their environment*. Smithsonian Books.
- [Rizvi, 2006] Rizvi, S. A. T. (2006). The sonnenschein-mantel-debreu results after thirty years. *History of Political Economy*, 38(Suppl.1):228–245.
- [Roemer, 2011] Roemer, J. E. (2011). The ethics of intertemporal distribution in a warming planet. *Environmental and resource economics*, 48(3):363–390.
- [Romer, 2016] Romer, P. (2016). The trouble with macroeconomics. *The American Economist*, 20:1–20.
- [Rosen, 1991] Rosen (1991). A comprehensive inquiry into the nature, origin, and fabrication of life.
- [Routh, 1877] Routh, E. J. (1877). *A treatise on the stability of a given state of motion: particularly steady motion*. Macmillan and Company.
- [Saint-Geours et al., 2011] Saint-Geours, N., Lavergne, C., Bailly, J.-S., and Grelot, F. (2011). Analyse de sensibilité de sobol d’un modèle spatialisé pour l’évaluation économique du risque d’inondation. *Journal de la Société Française de Statistique*, 152:24-46.

## BIBLIOGRAPHY

---

- [Saltelli, 2002] Saltelli, A. (2002). Making best use of model evaluations to compute sensitivity indices. *Applied Statistics, Institute for the Protection and the Safety of the Citizen (IPSC), The European Commission, Joint Research Centre, TP 361, 21020 Ispra (VA), Italy.*
- [Saltelli and Bolado, 1998] Saltelli, A. and Bolado, R. (1998). An alternative way to compute fourier amplitude sensitivity test (fast). *Computational Statistics and Data Analysis*, 26(4):445 – 460.
- [Saltelli et al., 2008a] Saltelli, A., Chan, K., and Scott, E. (2008a). Sensitivity analysis. *Chichester: Wiley.*
- [Saltelli et al., 2000] Saltelli, A., Chan, K., Scott, M., et al. (2000). Sensitivity analysis. probability and statistics series.
- [Saltelli et al., 2008b] Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., Saisana, M., and Tarantola, S. (2008b). *Global Sensitivity Analysis The Primer*. John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex PO19 8SQ, England.
- [Saltelli et al., 2004] Saltelli, A., Tarantola, S., Campolongo, F., and Ratto, M. (2004). Sensitivity analysis in practice: a guide to assessing scientific models. *Chichester, England.*
- [Saltelli et al., 1999] Saltelli, A., Tarantola, S., and Chan, K. P.-S. (1999). A quantitative model-independent method for global sensitivity analysis of model output. *Technometrics*, 41(1):39–56.
- [Samuelson, 1966] Samuelson, P. A. (1966). A summing up. *The quarterly journal of economics*, 80(4):568–583.
- [Saporta, 1990] Saporta, G. (1990). Probabilités, analyse des données et statistique. *editor Technip report.*
- [Sathaye and Shukla, 2013] Sathaye, J. and Shukla, P. (2013). Methods and models for costing carbon mitigation. *Annual Review of Environment and Resources*, 38:137–168.
- [Sbordone et al., 2010] Sbordone, A., Tambalotti, A., Rao, K., and Walsh, K. (2010). Policy analysis using dsge models: An introduction. *Economic Policy Review*, pages 23–43.
- [Schaibly and Shuler, 1973] Schaibly, J. H. and Shuler, K. E. (1973). Study of the sensitivity of coupled reaction systems to uncertainties in rate coefficients. ii applications. *The Journal of Chemical Physics*, 59(8):3879–3888.
- [Schorfheide, 2000] Schorfheide, F. (2000). Loss function-based evaluation of dsge models. *Journal of Applied Econometrics*, 15(6):645–670.
- [Schumpeter and Nichol, 1934] Schumpeter, J. A. and Nichol, A. J. (1934). Robinson’s economics of imperfect competition. *Journal of political economy*, 42(2):249–259.
- [Shannon, 1949] Shannon, C. E. (1949). Communication in the presence of noise. *Proceedings of the Institute of Radio Engineers*. 37 (1): Reprint as classic paper in: *Proc. IEEE, Vol. 86, No. 2, (Feb 1998).*
- [Sheng, 2013] Sheng, A. (2013). The end of financialization. <https://www.ineteconomics.org/perspectives/blog/the-end-of-financialization>. 2013-09-18.

- [Smets and Wouters, 2003] Smets, F. and Wouters, R. (2003). An estimated dynamic stochastic general equilibrium model of the euro area. *Journal of the European Economic Association*, 1(5):1123–1175.
- [Smets and Wouters, 2007] Smets, F. and Wouters, R. (2007). Shocks and frictions in us business cycles: A bayesian dsge approach. *American economic review*, 97(3):586–606.
- [Smith and Thieme, 2011] Smith, H. L. and Thieme, H. R. (2011). *Dynamical systems and population persistence*, volume 118. American Mathematical Soc.
- [Sobol, 1967] Sobol, I. (1967). On the distribution of points in a cube and the approximate evaluation of integrals. *USSR Computational Mathematics and Mathematical Physics*, 7(4):86 – 112.
- [Sobol, 1998] Sobol, I. (1998). On quasi-monte carlo integrations. *Mathematics and Computers in Simulation*, 47(2):103 – 112.
- [Sobol, 1993] Sobol, I. M. (1993). Sensitivity analysis for non-linear mathematical models. *Mathematical modelling and computational experiment*, 1:407–414.
- [Sobol’ and Kucherenko, 2009] Sobol’, I. and Kucherenko, S. (2009). Derivative based global sensitivity measures and their link with global sensitivity indices. *Mathematics and Computers in Simulation*, 79(10):3009 – 3017.
- [Solow, 1956] Solow, R. (1956). A contribution to the theory of economic growth. *The Quarterly Journal of Economics*. 70 (1): 65-94. doi:10.2307/1884513. JSTOR 1884513.
- [Stansbury and Summers, 2018] Stansbury, A. M. and Summers, L. H. (2018). Productivity and pay: Is the link broken? Technical report, National Bureau of Economic Research.
- [Stiglitz, 1974a] Stiglitz, J. (1974a). Growth with exhaustible natural resources: efficient and optimal growth paths. *The review of economic studies*, 41:123–137.
- [Stiglitz, 2015] Stiglitz, J. (2015). *The great divide*. Penguin UK.
- [Stiglitz, 1974b] Stiglitz, J. E. (1974b). The cambridge-cambridge controversy in the theory of capital; a view from new haven: A review article. *Journal of Political Economy*, 82(4):893–903.
- [Stiglitz, 2018] Stiglitz, J. E. (2018). Where Modern Macroeconomics Went Wrong. *Oxford Review of Economic Policy*, Volume 34, Issue 1-2, pages 70–106.
- [Stiglitz et al., 2017] Stiglitz, J. E., Stern, N., Duan, M., Edenhofer, O., Giraud, G., Heal, G. M., la Rovere, E. L., Morris, A., Moyer, E., Pangestu, M., et al. (2017). Report of the high-level commission on carbon prices.
- [Sudret, 2008] Sudret, B. (2008). Global sensitivity analysis using polynomial chaos expansions. *Reliability Engineering & System Safety*, 93(7):964 – 979. Bayesian Networks in Dependability.
- [Tarantola et al., 2007] Tarantola, S., Gatelli, D., Kucherenko, S., Mauntz, W., et al. (2007). Estimating the approximation error when fixing unessential factors in global sensitivity analysis. *Reliability engineering & system safety*, 92(7):957–960.
- [Tarantola and Mara, 2017] Tarantola, S. and Mara, T. A. (2017). Variance-based sensitivity indices of computer models with dependent inputs: The Fourier Amplitude Sensitivity Test. *International Journal for Uncertainty Quantification*, 7(6):511–523.



## BIBLIOGRAPHY

---

- [Taylor, 1993] Taylor, J. B. (1993). Discretion versus policy rules in practice. In *Carnegie-Rochester conference series on public policy*, volume 39, pages 195–214. Elsevier.
- [Taylor, 2009] Taylor, L. (2009). *Reconstructing macroeconomics: Structuralist proposals and critiques of the mainstream*. Harvard University Press.
- [Tian, 2013] Tian, W. (2013). A review of sensitivity analysis methods in building energy analysis. *Renewable and Sustainable Energy Reviews*, 20, pp. 411–419.
- [Tirole and Gollier, 2015] Tirole, J. and Gollier, C. (2015). The end of financialization. 2013-06-03.
- [Tissot and Prieur, 2015] Tissot, J.-Y. and Prieur, C. (2015). A randomized orthogonal array-based procedure for the estimation of first-and second-order sobol’indices. *Journal of Statistical Computation and Simulation*, 85(7):1358–1381.
- [Tol, 1965] Tol, R. (1965). The damage costs of climate change toward more comprehensive calculations. *Environmental & Resource Economics* 5, 353–374.
- [Turchin and Nefedov, 2009] Turchin, P. and Nefedov, S. A. (2009). *Secular cycles*. Princeton University Press.
- [Turner, 2014] Turner, G. (2014). *Is global collapse imminent?* University of Melbourne, Melbourne Sustainable Society Institute.
- [Turner, 2008] Turner, G. M. (2008). A comparison of the limits to growth with 30 years of reality. *Global environmental change*, 18(3):397–411.
- [Turner, 2012] Turner, G. M. (2012). On the cusp of global collapse? updated comparison of the limits to growth with historical data. *GAIA-Ecological Perspectives for Science and Society*, 21(2):116–124.
- [UNEP, 2021] UNEP, . (2021). *Emissions Gap Report 2021*. United Nations Environment Program.
- [UNFCCC, 2021] UNFCCC, . (2021). *Nationally determined contributions under the Paris Agreement Synthesis report by the secretariat*. United Nations Framework Convention on Climate Change Secretariat.
- [Van der Ploeg, 1985] Van der Ploeg, F. (1985). Classical growth cycles. *Metroeconomica* 37 (2), 221–230.
- [Volterra, 1928] Volterra, V. (1928). Variations and fluctuations of the number of individuals in animal species living together. *ICES Journal of Marine Science*, 3(1):3–51.
- [Wang, 2014] Wang, M. (2014). Sensitivity analysis and evolutionary optimization for building design. *PhD thesis Loughborough University Institutional Repository*.
- [Weitzman, 2012] Weitzman, M. L. (2012). Ghg targets as insurance against catastrophic climate damages. *Journal of Public Economic Theory*, 14(2):221–244.
- [Werner, 2014a] Werner, R. A. (2014a). Can banks individually create money out of nothing? the theories and the empirical evidence. *International Review of Financial Analysis*, 36:1–19.
- [Werner, 2014b] Werner, R. A. (2014b). Enhanced debt management: Solving the eurozone crisis by linking debt management with fiscal and monetary policy. *Journal of International Money and Finance*, 49:443–469.

- [Werner, 2014c] Werner, R. A. (2014c). How do banks create money, and why can other firms not do the same? an explanation for the coexistence of lending and deposit-taking. *International Review of Financial Analysis*, 36:71–77.
- [Werner, 2016] Werner, R. A. (2016). A lost century in economics: Three theories of banking and the conclusive evidence. *International Review of Financial Analysis*, 46:361–379.
- [Wiener, 1938] Wiener, N. (1938). The homogeneous chaos. *American Journal of Mathematics*, 60(4):897–936.
- [Wood, 1998] Wood, J. W. (1998). A theory of preindustrial population dynamics demography, economy, and well-being in malthusian systems. *Current Anthropology*, 39(1):99–135.
- [Wright, 2004] Wright, R. (2004). A short history of progress: House of anansi press.
- [Xu and Gertner, 2008] Xu, C. and Gertner, G. Z. (2008). Uncertainty and sensitivity analysis for models with correlated parameters. *Reliability Engineering and System Safety*, 93(10):1563 – 1573.
- [Zádor and Turány, 2006] Zádor, J. Z. I. and Turány, T. (2006). Local and global uncertainty analysis of complex chemical kinetic systems. *Reliability Engineering & System Safety*, 91(10):1232 – 1240. The Fourth International Conference on Sensitivity Analysis of Model Output (SAMO 2004).
- [Zaldivar et al., 2000] Zaldivar, J., Campolongo, F., et al. (2000). An application of sensitivity analysis to fish population dynamics. *Saltelli et al., editors. Sensitivity analysis. New York: Wiley*, pages 367–83.
- [Zaldivar et al., 1998] Zaldivar, J., Kourti, N., Villacastin, C., Strozzi, F., and Campolongo, F. (1998). Analysing dynamics of complex ecologies from natural recordings: an application to fish population models. *JRC-ISIS Technical Note I*, 98.