Anneau de Cox de variétés avec action de groupe

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Résumé

L'anneau de Cox d'une variété algébrique (satisfaisant des conditions naturelles) est un invariant très riche. Il est introduit par Cox en 1995 pour l'étude des variétés toriques, puis généralisé aux variétés normales par Arzhantsev, Berchtold et Hausen. Plus tard, Hu et Keel découvrent que les variétés normales dont l'anneau de Cox est de type fini définissent une classe de variétés dont la géométrie birationnelle est particulièrement bien comprise. Ils les nomment les Mori Dream Spaces (MDS), du fait de leur bon comportement vis-à-vis du programme de Mori. Un premier problème est alors de trouver des conditions naturelles pour qu'une variété normale soit un MDS. Un second est de décrire l'anneau de Cox d'un MDS donné: trouver une présentation par générateurs et relations, donner la nature de ses singularités, etc...

Parmi les variétés algébriques munies d'une action d'un groupe algébrique (affine), une classe particulièrement bien comprise est constituée des variétés normales de complexité au plus un : un groupe réductif connexe agit de telle sorte que la codimension minimale d'une orbite d'un sous-groupe de Borel est au plus un. Les variétés normales de complexité zéro sont dites sphériques (par exemple une variété torique est sphérique). En 2007, Brion montre que les variétés sphériques sont des MDS et donne une description par générateurs et relations de leur anneau de Cox. Une variété normale de complexité un est un MDS si et seulement si c'est une variété rationnelle (par exemple une surface normale rationnelle sur laquelle le groupe multiplicatif $\mathbb{G}_m$ agit effectivement, ou encore une variété normale de dimension trois sur laquelle SL_2 agit avec une orbite dense). Cela définit une classe naturelle de MDS avec action de groupe pour laquelle le second problème n'a été étudié que dans des cas très particuliers.

Dans cette thèse, on commence par définir l'anneau de Cox équivariant d'une variété normale munie d'une action d'un groupe algébrique. Puis, on étudie ses propriétés générales et on le relie à l'anneau de Cox ordinaire. On s'appuie alors sur ce travail préliminaire pour étudier différents aspects de l'anneau de Cox d'une variété normale rationnelle de complexité un. On obtient en particulier un résultat de finitude pour l'itération des anneaux de Cox, ainsi qu'une caractérisation de la log terminalité des singularités. On étudie en détail une classe d'exemples: les anneaux de Cox de variétés de dimension trois presque homogènes sous SL_2 (c.à.d. normales et admettant une orbite dense). Prolongeant les résultats obtenus sur ces exemples, on donne une présentation par générateurs et relations de l'anneau de Cox d'une variété presque homogène de complexité un. Pour obtenir ce dernier résultat, on introduit et étudie une classe intéressante de complétions équivariantes de certains espaces homogènes de complexité un.

Abstract

The Cox ring of an algebraic variety (satisfying some natural conditions) is a very rich invariant. It was introduced by Cox in 1995 for the study of toric varieties, and then generalized to normal varieties by Arzhantsev, Berchtold and Hausen. Later, Hu and Keel discovered that the normal varieties with a finitely generated Cox ring define a class of varieties whose birational geometry is particularly well understood. They called them the Mori Dream Spaces (MDS) by virtue of their good behaviour with respect to the minimal model program of Mori. A first problem is to find natural conditions for a normal variety to be an MDS. A second one is to describe the Cox ring of a given MDS: find a presentation by generators and relations, give the nature of its singularities, etc...

Among algebraic varieties equipped with an action of an (affine) algebraic group, a particularly well understood class consists of normal varieties of complexity at most one: a connected reductive group is acting in such a way that the minimal codimension of an orbit of a Borel subgroup is at most one. The normal varieties of complexity zero are the spherical varieties (e.g. a toric variety is spherical). In 2007, Brion proved that spherical varieties are MDS, and gave a description of their Cox ring by generators and relations. A normal variety of complexity one is an MDS if and only if it is a rational variety (e.g. a normal rational surface with an effective $\mathbb{G}_m$-action or a normal SL_2-threefold with a dense orbit). This provides a natural class of MDS with group action for which the second problem has only been solved in very particular cases.

In this thesis we begin by defining the equivariant Cox ring of a normal variety equipped with an action of an algebraic group. We study its general properties and relate it to the ordinary Cox ring. We then build on this preliminary work to study various aspects of the Cox ring of a normal rational variety of complexity one. Notably, we obtain a finiteness result for the iteration of Cox rings and we characterize the log terminality of singularities. Then, we study in detail a class of examples: the Cox rings of the SL_2-threefolds which are almost homogeneous (i.e. normal and having a dense orbit). Extending the results obtained on these examples, we give a presentation by generators and relations of the Cox ring of an almost homogeneous variety of complexity one. To obtain this last result, we introduce and study an interesting class of equivariant completions of certain complexity one homogeneous spaces.

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Chapter 1

Introduction

Dans cette introduction, on présente différents aspects des anneaux de Cox et on explique leur importance dans l'étude des variétés algébriques. On dresse également, sans chercher l'exhaustivité, un certain état des connaissances sur le sujet. Notre point de vue est orienté par le sujet de ce travail de thèse : la description de l'anneau de Cox d'une certaine classe de variétés algébriques avec action de groupe algébrique. On conclut en résumant les résultats obtenus et en esquissant quelques pistes de recherche.

Les chapitres suivant cette introduction sont organisés comme suit. Dans le Chapitre 2, on donne des définitions et résultats préliminaires en théorie géométrique des invariants. Après ce chapitre, la suite du document est la reproduction de trois articles [V1], [V2], [V3] dont le premier est à paraître dans Transformation Groups. Ceci explique que la rédaction passe du français à l'anglais à cet endroit.

Conventions. La lettre k désigne un corps algébriquement clos de caractéristique nulle. On travaille dans la catégorie des variétés algébriques (c.à.d. les k-schémas de type fini séparés et intègres et les k-morphismes entre eux). Sans précision supplémentaire, un point d'une variété algébrique désigne un point fermé. Un groupe algébrique est un k-schéma en groupes affine de type fini. Pour les définitions et résultats de base concernant les variétés algébriques munies d'une action d'un groupe algébrique, voir le Chapitre 2. Soit M un groupe abélien de type fini, on note $\mathbb{D}(M)$ le groupe diagonalisable dont le groupe des caractères est M (voir le Chapitre 2).

1.1 Panorama du sujet

Dans les fondements de la géométrie algébrique moderne, on commence par formaliser la dualité entre algèbre et géométrie. On peut énoncer une version de cette dualité de la manière suivante : toute k-algèbre intègre de type fini R définit intrinsèquement une *variété algébrique affine* X (on dit que X est le spectre de R et on note $X = \operatorname{Spec} R$). De plus, R s'identifie canoniquement à l'algèbre $\mathcal{O}(X)$ des *fonctions régulières* sur X (ou *algèbre de coordonnées*). Avec ceci, les propriétés géométriques des variétés algébriques affines se traduisent dans le langage de l'algèbre commutative.

Les *variétés algébriques* (ou plus simplement variétés) sont obtenues à partir des variétés affines par recollement. Une classe très importante de variétés est constituée des *variétés projectives*, c'est-à-dire les variétés X telles qu'il existe une immersion fermée

$$\iota : X \hookrightarrow \mathbb{P}^n_k$$

dans un espace projectif. Ces variétés sont complètes, par suite leur algèbre des fonctions régulières ne contient que les constantes. On ne peut donc pas espérer pour celles-ci un analogue naïf de l'algèbre de coordonnées d'une variété affine. Une alternative consiste à considérer l'*algèbre des coordonnées homogènes* S_ι associée à l'immersion fermée ι, c'est-à-dire l'algèbre des coordonnées du cône $\tilde{X}$ sur X dans l'espace affine $\mathbb{A}_k^{n+1}$. On peut encore voir cette algèbre comme l'image du morphisme gradué

$$k[x_0, ..., x_n] \to \bigoplus_{m\geqslant 0} \Gamma(X, \mathcal{L}^{\otimes m}),\ x_i \mapsto s_i,$$

où le faisceau inversible très ample $\mathcal{L} := \iota^*\mathcal{O}_{\mathbb{P}_k^n}(1)$ sur X et le uplet $(s_0, ..., s_n)$ de sections globales définissent le morphisme ι (voir [Har2, II.7]). Avec cette interprétation, les coordonnées homogènes sur une variété projective, ou plus généralement quasi-projective, sont des sections de faisceaux inversibles.

La $\mathbb{Z}_{\geqslant 0}$-graduation de S_ι définit une action du *groupe multiplicatif* $\mathbb{G}_m$ (le groupe algébrique tel que $\mathbb{G}_m(k) = k^*$) sur $\tilde{X} = \operatorname{Spec} S_\iota$. Le lieu $\mathcal{V}(S_\iota^+)$ des zéros des coordonnées homogènes de degré strictement positif est l'ensemble des points fixes pour cette action. L'ouvert $\hat{X} := \tilde{X} \setminus \mathcal{V}(S_\iota^+)$ est alors une réunion d'orbites fermées (dans $\hat{X}$) et l'ensemble $\operatorname{Proj} S_\iota$ de ces orbites est naturellement muni d'une structure de variété isomorphe à X. Aussi, l'algèbre S_ι joue pour X un rôle analogue à celui joué par l'algèbre de coordonnées d'une variété affine. Par exemple, les sous-variétés fermées de X sont définies par des idéaux premiers homogènes, et les faisceaux quasi-cohérents par des modules gradués sur S_ι [Har2, II.2 et II.5].

Toutefois, deux plongements différents peuvent définir des algèbres de coordonnées homogènes non isomorphes. On peut par exemple plonger $\mathbb{P}_k^1$ dans lui-même par l'identité, et dans $\mathbb{P}_k^2$ comme une conique. Le cône affine sur $\mathbb{P}_k^1$ est $\mathbb{A}_k^2$, tandis que le cône affine sur la conique est une quadrique dans $\mathbb{A}_k^3$. Cette dernière admettant une singularité à l'origine, les algèbres de coordonnées de ces deux variétés affines ne peuvent être isomorphes. A l'inverse des variétés affines pour lesquelles l'algèbre des coordonnées ne dépend pas du choix d'une immersion fermée dans un espace affine, il n'y a pas a priori pour les variétés projectives de choix canonique d'une algèbre de coordonnées homogènes.

Une heuristique pour remédier à ce problème serait de construire une « algèbre de coordonnées totale » contenant toutes les sections de (classes d'isomorphisme de) faisceaux inversibles sur X. Une telle algèbre R devrait être graduée par le *groupe de Picard* de X (le groupe abélien noté $\operatorname{Pic}(X)$ constitué des classes d'isomorphisme de faisceaux inversibles sur X avec pour opération le produit tensoriel sur $\mathcal{O}_X$). De plus, toute algèbre de coordonnées homogènes devrait être munie d'un morphisme gradué vers R. Supposons que $\operatorname{Pic}(X)$ soit engendré librement par un nombre fini de classes de faisceaux inversibles $[\mathcal{L}_1], ..., [\mathcal{L}_r]$. Puis posons

$$R := \bigoplus_{(n_i)\in\mathbb{Z}^r} \Gamma(X, \mathcal{L}_1^{\otimes n_1} \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} \mathcal{L}_r^{\otimes n_r}). \tag{1.1.1}$$

C'est une k-algèbre $\operatorname{Pic}(X)$-graduée ayant les propriétés recherchées. Par ailleurs, on a manifestement $R = \Gamma(X, \mathcal{R})$, où $\mathcal{R}$ est la $\mathcal{O}_X$-algèbre quasi-cohérente $\operatorname{Pic}(X)$-graduée

$$\mathcal{R} := \bigoplus_{(n_i)\in\mathbb{Z}^r} \mathcal{L}_1^{\otimes n_1} \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} \mathcal{L}_r^{\otimes n_r}.$$

On explique maintenant comment ces algèbres s'incarnent géométriquement. Tout faisceau inversible $\mathcal{L}$ sur X définit naturellement un fibré en droites L sur X en prenant le spectre relatif de son algèbre symétrique

$$L := \operatorname{Spec}_X(\bigoplus_{n\in\mathbb{N}} \mathcal{L}^{\otimes n}) \to X.$$

Le faisceau des sections de ce morphisme s'identifie canoniquement avec le faisceau inversible dual $\mathcal{L}^\vee := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, noté également $\mathcal{L}^{-1}$ en raison de l'isomorphisme canonique $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$ (la classe $[\mathcal{L}^\vee]$ est donc l'inverse de $[\mathcal{L}]$ dans $\operatorname{Pic}(X)$). L'image L_0 de la section nulle coïncide avec l'ensemble des points fixes sous l'action naturelle de $\mathbb{G}_m$ dans les fibres, et on a un $\mathbb{G}_m$-*torseur* (ou $\mathbb{G}_m$-*fibré principal*) $L^\times$ sur X défini par

$$L^\times := L \setminus L_0 = \mathrm{Spec}_X(\bigoplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}) \to X.$$

Lorsque R est de type fini, on a ainsi un diagramme de variétés

$$\begin{tikzcd} \hat{X} = \mathrm{Spec}_X \mathcal{R} = L_1^\times \times_X \dots \times_X L_r^\times \arrow[r, "j"] \arrow[d, "q"] & \tilde{X} = \mathrm{Spec}(R) \\ X & \end{tikzcd}, \tag{1.1.2}$$

où q est un $\mathbb{G}_m^r$-torseur sur X (le morphisme structural de $\hat{X}$ comme variété sur X), et j est le morphisme d'*affinisation* de $\hat{X}$ (en tant que morphisme vers une variété affine, ce morphisme est canoniquement défini à partir du morphisme identité $\mathrm{Id} : R \to \mathcal{O}(\hat{X}) = R$).

1.1.1 Les origines

Une *variété torique* est une variété normale X munie d'une action d'un tore T telle qu'il existe un point $x \in X$ dont le morphisme d'orbite associé

$$T \to X,\ t \mapsto t.x$$

soit une immersion ouverte. Par exemple, les espaces affines ou les espaces projectifs sont naturellement des variétés toriques. Pour l'étude de la géométrie de ces variétés, Cox leur associe un « anneau de coordonnées homogènes », désormais appelé anneau de Cox, ou anneau de coordonnées total [Co]. Cet anneau est une algèbre de polynômes, graduée par le *groupe des classes de diviseurs*, et joue pour une variété torique un rôle analogue à celui joué par l'algèbre des coordonnées homogènes d'un espace projectif. Lorsque la variété torique est projective lisse avec un groupe de Picard libre de rang fini, l'anneau de Cox est isomorphe comme k-algèbre graduée à l'algèbre R définie en (1.1.1).

L'article de Cox est sans doute le point de départ de la théorie, bien que la construction (1.1.2) vue plus haut soit antérieure. En effet, la variété $\hat{X}$ n'est autre que le *torseur universel* introduit par Colliot-Thélène et Sansuc en géométrie arithmétique [CTS]. Ce torseur $\hat{X} \xrightarrow{q} X$ ne dépend pas du choix des $\mathcal{L}_i$ à isomorphisme près, et il est universel en ce sens que q se factorise à travers tout torseur sur X sous un groupe diagonalisable. Plus généralement, considérons une forme Y de X définie sur un corps de nombres K, et le K-tore T correspondant au $\mathrm{Gal}(k/K)$-module $\mathrm{Pic}(X)$. Un torseur universel sur Y est un T-torseur $\hat{Y}$ sur Y tel que le changement de base $\hat{Y}_k \to X$ est isomorphe au torseur universel $\hat{X}$ sur X. Les faits suivants dus à Colliot-Thélène et Sansuc justifient l'importance de cette notion : lorsque X est projective et lisse,

- tout point rationnel $x \in Y(K)$ provient (au sens de l'application naturelle $\hat{Y}(K) \to Y(K)$) d'un torseur universel uniquement determiné par x,
- il n'y a qu'un nombre fini de torseurs universels à isomorphisme près.

De plus, les torseurs universels devraient être plus simples que Y du point de vue arithmétique (voir [Pe, Section 8]).

1.1.2 L'anneau de Cox d'une variété normale

Le cadre naturel généralisant la construction de Cox est de considérer une **variété normale** X dont le groupe des classes de diviseurs $\mathrm{Cl}(X)$ **est de type fini**, par exemple une variété normale et rationnelle. On va voir que cette hypothèse de finitude est nécessaire pour espérer obtenir un anneau de Cox de type fini (comme k-algèbre). On rappelle brièvement les idées principales de la construction (pour les détails, on pourra consulter le livre de synthèse [ADHL]).

Lorsque $\mathrm{Cl}(X)$ est libre de rang fini, on peut considérer un sous-groupe M du groupe des diviseurs de Weil $\mathrm{WDiv}(X)$ tel que la restriction $M \to \mathrm{Cl}(X)$ de la projection canonique soit un isomorphisme. On pose alors

$$\mathcal{R} := \bigoplus_{D\in M} \mathcal{O}(D) \text{ et } \mathrm{Cox}(X) := \Gamma(X,\mathcal{R}) = \bigoplus_{D\in M} \Gamma(X,\mathcal{O}(D)),$$

où $\mathcal{O}(D)$ est le $\mathcal{O}_X$-module dont les sections sur un ouvert $U \subset X$ sont par définition

$$\Gamma(U,\mathcal{O}(D)) := \{f \in k(X)^*; (\mathrm{div}(f)+D)_{|U} \geqslant 0\} \cup \{0\}.$$

Ce sont respectivement le *faisceau de Cox* et l'*anneau de Cox* de X. Le faisceau de Cox est une $\mathcal{O}_X$-algèbre quasi-cohérente $\mathrm{Cl}(X)$-graduée, la multiplication étant définie via la flèche naturelle

$$\Gamma(U,\mathcal{O}(D_1)) \otimes_k \Gamma(U,\mathcal{O}(D_2)) \to \Gamma(U,\mathcal{O}(D_1+D_2)),$$

pour tout ouvert $U \subset X$, et $D_1, D_2 \in M$. On peut montrer que cette algèbre ne dépend pas du choix de M à isomorphisme gradué près [ADHL, 1.4.1.1]. En particulier lorsque X est lisse, on a un isomorphisme d'algèbres $\mathrm{Pic}(X)$-graduées $\mathrm{Cox}(X) \simeq R$, où R est l'algèbre construite en (1.1.1).

Lorsque $\mathrm{Cl}(X)$ admet un sous-groupe de torsion non trivial, on choisit un sous-groupe de type fini $M \subset \mathrm{Cl}(X)$ tel que la (restriction de la) projection canonique $\pi : M \to \mathrm{Cl}(X)$ soit surjective. Puis on pose

$$\mathcal{S} := \bigoplus_{D\in M} \mathcal{O}(D),$$

que l'on munit de la $\mathrm{Cl}(X)$-graduation induite par π. Ensuite on choisit une $\mathbb{Z}$-base $(E_1 = \mathrm{div}(f_1), ..., E_r = \mathrm{div}(f_r))$ du noyau de π. On définit alors

$$\mathcal{R} := \mathcal{S}/\mathcal{I} \text{ et } \mathrm{Cox}(X) := \Gamma(X,\mathcal{R}),$$

où $\mathcal{I}$ est le faisceau d'idéaux engendré par les éléments

$$1 - f_i, \text{ où } 1 \in \mathcal{O}(X) \text{ et } f_i \in \Gamma(X,\mathcal{O}(-E_i)).$$

Imposer ces relations revient à identifier les sections homogènes $s_1 \in \Gamma(U,\mathcal{S}_{D_1})$ et $s_2 \in \Gamma(U,\mathcal{S}_{D_2})$ sur un ouvert U chaque fois que l'on a $D_1 = D_2 + E$ dans M avec E principal, et $s_2 = s_1 f$ où f est un monôme de Laurent en les f_i tel que $E = \mathrm{div}(f)$. L'idéal $\mathcal{I}$ est $\mathrm{Cl}(X)$-homogène, ce qui munit $\mathcal{R}$ et $\mathrm{Cox}(X)$ d'une $\mathrm{Cl}(X)$-graduation. Enfin, pour toute composante M-homogène $\Gamma(X,\mathcal{O}(D))$ de $\Gamma(X,\mathcal{S})$, la projection canonique $\Gamma(X,\mathcal{S}) \to \mathrm{Cox}(X)$ se restreint en un isomorphisme d'espaces vectoriels [ADHL, Lem. 1.4.3.4]

$$\Gamma(X,\mathcal{O}(D)) \to \mathrm{Cox}(X)_{[D]}.$$

Lorsque **les fonctions régulières inversibles sur X sont constantes** (par exemple lorsque X est complète), on peut montrer que cette construction de l'anneau de Cox est indépendante à isomorphisme près des choix réalisés [ADHL, Sec. 1.4.3].

Lorsque l'anneau de Cox est de type fini, le faisceau d'algèbres $\mathcal{R}$ est également de type fini et on a un diagramme de variétés

$$\begin{tikzcd} \hat{X} = \mathrm{Spec}_X \mathcal{R} \arrow[r, hook, "j"] \arrow[d, "q"] & \tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X)) \\ X & \end{tikzcd}, \tag{1.1.3}$$

où $\hat{X}$ est l'*espace caractéristique* de X, et $\tilde{X}$ l'*espace de coordonnées total* de X. De plus la graduation de $\mathcal{R}$ (resp. $\mathrm{Cox}(X)$) définit une action du groupe diagonalisable $\mathbb{D}(\mathrm{Cl}(X))$ dont le groupe des caractères est $\mathrm{Cl}(X)$ sur $\hat{X}$ (resp. $\tilde{X}$). Le morphisme d'affinisation j est une immersion ouverte équivariante pour ces actions, et le morphisme q est un *bon quotient* (c-à-d que q est affine, $\mathbb{D}(\mathrm{Cl}(X))$-invariant, et induit un isomorphisme $\mathcal{O}_X \simeq (q_*\mathcal{O}_{\hat{X}})^{\mathbb{D}(\mathrm{Cl}(X))}$; voir le

Chapitre 2) de $\hat{X}$ sous l'action de $\mathbb{D}(\mathrm{Cl}(X))$. Ainsi, on retrouve X comme quotient d'un ouvert de $\tilde{X}$ sous l'action de $\mathbb{D}(\mathrm{Cl}(X))$.

Les variétés partageant le même anneau de Cox sont donc dans la même classe birationnelle. De plus, celles qui sont projectives sont en nombre fini. Lorsque $\mathcal{O}(X) = k$, l'ensemble de ces variétés projectives est en bijection naturelle avec un ensemble de cônes convexes polyédraux vivant dans l'espace vectoriel $\mathrm{Cl}(X)_{\mathbb{Q}} := \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (voir la Section 1.1.5 ci-dessous). Ceci révèle des liens importants entre la théorie des anneaux de Cox et la géométrie birationnelle.

Remarquons que pour tout ouvert $U \subset X$ dont le complémentaire dans X est de codimension au moins deux, la restriction induit un isomorphisme canonique

$$\mathrm{Cox}(X) \simeq \mathrm{Cox}(U).$$

En particulier, X et son lieu lisse X_{sm} partagent le même anneau de Cox. Par ailleurs, comme les faisceaux $\mathcal{O}_D$ sont inversibles sur X_{sm}, le bon quotient q induit un $\mathbb{D}(\mathrm{Cl}(X))$-torseur au-dessus de X_{sm}. De plus, le complémentaire de $q^{-1}(X_{\mathrm{sm}})$ est de codimension au moins deux dans $\hat{X}$ [ADHL, Prop. 1.6.1.6]. On dit que q est un *presque torseur diagonalisable* (voir Section 1.1.4). Enfin, notons que lorsque X est lisse avec un groupe de Picard libre, le diagramme ci-dessus s'identifie avec le diagramme (1.1.2).

1.1.3 Quelques exemples

L'espace de coordonnées total comme cône affine sur une variété projective

L'exemple le plus simple est celui de l'espace projectif $\mathbb{P}^n_k = \mathrm{Proj}(k[x_0, ..., x_n])$. On a un isomorphisme de groupes $\mathbb{Z} \to \mathrm{Pic}(\mathbb{P}^n_k), m \mapsto \mathcal{O}(mD)$, où D est l'hyperplan $x_0 = 0$. En suivant la construction ci-dessus, on trouve

$$\mathrm{Cox}(\mathbb{P}^n_k) = \bigoplus\nolimits_{m\in\mathbb{Z}} \Gamma(\mathbb{P}^n_k, \mathcal{O}(mD)) \simeq \bigoplus\nolimits_{m\geqslant 0} k[x_0, ..., x_n]_m = k[x_0, ..., x_n],$$

où l'isomorphisme identifie $\Gamma(\mathbb{P}^n_k, \mathcal{O}(D)) = \mathrm{Vect}(1, x_1/x_0, ..., x_n/x_0)$ avec la composante homogène de degré un de $k[x_0, ..., x_n]$. On voit ainsi que l'anneau de Cox coïncide ici avec l'algèbre de coordonnées homogènes associée à l'identité de $\mathbb{P}^n_k$. En particulier, l'espace de coordonnées total $\tilde{X} = \mathrm{Spec}(\mathrm{Cox}(X))$ est le cône affine $\mathbb{A}^{n+1}_k$ sur $\mathbb{P}^n_k$. La graduation de l'anneau de Cox se traduit géométriquement comme l'action de $\mathbb{G}_m$ sur $\mathbb{A}^{n+1}_k$ par homothéties. Le diagramme (1.1.3) s'écrit alors

$$\begin{tikzcd} \mathbb{A}^{n+1}_k \setminus \{0\} \arrow[r, hook] \arrow[d, "/\mathbb{G}_m"] & \mathbb{A}^{n+1}_k \\ \mathbb{P}^n_k & \end{tikzcd}.$$

Plus généralement, soit X une intersection complète lisse de dimension au moins trois dans $\mathbb{P}^n_k$. Alors, le morphisme de restriction $\mathrm{Cl}(\mathbb{P}^n_k) \to \mathrm{Cl}(X)$ est un isomorphisme, et $\tilde{X}$ est le cône affine sur X dans $\mathbb{A}^{n+1}_k$ [ADHL, 4.1.1.7].

L'espace de coordonnées total d'une variété torique

Suivant Cox, l'anneau de Cox d'une variété torique est une algèbre de polynômes en les sections canoniques des diviseurs premiers dans le complémentaire de l'orbite dense. Ainsi, une variété torique peut être décrite comme quotient d'un ouvert d'un espace affine sous l'action d'un groupe diagonalisable. Réciproquement, une variété normale projective dont l'anneau de Cox est une algèbre de polynômes est une variété torique [ADHL, Ex. 3.2.1.5]. L'hypothèse de projectivité est importante car elle garantit une certaine maximalité parmi les variétés partageant le même anneau de Cox. Par exemple, le plan projectif moins quatre points en position générale n'est pas torique, toutefois son anneau de Cox est bien une algèbre de polynômes (la réunion d'un nombre fini de points est un fermé de codimension deux dans $\mathbb{P}^2_k$).

L'anneau de Cox d'un espace homogène

Soit G un groupe algébrique semisimple, et G/H un espace homogène. Comme G est semisimple, les fonctions régulières inversibles sur G/H sont constantes, si bien que l'anneau de Cox est bien défini à isomorphisme près. Quitte à remplacer G par un revêtement fini, on peut supposer que son groupe de Picard est trivial. Ceci implique un isomorphisme $\mathrm{Pic}(G/H) \simeq \hat{H}$, où $\hat{H}$ est le groupe des caractères de H [ADHL, Thm. 4.5.1.2]. Notons K l'intersection des noyaux des caractères de H, c'est un sous-groupe distingué de H, et H/K est le groupe diagonalisable dont le groupe des caractères est $\hat{H}$. Le H/K-torseur

$$G/K \to G/H$$

n'est autre que l'espace caractéristique de G/H [ADHL, Thm. 4.5.1.8].

On verra plus loin (1.1.5) que $\mathrm{Cox}(G/H) = \mathcal{O}(G/K)$ n'est pas en général de type fini. Néanmoins, G/K est toujours un espace homogène quasi-affine [ADHL, Lem. 4.5.1.10]. Lorsque

$$\mathrm{Cox}(G/H) = \mathcal{O}(G/K)$$

est de type fini, on a ainsi une immersion ouverte G-équivariante $G/K \to \operatorname{Spec} \mathcal{O}(G/K)$ dans une variété affine, c'est le *plongement canonique*.

1.1.4 Caractérisation algébrico-géométrique de l'anneau de Cox

Soit R une k-algèbre de type fini graduée par un groupe abélien de type fini M. Dans cette partie, on donne des conditions nécessaires et suffisantes pour que R soit l'anneau de Cox d'une variété normale X. On donne également la traduction géométrique de ces conditions en termes de propriétés de l'espace de coordonnées total. Commençons par dégager des conditions nécessaires en considérant une telle variété X.

Premières propriétés

Rappelons que l'on a une immersion ouverte $\hat{X} \hookrightarrow \tilde{X}$ et que le bon quotient $q : \hat{X} \to X$ induit un $\mathbb{D}(\mathrm{Cl}(X))$-torseur au-dessus du lieu lisse X_{sm}. De ces propriétés on tire facilement

$$\dim \mathrm{Cox}(X) = \dim \tilde{X} = \dim X + \dim \mathrm{Cl}(X)_{\mathbb{Q}},$$

où $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Si l'on remplace X par son lieu lisse, le morphisme q (et donc $\hat{X}$) devient lisse (en revanche l'anneau de Cox n'est pas modifié). En montrant par ailleurs que $\hat{X}$ est irréductible on obtient alors que $\mathrm{Cox}(X)$ est une k-algèbre intègre et normale [ADHL, Thm. 1.5.1.1].

Il est intéressant de remarquer que l'anneau de Cox se comporte bien vis à vis de la localisation en un élément homogène s. On a en effet

$$\mathrm{Cox}(X_s) = \mathrm{Cox}(X)[1/s],$$

où X_s désigne le complémentaire dans X du diviseur des zéros de s, et $\mathrm{Cox}(X)[1/s]$ est la localisation de $\mathrm{Cox}(X)$ en s [ADHL, Prop. 1.5.2.4].

Unités

Toute unité homogène de $\mathrm{Cox}(X)$ doit être constante par construction. Lorsque le groupe des classes de diviseurs (autrement dit le groupe graduant) est libre, toute unité est homogène donc en particulier constante. Par ailleurs, on montre que lorsque $\mathcal{O}(X) = k$ toute unité de l'anneau de Cox est constante [ADHL, Prop. 1.5.2.5]. Signalons pour finir que lorsque le groupe des classes admet un sous-groupe de torsion non trivial, on a en général $\mathrm{Cox}(X)^* \neq k^*$ [ADHL, Ex. 1.4.4.2].

$\mathrm{Cl}(X)$-factorialité

Lorsque $\mathrm{Cl}(X)$ est libre, il n'est pas difficile de montrer que $\mathrm{Cox}(X)$ est un anneau factoriel [ADHL, Prop. 1.4.1.5]. Cette hypothèse est toutefois nécessaire [ADHL, Ex. 1.4.4.1]. En revanche, l'anneau de Cox possède vis-à-vis de ses éléments homogènes une propriété analogue à la factorialité. Pour rendre cela précis, on a besoin de la définition suivante :

Définition. [ADHL, 1.5.3] Soient M un groupe abélien de type fini, et R une k-algèbre M-graduée intègre.

- Un élément non-nul et non-inversible $f \in R$ est *M-premier* s'il est homogène et $(f \mid gh)$ avec g, h homogènes implique $f \mid g$ ou $f \mid h$.
- R est *factoriellement M-gradué* si tout élément homogène non-nul et non-inversible f est un produit d'éléments M-premiers.
- Une $\mathbb{D}(M)$-variété normale X est *$\mathbb{D}(M)$-factorielle* si tout diviseur de Weil $\mathbb{D}(M)$-invariant sur X est principal.

L'anneau de Cox est $\mathrm{Cl}(X)$-factoriel [ADHL, Thm. 1.5.3.7], et donc l'espace de coordonnées total associé est une variété affine $\mathbb{D}(\mathrm{Cl}(X))$-factorielle. La décomposition dans $\mathrm{Cox}(X)$ d'un élément homogène s en produit de facteurs $\mathrm{Cl}(X)$-premiers reflète la décomposition du diviseur de Weil effectif $\mathrm{div}_X(s)$ en somme de diviseurs premiers.

Graduation presque libre

Le groupe diagonalisable $\mathbb{D}(\mathrm{Cl}(X))$ agit librement sur un ouvert de $\tilde{X}$ dont le complémentaire est de codimension au moins deux (par exemple $q^{-1}(X_{\mathrm{sm}})$ comme vu plus haut). Cette propriété admet une traduction en termes de graduation de l'algèbre de coordonnées [ADHL, Prop. 3.2.2.2] :

Définition 1.1.4.1. [ADHL, 3.2.1.1] La M-graduation de R est *presque libre* si R admet un ensemble $\{s_1, ..., s_m\}$ de générateurs M-premiers deux-à-deux non-associés (c.à.d. deux-à-deux distincts dans le monoïde quotient R/R^*) tel que tout ensemble formé de $m-1$ degrés de ces générateurs engendre $\mathrm{Cl}(X)$.

En vertu de sa caractérisation géométrique, cette propriété est indépendante du choix du système de générateurs. C'est donc bien une caractéristique intrinsèque de la graduation.

Caractérisation

D'après ce qui précède, les conditions ci-dessous sont nécessaires pour que R soit l'anneau de Cox d'une variété :

- R est intègre et normale,
- R est M-factorielle,
- la M-graduation de R est presque libre,
- les éléments homogènes inversibles de R sont constants.

Réciproquement, si R possède ces propriétés, alors un choix d'une certaine donnée combinatoire sur R permet de produire une variété algébrique normale dont le groupe des classes est M, et admettant R pour anneau de Cox (voir [ADHL, Thm. 3.2.1.4], et la Section 1.1.5 ci-dessous). Pour énoncer une traduction géométrique de cette caractérisation, on aura besoin de la définition suivante:

Définition. Soit G un groupe algébrique agissant sur une variété X et $q : X \to Y$ un morphisme G-invariant. On dit que q est un *presque torseur* s'il existe des ouverts (G-stables) $X_0 \subset X$, $Y_0 \subset Y$ dont les complémentaires respectifs sont de codimension au moins deux et tels que q induise un G-torseur $X_0 \to Y_0$. Un *presque torseur diagonalisable* est un presque torseur sous un groupe diagonalisable qui soit également un bon quotient.

Remarque. La notion de presque torseur est introduite dans une version beaucoup plus générale par Hashimoto dans [Has, Def. 0.4]. Notre version correspond à la notion d'action *fortement stable* d'un groupe diagonalisable (voir [ADHL, 1.6.4.1] et [V1, Rem. 2.6.2]).

Théorème. *[ADHL, Cor. 1.6.4.4] Soit Γ un groupe diagonalisable agissant sur une variété Z affine et normale. Supposons que les fonctions inversibles et homogènes sur Z soient constantes, que Z soit Γ-factorielle, et qu'il existe un ouvert Γ-stable $W \subset Z$ tel que $\operatorname{codim}_Z Z \setminus W \geqslant 2$ et le quotient $W \to X$ est un presque torseur diagonalisable sous Γ. Alors Z est l'espace de coordonnées total de X.*

1.1.5 Mori Dream Spaces

Un MDS (*Mori Dream Space*) est une variété normale dont l'anneau de Cox est de type fini. Cette notion introduite (avec des hypothèses plus restrictives) par Hu et Keel dans [HK] caractérise une classe de variétés dont la géométrie birationnelle est bien comprise. En voici des exemples :

- la droite projective et ses ouverts sont les seuls MDS de dimension un,
- une surface K3 est MDS si et seulement si son groupe d'automorphismes est fini [ADHL, Thm. 5.1.5.1],
- les variétés toriques et plus généralement les variétés sphériques (voir 1.1.7 ci-dessous),
- les variétés normales rationnelles de complexité un (voir 1.1.8 ci-dessous),
- les variétés de Fano (voir 1.1.9 ci-dessous).

Avant de passer en revue quelques idées importantes liées à cette notion, on rappelle quelques définitions.

Quelques définitions utiles en géométrie birationnelle

Pour une introduction détaillée aux notions introduites dans cette section, on pourra consulter [Deb, Chap. 1]. Un diviseur de Weil D sur une variété normale X est dit *effectif* s'il est à coefficients positifs. Les classes de diviseurs effectifs engendrent un cône convexe $\mathrm{Eff}(X)$ dans $\mathrm{Cl}(X)_{\mathbb{Q}}$, appelé *cône effectif*.

Le *lieu de base* d'un diviseur de Weil D est l'intersection des supports des diviseurs linéairement équivalents à D. Le *lieu de base stable* de D est l'intersection des lieux de base des nD pour $n \geqslant 1$. Le diviseur D est dit *mobile* si son lieu de base stable est de codimension au moins deux dans X. On note $\mathrm{Mov}(X)$ le cône (convexe) dans $\mathrm{Cl}(X)_{\mathbb{Q}}$ engendré par les classes de tels diviseurs. Par ailleurs, D est dit *semi-ample* si son lieu de base stable est vide, et on note $\mathrm{SAmple}(X)$ le cône (convexe) dans $\mathrm{Cl}(X)_{\mathbb{Q}}$ engendré par les classes de tels diviseurs.

Supposons que X soit projective avec un groupe des classes de diviseurs de type fini. Le groupe de Picard modulo son sous-groupe de torsion s'identifie avec le groupe des diviseurs de Cartier modulo l'équivalence numérique; le rang de ce groupe en tant que $\mathbb{Z}$-module libre est *le nombre de Picard* de X, on le note $\rho(X)$. Le cône $\mathrm{Nef}(X)$ est le cône (convexe fermé) de $\mathrm{Pic}(X)_{\mathbb{Q}}$ engendré par les classes de diviseurs de Cartier dont le degré sur toute courbe de X est positif. On a les inclusions

$$\mathrm{SAmple}(X) \subset \mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}}(X),$$

et le cône mobile est d'intérieur non-vide [ADHL, 4.3.3.2]. D'après le critère de Kleiman [Deb, Thm. 1.27], le cône ample est l'intérieur du cône nef :

$$\mathrm{Amp}(X) = \mathrm{Nef}(X)^{\circ}.$$

On verra plus loin que pour un MDS, les cônes nef et semi-ample coïncident.

On dit que X est $\mathbb{Q}$-*factorielle* si tout diviseur de Weil admet un multiple non-nul de Cartier, on a alors $\mathrm{Cl}(X)_{\mathbb{Q}} = \mathrm{Pic}(X)_{\mathbb{Q}}$. Modulo l'équivalence numérique, les 1-cycles engendrent le $\mathbb{Q}$-espace vectoriel $N_1(X)$ qui est en dualité avec $\mathrm{Pic}(X)_{\mathbb{Q}}$ via le produit d'intersection. Le *cône des courbes* est le cône (convexe) $\mathrm{NE}(X)$ de $N_1(X)$ engendré par les classes de courbes irréductibles, son adhérence est le cône dual de $\mathrm{Nef}(X)$.

Une *contraction* est un morphisme surjectif à fibres connexes entre deux variétés normales projectives. Le *lieu exceptionnel* $\mathrm{Ex}(\pi)$ d'un morphisme birationnel $\pi : X \to Y$ est l'ensemble des points de X où π n'est pas un isomorphisme local. Pour une contraction $\pi : X \to Y$, seuls trois cas se présentent :

- π est birationnel et $\mathrm{Ex}(\pi)$ est un diviseur premier (*contraction divisorielle*),
- π est birationnel et $\mathrm{Ex}(\pi)$ est de codimension au moins deux (*petite contraction*),
- $\dim X > \dim Y$ (*contraction fibrée*).

Le quatorzième problème de Hilbert

Le quatorzième problème de Hilbert pose la question suivante: Soit $G \subset \mathrm{GL}_n$ un sous-groupe algébrique agissant linéairement sur l'algèbre de polynômes $k[t_1, ..., t_n]$ par substitution des variables. La sous-algèbre $k[t_1, ..., t_n]^G$ constituée des polynômes G-invariants est-elle de type fini ? Si G est réductif, la réponse est positive d'après un résultat de Hilbert (G fini, $G = \mathrm{SL}_n$), puis Nagata. Toutefois, Nagata produit ensuite un contre-exemple en considérant une action bien choisie d'un groupe unipotent. L'idée derrière ce contre-exemple est de voir l'algèbre des invariants comme l'anneau de Cox de l'éclatement du plan projectif en certains points. On montre ensuite que le cône effectif de cet éclatement n'est pas polyédral. En conséquence, l'anneau de Cox ne peut être de type fini. Notons que le contre-exemple de Nagata permet de construire un espace homogène X sous SL_n tel que $\mathrm{Cox}(X)$ n'est pas de type fini [ADHL, Rem. 4.5.1.18].

En fait le problème originel de Hilbert est formulé de manière plus générale : soit $k(t_1, ..., t_n)$ une extension transcendante pure et $k \subset K \subset k(t_1, ..., t_n)$ une extension de corps intermédiaire. L'algèbre $R := K \cap k[t_1, ..., t_n]$ est-elle de type fini ? Zariski donne (quelques années avant le contre-exemple de Nagata) une interprétation algébrico-géométrique de cette version du problème [Z] : il existe une variété projective normale X et un diviseur de Weil effectif D sur X tels que R s'identifie à l'algèbre $S(D)$ des fonctions rationnelles sur X n'ayant des pôles que sur D, autrement dit

$$R \simeq S(D) = \bigcup_{n \geqslant 0} \Gamma(X, \mathcal{O}(nD)). \tag{1.1.4}$$

Il remarque également que toute *algèbre de sections* s'identifie naturellement à une algèbre de la forme (1.1.4). Une algèbre de sections est par définition une algèbre $\mathbb{N}^r$-graduée de la forme

$$S^+(D_1, ..., D_r) = \bigoplus\nolimits_{(n_i) \in \mathbb{N}^r} \Gamma(X, \mathcal{O}(\textstyle\sum_{i=1}^r n_i D_i)),$$

où $D_1, ..., D_r$ sont des diviseurs de Weil sur une variété normale X. Dans le contre-exemple de Nagata, l'anneau de Cox est une algèbre de sections au sens ci-dessus. Plus tard, Zariski montre que la non-finitude des algèbres de sections est un phénomène assez courant déjà en dimension deux. Pour citer Mumford [Mu], la réponse négative au problème de Hilbert révèle de profondes subtilités de la catégorie des variétés algébriques. Un aspect fondamental d'un MDS est que toute algèbre de sections lui étant associée est de type fini [ADHL, Sec. 4.3.3].

Remarque. Soit D un diviseur de Weil sur une variété normale $\mathbb{Q}$-factorielle X. Lorsque l'algèbre de sections associée est de type fini, on a une application rationnelle dominante canonique

$$\varphi_D : X \dashrightarrow X(D) := \operatorname{Proj} S^+(D).$$

Cette application est

- birationnelle si et seulement si $[D] \in \operatorname{Eff}(X)^\circ$,
- une *petite modification* (une application birationnelle induisant un isomorphisme en codimension un) si et seulement si $[D] \in \operatorname{Mov}(X)^\circ$,
- un morphisme si et seulement si $[D] \in \operatorname{SAmple}(X)$,
- un isomorphisme si et seulement si $[D] \in \operatorname{Amp}(X)$.

Variation des quotients (VGIT)

Soit $\tilde{X} = \operatorname{Spec} R$ un espace de coordonnées total. En tant qu'anneau de Cox, R est muni d'une graduation par un groupe abélien de type fini M. On cherche à déterminer les variétés projectives admettant R comme anneau de Cox (et M comme groupe des classes de diviseurs). On doit nécessairement avoir $R_0 = R^{\mathbb{D}(M)} = k$, sinon une telle variété n'existe pas. En particulier, on en déduit que $\tilde{X}$ est un *quasi-cône* (une $\mathbb{G}_m$-variété affine telles que les adhérences d'orbites se rencontrent en un unique point).

On commence par chercher les ouverts $\mathbb{D}(M)$-stables $U \subset \tilde{X}$ admettant un bon quotient projectif. D'après [ADHL, Thm. 3.1.4.3], ceux-ci sont en nombre fini et correspondent à des objets combinatoires vivant dans l'espace vectoriel $M_{\mathbb{Q}}$. Plus précisément, pour tout $x \in \tilde{X}$ on définit le cône convexe polyédral $\omega_x \subset M_{\mathbb{Q}}$ engendré par les $m \in M$ tel qu'il existe $f \in R_m$ satisfaisant $f(x) \neq 0$. Puis, pour tout élément m dans le cône ω_R des poids de R (le cône de $M_{\mathbb{Q}}$ engendré par les degrés des éléments homogènes non-nuls), on définit le *cône GIT*

$$\lambda(m) := \bigcap_{x \in \tilde{X}, m \in \omega_x} \omega_x$$

associé à m. Ces cônes GIT sont convexes polyédraux, en nombre fini, et leur réunion forme un éventail dans $M_{\mathbb{Q}}$ dont le support est ω_R. De plus chaque cône GIT λ définit un ouvert $\mathbb{D}(M)$-stable

$$\tilde{X}^{ss}(\lambda) := \{x \in \tilde{X}; \lambda \subset \omega_x\} \subset \tilde{X}$$

tel que l'on ait un bon quotient $\varphi_\lambda : \tilde{X}^{ss}(\lambda) \to X_\lambda$ avec X_λ projectif, et tout ouvert jouissant de ces propriétés est obtenu ainsi.

Remarque. Ces ouverts correspondent, dans le cadre de la *variation des quotients* (VGIT), aux ouverts *semi-stables* associés aux $\mathbb{D}(M)$*-linéarisations* du fibré trivial (voir le livre [MFK]).

Il reste à déterminer les quotients X_λ admettant R pour anneau de Cox. Soit $(m_i)_{1 \leqslant i \leqslant r}$ la famille des degrés correspondant à une famille de générateurs M-premiers deux à deux non-associés de R. Définissons $\operatorname{Mov}(R)$ comme l'intersection de tous les cônes engendrés par $r-1$ des m_i. Les variétés recherchées sont exactement celles dont l'intérieur de λ est inclus dans l'intérieur de $\operatorname{Mov}(R)$ [ADHL, Rem. 3.3.4.2]. Pour un tel λ, on a de plus [ADHL, Prop. 3.3.2.9]

$$\operatorname{Cl}(X_\lambda) = M,\ \operatorname{Eff}(X_\lambda) = \omega_R,\ \operatorname{Mov}(X_\lambda) = \operatorname{Mov}(R),\ \text{et } \operatorname{SAmple}(X_\lambda) = \lambda. \tag{1.1.5}$$

Théorie de Mori

Considérons un MDS complet X. Dans cette situation, Hu et Keel observent que la structure d'éventail sur $\mathrm{Eff}(X)$ définie par les cônes GIT encode également une paramétrisation des applications rationnelles φ_D (définies plus haut) vers les variétés projectives $X(D)$ associées aux algèbres de sections. Plus précisément, on dit que deux diviseurs effectifs D_1, D_2 sur X sont *Mori équivalents* [HK, 1.3] si leurs lieux de base stable coïncident et que l'on a un triangle commutatif

$$\begin{tikzcd} & X \arrow[dl, dashed, "\varphi_{D_1}"'] \arrow[dr, dashed, "\varphi_{D_2}"] & \\ X(D_1) \arrow[rr, "\simeq"] & & X(D_2). \end{tikzcd}$$

Cette relation est compatible avec l'équivalence linéaire, et définit une partition du cône effectif dite en *chambres de Mori*. En remarquant que l'on a $X(D) \simeq X_\lambda$, où $[D] \in \lambda^\circ$, on conclut que la décomposition en chambres de Mori coïncide avec la décomposition du cône effectif en cônes GIT. Considérons les cônes λ_i dont l'intérieur est non-vide et inclus dans $\mathrm{Mov}(X)^\circ$. Alors les applications rationnelles $\varphi_i : X \dashrightarrow X_{\lambda_i}$ définies par ces cônes sont exactement (à automorphisme près sur la cible) les petites modifications de X vers des variétés projectives et $\mathbb{Q}$-factorielles [ADHL, Rem. 3.3.4.2]. De plus on a d'après (1.1.5) la décomposition

$$\mathrm{Mov}(X) = \cup_{i=1}^r \varphi_i^* \mathrm{SAmple}(X_i)$$

du cône mobile. Réciproquement, une variété complète normale dont le groupe des classes est de type fini et dont le cône mobile admet une telle décomposition est un MDS [ADHL, Thm. 4.3.3.1].

Notons que la définition originelle de Hu et Keel d'un MDS est plus restrictive: c'est une variété projective normale $\mathbb{Q}$-factorielle dont le groupe de Picard est de type fini et telle que

1. le cône nef est engendré par les classes d'un nombre fini de diviseurs semi-amples,

2. il existe un nombre fini de petites modifications $\varphi_i : X \dashrightarrow X_i$, $i = 1, ..., r$ avec X_i projective $\mathbb{Q}$-factorielle telles que $\mathrm{Nef}(X_i)$ vérifie la condition précédente, et on a la décomposition

$$\mathrm{Mov}(X) = \cup_{i=1}^r \varphi_i^* \mathrm{Nef}(X_i).$$

Signalons tout de suite que pour un MDS projectif, on montre que les cônes nef et semi-ample coïncident [BeHa07, Prop. 7.4]; cela résulte essentiellement du critère d'amplitude de Kleiman et du fait que pour une telle variété le cône ample est l'intérieur du cône semi-ample (à nouveau [BeHa07, Prop. 7.4]). On constate ainsi qu'un MDS projectif $\mathbb{Q}$-factoriel vérifie la définition de Hu et Keel. En fait, leur version permet de caractériser une classe de variétés se comportant bien du point de vue du *programme du Modèle Minimal de Mori* (d'où le nom).

Partant d'une variété projective $\mathbb{Q}$-factorielle X, ce programme vise à transformer $X = X^0$ via une suite de contractions *extrémales*

$$\pi_j : X^j \to X^{j+1},\ j \geqslant 0$$

(c.à.d. contractant les courbes dont la classe engendre une arête du cône $\mathrm{NE}(X^j)$ ayant une intersection négative avec le diviseur canonique K_{X^j}). C'est un fait non-trivial que de telles contractions existent. Si la contraction π_j est divisorielle, on continue le programme avec X^{j+1} qui est alors $\mathbb{Q}$-factorielle. Si la contraction π_j est fibrée ou que le diviseur canonique $K_{X^{j+1}}$ est nef, la procédure se termine. Dans ce cas, on obtient un modèle birationnel de X dont l'étude devrait être simplifiée. Une difficulté technique survient chaque fois que l'on obtient une petite contraction $\pi_j : X^j \to Y$ car alors K_Y n'est pas $\mathbb{Q}$-Cartier et on ne peut plus a priori poursuivre le programme. Selon l'idée de Mori, il devrait alors exister une petite contraction

$$\pi_j^+ : (X^j)^+ \to Y \text{ avec } (X^j)^+ \text{ projective } \mathbb{Q}\text{-factorielle}$$

telle que $K_{(X^j)^+}$ intersecte positivement les courbes contractées par π_j^+. On dit que l'application birationnelle $\pi_j \circ (\pi_j^+)^{-1}$ est un *flip*, et on poursuit le programme avec $X^{j+1} := (X^j)^+$.

$$\begin{tikzcd} X^j \arrow[rr, dashed] \arrow[dr, "\pi_j"'] & & (X^j)^+ \arrow[dl, "\pi_j^+"] \\ & Y & \end{tikzcd}$$

L'existence des flips ainsi que la terminaison d'une suite de flips n'est pas acquise en général et c'est aujourd'hui l'objet de recherches très actives.

Soit X un MDS (projectif $\mathbb{Q}$-factoriel), et supposons avoir atteint la j^e étape du programme avec $X^0 = X, ..., X^j$ des MDS projectifs $\mathbb{Q}$-factoriels. Notons comme ci-dessus $X_1^j, ..., X_r^j$ les petites modifications projectives $\mathbb{Q}$-factorielles de X^j (avec $\varphi_1 = \mathrm{Id}$ par convention), ce sont tous des MDS. Si $\pi_j : X^j \to X^{j+1}$ est fibrée ou bien si K_{X^j} est nef, le programme est terminé. Si π_j est une contraction divisorielle, alors X^{j+1} est un MDS projectif $\mathbb{Q}$-factoriel [Ca, Prop. 4.1], et par récurrence le programme se termine (on a $\rho(X^{j+1}) = \rho(X^j) - 1$). Enfin, si $\pi_j : X^j \to Y$ est une petite contraction, alors $\pi_j^*(\mathrm{Nef}(Y))$ est une face de codimension 1 commune aux cônes $\mathrm{Nef}(X^j)$ et $\varphi_i^*(\mathrm{Nef}(X_i^j))$ pour un certain i ([Ca, Prop. 4.1]; on a en fait $\pi_j = \varphi_D$ pour un $[D]$ dans cette face). La petite modification φ_i est alors le flip de π_j. Si un nouveau flip est nécessaire il est alors de la forme $\varphi_k \varphi_i^{-1}$ pour un certain k et ainsi de suite [Ca, Cor. 3.12]. Une telle suite de flips se termine puisqu'il y a un nombre fini de petites modifications projectives $\mathbb{Q}$-factorielles de X et qu'une composition de flips ne peut être un isomorphisme [Ca, Thm. 4.2].

1.1.6 Description par générateurs et relations de l'anneau de Cox d'un MDS

La description de l'anneau de Cox d'un MDS par générateurs et relations est un problème naturel et important, mais en général ardu. Par exemple, considérons 8 points du plan projectif *en position générale* (c.à.d. aucun triplet de points colinéaires, aucun 6-uplet de points sur une conique, et aucun 8-uplet de points sur une cubique), et X l'éclatement du plan projectif en ces points. Alors X est une surface de Del Pezzo (lisse projective avec un diviseur anticanonique ample), son anneau de Cox est de type fini, et ce dernier admet pour système minimal de générateurs les sections canoniques des courbes irréductibles dont l'autointersection est égale à -1, soit 242 éléments [ADHL, Thm. 5.2.2.1]. De plus, l'idéal des relations entre ces générateurs est minimalement engendré par 17399 formes quadratiques [ADHL, Thm. 5.2.3.1].

Lorsqu'un MDS est muni d'une action d'un groupe algébrique, on peut tenter de tirer profit de cette structure supplémentaire pour obtenir une description de l'anneau de Cox. Ceci est bien sûr illustré par l'exemple historique des variétés toriques. Parmi les variétés normales munies d'une action d'un groupe algébrique G, une classe particulièrement bien comprise est celle des variétés de *complexité* au plus un : G est réductif connexe, et la codimension minimale d'une orbite d'un sous-groupe de Borel est au plus un. La description par générateurs et relations de l'anneau de Cox d'une variété sphérique est connue depuis l'article de Brion [Bri07] (voir aussi l'article de Gagliardi [G] pour une approche différente). Beaucoup reste à faire en complexité un quant à la description de l'anneau de Cox, c'est dans cette perspective que s'inscrit ce travail de thèse.

1.1.7 Variétés sphériques

Les *variétés sphériques* (c.à.d. les variétés normales de complexité zéro) forment une classe de variétés avec action de groupe particulièrement bien comprise (voir [T11, Chap. 5]). Cette classe inclut notamment les variétés toriques et les espaces homogènes projectifs rationnels.

Anneau de Cox d'une variété magnifique

Dans [Bri07], Brion commence par étudier l'anneau de Cox d'une *variété magnifique*. Ces variétés généralisent les espaces homogènes projectifs rationnels et constituent, d'après un résultat de Luna [T11, Thm. 30.15], une classe remarquable de variétés sphériques. Soit G un groupe semisimple simplement connexe, B un sous-groupe de Borel et $T \subset B$ un tore maximal. Une G-variété est dite magnifique de *rang* r si elle satisfait les trois conditions suivantes :

- c'est une variété complète et lisse,
- il existe une G-orbite ouverte dont le complémentaire est constitué de r diviseurs premiers lisses à croisement normaux,
- les adhérences des G-orbites sont exactement les intersections entre ces diviseurs.

La B-orbite ouverte d'une variété magnifique X est un espace affine dont le complémentaire est la réunion d'un nombre fini de diviseurs premiers B-invariants, les *couleurs* de X. Les classes des couleurs engendrent librement le groupe de Picard de X, ce dernier s'identifiant au groupe des classes de diviseurs de X.

Remarque. Soit G un groupe réductif connexe et B un sous-groupe de Borel. Le *rang* d'une G-variété normale X est le rang du groupe abélien libre $\Lambda(X)$ des B-poids de $k(X)$. Pour une variété magnifique, les deux notions de rang coïncident.

De plus, G étant semisimple simplement connexe, il existe une unique structure de G-algèbre sur $\mathrm{Cox}(X)$ (c.à.d. $\mathrm{Cox}(X)$ est un G-module rationnel sur lequel G agit par automorphismes d'algèbre) compatible avec la $\mathrm{Pic}(X)$-graduation. Autrement dit, pour tout faisceau inversible $\mathcal{L}$ sur X, il existe une unique *G-linéarisation* [Bri18, Thm. 4.2.2 et Thm. 5.2.1]. Géométriquement, cela signifie qu'il existe une unique action de G sur le fibré en droites associé $L \to X$ qui soit de plus compatible avec l'action de G sur X et commute avec l'action de $\mathbb{G}_m$ dans les fibres.

Pour étudier une G-algèbre R, une approche classique est d'étudier en premier lieu sa sous-algèbre R^U des invariants par la partie unipotente U de B. Cette dernière algèbre est naturellement $\hat{T}$-graduée et tout système de générateurs $\hat{T}$-homogène de R^U fournit des générateurs de R. En effet, des résultats classiques en théorie des représentations des groupes algébriques assurent que chaque générateur $\hat{T}$-homogène engendre un sous-G-module simple de R et que la réunion de ces G-modules simples engendre R comme k-algèbre. Ici, $\mathrm{Cox}(X)^U$ est engendré librement par les sections canoniques des couleurs et des diviseurs premiers G-stables, et on obtient ainsi des générateurs $\mathrm{Pic}(X)$-homogènes de $\mathrm{Cox}(X)$ en considérant les G-modules simples engendrés par ces sections [Bri07, Prop. 3.1.1]. Comme l'illustre l'exemple suivant, la description de l'idéal des relations entre ces générateurs est cependant plus délicate.

Exemple. La variété de drapeaux $X = G/B$ est par définition une variété magnifique de rang nul. En vertu de la décomposition de Bruhat, l'orbite du point de base $x := B/B$ par le Borel opposé B^- est ouverte. De plus, cette orbite est un espace affine dont le complémentaire est la réunion des *diviseurs de Schubert* $D_1, ..., D_r$, qui ne sont autres que les couleurs de X. On obtient ainsi des isomorphismes

$$\mathrm{Cl}(X) \simeq \mathrm{Pic}(X) \simeq \mathbb{Z}^r,$$

le dernier étant donné explicitement par $(m_i)_{1 \leqslant i \leqslant r} \mapsto \sum_{i=1}^r m_i[D_i]$. Par ailleurs, tout fibré en droites G-linéarisé sur G/B est à isomorphisme près de la forme

$$L_\chi := G \times^B \mathbb{A}^1_k$$

où B agit sur $\mathbb{A}^1_k$ via un caractère $\chi \in \hat{B}$ (voir [T11, Chap. 2]). De plus, comme G est semisimple simplement connexe, le morphisme $\hat{B} \to \mathrm{Pic}(X)$, $\chi \mapsto [L_\chi]$ est un isomorphisme. L'espace des sections du fibré L_χ est non-nul si et seulement si $-\chi$ est dominant, et il s'identifie au B-espace propre $\mathcal{O}(G)^{(B)}_{-\chi}$ pour l'action de B à droite. De plus, ces espaces de sections sont des G-modules simples (ce sont des G-modules pour la multiplication à gauche) et tout G-module simple est isomorphe à un tel espace. En accord avec la section 1.1.3, on a un isomorphisme canonique

$$\mathrm{Cox}(G/B) \simeq \mathcal{O}(G/U) \simeq \mathcal{O}(G)^U = \bigoplus\nolimits_{(m_i)\in\mathbb{N}^r} \mathcal{O}(G)^{(B)}_{\sum_i m_i\omega_i},$$

où les ω_i sont les poids fondamentaux. En notant $s_i \in \mathcal{O}(G)^{(B^-\times B)}_{(-\omega_i,\omega_i)}$ la section canonique de D_i, les composantes homogènes de $\mathrm{Cox}(G/B)$ s'identifient donc aux G-modules simples

$$V(\textstyle\sum_i m_i\omega_i) := \langle G.s_1^{m_1}...s_r^{m_r}\rangle = \mathcal{O}(G)^{(B)}_{\sum_i m_i\omega_i}, \quad (m_i) \in \mathbb{N}^r,$$

dont les vecteurs de plus haut poids (pour B^-) sont respectivement $-\sum_i m_i\omega_i$. En particulier, $\mathrm{Cox}(G/B)^U$ est l'algèbre de polynômes $k[s_1, ..., s_r]$, et on obtient un morphisme surjectif de G-algèbres $\mathbb{N}^r$-graduées

$$\varphi : (S := \mathrm{Sym}_k(V(\omega_1) \oplus ... \oplus V(\omega_r))) \to \mathrm{Cox}(G/B).$$

Pour $1 \leqslant i < j \leqslant r$, notons M_{ij} l'unique supplémentaire G-stable de la composante de Cartan dans $V(\omega_i) \otimes_k V(\omega_j) \subset S$ (la *composante de Cartan* d'un tel produit tensoriel est l'unique sous-G-module simple de plus haut poids $\omega_i + \omega_j$). Comme tout produit $V(\omega_i)V(\omega_j)$ dans $\mathrm{Cox}(G/B)$ est un G-module simple, on a $M_{ij} \subset \ker\varphi$. En fait les G-modules M_{ij} engendrent $\ker\varphi$ comme idéal de S [Bri07, Prop. 3.3.1].

Exemple. Prenons par exemple $G = \mathrm{SL}_3$, le sous-groupe B des matrices triangulaires supérieures et T le sous-groupe des matrices diagonales. En notant ε_i les caractères du tore, les poids fondamentaux sont $\omega_1 = \varepsilon_1$ et $\omega_2 = \varepsilon_1 + \varepsilon_2$. On a de plus

$$V(\omega_1) \simeq k^3 \text{ et } V(\omega_1) \simeq (k^3)^*,$$

où ces espaces vectoriels sont munis de l'action standard de G. Avec ces identifications, l'unique supplémentaire G-stable de la composante de Cartan dans $V(\omega_1) \otimes_k V(\omega_2)$ est la droite G-invariante engendrée par le tenseur $e_1 \otimes e_1^* + e_2 \otimes e_2^* + e_3 \otimes e_3^*$. On a donc un isomophisme

$$\mathrm{Cox}(G/B) \simeq k[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_1 + x_2y_2 + x_3y_3)$$

où la $\hat{T}$-graduation sur le membre de droite est définie par $\deg(x_i) = \omega_1$ et $\deg(x_i) = \omega_2$.

Anneau de Cox équivariant d'une variété sphérique

Considérons une G-variété sphérique X où G est un groupe réductif connexe. On identifie l'orbite ouverte de G à l'espace homogène G/H où H désigne le stabilisateur d'un point de base de X. Pour garder la trace de l'action de G au niveau de l'anneau de Cox, on cherche une structure de G-algèbre sur $\mathrm{Cox}(X)$ compatible avec la $\mathrm{Cl}(X)$-graduation. Ceci revient à munir les composantes homogènes du faisceau de Cox de G-linéarisations de sorte que celles-ci soient compatibles avec la structure de groupe de $\mathrm{Cl}(X)$. Autrement dit, on cherche une section du morphisme d'oubli

$$\mathrm{Cl}_G(X) \to \mathrm{Cl}(X),$$

où $\mathrm{Cl}_G(X)$ est le *groupe des classes équivariant* (le groupe des classes d'isomorphisme de faisceaux divisoriels G-linéarisés; voir [Bri07, Sec. 4.1]), et le morphisme consiste en l'oubli des linéarisations. Malheureusement, même dans le cas où ce morphisme est surjectif (par exemple lorsque G est factoriel) une telle section n'existe pas en général. De plus, lorsqu'une section existe, celle-ci n'est en général pas canonique. Rappelons également que l'anneau de Cox lui-même n'est défini canoniquement que sous l'hypothèse $\mathcal{O}(X)^* \simeq k^*$. Ces difficultés disparaissent si l'on considère l'*anneau de Cox équivariant* de la variété sphérique introduit dans [Bri07, Sec. 4]. L'idée est de prendre en compte toute les G-linéarisations possibles de faisceaux divisoriels afin d'obtenir une structure canonique de G-algèbre. On écrit alors

$$\mathrm{Cox}_G(X) = \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}_G(X)} \Gamma(X, \mathcal{F})$$

et on doit montrer qu'il existe des représentants $\mathcal{F}$ et une opération de multiplication faisant de cet espace vectoriel gradué une G-algèbre. Ceci est fait dans [Bri07, Prop. 4.2.2] en tirant profit de l'existence d'orbites ouvertes par G et B.

Cet anneau de Cox équivariant est ensuite obtenu à partir de l'anneau de Cox d'une certaine variété magnifique via un changement de base [Bri07, Thm. 4.3.2]. Plus précisément, quitte à remplacer G par un revêtement on peut le supposer de la forme $\boldsymbol{G} \times C$ où $\boldsymbol{G}$ est semisimple simplement connexe et C un tore. Le groupe des automorphismes G-équivariants de G/H s'identifie à $N_G(H)/H$ et ce groupe agit sur l'ensemble des couleurs de G/H. Le noyau de cette action s'écrit $\boldsymbol{H} \times C$ avec $\boldsymbol{H} \subset \boldsymbol{G}$, c'est la *clôture sphérique* de H. L'espace homogène $\boldsymbol{G}/\boldsymbol{H}$ est sphérique et admet une complétion magnifique $\boldsymbol{X}$. De plus, on a un morphisme G-équivariant

$$f : X_{\leqslant 1} \to \boldsymbol{X}$$

où $X_{\leqslant 1}$ désigne l'ouvert G-stable constitué des orbites de codimension au plus 1. Ce dernier ouvert admet un complémentaire dans X de codimension au moins deux si bien que l'on obtient (en prenant l'image inverse par f des sections de faisceaux inversibles sur $\boldsymbol{X}$) un morphisme canonique

$$f^* : \mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X}) \to \mathrm{Cox}_G(X).$$

Ce morphisme induit un morphisme entre les sous-algèbres d'invariants par $\boldsymbol{G}$. On a alors un isomorphisme ([Bri07, Thm. 4.3.2])

$$\mathrm{Cox}_G(X) \simeq \mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X}) \otimes_{\mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X})^{\boldsymbol{G}}} \mathrm{Cox}_G(X)^{\boldsymbol{G}}.$$

1.1.8 Variétés normales rationnelles de complexité un

Une variété normale X de complexité un sous un groupe réductif connexe G. On est alors dans l'une des alternatives suivantes :

- X est *presque homogène* (c-à-d normale avec une orbite dense).
- X admet une famille à un paramètre d'orbites sphériques.

Dans tous les cas, on a par un théorème de Rosenlicht [T11, 5.1] un quotient rationnel

$$\pi : X \dashrightarrow C,$$

par (le choix d') un sous groupe de Borel $B \subset G$. La courbe C est la courbe projective lisse telle que $k(C) = k(X)^B$. Ainsi les B-orbites générales définissent une famille à un paramètre de diviseurs premiers stables par B. Plus précisément, un théorème de stucture locale (de l'action de G) dû à Brion, Luna, Vust et Knop [T11, Thm. 4.7] assure qu'il existe un ouvert non-vide B-stable de X isomorphe à un produit de la forme

$$P_u \times \mathbb{T} \times C_0,$$

où P_u est le radical unipotent d'un sous-groupe parabolique de G contenant B, le facteur $\mathbb{T}$ est un tore, et C_0 est un ouvert non-vide de C. Au travers de cet isomorphisme, la projection sur C_0 s'identifie à la restriction de π. L'existence de cet ouvert implique aisément que le groupe des classes de diviseurs est de type fini si et seulement si X est rationnelle. On montre que ces conditions sont encore équivalentes à la génération finie de l'anneau de Cox [ADHL, Thm. 4.3.1.5].

En supposant donc que X est rationnelle, on a $C \simeq \mathbb{P}^1_k$. Comme X est normale et $\mathbb{P}^1_k$ est complet, π se restreint en un morphisme $V \to \mathbb{P}^1_k$ où V est un ouvert dont le complémentaire est de codimension au moins deux dans X. Ainsi, il existe à isomorphisme près un unique faisceau divisoriel $\mathcal{F}$ sur X dont la restriction à V est isomorphe à $\pi^*_{|V}\mathcal{O}_{\mathbb{P}^1_k}(1)$. Après le choix de coordonnées homogènes z_1, z_2 sur $\mathbb{P}^1_k$, les sections $\pi^*_{|V}(z_1)$, $\pi^*_{|V}(z_2)$ définissant le morphisme $\pi_{|V}$ s'étendent de manière unique en des sections $a, b \in \mathcal{F}(X)$. On a ainsi un morphisme d'image inverse

$$\pi^* : \mathrm{WDiv}(\mathbb{P}^1_k) \to \mathrm{WDiv}(X)^B,\ p = [\alpha : \beta] \mapsto \mathrm{div}(\beta a - \alpha b),$$

où $[\alpha : \beta]$ désigne les coordonnées homogènes de $p \in \mathbb{P}^1_k$. Les diviseurs premiers dans l'image de π^* sont appelés *diviseurs paramétriques*. Les autres diviseurs premiers B-stables sont appelés *diviseurs exceptionnels*, ils sont en nombre fini. L'image par π d'un diviseur exceptionnel est soit dense dans $\mathbb{P}^1_k$, soit un point de $\mathbb{P}^1_k$. Dans le second cas, le point est appelé *exceptionnel*.

A l'instar des variétés sphériques, les G-variétés dans la même classe G-birationnelle que X sont classifiées à isomorphisme G-équivariant près par des objets de nature combinatoire. En gros, X est entièrement décrite par un ensemble fini d'ouverts affines stables par B, et un tel ouvert est caractérisé par le choix d'un ensemble de valuations discrètes invariantes par B du corps des fonctions rationnelles $k(X)$. Plus précisément, on a une suite exacte

$$1 \to k(\mathbb{P}^1_k)^* \to k(X)^{(B)} \to \Lambda(X) \to 1,$$

où l'on rappelle que $k(\mathbb{P}^1_k) \simeq k(X)^B$ et $\Lambda(X)$ est le groupe des B-poids de $k(X)$. On choisit un scindage à droite de cette suite de manière à identifier $\Lambda(X)$ à un sous-groupe de $k(X)^{(B)}$. Pour toute *valuation géométrique* v (c.à.d. de la forme αv_D, où v_D est la valuation discrète normalisée associée à un diviseur premier D dans un modèle de $k(X)$, et $\alpha \in \mathbb{Q}_+$), la restriction $v_{|k(X)^{(B)}}$ est alors décrite par un triplet $(x, h, l) \in \mathbb{P}^1_k \times \mathbb{Q}_+ \times \mathcal{E}$, où $l = v_{|\Lambda(X)} \in \mathcal{E} := \mathrm{Hom}(\Lambda(X), \mathbb{Q})$ et $v_{|k(\mathbb{P}^1_k)} = hv_x$. Cela définit une application ϱ de l'ensemble des valuations géométriques de $k(X)$ dans l'ensemble $\mathbb{P}^1_k \times \mathbb{Q}_+ \times \mathcal{E}$. Tout ouvert affine B-stable $\mathring{X}$ d'une variété dans la même classe G-birationnelle que X définit une famille indexée par $\mathbb{P}^1_k$ de cônes convexes polyédraux strictement convexes dans $\mathbb{Q}_+ \times \mathcal{E}$: la famille des cônes convexes engendrés par les images par ϱ des valuations géométriques B-invariantes associées aux diviseurs premiers B-stables rencontrant $\mathring{X}$ (c.à.d. les couleurs et les diviseurs premiers G-stables). Une telle famille est appelée un *hypercône*. La donnée de l'hypercône et de l'ensemble des couleurs rencontrant $\mathring{X}$ est appelée un *hypercône colorié*. Un ensemble fini d'hypercônes coloriés satisfaisant certaines conditions de recollement (on parle d'*hyperéventail colorié*) définit une G-variété normale de complexité un dans la même classe que X. Inversement, toute telle G-variété est définie par un hyperéventail colorié. Cette classification combinatoire, se situant dans le prolongement des travaux de Luna et Vust [LV], est due à Timashev [T11, Chap. 3].

U-invariants de l'anneau de Cox et produit tensoriel de G-modules

Soit G un groupe semisimple. Un problème fondamental en théorie des représentations est de comprendre la décomposition du produit tensoriel de deux G-modules simples en somme directe de G-modules simples.

Supposons que G soit simplement connexe et choisissons un sous-groupe de Borel B contenant un tore maximal T. Notons $\Pi = \{\alpha_1, ..., \alpha_r\}$ l'ensemble des racines simples associé à ce choix et $\{\omega_1, ..., \omega_r\}$ l'ensemble des poids fondamentaux. Considérons la *variété de drapeaux double*

$$X := G/P_I^- \times G/P_J^-$$

où P_I^-, P_J^- sont les deux sous-groupes paraboliques respectivement opposés aux deux sous-groupes paraboliques standards définis par les sous-ensembles $I = \{\alpha_{i_1}, ..., \alpha_{i_s}\}, J = \{\alpha_{j_1}, ..., \alpha_{j_t}\} \subset \Pi$. En notant $V(\lambda)$ un G-module simple de plus haut poids λ, on a au regard de l'exemple 1.1.7

$$\operatorname{Cox}(X) = \operatorname{Cox}(G/P_I^-) \otimes_k \operatorname{Cox}(G/P_J^-) \simeq \bigoplus_{\lambda = m_{i_1}\omega_{i_1} + ... + m_{i_r}\omega_{i_s}} \bigoplus_{\mu = n_{i_1}\omega_{i_1} + ... + n_{i_r}\omega_{i_t}} V(\lambda) \otimes_k V(\mu).$$

Notons que cet anneau de Cox est de type fini et naturellement gradué par $\mathbb{Z}^s \times \mathbb{Z}^t$ (voir 1.1.7). Soient

$$\lambda := \sum m_{i_k}\omega_{i_k}, \quad \mu := \sum n_{j_l}\omega_{j_l}.$$

Le sous-G-module $V(\lambda) \otimes_k V(\mu) \subset \operatorname{Cox}(X)$ consiste en les éléments homogènes de degré $(\bar{m}, \bar{n}) := (m_{i_1}, ..., m_{i_s}, n_{j_1}, ..., n_{j_t})$. Soit U la partie unipotente de B, l'algèbre $\operatorname{Cox}(X)^U$ est naturellement $(\operatorname{Pic}(X) \times \hat{T})$-graduée et la multiplicité de $V(\nu)$ dans $V(\lambda) \otimes_k V(\mu)$ est la dimension de l'espace vectoriel $\operatorname{Cox}(X)^U_{(\bar{m},\bar{n}),\nu}$ constitué des éléments $(\operatorname{Pic}(X) \times \hat{T})$-homogènes de poids $((\bar{m}, \bar{n}), \nu)$. Lorsque $\operatorname{Cox}(X)^U$ admet une description agréable, par exemple une algèbre de polynômes engendrée par des éléments homogènes ou bien le quotient d'une telle algèbre par une unique équation, on peut alors espérer calculer facilement ces multiplicités.

Dans [Pon], Ponomareva décrit $\operatorname{Cox}(X)^U$ par générateurs et relations pour toutes les variétés de drapeaux doubles de complexité $\leqslant 1$ (voir aussi [Pa99] pour des exemples antérieurs). En complexité un, Ponomareva décrit plus généralement $\operatorname{Cox}(X)^U$ lorsque X est n'importe quelle G-variété rationnelle lisse complète de complexité un. Soit $(s_i)_i$ la famille finie des sections canoniques des diviseurs exceptionnels E_i de X, et $(x_i)_i = ([\alpha_i : \beta_i])_i$ la famille finie des points exceptionnels associée au quotient rationnel $X \dashrightarrow \mathbb{P}^1_k$. En notant

$$\pi^*(x_i) = \textstyle\sum_i n_i E_i,$$

l'algèbre $\operatorname{Cox}(X)^U$ est engendrée par les éléments a, b, s_i et l'idéal des relations est engendré par les suivantes

$$\beta_i a - \alpha_i b = \textstyle\prod_i s_i^{n_i}.$$

Cas d'une variété à orbites toriques

Dans leurs travaux [AH06] et [AH08], Altmann, Hausen et Süss étudient les variétés normales avec action d'un tore. Ils décrivent une telle variété X par un certain objet combinatoire appelé *éventail divisoriel*. La description de cet éventail divisoriel utilise une variété auxiliaire Y qui est un quotient rationnel de X par le tore agissant. Dans [HS], Hausen et Süss donnent une description de l'anneau de Cox d'une telle variété X lorsque Y est projective, le groupe des classes de X de type fini et les fonctions régulières sur X constantes. Ils construisent un morphisme injectif de k-algèbres graduées $\operatorname{Cox}(Y^\circ) \hookrightarrow \operatorname{Cox}(X)$, où $Y^\circ \subset Y$ est un ouvert non-vide (obtenu à partir de l'éventail divisoriel définissant X). Puis, ils donnent une description par générateurs et relations de $\operatorname{Cox}(X)$ vu comme algèbre sur $\operatorname{Cox}(Y^\circ)$. En particulier, l'anneau de Cox de X est de type fini sur k si et seulement si $\operatorname{Cox}(Y)$ l'est.

Lorsque X est rationnelle de complexité un, on peut prendre $Y = \mathbb{P}^1_k$ et cette description donne une présentation de $\operatorname{Cox}(X)$ par générateurs et relations comme k-algèbre; ces relations sont alors des trinômes [HS, Cor. 4.9].

Cas d'une variété à orbites horosphériques

Soit G un groupe réductif connexe et G/H un espace homogène. Cet espace homogène est dit *horosphérique* si H contient un sous groupe unipotent maximal de G. D'après la décomposition de Bruhat, G/H est sphérique. De plus on a un torseur $G/H \to G/P$ sous le tore $\mathbb{T} = P/H$ où $P = N_G(H)$ est un sous-groupe parabolique de G. Une *variété horosphérique* est une G-variété normale dont toutes les orbites sont horosphériques. Dans [LT], Langlois et Terpereau donnent une description de l'anneau de Cox d'une variété horosphérique rationnelle X de complexité un. Bien sûr, les orbites générales forment une famille à un paramètre d'orbites horosphériques. En fait, un résultat de Knop [Kn90] montre qu'il existe un espace homogène horosphérique G/H, et un ouvert G-stable de X qui est G-isomorphe à $G/H \times U$, où U est un ouvert de $\mathbb{P}^1_k$. Langlois et Terpereau montrent que X est naturellement une $\mathbb{T}$-variété avec quotient rationnel $X \dashrightarrow G/P \times \mathbb{P}^1_k$. De plus, ils donnent la description de X en termes d'éventail divisoriel sur $G/P \times \mathbb{P}^1_k$. En se basant sur le résultat de Hausen et Süss mentionné dans la section précédente, ils en déduisent une présentation de l'anneau de Cox par générateurs et relations (comme k-algèbre) en utilisant la description de l'anneau de Cox d'une variété de drapeaux.

Cas d'une SL_2-variété normale affine de dimension trois presque homogène

Dans [BatHad], Batyrev et Haddad décrivent l'anneau de Cox d'une variété comme dans le titre. La classification de Timashev montre qu'une telle variété X est un plongement normal de SL_2/μ_n, où μ_n est le groupe des racines $n$ièmes de l'unité. De plus, le complémentaire de SL_2/μ_n dans X est un diviseur premier G-stable. La classification de ces plongements affines remonte à l'article de Popov [Pop] dans lequel il montre que X est entièrement déterminée par l'entier n et un certain nombre rationnel $h_X = p/q \in [0;1[$, $p \wedge q = 1$, appelé *hauteur* de X.

Batyrev et Haddad proposent pour l'espace de coordonnées total $\tilde{X}$ l'hypersurface d'équation $y^b = x_1x_4 - x_2x_3$ dans l'espace affine $\mathbb{A}^5_k$ muni des coordonnées x_1, x_2, x_3, x_4, y, où $b = (q-p)/k$ avec $k = (q-p) \wedge n$. Ils montrent que ce candidat est le bon en vérifiant qu'il satisfait les propriétés caractéristiques d'un espace de coordonnées total.

1.1.9 Itération d'anneaux de Cox

L'espace de coordonnées total $\tilde{X}$ d'un MDS X peut très bien avoir un groupe des classes non-trivial (sauf lorsque $\mathrm{Cl}(X)$ est libre). Dans un tel cas, à supposer que les fonctions régulières inversibles sur $\tilde{X}$ soient constantes, on peut définir $\mathrm{Cox}(\tilde{X})$. Si ce dernier est de type fini, on obtient un espace de coordonnées total $\tilde{X}^{(2)}$ et ainsi de suite. Ce procédé itératif s'intitule l'*itération des anneaux de Cox*, et s'arrête dans l'un des trois cas suivants :

- on obtient un espace de coordonnées total dont l'anneau de Cox n'est pas bien défini ($\mathrm{Cl}(\tilde{X}^{(n)})$ admet un sous-groupe de torsion non-trivial et $\mathcal{O}(\tilde{X}^{(n)})^* \not\simeq k^*$),
- on obtient un espace de coordonnées total dont l'anneau de Cox n'est pas de type fini,
- on obtient un espace de coordonnées total factoriel ($\mathrm{Cl}(\tilde{X}^{(n)}) = 0$).

La première alternative ne survient jamais si l'on considère un MDS complet [Bra, Prop. 4.4]. Par ailleurs, on ne sait pas actuellement si l'itération d'un MDS disons complet se termine. Lorsque c'est le cas, le dernier anneau de Cox obtenu est appelé l'*anneau de Cox maître*. Si celui ci est de type fini (troisième alternative), l'espace caractéristique itéré $q : \hat{X}^{(n)} \to X$ satisfait une propriété universelle: ce morphisme se factorise à travers tout presque torseur sur X par un groupe réductif résoluble (la composante neutre est un tore et le groupe des composantes est un groupe résoluble fini).

L'itération des anneaux de Cox est introduite par Arzhantsev, Braun, Hausen et Wrobel dans [ABHW]. Ils prouvent que pour une variété normale rationnelle de complexité un sous l'action d'un tore ayant des singularités au plus *log terminales* (voir Section 1.1.10), l'itération des anneaux de Cox se termine avec un anneau de Cox maître de type fini et factoriel.

Les singularités log terminales d'une surface normale sont exactement les singularités quotients $\mathbb{A}_k^2/F$, où F est un sous-groupe fini de GL_2. Remarquons que le plan affine est naturellement muni de l'action de $\mathbb{G}_m$ par homothéties et cette action passe au quotient par F. Il suit que ces singularités quotients sont des $\mathbb{G}_m$-variétés normales de complexité un.

Les groupes binaires polyédraux $F_{\mathbb{T}}, F_{\mathbb{O}}, F_{\mathbb{I}}, F_{\mathbb{D}_n}, n \geqslant 2$ et les groupes cycliques, qui sont à conjugaison près les sous-groupes finis de SL_2, définissent en particulier les singularités E_6, E_7, E_8, D_n, A_n de *du Val*. Par exemple, l'itération des anneaux de Cox de E_7 donne lieu à une suite de morphismes structuraux d'espaces caractéristiques

$$\mathbb{A}_k^2 \xrightarrow{/\mu_2} A_1 \xrightarrow{/\mu_2\times\mu_2} D_4 \xrightarrow{/\mu_3} E_6 \xrightarrow{/\mu_2} E_7, \tag{1.1.6}$$

dont la composition est le quotient du plan affine par $F_{\mathbb{O}}$. Cette suite reflète la suite $(D^n(F_{\mathbb{O}}))_n$ des sous-groupes dérivés:

$$F_{\mathbb{O}} \supset F_{\mathbb{T}} \supset F_{\mathbb{D}_2} \supset \mu_2 \supset 1.$$

Les quotients successifs dans (1.1.6) sont les quotients par les facteurs $D^n(F_{\mathbb{O}})/D^{n+1}(F_{\mathbb{O}})$, $n \leqslant 3$.

Une des motivations des auteurs de [ABHW] est de réaliser en dimension supérieure les singularités log terminales (avec action d'un tore en complexité un) comme des singularités quotients. Grossièrement, le plan affine est remplacé par une variété affine normale factorielle de complexité un sous un tore et le sous-groupe fini de GL_2 par un groupe réductif résoluble.

Gagliardi montre dans [G] que les variétés sphériques admettent une itération des anneaux de Cox finie (de longueur au plus deux) avec un anneau de Cox maître de type fini et factoriel. Une *variété de Fano* est une variété projective lisse dont le faisceau anticanonique est ample. Plus généralement, une *variété de type Fano* est une variété projective normale X admettant un $\mathbb{Q}$-diviseur Δ tel que $-K_X + \Delta$ est ample et le couple (X, Δ) est *Kawamata log terminal* (voir [Deb, Def. 7.24]). Il est montré dans [BCHM], que les variétés de type Fano sont des MDS. De plus, Braun obtient dans [Bra] que ces variétés admettent une itération des anneaux de Cox finie avec un anneau de Cox maître de type fini et factoriel.

1.1.10 Singularités de l'espace de coordonnées total

Bien que l'anneau de Cox d'un MDS soit une k-algèbre normale, il n'est pas nécessairement de Cohen-Macaulay [Bro]. Toutefois, un résultat récent de Braun [Bra] montre que l'anneau de Cox d'un MDS X est *Gorenstein* (le diviseur canonique $K_{\tilde{X}}$ est de Cartier). L'étude des singularités de l'anneau de Cox est motivée par le résultat suivant dû à Gongyo, Okawa, Sannai et Takagi [GOST] : Soit X une variété projective $\mathbb{Q}$-factorielle. Alors les deux conditions suivantes sont équivalentes

- X est de type Fano.
- X est un MDS avec au plus des singularités log terminales.

Rappelons que X a des singularités au plus *log terminales* si les conditions suivantes sont satisfaites

- X est $\mathbb{Q}$-Gorenstein (un multiple de K_X est de Cartier).
- Il existe une résolution des singularités $\varphi : Z \to X$ telle que

$$K_Z = \varphi^* K_X + \sum \alpha_i E_i,\ \alpha_i > -1,$$

où l'on somme sur les diviseurs exceptionnels E_i de φ.

Soit X un MDS projectif de complexité un sous l'action d'un tore. On peut montrer que $\tilde{X}$ est encore de complexité un sous un certain tore $\mathbb{T}$ [ADHL, Thm. 4.2.3.2]. Supposons que $\tilde{X}$ soit un MDS (autrement dit $\tilde{X}$ est rationnel). Dans [ABHW], les auteurs caractérisent la log terminalité de $\tilde{X}$ en termes de propriétés arithmétiques des exposants associés aux relations trinomiales entre les générateurs de $\mathrm{Cox}(\tilde{X})$, de tels anneaux de Cox sont appelés *Platoniciens*.

1.2 Résultats obtenus dans la thèse

1.2.1 Construction et étude de l'anneau de Cox équivariant

Comme évoqué plus haut nous avons étudié dans cette thèse l'anneau de Cox des variétés normales avec action de groupe et plus particulièrement des variétés normales de complexité un. Dans cette perspective, on a commencé par généraliser la construction de Brion de l'anneau de Cox équivariant d'une variété sphérique.

Soit G un groupe algébrique et X une G-variété normale dont les fonctions régulières G-invariantes sont constantes. Après avoir choisi un point rationnel x dans le lieu lisse de X on obtient pour chaque classe de faisceau divisoriel G-linéarisé $[\mathcal{F}] \in \mathrm{Cl}_G(X)$ un représentant canonique $\mathcal{F}^x$ (Sec. 3.2.3). Puis, on parvient à munir le G-module gradué

$$\mathrm{Cox}_G(X) := \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}_G(X)} \Gamma(X, \mathcal{F}^x)$$

d'une multiplication telle que l'on obtienne une k-algèbre graduée sur laquelle G opère par automorphismes. On montre que l'algèbre obtenue ne dépend pas du choix de x à isomorphisme de G-algèbre graduée près, et on l'appelle l'anneau de Cox équivariant de X (Sec. 3.2.3 et Rem. 3.2.5.12). On montre ensuite que c'est une algèbre intègre et normale (Cor. 3.2.6.7). Supposons de plus qu'elle est de type fini, on a alors un diagramme de variétés

$$\begin{tikzcd} \hat{X}_G \arrow[r, hook] \arrow[d, "q"] & \tilde{X}_G \\ X & \end{tikzcd}$$

analogue à celui de la construction de Cox usuelle. Ici, $\hat{X}_G := \mathrm{Spec}_X(\bigoplus_{[\mathcal{F}]\in \mathrm{Cl}_G(X)} \mathcal{F}^x)$ est l'espace caractéristique équivariant, et $\tilde{X}_G := \mathrm{Spec}\,\mathrm{Cox}_G(X)$ l'espace de coordonnées total équivariant. Ce sont des $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-variétés et le morphisme d'affinisation $\hat{X}_G \hookrightarrow \tilde{X}_G$ est une immersion ouverte équivariante. De plus, le morphisme structural q est un presque torseur diagonalisable G-équivariant sur X. Ce morphisme est universel dans le sens où il se factorise à travers tout tel morphisme vers X (Sec. 3.2.5).

On a un morphisme canonique d'algèbres graduées

$$\varphi : \mathrm{Cox}_G(X) \to \mathrm{Cox}(X)$$

dont le morphisme induit entre les groupes graduant est l'oubli

$$\mathrm{Cl}_G(X) \to \mathrm{Cl}(X)$$

des G-linéarisations. Supposons G connexe et de groupe de Picard trivial; cette dernière hypothèse pouvant être satisfaite quitte à remplacer G par un revêtement fini. Ces morphismes sont alors surjectifs (Rem. 3.2.2.2), et on obtient que $\ker \varphi$ est engendré comme idéal par une suite régulière très simple (Cor. 3.2.9.3). Avec ceci, si l'on obtient une présentation de l'anneau de Cox équivariant on en déduit aisément une présentation de l'anneau de Cox usuel.

1.2.2 U-invariants en complexité un

Soit G un groupe semisimple simplement connexe, et X une G-variété rationnelle lisse complète de complexité un. On a alors un isomorphisme canonique $\mathrm{Cox}_G(X) \simeq \mathrm{Cox}(X)$ et on a déjà évoqué un résultat de Ponomareva décrivant la sous-algèbre $\mathrm{Cox}(X)^U$ des U-invariants, où U est la partie unipotente d'un sous-groupe de Borel.

On généralise ce résultat au cas où G est réductif connexe et X une G-variété normale rationnelle de complexité un telle que $\mathcal{O}(X)^{*G} \simeq k^*$. Plus précisément, on commence par montrer que l'anneau de Cox équivariant est de type fini dans cette situation (Prop. 3.3.1.4). Puis, on donne des générateurs de $\mathrm{Cox}_G(X)^U$ et on trouve des générateurs de l'idéal des relations dans les cas suivants (Thm. 3.3.2.3 et Prop. 3.3.2.4):

- $\mathrm{Cl}_G(X)$ est libre (par exemple lorsque X est lisse complète (Prop. 3.3.1.7)),
- $G = \mathbb{T}$ est un tore et $\mathcal{O}(X)^{\mathbb{T}} \simeq k$,
- X est presque homogène.

Dans le cas $G = \mathbb{T}$ cela donne une généralisation du résultat de Hausen et Süss mentionné en 1.1.8 au cas où l'on a seulement $\mathcal{O}(X)^{\mathbb{T}} \simeq k$.

1.2.3 $\mathrm{Cox}(X)^U$ comme anneau de Cox d'une variété de complexité un sous l'action d'un tore

Soit G un groupe réductif connexe dont le groupe de Picard est trivial, X une G-variété normale, et U la partie unipotente d'un sous-groupe de Borel. Alors X est en particulier une U-variété et on a un isomorphisme canonique $\mathrm{Cox}_U(X) \simeq \mathrm{Cox}(X)$. Il suit que $\mathrm{Cox}(X)$ est canoniquement muni d'une structure de U-algèbre. Supposons que X soit rationnelle de complexité un et satisfait une des trois conditions données dans la section précédente. En combinant le résultat sur la description de $\mathrm{Cox}_G(X)^U$ et l'expression de l'anneau de Cox ordinaire comme quotient de l'anneau de Cox équivariant, on obtient une description par générateurs et relations de $\mathrm{Cox}(X)^U$.

En se basant sur des travaux de Hausen et Herppich [ADHL, 3.4.2 and 3.4.3], on déduit de cette présentation que $\mathrm{Cox}(X)^U$ est l'anneau de Cox d'une variété de complexité un sous l'action d'un tore (Prop. 3.3.3.2). En particulier, cela implique que $\mathrm{Cox}(X)^U$ est une intersection complète (Cor. 3.3.3.3).

1.2.4 Singularités de l'anneau de Cox

Soit G un groupe réductif connexe dont le groupe de Picard est trivial, et U la partie unipotente d'un sous-groupe de Borel. Dans la lignée des résultats évoqués en 1.1.10 on obtient (Thm. 3.3.4.3)

Théorème. *Soit X une G-variété presque homogène de complexité un. Alors,*

$$\tilde{X} \text{ a des singularités au plus log terminales} \iff \mathrm{Cox}(X)^U \text{ est un anneau Platonicien.}$$

Essentiellement par le résultat de Braun assurant que l'anneau de Cox d'un MDS est Gorenstein, la log terminalité est équivalente à la *rationalité* des singularités. Pour obtenir le résultat on utilise alors les ingrédients suivants

- les singularités rationnelles se comportent bien par passage aux U-invariants [T11, Thm. D.5],
- $\mathrm{Cox}(X)^U$ est l'anneau de Cox d'une variété de complexité un sous un tore,
- les résultats de Arzhantsev, Braun, Hausen et Wrobel caractérisant la log terminalité de l'anneau de Cox d'une variété de complexité un sous un tore [ABHW].

1.2.5 Itération d'anneaux de Cox

Soit G un groupe réductif connexe et X une G-variété normale rationnelle de complexité un. On montre, lorsque X est complète ou presque homogène, que l'itération des anneaux de Cox est bien définie dans le sens où chaque espace de coordonnées total itéré admet un anneau de Cox bien défini. Avec cette hypothèse, on obtient de plus (Thm. 3.3.5.2)

Théorème. *L'itération des anneaux de Cox de X est finie. Dans le cas presque homogène, l'anneau de Cox maître est de type fini et factoriel.*

On esquisse la preuve en expliquant les principaux ingrédients. Tout d'abord on remarque que le morphisme structural $q_1 : \hat{X} \to X$ d'un espace caractéristique admet une factorisation canonique

$$\hat{X} \xrightarrow{g_1} X' \xrightarrow{f_1} X$$

où g_1, f_1 sont respectivement un presque torseur sous un tore et un presque torseur sous un groupe fini diagonalisable. Un fait très utile est que si les fonctions inversibles sur $\hat{X}$ sont constantes, alors X' et $\hat{X}$ ont même espace caractéristique (Prop. 3.2.10.1). Avec nos hypothèses ceci permet de construire une suite de presque torseurs par des groupes diagonalisables finis

$$X'^{(n)} \xrightarrow{f_n} X'^{(n-1)} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} X' \xrightarrow{f_1} X$$

telle que l'espace caractéristique de $X'^{(i)}$, $1 \leqslant i \leqslant n$ est $\hat{X}^{(i+1)}$. On peut supposer que $\hat{X}^{(n)}$ est rationnel, sinon son anneau de Cox n'est pas de type fini et l'itération s'arrête. On obtient alors à chaque étape n un carré commutatif

$$\begin{tikzcd} X'^{(n)} \arrow[r, dashed, "\pi_n"] \arrow[d, "f_n"] & \mathbb{P}^1_k \arrow[d, "\varphi_n"] \\ X'^{(n-1)} \arrow[r, dashed, "\pi_{n-1}"] & \mathbb{P}^1_k, \end{tikzcd}$$

où φ_n est un quotient géométrique par un groupe fini diagonalisable. On montre ensuite que l'itération se poursuit tant que φ_n est ramifié. On conclut par contradiction en montrant qu'on ne peut avoir une tour infinie de tels revêtement ramifiés.

1.2.6 Anneau de Cox d'un plongement de SL_2 modulo un sous-groupe fini

Les premiers exemples de variétés presque homogènes de complexité un sont les plongements normaux de SL_2/F, où F est un sous-groupe fini. Comme évoqué plus haut, ces sous-groupes sont à conjugaison près les sous-groupes cycliques du tore maximal T formé des matrices diagonales, ou bien les groupes binaires polyédraux. L'anneau de Cox équivariant d'un tel plongement X est canoniquement isomorphe à l'anneau de Cox ordinaire car SL_2 est semisimple et simplement connexe. De plus, cet anneau est de type fini puisqu'une variété presque homogène de complexité un est automatiquement rationnelle.

L'algèbre des invariants $\mathrm{Cox}(X)^{\mathrm{SL}_2}$ est la sous-algèbre de polynômes engendrée par les sections canoniques associées aux diviseurs premiers $X_1, ..., X_r$ dans le complémentaire de SL_2/F (Prop. 3.2.8.5). En tant que SL_2-variété affine on a alors un bon quotient

$$f : \mathrm{Cox}(X) \to \mathbb{A}^r_k$$

par SL_2 qui induit un quotient géométrique au dessus de $\mathbb{G}^r_m$. Ce dernier quotient géométrique est une fibration sur $\mathbb{G}^r_m$ ayant pour fibre l'espace de coordonnées total de SL_2/F (Prop. 3.2.8.1). La *fibre spéciale* est par définition la fibre schématique de f en 0. Cette dernière notion ainsi que les faits évoqués dans ce paragraphe sont valables dans le cas général d'un MDS presque homogène sous un groupe réductif connexe (Sec. 3.2.8).

Pour une variété sphérique, Brion montre que le morphisme f est fidèlement plat et la fibre spéciale est une variété normale [Bri07, Prop. 4.2.3]. De plus, pour une variété magnifique, cette fibre est horosphérique [Bri07, Sec. 3.2]. Ces propriétés sont des ingrédients importants pour la description de l'idéal des relations entre les générateurs de l'anneau de Cox d'une variété magnifique. Revenant à la SL_2-variété X, on donne un exemple où la fibre spéciale de f n'est pas irréductible, et un autre où elle n'est pas réduite (Sec. 4.4.2). On donne aussi des critères pour que cette fibre spéciale soit une variété normale en fonction du sous-groupe F et des données combinatoires du plongement.

On étudie ensuite l'itération des anneaux de Cox associée à X. D'après la section précédente, cette itération est finie avec un anneau de Cox maître de type fini et factoriel. Notant m le nombre d'itérations, on obtient les bornes suivantes en fonction du sous-groupe F (Prop. 4.4.4.13)

- $m = 0$ (c.à.d. X est factoriel) exactement quand $X = G$ ou $X = G/F_{\mathbb{I}}$,
- $m \leqslant 1$ si F est binaire icosaédral ou cyclique d'ordre $\leqslant 2$,
- $m \leqslant 2$ si F est cyclique d'ordre $\geqslant 3$,
- $m \leqslant 3$ si F est binaire diédral ou binaire tétraédral,
- $m \leqslant 4$ si F est binaire octaédral.

Dans le cas où F est cyclique on montre en fait que le groupe des classes de diviseurs de l'espace caractéristique est libre et on donne son rang (Lem. 4.4.4.8). Par ailleurs, notant U le groupe des matrices triangulaires supérieures unipotentes, on obtient un diagramme commutatif

$$\begin{tikzcd}
\tilde{X}^{(m)} \arrow[r] \arrow[d] & \ldots \arrow[r] & \tilde{X}^{(2)} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
\tilde{Y}^{(m)} \arrow[r] & \ldots \arrow[r] & \tilde{Y}^{(2)} \arrow[r] & \tilde{Y},
\end{tikzcd}$$

où les flèches horizontales sont des morphismes structuraux d'espaces caractéristiques, les flèches verticales sont des presque U-torseurs, et tous les carrés sont cartésiens (Cor. 4.4.4.5). De plus, $\tilde{Y}$ est l'espace de coordonnées total d'une T-variété de complexité un que l'on décrit explicitement comme un quotient (plus précisément un torseur) sous U d'un ouvert de X (Sec. 4.4.3).

On donne ensuite une description de l'anneau de Cox par générateurs et relations lorsque X est un plongement normal de SL_2/μ_n satisfaisant les deux conditions suivantes (Sec. 4.4.5):

- la fibre spéciale est une variété horosphérique,
- $\mathrm{Cl}(X)$ est libre.

On explique que l'on peut toujours se ramener à cette situation en voyant X comme un ouvert G-stable d'un plongement X_1 satisfaisant ces propriétés. L'anneau de Cox de X se décrit ensuite aisément comme un quotient de celui de X_1 (Prop. 3.2.8.1).

Remarque. Bien que cette réduction fonctionne théoriquement, décrire explicitement un tel plongement X_1 n'est pas évident. En effet, pour satisfaire la deuxième condition on utilise une complétion équivariante suivie d'une résolution équivariante des singularités. Dans la section suivante on trouve une méthode plus générale qui de plus contourne cette difficulté.

Finalement, on compare notre résultat avec l'approche de Batyrev et Haddad dans le cas d'un plongement affine (Sec. 4.4.5 et 4.4.5).

1.2.7 Anneau de Cox d'une variété presque homogène de complexité un

Soit G un groupe réductif connexe, B un sous-groupe de Borel, et G/H un espace homogène sphérique. Cet espace homogène est dit *sobre* s'il existe un plongement normal complet simple (il y a une unique orbite fermée) et toroïdal (tout diviseur premier B-stable contenant une G-orbite est G-stable). Lorsqu'un tel plongement existe, il est unique et on l'appelle le *plongement standard*. Par exemple, toute variété magnifique est un plongement standard d'un espace homogène sobre. L'existence du plongement standard se traduit aisément en termes combinatoires: le cône des *G-valuations* (c.à.d. le cône des valuations géométriques G-invariantes) doit être saillant (c.à.d. ne contient pas de droite). Cette dernière caractérisation d'un espace homogène sphérique sobre se traduit immédiatement en complexité supérieure (Def. 5.2.1.2). En s'appuyant sur les travaux de Knop relatifs aux cônes de G-valuations et sur la classification combinatoire de Timashev, on définit une famille canonique de plongements normaux complets d'un espace homogène sobre G/H de complexité un (Prop. 5.4.0.2). Cette famille est indexée par l'ensemble des parties finies de $\mathbb{P}^1_k$ (contenant l'ensemble des points exceptionnels associés à G/H), et on appelle ses éléments les *bonnes complétions* de G/H. Les bonnes complétions jouissent de propriétés géométriques remarquables: elles sont projectives, $\mathbb{Q}$-factorielles, et sont recouvertes par des ouverts affines isomorphes de groupe de Picard trivial et dont les fonctions régulières inversibles sont constantes.

Ces propriétés géométriques nous permettent d'obtenir une présentation de l'anneau de Cox équivariant d'une bonne complétion par générateurs et relations (Sec. 5.5). Cette G-algèbre est engendrée par un nombre fini de G-modules simples $V_1, ..., V_m$. En gros, ce sont les G-modules engendrés par les sections canoniques associées aux diviseurs exceptionnels. Pour décrire l'idéal des relations on mobilise à nouveau les techniques mises en œuvre dans le cas $G = \mathrm{SL}_2$. En particulier, on montre que la fibre spéciale est une variété horosphérique (Prop. 5.5.0.4). Notons que pour calculer effectivement l'idéal des relations en se basant sur notre résultat, il faut

- connaître les décompositions des $V_i \otimes_k V_j$ et $\mathrm{Sym}^2_k V_i$ en G-modules simples,
- calculer dans l'algèbre $\mathcal{O}(G)$ les décompositions des produits V_iV_j.

On effectue ces calculs sur un exemple issu de la géométrie algébrique classique: l'*espace des triangles ordonnés* (Sec. 5.8). Cette variété X est un plongement projectif normal de l'espace homogène sobre SL_3/T, où T est le tore maximal standard. Son anneau de Cox équivariant est isomorphe à l'anneau de Cox équivariant d'une certaine bonne complétion. On montre que cette bonne complétion est en fait une résolution des singularités de X (Sec. 5.8.4).

Finalement, on explique comment associer à toute G-variété presque homogène de complexité un X une bonne complétion $\boldsymbol{X}$ d'un espace homogène sobre $\boldsymbol{G}/\boldsymbol{H}$ (Sec. 5.6). Pour cela on utilise une traduction de la condition de sobriété en termes d'une condition sur un certain sous-groupe du normalisateur $N_{\boldsymbol{G}}(\boldsymbol{H})$. Cette traduction est due à Brion et Pauer dans le cas sphérique [BP] et à Knop en toute généralité [Kn93b]. On obtient alors un morphisme canonique d'algèbres graduées

$$\mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X}) \to \mathrm{Cox}_G(X).$$

L'anneau de Cox équivariant de X est ensuite obtenu à partir de celui de $\boldsymbol{X}$ par un changement de base (Thm. 5.6.0.2).

1.3 Quelques pistes de recherches

1.3.1 Caractéristique positive

La construction de l'anneau de Cox (resp. équivariant) fonctionne sans changement lorsque k est de caractéristique $p > 0$ (avec toujours $k = \bar{k}$). Toutefois, nous n'avons pas réussi à statuer quant à la normalité de l'anneau de Cox. En effet, considérons par exemple une variété lisse X dont le groupe de Picard admet de la p-torsion, et dont l'anneau de Cox est de type fini. On a alors un $\mathbb{D}(\mathrm{Pic}(X))$-torseur $q : \hat{X} \to X$, mais ce morphisme n'est pas lisse. Cette difficulté ne se présente pas en caractéristique nulle et la preuve de la normalité de l'anneau de Cox utilise alors de manière essentielle la lissité de q. Alternativement, il est tentant d'utiliser le critère de Serre mais cette piste n'a pour l'instant pas abouti.

1.3.2 Itération des anneaux de Cox

Un problème évident est de savoir si l'itération des anneaux de Cox d'un MDS est toujours finie. Un contre-exemple serait à chercher en complexité $\geqslant 2$. Par ailleurs, il serait intéressant de trouver des conditions naturelles assurant que l'itération est bien définie (cela signifie qu'à chaque étape de l'itération l'anneau de Cox est bien défini). De manière analogue, chercher des conditions pour que la propriété de MDS soit préservée tout au long de l'itération est une question intéressante. En complexité un par exemple, cela signifie trouver des conditions pour que les espaces caractéristiques itérés soient toujours rationnels (voir [HW] pour une telle condition dans le cas de l'action d'un tore).

1.3.3 Variété de complexité un à orbites sphériques

La description de l'anneau de Cox d'une G-variété normale rationnelle X de complexité un à orbites sphériques reste un problème ouvert en général. Un premier pas consiste à étudier le cas où la classe G-birationnelle de X est triviale, c'est-à-dire qu'il existe une application birationnelle G-équivariante

$$\mathbb{P}^1_k \times G/H \dashrightarrow X,$$

où G agit trivialement sur $\mathbb{P}^1_k$, et G/H est un espace homogène sphérique. En fait, on se ramène toujours à cette situation à quotient par un groupe fini abélien près. En effet, revenant au cas où la classe de X n'est à priori pas triviale, il existe un espace homogène sphérique G/H et une application rationnelle G-équivariante

$$\mathbb{P}^1_k \times G/H \dashrightarrow X,$$

qui induit un morphisme fini G-équivariant sur un ouvert dense G-stable de X. De plus, l'extension finie $k(X) \hookrightarrow k(\mathbb{P}^1_k \times G/H)$ est galoisienne de groupe de Galois abélien F. La normalisation X_1 de X dans $k(\mathbb{P}^1_k \times G/H)$ est alors une G-variété normale de même classe G-birationnelle que $\mathbb{P}^1_k \times G/H$. De plus, X_1 est muni d'une action de F telle que l'on ait un bon quotient

$$\gamma : X_1 \to X.$$

Pour ces faits, on pourra consulter l'article [L]. La tâche pourrait alors se découper ainsi:

1. étudier l'anneau de Cox équivariant des variétés à orbites sphériques de classe G-birationnelle triviale,

2. étudier le quotient γ. A-t-il de bonnes propriété ? par exemple est-ce un presque torseur ?

3. étudier le comportement de l'anneau de Cox équivariant par un quotient (de la base) tel que γ.

1.3.4 Théorie de Mori

Un MDS complet admet un nombre fini de petites modifications projectives $\mathbb{Q}$-factorielles (Sec. 1.1.5). Soient $X_1, ..., X_r$ de tels modèles que l'on suppose de plus être de complexité un sous un groupe réductif connexe. Ces modèles correspondent aux cônes maximaux dans la structure d'éventail sur le cône mobile de X donnée par la décomposition en chambres de Mori (Sec. 1.1.5). On peut supposer que les cônes correspondant à X_1 et X_2 se rencontrent le long d'une face de codimension un, et on considère la petite modification

$$\varphi_2 : X_1 \dashrightarrow X_2.$$

Soit D un diviseur de Cartier sur X_1 tel que sa classe soit contenue dans cette face, alors

$$\varphi_D : X_1 \to Y$$

est une petite contraction et φ_2 est son flip. Dans la lignée de [Bri93], il serait intéressant de comprendre ces modèles et ces contractions en termes de classification combinatoire des variétés G-birationnellement équivalentes à X. Un préalable serait de développer davantage la combinatoire des morphismes de G-modèles dans le langage de Timashev.

1.3.5 Détermination d'invariants de MDS de complexité un

Soit X un G-variété normale de complexité $\leqslant 1$. Un problème naturel est d'exprimer les invariants géométriques de X en termes de ses données combinatoires. Beaucoup reste à faire en complexité un dans cette direction. Pour certains invariants, la description de l'anneau de Cox peut être un atout. Notons par exemple

- les cônes de diviseurs: effectif, mobile, ample, semi-ample (voir Sec. 1.1.5),
- le groupe d'automorphismes de X (voir [ADHL, Sec. 4.2.4]).

Chapter 2

Préliminaires

On rappelle quelques notions et résultats de théorie géométrique des invariants. On utilise [MFK] comme référence générale et on travaille dans la catégorie des schémas algébriques, c'est-à-dire les k-schémas séparés de type fini et les k-morphismes. On fixe un groupe algébrique (affine) G.

2.1 Théorie géométrique des invariants

2.1.1 Quotient d'un G-schéma algébrique

Définition 2.1.1.1. Une *action* de G sur un schéma algébrique X est un morphisme $\sigma : G \times X \to X$ tel que les diagrammes

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{Id}_G \times \sigma"] \arrow[d, "m_G \times \mathrm{Id}_X"] & G \times X \arrow[d, "\sigma"'] \\ G \times X \arrow[r, "\sigma"'] & X \end{tikzcd} \quad \text{et} \quad \begin{tikzcd} \operatorname{Spec} k \times X \arrow[r, "e_G \times \mathrm{Id}_X"] \arrow[dr, "\simeq"'] & G \times X \arrow[d, "\sigma"'] \\ & X \end{tikzcd}$$

commutent. On dit alors que X est un G-schéma algébrique.

Définition 2.1.1.2. Soient X, Y deux G-schémas algébriques. Notons σ_X et σ_Y les actions respectives de G sur X et Y. Un morphisme $f : X \to Y$ est dit *G-équivariant* si $f \circ \sigma_X = \sigma_Y \circ (\mathrm{Id}_G \times f)$.

Définition 2.1.1.3. Soit X un G-schéma algébrique. Un morphisme $f : X \to Y$ est dit *G-invariant* s'il est équivariant lorsque l'on munit Y de l'action triviale de G.

Définition 2.1.1.4. Soit X un G-schéma algébrique. Un *quotient catégorique* de X par G est un couple (Y, f), où Y est un schéma algébrique et $f : X \to Y$ un morphisme, tel que la suite ci-dessous soit exacte

$$\begin{tikzcd} G \times X \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "p_X"'] & X \arrow[r, "f"] & Y \end{tikzcd} .$$

Cela signifie que pour tout couple (Z, g) où Z est un schéma algébrique et $g : X \to Z$ un morphisme tel que $g\sigma = gp_X$, le morphisme g se factorise de manière unique à travers Y via f.

Remarque 2.1.1.5. Autrement dit, un quotient catégorique (Y, f) de X par G est un morphisme $X \xrightarrow{f} Y$ qui est G-invariant et universel pour cette propriété. En particulier, un quotient catégorique est unique à unique isomorphisme près. Un quotient catégorique n'existe pas toujours, on a néanmoins le critère utile suivant.

Proposition 2.1.1.6. *Soit X un G-schéma algébrique, et $f : X \to Y$ un morphisme de schémas algébriques. Pour que (Y, f) soit un quotient catégorique il suffit que les conditions suivantes soient vérifiées*

1. *f est G-invariant.*
2. *La flèche naturelle $f^\sharp : \mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$ est un isomorphisme.*
3. *Pour tout fermé G-invariant W de X, l'image $f(W)$ est fermée dans Y.*
4. *Pour toute famille $(W_i)_{i\in I}$ de fermés G-invariants de X, on a $f(\cap_{i\in I} W_i) = \cap_{i\in I} f(W_i)$.*

De plus, si ces conditions sont satisfaites, la topologie de Zariski sur Y coïncide avec la topologie quotient induite par f.

Démonstration. Voir [MFK, 0.§2.Rq (6)]. □

Définition 2.1.1.7. Soit X un G-schéma algébrique. Un quotient catégorique (Y, f) de X par G vérifiant les conditions de la proposition précédente est appelé un *bon quotient.*

Remarque 2.1.1.8. Dans la littérature, on trouve aussi la définition suivante de bon quotient: Le couple (Y, f) est un bon quotient si G est réductif, f est affine, G-invariant, et induit un isomorphisme $f^\sharp : \mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$. Cette version est un cas particulier de la proposition précédente. En effet, après réduction au cas affine, on montre que la flèche $X \to \mathrm{Spec}(\mathcal{O}(X)^G)$ existe dans la catégorie des schémas algébriques et qu'elle vérifie les conditions de la proposition précédente.

Proposition 2.1.1.9. *Soit X un G-schéma algébrique et (Y, f) un quotient catégorique de X par G. Alors Y jouit des propriétés suivantes si c'est le cas pour X:*

1. *Réduit*
2. *Irréductible*
3. *Normal*

Démonstration. Voir [MFK, 0.§2.Rq (2)]. □

Définition 2.1.1.10. Soit X un G-schéma algébrique et $f : X \to Y$ un morphisme G-invariant de schémas algébriques. On dit que (Y, f) est un *quotient géométrique* de X par G si les conditions suivantes sont satisfaites

1. f est surjectif.
2. Le morphisme $(p_X, \sigma) : G \times X \to X \times_Y X$ est surjectif.
3. La flèche naturelle $f^\sharp : \mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$ est un isomorphisme.
4. La topologie de Zariski sur Y coïncide avec la topologie quotient induite par f.

Proposition 2.1.1.11. *Soit X un G-schéma algébrique et (Y, f) un quotient géométrique de X par G. Alors (Y, f) est un quotient catégorique.*

Démonstration. Voir proposition [MFK, 0.§2.0.1]. □

Définition 2.1.1.12. Soit X un G-schéma algébrique et $f : X \to Y$ un morphisme G-invariant de schémas algébriques. On dit que (X, f) est un *G-torseur* sur Y (où que X est un *G-fibré principal* sur Y) si les conditions suivantes sont satisfaites

1. f est fidèlement plat.

2. Le carré

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_X"] & X \arrow[d, "f"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

est cartésien.

Un morphisme de G-torseurs $(X', f') \to (X, f)$ est un couple (φ, ψ), où φ est un morphisme G-équivariant $X' \to X$, et ψ un morphisme $Y' \to Y$ tel que le carré

$$\begin{tikzcd} X' \arrow[r, "\varphi"] \arrow[d, "f'"] & X \arrow[d, "f"'] \\ Y' \arrow[r, "\psi"'] & Y \end{tikzcd}$$

est cartésien.

Exemple 2.1.1.13. 1. Soit X un schéma algébrique. Le torseur trivial sur X est le X-schéma $G \times X \xrightarrow{p_X} X$ où G agit sur $G \times X$ par multiplication à gauche sur G.

2. Soit $X \xrightarrow{f} Y$ un G-torseur, et $Y' \xrightarrow{\varphi} Y$ un Y-schéma. Alors, le changement de base $X_{Y'}$ de X par φ est naturellement un G-schéma, et $X_{Y'} \xrightarrow{f_{Y'}} Y'$ est un G-torseur. En effet, les conditions 1 et 2 de la définition sont préservées par changement de base. On dit que $(X_{Y'}, f_{Y'})$ est le G-torseur sur Y' image inverse de (Y, f) par φ.

Proposition 2.1.1.14. *Soit X un G-schéma algébrique et $f : X \to Y$ un morphisme de schémas algébriques G-invariant. Alors (X, f) est un G-torseur sur X si et seulement si tout point de Y admet un voisinage ouvert V pour lequel il existe un morphisme fidèlement plat $U \to V$ tel que X_U soit isomorphe au torseur trivial sur U.*

Démonstration. L'implication directe est immédiate car étant donné un G-torseur $X \xrightarrow{f} Y$ sur Y il suffit de prendre, pour un point quelconque de Y, le voisinage Y et le morphisme f.

Réciproquement, considérons un morphisme G-invariant $f : X \to Y$. Par hypothèse on obtient un recouvrement fini $(U_i \to Y)_{i \in I}$ de Y tel que pour tout $i \in I$, le changement de base X_{U_i} soit isomorphe en tant que G-schéma sur U_i au G-torseur trivial $G_{U_i} \to U_i$ sur U_i. Alors le schéma algébrique $Y' := \bigsqcup_{i \in I} U_i$ est fidèlement plat sur Y, et $X_{Y'}$ est isomorphe en tant que G-schéma sur Y' au G-torseur trivial $G_{Y'}$ sur Y'. On a en particulier un isomorphisme $G_{Y'} \times_{Y'} X_{Y'} \to X_{Y'} \times_{Y'} X_{Y'}$, cet isomorphisme étant l'image inverse du Y-morphisme $(\sigma, p_X) : G \times X \to X \times_Y X$ par le changement de base $Y' \to Y$. De plus, G est plat en tant que schéma algébrique, et donc $X_{Y'} \simeq G_{Y'}$ est fidèlement plat de présentation finie sur Y' car ces propriétés sont préservées par changement de base. On souhaite maintenant vérifier que ces propriétés descendent à $X \xrightarrow{f} Y$, de sorte que ce morphisme vérifiera les deux conditions de la définition d'un torseur sur Y. Pour cela on utilise les résultats de descente [EGA, IV.2.7.1] et [EGA, IV.17.7.4] qui s'appliquent dans ce contexte. □

Remarque 2.1.1.15. La preuve de la proposition précédente montre en particulier qu'un morphisme G-invariant $f : X \to Y$ de schémas algébriques est un G-torseur si et seulement si il existe un morphisme $Y' \to Y$ fidèlement plat de schémas algébriques tel que l'on obtienne à isomorphisme près le G-torseur trivial sur Y' après changement de base.

Proposition 2.1.1.16. *Soit X un G-schéma algébrique et $f : X \to Y$ un morphisme G-invariant de schémas algébriques. Si (X, f) est un G-torseur alors (Y, f) est un quotient géométrique de X par G.*

Démonstration. Soit (X, f) un G-torseur sur Y. Alors les conditions 1 et 2 de la définition de quotient géométrique sont vérifiées par définition. La condition 4 l'est également en vertu de [EGA, IV.2.3.12] car f est quasi-compact. Pour finir il faut vérifier que $f^\sharp$ induit un isomorphisme $\mathcal{O}_Y \simeq (f_*\mathcal{O}_X)^G$. Comme f est fidèlement plat, il suffit [EGA, IV.2.2.7] de vérifier que la flèche naturelle

$$u : \mathcal{O}_X = f^*\mathcal{O}_Y \to f^*(f_*(\mathcal{O}_X)^G)$$

est un isomorphisme. Or par [EGA, III.1.4.15] on a un isomorphisme canonique $f^*(f_*(\mathcal{O}_X)) \simeq p_{X*}(\sigma^*\mathcal{O}_X)$, qui s'écrit encore $f^*(f_*(\mathcal{O}_X)) \simeq p_{X*}\mathcal{O}_{G\times X}$. De plus pour tout ouvert $U \subset X$, on a les identités

$$\Gamma(U, p_{X*}(\mathcal{O}_{G\times X})) = \Gamma(G \times U, \mathcal{O}_{G\times X}) = \mathcal{O}(G) \otimes \mathcal{O}(U),$$

par application de [DG, I.§2.2.6] pour la dernière égalité. On en déduit un isomorphisme

$$p_{X*}(\mathcal{O}_{G\times X}) \simeq \mathcal{O}(G) \otimes \mathcal{O}_X.$$

En composant ces isomorphismes, on obtient un isomorphisme

$$f^*(f_*\mathcal{O}_X) \simeq \mathcal{O}(G) \otimes \mathcal{O}_X,$$

qui est G-équivariant en considérant l'action de G sur $\mathcal{O}(G) \otimes \mathcal{O}_X$ par multiplication à gauche sur $\mathcal{O}(G)$. En prenant les invariants dans les deux membres on obtient un isomorphisme v dont on vérifie qu'il est l'inverse de u. □

Remarque 2.1.1.17. Ainsi, on déduit de la proposition précédente que la définition 2.1.1.12 d'un G-torseur coïncide avec la définition [MFK, 0.§4.0.10].

Proposition 2.1.1.18. *Soit X un G-schéma algébrique, et (Y, f) un quotient géométrique de X par G. Si G agit librement sur X, alors (X, f) est un G-torseur sur X.*

Démonstration. Vor [MFK, 0.§4.0.9]. □

Théorème 2.1.1.19. *Soit G un groupe algébrique et H un sous-groupe algébrique agissant sur G par multiplication à droite. Alors le quotient catégorique $\pi : G \to G/H$ existe en tant que schéma algébrique, et c'est un H-torseur. Si de plus H est distingué dans G alors G/H est un groupe algébrique (affine) tel que π soit un morphisme de groupes algébriques.*

Démonstration. Voir [DG, III.§3.5.4] et [DG, III.§3.5.6]. □

Définition 2.1.1.20. Une suite $1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1$ de groupes algébriques est exacte si q est fidèlement plat, et i induit un isomorphisme avec le noyau de q.

Corollaire 2.1.1.21. *Soit G un groupe algébrique et H un sous-groupe algébrique distingué. Alors la suite*

$$1 \to H \xrightarrow{i} G \xrightarrow{\pi} G/H \to 1$$

est exacte.

Définition 2.1.1.22. Soit (X, f) un G-torseur sur Y, et Z un G-schéma algébrique. Considérons l'action diagonale de G sur $X \times Z$ et supposons que le quotient catégorique $q : X \times Z \to W$ existe. On dit que W est le *fibré associé à Z sur Y* si il existe un morphisme φ tel que l'on ait un carré cartésien

$$\begin{tikzcd} X \times Z \arrow[r, "p_X"] \arrow[d, "q"] & X \arrow[d, "f"'] \\ W \arrow[r, "\varphi"'] & Y. \end{tikzcd}$$

Remarque 2.1.1.23. Le fibré associé existe sous des hypothèses assez générales, par exemple lorsque Z s'identifie à une sous-variété (localement fermée) G-stable de $\mathbb{P}(M)$, où M est un G module de dimension finie (voir [Bri18, 3.3.3]). Lorsqu'il existe, le fibré associé est noté $X \times^G Z$.

2.1.2 Action d'un groupe diagonalisable sur un schéma algébrique

Définition 2.1.2.1. Soit M un groupe abélien de type fini. On définit l'*algèbre de groupe* $k[M]$ comme la k-algèbre admettant pour k-base les éléments de M et ayant pour multiplication celle de M étendue à $k[M]$ par linéarité. C'est une k-algèbre de type fini dont on note $\mathbb{D}(M)$ le spectre.

Le foncteur $R \mapsto \mathrm{Hom}_{gp}(M, R^*)$ de la catégorie des k-algèbres de type fini dans la catégorie des groupes abstraits est représenté par $\mathbb{D}(M)$. En effet, on a une correspondance naturelle entre les applications k-linéaires $k[M] \to R$ et les applications $M \to R$. Sous cette correspondance, les morphismes de k-algèbres correspondent aux morphismes de groupes $M \to R^*$. On en déduit que $\mathbb{D}(M)$ est naturellement muni d'une structure de groupe algébrique. Remarquons que le choix d'un isomorphisme $M \simeq \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \times ... \times \mathbb{Z}/n_s\mathbb{Z}$ définit un isomorphisme de groupes algébriques $\mathbb{D}(M) \to \mathbb{G}_{m,k}^r \times \mu_{n_1,k} \times ... \times \mu_{n_s,k}$, où $\mu_{n_i,k}$ est le groupe $\mathbb{D}(\mathbb{Z}/n_i\mathbb{Z})$ des racines n_i-ièmes de l'unité.

Définition 2.1.2.2. Un *groupe algébrique diagonalisable* est un groupe algébrique G tel qu'il existe un groupe abélien de type fini M et un isomorphisme de groupes algébriques $\mathbb{D}(M) \simeq G$.

Proposition 2.1.2.3. *Les foncteurs $M \mapsto \mathbb{D}(M)$ et $\Gamma \mapsto \mathrm{Hom}(\Gamma, \mathbb{G}_m)$ réalisent une anti-équivalence de catégories entre la catégorie des groupes abéliens de type fini et la catégorie des groupes diagonalisables.*

Démonstration. Voir [DG, II.§1.2.11]. □

Définition 2.1.2.4. Soit G un groupe algébrique, le groupe $\mathrm{Hom}_{\mathrm{gp}}(G, \mathbb{G}_m)$ est *le groupe de caractères de* G. On le note $\hat{G}$, ou $X^*(G)$.

La proposition précédente nous dit en particulier que pour tout groupe abélien de type fini M, le groupe $\mathbb{D}(M)$ est l'unique (à unique isomorphisme près) groupe diagonalisable dont le groupe des caractères est M.

Proposition 2.1.2.5. *Soit G un groupe algébrique. Les assertions suivantes sont équivalentes*

1. *G est diagonalisable.*
2. *Les caractères de G forment une base de $\mathcal{O}(G)$.*
3. *Tout G-module est somme directe de G-modules de dimension* 1.

Démonstration. Voir [DG, IV.§1.1.6] □

Soit M un groupe abélien de type fini, et V un $\mathbb{D}(M)$-module. En vertu de la proposition précédente on a une unique décomposition $V = \bigoplus_{\chi \in M} V_\chi$ où $\mathbb{D}(M)$ agit sur chaque V_χ via le caractère χ. Cette association définit un foncteur de la catégorie des $\mathbb{D}(M)$-modules vers la catégorie des k-espaces vectoriels gradués par un groupe abélien de type fini.

Proposition 2.1.2.6. *Le foncteur $V \mapsto \bigoplus_{\chi \in M} V_\chi$ est une équivalence de catégories.*

Démonstration. Voir [DG, IV.§2.2.5] □

Corollaire 2.1.2.7. *On considère d'une part la catégorie des schémas algébriques affines munis d'une action d'un groupe diagonalisable et morphismes équivariants, d'autre part la catégorie des k-algèbres de type fini graduées par un groupe abélien de type fini et morphismes gradués. Alors les foncteurs* $\operatorname{Spec}(k[.]) \times \operatorname{Spec}$ *et* $X^* \times \mathcal{O}(.)$ *réalisent une anti-équivalence de catégories entre ces deux catégories.*

Démonstration. Soit X un $\mathbb{D}(M)$-schéma algébrique affine. Comme $\mathcal{O}(X)$ est un $\mathbb{D}(M)$-module, c'est naturellement un k-espace vectoriel M-gradué et on vérifie immédiatement que c'est en fait une k-algèbre M-graduée. De plus on vérifie que cette association est fonctorielle. Réciproquement, considérons une k-algèbre A de type fini M-graduée, où M est un groupe abélien de type fini. On considère le morphisme naturel de k-algèbres $A \to k[M] \otimes A$ défini sur les éléments homogènes par $f_m \mapsto \chi^m \otimes f_m$. Cela définit un morphisme $\sigma : \mathbb{D}(M) \times \operatorname{Spec} A \to \operatorname{Spec} A$ dont on vérifie qu'il s'agit d'une action de $\mathbb{D}(M)$. À nouveau on vérifie que cette association est fonctorielle. Enfin, il est immédiat que ces deux associations sont quasi-inverses l'une de l'autre. □

Soit S un schéma algébrique et M un groupe abélien de type fini. Alors un $\mathbb{D}(M)$-schéma algébrique X qui est affine sur S correspond (à un isomorphisme près) à une $\mathcal{O}_S$-algèbre quasi-cohérente de type fini M-graduée. C'est en effet une conséquence de la proposition précédente (voir aussi [SGA3, Exp I, 4.7.3]). On peut ainsi étendre la correspondance précédente de la manière suivante.

Corollaire 2.1.2.8. *On considère la catégorie ayant pour objets les couples (X, Γ), où Γ est un groupe diagonalisable, et X un Γ-schéma algébrique qui est affine sur S. Les flèches de cette catégorie sont les S-morphismes équivariants. On a alors une anti-équivalence de catégories avec la catégorie des $\mathcal{O}_S$-algèbres quasi-cohérentes de type fini graduées par un groupe abélien de type fini et morphismes gradués. Cette anti-équivalence est réalisée via le foncteur* $\mathcal{A} \mapsto \operatorname{Spec}_S \mathcal{A}$.

Proposition 2.1.2.9. *Soit S un schéma algébrique, M un groupe abélien de type fini, et $\mathcal{A}$ une $\mathcal{O}_S$-algèbre quasi-cohérente de type fini M-graduée telle que le morphisme naturel $\mathcal{O}_S \to \mathcal{A}_0$ soit un isomorphisme. Alors $X := \operatorname{Spec}_S \mathcal{A}$ est un $\mathbb{D}(M)$-schéma affine sur S tel que le morphisme structural $\pi : X \to S$ soit un bon quotient.*

Démonstration. Par hypothèse, X est un $\mathbb{D}(M)$-schéma algébrique qui est affine sur S avec un morphisme structural $\mathbb{D}(M)$-invariant (2.1.2.8). De plus, l'hypothèse $\mathcal{O}_S \simeq \mathcal{A}_0$ se reformule en disant que la flèche naturelle $\mathcal{O}_S \to (\pi_* \mathcal{O}_X)^{\mathbb{D}(M)}$ est un isomorphisme. D'autre part, on affirme que π est surjectif. Pour montrer cela, il suffit de le vérifier sur les points fermés. Soit y un point fermé et $U \subset S$ un ouvert affine qui le contient. Pour tout idéal I de $\mathcal{O}(U)$, on a $I\mathcal{A}(U) \cap \mathcal{O}(U) = I$ car $I\mathcal{A}(U)$ est un idéal homogène de $\mathcal{A}(U)$ dont la composante homogène de degré zéro est I. On en déduit que $m_y \mathcal{A}(U)$ est contenu dans un idéal maximal, ce qui prouve que y admet un antécédent. Considérons maintenant un fermé invariant $Z \subset X$, et une famille $(Z_i)_{i \in I}$ de fermés invariants de X. Pour montrer que les conditions 3 et 4 de la proposition 2.1.1.6 sont satisfaites, on peut le faire localement sur S. On suppose ainsi S affine, et donc X également. Notons $I := \mathcal{I}_X(Z)$ l'idéal homogène définissant Z, et $(J_i)_{i \in I}$ la famille d'idéaux homogènes définissant $(Z_i)_{i \in I}$. Alors le morphisme surjectif d'algèbres graduées $\mathcal{O}(X) \to \mathcal{O}(X)/I$ donne en degré zéro un morphisme surjectif d'algèbres $\mathcal{O}(S) \to \mathcal{O}(S)/I_0$. On en déduit un diagramme commutatif

$$\begin{tikzcd} Z \arrow[r, hook, "\subset"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ \overline{\pi(Z)} \arrow[r, hook, "\subset"'] & S \end{tikzcd}$$

dans lequel π' est surjectif car les arguments utilisés pour montrer la surjectivité de π s'appliquent à π'. Par le diagramme, on obtient que $\pi(Z) = \overline{\pi(Z)}$. Autrement dit, $\pi(Z)$ est fermé dans S, on a donc montré que la condition 3 est vérifiée. Enfin, on vérifie immédiatement que l'on a $(\sum_{i\in I} J_i)_0 = \sum_{i\in I}(J_i)_0$ ce qui se traduit géométriquement par

$$\overline{\pi(\cap_{i\in I} Z_i)} = \cap_{i\in I}\overline{\pi(Z_i)}.$$

On obtient alors la condition 4 d'après la condition 3 de la proposition 2.1.1.6. Par application de cette dernière, (S, π) est un bon quotient de X par $\mathbb{D}(M)$. $\square$

Chapter 3

Equivariant Cox ring

Abstract

We define the equivariant Cox ring of a normal variety with algebraic group action. We study algebraic and geometric aspects of this object and show how it is related to the ordinary Cox ring. Then we specialize to the case of normal rational varieties of complexity one under the action of a connected reductive group G. We show that the G-equivariant Cox ring is then a finitely generated integral normal G-algebra. Under a mild additional condition, we give a presentation of its subalgebra of U-invariants, where U is the unipotent part of a Borel subgroup of G. The ordinary Cox ring is also finitely generated and inherits a canonical structure of U-algebra. Using a work of Hausen and Herppich, we prove that the subalgebra of U-invariants of the Cox ring is a finitely generated Cox ring of a variety of complexity one under the action of a torus. As an application, we provide a criterion of combinatorial nature for the Cox ring of an almost homogeneous G-variety of complexity one to have log terminal singularities. Finally, we prove that for a normal rational G-variety of complexity one satisfying a mild additional condition (e.g. complete or almost homogeneous), the iteration of Cox rings is finite.

3.1 Introduction

Let X be a normal algebraic variety over an algebraically closed field k of characteristic zero. The *Cox ring* of X, denoted by $\mathrm{Cox}(X)$, is an important invariant that encodes a lot of geometric information, see [ADHL] for a comprehensive reference on this rich subject. It is the ring of global sections of the *Cox sheaf*, which is the direct sum

$$\mathcal{R}_X := \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}(X)} \mathcal{F}$$

indexed by the elements of the *class group* of X, i.e. the group of isomorphism classes of divisorial sheaves on X. Under some additional assumptions, $\mathcal{R}_X$ can be endowed with a structure of quasi-coherent $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra, whence a structure of $\mathrm{Cl}(X)$-graded ring on $\mathrm{Cox}(X)$. When $\mathcal{R}_X$ is of finite type as an $\mathcal{O}_X$-algebra, its relative spectrum $\hat{X}$ is a normal algebraic variety over X called the *characteristic space* of X. The characteristic space is endowed with an action of the diagonalizable algebraic group $\mathbb{D}(\mathrm{Cl}(X))$ whose group of characters is $\mathrm{Cl}(X)$ and the structural morphism $\hat{X} \to X$ is a good quotient for this action. When X is smooth, this morphism is a $\mathbb{D}(\mathrm{Cl}(X))$-torsor called the *universal torsor* of X. When the Cox ring is finitely generated as a k-algebra, its spectrum $\tilde{X}$ is called the *total coordinate space* of X, and the affinization morphism $\hat{X} \to \tilde{X}$ is a $\mathbb{D}(\mathrm{Cl}(X))$-equivariant open immersion.

Given a variety with finitely generated Cox ring, it is an important problem to find a presentation of this k-algebra. When the variety has a lot of symmetries, one can hope to answer this problem by taking advantage of this additional information. This can already be seen in the case of *toric varieties* which is at the origin of the theory [ADHL, 2.2]. In this case, the Cox ring is a polynomial k-algebra in the canonical sections associated to the prime divisors lying in the complement of the torus. Hence, a toric variety admits a description as a quotient by a diagonalizable group of an open subvariety in an ambient affine space. This provides a vast generalization of the classical construction of the projective space. Toric varieties belong to the wider class of *spherical varieties*, i.e. normal varieties endowed with an action of a connected reductive group such that a Borel subgroup acts with a dense orbit. The Cox ring of a spherical variety is also finitely generated with a known explicit description. However, this description is already much more complicated than in the case of toric varieties [Bri07].

More generally, when X is endowed with an action of an algebraic group G, the general idea is to try to keep track of the G-action in $\mathrm{Cox}(X)$ in order to obtain a graded G-algebra and use methods from representation theory to approach its structure. Geometrically, this means lifting equivariantly the G-action to the characteristic space. Unfortunately, such a lift does not exist in general, and even when it exists, it is in general not unique [ADHL, 4.2.3]. An alternative approach which has proven useful [Bri07, 4.2] is to define an equivariant analogue of the Cox ring and to relate it to the ordinary Cox ring. Building on Brion's ideas in loc. cit., we give an intrinsic construction of the *G-equivariant Cox sheaf*. This is a direct sum

$$\mathcal{R}_{G,X} := \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}_G(X)} \mathcal{F}$$

indexed by the elements of the *equivariant class group* $\mathrm{Cl}_G(X)$, i.e. the group of isomorphism classes of G-linearized divisorial sheaves on X. The main difficulty is then to endow this sheaf with a structure of quasi-coherent $\mathrm{Cl}_G(X)$-graded $\mathcal{O}_X$-algebra. This requires in particular to revisit the construction of the ordinary Cox sheaf (Section 3.2.1). The *G-equivariant Cox ring*, denoted by $\mathrm{Cox}_G(X)$, is the ring of global sections of $\mathcal{R}_{G,X}$. When $\mathcal{R}_{G,X}$ is of finite type as an $\mathcal{O}_X$-algebra, its relative spectrum $\hat{X}_G$ is a normal $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-variety called the *G-equivariant characteristic space* of X. When moreover $\mathrm{Cox}_G(X)$ is a finitely generated k-algebra, its spectrum $\tilde{X}_G$ is a normal $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-variety called the *G-equivariant total coordinate space*. These varieties sit in the diagram

$$\begin{tikzcd} \hat{X}_G \arrow[r, hook] \arrow[d, "q"'] & \tilde{X}_G \\ X & \end{tikzcd},$$

where the affinization morphism $\hat{X}_G \hookrightarrow \tilde{X}_G$ is a $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-equivariant open immersion. Moreover, q is a G-equivariant good quotient of $\hat{X}_G$ by $\mathbb{D}(\mathrm{Cl}_G(X))$ which induces a $\mathbb{D}(\mathrm{Cl}_G(X))$-torsor between open subvarieties whose complements have codimension at least two. More generally, G-equivariant good quotients by diagonalizable groups satisfying this last property are called *G-equivariant diagonalizable almost principal bundles*. In Sections 3.2.4 to 3.2.7, we study these bundles and derive properties of $\hat{X}_G$ and $\tilde{X}_G$ all along.

Assuming that G is connected, we establish a localization exact sequence involving the natural morphism $\mathrm{Cl}_G(X) \to \mathrm{Cl}_G(X_0)$ associated to the restriction to a G-stable open subvariety $X_0 \subset X$ (Prop. 3.2.2.4). This exact sequence is our basic tool to obtain a description of $\mathrm{Cox}_G(X_0)$ as a quotient of $\mathrm{Cox}_G(X)$ (Prop. 3.2.8.1). As an application of this description, we consider the case where X is *almost homogeneous*, that is, admits an open G-orbit X_0. The equivariant Cox ring of X_0 is the coordinate algebra of a homogeneous space which can be described explicitly (Ex. 3.2.5.13). There is also a categorical quotient

$$\tilde{X}_G \to \mathbb{A}^n_k$$

by G, where n is the number of G-stable prime divisors in X (Prop. 3.2.8.5), and the general fibers of this morphism are isomorphic to the equivariant total coordinate space of the open orbit X_0 (Prop. 3.2.8.1). This yields a description of the open $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-orbit $q^{-1}(X_0)$ as a fiber bundle over $\mathbb{G}_m^n$ with fiber the total coordinate space of X_0 (Cor. 3.2.8.6). This description is useful for doing computations in $\mathrm{Cox}_G(X)$, because it can be embedded as a graded subalgebra of a coordinate algebra admitting an explicit description.

Still assuming G connected, we uncover the link between $\mathrm{Cox}_G(X)$ and $\mathrm{Cox}(X)$ in Section 3.2.9. Geometrically, there is a commutative diagram

$$\begin{tikzcd} \hat{X}_G \arrow[r, hook] & \tilde{X}_G \\ \hat{X} \arrow[u] \arrow[r, hook] & \tilde{X}, \arrow[u] \end{tikzcd}$$

where horizontal arrows are affinization morphisms, and vertical arrows are finite morphisms. Moreover, these finite morphisms are equivariant with respect to the morphism $\mathbb{D}(\mathrm{Cl}(X)) \to \mathbb{D}(\mathrm{Cl}_G(X))$ dually defined by the forgetful morphism $\mathrm{Cl}_G(X) \to \mathrm{Cl}(X)$. When this last morphism is surjective (which can always be achieved up to replacing G with a finite cover), the vertical arrows are closed immersions. In this case, we determine explicitly the ideal of $\mathrm{Cox}_G(X)$ defining the subvariety $\tilde{X}$ of $\tilde{X}_G$ (Cor. 3.2.9.3).

Starting from Section 3.3.1, we specialize to the case of a normal variety X of complexity one under the action of a connected reductive group G, i.e. the minimal codimension of an orbit of a Borel subgroup $B \subset G$ is one. We show that the finite generation of $\mathrm{Cox}_G(X)$ is equivalent to the rationality of X (Prop. 3.3.1.4). Assuming that X is rational, a classical result in invariant theory says that the subalgebra of invariants $\mathrm{Cox}_G(X)^U$ by the unipotent part U of B is finitely generated as well. Under a mild additional condition, we provide a presentation of this subalgebra (Thm. 3.3.2.3) which generalizes a result of Ponomareva [Pon, Thm. 4]. When G is a torus, this yields a description of the Cox ring which generalizes a result of Hausen and Süss [ADHL, Thm. 4.4.1.6].

There is a canonical isomorphism between the U-equivariant Cox ring and the ordinary Cox ring (Rem. 3.2.3.4), whence a canonical structure of U-algebra on $\mathrm{Cox}(X)$. When X is spherical, it follows from [Bri07, 4.2.3] and Corollary 3.2.9.3 that $\mathrm{Cox}(X)^U$ is a finitely generated polynomial k-algebra. When X is rational of complexity one, we show that the k-algebra $\mathrm{Cox}(X)^U$ is a finitely generated Cox ring of a normal variety of complexity one under the action of a torus (Prop. 3.3.3.2). To prove this property, we rely on work of Hausen and Herppich [ADHL, 3.4.2 and 3.4.3].

In Section 3.3.4, we turn to the singularities of the total coordinate space of an almost homogeneous G-variety of complexity at most one. It is well known that Cox rings are normal integral domains. However, they can be quite singular in general. They are not necessarily Cohen-Macaulay although it has been recently shown by Braun that they are 1-*Gorenstein* [Bra, Thm. 1] (i.e. the canonical sheaf over the total coordinate space is invertible). By [GOST, Thm 1.1], a $\mathbb{Q}$-factorial normal projective variety X is of *Fano type* (i.e. there exists an effective $\mathbb{Q}$-divisor Δ such that $-(K_X + \Delta)$ is ample and the pair (X, Δ) is Kawamata log terminal) if and only if its Cox ring is finitely generated with log terminal singularities. It turns out that the total coordinate space of a spherical variety is always log terminal (Prop. 3.3.4.1). In complexity one, we obtain the following criterion (Thm. 3.3.4.3):

Theorem. *Let G be a connected reductive group and X be an almost homogeneous G-variety of complexity one with only constant invertible regular functions. Then,*

$$\tilde{X} \textit{ has log terminal singularities} \iff \mathrm{Cox}(X)^U \textit{ is a Platonic ring.}$$

The above condition on $\mathrm{Cox}(X)^U$ is of combinatorial nature, and has been introduced in [ABHW]. It translates into a condition on a geometric datum attached to X (see Remarks 3.3.3.6 and 3.3.3.7).

In Section 3.3.5, we study the iteration of Cox rings for normal rational varieties of complexity one. Except when $\mathrm{Cl}(X)$ is free, the total coordinate space $\tilde{X}$ need not be factorial. Hence this total coordinate space may have itself a non-trivial total coordinate space. Iteration of Cox rings consists in studying the sequence of total coordinate spaces

$$\dots \to \tilde{X}^{(n)} \to \tilde{X}^{(n-1)} \to \dots \to \tilde{X}^{(2)} \to \tilde{X},$$

where $\tilde{X}^{(n)}$ denotes the total coordinate space of $\tilde{X}^{(n-1)}$. A basic question is whether this sequence is finite, that is, there exists an integer n for which $\tilde{X}^{(n)}$ has either trivial class group, or a non-finitely generated Cox ring. If the sequence is finite, X is said to have *finite iteration of Cox rings*, and the last obtained Cox ring is called *the master Cox ring*. This question has been studied in [ABHW] and [HW] for varieties of complexity one under a torus action. Finite iteration of Cox rings is proved in this setting, though with possibly a non-finitely generated master Cox ring. Further advances on this problem are works of Gagliardi in the spherical case [G], and of Braun for Fano type varieties [Bra]; the main result of this last article has been recently extended by Braun and Moraga [BM]. In this direction we obtain (Thm. 3.3.5.2):

Theorem. *Let G be a connected reductive group and X be a normal rational G-variety of complexity one with only constant invertible regular functions. Suppose moreover that X is complete or that X is almost homogeneous. Then X admits finite iteration of Cox rings with well defined iterated Cox rings. In the almost homogeneous case, the master Cox ring is factorial and finitely generated.*

To finish, we mention that many results in this text find applications in a further article [V2] dedicated to almost homogeneous SL_2-threefolds, for which ordinary and equivariant Cox rings are canonically isomorphic.

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Conventions. Recall that k denotes an algebraically closed base field of characteristic zero. This last assumption is not necessary for the construction of the equivariant Cox ring, but it will simplify the exposition, and it will be needed starting from Section 3.3.1.

In this text, we work in the category of *algebraic schemes*, that is, separated k-schemes of finite type. Hence, a morphism $(f, f^\sharp) : X \to Y$ of algebraic schemes is meant to be a k-morphism. A *variety* is an integral algebraic scheme. Without further precision, a *point* of a variety is meant to be a closed point. A *subvariety* of a variety is a locally closed subset equipped with its reduced scheme structure. The sheaf of *regular functions* (or structure sheaf) on an algebraic scheme X is denoted by $\mathcal{O}_X$ or simply by $\mathcal{O}$ if there is no risk of confusion. The dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is denoted by $\mathcal{F}^\vee$. The isomorphism class of an object A in a category is denoted by $[A]$. The group of invertible elements of a ring R is denoted by R^*.

An *algebraic group* G is an affine k-group scheme of finite type. The multiplication morphism is denoted by $m_G : G \times G \to G$, and the neutral element by $e_G : \operatorname{Spec} k \to G$. The *character group* of G is denoted by $\mathbb{X}(G)$. Given a finitely generated abelian group M, let $\mathbb{D}(M)$ denote the *diagonalizable (algebraic) group* $\mathrm{Spec}(k[M])$ with character group M. Without further precision, the letter $\mathbb{T}$ denotes an arbitrary (algebraic) torus. A graded G-algebra is a graded k-algebra which is a G-module such that G acts by automorphisms of graded algebra.

Let G be an algebraic group, X an algebraic G-scheme, and $q : X \to Y$ a G-invariant morphism. We say that q is a *G-torsor* (or *principal G-bundle*) over Y if q is faithfully flat and the natural morphism $G \times X \to X \times_Y X$ is an isomorphism. As algebraic groups are smooth in characteristic zero, q is faithfully flat if and only if it is smooth and surjective.

Let G be an algebraic group and X an affine algebraic G-scheme. If $\mathcal{O}(X)^G$ is a finitely generated k-algebra, then the affine algebraic scheme $X//G := \mathrm{Spec}(\mathcal{O}(X)^G)$ is called the *categorical quotient* of X by G. It is a universal object in the category of G-invariant morphisms from X to affine algebraic schemes.

Our convention is that a *reductive group* is a *linearly reductive* algebraic group, that is, every finite dimensional representation of such group is semisimple. In particular, a reductive group is not necessarily connected. Let G be a reductive group and X an algebraic G-scheme. A *good quotient* of X by G is an affine G-invariant morphism $q : X \to Y$ such that $q^\sharp$ induces an isomorphism $\mathcal{O}_Y \to (q_*\mathcal{O}_X)^G$. It is a universal object in the category of G-invariant morphisms from X to algebraic schemes.

An *almost homogeneous variety* is a normal variety on which a connected algebraic group acts with a dense orbit. A *quasicone* is an affine $\mathbb{G}_m$-variety such that the orbit closures meet in a common point.

For a normal variety X, the *group of Weil divisors* is denoted by $\mathrm{WDiv}(X)$. Every Weil divisor D defines a coherent sheaf $\mathcal{O}_X(D)$ whose non-zero sections over an open subset U are rational functions $f \in k(X)^*$ such that $\mathrm{div}(f)+D$ defines an effective divisor on U. The *canonical sheaf* ω_X is the pushforward on X of the sheaf of differential forms of maximal degree on the smooth locus X_{sm}. It is a divisorial sheaf, and any Weil divisor K_X such that $\mathcal{O}_X(K_X) \simeq \omega_X$ is a *canonical divisor*. We say that X is $\mathbb{Q}$-*Gorenstein* if some non-zero power of the canonical sheaf is invertible. If the canonical sheaf is invertible, X is said to be $\mathbb{Q}$-*Gorenstein of index one*, or simply 1-*Gorenstein*. A *Gorenstein variety* is a normal Cohen-Macaulay variety whose canonical sheaf is invertible. We say that X has *rational singularities* if there exists a proper birational morphism

$$\varphi : Z \to X,$$

where Z is a smooth variety (a resolution of singularities), and such that $R^i\varphi_*\mathcal{O}_Z = 0$, $\forall i > 0$. This last property does not depend on the choice of a resolution.

We say that X has *log terminal singularities* if it is $\mathbb{Q}$-Gorenstein and there is a resolution of singularities $\varphi : Z \to X$ such that

$$K_Z = \varphi^* K_X + \sum \alpha_i E_i,\ \alpha_i > -1,$$

where the sum runs over the exceptional divisors E_i of φ.

3.2 Equivariant Cox ring

3.2.1 Cox ring of a pointed variety

Considering spaces of sections of line bundles over a G-variety X, one doesn't necessarily want to see these as spaces of rational functions satisfying vanishing conditions with respect to some Cartier divisor on X. A prominent example of this situation is given by spaces of sections of line bundles over a flag variety, which naturally occur as simple G-submodules of the left regular representation. This motivates a definition of the Cox ring which is quite different from the usual one in the literature. In [ADHL, 1.4.2.3], a canonical construction for the Cox ring of a pointed normal variety is presented. We push this idea one step further by considering rigidified sheaves at the base point [FGA, 9.2.8]. This leads to an intrinsic construction of the Cox ring which will be well suited for the construction of the equivariant analogue.

Definition 3.2.1.1. A *pointed variety* is a variety X together with a smooth point $x \in X$. We also denote a pointed variety by (X, x) or by $(X, x \overset{\iota}{\hookrightarrow} X)$.

Let $(X, x \overset{\iota}{\hookrightarrow} X)$ be a pointed normal variety such that $\mathcal{O}(X)^* \simeq k^*$. It will be clear from the construction below that this last hypothesis is required in order to equip the Cox ring of X with a canonical structure of algebra. Typical examples of varieties satisfying this condition are normal complete varieties and almost homogeneous varieties under a semisimple algebraic group. A *divisorial sheaf* on X is a coherent reflexive $\mathcal{O}_X$-module of rank one. On a smooth variety, a divisorial sheaf is simply an invertible sheaf. Moreover, every divisorial sheaf on X is isomorphic to the pushforward on X of an invertible sheaf on the smooth locus X_{sm} [Har1, 1.12]. Given two divisorial sheaves $\mathcal{F}_1, \mathcal{F}_2$, the double dual of their tensor product

$$\mathcal{F}_1 \star \mathcal{F}_2 := (\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2)^{\vee\vee}$$

is a divisorial sheaf. The operation $\star$ is compatible with taking isomorphism classes and endows the set $\mathrm{Cl}(X)$ of isomorphism classes of divisorial sheaves with a structure of abelian group called the *class group*, the inverse map being $[\mathcal{F}] \mapsto [\mathcal{F}^\vee]$. The class group identifies with the group of Weil divisors modulo linear equivalence through the natural isomorphism

$$[D] \mapsto [\mathcal{O}_X(D)] \qquad \text{[Har1, 2.8]}.$$

By [Har1, 1.12] again, the pullback functor associated to the open immersion of the smoooth locus $X_{\mathrm{sm}} \hookrightarrow X$ induces an isomorphism of $\mathrm{Cl}(X)$ with the *Picard group* of X_{sm}, denoted by $\mathrm{Pic}(X_{\mathrm{sm}})$. In particular, the smoothness of x implies that there exists an isomorphism of k-vector spaces $\iota^*\mathcal{F} \to k$.

Definition 3.2.1.2. Let $(X, x \overset{\iota}{\hookrightarrow} X)$ be a pointed normal variety. A *rigidified divisorial* sheaf is a pair $(\mathcal{F}, f)$ where $\mathcal{F}$ is a divisorial sheaf on X, and $f : \iota^*\mathcal{F} \to k$ is an isomorphism of k-vector spaces. The isomorphism f is a *rigidification* of $\mathcal{F}$. A *morphism* of rigidified divisorial sheaves $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ is an $\mathcal{O}_X$-module morphism $v : \mathcal{F} \to \mathcal{G}$ such that the following diagram is commutative:

$$\begin{tikzcd} \iota^*\mathcal{F} \arrow[r, "\iota^* v"] \arrow[d, "f"] & \iota^*\mathcal{G} \arrow[dl, "g"] \\ k & \end{tikzcd}$$

The set E of isomorphism classes of rigidified divisorial sheaves carries a natural structure of abelian group. Indeed, the double dual of the tensor product of two rigidified divisorial sheaves is naturally equipped with a rigidification, and this operation is compatible with taking isomorphism classes. In fact, the next proposition says that we can identify the groups E and $\mathrm{Cl}(X)$.

Proposition 3.2.1.3. *[FGA, 9.2.9] The group morphism $\rho : E \to \mathrm{Cl}(X), [(\mathcal{F}, f)] \mapsto [\mathcal{F}]$ is an isomorphism.*

Also, an important feature of these rigidified divisorial sheaves is that they have no non-trivial automorphism. Indeed, similarly as for an invertible sheaf, an automorphism of a divisorial sheaf $\mathcal{F}$ is defined by an invertible regular function on X. Thus, $\mathrm{Aut}(\mathcal{F}) \simeq k^*$ by hypothesis, and it follows that the identity is the only automorphism preserving the additional data of a rigidification. Hence, if an isomorphism exists between two rigidified divisorial sheaves, it is unique. In particular, this allows us to canonically identify sections over an open set $U \subset X$ of rigidified divisorial sheaves in a given class $[(\mathcal{F}, f)] \in E$. Formally, we have for each open subset $U \subset X$ an equivalence relation

$$s_1 \sim_U s_2 \iff s_2 = v(U)(s_1), \text{ where } s_1 \in \mathcal{F}_1(U), s_2 \in \mathcal{F}_2(U), \text{ and } v : (\mathcal{F}_1, f_1) \xrightarrow{\simeq} (\mathcal{F}_2, f_2).$$

This relation is compatible with the $\mathcal{O}(U)$-module structure, and with the restrictions to open subsets. Performing the quotients with respect to the various relations $\sim_U$, we obtain a well-defined divisorial sheaf $\mathcal{F}^x$ lying in the class $[\mathcal{F}] \in \mathrm{Cl}(X)$. Indeed, let us denote by θ_U the canonical projection associated to the equivalence relation $\sim_U$. Then the isomorphisms

$$\mathcal{F}(U) \to \mathcal{F}^x(U), s \mapsto \theta_U(s)$$

for the various open subsets induce an isomorphism of sheaves $\mathcal{F} \xrightarrow{\simeq} \mathcal{F}^x$. In other words, for any element $[\mathcal{F}] \in \mathrm{Cl}(X)$ we have constructed a canonical representative $\mathcal{F}^x$, which we call the *rigidified sheaf associated to* $[\mathcal{F}]$.

Proposition 3.2.1.4. *Let (X, x) be a pointed normal variety with only constant invertible regular functions. Then the sheaf*

$$\mathcal{R}_X := \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}(X)} \mathcal{F}^x$$

has a natural structure of quasi-coherent $\mathrm{Cl}(X)$*-graded* $\mathcal{O}_X$*-algebra.*

Proof. By construction, $\mathcal{R}_X$ is a $\mathrm{Cl}(X)$-graded quasi-coherent $\mathcal{O}_X$-module. The multiplication between homogeneous sections is defined by the natural morphism

$$\mathcal{F}^x(U) \otimes_{\mathcal{O}(U)} \mathcal{G}^x(U) \to (\mathcal{F}^x \star \mathcal{G}^x)^x(U)$$

for every open subset $U \subset X$. This multiplication endows $\mathcal{R}_X$ with a structure of quasi-coherent $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra. □

Definition 3.2.1.5. [ADHL, 1.6.1] With the assumptions of the last proposition, $\mathcal{R}_X$ is the *Cox sheaf* of the pointed variety X. The *Cox ring* $\mathrm{Cox}(X)$ is the ring of global sections of $\mathcal{R}_X$. When $\mathcal{R}_X$ is of finite type as an $\mathcal{O}_X$-algebra, the relative spectrum $\hat{X}$ of the Cox sheaf is the *characteristic space* of X. When $\mathrm{Cox}(X)$ is finitely generated as a k-algebra, its spectrum $\tilde{X}$ is the *total coordinate space* of X.

Remark 3.2.1.6. It is customary to suppose that $\mathrm{Cl}(X)$ is finitely generated in the above definition. This is indeed a necessary condition to have a Cox sheaf of finite type. A sufficient condition is that the Cox ring be a finitely generated k-algebra [ADHL, 1.6.1.1]. Normal varieties with finitely generated Cox ring are called *Mori Dream Spaces (MDS)*, due to their good behaviour with respect to Mori's minimal model program [HK].

Now we compare the construction above with the construction of the Cox sheaf of a pointed normal variety as defined in [ADHL, 1.4.2.3]. In particular, we produce an isomorphism of graded $\mathcal{O}_X$-algebras between both constructions. We begin by recalling this last construction. Regard $\mathrm{Cl}(X)$ as the group of Weil divisors on X modulo linear equivalence and suppose it is finitely generated. Consider the subgroup $K^x \subset \mathrm{WDiv}(X)$ whose elements are Weil divisors on X whose support doesn't contain x. The restriction $K^x \to \mathrm{Cl}(X)$ of the natural projection is still surjective. Choose a finitely generated subgroup $K \subset K^x$ which again projects surjectively on $\mathrm{Cl}(X)$, and denote by $\pi : K \to \mathrm{Cl}(X)$ this projection. For each $E \in \ker \pi$, there exists by assumption a unique $f_E \in k(X)$ defined on a neighborhood of x such that $E = \mathrm{div}(f_E)$ and $f_E(x) = 1$. This allows to define a group morphism $\chi : \ker \pi \to k(X)^*$ such that $\mathrm{div}(\chi(E)) = E$, for all $E \in \ker \pi$. The Cox sheaf of X is then defined as the quotient of the $\mathcal{O}_X$-algebra $\mathcal{S} := \bigoplus_{D\in K} \mathcal{O}_X(D)$ by the sheaf of ideals $\mathcal{I}$ generated by the elements $(1 - \chi(E))$, where 1 and $\chi(E)$ are sections of $\mathcal{O}_X$ and $\mathcal{O}_X(-E)$, respectively, and E runs through $\ker \pi$.

Fix a $\mathbb{Z}$-basis $(D_1, ..., D_r)$ for K and rational functions $f_1, ..., f_r$ with $f_i(x) = 1$ and $\mathrm{div}(f_i) = D_i$ near x. For $D = a_1D_1 + ... + a_rD_r$, set $f_D = f_1^{a_1}...f_r^{a_r}$. This gives rigidifications being compatible with the multiplicative structure of $\mathcal{S}$, namely

$$\iota^*\mathcal{O}_X(D) \to k, \quad \iota^*s \mapsto sf_D(x).$$

The isomorphisms $\theta : \mathcal{O}_X(D) \to \mathcal{O}_X(D)^x$ fit together to a morphism $\theta : \mathcal{S} \to \mathcal{R}_X$ of graded $\mathcal{O}_X$-algebras, having $\pi : K \to \mathrm{Cl}(X)$ as its morphism of grading groups. Two homogeneous sections s_1, s_2 over U have the same projection $\theta_U(s_1) = \theta_U(s_2)$ if and only if there exists $E \in \ker \pi$ such that $s_2 = \chi(E)s_1$. This implies that $\ker \theta = \mathcal{I}$, which yields the desired isomorphism.

3.2.2 Equivariant class group

Let G be an algebraic group, $\sigma : G \times X \to X$ an action of G on a normal variety X, and $p_X : G \times X \to X$ the second projection. Following [SGA3, Exp. I, 6.5], a G-linearization of a quasi-coherent sheaf $\mathcal{F}$ on X is an isomorphism $\Phi : \sigma^*\mathcal{F} \xrightarrow{\simeq} p_X^*\mathcal{F}$ verifying a certain cocycle condition. The geometric incarnation of $\mathcal{F}$ is the vector fibration $\mathbb{V}(\mathcal{F}) := \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{F}))$ over X [EGA, II.1.7.8]. A G-linearization of $\mathcal{F}$ is then equivalent to an action of G on $\mathbb{V}(\mathcal{F})$ that lifts the G-action on X and commutes with the natural $\mathbb{G}_m$-action on $\mathbb{V}(\mathcal{F})$. Because σ and p_X are flat morphisms, the associated pullbacks of coherent sheaves commute with the operation of taking duals [EGA, III.12.3.5]. It follows that given two G-linearized divisorial sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X, there is a canonical G-linearization induced on $\mathcal{F}_1 \star \mathcal{F}_2$ and $\mathcal{F}_1^\vee$. Also, there is an obvious notion of isomorphism of G-linearized sheaves, and the operation $\star$ is compatible with this notion. The induced operation on the set of isomorphism classes of G-linearized divisorial sheaves on X yields an abelian group structure. This defines the *equivariant class group*, denoted by $\mathrm{Cl}_G(X)$. This group contains the *equivariant Picard group* $\mathrm{Pic}_G(X)$, that is, the group of isomorphism classes of G-linearized invertible sheaves on X.

Proposition 3.2.2.1. *Let X be a G-variety and $X_0 \subset X$ an open G-stable subvariety with complement of codimension at least two in X. Then restricting divisorial sheaves induces isomorphisms*

$$\mathrm{Cl}(X_0) \simeq \mathrm{Cl}(X) \simeq \mathrm{Pic}(X_{\mathrm{sm}}) \text{ and } \mathrm{Cl}_G(X_0) \simeq \mathrm{Cl}_G(X) \simeq \mathrm{Pic}_G(X_{\mathrm{sm}}).$$

Proof. The isomorphism $\mathrm{Cl}(X) \simeq \mathrm{Cl}(X_0)$ is a direct consequence of [Har1, 1.12]. Furthermore, denoting $j : X_0 \to X$ the inclusion, the pullback functor j^* on coherent sheaves induces a well defined morphism $\mathrm{Cl}_G(X) \to \mathrm{Cl}_G(X_0)$ because j is an equivariant flat morphism between normal G-varieties. Also, we have an open immersion $\mathrm{Id}_G \times j : G \times X_0 \to G \times X$ of normal varieties whose image has a complement of codimension at least two in $G \times X$. By [Har1, 1.12] again, the pullback functor $(\mathrm{Id}_G \times j)^*$ induces for each divisorial sheaf $\mathcal{F}$ on X a bijection between the set of G-linearizations of $\mathcal{F}$ and the set of G-linearizations of $j^*\mathcal{F}$. Thus, the morphism $\mathrm{Cl}_G(X) \to \mathrm{Cl}_G(X_0)$ is bijective. $\square$

Assuming from now that G is connected, we have an exact sequence

$$0 \to \mathcal{O}(X)^{*G} \to \mathcal{O}(X)^* \xrightarrow{\chi} \mathbb{X}(G) \xrightarrow{\gamma} \mathrm{Cl}_G(X) \xrightarrow{\phi} \mathrm{Cl}(X) \tag{3.2.1}$$

relating the equivariant class group to the ordinary one through the forgetful morphism ϕ that "forgets" the linearizations. This sequence is obtained from the analogous exact sequence for $\mathrm{Pic}(X_{\mathrm{sm}})$ [Bri18, 4.2.2], by means of the above isomorphisms and $\mathcal{O}(X)^* \simeq \mathcal{O}(X_{\mathrm{sm}})^*$. The morphism γ maps a character $\lambda \in \mathbb{X}(G)$ to the associated G-linearization $\mathcal{O}_X(\lambda)$ of the structure sheaf [Bri18, 4.1.7], and χ maps an invertible regular function f to its weight χ_f relative to the G-action [Bri18, 4.1.6].

Remark 3.2.2.2. By [Bri18, 5.2.1] and 3.2.2.1, the cokernel of ϕ has finite exponent bounded by the order of the Picard group of G (which is finite, see [Bri18, 5.1.3]). It follows that finite generation of $\mathrm{Cl}_G(X)$ implies finite generation of $\mathrm{Cl}(X)$. By the above exact sequence, the converse is true.

On several occasions, we will have to suppose that the forgetful morphism is surjective. It is apparent from the above that this condition is satisfied if we suppose that $\mathrm{Pic}(G) = 0$. This latter assumption is not a serious restriction as there exists a central isogeny $\tilde{G} \to G$ of connected algebraic groups such that $\mathrm{Pic}(\tilde{G}) = 0$ [Mi, 18.22].

Consider a non-empty open subset $X_0 \overset{j}{\hookrightarrow} X$ and denote by $(D_i)_{i=1,..,r}$ the possibly empty family of prime divisors lying in the complement. A very useful tool for the study of class groups of normal varieties is the localization exact sequence

$$0 \to \mathcal{O}(X)^* \to \mathcal{O}(X_0)^* \xrightarrow{\mathrm{div}} \bigoplus_{i=1}^{r} \mathbb{Z}D_i \xrightarrow{D \mapsto [D]} \mathrm{Cl}(X) \xrightarrow{j^*} \mathrm{Cl}(X_0) \to 0. \tag{3.2.2}$$

Suppose that X_0 is G-stable. Then, it is natural to look for an analogous exact sequence involving the restriction morphism $\mathrm{Cl}_G(X) \xrightarrow{j^*} \mathrm{Cl}_G(X_0)$. Again, we can suppose that X is smooth and consider (equivariant) Picard groups instead of (equivariant) class groups. The main idea is then to use a construction from Edidin and Graham in order to realize $\mathrm{Pic}_G(X)$ as the Picard group of a variety built from X in a natural way. Following [EG, Lemma 9], there exists a finite dimensional G-module E and a non-empty G-stable open subset $E_0 \subset E$ whose complement in E is of codimension at least two, and such that we have a G-torsor $E_0 \to E_0/G$.

Proposition 3.2.2.3. *The associated fiber bundle $f : E_0 \times X \to E_0 \times^G X$ exists in the category of varieties and we have an isomorphism*

$$\mathrm{Pic}(E_0 \times^G X) \xrightarrow{\simeq} \mathrm{Pic}_G(X)$$

obtained from the isomorphisms $\mathrm{Pic}_G(X) \xrightarrow{p_X^*} \mathrm{Pic}_G(E_0 \times X)$ and $\mathrm{Pic}(E_0 \times^G X) \xrightarrow{f^*} \mathrm{Pic}_G(E_0 \times X)$.

Proof. That the associated fiber bundle f exists in the category of varieties is a consequence of a theorem of Sumihiro [Bri18, 5.3.4]. The isomorphism $\mathrm{Pic}_G(X) \xrightarrow{p_X^*} \mathrm{Pic}_G(E_0 \times X)$ follows from [EG, Lemma 2], and the isomorphism $\mathrm{Pic}(E_0 \times^G X) \xrightarrow{f^*} \mathrm{Pic}_G(E_0 \times X)$ is a consequence of a descent result from Grothendieck [Bri18, 3.3.1]. Considering $(p_X^*)^{-1} \circ f^*$ yields the desired isomorphism. □

Corollary 3.2.2.4. *Let G be a connected algebraic group, X a normal G-variety, $X_0 \overset{j}{\hookrightarrow} X$ a non-empty G-stable open subvariety, and $(D_i)_{i=1,..,r}$ the family of prime divisors lying in the complement of X_0. Then the sequence*

$$0 \to \mathcal{O}(X)^{*G} \to \mathcal{O}(X_0)^{*G} \xrightarrow{\mathrm{div}} \bigoplus_{i=1}^{r} \mathbb{Z}D_i \xrightarrow{D \mapsto [D]} \mathrm{Cl}_G(X) \xrightarrow{j^*} \mathrm{Cl}_G(X_0) \to 0 \tag{3.2.3}$$

is exact.

Proof. Notice already that the arrow $\bigoplus_{i=1}^{r} \mathbb{Z}D_i \to \mathrm{Cl}_G(X)$ is well-defined because each prime divisor D_i is G-invariant, hence the sheaves $\mathcal{O}_X(D_i)$ are canonically linearized. We can replace X by its smooth locus and consider (equivariant) Picard groups. Consider the diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{O}(E_0 \times^G X)^* \arrow[r] \arrow[d] & \mathcal{O}(X)^{*G} \arrow[d] \\
\mathcal{O}(E_0 \times^G X_0)^* \arrow[r] \arrow[d, "E_0 \times^G D_i \mapsto D_i"] & \mathcal{O}(X_0)^{*G} \arrow[d] \\
\bigoplus_{i=1}^r \mathbb{Z}(E_0 \times^G D_i) \arrow[r] \arrow[d] & \bigoplus_{i=1}^r \mathbb{Z} D_i \arrow[d] \\
\mathrm{Pic}(E_0 \times^G X) \arrow[r] \arrow[d] & \mathrm{Pic}_G(X) \arrow[d] \\
\mathrm{Pic}(E_0 \times^G X_0) \arrow[r] \arrow[d] & \mathrm{Pic}_G(X_0) \arrow[d] \\
0 & 0
\end{tikzcd}$$

where the first vertical line is the exact sequence (3.2.2) applied to $E_0 \times^G X_0 \hookrightarrow E_0 \times^G X$, and the second vertical line is the sequence (3.2.3). The fourth and fifth horizontal arrows are the isomorphisms described in the last proposition, and the first and second horizontal arrows are defined as follows: notice that because f is a torsor, there is an isomorphism $f^\sharp : \mathcal{O}_{E_0 \times^G X} \to (f_* \mathcal{O}_{E_0 \times X})^G$, hence the ring morphism $f^\sharp(E_0 \times^G X)$ induces an isomorphism $\mathcal{O}(E_0 \times^G X)^* \simeq \mathcal{O}(E_0 \times X)^{*G}$. Also, because E_0 has a complement in E of codimension at least two and E is an affine space, the projection $p_X : E_0 \times X \to X$ induces an isomorphism

$$p_X^\sharp(X) : \mathcal{O}(X)^{*G} \to \mathcal{O}(E_0 \times X)^{*G}$$

between groups of G-invariant units. The first horizontal arrow is then the isomorphism

$$(p_X^\sharp(X))^{-1} \circ f^\sharp(E_0 \times^G X).$$

The second horizontal arrow is the isomorphism obtained the same way but replacing X by X_0. The third horizontal arrow is obviously an isomorphism between these two free abelian groups of rank r. The diagram is commutative by construction, and the second vertical arrow is exact because all horizontal arrows are isomorphisms. $\square$

3.2.3 Construction of the equivariant Cox ring

Let G be an algebraic group, $\sigma : G \times X \to X$ an action of G on a pointed normal variety $(X, x \overset{\iota}{\hookrightarrow} X)$ such that $\mathcal{O}(X)^{*G} \simeq k^*$, and $p_X : G \times X \to X$ the second projection. Natural examples of such varieties are almost homogeneous varieties, and normal complete G-varieties. Below, we follow the lines of the construction presented in Section 3.2.1, but considering G-linearized divisorial sheaves on X. In particular, this will motivate the assumption $\mathcal{O}(X)^{*G} \simeq k^*$.

Similarly as before, consider the category of rigidified G-linearized divisorial sheaves on X whose morphisms preserve linearizations and rigidifications. Let E^G be the set of isomorphism classes associated to this category, which again carries a natural structure of abelian group.

Proposition 3.2.3.1. *The group morphism $\rho : E^G \to \mathrm{Cl}_G(X), [(\mathcal{F}, f)] \mapsto [\mathcal{F}]$ is an isomorphism.*

Proof. We adapt the proof [FGA, 9.2.9] of Proposition 3.2.1.3. The map ρ obviously defines a surjective group morphism. To check injectivity, consider an element $[(\mathcal{F}, f)] \in \ker \rho$. Then, there is an isomorphism of G-linearized divisorial sheaves $v : \mathcal{F} \to \mathcal{O}_X$, where $\mathcal{O}_X$ is endowed with the trivial linearization. Consider the automorphism $w := \iota^* v \circ f^{-1}$ of k viewed as a k-vector space. Let $\varphi \to \operatorname{Spec} k$ be the structural morphism of X, the pullback $\varphi^* w : \mathcal{O}_X \to \mathcal{O}_X$ of w is the automorphism of $\mathcal{O}_X$ (as a G-linearized sheaf) defined by the multiplication by the constant function $w(1_k)$. By construction, we have $(\mathcal{F}, f) \simeq (\mathcal{O}_X, w^{-1})$ through v, and $(\mathcal{O}_X, \mathrm{Id}_k) \simeq (\mathcal{O}_X, w^{-1})$ through $\varphi^* w$. This proves $[(\mathcal{F}, f)] = [(\mathcal{O}_X, \mathrm{Id}_k)]$. □

Given a G-linearized divisorial sheaf $\mathcal{F}$ on X, we have $\mathrm{Aut}(\mathcal{F}) \simeq \mathcal{O}(X)^{*G}$. Because $\mathcal{O}(X)^{*G} \simeq k^*$ by assumption, we can identify sections of rigidified G-linearized divisorial sheaves lying in a given class $[(\mathcal{F}, f)] \in E^G$. Similarly as in 3.2.1, this identification is done using the equivalence relations $\sim_U^G$ indexed by open subsets $U \subset X$. This builds up a canonical representative $\mathcal{F}^x$ of the corresponding class $[\mathcal{F}] \in \mathrm{Cl}_G(X)$ which we call the *rigidified G-linearized sheaf associated to* $[\mathcal{F}]$.

Proposition 3.2.3.2. *The sheaf*

$$\mathcal{R}_{G,X} := \bigoplus\nolimits_{[\mathcal{F}] \in \mathrm{Cl}_G(X)} \mathcal{F}^x$$

has a natural structure of quasi-coherent $\mathrm{Cl}_G(X)$-graded $\mathcal{O}_X$-algebra. If this $\mathcal{O}_X$-algebra is of finite type, its relative spectrum $\hat{X}_G := \mathrm{Spec}_X(\mathcal{R}_{G,X})$ is an algebraic scheme endowed with a G-action that commutes with the natural $\mathbb{D}(\mathrm{Cl}_G(X))$-action, and that lifts the G-action on X.

Proof. By construction, $\mathcal{R}_{G,X}$ is a $\mathrm{Cl}_G(X)$-graded quasi-coherent $\mathcal{O}_X$-module. The multiplication between homogeneous sections is defined by the natural morphism

$$\mathcal{F}^x(U) \otimes_{\mathcal{O}(U)} \mathcal{G}^x(U) \to (\mathcal{F}^x \star \mathcal{G}^x)^x(U)$$

for every open subset $U \subset X$. This multiplication equips $\mathcal{R}_{G,X}$ with a structure of quasi-coherent $\mathrm{Cl}_G(X)$-graded $\mathcal{O}_X$-algebra.

Suppose that $\mathcal{R}_{G,X}$ is of finite type. Then, the relative spectrum $\hat{X}_G$ is by definition an algebraic scheme which comes with a morphism $q : \hat{X}_G \to X$. Moreover, the grading group $\mathrm{Cl}_G(X)$ has to be finitely generated, thus $\mathbb{D}(\mathrm{Cl}_G(X))$ is a diagonalizable group having a natural action on $\hat{X}_G$. Now, consider the direct sum

$$\Phi := \bigoplus\nolimits_{[\mathcal{F}] \in \mathrm{Cl}_G(X)} \Phi_{\mathcal{F}^x}$$

of the isomorphisms $\Phi_{\mathcal{F}^x} : \sigma^* \mathcal{F}^x \to p_X^* \mathcal{F}^x$ given by G-linearizations of the divisorial sheaves $\mathcal{F}^x$. By construction, Φ defines an isomorphism of $\mathrm{Cl}_G(X)$-graded $\mathcal{O}_{G \times X}$-algebra $\sigma^* \mathcal{R}_{G,X} \to p_X^* \mathcal{R}_{G,X}$. By virtue of [EGA, II.1.5.5], we can identify the morphism $\mathrm{Spec}_{G \times X}(\sigma^* \mathcal{R}_{G,X}) \to G \times X$ with the base change of q along the morphism $G \times X \xrightarrow{\sigma} X$. In the same way, we identify $\mathrm{Spec}_{G \times X}(p_X^* \mathcal{R}_{G,X})$ with $G \times \hat{X}_G$ as schemes over $G \times X$. Hence, we obtain a cartesian square

$$\begin{tikzcd} G \times \hat{X}_G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \hat{X}_G \arrow[d, "q"] \\ G \times X \arrow[r, "\sigma"'] & X . \end{tikzcd}$$

The G-linearizations of the homogeneous components yield a cocycle condition for the isomorphism Φ, hence a G-linearization of $\mathcal{R}_{G,X}$. It follows that the morphism $G \times \hat{X}_G \to \hat{X}_G$ satisfies the associativity axiom of a group action. By the diagram, this action lifts the G-action on the base X. Finally, this action commutes with the $\mathbb{D}(\mathrm{Cl}_G(X))$-action because the automorphism of $\mathrm{Cl}_G(X)$ induced by Φ through its action on the grading group is the identity. □

Definition 3.2.3.3. The *equivariant Cox sheaf* of the pointed variety X is the $\mathrm{Cl}_G(X)$-graded $\mathcal{O}_X$-algebra $\mathcal{R}_{G,X}$. The *equivariant Cox ring* $\mathrm{Cox}_G(X)$ is the ring of global sections of $\mathcal{R}_{G,X}$. When $\mathcal{R}_{G,X}$ is of finite type as an $\mathcal{O}_X$-algebra, the relative spectrum $\hat{X}_G$ over X of the Cox sheaf is the *equivariant characteristic space* of X. When $\mathrm{Cox}_G(X)$ is a finitely generated k-algebra, its spectrum $\tilde{X}_G$ is the *equivariant total coordinate space* of X.

Remark 3.2.3.4. By the exact sequence (3.2.1) and 3.2.2.2, the equivariant Cox ring and the ordinary Cox ring are canonically isomorphic if $\mathrm{Pic}(G)$ and $\mathbb{X}(G)$ are trivial (e.g. when G is semisimple and simply connected).

3.2.4 Equivariant diagonalizable torsors

Let G be an algebraic group, $\mathscr{D}$ a diagonalizable group, and (X, x) a pointed normal G-variety.

Definition 3.2.4.1. A *diagonalizable torsor* is a torsor (or principal bundle) under a diagonalizable group. Let Y be a $G \times \mathscr{D}$-variety, and $q : Y \to X$ a diagonalizable $\mathscr{D}$-torsor. Then q is a *G-equivariant diagonalizable $\mathscr{D}$-torsor* over X if the morphism q is G-equivariant.

The G-equivariant diagonalizable $\mathscr{D}$-torsors over X naturally form a category, taking for arrows the $G \times \mathscr{D}$-equivariant morphisms over X. In this Section, we classify G-equivariant diagonalizable torsors over X up to isomorphism, assuming X admits only constant invariant invertible regular functions. The next proposition gives the general form of such a torsor. Recall that a torsor by an (affine) algebraic group is necessarily an affine invariant morphism.

Proposition 3.2.4.2. *Let Y be an algebraic $\mathscr{D}$-scheme and $q : Y \to X$ be a $\mathscr{D}$-invariant affine morphism. Then, Y is the relative spectrum over X of a quasi-coherent $\mathbb{X}(\mathscr{D})$-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type. Moreover, q defines a G-equivariant $\mathscr{D}$-torsor over X if and only if the homogeneous components of $\mathcal{A}$ are G-linearized invertible sheaves and the morphisms*

$$\mathcal{A}_\chi \otimes_{\mathcal{O}_X} \mathcal{A}_\lambda \to \mathcal{A}_{\chi+\lambda}, \quad \chi, \lambda \in \mathbb{X}(\mathscr{D})$$

are isomorphisms of G-linearized invertible sheaves.

Proof. By [SGA3, Exp. 8, 4.1], the morphism q is a $\mathscr{D}$-torsor if and only if the homogeneous components of $\mathcal{A}$ are invertible sheaves and the natural morphisms $\mathcal{A}_\chi \otimes_{\mathcal{O}_X} \mathcal{A}_\lambda \to \mathcal{A}_{\chi+\lambda}$ are isomorphisms of invertible sheaves. Proceeding as in the proof of 3.2.3.2, we obtain that the conditions on G-linearizations are necessary and sufficient to obtain a G-action on Y with the desired properties. □

Denote by $H^1_G(X, \mathscr{D})$ the set of isomorphism classes of G-equivariant $\mathscr{D}$-torsor over X. The assignment

$$\mathscr{D} \mapsto H^1_G(X, \mathscr{D})$$

naturally defines a functor from the category of diagonalizable groups to the category of pointed sets, where origins are taken to be the classes of trivial G-equivariant diagonalizable torsors . Indeed, consider a morphism of diagonalizable groups $\varphi_1 : \mathscr{D}_1 \to \mathscr{D}_2$, and define $H^1_G(X, \varphi_1)$ as follows: let $q : Y \to X$ be a G-equivariant $\mathscr{D}_1$-torsor over X and $Y \times^{\mathscr{D}_1} \mathscr{D}_2$ the associated fiber bundle over X, where $\mathscr{D}_1$ acts on $\mathscr{D}_2$ via φ_1. This fiber bundle exists as a (normal) algebraic scheme [Bri18, 3.3.3] and sits in the cartesian square

$$\begin{tikzcd} Y \times \mathscr{D}_2 \arrow[r, "p_Y"] \arrow[d, "/\mathscr{D}_1"] & Y \arrow[d, "q"] \\ Y \times^{\mathscr{D}_1} \mathscr{D}_2 \arrow[r, "\psi"'] & X , \end{tikzcd}$$

in which all arrows are G-equivariant. Moreover, ψ defines a $\mathscr{D}_2$-torsor as it pullbacks to the trivial torsor along the faithfully flat morphism q. This construction is compatible with taking isomorphism classes, thus it defines a map

$$H^1_G(X, \varphi_1) : H^1_G(X, \mathscr{D}_1) \to H^1_G(X, \mathscr{D}_2).$$

Moreover, considering a second morphism $\varphi_2 : \mathscr{D}_2 \to \mathscr{D}_3$ of diagonalizable groups, we have

$$H^1_G(X, \varphi_2 \circ \varphi_1) = H^1_G(X, \varphi_2) \circ H^1_G(X, \varphi_1).$$

The next proposition says that the functor $H^1_G(X, .)$ is a group functor.

Proposition 3.2.4.3. *Let $\mathscr{D}, \mathscr{D}_1, \mathscr{D}_2$ be diagonalizable groups. We have a canonical bijection*

$$H^1_G(X, \mathscr{D}_1) \times H^1_G(X, \mathscr{D}_2) \to H^1_G(X, \mathscr{D}_1 \times \mathscr{D}_2),\ ([Y_1], [Y_2]) \mapsto [Y_1 \times_X Y_2],$$

where $\mathscr{D}_1 \times \mathscr{D}_2$ acts on $Y_1 \times_X Y_2$ componentwise. Moreover the canonical map

$$H^1_G(X, \mathscr{D}) \times H^1_G(X, \mathscr{D}) \to H^1_G(X, \mathscr{D} \times \mathscr{D}) \xrightarrow{H^1_G(X, m_{\mathscr{D}})} H^1_G(X, \mathscr{D})$$

endow $H^1_G(X, \mathscr{D})$ with the structure of an abelian group for which the neutral element is the class of the trivial G-equivariant $\mathscr{D}$-torsor over X. Finally, considering a morphism of diagonalizable groups $\varphi : \mathscr{D}_1 \to \mathscr{D}_2$, the map $H^1_G(X, \varphi)$ is a group morphism.

Proof. The first map is well defined and has an inverse. Indeed, denote by p_i the two projections $\mathscr{D}_1 \times \mathscr{D}_2 \to \mathscr{D}_i$. Then, the map which associates to a class $[Y]$ of G-equivariant $\mathscr{D}_1 \times \mathscr{D}_2$-torsor over X, the pair

$$(H^1_G(X, p_1)([Y]), H^1_G(X, p_2)([Y]))$$

is the sought inverse. Hence, $H^1_G(X, .)$ commutes with finite products. As a consequence, this functor sends groups to groups. But diagonalizable groups are group objects in their category, whence the claimed structure of abelian group on $H^1_G(X, \mathscr{D})$. The last assertion is directly checked. □

In order to determine the group $H^1_G(X, \mathscr{D})$, the last proposition allows to reduce to the situations where $\mathscr{D} = \mathbb{G}_m$ and $\mathscr{D} = \mu_n$. As a consequence of 3.2.4.2, we have an isomorphism

$$\mathrm{Pic}_G(X) \xrightarrow{\simeq} H^1_G(X, \mathbb{G}_m), [\mathcal{L}] \mapsto [\mathrm{Spec}_X(\bigoplus_{k \in \mathbb{Z}} \mathcal{L}^{\otimes k})]. \tag{3.2.4}$$

On the other hand, the exact sequence

$$1 \to \mu_n \xrightarrow{i} \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \to 1$$

gives rise to an exact sequence of abstract groups

$$1 \to \mathcal{O}(X)^*/(\mathcal{O}(X)^*)^n \to H^1(X, \mu_n) \to \mathrm{Pic}(X)[n] \to 1,$$

where $H^1(X, \mu_n)$ denotes the group of isomorphism classes of μ_n-torsors over X, and $\mathrm{Pic}(X)[n]$ denotes the n-torsion subgroup of $\mathrm{Pic}(X)$ [SGA1, Exp. XI, 6.4]. It follows that a μ_n-torsor Y over X is defined by choosing an n-torsion element $[\mathcal{L}]$ in $\mathrm{Pic}(X)$, an isomorphism $f : \mathcal{L}^{\otimes n} \to \mathcal{O}_X$, and setting

$$Y := \mathrm{Spec}_X \bigoplus_{i=0}^{n-1} \mathcal{L}^{\otimes i},$$

where the $\mathcal{O}_X$-algebra structure is defined by the choice of f. From this fact and 3.2.4.2, we deduce the exact sequence

$$1 \to \mathcal{O}(X)^{*G}/(\mathcal{O}(X)^{*G})^n \to H^1_G(X,\mu_n) \to \mathrm{Pic}_G(X)[n] \to 1.$$

Assuming $\mathcal{O}(X)^{*G} \simeq k^*$, this exact sequence yields an isomorphism

$$\mathrm{Pic}_G(X)[n] \xrightarrow{\simeq} H^1_G(X,\mu_n), [\mathcal{L}] \mapsto [\mathrm{Spec}_X(\bigoplus_{i=0}^{n-1}(\mathcal{L}^{\otimes i})^x], \tag{3.2.5}$$

where the algebra structure is given by the multiplication between sections of rigidified G-linearized invertible sheaves. Using (3.2.4) and (3.2.5), we give a refined version of 3.2.4.2.

Proposition 3.2.4.4. *Suppose that $\mathcal{O}(X)^{*G} \simeq k^*$, and fix an isomorphism $\mathbb{X}(\mathscr{D}) \simeq \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_s\mathbb{Z}$. Then a G-equivariant $\mathscr{D}$-torsor Y over X is defined up to isomorphism by choosing*

- *a r-tuple $([\mathcal{L}_1], \ldots, [\mathcal{L}_r]) \in \mathrm{Pic}_G(X) \times \ldots \times \mathrm{Pic}_G(X)$,*
- *a s-tuple $([\mathcal{L}_{r+1}], \ldots, [\mathcal{L}_{r+s}]) \in \mathrm{Pic}_G(X)[n_1] \times \ldots \times \mathrm{Pic}_G(X)[n_s]$,*

and setting $Y := \mathrm{Spec}_X \mathcal{A}$ where $\mathcal{A}$ is the $\mathbb{X}(\mathscr{D})$-graded $\mathcal{O}_X$-algebra

$$\bigoplus_{(k_i)\in\mathbb{Z}^r\times\mathbb{Z}/n_1\mathbb{Z}\times\ldots\times\mathbb{Z}/n_s\mathbb{Z}} (\mathcal{L}_1^{\otimes k_1} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{L}_{r+s}^{\otimes k_{r+s}})^x,$$

the algebra structure being given by the multiplication between sections of rigidified G-linearized invertible sheaves.

Corollary 3.2.4.5. *Suppose that $\mathcal{O}(X)^{*G} \simeq k^*$. Then, there is a natural isomorphism of abelian groups*

$$H^1_G(X,\mathscr{D}) \simeq \mathrm{Hom}(\mathbb{X}(\mathscr{D}), \mathrm{Pic}_G(X)).$$

3.2.5 Equivariant almost principal bundles

Let G be an algebraic group, $\mathscr{D}$ a diagonalizable group, and (X,x) a pointed normal G-variety. In [Has, 0.4], Hashimoto introduced the notion of an *almost principal bundle* under an algebraic group. We recall this notion and define a G-equivariant analogue.

Definition 3.2.5.1. Let H be an algebraic group, Y an algebraic H-scheme, and $q : Y \to X$ an H-invariant morphism. We say that q is an *almost principal H-bundle* over X if there exists (H-stable) open subschemes $Y_0 \subset Y$, $X_0 \subset X$ whose respective complements are of codimension at least two and such that q induces an H-torsor $Y_0 \to X_0$. If moreover, Y is a $G \times H$-scheme and q is G-equivariant, we say that q is a *G-equivariant almost principal H-bundle over X.*

Proposition 3.2.5.2. *Let H be an algebraic group and $q : Y \to X$ a G-equivariant almost principal H-bundle. Suppose that G is connected, and Y is a normal variety. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Cl}_{G\times H}(Y) & \xrightarrow{\phi_1} & \mathrm{Cl}_H(Y) \\ {\scriptstyle q^*}\uparrow & & {\scriptstyle q^*}\uparrow \\ \mathrm{Cl}_G(X) & \xrightarrow{\phi_2} & \mathrm{Cl}(X) \end{array},$$

where vertical arrows are induced by the pullback q^* of (equivariant) divisorial sheaves, and horizontal arrows are given by forgetting linearizations. Moreover, vertical arrows are isomorphisms, with inverse morphisms q^H_* explicitly given by

$$[\mathcal{F}] \mapsto [(q_*\mathcal{F})^H].$$

Proof. Using 3.2.2.1, we can suppose that q is a H-torsor, that X, Y are smooth, and consider (equivariant) Picard groups instead of (equivariant) class groups. Notice that because q is H-invariant and G-equivariant, the two vertical arrows are well defined. Also, the commutativity of the diagram is directly checked. The isomorphism $q^* : \mathrm{Pic}(X) \to \mathrm{Pic}_H(Y)$ is a consequence of a descent result from Grothendieck [Bri18, 3.3.1]. Now, we show that the left vertical arrow is an isomorphism. Let $\mathcal{L} \in \ker q^*$, then we have that $\mathcal{L} \simeq \mathcal{O}_X$ as invertible sheaves. Because of the exact sequence (3.2.1), we have $\mathcal{L} \simeq \mathcal{O}_X(\lambda)$ as G-linearized invertible sheaves, for a certain character $\lambda \in \mathbb{X}(G)$. But as $q^*\mathcal{O}_X(\lambda) \simeq \mathcal{O}_Y(\lambda)$, it follows that $\lambda = 0$ which proves injectivity. For surjectivity it suffices, by the exact sequence (3.2.1) again, to check that all the elements $[\mathcal{O}_Y(\lambda)] \in \mathrm{Pic}_{G\times H}(Y)$ with $\lambda \in \mathbb{X}(G)$ are reached by q^*. This is obvious as $q^*\mathcal{O}_X(\lambda) \simeq \mathcal{O}_Y(\lambda)$. The last claim is a direct consequence of the fact that q is a torsor and that invertible sheaves are locally trivial. $\square$

In the framework of Cox rings, we consider almost principal bundles under diagonalizable groups. In fact, we need a stronger notion in order to obtain a generalization of 3.2.4.4.

Definition 3.2.5.3. A *diagonalizable almost principal bundle over* X is a good quotient $q : Y \to X$ of a normal algebraic scheme Y by a diagonalizable group such that q is an almost principal bundle.

Proposition 3.2.5.4. *Consider a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle $q : Y \to X$. Then, Y is the relative spectrum over X of a $\mathbb{X}(\mathscr{D})$-graded quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ of finite type whose homogeneous components are G-linearized divisorial sheaves and such that the morphisms*

$$\mathcal{A}_\chi \otimes_{\mathcal{O}_X} \mathcal{A}_\lambda \to \mathcal{A}_{\chi+\lambda}, \quad \chi, \lambda \in \mathbb{X}(\mathscr{D})$$

induce isomorphisms of G-linearized sheaves over X_{sm}. Conversely, the relative spectrum of such a graded $\mathcal{O}_X$-algebra is a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle.

Proof. Because q is a good quotient by $\mathscr{D}$, we have an isomorphism $Y \simeq \mathrm{Spec}_X \mathcal{A}$, where $\mathcal{A} := q_*\mathcal{O}_Y$ is a $\mathbb{X}(\mathscr{D})$-graded quasi-coherent $\mathcal{O}_X$-algebra. By 3.2.4.2, the homogeneous components of $\mathcal{A}$ define invertible sheaves over the locus $X_0 \overset{i}{\hookrightarrow} X$ where q is a torsor, and the natural morphisms $\mathcal{A}_\chi \otimes_{\mathcal{O}_X} \mathcal{A}_\lambda \to \mathcal{A}_{\chi+\lambda}$ induce isomorphisms of invertible sheaves over X_0. Also, because Y is a normal algebraic scheme, we have an isomorphism $i_*i^*\mathcal{A}_\lambda \simeq \mathcal{A}_\lambda$ for each homogeneous component. It follows that

$$(\mathcal{A}_\lambda)^{\vee\vee} \simeq i_*i^*(\mathcal{A}_\lambda)^{\vee\vee} \simeq i_*((i^*\mathcal{A}_\lambda)^{\vee\vee}) \simeq i_*i^*\mathcal{A}_\lambda \simeq \mathcal{A}_\lambda,$$

where the first isomorphism comes from the fact that $(\mathcal{A}_\lambda)^{\vee\vee}$ is reflexive [Har1, 1.11], the second holds because the pullback by a flat morphism commutes with taking duals, and the third holds because $i^*\mathcal{A}_\lambda$ is invertible, hence reflexive. Thus, the homogeneous components of $\mathcal{A}$ are divisorial sheaves on X. Also, we see that X_0 can be taken to be the smooth locus of X. Indeed, the divisorial sheaves $\mathcal{A}_\lambda$ are invertible over X_{sm} and the natural morphisms $\mathcal{A}_\chi \otimes_{\mathcal{O}_X} \mathcal{A}_\lambda \to \mathcal{A}_{\chi+\lambda}$ induce isomorphisms of G-linearized invertible sheaves over X_{sm}. Finally, the conditions on G-linearizations are necessary and sufficient to have a G-action on Y with the desired properties.

Conversely, consider a quasi-coherent $\mathbb{X}(\mathscr{D})$-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ satisfying the conditions of the statement, and denote by $q : Y \to X$ the affine morphism defined by its relative spectrum over X. By 3.2.4.2, the restriction of q to $q^{-1}(X_{\mathrm{sm}})$ defines a G-equivariant $\mathscr{D}$-torsor over X_{sm}. This yields an isomorphism $\mathcal{O}_X \simeq \mathcal{A}_0$, thus q is a good quotient. To prove that we have a $G \times \mathscr{D}$-action on Y such that q is G-equivariant, we proceed exactly as in the proof of 3.2.3.2. Now, we show that Y is a normal algebraic scheme. For this, we verify that $\mathcal{A}$ is a sheaf of normal algebras, thus we can suppose that X is smooth. Then, q is smooth as well because it is a torsor (under a smooth algebraic group). As a consequence, the stalks of $\mathcal{A}$ are regular local rings, hence are integrally closed. It remains to prove that the complement of $q^{-1}(X_{\mathrm{sm}})$ in Y is of codimension $\geqslant 2$. Because q is affine, we can suppose that X and Y are affine. Also, as Y is normal, it is a disjoint union $Y = \cup_{i=1}^{r} Y_i$ of normal affine varieties, and we have canonical isomorphisms

$$\begin{aligned}\mathcal{O}_Y(q^{-1}(X_{\mathrm{sm}})) &\simeq \prod_{i=1}^{r} \mathcal{O}_{Y_i}(q^{-1}(X_{\mathrm{sm}}) \cap Y_i)\\ &\simeq \mathcal{A}(X_{\mathrm{sm}})\\ &\simeq \mathcal{A}(X)\\ &\simeq \mathcal{O}_Y(Y)\\ &\simeq \prod_{i=1}^{r} \mathcal{O}_{Y_i}(Y_i).\end{aligned}$$

It follows from these isomorphisms and the next lemma that $q^{-1}(X_{\mathrm{sm}}) \cap Y_i$ has a complement of codimension at least two in Y_i, $i = 1, ..., r$. Hence, $q^{-1}(X_{\mathrm{sm}})$ has a complement of codimension at least two in Y, and q is a G-equivariant almost principal $\mathscr{D}$-bundle over X. $\square$

Lemma 3.2.5.5. *Let X be an affine variety and $Z \subset X$ a closed subvariety such that the restriction morphism $\mathcal{O}(X) \to \mathcal{O}(X \setminus Z)$ is an isomorphism. Then, Z is of codimension $\geqslant 2$ in X.*

Proof. We consider the local cohomology groups $H^i_Z(X, \mathcal{O}_X)$ associated with the stucture sheaf $\mathcal{O}_X$ and the closed subset $Z \subset X$. By [Har2, Ex. III.2.3], we have a long exact sequence of local cohomology

$$0 \to H^0_Z(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X) \to H^0(X \setminus Z, \mathcal{O}_X) \to H^1_Z(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X) \to ...$$

By our assumptions, the groups $H^0_Z(X, \mathcal{O}_X)$ and $H^1_Z(X, \mathcal{O}_X)$ are trivial. On the other hand, recall the cohomological interpretation of the depth of an ideal $I \subset \mathcal{O}(X)$ (see [Har2, Ex. III.3.4]):

$$\mathrm{prof}_I(\mathcal{O}(X)) \geqslant n \iff H^i_{\mathcal{V}(I)}(X, \mathcal{O}_X) = 0, \forall i < n,$$

where $\mathcal{V}(I)$ denotes the zero set of I in X. Applying this to the ideal I defining Z, we obtain the result by virtue of the inequalities

$$\mathrm{codim}_X(Z) \geqslant \mathrm{prof}_I(\mathcal{O}(X)) \geqslant 2.$$

$\square$

Remark 3.2.5.6. The proof of the last proposition shows that given a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle $q : Y \to X$, the restriction $q^{-1}(X_{\mathrm{sm}}) \to X_{\mathrm{sm}}$ is a G-equivariant $\mathscr{D}$-torsor, and $q^{-1}(X_{\mathrm{sm}})$ has a complement of codimension at least two in Y.

Definition 3.2.5.7. Let $q : Y \to X$ be a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle. Then, the natural morphism

$$\mathrm{type}_G(Y) : \mathbb{X}(\mathscr{D}) \to \mathrm{Cl}_G(X),\ \chi \mapsto [(q_*\mathcal{O}_Y)_\chi],$$

is called the *G-equivariant type* of Y. The morphism

$$\mathrm{type}(Y) := \phi \circ \mathrm{type}_G(Y) : \mathbb{X}(\mathscr{D}) \to \mathrm{Cl}(X),$$

where $\phi : \mathrm{Cl}_G(X) \to \mathrm{Cl}(X)$ is the forgetful morphism, is the *type* of Y.

Proposition 3.2.5.4 gives the general form of a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle over X. These objects naturally form a category, taking for arrows the $G \times \mathscr{D}$-equivariant morphisms over X. Assume that $\mathcal{O}(X)^{*G} \simeq k^*$, and let $q : Y \to X$ be a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle. Using 3.2.2.1, 3.2.4.4, and 3.2.5.4, we see that the G-equivariant type of Y determines its isomorphism class. However, we don't know a priori if conversely any morphism

$$\mathbb{X}(\mathscr{D}) \to \mathrm{Cl}_G(X)$$

defines a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle over X. Indeed, such a morphism defines a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ satisfying the properties of Proposition 3.2.5.4 with the exception that it might not be of finite type. The next proposition gives a sufficient condition to guarantee that the algebra $\mathcal{A}$ defined by any morphism as above is of finite type. Moreover, its proof shows how to construct any G-equivariant diagonalizable almost principal bundle over X from the equivariant characteristic space of X.

Proposition 3.2.5.8. *Suppose that $\mathcal{O}(X)^{*G} \simeq k^*$ and that the equivariant Cox sheaf of X is of finite type. Consider a quasi-coherent $\mathbb{X}(\mathscr{D})$-graded $\mathcal{O}_X$-algebra $\mathcal{A}$ satisfying the conditions of Proposition 3.2.5.4. Then, $\mathcal{A}$ is of finite type as an $\mathcal{O}_X$-algebra.*

Proof. We are going to realize $\mathcal{A}$ as a subalgebra of invariants by a diagonalizable group. Consider the morphism of diagonalizable groups $\varphi : \mathbb{D}(\mathrm{Cl}_G(X)) \to \mathscr{D}$ dually defined by the morphism

$$\mathrm{type}_G(Y) : \mathbb{X}(\mathscr{D}) \to \mathrm{Cl}_G(X),$$

and the $\mathbb{D}(\mathrm{Cl}_G(X)) \times \mathscr{D}$-action on $\hat{X}_G \times \mathscr{D}$ defined on schematic points by $(g, h_1) \cdot (x, h_2) := (g \cdot x, \varphi(g)^{-1} h_1 h_2)$. Then, we have a good quotient $f : \hat{X}_G \times \mathscr{D} \to X$ by this action. Indeed, consider the morphisms

$$\hat{X}_G \times \mathscr{D} \xrightarrow{p_{\hat{X}_G}} \hat{X}_G \xrightarrow{q_1} X$$

where $p_{\hat{X}_G}$ is the projection, and q_1 is the structural morphism of the equivariant characteristic space $\hat{X}_G$ over X. Both morphisms are good quotients, and the composition is a good quotient by $\mathbb{D}(\mathrm{Cl}_G(X)) \times \mathscr{D}$, hence $f = q_1 \circ p_{\hat{X}_G}$. As a consequence of this factorization, f is in addition a G-equivariant diagonalizable almost principal $\mathbb{D}(\mathrm{Cl}_G(X)) \times \mathscr{D}$-bundle over X. Performing first the quotient by $\mathbb{D}(\mathrm{Cl}_G(X))$, we obtain another factorization

$$\hat{X}_G \times \mathscr{D} \xrightarrow{q_2} Y \xrightarrow{q_3} X$$

of f into G-equivariant diagonalizable almost principal bundles. The $\mathcal{O}_X$-algebra $q_{3*}\mathcal{O}_Y$ is obtained by computing $\mathbb{D}(\mathrm{Cl}_G(X))$-invariants:

$$(f_* \mathcal{O}_{\hat{X}_G \times \mathscr{D}})^{\mathbb{D}(\mathrm{Cl}_G(X))} = (\bigoplus_{(\lambda, [\mathcal{F}]) \in \mathbb{X}(\mathscr{D}) \times \mathrm{Cl}_G(X)} \mathcal{F}^x e^\lambda)^{\mathbb{D}(\mathrm{Cl}_G(X))} = \bigoplus_{\lambda \in \mathbb{X}(\mathscr{D})} \mathcal{F}^x_\lambda,$$

where $[\mathcal{F}_\lambda] := \mathrm{type}_G(Y)(\lambda)$, $\forall \lambda \in \mathbb{X}(\mathscr{D})$. Hence, we obtain

$$q_{3*}\mathcal{O}_Y = \mathcal{A},$$

which proves that this $\mathcal{O}_X$-algebra is of finite type. □

Assume from now that $\mathcal{O}(X)^{*G} \simeq k^*$, and that the equivariant Cox sheaf of (X, x) is of finite type. By the last proposition and the above discussion, we obtain a one-to-one correspondence between G-equivariant diagonalizable almost principal bundles over X and G-equivariant diagonalizable torsors over X_{sm}. This yields in particular the

Proposition 3.2.5.9. *Suppose that $\mathcal{O}(X)^{*G} \simeq k^*$, that the equivariant Cox sheaf of X is of finite type, and fix an isomorphism $\mathbb{X}(\mathscr{D}) \simeq \mathbb{Z}^r \times \mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_s\mathbb{Z}$. Then a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle Y over X is defined up to isomorphism by choosing*

- *a r-tuple* $([\mathcal{F}_1], ..., [\mathcal{F}_r]) \in \mathrm{Cl}_G(X) \times ... \times \mathrm{Cl}_G(X)$,
- *a s-tuple* $([\mathcal{F}_{r+1}], ..., [\mathcal{F}_{r+s}]) \in \mathrm{Cl}_G(X)[n_1] \times ... \times \mathrm{Cl}_G(X)[n_s]$,

and setting $Y := \mathrm{Spec}_X \mathcal{A}$ *where* $\mathcal{A}$ *is the* $\mathbb{X}(\mathscr{D})$*-graded* $\mathcal{O}_X$*-algebra*

$$\bigoplus_{(k_i)\in\mathbb{Z}^r\times\mathbb{Z}/n_1\mathbb{Z}\times...\times\mathbb{Z}/n_s\mathbb{Z}} (\mathcal{F}_1^{\star k_1} \star ... \star \mathcal{F}_{r+s}^{\star k_{r+s}})^x.$$

Remark 3.2.5.10. Let $q : (Y := \mathrm{Spec}_X \mathcal{A}) \to X$ be a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle as defined by the last proposition. There is a canonical morphism of graded $\mathcal{O}_X$-algebras

$$\mathcal{A} \to \mathcal{R}_{G,X}$$

whose associated morphism between grading groups is $\mathrm{type}_G(Y)$. Indeed, the proof of Proposition 3.2.5.8 shows that q sits in the following commutative square:

$$\begin{tikzcd} \hat{X}_G \times \mathscr{D} \arrow[r, "p_{\hat{X}_G}"] \arrow[d, "//\mathbb{D}(\mathrm{Cl}_G(X))"] & \hat{X}_G \arrow[d, "//\mathbb{D}(\mathrm{Cl}_G(X))"] \\ Y \arrow[r, "q"'] & X \end{tikzcd}$$

Furthermore, the section $\hat{X}_G \to \hat{X}_G \times \mathscr{D}$, $t \mapsto (t, e_{\mathscr{D}})$ of the projection $p_{\hat{X}_G}$ yields an equivariant morphism $\hat{X}_G \to Y$ of varieties over X. This last morphism corresponds to a morphism of graded $\mathcal{O}_X$-algebras $\mathcal{A} \to \mathcal{R}_{G,X}$ as claimed. It is directly checked that this morphism is surjective when $\mathrm{type}_G(Y)$ is surjective. Finally, this construction shows that the structural morphism of the G-equivariant characteristic space over X factors through any G-equivariant diagonalizable almost principal bundle over X.

As a consequence of the last proposition, the equivariant Cox sheaf of the pointed variety X doesn't depend on the choice of the base point up to isomorphism. Hence, we can speak of the equivariant Cox sheaf (resp. the equivariant characteristic space) of X viewed as a variety. In fact, we add some flexibility by introducing the following definition.

Definition 3.2.5.11. A *G-equivariant characteristic space* over X is an algebraic scheme Z over X endowed with an action of a diagonalizable group $\mathscr{D}$ such that there exists a pair (f, φ) consisting of an isomorphism $f : Z \to \hat{X}_G$ of varieties over X and an isomorphism $\varphi : \mathscr{D} \to \mathbb{D}(\mathrm{Cl}(X))$ such that the square

$$\begin{tikzcd} \mathscr{D} \times Z \arrow[r] \arrow[d, "\varphi\times f"] & Z \arrow[d, "f"'] \\ \mathbb{D}(\mathrm{Cl}(X)) \times \hat{X}_G \arrow[r] & \hat{X}_G \end{tikzcd}$$

commutes, the horizontal arrows being the respective actions. A *G-equivariant total coordinate space* of X is the affinization of a G-equivariant charateristic space. A *G-equivariant Cox ring* of X is the (graded) coordinate ring of a G-equivariant total coordinate space.

Remark 3.2.5.12. Equivalently, a characteristic space over X is a G-equivariant diagonalizable almost principal bundle over X whose G-equivariant type is an isomorphism onto $\mathrm{Cl}_G(X)$.

Example 3.2.5.13. Let G be a connected algebraic group, and G/H be a homogeneous space. Consider the faithfully flat morphism $\varphi : H \to \mathbb{D}(\mathbb{X}(H))$ of algebraic groups induced by the inclusion $k[\mathbb{X}(H)] \subset \mathcal{O}(H)$. This yields an exact sequence of algebraic groups

$$1 \to K := \ker\varphi \to H \xrightarrow{\varphi} \mathbb{D}(\mathbb{X}(H)) \to 1,$$

where the normal subgroup K of H is the intersection of the kernels of all characters of H. Hence $H/K \simeq \mathbb{D}(\mathbb{X}(H))$ is the diagonalizable group whose character group is $\mathbb{X}(H)$. It follows that we have a factorization

$$G \xrightarrow{q_1} G/K \xrightarrow{q_2} G/H$$

of the canonical projection $\pi : G \to G/H$, and that $q_2 : G/K \to G/H$ is a G-equivariant diagonalizable $\mathbb{D}(\mathbb{X}(H))$-torsor.

On the other hand, there is a natural isomorphism $\mathbb{X}(H) \to \mathrm{Pic}_G(G/H)$, $\lambda \mapsto [\mathcal{L}_\lambda]$, where $\mathcal{L}_\lambda$ denotes the G-linearized invertible sheaf of sections of the G-line bundle $G \times^\lambda \mathbb{A}^1$ over G/H. Recall that for an open subset $V \subset G/H$ we have $\mathcal{L}_\lambda(V) \simeq \mathcal{O}_G(\pi^{-1}(V))^{(H)}_\lambda$. This yields the equalities

$$\begin{aligned}
q_{2*}\mathcal{O}_{G/K}(V) &= \mathcal{O}_{G/K}(q_2^{-1}(V)) \\
&= (q_{1*}\mathcal{O}_G(q_2^{-1}(V)))^K \\
&= \mathcal{O}_G(\pi^{-1}(V))^K \\
&= \bigoplus_{\lambda\in\mathbb{X}(H)} \mathcal{O}_G(\pi^{-1}(V))^{(H)}_\lambda \\
&\simeq \bigoplus_{\lambda\in\mathbb{X}(H)} \mathcal{L}_\lambda(V),
\end{aligned}$$

where the last equality follows from the fact that every H/K-module is a direct sum of eigenspaces. Thus, we have obtained

$$q_{2*}\mathcal{O}_{G/K} = \textstyle\bigoplus_{\lambda\in\mathbb{X}(H)} \mathcal{L}_\lambda.$$

Hence, the G-equivariant type of G/K defines an isomorphism with $\mathrm{Pic}_G(G/H)$, whence G/K is a G-equivariant characteristic space of G/H. When G is semisimple and simply connected, the equivariant Cox sheaf is canonically isomorphic to the ordinary one, and we retrieve [ADHL, Thm 4.5.1.8].

3.2.6 Equivariant quotient presentations

When a normal variety X admits a Cox sheaf of finite type, the characteristic space $\hat{X}$ is naturally a diagonalizable almost principal $\mathbb{D}(\mathrm{Cl}(X))$-bundle over X such that the regular invertible $\mathbb{D}(\mathrm{Cl}(X))$-homogeneous functions are constant. This leads to the notion of a *quotient presentation* of X [ADHL, 4.2.1.1]. We recall this notion and introduce a G-equivariant analogue.

Definition 3.2.6.1. Let $\mathscr{D}$ be a diagonalizable group and X a normal variety with only constant invertible regular functions. A *quotient presentation* of X is a diagonalizable almost principal $\mathscr{D}$-bundle $q : Y \to X$ such that the invertible $\mathscr{D}$-homogeneous regular functions on Y are constant.

Remark 3.2.6.2. In fact, the definition [ADHL, 4.2.1.1] of a quotient presentation is different but equivalent to the one given above. Indeed, a quotient presentation in loc. cit. is a normal variety Y together with a morphism $q : Y \to X$ verifying

1. q is a good quotient by a diagonalizable group $\mathscr{D}$,
2. The $\mathscr{D}$-homogeneous invertible regular functions on Y are constant,
3. There exists a $\mathscr{D}$-stable open $Y_0 \subset Y$ whose complement is of codimension $\geqslant 2$, and on which $\mathscr{D}$ acts freely with closed orbits.

Suppose that $q : Y \to X$ verifies the above conditions, then $q(Y_0)$ is open with a complement of codimension at least two in X. On the other hand, as $\mathscr{D}$ acts with closed orbits on Y_0 and q is a good quotient, the fibers of $q_{|Y_0}$ are orbits. Moreover, Y_0 has the quotient topology with respect to $q_{|Y_0}$. Indeed, consider an open subset of the form $q^{-1}(V)$, where V is a subset of $q(Y_0)$. Because q is a good quotient, we have that $q(Y_0 \setminus q^{-1}(V)) = q(Y_0) \setminus V$ is closed in $q(Y_0)$, thus V is open in $q(Y_0)$. It follows that q induces a geometric quotient $Y_0 \to q(Y_0)$. Thus, it induces a $\mathscr{D}$-torsor $Y_0 \to q(Y_0)$ because $\mathscr{D}$ acts freely on Y_0 [MFK, Chap. 0, 4.0.9]. We have verified $q : Y \to X$ is a diagonalizable almost principal $\mathscr{D}$-bundle with only constant $\mathscr{D}$-homogeneous invertible regular functions.

Conversely, consider a diagonalizable almost principal $\mathscr{D}$-bundle $q : Y \to X$ with only constant $\mathscr{D}$-homogeneous invertible regular functions. By definition, it satisfies all the above conditions, except that Y is not a priori a variety. However, this will be verified in Proposition 3.2.6.6 below.

Definition 3.2.6.3. Let $\mathscr{D}$ be a diagonalizable group and X a normal G-variety with only constant invertible G-invariant regular functions. A *G-equivariant quotient presentation* of X is a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle $q : Y \to X$ such that the G-invariant $\mathscr{D}$-homogeneous invertible regular functions on Y are constant.

Proposition 3.2.6.4. *Let $\mathscr{D}$ be a diagonalizable group, X a normal G-variety with only constant invertible G-invariant regular functions, and $q : Y \to X$ a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle. Then q is a G-equivariant quotient presentation if and only if the G-equivariant type of Y is injective.*

Proof. Indeed, $\mathrm{type}_G(Y)$ is injective if and only if its kernel is trivial. But this kernel contains a non-trivial element if and only if $q_*\mathcal{O}_Y$ admits an homogeneous component isomorphic to $\mathcal{O}_X$ endowed with the trivial G-linearization. The latter means that this homogeneous component admits an invertible G-invariant global section. $\square$

Let G be an algebraic group and (X, x) a pointed normal G-variety with only constant invertible G-invariant regular functions. We define the category of quotient presentations of X by defining a morphism between two quotient presentations $q_i : Y_i \xrightarrow{//\mathscr{D}_i} X$ to be a pair (f, φ) consisting of a morphism $f : Y_1 \to Y_2$ over X and a morphism $\varphi : \mathscr{D}_1 \to \mathscr{D}_2$, such that the square

$$\begin{tikzcd} \mathscr{D}_1 \times Y_1 \arrow[r] \arrow[d, "\varphi \times f"] & Y_1 \arrow[d, "f"'] \\ \mathscr{D}_2 \times Y_2 \arrow[r] & Y_2 \end{tikzcd}$$

commutes, where the horizontal arrows denote the respective actions. By the preceding section, a G-equivariant quotient presentation of X is defined up to isomorphism by a finitely generated subgroup of $\mathrm{Cl}_G(X)$. Indeed, the injectivity of the G-equivariant type says that a quotient presentation is up to isomorphism the relative spectrum $\hat{X}_G(M)$ of the M-graded $\mathcal{O}_X$-algebra (assumed to be of finite type)

$$\mathcal{A}(M) := \bigoplus_{[\mathcal{F}] \in M} \mathcal{F}^x,$$

where M is a finitely generated subgroup of $\mathrm{Cl}_G(X)$. The following Proposition is a generalization of [ADHL, 4.2.1.4].

Proposition 3.2.6.5. *Suppose that the equivariant Cox sheaf of X is of finite type and consider the category of subgroups of $\mathrm{Cl}_G(X)$ with morphisms being the inclusions arrows. The association $M \mapsto \hat{X}_G(M)$ defines an essentially surjective contravariant functor to the category of G-equivariant quotient presentations of X.*

Proof. We have only to check the functoriality. Consider two finitely generated subgroups $M' \subset M \subset \mathrm{Cl}_G(X)$. This gives an exact sequence of diagonalizable groups

$$1 \to \mathbb{D}(M/M') \to \mathbb{D}(M) \to \mathbb{D}(M') \to 1,$$

which translates into the factorization of the G-equivariant diagonalizable almost principal $\mathbb{D}(M)$-bundle $\hat{X}_G(M) \to X$ in G-equivariant diagonalizable almost principal bundles

$$\hat{X}_G(M) \xrightarrow{//\mathbb{D}(M/M')} \hat{X}_G(M') \xrightarrow{//\mathbb{D}(M')} X.$$

The morphism $\hat{X}_G(M) \xrightarrow{//\mathbb{D}(M/M'))} \hat{X}_G(M')$ is equivariant with respect to the morphism $\mathbb{D}(M) \to \mathbb{D}(M')$, and with respect to the G-action, whence the claim. □

Proposition 3.2.6.6. *Consider a G-equivariant quotient presentation $q : Y \to X$ of X. Then Y is a normal variety.*

Proof. Because Y is a normal algebraic scheme, it is a variety if and only if it is connected. This is verified if and only if $q_*\mathcal{O}_Y$ is a sheaf of integral algebras, so that we can suppose that X (hence Y) is smooth. By the last proposition $Y \simeq \hat{X}_G(M)$ for a finitely generated subgroup $M \subset \mathrm{Pic}_G(X)$ that we can suppose non-trivial. To verify that Y is connected, we adapt the arguments from [BeHa03, 6.3]. Let $Y_1 \cup \ldots \cup Y_r$ be the decomposition of Y as a union of its connected components and suppose by contradiction that $r > 1$. These components are transitively permuted by $\mathbb{D}(M)$ because X is connected. Also by $\mathbb{D}(M)$-invariance, we have $q(Y_1) = X$. Let $\mathscr{D}$ be the stabilizer of Y_1 in $\mathbb{D}(M)$, which is by assumption a proper subgroup. We let

$$j : \mathscr{D} \hookrightarrow \mathbb{D}(M)$$

be the inclusion. Consider the $\mathbb{D}(M)$-equivariant morphism $\psi : Y \to \mathbb{D}(M)/\mathscr{D}$ which maps a point $y \in Y$ to the element of $\mathbb{D}(M)/\mathscr{D}$ corresponding to the connected component of Y containing y. It follows that we have a cartesian square

$$\begin{tikzcd} Y_1 \times \mathbb{D}(M) \arrow[r, "p_2"] \arrow[d, "/\mathscr{D}"] & \mathbb{D}(M) \arrow[d, "\pi"] \\ Y \arrow[r, "\psi"] & \mathbb{D}(M)/\mathscr{D} \end{tikzcd},$$

where vertical arrows are $\mathscr{D}$-torsors. Restricting $q : Y \to X$ to Y_1 yields a $\mathscr{D}$-torsor $Y_1 \to X$ and we obtain a cartesian square

$$\begin{tikzcd} Y_1 \times \mathbb{D}(M) \arrow[r, "p_1"] \arrow[d, "/\mathscr{D}"] & Y_1 \arrow[d, "/\mathscr{D}"] \\ Y \arrow[r, "q"] & X \end{tikzcd},$$

where all arrows are torsors. Proceeding as in the proof of 3.2.5.8, we obtain the equality

$$\mathrm{type}_G(Y) = \mathrm{type}_G(Y_1) \circ j^\sharp.$$

But $\mathrm{type}_G(Y)$ is injective by assumption, thus $j^\sharp$ is necessarily injective. Now $j^\sharp$ is surjective, being the restriction of characters. It follows that $\mathscr{D} = \mathbb{D}(M)$, a contradiction. □

Corollary 3.2.6.7. *The G-equivariant Cox ring is a normal integral $\mathrm{Cl}_G(X)$-graded G-algebra.*

Under the assumption that $\mathcal{O}(X)^* \simeq k^*$, we describe the group of units of $\mathrm{Cox}_G(X)$. As $\mathrm{Cox}_G(X)$ is a graded algebra, we also consider the subgroup of units which are homogeneous. For example, every unit is homogeneous if e.g. $\mathrm{Cl}_G(X)$ is torsion-free. Because of the assumption $\mathcal{O}(X)^* \simeq k^*$ and the exact sequence (3.2.1), we have the $\mathbb{X}(G)$-graded k-subalgebra

$$A(\mathbb{X}(G)) := \bigoplus_{\lambda\in\mathbb{X}(G)} \Gamma(X, \mathcal{O}_X(\lambda)^x)$$

of $\mathrm{Cox}_G(X)$. The group of units of this algebra is obviously isomorphic to $\mathbb{X}(G) \times k^*$ via the multiplication map.

Proposition 3.2.6.8. *Suppose that $\mathcal{O}(X)^* \simeq k^*$. Then*

1. *The group of homogeneous units of $\mathrm{Cox}_G(X)$ is isomorphic to $\mathbb{X}(G) \times k^*$ via the multiplication map.*

2. *If $\mathcal{O}(X) \simeq k$, then every unit is homogeneous.*

Proof. If a homogeneous component $\mathcal{F}^x$ of $\mathcal{R}_{G,X}$ admits an invertible global section f, then $\mathcal{F}^x$ and $\mathcal{O}_X$ are isomorphic as divisorial sheaves. It follows that $f \in A(\mathbb{X}(G))$, whence the first assertion. Now assume $\mathcal{O}(X) \simeq k$ and let $f \in \mathrm{Cox}_G(X)^*$. Consider the $\mathrm{Im}\,\phi$-grading on $\mathrm{Cox}_G(X)$ induced by the forgetful morphism ϕ. Then proceeding as in [ADHL, 1.5.2.5], we obtain that f is $\mathrm{Im}\,\phi$-homogeneous of degree zero. But the $\mathrm{Im}\,\phi$-homogeneous component of degree zero is the $\mathbb{X}(G)$-graded k-subalgebra $A(\mathbb{X}(G))$ so we conclude as before. □

3.2.7 Recognizing an equivariant characteristic space

Let G be an algebraic group and (X, x) a pointed normal G-variety with only constant invertible G-invariant regular functions. Suppose moreover that the G-equivariant Cox sheaf is of finite type. In this section, we give a criterion to recognize a G-equivariant characteristic space $Y \to X$, under some mild assumptions on X. As defined in Section 3.2.5, a G-equivariant characteristic space is characterized by its equivariant type being an isomorphism onto $\mathrm{Cl}_G(X)$. The equivariant type of Y is injective if and only if Y is a G-equivariant quotient presentation of X. We now look for an interpretation of the surjectivity. This turns out to be related to the factoriality of the grading on $\mathcal{O}(Y)$, we start by recalling this notion.

Definition 3.2.7.1. [ADHL, 1.5.3] Let M be a finitely generated abelian group and A be a M-graded integral k-algebra.

- A non-zero non-invertible element $f \in A$ is *M-prime* if it is homogeneous and $(f \mid gh)$ with g, h homogeneous implies $f \mid g$ or $f \mid h$.

- A is *factorially M-graded* if every non-zero non-invertible homogeneous element f is a product of M-primes.

- A normal $\mathbb{D}(M)$-variety X is *$\mathbb{D}(M)$-factorial* if every $\mathbb{D}(M)$-invariant Weil divisor on X is principal.

Let $\mathscr{D}$ be a diagonalizable group, and Y a normal quasi-affine $\mathscr{D}$-variety. By [ADHL, 1.5.3.3], Y is $\mathscr{D}$-factorial if and only if $\mathcal{O}(Y)$ is $\mathbb{X}(\mathscr{D})$-factorial. In the next proposition, we give further characterizations of the $\mathbb{X}(\mathscr{D})$-factoriality.

Proposition 3.2.7.2. *Let $\mathscr{D}$ be a diagonalizable group and $q : Y \to X$ a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle. The following assertions are equivalent*

1. *Y is $\mathscr{D}$-factorial.*

2. *$\mathrm{type}(Y)$ is surjective.*

3. *The morphism $\gamma : \mathbb{X}(\mathscr{D}) \to \mathrm{Cl}_{\mathscr{D}}(Y)$, $\lambda \mapsto [\mathcal{O}_Y(\lambda)]$ is surjective.*

Moreover, if one of these conditions is satisfied, then Y is quasi-affine.

Proof. The assertions 2 and 3 are equivalent because of the lemma below. We suppose that the condition 1 is verified and show 2. For this, we can suppose that X is smooth, and that q is a $\mathscr{D}$-torsor. With this assumption, the pullback morphism $q^* : \mathrm{WDiv}(X) \to \mathrm{WDiv}(Y)$ induces an isomorphism onto the subgroup $\mathrm{WDiv}(Y)^{\mathscr{D}}$ which consists of $\mathscr{D}$-invariant Weil divisors on Y. Denote by $q_*^{\mathscr{D}}$ the inverse of this isomorphism and consider a $\mathscr{D}$-homogeneous rational function f on Y. We view f as a rational section of a divisorial sheaf on X. Then, we have $q^*(\mathrm{div}_X(f)) = \mathrm{div}(f)$ and thus $\mathrm{div}_X(f) = q_*^{\mathscr{D}}(\mathrm{div}(f))$. By this last remark we can write explicitly

$$\mathrm{type}(Y) : \mathbb{X}(\mathscr{D}) \to \mathrm{Pic}(X),\ \lambda \mapsto [\mathcal{O}_X(q_*^{\mathscr{D}}(\mathrm{div}(f)))],$$

where f is an arbitrary $\mathscr{D}$-homogeneous rational function of degree λ on Y. But this morphism is surjective because every Weil divisor on X is the direct image by $q_*^{\mathscr{D}}$ of a $\mathscr{D}$-invariant Weil divisor on Y, which is principal by hypothesis. Now we suppose 3 and show 1. Consider a $\mathscr{D}$-invariant Weil divisor D on Y. The divisorial sheaf $\mathcal{O}_Y(D)$ is naturally $\mathscr{D}$-linearized. Thus, its class in $\mathrm{Cl}(Y)$ is trivial by hypothesis. Hence, D is the divisor of a rational function which is necessarily $\mathscr{D}$-homogeneous. We have shown that Y is $\mathscr{D}$-factorial. It remains to show that Y is quasi-affine. As $\mathrm{type}(Y)$ is surjective, we can cover X by affine open subsets of the form $X_f := X \setminus \mathrm{Supp}(\mathrm{div}_X(f))$, where f is a $\mathscr{D}$-homogeneous regular function on Y. Indeed, the complement in X of any affine open subvariety is the zero set of a global section of some divisorial sheaf. Because of the surjectivity of $\mathrm{type}(Y)$, this section can be viewed as a homogeneous element of $\mathcal{O}(Y)$. It follows that Y is covered by the affine open subsets $Y_f = q^{-1}(X_f)$, whence the claim by [EGA, II.5.1.2]. ☐

Lemma 3.2.7.3. *Let $\mathscr{D}$ be a diagonalizable group and $q : Y \to X$ a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle. Consider the morphism $\gamma : \mathbb{X}(\mathscr{D}) \to \mathrm{Cl}_{\mathscr{D}}(Y), \lambda \mapsto [\mathcal{O}_Y(\lambda)]$. Then, we have the identity*

$$\mathrm{type}(Y) = q_*^{\mathscr{D}} \circ \gamma,$$

where $q_*^{\mathscr{D}} : \mathrm{Cl}_{\mathscr{D}}(Y) \to \mathrm{Cl}(X)$ is the isomorphism provided by 3.2.5.2.

Proof. To prove this, we can suppose that X is smooth and consider (equivariant) Picard groups instead of (equivariant) class groups. For each character $\lambda \in \mathbb{X}(\mathscr{D})$, consider the cartesian square

$$\begin{tikzcd} Y \times \mathbb{G}_m \arrow[r, "p_1"] \arrow[d] & Y \arrow[d, "q"] \\ Z := Y \times^{\lambda} \mathbb{G}_m \arrow[r] & X \end{tikzcd},$$

defined by the fiber bundle associated with $\mathbb{G}_m$ endowed with the $\mathscr{D}$-action defined by λ. Then, $Z \to X$ is a $\mathbb{G}_m$-torsor which by construction corresponds to $q_*^{\mathscr{D}} \circ \gamma(\lambda) \in \mathrm{Pic}(X)$. But this is also $\mathrm{type}(Y)(\lambda)$, by definition of the type of Y. ☐

Proposition 3.2.7.4. *Suppose that G is connected and that $\mathrm{Cox}_G(X)$ is a finitely generated k-algebra. Then, the affinization morphism $\hat{X}_G \to \tilde{X}_G$ is an equivariant open immersion whose image has a complement of codimension at least two in $\tilde{X}_G$.*

Proof. Using that the cokernel of $\mathrm{type}(\hat{X}_G)$ has finite exponent (3.2.2.2), and that $X_f = X_{f^n}$ for any power of a homogeneous section $f \in \mathcal{R}_X(X)$, we can proceed as in the end of the proof of 3.2.7.2 to show that $\hat{X}_G$ is quasi-affine. The claim on codimension is obtained by applying 3.2.5.5. ☐

Theorem 3.2.7.5. *Let G be a connected algebraic group with trivial Picard group, (X, x) be a pointed normal G-variety with only constant invertible regular functions, $\mathscr{D}$ be a diagonalizable group, and $q : Y \to X$ be a G-equivariant diagonalizable almost principal $\mathscr{D}$-bundle. The conditions*

1. *The group of $\mathscr{D}$-homogeneous invertible regular functions on Y is isomorphic to $\mathbb{X}(G)\times k^*$,*

2. *Y is $\mathscr{D}$-factorial,*

are necessary and sufficient for Y to be a G-equivariant characteristic space of X.

Proof. Suppose that the above conditions are fulfilled. By the first one, q is a quotient presentation of X, hence $\mathrm{type}_G(Y)$ is injective. Moreover, with the assumptions of the statement the exact sequence (3.2.1) can be written as

$$0 \to \mathbb{X}(G) \xrightarrow{\gamma} \mathrm{Cl}_G(X) \xrightarrow{\phi} \mathrm{Cl}(X) \to 0.$$

By the second condition, $\mathrm{type}(Y)$ is surjective (3.2.7.2). Also, the degree zero part of $\mathcal{O}(Y)$ with respect to the induced $\mathrm{Cl}(X)$-grading is

$$A(M) := \bigoplus\nolimits_{\lambda\in\mathbb{X}(G)} \Gamma(X, \mathcal{O}_X(\lambda)^x),$$

for a certain subgroup M of $\mathbb{X}(G)$. Proceeding as in the proof of 3.2.6.8, we obtain that the group of homogeneous units of $\mathcal{O}(Y)$ is isomorphic with $M \times k^*$ via multiplication map. Hence, we have in fact $M = \mathbb{X}(G)$ by the first condition, so that $\mathbb{X}(G) \simeq \ker\phi \subset \mathrm{Im}(\mathrm{type}_G(Y))$. In view of the equality $\mathrm{type}(Y) = \phi\circ\mathrm{type}_G(Y)$, we obtain the surjectivity of $\mathrm{type}_G(Y)$. Conversely, if $Y \to X$ is a G-equivariant characteristic space, then the first condition is satisfied by 3.2.6.8. The second is also satisfied because $\mathrm{type}_G(Y)$ and ϕ are surjective by hypothesis (3.2.7.2). □

3.2.8 Equivariant Cox ring of an open G-stable subvariety

Let G be a connected algebraic group, (X, x) a pointed normal G-variety, and $X_0 \overset{j}{\hookrightarrow} X$ a G-stable open subvariety containing x. Suppose moreover that $\mathcal{O}_X(X_0)^{*G} \simeq k^*$, so that both equivariant Cox rings of X and X_0 are well-defined. Let $(D_i)_{i=1,..,n}$ be the possibly empty family of prime divisors lying in the complement of X_0 and $(r_i)_{i=1,..,n}$ the associated family of respective canonical sections of the G-linearized divisorial sheaves $\mathcal{O}_X(D_i)^x$, $i = 1, .., n$.

Proposition 3.2.8.1. *The k-subalgebra of $\mathrm{Cox}_G(X)$ generated by $r_1, ..., r_n$ is polynomial, and there is an isomorphism of graded k-algebras*

$$\mathrm{Cox}_G(X_0) \simeq \mathrm{Cox}_G(X)/((r_i - 1)_{1\leqslant i\leqslant n}),$$

where the right-hand side algebra is endowed with the $\mathrm{Cl}_G(X_0)$-grading induced by the surjective morphism $j^* : \mathrm{Cl}_G(X) \to \mathrm{Cl}_G(X_0)$. Moreover, $(r_i - 1)_{i=1,...,n}$ is a regular sequence in $\mathrm{Cox}_G(X)$.

Proof. With these assumptions, the exact sequence (3.2.3) becomes

$$0 \to \bigoplus\nolimits_{i=1}^n \mathbb{Z}D_i \xrightarrow{D\mapsto[D]} \mathrm{Cl}_G(X) \xrightarrow{j^*} \mathrm{Cl}_G(X_0) \to 0.$$

For all $[\mathcal{G}] \in \mathrm{Cl}_G(X_0)$, choose $[\mathcal{F}] \in \mathrm{Cl}_G(X)$ such that $j^*([\mathcal{F}]) = [\mathcal{G}]$. By the above exact sequence, we can write

$$\mathcal{R}_{G,X} = \bigoplus\nolimits_{[\mathcal{G}]\in\mathrm{Cl}_G(X_0)} \mathcal{M}_{[\mathcal{G}]}, \text{ where } \mathcal{M}_{[\mathcal{G}]} := \bigoplus\nolimits_{(n_i)\in\mathbb{Z}^n} (\mathcal{F}\otimes\mathcal{O}_X(\textstyle\sum_{i=1}^n n_iD_i))^x.$$

Notice that $\mathcal{M}_{[\mathcal{O}_{X_0}]}$ is a $\mathbb{Z}^n$-graded $\mathcal{O}_X$-subalgebra of $\mathcal{R}_{G,X}$. Because $k[r_1, ..., r_n]$ is a $\mathbb{Z}^n$-graded k-subalgebra of $\mathcal{M}_{[\mathcal{O}_{X_0}]}(X)$ spanned by elements of linearly independent degrees, it immediately follows that it is a polynomial subalgebra of $\mathrm{Cox}_G(X)$. Also, each $\mathcal{M}_{[\mathcal{G}]}(X)$ is a $\mathbb{Z}^n$-graded $k[r_1, ..., r_n]$-module, and $\mathcal{M}_{[\mathcal{G}]}(X_0)$ is the localisation of $\mathcal{M}_{[\mathcal{G}]}(X)$ at the homogeneous element $r_1...r_n$. Indeed, up to removing a G-stable closed subvariety of codimension $\geqslant 2$, we can suppose that X is smooth and $X_0 = X \setminus \cup_{i=1}^n D_i$. As the support of the divisor of zeroes of $r_1...r_n$ on X is $\cup_{i=1}^n D_i$, we conclude by applying [EGA, I.9.3.1]. It follows that $\mathcal{R}_{G,X}(X_0)$ is the localization of $\mathrm{Cox}_G(X)$ at the homogeneous element $r_1...r_n$.

We use the notation of 3.2.3 for the equivalence relations between sections over open subsets of X_0 of G-linearized divisorial sheaves on X_0. Given two homogeneous sections $s_1, s_2 \in \mathcal{R}_{G,X}(X_0)$, we have $s_1 \sim_{X_0}^G s_2$ if and only if both are $\mathbb{Z}^n$-homogeneous elements of some $\mathcal{M}_{[\mathcal{G}]}(X_0)$ and there exists a Laurent monomial $r_1^{\alpha_1}..r_n^{\alpha_n}$ such that $s_1 = s_2 r_1^{\alpha_1}..r_n^{\alpha_n}$. By Lemma 3.2.8.2, imposing the relations $r_i = 1$ in $\mathcal{R}_{G,X}(X_0)$ is the same as passing to the quotient modulo $\sim_{X_0}^G$. Thus, we obtain an isomorphism

$$\mathrm{Cox}_G(X)[\tfrac{1}{r_1..r_n}]/((r_i-1)_{1\leqslant i\leqslant n}) \simeq \mathrm{Cox}_G(X_0),$$

which implies the isomorphism in the statement above.

To prove that the sequence $((r_i-1)_{i=1,\dots,n})$ is regular in $\mathrm{Cox}_G(X)$, we first prove that each $\mathcal{M}_{[\mathcal{G}]}(X_0)$ is a free $k[r_1^{\pm 1}, \dots, r_n^{\pm 1}]$-module. An element $t \in \mathcal{M}_{[\mathcal{G}]}(X_0)$ decomposes uniquely as a sum of homogeneous elements so that we can suppose t homogeneous, say of degree $(d_i) \in \mathbb{Z}^n$. The multiplication by $r_1^{d_1} \dots r_n^{d_n}$ defines an isomorphism of k-vector spaces

$$\mathcal{G}^x(X_0) \to (\mathcal{G} \otimes \mathcal{O}_X(\textstyle\sum_{i=1}^n d_i D_i))^x(X_0).$$

Let $(f_i)_i$ be a k-basis of $\mathcal{G}^x(X_0)$, then t decomposes uniquely as a linear combination $t = \sum_i \alpha_i r^{d_1} \dots r^{d_n} f_i$, whence the free module structure. Now, we claim that for $j = 1, \dots, n$, there is an injective morphism of $k[r_1, \dots, r_n]$-modules

$$\mathcal{M}_{[\mathcal{G}]}(X)/((r_i-1)_{i=1,\dots,j}) \hookrightarrow \mathcal{M}_{[\mathcal{G}]}(X_0)/((r_i-1)_{i=1,\dots,j}).$$

For this, it suffices to apply Lemma 3.2.8.3 j times, considering for the kth step $(1 \leqslant k \leqslant j)$ the $\mathbb{Z}$-grading induced by the projection

$$\mathbb{Z}^n \to \mathbb{Z},\ (d_1, \dots, d_n) \mapsto d_k.$$

Now, the result follows from the claim, the fact that $\mathcal{M}_{[\mathcal{G}]}(X_0)$ is a free $k[r_1^{\pm 1}, \dots, r_n^{\pm 1}]$-module, and that the sequence $((r_i-1)_{1\leqslant i\leqslant n})$ is regular in $k[r_1^{\pm 1}, \dots, r_n^{\pm 1}]$. □

Lemma 3.2.8.2. *Let M be a graded module over an algebra of Laurent polynomials $k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ endowed with the standard $\mathbb{Z}^n$-grading. For all $x \in M$, the submodule generated by the family $(t_1^{d_1} \dots t_n^{d_n}.x - x)_{(d_i)\in\mathbb{Z}^n}$ equals the submodule generated by the family $(t_i.x - x)_{1\leqslant i\leqslant n}$.*

Proof. This comes from the identities

$$t_i t_j.x - x = t_i.(t_j.x - x) + (t_i.x - x) \text{ and } t_i^{-1}.x - x = (-t_i^{-1}).(t_i.x - x).$$

□

Lemma 3.2.8.3. *Let $k[r]$ be a polynomial k-algebra with its standard $\mathbb{Z}$-grading, and M a $\mathbb{Z}$-graded $k[r]$-module such that r is a non-zero divisor in M. Then, $(r-1)M[1/r] \cap M = (r-1)M$.*

Proof. The localization morphism provides a graded injective morphism of $\mathbb{Z}$-graded $k[r]$-module

$$M = \textstyle\bigoplus_{d\in\mathbb{Z}} M_d \hookrightarrow M[1/r] = \bigoplus_{d\in\mathbb{Z}} (M[1/r])_d.$$

It suffices to prove that for an element $m \in M[1/r]$ such that $(r-1)m \in M$, we have that $m \in M$. Consider the decomposition $m = m_d + \dots + m_{d+l}$, $l \geqslant 0$ as a sum of homogeneous elements. Then, the decomposition of $(r-1)m$ reads

$$(r-1)m = rm_{d+l} + (rm_{d+l-1} - m_{d+l}) + \dots + (rm_d - m_{d+1}) - m_d.$$

Necessarily, each term of this sum lies in M. It follows that $m_{d+i} \in M$, $i = 0, .., l$ by an immediate induction, whence the result. □

Corollary 3.2.8.4. *There exists a smooth complete G-variety X' containing X_{sm} as a an open G-stable variety. As a consequence, we have*

$$\mathrm{Cox}_G(X) \simeq \mathrm{Cox}_G(X')/((r_i - 1)_i),$$

where the r_i are the canonical sections in the G-linearized divisorial sheaves associated to the G-stable prime divisors lying in $X' \setminus X_{\mathrm{sm}}$.

Proof. By a theorem of Sumihiro [Su, Thm. 3], X can be embedded equivariantly as an open subvariety of a normal complete G-variety $\overline{X}$. Now, there is a G-equivariant resolution of singularities $\varphi : X' \to \overline{X}$, that is, X' is a smooth G-variety and φ is a projective birational equivariant morphism that induces an isomorphism over $\overline{X}_{\mathrm{sm}}$ [Ko, Proposition 3.9.1 and Theorem 3.36]. As we have $\mathrm{Cox}_G(X) \simeq \mathrm{Cox}_G(X_{\mathrm{sm}})$, the statement follows from 3.2.8.1. □

Proposition 3.2.8.5. *Suppose that X is almost homogeneous. Then the polynomial algebra $k[r_1, ..., r_n]$ is the subalgebra of invariants $(\mathrm{Cox}_G(X))^G$.*

Proof. Let $f \in (\mathrm{Cox}_G(X))^G$ that we can suppose homogeneous because G acts on homogeneous components. Then $\mathrm{div}_X(f)$ is a G-invariant effective divisor on X, hence is a positive linear combination of the D_i. It follows that f is the product of a monomial in the r_i with a G-invariant homogeneous unit. However, a G-invariant homogeneous unit is necessarily of trivial $\mathrm{Cl}_G(X)$-degree, otherwise the multiplication by such a section would provide an isomorphism of some non-trivial G-linearized divisorial sheaf with $\mathcal{O}_X$ endowed with the trivial G-linearization, a contradiction. We conclude, that f is a non-zero scalar multiple of a monomial in the r_i. □

Suppose that X is almost homogeneous, let $X_0 \simeq G/H$ be the dense orbit, and consider the equivariant characteristic space $q : \hat{X}_G \to X$. In the corollary below, we describe the open subvariety $q^{-1}(X_0)$. In order to state it, we need some preliminary observations. Clearly, $q^{-1}(X_0)$ is a homogeneous space for $G \times \mathbb{D}(\mathrm{Cl}_G(X))$ since X_0 is a homogeneous space for G and q induces is a $\mathbb{D}(\mathrm{Cl}_G(X))$-torsor over X_0. Also, the last proposition implies that we have a $\mathbb{D}(\mathrm{Cl}_G(X))$-equivariant categorical quotient $\pi : \tilde{X}_G \to \mathbb{A}^n_k$. Finally, the exact sequence (3.2.3) yields by duality an exact sequence of algebraic groups

$$1 \to \mathbb{D}(\mathbb{X}(H)) \to \mathbb{D}(\mathrm{Cl}_G(X)) \to \mathbb{G}_m^n \to 1,$$

whence a $\mathbb{D}(\mathbb{X}(H))$-torsor $\mathbb{D}(\mathrm{Cl}_G(X)) \to \mathbb{G}_m^n$.

Corollary 3.2.8.6. *The categorical quotient π restricts to a $\mathbb{D}(\mathrm{Cl}_G(X))$-equivariant morphism $q^{-1}(X_0) \to \mathbb{G}_m^n$. Moreover, $q^{-1}(X_0)$ is isomorphic as a $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-variety over $\mathbb{G}_m^n$ to the associated fiber bundle*

$$G/K \times^{\mathbb{D}(\mathbb{X}(H))} \mathbb{D}(\mathrm{Cl}_G(X)) \to \mathbb{G}_m^n,$$

where G/K is the equivariant characteristic space of G/H (3.2.5.13).

Proof. The morphism π pulls back the standard coordinates of $\mathbb{A}^n_k$ to the r_i, whence a $\mathbb{D}(\mathrm{Cl}_G(X))$-equivariant restriction $q^{-1}(X_0) \to \mathbb{G}_m^n$. By virtue of 3.2.8.1 and 3.2.5.13, the fiber at $(1, ..., 1)$ of this restriction is isomorphic to G/K as a $G \times \mathbb{D}(\mathbb{X}(H))$-variety. By $\mathbb{D}(\mathrm{Cl}_G(X))$-equivariance of π and surjectivity of the morphism $\mathbb{D}(\mathrm{Cl}_G(X)) \to \mathbb{G}_m^n$, the same holds for any fiber of this restriction. By standard properties of associated fiber bundles, $q^{-1}(X_0)$ is isomorphic to $G/K \times^{\mathbb{D}(\mathbb{X}(H))} \mathbb{D}(\mathrm{Cl}_G(X))$ as a $G \times \mathbb{D}(\mathrm{Cl}_G(X))$-variety over $\mathbb{G}_m^n$. □

The inclusion $k[\mathbb{X}(H)] \subset \mathcal{O}(H)$ yields a morphism of algebraic groups $H \to \mathbb{D}(\mathbb{X}(H))$. Composed with the injective morphism $\mathbb{D}(\mathbb{X}(H)) \to \mathbb{D}(\mathrm{Cl}_G(X))$ described above, we obtain a morphism $\varphi : H \to \mathbb{D}(\mathrm{Cl}_G(X))$. In the next proposition, we view H as a subgroup of $G \times \mathbb{D}(\mathrm{Cl}_G(X))$ by identifying it with the image of the morphism $h \mapsto (h, \varphi(h))$.

Proposition 3.2.8.7. *Let $\hat{x}$ be a point in the fiber $q^{-1}(x)$, where $x \in X_0$ is the point identified with H/H. Then, H is the stabilizer of $\hat{x}$ in $G \times \mathbb{D}(\mathrm{Cl}_G(X))$.*

Proof. Let $\mathcal{F}_1, ..., \mathcal{F}_m$ be G-linearized divisorial sheaves on X whose classes generate the abelian group $\mathrm{Cl}_G(X)$. Via restriction of this sheaves to $X_0 \simeq G/H$, we obtain G-linearized invertible sheaves $\mathcal{L}_{\chi_1}, ..., \mathcal{L}_{\chi_m}$ corresponding to certain characters $\chi_i \in \mathbb{X}(H) \simeq \mathrm{Pic}_G(G/H)$. Let L_{χ_i} be the G-linearized line bundle over X_0 corresponding to $\mathcal{L}_{\chi_i}$, and view $q^{-1}(X_0)$ as a subvariety of $L_{\chi_1} \times_{X_0} ... \times_{X_0} L_{\chi_m}$. Obviously, the image of the projection to G of $\mathrm{Stab}_{G\times\mathbb{D}(\mathrm{Cl}_G(X))}(\hat{x})$ has to be included in H. Then, consider an element $(h,t) \in \mathrm{Stab}_{G\times\mathbb{D}(\mathrm{Cl}_G(X))}(\hat{x}) \subset H \times \mathbb{D}(\mathrm{Cl}_G(X))$. We have $h.\hat{x} = (\chi_1(h)\hat{x}_1, ..., \chi_m(h)\hat{x}_m) = \varphi(h^{-1}).\hat{x}$, which forces $t = \varphi(h)$. $\square$

3.2.9 Relation between the equivariant Cox ring and the ordinary Cox ring

Let G be a connected algebraic group acting on a pointed normal G-variety (X, x) such that $\mathcal{O}(X)^* \simeq k^*$, so that both equivariant and ordinary Cox sheaves are well defined. In view of Remark 3.2.5.10, there is a canonical morphism of graded $\mathcal{O}_X$-algebras

$$\varphi : \mathcal{R}_{G,X} \to \mathcal{R}_X,$$

whose associated morphism between grading groups is $\mathrm{Cl}_G(X) \xrightarrow{\phi} \mathrm{Cl}(X)$, this is the *forgetful morphism.*

On the other hand, $\mathbb{X}(G)$ is a free $\mathbb{Z}$-module of finite rank acting by automorphisms of graded k-vector space on $\mathcal{R}_{G,X}(U)$, for any open subset $U \subset X$. To see this, consider $[\mathcal{F}] \in \mathrm{Cl}_G(X)$, $\lambda \in \mathbb{X}(G)$, and notice that $(\mathcal{F}^x \otimes \mathcal{O}_X(\lambda))^x \simeq \mathcal{F}^x$ as divisorial sheaves. Hence, for a section $s \in \mathcal{F}^x(U)$, we define the section

$$\lambda.s \in (\mathcal{F}^x \otimes \mathcal{O}_X(\lambda))^x(U)$$

to be the unique section satisfying $\varphi(U)(s) = \varphi(U)(\lambda.s)$. This assignment defines an action of $k[\mathbb{X}(G)]$ on $\mathcal{R}_{G,X}(U)$ that respects multiplication, so that we get a $k[\mathbb{X}(G)]$-algebra structure on $\mathcal{R}_{G,X}(U)$. In the next propositions we make this structure more explicit, and study the forgetful morphism.

Proposition 3.2.9.1. *The equivariant Cox ring is a free $k[\mathbb{X}(G)]$-module. More precisely, we have*

$$\mathrm{Cox}_G(X) = \bigoplus\nolimits_{[\mathcal{F}]\in \mathrm{Im}\,\phi} M_{[\mathcal{F}]},$$

where $M_{[\mathcal{F}]} := \bigoplus_{\mu\in\mathbb{X}(G)}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(\mu))^x(X)$ is $\mathbb{X}(G)$-graded and free of rank $\dim_k(\mathcal{F}(X))$ over $k[\mathbb{X}(G)]$.

Proof. The action of $k[\mathbb{X}(G)]$ stabilizes the k-vector spaces $M_{[\mathcal{F}]}$. With this action, they inherit a structure of $\mathbb{X}(G)$-graded $k[\mathbb{X}(G)]$-module. Moreover the exact sequence (3.2.1) becomes

$$0 \to \mathbb{X}(G) \xrightarrow{\gamma} \mathrm{Cl}_G(X) \xrightarrow{\phi} \mathrm{Cl}(X),$$

which justifies the decomposition of $\mathrm{Cox}_G(X)$ as the direct sum of the modules $M_{[\mathcal{F}]}$. In order to check that they are free, let us consider $[\mathcal{F}] \in \mathrm{Cl}_G(X)$ and the k-vector space $\mathcal{F}^x(X)$. For any $\lambda \in \mathbb{X}(G)$, the multiplication by λ defines an isomorphism of k-vector spaces with $(\mathcal{F}\otimes_{\mathcal{O}_X} \mathcal{O}(\lambda))^x(X)$, the inverse being the multiplication by λ^{-1}. By a simple verification as in the proof of 3.2.8.1, we obtain that $M_{[\mathcal{F}]}$ is freely generated over $k[\mathbb{X}(G)]$ by any k-basis of $\mathcal{F}^x(X)$. $\square$

Proposition 3.2.9.2. *The forgetful morphism $\varphi : \mathcal{R}_{G,X} \to \mathcal{R}_X$ is an integral extension of graded $\mathcal{O}_X$-algebras, whose kernel is the sheaf of ideals generated by the family of global sections $(1 - \lambda_j.1)_{1\leqslant j\leqslant n}$, where $(\lambda_j)_{1\leqslant j\leqslant n}$ is an arbitrary $\mathbb{Z}$-basis of $\mathbb{X}(G)$, and $1 \in \mathcal{O}(X)$ has trivial degree. Moreover, this family defines a regular sequence in $\mathcal{R}_{G,X}(U)$ for any open subset $U \subset X$, and we have*

$$\operatorname{Im}\varphi(U) \simeq \mathcal{R}_{G,X}(U)/(1_{|U} - (\lambda_j.1)_{|U})_{1\leqslant j\leqslant n}.$$

Proof. Let $U \subset X$ be a non-empty open subset. Replacing X by U in the last proposition and considering the action of $k[\mathbb{X}(G)]$ on $\mathcal{R}_{G,X}(U)$, we see that the forgetful morphism corresponds to the quotient of $\mathcal{R}_{G,X}(U)$ by the sub-$k[\mathbb{X}(G)]$-module spanned by $(f_{\mathcal{F},i} - \lambda.f_{\mathcal{F},i})_{[\mathcal{F}]\in\operatorname{Im}\phi,i,\lambda\in\mathbb{X}(G)}$, where $(f_{\mathcal{F},i})_i$ is a k-basis of $\mathcal{F}^x(U)$ for all $[\mathcal{F}] \in \operatorname{Im}\phi$. Using Lemma 3.2.8.2, this submodule is spanned by the elements of the family $(f_{\mathcal{F},i} - \lambda_j.f_{\mathcal{F},i})_{[\mathcal{F}]\in\operatorname{Im}\phi,i,1\leqslant j\leqslant n}$ where $(\lambda_j)_{1\leqslant j\leqslant n}$ is an arbitrary $\mathbb{Z}$-basis of $\mathbb{X}(G)$. Finally, we observe that this sub-$k[\mathbb{X}(G)]$-module is the ideal of $\mathcal{R}_{G,X}(U)$ generated by $(1_{|U} - (\lambda_j.1)_{|U})_{1\leqslant j\leqslant n}$ where $1 \in \mathcal{O}(X)$ has trivial degree.

The sequence $(1_{|U} - (\lambda_j.1)_{|U})_{1\leqslant j\leqslant n}$ is regular because the sequence $(1-\lambda_j)_{1\leqslant j\leqslant n}$ is regular in $k[\mathbb{X}(G)]$ and $\mathcal{R}_{G,X}(U)$ is a free $k[\mathbb{X}(G)]$-module. Now, we show that an element $f \in \mathcal{R}_X(U)$ is integral over $\mathcal{R}_{G,X}(U)$. We may assume this element homogeneous, say $f \in \mathcal{F}^x(U)$. Then we use the fact that a non-zero power of $\mathcal{F}^x$ is linearizable (3.2.2.2). For the last assertion, we argue exactly as in [ADHL, 1.4.3.5] to prove the isomorphism. □

Corollary 3.2.9.3. *Suppose that the Picard group of G is trivial, then*

$$\operatorname{Cox}(X) \simeq \operatorname{Cox}_G(X)/(1-\lambda_j.1)_{1\leqslant j\leqslant n},$$

where $(\lambda_j)_{1\leqslant j\leqslant n}$ is an arbitrary $\mathbb{Z}$-basis of $\mathbb{X}(G)$, and $1 \in \mathcal{O}(X)$ has trivial degree.

Proof. With this assumption, $\operatorname{Cl}_G(X) \xrightarrow{\phi} \operatorname{Cl}(X)$ is surjective (3.2.2.2). It follows that φ is also surjective. We conclude by applying the last proposition. □

3.2.10 Pullback of equivariant characteristic spaces

In this section, we study the pullback of an equivariant characteristic space along an equivariant almost principal bundle. Let G, H be algebraic groups with G connected, (X, x) a pointed normal G-variety, (Y, y) a pointed normal $G\times H$-variety, and $\pi_1 : Y \to X$ a G-equivariant almost principal H-bundle. Suppose that $\mathcal{O}(X)^{*G} \simeq \mathcal{O}(Y)^{*G\times H} \simeq k^*$, and that X has a G-equivariant Cox sheaf of finite type. Then, we claim that there is a cartesian square

$$\begin{array}{ccc} \hat{Y}_{G\times H} & \xrightarrow{\pi_2} & \hat{X}_G \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_1} \\ Y & \xrightarrow{\pi_1} & X, \end{array}$$

where q_1 is a G-equivariant characteristic space over X, q_2 is a $G\times H$-equivariant characteristic space over Y, and the horizontal arrows are G-equivariant almost principal H-bundles. Indeed, because of 3.2.5.2 we can write

$$\mathcal{R}_{G\times H,Y} = \bigoplus\nolimits_{[\mathcal{F}]\in\operatorname{Cl}_G(X)} (\pi_1^*\mathcal{F})^y.$$

Thus, $\hat{Y}_{G\times H} \to Y$ is the pullback of $\hat{X}_G \to X$ along π_1. As π_2 induces a G-equivariant H-torsor over $q_1^{-1}(X_{\mathrm{sm}})$, it follows that π_2 is a G-equivariant almost principal H-bundle. In fact, π_2 corresponds to the graded $\mathcal{O}_X$-algebra morphism

$$\mathcal{R}_{G,X} = \bigoplus\nolimits_{[\mathcal{F}]\in\operatorname{Cl}_G(X)} \mathcal{F}^x \to \pi_{1*}\mathcal{R}_{G\times H,Y}$$

defined by the natural morphisms $\mathcal{F}^x \to \pi_{1*}\pi_1^*\mathcal{F}^x$ between homogeneous components. As Y admits a $G\times H$-equivariant Cox sheaf of finite type, it also admits a G-equivariant Cox sheaf of finite type when H is connected and $\mathcal{O}(Y)^{*G} \simeq k^*$ (3.2.9.2). When $H = \mathbb{T}$ is a torus, we can say more (see also [Bra, 2.2]):

Proposition 3.2.10.1. *Suppose that $H = \mathbb{T}$ is a torus and that $\mathcal{O}(Y)^{*G} \simeq k^*$. Then $\hat{X}_G \xrightarrow{q_1} X$ factors through Y via the morphism π_1, and this factorization realizes $\hat{X}_G$ as the G-equivariant characteristic space of Y.*

Proof. In this situation, the exact sequence (3.2.1) reads

$$0 \to \mathbb{X}(\mathbb{T}) \to \mathrm{Cl}_{G\times\mathbb{T}}(Y) \xrightarrow{\phi} \mathrm{Cl}_G(Y) \to 0,$$

and we have an isomorphism $\pi_{1*}^{\mathbb{T}} : \mathrm{Cl}_{G\times\mathbb{T}}(Y) \to \mathrm{Cl}_G(X)$, $[\mathcal{F}] \mapsto [(\pi_{1*}\mathcal{F})^{\mathbb{T}}]$ (3.2.5.2). This allows us to write

$$\mathcal{R}_{G,X} = q_{1*}\mathcal{O}_{\hat{X}_G} = \bigoplus\nolimits_{[\mathcal{G}]\in\mathrm{Cl}_G(Y)} \bigoplus\nolimits_{\lambda\in\mathbb{X}(\mathbb{T})} (\pi_{1*}(\mathcal{F}\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(\lambda))^{\mathbb{T}})^x,$$

where for each $[\mathcal{G}] \in \mathrm{Cl}_G(Y)$, we choose $[\mathcal{F}] \in \mathrm{Cl}_{G\times\mathbb{T}}(Y)$ such that $\phi([\mathcal{F}]) = [\mathcal{G}]$. This yields a factorization of q_1 through the relative spectrum over X of the degree zero part of the graded algebra $\mathcal{R}_{G,X}$ with respect to the induced $\mathrm{Cl}_G(Y)$-grading. This degree zero part is

$$\bigoplus\nolimits_{\lambda\in\mathbb{X}(\mathbb{T})} \pi_{1*}(\mathcal{O}_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(\lambda))^{\mathbb{T}} = \bigoplus\nolimits_{\lambda\in\mathbb{X}(\mathbb{T})} (\pi_{1*}\mathcal{O}_Y)_\lambda^{(\mathbb{T})} = \pi_{1*}\mathcal{O}_Y.$$

Hence, the factorization is

$$\hat{X} \xrightarrow{\varphi} Y \xrightarrow{\pi_1} X,$$

where φ is a good quotient by $\mathbb{D}(\mathrm{Cl}_G(Y))$. In fact, because q_1 is a G-equivariant almost principal $\mathbb{D}(\mathrm{Cl}_G(X))$-bundle, the good quotient φ is a G-equivariant almost principal $\mathbb{D}(\mathrm{Cl}_G(Y))$-bundle. For every open subset $V \subset X$, we have

$$\begin{aligned}
\varphi_*\mathcal{O}_{\hat{X}}(\pi_1^{-1}(V)) &= q_{1*}\mathcal{O}_{\hat{X}}(V)\\
&= \bigoplus_{[\mathcal{G}]\in\mathrm{Cl}_G(Y)} \bigoplus_{\lambda\in\mathbb{X}(\mathbb{T})} (\mathcal{F}\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(\lambda))^y(\pi_1^{-1}(V))^{\mathbb{T}}\\
&= \bigoplus_{[\mathcal{G}]\in\mathrm{Cl}_G(Y)} \bigoplus_{\lambda\in\mathbb{X}(\mathbb{T})} \mathcal{F}^y(\pi_1^{-1}(V))_\lambda^{(\mathbb{T})}\\
&= \bigoplus_{[\mathcal{G}]\in\mathrm{Cl}_G(Y)} \mathcal{F}^y(\pi_1^{-1}(V))\\
&\simeq \bigoplus_{[\mathcal{G}]\in\mathrm{Cl}_G(Y)} \mathcal{G}^y(\pi_1^{-1}(V)).
\end{aligned}$$

As $\varphi_*\mathcal{O}_{\hat{X}}$ is a quasi-coherent sheaf on Y, and π_1 is affine, this shows that the G-equivariant type of $\hat{X}_G$ viewed as a G-equivariant almost principal $\mathbb{D}(\mathrm{Cl}_G(Y))$-bundle over Y is the identity of $\mathrm{Cl}_G(Y)$. □

3.3 Normal rational G-varieties of complexity one

Let G be a connected reductive algebraic group, B a Borel subgroup, T a maximal torus in B, and U the unipotent part of B. In this section, we begin by recalling basic facts on the geometry of normal G-varieties of complexity one. Then, we obtain a necessary and sufficient condition for the (equivariant) Cox ring to exist and be finitely generated. Then, we give a presentation of the subalgebra of U-invariants of the equivariant Cox ring.

3.3.1 Generalities

The *complexity* of a normal G-variety X is the minimal codimension of a B-orbit. As any two Borel subgroups of G are conjugated, the complexity doesn't depend on the choice of B. Suppose that X is of complexity one. By a theorem of Rosenlicht [T11, 5.1] there exists a unique up to isomorphism smooth projective algebraic curve C whose field of rational functions is $k(X)^B$. Let us consider the rational quotient

$$\pi : X \dashrightarrow C$$

by B given by the inclusion $k(X)^B \subset k(X)$. We show that this quotient admits a rational section. For this, we use the local structure theorem of Brion, Luna and Vust [BLV, Thm 1.4]. This is a very useful tool for the study of varieties with group action. We present a variant of this theorem and a corollary, both due to Knop.

Theorem 3.3.1.1. *[Kn93b, 1.2] Let G be a connected reductive group, X a G-variety equipped with an ample G-linearized invertible sheaf $\mathcal{L}$, and $s \in \mathcal{L}(X)^{(B)}$ a B-semi-invariant section. Let P be the parabolic subgroup stabilizer of s in $\mathbb{P}(\mathcal{L}(X))$, with Levi decomposition $P_u \rtimes L$, $T \subset L$. Then the open subvariety X_s is P-stable, and there exists a closed L-stable subvariety $Z \subset X_s$ such that the morphism induced by the P-action*

$$P_u \times Z \to X_s$$

is a P-equivariant isomorphism.

Corollary 3.3.1.2. *[Kn93b, §2] Let G be a connected reductive group, X a normal G-variety, and $P(X)$ the smallest stabilizer of a B-stable divisor in X, with Levi decomposition $P(X)_u \rtimes L(X)$, $T \subset L(X)$. Then, there exists an affine $L(X)$-stable subvariety Z such that $X_0 := P(X)Z$ is an open affine subvariety of X. The natural morphism $P(X)_u \times Z \to X_0$ is an isomorphism and the derived subgroup of $L(X)$ acts trivially on Z.*

Let $X_0 \subset X$ be as in the corollary, and H be the kernel of the $L(X)$-action on Z. Then $\mathbb{T} := L(X)/H$ is a torus acting faithfully on Z. The local structure of Z for such an action is well known: up to replacing Z by a smaller $L(X)$-stable subvariety, we have an $L(X)$-equivariant isomorphism $Z \simeq \mathbb{T} \times C_0$, where $L(X)$ acts on $\mathbb{T}$ by left multiplication via the projection $L(X) \to \mathbb{T}$, and C_0 is an algebraic curve. We end up with a $P(X)$-equivariant isomorphism

$$X_0 \simeq P(X)_u \times \mathbb{T} \times C_0,$$

which identifies $\pi_{|X_0}$ with the projection to C_0, and yields the desired rational section.

Recall that the (equivariant) class group being finitely generated is a necessary condition for the (equivariant) Cox ring to be finitely generated (3.2.1.6). In fact, we show in the two next propositions that for complexity one normal varieties, this is also sufficient.

Proposition 3.3.1.3. *The (equivariant) class group of X is finitely generated if and only if X is a rational variety.*

Proof. By 3.2.2.2, the equivariant class group is finitely generated if and only if the class group is too. By the localization sequence (3.2.2), $\mathrm{Cl}(X)$ is finitely generated if and only if $\mathrm{Cl}(X_0)$ is so. But because $P(X)_u \times \mathbb{T}$ is a rational variety with trivial class group, thus $\mathrm{Cl}(X_0)$ is isomorphic to $\mathrm{Cl}(C_0)$ via the pullback associated with the projection to C_0 [Bri18, 5.1.2]. By the localisation sequence again, we deduce that $\mathrm{Cl}(X)$ is finitely generated if and only if $\mathrm{Cl}(C)$ is too. This last condition is equivalent to $C \simeq \mathbb{P}^1$, which yields the statement. □

Proposition 3.3.1.4. *Suppose that X is a rational normal variety of complexity one satisfying $\mathcal{O}(X)^{*G} \simeq k^*$. Then $\mathrm{Cox}_G(X)$ is finitely generated. If moreover $\mathcal{O}(X)^* \simeq k^*$, then $\mathrm{Cox}(X)$ is finitely generated.*

Proof. Replacing X by its smooth locus doesn't change the equivariant Cox ring, hence we can suppose that X is smooth. With this assumption, the Cox sheaf is of finite type and the equivariant characteristic space exists as a normal variety. Let us embed $\mathbb{D}(\mathrm{Pic}_G(X))$ in a torus $\mathbb{T}$ by means of an injective morphism of diagonalizable groups $\varphi : \mathbb{D}(\mathrm{Pic}_G(X)) \to \mathbb{T}$ dually defined as follows: choose representatives $(\mathcal{L}_i)_{i=1,..,n}$ of elements in a generating set of $\mathrm{Pic}_G(X)$ and consider the free $\mathbb{Z}$-module $\mathbb{Z}^n$ generated by these representatives. Then, define $\varphi^\sharp$ to be the natural surjective morphism

$$\varphi^\sharp : \mathbb{Z}^n \to \mathrm{Pic}_G(X),\ (d_1, ..., d_n) \mapsto [\mathcal{L}_1^{\otimes d_1} \otimes_{\mathcal{O}_X} ... \otimes_{\mathcal{O}_X} \mathcal{L}_n^{\otimes d_n}].$$

Let $Y = \mathrm{Spec}_X \mathcal{A}$ be the G-equivariant $\mathbb{T}$-torseur over X whose G-equivariant type is $\varphi^\sharp$. By 3.2.5.10, there is a canonical morphism of graded $\mathcal{O}_X$-algebras

$$\mathcal{A} \to \mathcal{R}_{G,X}$$

whose induced morphism between grading groups is $\varphi^\sharp$. Hence, the above morphism is surjective over any open subset of X. As a consequence, proving that $\mathrm{Cox}_G(X)$ is finitely generated amounts to prove that the algebra of regular functions on Y is finitely generated. Because Y is a G-equivariant $\mathbb{T}$-torsor over X, it is a smooth rational $G \times \mathbb{T}$-variety of complexity one. We now apply a result of Knop [Kn93a, Satz 1] saying that $\mathcal{O}(Y)$ is finitely generated as a k-algebra. The second assertion is then an application of 3.2.9.2. □

Remark 3.3.1.5. The second assertion in the last statement is [ADHL, Thm 4.3.1.5]. Also, notice that almost homogeneous varieties of complexity one are automatically rational. Indeed, we have in this situation a natural injection $k(C) \to k(G)$ from which we see that C is unirational because G is a rational variety. By Lüroth's theorem, one deduces that C is rational and hence so is X in view of the isomorphism after Corollary 5.3.3.2. Moreover, we have $\mathcal{O}(X)^{*G} \simeq k^*$ because G has an open orbit. As a consequence, the equivariant Cox ring of X is well-defined and finitely generated.

Suppose that X is rational of complexity one with $\mathcal{O}(X)^* \simeq k^*$, so that $\mathrm{Cox}(X)$ is well defined and finitely generated. Because U has trivial character group and Picard group, every divisorial sheaf on X has a unique U-linearization, whence a canonical isomorphism

$$\mathrm{Cox}_U(X) \simeq \mathrm{Cox}(X) \qquad (3.2.3.4).$$

We consider $\mathrm{Cox}(X)$ endowed with this canonical structure of a $\mathrm{Cl}(X)$-graded U-algebra.

Proposition 3.3.1.6. *Suppose that* $\mathrm{Pic}(G) = 0$. *Then, the k-algebra* $\mathrm{Cox}(X)^U$ *is a finitely generated normal domain and there is an isomorphism of* $\mathrm{Cl}(X)$*-graded k-algebras*

$$\mathrm{Cox}(X)^U \simeq \mathrm{Cox}_G(X)^U/(1 - \lambda_j.1)_{1 \leqslant j \leqslant n},$$

where $(\lambda_j)_{1 \leqslant j \leqslant n}$ *is an arbitrary $\mathbb{Z}$-basis of* $\mathbb{X}(G)$. *From this isomorphism,* $\mathrm{Cox}(X)^U$ *is endowed with a structure of a* $\mathrm{Cl}(X) \times \mathbb{X}(T)$*-graded algebra.*

Proof. The k-algebra $\mathrm{Cox}(X)^U$ is a normal domain because $\mathrm{Cox}(X)$ is so. Moreover, applying Corollary 3.2.9.3 yields the isomorphism of the statement, whence the finite generation of $\mathrm{Cox}(X)^U$. The right hand side is naturally $\mathrm{Cl}(X) \times \mathbb{X}(T)$-graded, so is the left hand side by transport of structure. □

Corollary 3.2.8.4 can be useful in order to obtain a presentation of $\mathrm{Cox}_G(X)$. Indeed, reusing the notations of this result, X' is a smooth complete rational G-variety of complexity one and $\mathrm{Cox}_G(X)$ is the quotient of (the finitely generated) $\mathrm{Cox}_G(X')$ by a particularly simple regular sequence. So one may rather look for a description of $\mathrm{Cox}_G(X')$. This task might be easier thanks to the following proposition (which is probably known but for which we could not find a reference).

Proposition 3.3.1.7. *Suppose that X is a smooth complete rational G-variety of complexity one. Then, the abelian groups* $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}_G(X)$ *are free of finite rank.*

Proof. By [Su, Thm. 2] there is a G-equivariant projective birational morphism $\varphi : X' \to X$, where X' is a (projective) normal G-variety. This implies that the pullback morphism $\varphi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is injective. Thus we can suppose that X is projective. Also, using the exact sequence (3.2.1), we are reduce to prove that $\mathrm{Pic}(X)$ is free of finite rank. We look for a non-empty open subvariety $U \subset X$ such that $\mathcal{O}(U)^* \simeq k^*$ and $\mathrm{Pic}(U) = 0$. By the exact sequence (3.2.2), finding such a subvariety U yields the result.

Choose a maximal torus $T \subset G$ and a 1-parameter subgroup λ of T having the same fixed points in X as T. By the next lemma, each G-orbit admit only finitely many fixed points. Because X is of complexity one, it follows that the connected components $(C_i)_{0\leqslant i\leqslant r}$ of the fixed points scheme $X^T = X^\lambda$ are points or curves. Also, the Bialynicki-Birula decomposition [Mi, Thm. 13.47] says that X is the disjoint union of the subvarieties X_i, $i = 0, ..., r$ such that for all $x \in X_i$, the limit $\lim_{t\to 0} \lambda(t) \cdot x$ exists and lies in C_i. Moreover, the maps

$$f_i : X_i \to C_i, x \mapsto \lim_{t\to 0} \lambda(t) \cdot x, \quad i = 0, ..., r$$

are good quotients for the $\mathbb{G}_m$-action. We can assume that X_0 is an open subvariety. If C_0 is a curve, then it is isomorphic to $\mathbb{P}^1_k$. Indeed, it is a projective normal curve which is rational because $k(C_0) \subset k(X)$. Consider $U := \pi^{-1}(\mathbb{A}^1_k)$. Then U is affine and its coordinate algebra is $\mathbb{Z}_{\geqslant 0}$-graded. Moreover, the homogeneous component of degree zero is $\mathcal{O}(\mathbb{A}^1_k)$, whence $\mathcal{O}(U)^* \simeq k^*$. Let $\mathcal{L}$ be a $\mathbb{G}_m$-linearized invertible sheaf on U. By [BasHab, Cor. 6.4], we have $f^*i^*\mathcal{L} \simeq \mathcal{L}$ as invertible sheaves on U, where i denotes the closed immersion $\mathbb{A}^1_k \to U$ and f is the restriction of f_0 to U. Thus, $\mathcal{L}$ is trivial as an invertible sheaf. This implies that $\mathrm{Pic}(U)$ is trivial since the forgetful morphism $\mathrm{Pic}_{\mathbb{G}_m}(U) \to \mathrm{Pic}(U)$ is surjective. If C_0 is a point, we let $U := \pi^{-1}(C_0)$ and one proves similarly as above that $\mathcal{O}(U)^* \simeq k^*$ and $\mathrm{Pic}(U) = 0$. □

Lemma 3.3.1.8. *Consider an homogeneous space G/H, and a maximal torus $T \subset G$. Then there are only finitely many T-fixed points in G/H.*

Proof. We can suppose that T admit at least one fixed point x_0. Choosing x_0 as base point, we may suppose $T \subset H$. Consider a T-fixed point $g \cdot x_0 = gH$ in G/H. We have $g^{-1}Tg = h^{-1}Th$ for a certain $h \in H$, because any two maximal tori in H are conjugate by an element of H. As we can translate g by any element of H, we can suppose that $g \in N_G(T)$. Thus, $N_G(T)$ acts transitively on the set of fixed points. As the stabilizer of x_0 contains T we are are done. □

3.3.2 The algebra $\mathrm{Cox}_G(X)^U$

Let (X, x) be a pointed normal rational G-variety of complexity one such that $\mathcal{O}(X)^{*G} \simeq k^*$. Suppose that G has trivial Picard group. In order to study the G-algebra $\mathrm{Cox}_G(X)$, it is classical in invariant theory to investigate the more accessible structure of the subalgebra $\mathrm{Cox}_G(X)^U$ of U-invariants . By [T11, D.5], 3.3.1.4, and 3.2.6.7, this subalgebra is a finitely generated normal domain. Under some mild assumptions we give a presentation of this subalgebra, which generalizes a result of Ponomareva [Pon, Thm 4].

Notice that $\mathrm{Cox}_G(X)^U$ is naturally $\mathrm{Cl}_G(X) \times \mathbb{X}(T)$-graded, the homogeneous elements being global sections of G-linearized divisorial sheaves whose divisors of zeroes are the B-stable effective divisors of X. To see this, consider a B-invariant effective divisor D, and the divisorial sheaf $\mathcal{O}_X(D)^x$ endowed with a G-linearization (such a linearization exists by assumption on $\mathrm{Pic}(G)$, see 3.2.2.2). We have to verify that the canonical section associated to D spans a B-stable line in $\mathrm{Cox}_G(X)$. As $\mathcal{O}(X)^{*G} \simeq k^*$, every automorphism of $\mathcal{O}_X(D)^x$ preserving the G-linearization is given by the multiplication of sections by a non-zero constant. It follows that the canonical section associated with D is, up to multiplication by a non-zero constant, the unique global section of this G-linearized divisorial sheaf whose divisor of zeroes is D. Hence, this section is a B-eigenvector.

Consider the maximal open subvariety $V \subset X$ on which the rational quotient $\pi : X \dashrightarrow \mathbb{P}^1_k$ by B defines a morphism. By [EGA, II.7.3.6], the complement of V in X is of codimension at least two. Hence, there exists up to isomorphism a unique divisorial sheaf $\mathcal{F}$ whose restriction to V is isomorphic to $\pi^*_{|V}\mathcal{O}_{\mathbb{P}^1_k}(1)$. By hypothesis on the Picard group of G, we endow the sheaf $\mathcal{F}$ with a G-linearization and consider the canonical representative $\mathcal{F}^x$ associated to $[\mathcal{F}]$ (see Section 3.2.3). After choosing homogeneous coordinates z_1, z_2 on $\mathbb{P}^1_k$, the sections $\pi^*_{|V}(z_1)$ and $\pi^*_{|V}(z_2)$ defining the morphism $\pi_{|V}$ extend uniquely as global sections $a, b \in \mathcal{F}^x(X)$ which are B-semi-invariant of the same weight. The pullback of Weil divisors on $\mathbb{P}^1_k$ is well-defined, this is just the usual pullback of Cartier divisors

$$\pi^* : \mathrm{WDiv}(\mathbb{P}^1_k) \to \mathrm{WDiv}(X), p = [\alpha : \beta] \mapsto \mathrm{div}_X(\beta a - \alpha b).$$

As in the preceding section, consider a $P(X)$-stable open affine subvariety X_0 given by the local structure theorem. The complement of X_0 in X is a finite union of prime B-stable divisors. On the other hand, $\pi_{|X_0}$ is a geometric quotient by B whose schematic fibers are subvarieties of codimension one in X_0. We deduce that schematic fibers of $\pi_{|V}$ in general position define B-stable prime divisors of X by taking closure. These divisors are nothing but the prime divisors in the image of π^*.

Definition 3.3.2.1. [Pon, 3] The prime divisors in the image of π^* are the *parametric divisors*. The finite set of B-stable prime divisors that are not parametric is the set of *exceptional divisors*. For an exceptional divisor E, the image of $\pi_{|E}$ is either dense or a point in $\mathbb{P}^1_k$. In the former case, the divisor E is called *central*. In the latter case, the image point is called *exceptional*.

Notation 3.3.2.2. Let $(x_i)_{i \in I}$ be the finite family of exceptional points with respective homogeneous coordinates $[\alpha_i : \beta_i]$, $i \in I$. For all $i \in I$, let $(E_j^{x_i})_j$ be the finite family of exceptional divisors sent to x_i by π and let n_{ij} denote the coefficient of $E_j^{x_i}$ in the divisor $\pi^*(x_i)$. Let $(E_k)_k$ be the finite family of central exceptional divisors. Finally, equip $\mathcal{O}(E_j^{x_i})$ (resp. $\mathcal{O}(E_k)$) with arbitrary G-linearizations and let s_{ij} (resp. s_k) denote the canonical sections associated with $E_j^{x_i}$ (resp. E_k) in the G-linearized sheaves $\mathcal{O}(E_j^{x_i})^x$ (resp. $\mathcal{O}(E_k)^x$).

Theorem 3.3.2.3. *Suppose that $\mathcal{O}(X)^* \simeq k^*$. Then $\mathrm{Cox}_G(X)^U$ is generated as a $k[\mathbb{X}(G)]$-algebra by the elements $a, b, (s_{ij})_{ij}, (s_k)_k$. The ideal of relations contains the following identities*

$$\beta_i a - \alpha_i b = \lambda_i \textstyle\prod_j s_{ij}^{n_{ij}},$$

where for all $i \in I$, λ_i is a certain character of G. If moreover, the condition

- ($\star$) *the common degree of the homogeneous elements a, b is not a torsion element of $\mathrm{Cl}_G(X) \times \mathbb{X}(T)$*

is satisfied, then the above relations generate the whole ideal.

Proof. We first prove that the elements $a, b, (s_{ij})_{ij}, (s_k)_k$ generate $\mathrm{Cox}_G(X)^U$ as a $k[\mathbb{X}(G)]$-algebra. The divisor of zeroes of a $\mathrm{Cl}_G(X) \times \mathbb{X}(T)$-homogeneous element s can be written as

$$\mathrm{div}_X(s) = \textstyle\sum_{ij} p_{ij} E_j^{x_i} + \sum_k p_k E_k + \sum_l q_l D_l,$$

where the D_l are B-stable parametric divisors. We infer that s is associated, by mean of an homogeneous unit, with an element

$$\textstyle\prod_{ij} s_{ij}^{p_{ij}} \prod_k s_k^{p_k} \prod_l (v_l a - u_l b)^{q_l},$$

where $[u_l : v_l]$ is a homogeneous coordinate of the point $\pi(D_l) \in \mathbb{P}^1_k$. By virtue of 3.2.6.8, it follows that the elements $a, b, (s_{ij})_{ij}, (s_k)_k$ generate $\mathrm{Cox}_G(X)^U$ as $k[\mathbb{X}(G)]$-algebra.

By the same arguments, we see that the relations of the statement hold. Now suppose that the common degree of a, b is not a torsion element. We show that these relations generate the whole ideal. For this, consider an arbitrary relation

$$\lambda \prod_{ij} s_{ij}^{p_{ij}} \prod_k s_k^{p_k} f(a,b) + \mu \prod_{ij} s_{ij}^{q_{ij}} \prod_k s_k^{q_k} g(a,b) + ... = 0,$$

where $f, g...$ are $\mathbb{Z}$-homogeneous polynomials in two indeterminates (by assumption on torsion-freeness of the degree of a, b), and $\lambda, \mu, ...$ are characters of G. We claim that by substituting relations from the statement, one can transform a sum

$$\lambda \prod_{ij} s_{ij}^{p_{ij}} \prod_k s_k^{p_k} f(a,b) + \mu \prod_{ij} s_{ij}^{q_{ij}} \prod_k s_k^{q_k} g(a,b),$$

in an expression of the form $\nu \prod_{ij} s_{ij}^{r_{ij}} \prod_k s_k^{r_k} h(a,b)$, where h is a $\mathbb{Z}$-homogeneous polynomial in two indeterminates, and $\nu \in \mathbb{X}(G)$. Iterating this process, we obtain the trivial relation in a finite number of steps which finishes the proof. We set $A = \lambda \prod_{ij} s_{ij}^{p_{ij}} \prod_k s_k^{p_k} f(a,b)$, and $B = \mu \prod_{ij} s_{ij}^{q_{ij}} \prod_k s_k^{q_k} g(a,b)$, then $A/B \in k(X)^B = k(a/b)$. Taking the divisor of zeroes of A/B, we obtain an equation of the form

$$\sum_{ij} p_{ij} E_j^{x_i} + \sum_k p_k E_k + \mathrm{div}_X(f_A(a,b)) = \sum_{ij} q_{ij} E_j^{x_i} + \sum_k q_k E_k + \mathrm{div}_X(f_B(a,b))$$

where f_A and f_B are $\mathbb{Z}$-homogeneous polynomials in two indeterminates. We see right away that $p_k = q_k$ holds necessarily for all k, so that we can put in factor $\prod_k s_k^{p_k}$ in $A+B$. On the other hand, for each exceptional point x_i we get an equality

$$\sum_j (p_{ij} - q_{ij}) E_j^{x_i} = \mathrm{div}_X(\beta_i a - \alpha_i b)^{d_i}, \text{ where } [\alpha_i : \beta_i] = x_i.$$

We deduce that $p_{ij} - q_{ij} = d_i n_{ij}$ for all i, j. We can suppose that $d_i \geqslant 0$, so that $p_{ij} \geqslant q_{ij}$ for all j. Hence by using relations of the statement we write

$$\prod_j s_{ij}^{p_{ij}} = \prod_j s_{ij}^{q_{ij}} \prod_j s_{ij}^{p_{ij} - q_{ij}} = \lambda_i^{-d_i} (\beta_i a - \alpha_i b)^{d_i} \prod_j s_{ij}^{q_{ij}}.$$

This allows us to factor $\prod_j s_{ij}^{q_{ij}}$ in $A+B$. Doing this for each exceptional point, we see that the remaining factor is an homogeneous expression of the form

$$\lambda' f'(a,b) + \mu' g'(a,b),$$

where $\lambda', \mu' \in \mathbb{X}(G)$ and f', g' are $\mathbb{Z}$-homogeneous polynomials in two indeterminates. We necessarily have $\lambda' = \mu'$, and $\deg f' = \deg g'$, whence the sought reduction. $\square$

Proposition 3.3.2.4. *The condition* $(\star)$ *is satisfied in the following situations*

- X *is smooth and complete,*
- $G = \mathbb{T}$ *is a torus and* $\mathcal{O}(X)^{\mathbb{T}} \simeq k$*,*
- X *is almost homogeneous.*

Proof. In the first situation, $\mathrm{Pic}_G(X)$ is free of finite rank (3.3.1.7). Consequently, it suffices to check that the common $\mathrm{Pic}_G(X)$-degree of a, b is non-trivial. This is verified because a, b are linearly independent and $\mathcal{O}(X) \simeq k$.

In the second situation, suppose that there exists an integer $n \geqslant 1$ such that the common degree of a^n, b^n is trivial. Then, consider a non-empty $\mathbb{T}$-stable open subvariety X_0 equivariantly isomorphic to the product $\mathbb{T} \times Z$, where $\mathbb{T}$ acts on itself by left multiplication, and Z is an open subvariety of $\mathbb{P}^1_k$ on which $\mathbb{T}$ acts trivially. Restricting the homogeneous sections a, b to X_0, we have

$$\begin{cases} a, b \in \bigoplus_{\mathrm{Cl}_{\mathbb{T}}(X)} \mathcal{O}(Z)[\mathbb{X}(\mathbb{T})] \\ a^n, b^n \in k, \text{ of } \mathrm{Cl}_{\mathbb{T}}(X)\text{-degree zero} \end{cases}$$

Hence, we necessarily have $a, b \in k$, in a certain $\mathrm{Cl}_{\mathbb{T}}(X)$-homogeneous component. This is a contradiction as these two sections are linearly independent.

In the third situation, suppose again that there exists an integer $n \geqslant 1$ such that the common degree of a^n, b^n is trivial. Then we obtain $a^n, b^n \in \mathcal{O}(X)^B$, hence $a^n, b^n \in \mathcal{O}(X)^G$. As G is connected, $\mathrm{Cox}_G(X)^G$ is integrally closed in $\mathrm{Cox}_G(X)$, whence $a, b \in \mathrm{Cox}_G(X)^G$. But the homogeneous components of this last algebra are one dimensional k-vector spaces (3.2.8.5), and a, b are linearly independent, a contradiction. $\square$

Remark 3.3.2.5. In [ADHL, Thm 4.4.1.6], Hausen and Süss give a presentation of the Cox ring of a normal rational $\mathbb{T}$-variety of complexity one satisfying $\mathcal{O}(X) \simeq k$. In view of the preceding proposition, Theorem 3.3.2.3 combined with 3.2.9.3 yields a similar presentation for the slightly more general situation of a normal rational $\mathbb{T}$-variety of complexity one satisfying $\mathcal{O}(X)^{\mathbb{T}} \simeq k$.

3.3.3 $\mathrm{Cox}(X)^U$ as the Cox ring of a T-variety of complexity one

Let (X, x) be a pointed normal rational G-variety of complexity one such that $\mathcal{O}(X)^* \simeq k^*$. Suppose moreover that G has trivial Picard group, and that the condition $(\star)$ of 3.3.2.3 is satisfied, e.g. X is almost homogeneous (3.3.2.4).

In [ADHL, 3.4.2], are studied the finitely generated integral normal factorially graded k-algebras such that the grading is effective, pointed, and of complexity one. Geometrically, these are the coordinate algebras of normal affine varieties Z endowed with an effective action of a diagonalizable group $\mathscr{D}$ such that Z is $\mathscr{D}$-factorial, of complexity one under the action of $\mathscr{D}^\circ$, and the invariant regular functions as well as the invertible homogeneous regular functions on Z are constant. By [ADHL, 4.4.2.3], an algebra satisfying these conditions is up to graded isomorphism obtained by the

Construction 3.3.3.1. [ADHL, 3.4.2.1] Fix integers $r \in \mathbb{Z}_{\geqslant 1}$, $m \in \mathbb{Z}_{\geqslant 0}$, a sequence of integers $n_0, \ldots, n_r \in \mathbb{Z}_{\geqslant 1}$, and set $n := n_0 + \ldots + n_r$. Consider as inputs

- A matrix $A := [a_0, \ldots, a_r]$ with pairwise linearly independent column vectors $a_0, \ldots, a_r \in k^2$.
- An $r \times (n+m)$ block matrix $P_0 = [L \;\; 0_{r,m}]$, where L is an $r \times n$ matrix built from the n_i-tuples $l_i := (l_{i1}, \ldots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geqslant 1}$, $0 \leqslant i \leqslant r$, called *exponent vectors*, as below

$$L = \begin{bmatrix} -l_0 & l_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \ldots & l_r \end{bmatrix}$$

Now consider the polynomial algebra $k[T_{ij}, S_k]$, where $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$, and $1 \leqslant k \leqslant m$. For every $0 \leqslant i \leqslant r$, define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \ldots T_{in_i}^{l_{in_i}},$$

whence the name "exponent vector". Denote by $\mathscr{I}$ the set of triples (i_1, i_2, i_3) with $0 \leqslant i_1 < i_2 < i_3 \leqslant r$, and for all $I \in \mathscr{I}$, consider the trinomial

$$g_I := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

We introduce a grading on $k[T_{ij}, S_k]$ by the abelian group $K_0 := \mathbb{Z}^{n+m}/\operatorname{Im}({}^tP_0)$, where ${}^tP_0$ is the transpose of P_0. Let $Q_0 : \mathbb{Z}^{n+m} \to K_0$ be the projection, and set

$$\deg T_{ij} := Q_0(e_{ij}), \text{ and } \deg S_k := Q_0(e_k),$$

where (e_{ij}, e_k) is the standard basis of $\mathbb{Z}^{n+m}$. Finally consider the K_0-graded k-algebra

$$R(A, P_0) := k[T_{ij}, S_k]/(g_I)_{I\in\mathscr{I}}.$$

Proposition 3.3.3.2. *The k-algebra $\operatorname{Cox}(X)^U$ is isomorphic to an algebra $R(A, P_0)$ constructed as in 3.3.3.1.*

Proof. Consider a presentation of $\operatorname{Cox}(X)^U$ as provided by 3.3.2.3 and 3.3.1.6. Suppose first that X admits at least two exceptional points. Consider the algebra $R(A, P_0)$ built from the following input data

- m is the number of exceptional divisors in X dominating $\mathbb{P}^1_k$,
- $a_0, ..., a_r$ are homogemeous coordinates on $\mathbb{P}^1_k$ of the exceptional points, and we can suppose that these coordinates are of the form $[1 : 0], [0 : -1], [\alpha_2 : \beta_2], ..., [\alpha_r : \beta_r]$,
- the matrix A is
$$\begin{bmatrix} 1 & 0 & \alpha_2 & \dots & \alpha_r \\ 0 & -1 & \beta_2 & \dots & \beta_r \end{bmatrix},$$
- for $0 \leqslant i \leqslant r$, set
$$l_i := (n_{i,j})_j \text{ (notation of 3.3.2.3)}.$$

On the other hand, using the relations $a = \prod_j s_{0j}^{n_{0j}}$, and $b = \prod_j s_{1j}^{n_{1j}}$ given by 3.3.2.3, we can remove a, b from our generating set of $\operatorname{Cox}(X)^U$. We end up with the same number of generators as for the k-algebras $R(A, P_0)$. The ideal of relations is generated by the following ones

$$\alpha_i \prod_j s_{0j}^{n_{0,j}} - \beta_i \prod_j s_{1j}^{n_{1,j}} = \prod_j s_{ij}^{n_{i,j}},\ i = 2, ..., r.$$

With regard to the algebra $R(A, P_0)$, we have the identities

$$g_{(i,j,l)} = \alpha_{jl} g_{(i,j,k)} - \alpha_{ij} g_{(j,k,l)} \quad \text{[ADHL, 3.4.2.4]},$$

where $0 \leqslant i < j < k < l \leqslant r$, and $\alpha_{ij} := \det(a_i, a_j)$. From this, we infer that the trinomials $g_{(0,1,i)}$, $2 \leqslant i \leqslant r$ generate the ideal $((g_I)_{I\in\mathscr{I}})$. But these trinomials are, up to the change of variables $T_{ij} \mapsto s_{ij}$, the trinomials appearing in relations listed above for $\operatorname{Cox}(X)^U$. This finishes the proof in this case.

If X admits at most two exceptional points, then $\operatorname{Cox}(X)^U$ is a polynomial algebra. In this case we use almost the same input data for $R(A, P_0)$, except that we consider two column vectors $a_0, a_1 \in k^2$ which are homogeneous coordinates of two distinct points in $\mathbb{P}^1_k$ that contain all the exceptional points, and for the a_i which are not exceptional, we set $l_i := (1)$. □

Corollary 3.3.3.3. *The spectrum of $\operatorname{Cox}(X)^U$ is a complete intersection.*

Proof. By [ADHL, 3.4.2.3], the spectrum of any algebra $R(A, P_0)$ constructed as in 3.3.3.1 is a complete intersection, whence the result. □

Corollary 3.3.3.4. *The total coordinate space $\tilde{X}$ has only constant invertible functions.*

Proof. An invertible function $f \in \operatorname{Cox}(X)^*$ is U-invariant [Bri18, 4.1.6]. Thus, it is constant because any algebra $R(A, P_0)$ has only constant invertible functions [ADHL, 3.4.2.3]. □

Corollary 3.3.3.5. *The k-algebra $\mathrm{Cox}(X)^U$ is the Cox ring of a normal variety of complexity one under a torus action.*

Proof. Proceeding as in [ABHW, Prop. 5.1], we construct a normal affine rational variety of complexity one under a torus action, whose Cox ring is $\mathrm{Cox}(X)^U$. □

Remark 3.3.3.6. Following the proof of 3.3.3.2, the input data to be used in Construction 3.3.3.1 in order to obtain $\mathrm{Cox}(X)^U$ can interpreted geometrically:

- m is the number of exceptional divisors in X dominating $\mathbb{P}^1_k$,
- $a_0, ..., a_r$ are homogemeous coordinates on $\mathbb{P}^1_k$ of the exceptional points $x_0, ..., x_r$,
- the exponent vectors are the vectors formed by the multiplicities of the exceptional divisors in the pullbacks $\pi^*(x_i)$, $i = 0, ..., r$.

Remark 3.3.3.7. Consider the spectrum $\tilde{Y}$ of an algebra $R(A, P_0)$ viewed as the total coordinate space of a normal rational variety Y of complexity one under a torus action. The geometries of $\tilde{Y}$ and Y highly depend on the exponent vectors l_i, $i = 0, ..., r$ involved in the equations. In [ABHW], the study of singularities on Y leads to the following notion that will be used later: a ring $R(A, P_0)$ is a *Platonic ring* either if $r \leqslant 1$, or if every tuple $(l_{0i_0}, ..., l_{ri_r})$ is *Platonic*, i.e. after ordering it decreasingly, the first triple is one of the *Platonic triples*

$$(5, 3, 2), (4, 3, 2), (3, 3, 2), (x, 2, 2), (x, y, 1),\ x \geqslant y \geqslant 1,$$

and the remaining integers of the tuple equal one.

3.3.4 Log terminality in the total coordinate space

In this section, we characterize the log terminality of singularities in the total coordinate spaces of almost homogeneous G-varieties of complexity $\leqslant 1$. One key ingredient is a theorem of Braun [Bra, Thm 3.1] saying that a total coordinate space $\tilde{X}$ is always 1-Gorenstein. Thus, applying [I, 6.2.12, 6.2.15] yields the equivalence

$$\tilde{X} \text{ has log terminal singularities} \iff \tilde{X} \text{ has rational singularities.}$$

Proposition 3.3.4.1. *Let X be a spherical G-variety with only constant invertible regular functions. Then, the total coordinate space $\tilde{X}$ has log terminal singularities.*

Proof. The total coordinate space $\tilde{X}$ is a spherical variety under a connected reductive group $G_1 \times \mathbb{D}(\mathrm{Cl}(X))^\circ$, where G_1 is a finite cover of G (see [ADHL, Thm 4.2.3.2]). As spherical varieties have rational singularities [T11, Thm. 15.20], this completes the proof. □

Corollary 3.3.4.2. *Let X be a $\mathbb{Q}$-factorial projective spherical variety. Then X is of Fano type.*

Proof. Apply [GOST, Thm 1.1] together with the last proposition. □

Choose a Borel subgroup B, a maximal torus T in B, and denote by U the unipotent part of B. We now prove the

Theorem 3.3.4.3. *Let X be an almost homogeneous G-variety of complexity one with only constant invertible regular functions. Then*

$$\tilde{X} \textit{ has log terminal singularities} \iff \mathrm{Cox}(X)^U \textit{ is a Platonic ring} \quad \textit{(see 3.3.3.7).}$$

The Cox ring of X is well defined and finitely generated (3.3.1.5). Moreover, up to replacing G by a finite cover, we can suppose that $\mathrm{Pic}(G) = 0$, and that $\mathrm{Cox}(X)$ is endowed with a compatible structure of $\mathrm{Cl}(X)$-graded G-algebra (see [ADHL, Thm. 4.2.3.2]); this doesn't change U and replaces B and T by finite covers.

Proving that a spherical variety has rational singularities usually relies on a reduction to the toric case. Similarly we are going to reduce to the case of a normal variety of complexity one under a torus action, and use the following result of Arzhantsev, Braun, Hausen and Wrobel:

Theorem 3.3.4.4. *[ABHW, Thm. 1] Let $\tilde{Y}$ be a normal affine $\mathbb{Q}$-Gorenstein rational $\mathbb{T}$-variety of complexity one such that $\mathcal{O}(\tilde{Y})^{\mathbb{T}} \simeq k$. Then $\tilde{Y}$ has log terminal singularities if and only if its Cox ring is Platonic.*

Consider the categorical quotient $\tilde{Y} := \operatorname{Spec}\mathrm{Cox}(X)^U$ of $\tilde{X}$ by U. By [T11, Thm. D.5], $\tilde{Y}$ is a normal variety. Moreover, it is of complexity one under the induced action of the torus $\mathbb{T} := T \times \mathbb{D}(\mathrm{Cl}(X))^{\circ}$. Indeed, $\tilde{X}$ is almost homogeneous of complexity one under $G \times \mathbb{D}(\mathrm{Cl}(X))^{\circ}$.

Lemma 3.3.4.5. *The variety $\tilde{Y}$ satisfies the assumptions of Theorem 3.3.4.4. Moreover, it is 1-Gorenstein.*

Proof. As $\tilde{X}$ is almost homogeneous of complexity one under $G \times \mathbb{D}(\mathrm{Cl}(X))^{\circ}$, it is a rational variety (3.3.1.5). Then, we have

$$k(\mathbb{P}^1_k) \simeq k(\tilde{X})^{B\times\mathbb{D}(\mathrm{Cl}(X))^{\circ}} = k(\tilde{X})^{U.\mathbb{T}} = \mathrm{Frac}(\mathrm{Cox}(X)^U)^{\mathbb{T}} = k(\tilde{Y})^{\mathbb{T}},$$

whence $\tilde{Y}$ is a rational variety (for a proof of the second equality, proceed similarly as in [T11, D.7]). By the result of Braun again, $\tilde{Y}$ is 1-Gorenstein because it can be viewed as a total coordinate space (3.3.3.5). Finally, we verify that $\mathcal{O}(\tilde{Y})^{\mathbb{T}} \simeq k$. We have identifications

$$\mathcal{O}(\tilde{Y})^{\mathbb{T}} \simeq \mathcal{O}(\tilde{X})^{B\times\mathbb{D}(\mathrm{Cl}(X))^{\circ}} \simeq \mathcal{O}(\tilde{X})^{G\times\mathbb{D}(\mathrm{Cl}(X))^{\circ}} \simeq k[(r_i)_i]^{\mathbb{D}(\mathrm{Cl}(X))^{\circ}},$$

where $k[(r_i)_i]$ is a polynomial k-subalgebra in the canonical sections associated to the G-stable prime divisors in X. Indeed, because X is almost homogeneous, a G-fixed homogeneous element in $\mathrm{Cox}(X)$ is the product of a monomial in the r_i with a homogeneous unit, and the homogeneous units are constant [ADHL, 1.5.2.5]. The algebra $k[(r_i)_i]$ is $\mathrm{Cl}(X)$-graded, and any non-zero degree associated with a homogeneous element is torsion free in $\mathrm{Cl}(X)$. It follows that

$$k[(r_i)_i]^{\mathbb{D}(\mathrm{Cl}(X))^{\circ}} \simeq k[(r_i)_i]_0 \simeq k.$$

$\square$

By [T11, Thm D.5], $\tilde{X}$ has rational singularities if and only if $\tilde{Y}$ do so. Using Theorem 3.3.4.4 together with the above lemma yields

$$\begin{aligned} \tilde{X} \text{ has log terminal singularities} &\iff \tilde{Y} \text{ has log terminal singularities} \\ &\iff \mathrm{Cox}(\tilde{Y}) \text{ is a Platonic ring.} \end{aligned}$$

To finish, we claim that

$$\mathrm{Cox}(\tilde{Y}) \text{ is a Platonic ring} \iff \mathrm{Cox}(X)^U \text{ is a Platonic ring.}$$

The algebra $\mathrm{Cox}(\tilde{Y})$ is again of the form given by Construction 3.3.3.1, and the associated exponent vectors have an explicit description in term of the ones associated with $\mathrm{Cox}(X)^U$ (see [HW, 2.7]). Using this description, the claim is verified by means of an easy case by case check. This finishes the proof of the Theorem 3.3.4.3.

3.3.5 Iteration of (equivariant) Cox rings

In [ABHW], Arzhantsev, Braun, Hausen and Wrobel introduced the notion of *iteration of Cox rings*: Let Z be a normal variety with finitely generated Cox ring. If the total coordinate space $\tilde{Z}$ has a non-trivial class group and satisfies $\mathcal{O}(\tilde{Z})^* \simeq k^*$, then it has a non-trivial well-defined Cox ring. If the latter is finitely generated, we get a new total coordinate space $\tilde{Z}^{(2)}$, and so on. This iteration process yields a sequence of Cox rings which stops if only if one of the following cases occurs at some step:

- we obtain a total coordinate space whose Cox ring is not well defined (i.e. there exists $n \geqslant 0$ such that $\mathrm{Cl}(\tilde{Z}^{(n)})$ has a non-trivial torsion subgroup, and $\mathcal{O}(\tilde{Z}^{(n)})^* \not\simeq k^*$).
- we obtain a total coordinate space whose Cox ring is not finitely generated.
- we obtain a factorial total coordinate space (i.e. with trivial class group).

If we never fall in one of the cases above, Z is said to have *infinite iteration of Cox rings*. Otherwise, Z is said to have *finite iteration of Cox rings*, and the last obtained Cox ring is the *master Cox ring*. Toric varieties obviously admit finite iteration of Cox rings with a factorial master Cox ring in one step. More generally, spherical varieties admit finite iteration of Cox rings with a factorial (finitely generated) master Cox ring in at most two steps [G, Thm 1.1]. In this section, we prove the finite iteration of Cox rings in the complexity one case, assuming some technical condition (3.3.5.2). We use a similar approach as in [Bra], namely we systematically decompose a diagonalizable torsor into a torsor by a torus followed by a torsor by a finite diagonalizable group, and use the fact that characteristic spaces are preserved by torsors under tori (3.2.10.1).

The above definitions have obvious analogues in the equivariant setting. We then speak of *iteration of equivariant Cox rings*. For a variety X endowed with an action of an algebraic group G, we denote by $\hat{X}_G^{(i)}$ the ith G-equivariant characteristic space (if it exists), and $\tilde{X}_G^{(i)}$ the ith G-equivariant total coordinate space. The next theorem establishes the finiteness of iteration of equivariant Cox rings for almost homogeneous varieties under a connected reductive algebraic group. This yields in particular the finiteness of iteration of Cox rings for almost homogeneous varieties under a semisimple algebraic group (3.2.3.4).

Theorem 3.3.5.1. *Let G be a connected reductive group and X be an almost homogeneous G-variety of complexity one. Then X admits finite iteration of equivariant Cox rings. Moreover, each equivariant Cox ring occuring in the sequence is the Cox ring of a complexity one almost homogeneous G-variety. In particular, the equivariant master Cox ring is finitely generated and factorial.*

Proof. We can suppose that X is a smooth G-variety, up to replacing it by its smooth locus. By 3.3.1.4 and 3.3.1.5, the G-equivariant characteristic space $\hat{X}_G$ exists and $q_1 : \hat{X}_G \to X$ is a G-equivariant $\mathbb{D}(\mathrm{Pic}_G(X))$-torsor. Consider the exact sequence of abelian groups

$$0 \to \mathrm{Pic}_G(X)_{\mathrm{tor}} \to \mathrm{Pic}_G(X) \to M \to 0,$$

where $\mathrm{Pic}_G(X)_{\mathrm{tor}}$ denotes the torsion part of $\mathrm{Pic}_G(X)$. This defines a factorization

$$\hat{X}_G \xrightarrow{g_1} X' \xrightarrow{f_1} X$$

of q_1 into G-equivariant diagonalizable torsors. In particular, g_1 is a torsor under a torus, and f_1 is a torsor under a finite diagonalizable group. Denote by H the stabilizer of a base point in the open G-orbit of X. Then X is a normal G/H-embedding. Let $D_1, ..., D_r$ be the G-stable prime divisors lying in $X \setminus G/H$. Using the exact sequence (3.2.3) applied to the restriction $\mathrm{Pic}_G(X) \to \mathrm{Pic}_G(G/H)$, we infer that the group $\mathrm{Pic}_G(X)_{\mathrm{tor}}$ embeds in $\mathrm{Pic}_G(G/H) \simeq \mathbb{X}(H)$.

Let H_1 be the intersection of the kernels of the characters of H corresponding to elements of $\mathrm{Pic}_G(X)_{\mathrm{tor}}$. This is a normal subgroup of H, and $\mathrm{Pic}_G(X)_{\mathrm{tor}} \simeq \mathbb{X}(H/H_1)$ under the above embedding. Then $G/H_1 \to G/H$ is a diagonalizable torsor under the group H/H_1, and G/H_1 identifies with the G-open orbit in X'. Because X' is an almost homogeneous G-variety of complexity one, it has a well defined G-equivariant characteristic space. Proposition 3.2.8.5 guarantees that $\mathcal{O}(\hat{X}_G)^{*G} \simeq k^*$, thus we can apply 3.2.10.1 in order to obtain that $\hat{X}_G$ has also a well defined G-equivariant characteristic space which coincides with that of X'. This construction can be iterated so that we obtain a commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d, "f_3"] & \cdots \arrow[d] \arrow[dl, "q_3"'] & \cdots \arrow[d] \\
G/H_2 \arrow[r, hook] \arrow[d] & X'^{(2)} \arrow[d, "f_2"] & \hat{X}_G^{(2)} \arrow[l, "g_2"] \arrow[r, hook] \arrow[d] \arrow[dl, "q_2"'] & \tilde{X}_G^{(2)} \arrow[d] \\
G/H_1 \arrow[r, hook] \arrow[d] & X' \arrow[d, "f_1"] & \hat{X}_G \arrow[l, "g_1"] \arrow[r, hook] \arrow[dl, "q_1"'] & \tilde{X}_G \\
G/H \arrow[r, hook] & X & &
\end{tikzcd}
$$

Moreover, as each H_i, $i \geqslant 1$ is necessarily of finite index in H, it follows that this construction gives rise to a descending sequence $(H_i)_{i\geqslant 1}$ of subgroups of H, all of which containing the neutral component H°. Thus, this sequence is constant after a certain rank. As a consequence X admits finite iteration of equivariant Cox rings with a finitely generated factorial equivariant master Cox ring. $\square$

Let G be a connected reductive algebraic group, and X be a normal rational G-variety of complexity one satisfying the condition

$(\mathcal{C})$: X and each of its iterated characteristic spaces have only constant invertible functions.

By adapting the argument in the last proof, we show in Theorem 3.3.5.2 that X admits finite iteration of Cox rings. The condition $(\mathcal{C})$ guarantees that X has well defined iterated Cox rings, and is satisfied in the following situations

- $\mathcal{O}(X) \simeq k$ (e.g. X is complete),
- X is a quasicone,
- $\mathcal{O}(X)^* \simeq k^*$, and X is almost homogeneous,
- $G = \mathbb{T}$ is a torus, $\mathcal{O}(X)^* \simeq k^*$, and $\mathcal{O}(X)^{\mathbb{T}} \simeq k$.

Indeed, for the first two cases, apply [Bra, 4.4], and for the remaining cases, use repeatedly 3.3.3.4.

Theorem 3.3.5.2. *Let G be a connected reductive group, and X be a normal rational G-variety of complexity one satisfying condition $(\mathcal{C})$. Then, X admits finite iteration of Cox rings with well defined iterated Cox rings. If moreover X is almost homogeneous, then the master Cox ring is factorial and finitely generated.*

Proof. Up to replacing X by its smooth locus, we can suppose that X is smooth as this doesn't affect the assumptions and the iteration of Cox rings. By virtue of the condition $(\mathcal{C})$ and 3.3.1.3, the iteration stops at the step $n \geqslant 0$ if and only if

$$\text{either } \tilde{X}^{(n)} \text{ is not rational, or } \tilde{X}^{(n)} \text{ is factorial.}$$

To begin with, let us examine the first iteration in detail. We can suppose that $\mathrm{Pic}(X)$ has a non-trivial torsion subgroup, as otherwise $\hat{X}$ is factorial and iteration stops [ADHL, 1.4.1.5]. Consider the diagonalizable torsor $f_1 : X' \to X$ associated to the inclusion morphism $\mathrm{Pic}(X)_{\mathrm{tor}} \hookrightarrow \mathrm{Pic}(X)$ (3.2.4.5). Also, consider the exact sequence of abelian groups

$$0 \to \mathrm{Pic}(X)_{\mathrm{tor}} \to \mathrm{Pic}(X) \to M \to 0.$$

This defines a factorization

$$\hat{X} \xrightarrow{g_1} X' \xrightarrow{f_1} X$$

of the characteristic space $q_1 : \hat{X} \to X$ into diagonalizable torsors. In particular, g_1 is a torsor under the torus $\mathbb{D}(M)$, and f_1 is a torsor under the finite diagonalizable group $\mathscr{D}_1$ with character group $\mathrm{Pic}(X)_{\mathrm{tor}}$. In fact we can suppose, up to replacing G by a connected finite cover, that f_1, g_1 are a G-equivariant diagonalizable torsors [ADHL, 4.2.3.2]. Also, we can suppose that X' is rational, otherwise $\hat{X}$ is not rational and iteration stops. Then there is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, dashed, "\pi_1"] \arrow[d, "f_1"] & \mathbb{P}^1_k \arrow[d, "\varphi_1"] \\ X \arrow[r, dashed, "\pi_0"] & \mathbb{P}^1_k, \end{tikzcd}$$

where the π_i are the respective rational quotients by B, and φ_1 is a geometric quotient by $\mathscr{D}_1$. Notice that the π_i are defined at every codimension 1 point. Let $\mathscr{D}_{1,\mathrm{eff}}$ be the quotient of $\mathscr{D}_1$ by the kernel of the $\mathscr{D}_1$-action on $\mathbb{P}^1_k$ and $d := \mid \mathscr{D}_{1,\mathrm{eff}} \mid$. The locus where $\mathscr{D}_{1,\mathrm{eff}}$ acts freely is non-empty open, φ_1 is étale at each point of this locus and ramified at each point of its complement. Let x be a point in $\mathbb{P}^1_k$, and consider the pullback

$$\pi_0^*(x) = \sum_{i=1}^{N_x} n_{i,x} E_i^x, \tag{3.3.1}$$

where the E_i^x are pairwise distinct B-stable prime divisors in X, and $n_{i,x} \in \mathbb{Z}_{>0}$, $i = 1, ..., N_x$. Inspired by Construction 3.3.3.1, the vector $(n_{i,x})_{i=1,...,N_x}$ is called the *exponent vector* associated with x. On the other hand, considering the set theoretic fiber $\varphi_1^{-1}(x) = \{x'_1, ..., x'_{l_x}\}$, we have

$$\varphi_1^*(x) = \frac{d}{l_x} x'_1 + ... + \frac{d}{l_x} x'_{l_x}. \tag{3.3.2}$$

Indeed, the ramification indices associated to the x'_k, $k = 1, ..., l_x$ are equal and their sum is d since the finite extension of fields of rational functions defined by φ_1 is Galois of degree d. By definition, φ_1 is ramified over x if and only if $d/l_x > 1$.

Now we compute the pullbacks $\pi_1^*(x'_k)$: since f_1 is étale, there are no multiplicities for the prime divisors occuring in $f_1^*(E_i^x)$. Moreover, denoting $\Sigma_{x,i}$ the set of prime divisors occuring in the support of $f_1^*(E_i^x)$, the morphism π_1 induces an equivariant map of transitive $\mathscr{D}_1$-sets

$$\Sigma_{x,i} \to \{x'_1, ..., x'_{l_x}\}.$$

It follows that we have $f_1^*(E_i^x) = \sum_{k=1}^{l_x} (E_i^{x'_k,1} + ... + E_i^{x'_k,m_i})$, where the $E_i^{x'_k,j}$ are pairwise distinct B-stable prime divisors in X' which are respectively sent to x'_k by π_1, and m_i is the cardinality of the orbit $\mathrm{Stab}_{\mathscr{D}_1}(x'_1).E_i^{x'_k,1}$. Using the commutativity of the above square and the formulas (3.3.1) and (3.3.2), we obtain

$$\pi_1^*(x'_k) = \sum_{i=1}^{N_x} \frac{n_{i,x}}{d/l_x} (E_i^{x'_k,1} + ... + E_i^{x'_k,m_i}). \tag{3.3.3}$$

It follows from this computation that if φ_1 is ramified over x, then it is an exceptional point, and the coordinates of the associated exponent vector have a non-trivial greatest common divisor. We call *primitive* an exponent vector whose coordinates have a trivial greatest common divisor. By the above formula, the exponent vectors associated to the points x'_k lying in the fiber $\varphi_1^{-1}(x)$ are equal to each other. Also, if the exponent vector associated with x is primitive, then the exponent vectors associated with the points x'_k are also primitive. The idea of the proof is roughly to observe that there is a finite number of non-primitive exponent vectors, and that this number decreases along the iteration process (up to identifying equal exponent vectors). Then, assuming infinite iteration Cox rings easily yields a contradiction. We now give the details

Thanks to condition $(\mathcal{C})$ and Proposition 3.2.10.1, the above construction can be iterated together with the iteration of Cox rings. At the nth step of the iteration of Cox rings, we obtain a diagram

$$\begin{tikzcd}
X'^{(n)} \arrow[r, "f_n"] \arrow[d, dashed, "\pi_n"] & \dots \arrow[r, "f_3"] & X'^{(2)} \arrow[r, "f_2"] \arrow[d, dashed, "\pi_2"] & X' \arrow[r, "f_1"] \arrow[d, dashed, "\pi_1"] & X \arrow[d, dashed, "\pi_0"] \\
\mathbb{P}^1_k \arrow[r, "\varphi_n"] & \dots \arrow[r, "\varphi_3"] & \mathbb{P}^1_k \arrow[r, "\varphi_2"] & \mathbb{P}^1_k \arrow[r, "\varphi_1"] & \mathbb{P}^1_k,
\end{tikzcd}$$

where $X'^{(i)}$ and $\hat{X}^{(i)}$ have the common characteristic space $\hat{X}^{(i+1)}$ $(i = 1, ..., n)$, and the f_i are G-equivariant $\mathscr{D}_i$-torsors ($\mathscr{D}_i := \mathrm{Pic}(X'^{(i-1)})_{\mathrm{tor}}$, $i = 1, ..., n-1$). Here, the $\mathscr{D}_i$ must be non-trivial because otherwise iteration would stop before the nth step. For the torsors f_i to be G-equivariant, we have possibly replaced G by the choice of a finite cover, however the morphisms in the above diagram don't depend on this choice. By repeatedly applying Lemmas 3.3.5.3 and 3.3.5.4 below, we obtain respectively the following facts:

- The morphisms φ_i are ramified for $i = 1, ..., n-1$.
- The composition $F_n := f_1 \circ ... \circ f_{n-1} \circ f_n$ is a G-equivariant torsor under the action of a finite group S_n. Moreover, the actions of S_n and G commute.

In particular, F_n is finite étale and $\Phi_n := \varphi_1 \circ ... \circ \varphi_{n-1} \circ \varphi_n$ is a geometric quotient by S_n. Now, in view of the commutativity of the diagram

$$\begin{tikzcd}
X'^{(n)} \arrow[r, dashed, "\pi_n"] \arrow[d, "F_n"] & \mathbb{P}^1_k \arrow[d, "\Phi_n"] \\
X \arrow[r, dashed, "\pi_0"] & \mathbb{P}^1_k,
\end{tikzcd}$$

the exponent vector associated with a point in a fiber $\Phi_n^{-1}(x)$ can be obtained by using the same computation as the one leading to formula (3.3.3). It follows that the exponent vectors associated to points in a common fiber $\Phi_n^{-1}(x)$ are equal to each other. We say that two exponent vectors are *equivalent* if they are associated with points in a common fiber $\Phi_n^{-1}(x)$, this defines an equivalence relation. Consider the exceptional points $x_1, ..., x_r \in \mathbb{P}^1_k$ associated with π_0 and such that each associated exponent vector is not primitive. Let u_n be the number of classes of non-primitive exponent vectors defined by pullbacks of points of $\mathbb{P}^1_k$ by π_n. We have $u_0 = r$ and $(u_n)_n$ is clearly decreasing with values in $\mathbb{N}$. Suppose by contradiction that we have infinite iteration of Cox rings, which implies that u_n is defined for all $n \in \mathbb{N}$. Then, this sequence clearly admits a limit $l \geqslant 0$. We cannot have $l = 0$ because there would exist an integer $n \geqslant 0$ such that φ_n is unramified. Thus, there is an exceptional point x_k, $1 \leqslant k \leqslant r$, for which the greatest common divisor of the coordinates of the associated exponent vector is unbounded, a contradiction.

To finish the proof of the finiteness of the iteration of Cox rings, we now give the two lemmas mentionned above:

Lemma 3.3.5.3. *Suppose that φ_1 is not ramified. Then φ_1 is the identity of $\mathbb{P}^1_k$. Moreover, X' and $\hat{X}$ have torsion-free Picard groups.*

Proof. If φ_1 is not ramified, it is a finite surjective étale morphism, whence an isomorphism since $\mathbb{P}^1_k$ is simply connected. This means that $\mathscr{D}_1$ acts trivially on $\mathbb{P}^1_k$, thus $\varphi_1 = \mathrm{Id}_{\mathbb{P}^1_k}$. For the second statement, we show first that the torsion freeness of $\mathrm{Pic}(\hat{X})$ implies the torsion freeness of $\mathrm{Pic}(X')$. Using condition $(\mathcal{C})$, the exact sequence (3.2.1) applied to $\hat{X}$ viewed as a $\mathbb{D}(M)$-variety gives

$$0 \to M \to \mathrm{Pic}_{\mathbb{D}(M)}(\hat{X}) \to \mathrm{Pic}(\hat{X}) \to 0.$$

This exact sequence guarantees that the torsion subgroup of $\mathrm{Pic}_{\mathbb{D}(M)}(\hat{X})$ embeds in that of $\mathrm{Pic}(\hat{X})$. To finish, we have an isomorphism $\mathrm{Pic}_{\mathbb{D}(M)}(\hat{X}) \simeq \mathrm{Pic}(X')$ (3.2.5.2).

Now we prove that $\mathrm{Pic}(\hat{X})$ is torsion free. By hypothesis, we have a commutative triangle

$$\begin{tikzcd} X' \arrow[d, "f_1"'] \arrow[dr, dashed, "\pi_1"] & \\ X \arrow[r, dashed, "\pi_0"] & \mathbb{P}^1_k, \end{tikzcd}$$

and we claim that all but a finite number of B-stable prime divisors E in X satisfy $f_1^*(E) = E'$ where E' is a $B \times \mathscr{D}_1$-stable prime divisor in X'. Indeed, it suffices to consider an affine B-stable open subvariety $X_0 \subset X$ as in the paragraph after Corollary 5.3.3.2 and the geometric quotient $X_0 \to C_0 \subset \mathbb{P}^1_k$ by B induced by the restriction of π_0. Then π_1 also induces a geometric quotient $f_1^{-1}(X_0) \to C_0$ by B. For all $x \in C_0$, the pullbacks $\pi_0^*(x) = E^x$ and $\pi_1^*(x) = E'^x$ are B-stable prime divisors. It follows that the support of $f_1^*(E^x)$ is E'^x and there is no multiplicity as f_1 is étale. As all but a finite number of B-stable prime divisors in X are pullbacks of points of C_0, the claim follows. Now, because g_1 is a torsor under a torus, we conclude that for all but a finite number of B-stable prime divisors E in X, we have the property

$$q_1^*(E) = \hat{E}, \text{ where } \hat{E} \text{ is a } B \times \mathbb{D}(M)\text{-stable prime divisor in } \hat{X}.$$

For any prime divisor E in X, we have $q_1^*(E) = \mathrm{div}(s_E)$, where we view the canonical section s_E associated with E as a regular function on $\hat{X}$. Hence, all but a finite number of $B \times \mathbb{D}(M)$-stable prime divisors in $\hat{X}$ are principal. Let $\{E_1, ..., E_r\}$ be the finite set of B-stable prime divisors in X not satisfying the above property. We have $\mathrm{div}(s_{E_i}) = q_1^*(E_i) = \hat{E}_i^1 + ... + \hat{E}_i^{l_i}$ where the $\hat{E}_i^k$ are $B \times \mathbb{D}(M)$-stable prime divisors in $\hat{X}$ ($1 \leqslant k \leqslant l_i$). Consider the $B \times \mathbb{D}(M)$-stable open subvariety $\hat{X}_0 \subset \hat{X}$ obtained by removing the supports of the $q_1^*(E_i)$, $i = 1, ..., r$. The exact sequence (3.2.2) reads

$$0 \to \mathcal{O}(\hat{X}_0)^* \to \bigoplus_{i,k} \mathbb{Z}\hat{E}_i^k \to \mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(\hat{X}_0) = 0.$$

Indeed, all the $B \times \mathbb{D}(M)$-stable prime divisors in $\hat{X}_0$ are principal whence $\mathrm{Pic}(\hat{X}_0) = 0$. Up to multiplication by a non-zero constant, the units of $\mathcal{O}(\hat{X}_0)$ are Laurent monomials in the canonical sections s_{E_i}, $i = 1, ..., r$. Hence, the cokernel of the morphism $\mathcal{O}(\hat{X}_0)^* \to \bigoplus_{i,k} \mathbb{Z}\hat{E}_i^k$ is a torsion-free abelian group isomorphic to $\mathrm{Pic}(\hat{X})$. $\square$

Lemma 3.3.5.4. *Consider as above the G-equivariant diagonalizable $\mathscr{D}_1$-torsor $f_1 : X' \to X$. Suppose that there is a G-equivariant torsor $f_0 : X \to Y$ under a finite group S_0 such that the actions of G and S_0 commute. Then $F_1 := f_0 \circ f_1$ is a torsor under a finite group S_1 which is an extension of S_0 by $\mathscr{D}_1$. Moreover, F_1 is G-equivariant and the respective actions of S_1 and G on X' commute.*

Proof. We adapt the arguments of [ADHL, 4.2.4.1]. Let $N_{\mathrm{Aut}(X')}(\mathscr{D}_1)$ be the normalizer of $\mathscr{D}_1$ viewed as a subgroup of $\mathrm{Aut}(X')$, and $\Phi \in N_{\mathrm{Aut}(X')}(\mathscr{D}_1)$. Then, $f_1\Phi : X' \to X$ is a $\mathscr{D}_1$-invariant morphism, whence the existence of a morphism φ such that following diagram commutes

$$\begin{tikzcd} X' \arrow[r, "\Phi"] \arrow[d, "f_1"] & X' \arrow[d, "f_1"] \\ X \arrow[r, "\varphi"] & X. \end{tikzcd}$$

It is readily checked that $\varphi \in \mathrm{Aut}(X)$, and that the map $\beta : N_{\mathrm{Aut}(X')}(\mathscr{D}_1) \to \mathrm{Aut}(X), \Phi \mapsto \varphi$ is a group morphism. We show that this morphism is surjective. Consider an automorphism $\varphi \in \mathrm{Aut}(X)$, and let φ^*X' denote the pullback of X' along φ. This defines an isomorphism $u : \varphi^*X' \to X$. Moreover, the structural morphism $\varphi^*X' \to X$ is a $\mathscr{D}_1$-torsor which has the same type (3.2.5.7) as f_1 up to an automorphism of $\mathbb{X}(\mathscr{D}_1)$. Indeed, the pullback by φ induces an automorphism of $\mathrm{Pic}(X)_{\mathrm{tor}} = \mathbb{X}(\mathscr{D}_1)$. It follows from this last fact and 3.2.5.9 that there exists an isomorphism $v : X' \to \varphi^*X'$ over X compatible with the $\mathscr{D}_1$-action (here we used that $\mathcal{O}(X)^* \simeq k^*$). Summarizing, we obtain a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[dr] & \varphi^*X' \arrow[r, "u"] \arrow[d] & X' \arrow[d, "f_1"] \\ & X \arrow[r, "\varphi"] & X, \end{tikzcd}$$

where $uv \in N_{\mathrm{Aut}(X')}(\mathscr{D}_1)$ and $\beta(uv) = \varphi$, which proves the surjectivity. The kernel of β consists of the elements of $N_{\mathrm{Aut}(X')}(\mathscr{D}_1)$ which are automorphisms of X' viewed as a variety over X. Obviously, $\mathrm{Ker}\,\beta$ contains $\mathscr{D}_1$. Conversely, any element $v \in \mathrm{Ker}\,\beta$ induces an automorphism of the field of rational functions $k(X')$ viewed as a $k(X)$-algebra. As the extension $k(X) \hookrightarrow k(X')$ is Galois with Galois group $\mathscr{D}_1$, we conclude that $\ker\beta = \mathscr{D}_1$. We have obtained an exact sequence of groups (see also [Ma, Proposition 2.1])

$$1 \to \mathscr{D}_1 \to N_{\mathrm{Aut}(X')}(\mathscr{D}_1) \xrightarrow{\beta} \mathrm{Aut}(X) \to 1.$$

Let S_1 be the preimage of $S_0 \subset \mathrm{Aut}(X)$ by β. This is a finite group acting on X' which is an extension of S_0 by $\mathscr{D}_1$. Also, it is directly checked that the quotient of X' by S_1 exists and is a torsor which identifies with the morphism F_1.

It remains to prove that the actions of G and S_1 commute. Let $s_1 \in S_1(k)$, $g \in G(k)$, $x \in X(k)$. Then we have

$$f_1 s_1 g s_1^{-1} g^{-1}(x) = s_0 g s_0^{-1} g^{-1} f_1(x) = f_1(x),$$

where $s_0 := \beta(s_1)$. It follows that $s_1 g s_1^{-1} g^{-1} \in \mathscr{D}_1(k)$, whence a continuous map $G(k) \to \mathscr{D}_1(k)$ taking an element to its commutator with s_1. Because G is connected, this map is constant. Taking its value at e_G, we conclude that the actions of G and S_1 commute. □

We finally have to prove the last claim of the theorem. By construction, the G-varieties $X'^{(i)}$ are of complexity one. They are moreover almost homogeneous if X is so. The claim follows from this observation and 3.3.1.4, 3.3.1.5. □

Remark 3.3.5.5. There exists a normal rational $\mathbb{G}_m$-variety of complexity one with only constant regular functions, such that the master Cox ring is not finitely generated. Consider indeed the two normal affine rational $\mathbb{G}_m$-surfaces defined by the $\mathbb{Q}$-divisors

$$D_1 := \tfrac{1}{3}[0] + \tfrac{1}{3}[1] + \tfrac{1}{3}[\infty], \text{ and } D_2 := -\tfrac{1}{3}[0] - \tfrac{1}{3}[1] - \tfrac{1}{3}[\infty].$$

on $\mathbb{P}^1_k$ (see [FZ]). It is easy to verify that these two surfaces glue together to form a normal rational $\mathbb{G}_m$-surface X with only constant regular functions. The exponent vectors defined by the rational quotient $X \dashrightarrow \mathbb{P}^1_k$ by $\mathbb{G}_m$ are $(3,3)$, $(3,3)$, $(3,3)$. Hence, $\tilde{X}$ is not rational in view of the criterion [ABHW, 5.8]. This implies that $\mathrm{Cox}(\tilde{X})$ is not finitely generated.

Remark 3.3.5.6. Let G be a connected reductive group, and X be a normal rational G-variety of complexity one satisfying the condition

$$(\mathcal{C}^G) : \mathcal{O}(\hat{X}_G^{(i)})^{*G} \simeq k^*, \text{ for } i \geqslant 0.$$

For instance, a normal rational G-variety of complexity one satisfies $(\mathcal{C}^G)$ if it is complete, or a quasicone, or almost homogeneous. The Theorem 3.3.5.2 has an obvious equivariant analogue in this setting and its proof adapts as well. More precisely, we obtain that X admits finite iteration of G-equivariant Cox rings, with well defined G-equivariant iterated Cox rings. The proof is in fact easier as the iterated equivariant characteristic spaces are canonically endowed with a G-action.

Chapter 4

Cox ring of almost homogeneous SL_2-threefolds

Abstract

We study Cox rings of normal threefolds on which SL_2 acts with a dense orbit. Exploiting the method of U-invariants, we obtain combinatorial criteria for the total coordinate space and the base variety to have log terminal singularities. Then, we investigate the iteration of Cox rings and obtain a bound on its length. Finally, we develop a general approach to the description of the Cox ring by generators and relations which is effective for normal SL_2/μ_n-embeddings.

4.1 Introduction

The *complexity* of the action of a connected reductive group is a very important birational invariant in studying the geometry of the action. This is the minimal codimension of an orbit of a Borel subgroup. The normal algebraic varieties of complexity zero are the *spherical varieties*, they constitute a natural generalization of toric varieties and have a well understood geometry [T11, Chap. 5]. The next step is the study of varieties of complexity one, which is not so well developed, except for the case of actions by tori [KKMS], [ABHW], [LS]. In general, two cases are possible for a normal variety of complexity one: either it is *almost homogeneous* (i.e. there exists a dense orbit), or it admits a one-parameter family of spherical orbits. The first examples of the former case consist of almost homogeneous SL_2-threefolds, that is, normal SL_2-varieties of dimension three with a dense orbit. Various aspects make this class of examples especially important, and one can view their study as a preliminary step toward a better understanding of general almost homogeneous varieties of complexity one. For example, these examples yield all homogeneous spaces of complexity and rank one via "parabolic induction", as shown by Panyushev in [Pa95].

An almost homogeneous SL_2-threefold X can be viewed as a *normal embedding* of a homogeneous space of the form SL_2/F where F is a finite subgroup. This means that the orbit morphism associated to a point in the dense orbit factors through a SL_2-equivariant open immersion $\mathrm{SL}_2/F \hookrightarrow X$. This point of view is interesting because one can take advantage of the combinatorial data defining the embedding to approach various questions. The setup of this combinatorial framework goes back to the seminal work of Luna and Vust [LV]. It has been considerably clarified in the complexity zero case by Brion, Knop, Luna, Vust, and by Timashev in the complexity one case [T97]. In fact Timashev's description doesn't restrict to almost homogeneous spaces but allows to classify varieties in any fixed equivariant birational class in terms of objects of convex geometry. The combinatorial description of normal SL_2/F-embeddings has been obtained by Luna and Vust [LV] when F is trivial, Moser-Jauslin [Mo] who extended the classification of Luna-Vust for arbitrary F, and Timashev [T97] who put these results in his framework.

In the present work, we mainly focus on *Cox rings* of normal SL_2/F-embeddings. The Cox ring of a normal variety X, denoted $\mathrm{Cox}(X)$, is an important invariant that encodes a lot of geometric information, see [ADHL] for a comprehensive reference on this rich subject. It is the ring of global sections of the *Cox sheaf*, which is, roughly speaking, the direct sum

$$\mathcal{R}_X := \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}(X)} \mathcal{F}$$

indexed by elements of the *class group* of X, i.e. the group of isomorphism classes of divisorial sheaves on X. Under mild assumptions, $\mathcal{R}_X$ can be endowed with a structure of quasi-coherent $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra, whence a structure of $\mathrm{Cl}(X)$-graded ring on $\mathrm{Cox}(X)$. When $\mathcal{R}_X$ is of finite type as an $\mathcal{O}_X$-algebra, its relative spectrum $\hat{X}$ is a normal $\mathbb{D}(\mathrm{Cl}(X))$-variety over X, where $\mathbb{D}(\mathrm{Cl}(X))$ is the diagonalizable group with character group $\mathrm{Cl}(X)$. This is called the *characteristic space* of X, and the structural morphism $\hat{X}\to X$ is a good quotient by $\mathbb{D}(\mathrm{Cl}(X))$. For a finitely generated Cox ring, its spectrum $\tilde{X}$ is called the *total coordinate space* of X, and the affinization morphism $\hat{X}\to\tilde{X}$ is a $\mathbb{D}(\mathrm{Cl}(X))$-equivariant open immersion whose image has a complement of codimension $\geqslant 2$ in $\tilde{X}$. When X is smooth, the characteristic space is a $\mathbb{D}(\mathrm{Pic}(X))$-torsor over X called the *universal torsor*.

Let X be a normal SL_2/F-embedding. In fact, we consider the *equivariant Cox ring* of X introduced and studied in [V1]. For the case of SL_2, it is canonically isomorphic to the ordinary Cox ring [V1, 2.3.4]. However, we take advantage of the structure of graded SL_2-algebra provided by the construction of the equivariant Cox ring. Also, we occasionally use general results from [V1]. In particular, $\mathrm{Cox}(X)$ is a finitely generated normal domain, and we use the description of the k-subalgebra of U-invariants $\mathrm{Cox}(X)^U$, where U is the unipotent part of the standard Borel subgroup B of SL_2. For the convenience of the reader, we recall in Section 4.2 useful facts on normal rational varieties of complexity one and their (equivariant) Cox ring. In Section 4.3, we describe the class group of X by generators and relations. Then, we provide a combinatorial criterion for X to have log terminal singularities, extending a previous result of Degtyarev [Deg].

The study of the Cox ring of X starts in Section 4.4.1. As an application of [V1, 3.4.3], we obtain a criterion of combinatorial nature for the total coordinate space $\tilde{X}$ to have log terminal singularities. This is an interesting question, particularly since the work of Gongyo, Okawa, Sannai and Takagi characterizing varieties of Fano type via singularities of Cox rings [GOST].

In Section 4.4.2, we put to light some interesting new phenomena with regard to the *special fiber*, i.e. the schematic fiber at zero of the quotient morphism

$$\tilde{X} \xrightarrow{//\mathrm{SL}_2} \mathbb{A}^N_k,$$

where N is the number of SL_2-invariant prime divisors in X [V1, 2.8.5]. The general fibers of this morphism are isomorphic to the total coordinate space of the dense orbit [V1, 2.8.1]. For a spherical variety, the quotient morphism is faithfully flat, and the special fiber is a normal affine variety which is moreover *horospherical*, in the sense that the product of two irreducible representations, say V_λ, V_μ, in its coordinate algebra is their Cartan product $V_{\lambda+\mu}$ [Bri07, 3.2.3]. These nice geometric properties are important ingredients for the determination of a presentation of the Cox ring of a spherical variety in loc. cit. We show by examples that these properties do not extend to the complexity one world. Nevertheless, we give a criterion for the special fiber to be a normal variety, in terms of the basic geometry of X.

Pursuing our investigation, we turn to the study of the *iteration of Cox rings* for X in Sections 4.4.3 and 4.4.4. Roughly, iteration of Cox rings consists in studying the sequence of total coordinate spaces

$$\ldots\to\tilde{X}^{(n)}\to\tilde{X}^{(n-1)}\to\ldots\to\tilde{X}^{(2)}\to\tilde{X},$$

where $\tilde{X}^{(n)}$ denotes the total coordinate space of $\tilde{X}^{(n-1)}$. A basic question is whether this sequence is finite, in which case X is said to have *finite iteration of Cox rings*, and the last obtained Cox ring is called *the master Cox ring*. By virtue of [V1, 3.5.1], X has finite iteration of Cox rings with a factorial master Cox ring. More precisely, we obtain

Proposition. *The length of the iteration of Cox rings sequence of X is bounded by* 4.

See the statement 4.4.4.13 where more precise bounds are given depending on the finite subgroup $F \subset \mathrm{SL}_2$. Also, we obtain a commutative diagram

$$\begin{tikzcd}
\tilde{X}^{(m)} \arrow[r] \arrow[d] & \dots \arrow[r] & \tilde{X}^{(2)} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
\tilde{Y}^{(m)} \arrow[r] & \dots \arrow[r] & \tilde{Y}^{(2)} \arrow[r] & \tilde{Y},
\end{tikzcd}$$

where the horizontal arrows are structural morphisms of characteristic spaces, the vertical arrows are *almost principal U-bundles*, and all squares are cartesian. Moreover, $\tilde{X}^{(m)}, \tilde{Y}^{(m)}$ are factorial and $\tilde{Y}$ is the total coordinate space of a certain variety of complexity one under the standard maximal torus of SL_2.

Finally, we develop in Section 4.4.5 an approach for the description of the Cox ring by generators and relations. In the same way as it is important for projective geometry to have explicit homogeneous coordinates, it is an important problem to find an explicit description by generators and relations of the Cox ring. Moreover, the shape of the equations defining the total coordinate space turns out to have important arithmetic consequences for the base variety, see for example the recent preprint [BBDG] related to the Manin-Peyre conjecture. Our approach is effective for normal SL_2/μ_n-embeddings. This eventually leads us to compare our treatment with previous work by Batyrev and Haddad in the particular case of affine almost homogeneous SL_2-threefolds [BatHad].

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Conventions. Let k denote an algebraically closed base field of characteristic zero. In this text, we work in the category of *algebraic schemes*, that is, separated k-schemes of finite type. Hence, a morphism $(f, f^\sharp) : X \to Y$ of algebraic schemes is meant to be a k-morphism. A *variety* is an integral algebraic scheme. Without further precision, a *point* of a variety is meant to be a closed point. The sheaf of *regular functions* (or structure sheaf) on an algebraic scheme X is denoted $\mathcal{O}_X$. A *subvariety* of a variety X is a locally closed subset equipped with its reduced scheme structure. The sheaf of units associated to $\mathcal{O}_X$ is denoted $\mathcal{O}_X^*$.

In this text, an *algebraic group* is an affine algebraic group scheme. The character group of an algebraic group G is denoted $\hat{G}$ or $X^*(G)$. Given a finitely generated abelian group M, we let $\mathbb{D}(M)$ denote the *diagonalizable (algebraic) group* $\mathrm{Spec}(k[M])$ with character group M. Without further precision, the letter $\mathbb{T}$ denotes an arbitrary torus. An *almost homogeneous variety* is a normal variety on which a connected algebraic group acts with a dense orbit.

Let G be an algebraic group, X an algebraic G-scheme, and $q : X \to Y$ a G-invariant morphism. We say that q is a *G-torsor* (or *principal G-bundle*) over Y if q is faithfully flat, and the natural morphism $G \times X \to X \times_Y X$ is an isomorphism. As algebraic groups are smooth in characteristic zero, q is faithfully flat if and only if it is smooth and surjective. We say that q is a *trivial G-torsor* over Y if there is a G-equivariant isomorphism $X \simeq G \times Y$ over Y, where G acts on $G \times Y$ via left multiplication on the first factor. A fact that will be used implicitly many times in the text is that a torsor under a torus or a unipotent algebraic group is locally trivial in the Zariski topology.

Let G be an algebraic group and X an affine algebraic G-scheme. If $\mathcal{O}(X)^G$ is a finitely generated k-algebra, then the affine algebraic scheme $X//G := \operatorname{Spec}(\mathcal{O}(X)^G)$ is called the categorical quotient of X by G. It is the universal object in the category of G-invariant morphisms from X to affine algebraic schemes.

Our convention is that a *reductive group* is a *linearly reductive group*, that is, every finite dimensional representation of such group is semisimple. In particular, a reductive group is not necessarily connected. Let G be a reductive group and X an algebraic G-scheme. A *good quotient* of X by G is an affine G-invariant morphism $q : X \to Y$ such that $q^\sharp$ induces an isomorphism $\mathcal{O}_Y \to (q_*\mathcal{O}_X)^G$. It is a universal object in the category of G-invariant morphisms from X to algebraic schemes.

Let X be a normal variety, the *group of Weil divisors* is denoted $\operatorname{WDiv}(X)$. Every Weil divisor defines a coherent sheaf $\mathcal{O}_X(D)$ whose non-zero sections over an open subset U are rational functions $f \in k(X)^*$ such that $\operatorname{div}(f) + D$ defines an effective divisor on U. A *divisorial sheaf* on X is a coherent reflexive sheaf of rank one. The *class group* (resp. *Picard group*) of X, denoted $\operatorname{Cl}(X)$ (resp. $\operatorname{Pic}(X)$), is the group of isomorphism classes of divisorial sheaves (resp. invertible sheaves) on X. It is isomorphic to the group of Weil divisors (resp. Cartier divisors) modulo linear equivalence through the morphism $[D] \mapsto [\mathcal{O}_X(D)]$. Let G be an algebraic group acting on a normal variety X. The *equivariant class group* (resp. *equivariant Picard group*) of a normal G-variety X is denoted $\operatorname{Cl}_G(X)$ (resp. $\operatorname{Pic}_G(X)$), and the equivariant Cox ring is denoted $\operatorname{Cox}_G(X)$ (see [V1, Sec. 2.2 and 2.3]). A *pointed normal variety* (X, x) consists of a normal variety X and a smooth point $x \in X$. For each class $[\mathcal{F}] \in \operatorname{Cl}_G(X)$, there exists a canonical representative $\mathcal{F}^x$ called the *rigidified G-linearized divisorial sheaf* associated to $[\mathcal{F}]$ (see [V1, Sec. 2.3]).

The *canonical sheaf* ω_X on a normal variety X is the pushforward on X of the sheaf of differental forms of maximal degree on the smooth locus X_{sm}. It is a divisorial sheaf, and any Weil divisor K_X such that $\mathcal{O}_X(K_X) \simeq \omega_X$ is a *canonical divisor*. A $\mathbb{Q}$*-Gorenstein variety* is a normal variety such that some non-zero power of the canonical sheaf is invertible. A *Gorenstein variety* is a normal Cohen-Macaulay variety whose canonical sheaf is invertible.

Let X be a normal variety. We say that X has *rational singularities* if there exists a proper birational morphism

$$\varphi : Z \to X,$$

where Z is a smooth variety (a resolution of singularities), and such that $R^i\varphi_*\mathcal{O}_Z = 0$, $\forall i > 0$. This last property doesn't depend on the choice of a resolution. We say that X has *log terminal singularities* if the following conditions are satisfied:

- X is $\mathbb{Q}$-Gorenstein.
- There is a resolution of singularities $\varphi : Z \to X$ such that

  $$K_Z = \varphi^* K_X + \sum \alpha_i E_i,\ \alpha_i > -1,$$

  where the sum runs over the exceptional divisors E_i of φ.

4.2 Normal rational varieties of complexity one

Let G be a connected reductive group, and (X, x) be a pointed normal G-variety of complexity one. Suppose moreover that X is a rational variety, so that its Cox ring is finitely generated ([V1, 3.1.4]; this is verified e.g. when X is almost homogeneous). Fix a Borel subgroup B, a maximal torus T in B, and denote by U the unipotent part of B. Suppose that G has trivial Picard group, so that any divisorial sheaf on X admits a G-linearization (see [V1, 2.2.2]; this can always be achieved by replacing G with a finite cover). In this section, we recall facts on the geometry of X following [T11], [Pon], and [V1].

4.2.1 B-stable divisors

By a theorem of Rosenlicht [T11, 5.1] there is a rational quotient

$$\pi : X \dashrightarrow \mathbb{P}^1_k$$

by B. Thus, general B-orbits determine a one-parameter family of B-stable prime divisors in X. This rational map is defined by two global sections a, b of a rigidified G-linearized divisorial sheaf $\mathcal{F}^x$ on X [V1, Sec. 3.2]. The pullback of Weil divisors on $\mathbb{P}^1_k$ corresponds to the usual pullback of Cartier divisors and is given by

$$\pi^* : \mathrm{WDiv}(\mathbb{P}^1_k) \to \mathrm{WDiv}(X), p = [\alpha : \beta] \mapsto \mathrm{div}(\beta a - \alpha b),$$

where $[\alpha : \beta]$ are homogeneous coordinates of $p \in \mathbb{P}^1_k$. All the B-stable prime divisors in X but a finite number lie in the image of π^* [V1, Sec. 3.1].

Definition 4.2.1.1. [Pon, 3] The prime divisors in X lying in the image of π^* are the *parametric divisors*. The finite set of B-stable prime divisors that are not parametric is the set of *exceptional divisors*. For an exceptional divisor E, the image of $\pi_{|E}$ is either dense or a point in $\mathbb{P}^1_k$. In the former case, we say that E *dominates* $\mathbb{P}^1_k$, in the latter case, the image point is called *exceptional*.

4.2.2 The algebras $\mathrm{Cox}_G(X)^U$ and $\mathrm{Cox}(X)^U$

Suppose that $\mathcal{O}(X)^* \simeq k^*$, so that both $\mathrm{Cox}_G(X)$ and $\mathrm{Cox}(X)$ are well defined and finitely generated [V1, 3.1.4]. We recall the description of the k-algebra $\mathrm{Cox}_G(X)^U$ of U-invariants obtained by Ponomareva in [Pon, Thm. 4] and generalized in [V1, 3.2.3]. Also, considering the canonical structure of U-algebra on $\mathrm{Cox}(X)$, we recall an interpretation of $\mathrm{Cox}(X)^U$ as the Cox ring of a complexity one T-variety [V1, 3.3.2].

Notation 4.2.2.1. Let $(x_i)_{i\in I}$ be the finite family of exceptional points with respective homogeneous coordinates $[\alpha_i : \beta_i]$. For all $i \in I$, let $(E_j^{x_i})_j$ be the finite family of exceptional divisors that are sent to x_i by π. Let $(E_k)_k$ be the finite family of exceptional divisors dominating $\mathbb{P}^1_k$. Equip $\mathcal{O}(E_j^{x_i})^x$ (resp. $\mathcal{O}(E_k)^x$) with arbitrary G-linearizations, and let s_{ij} (resp. s_k) denote the canonical sections of these sheaves associated with the divisors $E_j^{x_i}$ (resp E_k), and let h_{ij} denote the (integral) coefficient of $E_j^{x_i}$ in the divisor $\pi^*(x_i)$.

Theorem 4.2.2.2. *[V1, 3.2.3] The k-algebra $\mathrm{Cox}_G(X)^U$ is generated as a $k[\hat{G}]$-algebra by the elements $a, b, (s_{ij})_{ij}, (s_k)_k$. The ideal of relations contains the following identities*

$$\beta_i a - \alpha_i b = \lambda_i \textstyle\prod_j s_{ij}^{h_{ij}},$$

where for all $i \in I$, λ_i is a certain character of G. If moreover, the condition

($\star$) *the common degree of the sections a and b is $\mathbb{Z}$-torsion free in $\mathrm{Cl}_G(X) \times \hat{T}$*

is satisfied, then the above relations generate the whole ideal.

Remark 4.2.2.3. [V1, 3.2.4] Examples of situations where the condition ($\star$) is satisfied are given by

- rational normal $\mathbb{T}$-varieties of complexity one such that $\mathcal{O}(X)^{\mathbb{T}} \simeq k$.

- almost homogeneous varieties of complexity one.

In [ADHL, 3.4.2], are studied the coordinate algebras of certain *trinomial varieties*, that is, affine varieties which are the intersection in an affine space of hypersurfaces defined by trinomial equations. These algebras are obtained via a construction taking in input certain matrices A and P_0 storing the coefficients and exponents of the trinomials. Their spectrum defines normal affine varieties of complexity one under the action of (the connected component) of a diagonalizable group, and such that the invariant regular functions are constant. Moreover, these algebras turn out to be Cox rings of varieties of complexity one under the action of a torus [ADHL, 3.4.3].

Construction 4.2.2.4. [ADHL, 3.4.2.1] Fix integers $r \in \mathbb{Z}_{\geqslant 1}$, $m \in \mathbb{Z}_{\geqslant 0}$, a sequence of integers $n_0, ..., n_r \in \mathbb{Z}_{\geqslant 1}$, and let $n := n_0 + ... + n_r$. Consider as inputs

- A matrix $A := [a_0, ..., a_r]$ with pairwise linearly independent column vectors $a_0, ..., a_r \in k^2$.
- An $r \times (n+m)$ block matrix $P_0 := [L \;\; 0_{r,m}]$, where L is an $r \times n$ matrix built from the n_i-tuples $l_i := (l_{i1}, ..., l_{in_i}) \in \mathbb{Z}^{n_i}_{\geqslant 1}$, $0 \leqslant i \leqslant r$, called *exponent vectors*, as below

$$L = \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}$$

Now consider the polynomial algebra $k[T_{ij}, S_k]$, where $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n_i$, and $1 \leqslant k \leqslant m$. For every $0 \leqslant i \leqslant r$, define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} ... T_{in_i}^{l_{in_i}},$$

whence the name "exponent vector". Denote $\mathscr{I}$ the set of triples (i_1, i_2, i_3) with $0 \leqslant i_1 < i_2 < i_3 \leqslant r$, and for all $I \in \mathscr{I}$, consider the trinomial

$$g_I := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

We introduce a grading on $k[T_{ij}, S_k]$ by the abelian group $K_0 := \mathbb{Z}^{n+m} / \operatorname{Im}({}^tP_0)$, where ${}^tP_0$ is the transpose of P_0. Let $Q_0 : \mathbb{Z}^{n+m} \to K_0$ be the projection, and set

$$\deg T_{ij} := Q_0(e_{ij}), \text{ and } \deg S_k := Q_0(e_k),$$

where (e_{ij}, e_k) is the standard basis of $\mathbb{Z}^{n+m}$. Finally consider the K_0-graded k-algebra

$$R(A, P_0) := k[T_{ij}, S_k]/(g_I)_{I \in \mathscr{I}}.$$

Proposition 4.2.2.5. *[V1, 3.3.2] Suppose that the condition* $(\star)$ *of 4.2.2.2 is satisfied. Then, the* k*-algebra* $\operatorname{Cox}(X)^U$ *is isomorphic to an algebra* $R(A, P_0)$ *constructed as in 4.2.2.4.*

Remark 4.2.2.6. By [V1, 3.3.2], the input data to be used in Construction 4.2.2.4 in order to obtain $\operatorname{Cox}(X)^U$ can interpreted geometrically:

- m is the number of exceptional divisors in X dominating $\mathbb{P}^1_k$,
- $a_0, ..., a_r$ are homogemeous coordinates on $\mathbb{P}^1_k$ of the exceptional points $x_0, ..., x_r$.
- the exponent vectors are the vectors formed by the multiplicities of the exceptional divisors in the pullbacks $\pi^*(x_i)$, $i = 0, ..., r$.

Remark 4.2.2.7. Consider the spectrum $\tilde{Y}$ of an algebra $R(A, P_0)$ viewed as the total coordinate space of a normal rational variety Y of complexity one under a torus action. The geometries of $\tilde{Y}$ and Y highly depend on the exponent vectors l_i, $i = 0, ..., r$, involved in the equations. In [ABHW], the study of singularities on Y leads to the following notion that will be used later: a ring $R(A, P_0)$ is a *Platonic ring* either if $r \leqslant 1$, or if every tuple $(l_{0i_0}, ..., l_{ri_r})$ is *Platonic*, i.e. after ordering it decreasingly, the first triple is one of the *Platonic triples*

$$(5, 3, 2), (4, 3, 2), (3, 3, 2), (x, 2, 2), (x, y, 1),\ x \geqslant y \geqslant 1,$$

and the remaining integers of the tuple equal one.

4.2.3 Combinatorial classification

In this section, we present the combinatorial framework for the classification of complexity one normal rational G-varieties lying in a fixed G-birational class. Our reference for this material is [T11]. Let K be the field of rational functions of X. Recall that a *geometric valuation* of K is a discrete valuation $K^* \to \mathbb{Q}$ of the form αv_D, where $\alpha \in \mathbb{Q}_+$, and v_D is the normalized discrete valuation associated to a prime divisor D in a normal variety Z whose field of fractions is identified with K.

Notation 4.2.3.1. We set $\mathcal{D} = \mathcal{D}(K)$ for the set of prime divisors of X that are not G-stable, and $\mathcal{D}^B$ the subset of B-stable ones. This last subset consists of the so-called *colors* of K. These sets don't depend on X up to G-birational equivalence and we identify them with the corresponding sets of normalized geometric valuations of K. Let $\mathcal{V}$ denote the set of G-valuations of K, that is, the geometric G-invariant valuations of K. Let $\mathcal{V}(X)$ denote the subset of normalized G-valuations corresponding to G-stable prime divisors of X. The element of $\mathcal{V}(X)$ corresponding to a G-stable prime divisor D is denoted v_D.

There is an exact sequence of abelian groups

$$1 \to (K^B)^* \simeq k(\mathbb{P}^1_k)^* \to K^{(B)} \to \Lambda(X) \to 1,$$

where $K^{(B)}$ is the multiplicative group of rational B-semi-invariant functions and $\Lambda(X)$ is the associated group of weights. The abelian group $\Lambda(X)$ is the *weight lattice* of the G-variety X. It is a subgroup of $\hat{T}$ whose rank is the *rank* of the G-variety X. After choosing a splitting $\lambda \mapsto f_\lambda$ of the above sequence, we view $\Lambda(X)$ as a submodule of $K^{(B)}$. Then, considering a geometric valuation v of K, the restriction $v_{|K^{(B)}}$ is determined by a triple (x, h, l), where $x \in \mathbb{P}^1_k$, $h \in \mathbb{Q}_+$, and $l \in \mathcal{E} := \mathrm{Hom}(\Lambda(X), \mathbb{Q})$. Indeed, the restriction of v to $k(\mathbb{P}^1_k)$ is again a geometric valuation [T11, B.8], hence of the form hv_x, where $x \in \mathbb{P}^1_k$. On the other hand, $v_{|\Lambda(X)}$ yields an element $l \in \mathcal{E}$.

Definition 4.2.3.2. For all $x \in \mathbb{P}^1_k$, consider the closed half-space $\mathcal{E}_x = \mathbb{Q}_+ \times \mathcal{E}$. The *hyperspace* $\breve{\mathcal{E}}$ associated to K is the union of the $\mathcal{E}_x$ glued together along $\mathcal{E}$.

The set $\mathcal{V}$ embeds in $\breve{\mathcal{E}}$, and $\mathcal{V}_x := \mathcal{V} \cap \mathcal{E}_x$ is a simplicial convex polyhedral cone for all $x \in \mathbb{P}^1_k$. Also, the natural map $\varrho : \mathcal{D}^B \to \breve{\mathcal{E}}$ is not injective in general. Notice that from the preceding section, all but a finite number of B-stable prime divisors are sent in $\breve{\mathcal{E}}$ to a vector of the form $\varepsilon_x := (x, 1, 0) \in \mathcal{E}_x$ for a certain $x \in \mathbb{P}^1_k$.

Definition 4.2.3.3. The pair $(\mathcal{V}, \mathcal{D}^B)$ is the *colored equipment* of K. We say that $(\breve{\mathcal{E}}, \mathcal{V}, \mathcal{D}^B, \varrho)$ is the *colored hyperspace* of K.

Definition 4.2.3.4. A *cone* in $\breve{\mathcal{E}}$ is a cone in some $\mathcal{E}_x$, $x \in \mathbb{P}^1_k$. A *hypercone* in $\breve{\mathcal{E}}$ is a union $\mathcal{C} = \cup_{x \in \mathbb{P}^1_k} \mathcal{C}_x$ of convex cones, each generated by a finite number of vectors, and such that

1. $\mathcal{C}_x = \mathcal{K} + \mathbb{Q}_+\varepsilon_x$ for all $x \in \mathbb{P}^1_k$ but a finite number, where $\mathcal{K} := \mathcal{C} \cap \mathcal{E}$.

2. One of the following cases occurs

   (A) $\exists x \in \mathbb{P}^1_k$, $\mathcal{C}_x = \mathcal{K}$.

   (B) $\emptyset \neq \mathcal{B} := \sum \mathcal{B}_x \subset \mathcal{K}$, where $\varepsilon_x + \mathcal{B}_x = \mathcal{C}_x \cap (\varepsilon_x + \mathcal{E})$.

The hypercone is said of *type* (A) (resp. (B)) depending on the alternative of condition 2. The hypercone is said *strictly convex* if each $\mathcal{C}_x$ is, and $0 \notin \mathcal{B}$.

Definition 4.2.3.5. Let $\mathcal{Q} \subset \breve{\mathcal{E}}$ a subset all of whose elements but a finite number are of the form ε_x for some $x \in \mathbb{P}^1_k$. Let $\varepsilon_x + \mathcal{P}_x$ the convex hull of intersection points of the half-lines $\mathbb{Q}_+q, q \in \mathcal{Q}$ with $\varepsilon_x + \mathcal{E}$. We say that the hypercone $\mathcal{C} = \mathcal{C}(\mathcal{Q})$, where the $\mathcal{C}_x$ are generated by $\mathcal{Q} \cap \mathcal{E}_x$ and $\mathcal{P} := \sum \mathcal{P}_x$, is *generated* by $\mathcal{Q}$.

The fundamental result is that strictly convex hypercones $(\mathcal{C}, \mathcal{R}) := \mathcal{C}(\mathcal{W} \cup \varrho(\mathcal{R}))$, where $\mathcal{W} \subset \mathcal{V}$, $\mathcal{R} \subset \mathcal{D}^B$, and $0 \notin \varrho(\mathcal{R})$, classify affine B-stable open subvarieties of normal G-models of K (the so-called *B-charts*). These hypercones are called *colored hypercones*.

Remark 4.2.3.6. Let X_0 be a B-chart defined by a colored hypercone $(\mathcal{C}, \mathcal{R})$. Then $\mathcal{O}(X_0)^B \simeq k$ if and only if $(\mathcal{C}, \mathcal{R})$ is of type (B).

A finite set of colored hypercones $(\mathcal{C}, \mathcal{R})$ defines B-charts $\mathring{X}_i$ and G-models $X_i := G\mathring{X}_i$. These G-models can be glued together into a G-model if and only if these colored hypercones defines a *colored hyperfan*. In turn, the colored hyperfans classify normal G-models of K. To make this last notion precise we need the notion of a *hyperface* of a hypercone, which is defined through the notion of a *linear functional* on the hyperspace. We can think of the abelian group $K^{(B)}$ as the dual object to the hyperspace. Indeed, every $f = f_0 f_\lambda \in K^{(B)}$ defines a so-called linear functional on $\breve{\mathcal{E}}$, namely the restriction to each $\mathcal{E}_x$ is a $\mathbb{Q}$-linear form $((h, \gamma) \mapsto hv_x(f_0) + \gamma(\lambda))$. Conversely, considering a linear functional on $\breve{\mathcal{E}}$, a multiple of it is given by a rational B-semi-invariant function.

Definition 4.2.3.7. A *face* of a hypercone $\mathcal{C}$ is a face $\mathcal{C}'$ of a certain cone $\mathcal{C}_x$ such that $\mathcal{C}' \cap \mathcal{B} = \emptyset$. A *hyperface* of $\mathcal{C}$ is a hypercone $\mathcal{C}' = \mathcal{C} \cap \ker \varphi$, where φ is a linear functional on $\mathcal{C}$ such that $\varphi(\mathcal{C}) \geqslant 0$. If $\mathcal{C}$ is of type B, its *interior* is by definition $\operatorname{int} \mathcal{C} := \cup_{x \in \mathbb{P}^1_k} \operatorname{int} \mathcal{C}_x \cup \operatorname{int} \mathcal{K}$.

Definition 4.2.3.8. A colored hypercone $(\mathcal{C}, \mathcal{R})$ of type (B) is *supported* if $\operatorname{int} \mathcal{C} \cap \mathcal{V} \neq \emptyset$. A *(hyper)face* of $(\mathcal{C}, \mathcal{R})$ is a colored (hyper)cone $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{C}'$ is a (hyper)face of $\mathcal{C}$, and $\mathcal{R}' = \mathcal{R} \cap \varrho^{-1}(\mathcal{C}')$. A *colored hyperfan* is a set of supported colored cones and hypercones of type (B) whose interiors are disjoint inside $\mathcal{V}$, and which is obtained as the set of all supported colored (hyper)faces of finitely many colored hypercones.

An interesting feature of this classification framework is that the lattice of G-stable subvarieties of X can be read from the combinatorial representation of X as a colored hyperfan. Indeed, consider a G-stable subvariety $Y \subset X$, and a B-chart $\mathring{X}$ intersecting Y which is defined by a hypercone $\mathcal{C}(\mathcal{W} \cup \varrho(\mathcal{R}))$. Denote $\mathcal{V}_Y \subset \mathcal{W}$, $\mathcal{D}^B_Y \subset \mathcal{R}$ the respective subsets which correspond to the B-stable prime divisors in $\mathring{X}$ whose closure in X contains Y, and let $\mathcal{C}_Y$ be the (hyper)cone spanned by $\mathcal{V}_Y \cup \varrho(\mathcal{D}^B_Y)$. Then for any other G-stable subvariety Z, we have $Y \subset \overline{Z}$ if and only if $(\mathcal{C}_Z, \mathcal{D}^B_Z)$ is a (hyper)face of $(\mathcal{C}_Y, \mathcal{D}^B_Y)$.

4.2.4 Structure of B-charts

The local structure theorem of Brion, Luna and Vust [BLV, Thm 1.4] is a very useful tool for the study of varieties with group action, the following variant is due to Knop.

Theorem 4.2.4.1. *[Kn93b, 1.2] Let G a connected reductive group, X a G-variety equipped with an ample G-linearized invertible sheaf $\mathcal{L}$, and $s \in \mathcal{L}(X)^{(B)}$ a B-semi-invariant section. Let P the parabolic subgroup stabilizing s in $\mathbb{P}(\mathcal{L}(X))$, with Levi decomposition $P_u \rtimes L$, $T \subset L$. Then the open subvariety X_s is P-stable, and there exists a closed L-stable subvariety $Y \subset X_s$ such that the morphism induced by the P-action*

$$P_u \times Y \to X_s$$

is a P-equivariant isomorphism.

Consider a B-chart $V \subset X$, the G-variety $X_0 := GV$, and the B-stable effective divisor $D := X_0 \setminus V$. By [T00, Lemma 2], $\mathcal{O}_{X_0}(D)$ is an ample (G-linearized) line bundle on X_0. The canonical section associated with D is B-semi-invariant. Thus, its projective stabilizer P is a parabolic subgroup containing B. Then, the above theorem yields a P-equivariant isomorphism

$$P_u \times Y \to V.$$

Corollary 4.2.4.2. *Suppose that $G = \mathrm{SL}_2$ and V is not G-stable. Then, there is a B-equivariant isomorphism*

$$U \times Y \to V,$$

where Y is a normal affine T-variety of complexity one.

4.3 Almost homogenous SL_2-threefolds

4.3.1 Generalities

Consider the algebraic group $G := \mathrm{SL}_2$ identified with its group of rational points, namely the matrices

$$g = \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix}$$

such that $g_i \in k$ and $\det(g) = 1$. By abuse of notation, we also let g_i denote the corresponding matrix coordinates on G. Consider also the Borel subgroup B whose elements are upper triangular matrices, and the maximal torus $T \subset B$ whose elements are diagonal matrices. The character group of T is free of rank one generated by

$$\omega := (\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \mapsto t).$$

In the sequel, we fix a normal embedding $i : G/F \hookrightarrow X$, where F is a finite subgroup of G. This means that there exists a point x in the G-variety X such that the associated orbit morphism factors through a G-equivariant open immersion $i : G/F \hookrightarrow X$. In the sequel, we identify G/F with this dense orbit. It is well known that F is conjugate either to a cyclic subgroup $\mu_n \subset T$ of order $n \geqslant 1$, or to one of the famous binary polyhedral groups [Sp, 4.4]:

- $F_{\mathbb{D}_n}$:= Binary dihedral group of order $4n$, $n > 1$,
- $F_{\mathbb{T}}$:= Binary tetrahedral group,
- $F_{\mathbb{O}}$:= Binary octahedral group,
- $F_{\mathbb{I}}$:= Binary icosahedral group.

The pointed normal G-variety $(X, x := i(F/F))$ is rational of complexity one, and the rational quotient $\pi : X \dashrightarrow \mathbb{P}^1_k$ by B induces a geometric quotient $\pi_{|G/F} : G/F \to \mathbb{P}^1_k$. The situation is summarized in the commutative diagram

$$\begin{tikzcd} G \arrow[r, "B\backslash"] \arrow[d, "/F"] & B\backslash G \simeq \mathbb{P}^1_k \arrow[d, "/F"] \\ G/F \arrow[r, "B\backslash"] \arrow[d, hook, "i"] & \mathbb{P}^1_k/F \simeq \mathbb{P}^1_k \\ X \arrow[ur, dashed, "\pi"] & \end{tikzcd}$$

where B acts on G/F by left multiplication. Denote $\mathcal{F}^x$ the unique G-linearized rigidified divisorial sheaf associated to $[\pi^*\mathcal{O}_{\mathbb{P}^1_k}(1)]$, and $a, b \in \mathcal{F}^x(X)^{(B)}$ the two global sections corresponding to the pullbacks of (a choice) of coordinates on $\mathbb{P}^1_k$. Also, let $n_0\omega$ be the common B-weight of a, b.

The colored equipment associated to $k(G/F)$ has been described by Timashev in [T97, Sec. 5] for the various finite subgroups. For convenience, we recall the associated basic facts in Appendix 4.5. When $F = \mu_n, n \geqslant 3$, the morphism $\pi_{|G/F}$ defines exactly two exceptional colors E^{x_0}, E^{x_∞} corresponding to the two μ_n-fixed points x_0, x_∞ of $\mathbb{P}^1_k$. When F is binary polyhedral, $\pi_{|G/F}$ defines exactly three exceptional colors E^{x_v}, E^{x_e}, and E^{x_f}. The subscripts are aimed to suggest (when $k = \mathbb{C}$) vertices, edges and faces of the corresponding Platonic solids inscribed in $\mathbb{P}^1_k$ identified with the Riemann sphere. These exceptional colors correspond exactly to the degenerate F-orbits in $\mathbb{P}^1_k$ (see [Sp, 4.4]). The homogeneous space G/F being affine, the complement of G/F in X is a finite union of G-stable prime divisors and every G-stable prime divisor lies in this complement. By Section 4.2.3, there is at most one G-stable prime divisor X^∞ in X dominating $\mathbb{P}^1_k$. It consists of infinitely many G-orbits, all isomorphic to $\mathbb{P}^1_k$.

4.3.2 Class group of X

Because the divisor class group of X is the grading group of the Cox ring, a good understanding of its structure is a preliminary to the approach of the structure of $\mathrm{Cox}(X)$. In this section, we give a description of $\mathrm{Cl}(X)$ by generators and relations. There are two classes of embeddings where this description is easy:

Proposition 4.3.2.1. *Suppose that $F = F_{\mathbb{I}}$ or is trivial. Then, $\mathrm{Cl}(X)$ is freely generated by the classes of the G-stable prime divisors.*

Proof. Let $(D_i)_i$ denote the family of G-stable prime divisors in X. In our situation, the localization exact sequence [V1, 2.2.4] reads

$$0 \to \bigoplus_{(D)_i} \mathbb{Z}D_i \to \mathrm{Cl}_G(X) \to \mathrm{Pic}_G(G/F) \to 0.$$

Using the exact sequence [V1, 2.2 (1)], the remark [V1, 2.2.2], and the fact that G is a semisimple simply connected algebraic group, we obtain an isomorphism $\mathrm{Cl}_G(X) \simeq \mathrm{Cl}(X)$. Also, there is an isomorphism $\mathrm{Pic}_G(G/F) \simeq \hat{F}$ [ADHL, 4.5.1.2]. Finally, it suffices to remark that $\hat{F}$ is trivial (4.5.4). □

For the description of $\mathrm{Cl}(X)$ in the general case, the ideas in [Bri89] apply well (see also [PS] for a similar approach in the context of $\mathbb{T}$-varieties of complexity one). The fundamental remark is that in virtue of the G-module structure on the spaces of sections of divisorial sheaves on X, every Weil divisor is linearly equivalent to a B-stable one. From this we get an isomorphism

$$\mathrm{Cl}(X) \simeq \frac{\bigoplus_{D \in \mathcal{D}^B \sqcup \mathcal{V}(X)} \mathbb{Z}D}{\mathrm{PDiv}(X)^B},$$

where $\mathrm{PDiv}(X)^B$ is the group of B-invariant principal divisors. These divisors are the divisors of B-semi-invariant rational functions. After the choice of a section

$$\Lambda(X) = \alpha_0\mathbb{Z}\omega \to k(X)^{(B)},\ k\alpha_0\omega \mapsto f^k_{\alpha_0\omega}$$

of the exact sequence

$$1 \to k(\mathbb{P}^1_k)^* \to k(X)^{(B)} \to \Lambda(X) \to 1,$$

every such semi-invariant rational function can be written uniquely as a product $gf^k_{\alpha_0\omega}$ for a certain $g \in k(\mathbb{P}^1_k)^*$, and $k \in \mathbb{Z}$. The divisor of this function is then

$$\mathrm{div}(gf^k_{\alpha_0\omega}) = \sum_{z\in\mathbb{P}^1_k} v_z(g)\pi^*(z) + \sum_{D\in\mathcal{D}^B\sqcup\mathcal{V}(X)} kl_D D, \tag{4.3.1}$$

where $l_D := v_D(f_{\alpha_0\omega})$ is the coordinate on $\mathcal{E} \simeq \mathbb{Q}$ of v_D seen in $\breve{\mathcal{E}}$, and π^* is the pullback of Weil divisors associated to the rational quotient $\pi : X \dashrightarrow \mathbb{P}^1_k$ by B.

Proposition 4.3.2.2. *The class group of X is generated by the classes of exceptional divisors, and the classes of parametric divisors whose projection on $\mathcal{E}$ is non-zero. Moreover, after choosing an exceptional point $x_0 \in \mathbb{P}^1_k$, the relations are generated by*

$$\begin{cases} [\pi^*(x_0)] = [\pi^*(x)], \forall x \in \mathbb{P}^1_k \textit{ exceptional},\\ \sum_{D\in\mathcal{D}^B\sqcup\mathcal{V}(X)} l_D[D] = 0. \end{cases}$$

Proof. For any parametric divisor Z, and exceptional point x, we have $[Z] = [\pi^*(x)]$ in $\mathrm{Cl}(X)$. Also, $\pi^*(x)$ is a linear combination of exceptional divisors. It follows that $\mathrm{Cl}(X)$ is indeed generated by the elements listed in the statement. By the general form (4.3.1) of a principal divisor, every relation between these generators is a $\mathbb{Z}$-linear combination of the relations of the statement. □

4.3.3 Singularities of X

In this section, we give a combinatorial criterion for the singularities of X to be log terminal. In fact we extend a previous result from Degtyarev for the case of a normal G-embedding [Deg, Thm 1]. Our method is however quite different, namely we reduce to studying singularities of normal rational affine T-varieties of complexity one using the particular structure of B-charts. This allows us to take advantage of a previous work of Liendo and Süss [LS], and of the description of the T-equivariant Cox ring for these varieties.

Consider a G-orbit $\mathcal{O}$ in X. To study the singularities along $\mathcal{O}$, it suffices to consider a B-chart $X_{\mathcal{O}}$ intersecting $\mathcal{O}$ because the translates by elements of G of this chart cover $\mathcal{O}$ and are isomorphic. We examine the different types of orbits case by case, and use the classification and terminology for orbits by Luna and Vust in [LV, Section 9], that is, we consider orbits of type A_l $(l \geqslant 1), AB, B_+, B_-, B_0$, and C.

Consider the colored hyperspace $(\breve{\mathcal{E}}, \mathcal{V}, \mathcal{D}^B, \varrho)$ associated with X, and choose coordinates on $\breve{\mathcal{E}}$ as defined in Appendix 4.5. In general, a G-orbit is characterized by the colors and G-stable prime divisors containing it [T11, 16.19]. For example, an orbit $\mathcal{O}$ of type A_l $(l \geqslant 1)$ is defined by the G-valuations $v_{X^{x_i}} = (x_i, h_i, l_i) \in \breve{\mathcal{E}}$, $i = 1, ..., l$ corresponding to the G-stable exceptional divisors X^{x_i} containing $\mathcal{O}$ (all the colors but the exceptional colors E^{x_i} contain $\mathcal{O}$, and the exceptional points x_i, $i = 1, .., l$ are pairwise distinct). We say that an orbit of type A_l $(l \geqslant 1)$ is *Platonic* if either $l \leqslant 2$ or the associated tuple $(h_1, ..., h_l)$ is Platonic. Furthermore, X can contain at most one orbit of type A_l.

Proposition 4.3.3.1. *The singularities of X are log terminal if and only if it has no G-fixed point and the orbit of type A_l, if it exists, is Platonic.*

Proof. For $\mathscr{O}$ a fixed point (type B_0), F is necessarily a cyclic group μ_n, and any B-chart $X_\mathscr{O}$ intersects all the colors, so is G-stable [T97, Sec. 5]. Hence, $X_\mathscr{O}$ is a normal affine G/μ_n-embedding. By [Pa92, Thm 2 and Prop 4], the canonical class is torsion-free in $\mathrm{Cl}(X_\mathscr{O})$, and $\mathrm{Pic}(X_\mathscr{O})$ is trivial. From this, we conclude that $X_\mathscr{O}$ is not $\mathbb{Q}$-Gorenstein. In particular, its singularities are not log terminal.

For orbits of type C, AB and B_+, the chart $X_\mathscr{O}$ is given by a colored hypercone of type (A). By 4.2.4.2, there is a (trivial) U-torsor $X_\mathscr{O} \to Y_\mathscr{O}$, and $Y_\mathscr{O}$ is *toroidal* [T11, 16.21] in the sense of [KKMS, Chap IV]. As we consider an isolated singularity of a toroidal surface, we can suppose that $Y_\mathscr{O}$ is toric [I, Thm 4.2.4]. We conclude by using the fact that toric surface singularities are log terminal [I, 7.4.11 and 7.4.17].

Now suppose that $\mathscr{O}$ is an orbit of type A_l, $l \geqslant 1$. Then, there are finitely many G-stable exceptional divisor $X^{x_1}, ..., X^{x_l}$ containing $\mathscr{O}$, with $x_1, ..., x_l$ pairwise distinct exceptional points. We consider the B-chart $X_\mathscr{O}$ of $\mathscr{O}$ given by the colored hypercone of type (B) generated by the X^{x_i} and the colors associated to the points of $\mathbb{P}^1_k \setminus \{x_1, ..., x_l\}$. Again, we have a U-torsor $X_\mathscr{O} \to Y_\mathscr{O}$. Moreover, since $Y_\mathscr{O}$ has an attractive fixed point for the T-action (4.2.3.6), its Picard group is trivial. It follows that $Y_\mathscr{O}$ is $\mathbb{Q}$-Gorenstein if and only if a multiple of $K_{Y_\mathscr{O}}$ is a principal divisor. By [LS, Prop 4.3], this is in turn equivalent to asking for a certain system of linear equations $Ax = y$ written in matrix form to have a solution, where A has linearly independant columns [LS, Prop 4.6]. But in our case, A is a square matrix so that $Y_\mathscr{O}$ is $\mathbb{Q}$-Gorenstein. Now, we can apply the criterion [LS, Cor 5.8] to obtain that $Y_\mathscr{O}$ (hence $X_\mathscr{O}$) has log terminal singularities if and only if the tuple $(h_1, ..., h_l)$ is Platonic, where h_i is the multiplicity of X^{x_i} in $\pi^*(x_i)$.

Orbits of type B_- can only occur when $F = \mu_n$ [T97, 5], we suppose that $\mathscr{O}$ is of this type. If $n \geqslant 3$, there are two exceptional points x_0, x_∞ associated to the dense orbit and $\mathscr{O}$ lies in exactly one of the two exceptional colors, say E^{x_0} [T97, 5.2]. Consider the B-chart $X_\mathscr{O}$ given by the colored hypercone of type (B) generated by all the colors but E^{x_∞}, and by the G-stable exceptional divisor sent to x_∞ which contains $\mathscr{O}$. As before, we are reduced to studying the singularities of a normal rational affine T-surface $Y_\mathscr{O}$ of complexity one admitting an attractive fixed point, whence $\mathcal{O}(Y_\mathscr{O})^* \simeq k^*$. By 4.2.2.2 and 4.2.2.3, the T-equivariant Cox ring $\mathrm{Cox}_T(Y_\mathscr{O})$ is a polynomial ring over $k[\hat{T}]$. As a consequence, $Y_\mathscr{O}$ is toric and we conclude as above. The same method applies when $n \leqslant 2$. □

4.4 Cox ring of an almost homogeneous SL_2-threefold

Keep the notation of section 4.3.1. By [V1, 2.3.4 and 3.1.4], the G-equivariant Cox ring of X is well-defined, finitely generated and canonically isomorphic to the ordinary Cox ring. We slightly modify the notation of 4.2.2.2 in order to distinguish G-stable prime divisors and colors. It is natural to make this distinction as colors don't depend on the embedding, whereas G-stable prime divisors do. We choose homogeneous coordinates on $\mathbb{P}^1_k/F \simeq \mathbb{P}^1_k$, and let

- $(x_i = [\alpha_i : \beta_i])_i$ be the family of exceptional points of $\pi_{|G/F} \to \mathbb{P}^1_k/F$. Possible families are $\emptyset$ when F is cyclic of order $n \leqslant 2$, (x_0, x_∞) when F is cyclic of order $n \geqslant 3$, and (x_v, x_f, x_e) for the others F,
- $(x'_i = [\alpha'_i : \beta'_i])_i$ be the family whose elements are the others exceptional points of $\pi : X \dashrightarrow \mathbb{P}^1_k/F$,
- $\pi^*(x_i) = n_i E^{x_i} + \sum_j h_{ij} X^{x_i}_j$, $n_i > 1$,

- $\pi^*(x'_i) = E^{x'_i} + \sum_j h'_{ij} X_j^{x'_i}$,
- $(s_i)_i$ (resp. $(s'_i)_i$) the family of canonical sections corresponding to the family $(E^{x_i})_i$ (resp. $(E^{x'_i})_i$),
- $(r_{ij})_{ij}$ (resp. $(r'_{ij})_{ij}$) the family of canonical sections corresponding to the family $(X_j^{x_i})_{ij}$ (resp. $(X_j^{x'_i})_{ij}$),
- $N := \sharp(X_j^{x_i})_{ij}$ if $v_{X^\infty} \notin \mathcal{V}(X)$, $N := \sharp(X_j^{x_i})_{ij} + 1$ otherwise,
- $N' := \sharp(X_j^{x'_i})_{ij}$,
- $(D^x)_{x \in \mathbb{P}^1_k \setminus \{(x_i),(x'_i)\}}$ the family of parametric colors of X.

With this notation, the number of G-stable prime divisors in X is $N + N'$, and we identify the subgroup of $\mathrm{WDiv}(X)$ generated by these divisors with $\mathbb{Z}^{N+N'}$. By 4.2.2.2, we have the following presentation of $\mathrm{Cox}(X)^U$:

- Generators: $a, b, (s_i)_i, (s'_i)_i, (r_{ij})_{ij}, (r'_{ij})_{ij}$.
- Relations: $(\beta_i a - \alpha_i b - s_i^{n_i} \prod_j (r_{ij})^{h_{ij}})_i$, $1 \leqslant i \leqslant \sharp(x_i)_i$, and $(\beta'_i a - \alpha'_i b - s'_i \prod_j (r'_{ij})^{h'_{ij}})_i$, $1 \leqslant i \leqslant \sharp(x'_i)_i$.

4.4.1 Characterizing log terminality in the total coordinate space

In this section, we provide a condition of combinatorial nature for the total coordinate space $\tilde{X}$ to have log terminal singularities. This is an interesting question, for example a $\mathbb{Q}$-factorial normal projective variety Z is of *Fano type* (i.e. there exists an effective $\mathbb{Q}$-divisor Δ such that $-(K_Z + \Delta)$ is ample and the pair (Z, Δ) is Kawamata log terminal) if and only if its Cox ring is finitely generated with log terminal singularities [GOST, Thm 1.1]. In the following proposition, the condition on $\mathrm{Cox}(X)^U$ translates into a condition on a geometric data attached to X (see 4.2.2.6, 4.2.2.7).

Proposition 4.4.1.1. *The total coordinate space $\tilde{X}$ has log terminal singularities if and only if $\mathrm{Cox}(X)^U$ is a Platonic ring.*

Proof. Apply [V1, 3.4.3]. □

4.4.2 Geometry of the special fiber

The good quotient

$$f : \tilde{X} \xrightarrow{//G} \mathbb{A}_k^{N+N'},$$

is a $\mathbb{D}(\mathrm{Cl}(X))$-equivariant morphism, where $\mathbb{D}(\mathrm{Cl}(X))$ acts on $\mathbb{A}_k^{N+N'}$ through the surjective morphism

$$\mathbb{D}(\mathrm{Cl}(X)) \to \mathbb{G}_m^{N+N'}$$

dually defined by the natural injective morphism $\mathbb{Z}^{N+N'} \hookrightarrow \mathrm{Cl}(X)$ which sends a G-stable divisor in X to its class. The morphism f pulls back the standard coordinates of the affine space to the canonical sections associated with the corresponding G-stable prime divisors. It follows from [V1, 2.8.1] that the general schematic fibers of f are normal varieties isomorphic to the total coordinate space of the open orbit G/F.

In this section, we study the geometry of the *special fiber*, that is, the schematic zero fiber $\tilde{X}_0 := f^{-1}(0)$. By virtue of the permanency of a lot of properties when taking U-invariants [T11, D.5], we in fact study the zero fiber $\tilde{Y}_0 := \tilde{X}_0//U$ of the induced morphism

$$f_U : \tilde{Y} \to \mathbb{A}_k^{N+N'},$$

where $\tilde{Y} := \tilde{X}//U$. We suppose that X doesn't admit an exceptional divisor dominating $\mathbb{P}_k^1$ ($v_{X^\infty} \notin \mathcal{V}(X)$). This is indeed harmless for our purpose as if $v_{X^\infty} \in \mathcal{V}(X)$, then one has to add the associated canonical section to the generating set of $\mathrm{Cox}(X)^U$ from 4.2.2.2, but this generator doesn't appear in any relation.

$F = \mu_n$, $n \leqslant 2$

In this case, we have $N = 0$ and the presentation of $\mathrm{Cox}(X)^U$ reads

- Generators: $a, b, (s'_i)_i, (r'_{ij})_{ij}$.
- Relations: $(\beta'_i a - \alpha'_i b - s'_i \prod_j (r'_{ij})^{h'_{ij}})_i$, $1 \leqslant i \leqslant \sharp(x'_i)_i$.

If $\sharp(x'_i)_i \leqslant 2$, then $\mathrm{Cox}(X)^U$ is a polynomial k-algebra. Indeed, each relation can be used to remove a generator (first a and then possibly b) from the generating set, so that we end up with a polynomial algebra. If $\sharp(x'_i)_i > 2$, each new exceptional point starting from the third defines a new relation between the remaining generators. In any case, denote Σ the new generating set. The coordinate algebra of $\tilde{Y}_0$ is $\mathrm{Cox}(X)^U/((r'_{ij})_{ij})$, and is freely generated by the elements of the set $\Sigma \setminus \{(r'_{ij})_{ij}\}$. Indeed, all the relations become trivial in this quotient. This yields the

Proposition 4.4.2.1. *The special fiber $\tilde{X}_0$ is a normal variety. Moreover, $\tilde{Y}_0 = \tilde{X}_0//U$ is an affine space.*

Remark 4.4.2.2. For a spherical variety Z under a connected reductive group G_1, the quotient morphism

$$\tilde{Z}_{G_1} \to \tilde{Z}_{G_1}//G_1$$

is faithfully flat, and the special fiber is a normal horospherical variety [Bri07]. Both results does not extend to varieties of complexity one. Indeed, the general fibers of f are isomorphic to G, thus of dimension three. On the other hand, when $\sharp(x'_i)_i \geqslant 2$, the quotient by U of the special fiber is an affine space of dimension $\sharp(x'_i)_i$, thus f is not flat when $\sharp(x'_i)_i \geqslant 3$. For a non-horospherical example, consider the case where $X = G$. Then $\tilde{X} = \tilde{X}_0 = X$ which is not horospherical.

$F = \mu_n$, $n \geqslant 3$

Recall (4.5.1), that we defined $\bar{n} := n$ if n is odd, and $\bar{n} := n/2$ otherwise, and that the morphism $\pi_{|G/\mu_n}$ defines two exceptional points $x_0 = [0 : 1], x_\infty = [-1 : 0] \in \mathbb{P}_k^1$ with respect to the homogeneous coordinates $g_3^{\bar{n}}, g_4^{\bar{n}}$. These points define the two relations

$$a = s_0^{\bar{n}} \prod_j r_{0j}^{h_{0j}}, \text{ and } b = s_\infty^{\bar{n}} \prod_j r_{\infty j}^{h_{\infty j}}.$$

Using these relations, we remove a and b from the set of generators. This yields the following presentation of $\mathrm{Cox}(X)^U$:

- Generators: $s_0, s_\infty, (r_{0j})_j, (r_{\infty j})_j, (s'_i)_i, (r'_{ij})_{ij}$.
- Relations: $(\beta'_i s_0^{\bar{n}} \prod_j r_{0j}^{h_{0j}} - \alpha'_i s_\infty^{\bar{n}} \prod_j r_{\infty j}^{h_{\infty j}} = s'_i \prod_j (r'_{ij})^{h'_{ij}})_i$, $1 \leqslant i \leqslant \sharp(x'_i)_i$.

Consider the coordinate algebra $\mathrm{Cox}(X)^U/((r_{0j})_j, (r_{\infty j})_j, (r'_{ij})_{ij})$ of $\tilde{Y}_0$. The elements $s_0, s_\infty, (s'_i)_i$ generate this algebra, and we obtain the following criterion via a case by case analysis.

Proposition 4.4.2.3. *We have the following equivalences:*

$$\begin{aligned}\tilde{X}_0 \text{ is a normal variety} &\iff \tilde{Y}_0 \text{ is a normal variety}\\ &\iff \tilde{Y}_0 \text{ is an affine space}\\ &\iff ((X_j^{x_0})_j \neq \emptyset \text{ and } (X_j^{x_\infty})_j \neq \emptyset) \text{ or } (x_i')_i = \emptyset\end{aligned}$$

Example 4.4.2.4. Suppose that $(X_j^{x_0})_j = \emptyset$, $(X_j^{x_\infty})_j = \emptyset$, and $(x_i')_i$ consist of a unique point $x_1' = [\alpha_1' : \beta_1']$. Then, the ideal of relations is principal, generated by the relation $\beta_1' s_0^{\bar{n}} - \alpha_1' s_\infty^{\bar{n}} = 0$. It follows that the special fiber is a reducible reduced non-normal algebraic scheme. Indeed, $\tilde{Y}_0$ is the union of $\bar{n}$ planes intersecting along the line of equation $s_0 = s_\infty = 0$ in k^3 with coordinates s_0, s_∞, s_1'.

Example 4.4.2.5. Suppose that $(X_j^{x_0})_j = \emptyset$, $(X_j^{x_\infty})_j = \emptyset$, and $(x_i')_i$ consists of at least two points. Then, the ideal of relations is generated by $s_0^{\bar{n}} = 0$ and $s_\infty^{\bar{n}} = 0$. It follows that the special fiber is an irreducible non-reduced algebraic scheme.

F is binary polyhedral

As a typical example, we give the generators and relations when $F = F_{\mathbb{T}}$ is the binary tetrahedral group. We can assume that the three exceptional points x_v, x_e, x_f have homogeneous coordinates

$$[0:1], [1:0], [-1:-1],$$

and obtain the following presentation of $\mathrm{Cox}(X)^U$:

- Generators: $s_v, s_e, s_f, (r_{v,j})_j, (r_{e,j})_j, (r_{f,j})_j, (s_i')_i, (r_{ij}')_{ij}$.
- Relations: $s_v^3 \prod_j r_{v,j}^{h_{v,j}} + s_e^2 \prod_j r_{e,j}^{h_{e,j}} + s_f^3 \prod_j r_{f,j}^{h_{f,j}} = 0$, $(\beta_i' s_v^3 \prod_j r_{v,j}^{h_{v,j}} + \alpha_i' s_e^2 \prod_j r_{e,j}^{h_{e,j}} = s_i' \prod_j (r_{ij}')^{h_{ij}'})_i$.

Consider the coordinate algebra

$$(\mathrm{Cox}(X))^U / ((r_{v,j})_j, (r_{e,j})_j, (r_{f,j})_j, (r_{ij}')_{ij})$$

of $\tilde{Y}_0$. The elements $s_v, s_e, s_f, (s_i')_i$ generate this algebra, and we obtain the following criterion by again analyzing the different cases.

Proposition 4.4.2.6. *Suppose that F is binary polyhedral. Then $\tilde{X}_0$ is a normal variety if and only if one of the following conditions is satisfied:*

- $(X_j^{x_v})_j \neq \emptyset$ *and* $(X_j^{x_e})_j \neq \emptyset$ *and* $(X_j^{x_f})_j \neq \emptyset$
- $(x_i')_i = \emptyset$ *and* $(X_j^{x_v})_j = (X_j^{x_e})_j = (X_j^{x_f})_j = \emptyset$

Example 4.4.2.7. Suppose that $(X_j^{x_v})_j = (X_j^{x_e})_j = (X_j^{x_f})_j = \emptyset$, and $(x_i')_i$ consists of a unique point x_1' of homogeneous coordinates $[\alpha_1' : \beta_1']$ (distinct from $[1:0], [0:1], [-1:-1]$). Then, the coordinate algebra of $\tilde{Y}_0$ is generated by s_v, s_e, s_f, s_1' with the two relations $s_v^3 + s_e^2 + s_f^3 = 0$ and $\beta_1' s_v^3 + \alpha_1' s_e^2 = 0$. We check that the special fiber is a non-normal variety.

Let $I = (s_v^3 + s_e^2 + s_f^3, \beta_1' s_v^3 + \alpha_1' s_e^2)$ be the ideal of the polynomial algebra $k[s_v, s_e, s_f, s_1']$, we claim that it is a prime ideal. To prove this, we can work in $k[s_v, s_e, s_f]$ and replace I by $I \cap k[s_v, s_e, s_f]$. Then, consider $A := k[s_v, s_e]$, and $J := I \cap A = (\beta_1' s_v^3 + \alpha_1' s_e^2)$. The algebra $B := A/J$ is integral, and to prove the claim, it now suffices to prove that $\bar{I} = (\bar{s_v}^3 + \bar{s_e}^2 + s_f^3)$ is prime in $B[s_f]$. But it is clear that $\bar{s_v}^3 + \bar{s_e}^2 + s_f^3$ is irreducible in $\mathrm{Frac}(B)[s_f]$, hence a prime element of this polynomial algebra. As $((\bar{s_v}^3+\bar{s_e}^2+s_f^3)\operatorname{Frac}(B)[s_f])\cap B[s_f]=(\bar{s_v}^3+\bar{s_e}^2+s_f^3)$ we obtain that $(\bar{s_v}^3+\bar{s_e}^2+s_f^3)$ is a prime ideal. It follows that the special fiber is an affine variety. As a surface in k^4 with coordinates s_v, s_e, s_f, s'_1 having a one-dimensional singular locus, $\tilde{Y}_0$ is not normal. Indeed, the line of equation $s_v = s_e = s_f = 0$ is the singular locus. Hence, the special fiber is not normal either.

4.4.3 $\operatorname{Cox}(X)^U$ is the Cox ring of a T-variety of complexity one

By Section 4.2.2, $\operatorname{Cox}(X)^U$ can be interpreted as the Cox ring of a T-variety Y of complexity one. We build such a variety Y in a natural way from X. The idea is to find a B-stable open subvariety V of X whose complement is of codimension $\geqslant 2$ in X, and which is a U-torsor over a normal rationa T-surface Y of complexity one. Notice that we can always suppose that V (hence Y) is smooth, up to replacing V by its smooth locus. For such V and Y, both Cox rings are well-defined, finitely generated [V1, 3.1.4], and $\operatorname{Cox}(X)\simeq\operatorname{Cox}(V)$. Moreover, using [V1, 2.2 (2)], [V1, 2.2.2], [V1, 2.2.4], and [V1, 2.5.2], we obtain isomorphisms

$$\operatorname{Cl}(X)\simeq\operatorname{Cl}_U(X)\simeq\operatorname{Pic}_U(V)\simeq\operatorname{Pic}(Y).$$

Also by [V1, Sec. 2.10], we have a cartesian square

$$\begin{array}{ccc} \hat{V} & \xrightarrow{/U} & \hat{Y} \\ {\scriptstyle /\mathbb{D}(\operatorname{Pic}(V))}\downarrow & & \downarrow{\scriptstyle /\mathbb{D}(\operatorname{Pic}(Y))} \\ V & \xrightarrow{/U} & Y, \end{array} \tag{4.4.1}$$

where the horizontal arrows are U-torsors and the vertical arrows are universal torsors. This implies that we have $\operatorname{Cox}(Y)\simeq\operatorname{Cox}(X)^U$, as desired.

Now, we proceed to the construction of V. For simplicity, denote $x_1,\ldots,x_r\in\mathbb{P}^1_k$ the exceptional points of X. For each G-stable prime divisor $X_j^{x_i}$ in X, we consider a B-chart V_{ij} intersecting the open G-orbit in $X_j^{x_i}$, the open G-orbit in X, and no other orbits. Such a B-chart is given by the colored hypercone of type (A) spanned by $X_j^{x_i}$, and by all the colors but E^{x_i} and D^{x_d}, where x_d is an arbitrary fixed (distinguished) non-exceptional point. If $v_{X^\infty}\in\mathcal{V}(X)$, that is, X contains a G-stable prime divisor X^∞ dominating $\mathbb{P}^1_k$, we consider the B-chart V_∞ defined by the colored hypercone of type (A) spanned by v_{X^∞} and all the colors but $E^{x_1},\ldots,E^{x_r}$, and D^{x_d}. Otherwise, we set $V_\infty=\emptyset$. By 4.2.4.2, we have trivial U-torsors

$$\pi_{ij}: V_{ij}\simeq U\times Y_{ij}\to Y_{ij},$$

where the Y_{ij} are normal affine T-surfaces of complexity one. If $V_\infty=\emptyset$, we set $Y_\infty=\emptyset$, otherwise we also have a trivial U-torsor

$$\pi_\infty: V_\infty\simeq U\times Y_\infty\to Y_\infty,$$

where Y_∞ is a normal affine T-surface of complexity one. Finally, the open G-orbit $V_0\simeq G/F$ is a U-torsor over an affine normal T-surface Y_0 of complexity one

$$\pi_0: V_0\to Y_0.$$

Indeed, F acts freely on $G/U\simeq\mathbb{A}^2_k\setminus\{0\}$ with closed orbits. Alternatively, we can consider two covering B-charts of V_0. Proceeding in this way, Y_0 is obtained by gluing two affine T-surfaces of complexity one $Y_{0,1}$ and $Y_{0,2}$.

Proposition 4.4.3.1. *The varieties $(Y_{ij})_{ij}$, Y_∞, $Y_{0,1}$ and $Y_{0,2}$ glue together to a normal rational T-variety Y of complexity one. The morphisms $(\pi_{ij})_{ij}$, π_∞, π_0 glue together to a morphism*

$$\pi : V := \cup_{ij} V_{ij} \cup V_\infty \cup V_0 \to Y,$$

which is a U-torsor over Y. Moreover, V is B-stable, with a complement in X of codimension $\geqslant 2$.

Proof. Denote $(\breve{\mathcal{E}}_T, \mathcal{V}_T)$ the hyperspace of $k(X)^U$ in which live the hypercones defining the T-charts $(Y_{ij})_{ij}$, Y_∞. These hypercones are obtained from the colored hypercones associated with the B-charts $(V_{ij})_{ij}$, V_∞. Indeed, they are respectively spanned by the T-stable divisors in these T-charts, and these T-stable divisors are the images by π_{ij} (resp. π_∞) of the respective intersections of the colors and G-stable prime divisors of X with V_{ij} (resp. V_∞). We can define the varieties $Y_{0,1}$ and $Y_{0,2}$ in the following way: consider the subset $\mathcal{D}_T^B \subset \mathcal{V}_T$ of all the T-valuations obtained via π_0 from the colors of G/F. Then choose two distinct elements $v_{0,1}$ and $v_{0,2}$ in this set and consider the two varieties defined by the hypercones of type (A) generated respectively by $\mathcal{D}_T^B \setminus \{v_{0,1}\}$ and $\mathcal{D}_T^B \setminus \{v_{0,2}\}$. The proper non-trivial supported (hyper)faces of the hypercones defining the varieties $(Y_{ij})_{ij}$, Y_∞, $Y_{0,1}$, $Y_{0,2}$ are cones which by construction don't overlap in $\mathcal{V}_T$. By Section 4.2.3, the latter varieties glue together into a normal T-variety of complexity one. The morphisms $(\pi_{ij})_{ij}$, π_∞, π_0 coincide on intersections so that they glue together. The assertion that $\pi : X \to Y$ is a U-torsor has already been checked locally above, and it implies that Y is rational. For the last assertion, it suffices to notice that V contains the open G-orbit and meets every boundary divisor, whence the claim on the codimension of $X \setminus V$ in X. □

4.4.4 Iteration of Cox rings

In this section, we use the construction of the preceding section to uncover a connection between iterations of Cox rings for X and Y. We first recall the definition of an almost principal bundle under an algebraic group. Hashimoto introduced this notion in [Has, Def. 0.4] where he systematically studies properties preserved by almost principal bundles.

Definition 4.4.4.1. Let H be an algebraic group, and let Z_1, Z_2 be normal H-varieties such that H acts trivially on Z_2. We say that a H-equivariant morphism $\varphi : Z_1 \to Z_2$ is an *almost principal H-bundle* over Z_2 if there exists H-stable open subvarieties $V_1 \subset Z_1$, $V_2 \subset Z_2$ whose respective complements are of codimension $\geqslant 2$ and such that φ induces a H-torsor $V_1 \to V_2$.

Example 4.4.4.2. In the framework of Cox rings, an almost principal bundle $\varphi : Z_1 \to Z_2$ under a diagonalizable group such that

- Z_2 is a normal variety with finitely generated class group and only constant invertible regular functions,
- φ is a good quotient,
- Z_1 is a normal variety with only constant invertible homogeneous regular functions,

is precisely a *quotient presentation* in the sense of [ADHL, 4.2.1.1]. For example, the structural morphism of the characteristic space $\hat{Z}_2 \to Z_2$, if it exists, is a quotient presentation of Z_2.

In [ABHW], the authors introduce the notion of *iteration of Cox rings*: Let Z be a normal variety with finitely generated Cox ring. If the total coordinate space $\tilde{Z}$ has non-trivial class group and satisfies $\mathcal{O}(\tilde{Z})^* \simeq k^*$, then it has a non-trivial well-defined Cox ring. If the latter is finitely generated, we get a new total coordinate space $\tilde{Z}^{(2)}$, and so on. This iteration process yields a sequence of Cox rings which stops if and only if one of the following cases occurs at some step:

- we obtain a total coordinate space whose Cox ring is not well defined (i.e. there exists $n \geqslant 0$ such that $\mathrm{Cl}(\tilde{Z}^{(n)})$ has a non-trivial torsion subgroup, and $\mathcal{O}(\tilde{Z}^{(n)})^* \not\simeq k^*$).
- we obtain a total coordinate space whose Cox ring is not finitely generated.
- we obtain a factorial total coordinate space (i.e. with trivial class group).

If we never fall in one of the cases above, Z is said to have *infinite iteration of Cox rings*. Otherwise, Z is said to have *finite iteration of Cox rings*, and the last obtained Cox ring is the *master Cox ring*. By virtue of [V1, 3.4.1], X admits finite iteration of Cox rings with a factorial finitely generated master Cox ring $\tilde{X}^{(m)}$, $m \geqslant 1$. In the following proposition, $Y = \tilde{Y}^{(0)}$ is the T variety of complexity one constructed in the preceding section.

Proposition 4.4.4.3. *For $1 \leqslant i \leqslant m$, the categorical quotient of $\tilde{X}^{(i)}$ by U identifies $\tilde{X}^{(i)}//U$ with the total coordinate space of $\tilde{Y}^{(i-1)}$. Moreover, the categorical quotient*

$$\pi_i : \tilde{X}^{(i)} \xrightarrow{//U} \tilde{Y}^{(i)}$$

is an almost principal U-bundle.

Proof. By virtue of the cartesian square (4.4.1), the categorical quotient $\pi_1 : \tilde{X} \xrightarrow{//U} \tilde{X}//U$ is an almost principal U-bundle, and $\tilde{X}//U$ identifies with $\tilde{Y}$. Consider the categorical quotient

$$\pi_2 : \tilde{X}^{(2)} \xrightarrow{//U} \tilde{X}^{(2)}//U,$$

where both are affine normal varieties [T11, D.5]. We claim that $\tilde{X}^{(2)}//U$ is the total coordinate space of $\tilde{Y}$. Indeed, $\tilde{X}^{(2)}//U$ is naturally a variety over $\tilde{Y}$ with an affine structural morphism. Moreover, the morphism π_2 can be viewed as the morphism corresponding to the graded $\mathcal{O}_{\tilde{Y}}$-algebras morphism

$$(\pi_{1*}\mathcal{R}_{\tilde{X}})^U \hookrightarrow \pi_{1*}\mathcal{R}_{\tilde{X}}.$$

Using the exact sequence [V1, 2.2 (1)], and the Proposition [V1, 2.5.2], we can write $\mathcal{R}_{\tilde{X}} = \bigoplus_{[\mathcal{F}]\in \mathrm{Cl}(\tilde{Y})} \pi_1^*\mathcal{F}$, which yields an isomorphism of graded $\mathcal{O}_{\tilde{Y}}$-algebras

$$(\pi_{1*}\mathcal{R}_{\tilde{X}})^U \simeq \mathcal{R}_{\tilde{Y}},$$

again by [V1, Proposition 2.5.2]. This proves the claim, and we have a cartesian square (see [V1, Sec. 2.10])

$$\begin{CD}
\tilde{X}^{(2)} @>{\pi_2}>> \tilde{Y}^{(2)} \\
@V{//\mathbb{D}(\mathrm{Cl}(\tilde{X}))}VV @VV{//\mathbb{D}(\mathrm{Cl}(\tilde{Y}))}V \\
\tilde{X} @>{\pi_1}>> \tilde{Y},
\end{CD}$$

where horizontal arrows are almost principal U-bundles, and vertical arrows are structural morphisms of characteristic spaces. Iterating this construction, we obtain the result. □

Corollary 4.4.4.4. *For $i = 1, ..., m$, the total coordinate space $\tilde{Y}^{(i)}$ is an affine normal rational variety of complexity one under a torus action, and the regular invariant functions on $\tilde{Y}^{(i)}$ are constant.*

Proof. Everything stems from the fact that $\tilde{X}^{(i)} \xrightarrow{//U} \tilde{Y}^{(i)}$ is an almost principal bundle, and that $\tilde{X}^{(i)}$ is almost homogeneous of complexity one under the action of a connected reductive group of the form $G \times \mathbb{T}_i$. □

Corollary 4.4.4.5. *There is a commutative diagram*

$$\begin{tikzcd}
\tilde{X}^{(m)} \arrow[r] \arrow[d] & \dots \arrow[r] & \tilde{X}^{(2)} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\
\tilde{Y}^{(m)} \arrow[r] & \dots \arrow[r] & \tilde{Y}^{(2)} \arrow[r] & \tilde{Y},
\end{tikzcd}$$

where $\tilde{X}^{(m)}, \tilde{Y}^{(m)}$ are factorial, the horizontal arrows are structural morphisms of characteristic spaces, the vertical arrows are almost principal U-bundles, and all squares are cartesian.

In [HW], Hausen and Wrobel prove that a trinomial variety obtained from Construction 4.2.2.4 admits finite iteration of Cox rings with a finitely generated factorial master Cox ring if and only if it is rational and the tuple $(\mathfrak{l}_0, \dots, \mathfrak{l}_r)$ is Platonic, where $\mathfrak{l}_i$ is the greatest common divisor of the integers appearing in the exponent vector l_i. It is immediate to check that $\tilde{Y}$ indeed satisfies these properties. Also by [HW, Cor. 1.4], the length of the Cox ring iteration sequence of Y (hence of X) is determined by the tuple $(\mathfrak{l}_0, \dots, \mathfrak{l}_r)$, and is bounded by 4. Below, we give another proof for the bound $m \leqslant 4$ which gives more precise bounds depending on the finite subgroup $F \subset G$, and uses arguments of different nature (4.4.4.13). This yields interesting intermediate results (4.4.4.8, 4.4.4.10, 4.4.4.11, 4.4.4.12). We start by recalling a useful construction for the study of iteration (see also [Bra], and [V1, Sec. 3.5]).

Construction 4.4.4.6. Using the isomorphism $\mathrm{Pic}_G(G/F) \simeq \hat{F}$, and that G is semisimple and simply connected, the localization exact sequence [V1, 2.2.4] applied to the open orbit in X reads

$$0 \to \mathbb{Z}^{N+N'} \to \mathrm{Cl}(X) \to \hat{F} \to 0. \tag{4.4.2}$$

From this sequence we obtain that $\mathrm{Cl}(X)_{\mathrm{tor}}$ embeds in $\hat{F}$ via restriction of divisorial sheaves to G/F. Hence, $\mathrm{Cl}(X)_{\mathrm{tor}}$ is canonically identified with a subgroup of $X^*(F/D(F)) \simeq F/D(F)$, where $D(F)$ denotes the derived subgroup of F. The exact sequence

$$0 \to \mathrm{Cl}(X)_{\mathrm{tor}} \to \mathrm{Cl}(X) \to M \to 0$$

translates into the factorization

$$\hat{X} \xrightarrow{g_1 = /\mathbb{T}_1} X' \xrightarrow{f_1 = /\mathbb{D}(\mathrm{Cl}(X)_{\mathrm{tor}})} X$$

of the characteristic space $q : \hat{X} \to X$, where $\mathbb{T}_1 := \mathbb{D}(M)$ is a torus, and X' is a normal G/F_1-embedding. Similarly to [V1, 2.5.12], one shows that F_1 is the intersection of the kernels of characters in $\hat{F}$ corresponding to elements of $\mathrm{Cl}(X)_{\mathrm{tor}}$. This yields a canonical isomorphism $\mathrm{Cl}(X)_{\mathrm{tor}} \simeq X^*(F/F_1)$. Applying [V1, 2.10.1], we obtain that $\hat{X}^{(2)}$ is a characteristic space of both $\hat{X}$ and X'. Iterating this construction, one obtains a commutative diagram

$$\begin{tikzcd}
\hat{X}^{(m)} \arrow[r, "q_m"] \arrow[d, "g_m"] \arrow[dr, "q'_m"] & \dots \arrow[r, "q_3"] & \hat{X}^{(2)} \arrow[r, "q_2"] \arrow[d, "g_2"] \arrow[dr, "q'_2"] & \hat{X} \arrow[d, "g_1"] \arrow[dr, "q"] & \\
X'^{(m)} \arrow[r, "f_m"] & \dots \arrow[r, "f_3"] & X'^{(2)} \arrow[r, "f_2"] & X' \arrow[r, "f_1"] & X \\
G/F_m \arrow[r] \arrow[u] & \dots \arrow[r] & G/F_2 \arrow[r] \arrow[u] & G/F_1 \arrow[r] \arrow[u] & G/F \arrow[u]
\end{tikzcd}$$

where the q_i, q'_i and g_m are structural morphisms of characteristic spaces, the f_i are quotient presentations by the finite diagonalizable groups F_i/F_{i-1}, and $\hat{X}^{(m)}$ is a factorial characteristic space. Notice that $X'^{(m)} = X'^{(m-1)}$ if $\mathrm{Pic}(X'^{(m-1)})$ is torsion-free, and $\hat{X}^{(m)} = X'^{(m)}$ if $\mathrm{Pic}(X'^{(m-1)})$ is finite. This latter case occurs if and only if $X = G/F$. The sequence of subgroups $F, F_1 \dots, F_m$ yields a normal series of F with abelian quotients, and each $X'^{(i)}$ is a normal G/F_i-embedding.

Proposition 4.4.4.7. *Suppose that X is a normal G/μ_n-embedding, and let d be the order of the torsion subgroup of* $\mathrm{Cl}(X)$*. Denote $\bar{n} := n/2$ if n is even, and $\bar{n} := n$ otherwise. Similarly let $\overline{n/d} := n/2d$ if n/d is even, and $\overline{n/d} := n/d$ otherwise. Finally, let $\tilde{d} := \frac{\bar{n}}{\overline{n/d}}$. Consider a point $x \in \mathbb{P}^1_k/\mu_n$, and a B-stable prime divisor E^x in X such that $\pi(E^x) = x$. Then,*

$$q^*(E^x) = \begin{cases} \hat{E}^x_1, \text{ if } x = x_0 \text{ or } x = x_\infty, \\ \hat{E}^x_1 + ... + \hat{E}^x_{\tilde{d}}, \text{ otherwise,} \end{cases}$$

where the $\hat{E}^x_i$ are pairwise distinct B-stable prime divisors in $\hat{X}$.

Proof. Consider the commutative square

$$\begin{array}{ccc} X' & \overset{\pi_1}{\dashrightarrow} & \mathbb{P}^1_k/\mu_{n/d} \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle \varphi_1} \\ X & \overset{\pi}{\dashrightarrow} & \mathbb{P}^1_k/\mu_n, \end{array}$$

where π_1, π are the rational quotients by B, and φ_1 is the geometric quotient of $\mathbb{P}^1_k/\mu_{n/d}$ by μ_d. In view of 4.5.1, we have

$$\varphi_1^*(x) = \begin{cases} \tilde{d}x'_1, \text{ if } x = x_0 \text{ or } x = x_\infty, \\ x'_1 + ... + x'_{\tilde{d}}, \text{ otherwise,} \end{cases}$$

where the x'_i are pairwise distinct points of $\mathbb{P}^1_k/\mu_{n/d}$. Moreover, denoting Σ_x the set of B-stable prime divisors in the support of $f_1^*(E^x)$, the morphism π_1 defines an equivariant map of transitive μ_d-sets

$$\Sigma_x \to \varphi^{-1}(x).$$

Using this and the fact that f_1 is étale, we obtain that

$$f_1^*(E^x) = \textstyle\sum_{k=1}^{|\varphi^{-1}(x)|} E^{x'_k,1} + ... + E^{x'_k,l},$$

where the $E^{x'_k,i}$ are pairwise distinct B-stable prime divisors in X' with $\pi_1(E^{x'_k,i}) = x'_k$ respectively, and l is the cardinality of the orbit $\mathrm{Stab}_{\mu_d}(x'_1).E^{x'_k,1}$. In fact, we have $l = 1$ because for a given k, each $E^{x'_k,i}$ $i = 1, ..., l$ defines the same B-stable valuation (see Section 4.2.3 and notice that the map $\varrho : \mathcal{D}^B \to \check{\mathcal{E}}$ is injective for almost homogeneous G-threefolds). Now, the statement follows from the observation that $q = f_1 g_1$ and that g_1 is a torsor under a torus. $\square$

Lemma 4.4.4.8. *Suppose that X is a normal G/μ_n-embedding. Then,* $\mathrm{Cl}(\hat{X})$ *is free of rank $(\tilde{d}-1)N'$, where d is the order of the torsion subgroup of* $\mathrm{Cl}(X)$*, and $\tilde{d}$ is defined as in Proposition 4.4.4.7.*

Proof. By [V1, 2.8.6], the open $G \times \mathbb{D}(\mathrm{Cl}(X))$-orbit $\hat{X}_0$ in $\hat{X}$ is isomorphic to $G \times^{\mu_n} \mathbb{D}(\mathrm{Cl}(X))$, where μ_n is identified with a subgroup of $\mathbb{D}(\mathrm{Cl}(X))$ and acts on the latter by translation. Using [V1, 2.5.2] and that G is semisimple and simply connected, we obtain isomorphisms

$$\mathrm{Pic}(\hat{X}_0) \simeq \mathrm{Pic}_G(G \times^{\mu_n} \mathbb{D}(\mathrm{Cl}(X))) \simeq \mathrm{Pic}(\mathbb{D}(\mathrm{Cl}(X))/\mu_n) \simeq \mathrm{Pic}(\mathbb{G}_m^{N+N'}) = 0.$$

We deduce that $\mathrm{Cl}(\hat{X})$ is generated by the classes of the prime divisors lying in $\hat{X} \setminus \hat{X}_0$. Consider the free $\mathbb{Z}$-module K on these prime divisors. Using the notation introduced at the beginning of Section 4.4, we claim that the relations in K defining $\mathrm{Cl}(\hat{X})$ are given by

- $\mathrm{div}(r'_{ij}) = q^*(X_j^{x_i}) = 0$, $\forall i, j$,

- $\mathrm{div}(r_{0j}) = q^*(X_j^{x_0}) = 0$, $\forall j$,
- $\mathrm{div}(r_{\infty j}) = q^*(X_j^{x_\infty}) = 0$, $\forall j$,
- $\mathrm{div}(r_\infty) = q^*(X^\infty)$, if $v_{X^\infty} \in \mathcal{V}(X)$.

Indeed, on one hand we have the exact sequence [V1, 2.2.4]

$$0 \to \mathcal{O}(\hat{X})^{*G} \to \mathcal{O}(\hat{X}_0)^{*G} \xrightarrow{\mathrm{div}} \bigoplus_{(D)_i} \mathbb{Z}D_i \xrightarrow{D\mapsto[\mathcal{O}_X(D)]} \mathrm{Cl}_G(\hat{X}) \simeq \mathrm{Cl}(\hat{X}) \to 0,$$

where (D_i) is the family of prime divisors lying in $\hat{X} \setminus \hat{X}_0$. In view of Proposition 4.4.4.7, the cardinality of this family is $\tilde{d}N' + N$. On the other hand, using [V1, 2.8.5], the G-invariant units in $\mathcal{O}(\hat{X})$ are constant, and the G-invariant units in $\mathcal{O}(\hat{X}_0)$ are Laurent monomials in the canonical sections associated to the G-stable prime divisors in X (namely, the $r'_{ij}, r_{0j}, r_{\infty j}, r_\infty$). By Proposition 4.4.4.7 again, we see that the submodule of K generated by the above relations is a direct factor of rank $N + N'$. It follows that $\mathrm{Cl}(\hat{X})$ is free of rank $\tilde{d}N' + N - (N + N') = (\tilde{d} - 1)N'$. □

Remark 4.4.4.9. Proposition 4.3.2.2 provides an effective method for the description of the class group of an almost homogeneous G-threefold X. Hence, the order d of the torsion subgroup of $\mathrm{Cl}(X)$ can be computed in practice. Also, for the computation of $\mathrm{Cl}(\tilde{X})$ in the general case, recall that we have $\mathrm{Cl}(\tilde{X}) \simeq \mathrm{Cl}(\tilde{Y})$ because $\tilde{X} \to \tilde{Y}$ is an almost principal U-bundle. Now it suffices to use Wrobel's computations of class groups of affine trinomial varieties in term of arithmetic data from the exponent vectors [W].

Lemma 4.4.4.10. *With the notation of Construction 4.4.4.6, there is a natural identification of* $\mathrm{Cl}(X')_{\mathrm{tor}}$ *with a subgroup of* $\mathrm{Cl}(\hat{X})_{\mathrm{tor}}$.

Proof. Using [V1, 2.5.2] and [V1, 2.2 (1)], we obtain an exact sequence

$$0 \to \hat{\mathbb{T}}_1 \to \mathrm{Cl}_{\mathbb{T}_1}(\hat{X}) \simeq \mathrm{Cl}(X') \to \mathrm{Cl}(\hat{X}) \to 0,$$

from which we deduce that the torsion subgroup of $\mathrm{Cl}(X')$ embeds in that of $\mathrm{Cl}(\hat{X})$. □

Lemma 4.4.4.11. *With the notation of Construction 4.4.4.6, suppose that X is a normal $G/F_{\mathbb{D}_n}$-embedding. Then,* $\mathrm{Cl}(X')_{\mathrm{tor}}$ *is identified with a subgroup of* $\mathbb{Z}/n\mathbb{Z}$.

Proof. As $X' \to X$ is a quotient presentation, we can suppose that X, X' are smooth, and consider Picard groups instead of class groups. By [V1, 2.8.6], the open $G \times \mathbb{D}(\mathrm{Pic}(X))$-orbit $\hat{X}_0$ in $\hat{X}$ can be identified with $G/\mu_n \times^{\mu_2\times\mu_2} \mathbb{D}(\mathrm{Pic}(X))$, where $\mu_2 \times \mu_2$ is identified with a subgroup of $\mathbb{D}(\mathrm{Pic}(X))$ and acts on the latter by translation. By [V1, 2.5.2], we have $\mathrm{Pic}(\hat{X}_0) \simeq \mathrm{Pic}_{\mu_2\times\mu_2}(G/\mu_n \times \mathbb{D}(\mathrm{Pic}(X)))$, and we show that the forgetful morphism

$$\phi : \mathrm{Pic}_{\mu_2\times\mu_2}(G/\mu_n \times \mathbb{D}(\mathrm{Pic}(X))) \to \mathrm{Pic}(G/\mu_n \times \mathbb{D}(\mathrm{Pic}(X))) \simeq \mathbb{Z}/n\mathbb{Z}$$

has a trivial kernel. By [KKV, Lemma 2.2], this kernel is identified with the group of classes of algebraic cocycles $H^1_{alg}(\mu_2 \times \mu_2, \mathcal{O}(G/\mu_n \times \mathbb{D}(\mathrm{Pic}(X)))^*)$. Now, it follows from [Bri18, 4.1.3] that $\mathcal{O}(G/\mu_n \times \mathbb{D}(\mathrm{Pic}(X)))^* \simeq \mathcal{O}(\mathbb{D}(\mathrm{Pic}(X)))^*$. This last fact allows to view $\ker\phi$ as a subgroup of

$$\mathrm{Pic}_{\mu_2\times\mu_2}(\mathbb{D}(\mathrm{Pic}(X))) \simeq \mathrm{Pic}(\mathbb{D}(\mathrm{Pic}(X))/\mu_2 \times \mu_2) = 0.$$

As a consequence, $\ker\phi$ is trivial and we obtain an injective morphism $\mathrm{Pic}(\hat{X}_0) \to \mathbb{Z}/n\mathbb{Z}$.

On the other hand, the localization exact sequence [V1, 2.2.4] reads in our situation

$$0 \to \mathcal{O}(\hat{X})^{*G} \to \mathcal{O}(\hat{X}_0)^{*G} \xrightarrow{\mathrm{div}} \bigoplus_{(D)_i} \mathbb{Z}D_i \xrightarrow{D\mapsto[\mathcal{O}_X(D)]} \mathrm{Pic}(\hat{X}) \to \mathrm{Pic}(\hat{X}_0) \to 0,$$

where $(D_i)_i$ denotes the family of prime divisors lying in the complement of $\hat{X}_0$ in $\hat{X}$. By [V1, 2.8.5], the G-invariant units in $\mathcal{O}(\hat{X})$ are constant, and the G-invariant units in $\mathcal{O}(\hat{X}_0)$ are Laurent monomials in the canonical sections associated to the G-stable prime divisors in X. Because q is the composition of torsor by a torus followed by a torsor by a finite diagonalizable group, the divisor of such a canonical section r (seen as a regular function on $\hat{X}$) is a sum of pairwise distinct elements of $(D_i)_i$. Moreover, the intersection of the supports of any two such divisors is empty. It follows that the cokernel of the morphism $\mathcal{O}(\hat{X}_0)^{*G} \xrightarrow{\text{div}} \bigoplus_{(D)_i} \mathbb{Z}D_i$ is a free abelian group. This implies that $\mathrm{Pic}(\hat{X})_{\mathrm{tor}}$ embeds as a subgroup of $\mathrm{Pic}(\hat{X}_0)$, thus of $\mathbb{Z}/n\mathbb{Z}$. By Lemma 4.4.4.10, the same holds for $\mathrm{Pic}(X')_{\mathrm{tor}}$. $\square$

Lemma 4.4.4.12. *With the notation of Construction 4.4.4.6, suppose that $\mathrm{Cl}(X)_{\mathrm{tor}}$ is cyclic. Then $\mathrm{Cl}(X')_{\mathrm{tor}}$ is not cyclic of even order.*

Proof. As before we can suppose that X, X' are smooth, and consider Picard groups instead of class groups. By contradiction, suppose that $\mathrm{Pic}(X')_{\mathrm{tor}}$ is cyclic of even order. Then, there exists a unique subgroup of order two generated by a non-trivial element $[\mathcal{L}] \in \mathrm{Pic}(X')_{\mathrm{tor}}$. Consider the natural action of $\Gamma := \mathbb{D}(\mathrm{Pic}(X)_{\mathrm{tor}}) \simeq \mu_n$ on X'. Let L be the line bundle over X' associated with $\mathcal{L}$, and let g_1 denote a generator of $\Gamma(k)$ viewed as an automorphism of the variety X'. As the pullback by g_1 induces an automorphism of $\mathrm{Pic}(X')_{\mathrm{tor}}$, we have $g_1^*\mathcal{L} \simeq \mathcal{L}$. By [Bri18, 3.4.1 (ii)], there exists an automorphism $\tilde{g}_1$ of L viewed as a variety such that $\tilde{g}_1$ commutes with the $\mathbb{G}_m$-action on fibers, and restricts on X' to g_1. By [Bri18, 3.4.1 (ii)] again, $\tilde{g}_1^n$ is the multiplication in the fibers of L by a non-zero scalar (recall that $\mathcal{O}(X')^* \simeq k^*$ because X' is almost homogeneous under G). Hence, up to compose $\tilde{g}_1$ with an automorphism of L viewed as a line bundle, we can suppose that $\tilde{g}_1^n = \mathrm{Id}_L$. By abuse, we also let Γ denote the subgroup of the automorphism group of the variety L spanned by $\tilde{g}_1$. This is a finite group acting on L compatibly with the $\mathbb{G}_m$-action on the fibers and the Γ-action on the base. By definition, we have built a Γ-linearization of $\mathcal{L}$, and we now prove that this yields a contradiction. Using [KKV, 2.3] and the fact that $\mathcal{O}(X')^* \simeq k^*$, we obtain an exact sequence

$$1 \to \hat{\Gamma} \to \mathrm{Pic}_{\Gamma}(X') \xrightarrow{\phi} \mathrm{Pic}(X'),$$

where the image of the morphism $\hat{\Gamma} \to \mathrm{Pic}_{\Gamma}(X')$ is the subgroup of the (classes of) linearizations of $\mathcal{O}_{X'}$, and ϕ is defined by forgetting the linearization. On the other hand, we have $\mathrm{Pic}_{\Gamma}(X')_{\mathrm{tor}} \simeq \mathrm{Pic}(X)_{\mathrm{tor}} \simeq \hat{\Gamma}$ [V1, 2.5.2]. It follows that the image of ϕ is a free abelian subgroup of $\mathrm{Pic}(X')$. This is a contradiction as $[\mathcal{L}]$ is of order two in $\mathrm{Pic}(X')$ and linearizable. $\square$

Proposition 4.4.4.13. *We have the following upper bounds for the length m of the iteration of Cox rings:*

- *$m = 0$ (i.e. X is factorial) exactly when $X = G$ or $X = G/F_{\mathbb{I}}$,*
- *$m \leqslant 1$ if F is binary icosahedral or cyclic of order $\leqslant 2$,*
- *$m \leqslant 2$ if F is cyclic of order $\geqslant 3$,*
- *$m \leqslant 3$ if F is binary dihedral or binary tetrahedral,*
- *$m \leqslant 4$ if F is binary octahedral.*

Moreover, $\hat{X}^{(m)}$ is the characteristic space of a normal almost homogeneous G-threefold ${X'}^{(m)}$ with torsion-free class group.

Proof. Except for the upper bounds on m, the statement is clear from Construction 4.4.4.6. We have $m = 0$ if and only if X is factorial (i.e. $\mathrm{Cl}(X) = 0$). In view of the exact sequence (4.4.2), this amount to $X = G/F$ with $\hat{F} = 0$, whence F is trivial or icosahedral. If F is trivial or icosahedral, then $\mathrm{Cl}(X)$ is torsion-free by 4.3.2.1. This in turn implies that $\hat{X}$ is factorial by [ADHL, 1.4.1.5], whence $m \leqslant 1$ in this case. If $F = \mu_2$, then $m = 1$ by virtue of Lemma 4.4.4.8. Also, if $F = \mu_n$, $n \geqslant 3$, then $m \leqslant 2$ by Lemma 4.4.4.8 and [ADHL, 1.4.1.5].

Now suppose that $F = F_{\mathbb{O}}$. Then, either X has torsion-free class group and $m = 1$, or we obtain a normal $G/F_{\mathbb{T}}$-embedding X'. Indeed, there is no other possibility as $F_{\mathbb{T}}$ is the derived subgroup of $F_{\mathbb{O}}$ and $F_{\mathbb{O}}/F_{\mathbb{T}} \simeq \mu_2$ is a simple group. Then, either X' has torsion-free class group and $m = 2$, or we obtain a normal $G/F_{\mathbb{D}_2}$-embedding $X'^{(2)}$. Indeed, there is no other possibility as $F_{\mathbb{D}_2}$ is the derived subgroup of $F_{\mathbb{T}}$ and $F_{\mathbb{T}}/F_{\mathbb{D}_2} \simeq \mu_3$ is a simple group. The derived subgroup of $F_{\mathbb{D}_2}$ is μ_2 and $F_{\mathbb{D}_2}/\mu_2 \simeq \mu_2 \times \mu_2$. In view of Lemma 4.4.4.12, we now have only two possibilities

1. $\mathrm{Cl}(X'^{(2)})$ is torsion-free, whence $m = 3$,

2. $\mathrm{Cl}(X'^{(2)}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In the second case, we obtain a normal G/μ_2-embedding $X'^{(3)}$, and Lemma 4.4.4.8 gives that $\hat{X}^{(4)}$ is factorial, thus $m = 4$ in this case.

In the case $F = F_{\mathbb{T}}$, we have $m \leqslant 3$ by the last paragraph. It remains to treat the case $F = F_{\mathbb{D}_n}$, $n \geqslant 2$. We have $D(F_{\mathbb{D}_n}) \simeq \mu_n$ and $F_{\mathbb{D}_n}/\mu_n \simeq \mu_2 \times \mu_2$ when n is even, and $F_{\mathbb{D}_n}/\mu_n \simeq \mu_4$ when n is odd. In this last case, the proper subgroups of $F_{\mathbb{D}_n}$ containing its derived subgroup are cyclic. Hence, either $\mathrm{Cl}(X)$ is torsion-free and $m = 1$, or we obtain a normal embedding X' of G modulo a cyclic group, whence $m \leqslant 3$. If n is even, the subgroups of $F_{\mathbb{D}_n}$ containing its derived subgroup are $F_{\mathbb{D}_n}$, two copies of $F_{\mathbb{D}_{n/2}}$, μ_{2n}, and μ_n. Hence, either $\mathrm{Cl}(X)$ is torsion-free ($m = 1$), or we obtain a normal embedding X' of G modulo a cyclic group ($m \leqslant 3$), or we obtain a normal $G/F_{\mathbb{D}_{n/2}}$-embedding X'. In this last case, $\mathrm{Cl}(X')_{\mathrm{tor}}$ must be trivial, thus $m = 2$. Indeed, the order of $\mathrm{Cl}(X')_{\mathrm{tor}}$ cannot be 2 by Lemma 4.4.4.12. Suppose by contradiction that this order is 4. By Lemma 4.4.4.12 again, we must have $\mathrm{Cl}(X')_{\mathrm{tor}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but this is impossible in view of Lemma 4.4.4.11. □

4.4.5 Presentation of $\mathrm{Cox}(X)$ by generators and relations

In this section, we develop a general strategy for the description of $\mathrm{Cox}(X)$ by generators and relations. In the sequel, we always suppose that X doesn't admit a G-stable divisor dominating $\mathbb{P}^1_k$, and admits at least two exceptional points. These assumptions are aimed to simplify the notation, the cases left aside can be treated with the same method. To begin with, an easy consequence of Theorem 4.2.2.2 is the

Proposition 4.4.5.1. *The Cox ring of X is generated as a k-algebra by the simple G-modules spanned by the canonical sections of the exceptional divisors.*

This result provides us with homogeneous generators for $\mathrm{Cox}(X)$, namely the elements of k-bases of these simple G-modules. The more difficult task is now to describe the ideal of relations between them. We use as basic tools some elementary facts from the representation theory of G. For each $n \geqslant 0$, denote V_n the space of binary forms of degree n. These spaces are naturally G-modules, G acting by linear change of variables. Moreover, they are up to isomorphism the simple G-modules (see [Sp]). Also, recall the Clebsch-Gordan decomposition

$$V_n \otimes_k V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus V_{n+m-4} \oplus \dots \oplus V_{n-m}, \qquad n \geqslant m.$$

For a B-stable effective divisor E in X, let $V_E \subset \mathrm{Cox}(X)$ denote the simple G-module spanned by the canonical section associated with E.

Normal G/μ_n-embeddings, $n \geqslant 1$

In this section, X is a normal G/μ_n-embedding ($n \geqslant 1$). Remark that if $n \leqslant 2$, we can still choose two distinct exceptional points $x_0, x_\infty \in \mathbb{P}^1_k$ which play the same role as in the case $n \geqslant 3$. Indeed, we can suppose up to a G-equivariant automorphism of X that the pairwise distinct exceptional points on $B\backslash G/\mu_n \simeq \mathbb{P}^1_k$ are

$$x_0 = [0:1], x_\infty = [1:0], x_1 = [\alpha_1 : \beta_1], ..., x_r = [\alpha_r : \beta_r],$$

with respect to the homogeneous coordinates $g_3^{\bar{n}}, g_4^{\bar{n}}$ (4.5.1). The last proposition provides a surjective morphism of graded k-algebras

$$\varphi : S := \mathrm{Sym}_k(V_{E^{x_0}} \oplus V_{E^{x_\infty}} \oplus (\textstyle\bigoplus_{i=1}^r V_{E^{x_i}})) \otimes_k k[(r_{ij})_{i\in\{0,\infty,1,...,r\};j}] \to \mathrm{Cox}(X),$$

where $k[(r_{ij})_{i\in\{0,\infty,1,...,r\};j}] \subset \mathrm{Cox}(X)$ is a polynomial k-algebra, $V_{E^{x_0}} \simeq V_{E^{x_\infty}} \simeq V_1$ and $V_{E^{x_i}} \simeq V_{\bar{n}}$. We view $\mathrm{Cox}(X)$ as a graded k-subalgebra of the coordinate algebra of the open $G \times \mathbb{D}(\mathrm{Cl}(X))$-orbit $\hat{X}_0$ in $\hat{X}$, and use the description of this orbit provided by [V1, 2.8.6]

$$\hat{X}_0 \simeq G \times^{\mu_n} \mathbb{D}(\mathrm{Cl}(X)).$$

This implies an isomorphism of graded k-algebras

$$\mathcal{O}(\hat{X}_0) \simeq \bigoplus_{(k\omega,[D])\in\hat{T}\times\mathrm{Cl}(X),\, k \bmod n=[D]_{|G/\mu_n}} \mathcal{O}(G)^{(T)}_{k\omega} e^{[D]},$$

where $e^{[D]}$ denote the character of $\mathbb{D}(\mathrm{Cl}(X))$ associated with $[D]$, and $\mathrm{Cl}(G/\mu_n)$ is identified with $\mathbb{Z}/n\mathbb{Z}$. The canonical sections

$$s_0, s_\infty, s_i, r_{ij}$$

are respectively identified with

$$g_3 e^{[E^{x_0}]}, g_4 e^{[E^{x_\infty}]}, (\beta_i g_3^{\bar{n}} - \alpha_i g_4^{\bar{n}}) e^{[E^{x_i}]}, e^{[X_j^{x_i}]}.$$

We consider respectively the following k-bases of T-eigenvectors

- $(s_0, t_0) := (g_3 e^{[E^{x_0}]}, g_1 e^{[E^{x_0}]})$
- $(s_\infty, t_\infty) := (g_4 e^{[E^{x_\infty}]}, g_2 e^{[E^{x_\infty}]})$

of the simple G-modules $V_{E^{x_0}}$ and $V_{E^{x_\infty}}$. To start with, we have the simple case where the only exceptional points are x_0 and x_∞:

Proposition 4.4.5.2. *If $r = 0$, the ideal* $\ker\varphi$ *is principal, generated by the G-invariant relation*

$$s_\infty t_0 - s_0 t_\infty = \textstyle\prod_j r_{0j}^{m_{0j}} \prod_j r_{\infty j}^{m_{\infty j}},$$

where $(m_{0j})_j$,$(m_{\infty j})_j$ are the families of non-negative integers defined by

$$[E^{x_0}] + [E^{x_\infty}] = \textstyle\sum_j m_{0j}[X_j^{x_0}] + \sum_j m_{\infty j}[X_j^{x_\infty}].$$

Proof. In this situation, S is a polynomial k-algebra of dimension $4 + \sharp((r_{0j})_j) + \sharp((r_{\infty j})_j)$, and the dimension of $\mathrm{Cox}(X)$ is $3 + \sharp((r_{0j})_j) + \sharp((r_{\infty j})_j)$. It follows that $\ker\varphi$ is a principal ideal generated by a G-invariant relation. The determinant $s_\infty t_0 - s_0 t_\infty = e^{[E^{x_0}]+[E^{x_\infty}]}$ is a non-zero G-invariant homogeneous element in $\mathrm{Cox}(X)$. Using [V1, 2.8.5], we obtain a G-invariant irreducible relation as in the above statement which necessarily generates $\ker\varphi$. □

We now treat the case where $r \geqslant 1$ (i.e. X admits at least three exceptional points). For this, we first show that X can be assumed to satisfy the following technical conditions:

- the special fiber is a normal variety,
- the class group of X is torsion-free.

Recall that the special fiber is a normal variety if and only if

$$n \leqslant 2, \text{ or } (X_j^{x_0})_j \neq \emptyset \text{ and } (X_j^{x_\infty})_j \neq \emptyset \quad \text{(Propositions 4.4.2.1 and 4.4.2.3).}$$

Consider the G-stable open subvariety V consisting of the orbits of codimension $\leqslant 1$. We have $\mathrm{Cox}(V) \simeq \mathrm{Cox}(X)$ because V has a complement of codimension $\geqslant 2$ in X. Thus, we can replace X by V. Also, consider an almost homogeneous G-threefold Y having three orbits, namely the open orbit and two G-stable prime divisors Y^0 and Y^∞. We can choose Y so that $v_{Y^0}, v_{Y^\infty} \notin \mathcal{V}(X)$ and $\pi(Y^0) = x_0$, $\pi(Y^\infty) = x_\infty$. Using the valuative criterion of separation [T11, App. B], it is immediate to verify that X and Y glue together into an almost homogeneous G-threefold X_1. By [V1, 2.8.1], we have an isomorphism

$$\mathrm{Cox}(X) \simeq \mathrm{Cox}(X_1)/(r_0 - 1, r_\infty - 1),$$

where r_0, r_∞ are the canonical sections respectively associated to Y^0, Y^∞. Also, X_1 has by construction at least three exceptional points and an associated special fiber which is a normal variety.

Now, if X_1 has a class group with a non-trivial torsion subgroup. Then, we can consider a smooth complete almost homogeneous G-threefold X_1' containing the smooth locus of X_1 as an open G-stable subvariety [V1, 2.8.4]. By [V1, 3.1.7], $\mathrm{Pic}(X_1')$ is free of finite rank, and it is directly checked that X_1' has at least three exceptional points, and an associated special fiber which is a normal variety. As above, we have an isomorphism

$$\mathrm{Cox}(X_1) \simeq \mathrm{Cox}(X_1')/((r_i - 1)_i),$$

where the r_i are the canonical sections associated to the G-stable prime divisors lying in $X_1' \setminus (X_1)_{\mathrm{sm}}$.

Possibly replacing X by X_1 or X_1', we can suppose that X has a torsion free class group, at least three exceptional points, and a special fiber which is a normal variety. In Proposition 4.4.5.3, we prove that certain G-submodules of S generate the ideal $\ker \varphi$, we start by defining these G-submodules. For $k, l \in \{0, \infty, 1, ..., r\}$, consider the natural surjective morphisms of G-modules

$$\begin{cases} \varphi_{kl} : V_{E^{x_k}} \otimes_k V_{E^{x_l}} \to V_{E^{x_k}} V_{E^{x_l}} \subset \mathrm{Cox}(X)_{[E^{x_k}]+[E^{x_l}]}, \; k < l \\ \varphi_{kk} : \mathrm{Sym}_k^2(V_{E^{x_k}}) \to V_{E^{x_k}}^2 \subset \mathrm{Cox}(X)_{2[E^{x_k}]}, \end{cases}$$

obtained via restriction of φ. Using the Clebsch-Gordan decomposition, we obtain via a direct computation that any B-semi-invariant in a product $V_{E^{x_k}} V_{E^{x_l}}$ is a non-zero scalar multiple of an element of the form

$$g_3^{n_{0,kl}} g_4^{n_{\infty,kl}} e^{[E^{x_k}]+[E^{x_l}]} \in \mathrm{Cox}(X)_{[E^{x_k}]+[E^{x_l}]} \subset \mathcal{O}(G) e^{[E^{x_k}]+[E^{x_l}]}.$$

As $\mathrm{Cox}(X)^U$ is generated as a k-algebra by the elements

$$(s_i)_{i \in \{0,\infty,1,\ldots,r\}}, (r_{ij})_{i \in \{0,\infty,1,\ldots,r\};j},$$

it follows from the form of this B-semi-invariant that it equals a monomial

$$s_0^{n_{0,kl}} s_\infty^{n_{\infty,kl}} \textstyle\prod_{i \in \{0,\infty,1,\ldots,r\};j} r_{ij}^{a_{i,klj}},$$

where the $(a_{i,klj})_{i \in \{0,\infty,1,\ldots,r\};j}$ is the family of non-negative integers defined by

$$[E^{x_k}] + [E^{x_l}] = n_{0,kl}[E^{x_0}] + n_{\infty,kl}[E^{x_\infty}] + \textstyle\sum_{i \in \{0,\infty,1,\ldots,r\};j} a_{i,klj}[X_j^{x_i}].$$

The table below lists the B-weights occurring in the products $V_{E^{x_k}}V_{E^{x_l}}$. For each B-weight $m\omega$ in the table, the third column provides an explicit B-semi-invariant $x_{kl,m\omega}$. To shorten the notation, we let

$$\mathbf{r}^{\mathbf{a}_{kl,m\omega}} := \prod_{i\in\{0,\infty,1,\dots,r\};j} r_{ij}^{a_{i,klj}}.$$

G-module	B-weight: $m\omega$	B-semi-invariant: $x_{kl,m\omega}$
$V_{E^{x_0}}V_{E^{x_\infty}}$	2ω 0	s_0s_∞ $\mathbf{r}^{\mathbf{a}_{0\infty,0}}$
$V_{E^{x_0}}^2$	2ω	s_0^2
$V_{E^{x_\infty}}^2$	2ω	s_∞^2
$V_{E^{x_0}}V_{E^{x_i}}$ $(1\leqslant i\leqslant r)$	$(\bar{n}+1)\omega$ $(\bar{n}-1)\omega$	s_0s_i $s_\infty^{\bar{n}-1}\mathbf{r}^{\mathbf{a}_{0i,(\bar{n}-1)\omega}}$
$V_{E^{x_\infty}}V_{E^{x_i}}$ $(1\leqslant i\leqslant r)$	$(\bar{n}+1)\omega$ $(\bar{n}-1)\omega$	$s_\infty s_i$ $s_0^{\bar{n}-1}\mathbf{r}^{\mathbf{a}_{\infty i,(\bar{n}-1)\omega}}$
$V_{E^{x_i}}^2$ $(1\leqslant i\leqslant r)$	$2\bar{n}\omega$ $m_p\omega, m_p=2(\bar{n}-2p), p=1,\dots,\lfloor\frac{\bar{n}}{2}\rfloor$	s_i^2 $s_0^{\bar{n}-2p}s_\infty^{\bar{n}-2p}\mathbf{r}^{\mathbf{a}_{ii,m_p\omega}}$
$V_{E^{x_i}}V_{E^{x_j}}$ $(1\leqslant i<j\leqslant r, \alpha_i\beta_j+\alpha_j\beta_i\neq 0)$	$2\bar{n}\omega$ $m_p\omega, m_p=2(\bar{n}-p), p=1,\dots,\bar{n}$	s_is_j $s_0^{\bar{n}-p}s_\infty^{\bar{n}-p}\mathbf{r}^{\mathbf{a}_{ij,m_p\omega}}$
$V_{E^{x_i}}V_{E^{x_j}}$ $(1\leqslant i<j\leqslant r, \alpha_i\beta_j+\alpha_j\beta_i=0)$	$2\bar{n}\omega$ $m_p\omega, m_p=2(\bar{n}-(2p+1)), p=0,\dots,\lfloor\frac{\bar{n}-1}{2}\rfloor$	s_is_j $s_0^{\bar{n}-(2p+1)}s_\infty^{\bar{n}-(2p+1)}\mathbf{r}^{\mathbf{a}_{ij,m_p\omega}}$

For $k,l\in\{0,\infty,1,\dots,r\}$, $0\leqslant k\leqslant l\leqslant\infty$, and $m\omega$ a B-weight occurring in $V_{E^{x_k}}V_{E^{x_l}}$, let $y_{kl,m\omega}$ be the unique B-semi-invariant in S sent to $x_{kl,m\omega}$ by φ_{kl}. Then, consider the following G-submodules of $\ker\varphi$:

$$M_{kl} := (\bigoplus_{m\omega} < G.(x_{kl,m\omega}-y_{kl,m\omega}) >)\oplus\ker\varphi_{kl},$$

where $x_{kl,m\omega}$ is naturally identified with an element of S. Also, in view of the description of $\mathrm{Cox}(X)^U$ by generators and relations (Section 4.4.2), define for $i=1,\dots,r$ the following G-submodules of $\ker\varphi$:

$$N_i := < G.(\beta_is_0^{\bar{n}}\otimes\prod_j r_{0j}^{h_{0j}}-\alpha_is_\infty^{\bar{n}}\otimes\prod_j r_{\infty j}^{h_{\infty j}}-s_i\otimes\prod_j(r_{ij})^{h_{ij}}) >\simeq V_{\bar{n}}.$$

Proposition 4.4.5.3. *The ideal* $\ker\varphi$ *is generated by the G-modules M_{kl} and N_i.*

Proof. Consider the natural $\mathrm{Cl}(X)$-grading on S induced by the canonical projection $\mathrm{WDiv}(X)\to\mathrm{Cl}(X)$, and let I be the homogeneous ideal generated by the G-modules M_{kl} and N_i. We prove that the surjective morphism of graded k-algebras

$$\bar{\varphi}: S/I\to\mathrm{Cox}(X)$$

induced by φ is an isomorphism. This follows from the claim that $(S/I)^U$ is generated as a k-algebra by the images of the s_i, r_{ij}, $i\in\{0,\infty,1,\dots,r\}$. Indeed, the algebra $(S/I)^U$ can then be presented as the quotient of the polynomial k-algebra in the elements s_i, r_{ij} modulo an ideal J of relations. In view of the presentation of $\mathrm{Cox}(X)^U$ and of the morphism

$$\bar{\varphi}_U:(S/I)^U\to\mathrm{Cox}(X)^U$$

induced by $\bar{\varphi}$, the ideal J is contained in the ideal generated by $\oplus_{i=1}^{r} N_i^U$. As we have the reverse inclusion by definition of I, it follows that the morphism $\bar{\varphi}_U$ is an isomorphism, thus $\bar{\varphi}$ is so.

We now prove the above claim. Remark that because the special fiber is a normal variety, the G-modules N_i become zero modulo $((r_{ij})_{i\in\{0,\infty,1,\dots,r\};j})$. The k-algebra $S/(I,(r_{ij})_{ij})$ is generated by the simple G-modules $V_{E^{x_i}}$, $i \in \{0,\infty,1,\dots,r\}$, and it follows from Lemma 4.4.5.5, the preceding remark, and the definition of the G-modules M_{kl}, that the product of any two of them in $S/(I,(r_{ij})_{ij})$ is their Cartan product. As a consequence of a result of Kostant (see e.g. [Bri85, 4.1 Lemme]), the product of any two simple G-modules in $S/(I,(r_{ij})_{ij})$ is again their Cartan product or zero. This implies that $(S/(I,(r_{ij})_{ij}))^U$ is generated as a k-algebra by the images of the s_i, $i \in \{0,\infty,1,\dots,r\}$. Let A be the k-subalgebra of $(S/I)^U$ generated by the images of the s_i, r_{ij}, $i \in \{0,\infty,1,\dots,r\}$. Viewing A and $(S/I)^U$ as $\mathrm{Cl}(X)$-graded $k[(r_{ij})_{ij}]$-modules, the above description of $(S/(I,(r_{ij})_{ij}))^U$ yields

$$(S/I)^U = A + ((r_{ij})_{ij})(S/I)^U.$$

By Lemma 4.4.5.6, the regular functions on X are constant. This implies that S/I (hence $(S/I)^U$ and A) can be endowed with a coarser positive grading such that the elements r_{ij} have non-zero degree. Indeed, S/I is naturally graded by the cone of effective divisors in $\mathrm{Cl}(X)_{\mathbb{Q}}$, but this cone contains no line as $\mathcal{O}(X) \simeq k$. Thus, we can choose convenient positive integers a_{ij} such that the linear map $\mathrm{Cl}(X) \to \mathbb{Z}$ defined by $[X_j^{x_i}] \mapsto a_{ij}$ is positive on generators. Now, the graded Nakayama lemma yields $(S/I)^U = A$, whence the claim. $\square$

Corollary 4.4.5.4. *Suppose that $n \leqslant 2$. Then, the ideal $\ker \varphi$ is generated by the G-invariant relations*

$$s_k t_l - s_l t_k = (\alpha_k \beta_l - \alpha_l \beta_k) \prod_{i\in\{0,\infty,1,\dots,r\};j} r_{ij}^{m_{klij}},\ k,l \in \{0,\infty,1,\dots,r\},\ 0 \leqslant k < l \leqslant \infty,$$

where $(m_{klij})_{ij}$ are the families of non-negative integers defined by

$$[E^{x_k}] + [E^{x_l}] = \textstyle\sum_{ij} m_{klij}[X_j^{x_i}],$$

and the simple G-modules N_i.

Proof. It suffices to notice that in this case, the M_{kk} are trivial, and the M_{kl}, $k < l$ are the G-invariant lines spanned by the elements $s_k t_l - s_l t_k - (\alpha_k \beta_l - \alpha_l \beta_k)\mathbf{r}^{\mathbf{a}_{kl,0}}$. $\square$

Lemma 4.4.5.5. *Let Z be a normal G/μ_n-embedding ($n \geqslant 1$) whose associated special fiber is a normal variety, and admitting three or more exceptional points. Then, the special fiber is a horospherical variety.*

Proof. In view of the table above, it suffices to show that the total degrees of the monomials $\mathbf{r}^{\mathbf{a}_{kl,m\omega}}$ are non-zero. This is equivalent to prove that the divisors

- $E^{x_0} + E^{x_\infty}$,
- $E^{x_0} + E^{x_i} - (\bar{n}-1)E^{x_\infty}$, $i = 1,\dots,r$,
- $E^{x_\infty} + E^{x_i} - (\bar{n}-1)E^{x_0}$, $i = 1,\dots,r$,
- $2E^{x_i} - (\bar{n}-2p)(E^{x_0} + E^{x_\infty})$, $1 \leqslant i \leqslant r$, $p = 1,\dots,\lfloor \frac{\bar{n}}{2} \rfloor$,
- $E^{x_i} + E^{x_j} - (\bar{n}-p)(E^{x_0} + E^{x_\infty})$, $1 \leqslant i < j \leqslant r$, $p = 1,\dots,\bar{n}$.

are not principal. We verify this for the divisor $E^{x_1}+E^{x_2}-k(E^{x_0}+E^{x_\infty})$, $0\leqslant k\leqslant \bar{n}-1$, where $\bar{n}=n$ is assumed to be odd. We skip the proof for the other cases which are treated similarly. We look for a B-semi-invariant rational function gf_ω^γ, $\gamma\in\mathbb{Z}$, where f_ω is defined in 4.5.1, and $g\in k(g_3^n/g_4^n)^*$, whose divisor is the above one. Necessarily, zeroes and poles of g are located on the points $x_0, x_\infty, x_1, x_2, x_d$, and their orders are respectively $\alpha,\alpha,1,1,\beta$. In order to simplify the notation, we suppose that for each of the points x_0,x_∞,x_1,x_2, there is exactly one G-stable prime divisor sent to this point, the general case being treated the same way. Coordinates on the hyperspace are defined as in Section 4.5.1, and we set $v_{X^{x_i}}=(x_i,h_i,l_i)$, $i\in\{0,\infty,1,2\}$. This gives

$$\begin{aligned}\operatorname{div}(gf_\omega^\gamma)&=\alpha(n(E^{x_0}+E^{x_\infty})+h_0X^{x_0}+h_\infty X^{x_\infty})+E^{x_1}+h_1X^{x_1}+E^{x_2}+h_2X^{x_2}+\beta D^{x_d}\\&\quad+\gamma(D^{x_d}-\frac{n-1}{2}(E^{x_0}+E^{x_\infty})+l_0X^{x_0}+l_\infty X^{x_\infty}+l_1X^{x_1}+l_2X^{x_2}).\end{aligned}$$

It follows that we must have $\beta=-\gamma$, and $\alpha n-\gamma\frac{n-1}{2}=-k$, the general solution of this last equation being of the form

$$(\alpha,\gamma)=(-k,-2k)+u(\tfrac{n-1}{2},n),\ u\in\mathbb{Z}.$$

Recall that we must have $2\alpha+2+\beta=0$, so that $u=2$. Now, this principal divisor reads

$$\operatorname{div}(gf_\omega^\gamma)=$$
$$E^{x_1}+E^{x_2}-k(E^{x_0}+E^{x_\infty})+(\alpha h_0+\gamma l_0)X^{x_0}+(\alpha h_\infty+\gamma l_\infty)X^{x_\infty}+(h_1+\gamma l_1)X^{x_1}+(h_2+\gamma l_2)X^{x_2}.$$

It follows that we must impose $\frac{l_1}{h_1}=-\frac{1}{\gamma}=-\frac{1}{2(n-k)}$. This condition cannot be satisfied as soon as $k<n-1$, because we must have $\frac{l_1}{h_1}\leqslant-\frac{1}{2}$. Thus, we suppose that $k=n-1$, and we look at the condition $\frac{l_0}{h_0}=-\frac{\alpha}{\gamma}=0$ which cannot be satisfied. $\square$

Lemma 4.4.5.6. *Let Z be a normal G/μ_n-embedding ($n\geqslant 1$) whose associated special fiber is a normal variety and with three or more exceptional points. Then, $\mathcal{O}(Z)\simeq k$.*

Proof. It suffices to prove that every B-semi-invariant regular function $u\in\mathcal{O}(X)$ is necessarily constant. Let $\alpha:=1$ if n is odd, and $\alpha:=2$ otherwise. We can write $u=gf_{\alpha\omega}^m$, where $g\in k(g_3^{\bar{n}}/g_4^{\bar{n}})^*$, and $f_{\alpha\omega}$ is defined in Section 4.5.1. Moreover the divisor of u is effective by assumption, so that we have $v_D(u)\geqslant 0$, $\forall D\in \operatorname{WDiv}(X)^B$. By summing these inequalities over the set of colors we obtain the inequality

$$m(-\tfrac{1}{2}+\tfrac{1}{2\bar{n}}-\tfrac{1}{2}+\tfrac{1}{2\bar{n}}+1)\geqslant 0,$$

which implies that $m\geqslant 0$. In view of Proposition 4.4.2.3, Z admits at least three G-stable prime divisors $X^{x_0},X^{x_\infty},X^{x_1}$ sent to pairwise distinct exceptional points x_0,x_∞,x_1. Consider the set of B-stable prime divisors consisting of $X^{x_0},X^{x_\infty},X^{x_1}$, and all the colors except E^{x_0},E^{x_∞}, and E^{x_1}. By summing the inequalities over this set we obtain the inequality

$$m(\tfrac{l_0}{h_0}+\tfrac{l_\infty}{h_\infty}+\tfrac{l_1}{h_1}+1)\geqslant 0,$$

where $(x_0,l_0,h_0),(x_\infty,l_\infty,h_\infty)$, and (x_1,l_1,h_1) are the respective coordinates in the hyperspace associated with X of the valuations $v_{X^{x_0}},v_{X^{x_\infty}}$ and $v_{X^{x_1}}$. Because $\frac{l_0}{h_0}+\frac{l_\infty}{h_\infty}+\frac{l_1}{h_1}+1<0$ (see 4.5.1), the above inequality is equivalent to $m\leqslant 0$, and we obtain that $m=0$. Thus, we have that $g\in k^*$ because the divisor of g has to be effective. Indeed, any non-constant rational function on $\mathbb{P}^1_k/\mu_n$ admits at least one pole. We conclude that u is constant. $\square$

Remark 4.4.5.7. Suppose that $n\leqslant 2$. From the above description of $\operatorname{Cox}(X)$, we have

$$\tilde{X}\text{ is a complete intersection }\iff \sharp(x_i)_i\leqslant 2\iff \tilde{X}\text{ is a hypersurface.}$$

Furthermore these equivalent conditions characterize when the good quotient morphism $\tilde{X} \xrightarrow{//G} \mathbb{A}^N$ is faithfully flat. Indeed, this morphism has equidimensional fibers if and only if $\sharp(x_i)_i \leqslant 2$ (see Section 4.4.2.1). Moreover, $\tilde{X}$ is Cohen-Macaulay under these assumptions, so that the quotient morphism is flat [Har2, III, Ex 10.9].

Example 4.4.5.8. Consider a normal G-embedding X with four exceptional points

$$x_1 = [1:0], x_2 = [0:1], x_3 = [1:1], x_4 = [2:1] \in \mathbb{P}^1_k,$$

to which are sent the exceptional divisors

$$E^{x_1},\ X^{x_1},\ E^{x_2},\ X^{x_2},\ E^{x_3},\ X^{x_3},\ E^{x_4}, \text{ and } X^{x_4}.$$

We choose the section $\omega \mapsto f_\omega$ of the exact sequence

$$1 \to k(\mathbb{P}^1_k)^* \to k(X)^{(B)} \to \mathbb{Z}\omega \to 1,$$

such that $\mathrm{div}(f_\omega)$ is the color E^{x_1} on G. This provides coordinates on $\breve{\mathcal{E}}$ for which we have

$$v_{X^{x_1}} = (x_1, 2, -1), v_{X^{x_2}} = (x_2, 3, -5), v_{X^{x_3}} = (x_3, 1, -1), \text{ and } v_{X^{x_4}} = (x_4, 5, -4).$$

By 4.3.2.2, the generators $([E^{x_1}], [X^{x_1}], ..., [E^{x_4}],\ [X^{x_4}])$ of the class group give the following presentation matrix

$$P := \begin{bmatrix} -1 & -2 & 1 & 3 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 1 & 5 \\ 1 & -1 & 0 & -5 & 0 & -1 & 0 & -4 \end{bmatrix}.$$

From this presentation of $\mathrm{Cl}(X)$ we obtain

- $[E^{x_1}] = [X^{x_1}] + 5[X^{x_2}] + [X^{x_3}] + 4[X^{x_4}]$
- $[E^{x_2}] = 3[X_1^{x_1}] + 2[X^{x_2}] + [X^{x_3}] + 4[X^{x_4}]$
- $[E^{x_3}] = 3[X_1^{x_1}] + 5[X^{x_1}] + 4[X^{x_4}]$
- $[E^{x_4}] = 3[X_1^{x_1}] + 5[X^{x_1}] + [X^{x_3}] - [X^{x_4}]$

Finally, we find

$$\mathrm{Cox}(X) = k[s_1, t_1, s_2, t_2, s_3, t_3, s_4, t_4, r_1, r_2, r_3, r_4],$$

with ideal of relations generated by the following

- $s_1t_2 - s_2t_1 - r_1^4r_2^7r_3^2r_4^8$
- $s_1t_3 - s_3t_1 - r_1^4r_2^{10}r_3r_4^8$
- $s_1t_4 - s_4t_1 - r_1^4r_2^{10}r_3^2r_4^3$
- $s_2t_3 - s_3t_2 + r_1^6r_2^7r_3r_4^8$
- $s_2t_4 - s_4t_2 + 2r_1^6r_2^7r_3^2r_4^3$
- $s_3t_4 - s_4t_3 + r_1^6r_2^{10}r_3r_4^3$
- $s_2r_2^3 + s_1r_1^2 - s_3r_3$
- $t_2r_2^3 + t_1r_1^2 - t_3r_3$
- $s_2r_2^3 + 2s_1r_1^2 - s_4r_4^5$
- $t_2r_2^3 + 2t_1r_1^2 - t_4r_4^5$

Example 4.4.5.9. Consider a normal G/μ_3-embedding X with three pairwise distinct exceptional points $x_0 = [0:1], x_\infty = [1:0], x_1 = [\alpha:\beta] \in \mathbb{P}^1_k$, to which are sent the following exceptional divisors:

$$E^{x_0},\ X^{x_0},\ E^{x_\infty},\ X^{x_\infty},\ E^{x_1}, \text{ and } X^{x_1}.$$

We choose coordinates on the hyperspace as described in Section 4.5.1, and the G-valuations associated to the G-invariant exceptional divisors have the following coordinates in $\breve{\mathcal{E}}$:

$$v_{X^{x_0}} = (x_0, 1, -1), v_{X^{x_\infty}} = (x_\infty, 1, -1), \text{ and } v_{X^{x_1}} = (x_1, 1, -1).$$

The class group of X is free of rank 3 and the special fiber is a normal variety, so the preceding proposition applies. We consider the following k-bases for the simple G-modules generating $\mathrm{Cox}(X)$:

- $V_{E^{x_0}} : (s_0, t_0) := (s_0, g_1 e^{[E^{x_0}]})$,
- $V_{E^{x_\infty}} : (s_\infty, t_\infty) := (s_\infty, g_2 e^{[E^{x_\infty}]})$,
- $V_{E^{x_1}} : (s_1, t_1, u_1, v_1) := (s_1, (\beta_1 g_3^2 g_1 - \alpha_1 g_4^2 g_2) e^{[E^{x_1}]}, (\beta_1 g_3 g_1^2 - \alpha_1 g_4 g_2^2) e^{[E^{x_1}]}, (\beta_1 g_1^3 - \alpha_1 g_2^3) e^{[E^{x_1}]})$,
- $V_{X^{x_i}}$, $i = 0, \infty, 1 : (r_i) := (e^{[X^{x_i}]})$.

We have

$$\mathrm{Cox}(X) \simeq \mathrm{Sym}_k(V_{E^{x_0}} \oplus V_{E^{x_\infty}} \oplus V_{E^{x_1}}) \otimes_k k[r_0, r_\infty, r_1] \mod I,$$

where $k[r_0, r_\infty, r_1]$ is a polynomial k-algebra, and I is generated by the simple G-modules of relations given in the following table:

G-module	B-semi-invariant	B-weight
$V_0 \simeq M_{0\infty} \subset V_{E^{x_0}} \otimes_k V_{E^{x_\infty}} \simeq V_0 \oplus V_2$	$t_0 s_\infty - s_0 t_\infty - r_0 r_\infty r_1^2$	0
$V_2 \simeq M_{01} \subset V_{E^{x_0}} \otimes_k V_{E^{x_1}} \simeq V_4 \oplus V_2$	$t_1 s_0 - s_1 t_0 - \alpha s_\infty^2 \otimes r_0 r_\infty^2 r_1$	2ω
$V_2 \simeq M_{\infty 1} \subset V_{E^{x_\infty}} \otimes_k V_{E^{x_1}} \simeq V_4 \oplus V_2$	$t_1 s_\infty - s_1 t_\infty - \beta s_0^2 \otimes r_0^2 r_\infty r_1$	2ω
$V_2 \simeq M_{11} \subset \mathrm{Sym}_k^2(V_{E^{x_1}}) \simeq V_6 \oplus V_2$	$t_1^2 - s_1 u_1 - \alpha\beta s_0 s_\infty \otimes r_0^3 r_\infty^3 r_1^2$	2ω
$N_1 \simeq V_3$	$\beta s_0^3 \otimes r_0 - \alpha s_\infty^3 \otimes r_\infty - s_1 \otimes r_1$	3ω

Example of affine almost homogeneous G-threefolds

Consider the case where X is affine. In [Pop], Popov classifies these varieties up to isomorphism by means of numerical invariants. These results can be reinterpreted in term of the combinatorial description of these varieties. Indeed, the colored hypercone defining X must have all the colors among its generators, which is only possible if $F \simeq \mu_n$ [T97, 5.2]. We use coordinates on the hyperspace as defined in Section 4.5.1, and define $u := 2$ if n is even, and $u := 1$ otherwise. The combinatorial description of the normal affine G/μ_n-embeddings shows that X admits a unique G-orbit of codimension 1 corresponding to a G-stable prime divisor X^{x_0}, which is sent to x_0 by the rational quotient $\pi : X \dashrightarrow \mathbb{P}_k^1$ by B. Moreover, the condition

$$v_{X^{x_0}} = (x_0, h, l) \in \mathbb{P}_k^1 \times \mathbb{Z}_{>0} \times \tfrac{1}{u}\mathbb{Z}, \text{ with } -\tfrac{1}{2} - \tfrac{1}{2\bar{n}} < \tfrac{l}{h} \leqslant -\tfrac{1}{2}, \text{ and } h \wedge ul = 1$$

is satisfied. Remark that if $n \leqslant 2$, we can still choose two distinct points $x_0, x_\infty \in \mathbb{P}_k^1$ which play the same role as in the case $n \geqslant 3$. In general, X admits a G-fixed point, except in the case where $l/h = -1/2$. Concretely, this exceptional embedding is realized as the G-linearized line bundle

$$G \times^T \mathbb{A}_k^1 \to G/T,$$

associated to the T-action on $\mathbb{A}_k^1$ defined by the character ω^n.

The generators $([E^{x_d}], [E^{x_0}], [X^{x_0}], [E^{x_\infty}])$ of the class group give the presentation matrix

$$P := \begin{bmatrix} -1 & \bar{n} & h & 0 \\ -1 & 0 & 0 & \bar{n} \\ u & -u\frac{\bar{n}-1}{2} & ul & -u\frac{\bar{n}-1}{2} \end{bmatrix}.$$

Transforming this matrix to its Smith normal form by means of elementary operations on the rows and columns yields an isomorphism

$$\mathrm{Cl}(X) \simeq \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}, \tag{4.4.3}$$

where $d := n \wedge h$ if n is odd or $h + 2l$ is even, and $d := (\bar{n} + h) \wedge (\bar{n} - h)$ otherwise. Panyushev computed this class group in [Pa92, Thm 2] taking as input Popov's numerical invariants.

By Proposition 4.4.5.2, the Cox ring is generated by the elements $s_0, t_0, s_\infty, t_\infty, r_0$, and the relations are generated by a relation of the form

$$r_0^m = s_\infty t_0 - s_0 t_\infty.$$

On the other hand, computing $L_1 + L_2 + \frac{2}{u} L_3$ on the rows of P, we obtain

$$[E^{x_0}] + [E^{x_\infty}] = -(h + 2l)[X^{x_0}].$$

It follows that the sought relation is $r_0^{-(h+2l)} = s_\infty t_0 - s_0 t_\infty$. This presentation of the Cox ring is similar to the one obtained by Batyrev and Haddad in [BatHad].

Comparison with the results of Batyrev and Haddad

In [Pop], Popov associates to the isomophism class of X a unique pair (h_P, n) where $h_P \in]0, 1]$ is a rational number called the *height* of X, and n is the order of the cyclic group stabilizing a point in the open orbit. With [K, III.4.3], the height has the following useful interpretation: the algebra of U-invariant regular functions on X identifies with the algebra of the monoid

$$M_{h,n} = \{(i, j) \in \mathbb{Z}^2_{>0} \,; j \leqslant h_P i \text{ and } n | i - j\}.$$

More precisely, by considering the injective graded morphism

$$\varphi : \mathcal{O}(X)^U \hookrightarrow \mathcal{O}(G/\mu_n)^U = \bigoplus\nolimits_{i,j \geqslant 0, n|i-j} \mathrm{Vect}_k(g_3^i g_4^j)$$

given by the restriction of functions, we identify elements of the monoid $M_{h,n}$ with the monomials in the image of φ. As in the preceding section, consider the G-valuation $v_{X^{x_0}} = (x_0, h, l) \in \check{\mathcal{E}}$. By normality of X, a monomial $g_3^i g_4^j \in \mathcal{O}(G/\mu_n)^U$ belongs to $\mathcal{O}(X)^U$ if and only if

$$\begin{aligned} v_{X^{x_0}}(g_3^i g_4^j) \geqslant 0 \iff & v_{X^{x_0}}\Big(\frac{g_3^i g_4^j}{f_{u\omega}^{(i+j)/u}} f_{u\omega}^{(i+j)/u}\Big) \geqslant 0 \\ \iff & h v_{x_0}\Big(\frac{g_3^i g_4^j}{f_{u\omega}^{(i+j)/u}}\Big) + l(i + j) \geqslant 0 \\ \iff & \frac{h}{\bar{n}}\Big(i + u\frac{\bar{n} - 1}{2}\frac{i + j}{u}\Big) + l(i + j) \geqslant 0 \\ \iff & j \leqslant \alpha i, \text{ where } \alpha := \frac{h(\bar{n} + 1) + 2l\bar{n}}{h(1 - \bar{n}) - 2l\bar{n}}. \end{aligned}$$

Remark that together with the condition $-\frac{1}{2} - \frac{1}{2\bar{n}} < \frac{l}{h} \leqslant -\frac{1}{2}$, we verify that $0 < \alpha \leqslant 1$, with $\alpha = 1$ exactly when $l/h = -1/2$. By unicity of the cone in $\mathbb{Z}^2$ defined by the monomials of $\mathcal{O}(X)^U$, we have $\alpha = h_P$.

In [BatHad], Batyrev and Haddad consider in the affine space $\mathbb{A}^5_k$ endowed with coordinates y, t_1, t_2, t_3, t_4 the hypersurface H_b defined by the equation

$$y^b = t_1 t_4 - t_2 t_3,$$

where $b := \frac{q-p}{k}$, $k := (q - p) \wedge n$, and $p/q = h_P$ with $p \wedge q = 1$. Then, they set $a := \frac{n}{k}$, and let $\mathbb{G}_m \times \mu_a$ act on $\mathbb{A}^5_k$ by allocating the following weights to coordinates

$\deg(y) = (k,0), \deg(t_1) = (-p,-1), \deg(t_2) = (-p,-1), \deg(t_3) = (q,1)$, and $\deg(t_4) = (q,1)$.

This action stabilizes H_b and realizes it as the total coordinate space of X [BatHad, 2.6].

We now verify that this coincides with the presentation of the Cox ring given in the last section. For simplicity, we assume that n is odd. By identification, we have

$$\begin{cases} p = \frac{1}{2(n\wedge h)}(h(n+1)+2nl) \\ q = \frac{1}{2(n\wedge h)}(h(1-n)-2nl) \end{cases}.$$

Then, the following identity holds

$$\begin{aligned} b = \frac{q-p}{k} &= \frac{-(h+2l)n}{(n\wedge h)((q-p)\wedge n)} \\ &= \frac{-(h+2l)n}{(n\wedge h)(\frac{-n(h+2l)}{n\wedge h}\wedge n)} \\ &= \frac{-(h+2l)n}{n(-(h+2l)\wedge n\wedge h)} \\ &= \frac{-(h+2l)}{-2l\wedge n\wedge h} \\ &= -(h+2l). \end{aligned}$$

Turning to the grading of the Cox ring, recall that the isomorphism

$$\mathrm{Cl}(X) \simeq \mathbb{Z}\times\mathbb{Z}/(n\wedge h)\mathbb{Z}$$

has been obtained by considering the four generators $[E^{x_d}], [E^{x_0}], [X^{x_0}], [E^{x_\infty}]$ of $\mathrm{Cl}(X)$, and looking for a $\mathbb{Z}$-basis of the free $\mathbb{Z}$-module on these generators adapted to the submodule of relations. Let $u,v\in\mathbb{Z}$ be such that

$$-qu+kv=1.$$

By the above isomorphism, the elements $[E^{x_0}], [X^{x_0}]$, and $[E^{x_\infty}]$ are sent respectively to $(-p, ub-v), (k,u)$, and (q,v), where the second coordinate is computed in $\mathbb{Z}/(n\wedge h)\mathbb{Z}$. This gives us the corresponding degrees for the generators of the Cox ring. This degrees correspond to the degrees of Batyrev and Haddad up to an automorphism f of $\mathbb{Z}\times\mathbb{Z}/(n\wedge h)\mathbb{Z}$. Indeed, it suffices to set $f((1,0)) = (1, k^{-1}u)$ and $f((0,1)) = (0,k^{-1})$.

4.5 Appendix: Colored equipment of $k(G/F)$

In this appendix, we recall from [T97] the colored equipment of $k(G/F)$ for F either cyclic or one of the binary polyhedral groups in $G = \mathrm{SL}_2$. There is a natural bijection between the monoid of effective B-stable divisors in G/F, and the monoid $\mathcal{O}(G)^{(B\times F)}/k^*$. Indeed, it suffices to consider the well-defined morphism

$$\mathcal{O}(G)^{(B\times F)}/k^* \to \mathrm{WDiv}(X)^B, \bar{f}\mapsto \mathrm{div}_{G/F}(f).$$

Via this bijection, B-stable prime divisors in G/F correspond to indecomposable elements of $\mathcal{O}(G)^{(B\times F)}/k^*$. Remark that we can again see these elements as indecomposable homogeneous elements of $\mathrm{Cox}(G/F)^{(B)}/k^*$. There exists a linear system $\mathcal{O}(G)^{(B\times F)}_{(n_0\omega,\lambda_0)}$ of dimension two over k that defines the morphism $\pi_{|G/F}$, and a non-empty open subset of $\mathbb{P}(\mathcal{O}(G)^{(B\times F)}_{(n_0\omega,\lambda_0)}) \simeq \mathbb{P}^1_k$ that gives the indecomposable elements of $\mathcal{O}(G)^{(B\times F)}/k^*$ corresponding to the parametric colors of G/F. The finite set of exceptional colors of G/F is obtained by taking divisors of zeroes of the non-invertible $\mathrm{Cl}(G/F)$-prime elements appearing in the decompositions of the decomposable elements of $\mathcal{O}(G)^{(B\times F)}_{(n_0\omega,\lambda_0)} \subset \mathrm{Cox}(G/F)$.

Definition 4.5.0.1. [T11, 16.1] The $\mathrm{Cl}(G/F)$-prime elements of $\mathcal{O}(G)^{(B\times F)}_{(n_0\omega,\lambda_0)} \subset \mathrm{Cox}(G/F)$ are called the *regular semi-invariants*. The others elements $\mathcal{O}(G)^{(B\times F)}_{(n_0\omega,\lambda_0)}$ are called the *exceptional semi-invariants*. The *subregular semi-invariants* are the non-invertible $\mathrm{Cl}(G/F)$-prime B-semi-invariants appearing in the decomposition of the exceptional semi-invariants as a product of $\mathrm{Cl}(G/F)$-prime elements.

Any G-valuation or color v of $k(G/F)$ is located in the hyperspace $\check{\mathcal{E}}$ by a triple

$$(x,h,l) \in \mathbb{P}^1_k \times \mathbb{Q}_+ \times \mathbb{Q},$$

where x and h are defined by the restriction of v to $k(\mathbb{P}^1_k)$, and the coordinate l is obtained by evaluating v at a fixed generator of the weight group $\Lambda(G/F)$ viewed in $k(G/F)^{(B)}$ via the choice of a section $\lambda \mapsto f_\lambda$ (4.2.3).

4.5.1 F is cyclic of order $n \geqslant 1$

We identify the character group of $F = \mu_n$ with $\mathbb{Z}/n\mathbb{Z}$. If $n \leqslant 2$, there is no exceptional semi-invariant. If $n \geqslant 3$, there are, up to non-zero scalar multiple, two subregular semi-invariants g_3 and g_4 whose respective weights are $(\omega, 1 \mod n), (\omega, -1 \mod n)$. Let

$$\bar{n} = \begin{cases} n, & n \text{ odd}, \\ n/2, & n \text{ even}. \end{cases}$$

We have $\mathcal{O}(G)^{(B\times\mu_n)}_{(n_0\omega,\lambda_0)} = \mathcal{O}(G)^{(B\times\mu_n)}_{(\bar{n}\omega,\bar{n} \mod n)}$ generated by the two exceptional semi-invariant $g_3^{\bar{n}}$ and $g_4^{\bar{n}}$. There are two exceptional points x_0, x_∞ and two exceptional colors $\pi^*(x_0) = \bar{n}E^{x_0}$, $\pi^*(x_\infty) = \bar{n}E^{x_\infty}$. Let A be an arbitrary regular semi-invariant, and x_d be the corresponding *distinguished point* of $\mathbb{P}^1_k/\mu_n \simeq \mathbb{P}^1_k$. We define a section of $\Lambda(G/\mu_n) \to k(G/\mu_n)^{(B)}$ by the choice of the generator

$$\begin{cases} f_\omega := \frac{A}{(g_3g_4)^{(n-1)/2}}, \text{ of weight } \omega, & (n \text{ odd}), \\ f_{2\omega} := \frac{A^2}{(g_3g_4)^{(n/2-1)}}, \text{ of weight } 2\omega, & (n \text{ even}). \end{cases}$$

In order to have a uniform description of the hyperspace, the third coordinate l_v of a G-valuation (or color) v is defined by $l_v := v(f_\omega)$ if n is odd, and $l_v := v(f_{2\omega})/2$ if n is even. The elements of $\mathcal{V}_x \subset \mathcal{E}_x$ are the vectors (x,h,l) in $\check{\mathcal{E}}$ whose coordinates satisfy the inequalities $2l + h \leqslant 0$ for $x \neq x_d$ and $2l - h \leqslant 0$ for $x = x_d$. The colors are sent to the vectors ε_x for $x \neq x_d, x_0, x_\infty$, to $(x_d, 1, 1)$ for x_d, and to $(x_0, \bar{n}, -(\bar{n}-1)/2)$ (resp. $(x_\infty, \bar{n}, -(\bar{n}-1)/2)$) for x_0 (resp. for x_∞).

4.5.2 F is binary tetrahedral

The character group of $F_{\mathbb{T}}$ identifies with the cyclic group of order 3, for which we choose a generator ζ. Up to a constant, there are three subregular semi-invariants f_v, f_e, f_f whose respective weights are $(4\omega, \zeta), (6\omega, 1), (4\omega, \zeta^{-1})$. We have $\mathcal{O}(G)^{(B\times F_{\mathbb{T}})}_{(n_0\omega,\lambda_0)} = \mathcal{O}(G)^{(B\times F_{\mathbb{T}})}_{(12\omega,1)}$ generated by the three exceptional semi-invariant f_v^3, f_e^2, f_f^3 with the relation $f_v^3 + f_e^2 + f_f^3 = 0$. This defines three exceptional points x_v, x_e, x_f, and three exceptional colors $\pi^*(x_v) = 3E^{x_v}$, $\pi^*(x_e) = 2E^{x_e}$, and $\pi^*(x_f) = 3E^{x_f}$. We define a section of $\Lambda(G/F_{\mathbb{T}}) \to k(G/F_{\mathbb{T}})^{(B)}$ by the choice of the generator $f_{2\omega} := \frac{f_vf_f}{f_e}$. The elements of $\mathcal{V}_x \subset \mathcal{E}_x$ are the vectors (x,h,l) of $\check{\mathcal{E}}$ whose coordinates satisfy the inequalities $l + h \leqslant 0$ for $x \neq x_f, x_v$, and $l \leqslant 0$ for $x = x_f$ or $x = x_v$. The colors are sent to the vectors ε_x for $x \neq x_v, x_e, x_f$, and to $(x_v, 3, 1)$, $(x_e, 2, -1)$, and $(x_f, 3, 1)$ for $\pi^*(x_v)$, $\pi^*(x_e)$ and $\pi^*(x_f)$.

4.5.3 F is binary octahedral

The character group of $F_{\mathbb{O}}$ identifies with the cyclic group of order 2. Up to a constant, there are three subregular semi-invariants f_v, f_e, f_f whose respective weights are $(8\omega, \zeta), (12\omega, 1), (6\omega, \zeta^{-1})$. We have $\mathcal{O}(G)^{(B\times F_{\mathbb{O}})}_{(n_0\omega,\lambda_0)} = \mathcal{O}(G)^{(B\times F_{\mathbb{O}})}_{(24\omega,1)}$ generated by the three exceptional semi-invariant f_v^3, f_e^2, f_f^4 with the relation $f_v^3 + f_e^2 + f_f^4 = 0$. This defines three exceptional points x_v, x_e, x_f, and three exceptional colors $\pi^*(x_v) = 3E^{x_v}$, $\pi^*(x_e) = 2E^{x_e}$, and $\pi^*(x_f) = 4E^{x_f}$. We define a section of $\Lambda(G/F_{\mathbb{O}}) \to k(G/F_{\mathbb{O}})^{(B)}$ by the choice of the generator $f_{2\omega} := \frac{f_v f_f}{f_e}$. The elements of $\mathcal{V}_x \subset \mathcal{E}_x$ are the vectors (x, h, l) of $\check{\mathcal{E}}$ whose coordinates satisfy the inequalities $l + h \leqslant 0$ for $x \neq x_f, x_v$, and $l \leqslant 0$ for $x = x_f$ or $x = x_v$. The colors are sent to the vectors ε_x for $x \neq x_v, x_e, x_f$, and to $(x_v, 3, 1)$, $(x_e, 2, -1)$, and $(x_f, 4, 1)$ for $\pi^*(x_v)$, $\pi^*(x_e)$ and $\pi^*(x_f)$.

4.5.4 F is binary icosahedral

The character group of $F_{\mathbb{I}}$ is trivial. Up to a constant, there are three subregular semi-invariants f_v, f_e, f_f whose respective weights are $(12\omega, \zeta), (30\omega, 1), (20\omega, \zeta^{-1})$. We have $\mathcal{O}(G)^{(B\times F_{\mathbb{I}})}_{(n_0\omega,\lambda_0)} = \mathcal{O}(G)^{(B\times F_{\mathbb{I}})}_{(60\omega,1)}$ generated by the three exceptional semi-invariant f_v^5, f_e^2, f_f^3 with the relation $f_v^5 + f_e^2 + f_f^3 = 0$. This defines three exceptional points x_v, x_e, x_f, and three exceptional colors $\pi^*(x_v) = 5E^{x_v}$, $\pi^*(x_e) = 2E^{x_e}$, and $\pi^*(x_f) = 3E^{x_f}$. We define a section of $\Lambda(G/F_{\mathbb{I}}) \to k(G/F_{\mathbb{I}})^{(B)}$ by the choice of the generator $f_{2\omega} := \frac{f_v f_f}{f_e}$. The elements of $\mathcal{V}_x \subset \mathcal{E}_x$ are the vectors (x, h, l) of $\check{\mathcal{E}}$ whose coordinates satisfy the inequalities $l + h \leqslant 0$ for $x \neq x_f, x_v$, and $l \leqslant 0$ for $x = x_f$ or $x = x_v$. The colors are sent to the vectors ε_x for $x \neq x_v, x_e, x_f$, and to $(x_v, 5, 1)$, $(x_e, 2, -1)$, and $(x_f, 3, 1)$ for $\pi^*(x_v)$, $\pi^*(x_e)$ and $\pi^*(x_f)$.

4.5.5 F is binary dihedral of order $4n$, $n > 1$

The group $F_{\mathbb{D}_n}$ is generated by the elements

$$h = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \text{ et } r = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

where ζ is a primitive $2n^{th}$ root of unity. A character of $F_{\mathbb{D}_n}$ is determined by a pair of values $h \mapsto \zeta^k$, $r \mapsto i^l$, where $i^2 = -1$. The subregular semi-invariants are $f_f = g_3 g_4$, $f_e = g_3^n - (-ig_4)^n$, and $f_v = g_3^n + (ig_4)^n$ of respective weights $(2\omega, (1, -1)), (n\omega, (-1, -i^n))$, and $(n\omega, (-1, i^n))$. We have $\mathcal{O}(G)^{(B\times F_{\mathbb{D}_n})}_{(n_0\omega,\lambda_0)} = \mathcal{O}(G)^{(B\times F_{\mathbb{D}_n})}_{(2n\omega,(1,(-1)^n))}$ generated by the three exceptional semi-invariants f_f^n, f_e^2, f_v^2 with the relation $4(-i)^n f_f^n + f_e^2 - f_v^2 = 0$. This defines three exceptional points x_v, x_e, x_f, and three exceptional colors $\pi^*(x_v) = 2E^{x_v}$, $\pi^*(x_e) = 2E^{x_e}$, and $\pi^*(x_f) = nE^{x_f}$. We define a section of $\Lambda(G/F_{\mathbb{D}_n}) \to k(G/F_{\mathbb{D}_n})^{(B)}$ by the choice of the generator $f_{2\omega} := \frac{f_v f_f}{f_e^{n-1}}$. The elements of $\mathcal{V}_x \subset \mathcal{E}_x$ are the vectors (x, h, l) of $\check{\mathcal{E}}$ whose coordinates satisfy the inequalities $l + h \leqslant 0$ for $x \neq x_f, x_v$, and $l \leqslant 0$ for $x = x_f$ or $x = x_v$. The colors are sent to the vectors ε_x for $x \neq x_v, x_e, x_f$, and to $(x_v, 2, 1)$, $(x_e, 2, 1)$, and $(x_f, n, 1 - n)$ for $\pi^*(x_v)$, $\pi^*(x_e)$ and $\pi^*(x_f)$.

Chapter 5

The Cox ring of a complexity one almost homogeneous variety

Abstract

We give a description by generators and relations of the Cox ring of a normal variety as in the title, that is, equipped with an action of a connected reductive group such that there is a dense orbit and the minimal codimension of an orbit of a Borel subgroup is one. To obtain this description, we introduce and study an interesting class of equivariant completions of certain complexity one homogeneous spaces. Finally, we illustrate our method by treating an example of classical algebraic geometry: the space of ordered triangles.

5.1 Introduction

The normal varieties of complexity at most one (i.e. equipped with an action of a connected reductive group such that the minimal codimension of an orbit of a Borel subgroup is at most one) form a rich class of varieties whose geometry is rather accessible. This class comprises for instance the *toric varieties*, and more generally the *spherical varieties* (i.e. the normal varieties of complexity zero), the normal surfaces with an effective $\mathbb{G}_m$-action and the normal SL_2-threefolds with a dense orbit. These last two examples illustrate the general fact that a normal variety of complexity one is either almost homogeneous (i.e. there exists a dense orbit), or admits a one-parameter family of spherical orbits. All these varieties admit a combinatorial classification similar to the one of toric varieties (see [T11]), and one can try to describe their geometric invariants in terms of their defining data. In this direction, a lot of work remains to be done for varieties of complexity one.

We work over an algebraically closed field k of characteristic zero and let G be a connected reductive group with trivial Picard group; this last assumption can always be satisfied up to replacing G by a finite cover. The goal of this paper is to describe an important invariant of an almost homogeneous G-variety X of complexity one: its Cox ring $\mathrm{Cox}(X)$. This is a finitely generated k-algebra graded by the divisor class group $\mathrm{Cl}(X)$, and one can think of it as an algebra of homogeneous coordinates generalizing the classical notion of homogeneous coordinates algebra on a projective space (see the book [ADHL] for a comprehensive and exhaustive presentation). The Cox ring of a normal variety need not be finitely generated in general. In fact, it is known since the work of Hu and Keel [HK] that normal varieties with finitely generated Cox ring have a particularly well understood birational geometry (see also [ADHL, Section 4.3.3]). For this reason, such varieties are called *Mori Dream Spaces* (MDS).

A normal variety of complexity at most one is an MDS if and only if it is spherical or rational of complexity one. Having identified such a natural class of MDS, it is then a natural and important problem to describe their Cox rings by generators and relations. Whereas the

Cox ring has a quite straightforward definition, providing such a description turns out to be a rather challenging task. This has been overcome for spherical varieties by Brion's work [Bri07], but the problem remains open in general for varieties of complexity one. Some advances have been done recently for some specific classes of normal rational varieties of complexity one with spherical orbits [HS], [LT]. Regarding almost homogeneous varieties of complexity one (which are automatically rational), only the particular situation of a SL_2-variety has been treated [BatHad], [V2].

Following the general philosophy introduced in [Bri07] and developed in [V1], we rather try to describe the equivariant Cox ring, denoted $\mathrm{Cox}_G(X)$, by taking advantage of its canonical structure of G-algebra. It is finitely generated as k-algebra, and the ordinary Cox ring is obtained as a quotient of the former by particularly simple relations (see Section 5.3.2 below). Our strategy is then analogous to Brion's one in [Bri07], where he studies the equivariant Cox ring of a spherical variety by relating it to the equivariant Cox ring of a wonderful variety. More specifically, we first associate to any *sober homogeneous space* of complexity one a natural family of equivariant completions, called *good completions*. The notion of a sober homogeneous space originates in the theory of spherical embeddings (see [T11, Chap. 5]). A spherical homogeneous space G/H (with a dense orbit of a Borel subgroup $B \subset G$) is sober if there exists a normal complete G/H-embedding that is simple (there is a unique closed G-orbit) and toroidal (any B-stable prime divisor containing a G-orbit is G-stable). When it exists, this embedding is unique up to isomorphism and called the *standard embedding*. For instance, the wonderful completion of a spherical homogeneous space, when it exists, is a particular standard embedding. From the combinatorial perspective, the soberness condition easily translates into a condition on the combinatorial data attached with the spherical homogeneous space, and this formulation generalizes well to higher complexity (see Section 5.2.1). Building on Knop's description of valuation cones [T11, Chap. 4] and Timashev's combinatorial framework for the classification of embeddings of complexity one homogeneous spaces [T11, Chap. 3], we define a canonical family of equivariant completions of a complexity one sober homogeneous space G/H. This family is indexed by the set of finite subsets of $\mathbb{P}^1_k$ containing a fixed finite subset $\mathcal{S}_{G/H}$ (which is prescribed by G/H). We then study the geometry of these so-called good completions. We prove that any such completion is projective, $\mathbb{Q}$-factorial and covered by isomorphic affine open subvarieties with trivial Picard group and only constant invertible functions. These nice geometric properties allow us to obtain a description of the equivariant Cox ring of any good completion by generators and relations. Then, $\mathrm{Cox}_G(X)$ is obtained via a base change from the equivariant Cox ring of a certain good completion.

This article is organized as follows. In Section 5.2, we present the combinatorial classification of rational G-varieties of complexity one following [T11] and define the notion of a sober G-homogeneous space. Preliminary results, definitions and notations on almost homogeneous varieties of complexity one are given in Section 5.3. In Section 5.4, we define the so-called good completions of a sober homogeneous space of complexity one and obtain some of their geometric properties. In Section 5.5, we describe the equivariant Cox ring of a good completion by generators and relations. The equivariant Cox ring of an almost homogeneous variety of complexity one is described in Section 5.6, building on the result of the preceding section. In Section 5.7, we provide a method of geometrical nature to determine the *exceptional colors* (see Section 5.3) of a homogeneous space of complexity one; as explained all along the article, this is a crucial step towards the description of the Cox ring of the various embeddings of this homogeneous space. Finally, Section 5.8 is dedicated to the study of an interesting example from classical algebraic geometry, namely the space of ordered triangles implicitly used by Schubert in its work on enumerative geometry (see [CF]).

Conventions. We fix an algebraically closed field k of characteristic zero and work in the category of *varieties*, that is, integral separated k-schemes of finite type, and the k-morphisms between them. Without further precision, a *point* of a variety is meant to be a closed point. A *subvariety* of a variety X is a locally closed subset equipped with its reduced subscheme structure.

An *algebraic group* is an affine group scheme of finite type over k. The character group of an algebraic group G is denoted $\hat{G}$. Given a finitely generated abelian group M, we let $\mathbb{D}(M)$ denote the *diagonalizable (algebraic) group* $\operatorname{Spec}(k[M])$ with character group M. We say that X is a G-variety if it is equipped with a G-action

$$\alpha : G \times X \to X, (g, x) \mapsto g.x.$$

Our convention is that a *reductive group* is a *linearly reductive group*, that is, every finite dimensional representation of such group is semisimple. In particular, a reductive group is not necessarily connected. If G is reductive and acts on X, a *good quotient* of X by G is an affine G-invariant morphism $q : X \to Y$ such that $q^\sharp$ induces an isomorphism $\mathcal{O}_Y \to (q_*\mathcal{O}_X)^G$. It is a universal object in the category of G-invariant morphisms from X to varieties.

In all the text, G denotes a connected reductive group with trivial Picard group. Also, we choose once for all a maximal torus $T \subset G$, a Borel subgroup B containing T, and let U denote the unipotent part of B.

An *almost homogeneous G-variety* is a normal G-variety X with a dense orbit. In particular, X is a normal G/H*-embedding*, that is, there is a base point $x \in X$ with stabilizer $G_x = H$, and such that the associated orbit morphism factors through a G-equivariant open immersion $G/H \hookrightarrow X$. To keep track of the base point, we sometimes refer to X as a pointed variety (X, x). A morphism $(X, x) \to (Y, y)$ of normal G/H-embeddings is a G-equivariant morphism $X \to Y$ mapping x to y. An *equivariant completion* of a homogeneous space G/H is a normal complete G/H-embedding.

For a normal variety X, the *group of Weil divisors* is denoted $\operatorname{WDiv}(X)$. Every Weil divisor defines a coherent sheaf $\mathcal{O}(D)$ whose non-zero sections over an open subset U are the rational functions $f \in k(X)^*$ such that $\operatorname{div}(f) + D$ defines an effective divisor on U. A *divisorial sheaf* on X is a coherent reflexive sheaf of rank one. The *divisor class group* (resp. *Picard group*) of X, denoted $\operatorname{Cl}(X)$ (resp. $\operatorname{Pic}(X)$), is the group of isomorphism classes of divisorial sheaves (resp. invertible sheaves) on X. It is isomorphic to the group of Weil divisors (resp. Cartier divisors) modulo linear equivalence through the morphism $[D] \mapsto [\mathcal{O}(D)]$. The *equivariant class group* (resp. *equivariant Picard group*) of a normal G-variety X is denoted $\operatorname{Cl}_G(X)$ (resp. $\operatorname{Pic}_G(X)$) (see [V1, Section 2.2]).

5.2 Combinatorial framework

In this section, we present Timashev's combinatorial framework for the classification of complexity one normal rational G-varieties (up to G-equivariant isomorphism) lying in a fixed G-birational class. We start by recalling definitions and results around the *central valuation cone* associated with a normal G-variety X of arbitrary complexity. This allows us to define the notion of soberness for arbitrary homogeneous G-varieties. Our reference for this material is [T11].

5.2.1 The central valuation cone

Let $K = k(X)$ be the field of rational functions on X, equipped with its natural G-action. A *geometric valuation* of K is a discrete valuation $K^* \to \mathbb{Q}$ of the form αv_D, where $\alpha \in \mathbb{Q}_+$, and v_D is the normalized discrete valuation associated to a prime divisor D in a normal variety whose field of rational functions is identified with K.

Notation 5.2.1.1. We set $\mathcal{D} = \mathcal{D}(K)$ for the set of prime divisors of X that are not G-stable, and $\mathcal{D}^B$ the subset of B-stable ones. This last subset consists of the so-called *colors* of K. These sets don't depend on X up to G-birational equivalence and we identify them with the corresponding sets of normalized geometric valuations of K. Let $\mathcal{V}$ denote the set of G-*valuations* of K, that is, the geometric G-invariant valuations of K. Let $\mathcal{V}(X) \subset \mathcal{V}$ be the subset consisting of the normalized G-valuations that correspond to the G-stable prime divisors of X.

There is an exact sequence of abelian groups

$$1 \to (K^B)^* \to K^{(B)} \to \Lambda(K) \to 1, \tag{5.2.1}$$

where $K^{(B)}$ is the multiplicative group of rational B-semi-invariant functions and $\Lambda(K)$ is the associated group of weights. The abelian group $\Lambda(K) = \Lambda(X)$ is a birational invariant of the G-variety X, called the *weight lattice*. It is a subgroup of $\hat{T}$ whose rank is the *rank* of the G-variety X.

The G-valuations are uniquely determined by their restriction to $K^{(B)}$ [T11, Cor. 19.13]. Those vanishing on $(K^B)^*$ are called *central*. The restriction of a central G-valuation to $K^{(B)}$ factors through a linear form on $\Lambda(K)$. Thus, the set of central G-valuations is identified with a subset $\mathscr{L} \subset \mathcal{E} := \mathrm{Hom}(\Lambda(K), \mathbb{Q})$. By [T11, Thm. 21.1 and Thm. 22.13], this subset $\mathscr{L}$ is a cosimplicial polyhedral solid cone in $\mathcal{E}$.

Definition 5.2.1.2. A homogeneous space G/H is sober if the associated cone $\mathscr{L}$ of central G-valuations is strictly convex.

Since the work of Brion and Pauer [BP] in the spherical case and of Knop in full generality [Kn93b], the soberness condition translates geometrically into a condition on a certain subgroup of the equivariant automorphism group of G/H. Recall that this equivariant automorphism group is naturally identified with $N_G(H)/H$ acting on G/H via $nH.gH = gn^{-1}H$, and that this group also acts on K and K^B. We define the *central automorphism group* $\mathrm{CAut}(G/H)$ as the kernel of the action on K^B. This is a diagonalizable group lying in the center of $N_G(H)/H$. Moreover, its dimension is the dimension of the linear part of the central valuation cone [T11, Thm. 21.6]. Hence, G/H is sober if and only if $\mathrm{CAut}(G/H)$ is a finite group.

Examples of sober homogeneous spaces of complexity one are those of rank one. Indeed, any such homogeneous space is obtained by parabolic induction from a homogenous SL_2-variety of complexity one [Pa95], and the dimension of the linear part of the valuation cone is preserved under such an operation [T11, Prop. 14.4]. Also, any complexity one homogeneous space G/H such that $(N_G(H)/H)^\circ$ is semisimple is sober (e.g. SL_3/T, where T is the standard maximal torus).

5.2.2 Classification

Now, suppose that X is rational of complexity one. Then, the exact sequence (5.2.1) writes

$$1 \to k(\mathbb{P}^1_k)^* \to K^{(B)} \to \Lambda(X) \to 1,$$

and the choice of a splitting $\lambda \mapsto f_\lambda$ allows us to view $\Lambda(X)$ as a subgroup of $K^{(B)}$. Then, considering a geometric valuation v of K, the restriction $v_{|K^{(B)}}$ is determined by a triple (x, h, l), where $x \in \mathbb{P}^1_k$, $h \in \mathbb{Q}_+$, and $l \in \mathcal{E} := \mathrm{Hom}(\Lambda(X), \mathbb{Q})$. Indeed, the restriction of v to $k(\mathbb{P}^1_k)$ is again a geometric valuation [T11, B.8], hence of the form hv_x, where $x \in \mathbb{P}^1_k$. On the other hand, $v_{|\Lambda(X)}$ yields an element $l \in \mathcal{E}$.

Definition 5.2.2.1. For all $x \in \mathbb{P}^1_k$, consider the closed half-space $\mathcal{E}_x = \mathbb{Q}_+ \times \mathcal{E}$. The *hyperspace* $\breve{\mathcal{E}}$ associated to K is the union of the $\mathcal{E}_x$ glued together along $\mathcal{E}$.

The set $\mathcal{V}$ embeds in $\breve{\mathcal{E}}$, and $\mathcal{V}_x := \mathcal{V} \cap \mathcal{E}_x$ is a cosimplicial convex polyhedral cone for all $x \in \mathbb{P}^1_k$. The natural map $\varrho : \mathcal{D}^B \to \breve{\mathcal{E}}$ is not injective in general. All but a finite number of B-stable prime divisors are sent in $\breve{\mathcal{E}}$ to vectors of the form $\varepsilon_x := (x, 1, 0) \in \mathcal{E}_x$ for a certain $x \in \mathbb{P}^1_k$.

Definition 5.2.2.2. The pair $(\mathcal{V}, \mathcal{D}^B)$ is the *colored equipment* of K. We say that $(\breve{\mathcal{E}}, \mathcal{V}, \mathcal{D}^B, \varrho)$ is the *colored hyperspace* of K.

Definition 5.2.2.3. A *cone* in $\breve{\mathcal{E}}$ is a cone in some $\mathcal{E}_x$, $x \in \mathbb{P}^1_k$. A *hypercone* in $\breve{\mathcal{E}}$ is a union $\mathcal{C} = \cup_{x \in \mathbb{P}^1_k} \mathcal{C}_x$ of convex cones, each generated by a finite number of vectors, and such that

1. $\mathcal{C}_x = \mathcal{K} + \mathbb{Q}_+ \varepsilon_x$ for all $x \in \mathbb{P}^1_k$ but a finite number, where $\mathcal{K} := \mathcal{C} \cap \mathcal{E}$.
2. One of the following cases occurs

   (A) $\exists x \in \mathbb{P}^1_k$, $\mathcal{C}_x = \mathcal{K}$.

   (B) $\emptyset \neq \mathcal{B} := \sum \mathcal{B}_x \subset \mathcal{K}$, where $\varepsilon_x + \mathcal{B}_x = \mathcal{C}_x \cap (\varepsilon_x + \mathcal{E})$.

The hypercone is said of *type* (A) (resp. (B)) depending on the alternative of condition 2. The hypercone is said *strictly convex* if each $\mathcal{C}_x$ is, and $0 \notin \mathcal{B}$.

Definition 5.2.2.4. Let $\mathcal{Q} \subset \breve{\mathcal{E}}$ a subset all of whose elements but a finite number are of the form ε_x for some $x \in \mathbb{P}^1_k$. Let $\varepsilon_x + \mathcal{P}_x$ the convex hull of intersection points of the half-lines $\mathbb{Q}_+ q, q \in \mathcal{Q}$ with $\varepsilon_x + \mathcal{E}$. We say that the hypercone $\mathcal{C} = \mathcal{C}(\mathcal{Q})$, where the $\mathcal{C}_x$ are generated by $\mathcal{Q} \cap \mathcal{E}_x$ and $\mathcal{P} := \sum \mathcal{P}_x$, is *generated* by $\mathcal{Q}$.

The fundamental result is that strictly convex hypercones $(\mathcal{C} := \mathcal{C}(\mathcal{W} \cup \varrho(\mathcal{R})), \mathcal{R})$, where $\mathcal{W} \subset \mathcal{V}$, $\mathcal{R} \subset \mathcal{D}^B$, and $0 \notin \varrho(\mathcal{R})$, classify affine B-stable open subvarieties of normal G-models of K (the so-called B-*charts*). These hypercones are called *colored hypercones*.

Remark 5.2.2.5. Let X_0 be a B-chart defined by a colored hypercone $(\mathcal{C}, \mathcal{R})$. Then $\mathcal{O}(X_0)^B \simeq k$ if and only if $(\mathcal{C}, \mathcal{R})$ is of type (B).

A finite set of colored hypercones $(\mathcal{C}, \mathcal{R})$ defines B-charts $\mathring{X}_i$ and G-models $X_i := G\mathring{X}_i$. These G-models can be glued together into a G-model if and only if these colored hypercones define a *colored hyperfan*. In turn, the colored hyperfans classify normal G-models of K. To make this last notion precise we need the notion of a *hyperface* of a hypercone, which is defined through the notion of a *linear functional* on the hyperspace. We can think of the abelian group $K^{(B)}$ as the dual object to the hyperspace. Indeed, every $f = f_0 f_\lambda \in K^{(B)}$ defines a so-called linear functional on $\breve{\mathcal{E}}$, namely the restriction of f to each $\mathcal{E}_x$ is a $\mathbb{Q}$-linear form $((h, \gamma) \mapsto hv_x(f_0) + \gamma(\lambda))$. Conversely, considering a linear functional on $\breve{\mathcal{E}}$, a non-zero multiple of it is given by a rational B-semi-invariant function.

Definition 5.2.2.6. A *face* of a hypercone $\mathcal{C}$ is a face $\mathcal{C}'$ of a certain cone $\mathcal{C}_x$ such that $\mathcal{C}' \cap \mathcal{B} = \emptyset$. A *hyperface* of $\mathcal{C}$ is a hypercone $\mathcal{C}' = \mathcal{C} \cap \ker \varphi$, where φ is a linear functional on $\mathcal{C}$ such that $\varphi(\mathcal{C}) \geqslant 0$. If $\mathcal{C}$ is of type B, its *interior* is by definition $\operatorname{int} \mathcal{C} := \cup_{x \in \mathbb{P}^1_k} \operatorname{int} \mathcal{C}_x \cup \operatorname{int} \mathcal{K}$.

Definition 5.2.2.7. A colored hypercone $(\mathcal{C}, \mathcal{R})$ of type (B) is *supported* if $\operatorname{int} \mathcal{C} \cap \mathcal{V} \neq \emptyset$. A *(hyper)face* of $(\mathcal{C}, \mathcal{R})$ is a colored (hyper)cone $(\mathcal{C}', \mathcal{R}')$, where $\mathcal{C}'$ is a (hyper)face of $\mathcal{C}$, and $\mathcal{R}' = \mathcal{R} \cap \varrho^{-1}(\mathcal{C}')$. A *colored hyperfan* is a set of supported colored cones and hypercones of type (B) whose interiors are disjoint inside $\mathcal{V}$, and which is obtained as the set of all supported colored (hyper)faces of finitely many colored hypercones.

An interesting feature of this classification framework is that the lattice of G-stable subvarieties of X can be read from the combinatorial representation of X as a colored hyperfan. Indeed, consider a G-stable subvariety $Y \subset X$, and a B-chart $\mathring{X}$ intersecting Y which is defined by a hypercone $\mathcal{C}(\mathcal{W} \cup \varrho(\mathcal{R}))$. Denote $\mathcal{V}_Y \subset \mathcal{W}$, $\mathcal{D}^B_Y \subset \mathcal{R}$ the respective subsets which correspond to the B-stable prime divisors in $\mathring{X}$ whose closure in X contains Y, and let $\mathcal{C}_Y$ be the (hyper)cone spanned by $\mathcal{V}_Y \cup \varrho(\mathcal{D}^B_Y)$. Then for any other G-stable subvariety Z, we have $Y \subset \overline{Z}$ if and only if $(\mathcal{C}_Z, \mathcal{D}^B_Z)$ is a (hyper)face of $(\mathcal{C}_Y, \mathcal{D}^B_Y)$.

5.3 Almost homogeneous varieties of complexity one

A variety X as in the title is naturally a normal embedding $(X, x_0 = H/H)$ of a complexity one homogeneous space G/H. We recall some results on the geometry of X and on the structure of its equivariant Cox ring.

5.3.1 Colors

It follows from Lüroth's theorem that X is a rational variety. Hence, Rosenlicht's theorem [T11, 5.1] yields a rational quotient

$$\pi_X : X \dashrightarrow \mathbb{P}^1_k$$

by B. Thus, general B-orbits determine a one-parameter family of colors. Since π_X is a rational map from a normal variety to a complete one, it restricts to a morphism $\pi_V : V \to X$ where V is an open subvariety whose complement is of codimension $\geqslant 2$ in X. Hence, there exists up to isomorphism a unique divisorial sheaf $\mathcal{F}$ on X whose restriction to V is isomorphic to $\pi_V^* \mathcal{O}_{\mathbb{P}^1_k}(1)$. The Picard group of G being trivial, we can endow the sheaf $\mathcal{F}$ with a G-linearization. After choosing homogeneous coordinates z_1, z_2 on $\mathbb{P}^1_k$, the sections $\pi_V^*(z_1)$ and $\pi_V^*(z_2)$ defining the morphism π_V uniquely extend to global sections $a, b \in \mathcal{F}(X)$ which are B-semi-invariant of the same weight λ_0.

The pullback of Weil divisors on $\mathbb{P}^1_k$ corresponds to the usual pullback of Cartier divisors and is given by

$$\pi_X^* : \mathrm{WDiv}(\mathbb{P}^1_k) \to \mathrm{WDiv}(X), p = [\alpha : \beta] \mapsto \mathrm{div}(\beta a - \alpha b),$$

where $[\alpha : \beta]$ are homogeneous coordinates of $p \in \mathbb{P}^1_k$. All the B-stable prime divisors in X but a finite number lie in the image of π_X^* [V1, Sec. 3.1].

Definition 5.3.1.1. [Pon, 3] The prime divisors in X lying in the image of π_X^* are the *parametric divisors*. The finite set of B-stable prime divisors that are not parametric is the set of *exceptional divisors* of X. For an exceptional divisor E, the image of $\pi_{X|E}$ is either dense or a point in $\mathbb{P}^1_k$. In the former case, we say that E is *central*, in the latter case, the image point is called *exceptional*.

Obviously, every G-stable prime divisor is exceptional, and is central if and only if its associated G-valuation is central.

5.3.2 Equivariant Cox ring

Recall from [V1, Sec. 2.3] that for each element $[\mathcal{F}] \in \mathrm{Cl}_G(X)$, there exists a canonical representative $\mathcal{F}^{x_0}$ which is obtained by identifying sections of G-linearized divisorial sheaves (in the class $[\mathcal{F}]$) *rigidified* over the base point $x_0 = H/H$. This allows us to define a canonical multiplication law which turns the graded vector space

$$\mathrm{Cox}_G(X) = \bigoplus_{[\mathcal{F}] \in \mathrm{Cl}_G(X)} \mathcal{F}^{x_0}(X)$$

into a graded k-algebra, the multiplicative identity element being the constant function $1 \in \mathcal{O}(X)$. Moreover, this algebra is finitely generated, integral, normal, and canonically equipped with a G-action compatible with the grading [V1, Cor. 2.6.7 and Prop. 3.1.4]. Since X is almost homogeneous, we can conveniently restrict sheaves to the open orbit, and thus view $\mathrm{Cox}_G(X)$ as a graded subalgebra of

$$\bigoplus_{[\mathcal{F}] \in \mathrm{Cl}_G(X)} \mathcal{O}(G)^{(H)}_{\chi(\mathcal{F})},$$

where $\chi(\mathcal{F}) \in \hat{H}$ is the character defined by the restriction $\mathrm{Cl}_G(X) \to \mathrm{Cl}_G(G/H)$ followed by the canonical identification $\mathrm{Cl}_G(G/H) \simeq \hat{H}$.

A description of the subalgebra of U-invariants $\mathrm{Cox}_G(X)^U$ is known [V1, 3.2.3]. Before stating this result, we need to recall the natural $k[\hat{G}]$-algebra structure on $\mathrm{Cox}_G(X)$: given a section $s \in \mathcal{F}^{x_0}(X)$ and a character $\chi \in \hat{G}$, the product $\chi.s$ is the unique section of the twisted linearization $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}(\chi))^{x_0}$ that correspond to s by forgetting the linearizations (see [V1, Sec. 2.9] for more details); the elements $\chi.1$, $\chi \in \hat{G}$ are simply denoted χ. We also need the following general notations:

- let $\mathcal{S}_X \subset \mathbb{P}^1_k$ be the finite set of exceptional points associated with π_X,
- for a point $x \in \mathcal{S}_X$, write $[\alpha_x : \beta_x]$ its homogeneous coordinates on $\mathbb{P}^1_k$,
- let $(E_i)_i$ be the family of exceptional colors of X,
- let $(D^x)_{x\in\mathbb{P}^1_k\setminus\mathcal{S}_X}$ be the family of parametric colors,
- let $(X_i^x)_{x\in\mathcal{S}_X,i}$ be the family of non-central G-stable prime divisors,
- let $(X_{\infty,i})_i$ be the family of central G-stable prime divisors,
- for $x \in \mathcal{S}_X$, write $\pi_X^*(x) = \sum_{\pi_X(E_i)=x} n_{x,i}E_i + \sum_i h_{x,i}X_i^x$,
- let $s_i, s_x, r_{x,i}, r_{\infty,i}$ be the canonical sections of the respective divisorial sheaves $\mathcal{O}(E_i)$, $\mathcal{O}(D^x)$, $\mathcal{O}(X_i^x)$, $\mathcal{O}(X_{\infty,i})$ endowed with (a choice of) G-linearizations,
- let λ_i (resp. λ_0) be the B-weights associated with the elements s_i (resp. s_x),
- for a B-semi-invariant $s \in \mathrm{Cox}_G(X)$, let V_s be the simple G-submodule of $\mathrm{Cox}_G(X)$ generated by s.

Now, $\mathrm{Cox}_G(X)^U$ is generated as a $k[\hat{G}]$-algebra by the

$$a, b, (s_i)_i, (r_{x,i})_{x\in\mathcal{S}_X,i}, (r_{\infty,i})_i,$$

with relations generated by the

$$\beta_x a - \alpha_x b = \chi_x \prod_i s_i^{n_{x,i}} \prod_i r_{x,i}^{h_{x,i}} \quad x \in \mathcal{S}_X,$$

where for $x \in \mathcal{S}_X$, χ_x is a certain character of G. This description of the subalgebra of U-invariants immediately yields a natural set of homogeneous generators. More precisely, there is a surjective morphism of $k[\hat{G}]$-algebras

$$\varphi : k[\hat{G}] \otimes_k \mathrm{Sym}_k(V_a \oplus V_b \oplus (\bigoplus_i V_{s_i})) \otimes_k k[(r_{x,i})_{x\in\mathcal{S}_X,i}, (r_{\infty,i})_i] \to \mathrm{Cox}_G(X). \tag{5.3.1}$$

Suppose that X admits only constant invertible regular functions so that its ordinary Cox ring is well defined. The latter is also finitely generated and there is a canonical isomorphism [V1, 2.9.3]

$$\mathrm{Cox}(X) \simeq \mathrm{Cox}_G(X)/((\chi_i - 1)_i), \tag{5.3.2}$$

where $(\chi_i)_i$ denotes an arbitrary $\mathbb{Z}$-basis of $\hat{G}$.

5.3.3 Structure of B-charts

The local structure theorem of Brion, Luna and Vust [BLV, Thm 1.4] is a very useful tool for the study of varieties with group action. We present a variant of this theorem and a corollary, both due to Knop.

Theorem 5.3.3.1. *[Kn93b, 1.2] Let X be a G-variety equipped with an ample G-linearized invertible sheaf $\mathcal{L}$, and $s \in \mathcal{L}(X)^{(B)}$ a B-semi-invariant section. Let P the parabolic subgroup stabilizer of s in $\mathbb{P}(\mathcal{L}(X))$, with Levi decomposition $P_u \rtimes L$, $T \subset L$. Then, the open subvariety X_s is P-stable, and there exists a closed L-stable subvariety $Z \subset X_s$ such that the morphism induced by the P-action*

$$P_u \times Z \to X_s$$

is a P-equivariant isomorphism.

Corollary 5.3.3.2. *[Kn93b, §2] Let X be a normal G-variety, and $P(X)$ the smallest stabilizer of a B-stable divisor in X, with Levi decomposition $P(X)_u \rtimes L(X)$, $T \subset L(X)$. Then, there exists an affine $L(X)$-stable subvariety Z such that $X_0 := P(X)Z$ is an open affine subvariety of X. The natural morphism $P(X)_u \times Z \to X_0$ is an isomorphism and the derived subgroup of $L(X)$ acts trivially on Z.*

Let X be an almost homogeneous G-variety of complexity one. Consider a B-chart $\mathring{X} \subset X$, the open G-stable subvariety $X_0 := G\mathring{X}$ of X, and the B-stable effective divisor $D := X_0 \setminus \mathring{X}$. By [T00, Lemma 2], $\mathcal{O}_{X_0}(D)$ is an ample (G-linearized) line bundle on X_0. The canonical section associated with D is B-semi-invariant, hence its projective stabilizer P is a parabolic subgroup containing B. The above theorem yields a P-equivariant isomorphism

$$P_u \times Z \to \mathring{X},$$

where Z is a normal affine rational L-subvariety of $\mathring{X}$. Suppose moreover that $P = P(X)$ is the parabolic subgroup defined in the above corollary, and denote L_0 the kernel of the L-action on Z. Then, Z is a rational variety of complexity one under the torus $\mathbb{T} := L/L_0$.

5.4 Good equivariant completions of a sober homogeneous space

Consider a sober homogeneous space G/H of complexity one. The goal of this section is to define a certain family of equivariant completions of G/H with nice properties. We use the general notation introduced in Section 5.3.

Since the radical $\mathcal{R}(G)$ of G is a central torus, it normalizes H and its action on $k(G/H)^B$ is trivial. Thus we must have $\mathcal{R}(G) \subset H$ and we can therefore suppose that G is semisimple and simply connected. Define a finite set $\mathcal{S} \subset \mathbb{P}^1_k$ as the union of $\mathcal{S}_{G/H}$ with a (choice of a) finite set of points of $\mathbb{P}^1_k$. We define combinatorially a certain normal G/H-embedding $(X(\mathcal{S}), x_0 = H/H)$ whose set of exceptional points is $\mathcal{S}$. The description of $X(\mathcal{S})$ builds on the following proposition:

Proposition 5.4.0.1. *For $x \in \mathbb{P}^1_k$, the cone $\mathcal{V}_x \subset \check{\mathcal{E}}_x$ is a solid simplicial polyhedral cone. In particular, there is a unique extremal ray $\mathbb{Q}_+ v_x$ of $\mathcal{V}_x$ in $\mathcal{V}_x \setminus \mathscr{L}$.*

Proof. Since $\mathscr{L}$ is strictly convex by assumption, the cone $\mathcal{V}_x \subset \check{\mathcal{E}}_x$ is strictly convex as well. This implies that $\mathcal{V}_x^\vee$ is a solid cone. As $\mathcal{V}_x^\vee$ is simplicial by [T11, Cor. 22.14], we obtain that $\mathcal{V}_x = \mathcal{V}_x^{\vee\vee}$ is a solid simplicial cone in $\check{\mathcal{E}}_x$. The same arguments yield that $\mathscr{L}$ is a solid simplicial cone in $\mathcal{E}$. These two facts imply the statement. $\square$

Now, consider the category of normal G/H-embeddings. Remark that the objects of this category have no non-trivial endomorphisms. Indeed, any such endomorphism restricts to a G-equivariant automorphism on the open orbit which fixes the base point. Thus, this endomorphism must be the identity. As a consequence, the isomorphism classes of normal G/H-embeddings form an ordered set for the relation

$$[(X, x)] \leqslant [(Y, y)] \text{ if and only if there exists a morphism of embeddings } (Y, y) \to (X, x).$$

Proposition 5.4.0.2. *There exists a unique, up to isomorphism, normal complete G/H-embedding $(X(\mathcal{S}), x_0)$ satisfying*

- $\mathcal{S}_{X(\mathcal{S})} = \mathcal{S}$,
- *no exceptional color in $X(\mathcal{S})$ contains a G-orbit,*
- *$X(\mathcal{S})$ admits infinitely many closed G-orbits,*
- *$[(X(\mathcal{S}), x_0)]$ is minimal for the previous properties.*

Moreover, each color in $X(\mathcal{S})$ is a globally generated Cartier divisor.

Proof. For $x \in \mathcal{S}$, let v_x be the non-central normalized G-valuation spanning the extremal ray of $\mathcal{V}_x$ as defined in the last proposition. Choose two distinct points $x_a, x_b \in \mathbb{P}^1_k \setminus \mathcal{S}$, and define the two following normal G/H-embeddings:

- $X_a = G.\mathring{X}_a$, where $\mathring{X}_a$ is the B-chart defined by the colored hypercone

  $$(\mathcal{C}_a, \mathcal{R}_a := \{v_{D^x}; x \in \mathbb{P}^1_k \setminus (\{x_a\} \cup \mathcal{S})\}),$$

  where $\mathcal{C}_a$ is the hypercone spanned by $\varrho(\mathcal{R}_a)$, v_x, $x \in \mathcal{S}$, and $\mathscr{L}$.

- $X_b = G.\mathring{X}_b$, where $\mathring{X}_b$ is the B-chart defined by the colored hypercone

  $$(\mathcal{C}_b, \mathcal{R}_b := \{v_{D^x}; x \in \mathbb{P}^1_k \setminus (\{x_b\} \cup \mathcal{S})\}),$$

  where $\mathcal{C}_b$ is the hypercone spanned by $\varrho(\mathcal{R}_b)$, v_x, $x \in \mathcal{S}$, and $\mathscr{L}$.

By [T11, Sec. 14.2], the varieties X_a and X_b glue together to form a normal complete G/H-embedding $(X(\mathcal{S}), x_0)$. Using the combinatorial classification of G/H-embeddings (see Section 5.2.2), it is immediate to check that the first three properties of the statement are verified. Moreover, we see using [T11, Prop. 12.12], that the element $[(X(\mathcal{S}), x_0)]$ is minimal for these properties. For the final assertion, recall that a prime divisor that contains no G-orbit is globally generated Cartier [T11, Lem. 17.3]. In fact the associated invertible sheaf is globally generated by the G-translates of the canonical section. Hence, any exceptional color E is globally generated both on X_a and X_b. As X_a and X_b have complement of codimension at least two in X, generators for the restriction of E on X_a and X_b respectively extend to X, defining two sets of global sections. As X is the union of X_a and X_b, taking the union of these two sets of global sections, we obtain generators for $\mathcal{O}(E)$. Thus, E is globally generated and Cartier on X. Similarly, D^{x_a} is globally generated Cartier on X_a, and D^{x_b} on X_b. Since any two parametric colors are linearly equivalent this finishes the proof. $\square$

Example 5.4.0.3. Let $G = \mathrm{SL}_2$, $H = \{I_2\}$, $\mathcal{S} = \emptyset$, and consider the G-action on the variety of 2×2 matrices by left multiplication. Then $X(\mathcal{S})$ is the natural (normal) embedding of G in the subvariety of $\mathbb{P}(\mathrm{M}_2(k) \oplus k)$ defined by the homogeneous equation $ad - bc - t^2 = 0$, where a, b, c, d are the matrix coefficients and G acts trivially on t.

Fix a set $\mathcal{S}$ and choose a distinguished parametric point $x_d \in \mathbb{P}^1_k$ such that $x_d \notin \mathcal{S}$. As in the preceding proof, define the B-chart $\mathring{X} \subset X(\mathcal{S})$ via the colored hypercone

$$(\mathcal{C}, \mathcal{R} := \{v_{D^x}; x \in \mathbb{P}^1_k \setminus (\{x_d\} \cup \mathcal{S})\}),$$

where $\mathcal{C}$ is the hypercone spanned by $\varrho(\mathcal{R})$, v_x, $x \in \mathcal{S}$, and $\mathscr{L}$. This chart has a nice structure:

Lemma 5.4.0.4. *Let $P \supset B$ be the parabolic subgroup stabilizer of the divisor $D^{x_d} + \sum_i E_i$, and $P_u \rtimes L$ be the Levi decomposition of P such that $T \subset L$. Then, there exists an affine L-subvariety $Z \subset \mathring{X}$ and a P-equivariant isomorphism*

$$\mathring{X} \simeq P_u \times Z.$$

Moreover the kernel L_0 of the L-action on Z contains the derived subgroup $D(L)$, and Z is a variety of complexity one under the torus $\mathbb{T} := L/L_0$.

Proof. Let λ_i and λ_0 be the dominant weights respectively associated with the B-semi-invariants s_i and s_{x_d}. Also, let $P(\lambda_i), P(\lambda_0)$ be the respective stabilizers of the s_i, s_{x_d} (which depend only on the associated dominant weights). Then, we have

$$P = P(\lambda_0) \cap (\cap_i P(\lambda_i)).$$

It follows that P is the minimal parabolic subgroup stabilizer of a B-stable divisor in $X(\mathcal{S})$. Hence, the result follows from the local structure of B-charts (see Section 5.3.3). □

Lemma 5.4.0.5. *The hypercone defining Z as a rational affine $\mathbb{T}$-variety of complexity one is $\mathcal{C}$ viewed in the hyperspace associated with $k(Z)$.*

Proof. Every $(B \cap L)$-stable divisor in Z is also L-stable, thus the $\mathbb{T}$-stable prime divisors in Z are exactly the intersections with Z of the B-stable prime divisors of $X(\mathcal{S})$ meeting $\mathring{X}$. Also, there is a natural identification $k(X(\mathcal{S}))^{(B)} \simeq k(Z)^{(\mathbb{T})}$ preserving orders of functions along divisors. It follows that the hypercone defining Z identifies with $\mathcal{C}$. □

Lemma 5.4.0.6. *There is a good quotient*

$$\pi_Z : Z \to \mathbb{A}^1_k$$

by $\mathbb{T}$. This quotient admits a section which may be identified with the inclusion $Z^{\mathbb{T}} \subset Z$ of the variety of fixed points. Finally, the rational quotient $\pi_{X(\mathcal{S})} : X(\mathcal{S}) \dashrightarrow \mathbb{P}^1_k$ restricts to a categorical quotient $\pi_{\mathring{X}} : \mathring{X} \to \mathbb{A}^1_k$, the latter being identified with $\pi_Z p_Z$, where $p_Z : P_u \times Z \to Z$ is the projection on Z.

Proof. Because Z is a rational affine $\mathbb{T}$-variety of complexity one, the good quotient π_Z exists and maps to an open subset of $\mathbb{P}^1_k$. Using Lemma 5.4.0.5 and the definition of $\mathcal{C}$, this quotient maps to $\mathbb{A}^1_k = \mathbb{P}^1_k \setminus \{x_d\}$. The set-theoretic fibers $\pi_Z^{-1}(x)$, $x \in \mathbb{A}^1_k$ are $\mathbb{T}$-stable prime divisors with a dense $\mathbb{T}$-orbit. In particular, each such fiber has a natural structure of affine toric $\mathbb{T}$-variety. Moreover, the lattice of its $\mathbb{T}$-stable subvarieties is determined from the cone $\mathcal{C}_x$. Precisely, each $\mathbb{T}$-orbit in $\pi_Z^{-1}(x)$ corresponds to a face of $\mathcal{C}_x$ not contained in $\mathcal{E}$ (see [T08, Remark 3]). Because $\mathcal{C}_x$ is solid, the unique closed orbit in $\pi_Z^{-1}(x)$ is a $\mathbb{T}$-fixed point. This implies that the inclusion $Z^{\mathbb{T}} \subset Z$ is a section of π_Z.

By Lemma 5.4.0.4, the categorical quotient of $\mathring{X}$ by P exists in the category of varieties and identifies with $\pi_Z p_Z$. Since B and P have the same orbits in $\mathring{X}$, every B-invariant morphism from $\mathring{X}$ is also P-invariant. It follows that the categorical quotient of $\mathring{X}$ by B exists and identifies with $\pi_Z p_Z$. □

Lemma 5.4.0.7. *The B-chart $\mathring{X}$ has trivial Picard group and only constant invertible regular functions. Moreover, it is $\mathbb{Q}$-factorial.*

Proof. In view of the isomorphism $\mathring{X} \simeq P_u \times Z$, we are reduced to proving the assertions of the statement by replacing B by $\mathbb{T}$, and $\mathring{X}$ by Z (see [Bri18, 4.1.3 and 5.1.2]). Choose a one-parameter subgroup $\lambda(\mathbb{G}_m) \subset \mathbb{T}$ having the same fixed points as $\mathbb{T}$. By Lemma 5.4.0.6, the $\mathbb{G}_m$-quotient of Z identifies with π_Z. Thus, $\mathcal{O}(Z)$ is a $\mathbb{Z}_{\geqslant 0}$-graded $\mathcal{O}(\mathbb{A}^1_k)$-algebra with $\mathcal{O}(Z)_0 = \mathcal{O}(\mathbb{A}^1_k)$. Let $\mathcal{L}$ be a $\mathbb{G}_m$-linearized invertible sheaf on Z. By [BasHab, Cor. 6.4], we have $\pi_Z^* i^* \mathcal{L} \simeq \mathcal{L}$ as invertible sheaves on Z, where i denotes the closed immersion $\mathbb{A}^1_k \to Z$. Thus, $\mathcal{L}$ is trivial as an invertible sheaf. This implies that $\mathrm{Pic}(Z)$ is trivial since the forgetful morphism $\mathrm{Pic}_{\mathbb{G}_m}(Z) \to \mathrm{Pic}(Z)$ is surjective. The assertion on invertible regular functions is clear as any such function is necessarily homogeneous of degree zero. The $\mathbb{Q}$-factoriality of Z is proved by applying [LS, Thm. 4.4]. □

Corollary 5.4.0.8. *The normal variety $X(\mathcal{S})$ is $\mathbb{Q}$-factorial.*

Proof. Notice that $X(\mathcal{S})$ is covered by G-translates of open subvarieties isomorphic to $\mathring{X}$. By quasi-compactness of $X(\mathcal{S})$, there is a finite open covering by $\mathbb{Q}$-factorial varieties. This finishes the proof. □

Proposition 5.4.0.9. *The variety $X(\mathcal{S})$ is smooth if and only if the cones $\mathcal{C}_x$ are smooth, for $x \in \mathbb{P}^1_k \setminus \{x_d\}$.*

Proof. The variety $X(\mathcal{S})$ is smooth if and only if $\mathring{X}$ is smooth. This is again equivalent to the smoothness of the variety Z. By [LP, Cor. 9], the latter is equivalent to the cones $\mathcal{C}_x$ being smooth. □

The embedding $X(\mathcal{S})$ admits an explicit geometric realization. The construction is classical and analogous to the realization of the standard embedding of a sober spherical homogeneous space (see e.g. [Bri97, 4.4]).

Construction 5.4.0.10. The set $\mathcal{S}$ defines a family $(E_i)_i$ of colors in G/H which are the central colors and the non-central colors lying in the support of pullbacks of points of $\mathcal{S}$ by $\pi_{G/H}$. The canonical sections associated with these colors naturally identify with equations $s_i \in \mathcal{O}(G)^{(B\times H)}_{(\lambda_i,\chi_i)}$ of their preimages in G via the quotient $G \to G/H$. Recall that there is a $G \times G$-module isomorphism

$$\mathcal{O}(G) = \bigoplus (V^* \otimes_k V),$$

where V runs through all simple G-modules, see [T11, Thm. 2.15]. In particular, there exist simple G-modules V_i, linear forms $\eta_i \in (V_i^*)^{(B)}_{\lambda_i}$, and vectors $v_i \in (V_i)^{(H)}_{\chi_i}$ such that

$$s_i(g) = \langle \eta_i, g.v_i \rangle, \; \forall g \in G.$$

Likewise, there is a simple G-module V_0, a linear form $\eta_0 \in (V_0^*)^{(B)}_{\lambda_0}$, and two linealy independent vectors $v_a, v_b \in (V_0)^{(H)}_{\chi_0}$ defining the two functions a, b generating the two dimensional linear system $\mathcal{O}(G)^{(B\times H)}_{(\lambda_0,\chi_0)}$ (see Section 5.3). Now, consider the diagonal G-action on

$$\textstyle\prod_i \mathbb{P}(V_i) \times \mathbb{P}(V_0 \oplus V_0),$$

and let Y_1 be the G-orbit closure of the point $y := (([v_i])_i, [(v_a, v_b)])$; this is naturally a projective G/G_y-embedding $(Y_1, y = G_y/G_y)$.

We claim that G_y is a subgroup of $N_G(H)$ containing H. Consider the G_y-action on G by multiplication on the right. Obviously, the stabilizer G_y leaves the lines ka and kb invariant, and acts on them through the same character. Likewise, G_y leaves the lines ks_i invariant. Since any element $s \in \mathcal{O}(G)^{(B\times H)}$ is, up to multiplication by a non-zero constant, a finite product

$$\textstyle\prod_i s_i^{n_i} \prod_{p=[\alpha:\beta]\in\mathbb{P}^1_k} (\beta a - \alpha b)^{m_p},$$

we see that G_y acts on $\mathcal{O}(G)^{(B\times H)}$. More specifically, G_y acts on any element of the latter set through the multiplication by a character. As the G_y-action commutes with the left G-action, we infer that G_y acts on the equivariant Cox ring

$$\mathrm{Cox}_G(G/H) = \bigoplus_{\chi\in\hat{H}} \mathcal{O}(G)^{(H)}_{\lambda}$$

by automorphisms of graded G-algebra. This implies that G_y acts on G/H by G-automorphisms, whence the claim.

Since the group G_y/H fixes each B-stable prime divisor in G/H, it fixes $\mathbb{P}^1_k$ pointwise. Then, it is by definition a subgroup of $\mathrm{CAut}(G/H)$ (see Section 5.2.1). The homogeneous space G/H being sober by assumption, this implies that G_y/H is a finite diagonalizable group. Hence, the quotient $G/H \to G/G_y$ is a torsor under a finite group. This yields in turn that the normalization Y of Y_1 in $k(G/H)$ is a complete normal G/H-embedding and that the associated normalization morphism is a morphism of embeddings

$$(Y, x_0 = H/H) \to (Y_1, y).$$

Because each color of $X(\mathcal{S})$ is a globally generated Cartier divisor (Prop. 5.4.0.2), there is a natural morphism of embeddings $(X(\mathcal{S}), x_0) \to (Y_1, y)$, defined by considering the tuple of morphisms respectively defined by the linear systems V_{s_i}, and $V_a \oplus V_b$. By the universal property of normalization, this morphism factors through a morphism of embeddings $(X(\mathcal{S}), x_0) \to (Y, x_0)$. From this, we see that Y must admit infinitely many closed G-orbits, and that $\mathcal{S}_Y \subset \mathcal{S}$. The hyperplane $\eta_i^{\perp} \subset V_i$ contains no G-orbit (V_i^* is spanned by η_i as a G-module). As a consequence, the same property holds for E_i (still using the letter E_i to denote its closure in Y) and it is an exceptional color of Y. It follows that $\mathcal{S}_Y = \mathcal{S}$, and that no exceptional color in Y contains a G-orbit. By minimality of $X(\mathcal{S})$ (Prop. 5.4.0.2), $X(\mathcal{S})$ and Y are isomorphic G/H-embeddings.

Corollary 5.4.0.11. *The normal variety $X(\mathcal{S})$ is projective.*

Proof. This stems from the previous construction and the fact that the normalization of a projective variety is again projective. $\square$

5.5 The equivariant Cox ring of $X(\mathcal{S})$

We keep the assumptions and notations of the preceding section, except that we write X instead of $X(\mathcal{S})$ for simplicity. Recall from Section 5.3.2 that the k-algebra $\mathrm{Cox}_G(X)^U$ is generated by the elements

$$a, b, (s_i)_i, (r_x)_{x\in\mathcal{S}}, (r_{\infty,i})_i,$$

where for $x \in \mathcal{S}$, r_x is the canonical section defined by the prime divisor X^x whose associated valuation is v_x, and the $r_{\infty,i}$ are the canonical sections associated with the central G-stable prime divisors $X_{\infty,i}$. By definition of X, the cardinality of the family $(r_{\infty,i})_i$ is the rank of G/H. Moreover, the relations between the above generators are generated by the following

$$\beta_x a - \alpha_x b = r_x^{h_x} \prod_i s_i^{n_{x,i}}, \; x \in \mathcal{S}. \tag{5.5.1}$$

This yields in particular a natural surjective morphism of G-algebras

$$\varphi : S := \mathrm{Sym}_k(V_a \oplus V_b \oplus (\textstyle\bigoplus_i V_{s_i})) \otimes_k k[(r_x)_{x\in\mathcal{S}}, (r_{\infty,i})_i] \to \mathrm{Cox}_G(X).$$

Remark 5.5.0.1. If $\sharp\mathcal{S} \geqslant 2$, one can remove a, b from the above generating set of $\mathrm{Cox}_G(X)^U$ by using the relations (5.5.1). Hence, the simple G-submodules V_a, V_b can also be removed from the generators of $\mathrm{Cox}_G(X)$.

Our task is now to find generators for the ideal $\ker\varphi$ of relations; this has a natural structure of G-module. We suitably coarsen the grading on $\mathrm{Cox}_G(X)$: consider the free abelian group $A := \mathrm{Cl}_B(\mathring{X})/\mathrm{Cl}_B(\mathring{X})_{\mathrm{tor}}$, where $\mathrm{Cl}_B(\mathring{X})_{\mathrm{tor}}$ denotes the torsion subgroup of $\mathrm{Cl}_B(\mathring{X})$. The morphism

$$p : \mathrm{Cl}_G(X) \to A$$

obtained by composing the natural morphism $\mathrm{Cl}_G(X) \to \mathrm{Cl}_B(\mathring{X})$ with the projection $\mathrm{Cl}_B(\mathring{X}) \to A$, defines a A-grading on $\mathrm{Cox}_G(X)$.

Lemma 5.5.0.2. *The character group $\hat{T}$ embeds in A as a subgroup of finite index. Viewing $\mathrm{Cox}_G(X)$ with its A-grading, we have*

$$\deg(s_i) = \lambda_i,\ \deg(a) = \deg(b) = \lambda_0,\ \deg(r_{\infty,i}) \neq 0,\ \textit{and}\ \deg(r_x) = 0,\ \textit{for}\ x \in \mathcal{S}.$$

Moreover, the weights of A in $\mathrm{Cox}_G(X)$ generate a strictly convex polyhedral cone in $A_{\mathbb{Q}}$. Consequently, the A-grading on $\mathrm{Cox}_G(X)$ can be coarsened to a $\mathbb{Z}_{\geqslant 0}$-grading.

Proof. By Lemma 5.4.0.7 and [V1, Section 2.2], there is an exact sequence

$$0 \to \hat{T} \to \mathrm{Cl}_B(\mathring{X}) \to \mathrm{Cl}(\mathring{X}) \to 0,$$

and $\mathrm{Cl}(\mathring{X})$ is a finite abelian group. It follows that we can view $\hat{T}$ as subgroup of finite index of A.

We have

$$p([\mathcal{O}(E_i)]) = \lambda_i, \text{ and } p([\mathcal{O}(D^{x_d})]) = \lambda_0,$$

since the sections s_i (resp. s_{x_d}) yield trivializations of these sheaves over $\mathring{X}$. This yields the assertions on the A-degrees of a, b and the s_i. For $x \in \mathcal{S}$, consider the B-invariant rational function $f_x = \frac{\beta_x a - \alpha_x b}{\beta_{x_d} a - \alpha_{x_d} b}$, the divisor of which reads

$$\pi_X^*(x) - \pi_X^*(x_d) = h_x X^x + (\textstyle\sum_{\pi_X(E_i)=x} n_{x,i} E_i) - D^{x_d}.$$

Hence, the rational function f_x is regular on $\mathring{X}$ and its divisor is $h_x X^x$. It follows that

$$0 = p([\mathcal{O}(h_x X^x)]) = h_x p([\mathcal{O}(X^x)]),$$

whence $\deg(r_x) = 0$. Similarly, there exists $u_{\infty,i} \in \mathbb{Z}_{>0}$ such that the divisor $u_{\infty,i} X_{\infty,i}$ is principal on $\mathring{X}$. Hence, we can write

$$u_{\infty,i} p([\mathcal{O}(X_{\infty,i})]) = p([\mathcal{O}(u_{\infty,i} X_{\infty,i})]) = \gamma_{\infty,i},$$

where $\gamma_{\infty,i}$ is the opposite of the B-weight of an equation $f_{\infty,i}$ of $u_{\infty,i} X_{\infty,i}$ on $\mathring{X}$. It is easy to see that $f_{\infty,i} \notin k(X)^B$, and that its non-zero weight lies in $\mathscr{L}^\vee$. Thus, we obtain using [T11, Thm. 21.1] that $\gamma_{\infty,i}$ lies in the cone generated by the simple roots. As the weight monoid of $\mathrm{Cox}_G(X)$ is generated by the $\lambda_i, \gamma_x, \gamma_{\infty,i}$, it follows that this monoid is included in the cone generated by the simple roots. The remaining assertions follow readily. $\square$

Let $\tilde{X}_G := \mathrm{Spec}\,\mathrm{Cox}_G(X)$ be the equivariant total coordinate space. By [V1, 2.8.5], the invariant subalgebra $\mathrm{Cox}_G(X)^G$ identifies with the polynomial algebra in the r_x and the $r_{\infty,i}$, whence a good quotient

$$\tilde{X}_G \to \mathbb{A}_k^{\sharp\mathcal{S}+\mathrm{rk}(X)}$$

by G. The *special fiber* of this morphism is the schematic fiber $\tilde{X}_{G,0}$ at the origin. When $G = \mathrm{SL}_2$ and $H = \mu_n$, the Cox ring of a normal G/H-embedding whose special fiber is a horospherical variety (i.e. the product of any two simple G-modules in its coordinate algebra is their Cartan product) was computed in [V2].

Proposition 5.5.0.3. *The special fiber $\tilde{X}_{G,0}$ is a normal horospherical affine variety.*

Proof. The coordinate algebra of $\tilde{X}_{G,0}$ is

$$\mathrm{Cox}_G(X)/((r_x, r_{\infty,i})_{x\in\mathcal{S},1\leqslant i\leqslant \mathrm{rk}(X)}).$$

In view of the description of $\mathrm{Cox}_G(X)^U$ (see the relations (5.5.1)), this k-algebra is finitely generated, integral and normal [T11, Thm. D.5]. Moreover, this algebra is generated by the simple G-submodules V_a, V_b, $(V_{s_i})_i$. By a result of Kostant (see e.g. [Bri85, 4.1 Lemme]), it now suffices to verify that the product of any two of these G-modules is again simple. This is implied by the next lemma. □

Lemma 5.5.0.4. *Let V_1, V_2 be two simple G-modules chosen in the family $(V_a, V_b, (V_{s_i})_i)$. Then the product V_1V_2 in the algebra $\mathrm{Cox}_G(X)/((r_{\infty,i})_i)$ is simple.*

Proof. Denote λ_1, λ_2 the respective highest weights of V_1, V_2. View first the product V_1V_2 in $\mathrm{Cox}_G(X)$, and suppose that there exists a B-semi-invariant $t \in V_1V_2$ of B-weight $\lambda_t < \lambda_1 + \lambda_2$ for the usual ordering on weights (we can suppose this otherwise we are done). The latter semi-invariant can be chosen so that it satisfies an equality

$$t = \prod_i s_i^{k_i} \prod_i (\beta_i a - \alpha_i b)^{l_i} \prod_x r_x^{m_x} \prod_i r_{\infty,i}^{m_i}.$$

In view of Lemma 5.5.0.2, this yields the equalities

$$\begin{aligned}\deg(t) &= \sum_i k_i\lambda_i + (\sum_i l_i)\lambda_0 + \sum_i m_i \deg(r_{\infty,i})\\ &= \lambda_t + \sum_i m_i \deg(r_{\infty,i})\\ &= \lambda_1 + \lambda_2,\end{aligned}$$

from which it follows that we must have $\sum_i m_i > 0$. This implies that t is sent to zero in $\mathrm{Cox}_G(X)/((r_{\infty,i})_i)$. To finish the proof, it now suffices to check that the G-module V_1V_2 is non-zero in $\mathrm{Cox}_G(X)/((r_{\infty,i})_i)$. This is verified since this algebra is integral. □

Write the family of simple G-submodules $(V_a, V_b, (V_{s_i})_i)$ as $(V_1, ..., V_m)$. For $i, j \in \{1, ..., m\}$, $i \leqslant j$, consider the natural surjective morphisms of G-modules

$$\begin{cases}\varphi_{ij} : V_i \otimes_k V_j \to V_iV_j \subset \mathrm{Cox}_G(X), i < j,\\ \varphi_{ii} : \mathrm{Sym}_k^2(V_i) \to V_i^2 \subset \mathrm{Cox}_G(X)\end{cases}$$

obtained via restriction of φ. For each B-weight ω occurring in V_iV_j, let $x_{ij,\omega}$ be an associated B-semi-invariant. Writing $x_{ij,\omega}$ as a product $\prod_i s_i^{k_i} \prod_i (\beta_i a - \alpha_i b)^{l_i} \prod_x r_x^{m_x} \prod_i r_{\infty,i}^{m_i}$, this element is naturally identified with an element of S. Also, for each such $x_{ij,\omega}$, define $y_{ij,\omega}$ to be the unique B-semi-invariant in S sent to $x_{ij,\omega}$ via φ_{ij}. Then, define the G-submodules of $\ker\varphi$:

$$M_{ij} := (\textstyle\bigoplus_\omega \langle G.(x_{ij,\omega} - y_{ij,\omega})\rangle) \oplus \ker\varphi_{ij},\ i \leqslant j.$$

Finally, let N be the G-submodule of $\ker\varphi$ generated by the U-invariant relations (5.5.1).

Proposition 5.5.0.5. *The ideal $\ker\varphi$ is generated by the G-modules M_{ij} and N.*

Proof. We begin by choosing a $\mathbb{Z}_{\geqslant 0}$-grading for $\mathrm{Cox}_G(X)$ whose properties are as described by Lemma 5.5.0.2. Then, we endow S with a $\mathbb{Z}_{\geqslant 0}$-grading turning φ into a graded morphism. To do this, we simply define the degrees of the elements of $V_a, V_b, V_{s_i} \subset S$ to be the respective degrees $\deg(a), \deg(b), \deg(s_i)$ and likewise for the elements $r_x, r_{\infty,i} \subset S$. Let I be the homogeneous ideal generated by the G-modules M_{ij} and N. We prove that the surjective morphism of graded k-algebras

$$\bar{\varphi} : S/I \to \mathrm{Cox}_G(X)$$

induced by φ is an isomorphism. This follows from the claim that $(S/I)^U$ is generated as a k-algebra by the images of the elements a, b, s_i, r_x, $r_{\infty,i}$. Indeed, the algebra $(S/I)^U$ can then be presented as the quotient of the polynomial k-algebra in the elements a, b, s_i, r_x, $r_{\infty,i}$ modulo an ideal J of relations. In view of the presentation of $\mathrm{Cox}_G(X)^U$, and of the natural morphism

$$\bar{\varphi}^U : (S/I)^U \to \mathrm{Cox}_G(X)^U$$

induced by $\bar{\varphi}$, the ideal J is contained in the ideal generated by N^U. As we have the reverse inclusion by definition of I, it follows that the morphism $\bar{\varphi}_U$ is an isomorphism, thus $\bar{\varphi}$ is also an isomorphism.

We now prove this claim. The k-algebra $S/(I, (r_{\infty,i})_i)$ is generated by the simple G-modules $V_1, ..., V_m$, and the r_x. It follows from Lemma 5.5.0.4 and the definition of the G-modules M_{ij}, that the product of any two of them in $S/(I, (r_{\infty,i})_i)$ is their Cartan product. As a consequence of a result of Kostant (see again [Bri85, 4.1 Lemme]), the product of any two simple G-modules in $S/(I, (r_{\infty,i})_i)$ is again their Cartan product or zero. This implies that $(S/(I, (r_{\infty,i})_i))^U$ is generated as a k-algebra by the images of the s_i and r_x. Let R be the k-subalgebra of $(S/I)^U$ generated by the images of the $s_i, r_x, r_{\infty,i}$. Viewing R and $(S/I)^U$ as graded $k[(r_{\infty,i})_i]$-modules, the above description of $(S/(I, (r_{\infty,i})_i)^U$ yields

$$(S/I)^U = R + (r_{\infty,i})_i (S/I)^U.$$

Since the elements $r_{\infty,i}$ have strictly positive degrees, the graded Nakayama lemma implies that $(S/I)^U = R$, whence the claim. $\square$

5.6 The Cox ring of a complexity one almost homogeneous variety

Let G/H be a homogeneous space of complexity one, and X be a normal G/H-embedding. In this section, we describe the equivariant Cox ring of X by relating it to the equivariant Cox ring of a good completion of an associated sober homogeneous space. Up to replacing G by a finite cover, we can suppose that $G = \boldsymbol{G} \times C$, where $\boldsymbol{G}$ is semisimple and simply connected, and C is a torus. Thus, the Borel subgroup B can be written $\boldsymbol{B} \times C$ where $\boldsymbol{B}$ is a Borel subgroup of $\boldsymbol{G}$.

The normalizer $N_G(H)$ acts on G/H by equivariant automorphisms, thus it also acts on the set $\mathcal{D}^B$ of colors. The kernel Γ of the latter action contains C and H. Let $\boldsymbol{H} \subset \boldsymbol{G}$ be the subgroup such that $\Gamma = \boldsymbol{H} \times C$. The homogeneous space $\boldsymbol{G}/\boldsymbol{H}$ is of complexity one: we have indeed

$$k(\boldsymbol{G}/\boldsymbol{H})^{\boldsymbol{B}} \simeq k(G/H)^{\boldsymbol{B} \times \Gamma/H} \simeq (k(G/H)^B)^{\Gamma/CH} \simeq k(\mathbb{P}^1_k)^{\Gamma/CH} \simeq k(\mathbb{P}^1_k),$$

because Γ acts trivially on $k(\mathbb{P}^1_k)$. Also notice that Γ/H is of finite index in $\mathrm{CAut}(G/H)$ because it contains $\mathrm{CAut}(G/H)^\circ$. It follows that $\mathrm{CAut}(\boldsymbol{G}/\boldsymbol{H})$ is a finite group, whence $\boldsymbol{G}/\boldsymbol{H}$ is a sober homogeneous space of complexity one (see Section 5.2.1).

The quotient morphism $f_{G/H} : G/H \to \boldsymbol{G}/\boldsymbol{H}$ induces natural identifications

$$\mathcal{D}^B_{G/H} \simeq \mathcal{D}^{\boldsymbol{B}}_{\boldsymbol{G}/\boldsymbol{H}}, \text{ and } \mathcal{S}_{G/H} \simeq \mathcal{S}_{\boldsymbol{G}/\boldsymbol{H}}.$$

More precisely, since the left B-action and the right Γ-action on G/H commute, there is a commutative triangle

$$\begin{tikzcd} G/H \arrow[d, "f_{G/H}"'] \arrow[dr, dashed, "\pi_{G/H}"] & \\ \boldsymbol{G}/\boldsymbol{H} \arrow[r, dashed, "\pi_{\boldsymbol{G}/\boldsymbol{H}}"'] & \mathbb{P}^1_k \end{tikzcd}$$

and the bijection between the sets of colors is given by the pullback by $f_{G/H}$.

Define $\boldsymbol{X} := \boldsymbol{X}(\mathcal{S}_X)$ to be the good completion of $\boldsymbol{G}/\boldsymbol{H}$ constructed in Section 5.4. Since $\boldsymbol{X}$ is complete we can suppose, up to replacing X by a G-stable open subvariety whose complement is of codimension $\geqslant 2$, that the rational map $X \dashrightarrow \boldsymbol{X}$ defined by $f_{G/H}$ is everywhere defined. This yields a G-equivariant morphism

$$f : X \to \boldsymbol{X},$$

which by the following lemma induces a natural bijection between the set of colors of $\boldsymbol{X}$ and X.

Lemma 5.6.0.1. *For each color $\boldsymbol{D} \in \mathcal{D}_{\boldsymbol{X}}^{\boldsymbol{B}}$, the schematic preimage $f^{-1}(\boldsymbol{D})$ is a color in X.*

Proof. First notice that for any color $\boldsymbol{D} \in \mathcal{D}_{\boldsymbol{G}/\boldsymbol{H}}^{\boldsymbol{B}}$, the schematic preimage $f_{G/H}^{-1}(\boldsymbol{D})$ is a color in G/H. Now for a parametric color $\boldsymbol{D}$ in $\boldsymbol{X}$, the schematic preimage $f^{-1}(\boldsymbol{D})$ is a color in X since $\mathcal{S}_{\boldsymbol{X}} = \mathcal{S}_X$ by construction. For an exceptional color $\boldsymbol{E}_i \in \mathcal{D}_{\boldsymbol{X}}^{\boldsymbol{B}}$, it might be that the underlying set of $f^{-1}(\boldsymbol{E}_i)$ contains a G-stable divisor. However, this does not happen as $\boldsymbol{E}_i$ contains no $\boldsymbol{G}$-orbit. ☐

Likewise, the schematic preimages $f^{-1}(\boldsymbol{X}^x)$ (resp. $f^{-1}(\boldsymbol{X}_{\infty,i})$) define (possibly empty) G-stable divisors on X. This defines a pullback morphism

$$\mathrm{WDiv}(\boldsymbol{X})^{\boldsymbol{B}} \to \mathrm{WDiv}(X)^B$$

since $\mathrm{WDiv}(\boldsymbol{X})^{\boldsymbol{B}}$ is generated by the colors and the $\boldsymbol{G}$-stable prime divisors. As a consequence we obtain a pullback morphism

$$\mathrm{Cl}(\boldsymbol{X}) \to \mathrm{Cl}(X).$$

Since f is equivariant and $\mathrm{Cl}_{\boldsymbol{G}}(\boldsymbol{X}) \simeq \mathrm{Cl}(\boldsymbol{X})$ by assumption on $\boldsymbol{G}$, this finally yield a morphism

$$\mathrm{Cl}_{\boldsymbol{G}}(\boldsymbol{X}) \to \mathrm{Cl}_G(X), \tag{5.6.1}$$

together with a morphism of graded G-algebras

$$f^* : \mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X}) \to \mathrm{Cox}_G(X),$$

obtained by pulling back sections of $\boldsymbol{G}$-linearized divisorial sheaves on $\boldsymbol{X}$. The corresponding morphism between grading groups is given by (5.6.1).

Theorem 5.6.0.2. *The natural morphism*

$$\psi : \mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X}) \otimes_{\mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X})^{\boldsymbol{G}}} \mathrm{Cox}_G(X)^{\boldsymbol{G}} \to \mathrm{Cox}_G(X)$$

is an isomorphism.

Proof. We view ψ as a morphism of algebras over $R := \mathrm{Cox}_G(X)^{\boldsymbol{G}} = k[\hat{C}][(r_{x,i})_{x \in \mathcal{S}_X}, (r_{\infty,i})_i]$. It suffices to show that the induced morphism

$$\psi^U : \mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X})^U \otimes_{\mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X})^{\boldsymbol{G}}} R \to \mathrm{Cox}_G(X)^U \tag{5.6.2}$$

is an isomorphism of R-algebras. We saw in Section 5.3.2 that the right-hand side is generated by the elements a, b, s_i, with ideal J of relations generated by the elements

$$\beta_x a - \alpha_x b - \chi_x \textstyle\prod_i s_i^{n_{x,i}} \prod_i r_{x,i}^{h_{x,i}},\ x = [\alpha_x : \beta_x] \in \mathcal{S}_X,$$

where for $x \in \mathcal{S}_X$, χ_x is a certain character of C, and the $n_{x,i}$, $h_{x,i}$ are such that

$$\pi_X^*(x) = \textstyle\sum_{\pi_X(X_i^x)=x} h_{x,i} X_i^x + \sum_{\pi_X(E_i)=x} n_{x,i} E_i.$$

Similarly, the left-hand side of (5.6.2) is generated by $\boldsymbol{a}\otimes 1, \boldsymbol{b}\otimes 1$, and the $\boldsymbol{s_i}\otimes 1$, with ideal I of relations generated by the elements

$$(\beta_x\boldsymbol{a} - \alpha_x\boldsymbol{b} - \boldsymbol{r}_x^{\boldsymbol{h}_x}\prod_{i,\boldsymbol{\pi_X}(\boldsymbol{E}_i)=x}\boldsymbol{s}_i^{\boldsymbol{n}_{i,x}})\otimes 1,\ x = [\alpha_x:\beta_x]\in\mathcal{S}_{\boldsymbol{X}}.$$

By applying the lemma after this proof, these elements can be rewritten

$$(\beta_x\boldsymbol{a} - \alpha_x\boldsymbol{b})\otimes 1 - \prod_{i,\boldsymbol{\pi_X}(\boldsymbol{E}_i)=x}\boldsymbol{s}_i^{n_{i,x}}\otimes\chi_x\prod_i r_{x,i}^{h_{x,i}},\ x = [\alpha_x:\beta_x]\in\mathcal{S}_X.$$

Because $\psi^U(\boldsymbol{a}\otimes 1) = a$, $\psi^U(\boldsymbol{b}\otimes 1) = b$, and $\psi^U(\boldsymbol{s_i}\otimes 1) = s_i$, we obtain a commutative square

$$\begin{tikzcd}
R[\boldsymbol{a}\otimes 1, \boldsymbol{b}\otimes 1, \boldsymbol{s_i}\otimes 1] \arrow[r, "\gamma"] \arrow[d, "/I"] & R[a,b,s_i] \arrow[d, "/J"] \\
\mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X})^U\otimes_{\mathrm{Cox}_{\boldsymbol{G}}(\boldsymbol{X})^{\boldsymbol{G}}} R \arrow[r, "\psi^U"] & \mathrm{Cox}_G(X)^U
\end{tikzcd}$$

where γ is the obvious isomorphism between the above two polynomial algebras. As $\gamma(I) = J$, we readily obtain that ψ^U is an isomorphism. □

Lemma 5.6.0.3. *For all $x\in\mathcal{S}_{\boldsymbol{X}}$ and all exceptional colors $\boldsymbol{E}_i$ such that $\pi_{\boldsymbol{X}}(\boldsymbol{E}_i) = x$, we have $\boldsymbol{n}_{x,i} = n_{x,i}$. Also, we have*

$$f^*(\boldsymbol{X}^x) = \frac{1}{\boldsymbol{h}_x}\sum_{\pi_X(X_i^x)=x} h_{x,i}X_i^x.$$

Proof. This stems from the fact that $f^*(\boldsymbol{E}_i) = E_i$ and that $\pi_X = \boldsymbol{\pi_X} f$. □

5.7 Finding the exceptional colors

The description of the exceptional colors in a homogeneous space G/H of complexity one is obviously a crucial step toward the description of the Cox rings of the various G/H-embeddings. In this section, we present a method of geometrical nature to achieve this task. In view of the preceding section, we suppose that G is semisimple.

Choose $y_1\in G/H$ such that $B.y_1$ is of codimension one in G/H. Consider a minimal parabolic subgroup P_1 of G containing B such that $O_1 := P_1.y_1$ is open in X. Such a subgroup exists because the minimal parabolic subgroups generate G.

Lemma 5.7.0.1. *Identify the open orbit O_1 with $P_1/\operatorname{Stab}_{P_1}(y_1)$. Then, the rational quotient π_{O_1} of O_1 by B is everywhere defined and is a geometric quotient.*

Proof. Consider the geometric quotient

$$g : P_1/\operatorname{Stab}_{P_1}(y_1)\to P_1/\mathcal{R}(P_1)\operatorname{Stab}_{P_1}(y_1)$$

by the radical $\mathcal{R}(P_1)$ of P_1. This quotient is a homogeneous space of complexity one for the algebraic group $P_1/\mathcal{R}(P_1)$. This last algebraic group is semisimple of rank 1, hence isomorphic to SL_2 or PSL_2. This implies an isomorphism

$$P_1/\mathcal{R}(P_1)\operatorname{Stab}_{P_1}(y_1)\simeq\mathrm{SL}_2/F,$$

where F is a finite subgroup of SL_2. The rational quotient

$$h : P_1/\mathcal{R}(P_1)\operatorname{Stab}_{P_1}(y_1) \dashrightarrow \mathbb{P}_k^1,$$

by $B/\mathcal{R}(P_1)$ is then identified with the quotient of SL_2/F by a Borel subgroup of SL_2. Hence, the rational map h is in fact everywhere defined and moreover a geometric quotient morphism (see [V2, Sec. 3.1]). By construction, we have $\pi_{|O_1} = hg$ whence the claim. □

Hence, the morphism π_{O_1} defines a one-parameter family of colors in G/H by taking closures of (supports of) pullbacks of points of $\mathbb{P}^1_k$. Define Σ_1 to be the set of colors of G/H which are either contained in $G/H \setminus O_1$ or are the closure of the support of a divisor in O_1 of the form $\pi^*_{O_1}(x) = nE$, where $x \in \mathbb{P}^1_k$ and $n > 1$. By construction, the elements of Σ_1 are (some) exceptional colors of G/H.

Now, look for a point $y_2 \in G/H \setminus O_1$ such that $B.y_2$ is of codimension one in G/H, and define P_2, O_2, Σ_2 similarly as before. By induction continue this process in order to construct

- a sequence of points $y_1, ..., y_r$,
- a sequence of open subsets $O_i = P_i.y_i$, $i = 1, .., r$ such that $y_i \in G/H \setminus \cup_{j=1}^{i-1} O_j$,
- a sequence of sets of exceptional colors Σ_i, $i = 1, .., r$.

Moreover $Z := G/H \setminus \cup_{i=1}^{r} O_i$ is by construction a (closed) union of B-orbits of codimension $\geqslant 2$. Denote Σ_C the set of prime divisors contained in Z. The elements of Σ_C are by construction the central colors of G/H. Also

$$\Sigma := (\cup_{i=1}^{r} \Sigma_i) \cup \Sigma_C$$

is the set of exceptional colors of G/H. Write $(E^{n_i}_{J_i})_{i \in \Sigma \setminus \Sigma_C}$ the family of non-central exceptional colors, where $J_i \subset \{1, ..., r\}$ corresponds to the set of open subsets O_k intersecting $E^{n_i}_{J_i}$ non-trivially, and $n_i \in \mathbb{Z}_{>0}$ is the multiplicity of $E^{n_i}_{J_i}$ in the pullback of the associated exceptional point of $\mathbb{P}^1_k$ by $\pi_{G/H}$. Also, let s_i be a $B \times H$-semi-invariant element of $\mathrm{Cox}_G(G/H)$ whose divisor of zeroes is $E^{n_i}_{J_i}$ (it is unique up to multiplication by a non-zero constant). Then, consider all the products

$$s_{i_1}^{n_{i_1}} ... s_{i_l}^{n_{i_l}} \tag{5.7.1}$$

such that $J_{i_1}, ..., J_{i_l}$ defines a partition of $\{1, ..., r\}$. By [T11, 16.2], there exists a unique weight $(\lambda_0, -\chi_0)$ that occurs at least twice as the weight of such a product, and $\mathcal{O}(G)^{(B\times H)}_{(\lambda_0, -\chi_0)}$ is the two dimensional linear system defining $\pi_{G/H}$. This vector space is spanned by any two distinct products of the form (5.7.1) with weight $(\lambda_0, -\chi_0)$, and the divisor of zeroes of those products gives exactly the non-central exceptional colors of G/H with their multiplicities.

5.8 The space of ordered triangles

Consider the algebraic group $G := \mathrm{SL}_3$ identified with its group of rational points, namely the matrices

$$g = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

such that $g_{ij} \in k$ and $\det(g) = 1$. By abuse of notation, we also let g_{ij} denote the corresponding matrix coordinates on G. Consider also the Borel subgroup B of upper triangular matrices, the maximal torus $T \subset B$ of diagonal matrices, and the unipotent part U of B. The character group of T is free of rank two generated by $\varepsilon_i := g_{ii}$ with the relation $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$, and the fundamental weights are $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$.

The homogeneous space $\mathscr{O} := G/T$ is viewed as the space of nondegenerate ordered triangles by interpreting the matrix columns as points in a projective plane. The set of triples of lines and points with incidence relations yields a natural equivariant completion

$$X := \{(p_1, p_2, p_3, l_1, l_2, l_3) \mid p_j \in \mathbb{P}^2, l_i \in (\mathbb{P}^2)^*, p_i \in l_j \text{ whenever } i \neq j\},$$

where $(\mathbb{P}^2)^* := \mathbb{P}((k^3)^*)$. If z_{kj} are the homogeneous coordinates of p_j in $\mathbb{P}^2_k$, and y_{ik} are the dual coordinates of $l_i \in \mathbb{P}^{2*}_k$, then X is determined by six equations $(y.z)_{ij} = 0$, $(i \neq j)$, in $(\mathbb{P}^2_k)^3 \times (\mathbb{P}^{2*}_k)^3$. Using Serre's normality criterion, one verifies that X is a normal complete intersection. In fact, X is singular with singular locus contained in the unique closed G-orbit (see below).

Our goal is to describe the Cox ring of X by generators and relations following the strategy described in the above sections. Remark that because G is semisimple and simply connected, the Cox ring is canonically isomorphic to the equivariant Cox ring, and thus inherits a canonical structure of G-algebra.

5.8.1 Exceptional colors of $\mathscr{O}$

There is a direct geometric way to identify the exceptional colors of $\mathscr{O}$. Indeed, notice that the rational quotient $\pi_{\mathscr{O}} : \mathscr{O} \dashrightarrow \mathbb{P}^1_k$ by B sits in the commutative square

$$\begin{tikzcd} G \arrow[r, "/T"] \arrow[d, "B\backslash"'] & \mathscr{O} \arrow[d, dashed, "\pi_{\mathscr{O}}"] \\ B\backslash G \arrow[r, dashed, "/T"] & \mathbb{P}^1_k \end{tikzcd}$$

where the homogeneous space $B\backslash G$ is the variety of complete flags in k^3. Such a flag is defined by a pair $(k^*.v, k^*.\varphi) \in \mathbb{P}^2_k \times (\mathbb{P}^2_k)^*$, such that $\varphi(v) = 0$. Hence, there is a natural identification of this flag variety with the subvariety of $\mathbb{P}^2_k \times (\mathbb{P}^2_k)^*$ defined by the incidence relation

$$s_1\tilde{s}_1 + s_2\tilde{s}_2 + s_3\tilde{s}_3 = 0,$$

where the $s_i, \tilde{s}_i$ are homogeneous coordinates along the T-stable lines of respective weights ε_i, $-\varepsilon_i$ for the natural T-action on k^3 and $(k^3)^*$. Remark that the s_i and $\tilde{s}_i$ pull back to the functions $g_{3i} \in \mathcal{O}(G)$ and $\begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix} \in \mathcal{O}(G)$, where (i, j, k) is any cyclic permutation of $(1, 2, 3)$. In the sequel, we let $s_i = g_{3i}$, and $\tilde{s}_i = \begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix}$. It easy to verify that $k(\mathscr{O})^B = k(B\backslash G)^T$ is generated by any of the rational functions $s_i\tilde{s}_i/s_j\tilde{s}_j$, $i \neq j$. Hence, $\pi_{\mathscr{O}}$ is defined by the two dimensional linear system

$$\mathcal{O}(G)^{(B\times T)}_{(\omega_1+\omega_2,0)} \subset \mathcal{O}(\mathscr{O})$$

spanned by the $s_i\tilde{s}_i$ with the relation $s_1\tilde{s}_1 + s_2\tilde{s}_2 + s_3\tilde{s}_3 = 0$.

Now, we run the method described in Section 5.7, to see how we can recover these results. Consider the point

$$y_1 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \mathrm{mod}\ T \in \mathscr{O}.$$

Consider also the standard parabolic subgroup

$$P_{\alpha_1} = \left\{ a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \mid \det(a) = 1 \right\}.$$

Then we have

$$O_1 := P_{\alpha_1}.y_1 = \left\{ aT \mid a = \begin{bmatrix} -a_{13} & a_{12} + a_{13} & a_{11} + a_{13} \\ -a_{23} & a_{22} + a_{23} & a_{21} + a_{23} \\ -a_{33} & a_{33} & a_{33} \end{bmatrix}, a_{ij} \in k, \det(a) = 1 \right\}.$$

and we see that O_1 is open in $\mathscr{O}$ with complement the zero set of the P-semi-invariant regular function $s_1s_2s_3$ on $\mathscr{O}$. This complement is the union of the three P-stable prime divisors $E_i = \mathrm{div}_{\mathscr{O}}(s_i)$. An easy verification shows that the stabilizer of y_1 in P_{α_1} coincides with the center μ_3 of G. This latter group lies in the radical of P_{α_1}, thus we have identifications

$$O_1/\mathcal{R}(P_{\alpha_1}) \simeq P_{\alpha_1}/\mathcal{R}(P_{\alpha_1}) \simeq \mathrm{PGL}_2.$$

We deduce that for any $x \in \mathbb{P}^1_k$, the divisor $\pi^*_{O_1}(x)$ is prime (there is no multiplicity).

Likewise consider the point

$$y_2 := \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \mod T,$$

and the standard parabolic subgroup

$$P_{\alpha_2} = \left\{ a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \mid \det(a) = 1 \right\}.$$

Then we have

$$O_2 := P_{\alpha_2}.y_2 = \left\{ aT \mid a = \begin{bmatrix} a_{11}+a_{12} & a_{13} & a_{12}+a_{13} \\ a_{22} & a_{23} & a_{22}+a_{23} \\ a_{32} & a_{33} & a_{32}+a_{33} \end{bmatrix}, a_{ij} \in k, \det(a) = 1 \right\}.$$

It is easy to verify that O_2 is open in $\mathscr{O}$ with complement the zero set of the regular function $\tilde{s}_1\tilde{s}_2\tilde{s}_3 \in \mathcal{O}(\mathscr{O})$. This complement is the union of the three P-stable prime divisors $\tilde{E}_i := \mathrm{div}_{\mathscr{O}}(\tilde{s}_i)$. As before, the restriction π_{O_2} has only parametric B-stable divisors.

We have $\mathscr{O} = O_1 \cup O_2$, and three products

$$s_1\tilde{s}_1,\ s_2\tilde{s}_2,\ s_3\tilde{s}_3$$

with common weight $(\omega_1+\omega_2, 0)$. Hence, the two-dimensional linear system $\mathcal{O}(G)^{(B\times T)}_{(\omega_1+\omega_2,0)}$ is spanned by the regular functions $s_1\tilde{s}_1$, $s_2\tilde{s}_2$, $s_3\tilde{s}_3 \in \mathcal{O}(\mathscr{O})$ with the obvious relation $s_1\tilde{s}_1 + s_2\tilde{s}_2 + s_3\tilde{s}_3 = 0$. Let x_i be the point of $\mathbb{P}^1_k$ defined by the line $k^*s_i\tilde{s}_i$. The exceptional points are the x_i, the exceptional colors in $\mathcal{O}$ are the E_i, $\tilde{E}_i$, and we have

$$\pi^*_{\mathscr{O}}(x_i) = E_i + \tilde{E}_i.$$

5.8.2 Geometric description of X

Following [T11, 16.24], the projective $\mathscr{O}$-embedding X contains the following G-stable subvarieties of degenerate triangles

- $W_i : p_j = p_k$ and $l_j = l_k$, where (i, j, k) is any cyclic permutation of $(1, 2, 3)$ (codimension 1),
- $\tilde{W} : p_1, p_2, p_3$ are collinear and $l_1 = l_2 = l_3$ (codimension 1),
- $W : p_1 = p_2 = p_3$ and l_1, l_2, l_3 pass through this point (codimension 1),
- $\tilde{Y}_i : p_j = p_k$ and $l_1 = l_2 = l_3$ (codimension 2),
- $Y_i : l_j = l_k$ and $p_1 = p_2 = p_3$ (codimension 2),

- $Y : p_1 = p_2 = p_3$ and $l_1 = l_2 = l_3$ (codimension 3).

Moreover, Y is the unique closed G-orbit in X. The divisors W and $\tilde{W}$ are central and

$$\pi_X^*(x_i) = E_i + \tilde{E}_i + W_i, \ \ i = 1, 2, 3.$$

Hence, the exceptional colors of X are the $E_i, \tilde{E}_i$. Moreover, these exceptional colors don't contain a G-orbit. Indeed, a direct check shows that X is the orbit closure of the point

$$x = (([e_i])_i, ([e_i^*])_i) \in \prod_{i=1}^3 \mathbb{P}(V) \times \prod_{i=1}^3 \mathbb{P}(V^*),$$

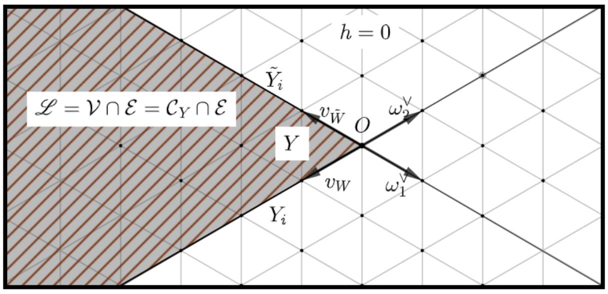
where $(e_i)_{1 \leqslant i \leqslant 3}$ be the canonical basis of the simple G-module $V = k^3$, and $(e_i^*)_{1 \leqslant i \leqslant 3}$ the dual basis of V^*.

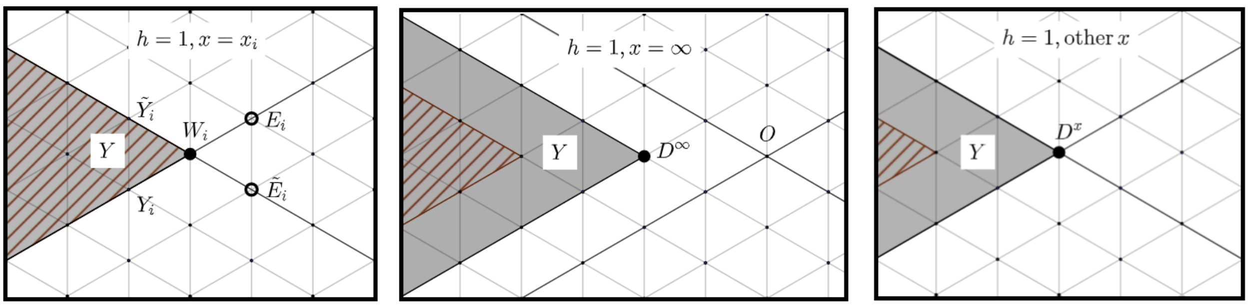
5.8.3 Combinatorial description of X

The colored hyperspace $(\check{\mathcal{E}}, \mathcal{V}, \mathcal{D}^B, \varrho)$ associated with $\mathscr{O}$ has been described in [T11, Ex. 16.24], we recall this description and the combinatorial data defining X. The weight lattice $\Lambda(\mathscr{O})$ coincides with the root lattice $\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ and we denote $(\omega_1^\vee, \omega_2^\vee)$ the dual basis to (α_1, α_2). We choose an arbitrary parametric color D^∞, $\infty \in \mathbb{P}_k^1$ with canonical section $s_\infty \in \mathcal{O}(G)^{(B\times T)}_{(\omega_1+\omega_2,0)}$. The B-semi-invariant rational functions

$$f_{\alpha_1} = \frac{\tilde{s}_1\tilde{s}_2\tilde{s}_3}{s_\infty} \text{ and } f_{\alpha_2} = \frac{s_1 s_2 s_3}{s_\infty} \tag{5.8.1}$$

define a section of the morphism $k(\mathscr{O})^{(B)} \to \Lambda(\mathscr{O})$, and thus provide coordinates on the hyperspace. Indeed, a valuation v associated with a B-stable prime divisor has coordinates $(x, h, (u_1, u_2))$, where $x \in \mathbb{P}_k^1$, $h \in \mathbb{Z}_{\geqslant 0}$, $u_1, u_2 \in \mathbb{Z}$, $v_{|k(\mathbb{P}^1)} = hv_x$, and $v(f_{\alpha_1}) = u_1$, $v(f_{\alpha_2}) = u_2$. The intersection of $\mathcal{V}$ with the hyperplane sections $\mathcal{E} = \{h = 0\}$ and $h = 1$ of $\mathcal{E}_x$ are hatched in the figures below.





Since X is complete and has a unique closed G-orbit Y, we have $X = G\mathring{X}$ where $\mathring{X}$ is the minimal B-chart of Y determined by the colored hypercone $(\mathcal{C}_Y, \mathcal{D}_Y^B)$. This hypercone is darked in the above figures.

5.8.4 Resolution of singularities

Consider the normal $\mathscr{O}$-embedding $X(\mathcal{S}_{\mathscr{O}})$ provided by Proposition 5.4.0.2, where $\mathcal{S}_{\mathscr{O}} = \{x_1, x_2, x_3\}$ by the above. This is a smooth projective variety by virtue of Proposition 5.4.0.9. Hence, the G-equivariant projection (see Construction 5.4.0.10)

$$X(\mathcal{S}_{\mathscr{O}}) \to X$$

is a resolution of singularities. Historically, another resolution of singularities $X_{\mathrm{Sch}} \to X$ was implicitly used by Schubert in his work on enumerative geometry. This resolution can be constructed as follows: assign to each non-degenerate triangle $(p_1, p_2, p_3, l_1, l_2, l_3)$ the net of conics passing through (p_1, p_2, p_3). This defines a rational map

$$X \to \mathrm{Gr}(2, \mathbb{P}(\mathrm{Sym}_2(k^{3*}))),$$

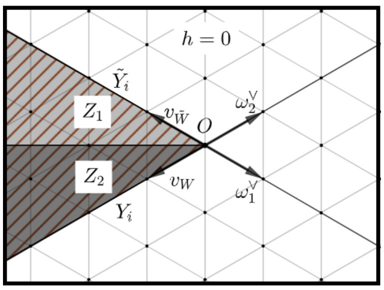
and X_{Sch} is the closure of its graph. Following [CF], X_{Sch} is smooth, whence the G-equivariant projection

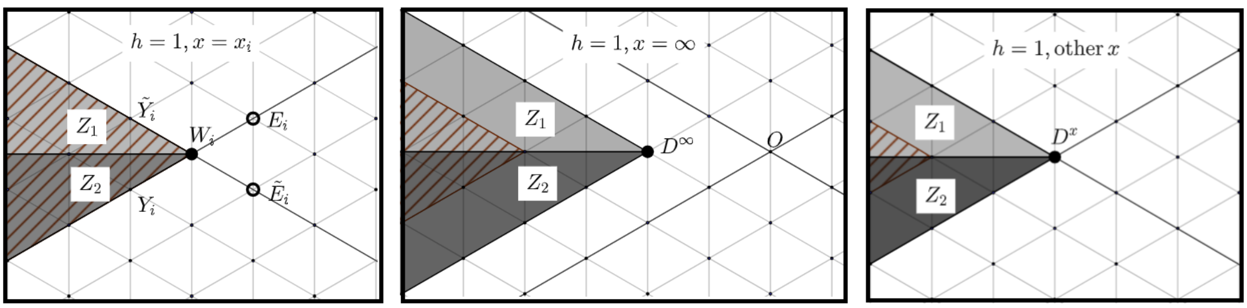
$$X_{\mathrm{Sch}} \to X$$

is another resolution of singularities. The G-stable subvarieties of degenerate triangles in X_{Sch} are described in [CF]. A combinatorial check proves that X and X_{Sch} are up to isomorphism the only two complete normal $\mathscr{O}$-embedding sharing the following properties

- $\mathcal{S}$ is the set of exceptional points,
- $W_1, W_2, W_3, W, \tilde{W}$ are the G-stable prime divisors,
- no exceptional color contains a G-orbit,
- there are finitely many closed G-orbits.

The variety X_{Sch} is obtained by gluing two G-varieties defined by the darked hypercones in the figure below. These hypercones are respectively generated by $v_W, (v_{W_i})_i, (v_{D^x})_{x \neq x_i}$, and $v_{\tilde{W}}, (v_{W_i})_i, (v_{D^x})_{x \neq x_i}$. In particular, it is easily seen that X_{Sch} has exactly two closed G-orbits Z_1 and Z_2, in accordance with [CF].





Finally, notice that the three varieties X, $X(\mathcal{S}_{\mathscr{O}})$, and X_{Sch} are isomorphic over open G-stable subvarieties whose complements are of codimension $\geqslant 2$. In particular, they have canonically isomorphic Cox rings.

5.8.5 The equivariant Picard group of $X(\mathcal{S}_{\mathscr{O}})$

Recall that there is a natural isomorphism $\mathrm{Pic}_G(\mathscr{O}) \simeq \hat{T}$, and that $\mathrm{Pic}_G(\mathscr{O}) \simeq \mathrm{Pic}(\mathscr{O})$ because G is semisimple and simply connected. Hence, we obtain an exact sequence

$$0 \to (\bigoplus_{i=1}^{3} \mathbb{Z}W_i) \oplus \mathbb{Z}W \oplus \mathbb{Z}\tilde{W} \to \mathrm{Pic}(X(\mathcal{S}_{\mathscr{O}})) \to \hat{T} \to 0$$

by restricting invertible sheaves to $\mathscr{O}$. In particular, $\mathrm{Pic}(X(\mathcal{S}_{\mathscr{O}}))$ is a free abelian group of rank 7 and the classes of the G-stable prime divisors generate a direct factor. Also, in view of Lemma 5.4.0.7, the Picard group is freely generated by the classes

$$[E_i],\ [\tilde{E}_i],\ [D^\infty].$$

Another description that will be useful is obtained by noticing that $\mathrm{Pic}(X(\mathcal{S}_{\mathscr{O}}))$ is generated by the classes of the exceptional divisors, namely the

$$[E_i],\ [\tilde{E}_i],\ [W_i],\ [W],\ [\tilde{W}].$$

with relations generated by the following ones:

$$\begin{cases} [E_i] + [\tilde{E}_i] + [W_i] = [E_j] + [\tilde{E}_j] + [W_j],\ i \neq j \\ [E_i] + [\tilde{E}_i] + [W_i] = [E_1] + [E_2] + [E_3] - [W], \\ [E_i] + [\tilde{E}_i] + [W_i] = [\tilde{E}_1] + [\tilde{E}_2] + [\tilde{E}_3] - [\tilde{W}]. \end{cases}$$

Indeed, any B-semi-invariant rational function can be written uniquely as a product

$$g f_{\alpha_1}^m f_{\alpha_2}^n, \text{ where } g \in k(X)^B \simeq k(\mathbb{P}^1_k),\ m, n \in \mathbb{Z} \quad (\text{see } (5.8.1)).$$

Moreover, f_{α_1} (resp. f_{α_2}) has zeroes of order 1 along the $\tilde{E}_i$ (resp. the E_i) and poles of order 1 along $\tilde{W}, D^\infty$ (resp. along W, D^∞). Since

$$\begin{cases} \pi^*_{X(\mathcal{S}_{\mathscr{O}})}(x_i) = E_i + \tilde{E}_i + W_i,\ i = 1,2,3, \text{ and} \\ \pi^*_{X(\mathcal{S}_{\mathscr{O}})}(x) = D^x,\ x \neq x_i, \end{cases} \tag{5.8.2}$$

we obtain that the above relations hold and generate all the relations. In particular, we deduce the following relations that we record for future use:

$$\begin{cases} [\tilde{E}_j] + [\tilde{E}_k] - [E_i] = [W_i] + [\tilde{W}], \{i,j,k\} = \{1,2,3\}, \\ [E_j] + [E_k] - [\tilde{E}_i] = [W_i] + [W], \{i,j,k\} = \{1,2,3\}, \\ [E_i] + [\tilde{E}_i] = [W_j] + [W_k] + [W] + [\tilde{W}], \{i,j,k\} = \{1,2,3\}. \end{cases} \tag{5.8.3}$$

5.8.6 Representation theory of G

The isomorphism classes of simple G-modules are parametrized by the dominant weights $a\omega_1 + b\omega_2$, $a, b \in \mathbb{N}$. For each such dominant weight, let $V_{a,b}$ be a simple G-module of highest weight $a\omega_1 + b\omega_2$. We have

$$V_{1,0} \simeq k^3, \text{ and } V_{0,1} \simeq (k^3)^*$$

with the standard G-action on these vector spaces. With these identifications, $V_{1,0} \otimes_k V_{0,1}$ identifies with the vector space of 3×3 matrices, with G acting by conjugation. The latter decomposes as the direct sum of the simple G-submodule $V_{1,1}$ of traceless matrices, with the trivial G-submodule $V_{0,0} \simeq kI_3$. Also, we have

$$\mathrm{Sym}^2_k(V_{1,0}) \simeq V_{2,0} \text{ and } \mathrm{Sym}^2_k(V_{0,1}) \simeq V_{0,2}.$$

Finally, observing that $(k^3)^* \simeq \wedge^2 k^3$ as G-modules, and that $V \otimes_k V \simeq \mathrm{Sym}^2_k V \oplus \wedge^2 V$ for any G-module V, we obtain

$$V_{1,0} \otimes_k V_{1,0} \simeq V_{2,0} \oplus V_{0,1} \text{ and } V_{0,1} \otimes_k V_{0,1} \simeq V_{0,2} \oplus V_{1,0}.$$

5.8.7 Description of $\mathrm{Cox}(X)$

Let $s_i, \tilde{s}_i$ be the canonical sections associated with the exceptional colors $E_i, \tilde{E}_i$. Also, let $r, \tilde{r}, r_i$ denote the canonical sections associated with the G-stable prime divisors $W, \tilde{W}, W_i$. In view of (5.5.1) and (5.8.2), the k-algebra $\mathrm{Cox}(X)^U$ is generated by the elements $s_i, \tilde{s}_i, r_i, r, \tilde{r}$ with the unique relation

$$s_1\tilde{s}_1r_1 + s_2\tilde{s}_2r_2 + s_3\tilde{s}_3r_3 = 0.$$

Viewing $\mathrm{Cox}(X)$ as a graded subalgebra of

$$\bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}(G)e^{[D]},$$

where $e^{[D]}$ denote the character of $\mathbb{D}(\mathrm{Cl}(X))$ associated with $[D]$, the canonical sections

$$s_i, \tilde{s}_i, r_i, r, \tilde{r}$$

are respectively identified with

$$g_{3i}e^{[E_i]}, \begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3jk} \end{vmatrix} e^{[\tilde{E}_i]} \text{ with } (i,j,k) = (1,2,3) \text{ as permutations, } e^{[W_i]}, e^{[W]}, e^{[\tilde{W}]}.$$

We consider respectively the following k-bases of T-eigenvectors

- $(s_i, t_i, u_i) := (g_{3i}e^{[E_i]}, g_{2i}e^{[E_i]}, g_{1i}e^{[E_i]})$
- $(\tilde{s}_i, \tilde{t}_i, \tilde{u}_i) := (\begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix} e^{[\tilde{E}_i]}, \begin{vmatrix} g_{1j} & g_{1k} \\ g_{3j} & g_{3k} \end{vmatrix} e^{[\tilde{E}_i]}, \begin{vmatrix} g_{1j} & g_{1k} \\ g_{2j} & g_{2k} \end{vmatrix} e^{[\tilde{E}_i]})$

of the simple G-modules $V_{s_i} \simeq V_{0,1}$ and $V_{\tilde{s}_i} \simeq V_{1,0}$. In view of Section 5.5, the k-algebra $\mathrm{Cox}(X)$ is generated by the simple G-modules V_{s_i}, $V_{\tilde{s}_i}$, and the $r_i, r, \tilde{r}$.

In order to determine the ideal of relations, we have to determine the B-weights occurring in products $V_{s_i}V_{s_j}$, $V_{\tilde{s}_i}V_{\tilde{s}_j}$, $V_{s_i}V_{\tilde{s}_j}$. We have seen in the preceding section that the G-modules $V_{s_i}^2$, $V_{\tilde{s}_i}^2$ are simple. For $i<j$ and $\{i,j,k\} = \{1,2,3\}$, the G-module $V_{s_i}V_{s_j}$ contains up to non-zero multiple the following B-semi-invariants

- s_is_j of B-weight $2\omega_2$,
- $s_jt_i - s_it_j = (g_{3j}g_{2i} - g_{3i}g_{2j})e^{[E_i+E_j]} = (g_{3j}g_{2i} - g_{3i}g_{2j})e^{[\tilde{E}_k]}e^{[E_i+E_j-\tilde{E}_k]} \overset{(5.8.3)}{=} \tilde{s}_kr_kr$ of B-weight ω_1.

Likewise, using the relations of (5.8.3) we obtain that the G-module $V_{\tilde{s}_i}V_{\tilde{s}_j}$, $i<j$, contain up to non-zero multiple the following B-semi-invariants

- $\tilde{s}_i\tilde{s}_j$ of B-weight $2\omega_1$,
- $\tilde{s}_j\tilde{t}_i - \tilde{s}_i\tilde{t}_j = s_kr_k\tilde{r}$ of B-weight ω_2.

Finally, the G-modules $V_{s_i}V_{\tilde{s}_j}$, $i<j$ are simple since we have

$$s_i\tilde{u}_j - t_i\tilde{t}_j + u_i\tilde{s}_j = 0,$$

and the G-modules $V_{s_i}V_{\tilde{s}_i}$ contain the G-stable line spanned by the element

$$s_i\tilde{u}_i - t_i\tilde{t}_i + u_i\tilde{s}_i \overset{(5.8.3)}{=} r_jr_kr\tilde{r}.$$

These computations, together with Proposition 5.5.0.5 yield the

Proposition 5.8.7.1. *The Cox ring of X is generated as a k-algebra by the homogeneous elements*

$$(s_i)_i, (t_i)_i, (u_i)_i, (\tilde{s}_i)_i, (\tilde{t}_i)_i, (\tilde{u}_i)_i, (r_i)_i, r, \tilde{r}.$$

The ideal of relations is generated as a G-stable ideal by the following B-semi-invariants

$$\begin{cases} s_i t_j - s_j t_i - \tilde{s}_k r_k r, \quad (i,j,k) = (1,2,3) \text{ as permutations, } (B\text{-weight: } \omega_2) \\ \tilde{s}_i \tilde{t}_j - \tilde{s}_j \tilde{t}_i - s_k r_k \tilde{r}, \quad (i,j,k) = (1,2,3) \text{ as permutations, } (B\text{-weight: } \omega_1) \\ s_i \tilde{u}_i - t_i \tilde{t}_i + u_i \tilde{s}_i - r_j r_k r \tilde{r}, \quad (i,j,k) = (1,2,3) \text{ as permutations, } (B\text{-weight: } 0) \\ s_i \tilde{u}_j - t_i \tilde{t}_j + u_i \tilde{s}_j, \quad i < j, \ (B\text{-weight: } 0) \\ s_1 \tilde{s}_1 r_1 + s_2 \tilde{s}_2 r_2 + s_3 \tilde{s}_3 r_3, \quad (B\text{-weight: } \omega_1 + \omega_2) \end{cases}$$

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