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Aya Oussaily

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Par Aya OUSSAILY

*Étude théorique et numérique des systèmes
modélisant la dynamique des
densités des dislocations*

Thèse présentée
pour l'obtention du grade
de Docteur de l'UTC



Soutenue le 11 octobre 2021

Spécialité : Mathématiques Appliquées : Laboratoire de
Mathématiques Appliquées de Compiègne (Unité de recherche
EA-2222)

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Thèse en vue d'obtention du grade de

DOCTEUR DE L'UNIVERSITE DE TECHNOLOGIE DE COMPIEGNE

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la dynamique des densités des dislocations**

Par Aya OUSSAILY

Spécialité : Mathématiques Appliquées

Thèse soutenue le 11 octobre 2021 devant le jury composé de :

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Résumé

Dans cette thèse, nous nous intéressons à l'analyse théorique et numérique de la dynamique des densités des dislocations. Les dislocations sont des défauts linéaires qui se déplacent dans les cristaux lorsque ceux-ci sont soumis à des contraintes extérieures. D'une manière générale, la dynamique des densités des dislocations est décrite par un système d'équations de transport, où les champs de vitesse dépendent de manière non-locale des densités des dislocations.

Au départ, notre travail se focalise sur l'étude d'un système unidimensionnel (2×2) de type Hamilton-Jacobi dérivé d'un système bidimensionnel proposé par Groma et Balogh en 1999. Pour ce modèle, nous montrons un résultat d'existence globale et d'unicité. En addition, nous nous intéressons à l'étude numérique de ce problème, complété par des conditions initiales croissantes, en proposant un schéma aux différences finies implicite dont on prouve la convergence. Ensuite, en s'inspirant du travail effectué pour la résolution de la dynamique des densités des dislocations, nous mettons en oeuvre une théorie plus générale permettant d'obtenir un résultat similaire d'existence et d'unicité d'une solution dans le cas des systèmes de type eikonal unidimensionnels. En considérant des conditions initiales croissantes, nous faisons une étude numérique pour ce système. Sous certaines conditions de monotonies sur la vitesse, nous proposons un schéma aux différences finies implicite permettant de calculer la solution discrète et simuler ainsi la dynamique des dislocations à travers ce modèle.

Abstract

In this thesis, we are interested in the theoretical and numerical studies of dislocations densities. Dislocations are linear defects that move in crystals when those are subjected to exterior stress. More generally, the dynamics of dislocations densities are described by a system of transport equations where the velocity field depends non locally on the dislocations densities.

First, we are interested in the study of a one dimensional submodel of a (2×2) Hamilton-Jacobi system introduced by Groma and Balogh in 1999, proposed in the two dimensional case. For this system, we prove global existence and uniqueness results. Adding to that, considering nondecreasing initial data, we study this problem numerically by proposing a finite difference implicit scheme for which we show the convergence.

Then, inspired by the first work, we show a more general theory which allows us to get similar results of existence and uniqueness of solution in the case of one dimensional eikonal systems. By considering nondecreasing initial data, we study this problem numerically. Under certain conditions on the velocity, we propose a finite difference implicit scheme allowing us to calculate the discrete solution and simulate then the dislocations dynamics via this model.

Publications issues de la thèse

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- (avec A. El Hajj), *Existence and uniqueness of continuous solution for a non-local coupled system modeling the dynamics of dislocations densities*, Journal of Nonlinear Science, (2021), p.20 (cf. Chapitre 3)
- (avec R. Boudjerada et A. El Hajj), *Convergence of an implicit scheme for diagonal non-conservative hyperbolic systems*, M2AN, (2020) (cf. Chapitre 6)
- (avec A. El Hajj), *Continuous solution for a non-linear eikonal system*, accepté dans Communications on Pure and Applied Analysis. (cf. Chapitre 5)

Article soumis

- (avec A. EL Hajj) *Convergent scheme for a non-local transport system modeling dislocations densities dynamics*, soumis à Journal of Numerical Mathematics (cf. Chapitre 4)

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1

Introduction générale

Cette thèse porte sur la modélisation, l'analyse théorique et numérique ainsi que la simulation de la dynamique des dislocations. Les dislocations sont des défauts microscopiques présents dans la plupart des matériaux cristallins, surtout dans les alliages métalliques.

D'un point de vue mathématique, la dynamique des dislocations se traduit par un système d'équations de transport ou Hamilton Jacobi couplées. Les travaux de cette thèse visent à l'étude de ces systèmes ainsi qu'à leurs résolutions numériques.

Cette introduction est constituée de cinq parties. La première partie porte sur une introduction physique à la dynamique des dislocations. Nous commençons ce chapitre par un rappel sur les dislocations et leurs propriétés. Ensuite, les Sections 2, 3, 4 et 5 sont dédiées à la présentation des résultats mathématiques obtenus.

Les hypothèses utilisées dans les théorèmes suivant ne sont pas optimales. Nous renvoyons les lecteurs de cette thèse aux chapitres correspondants pour plus de détails.

1 Motivation physique

Les dislocations sont des défauts linéaires mésoscopiques dont le déplacement et l'interaction sont à l'origine de la déformation plastique macroscopique des matériaux cristallins. Par, exemple, dans la Figure 1.1, nous observons les dislocations présentes dans un alliage (Ni_3Al), pris en microscopie électronique en transmission, à l'échelle $1\mu m = 10^{-6}m$.

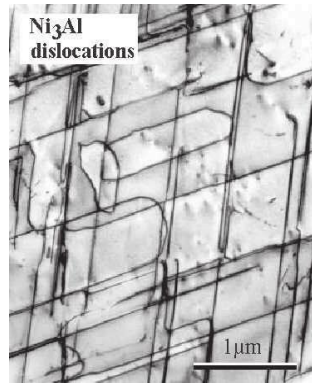


Figure 1.1: Observations de dislocations au microscope électronique en transmission dans un alliage Ni_3Al (<http://www.utc.fr/elhajjah/these.pdf>)

Dans cette thèse, nous nous intéressons à l'étude des systèmes modélisant la dynamique des densités des dislocations où la densité révèle la quantité des dislocations dans un cristal. La densité des dislocations est définie par le nombre de lignes de dislocations traversant une section unitaire. Notre motivation était de mieux comprendre le comportement mécanique issu de la déformation plastique du matériau en présence de forte concentration de dislocations.

1.1 Historique

En 1907, Volterra [71] a été le premier à analyser théoriquement les défauts linéaires qu'il a appelés "distortion". Le concept des dislocations a été introduit en 1934 par E. Orowan [61], M. Polanyi [65] and G. I. Taylor [68] qui ont expliqué la déformation plastique des matériaux, à l'échelle microscopique, par la présence des dislocations.

Les dislocations ont été observées pour la première fois dans les années 1950, grâce à l'évolution de la microscopie électronique (voir Figure 1.2), par Bollman [17], Hirsh, Horne et Whelan [41]. Ces observations ont permis de vérifier quelques prédictions théoriques, comme par exemple la nature microscopique du défaut, qui entraîne une distorsion élastique des liaisons dans une zone de taille très confinée autour de la dislocation (quelques longueurs inter-atomiques de largeur). Les dislocations jouent un rôle primordial dans les propriétés mécaniques des matériaux cristallins. Nous renvoyons les lecteurs aux traités classiques achevés dès la fin des années soixante voir Hirth et Lothe [42] et Nabarro [60]. La théorie des dislocations a permis aux mécaniciens de comprendre le comportement mécanique issu de la déformation plastique des milieux cristallins.

Grâce à l'essor prodigieux de la puissance de calcul des ordinateurs, la mécanique des matériaux a connu un développement des méthodes de simulation de la dynamique des lignes de dislocations, au milieu des années 80.



Figure 1.2: Micrographie obtenue en microscopie électronique en transmission montrant la présence de dislocations dans un acier inoxydable (<https://fr.wikipedia.org/wiki/Dislocation>)

1.2 Propriétés des dislocations

1.2.1 Vecteurs de Burgers

Soumises à des contraintes extérieures, les dislocations se déplacent dans un plan cristallographique appelé "plan de glissement". Le déplacement d'une dislocation se caractérise par un vecteur \vec{b} dit "vecteur de Burgers" [23].

Physiquement, le vecteur de Burgers représente l'amplitude de la déformation transportée par une dislocation.

1.2.2 Type de dislocations

Dans un solide continu nous réalisons une coupure selon une surface S en s'appuyant sur une ligne l . Nous translatons ensuite les lèvres ainsi formés l'une par rapport à l'autre selon le vecteur de Burgers \vec{b} de la dislocation. Cette ligne de dislocation l est caractérisée par son vecteur de Burgers \vec{b} et son vecteur tangent \vec{t} définissant le sens de parcours de la ligne. Nous distinguons alors les différents types de dislocations:

- Dislocation coin, si \vec{b} et \vec{t} sont perpendiculaires (Figure 1.3)

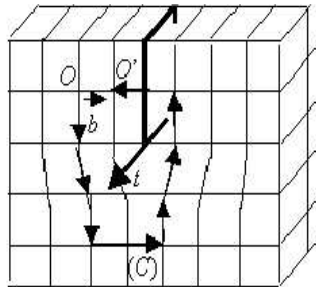
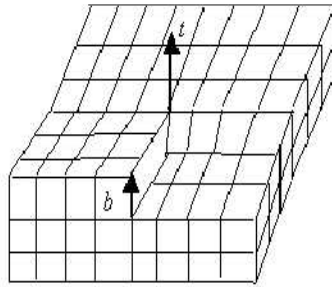


Figure 1.3: Dislocation coin

- Dislocation vis, si \vec{b} et \vec{t} sont parallèles (Figure 1.4)

Figure 1.4: *Dislocation vis*

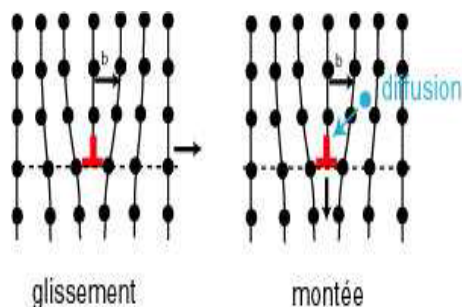
- Dislocation mixte où la ligne de dislocation forme un angle arbitraire avec le vecteur de Burgers

1.2.3 Déplacement des dislocations

Généralement, ils existent deux types de déplacement des dislocations: le glissement et la montée. En fait, le glissement d'une dislocation correspond à un mouvement conservatif, c'est-à-dire que le mouvement de la dislocation n'implique pas de déplacement d'atomes à longue distance. La dislocation se déplace dans le plan contenant son vecteur de Burgers \vec{b} et sa ligne de dislocation. L'ensemble {plan de glissement ; vecteur de Burgers} définit le système de glissement de la dislocation.

Les dislocations coins et mixtes ne peuvent ainsi glisser que dans un plan unique. Au contraire, les dislocations vis peuvent glisser dans plusieurs plans d'orientations différentes.

La montée d'une dislocation traduit le mouvement de cette dernière dans un plan ne comprenant pas son vecteur de Burgers \vec{b} . C'est un mécanisme impliquant la diffusion d'atomes, ce qui va entraîner le déplacement de la dislocation hors de son plan de glissement.

Figure 1.5: *Glissement et montée d'une dislocation*

2 Existence et unicité pour un système (2×2) modélisant la dynamique des densités des dislocations

Notre travail se focalise principalement sur l'étude théorique d'un problème unidimensionnel de type Hamilton-Jacobi modélisant la dynamique des densités des dislocations. Ce modèle a été introduit initialement par Groma et Balogh [40] dans le cas bidimensionnel. Nous renvoyons les lecteurs à la Section 2 du Chapitre 3 pour une dérivation physique du problème.

Plus précisément, nous admettons que le problème sur le domaine est 1-périodique en x et nous supposons que les densités des dislocations dépendent d'une seule variable $x = x_1 + x_2$ où (x_1, x_2) sont les coordonnées d'un point de \mathbb{R}^2 . Ce cadre particulier permet de simplifier le modèle bidimensionnel, introduit par Groma et Balogh, en un système unidimensionnel d'équations eikonales couplées non-linéaires et non-locales:

$$\begin{cases} \partial_t \rho^+(x, t) = - \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) |\partial_x \rho^+(x, t)| & \text{sur } \mathbb{R} \times (0, T), \\ \partial_t \rho^-(x, t) = - \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) |\partial_x \rho^-(x, t)| & \text{sur } \mathbb{R} \times (0, T), \end{cases} \quad (1.1)$$

où les inconnues sont les scalaires ρ^+ et ρ^- notées ρ^\pm . Ici, la différence $(\rho^+ - \rho^-)$ représente la déformation plastique et les dérivées partielles $\partial_x \rho^\pm$ représentent les densités des dislocations dont le vecteur de Burgers est $\pm \vec{b} = \pm(1, 0)$. Là, la constante α dépend des coefficients élastiques et de la taille du matériau et la fonction $a(t)$ constitue le champ des contraintes de cisaillement extérieures vérifiant la condition suivante:

$$a \in C(0, T). \quad (1.2)$$

Ce système est complété par les conditions initiales suivantes:

$$\rho^\pm(x, 0) = \rho_0^\pm(x) = P_0^\pm(x) + L_0 x \quad \text{sur } \mathbb{R}, \quad (1.3)$$

où P_0^\pm sont des fonctions 1-périodiques satisfaisant la condition suivante:

$$P_0^\pm \in L^\infty(\mathbb{T}) \text{ et } \partial_x P_0^\pm \in L \log L(\mathbb{T}), \quad (1.4)$$

avec $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ est l'intervalle périodique $[0, 1)$ et $L \log L(\mathbb{T})$ est l'espace de Zygmund défini par:

$$L \log L(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) \text{ tel que } \int_{\mathbb{T}} |f| \ln(e + |f|) < +\infty \right\}.$$

Cet espace est muni de la norme suivante:

$$\|f\|_{L \log L(\mathbb{T})} = \inf \left\{ \mu > 0 : \int_{\mathbb{T}} \frac{|f|}{\mu} \ln \left(e + \frac{|f|}{\mu} \right) \leq 1 \right\}.$$

Nous citons quelques résultats effectués en relation avec le problème (1.1). Dans [30], en utilisant une estimation d'énergie dans L^2 , les auteurs ont montré un résultat d'existence globale et d'unicité d'une solution croissante dans l'espace $W_{loc}^{1,2}(\mathbb{R} \times (0, T))$.

Tandis que, dans [33], en utilisant la théorie des solutions de viscosité, un résultat d'existence et d'unicité d'une solution Lipschitz a été montré sans prendre en considération la monotonie de la condition initiale.

Dans le cas des conditions initiales discontinues, nous citons le travail effectué dans [38] où un résultat d'existence a été établi.

De plus, un résultat d'existence globale et d'unicité a été démontré par Cannone et *al.* dans [24] dans le cadre des problèmes bidimensionnels en se basant sur une estimation entropique. Également, dans le même contexte, nous citons un résultat d'existence locale dans les espaces de Hölder démontré dans [31].

Pour étudier les solutions du système (1.1), nous nous sommes basés sur la théorie des solutions de viscosité. Cette théorie a été introduite par M. G. Crandall et P. L. Lions [25, 26] pour la résolution des équations d'Hamilton-Jacobi du premier ordre. Ensuite, cette théorie a été élargi pour considérer les équations du second ordre après les travaux de R. Jensen [51] et H. Ishii [45]. Nous envoyons les lecteurs à G. Barles [11, 12] et G. Crandall, H. Ishii, P. L. Lions [27] pour une introduction détaillée de la théorie des solutions de viscosité.

Dans ce qui suit, nous présentons une définition des solutions de viscosité proposée par H. Ishii, S. Koike [48]. Cette définition est indispensable dans l'annonce de nos résultats.

Definition 1.1. (Sous-solution, sur-solution et solution de viscosité)

Pour $i \in \{1, \dots, d\}$, soient $H_i : (0, T) \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ les Hamiltoniens continus.

Une fonction $w = (w^i)_{i=1, \dots, d} \in (C([0, T] \times \mathbb{R}))^d$ est une sous-solution de viscosité (respectivement sur-solution) du système suivant:

$$\begin{cases} \partial_t w^i(t, x) + H_i(t, w(t, x), \partial_x w^i(t, x)) = 0 & \text{sur } (0, T) \times \mathbb{R}, \\ w^i(0, x) = w_0^i(x) & \text{sur } \mathbb{R}, \end{cases} \quad (1.5)$$

si et seulement si

- $w^i(0, x) \leq w_0^i(x)$ (respectivement $w^i(0, x) \geq w_0^i(x)$),
- Pour tout $i \in \{1, \dots, d\}$ et pour toute fonction test $\phi \in C^1((0, T) \times \mathbb{R})$ telle que $w^i - \phi$ atteint un maximum local (respectivement minimum local) au point $(t_0, x_0) \in (0, T) \times \mathbb{R}$, alors nous avons

$$\begin{aligned} \partial_t \phi(t_0, x_0) + H_i(t_0, w(t_0, x_0), \partial_x \phi(t_0, x_0)) &\leq 0, \\ (\text{respectivement } \partial_t \phi(t_0, x_0) + H_i(t_0, w(t_0, x_0), \partial_x \phi(t_0, x_0)) &\geq 0). \end{aligned}$$

Finalemnt, une fonction w^i est une solution de viscosité du système (1.5) si et seulement si elle est une sous et sur-solution de (1.5).

Le premier résultat concernant le système non-local (1.1) est un résultat d'existence globale. Notre deuxième résultat est un résultat d'unicité de la solution uniquement quand le problème est local (c.à.d $\alpha = 0$). Ceci est présenté dans le théorème suivant:

Théorème 1.1. (Existence globale et unicité d'une solution)

Supposons que les conditions (1.2) et (1.4) sont vérifiées. Alors, nous aurons:

i) Existence d'une solution de viscosité continue pour le problème (1.1)-(1.3):

Il existe une solution de viscosité continue ρ^\pm pour le problème (1.1)-(1.3) telles que:

$$\rho^\pm(x, t) = P^\pm(x, t) + L_0 x \tag{1.6}$$

et

$$P^\pm \in L^\infty(\mathbb{T} \times (0, T)) \cap C([0, T]; L \log L(\mathbb{T})) \text{ et } \partial_x P^\pm \in L^\infty((0, T); L \log L(\mathbb{T})),$$

où P^\pm sont des fonctions 1-périodiques en espace et P^\pm vérifient, pour tout $T > 0$, l'estimation suivante

$$\max_{\pm} \left(\|P^\pm\|_{L^\infty(\mathbb{T} \times (0, T))} \right) \leq \Lambda_1 \tag{1.7}$$

où $\Lambda_1 = \Lambda_1 \left(T, \|a\|_{L^\infty(0, T)}, L_0, \alpha, \|P_0^\pm\|_{L^\infty(\mathbb{T})} \right)$. En plus, la fonction ρ^\pm satisfait l'estimation entropique du gradient suivante:

$$\max_{\pm} \left(\|\partial_x \rho^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))} + \|\partial_t \rho^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))} \right) \leq \Lambda_2 \tag{1.8}$$

où $\Lambda_2 = \Lambda_2 \left(T, \|a\|_{L^\infty(0, T)}, L_0, \alpha, \|P_0^\pm\|_{L^\infty(\mathbb{T})}, \|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})} \right)$.

ii) Unicité de la solution dans le cas local:

La solution, donnée par (i), est unique si $\alpha = 0$.

L'estimation (1.8) est primordial dans la preuve du Théorème 1.1. En fait, tout d'abord nous considérons un problème parabolique régularisé obtenu à partir du problème local. Puis, nous montrons que la solution de ce dernier satisfait l'estimation L^∞ (1.7) et l'inégalité fondamentale suivante:

$$\int_{\mathbb{T}} \left[f \left(\left| \partial_x \rho^+(x, t) \right| \right) + f \left(\left| \partial_x \rho^-(x, t) \right| \right) \right] dx \leq C, \quad \text{pour tout } t \in (0, T) \tag{1.9}$$

où

$$f(x) = \begin{cases} x \ln(x) + \frac{1}{e} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases} \tag{1.10}$$

et C est une constante qui dépend de $T, \|\partial_x \rho_0^+\|_{L \log L(\mathbb{T})}, \|\partial_x \rho_0^-\|_{L \log L(\mathbb{T})}$.

À partir des résultats énoncés par le Théorème 1.1, nous pouvons voir que la solution est continue vérifiant un module de continuité qui sera présenté par le corollaire suivant.

Corollaire 1.1. (Continuité de la solution)

La solution ρ^\pm construite dans (i) appartient à $C(\mathbb{R} \times [0, T])$ et il existe un module de continuité tel que pour tous $\delta, h \geq 0$ et pour tous $(x, t) \in \mathbb{R} \times (0, T - \delta)$, nous avons:

$$|\rho^\pm(x + h, t + \delta) - \rho^\pm(x, t)| \leq \Lambda_3 \frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)},$$

où $\Lambda_3 = \Lambda_3(T, \|a\|_{L^\infty(0, T)}, L_0, \alpha, \|P_0^\pm\|_{L^\infty(\mathbb{T})}, \|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})})$.

Nous présentons brièvement dans ce qui suit la stratégie de la preuve qui sera présentée en détails dans le Chapitre 3. En effet, nous commençons à geler le terme non-local dans le système (1.1) en considérant des données régulières ce qui nous ramène à étudier le problème local suivant:

$$\begin{cases} \partial_t v^+(x, t) = -((v^+ - v^-)(x, t) + L(t)) |\partial_x v^+(x, t)| & \text{sur } \mathbb{R} \times (0, T), \\ \partial_t v^-(x, t) = ((v^+ - v^-)(x, t) + L(t)) |\partial_x v^-(x, t)| & \text{sur } \mathbb{R} \times (0, T), \\ v^\pm(x, 0) = v_0^\pm(x) = Q_0^\pm(x) + L_0 x & \text{sur } \mathbb{R}, \end{cases} \quad (1.11)$$

où Q_0^\pm sont des fonctions 1-périodiques telles que

$$Q_0^\pm \in C^\infty(\mathbb{R}) \text{ et } L \in C^\infty(0, T). \quad (1.12)$$

Puis, nous considérons une régularisation parabolique du problème (1.11) en lui ajoutant le terme de viscosité $\eta \partial_{xx}^2 v_\eta^\pm$ pour $0 < \eta \leq 1$, ce qui nous conduit à considérer le problème suivant:

$$\begin{cases} \partial_t v_\eta^\pm(x, t) - \eta \partial_{xx}^2 v_\eta^\pm(x, t) = \mp ((v_\eta^+ - v_\eta^-)(x, t) + L(t)) |\partial_x v_\eta^\pm(x, t)| & \text{sur } \mathbb{R} \times (0, T), \\ v_\eta^\pm(x, 0) = v_0^\pm(x) = Q_0^\pm(x) + L_0 x & \text{sur } \mathbb{R}. \end{cases} \quad (1.13)$$

Nous régularisons le terme non-linéaire $|\partial_x v_\eta^\pm|$ en le remplaçant par la fonction $\beta_\delta(\partial_x v_\eta^\pm)$, $0 < \delta \leq 1$, où β_δ est la fonction définie par

$$\beta_\delta(x) = \sqrt{x^2 + \delta^2}.$$

Ceci nous conduit finalement à considérer le problème suivant:

$$\begin{cases} \partial_t v_\eta^{\pm, \delta}(x, t) - \eta \partial_{xx}^2 v_\eta^{\pm, \delta}(x, t) = \mp ((v_\eta^{+, \delta} - v_\eta^{-, \delta})(x, t) + L(t)) \beta_\delta (\partial_x v_\eta^{\pm, \delta}(x, t)) & \text{sur } \mathbb{R} \times (0, T), \\ v_\eta^{\pm, \delta}(x, 0) = v_0^\pm(x) & \text{sur } \mathbb{R}. \end{cases} \quad (1.14)$$

Théorème 1.2. (Existence et unicité d'une solution Lipschitz continue pour le problème (1.13))

Supposons que la condition (1.12) est satisfaite. Alors, pour tout $T > 0$, nous avons

i) **(Existence et unicité pour (1.14)):**

Le système (1.14) admet une solution unique $v_\eta^{\pm, \delta}$, telle que

$$v_\eta^{\pm, \delta}(x, t) = Q_\eta^{\pm, \delta}(x, t) + L_0 x,$$

où $Q_\eta^{\pm, \delta}$ sont des fonctions 1-périodiques, appartenant à $C^\infty(\mathbb{R} \times (0, T))$ et satisfaisant les estimations suivantes:

$$\begin{aligned} \max_{\pm} (\|Q_\eta^{\pm, \delta}\|_{L^\infty(\mathbb{T} \times (0, T))}) &\leq N_0^\delta \\ \max_{\pm} (\|\partial_x Q_\eta^{\pm, \delta}\|_{L^\infty(\mathbb{T} \times (0, T))}) &\leq \max_{\pm} (\|\partial_x Q_0^\pm\|_{L^\infty(\mathbb{T})}) \end{aligned}$$

avec

$$N_0^\delta = \max_{\pm} (\|Q_0^\pm\|_{L^\infty(\mathbb{T})}) + \beta_\delta(L_0) \|L\|_{L^\infty(0, T)} T.$$

ii) **(Convergence):**

A une sous-suite près, la fonction $v_\eta^{\pm, \delta}$ converge, quand δ tend vers 0, vers v_η^\pm localement et fortement dans $C([0, T]; W^{1, \infty}(\mathbb{R}))$. En plus,

$$v_\eta^\pm(x, t) = Q_\eta^\pm(x, t) + L_0 x,$$

où Q_η^\pm sont des fonctions 1-périodiques et v_η^\pm est l'unique solution pour (1.13) (au sens de distribution) appartenant à l'espace $C([0, T]; W_{loc}^{1, \infty}(\mathbb{R}))$ et satisfaisant, pour tout $0 < \eta \leq 1$, les estimations η -uniformes suivantes:

$$\begin{aligned} \max_{\pm} (\|Q_\eta^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) &\leq N_0 \\ \max_{\pm} (\|\partial_x Q_\eta^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) &\leq \max_{\pm} (\|\partial_x Q_0^\pm\|_{L^\infty(\mathbb{T})}) \end{aligned}$$

où

$$N_0 = \max_{\pm} (\|Q_0^\pm\|_{L^\infty(\mathbb{T})}) + |L_0| \|L\|_{L^\infty(0, T)} T.$$

En effet, en utilisant un argument de point fixe, nous montrons un résultat d'existence globale et d'unicité d'une solution régulière $v_{\eta, \delta}^\pm$ pour le problème (1.14) (résultat donné dans le Théorème 1.2). Ensuite, nous montrons que la solution $v_{\eta, \delta}^\pm$ vérifie une estimation entropique du gradient similaire à (1.9) et des estimations uniformes en η et en δ . Ceci nous permet de montrer que $v_\eta^\pm = \lim_{\delta \rightarrow 0} v_{\eta, \delta}^\pm$, solution du problème (1.13), satisfait également des estimations uniformes en η . Dans une deuxième étape, nous utilisons les résultats de stabilité des solutions de viscosité et les résultats de compacité de Simon pour passer à la limite quand $\eta \rightarrow 0$ et montrer ainsi l'existence globale d'une solution de viscosité continue pour le problème (1.11). En utilisant un argument de point fixe, nous prouvons le résultat d'existence pour le problème non-local. Finalement, en se basant sur le fait que notre Hamiltonien est quasi-monotone au sens de Ishii, Koike [48,

A.1], nous montrons un principe de comparaison afin de prouver l'unicité de la solution dans le cas du problème local, ce qui est reflété par les résultats suivants.

Théorème 1.3. (Principe de Comparaison)

Soient $\rho_1 = (\rho_1^+, \rho_1^-)$ et $\rho_2 = (\rho_2^+, \rho_2^-)$ deux sous et sur-solution de viscosité continues de (1.1) telles que $\rho_i^\pm = P_i^\pm + L_0 x$ pour $i = 1, 2$. Alors, si P_i^\pm vérifie l'estimation L^∞ suivante:

$$\max_{\pm} (\|P_i^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \overline{M}_0,$$

avec $\overline{M}_0 = \max_{\pm} (\|P_i^\pm(0, \cdot)\|_{L^\infty(\mathbb{T})}) + |L_0| \|a\|_{L^\infty(0, T)} T$.

Si pour tout $(x, t) \in \mathbb{R} \times (0, T - \delta)$, l'estimation de continuité suivante est vérifiée:

$$|\rho_i^\pm(x + h, t + \delta) - \rho_i^\pm(x, t)| \leq \overline{C} \left(\frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)} \right), \quad \text{pour } i = 1, 2$$

alors, si $\rho_1^\pm(\cdot, 0) \leq \rho_2^\pm(\cdot, 0)$ sur \mathbb{R} alors $\rho_1^\pm \leq \rho_2^\pm$ sur $\mathbb{R} \times (0, T)$.

3 Un schéma convergent pour un système de transport non-local modélisant la dynamique des densités des dislocations

Dans cette section, nous nous intéressons à l'étude numérique du système (1.1) mais en considérant des conditions initiales croissantes. Plus précisément, nous considérons un système hyperbolique d'équations de transport non-linéaires et non-locales pour lequel des résultats d'existence et d'unicité d'une solution croissante H_{loc}^1 ont été établis dans [30]. En particulier, ces résultats ont été prouvés en se basant sur une estimation d'énergie dans L^2 .

Nous proposons ainsi un schéma aux différences-finies de type "Upwind" implicite pour lequel nous montrons notre résultat de convergence en se basant sur une estimation L^2 discrète du gradient similaire à celle obtenue dans le cas continue.

3.1 Existence et unicité pour le problème continue

Dans cette sous-section, nous présentons brièvement les résultats d'existence et d'unicité effectué dans [30] pour le problème (1.1)-(1.3) en considérant des conditions initiales croissantes.

Théorème 1.4. (Existence et unicité), [30, Th 1.1]

Pour tous $T, L_0 \geq 0, \alpha \in \mathbb{R}$ et $\rho_0^\pm \in H_{loc}^1(\mathbb{R})$ et sous les conditions suivantes

(H1) $\rho_0^\pm(x + 1) = \rho_0^\pm(x) + L_0$ (fonction 1-periodique + fonction linéaire),

(H2) $\frac{\partial \rho_0^\pm}{\partial x} \geq 0$ presque partout sur \mathbb{R} (ρ_0^\pm est croissante),

(H3) $a \in C(0, T)$,

le système (1.1)-(1.3) admet une solution unique $\rho^\pm \in H_{loc}^1(\mathbb{R} \times [0, T])$, au sens de distribution, telle que, pour presque partout $t \in (0, T)$, la fonction $\rho^\pm(\cdot, t) : x \mapsto \rho^\pm(x, t)$ vérifie (H1) et (H2).

Par la suite, nous présentons une discrétisation compatible avec le problème continue.

3.2 Problème discret

Nous cherchons à approximer la solution du système (1.1)-(1.3) en considérant des conditions initiales croissantes. Étant donné une taille de discrétisation $\Delta x, \Delta t$, nous définissons

$$\Xi = \{i\Delta x, i \in \{0, \dots, M-1\}\} \quad \Xi_T = \Xi \times \{n\Delta t, n \in \{0, \dots, N_T\}\},$$

où $\Delta x = \frac{1}{M}$ et $\Delta t = \frac{T}{N_T}$. Notons $u_{\pm}(x_i, t_n)$ ou $u_{\pm,i}^n$ comme étant la valeur de l'approximation de la solution ρ^{\pm} au point (x_i, t_n) avec $x_i = i\Delta x$ et $t_n = n\Delta t$.

Nous proposons le schéma implicite de type "Upwind" suivant:

$$\forall k \in \{+, -\}, \quad \begin{cases} \frac{u_{k,i}^{n+1} - u_{k,i}^n}{\Delta t} + C_{k,i}^{n+1}[u^{n+1}] \left(\frac{u_{k,i+1}^{n+1} - u_{k,i}^{n+1}}{\Delta x} \right) = 0 & \text{si } C_{k,i}^{n+1}[u^{n+1}] \leq 0, \\ \frac{u_{k,i}^{n+1} - u_{k,i}^n}{\Delta t} + C_{k,i}^{n+1}[u^{n+1}] \left(\frac{u_{k,i}^{n+1} - u_{k,i-1}^{n+1}}{\Delta x} \right) = 0 & \text{si } C_{k,i}^{n+1}[u^{n+1}] \geq 0, \end{cases} \quad (1.15)$$

avec

$$u_i^0 = (u_{+,i}^0, u_{-,i}^0) = \rho_0(x_i) = (\rho_0^+(x_i), \rho_0^-(x_i)), \quad \text{pour } -1 \leq i \leq M, \quad (1.16)$$

$$C_{k,i}^{n+1}[u^{n+1}] = k \left(u_{+,i}^{n+1} - u_{-,i}^{n+1} + \alpha \sum_{j=0}^{M-1} \Delta x (u_{+,j}^{n+1} - u_{-,j}^{n+1}) + a(t_{n+1}) \right),$$

et les conditions de périodicité suivantes

$$\left| \begin{array}{l} u_{+,-1}^n = u_{+,M-1}^n - L_0, \\ u_{+,M}^n = u_{+,0}^n + L_0, \end{array} \right. \quad \text{et} \quad \left| \begin{array}{l} u_{-,-1}^n = u_{-,M-1}^n - L_0, \\ u_{-,M}^n = u_{-,0}^n + L_0, \end{array} \right. \quad \forall n \geq 0.$$

Dans la suite, nous notons

$$\theta_{i+\frac{1}{2}}^{\alpha,n} = \frac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}, \quad (1.17)$$

qui représente une discrétisation équivalente à $\partial_x \rho^k$.

Plusieurs résultats numériques ont été établis en relation avec ce problème. Dans [33], les auteurs ont démontré la convergence d'un schéma numérique explicite pour le système (1.1) dans le cas où $\alpha = 1$ en considérant des conditions initiales Lipschitz. Ils ont montré en plus une estimation d'erreur de type Crandall-Lions [26] entre la solution continue et la solution discrète en s'inspirant du travail effectué dans [7].

Nous nous référons à Jacobson, Karlsen [49] et Jakobsen, Karlsen, Risebro [50] où ils ont montré un résultat de convergence pour un système de type Hamilton-Jacobi faiblement couplé en considérant un algorithme de "splitting" semi-discret qui approche la solution continue.

Dans le cas de la dynamique des lignes de dislocations, l'analyse numérique du modèle a été faite, dans le cas des équations scalaires, par Alvarez, Carlini, Monneau et Rouy [6, 7].

Dans ce qui suit, nous présentons les résultats principaux établis dans ce travail.

Notons $u^n = (u_i^n)_{i=-1, \dots, M}$ où $u_i^n = (u_{k,+}^n, u_{k,-}^n)$. Nous définissons $P_0 = (P_0^+, P_0^-)$ tel que $\|P_0\|_{(L^\infty(\mathbb{T}))^2} = \|P_0^+\|_{L^\infty(\mathbb{T})} + \|P_0^-\|_{L^\infty(\mathbb{T})}$.

Nous introduisons la condition CFL suivante:

$$\frac{\Delta t}{\Delta x} \leq \min(\beta_1, \beta_2), \quad (1.18)$$

où

$$\beta_1 = \frac{1}{[(2\zeta + 2)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)}] (4(\zeta + 1) + L_0)},$$

et

$$\beta_2 = \frac{1}{3(8[(2\zeta + 2)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)}] + 4(1 + |\alpha|)(2\zeta + 2) + 2L_0(1 + |\alpha|))},$$

avec

$$\zeta = e^{4TL_0(1+|\alpha|)} \left(\|P_0\|_{(L^\infty(\mathbb{T}))^2} + \frac{\|a\|_{L^\infty(0,T)}}{2(1+|\alpha|)} \right). \quad (1.19)$$

Dans ce travail, trois résultats sont établis et seront présentés ensuite. En particulier, nous commençons par la résolution du schéma numérique afin de montrer l'existence et l'unicité d'une solution bornée pour ce schéma. Ensuite, nous montrons une estimation L^2 sur le gradient discret de la solution numérique. Cette dernière nous permet enfin de prouver la convergence de la solution numérique vers celle du problème continue.

Théorème 1.5. (Résolution du schéma implicite)

Supposons que les hypothèses (H1), (H2) et (H3) données par le Théorème 1.4 et que la condition CFL (1.18) sont vérifiées. Soit $u_{k,i}^n = w_{k,i}^n + L_0 x_i$ où, pour $-1 \leq i \leq M$, $w_{k,i}^n$ satisfait les conditions suivantes

$$\left| \begin{array}{l} w_{+,-1}^n = w_{+,M-1}^n, \\ w_{+,M}^n = w_{+,0}^n, \end{array} \right. \quad \text{et} \quad \left| \begin{array}{l} w_{-,-1}^n = w_{-,M-1}^n, \\ w_{-,M}^n = w_{-,0}^n, \end{array} \right. \quad (1.20)$$

Alors, nous aurons

(i) **(Existence)**

Il existe une solution unique u^{n+1} pour le schéma implicite (1.15)-(1.16) telle que $u_{k,i}^{n+1} = w_{k,i}^{n+1} + L_0 x_i$ où $w_{k,i}^{n+1}$ satisfait les conditions données par (1.20) et la borne L^∞ suivante

$$\max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} w_{k,i}^{n+1} \leq \zeta,$$

pour tout $k \in \{+, -\}$ et pour tout $i = 0, \dots, M-1$ et $n = 0, \dots, N_T - 1$, où ζ est donnée par (1.19).

(ii) **(Monotonie)**

Si u^n est croissante, c.à.d satisfait

$$u_{k,i+1}^n \geq u_{k,i}^n \quad \text{pour tout } i \in \{0, \dots, M-1\}, \quad \text{et } k \in \{+, -\},$$

alors u^{n+1} est croissante.

Théorème 1.6. (Estimation L^2 discrète du gradient)

Supposons que les hypothèses (H1), (H2) et (H3) sont vérifiées. Si u^n est la solution du schéma implicite (1.15)-(1.16) présentée par le Théorème 1.5, alors $\theta_{k,i+\frac{1}{2}}^n$ définie dans (1.17) est positive pour tout $n \in \{0, \dots, N_T - 1\}$ et satisfait l'estimation suivante:

$$\sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k,i+\frac{1}{2}}^{n+1})^2 + \Delta t \sum_{i=0}^{M-1} \left(\left[\theta_{+,i+\frac{1}{2}}^{n+1} - \theta_{-,i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+,i+\frac{1}{2}}^{n+1} + \theta_{-,i+\frac{1}{2}}^{n+1} \right] \right) \leq \sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k,i+\frac{1}{2}}^n)^2.$$

Nous présentons ensuite notre résultat de convergence.

Théorème 1.7. (Convergence de la solution du schéma)

Supposons que les hypothèses (H1), (H2), (H3) et la condition CFL (1.18) sont vérifiées. Considérons $(u^n)_{n \geq 0}$ la solution du schéma (1.15)-(1.16) pour un pas de temps $\Delta t > 0$ et un pas d'espace $\Delta x > 0$ présentée par le Théorème 1.5. Notons $\epsilon = (\Delta x, \Delta t)$ et u^ϵ la fonction définie par

$$u^\epsilon(i\Delta x, n\Delta t) = u_i^n \quad \text{pour } n \in \{0, \dots, N_T - 1\}, \quad i \in \{0, \dots, M - 1\}.$$

Alors, quand $\epsilon \rightarrow 0$, la suite $(u^\epsilon)_\epsilon$ converge vers l'unique solution ρ de (1.1)-(1.3). En plus, nous avons

$$\sup_{\substack{\mathbb{T} \times [0, T] \\ k \in \{+, -\}}} |u^\epsilon - \rho| \longrightarrow 0 \quad \text{quand } \epsilon \longrightarrow (0, 0).$$

4 Solution continue pour un système eikonal non-linéaire

Dans les Sections 2 et 3, nous avons présenté une étude théorique et numérique d'un système modélisant la dynamique des densités des dislocations se propageant dans une seule direction.

Dans cette section, nous citons des résultats concernant les systèmes $(d \times d)$ de type Hamilton-Jacobi modélisant la dynamique des densités des dislocations dans le cas de plusieurs directions de glissement. Nous renvoyons les lecteurs de cette thèse au Chapitre 2 pour une dérivation physique du modèle.

Notre travail se concentre sur les solutions $u(t, x) = (u^i(t, x))_{i=1, \dots, d}$, où d est un nombre entier, pour tout $T > 0$, du problème suivant:

$$\partial_t u^i(t, x) = \lambda^i(t, x, u) |\partial_x u^i(t, x)| \quad \text{sur } (0, T) \times \mathbb{R}, \quad \text{pour } i = 1, \dots, d, \quad (1.21)$$

avec des conditions initiales

$$u^i(0, x) = u_0^i(x), \quad x \in \mathbb{R}, \quad \text{pour } i = 1, \dots, d. \quad (1.22)$$

Notre étude vise à prouver l'existence et l'unicité d'une solution continue pour le système (1.21)-(1.22) en supposant que la vitesse $\lambda^i : (0, T) \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}$ vérifie, pour tout $i \in \{1, \dots, d\}$, les régularités suivantes:

$$(K1) \quad \left\{ \begin{array}{l} \lambda^i \in L^\infty((0, T) \times \mathbb{R} \times \mathcal{U}) \cap C((0, T) \times \mathbb{R} \times \mathcal{U}), \text{ pour } i = 1, \dots, d, \\ \text{il existe } M_1 > 0 \text{ tel que pour } i = 1, \dots, d, \\ \sum_{i=1}^d |\lambda^i(t, x, u) - \lambda^i(t, y, v)| \leq M_1 (|u - v| + |x - y|) \text{ pour tous } u, v \in \mathcal{U}, x, y \in \mathbb{R} \text{ et } t \in (0, T), \end{array} \right.$$

où $|w| = \sum_{i=1, \dots, d} |w^i|$, $w = (w^1, \dots, w^d)$ et \mathcal{U} est définie par la boîte suivante:

$$\mathcal{U} = \prod_{i=1}^d [-\|u_0^i\|_{L^\infty(\mathbb{R})}, \|u_0^i\|_{L^\infty(\mathbb{R})}].$$

De plus, nous supposons que la matrice Jacobienne de la vitesse λ^i notée par

$$(\lambda_{,j}^i(t, x, u))_{i,j=1, \dots, d} \quad \text{où} \quad \lambda_{,j}^i = \frac{\partial \lambda^i}{\partial u^j},$$

est négative, autrement dit

$$(K2) \quad \left| \sum_{i,j=1}^d w^i w^j \lambda_{,j}^i(t, x, u) \leq 0, \quad \forall (t, x, u) \in (0, T) \times \mathbb{R} \times \mathcal{U}, \quad \text{et} \quad \forall w = (w^1, \dots, w^d) \in \mathbb{R}^d. \right.$$

Chaque composante u_0^i de la condition initiale $u_0 = (u_0^i)_{i=1, \dots, d}$ vérifie la régularité suivante:

$$(K3) \quad \left\{ \begin{array}{l} \alpha^i \leq u_0^i \leq \beta^i, \\ \partial_x u_0^i \in L^1(\mathbb{R}) \cap L \log L(\mathbb{R}), \end{array} \right. \quad \text{pour } i = 1, \dots, d,$$

où $L \log L(\mathbb{T})$ est l'espace de Zygmund défini par:

$$L \log L(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) \text{ tel que } \int_{\mathbb{R}} |f| \ln(e + |f|) < +\infty \right\}.$$

Évoquons maintenant quelques résultats établis en relation avec le problème (1.21). Motivés par la dynamique des dislocations, El Hajj et Boudjerada [19] ont démontré l'existence globale des solutions de viscosité discontinues pour des équations eikonales unidimensionnelles non-linéaires et non-locales, considérant en particulier le cas $d = 1$.

De plus, citons notre travail récent [36], évoqué dans la Section 2, où des résultats d'existence et d'unicité d'une solution de viscosité sont établis pour un système (2×2) quasi-monotone qui correspond au cas où $(\lambda_{,j}^i)_{i,j=1,2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Nous rappelons que dans [36], nous avons pu développer une théorie d'existence et d'unicité d'une solution de viscosité pour un système quasi-monotone de type Hamilton-

Jacobi en basant sur les travaux de Ishii, Koikee [46, 47].

Pour le problème $(d \times d)$, nous citons le résultat d'existence de solutions BV discontinues qui a été prouvé dans [37] sous une condition de monotonie de la vitesse.

Notons que dans le cas d'une condition initiale croissante le système (1.21) devient un système diagonal hyperbolique pour lequel nous mentionnons par la suite quelques résultats connus. Dans le cas d'un système (2×2) Lax a montré dans [54] l'existence et l'unicité d'une solution croissante régulière. En plus, El Hajj, Monneau ont prouvé dans [35] des résultats d'existence et d'unicité d'une solution continue pour un système $(d \times d)$ diagonal strictement hyperbolique en supposant des conditions initiales croissantes. Récemment, dans [5], un résultat d'existence globale des solutions de viscosité discontinues pour un système $(d \times d)$ diagonal hyperbolique avec des conditions initiales BV a été montré.

Dans ce qui suit, nous énonçons le Théorème principal de cette section concernant l'existence et l'unicité d'une solution de viscosité dans le sens de la définition 1.1.

Théorème 1.8. (Existence globale et unicité d'une solution)

Supposons que les conditions (K1), (K2) et (K3) sont vérifiées. Alors, nous avons:

i) Existence d'une solution de viscosité continue:

Il existe une solution de viscosité continue u^i pour le système (1.21)-(1.22), pour tout $i \in \{1, \dots, d\}$, où

$$u^i \in L^\infty((0, T) \times \mathbb{R}) \quad \text{et} \quad \partial_x u^i \in L^\infty((0, T); L \log L(\mathbb{R})).$$

En plus, pour tout $i \in \{1, \dots, d\}$, u^i vérifie les estimations suivantes:

$$\|u^i\|_{L^\infty((0, T) \times \mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \tag{1.23}$$

$$\|\partial_x u^i\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq \|\partial_x u_0^i\|_{L^1(\mathbb{R})} \tag{1.24}$$

et l'estimation entropique du gradient

$$\sum_{i=1}^d \|\partial_x u^i\|_{L^\infty((0, T); L \log L(\mathbb{R}))} \leq \gamma_1, \tag{1.25}$$

où $\gamma_1 = \gamma_1(T, d, M_1, \Lambda, \|\partial_x u_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d})$ avec

$$\Lambda = \sum_{i=1}^d \Lambda^i \quad \text{et} \quad \Lambda^i = \sup_{\substack{t \in (0, T) \\ x \in \mathbb{R} \\ u \in \mathcal{U}}} |\lambda^i(t, x, u)| \quad \forall i \in \{1, \dots, d\}.$$

ii) Unicité de la solution:

En supposant que la vitesse vérifie la condition de monotonie suivante:

$$(K4) \quad \left\{ \begin{array}{l} \lambda^j(t, x, r) - \lambda^j(t, x, s) \leq 0, \quad \text{pour tous vecteurs } r = (r^i)_{i=1, \dots, d}, s = (s^i)_{i=1, \dots, d}, \text{ tels que} \\ r^j - s^j = \max_{i \in \{1, \dots, d\}} (r^i - s^i) \geq 0. \end{array} \right.$$

Alors, la solution définie dans (i), est unique.

L'estimation (1.25) est nécessaire dans la preuve du Théorème 1.8. En fait, nous considérons une régularisation parabolique du problème (1.21). Puis, nous montrons que la solution de ce dernier satisfait l'estimation L^∞ (1.23) et l'inégalité fondamentale suivante:

$$\int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x u^i(t, x)|) dx - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \lambda_{i,j}^i(t, x, u) \partial_x u^i(t, x) \partial_x u^j(t, x) \leq C, \quad \text{pour tout } t \in (0, T) \quad (1.26)$$

où f est définie dans la Section 2 par (1.10) et C est une constante positive qui dépend de T, d, M_1 et $\|\partial_x u_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d}$.

Dans ce qui suit, nous énonçons courtement la stratégie de la preuve qui sera détaillée dans le Chapitre 5. Tout d'abord, nous montrons un résultat d'existence locale et d'unicité d'une solution pour un problème parabolique obtenu en régularisant le problème (1.21) après avoir régulariser les données initiales et la vitesse. Ce qui nous ramène à étudier pour $0 < \eta \leq 1$, $i = 1, \dots, d$ et $v_\eta = (v_\eta^i)_{i=1, \dots, d}$ le problème suivant:

$$\begin{cases} \partial_t v_\eta^i(t, x) - \eta \partial_{xx}^2 v_\eta^i(t, x) = \tilde{\lambda}^i(t, x, v_\eta) |\partial_x v_\eta^i(t, x)| & \text{sur } (0, T) \times \mathbb{R}, \\ v_\eta^i(0, x) = v_0^i(x) & x \in \mathbb{R}, \end{cases} \quad (1.27)$$

où, pour tout $i \in \{1, \dots, d\}$, nous avons

$$v_0^i \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{et} \quad \partial_x v_0^i \in L^p(\mathbb{R}) \quad \text{pour tout } 1 \leq p \leq +\infty. \quad (1.28)$$

En plus, $\tilde{\lambda}^i : (0, T) \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}$ satisfait, pour tout $i \in \{1, \dots, d\}$, les hypothèses suivantes:

$$(\widetilde{K1}) \quad \left\{ \begin{array}{l} \tilde{\lambda}^i \in L^\infty((0, T) \times \mathbb{R} \times \mathcal{U}) \cap C^\infty((0, T) \times \mathbb{R} \times \mathbb{R}^d), \text{ pour } i = 1, \dots, d, \\ \text{Il existe } \widetilde{M}_1 > 0 \text{ tel que pour } i = 1, \dots, d, \\ \sum_{i=1}^d |\tilde{\lambda}^i(t, x, u) - \tilde{\lambda}^i(t, y, v)| \leq \widetilde{M}_1 (|u - v| + |x - y|) \text{ pour tous } u, v \in \mathcal{U}, x, y \in \mathbb{R} \text{ et } t \in (0, T), \end{array} \right.$$

et

$$(\widetilde{K2}) \quad \left| \sum_{i,j=1}^d w^i w^j \tilde{\lambda}_{i,j}^i(t, x, u) \leq 0 \quad \forall (t, x, u) \in (0, T) \times \mathbb{R} \times \mathcal{U}, \quad \text{et} \quad \forall w = (w^1, \dots, w^d) \in \mathbb{R}^d. \right.$$

Nous régularisons le terme non-linéaire $|\partial_x v_\eta^i|$ en le remplaçant par la fonction $\beta_\delta(\partial_x v_\eta^i)$, $0 < \delta \leq 1$, où β_δ est la fonction définie par

$$\beta_\delta(x) = \sqrt{x^2 + \delta^2}.$$

Puis, nous considérons une troncature de la fonction β_δ en introduisant:

$$\psi_\delta(\cdot) = \phi(\sqrt{\delta} \cdot),$$

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où $\phi \in C^\infty(\mathbb{R})$ est une fonction "cut-off" à valeurs dans $[0, 1]$ telle que $\text{supp}(\phi) \subset [-2, 2]$ et $\phi(x) \equiv 1$ sur $[-1, 1]$. Ceci nous conduit finalement à considérer le problème suivant:

$$\begin{cases} \partial_t v_{\eta,\delta}^i(t, x) - \eta \partial_{xx}^2 v_{\eta,\delta}^i(t, x) = \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta,\delta}^i(t, x)) & \text{sur } (0, T) \times \mathbb{R}, \\ v_{\eta,\delta}^i(0, x) = v_0^i(x) & x \in \mathbb{R}. \end{cases} \quad (1.29)$$

Dans un premier temps, nous prouvons un résultat préliminaire d'existence et d'unicité pour le problème (1.29). Ceci est évoqué par le théorème suivant.

Théorème 1.9. (Existence et unicité d'une solution Lipschitz continue pour (1.27))

Supposons que $(\widetilde{K1})$ and (1.28) sont vérifiées. Alors, pour tout $T > 0$ tel que, nous avons

i) (Existence et unicité pour (1.29)):

Le système (1.29) admet une solution unique $v_{\eta,\delta} = (v_{\eta,\delta}^i)_{i=1,\dots,d} \in (C^\infty((0, T) \times \mathbb{R}))^d \cap (W^{1,\infty}((0, T) \times \mathbb{R}))^d$, satisfaisant les estimations suivantes:

$$\|v_{\eta,\delta}^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq T \delta \widetilde{\Lambda}^i + \|v_0^i\|_{L^\infty(\mathbb{R})} \quad \text{pour tout } i = 1, \dots, d,$$

où $\widetilde{\Lambda}^i = \|\tilde{\lambda}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})}$.

En plus, il existe une constante ξ_p^δ qui dépend de $\|v_0^i\|_{L^\infty(\mathbb{R})}$, $\|\partial_x v_0^i\|_{L^p(\mathbb{R})}$, et T telle que

$$\|\partial_x v_{\eta,\delta}^i\|_{L^\infty((0,T); L^p(\mathbb{R}))} \leq \xi_p^\delta \quad \text{pour tout } 1 \leq p \leq +\infty.$$

ii) (Convergence):

A une sous-suite près, la fonction $v_{\eta,\delta}^i$ converge, quand $\delta \rightarrow 0$, vers une fonction v_η^i localement et fortement dans $C([0, T]; W^{1,\infty}(\mathbb{R}))$. En plus, v_η^i est l'unique solution de (1.27) (au sens de distribution) appartenant à $C([0, T]; W^{1,\infty}(\mathbb{R}))$ et satisfaisant, pour tout $0 < \eta \leq 1$ et pour tout $i \in \{1, \dots, d\}$, les estimations η -uniformes :

$$\|v_\eta^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|v_0^i\|_{L^\infty(\mathbb{R})},$$

et

$$\|\partial_x v_\eta^i\|_{L^\infty((0,T); L^p(\mathbb{R}))} \leq \xi_p^0 \quad \text{pour tout } 1 \leq p \leq +\infty.$$

En fait, en se basant sur un argument de point fixe, nous démontrons un résultat d'existence locale et d'unicité d'une solution régulière $v_{\eta,\delta}^i$ pour le problème (1.29) (résultat donné dans le Théorème 1.9). Ensuite, nous montrons que la solution $v_{\eta,\delta}^i$ vérifie une estimation entropique du gradient similaire à (1.26) et d'autres estimations uniformes en η et en δ parmi lesquelles l'estimation BV . En passant à la limite quand $\delta \rightarrow 0$, nous prouvons que la solution v_η^i du problème (1.27) satisfait des estimations uniformes en η . Puis, en utilisant ces dernières, nous démontrons que la solution locale peut être étendue en une solution globale. Dans une deuxième étape, nous utilisons les résultats de stabilité des solutions de viscosité et les résultats de compacité de Simon pour passer à la limite quand $(\delta, \eta) \rightarrow (0, 0)$ et montrer ainsi l'existence globale d'une solution de viscosité continue pour le problème (1.21).

Finalement, en utilisant le principe de comparaison énoncé par la suite, nous prouvons l'unicité de la solution.

Théorème 1.10. (Principe de comparaison)

Soient $p = (p^i)_{i=1,\dots,d}$ et $q = (q^i)_{i=1,\dots,d}$, respectivement, des sous et sur-solutions de viscosité continues

pour (1.21), avec $p^i(0, \cdot) = p_0^i(\cdot)$ et $q^i(0, \cdot) = q_0^i(\cdot)$. Supposons qu'il existe une constante $K > 0$ telle que

$$\|p^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq K \quad \text{et} \quad \|q^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq K \quad \text{pour tout } i \in \{1, \dots, d\}.$$

Alors, si $p^i(0, \cdot) \leq q^i(0, \cdot)$ sur \mathbb{R} nous aurons $p^i \leq q^i$ sur $(0, T) \times \mathbb{R}$ pour $i \in \{1, \dots, d\}$.

5 Convergence d'un schéma implicite pour un système diagonal hyperbolique

Dans cette section, notre travail se focalise sur l'analyse numérique d'un système $(d \times d)$ diagonal hyperbolique qui apparaît naturellement dans la modélisation de la dynamique des densités des dislocations. Pour réaliser l'étude numérique de ce système, nous proposons un schéma aux différences-finies de type "Upwind" implicite. Pour ce schéma, nous prouvons un résultat de convergence en se basant sur des estimations Lipschitz discrètes similaires à celles établies pour le problème continue dans [35]. Enfin, nous présentons quelques simulations numériques.

5.1 Problème continue

Dans cette sous-section, nous citons des résultats d'existence et d'unicité de solutions Lipschitz montrés dans [35]. Nous considérons le système diagonal hyperbolique suivant:

$$\begin{cases} \partial_t v^j + \lambda^j(v) \partial_x v^j = 0 & \text{sur } (0, +\infty) \times \mathbb{R}, \quad \text{pour } j = 1, \dots, d, \\ v^j(0, x) = v_0^j(x) & x \in \mathbb{R}, \quad \text{pour } j = 1, \dots, d, \end{cases} \quad (1.30)$$

où $v(t, x) = (v^j(t, x))_{j=1, \dots, d}$, avec $d \geq 1$ est un nombre entier.

Considérons la fonction $(\lambda^j)_{j=1, \dots, d} : \mathbb{R}^d \rightarrow \mathbb{R}$, qui satisfait la supposition suivante:

$$(L1) \begin{cases} \lambda^j \in C^1(\mathbb{R}^d), \quad \text{pour } j = 1, \dots, d, \\ \text{Il existe } M_1 > 0 \text{ tel que pour } j = 1, \dots, d, \\ |\lambda^j(u) - \lambda^j(v)| \leq M_1 |u - v| \quad \text{pour tous } u, v \in \mathbb{R}^d. \end{cases}$$

Notons

$$\Lambda^j = \sup_{u \in \mathcal{V}} |\lambda^j(u)|,$$

où \mathcal{V} est définie par la boîte:

$$\mathcal{V} = \prod_{\alpha=1}^d [-\|v_0^\alpha\|_{L^\infty(\mathbb{R})}, \|v_0^\alpha\|_{L^\infty(\mathbb{R})}].$$

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Dans (1.30), la condition initiale $v_0 = (v_0^1, \dots, v_0^d)$ satisfait la régularité suivante:

$$(L2) \left\{ \begin{array}{l} v_0^j \text{ est croissante,} \\ \partial_x v_0^j \in L^\infty(\mathbb{R}), \end{array} \right. \quad \left| \quad \text{pour } j = 1, \dots, d.$$

Nous introduisons quelques hypothèses sur la matrice Jacobienne $(\lambda_{,i}^j(v))_{i,j=1,\dots,d}$ qui impliquent l'existence et l'unicité d'une solution Lipschitz.

(Matrice triangulaire positive)

$$(\mathcal{M}_1) \left\{ \begin{array}{l} \lambda_{,i}^j(v) \leq 0 \text{ pour tous } v \in \mathcal{V} \text{ et } i \neq j \text{ avec } i, j \in \{1, \dots, d\}, \\ A_{i,j} = \inf_{v \in \mathcal{V}} (\lambda_{,i}^j(v)) \text{ et } \sum_{i,j=1,\dots,d} A_{i,j} \xi_j \xi_i \geq 0 \text{ pour tout } \xi = (\xi_1, \dots, \xi_d) \in [0, +\infty)^d. \end{array} \right.$$

(Matrice à diagonale dominante)

$$(\mathcal{M}_2) \quad \lambda_{,j}^j(v) \geq \sum_{i \neq j} (\lambda_{,i}^j(v))^- \text{ pour tout } v \in \mathcal{V} \text{ et } j = 1, \dots, d,$$

où nous notons $x^- = \max(0, -x)$.

(Matrice avec des termes sous-diagonaux positifs)

$$(\mathcal{M}_3) \quad \lambda_{,i}^j(v) \geq 0 \text{ pour tous } v \in \mathcal{V} \text{ et } i \geq j \text{ avec } i, j \in \{1, \dots, d\}.$$

Avant de présenter cette étude, nous rappelons les lecteurs du résultat principal effectué dans [35].

Théorème 1.11. (Existence et unicité d'une solution Lipschitz, [35, Th 1.3])

Supposons que (L1) et (L2) sont satisfaites. Supposons encore qu'une des suppositions (\mathcal{M}_1) , (\mathcal{M}_2) , ou (\mathcal{M}_3) est vérifiée. Alors, il existe une fonction unique $v \in \bigcap_{T>0} [W^{1,\infty}([0, T] \times \mathbb{R})]^d$ solution pour (1.30),

au sens de distribution, avec $v(t, x) \in \mathcal{V}$ pour tout (t, x) .

De plus, nous avons pour tout $t \in (0, +\infty)$:

$$\sum_{j=1,\dots,d} \|\partial_x v^j(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sum_{\alpha=1,\dots,d} \|\partial_x v_0^\alpha\|_{L^\infty(\mathbb{R})} \quad \text{si } (\mathcal{M}_1) \text{ est vérifiée,} \quad (1.31)$$

et

$$\max_{j=1,\dots,d} \|\partial_x v^j(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \max_{j=1,\dots,d} \|\partial_x v_0^j\|_{L^\infty(\mathbb{R})} \quad \text{si } (\mathcal{M}_2) \text{ est vérifiée.} \quad (1.32)$$

Par la suite, nous présentons une discrétisation convenable avec le problème continue.

5.2 Problème discret

Considérons un pas de temps $\Delta t > 0$ et un pas d'espace $\Delta x > 0$. Afin d'éviter l'ambiguïté dans les notations, notons $u_i^{j,n}$ la solution discrète définie comme étant une approximation de $v^j(t_n, x_i)$, pour

$n \in \mathbb{N}$ et $i \in \mathbb{Z}$. Nous proposons le schéma implicite de type "Upwind" suivant:

$$\forall j \in \{1, \dots, d\}, \begin{cases} \frac{u_i^{j,n+1} - u_i^{j,n}}{\Delta t} + \lambda^j(u_i^{n+1}) \left(\frac{u_{i+1}^{j,n+1} - u_i^{j,n+1}}{\Delta x} \right) = 0 & \text{si } \lambda^j(u_i^{n+1}) \leq 0, \\ \frac{u_i^{j,n+1} - u_i^{j,n}}{\Delta t} + \lambda^j(u_i^{n+1}) \left(\frac{u_i^{j,n+1} - u_{i-1}^{j,n+1}}{\Delta x} \right) = 0 & \text{si } \lambda^j(u_i^{n+1}) \geq 0, \\ u_i^{j,0} = u_0^j(x_i), \end{cases} \quad (1.33)$$

avec la vitesse $\lambda^j(u)$ est implicite en temps.

Avant d'énoncer les résultats principaux de cette section, nous évoquons quelques travaux numériques effectués dans le cadre de notre problème.

Les schémas numériques pour les systèmes hyperboliques sont principalement écrits pour des systèmes en forme conservative ce qui nous permet d'obtenir un choc correct de Rankine-Hugoniot. Nous renvoyons les lecteurs à [55] pour une présentation des classes principales des schémas existant. Parmi ces schémas, les résultats de convergence sont rarement trouvés pour les systèmes hyperboliques.

Le théorème de Lax-Wendroff [53] montre que tout schéma numérique consistant et conservatif converge vers une solution faible du système hyperbolique. Mais, la stabilité de la variation totale est ici nécessaire pour prouver la convergence du schéma numérique.

Pour les formes non-conservatives, nous référons les lecteurs à Monasse et Monneau [59] où la convergence d'un schéma semi-explicite en supposant que le problème est strictement hyperbolique et en utilisant une estimation entropique discrète du gradient compatible à celle établie dans le cas continue [35] a été montrée. Notons que le travail de Monasse et Monneau reste restreint à une classe de solutions dite "Vanishing viscosity solutions", qui a été introduite par Bianchini et Bressan dans [16].

Dans notre travail, nous considérons un système hyperbolique non nécessairement strictement hyperbolique et nous montrons la convergence d'un schéma implicite en se basant sur des estimations Lipschitz discrètes sans aucune restriction à la classe des solutions.

Notons $u_i^n = (u_i^{j,n})_{j=1,\dots,d}$, $u^n = (u_i^n)_{i \in \mathbb{Z}}$ et nous considérons que $u^n \in \mathcal{V}^{\mathbb{Z}}$ si $u_i^n \in \mathcal{V}$, pour tout $i \in \mathbb{Z}$.

Nous considérons la conditions CFL suivante:

$$\frac{\Delta t}{\Delta x} < \min(\beta_1, \beta_2), \quad (1.34)$$

avec

$$\beta_1 = \frac{1}{4\Lambda^j(\|v_0^j\|_{L^\infty(\mathbb{R})} + 1)} \quad \text{et} \quad \beta_2 = \frac{1}{3(\Lambda^j + M_1(\|v_0^j\|_{L^\infty(\mathbb{R})} + 1))}.$$

Nous énonçons maintenant les résultats établis dans ce travail. Il s'agit en fait de 3 résultats.

Le premier établit un résultat d'existence et d'unicité de la solution numérique. Ceci est reflété par le théorème suivant:

Théorème 1.12. (Résolution du schéma implicite)

Supposons que (L1), (L2) et la condition CFL (1.34) sont vérifiées. Soit $u^n \in \mathcal{V}^{\mathbb{Z}}$. Alors, nous avons

(i) **(Existence)**

Il existe une solution unique $u^{n+1} \in \mathcal{V}^{\mathbb{Z}}$ du schéma implicite (1.33).

(ii) **(Monotonie)**

Si u^n est croissante, c.à.d satisfait

$$u_{i+1}^{j,n} \geq u_i^{j,n} \quad \text{pour tout } i \in \mathbb{Z}, \quad \text{et } j = 1, \dots, d,$$

alors u^{n+1} est croissante.

Le deuxième résultat montre des estimations Lipschitz discrètes. Ceci est présenté par le théorème suivant:

Théorème 1.13. (Estimations discrètes du gradient)

Supposons que (L1) et (L2) sont vérifiées. Si u^n est la solution du schéma implicite donnée par le

Théorème 1.12, alors, $\theta_{i+\frac{1}{2}}^{j,n} = \frac{u_{i+1}^{j,n} - u_i^{j,n}}{\Delta x}$, est positive pour tout $n \in \mathbb{N}$ et vérifie les estimations suivantes:

$$\sum_{j=1}^d \sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{j,n} \leq \sum_{j=1}^d \sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{j,0} \quad \text{si } (\mathcal{M}_1) \text{ est vérifiée,}$$

$$\max_{j=1, \dots, d} \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{j,n} \right) \leq \max_{j=1, \dots, d} \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{j,0} \right) \quad \text{si } (\mathcal{M}_2) \text{ est vérifiée,}$$

$$\sup_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{j,n} \right) \leq \mathcal{K}_j(T) \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{j,0} \right) \quad \text{pour tout } n \geq n_0 \quad \text{si } (\mathcal{M}_3) \text{ est vérifiée,}$$

où \mathcal{K}_j est une constante qui dépend de T , M_1 et $\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{l,0}$, pour tous $l \leq j-1$ et $n_0 \in \mathbb{N}$ dépend de T .

Le troisième résultat prouve la convergence de la solution discrète vers la solution du problème continue. Ceci est illustré par le théorème suivant:

Théorème 1.14. (Convergence de la solution du schéma)

Supposons que (L1), (L2) et la condition CFL (1.34) sont vérifiées. Supposons de plus que l'une des conditions (\mathcal{M}_1) , (\mathcal{M}_2) ou (\mathcal{M}_3) est satisfaite. Considérons la solution $(u^n)_{n \geq 0}$ du schéma pour un pas de temps $\Delta t > 0$ et un pas d'espace $\Delta x > 0$ donnée par la Théorème 1.12. Nous notons $\epsilon = (\Delta t, \Delta x)$ et u^ϵ la fonction définie par

$$u^\epsilon(n\Delta t, i\Delta x) = u_i^n \quad \text{pour } n \in \mathbb{N}, \quad i \in \mathbb{Z}.$$

Alors, quand $\epsilon \rightarrow 0$, la suite $(u^\epsilon)_\epsilon$ converge vers l'unique solution Lipschitz v du système (1.30). En plus, pour n'importe quel compact $K \subset [0, +\infty) \times \mathbb{R}$, nous avons

$$\sup_{\substack{K \cap ((\Delta t \mathbb{N}) \times (\Delta x \mathbb{Z})) \\ j=1, \dots, d}} |u^\epsilon - v| \rightarrow 0 \quad \text{quand } \epsilon \rightarrow (0, 0).$$

Nous terminons cette section par une illustration numérique, basée sur le schéma (1.33), du problème (1.1).

5.3 Simulations numériques

Nous traitons d'un point de vue numérique le problème (1.1) dans un cadre particulier en considérant des conditions initiales croissantes et $a(t) = \alpha = 0$. Ceci revient à résoudre numériquement le problème

suitant:

$$\begin{cases} \partial_t \rho^+(t, x) = -((\rho^+ - \rho^-)(t, x)) \partial_x \rho^+(t, x) \text{ sur } (0, T) \times \mathbb{R}, \\ \partial_t \rho^-(t, x) = ((\rho^+ - \rho^-)(t, x)) \partial_x \rho^-(t, x) \text{ sur } (0, T) \times \mathbb{R}. \end{cases} \quad (1.35)$$

Le but ici est de calculer la solution numérique du système (1.35) via le schéma implicite (1.33). Pour se faire, nous choisissons des paramètres de discrétisation Δt et Δx vérifiant la condition CFL (1.34) et nous considérons des conditions initiales croissantes, 1-périodiques+ linéaires de la forme:

$$\rho^+(0, x) = \frac{1}{2\pi} \sin(2\pi x) + x, \quad \rho^-(0, x) = -\frac{1}{2\pi} \sin(2\pi x) + x, \quad \forall x \in \mathbb{R}.$$

La Figure 1.6 illustre le comportement de la fonction $\rho^+ - \rho^-$ qui représente en réalité dans le cas des dislocations la déformation viscoplastique. Intuitivement, d'après le système (1.35), nous pouvons voir que la fonction $\rho^+ - \rho^-$ atteindra un état stationnaire ($\rho^+ - \rho^- = 0$) comme le montre la Figure 1.6.

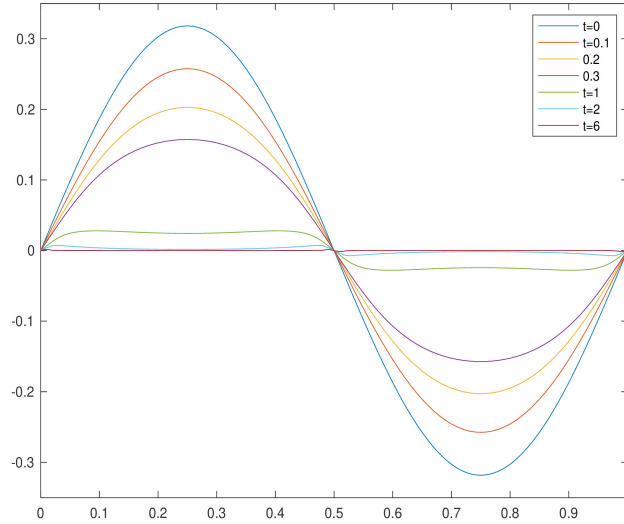


Figure 1.6: Comportement de $\rho^+ - \rho^-$ en fonction du temps

2

Modélisation

Dans ce chapitre, nous présentons la dérivation physique du modèle unidimensionnel décrivant la dynamique des densités des dislocations se propageant dans plusieurs directions de glissement. Ce chapitre est inspiré du travail effectué par El Hajj, Monneau [34, Section 5].

Initialement, la dynamique des densités des dislocations dérive d'un modèle tridimensionnel. Nous nous plaçons dans un cadre particulier où la géométrie du problème est invariante par la translation dans la direction de x_3 . Ceci nous conduit à analyser les densités des dislocations dans le plan (x_1, x_2) qui se propagent suivant un vecteur de Burgers \vec{b} appartenant au plan (x_1, x_2) .

Pour une présentation plus détaillée de la dérivation physique du système modélisant la dynamique des densités des dislocations dans le cas de plusieurs directions de glissement, nous référons les lecteurs à Yefimov, Van der Giessen [73] et à Yefimov [72] et à Groma, Balogh [40] pour le cas d'un modèle considérant une seule direction de glissement des dislocations. Pour une analyse mathématique du modèle de Groma et Balogh, nous renvoyons les lecteurs à Cannone et al. [24] et El Hajj [31].

Ce chapitre comporte 3 sections. Dans la Section 1, nous présentons le modèle bidimensionnel dans le cas de plusieurs directions de glissement. Dans la Section 2, en considérant une géométrie particulière où les dislocations dépendent d'une seule variable $x = x_1 + x_2$, nous réduisons ainsi le modèle bidimensionnel à un modèle unidimensionnel. Finalement, nous montrons dans la Section 3 que la dynamique des densités des dislocations est modélisée dans ce cadre par le système suivant:

$$\begin{cases} \partial_t u^i(t, x) + \left(\sum_{j=1}^d A_{ij} u^j \right) \partial_x u^i(t, x) = 0 & \text{sur } (0, +\infty) \times \mathbb{R}, \text{ pour } i = 1, \dots, d, \\ u^i(0, x) = u_0^i(x) & x \in \mathbb{R}, \text{ pour } i = 1, \dots, d, \end{cases} \quad (2.1)$$

avec $A = (A_{ij})_{i,j=1,\dots,d}$ est une matrice symétrique non-négative. Notons que le problème (2.1) correspond aux systèmes traités auparavant.

1 Le modèle bidimensionnel

Notons $X = (x_1, x_2)$ comme étant un vecteur de \mathbb{R}^2 et considérons un cristal se trouvant dans l'espace \mathbb{R}^2 . Notons $v = (v_1, v_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ comme étant le déplacement du cristal.

Nous supposons que les densités des dislocations étudiées correspondent à des dislocations coins. Nous considérons alors d directions de glissement telles que chaque direction est caractérisée par un vecteur de Burgers $\vec{b}^k = (b_1^k, b_2^k) \in \mathbb{R}^2$, avec $k = 1, \dots, d$. De ce fait, nous avons d types de dislocations qui se propagent dans le plan (x_1, x_2) suivant la direction de \vec{b}^k , pour $k = 1, \dots, d$.

Nous introduisons la déformation totale $\varepsilon(v) = (\varepsilon_{ij}(v))_{i,j=1,2}$ qui est une matrice symétrique définie par:

$$\varepsilon_{ij}(v) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

Notons que la déformation totale est donnée par la somme suivante:

$$\varepsilon_{ij}(v) = \varepsilon_{ij}^p(v) + \varepsilon_{ij}^e(v), \quad \text{avec } \varepsilon^p = \sum_{i=1, \dots, d} \varepsilon^{0,k} u^k,$$

où u^k est une fonction scalaire qui représente le déplacement plastique associé au k -ème système de glissement dont la matrice est définie par:

$$\varepsilon_{ij}^{0,k} = \frac{1}{2} (b_i^k n_j^k + n_i^k b_j^k), \quad (2.2)$$

où $\vec{n}^k = (n_1^k, n_2^k)$ est un vecteur unitaire orthogonal à \vec{b}^k .

Afin de simplifier la présentation, nous supposons que les inconnues vérifient les propriétés de périodicité suivantes.

Hypothèse (H):

i) La fonction v est \mathbb{Z}^2 -périodique avec $\int_{(0,1)^2} v \, dX = 0$.

ii) Pour tout $k = 1, \dots, d$, il existe $\vec{L}^k \in \mathbb{R}^2$ tel que $u^k(X) - \vec{L}^k \cdot X$ est \mathbb{Z}^2 -périodique.

iii) L'entier d est pair avec $d = 2N$ et nous avons pour $k = 1, \dots, N$:

$$\vec{L}^{k+N} = \vec{L}^k, \quad \vec{n}^{k+N} = \vec{n}^k, \quad \vec{b}^{k+N} = -\vec{b}^k, \quad \varepsilon^{0,k+N} = -\varepsilon^{0,k}.$$

iv) Notons par $\vec{\tau}^k = (\tau_1^k, \tau_2^k)$ le vecteur unitaire parallèle à \vec{b}^k tel que $\vec{\tau}^{k+N} = \vec{\tau}^k$. Nous imposons que \vec{L}^k soit choisi sachant que $\vec{\tau}^k \cdot \vec{L}^k \geq 0$.

Les contraintes d'élasticité σ_{ij} pour $i, j = 1, 2$ sont reliées à la déformation élastique par la loi de Hooke:

$$\sigma = \Lambda : \varepsilon^e(v), \quad \text{c'est-à-dire } \sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} \varepsilon_{kl}^e(v), \quad (2.3)$$

et vérifient l'équation d'élasticité d'Euler-Lagrange

$$\operatorname{div} \sigma = 0, \quad \text{c'est-à-dire } \sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{pour } i = 1, 2.$$

2. Modélisation

Dans (2.3), $\Lambda = (\Lambda_{ijkl})_{i,j,k,l=1,2}$ sont les coefficients d'élasticité du matériel vérifiant l'hypothèse de coercitivité suivante pour $m > 0$

$$\sum_{i,j,k,l=1,2} \Lambda_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq m \sum_{i,j=1,2} \varepsilon_{ij}^2, \quad (2.4)$$

pour toutes matrices $\varepsilon = (\varepsilon_{ij})_{i,j=1,2}$ symétriques, c'est-à-dire vérifiant $\varepsilon_{ij} = \varepsilon_{ji}$.

D'autre part, le déplacement plastique u^k est lié à la densité des dislocations coins associées au vecteur de Burgers \vec{b}^k . La k -ème densité des dislocations est donnée par

$$\theta^k = \vec{\tau}^k \cdot \nabla u^k \geq 0. \quad (2.5)$$

Le déplacement plastique u^k satisfait alors l'équation de transport suivante:

$$\partial_t u^k = c^k \vec{\tau}^k \cdot \nabla u^k \quad \text{avec} \quad c_k = \sum_{i,j=1,2} \sigma_{ij} \varepsilon_{ij}^{0,k},$$

où c_k représente la vitesse de Peach-Koehler résolue s'exerçant sur les dislocations.

En utilisant (2.5), nous remarquons que la densité des dislocations θ^k satisfait la loi de conservation suivante:

$$\partial_t \theta^k = \operatorname{div}(c^k \vec{\tau}^k \theta^k).$$

Finalement, pour $k = 1, \dots, d$, les fonctions u^k et v supposées dépendantes de $(t, X) \in (0, +\infty) \times \mathbb{R}^2$ sont des solutions du système couplé suivant (voir Yefimov, Van der Giessen [73] et à Yefimov [72]):

$$\left\{ \begin{array}{ll} \operatorname{div} \sigma = 0 & \text{sur } (0, +\infty) \times \mathbb{R}^2, \\ \sigma = \Lambda : (\varepsilon(v) - \varepsilon^p) & \text{sur } (0, +\infty) \times \mathbb{R}^2, \\ \varepsilon(v) = \frac{1}{2} (\nabla v + {}^t \nabla v) & \text{sur } (0, +\infty) \times \mathbb{R}^2, \\ \varepsilon^p = \sum_{k=1, \dots, d} \varepsilon^{0,k} u^k & \text{sur } (0, +\infty) \times \mathbb{R}^2, \\ \partial_t u^k = (\sigma : \varepsilon^{0,k}) \vec{\tau}^k \cdot \nabla u^k & \text{sur } (0, +\infty) \times \mathbb{R}^2, \quad \text{pour } k = 1, \dots, d, \end{array} \right. \quad (2.6)$$

c.à.d en coordonnées

$$\left\{ \begin{array}{ll} \sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 & \text{sur } (0, +\infty) \times \mathbb{R}^2, \quad \text{pour } i = 1, 2, \\ \sigma_{ij} = \sum_{k,l=1,2} \Lambda_{ijkl} (\varepsilon_{kl}(v) - \varepsilon_{kl}^p) & \text{sur } (0, +\infty) \times \mathbb{R}^2, \\ \varepsilon_{ij}(v) = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) & \text{sur } (0, +\infty) \times \mathbb{R}^2, \\ \varepsilon_{ij}^p = \sum_{k=1, \dots, d} \varepsilon_{ij}^{0,k} u^k & \text{sur } (0, +\infty) \times \mathbb{R}^2, \end{array} \right. \quad \text{pour } i, j = 1, 2 \quad (2.7)$$

$$\left. \begin{array}{l} \partial_t u^k = \left(\sum_{i,j \in \{1,2\}} \sigma_{ij} \varepsilon_{ij}^{0,k} \right) \vec{\tau}^k \cdot \nabla u^k \quad \text{sur } (0, +\infty) \times \mathbb{R}^2, \quad \text{pour } k = 1, \dots, d, \end{array} \right\}$$

où les inconnues du système sont u^k et le déplacement $v = (v_1, v_2)$ et $\varepsilon_{ij}^{0,k}$ est définie dans (2.2). Notons que ces équations sont accordables avec les hypothèses de périodicité (H), (i) – (ii).

2 Le modèle unidimensionnel

Dans cette section, nous nous plaçons dans une géométrie particulière où les densités des dislocations dépendent d'une seule variable $x = x_1 + x_2$. Ceci nous conduit à un modèle unidimensionnel.

Plus précisément, nous supposons les propriétés suivantes:

Hypothèse (H'):

i) Les fonctions $v(t, X)$ et $u^k(t, X) - \vec{L}^k \cdot X$ dépendent de la variable $x = x_1 + x_2$.

ii) Pour $k = 1, \dots, d$, le vecteur $\vec{\tau}^k = (\tau_1^k, \tau_2^k)$ satisfait $\tau_1^k + \tau_2^k = 1$.

iii) Pour $k = 1, \dots, d$, le vecteur $\vec{L}^k = (L_1^k, L_2^k)$ satisfait $L_1^k = L_2^k$.

Dans ce cadre particulier, nous désignons la fonction $v = v(t, x)$ qui est une fonction 1-périodique en x .

En posant $l^k = \frac{L_1^k + L_2^k}{2}$, nous avons

$$\vec{L}^k \cdot X = l^k \cdot x + \left(\frac{L_1^k - L_2^k}{2} \right) (x_1 - x_2).$$

En utilisant l'hypothèse (H'), (iii), nous constatons que $u = (u^k(t, x))_{k=1, \dots, d}$ est définie telle que pour $k = 1, \dots, d$, $u^k(t, x) - l^k \cdot x$ est une fonction 1-périodique en x .

Nous intégrons les équations d'élasticité, c.à.d, la première équation de (2.6). En se basant sur la \mathbb{Z}^2 -périodicité des inconnues (voir l'hypothèse (H), (i)-(ii)), et le fait que $\varepsilon^{0,k+N} = -\varepsilon^{0,k}$ (voir l'hypothèse (H), (iii)), nous déduisons que la déformation

$$\varepsilon^e \text{ est une fonction linéaire de } (u^j - u^{j+N})_{j=1, \dots, N} \text{ et de } \left(\int_0^1 (u^j - u^{j+N}) dx \right)_{j=1, \dots, N}. \quad (2.8)$$

Ceci nous conduit au lemme suivant.

Lemme 2.1. (Contrainte pour le modèle unidimensionnel)

Supposons que les hypothèses (H), (i)-(ii)-(iii) et (H'), (i)-(iii) et (2.4) sont vérifiées. Nous avons

$$-\sigma : \varepsilon^{0,i} = \sum_{j=1, \dots, d} A_{ij} u^j + \sum_{j=1, \dots, d} Q_{ij} \int_0^1 u^j dx, \quad \text{pour } i = 1, \dots, N, \quad (2.9)$$

où pour $i, j = 1, \dots, N$

$$\left\{ \begin{array}{l} A_{ij} = A_{ji} \quad \text{et} \quad A_{i+Nj} = -A_{ij} = A_{ij+N}, \\ Q_{ij} = Q_{ji} \quad \text{et} \quad Q_{i+Nj} = -Q_{ij} = Q_{ij+N}. \end{array} \right. \quad (2.10)$$

En plus, la matrice $A = (A_{ij})_{i,j=1, \dots, N}$ est non-négative.

La preuve du Lemme 2.1 sera présentée à la fin de cette section.

Enfin, en se basant sur le Lemme 2.1, nous éliminons la contrainte et nous transformons ainsi le système

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(2.7) en un problème unidimensionnel de d équations de transport dépendant seulement de u^i , pour $i = 1, \dots, d$. En s'appuyant sur (2.9) et (H') , (ii), le modèle unidimensionnel a la forme suivante

$$\partial_t u^i + \left(\sum_{j=1, \dots, d} A_{ij} u^j + \sum_{j=1, \dots, d} Q_{ij} \int_0^1 u^j dx \right) \partial_x u^i = 0, \quad \text{sur } (0, +\infty) \times \mathbb{R}, \quad \text{pour } i = 1, \dots, d, \quad (2.11)$$

avec

$$\partial_x u^i \geq 0 \quad \text{pour } i = 1, \dots, d, \quad (2.12)$$

d'après (2.5). Nous présentons par la suite la preuve du Lemme 2.1.

Preuve du Lemme 2.1:

Pour le problème bidimensionnel, en utilisant le fait que ε^e est \mathbb{Z}^2 -périodique, nous considérons l'énergie élastique

$$E^{el} = \frac{1}{2} \int_{(0,1)^2} \Lambda : \varepsilon^e : \varepsilon^e dX.$$

D'après la définition de σ et ε^e , nous avons pour $i = 1, \dots, d$

$$\sigma : \varepsilon^{0,i} = -\nabla_{u^i} E^{el}. \quad (2.13)$$

D'autre part, en utilisant (H') , (i)-(iii), (où $x = x_1 + x_2$), nous pouvons réécrire l'énergie élastique comme suivant:

$$E^{el} = \frac{1}{2} \int_0^1 \Lambda : \varepsilon^e : \varepsilon^e dx.$$

En remplaçant ε^e par son expression (2.8), nous avons:

$$\begin{aligned} E^{el} = & \frac{1}{2} \int_0^1 \sum_{i,j=1, \dots, N} A_{ij} (u^j - u^{j+N}) (u^i - u^{i+N}) dx \\ & + \frac{1}{2} \sum_{i,j=1, \dots, N} Q_{ij} \left(\int_0^1 (u^j - u^{j+N}) dx \right) \left(\int_0^1 (u^i - u^{i+N}) dx \right), \end{aligned}$$

où $A_{ij} = A_{ji}$, $Q_{ij} = Q_{ji}$ sont des matrices symétriques. Particulièrement, combinée avec (2.13), celle-ci implique (2.9) et (2.10).

Considérons les fonctions $w^i = u^i - u^{i+N}$ telle que

$$\int_0^1 w^i dx = 0 \quad \text{pour } i=1, \dots, N. \quad (2.14)$$

D'après (2.4), nous constatons que

$$0 \leq E^{el} = \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} w^i w^j dx.$$

D'une manière précise, pour tout $i = 1, \dots, N$ et pour tout $\bar{w}^i \in \mathbb{R}$, nous définissons

$$w^i = \begin{cases} \bar{w}^i & \text{sur } \left[0, \frac{1}{2}\right], \\ -\bar{w}^i & \text{sur } \left[\frac{1}{2}, 1\right], \end{cases}$$

qui vérifie (2.14). Enfin, nous obtenons

$$0 \leq E^{el} = \frac{1}{2} \int_0^1 \sum_{i,j=1,\dots,N} A_{ij} \bar{w}^i \bar{w}^j dx.$$

Comme ceci est vrai pour tout \bar{w}^i , nous pouvons conclure que A est une matrice non-négative. \square

3 Dérivation du modèle non-périodique

En partant du modèle (2.11)-(2.12) où pour $i = 1, \dots, d$, la fonction $u^i(t, x) - l^i \cdot x$ est 1-périodique en x , nous voulons la disparition de la périodicité. Plus précisément, nous avons le lemme suivant:

Lemme 2.2. (Modèle non-périodique)

Soit u une solution de (2.11)-(2.12). Supposons que le Lemme 2.1 est vérifié et que $u^i(t, x) - l^i \cdot x$ est 1-périodique en x . Soit

$$u_\delta^j(t, x) = u^j(\delta t, \delta x), \quad \text{pour } \delta > 0 \text{ et pour } j = 1, \dots, d,$$

sachant que, pour tout $j = 1, \dots, d$

$$u_\delta^j(0, \cdot) \rightarrow \bar{u}^j(0, \cdot), \quad \text{quand } \delta \rightarrow 0, \quad \text{et } \bar{u}^j(0, \pm\infty) = \bar{u}^{j+N}(0, \pm\infty). \quad (2.15)$$

Alors, $\bar{u} = (\bar{u}^j)_{j=1,\dots,d}$ est formellement une solution de

$$\partial_t \bar{u}^i + \left(\sum_{j=1,\dots,d} A_{ij} \bar{u}^j \right) \partial_x \bar{u}^i = 0, \quad \text{sur } (0, +\infty) \times \mathbb{R}, \quad (2.16)$$

où $\partial_x \bar{u}^i \geq 0$ pour $i = 1, \dots, d$.

Nous remarquons que le problème limite (2.16) est de même type que (2.1).

Nous démontrons par la suite le Lemme 2.2 d'une façon formelle.

Preuve formelle du Lemme 2.2:

Là, nous avons $u_\delta^i - \delta l^i \cdot x$ est $\frac{1}{\delta}$ -périodique en x , et satisfait pour $i = 1, \dots, d$

$$\partial_t u_\delta^i + \left(\sum_{j=1, \dots, d} A_{ij} u_\delta^j + \delta \sum_{j=1, \dots, d} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j dx \right) \partial_x u_\delta^i = 0, \quad \text{sur } (0, +\infty) \times \mathbb{R}. \quad (2.17)$$

Dans le but de simplifier, nous supposons que la condition initiale $u_\delta(0, \cdot)$ converge à une fonction $\bar{u}(0, \cdot)$ telle que la fonction $\partial_x u_\delta(0, \cdot)$ a un support dans $(-R, R)$, uniformément en δ , où R est une constante positive. En utilisant (2.15) et la propriété d'antisymétrie de la matrice Q donnée par (2.10), nous prévoyons que la vitesse dans (2.17) reste uniformément bornée quand $\delta \rightarrow 0$.

Alors, en utilisant la propriété de vitesse de propagation finie, nous constatons qu'il existe une constante C indépendante de δ , telle que $\partial_x u_\delta(t, \cdot)$ a un support dans $(-R - Ct, R + Ct)$ uniformément en δ . En plus, en utilisant (2.15) et le fait que

$$\sum_{j=1, \dots, d} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j dx = \sum_{j=1, \dots, N} Q_{ij} \int_0^{\frac{1}{\delta}} (u_\delta^j - u_\delta^{j+N}) dx,$$

nous déduisons que

$$\sum_{j=1, \dots, d} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j dx,$$

reste uniformément bornée en δ . Alors, le terme non-local disparaît et nous avons pour $i = 1, \dots, d$

$$\sum_{j=1, \dots, d} A_{ij} u_\delta^j + \delta \sum_{j=1, \dots, d} Q_{ij} \int_0^{\frac{1}{\delta}} u_\delta^j dx \rightarrow \sum_{j=1, \dots, d} A_{ij} \bar{u}^j, \quad \text{quand } \delta \rightarrow 0,$$

ce qui montre que \bar{u} est une solution de (2.16), avec A est une matrice non-négative. □

3

Existence et unicité d'une solution pour un système non-local couplé modélisant la dynamique des densités des dislocations

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Dans ce travail, nous étudions un problème couplé non-local et non-linéaire modélisant la dynamique des densités des dislocations. En utilisant une estimation entropique du gradient dans l'espace $L \log L$, nous montrons l'existence globale d'une solution de viscosité continue du problème. Afin d'aboutir à ce résultat nous régularisons tout d'abord le système en lui ajoutant un terme de viscosité puis nous passons à la limite en utilisant les résultats de stabilité des solutions de viscosité continues prouvés par Crandall et Lions. Finalement, nous démontrons l'unicité de la solution pour le problème local en se basant sur un principe de comparaison.

Existence and uniqueness of continuous solution for a non-local coupled system modeling the dynamics of dislocations densities

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Abstract

In this paper, we study a non-local coupled system that arises in the theory of dislocations densities dynamics. Based on a new gradient entropy estimate in $L \log L$ space, we prove the global existence of a continuous solution. The approach is made by adding a viscosity term and passing to the limit for vanishing viscosity. A comparison principle is used for proving uniqueness of the solution for the local problem.

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1 Introduction and main results

1.1 Presentation and physical motivations

Real crystals comprise certain defects in the organization of their crystalline structure called dislocations. This concept has been introduced by Polany, Taylor and Orowan in 1934 as the main explanation, at microscopic scale, of a plastic deformation.

Under the effect of exterior constraints, dislocations can move in a certain crystallographic direction called the slip direction. This slip is given by a vector called "Burger's vectors" whose norm represents the amplitude of plastic deformation.

In this work, we are interested in the study of 1D submodel of a problem introduced by Groma and Balogh in [40], initially proposed in the two dimensional case. This two dimensional model is characterized by the fact that dislocations propagate in the plane (x_1, x_2) following two Burger's vectors $\pm \vec{b} = \pm(1, 0)$. In this 1D submodel we suppose also that the dislocations densities depend only on the variable $x = x_1 + x_2$, that transforms the 2D into 1D model. We refer the reader to Section 2, for the physical modeling of the equations considered here, by highlighting the link with Groma-Balogh [40].

More precisely, this 1D model is given by the following coupled equations of non-local eikonal system

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type :

$$\begin{cases} \partial_t \rho^+(x, t) = - \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) |\partial_x \rho^+(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \partial_t \rho^-(x, t) = \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) |\partial_x \rho^-(x, t)| & \text{in } \mathbb{R} \times (0, T). \end{cases} \quad (3.1)$$

The unknowns ρ^+ and ρ^- are scalar valued functions, that we denote for simplicity by ρ^\pm . These quantities represent respectively the plastic distortions associated with the dislocations moving according to the vectors $(1, 0)$ and $(-1, 0)$. Their spacial derivatives $\partial_x \rho^\pm$ are the dislocations densities of Burger's vector $\pm \vec{b} = \pm(1, 0)$. From a physical viewpoint, in the context of the dynamics of dislocation densities, the functions ρ^\pm and $\partial_x \rho^\pm$ are positive reals. However, we do not make a sign restriction on these quantities in the mathematical analysis in order to deal with the problem in a more general framework and broaden the application areas. Here, the difference $(\rho^+ - \rho^-)$ represents the resolved plastic strain. The constant α depends on the elastic coefficients and the material size, while the function $a(\cdot)$ represents the exterior shear strain field, which is assumed to satisfy the following regularity:

$$a \in C(0, T). \quad (3.2)$$

The initial conditions associated to system (3.1) are defined as follows:

$$\rho^\pm(x, 0) = \rho_0^\pm(x) = P_0^\pm(x) + L_0 x \quad \text{on } \mathbb{R}, \quad (3.3)$$

where P_0^\pm are 1-periodic functions. We thus modelize a periodic distribution for the \pm dislocations, with a spatial period of length 1. Note that each type of \pm dislocations have a mean density equal to L_0 . In fact, the use of the periodic boundary conditions is a way of regarding what is going on the interior of the material away from its boundary.

Let us now mention some results related to this problem. In [30], based on an energy estimate the global existence and uniqueness of nondecreasing solution in $W_{loc}^{1,2}(\mathbb{R} \times (0, T))$ space has been established. However without monotony assumptions on initial data, in [33] using the notion of viscosity solutions in the case of system introduced by Ishii and Koike in [48] the authors show existence and uniqueness of a Lipschitz solution. They also proposed a convergent numerical scheme and proved a Crandall-Lions error type estimate between the continuous solution and the numerical one. By weakening the regularity of the initial data considering discontinuous solutions in particular *BV* functions, in [38] a result of existence has been shown.

In this paper, we are interested in proving existence and uniqueness results considering continuous initial data without monotony assumptions.

For a similar 1D model, to that of Groma-Balogh, taking into account the back stress and describing moreover the boundary layer effects, we refer the reader to Ibrahim et al. [43, 44] where a result of existence and uniqueness has been established under the framework of viscosity solutions for Hamilton-Jacobi equations.

Let us also mention that for an equivalent 2D model, a global existence result was proved in Cannone et

al. [24] using an entropy inequality similar to the one used in this paper. Beside this, a local existence and uniqueness result in Hölder spaces was shown in [31] basing on commutator estimates.

Note that the non-local system (3.1) studied in this paper is a nonlocal version of crystal plasticity, which can be seen as a sub-model of the 2D Groma-Balogh model proposed in [40] in order to study the shearing of a composite material with single slip deformations, from statistical-mechanics descriptions of collective behavior of dislocations (see Section 2 for more modeling details).

Several others continuous models were proposed in the literature for understanding plasticity of solids as it arises from the nucleation, the motion, and the interaction of defects in the elastic deformation of the material. We cite among this the interesting PDE-model developed in the work of Acharya (see for instance [1]) wherein, building on some fundamental laws of kinematics or conservation, a full time-dependent 3D system of equations governing the rate of plastic distortion tensor is formulated. This theory is called Field Dislocation Mechanics (FDM) and utilizes a tensorial description of dislocation density which is related to special gradients of the elastic distortion field.

A mathematical study using viscosity solution was established in Acharya *et al.* [3] for a 1D reduced model arising naturally from the 3D-FDM theory in the case of homogeneously sheared rectangular cylinder which contains some collections of planar screw dislocation walls. It is precisely a scalar problem of a nonlinear second order Hamilton-Jacobi equation describing the evolution of plastic shear strain with non-convex elasticity. The existence and nonexistence of equilibrium and traveling waves solutions of this equation have been analyzed in this work, providing interesting physical insight into dislocation wall equilibria. It is also shown that the comparison principle does not hold for the equilibrium equation (i.e. with respect to the space variable). This also remains true in our framework, since we can easily verify that the associated equilibrium system of (3.1) with zero exterior shear strain ($a(\cdot) = 0$) admits two intersecting solutions, namely the constants and all functions such as $\rho^+ = \rho^-$. However, we were able to prove here a comparison principle result for the non-stationary local problem (see Section 8) with respect to time, relying on some monotonicity properties of the system. Although this was not investigated in [3], we expect that such a result is invalid for the time-dependent 1D equation considered in [3], due to lack of monotonicity properties. A numerical approximation, based on two different schemes (of Upwind and Friedrichs types), allowing to solve the 1D model of Acharya not necessarily in traveling waves case, has been proposed and discussed in [28]. The influence of these numerical schemes on the behavior of equilibrium and quasi-equilibrium profiles as well as on the motion of dislocations was also analyzed. The performed simulations show many important physical phenomena such as the existence of non-periodic quasi-equilibrium cell-wall dislocation microstructures reflecting some common features to the persistent slip band phenomenon of fatigue experiments. A broader mathematical analysis concerning the global existence and uniqueness of solution for a one-dimensional model more general than that discussed in the previous paper, was provided by Acharya and Tartar in [2]. In their paper, they considered the study of a 1D reduced quasilinear system both in the stationary and non-stationary case giving a more large class of equilibria. Their proof relies on classical tools applied to a local second order equation, making an approximation by Galerkin method and then passing to the limit using some compactness arguments and regularity properties. These techniques do not apply in our framework, since it is a nonlocal and non-conservative Hamilton-Jacobi system with low regularity. Besides mentioned model, a 1D model taking into account the existence of edge dislocations, moving horizontally in a fixed layer of plane, was described in [74] and studied numerically as in [28]. The resulting equation brings up a non-local term

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according to the plastic strain, which, in the case of linear elasticity and under particular assumptions, is similar to our system but in the scalar case (i.e. $\rho^+ = 0$ or $\rho^- = 0$). Important observations like Peierls-Nabarro stress effects, dislocation annihilation and dissociation, were also explored in [74] through this 1D equation. Other model, called Mesoscale Field Dislocation Mechanics (MFDM), was also recently developed in order to derive the finite deformation mesoscale plasticity. This model allows us to understand several key issues involved in the fields of plasticity and dislocation mechanics, and it can be seen as an average (space-time) field model of the microscopic FDM model. Regarding this, we refer to [29] wherein a reduced 1D system, coupling dislocation transport and gradient hardening, was deduced from MFDM theory and studied numerically. We can also see [9] for an in-depth numerical analysis based on a finite element approximation of MFDM, allowing to recover the power law scaling of stress-strain behavior observed in micropillar compression experiments as well as predict the stress field of a progressively forming dislocation wall.

Let us mention that, our work focused on the mathematical study of system (3.1) arising from the dynamics of dislocation densities. However, following a different approach, let us quote some recent results on the dynamics of dislocation lines that are represented by non-local Hamilton-Jacobi equations. In this direction, we address the reader to [8] for local existence results, and to [14, 15] and to [19, 20] for global existence results.

1.2 Main results

In this paper, we present a global existence for the non-local Hamilton-Jacobi system (3.1). Taking the following regularity on the initial data given by (3.3):

$$P_0^\pm \in L^\infty(\mathbb{T}) \text{ and } \partial_x P_0^\pm \in L \log L(\mathbb{T}) \quad (3.4)$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the $[0, 1)$ periodic interval and $L \log L(\mathbb{T})$ is the following Zygmund space:

$$L \log L(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) \text{ such that } \int_{\mathbb{T}} |f| \ln(e + |f|) < +\infty \right\}. \quad (3.5)$$

This space is equipped with the following norm:

$$\|f\|_{L \log L(\mathbb{T})} = \inf \left\{ \mu > 0 : \int_{\mathbb{T}} \frac{|f|}{\mu} \ln \left(e + \frac{|f|}{\mu} \right) \leq 1 \right\}. \quad (3.6)$$

This norm is due to Luxemburg (see Adams [4], (13), Page 234).

We now present the main result.

Theorem 3.1. (Global existence and uniqueness of solution)

Assume that (3.2) and (3.4) are satisfied. Then the following hold:

i) Existence of a continuous viscosity solution of (3.1)-(3.3):

There exists a continuous viscosity solution ρ^\pm of (3.1)-(3.3) such that:

$$\rho^\pm(x, t) = P^\pm(x, t) + L_0 x \quad (3.7)$$

and

$$P^\pm \in L^\infty(\mathbb{T} \times (0, T)) \cap C([0, T]; L \log L(\mathbb{T})) \text{ and } \partial_x P^\pm \in L^\infty((0, T); L \log L(\mathbb{T})),$$

where P^\pm are 1-periodic functions (with respect to the space variable) and P^\pm satisfy, for all $T > 0$, the following estimate

$$\max_{\pm} (\|P^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \Lambda_1 \quad (3.8)$$

where $\Lambda_1(T, \|a\|_{L^\infty(0, T)}, L_0, \alpha, \|P_0^\pm\|_{L^\infty(\mathbb{T})})$.

Also the function ρ^\pm satisfies the following gradient entropy estimate:

$$\max_{\pm} (\|\partial_x \rho^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))} + \|\partial_t \rho^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))}) \leq \Lambda_2 \quad (3.9)$$

where $\Lambda_2(T, \|a\|_{L^\infty(0, T)}, L_0, \alpha, \|P_0^\pm\|_{L^\infty(\mathbb{T})}, \|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})})$.

ii) Uniqueness of solution in the local case:

The solution, given by (i), is unique if $\alpha = 0$.

The key point to establish Theorem 3.1 is the gradient entropy estimate (3.9). We first consider the parabolic regularization of the local problem ($\alpha = 0$) and then we show that the smooth solution admits the L^∞ bound (3.8) and the following fundamental gradient entropy inequality:

$$\int_{\mathbb{T}} [f(|\partial_x \rho^+(x, t)|) + f(|\partial_x \rho^-(x, t)|)] dx \leq C, \quad \text{for all } t \in (0, T) \quad (3.10)$$

where

$$0 \leq f(x) = \begin{cases} x \ln(x) + \frac{1}{e} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases} \quad (3.11)$$

with the constant C depends only on $T, \|\partial_x \rho_0^+\|_{L \log L(\mathbb{T})}, \|\partial_x \rho_0^-\|_{L \log L(\mathbb{T})}$.

This estimate allows us to pass to the limit when the regularization vanishes, using compactness argument and stability of viscosity solutions, which leads us to obtain the existence of a solution. In addition, by proving a comparison principle result in the local case, we can obtain the uniqueness of the solution. Finally, by a fixed point argument we get the existence for the non-local case.

Remark 3.1. Note that, thanks to the quasi-monotonicity of the system in the case $\alpha = 0$ we were able to show a comparison principle result with respect to time, which proves in particular the uniqueness of the solution. However, due to the presence of the non-local term when $\alpha \neq 0$ we lose this monotonicity property as well as the uniqueness except in the case of regular solutions (namely, Lipschitz solution).

Corollary 3.1. (Continuity of the solution)

The solution ρ^\pm constructed in (i) belongs to $C(\mathbb{R} \times [0, T])$ and there exists a modulus of continuity such that for all $\delta, h \geq 0$ and all $(x, t) \in \mathbb{R} \times (0, T - \delta)$, we have:

$$|\rho^\pm(x + h, t + \delta) - \rho^\pm(x, t)| \leq \Lambda_3 \frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)}, \quad (3.12)$$

where $\Lambda_3(T, \|a\|_{L^\infty(0, T)}, L_0, \alpha, \|P_0^\pm\|_{L^\infty(\mathbb{T})}, \|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})})$.

Remark 3.2. Note that, all presented results remain valid in the case where $a \in L^\infty(0, T)$. To do that, it is sufficient to use in Definition 3.1 the definition of viscosity solutions for discontinuous Hamiltonian instead of the definition of viscosity solutions for continuous Hamiltonian.

1.3 Organization of the paper

This paper is organized as follows: In Section 2, we briefly recall the physical derivation of system (3.1) and its link with the Groma-Balogh [40]. In Section 3, we regularize the approximated local problem ($\alpha = 0$) by adding the viscosity term $\eta \partial_{xx}^2 \rho^\pm$, replacing the non-linear term $|\partial_x \rho^\pm|$ by $\sqrt{(\partial_x \rho^\pm)^2 + \delta^2}$ and considering regular initial data. Then, we show that this system admits a global in time regular solution. In Section 4, we prove the gradient entropy inequality and some others (η, δ) -uniform *a priori* estimates. In Section 5, we prove some η -uniform *a priori* estimates similar to that obtained in Section 4 by passing to the limit as $\delta \rightarrow 0$. Then, in Section 6 we get by using the stability result of continuous viscosity solutions and by passing to the limit when η goes to 0, the global existence of continuous viscosity solution with regular initial data. Which, leads us to prove, in Section 7, Theorem 3.1 (i) in the particular case where $\alpha = 0$. Then, in the same section, using a fixed point argument, we prove the existence of solution for the non-local problem. Finally, by comparison principle, we obtain in Section 8 the uniqueness of the solution for the local problem given by Theorem 3.1 (ii).

2 Physical derivation

In this section we explain how to derive physically the system (3.1). We consider a crystal with periodic deformation, namely the case where the total displacement of the crystal

$$U = (U_1, U_2, U_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

can be decomposed in a 1-periodic displacement $u = (u_1, u_2, u_3)$ and a linear displacement $A(t)^t \mathbf{x}$ with $\mathbf{x} = (x_1, x_2, x_3)$ and $A(t)$ a given (3×3) matrix which represents the applied shear strain $A(t) = (A_{ij}(t))_{i,j=1,2,3}$. The displacement U is then given by

$$U(\mathbf{x}, t) = u(\mathbf{x}, t) + A(t)^t \mathbf{x}$$

and we define the total strain by

$$\varepsilon_{ij}(U) = \frac{1}{2} (\partial_{x_j} u_i + \partial_{x_i} u_j + A_{ij}(t) + A_{ji}(t)), \quad i, j = 1, 2, 3.$$

This total strain is decomposed as

$$\varepsilon_{ij}(U) = \varepsilon_{ij}^e + \varepsilon_{ij}^p,$$

with ε_{ij}^e is the elastic strain and ε_{ij}^p the plastic strain which is defined by:

$$\varepsilon_{ij}^p = \gamma \varepsilon_{ij}^0, \tag{3.13}$$

with the fixed matrix $\varepsilon_{ij}^0 = \frac{1}{2} (\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1)_{ij} = \frac{1}{2} (1 - \delta_{ij})$, where δ_{ij} is the Kronecker symbol, in the special case of a single slip system where dislocations move in the plane $\{x_2 = 0\}$ with Burgers vector $\pm \vec{e}_1 = \pm(1, 0)$. Here γ is the resolved plastic strain, and will be clarified later. In the case of linear

homogeneous and isotropic elasticity, the stress is given by

$$\sigma_{ij} = 2\mu\varepsilon_{ij}^e + \lambda\delta_{ij} \left(\sum_{k=1,2,3} \varepsilon_{kk}^e \right) \quad \text{for } i, j = 1, 2, 3, \quad (3.14)$$

where λ, μ are the constant Lamé coefficients of the crystal (satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$). Moreover the stress satisfies the equation of elasticity:

$$\sum_{j=1,2,3} \partial_{x_j} \sigma_{ij} = 0. \quad (3.15)$$

We now assume that we are in a particular geometry where the dislocations are straight lines parallel to the direction e_3 and that the problem is invariant by translation in the x_3 direction. Moreover we assume that $u_3 = A_{i3} = A_{3i} = \sigma_{i3} = 0$ for $i = 1, 2, 3$. Then, this problem reduces to a two-dimensional problem with u_1, u_2 only depending on (x_1, x_2) and so we can express the resolved plastic strain γ as

$$\gamma = \rho^+ - \rho^-,$$

where ρ^+ and ρ^- represents the plastic shearing in the direction of Burgers vectors \vec{e}_1 and $-\vec{e}_1$ respectively. Therefore, their derivatives with respect to x_1 , $\theta^+ = \partial_{x_1} \rho^+$ and $\theta^- = \partial_{x_1} \rho^-$ are respectively the densities of dislocations with Burgers vectors \vec{e}_1 and $-\vec{e}_1$. Let us mention that, the quantities ρ^\pm and θ^\pm are physically positive. However, we do not need to make a sign restriction in our work, the result presented in Theorem 3.1 remains valid for any real functions (in particular if they are positive).

Furthermore, these dislocation densities are transported in the direction of the Burgers vector at a given velocity. This velocity is indeed the resolved shear stress $\sum_{i,j=1,2,3} \sigma_{ij} \varepsilon_{ij}^0 = \sigma_{12}$, up to sign of the Burgers vectors. More precisely, we have:

$$\partial_t \theta^\pm = \pm \operatorname{div}(M \sigma_{12} \vec{e}_1 \theta^\pm), \quad (3.16)$$

where M is the dislocation mobility. The system (3.16) was already derived by Groma and Balogh in [40], showing that by solving the elasticity equation (3.15) we can express the resolved shear stress σ_{12} as a function of $\gamma = \rho^+ - \rho^-$ as follows (see [40, Eqs. (10)-(11)])

$$\sigma_{12}(\gamma) = -(\tau \star (\theta^+ - \theta^-) + \tau_{ext}) = -((\partial_{x_1} \tau) \star (\rho^+ - \rho^-) + \tau_{ext})$$

where τ is a explicit kernel depends only on the elasticity coefficients (λ, μ) and τ_{ext} is a time dependent function computed from the applied shear strain $A(t)$. It was also shown in [33, Lemme 3.1] that, if we assume that dislocation densities θ^\pm and the displacement only depend on one variable $x = x_1 + x_2$ then the Groma-Balogh model (3.16) can be reduced to the following 1D problem

$$\partial_t(\partial_x \rho^\pm) = \pm \partial_x(M \sigma_{12}(\partial_x \rho^\pm)) \quad (3.17)$$

with

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$$\sigma_{12}(\gamma) = -\frac{(\lambda + \mu)\mu}{\lambda + 2\mu} \left((\rho_+ - \rho_-) + \frac{\mu}{2(\lambda + \mu)} \int_0^1 (\rho_+ - \rho_-) - \frac{(\lambda + 2\mu)}{(\lambda + \mu)} (A_{12}(t) + A_{21}(t)) \right).$$

Here, the integral term appears due to the integration of the elasticity equation on the 1-periodic interval. Now, if we integrate the equations (3.17) with respect to x , we can then describe the evolution of ρ^\pm by the following system:

$$\partial_t \rho^\pm = \mp C_1 \left((\rho_+ - \rho_-) + C_2 \int_0^1 (\rho_+ - \rho_-) + a(t) \right) \partial_x \rho^\pm$$

where $C_1 = \frac{B(\lambda + \mu)\mu}{\lambda + 2\mu}$, $C_2 = \frac{\mu}{2(\lambda + \mu)}$ and $a(t) = -\frac{(\lambda + 2\mu)}{(\lambda + \mu)} (A_{12}(t) + A_{21}(t))$. Since the constant C_1 is positive we can always reduce the system to the case $C_1 = 1$ by rescaling in time. Which brings us to a system of form (3.1), wherein we have replaced $\partial_x \rho^\pm$ by $|\partial_x \rho^\pm|$ to ensure the positivity of dislocation densities.

Note that, system (3.1) allows us to obtain a simple 1D continuous model of the dislocation dynamics in which the intersection between two different dislocations (+ and -) is directly taken into account.

Remark 3.3. *Let us point out that, if we take $C_2 = 0$ we can see that the shear stress σ_{12} is locally defined and locally monotone, according to the plastic strain γ . Namely, if we consider two functions γ_1 and γ_2 corresponding to two solutions ρ_1^\pm and ρ_2^\pm such as $\gamma_1(x, \cdot) \leq \gamma_2(x, \cdot)$ at a fixed position x , then we have $\sigma_{12}(\gamma_1(x, \cdot)) \geq \sigma_{12}(\gamma_2(x, \cdot))$. Thanks to this local monotonicity property we prove later a comparison principle result in time and particularly the uniqueness of the solution when $C_2 = 0$. However, we cannot establish this result in the opposite case, since if $C_2 \neq 0$ the shear stress σ_{12} is non-local and therefore it is not possible to order the evolution only from local informations.*

3 Global solution for parabolic regularized equation

In this section, we present some preliminary global existence and uniqueness results for a parabolic equation obtained by regularization of the following problem:

$$\begin{cases} \partial_t v^+(x, t) = -((v^+ - v^-)(x, t) + L(t)) |\partial_x v^+(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \partial_t v^-(x, t) = ((v^+ - v^-)(x, t) + L(t)) |\partial_x v^-(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ v^\pm(x, 0) = v_0^\pm(x) = Q_0^\pm(x) + L_0 x & \text{in } \mathbb{R}, \end{cases} \quad (3.18)$$

where Q_0^\pm are 1-periodic functions, such that

$$Q_0^\pm \in C^\infty(\mathbb{R}) \text{ and } L \in C^\infty(0, T). \quad (3.19)$$

More precisely, we consider, for $0 < \eta \leq 1$, the following system:

$$\begin{cases} \partial_t v_\eta^\pm(x, t) - \eta \partial_{xx}^2 v_\eta^\pm(x, t) = \mp ((v_\eta^+ - v_\eta^-)(x, t) + L(t)) |\partial_x v_\eta^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ v_\eta^\pm(x, 0) = v_0^\pm(x) = Q_0^\pm(x) + L_0 x & \text{in } \mathbb{R}. \end{cases} \quad (3.20)$$

We also regularize the non-linear term $|\partial_x v_\eta^\pm|$ replacing it by function $\beta_\delta(\partial_x v_\eta^\pm)$, $0 < \delta \leq 1$, where β_δ is defined as follows:

$$\beta_\delta(x) = \sqrt{x^2 + \delta^2}. \quad (3.21)$$

This brings us, finally, to consider the following parabolic system:

$$\begin{cases} \partial_t v_\eta^{\pm, \delta}(x, t) - \eta \partial_{xx}^2 v_\eta^{\pm, \delta}(x, t) = \mp ((v_\eta^{+, \delta} - v_\eta^{-, \delta})(x, t) + L(t)) \beta_\delta(\partial_x v_\eta^{\pm, \delta}(x, t)) & \text{in } \mathbb{R} \times (0, T), \\ v_\eta^{\pm, \delta}(x, 0) = v_0^\pm(x) & \text{in } \mathbb{R}. \end{cases} \quad (3.22)$$

In the following two subsections, some preliminary results concerning the regularized equations (3.20) and (3.22) are presented.

Theorem 3.2. (Existence and uniqueness of Lipschitz continuous solution of (3.20))

Assume that (3.19) holds. Then, for all $T > 0$, we have

i) **(Existence and uniqueness of (3.22)):**

System (3.22) admits a unique solution $v_\eta^{\pm, \delta}$, such that

$$v_\eta^{\pm, \delta}(x, t) = Q_\eta^{\pm, \delta}(x, t) + L_0 x,$$

where $Q_\eta^{\pm, \delta}$ are 1-periodic functions (with respect to the space variable), belonging to $C^\infty(\mathbb{R} \times (0, T))$ and satisfying the following estimates:

$$\max_{\pm} (\|Q_\eta^{\pm, \delta}\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq N_0^\delta \quad (3.23)$$

$$\max_{\pm} (\|\partial_x Q_\eta^{\pm, \delta}\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \max_{\pm} (\|\partial_x Q_0^\pm\|_{L^\infty(\mathbb{T})}) \quad (3.24)$$

where

$$N_0^\delta = \max_{\pm} (\|Q_0^\pm\|_{L^\infty(\mathbb{T})}) + \beta_\delta(L_0) \|L\|_{L^\infty(0, T)} T. \quad (3.25)$$

ii) **(Convergence):**

Up to the extraction of a subsequence, the function $v_\eta^{\pm, \delta}$ converges, as δ goes to 0, to v_η^\pm locally and strongly in $C([0, T]; W^{1, \infty}(\mathbb{R}))$. Moreover,

$$v_\eta^\pm(x, t) = Q_\eta^\pm(x, t) + L_0 x,$$

where Q_η^\pm are 1-periodic functions and v_η^\pm is the unique solution of (3.20) (in the distributional sense) belonging to the space $C([0, T]; W_{loc}^{1, \infty}(\mathbb{R}))$ that satisfies, for all $0 < \eta \leq 1$, the following η -uniform estimates:

$$\max_{\pm} (\|Q_\eta^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq N_0 \quad (3.26)$$

$$\max_{\pm} (\|\partial_x Q_\eta^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \max_{\pm} (\|\partial_x Q_0^\pm\|_{L^\infty(\mathbb{T})}) \quad (3.27)$$

where

$$N_0 = \max_{\pm} (\|Q_0^\pm\|_{L^\infty(\mathbb{T})}) + |L_0| \|L\|_{L^\infty(0, T)} T. \quad (3.28)$$

To prove Theorem 3.2, we need to recall the following lemma.

Lemma 3.1. (Properties of the heat kernel)

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Let $G(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ be the standard heat kernel. Then, for all $t > 0$, we have:

$$(i) \quad \|G(t)\|_{L^1(\mathbb{R})} = 1,$$

$$(ii) \quad \|\partial_x G(t)\|_{L^1(\mathbb{R})} \leq \frac{1}{\sqrt{t}},$$

where $G(t) = G(\cdot, t)$.

For the proof of this lemma, we refer to Pazy [63, Theorem 5.2, Page 107].

Proof of Theorem 3.2: The proof is outlined in six steps.

Step 1 (Rewriting the equation in its integral form):

First, we remark that proving $v_\eta^{\pm, \delta}$ is a solution for (3.22) is equivalent to show that $Q_\eta^{\pm, \delta}$ verifies the following problem:

$$\begin{cases} \partial_t Q_\eta^{\pm, \delta}(x, t) - \eta \partial_{xx}^2 Q_\eta^{\pm, \delta}(x, t) = \mp ((Q_\eta^{+, \delta} - Q_\eta^{-, \delta})(x, t) + L(t)) \beta_\delta (\partial_x Q_\eta^{\pm, \delta}(x, t) + L_0) & \text{in } \mathbb{R} \times (0, T), \\ Q_\eta^{\pm, \delta}(x, 0) = Q_0^\pm(x) & \text{in } \mathbb{R}. \end{cases} \quad (3.29)$$

Let $R_\eta^{\pm, \delta}(x, t) = Q_\eta^{\pm, \delta}(\eta x, \eta t)$. Then, it is easy to show that $R_\eta^{\pm, \delta}$ satisfies the following problem:

$$\begin{cases} \partial_t R_\eta^{\pm, \delta}(x, t) - \partial_{xx}^2 R_\eta^{\pm, \delta}(x, t) = \mp ((R_\eta^{+, \delta} - R_\eta^{-, \delta})(x, t) + L^\eta(t)) \beta_{\delta\eta} (\partial_x R_\eta^{\pm, \delta}(x, t) + L_0^\eta) & \text{in } \mathbb{R} \times (0, T^\eta), \\ R_\eta^{\pm, \delta}(x, 0) = R_{0, \eta}^\pm(x) = Q_0^\pm(\eta x) & \text{in } \mathbb{R}, \end{cases} \quad (3.30)$$

where $L^\eta(t) = L(\eta t)$, $L_0^\eta = \eta L_0$ and $T^\eta = \frac{T}{\eta}$.

The previous problem can be written in its integral form as follows:

$$\begin{aligned} R_\eta^{\pm, \delta}(x, t) &= G(t) \star R_{0, \eta}^\pm(x) \mp \int_0^t G(t-s) \star [(R_\eta^{+, \delta}(x, s) - R_\eta^{-, \delta}(x, s)) \beta_{\delta\eta} (\partial_x R_\eta^{\pm, \delta}(x, s) + L_0^\eta)] ds \\ &\quad \mp \int_0^t G(t-s) \star \beta_{\delta\eta} (\partial_x R_\eta^{\pm, \delta}(x, s) + L_0^\eta) L^\eta(s) ds. \end{aligned}$$

In other words, we consider the following problem:

$$\begin{cases} R_\eta^\delta(x, t) = \begin{pmatrix} R_\eta^{+, \delta}(x, t) \\ R_\eta^{-, \delta}(x, t) \end{pmatrix}, \quad R_{0, \eta}(x) = \begin{pmatrix} R_{0, \eta}^+(x) \\ R_{0, \eta}^-(x) \end{pmatrix}, \\ R_\eta^\delta(x, t) = G(t) \star R_{0, \eta}(x) + B(R_\eta^\delta)(x, t), \end{cases} \quad (3.31)$$

where, for $r(x, t) = \begin{pmatrix} r_1(x, t) \\ r_2(x, t) \end{pmatrix}$, we have

$$\begin{aligned} B(r)(x, t) &= \int_0^t G(t-s) \star \begin{pmatrix} \beta_{\delta\eta}(\partial_x r_1(x, s) + L_0^\eta) & 0 \\ 0 & \beta_{\delta\eta}(\partial_x r_2(x, s) + L_0^\eta) \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} r_1(x, s) \\ r_2(x, s) \end{pmatrix} ds \\ &+ \int_0^t G(t-s) \star \begin{pmatrix} \beta_{\delta\eta}(\partial_x r_1(x, s) + L_0^\eta) & 0 \\ 0 & \beta_{\delta\eta}(\partial_x r_2(x, s) + L_0^\eta) \end{pmatrix} \begin{pmatrix} -L^\eta(s) \\ L^\eta(s) \end{pmatrix} ds. \end{aligned} \quad (3.32)$$

In what follows, we show some estimates to prove in particular the contraction of the operator B and therefore the existence and the uniqueness of the solution of (3.31) by applying a fixed point argument.

Step 2 (A priori estimates on B and $G(t) \star R_{0,\eta}$):

Taking, for any vector field $r(\cdot) = \begin{pmatrix} r_1(\cdot) \\ r_2(\cdot) \end{pmatrix}$ the norm of r in the Banach space E^2 as

$$\|r\|_{E^2} = \|r_1\|_E + \|r_2\|_E,$$

we can remark that

$$\begin{aligned} \|B(r)(\cdot, t)\|_{(W^{1,\infty}(\mathbb{R}))^2} &= \left\| \int_0^t G(t-s) \star \beta_{\delta\eta}(\partial_x r_1(\cdot, s) + L_0^\eta) (r_2(\cdot, s) - r_1(\cdot, s) - L^\eta(s)) ds \right\|_{W^{1,\infty}(\mathbb{R})} \\ &+ \left\| \int_0^t G(t-s) \star \beta_{\delta\eta}(\partial_x r_2(\cdot, s) + L_0^\eta) (r_1(\cdot, s) - r_2(\cdot, s) + L^\eta(s)) ds \right\|_{W^{1,\infty}(\mathbb{R})} \\ &\leq \int_0^t \|G(t-s) \star \beta_{\delta\eta}(\partial_x r_1(\cdot, s) + L_0^\eta) (r_2(\cdot, s) - r_1(\cdot, s) - L^\eta(s)) ds\|_{L^\infty(\mathbb{R})} \\ &+ \int_0^t \|G(t-s) \star \beta_{\delta\eta}(\partial_x r_2(\cdot, s) + L_0^\eta) (r_1(\cdot, s) - r_2(\cdot, s) + L^\eta(s)) ds\|_{L^\infty(\mathbb{R})} \\ &+ \int_0^t \|\partial_x G(t-s) \star \beta_{\delta\eta}(\partial_x r_1(\cdot, s) + L_0^\eta) (r_2(\cdot, s) - r_1(\cdot, s) - L^\eta(s)) ds\|_{L^\infty(\mathbb{R})} \\ &+ \int_0^t \|\partial_x G(t-s) \star \beta_{\delta\eta}(\partial_x r_2(\cdot, s) + L_0^\eta) (r_1(\cdot, s) - r_2(\cdot, s) + L^\eta(s)) ds\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

We employ Lemma 3.1 (i) to estimate the first two terms and then use Lemma 3.1 (ii) to estimate the last two terms. Then we get, for $0 < t \leq T^\eta$, the following estimate:

$$\|B(r)\|_F \leq 2 \left(T^\eta + 2(T^\eta)^{\frac{1}{2}} \right) (\delta\eta + |L_0^\eta| + \|r\|_F) (\|r\|_F + \|L^\eta\|_{L^\infty(0, T^\eta)}), \quad (3.33)$$

where $F = (L^\infty((0, T^\eta), W^{1,\infty}(\mathbb{R})))^2$.

In addition, we know by the properties of the heat kernel that

$$\|G(\cdot) \star R_{0,\eta}\|_F \leq \|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2}. \quad (3.34)$$

Step 3 (Fixed point argument):

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We choose to work with the Banach space $X = \{r \in F; \|r\|_F \leq \|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1\}$. Now, we define the mapping

$$\begin{aligned} \Lambda : X &\rightarrow X \\ r &\rightarrow \Lambda(r) = G(\cdot) \star R_{0,\eta} + B(r). \end{aligned}$$

We will show that Λ is a well-defined and contracting for T small enough. First, we will prove that Λ is well-defined.

Let $r \in X$, using (3.33) and (3.34) we get

$$\begin{aligned} &\|\Lambda(r)\|_F \\ &\leq \|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 2 \left(T^\eta + 2(T^\eta)^{\frac{1}{2}} \right) \left(\delta\eta + |L_0^\eta| + \|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1 \right) \left(\|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1 + \|L^\eta\|_{L^\infty(0,T^\eta)} \right) \\ &\leq \|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 2 \left(T^\eta + 2(T^\eta)^{\frac{1}{2}} \right) \left(\|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1 + \|L^\eta\|_{L^\infty(0,T)} + \delta\eta + |L_0^\eta| \right)^2. \end{aligned}$$

Then, taking

$$T^\eta \leq \frac{1}{\left[6 \left(\|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1 + \|L^\eta\|_{L^\infty(0,T^\eta)} + \delta\eta + |L_0^\eta| \right)^2 \right]^2} := T_1^*$$

implies that Λ is well-defined.

It remains to show that Λ is a contraction on X . Indeed, let $r, p \in X$,

$$\begin{aligned} \|\Lambda(r) - \Lambda(p)\|_F &= \|B(r) - B(p)\|_F \\ &\leq \left\| \int_0^t G(t-s) \star \begin{pmatrix} \beta_{\delta\eta}(\partial_x r_1(\cdot, s) + L_0^\eta) [(r_2(\cdot, s) - r_1(\cdot, s)) - (p_2(\cdot, s) - p_1(\cdot, s))] \\ \beta_{\delta\eta}(\partial_x r_2(\cdot, s) + L_0^\eta) [(r_1(\cdot, s) - r_2(\cdot, s)) - (p_1(\cdot, s) - p_2(\cdot, s))] \end{pmatrix} ds \right\|_F \\ &+ \left\| \int_0^t G(t-s) \star \begin{pmatrix} [\beta_{\delta\eta}(\partial_x r_1(\cdot, s) + L_0^\eta) - \beta_{\delta\eta}(\partial_x p_1(\cdot, s) + L_0^\eta)] (p_2(\cdot, s) - p_1(\cdot, s)) \\ [\beta_{\delta\eta}(\partial_x r_2(\cdot, s) + L_0^\eta) - \beta_{\delta\eta}(\partial_x p_2(\cdot, s) + L_0^\eta)] (p_1(\cdot, s) - p_2(\cdot, s)) \end{pmatrix} ds \right\|_F \\ &+ \left\| \int_0^t G(t-s) \star \begin{pmatrix} -L^\eta(s) [\beta_{\delta\eta}(\partial_x r_1(\cdot, s) + L_0^\eta) - \beta_{\delta\eta}(\partial_x p_1(\cdot, s) + L_0^\eta)] \\ L^\eta(s) [\beta_{\delta\eta}(\partial_x r_2(\cdot, s) + L_0^\eta) - \beta_{\delta\eta}(\partial_x p_2(\cdot, s) + L_0^\eta)] \end{pmatrix} ds \right\|_F \\ &\leq 2(T^\eta + 2(T^\eta)^{\frac{1}{2}})(\delta\eta + |L_0^\eta| + \|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1)\|r - p\|_F \\ &+ 2(T^\eta + 2(T^\eta)^{\frac{1}{2}})(\|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1)\|r - p\|_F \\ &+ 2(T^\eta + 2(T^\eta)^{\frac{1}{2}})\|L^\eta\|_{L^\infty(0,T^\eta)}\|r - p\|_F \\ &\leq 6(T^\eta + 2(T^\eta)^{\frac{1}{2}}) \left(\|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1 + \|L^\eta\|_{L^\infty(0,T^\eta)} + \delta\eta + |L_0^\eta| \right) \|r - p\|_F. \end{aligned}$$

Then, taking

$$T^\eta < \frac{1}{\left[18 \left(\|R_{0,\eta}\|_{(W^{1,\infty}(\mathbb{R}))^2} + 1 + \|L^\eta\|_{L^\infty(0,T^\eta)} + \delta\eta + |L_0^\eta| \right)^2 \right]^2} := T_2^*$$

implies that Λ is a contraction on the closed set X .

Hence, by fixed point theorem, there exists a unique solution R_η^δ to system (3.31) in X , for $0 < T^\eta < (T^\eta)^*$ where $(T^\eta)^* = \min(T_1^*, T_2^*) = T_2^*$. This shows that R_η^δ is a solution of system (3.30) in $\mathbb{R} \times (0, (T^\eta)^*)$ and therefore $Q_\eta^\delta(x, t) = R_\eta^\delta(\frac{x}{\eta}, \frac{t}{\eta})$ is a solution of system (3.29) in $\mathbb{R} \times (0, T^*)$ with $T^* = \eta(T^\eta)^*$.

Step 4 (Periodicity):

Let $Q_\eta^{\pm, \delta}$ be the solution of (3.29) given in the previous step. Our goal is to show that this solution is periodic, which amounts to proving that $Q_\eta^{\pm, \delta}(x+1, t) = Q_\eta^{\pm, \delta}(x, t)$, for all $(x, t) \in \mathbb{R} \times (0, T^*)$. It is easy to check that, if $\tilde{Q}_\eta^{\pm, \delta}(x, t) = Q_\eta^{\pm, \delta}(x+1, t)$, then

$$\partial_t \tilde{Q}_\eta^{\pm, \delta}(x, t) - \eta \partial_{xx}^2 \tilde{Q}_\eta^{\pm, \delta}(x, t) = \mp ((\tilde{Q}_\eta^{+, \delta} - \tilde{Q}_\eta^{-, \delta})(x, t) + L(t)) \beta_\delta (\partial_x \tilde{Q}_\eta^{\pm, \delta}(x, t) + L_0) \quad \text{in } \mathbb{R} \times (0, T^*).$$

Moreover, seen that Q_0^\pm is periodic, we have $\tilde{Q}_\eta^{\pm, \delta}(x, 0) = Q_0^\pm(x)$ and then $\tilde{Q}_\eta^{\pm, \delta}$ is a solution of (3.29). Which shows by means of the uniqueness of the fixed point our result.

Step 5 (Regularity, estimates and convergence):

Using the L^p -regularity of parabolic equations and the fact that β_δ is regular, we can show by classical bootstrap arguments that $Q_\eta^{\pm, \delta}$ belongs to $C^\infty(\mathbb{R} \times (0, T^*)) \cap L^\infty((0, T^*); W_{loc}^{1, \infty}(\mathbb{R}))$. In particular, it belongs to $C^\infty(\mathbb{T} \times (0, T^*)) \cap L^\infty((0, T^*); W^{1, \infty}(\mathbb{T}))$ by the periodicity of $Q_\eta^{\pm, \delta}$. Moreover, estimates (3.23)-(3.24) directly follow by applying the maximum principle for parabolic equations (see Liberman [57, Th 2.10]). To prove (ii), we use the compactness lemma of Simon [67, Corollary 4], that is inherited from estimates (3.23), (3.24), and the standard parabolic estimates. The last estimates (3.26), (3.27) come from (3.23), (3.24), by passing to the limit as $\delta \rightarrow 0$.

The uniqueness follows from the comparison principle. Namely, since the solution is smooth, it is easy to check that if we take two solutions of system (3.20) noted $v_\eta^{\pm, 1}, v_\eta^{\pm, 2}$ such that $v_\eta^{\pm, 1}(\cdot, 0) \leq v_\eta^{\pm, 2}(\cdot, 0)$ then, $v_\eta^{\pm, 1} \leq v_\eta^{\pm, 2}$ in $\mathbb{R} \times [0, T]$. This result will also be proved and detailed later in Section ?? in the case of continuous solutions.

Step 6 (Global existence and uniqueness):

We are going to prove that the local time solution obtained in the previous steps can be extended to global one. We argue by contradiction. Assume that there exists a maximum time T_{max} such that, we have the existence of solution of the system (3.29) in the function space $(L^\infty((0, T_{max}); W^{1, \infty}(\mathbb{T})))^2$.

For every $\mu > 0$, we consider the system (3.29) with the initial conditions

$$Q_0^{\mu, \pm}(x) = Q_\eta^{\pm, \delta}(x, T_{max} - \mu).$$

We apply for the second time the same technics of Step 3 to deduce that there exists a time T_μ^* such that system (3.29) admits a unique solution defined until the time

$$T_0 = (T_{max} - \mu) + T_\mu^*$$

with

$$T_\mu^* = \frac{\eta}{\left[18 \left(\|R_{0, \eta}^\mu\|_{(W^{1, \infty}(\mathbb{R}))^2} + 1 + \|L^\eta\|_{L^\infty(0, T^\eta)} + \delta\eta + |L_0^\eta|\right)\right]^2},$$

where $R_{0, \eta}^\mu(x) = \begin{pmatrix} R_{0, \eta}^{\mu, +}(x) \\ R_{0, \eta}^{\mu, -}(x) \end{pmatrix}$ such that $R_{0, \eta}^{\mu, \pm}(x) = Q_0^{\mu, \pm}(\eta x)$.

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According to (3.23) and (3.24), we know that $Q_0^{\mu,\pm}$ is μ -uniformly bounded in $W^{1,\infty}(\mathbb{T})$ and then there exists a constant

$$C(\delta, \eta, \|Q_0^\pm\|_{W^{1,\infty}(\mathbb{T})}, \|L\|_{L^\infty(0,T)}, |L_0|, T_{max}) > 0,$$

independent of μ such that $T_\mu^* \geq C > 0$. Passing to the limit $\mu \rightarrow 0$, we can see that $\lim_{\mu \rightarrow 0} T_\mu^* \geq C > 0$. This implies that $T_0 > T_{max}$ (for small μ). So we have contradiction and then we can construct a solution $Q_\eta^{\pm,\delta} \in X$ for all $T > 0$. \square

4 (η, δ) -Uniform *a priori* estimates on the solution of (3.22)

In this section we prove *a priori* estimates on the solution of (3.22), which will be useful for establishing the fundamental gradient entropy inequality in Section 3. We first start by obtaining a uniform estimate on $\partial_{xx}^2 v_\eta^{\pm,\delta}$.

Lemma 3.2. (*L^2 estimate on $\partial_{xx}^2 v_\eta^{\pm,\delta}$*)

Let $v_\eta^{\pm,\delta}$ be the solution of (3.22), given in Theorem 3.2 (i). Then,

$$\frac{1}{2} \int_0^t \int_{\mathbb{T}} (\partial_{xx}^2 v_\eta^{\pm,\delta}(x, s))^2 dx ds \leq N_\pm^\delta,$$

where N_\pm^δ is a constant uniformly bounded on δ , that is given by

$$N_\pm^\delta = \frac{1}{2\eta} \int_{\mathbb{T}} (\partial_x v_0^\pm(x))^2 dx + \frac{T}{2\eta^2} \left((2N_0^\delta + \|L\|_{L^\infty(0,T)}) \beta_\delta (\|\partial_x v_0^\pm\|_{L^\infty(\mathbb{T})}) \right)^2$$

with N_0^δ is defined in (3.25).

Proof of Lemma 3.2:

The proof of this lemma is very classical. In fact, differentiating the first equation in (3.22) with respect to x and then multiplying by $(\partial_x v_\eta^{\pm,\delta})$, gives

$$(\partial_{tx}^2 v_\eta^{\pm,\delta})(\partial_x v_\eta^{\pm,\delta}) - \eta(\partial_{xxx}^3 v_\eta^{\pm,\delta})(\partial_x v_\eta^{\pm,\delta}) = \mp \partial_x [(v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t)) \beta_\delta(\partial_x v_\eta^{\pm,\delta})] \partial_x v_\eta^{\pm,\delta}.$$

We integrate the previous equation with respect to x , to deduce

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}} (\partial_x v_\eta^{\pm,\delta})^2 + \eta \int_{\mathbb{T}} (\partial_{xx}^2 v_\eta^{\pm,\delta})^2 = \pm \int_{\mathbb{T}} (v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t)) \beta_\delta(\partial_x v_\eta^{\pm,\delta}) \partial_{xx}^2 v_\eta^{\pm,\delta},$$

where we have integrated by part the second and the first terms. We note that these computations (and the integration by parts) are justified, because $v_\eta^{\pm,\delta}$ is a smooth function.

Now, we integrate in time on $(0, t)$, to obtain

$$\int_0^t \int_{\mathbb{T}} (\partial_{xx}^2 v_\eta^{\pm,\delta})^2 = \frac{1}{2\eta} \int_{\mathbb{T}} (\partial_x v_0^\pm)^2 - \overbrace{\frac{1}{2\eta} \int_{\mathbb{T}} (\partial_x v_\eta^{\pm,\delta})^2}^{H_1} \pm \overbrace{\frac{1}{\eta} \int_0^t \int_{\mathbb{T}} (v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t)) \beta_\delta(\partial_x v_\eta^{\pm,\delta}) \partial_{xx}^2 v_\eta^{\pm,\delta}}^{H_2}. \quad (3.35)$$

We have

$$H_1 \leq 0. \quad (3.36)$$

In the following, we will estimate the term H_2 . Indeed, applying Young's inequality, (3.23) and (3.24) we

get

$$\begin{aligned} |H_2| &\leq \frac{1}{2\eta^2} \int_0^t \int_{\mathbb{T}} [(v_\eta^{+, \delta} - v_\eta^{-, \delta} + L(t)) \beta_\delta(\partial_x v_\eta^{\pm, \delta})]^2 + \frac{1}{2} \int_0^t \int_0^1 (\partial_{xx}^2 v_\eta^{\pm, \delta})^2 \\ &\leq \frac{T}{2\eta^2} \left((2N_0^\delta + \|L\|_{L^\infty(0, T)}) \beta_\delta(\|\partial_x v_0^\pm\|_{L^\infty(\mathbb{T})}) \right)^2 + \frac{1}{2} \int_0^t \int_{\mathbb{T}} (\partial_{xx}^2 v_\eta^{\pm, \delta})^2. \end{aligned} \quad (3.37)$$

By injecting (3.36), (3.37) in (3.35), we reach the desired result. \square

The following lemma presents a preliminary gradient entropy inequality, which is equivalent to (3.10) satisfied by $v_\eta^{\pm, \delta}$.

Lemma 3.3. (Preliminary gradient entropy inequality)

Let $v_\eta^{\pm, \delta}$ be the solution of (3.22) given in Theorem 3.2 (i). Then, there exists a constant C^δ (that is defined explicitly in (3.48)), such that, for all $t > 0$, we have

$$S^\delta(t) \leq (4T + S^\delta(0) + C^\delta) e^{\left(\frac{2}{\varepsilon} + 2\delta\pi\right)T} \quad (3.38)$$

where $S^\delta(t)$ is the following entropy

$$S^\delta(t) = \int_{\mathbb{T}} [f(\beta_\delta(\partial_x v_\eta^{+, \delta}(x, t)) + f(\beta_\delta(\partial_x v_\eta^{-, \delta}(x, t)))] dx$$

with f is defined in (3.11).

Proof of Lemma 3.3:

First, by (3.24), it is easy to check that $S^\delta(t)$ is well-defined. Now, we denote by

$$\partial_x v_\eta^{\pm, \delta} = \theta^{\pm, \delta} \quad \text{and} \quad S^{\pm, \delta}(t) = \int_{\mathbb{T}} f(\beta_\delta(\theta^{\pm, \delta})).$$

We compute, using system (3.22),

$$\begin{aligned} \frac{d}{dt} S^{\pm, \delta}(t) &= \int_{\mathbb{T}} f'(\beta_\delta(\theta^{\pm, \delta})) \beta_\delta'(\theta^{\pm, \delta}) (\partial_t \theta^{\pm, \delta}) \\ &= \int_{\mathbb{T}} f'(\beta_\delta(\theta^{\pm, \delta})) \beta_\delta'(\theta^{\pm, \delta}) \partial_x [\mp (v_\eta^{+, \delta} - v_\eta^{-, \delta} + L(t)) \beta_\delta(\theta^{\pm, \delta}) + \eta \partial_x \theta^{\pm, \delta}]. \end{aligned}$$

Integrating by part, we get

$$\begin{aligned} \frac{d}{dt} S^{\pm, \delta}(t) &= \pm \int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{\pm, \delta})) (\beta_\delta'(\theta^{\pm, \delta}))^2 + \beta_\delta''(\theta^{\pm, \delta}) f'(\beta_\delta(\theta^{\pm, \delta})) \right] \beta_\delta(\theta^{\pm, \delta}) (v_\eta^{+, \delta} - v_\eta^{-, \delta} + L(t)) \partial_x \theta^{\pm, \delta} \\ &\quad - \eta \int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{\pm, \delta})) (\beta_\delta'(\theta^{\pm, \delta}))^2 + \beta_\delta''(\theta^{\pm, \delta}) f'(\beta_\delta(\theta^{\pm, \delta})) \right] (\partial_x \theta^{\pm, \delta})^2. \end{aligned}$$

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Then, we get

$$\begin{aligned}
\frac{d}{dt}S^\delta(t) &= \frac{d}{dt}S^{\delta,+}(t) + \frac{d}{dt}S^{-,\delta}(t) \\
&= \int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{+,\delta}))(\beta'_\delta(\theta^{+,\delta}))^2 + \beta''_\delta(\theta^{+,\delta})f'(\beta_\delta(\theta^{+,\delta})) \right] \beta_\delta(\theta^{+,\delta})(v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t))\partial_x\theta^{+,\delta} \\
&\quad - \int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{-,\delta}))(\beta'_\delta(\theta^{-,\delta}))^2 + \beta''_\delta(\theta^{-,\delta})f'(\beta_\delta(\theta^{-,\delta})) \right] \beta_\delta(\theta^{-,\delta})(v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t))\partial_x\theta^{-,\delta} \\
&\quad - \eta \int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{+,\delta}))(\beta'_\delta(\theta^{+,\delta}))^2 + \beta''_\delta(\theta^{+,\delta})f'(\beta_\delta(\theta^{+,\delta})) \right] (\partial_x\theta^{+,\delta})^2 \\
&\quad - \eta \int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{-,\delta}))(\beta'_\delta(\theta^{-,\delta}))^2 + \beta''_\delta(\theta^{-,\delta})f'(\beta_\delta(\theta^{-,\delta})) \right] (\partial_x\theta^{-,\delta})^2.
\end{aligned}$$

Since the functions f, β_δ are convex and f is nondecreasing, we can see that the last two terms are negatives. Consequently,

$$\begin{aligned}
\frac{d}{dt}S^\delta(t) &\leq \overbrace{\int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{+,\delta}))(\beta'_\delta(\theta^{+,\delta}))^2 + \beta''_\delta(\theta^{+,\delta})f'(\beta_\delta(\theta^{+,\delta})) \right] \beta_\delta(\theta^{+,\delta})(v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t))\partial_x\theta^{+,\delta}}^{I_1} \\
&\quad - \overbrace{\int_{\mathbb{T}} \left[f''(\beta_\delta(\theta^{-,\delta}))(\beta'_\delta(\theta^{-,\delta}))^2 + \beta''_\delta(\theta^{-,\delta})f'(\beta_\delta(\theta^{-,\delta})) \right] \beta_\delta(\theta^{-,\delta})(v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t))\partial_x\theta^{-,\delta}}^{I_2}.
\end{aligned}$$

Step 1 (Estimate of I_1 and I_2):

Since I_1 and I_2 are similars, it is sufficient to bound I_1 . Indeed,

$$\begin{aligned}
I_1 &= \overbrace{\int_{\mathbb{T}} \left[\theta^{+,\delta} f''(\beta_\delta(\theta^{+,\delta}))\beta'_\delta(\theta^{+,\delta})\partial_x\theta^{+,\delta} \right] (v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t))}^{I_{11}} \\
&\quad + \delta^2 \overbrace{\int_{\mathbb{T}} \left[\frac{f'(\beta_\delta(\theta^{+,\delta}))}{(\beta_\delta(\theta^{+,\delta}))^2} \right] (v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t))\partial_x\theta^{+,\delta}}^{I_{12}}
\end{aligned}$$

where we have used the fact that $\beta_\delta(w)\beta'_\delta(w) = w$, in the first line, and $\beta''_\delta(w)\beta_\delta(w) = \frac{\delta^2}{(\beta_\delta(w))^2}$, in the second one. From the definitions of f and β_δ (see (3.11) and (3.21)), we can check that, for all $0 < \delta < \frac{1}{e}$, we have

$$I_{11} = \int_{\mathbb{T}} \left[g'_\delta(\theta^{+,\delta})\partial_x\theta^{+,\delta} \right] (v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t)) = \int_{\mathbb{T}} \partial_x(g_\delta(\theta^{+,\delta}))(v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t)),$$

where

$$g_\delta(x) = \begin{cases} h_\delta(x) - h_\delta(\sqrt{\frac{1}{e^2} - \delta^2}) & \text{if } x \geq \sqrt{\frac{1}{e^2} - \delta^2}, \\ 0 & \text{if } -\sqrt{\frac{1}{e^2} - \delta^2} \leq x \leq \sqrt{\frac{1}{e^2} - \delta^2}, \\ h_\delta(x) + h_\delta(\sqrt{\frac{1}{e^2} - \delta^2}) & \text{if } x \leq -\sqrt{\frac{1}{e^2} - \delta^2}, \end{cases} \quad \text{with } h_\delta(x) = x - \delta \arctan\left(\frac{x}{\delta}\right). \tag{3.39}$$

Integrating by part, we obtain

$$I_{11} = - \int_{\mathbb{T}} g_{\delta}(\theta^{+, \delta}) \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta} + L(t)).$$

Adding and subtracting $\theta^{+, \delta}$, we deduce that

$$I_{11} = - \int_{\mathbb{T}} [g_{\delta}(\theta^{+, \delta}) - \theta^{+, \delta}] \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta}) - \int_{\mathbb{T}} \theta^{+, \delta} \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta}). \quad (3.40)$$

Moreover, we know that

$$\begin{aligned} I_{12} &= \delta^2 \int_{\mathbb{T}} \left[\frac{f'(\beta_{\delta}(\theta^{+, \delta}))}{(\beta_{\delta}(\theta^{+, \delta}))^2} \right] (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta} + L(t)) \partial_x \theta^{+, \delta} \\ &= \delta^2 \int_{\mathbb{T}} \mathbb{1}_{\{\beta_{\delta}(\theta^{+, \delta}) \geq \frac{1}{\varepsilon}\}} \left[\frac{f'(\beta_{\delta}(\theta^{+, \delta}))}{(\beta_{\delta}(\theta^{+, \delta}))^2} \right] (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta} + L(t)) \partial_x \theta^{+, \delta}. \end{aligned}$$

We should estimate I_{12} . To do that, we integrate $|I_{12}|$ in time on $(0, t)$, for $0 < t < T$, then we use (3.23), (3.24) and the fact that f' is nondecreasing function, with support in $\{x \mid \beta_{\delta}(x) \geq \frac{1}{\varepsilon}\}$. We deduce that

$$\int_0^t |I_{12}| \leq (2N_0^{\delta} + \|L\|_{L^{\infty}(0, T)}) (\delta e)^2 f'(\beta_{\delta}(\|\partial_x v_0^+\|_{L^{\infty}(\mathbb{T})})) \int_0^t \int_{\mathbb{T}} \mathbb{1}_{\{\beta_{\delta}(\theta^{+, \delta}) \geq \frac{1}{\varepsilon}\}} |\partial_x \theta^{+, \delta}|$$

which imply, applying Hölder's inequality with Lemma 3.2, that

$$\int_0^t |I_{12}| \leq (2N_0^{\delta} + \|L\|_{L^{\infty}(0, T)}) (\delta e)^2 f'(\beta_{\delta}(\|\partial_x v_0^+\|_{L^{\infty}(\mathbb{T})})) (2TN_+^{\delta})^{\frac{1}{2}} := A_1^{\delta}. \quad (3.41)$$

Hence, from (3.40) and (3.41), we get

$$\begin{aligned} \int_0^t I_1 &\leq \int_0^t I_{11} + A_1^{\delta} \\ &= - \int_0^t \int_{\mathbb{T}} [g_{\delta}(\theta^{+, \delta}) - \theta^{+, \delta}] \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta}) - \int_0^t \int_{\mathbb{T}} \theta^{+, \delta} \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta}) + A_1^{\delta}. \end{aligned} \quad (3.42)$$

Similarly, proceeding as in the estimate of I_1 , we get

$$\int_0^t I_2 \leq \int_0^t \int_{\mathbb{T}} [g_{\delta}(\theta^{-, \delta}) - \theta^{-, \delta}] \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta}) + \int_0^t \int_{\mathbb{T}} \theta^{-, \delta} \partial_x (v_{\eta}^{+, \delta} - v_{\eta}^{-, \delta}) + A_2^{\delta}, \quad (3.43)$$

where

$$A_2^{\delta} := (2N_0^{\delta} + \|L\|_{L^{\infty}(0, T)}) (\delta e)^2 f'(\beta_{\delta}(\|\partial_x v_0^-\|_{L^{\infty}(\mathbb{T})})) (2TN_-^{\delta})^{\frac{1}{2}}. \quad (3.44)$$

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Step 2 (Estimate on the entropy S^δ): We recall that

$$\frac{d}{dt} S^\delta(t) \leq I_1 + I_2.$$

Integrate this inequality in time on $(0, t)$, for $0 < t < T$ and use (3.42), (3.43) we get:

$$\begin{aligned} & S^\delta(t) - S^\delta(0) \\ & \leq \int_0^t I_1 + \int_0^t I_2 \\ & \leq A_1^\delta + A_2^\delta + \int_0^t \int_{\mathbb{T}} [-(g_\delta(\theta^{+, \delta}) - \theta^{+, \delta}) + (g_\delta(\theta^{-, \delta}) - \theta^{-, \delta})] (\theta^{+, \delta} - \theta^{-, \delta}) - \int_0^t \int_{\mathbb{T}} (\theta^{+, \delta} - \theta^{-, \delta}) \partial_x (v_\eta^{+, \delta} - v_\eta^{-, \delta}) \\ & = A_1^\delta + A_2^\delta + \int_0^t \int_{\mathbb{T}} [-(g_\delta(\theta^{+, \delta}) - \theta^{+, \delta}) + (g_\delta(\theta^{-, \delta}) - \theta^{-, \delta})] (\theta^{+, \delta} - \theta^{-, \delta}) - \int_0^t \int_{\mathbb{T}} (\theta^{+, \delta} - \theta^{-, \delta})^2 \\ & \leq A_1^\delta + A_2^\delta + \int_0^t \int_{\mathbb{T}} [-(g_\delta(\theta^{+, \delta}) - \theta^{+, \delta}) + (g_\delta(\theta^{-, \delta}) - \theta^{-, \delta})] (\theta^{+, \delta} - \theta^{-, \delta}). \end{aligned}$$

By using the fact that $|g_\delta(x) - x| \leq \frac{1}{e} + \delta\pi$ for all $x \in \mathbb{R}$, we can verify that

$$S^\delta(t) - S^\delta(0) \leq A_1^\delta + A_2^\delta + 2 \left(\frac{1}{e} + \delta\pi \right) \int_0^t \int_{\mathbb{T}} (|\theta^{+, \delta}| + |\theta^{-, \delta}|). \quad (3.45)$$

Since $x \leq f(x)$ for $x \geq e$, we can show that

$$\int_{\mathbb{T}} |\theta^{\pm, \delta}| \leq \int_{\mathbb{T}} |\theta^{\pm, \delta}| \mathbb{1}_{\{\beta_\delta(\theta^{\pm, \delta}) \geq e\}} + \int_{\mathbb{T}} |\theta^{\pm, \delta}| \mathbb{1}_{\{\beta_\delta(\theta^{\pm, \delta}) \leq e\}} \leq e + \int_{\mathbb{T}} f(\beta_\delta(\theta^{\pm, \delta})). \quad (3.46)$$

From (3.45) and (3.46), we get

$$S^\delta(t) - S^\delta(0) \leq A_1^\delta + A_2^\delta + 2 \left(\frac{1}{e} + \delta\pi \right) \int_0^t \left[2e + \int_{\mathbb{T}} [f(\beta_\delta(\theta^{+, \delta})) + f(\beta_\delta(\theta^{-, \delta}))] \right].$$

Hence, we get

$$S^\delta(t) \leq S^\delta(0) + 4T + \left(A_1^\delta + A_2^\delta + 4\delta\pi eT \right) + \left(\frac{2}{e} + 2\delta\pi \right) \int_0^t S^\delta(t). \quad (3.47)$$

Finally, by applying Gronwall's Lemma to (3.47), we get

$$S^\delta(t) \leq (S^\delta(0) + 4T + C^\delta) e^{\left(\frac{2}{e} + 2\delta\pi\right)T}$$

with

$$C^\delta = A_1^\delta + A_2^\delta + 4\delta\pi eT, \quad (3.48)$$

where A_1^δ and A_2^δ are defined in (3.41) and (3.44) respectively.

Which gives the announced result. \square

Ending this subsection, by the following lemma which gives a sufficient time regularity, allowing the passage to limit by compactness.

Lemma 3.4. (Estimate on the time derivation of the solution uniform on η and δ)
 Assume that (3.19) holds, and let $W^{-1,\infty}(\mathbb{T})$ be the dual space of $W^{1,1}(\mathbb{T})$. Then, for all $0 < \eta < 1$ and $T > 0$, the solution $v_\eta^{\pm,\delta}$ of (3.22), given by Theorem 3.2 (i), satisfies the following estimate:

$$\left\| \partial_t v_\eta^{\pm,\delta} \right\|_{L^\infty((0,T);W^{-1,\infty}(\mathbb{T}))} \leq B_0 + (2N_0^\delta + \|L\|_{L^\infty(0,T)})\beta_\delta(B_0) \quad (3.49)$$

where $B_0 = \max_{\pm} \left(\left\| \partial_x v_0^\pm \right\|_{L^\infty(\mathbb{T})} \right)$ and N_0^δ is defined in (3.25).

Proof of Lemma 3.4:

The proof is given by duality. Multiplying the equation (3.22) by $\psi \in L^1((0,T);W^{1,1}(\mathbb{T}))$ and integrating on $\mathbb{T} \times (0,T)$, we get

$$\int_{\mathbb{T} \times (0,T)} \psi \partial_t v_\eta^{\pm,\delta} dx dt = \overbrace{\eta \int_{\mathbb{T} \times (0,T)} \psi \partial_{xx}^2 v_\eta^{\pm,\delta} dx dt}^{J_1} \mp \overbrace{\int_{\mathbb{T} \times (0,T)} \psi (v_\eta^{+,\delta} - v_\eta^{-,\delta} + L(t)) \beta_\delta(\partial_x v_\eta^{\pm,\delta}) dx dt}^{J_2}.$$

Integrating by parts in J_1 , for all $0 < \eta \leq 1$, we obtain

$$\begin{aligned} |J_1| &\leq \left| \int_{\mathbb{T} \times (0,T)} \partial_x \psi \partial_x v_\eta^{\pm,\delta} \right| \leq \|\partial_x \psi\|_{L^1(\mathbb{T} \times (0,T))} \|\partial_x v_\eta^{\pm,\delta}\|_{L^\infty(\mathbb{T} \times (0,T))}, \\ &\leq \|\psi\|_{L^1(0,T);W^{1,1}(\mathbb{T})} B_0 \end{aligned} \quad (3.50)$$

where we have used inequality (3.24) in the second line. Similarly, using (3.23), we have

$$\begin{aligned} |J_2| &\leq \|\psi\|_{L^1(\mathbb{R} \times (0,T))} \|(Q_\eta^{+,\delta} - Q_\eta^{-,\delta}) + L(\cdot)\|_{L^\infty(\mathbb{T} \times (0,T))} \|\beta_\delta(\partial_x v_\eta^{\pm,\delta})\|_{L^\infty(\mathbb{R} \times (0,T))}, \\ &\leq \|\psi\|_{L^1((0,T);W^{1,1}(\mathbb{T}))} (2N_0^\delta + \|L\|_{L^\infty(0,T)}) \beta_\delta(B_0). \end{aligned} \quad (3.51)$$

Collecting (3.50) and (3.51), we get

$$\int_{\mathbb{T} \times (0,T)} \psi \partial_t v_\eta^{\pm,\delta} \leq \left(B_0 + (2N_0^\delta + \|L\|_{L^\infty(0,T)}) \beta_\delta(B_0) \right) \|\psi\|_{L^1((0,T);W^{1,1}(\mathbb{T}))}.$$

Then, by duality, we deduce the announced result. \square

5 η -Uniform *a priori* estimates on the solution of (3.20)

In this section, we show, by passing to the limit as $\delta \rightarrow 0$, two η -uniform estimates (similar to that obtained in Section 4) on the solution v_η^\pm of system (3.20) obtained in Theorem 3.2 (ii).

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The first one concerns the $L \log L$ inequality, which is the key result.

Lemma 3.5. *For any $0 < \eta \leq 1$, we consider the solution v_η^\pm of (3.20), given by Theorem 3.2 (ii). Then, for any $T > 0$, v_η^\pm satisfies estimates (3.26), (3.27) and there exists a constant $C(T, \|\partial_x v_0^\pm\|_{L \log L(\mathbb{T})})$ (that is defined explicitly in (3.55)), such that*

$$S(t) \leq C, \quad \text{with } S(t) = \int_{\mathbb{T}} \left[f(|\partial_x v_\eta^+(x, t)|) + f(|\partial_x v_\eta^-(x, t)|) \right] dx \quad (3.52)$$

where f is defined in (3.11).

For the proof of Lemma 3.5 we need the following technical lemma, which allows us to control $\|f(\cdot)\|_{L^1(\mathbb{T})}$ by $\|\cdot\|_{L \log L(\mathbb{T})}$.

Lemma 3.6. (Bound of L^1 norm of f by the $L \log L$ norm)

Let f be the function defined in (3.11) and h be a nonnegative function.

If $h \in L \log L(\mathbb{T})$, then $\int_{\mathbb{T}} f(h) < +\infty$ and moreover we have the following estimate:

$$\int_{\mathbb{T}} f(h) \leq 1 + \|h\|_{L \log L(\mathbb{T})} + \frac{2}{\ln 2} \|h\|_{L \log L(\mathbb{T})} \ln \left(1 + \|h\|_{L \log L(\mathbb{T})} \right). \quad (3.53)$$

Proof of Lemma 3.6:

First, we remark that for all $h \geq 0$ and $\mu \in (0, 1]$, we have

$$\left(h \ln(h) + \frac{1}{e} \right) \mathbb{1}_{\{h \geq \frac{1}{e}\}} \leq h \ln(h + e) \leq h \ln(e + \mu h) + |\ln(\mu)| h.$$

Applying this inequality with $\mu = \frac{1}{\max(1, \|h\|_{L \log L(\mathbb{T})})}$ and integrating, we get

$$\begin{aligned} \int_{\mathbb{T}} f(h) &\leq \frac{1}{\mu} \int_{\mathbb{T}} \mu h \ln(e + \mu h) + |\ln(\mu)| \int_{\mathbb{T}} h \\ &\leq \frac{1}{\mu} \int_{\mathbb{T}} \mu h \ln(e + \mu h) + \frac{2}{\ln 2} |\ln(\mu)| \|h\|_{L \log L(\mathbb{T})} \\ &\leq \frac{1}{\mu} \int_{\mathbb{T}} \frac{h}{\|h\|_{L \log L(\mathbb{T})}} \ln \left(e + \frac{h}{\|h\|_{L \log L(\mathbb{T})}} \right) + \frac{2}{\ln 2} |\ln(\mu)| \|h\|_{L \log L(\mathbb{T})} \\ &\leq \frac{1}{\mu} + \frac{2}{\ln 2} |\ln(\mu)| \|h\|_{L \log L(\mathbb{T})}, \end{aligned}$$

where we have used the Generalized Hölder inequality (see Adams [4], (8.11), Page 234) in the second line and the definition of $\|h\|_{L \log L(\mathbb{T})}$ in the last one. This gives (3.53) using the fact that

$$\mu \geq \frac{1}{1 + \|h\|_{L \log L(\mathbb{T})}}.$$

□

Proof of Lemma 3.5:

Due to the uniqueness, the announced result will be proven by passing to the limit as δ goes to 0, in the approximated gradient entropy inequality (3.38).

Since, for all $0 < \delta < \frac{1}{e}$, the constant N_0^δ is uniformly bounded with respect to δ and $\theta^{\pm, \delta} = \partial_x v_\eta^{\pm, \delta}$ converges locally uniformly, as δ goes to 0, toward $\partial_x v_\eta^\pm$, we can pass to the limit in (3.38), as δ goes to 0, to obtain

$$\int_{\mathbb{T}} \left[f(|\partial_x v_\eta^+|) + f(|\partial_x v_\eta^-|) \right] dx \leq (4T + S(0)) e^{\frac{2T}{e}}, \quad (3.54)$$

where $S(0) = \int_{\mathbb{T}} \left[f(|\partial_x v_0^+|) + f(|\partial_x v_0^-|) \right]$.

The last estimate combined with (3.53), show that

$$S(t) \leq C,$$

with

$$C = \left[2 + 4T + \sum_{\pm} \|\partial_x v_0^\pm\|_{L \log L(\mathbb{T})} \left(1 + \frac{2}{\ln 2} \ln(1 + \|\partial_x v_0^\pm\|_{L \log L(\mathbb{T})}) \right) \right] e^{\frac{2T}{e}}, \quad (3.55)$$

which gives the stated result. □

The following estimate will provide the compactness in time uniformly with respect to η of the solution v_η^\pm , given in Theorem 3.2 (ii).

Lemma 3.7. (Estimate on the time derivative of the solution uniform on η)

Assume that (3.19) holds, and let $W^{-1, \infty}(\mathbb{T})$ be the dual space of $W^{1, 1}(\mathbb{T})$. Then, for all $0 < \eta \leq 1$ and $T > 0$, the solution v_η^\pm of (3.20), given by Theorem 3.2 (ii), satisfies the following estimate:

$$\|\partial_t v_\eta^\pm\|_{L^\infty((0, T); W^{-1, \infty}(\mathbb{T}))} \leq \widetilde{M}, \quad (3.56)$$

where \widetilde{M} depends only on T , L_0 , $\|\partial_x v_0^\pm\|_{L^\infty(\mathbb{T})}$, $\|Q_0^\pm\|_{L^\infty(\mathbb{T})}$ and $\|L\|_{L^\infty(0, T)}$.

Proof of Lemma 3.7:

As done before, thanks to the uniqueness of the solution, we prove this result by passing to the limit as δ goes to 0 in (3.49). Indeed, from (3.49) we know that

$$\|\partial_t v_\eta^{\pm, \delta}\|_{L^\infty((0, T); W^{-1, \infty}(\mathbb{T}))} \leq B_0 + (2N_0^\delta + \|L\|_{L^\infty(0, T)}) \beta_\delta(B_0).$$

Passing to the limit inferior in the previous inequality, as δ goes to 0, and using the lower semi-continuous of weakly- \star topology in $L^\infty((0, T); W^{-1, \infty}(\mathbb{T}))$, we get

$$\|\partial_t v_\eta^\pm\|_{L^\infty((0, T); W^{-1, \infty}(\mathbb{T}))} \leq \liminf_{\delta \rightarrow 0} \|\partial_t v_\eta^{\pm, \delta}\|_{L^\infty((0, T); W^{-1, \infty}(\mathbb{T}))} \leq \widetilde{M},$$

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where

$$\widetilde{M} = \lim_{\delta \rightarrow 0} \left[B_0 + (2N_0^\delta + \|L\|_{L^\infty(0,T)})\beta_\delta(B_0) \right] = B_0 + (2N_0 + \|L\|_{L^\infty(0,T)})B_0,$$

and N_0 is defined in (3.28). This terminates the proof. \square

Remark 3.4. *Thanks to the periodicity of the solution, it is possible to prove that the estimates announced in Lemmas 3.5 and 3.7 also remain valid on every interval of length 1. Consequently, we can obtain the same η -uniform estimates locally on any interval.*

6 Global solution for eikonal system with regular initial data

In this section, we state a global in time existence and uniqueness result for the following eikonal system:

$$\begin{cases} \partial_t v^\pm(x, t) = \mp ((v^+ - v^-)(x, t) + L(t)) |\partial_x v^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ v^\pm(x, 0) = v_0^\pm(x) = Q_0^\pm(x) + L_0 x & \text{in } \mathbb{R}. \end{cases} \quad (3.57)$$

We prove that this system admits a unique Lipschitz continuous viscosity solution v^\pm , when subjected to Lipschitz continuous initial data. Moreover, we show that v^\pm is the limit of v_η^\pm when $\eta \rightarrow 0$, where v_η^\pm is the solution of (3.20) given by Theorem 3.2 (ii). We also show that v^\pm satisfies the Lipschitz estimates (3.26), (3.27) and the gradient entropy estimate (3.52).

Before showing the main theorem of this section, let us recall the definition of the continuous viscosity solution, for system, that was proposed by Ishii and Koike in [48].

Definition 3.1. (Continuous viscosity sub-solution, super-solution and solution)

For $k \in \{+, -\}$, let $H_k : (0, T) \times \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous Hamiltonian.

A function $w = (w^+, w^-) \in (C(\mathbb{R} \times [0, T]))^2$ is a viscosity sub-solution of the following system:

$$\begin{cases} \partial_t w^k(x, t) + H_k(t, w(x, t), \partial_x w^k(x, t)) - \eta \partial_{xx}^2 w^k(x, t) = 0 & \text{in } \mathbb{R} \times (0, T), \\ w^k(x, 0) = w_0^k(x) & \text{in } \mathbb{R}, \end{cases} \quad (3.58)$$

if and only if

- $w^\pm(x, 0) \leq w_0^\pm(x)$,
- for all $k \in \{+, -\}$ and for any test-function $\phi \in C^2(\mathbb{R} \times (0, T))$ such that $w^k - \phi$ reaches a local maximum at a point $(x_0, t_0) \in \mathbb{R} \times (0, T)$, we have

$$\partial_t \phi(x_0, t_0) + H_k(t_0, w(x_0, t_0), \partial_x \phi(x_0, t_0)) - \eta \partial_{xx}^2 \phi(x_0, t_0) \leq 0.$$

In a similar way, a function $w = (w^+, w^-) \in (C(\mathbb{R} \times [0, T]))^2$ is a viscosity super-solution of (3.58) if and only if

- $w^\pm(x, 0) \geq w_0^\pm(x)$,

- for all $k \in \{+, -\}$ and for any test-function $\phi \in C^2(\mathbb{R} \times (0, T))$ such that $w^k - \phi$ reaches a local minimum at point $(x_0, t_0) \in \mathbb{R} \times (0, T)$, we have

$$\partial_t \phi(x_0, t_0) + H_k(t_0, w(x_0, t_0), \partial_x \phi(x_0, t_0)) - \eta \partial_{xx}^2 \phi(x_0, t_0) \geq 0.$$

Finally, a continuous function w^\pm is a viscosity solution of (3.58) if and only if it is a sub- and a super-solution of (3.58).

We have the following existence and uniqueness result for the local system (3.57).

Theorem 3.3. (Existence and uniqueness of Lipschitz continuous viscosity solution)

Assume that (3.19) holds. Then, for all $T > 0$, we have:

- i) Problem (3.57) admits a unique Lipschitz continuous viscosity solution v^\pm , such that

$$v^\pm(x, t) = Q^\pm(x, t) + L_0 x, \quad (3.59)$$

where Q^\pm are 1-periodic functions (with respect to the space), belonging to $W^{1,\infty}(\mathbb{R} \times (0, T))$ and satisfying the following estimates:

$$\max_{\pm} (\|Q^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq N_0, \quad (3.60)$$

$$\max_{\pm} (\|\partial_x v^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \max_{\pm} (\|\partial_x v_0^\pm\|_{L^\infty(\mathbb{T})}), \quad (3.61)$$

$$\max_{\pm} (\|\partial_t v^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq (2N_0 + \|L\|_{L^\infty(\mathbb{T})}) \max_{\pm} (\|\partial_x v_0^\pm\|_{L^\infty(\mathbb{T})}), \quad (3.62)$$

where N_0 is defined in (3.28).

- ii) The solution v^\pm satisfies the gradient entropy estimate

$$\max_{\pm} (\|\partial_x v^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))} + \|\partial_t v^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))}) \leq \tilde{C}, \quad (3.63)$$

where $\tilde{C}(T, \|L\|_{L^\infty(0, T)}, L_0, \|Q_0^\pm\|_{L^\infty(\mathbb{T})}, \|\partial_x v_0^\pm\|_{L \log L(\mathbb{T})})$.

For the sake of a clear presentation, we divide this section in two subsections. In the first subsection, we show some preliminary results that we need in the next subsection to prove the previous theorem.

6.1 Some useful results

First, we recall the following stability lemma for continuous viscosity solutions.

Lemma 3.8. (Stability of continuous viscosity solutions, Barles, [11, Th 2.3])

Let v_η^\pm be the solution of (3.20), constructed in Theorem 3.2 (ii) as the limit of $v_\eta^{\pm, \delta}$ when $\delta \rightarrow 0$. Then we have:

- i) The function v_η^\pm is a continuous viscosity solution of (3.20).
 ii) If v_η^\pm converges locally uniformly, as η goes to 0, to v^\pm , then v^\pm is a Lipschitz continuous viscosity solution of (3.57).

To prove this lemma, it is sufficient to apply the stability result for continuous viscosity solutions (see Barles [11, Th 2.3]), and the fact that all classical C^2 solutions of (3.22) are also viscosity solutions.

We recall the following compactness lemma.

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Lemma 3.9. (*Simon's Lemma [67, Corollary 4]*)

Let X , B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a uniformly bounded sequence in $L^\infty((0, T); X)$ and $(\partial_t \theta_n)_n$ is uniformly bounded in $L^r((0, T); Y)$ for $r > 1$, then $(\theta_n)_n$ is relatively compact in $C((0, T); B)$.

Finally, we also need to prove the following technical lemma, that allows us to bound $\|\cdot\|_{L \log L(\mathbb{T})}$ by $\|f(\cdot)\|_{L^1(\mathbb{T})}$.

Lemma 3.10. (*$L \log L$ estimate*)

Let f be the function defined in (3.11) and h be a nonnegative function.

If $\int_{\mathbb{T}} f(h) < +\infty$, then $h \in L \log L(\mathbb{T})$. Moreover, we have the following estimate:

$$\|h\|_{L \log L(\mathbb{T})} \leq 1 + e \ln(2e) + (1 + \ln(2)) \int_{\mathbb{T}} f(h). \quad (3.64)$$

Proof of Lemma 3.10:

First, we remark, using the definition of $\|\cdot\|_{L \log L(\mathbb{T})}$ (see (3.6)), that $\|h\|_{L \log L(\mathbb{T})} \leq 1 + \int_{\mathbb{T}} h \ln(e + h)$.

Then, we have

$$\begin{aligned} 1 + \int_{\mathbb{T}} h \ln(e + h) &\leq 1 + \int_{\mathbb{T} \cap \{h \leq e\}} h \ln(e + h) + \int_{\mathbb{T} \cap \{h \geq e\}} h \ln(e + h) \leq 1 + e \ln(2e) + \int_{\mathbb{T} \cap \{h \geq e\}} h \ln(2h) \\ &\leq 1 + e \ln(2e) + (1 + \ln(2)) \int_{\mathbb{T} \cap \{h \geq e\}} h \ln(h) \leq 1 + e \ln(2e) + (1 + \ln(2)) \int_{\mathbb{T}} f(h) \end{aligned}$$

where we have used the fact that for $h \geq e$, $h \leq h \ln(h)$. Which gives (3.64). \square

6.2 Proof of Theorem 3.3

Proof of i):

Let v_η^\pm be the solution of (3.20) constructed in Theorem 3.2 (ii). First, by Lemma 3.8 (i), we know that v_η^\pm is a continuous viscosity solution of (3.20). Let K be a compact subset in \mathbb{R} . Using estimates (3.26), (3.27), (3.56) and applying Lemma 3.9 in the particular case where $X = W^{1,\infty}(K)$, $B = L^\infty(K)$ and $Y = W^{-1,\infty}(K) = (W^{1,1}(K))'$, we can show according to Remark 3.4 that v_η^\pm is relatively compact in $L^\infty(K \times (0, T))$. Hence, as $\eta \rightarrow 0$, we can extract a subsequence, still denoted by v_η^\pm , that converges locally uniformly to some limit v^\pm . Then, by Lemma 3.8 (ii), we can see that v^\pm is a Lipschitz continuous viscosity solution of (3.57), and consequently the existence follows.

Finally, using the lower semi-continuity of the L^∞ norm with respect to the weakly- \star topology, we can pass to the limit inferior as $\eta \rightarrow 0$ in estimates (3.26) and (3.27) in order to get (3.60) and (3.61). Estimate (3.62) follows from the fact that v^\pm satisfies (3.57) almost everywhere knowing that it is a Lipschitz continuous viscosity solution of (3.57) on $\mathbb{R} \times (0, T)$.

Proof of ii):

From Lemma 3.5, we deduce that $S(t) \leq C$ where C is given by (3.55). This implies, using estimate

(3.64), that

$$\begin{aligned} \|\partial_x v_\eta^\pm\|_{L^\infty((0,T);L \log L(\mathbb{T}))} &\leq 1 + e \ln(2e) + (1 + \ln(2)) \sup_{0 \leq t \leq T} \int_{\mathbb{T}} f(|\partial_x v_\eta^\pm|) \\ &\leq 1 + e \ln(2e) + (1 + \ln(2))C := \tilde{C}_1. \end{aligned}$$

By weakly- \star convergence and from the fact that $L^\infty((0, T); L \log L(\mathbb{T}))$ is the dual of $L^1((0, T); E_{exp}(\mathbb{T}))$ (see Adams [4] for the definition of the Banach space $E_{exp}(\mathbb{T})$), we can check that we get the following estimate:

$$\|\partial_x v^\pm\|_{L^\infty((0,T);L \log L(\mathbb{T}))} \leq \liminf_{\eta \rightarrow 0} \|\partial_x v_\eta^\pm\|_{L^\infty((0,T);L \log L(\mathbb{T}))} \leq \tilde{C}_1. \quad (3.65)$$

Moreover, knowing that the Lipschitz continuous viscosity solution v^\pm satisfies equation (3.57) almost everywhere and by using Hölder's inequality and estimate (3.60), we obtain

$$\|\partial_t v^\pm\|_{L^\infty((0,T);L \log L(\mathbb{T}))} \leq (2N_0 + \|L\|_{L^\infty(0,T)})\tilde{C}_1. \quad (3.66)$$

Adding this to (3.65), we get (3.63) with $\tilde{C} = \tilde{C}_1 + (2N_0 + \|L\|_{L^\infty(0,T)})\tilde{C}_1$.

The uniqueness here derives from a comparison principle result, proved and detailed in Section 8 in the case of continuous solution. \square

7 Existence of continuous viscosity solution

In this section we prove the existence result announced in Theorem 3.1 (i). We proceed in two stages, first we show the existence in the local case and then we treat the non-local case.

7.1 Existence of a continuous viscosity solution for the local problem ($\alpha = 0$)

In this subsection, we prove the global existence of a Lipschitz continuous viscosity solution of the following system:

$$\begin{cases} \partial_t \rho^\pm(x, t) = \mp ((\rho^+ - \rho^-)(x, t) + \bar{L}(t)) |\partial_x \rho^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \rho^\pm(x, 0) = \rho_0^\pm(x) = P_0^\pm(x) + L_0 x & \text{in } \mathbb{R}, \end{cases} \quad (3.67)$$

where P_0^\pm is 1-periodic function satisfying (3.4) and

$$\bar{L} \in C(0, T). \quad (3.68)$$

More precisely, we will show that the following theorem holds:

Theorem 3.4. *Assume (3.4) and (3.68). Then, for all $T > 0$, we have:*

i) *Problem (3.67) admits a continuous viscosity solution ρ^\pm , such that*

$$\rho^\pm(x, t) = P^\pm(x, t) + L_0 x,$$

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where P^\pm are 1-periodic functions (with respect to the space), belonging to $C(\mathbb{R} \times (0, T))$ and satisfying the following estimates:

$$\max_{\pm} (\|P^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \overline{M}_0, \quad (3.69)$$

where

$$\overline{M}_0 = \max_{\pm} (\|P_0^\pm\|_{L^\infty(\mathbb{T})}) + |L_0| \|\overline{L}\|_{L^\infty(0, T)} T, \quad (3.70)$$

and

$$\max_{\pm} (\|\partial_x \rho^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))} + \|\partial_t \rho^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))}) \leq \overline{C}_1. \quad (3.71)$$

where \overline{C}_1 is defined explicitly in (3.78).

First, we recall the following results.

Lemma 3.11. (Compact embedding)

Let I be an open bounded interval of \mathbb{R} and $L \log L(I)$ be the Zygmund space. If we denote by

$$W^{1, L \log L}(I) = \{h \in L^1(I) \text{ such that } \partial_x h \in L \log L(I)\},$$

then the following injection:

$$W^{1, L \log L}(I) \hookrightarrow C(I),$$

is compact.

For the proof of this Lemma see R. A. Adams [4, Th 8.32].

Lemma 3.12. Let $(\eta_\epsilon)_{\epsilon \in (0, 1]}$ be the standard mollifier satisfying $\eta_\epsilon(\cdot) = \frac{1}{\epsilon} \eta(\frac{\cdot}{\epsilon})$, such that, $\eta \in C_c^\infty(\mathbb{R})$, $\text{supp}\{\eta\} \subseteq B(0, 1)$, $\eta \geq 0$ and $\int_{\mathbb{R}} \eta = 1$. If $h \in L \log L(\mathbb{T})$, then for every $\epsilon \in (0, 1]$ the function $h_\epsilon = h * \eta_\epsilon \in L \log L(\mathbb{T})$ and satisfies

$$\|h_\epsilon\|_{L \log L(\mathbb{T})} \leq \mathcal{C} \|h\|_{L \log L(\mathbb{T})} \quad \text{and} \quad \|h - h_\epsilon\|_{L \log L(\mathbb{T})} \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0,$$

where \mathcal{C} is a universal constant.

For the proof of this lemma see Adams [4, Th 8.20].

Lemma 3.13. (Continuity)

Let $T > 0$. Assume that $h \in L^\infty(\mathbb{T} \times (0, T))$ such that

$$\|\partial_x h\|_{L^\infty((0, T); L \log L(\mathbb{T}))} + \|\partial_t h\|_{L^\infty((0, T); L \log L(\mathbb{T}))} \leq \Lambda.$$

Then, for all $\delta, \alpha \geq 0$ and all $(x, t) \in \mathbb{T} \times (0, T - \delta)$, we have:

$$|h(x + \alpha, t + \delta) - h(x, t)| \leq 6\Lambda \left(\frac{1}{\ln(\frac{1}{\alpha} + 1)} + \frac{1}{\ln(\frac{1}{\delta} + 1)} \right).$$

For the proof of this lemma see [34, Lemma 4.3].

Proof of Theorem 3.4:

To simplify the presentation, we make the proof in two steps.

Step 1 (Approximate problem):

We start by regularization of the function \overline{L} , and the initial data ρ_0^\pm . Here, the approximation brings us

to consider, for every $\epsilon > 0$, the following equation:

$$\begin{cases} \partial_t \rho_\epsilon^\pm(x, t) = \mp ((\rho_\epsilon^+ - \rho_\epsilon^-)(x, t) + L_\epsilon(t)) |\partial_x \rho_\epsilon^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \rho_\epsilon^\pm(x, 0) = \rho_{0,\epsilon}^\pm(x) = P_{0,\epsilon}^\pm(x) + L_0 x & \text{in } \mathbb{R}, \end{cases} \quad (3.72)$$

where

$$P_{0,\epsilon}^\pm(x) = P_0^\pm \star \eta_\epsilon(x) \quad \text{and} \quad L_\epsilon(t) = \hat{L} \star \eta_\epsilon(t) \quad \forall x \in \mathbb{R}, t \in \mathbb{R}. \quad (3.73)$$

Here, the function \hat{L} is the extension by 0 of the function \bar{L} over \mathbb{R} , and η_ϵ is the standard mollifier defined as follows:

$$\eta_\epsilon(\cdot) = \frac{1}{\epsilon} \eta\left(\frac{\cdot}{\epsilon}\right), \quad \text{such that } \eta \in C_c^\infty(\mathbb{R}), \text{ supp}\{\eta\} \subseteq (-1, 1), \eta \geq 0, \text{ and } \int_{\mathbb{R}} \eta = 1.$$

By assumptions (3.4), (3.68), and by classical properties of the mollifiers, we can see that, for all $\epsilon > 0$, the functions $P_{0,\epsilon}^\pm$ and L_ϵ satisfy assumption (3.19). Thus, we can apply Theorem 3.3-(i) with $L = L_\epsilon$ and $v_0^\pm = \rho_{0,\epsilon}^\pm$ in order to prove that equation (3.72) admits a unique Lipschitz continuous viscosity solution

$$\rho_\epsilon^\pm(x, t) = P_\epsilon^\pm(x, t) + L_0 x, \quad (3.74)$$

where P_ϵ^\pm are 1-periodic functions (with respect to the space). By classical properties of mollifiers, we know that

$$\begin{aligned} \|P_{0,\epsilon}^\pm\|_{L^\infty(\mathbb{T})} &\leq \|P_0^\pm\|_{L^\infty(\mathbb{T})}, \\ \|L_\epsilon\|_{L^\infty(0,T)} &\leq \|\bar{L}\|_{L^\infty(0,T)}. \end{aligned}$$

Then, for all $T > 0$, we have the following uniform *a priori* estimates:

$$\begin{aligned} \max_{\pm} (\|P_\epsilon^\pm\|_{L^\infty(\mathbb{T} \times (0,T))}) &\leq \max_{\pm} (\|P_{0,\epsilon}^\pm\|_{L^\infty(\mathbb{T})}) + |L_0| \|L_\epsilon\|_{L^\infty(0,T)} T \\ &\leq \bar{M}_0, \end{aligned} \quad (3.75)$$

where \bar{M}_0 is given by (3.70).

We also have the following estimate (depending on ϵ):

$$\max_{\pm} \left(\|\partial_x \rho_\epsilon^\pm\|_{L^\infty(\mathbb{T} \times (0,T))} \right) \leq \max_{\pm} \left(\|\partial_x \rho_{0,\epsilon}^\pm\|_{L^\infty(\mathbb{T})} \right). \quad (3.76)$$

Moreover, from Theorem 3.3-(ii), we know that

$$\max_{\pm} \left(\|\partial_x \rho_\epsilon^\pm\|_{L^\infty((0,T); L \log L(\mathbb{T}))} + \|\partial_t \rho_\epsilon^\pm\|_{L^\infty((0,T); L \log L(\mathbb{T}))} \right) \leq \tilde{C}_\epsilon,$$

where

$$\tilde{C}_\epsilon = (1 + 2\bar{M}_0 + \|\bar{L}\|_{L^\infty(0,T)}) (1 + e \ln(2e)) + (1 + \ln(2)) C_\epsilon,$$

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with

$$C_\varepsilon = \left[2 + 4T + \sum_{\pm} \|\partial_x \rho_{0,\varepsilon}^\pm\|_{L \log L(\mathbb{T})} \left(1 + \frac{2}{\ln 2} \ln(1 + \|\partial_x \rho_{0,\varepsilon}^\pm\|_{L \log L(\mathbb{T})}) \right) \right] e^{\frac{2T}{\varepsilon}}.$$

This shows, using Lemma 3.12 and the classical properties of the mollifiers, that there exists a constant \bar{C}_1 which depends only on T , $\|\bar{L}\|_{L^\infty(0,T)}$, \bar{M}_0 and $\|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})}$, such that

$$\max_{\pm} \left(\|\partial_x \rho_\varepsilon^\pm\|_{L^\infty((0,T);L \log L(\mathbb{T}))} + \|\partial_t \rho_\varepsilon^\pm\|_{L^\infty((0,T);L \log L(\mathbb{T}))} \right) \leq \bar{C}_1, \quad (3.77)$$

where

$$\bar{C}_1 = (1 + 2\bar{M}_0 + \|\bar{L}\|_{L^\infty(0,T)}) (1 + e \ln(2e) + (1 + \ln(2))\bar{C}), \quad (3.78)$$

with

$$\bar{C} = \left[2 + 4T + \sum_{\pm} C \|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})} \left(1 + \frac{2}{\ln 2} \ln(1 + C \|\partial_x \rho_0^\pm\|_{L \log L(\mathbb{T})}) \right) \right] e^{\frac{2T}{\varepsilon}}.$$

Step 2 (Passage to the limit):

First, we remark that, from estimate (3.75), the solution $\rho_\varepsilon^\pm(x, t) = P_\varepsilon^\pm(x, t) + L_0 x$ of (3.72) obtained in the previous step, is ε -uniformly bounded in $L^\infty(\mathbb{T} \times (0, T))$, for any $T > 0$. Hence, as ε goes to zero, we can extract a subsequence still denoted by ρ_ε^\pm , that converges weakly- \star in $L^\infty(\mathbb{T} \times (0, T))$ to some limit $\rho^\pm(x, t) = P^\pm(x, t) + L_0 x$.

From the compactness of $W^{1, L \log L}(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ (cf. Lemma 3.11), we can apply Simon's Lemma (i.e. Lemma 3.9), with $X = W^{1, L \log L}(\mathbb{T})$, $B = L^\infty(\mathbb{T})$ and $Y = L \log L(\mathbb{T})$, which shows in particular that we can extract a subsequence, still denoted by $(P_\varepsilon^\pm)_\varepsilon$, satisfying the following convergences:

$$\left\{ \begin{array}{ll} P_\varepsilon^\pm \longrightarrow P^\pm, & \text{strongly in } L^\infty(\mathbb{T} \times (0, T)), \\ P_\varepsilon^\pm \longrightarrow P^\pm, & \text{weakly-} \star \text{ in } L^\infty(\mathbb{T} \times (0, T)), \\ \partial_x P_\varepsilon^\pm \longrightarrow \partial_x P^\pm, & \text{weakly-} \star \text{ in } L^\infty((0, T); L \log L(\mathbb{T})), \\ \partial_t P_\varepsilon^\pm \longrightarrow \partial_t P^\pm, & \text{weakly-} \star \text{ in } L^\infty((0, T); L \log L(\mathbb{T})). \end{array} \right. \quad (3.79)$$

Taking the \liminf in estimates (3.75), (3.77) and using the lower semi-continuity of $\|\cdot\|_{L^\infty(\mathbb{T} \times (0, T))}$ and $\|\cdot\|_{L^\infty((0, T); L \log L(\mathbb{T}))}$ with respect to weakly- \star topology, we can prove that ρ^\pm satisfies (3.8) and (3.9) (replacing the function $a(\cdot)$ by $\bar{L}(\cdot)$).

The Theorem 3.1 (i) and the Corollary 3.1 are proved in the particular case where $\alpha = 0$. More precisely, from the uniform local convergence of ρ_ε^\pm to ρ^\pm , we can apply the stability of continuous viscosity solutions, which implies that ρ^\pm is continuous viscosity solution of the local problem in the sense of Definition 3.1. The proof of the continuity of the solution comes directly from the gradient entropy estimate (3.9) using Lemma 3.13. \square

7.2 Existence of continuous viscosity solution for the non-local problem ($\alpha \neq 0$)

In this subsection, we study the non-local system (3.1) describing the dynamics of dislocation densities. This is a simplified 1D model derived from the original one of Groma and Balogh [40], initially introduced in 2D setting (see El Hajj and Forcadel [33, Lemme 3.1] for modeling details). Indeed, we consider the following non-local eikonal system:

$$\begin{cases} \partial_t \rho^\pm(x, t) = \mp \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) |\partial_x \rho^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \rho^\pm(x, 0) = \rho_0^\pm(x) = P_0^\pm(x) + L_0 x & \text{in } \mathbb{R}, \end{cases} \quad (3.80)$$

where P_0^\pm are 1-periodic functions, L_0 and α are given positive constants. We will prove a global existence result of this system with continuous initial data without monotony assumptions. To do this, we divide this subsection into two parts. In the first one, after regularizing the initial data and the function $a(\cdot)$, we prove the existence of a Lipschitz continuous viscosity solution of the regularized problem. Then, in the second part, we use (3.2) and the stability result of continuous viscosity solutions, in order to pass to the limit.

7.2.1 Existence, uniqueness and *a priori* estimates of the regularized problem

We consider, for $\epsilon > 0$, the following regularized problem of (3.80)

$$\begin{cases} \partial_t \rho_\epsilon^\pm(x, t) = \mp \left((\rho_\epsilon^+ - \rho_\epsilon^-)(x, t) + \alpha \int_0^1 (\rho_\epsilon^+ - \rho_\epsilon^-)(y, t) dy + a_\epsilon(t) \right) |\partial_x \rho_\epsilon^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \rho_\epsilon^\pm(x, 0) = \rho_{0,\epsilon}^\pm(x) = P_{0,\epsilon}^\pm(x) + L_0 x & \text{in } \mathbb{R}, \end{cases} \quad (3.81)$$

where $P_{0,\epsilon}^\pm$ is the regularization of P_0^\pm and $a_\epsilon(\cdot)$ is the regularization of $a(\cdot)$ defined as in (3.73) by replacing $\bar{L}(\cdot)$ by $a(\cdot)$. We define, for all $\epsilon > 0$ and $T > 0$, the set

$$X_T^\epsilon = \left\{ v_\epsilon^\pm = Q_\epsilon^\pm + L_0 x, \text{ such that } \begin{cases} Q_\epsilon^\pm \in L^\infty(\mathbb{T} \times (0, T)), \\ \|Q_\epsilon^\pm\|_{L^\infty(\mathbb{T} \times (0, T))} \leq I, \\ \|\partial_x v_\epsilon^\pm\|_{L^\infty(\mathbb{T} \times (0, T))} \leq J_\epsilon, \\ \max_{\pm} (\|\partial_x v_\epsilon^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))} + \|\partial_t v_\epsilon^\pm\|_{L^\infty((0, T); L \log L(\mathbb{T}))}) \leq C_1. \end{cases} \right\}$$

with

$$I = 2 \left(\max_{\pm} (\|P_0^\pm\|_{L^\infty(\mathbb{T})}) + |L_0| \|a\|_{L^\infty(0, T)T} \right),$$

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$$J_\varepsilon = \max_{\pm} \left(\|\partial_x \rho_{0,\varepsilon}^\pm\|_{L^\infty(\mathbb{T})} \right),$$

and C_1 is equal to the constant \bar{C}_1 given in (3.78) by substituting $\|\bar{L}\|_{L^\infty(0,T)}$ by $(\|a\|_{L^\infty(0,T)} + 2\alpha I)$. Now, for $v_\varepsilon^\pm \in X_T^\varepsilon$, we define $\rho_\varepsilon^\pm = G(v_\varepsilon^\pm)$ as the unique Lipschitz continuous viscosity solution of the following local problem

$$\begin{cases} \partial_t \rho_\varepsilon^\pm(x, t) = \mp \left((\rho_\varepsilon^+ - \rho_\varepsilon^-)(x, t) + \alpha \int_0^1 (v_\varepsilon^+ - v_\varepsilon^-)(y, t) dy + a_\varepsilon(t) \right) |\partial_x \rho_\varepsilon^\pm(x, t)| & \text{in } \mathbb{R} \times (0, T), \\ \rho_\varepsilon^\pm(x, 0) = \rho_{0,\varepsilon}^\pm(x) = P_{0,\varepsilon}^\pm(x) + L_0 x & \text{in } \mathbb{R}. \end{cases} \quad (3.82)$$

We will show that $G : X_T^\varepsilon \rightarrow X_T^\varepsilon$ is a well-defined contraction for T small enough. First, we will prove that G is well-defined. We consider

$$\bar{L}(t) = \alpha \int_0^1 (v_\varepsilon^+ - v_\varepsilon^-)(y, t) dy + a_\varepsilon(t).$$

Then, we can see that,

$$\|\bar{L}\|_{L^\infty(0,T)} \leq (\|a\|_{L^\infty(0,T)} + 2\alpha I). \quad (3.83)$$

Let $v_\varepsilon^\pm \in X_T^\varepsilon$. Using (3.2) and (3.83), we can show that the system (3.82) admits a Lipschitz continuous viscosity solution ρ_ε^\pm satisfying (3.74), (3.75), (3.76) and (3.77). This shows, by Young's inequality and by classical properties of the mollifiers, that $\rho_\varepsilon^\pm \in X_T^\varepsilon$ for $T \leq \frac{1}{4\alpha|L_0|}$. It remains to show that G is a contraction. Let $\rho_\varepsilon^{\pm,i} = G(v_\varepsilon^{\pm,i})$, $i = 1, 2$, and set

$$L_c = \max_{\pm} (\|v_\varepsilon^{\pm,1} - v_\varepsilon^{\pm,2}\|_{L^\infty(\mathbb{T} \times (0,T))}).$$

We remark that $\rho_\varepsilon^{\pm,2}$ is a viscosity sub-solution of

$$\partial_t \rho_\varepsilon^\pm(x, t) = \mp \left((\rho_\varepsilon^+ - \rho_\varepsilon^-)(x, t) + \alpha \int_0^1 (v_\varepsilon^{\pm,1} - v_\varepsilon^{\pm,2})(y, t) dy + a_\varepsilon(t) \right) |\partial_x \rho_\varepsilon^\pm(x, t)| + 2\alpha L_c J_\varepsilon.$$

Moreover, $\rho_\varepsilon^{\pm,1} + 2\alpha L_c J_\varepsilon t$ is a viscosity solution of the same equation. Using the comparison principle of this local system (see Section 7) and interchanging $\rho_\varepsilon^{\pm,1}$, $\rho_\varepsilon^{\pm,2}$, we get

$$\max_{\pm} (\|\rho_\varepsilon^{\pm,1} - \rho_\varepsilon^{\pm,2}\|_{L^\infty(\mathbb{T} \times (0,T))}) \leq 2\alpha J_\varepsilon T \max_{\pm} (\|v_\varepsilon^{\pm,1} - v_\varepsilon^{\pm,2}\|_{L^\infty(\mathbb{T} \times (0,T))}).$$

This shows that, for T small enough, G is a contraction on X_T^ε which is a closed set. Hence, by the fixed point theorem, there exists a unique Lipschitz continuous viscosity solution of (3.81) in X_T^ε , for $0 < T \leq \min(\frac{1}{4\alpha|L_0|}, \frac{1}{4\alpha J_\varepsilon})$. By iterating this process, we can construct a solution for all $T > 0$ (proceeding as in the Step 6 of the proof of Theorem 3.2). It is easy to verify also that this viscosity Lipschitz solution ρ_ε^\pm satisfies estimates (3.8) and (3.9).

7.2.2 Passage to the limit

We repeat the same approach done in Section 6. First, we remark that, from the classical properties of mollifiers, the solution ρ_ϵ^\pm of (3.81) obtained in the previous subsection, is ϵ -uniformly bounded in $L^\infty(\mathbb{R} \times (0, T))$ and $\partial_x \rho^\pm, \partial_t \rho^\pm$ are also bounded in $L^\infty((0, T); L \log L(\mathbb{T}))$. Hence, as ϵ goes to zero, we can extract a subsequence still denoted ρ_ϵ^\pm , that converges locally and uniformly to some limit $\rho^\pm(x, t) = P^\pm(x, t) + L_0 x$ in the sense of (3.79).

Now, we can apply the stability of continuous viscosity solutions, which implies that ρ^\pm is continuous viscosity solution of (3.80), that satisfies estimates (3.8) and (3.9). The proof of the continuity of the solution (Corollary 3.1) comes directly from the gradient entropy estimate using Lemma 3.13.

8 Uniqueness of the solution (Theorem 3.1-(ii))

In this section, we will show a uniqueness result for the local system (3.67) in the framework of viscosity solution. Indeed, in the case where $\rho^\pm(x, t) = P^\pm(x, t) + L_0 x$, this is equivalent to study the uniqueness of the following system:

$$\begin{cases} \partial_t P^\pm(x, t) = \mp ((P^+ - P^-)(x, t) + \bar{L}(t)) |\partial_x P^\pm(x, t) + L_0| & \text{in } \mathbb{R} \times (0, T), \\ P^\pm(x, 0) = P_0^\pm(x) & \text{in } \mathbb{R}. \end{cases} \quad (3.84)$$

For $k \in \{+, -\}$, we define the following Hamiltonian

$$H_k(t, p, q) = k(p^+ - p^- + \bar{L}(t)) |q + L_0|, \quad \text{for } p = (p^+, p^-).$$

The key point is that our system is quasi-monotone in the sense of Ishii, Koike [48, A.1] which will be announced in the following lemma.

Lemma 3.14. (Quasi-monotony of the Hamiltonian)

The Hamiltonian H_k is quasi-monotone, i.e., for all vectors $r = (r^+, r^-)$ and $s = (s^+, s^-)$ such that

$$r^j - s^j = \max_{k \in \{+, -\}} (r^k - s^k) \geq 0,$$

then

$$H_j(t, r, q) - H_j(t, s, q) \geq 0. \quad (3.85)$$

Proof of Lemma 3.14:

Let r and s be two vectors such that $r^j - s^j = \max_{k \in \{+, -\}} (r^k - s^k)$. We have

$$\begin{aligned} H_j(t, r, q) - H_j(t, s, q) &= j(r^+ - r^- + \bar{L}(t)) |q + L_0| - j(s^+ - s^- + \bar{L}(t)) |q + L_0| \\ &= j |q + L_0| ((r^+ - s^+) - (r^- - s^-)) \\ &= |q + L_0| \text{sign}((r^+ - s^+) - (r^- - s^-)) ((r^+ - s^+) - (r^- - s^-)) \\ &= |q + L_0| |(r^+ - s^+) - (r^- - s^-)| \geq 0. \end{aligned}$$

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This ends the proof. □

Proposition 3.1. (Comparison principle)

Assume that (3.68) holds. Let $P = (P^+, P^-)$ and $Q = (Q^+, Q^-)$ be 1-periodic viscosity continuous sub-solution and super-solution of (3.84) respecting to the initial conditions $P_0 = (P_0^+, P_0^-)$ and $Q_0 = (Q_0^+, Q_0^-)$, respectively. Then, if P^\pm and Q^\pm satisfy the following L^∞ estimates:

$$\max_{\pm} (\|P^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \bar{M}_0, \quad \max_{\pm} (\|Q^\pm\|_{L^\infty(\mathbb{T} \times (0, T))}) \leq \bar{M}_0 \quad (3.86)$$

and, for all $(x, t) \in \mathbb{R} \times (0, T - \delta)$, the following continuity estimates:

$$|P^\pm(x + h, t + \delta) - P^\pm(x, t)| \leq \bar{C}_2 \left(\frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)} \right), \quad (3.87)$$

$$|Q^\pm(x + h, t + \delta) - Q^\pm(x, t)| \leq \bar{C}_2 \left(\frac{1}{\ln(\frac{1}{\delta} + 1)} + \frac{1}{\ln(\frac{1}{h} + 1)} \right). \quad (3.88)$$

Then, if $P^k(\cdot, 0) \leq Q^k(\cdot, 0)$ in \mathbb{R} then $P^k \leq Q^k$ in $\mathbb{R} \times (0, T)$ for $k \in \{+, -\}$.

Proof of Proposition 3.1:

Let us denote by $M_{sup} = \max_{k \in \{+, -\}} \sup_{\mathbb{R} \times [0, T]} (P^k - Q^k) = \max_{k \in \{+, -\}} \sup_{[0, 1] \times [0, T]} (P^k - Q^k)$. It is sufficient to prove that $M_{sup} \leq 0$. Let us suppose by contradiction that $M_{sup} > 0$. We duplicate the variables by considering, for all ϵ, β, η and α positives

$$\psi(x, y, s, t, k) = P^k(x, t) - Q^k(y, s) - \frac{|x - y|^2}{2\epsilon} - \frac{|t - s|^2}{2\beta} - \frac{\eta}{T - t} - \alpha(|x|^2 + |y|^2).$$

We note that $\psi(x, y, s, t, k)$ is locally bounded continuous function in $\mathbb{R}^2 \times [0, T]^2$.

We can think that the maximum of ψ noted $M(\epsilon, \beta, \alpha, \eta) = \sup_{\substack{x \in \mathbb{R}, y \in \mathbb{R} \\ t \in [0, T], s \in [0, T] \\ k \in \{+, -\}}} \psi(x, y, t, s, k)$, is similar with

M_{sup} .

The idea is justified by the following lemma.

Lemma 3.15. *Let $(\bar{x}, \bar{y}, \bar{t}, \bar{s}, \bar{k})$ be a maximum of ψ . Then, the following properties hold:*

1. $\alpha|\bar{x}|, \alpha|\bar{y}| \leq \sqrt{2\alpha\bar{M}_0}$.
2. $\lim_{\beta \rightarrow 0} |\bar{t} - \bar{s}| = \lim_{\epsilon \rightarrow 0} |\bar{x} - \bar{y}| = 0$.
3. $\bar{t} < T - \frac{\eta}{2\bar{M}_0} < T$.
4. $\lim_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) = \lim_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) = M_{sup}$,
 where $\lim_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} = \limsup_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0}$ and $\lim_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} = \liminf_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0}$.
5. \bar{t}, \bar{s} are positives if ϵ, β, α and η are sufficiently small.

We first show the proof of this lemma and then we complete the proof of Proposition 3.1.

Proof of Lemma 3.15:

Using the fact that P^k and Q^k , for $k \in \{+, -\}$, are bounded, we deduce that

$$\lim_{|x|, |y| \rightarrow \infty} \psi(x, y, t, s) = -\infty,$$

and so the function ψ reaches a maximum at a point $(\bar{x}, \bar{y}, \bar{t}, \bar{s}, \bar{k}) \in \mathbb{R}^2 \times (0, T)^2 \times \{+, -\}$.

We know that $M_{sup} = \max_{k \in \{+, -\}} \sup_{\mathbb{R} \times [0, T]} (P^k - Q^k) = \max_{k \in \{+, -\}} \sup_{[0, 1] \times [0, T]} (P^k - Q^k) > 0$, then there exists $(x^*, t^*, k^*) \in (0, 1) \times (0, T) \times \{+, -\}$ such that $P^{k^*}(x^*, t^*) - Q^{k^*}(x^*, t^*) > 0$. Then, we have

$$\psi(x^*, t^*, x^*, t^*, k^*) \leq M(\epsilon, \alpha, \beta, \eta),$$

which implies

$$0 < P^{k^*}(x^*, t^*) - Q^{k^*}(x^*, t^*) \leq M(\epsilon, \alpha, \beta, \eta) + \frac{\eta}{T - t^*} + 2\alpha|x^*|^2,$$

$$0 < M(\epsilon, \alpha, \beta, \eta) + \frac{\eta}{T - t^*} + 2\alpha|x^*|^2.$$

Then, for η and α small enough, we get

$$M(\epsilon, \beta, \alpha, \eta) > 0.$$

We then deduce

$$\alpha(|\bar{x}|^2 + |\bar{y}|^2) \leq P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) < 2\bar{M}_0,$$

where we have used (3.86) for the second inequality. Multiplying the previous inequality by α , yields (1) and

$$\lim_{\alpha \rightarrow 0} \alpha|\bar{x}| = \lim_{\alpha \rightarrow 0} \alpha|\bar{y}| = 0. \quad (3.89)$$

In the same way, we have

$$\frac{|\bar{x} - \bar{y}|^2}{2\epsilon} + \frac{|\bar{t} - \bar{s}|^2}{2\beta} < 2\bar{M}_0 \quad (3.90)$$

and so

$$\lim_{\epsilon \rightarrow 0} |\bar{x} - \bar{y}|^2 = \lim_{\beta \rightarrow 0} |\bar{t} - \bar{s}|^2 = 0. \quad (3.91)$$

We also deduce that

$$\frac{\eta}{T - \bar{t}} < P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) < 2\bar{M}_0,$$

We, then, have

$$\bar{t} < T - \frac{\eta}{2\bar{M}_0} < T.$$

Now, let us prove (4). Indeed, for all $(x, t, k) \in \mathbb{R} \times (0, T) \times \{+, -\}$, we have

$$\psi(x, x, t, t, k) \leq \psi(\bar{x}, \bar{y}, \bar{t}, \bar{s}, \bar{k}),$$

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then,

$$\begin{aligned} P^k(x, t) - Q^k(x, t) &\leq P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) - \frac{|\bar{x} - \bar{y}|^2}{2\epsilon} - \frac{|\bar{t} - \bar{s}|^2}{2\beta} - \frac{\eta}{T - \bar{t}} - \alpha(|\bar{x}|^2 + |\bar{y}|^2) + \frac{\eta}{T - t} + 2\alpha|x|^2 \\ &\leq P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) + \frac{\eta}{T - t} + 2\alpha|x|^2. \end{aligned}$$

Then, we get

$$P^k(x, t) - Q^k(x, t) - 2\alpha|x|^2 - \frac{\eta}{T - t} \leq \left(P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) \right),$$

which implies that

$$\begin{aligned} P^k(x, t) - Q^k(x, t) - 2\alpha|x|^2 - \frac{\eta}{T - t} &\leq \left(P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{x}, \bar{t}) + Q^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) \right) \\ &\leq M_{sup} + \left(Q^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) \right) \\ &\leq M_{sup} + \bar{C}_2 \left(\frac{1}{\ln\left(\frac{1}{|\bar{x} - \bar{y}|} + 1\right)} + \frac{1}{\ln\left(\frac{1}{|\bar{t} - \bar{s}|} + 1\right)} \right) \\ &\leq M_{sup} + \bar{C}_2 \left(\frac{1}{\ln\left(\frac{1}{2\sqrt{\epsilon\bar{M}_0}} + 1\right)} + \frac{1}{\ln\left(\frac{1}{2\sqrt{\beta\bar{M}_0}} + 1\right)} \right). \end{aligned}$$

Now, passing to $\overline{\lim}$ and $\underline{\lim}$ in the above inequality as $(\epsilon, \beta, \alpha, \eta) \rightarrow 0$, we obtain

$$\begin{aligned} P^k(x, t) - Q^k(x, t) &\leq \underline{\lim}_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} \left(P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) \right) \\ &\leq \overline{\lim}_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} \left(P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) \right) \\ &\leq M_{sup}. \end{aligned}$$

Finally, for proving (5) let us suppose, for example, that $\bar{t} = 0$, then, because of $P^k(\cdot, 0) \leq Q^k(\cdot, 0)$, we have

$$\begin{aligned} P^{\bar{k}}(\bar{x}, 0) - Q^{\bar{k}}(\bar{y}, \bar{s}) &\leq P^{\bar{k}}(\bar{x}, 0) - Q^{\bar{k}}(\bar{x}, 0) + Q^{\bar{k}}(\bar{x}, 0) - Q^{\bar{k}}(\bar{y}, \bar{s}) \\ &\leq Q^{\bar{k}}(\bar{x}, 0) - Q^{\bar{k}}(\bar{y}, \bar{s}) \\ &\leq \bar{C}_2 \left(\frac{1}{\ln\left(\frac{1}{2\sqrt{\epsilon\bar{M}_0}} + 1\right)} + \frac{1}{\ln\left(\frac{1}{2\sqrt{\beta\bar{M}_0}} + 1\right)} \right). \end{aligned}$$

Then, $M_{sup} = \overline{\lim}_{(\epsilon, \beta, \alpha, \eta) \rightarrow 0} P^{\bar{k}}(\bar{x}, 0) - Q^{\bar{k}}(\bar{y}, \bar{s}) \leq 0$. However $M_{sup} > 0$ and when ϵ, β, α , and η are sufficiently small, we get a contradiction. When $\bar{s} = 0$, a similar proof could be given, which ends the proof of the Lemma 3.15. \square

Let's go back now to the proof of Proposition 3.1. Indeed, we take ϵ, β, α and η small enough such that

$\bar{t} > 0$ and $\bar{s} > 0$ (see Lemma 3.15-(5)). We can remark that the function

$$(x, t) \rightarrow P^{\bar{k}}(x, t) - \left(Q^{\bar{k}}(\bar{y}, \bar{s}) + \frac{|x - \bar{y}|^2}{2\epsilon} + \frac{|t - \bar{s}|^2}{2\beta} + \frac{\eta}{T - t} + \alpha(|x|^2 + |\bar{y}|^2) \right)$$

reaches a maximum at (\bar{x}, \bar{t}) . By using the test-function

$$\phi(x, t) = Q^{\bar{k}}(\bar{y}, \bar{s}) + \frac{|x - \bar{y}|^2}{2\epsilon} + \frac{|t - \bar{s}|^2}{2\beta} + \frac{\eta}{T - t} + \alpha(|x|^2 + |\bar{y}|^2)$$

and the fact that $P = (P^+, P^-)$ is a subsolution of (3.84), we deduce that

$$\frac{(\bar{t} - \bar{s})}{\beta} + \frac{\eta}{(T - \bar{t})^2} + H_{\bar{k}} \left(\bar{t}, P(\bar{x}, \bar{t}), \frac{(\bar{x} - \bar{y})}{\epsilon} + 2\alpha\bar{x} \right) \leq 0. \quad (3.92)$$

In the other hand, we deduce that the function

$$(y, s) \rightarrow Q^{\bar{k}}(y, s) - \left(P^{\bar{k}}(\bar{x}, \bar{t}) - \frac{|\bar{x} - y|^2}{2\epsilon} - \frac{|\bar{t} - s|^2}{2\beta} - \frac{\eta}{T - \bar{t}} - \alpha(|\bar{x}|^2 + |y|^2) \right)$$

reaches a minimum at (\bar{y}, \bar{s}) . By using the test-function

$$\phi(x, t) = P^{\bar{k}}(\bar{x}, \bar{t}) - \frac{|\bar{x} - y|^2}{2\epsilon} - \frac{|\bar{t} - s|^2}{2\beta} - \frac{\eta}{T - \bar{t}} - \alpha(|\bar{x}|^2 + |y|^2)$$

and the fact that $Q = (Q^+, Q^-)$ is a supersolution of (3.84), we deduce that

$$\frac{(\bar{t} - \bar{s})}{\beta} + H_{\bar{k}} \left(\bar{s}, Q(\bar{y}, \bar{s}), \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right) \geq 0. \quad (3.93)$$

By subtracting (3.93) to (3.92), we deduce that

$$\frac{\eta}{(T - \bar{t})^2} + H_{\bar{k}} \left(\bar{t}, P(\bar{x}, \bar{t}), \frac{(\bar{x} - \bar{y})}{\epsilon} + 2\alpha\bar{x} \right) - H_{\bar{k}} \left(\bar{s}, Q(\bar{y}, \bar{s}), \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right) \leq 0. \quad (3.94)$$

By using Lemma 3.15, in particular the fact that $\bar{x}, \bar{y}, \bar{t}, \bar{s}$ and \bar{k} are uniformly bounded with respect to β (using Lemma 3.15-(1) and Lemma 3.15-(2)), we have, up to extract a subsequence

$$\lim_{\beta \rightarrow 0} \bar{x} = \tilde{x}, \quad \lim_{\beta \rightarrow 0} \bar{y} = \tilde{y},$$

$$\lim_{\beta \rightarrow 0} \bar{t} = \lim_{\beta \rightarrow 0} \bar{s} = \tau.$$

$$\bar{k} = \lim_{\beta \rightarrow 0} \bar{k} = \tilde{k}.$$

Sending $\beta \rightarrow 0$ in (3.94), we deduce using Lemma 3.15-(3) ($\tau < T$) that

$$\frac{\eta}{(T - \tau)^2} + H_{\tilde{k}} \left(\tau, P(\tilde{x}, \tau), \frac{(\tilde{x} - \tilde{y})}{\epsilon} + 2\alpha\tilde{x} \right) - H_{\tilde{k}} \left(\tau, Q(\tilde{y}, \tau), \frac{(\tilde{x} - \tilde{y})}{\epsilon} - 2\alpha\tilde{y} \right) \leq 0. \quad (3.95)$$

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Moreover, we have

$$\psi(\bar{x}, \bar{t}, \bar{y}, \bar{t}, k) \leq \psi(\bar{x}, \bar{t}, \bar{y}, \bar{s}, \bar{k}),$$

which implies

$$P^k(\bar{x}, \bar{t}) - Q^k(\bar{y}, \bar{t}) \leq P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) = P^{\bar{k}}(\bar{x}, \bar{t}) - Q^{\bar{k}}(\bar{y}, \bar{s}) \geq 0.$$

Hence, we obtain

$$P^{\bar{k}}(\tilde{x}, \tau) - Q^{\bar{k}}(\tilde{y}, \tau) \geq P^k(\tilde{x}, \tau) - Q^k(\tilde{y}, \tau).$$

By adding and subtracting the term $H_{\bar{k}}\left(\tau, P(\tilde{x}, \tau), \frac{(\tilde{x} - \tilde{y})}{\epsilon} - 2\alpha\tilde{y}\right)$ in the inequality (3.95), using Lemma 3.14 and (3.68), we deduce that

$$\frac{\eta}{(T - \tau)^2} + H_{\bar{k}}\left(\tau, P(\tilde{x}, \tau), \frac{(\tilde{x} - \tilde{y})}{\epsilon} + 2\alpha\tilde{x}\right) - H_{\bar{k}}\left(\tau, P(\tilde{x}, \tau), \frac{(\tilde{x} - \tilde{y})}{\epsilon} - 2\alpha\tilde{y}\right) \leq 0,$$

and so

$$\frac{\eta}{(T - \tau)^2} + \tilde{k}(P^+(\tilde{x}, \tau) - P^-(\tilde{x}, \tau) + \bar{L}(\tau)) \left(\left| \frac{(\tilde{x} - \tilde{y})}{\epsilon} + 2\alpha\tilde{x} + L_0 \right| - \left| \frac{(\tilde{x} - \tilde{y})}{\epsilon} - 2\alpha\tilde{y} + L_0 \right| \right) \leq 0.$$

Using the fact that $\tilde{k}(P^+(\tilde{x}, \tau) - P^-(\tilde{x}, \tau) + \bar{L}(\tau))$ is bounded (see (3.86)), and sending $\alpha \rightarrow 0$, we obtain using Lemma 3.15-(1) and (3.90), that

$$\frac{\eta}{(T - \tau)^2} \leq 0,$$

this contradiction ends the proof of the proposition. This result ensures the uniqueness of the solution in the local case and implies Theorem 3.1-(ii). \square

4

Un schéma convergent pour un système de transport non-local modélisant la dynamique des densités des dislocations

Ce chapitre est un article soumis à *Journal of Numerical Mathematics* écrit en collaboration avec Ahmad El Hajj

Dans ce travail, nous nous intéressons à l'analyse numérique d'un système diagonal hyperbolique, non nécessairement strictement hyperbolique, non-linéaire et non local qui intervient dans la modélisation de la dynamique des densités des dislocations. En considérant des conditions initiales croissantes, des résultats d'existence et d'unicité d'une solution H_{loc}^1 pour ce système ont été établis dans [30]. En particulier, dans [30], une estimation d'énergie du gradient dans L^2 a été montrée. Nous proposons ainsi un schéma implicite et nous montrons que pour ce dernier une estimation L^2 discrète du gradient est satisfaite. Cette inégalité nous permet ensuite de montrer la convergence de la solution numérique vers la solution du problème continue.

Convergent scheme for a non-local transport system modeling dislocations densities dynamics

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Abstract

In this paper, we consider a diagonal hyperbolic, not necessarily strictly hyperbolic, nonlinear and non-local system of transport equations that arise in the theory of dislocations densities dynamics. Under certain monotony assumptions on the initial data, global existence and uniqueness results of H^1_{loc} solutions for this system was proved in [30]. We propose a natural implicit scheme verifying a gradient L^2 estimate at the discrete level compatible with that obtained in the continuous case in [30]. Which leads us to prove the convergence of the numerical solution to the continuous one.

AMS Classification: 74S20, 65M06, 65M12, 35B45, 35Q49, 35L60, 35F20, 35A01, 74H25, 74A99.

Key words: Finite difference method, implicit Upwind scheme, convergence, discrete estimate, H^1 discrete solution, transport equations, nonlinear hyperbolic systems, existence and uniqueness, dislocations dynamics.

1 Introduction and main results

Before presenting our scheme, let us first, recall the setting of the continuous problem. In particular, we will restate the existence and uniqueness results of a H^1_{loc} solution for the continuous problem that was proved in [30].

1.1 Existence and uniqueness for the continuous problem

In this paper, we will propose an implicit numerical scheme for a system of coupled equations of non-conservative Burgers type. This system intervenes naturally in the modeling of dislocations densities dynamics where a dislocation is a crystal defect which corresponds to a discontinuity in the crystalline structure organization. This concept has been introduced as the main explanation, at the microscopic scale, of plastic deformation. For modeling details, we refer the readers to [36, Section 2] and [33, Section 2].

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More precisely, we consider the following system

$$\begin{cases} \frac{\partial \rho^+}{\partial t}(x, t) = - \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) \frac{\partial \rho^+}{\partial x}(x, t) & \text{in } D'(\mathbb{R} \times (0, T)), \\ \frac{\partial \rho^-}{\partial t}(x, t) = \left((\rho^+ - \rho^-)(x, t) + \alpha \int_0^1 (\rho^+ - \rho^-)(y, t) dy + a(t) \right) \frac{\partial \rho^-}{\partial x}(x, t) & \text{in } D'(\mathbb{R} \times (0, T)). \end{cases} \quad (4.1)$$

The unknowns ρ^+ and ρ^- are scalar valued functions, that we denote for simplicity by ρ^\pm . These quantities represent respectively the plastic distortions associated with the dislocations moving according to the vectors $(1, 0)$ and $(-1, 0)$. Their spacial derivatives $\partial_x \rho^\pm$ are the densities of dislocations propagating in the direction of the vector $\pm(1, 0)$ respectively. From a physical viewpoint, in the context of the dynamics of dislocation densities, the functions ρ^\pm and $\partial_x \rho^\pm$ are positive reals. Here, the difference $(\rho^+ - \rho^-)$ represents the resolved plastic strain. The constant α depends on the elastic coefficients and the material size, while the function $a(\cdot)$ represents the exterior shear strain field. We consider the following initial conditions for (4.1):

$$\rho^\pm(x, 0) = \rho_0^\pm(x) = \rho_0^{\pm,p}(x) + L_0 x \quad \text{on } \mathbb{R}, \quad (4.2)$$

where $\rho_0^{\pm,p}$ are 1-periodic functions. We thus have a model for a periodic distribution of the \pm dislocations, with a spatial period of length 1. In fact, the use of the periodic boundary conditions is a way of regarding what is going on in the interior of the material away from its boundary.

For this system, based on an energy estimate the global existence and uniqueness of nondecreasing solution in $W_{loc}^{1,2}(\mathbb{R} \times (0, T))$ space has been established in [30].

Theorem 4.1. (*Existence and uniqueness*), [30, Th 1.1])

For all $T, L_0 \geq 0$, $\alpha \in \mathbb{R}$, and $\rho_0^\pm \in H_{loc}^1(\mathbb{R})$ and under the assumptions

(H1) $\rho_0^\pm(x+1) = \rho_0^\pm(x) + L_0$ (1-periodic function + linear function),

(H2) $\frac{\partial \rho_0^\pm}{\partial x} \geq 0$ a.e in \mathbb{R} (ρ_0^\pm is nondecreasing),

(H3) $a \in L^\infty(0, T)$,

the system (4.1)-(4.2) admits a unique solution $\rho^\pm \in H_{loc}^1(\mathbb{R} \times [0, T])$, in distributional sense, such that, for a.e $t \in (0, T)$, the function $\rho^\pm(\cdot, t) : x \mapsto \rho^\pm(x, t)$ verifies (H1) and (H2).

In the following, we will propose a compatible discretization with the continuous problem.

1.2 The discrete problem

We want to approximate the solution of (4.1)-(4.2). Given a mesh size $\Delta x, \Delta t$, we define

$$\Xi = \{i\Delta x, i \in \{0, \dots, M-1\}\} \quad \Xi_T = \Xi \times \{n\Delta t, n \in \{0, \dots, N_T\}\},$$

with $\Delta x = \frac{1}{M}$ and $\Delta t = \frac{T}{N_T}$. The discrete running point is (x_i, t_n) with $x_i = i\Delta x$, $t_n = n\Delta t$. The approximation of the solution ρ^\pm at the node (x_i, t_n) is written indifferently as $u_\pm(x_i, t_n)$ or $u_{\pm,i}^n$ according to whether we view it as a function defined on the lattice or as a sequence.

We propose, then, the following Upwind implicit discretization of the system

$$\forall k \in \{+, -\}, \quad \begin{cases} \frac{u_{k,i}^{n+1} - u_{k,i}^n}{\Delta t} + C_{k,i}^{n+1}[u^{n+1}] \left(\frac{u_{k,i+1}^{n+1} - u_{k,i}^{n+1}}{\Delta x} \right) = 0 & \text{if } C_{k,i}^{n+1}[u^{n+1}] \leq 0, \\ 0 \leq i \leq M-1, \\ 0 \leq n \leq N_T - 1, \end{cases} \quad \begin{cases} \frac{u_{k,i}^{n+1} - u_{k,i}^n}{\Delta t} + C_{k,i}^{n+1}[u^{n+1}] \left(\frac{u_{k,i}^{n+1} - u_{k,i-1}^{n+1}}{\Delta x} \right) = 0 & \text{if } C_{k,i}^{n+1}[u^{n+1}] \geq 0, \end{cases}$$

with

$$u_i^0 = (u_{+,i}^0, u_{-,i}^0) = \rho_0(x_i) = (\rho_0^+(x_i), \rho_0^-(x_i)), \quad \text{for } -1 \leq i \leq M,$$

$$C_{k,i}^{n+1}[u^{n+1}] = k \left(u_{+,i}^{n+1} - u_{-,i}^{n+1} + \alpha \sum_{j=0}^{M-1} \Delta x (u_{+,j}^{n+1} - u_{-,j}^{n+1}) + a(t_{n+1}) \right),$$

and the following periodicity conditions

$$\left| \begin{array}{l} u_{+,-1}^n = u_{+,M-1}^n - L_0, \\ u_{+,M}^n = u_{+,0}^n + L_0, \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} u_{-,-1}^n = u_{-,M-1}^n - L_0, \\ u_{-,M}^n = u_{-,0}^n + L_0, \end{array} \right. \quad \text{for all } n \geq 0.$$

We define the positive and the negative parts $(C_{k,i}^{n+1}[u^{n+1}])_+$ and $(C_{k,i}^{n+1}[u^{n+1}])_-$ as follows:

$$(C_{k,i}^{n+1}[u^{n+1}])_+ = \frac{1}{2} (C_{k,i}^{n+1}[u^{n+1}] + |C_{k,i}^{n+1}[u^{n+1}]|),$$

$$(C_{k,i}^{n+1}[u^{n+1}])_- = \frac{1}{2} (|C_{k,i}^{n+1}[u^{n+1}]| - C_{k,i}^{n+1}[u^{n+1}]).$$

Both $(C_{k,i}^{n+1}[u^{n+1}])_+$ and $(C_{k,i}^{n+1}[u^{n+1}])_-$ are positive real numbers.

We can write the scheme in a more compact form:

$$\left\{ \begin{array}{l} \frac{u_{k,i}^{n+1} - u_{k,i}^n}{\Delta t} - (C_{k,i}^{n+1}[u^{n+1}])_- \left(\frac{u_{k,i+1}^{n+1} - u_{k,i}^{n+1}}{\Delta x} \right) + (C_{k,i}^{n+1}[u^{n+1}])_+ \left(\frac{u_{k,i}^{n+1} - u_{k,i-1}^{n+1}}{\Delta x} \right) = 0, \\ u_{k,i}^0 = \rho_0^k(x_i). \end{array} \right. \quad (4.3)$$

In the sequel, we set

$$\theta_{k,i+\frac{1}{2}}^n = \frac{u_{k,i+1}^n - u_{k,i}^n}{\Delta x}, \quad \text{for } k \in \{+, -\} \quad (4.4)$$

which is a discrete equivalent of $\partial_x \rho^k$.

1.3 Main results

First, we set $u^n = (u_i^n)_{i \in \{-1, \dots, M\}}$ where $u_i^n = (u_{+,i}^n, u_{-,i}^n)$. We define $\rho_0^p = (\rho_0^{+,p}, \rho_0^{-,p})$ such that $\|\rho_0^p\|_{(L^\infty(\mathbb{T}))^2} = \|\rho_0^{+,p}\|_{L^\infty(\mathbb{T})} + \|\rho_0^{-,p}\|_{L^\infty(\mathbb{T})}$ where \mathbb{T} is the $[0, 1)$ periodic interval.

We now introduce the following CFL condition:

$$\frac{\Delta t}{\Delta x} \leq \min(\beta_1, \beta_2), \quad (4.5)$$

where

$$\beta_1 = \frac{1}{[(2\zeta + 2)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)}] (4(\zeta + 1) + L_0)},$$

and

$$\beta_2 = \frac{1}{3(8[(2\zeta + 2)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)}] + 4(1 + |\alpha|)(2\zeta + 2) + 2L_0(1 + |\alpha|))},$$

with

$$\zeta = e^{4TL_0(1+|\alpha|)} \left(\|\rho_0^p\|_{(L^\infty(\mathbb{T}))^2} + \frac{\|a\|_{L^\infty(0,T)}}{2(1+|\alpha|)} \right). \quad (4.6)$$

We present in this paper three main results. The first one is proving that the implicit scheme (4.3) has a unique bounded solution at each time-step. Then, we show a discrete gradient L^2 estimate compatible with that obtained in the continuous case. The third one is showing the convergence of the solution of the scheme.

Theorem 4.2. (*Resolution of the implicit scheme on one time step*)

Suppose that assumptions (H1), (H2) and (H3) given by Theorem 4.1 and the CFL condition (4.5) are satisfied. Let $u_{k,i}^n = w_{k,i}^n + L_0 x_i$ where $w_{k,i}^n$, for $-1 \leq i \leq M$, satisfies the following conditions

$$\left| \begin{array}{l} w_{+,-1}^n = w_{+,M-1}^n, \\ w_{+,M}^n = w_{+,0}^n, \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} w_{-,-1}^n = w_{-,M-1}^n, \\ w_{-,M}^n = w_{-,0}^n, \end{array} \right. \quad (4.7)$$

Then, we get

(i) (**Existence**)

There exists a unique solution u^{n+1} to the implicit scheme (4.3) such that $u_{k,i}^{n+1} = w_{k,i}^{n+1} + L_0 x_i$ with $w_{k,i}^{n+1}$ satisfies the conditions given by (4.7) and the following L^∞ bound

$$\max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} w_{k,i}^{n+1} \leq e^{4TL_0(1+|\alpha|)} \left(\|\rho_0^p\|_{(L^\infty(\mathbb{T}))^2} + \frac{\|a\|_{L^\infty(0,T)}}{2(1+|\alpha|)} \right),$$

for all $k \in \{+, -\}$ and for all $i = 0, \dots, M-1$ and $n = 0, \dots, N_T - 1$.

(ii) (**Monotonicity**)

Moreover if u^n is nondecreasing, i.e. satisfies

$$u_{k,i+1}^n \geq u_{k,i}^n \quad \text{for all } i \in \{0, \dots, M-1\}, \quad \text{and } k \in \{+, -\},$$

then u^{n+1} is also nondecreasing.

Theorem 4.3. (Discrete gradient L^2 estimate)

Suppose that assumptions (H1), (H2) and (H3) are verified. If u^n is the solution of the implicit scheme (4.3) given by Theorem 4.2, then, $\theta_{k,i+\frac{1}{2}}^n$ defined in (4.4) is nonnegative for all $n \in \{0, \dots, N_T - 1\}$ and verifies the following estimate:

$$\sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k,i+\frac{1}{2}}^{n+1})^2 + \Delta t \sum_{i=0}^{M-1} \left(\left[\theta_{+,i+\frac{1}{2}}^{n+1} - \theta_{-,i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+,i+\frac{1}{2}}^{n+1} + \theta_{-,i+\frac{1}{2}}^{n+1} \right] \right) \leq \sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k,i+\frac{1}{2}}^n)^2. \quad (4.8)$$

Now, we present the convergence result of our implicit scheme.

Theorem 4.4. (Convergence of the solution of the scheme)

Assume that assumptions (H1), (H2) and the CFL condition (4.5) are satisfied and $a \in C(0, T)$. Let us consider the solution $(u^n)_{n \geq 0}$ of the scheme (4.3) for the time step $\Delta t > 0$ and the space step $\Delta x > 0$ given by Theorem 4.2. Let us call $\epsilon = (\Delta x, \Delta t)$ and u^ϵ the function defined by

$$u^\epsilon(i\Delta x, n\Delta t) = u_i^n \quad \text{for } n \in \{0, \dots, N_T - 1\}, \quad i \in \{0, \dots, M - 1\}.$$

Then, as ϵ goes to zero, the whole sequence $(u^\epsilon)_\epsilon$ converges to the unique solution ρ of (4.1)-(4.2). Moreover, we have

$$\sup_{\substack{\mathbb{T} \times [0, T] \\ k \in \{+, -\}}} |u^\epsilon - \rho| \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow (0, 0).$$

Remark 4.1.

1. Here, we choose the implicit scheme since it naturally preserves the gradient L^2 estimate at the discrete level, which is neither the case of the explicit scheme nor that of the semi-explicit scheme.
2. Note that, all presented results remain valid in the case where $a \in L^\infty(0, T)$. To do that, it is sufficient to regularize a and then pass to the limit as Δt tends to 0 so that the regularization vanishes. But, we have chosen to take $a \in C(0, T)$ for the sake of simplicity of the calculations.

1.4 Brief literature

Many numerical results were brightened up on systems which intervene in the dislocations densities dynamics. In this framework, we refer the readers to [33] where the authors proved a convergence result of an explicit scheme to the Lipschitz continuous solution for a particular (2×2) Hamilton-Jacobi system. Inspired by the results proved in [7], they showed also an error estimate of Crandall-Lions type [26] between the numerical solution and the continuous one.

We also refer to Jakobsen, Karlsen [49] and Jakobsen, Karlsen, Risebro [50] where they proved a convergence result for a weakly coupled Hamilton-Jacobi system considering a semi-discrete splitting algorithm that they proposed to approach the continuous solution.

In the framework of hyperbolic systems with Lipschitz and nondecreasing initial data, let us mention [21] (cf. Chapter 6) where the authors proposed a natural implicit scheme satisfying a Lipschitz estimate at the discrete level. This estimate allowed them to show the convergence of the discrete solution to the solution of the continuous problem. We draw the attention of the readers to the fact that in [21], results

were established without supposing that the system is strictly hyperbolic, unlike what was done in [59]. In fact, Monasse and Monneau proved in [59] a convergence result of a semi-explicit scheme for diagonal non-conservative hyperbolic systems assuming that the system is strictly hyperbolic and using a discrete gradient estimate that was proved in the continuous case in [35]. This result was achieved in a restricted class of vanishing viscosity solutions which was introduced initially by Bianchini and Bressan [16].

In the present work, we consider a hyperbolic system, not necessarily strictly hyperbolic, and we prove, basing on a discrete gradient L^2 estimate, the convergence of the discrete solution to the continuous one introduced in [30].

Beside this, for systems in conservative form, we refer to [55] for a review of the main classes of existing schemes. Among these schemes, convergence results are seldom found for hyperbolic systems. The Lax-Wendroff theorem [53] shows that if a consistent and conservative numerical scheme converges (in L^1 with bounded total variation), its limit is a weak solution to the hyperbolic system. However, in order to obtain convergence of the scheme, stability is needed, in general in the form of TV-stability. For the scalar Godunov scheme, convergence is obtained due to its total variation diminishing (TVD) property. This is no longer the case for systems [55]. Stability can still be proved for certain special systems of two equations, for instance in [56, 69, 70]. Similar results can be obtained for a class of nonlinear systems with straight-line fields [22, pp. 102-103]. Nonlinear stability can also be assessed through the use of invariant domains and entropy inequalities [18], for HLL, HLLC and kinetic solvers of Euler equations of gas dynamics.

In the case of conservative systems where the initial data has sufficiently small total variation, Glimm's random choice method [39] is probably convergent. A deterministic variant (replacing random with equidistributed sampling) has also been proved to converge under the same assumptions [58].

1.5 Organization of the paper

This paper is organized as follows. In Section 2, we prove some preliminary results on the existence and monotonicity of the solution of the scheme (Theorem 4.2). Section 3 is devoted to show a discrete gradient L^2 estimate (Theorem 4.3). Finally, based on this gradient estimate, we prove the convergence result of our scheme (Theorem 4.4) in Section 4.

2 Existence and monotonicity of the discrete solution

This section is devoted to prove the existence, the uniqueness and the monotonicity of a bounded solution for the implicit scheme (4.3). This section is divided into two subsections, in Subsection 2.1, we prove by a fixed point argument the existence and uniqueness of the discrete solution. Then, in Subsection 2.2, we show that the discrete solution is nondecreasing.

2.1 Proof of Theorem 4.2-(i):

Proving the existence of a solution $u_{k,i}^n = w_{k,i}^n + L_0 x_i$ of the scheme (4.3) is equivalent to prove the existence of a solution of the following scheme:

$$\begin{cases} \frac{w_{k,i}^{n+1} - w_{k,i}^n}{\Delta t} - (C_{k,i}^{n+1}[w^{n+1}])_- \left(\frac{w_{k,i+1}^{n+1} - w_{k,i}^{n+1}}{\Delta x} + L_0 \right) + (C_{k,i}^{n+1}[w^{n+1}])_+ \left(\frac{w_{k,i}^{n+1} - w_{k,i-1}^{n+1}}{\Delta x} + L_0 \right) \\ w_{k,i}^0 = \rho_0^{k,p}(x_i), \end{cases} \quad (4.9)$$

with the following conditions

$$\begin{cases} w_{+,-1}^n = w_{+,M-1}^n, \\ w_{+,M}^n = w_{+,0}^n, \end{cases} \quad \text{and} \quad \begin{cases} w_{-,-1}^n = w_{-,M-1}^n, \\ w_{-,M}^n = w_{-,0}^n, \end{cases} \quad \text{for all } n \geq 0.$$

We denote the vector:

$$\begin{aligned} v^{n+1} &= \begin{pmatrix} v_0^{n+1} & v_1^{n+1} & v_2^{n+1} & \dots & v_{M-1}^{n+1} & v_M^{n+1} & v_{M+1}^{n+1} & \dots & v_{2M-1}^{n+1} \end{pmatrix} \\ &= \begin{pmatrix} v_{+,0}^{n+1} & v_{+,1}^{n+1} & v_{+,2}^{n+1} & \dots & v_{+,M-1}^{n+1} & v_{-,0}^{n+1} & v_{-,1}^{n+1} & \dots & v_{-,M-1}^{n+1} \end{pmatrix}. \end{aligned}$$

We introduce now the function $F : \mathbb{R}^{2M} \rightarrow \mathbb{R}^{2M}$

$$\begin{aligned} F_{w^n}(v^{n+1}) &= \begin{pmatrix} F_{w^n}^+(v^{n+1}) & F_{w^n}^-(v^{n+1}) \end{pmatrix} \\ &= \begin{pmatrix} F_{0,w^n}^+(v^{n+1}) & F_{1,w^n}^+(v^{n+1}) & \dots & F_{M-1,w^n}^+(v^{n+1}) & F_{M,w^n}^-(v^{n+1}) & \dots & F_{2M-1,w^n}^-(v^{n+1}) \end{pmatrix}. \end{aligned}$$

For $v^{n+1} \in \mathbb{R}^{2M}$, we define the function F as follows

$$\begin{cases} \bullet \text{ For } i = 0 \\ \bullet \text{ For } i = M \\ \bullet \text{ For } 1 \leq i \leq M-2 \end{cases} \begin{cases} F_{0,w^n}^+(v^{n+1}) = w_{+,0}^n + \Delta t (C_{+,0}^{n+1}[v^{n+1}])_- \left[\frac{v_{+,1}^{n+1} - v_{+,0}^{n+1}}{\Delta x} + L_0 \right] \\ \quad - \Delta t (C_{+,0}^{n+1}[v^{n+1}])_+ \left[\frac{v_{+,0}^{n+1} - v_{+,M-1}^{n+1}}{\Delta x} + L_0 \right]. \\ F_{M,w^n}^-(v^{n+1}) = w_{-,0}^n + \Delta t (C_{-,0}^{n+1}[v^{n+1}])_- \left[\frac{v_{-,1}^{n+1} - v_{-,0}^{n+1}}{\Delta x} + L_0 \right] \\ \quad - \Delta t (C_{-,0}^{n+1}[v^{n+1}])_+ \left[\frac{v_{-,0}^{n+1} - v_{-,M-1}^{n+1}}{\Delta x} + L_0 \right]. \\ F_{i,w^n}^+(v^{n+1}) = w_{+,i}^n + \Delta t (C_{+,i}^{n+1}[v^{n+1}])_- \left[\frac{v_{+,i+1}^{n+1} - v_{+,i}^{n+1}}{\Delta x} + L_0 \right] \\ \quad - \Delta t (C_{+,i}^{n+1}[v^{n+1}])_+ \left[\frac{v_{+,i}^{n+1} - v_{+,i-1}^{n+1}}{\Delta x} + L_0 \right]. \end{cases}$$

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$$\bullet \text{ For } M+1 \leq i \leq 2M-2 \left| \begin{array}{l} F_{i,w^n}^-(v^{n+1}) = w_{-,i}^n + \Delta t(C_{-,i}^{n+1}[v^{n+1}])_- \left[\frac{v_{-,i+1}^{n+1} - v_{-,i}^{n+1}}{\Delta x} + L_0 \right] \\ -\Delta t(C_{-,i}^{n+1}[v^{n+1}])_+ \left[\frac{v_{-,i}^{n+1} - v_{-,i-1}^{n+1}}{\Delta x} + L_0 \right]. \end{array} \right.$$

$$\bullet \text{ For } i = M-1 \left| \begin{array}{l} F_{M-1,w^n}^+(v^{n+1}) = w_{+,M-1}^n + \Delta t(C_{+,M-1}^{n+1}[v^{n+1}])_- \left[\frac{v_{+,0}^{n+1} - v_{+,M-1}^{n+1}}{\Delta x} + L_0 \right] \\ -\Delta t(C_{+,M-1}^{n+1}[v^{n+1}])_+ \left[\frac{v_{+,M-1}^{n+1} - v_{+,M-2}^{n+1}}{\Delta x} + L_0 \right]. \end{array} \right.$$

$$\bullet \text{ For } i = 2M-1 \left| \begin{array}{l} F_{2M-1,w^n}^-(v^{n+1}) = w_{-,M-1}^n + \Delta t(C_{-,M-1}^{n+1}[v^{n+1}])_- \left[\frac{v_{-,0}^{n+1} - v_{-,M-1}^{n+1}}{\Delta x} + L_0 \right] \\ -\Delta t(C_{-,M-1}^{n+1}[v^{n+1}])_+ \left[\frac{v_{-,M-1}^{n+1} - v_{-,M-2}^{n+1}}{\Delta x} + L_0 \right]. \end{array} \right.$$

Then, we can see that the scheme (4.9) can be written as

$$\left\{ \begin{array}{l} w_{+,i}^{n+1} = F_{i,w^n}^+(w^{n+1}) \text{ for } i = 0, \dots, M-1, \\ w_{+,i}^0 = \rho_0^{+,p}(x_i). \end{array} \right. \quad (4.10)$$

and

$$\left\{ \begin{array}{l} w_{-,i-M+1}^{n+1} = F_{i,w^n}^-(w^{n+1}) \text{ for } i = M, \dots, 2M-1, \\ w_{-,i}^0 = \rho_0^{-,p}(x_i). \end{array} \right. \quad (4.11)$$

Let us now define the mapping

$$G_{w^n} : [-1 - \zeta, 1 + \zeta]^{2M} \longrightarrow [-1 - \zeta, 1 + \zeta]^{2M}$$

$$v^{n+1} \longrightarrow F_{w^n}(v^{n+1}).$$

Our goal is to show that the mapping G_{w^n} is a well-defined contraction supposing that $|w_{k,i}^n| \leq \zeta$. This leads to prove, first, the existence and uniqueness of a fixed point of (4.10)-(4.11) under a CFL condition. Then, we will show that the solution is uniformly bounded. To do that, we proceed in three steps.

Step 1 (G_{w^n} is well-defined):

It is sufficient to prove that $-\zeta - 1 \leq F_{i,w^n}^k(v^{n+1}) \leq \zeta + 1$, if $w^n \in [-\zeta, \zeta]^{2M}$. Indeed, we have for

$i = 1, \dots, M - 2$ and for $k = +$

$$\begin{aligned} & F_{i,w^n}^+(v^{n+1}) \\ &= w_{+,i}^n - \frac{\Delta t}{\Delta x} \left((C_{+,i}^{n+1}[v^{n+1}])_- + (C_{+,i}^{n+1}[v^{n+1}])_+ \right) v_{+,i}^{n+1} \\ &+ \frac{\Delta t}{\Delta x} (C_{+,i}^{n+1}[v^{n+1}])_- v_{+,i+1}^{n+1} + \frac{\Delta t}{\Delta x} (C_{+,i}^{n+1}[v^{n+1}])_+ v_{+,i-1}^{n+1} + -\frac{\Delta t}{\Delta x} L_0 \Delta x C_{+,i}^{n+1}[v^{n+1}]. \end{aligned}$$

Then, using the CFL condition (4.5), we obtain

$$|F_{i,w^n}^+(v^{n+1})| \leq \zeta + \frac{\Delta t}{\Delta x} \left[(2\zeta + 2)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)} \right] (4(\zeta + 1) + L_0) \leq \zeta + 1,$$

which implies that the mapping G_{w^n} is well-defined.

For the cases $\{i = M + 1, \dots, 2M - 2$ and for $k = -\}$, $\{i = M - 1$ and $k = +\}$, $\{i = 2M - 1$ and $k = -\}$, $\{i = 0$ and $k = +\}$ and $\{i = M$ and $k = -\}$, we proceed in a similar manner to prove that G_{w^n} is well-defined.

Step 2 (G_{w^n} is a contraction):

We define the sets $\mathcal{M}^+ = \{1, \dots, M - 2\}$ and $\mathcal{M}^- = \{M + 1, \dots, 2M - 2\}$.

We equipped $[-1 - \zeta, 1 + \zeta]^{2M}$ with the following norm $(l^\infty)^2$:

$$\|u^{n+1}\|_{(l^\infty)^2} = \sum_{k=+,-} \left(\sup_{i \in \mathcal{M}^k} |u_{k,i}^{n+1}| + \sup_{i=0, M-1} |u_{+,i}^{n+1}| + \sup_{i=M, 2M-1} |u_{-,i}^{n+1}| \right).$$

We observe that, for all u^{n+1} and v^{n+1} in $[-1 - \zeta, 1 + \zeta]^{2M}$, for all $k \in \{+, -\}$,

$$\begin{aligned} & \|G_{w^n}(u^{n+1}) - G_{w^n}(v^{n+1})\|_{(l^\infty)^2} \\ &= \overbrace{\sum_{k=+,-} \sup_{i \in \mathcal{M}^k} |F_{i,w^n}^k(u^{n+1}) - F_{i,w^n}^k(v^{n+1})|}^{I_1} \\ &+ \overbrace{\sum_{k=+,-} \sup_{i=0, M-1} |F_{i,w^n}^+(u^{n+1}) - F_{i,w^n}^+(v^{n+1})|}^{I_2} + \overbrace{\sum_{k=+,-} \sup_{i=M, 2M-1} |F_{i,w^n}^-(u^{n+1}) - F_{i,w^n}^-(v^{n+1})|}^{I_3}. \\ I_1 &= \frac{\Delta t}{\Delta x} \sum_{k=+,-} \sup_{i \in \overline{\mathcal{M}}^k} \left| \left[(C_{k,i}^{n+1}[u^{n+1}])_- - (C_{k,i}^{n+1}[v^{n+1}])_- \right] (u_{k,i+1}^{n+1} - u_{k,i}^{n+1}) \right. \\ &\quad \left. - \left[(C_{k,i}^{n+1}[u^{n+1}])_+ - (C_{k,i}^{n+1}[v^{n+1}])_+ \right] (u_{k,i}^{n+1} - u_{k,i-1}^{n+1}) \right. \\ &\quad \left. + (C_{k,i}^{n+1}[v^{n+1}])_- \left[(u_{k,i+1}^{n+1} - v_{k,i+1}^{n+1}) - (u_{k,i}^{n+1} - v_{k,i}^{n+1}) \right] \right. \\ &\quad \left. + (C_{k,i}^{n+1}[v^{n+1}])_+ \left[(v_{k,i}^{n+1} - u_{k,i}^{n+1}) - (v_{k,i-1}^{n+1} - u_{k,i-1}^{n+1}) \right] + \left(C_{k,i}^{n+1}[v^{n+1}] - C_{k,i}^{n+1}[u^{n+1}] \right) L_0 \right|, \end{aligned}$$

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where we have added and subtracted the terms $\frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[v^{n+1}]_-) u_{k,i+1}^{n+1}$ and $\frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[v^{n+1}]_+) u_{k,i-1}^{n+1}$. We obtain

$$\begin{aligned} & \|G_{w^n}(u^{n+1}) - G_{w^n}(v^{n+1})\|_{(l^\infty)^2} \\ & \leq 3 \frac{\Delta t}{\Delta x} (8 [(2\zeta + 2)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)}] + 4(1 + |\alpha|)(2\zeta + 2) + 2L_0(1 + |\alpha|)) \|u^{n+1} - v^{n+1}\|_{(l^\infty)^2} \end{aligned}$$

which gives by the CFL condition (4.5) that G_{w^n} is a contraction on $[-1 - \zeta, 1 + \zeta]^{2M}$.

Step 3 (The fixed point is uniformly bounded):

We assume now that w^{n+1} satisfies the following conditions:

$$\left\{ \begin{array}{l} w_{+,-1}^{n+1} = w_{+,M-1}^{n+1}, \\ w_{+,M}^{n+1} = w_{+,0}^{n+1}, \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} w_{-,-1}^{n+1} = w_{-,M-1}^{n+1}, \\ w_{-,M}^{n+1} = w_{-,0}^{n+1}. \end{array} \right.$$

In the following, we will prove that $|w_{k,i}^n| \leq \zeta$ which is true for $n = 0$, so we will prove it for $n > 0$. Indeed, we have for $0 \leq i \leq M - 1$

$$\begin{aligned} w_{k,i}^{n+1} &= w_{k,i}^n - \frac{\Delta t}{\Delta x} \left[(C_{k,i}^{n+1}[w^{n+1}]_-) + (C_{k,i}^{n+1}[w^{n+1}]_+) \right] w_{k,i}^{n+1} \\ &+ \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[w^{n+1}]_-) w_{k,i+1}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[w^{n+1}]_+) w_{k,i-1}^{n+1} \\ &- \Delta t C_{k,i}^{n+1}[w^{n+1}] L_0. \end{aligned} \tag{4.12}$$

We denote $|w_{k_0,j}^{n+1}| = \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} (|w_{k,i}^{n+1}|)$. Then, using (4.12), we obtain

$$\begin{aligned} \pm w_{k_0,j}^{n+1} &= \pm w_{k_0,j}^n \mp \frac{\Delta t}{\Delta x} \left[(C_{k_0,j}^{n+1}[w^{n+1}]_-) + (C_{k_0,j}^{n+1}[w^{n+1}]_+) \right] w_{k_0,j}^{n+1} \pm \frac{\Delta t}{\Delta x} (C_{k_0,j}^{n+1}[w^{n+1}]_-) w_{k_0,j+1}^{n+1} \\ &\pm \frac{\Delta t}{\Delta x} (C_{k_0,j}^{n+1}[w^{n+1}]_+) w_{k_0,j-1}^{n+1} \mp \Delta t C_{k_0,j}^{n+1}[w^{n+1}] L_0. \end{aligned}$$

Then, we get

$$\begin{aligned} \pm w_{k_0,j}^{n+1} &\leq \pm w_{k_0,j}^n \mp \frac{\Delta t}{\Delta x} \left[(C_{k_0,j}^{n+1}[w^{n+1}]_-) + (C_{k_0,j}^{n+1}[w^{n+1}]_+) \right] w_{k_0,j}^{n+1} \pm \frac{\Delta t}{\Delta x} (C_{k_0,j}^{n+1}[w^{n+1}]_-) w_{k_0,j}^{n+1} \\ &\pm \frac{\Delta t}{\Delta x} (C_{k_0,j}^{n+1}[w^{n+1}]_+) w_{k_0,j}^{n+1} \mp \Delta t C_{k_0,j}^{n+1}[w^{n+1}] L_0 \\ &\leq \pm w_{k_0,j}^n \mp k_0 \Delta t L_0 \left[w_{+,j}^{n+1} - w_{-,j}^{n+1} + \alpha \sum_{j=0}^{M-1} \Delta x (w_{+,j}^{n+1} - w_{-,j}^{n+1}) + a(t_{n+1}) \right]. \\ &\leq \pm w_{k_0,j}^n + \Delta t L_0 \|a\|_{L^\infty(0,T)} + 2\Delta t L_0 (1 + |\alpha|) |w_{k_0,j}^{n+1}|. \end{aligned}$$

Which implies that

$$|w_{k_0,j}^{n+1}| \leq |w_{k_0,j}^n| + \Delta t L_0 \|a\|_{L^\infty(0,T)} + 2\Delta t L_0 (1 + |\alpha|) |w_{k_0,j}^{n+1}|.$$

Then, we can see that

$$|w_{k_0,j}^{n+1}| \leq |w_{k_0,j}^n| + 2\Delta t L_0 (1 + |\alpha|) \left(|w_{k_0,j}^{n+1}| + \frac{\|a\|_{L^\infty(0,T)}}{2(1 + |\alpha|)} \right).$$

By adding the term $\frac{\|a\|_{L^\infty(0,T)}}{2(1 + |\alpha|)}$ on the left and on the right hand sides of the above inequality and by denoting $|\xi_{k_0,j}^n| = |w_{k_0,j}^n| + \frac{\|a\|_{L^\infty(0,T)}}{2(1 + |\alpha|)}$, we get

$$|\xi_{k_0,j}^{n+1}| \leq |\xi_{k_0,j}^n| + 2\Delta t L_0 (1 + |\alpha|) |\xi_{k_0,j}^{n+1}|.$$

Then, we obtain

$$\left(1 - \frac{2T}{n} L_0 (1 + |\alpha|)\right) \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |\xi_{k,i}^{n+1}| \leq \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |\xi_{k,i}^n|.$$

Now, using a recursion on n , we can deduce that

$$\begin{aligned} \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |\xi_{k,i}^n| &\leq \frac{1}{\left(1 - \frac{2T}{n} L_0 (1 + |\alpha|)\right)^n} \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |\xi_{k,i}^0| \\ &\leq e^{4TL_0(1+|\alpha|)} \left(\max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |w_{k,i}^0| + \frac{\|a\|_{L^\infty(0,T)}}{2(1 + |\alpha|)} \right). \end{aligned}$$

We can see that $\max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |w_{k,i}^n| \leq \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |\xi_{k,i}^n|$, which leads to the following estimate

$$\begin{aligned} \max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |w_{k,i}^n| &\leq e^{4TL_0(1+|\alpha|)} \left(\max_{\substack{k=+,- \\ 0 \leq i \leq M-1}} |w_{k,i}^0| + \frac{\|a\|_{L^\infty(0,T)}}{2(1 + |\alpha|)} \right) \\ &\leq e^{4TL_0|\alpha|} \left(\|\rho_0^p\|_{(L^\infty(\mathbb{T}))^2} + \frac{\|a\|_{L^\infty(0,T)}}{2(1 + |\alpha|)} \right) = \zeta. \end{aligned} \tag{4.13}$$

This implies that by applying a recursion on n , there exists a unique fixed point w^{n+1} , unique solution of the scheme (4.9), belonging to $[-\zeta, \zeta]^{2M}$. \square

2.2 Proof of Theorem 4.2-(ii):

In what follows, we derive an evolution in time of the discrete gradient $\theta_{k,i+\frac{1}{2}}^n$ defined by (4.4) from (4.3), which allows us to prove the monotonicity of discrete solution.

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Let $u_{k,i}^{n+1}$ be the solution of the implicit scheme (4.3). Using the definition of $\theta_{k,i+\frac{1}{2}}^n$, we observe that

$$\theta_{k,i+\frac{1}{2}}^{n+1} = \theta_{k,i+\frac{1}{2}}^n + \frac{\Delta t}{\Delta x} \frac{u_{k,i+1}^{n+1} - u_{k,i+1}^n}{\Delta t} - \frac{\Delta t}{\Delta x} \frac{u_{k,i}^{n+1} - u_{k,i}^n}{\Delta t}.$$

Inserting (4.3) at i and $i+1$, allows us to show that $\theta_{k,i+\frac{1}{2}}^n$ defined in (4.4) satisfies the following relation

$$\begin{aligned} \theta_{k,i+\frac{1}{2}}^{n+1} &= \theta_{k,i+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} \left[(C_{k,i+1}^{n+1}[u^{n+1}]_+ + C_{k,i}^{n+1}[u^{n+1}]_-) \theta_{k,i+\frac{1}{2}}^{n+1} \right. \\ &\quad \left. + \frac{\Delta t}{\Delta x} (C_{k,i+1}^{n+1}[u^{n+1}]_- \theta_{k,i+\frac{3}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[u^{n+1}]_+ \theta_{k,i-\frac{1}{2}}^{n+1} \right). \end{aligned} \quad (4.14)$$

It is sufficient to show that $\theta_{k,i+\frac{1}{2}}^{n+1} \geq 0$. Let us denote $\theta_{k,j+\frac{1}{2}}^{n+1} = \min_{i=0,\dots,M-1} \left(\theta_{k,i+\frac{1}{2}}^{n+1} \right)$. Then, from (4.14), we have

$$\begin{aligned} \theta_{k,j+\frac{1}{2}}^{n+1} &= \theta_{k,j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} \left[(C_{k,j+1}^{n+1}[u^{n+1}]_+ + C_{k,j}^{n+1}[u^{n+1}]_-) \theta_{k,j+\frac{1}{2}}^{n+1} \right. \\ &\quad \left. + \frac{\Delta t}{\Delta x} (C_{k,j+1}^{n+1}[u^{n+1}]_- \theta_{k,j+\frac{3}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,j}^{n+1}[u^{n+1}]_+ \theta_{k,j-\frac{1}{2}}^{n+1} \right. \\ &\geq \theta_{k,j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} \left[(C_{k,j+1}^{n+1}[u^{n+1}]_+ + C_{k,j}^{n+1}[u^{n+1}]_-) \theta_{k,j+\frac{1}{2}}^{n+1} \right. \\ &\quad \left. + \frac{\Delta t}{\Delta x} (C_{k,j+1}^{n+1}[u^{n+1}]_- \theta_{k,j+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,j}^{n+1}[u^{n+1}]_+ \theta_{k,j+\frac{1}{2}}^{n+1} \right). \end{aligned}$$

Then, we obtain

$$\theta_{k,j+\frac{1}{2}}^{n+1} \geq \theta_{k,j+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} \left(C_{k,j+1}^{n+1}[u^{n+1}] - C_{k,j}^{n+1}[u^{n+1}] \right) \theta_{k,j+\frac{1}{2}}^{n+1}.$$

Hence, we obtain

$$\left[1 + \frac{\Delta t}{\Delta x} \left(C_{k,j+1}^{n+1}[u^{n+1}] - C_{k,j}^{n+1}[u^{n+1}] \right) \right] \theta_{k,j+\frac{1}{2}}^{n+1} \geq \theta_{k,j+\frac{1}{2}}^n.$$

We argue by contradiction. Suppose that $\theta_{k,i+\frac{1}{2}}^{n+1} < 0$ for certain $i \in \{0, \dots, M-1\}$. Then, using the fact that $\theta_{k,i+\frac{1}{2}}^n \geq 0$ and taking

$$\frac{\Delta t}{\Delta x} \leq \frac{1}{8\zeta},$$

which is in accordance with the CFL condition (4.5), we obtain

$$\begin{aligned} \frac{1}{2} \theta_{k,i+\frac{1}{2}}^{n+1} &\geq \left[1 + \frac{\Delta t}{\Delta x} \left(C_{k,j+1}^{n+1}[u^{n+1}] - C_{k,j}^{n+1}[u^{n+1}] \right) \right] \theta_{k,i+\frac{1}{2}}^{n+1} \\ &\geq \left[1 + \frac{\Delta t}{\Delta x} \left(C_{k,j+1}^{n+1}[u^{n+1}] - C_{k,j}^{n+1}[u^{n+1}] \right) \right] \theta_{k,j+\frac{1}{2}}^{n+1} \\ &\geq \theta_{k,j+\frac{1}{2}}^n \geq 0. \end{aligned}$$

This implies that

$$\frac{1}{2}\theta_{k,i+\frac{1}{2}}^{n+1} \geq 0,$$

which implies a contradiction and then $\theta_{k,i+\frac{1}{2}}^{n+1} \geq 0$. This is equivalent to $u_{k,i}^{n+1}$ is also nondecreasing. This leads to Theorem 4.2-(ii), by applying a recursion on $n \geq 0$.

3 Proof of Theorem 4.3 (Discrete gradient L^2 estimate)

Our goal in this section is to prove that the discrete gradient enjoys a L^2 estimate.

Using (4.14) and the definition of the positive and the negative parts of $C_{k,i}^{n+1}[u^{n+1}]$, we can deduce that

$$\begin{aligned} \theta_{k,i+\frac{1}{2}}^{n+1} &= \theta_{k,i+\frac{1}{2}}^n - \frac{\Delta t}{\Delta x} [C_{k,i+1}^{n+1}[u^{n+1}] - C_{k,i}^{n+1}[u^{n+1}]] \theta_{k,i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i+1}^{n+1}[u^{n+1}])_- \theta_{k,i+\frac{3}{2}}^{n+1} \\ &\quad - \frac{\Delta t}{\Delta x} [(C_{k,i}^{n+1}[u^{n+1}])_+ + (C_{k,i+1}^{n+1}[u^{n+1}])_-] \theta_{k,i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[u^{n+1}])_+ \theta_{k,i-\frac{1}{2}}^{n+1}. \end{aligned} \quad (4.15)$$

Now, by using the explicit definition of $C_{k,i}^{n+1}[u^{n+1}]$, we can see that

$$\frac{\Delta t}{\Delta x} (C_{k,i+1}^{n+1}[u^{n+1}] - C_{k,i}^{n+1}[u^{n+1}]) = \pm \Delta t (\theta_{+,i+\frac{1}{2}}^{n+1} - \theta_{-,i+\frac{1}{2}}^{n+1}).$$

By injecting the previous equality in (4.15), we obtain

$$\begin{aligned} \theta_{k,i+\frac{1}{2}}^{n+1} &= \theta_{k,i+\frac{1}{2}}^n \mp \Delta t (\theta_{+,i+\frac{1}{2}}^{n+1} - \theta_{-,i+\frac{1}{2}}^{n+1}) \theta_{k,i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i+1}^{n+1}[u^{n+1}])_- \theta_{k,i+\frac{3}{2}}^{n+1} \\ &\quad - \frac{\Delta t}{\Delta x} [(C_{k,i}^{n+1}[u^{n+1}])_+ + (C_{k,i+1}^{n+1}[u^{n+1}])_-] \theta_{k,i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[u^{n+1}])_+ \theta_{k,i-\frac{1}{2}}^{n+1}. \end{aligned}$$

Now, by multiplying the previous equality by $\theta_{k,i+\frac{1}{2}}^{n+1}$, we get

$$\begin{aligned} (\theta_{k,i+\frac{1}{2}}^{n+1})^2 &= \theta_{k,i+\frac{1}{2}}^n \theta_{k,i+\frac{1}{2}}^{n+1} \mp \Delta t (\theta_{+,i+\frac{1}{2}}^{n+1} - \theta_{-,i+\frac{1}{2}}^{n+1}) (\theta_{k,i+\frac{1}{2}}^{n+1})^2 - \frac{\Delta t}{\Delta x} [(C_{k,i}^{n+1}[u^{n+1}])_+ + (C_{k,i+1}^{n+1}[u^{n+1}])_-] (\theta_{k,i+\frac{1}{2}}^{n+1})^2 \\ &\quad + \frac{\Delta t}{\Delta x} (C_{k,i+1}^{n+1}[u^{n+1}])_- \theta_{k,i+\frac{3}{2}}^{n+1} \theta_{k,i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[u^{n+1}])_+ \theta_{k,i-\frac{1}{2}}^{n+1} \theta_{k,i+\frac{1}{2}}^{n+1} \\ &\leq \frac{1}{2} (\theta_{k,i+\frac{1}{2}}^n)^2 + \frac{1}{2} (\theta_{k,i+\frac{1}{2}}^{n+1})^2 \mp \Delta t (\theta_{+,i+\frac{1}{2}}^{n+1} - \theta_{-,i+\frac{1}{2}}^{n+1}) (\theta_{k,i+\frac{1}{2}}^{n+1})^2 \\ &\quad - \frac{\Delta t}{\Delta x} [(C_{k,i}^{n+1}[u^{n+1}])_+ + (C_{k,i+1}^{n+1}[u^{n+1}])_-] (\theta_{k,i+\frac{1}{2}}^{n+1})^2 \\ &\quad + \frac{\Delta t}{\Delta x} (C_{k,i+1}^{n+1}[u^{n+1}])_- \theta_{k,i+\frac{3}{2}}^{n+1} \theta_{k,i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{k,i}^{n+1}[u^{n+1}])_+ \theta_{k,i-\frac{1}{2}}^{n+1} \theta_{k,i+\frac{1}{2}}^{n+1}. \end{aligned} \quad (4.16)$$

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Summing (4.16) over $k \in \{+, -\}$, we get

$$\begin{aligned}
& \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 \\
& \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 - \Delta t \left(\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right) \left[(\theta_{+, i+\frac{1}{2}}^{n+1})^2 - (\theta_{-, i+\frac{1}{2}}^{n+1})^2 \right] \\
& \quad - \frac{\Delta t}{\Delta x} \left[(C_{-, i}^{n+1}[u^{n+1}])_+ + (C_{-, i+1}^{n+1}[u^{n+1}])_- \right] (\theta_{-, i+\frac{1}{2}}^{n+1})^2 - \frac{\Delta t}{\Delta x} \left[(C_{+, i}^{n+1}[u^{n+1}])_+ + (C_{+, i+1}^{n+1}[u^{n+1}])_- \right] (\theta_{+, i+\frac{1}{2}}^{n+1})^2 \\
& \quad + \sum_{k \in \{+, -\}} \frac{\Delta t}{\Delta x} (C_{k, i+1}^{n+1}[u^{n+1}])_- \theta_{k, i+\frac{3}{2}}^{n+1} \theta_{k, i+\frac{1}{2}}^{n+1} + \sum_{k \in \{+, -\}} \frac{\Delta t}{\Delta x} (C_{k, i}^{n+1}[u^{n+1}])_+ \theta_{k, i-\frac{1}{2}}^{n+1} \theta_{k, i+\frac{1}{2}}^{n+1}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
& \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 \\
& \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 - \Delta t \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) \\
& \quad - \frac{\Delta t}{\Delta x} \left[(C_{-, i}^{n+1}[u^{n+1}])_+ + (C_{-, i+1}^{n+1}[u^{n+1}])_- \right] (\theta_{-, i+\frac{1}{2}}^{n+1})^2 - \frac{\Delta t}{\Delta x} \left[(C_{+, i}^{n+1}[u^{n+1}])_+ + (C_{+, i+1}^{n+1}[u^{n+1}])_- \right] (\theta_{+, i+\frac{1}{2}}^{n+1})^2 \\
& \quad + \frac{\Delta t}{\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}])_- \theta_{-, i+\frac{3}{2}}^{n+1} \theta_{-, i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}])_- \theta_{+, i+\frac{3}{2}}^{n+1} \theta_{+, i+\frac{1}{2}}^{n+1} \\
& \quad + \frac{\Delta t}{\Delta x} (C_{-, i}^{n+1}[u^{n+1}])_+ \theta_{-, i-\frac{1}{2}}^{n+1} \theta_{-, i+\frac{1}{2}}^{n+1} + \frac{\Delta t}{\Delta x} (C_{+, i}^{n+1}[u^{n+1}])_+ \theta_{+, i-\frac{1}{2}}^{n+1} \theta_{+, i+\frac{1}{2}}^{n+1} \\
& \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 - \Delta t \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) \\
& \quad - \frac{\Delta t}{\Delta x} \left[(C_{-, i}^{n+1}[u^{n+1}])_+ + (C_{-, i+1}^{n+1}[u^{n+1}])_- \right] (\theta_{-, i+\frac{1}{2}}^{n+1})^2 - \frac{\Delta t}{\Delta x} \left[(C_{+, i}^{n+1}[u^{n+1}])_+ + (C_{+, i+1}^{n+1}[u^{n+1}])_- \right] (\theta_{+, i+\frac{1}{2}}^{n+1})^2 \\
& \quad + \frac{\Delta t}{2\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}])_- (\theta_{-, i+\frac{3}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}])_- (\theta_{-, i+\frac{1}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}])_- (\theta_{+, i+\frac{3}{2}}^{n+1})^2 \\
& \quad + \frac{\Delta t}{2\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}])_- (\theta_{+, i+\frac{1}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{-, i}^{n+1}[u^{n+1}])_+ (\theta_{-, i-\frac{1}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{-, i}^{n+1}[u^{n+1}])_+ (\theta_{-, i+\frac{1}{2}}^{n+1})^2 \\
& \quad + \frac{\Delta t}{2\Delta x} (C_{+, i}^{n+1}[u^{n+1}])_+ (\theta_{+, i-\frac{1}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{+, i}^{n+1}[u^{n+1}])_+ (\theta_{+, i+\frac{1}{2}}^{n+1})^2.
\end{aligned}$$

Now, we can easily deduce the following inequality

$$\begin{aligned}
 & \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 \\
 & \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 \\
 & \quad - \Delta t \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) \\
 & \quad - \frac{\Delta t}{2\Delta x} \left[(C_{-, i}^{n+1}[u^{n+1}]_+ + (C_{-, i+1}^{n+1}[u^{n+1}]_-) (\theta_{-, i+\frac{1}{2}}^{n+1})^2 \right. \\
 & \quad \left. - \frac{\Delta t}{2\Delta x} \left[(C_{+, i}^{n+1}[u^{n+1}]_+ + (C_{+, i+1}^{n+1}[u^{n+1}]_-) (\theta_{+, i+\frac{1}{2}}^{n+1})^2 \right. \right. \\
 & \quad \left. \left. + \frac{\Delta t}{2\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}]_- (\theta_{-, i+\frac{3}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}]_- (\theta_{+, i+\frac{3}{2}}^{n+1})^2 \right. \right. \\
 & \quad \left. \left. + \frac{\Delta t}{2\Delta x} (C_{-, i}^{n+1}[u^{n+1}]_+ (\theta_{-, i-\frac{1}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{+, i}^{n+1}[u^{n+1}]_+ (\theta_{+, i-\frac{1}{2}}^{n+1})^2 \right). \right.
 \end{aligned} \tag{4.17}$$

Then, using (4.17) and the definition of the positive and the negative parts of $C_{k,i}^{n+1}[u^{n+1}]$, we get

$$\begin{aligned}
 & \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 \\
 & \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 + \frac{\Delta t}{2\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}]_- (\theta_{-, i+\frac{3}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}]_- (\theta_{+, i+\frac{3}{2}}^{n+1})^2 \\
 & \quad + \frac{\Delta t}{2\Delta x} (C_{-, i}^{n+1}[u^{n+1}]_+ (\theta_{-, i-\frac{1}{2}}^{n+1})^2 + \frac{\Delta t}{2\Delta x} (C_{+, i}^{n+1}[u^{n+1}]_+ (\theta_{+, i-\frac{1}{2}}^{n+1})^2 \\
 & \quad - \Delta t \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) \\
 & \quad - \frac{\Delta t}{2\Delta x} \left[C_{+, i}^{n+1}[u^{n+1}] - C_{+, i+1}^{n+1}[u^{n+1}] \right] (\theta_{+, i+\frac{1}{2}}^{n+1})^2 \\
 & \quad - \frac{\Delta t}{2\Delta x} \left[(C_{+, i}^{n+1}[u^{n+1}]_- + (C_{+, i+1}^{n+1}[u^{n+1}]_+)) (\theta_{+, i+\frac{1}{2}}^{n+1})^2 \right. \\
 & \quad \left. - \frac{\Delta t}{2\Delta x} \left[C_{-, i}^{n+1}[u^{n+1}] - C_{-, i+1}^{n+1}[u^{n+1}] \right] (\theta_{-, i+\frac{1}{2}}^{n+1})^2 \right. \\
 & \quad \left. - \frac{\Delta t}{2\Delta x} \left[(C_{-, i}^{n+1}[u^{n+1}]_- + (C_{-, i+1}^{n+1}[u^{n+1}]_+)) (\theta_{-, i+\frac{1}{2}}^{n+1})^2 \right]. \right.
 \end{aligned}$$

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Hence, we can see that

$$\begin{aligned} & \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 \\ & \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 + A_i^n - \Delta t \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) \\ & \quad + \frac{\Delta t}{2} \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) (\theta_{+, i+\frac{1}{2}}^{n+1})^2 - \frac{\Delta t}{2} \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right) (\theta_{-, i+\frac{1}{2}}^{n+1})^2, \end{aligned}$$

where

$$\begin{aligned} A_i^n &= -\frac{\Delta t}{2\Delta x} (C_{+, i}^{n+1}[u^{n+1}]_- (\theta_{+, i+\frac{1}{2}}^{n+1})^2) + \frac{\Delta t}{2\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}]_- (\theta_{+, i+\frac{1}{2}}^{n+1})^2) \\ & \quad - \frac{\Delta t}{2\Delta x} (C_{+, i+1}^{n+1}[u^{n+1}]_+ (\theta_{+, i+\frac{1}{2}}^{n+1})^2) + \frac{\Delta t}{2\Delta x} (C_{+, i}^{n+1}[u^{n+1}]_+ (\theta_{+, i-\frac{1}{2}}^{n+1})^2) \\ & \quad - \frac{\Delta t}{2\Delta x} (C_{-, i}^{n+1}[u^{n+1}]_- (\theta_{-, i+\frac{1}{2}}^{n+1})^2) + \frac{\Delta t}{2\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}]_- (\theta_{-, i+\frac{3}{2}}^{n+1})^2) \\ & \quad - \frac{\Delta t}{2\Delta x} (C_{-, i+1}^{n+1}[u^{n+1}]_+ (\theta_{-, i+\frac{1}{2}}^{n+1})^2) + \frac{\Delta t}{2\Delta x} (C_{-, i}^{n+1}[u^{n+1}]_+ (\theta_{-, i-\frac{1}{2}}^{n+1})^2). \end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned} \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 & \leq \frac{1}{2} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 + A_i^n \\ & \quad - \frac{\Delta t}{2} \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right). \end{aligned} \tag{4.18}$$

Summing (4.18) over $i = 0, \dots, M-1$, we obtain the following inequality

$$\begin{aligned} \sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 & \leq \sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 + 2 \sum_{i=0}^{M-1} A_i^n \\ & \quad - \Delta t \sum_{i=0}^{M-1} \left(\left[\theta_{+, i+\frac{1}{2}}^{n+1} - \theta_{-, i+\frac{1}{2}}^{n+1} \right]^2 \left[\theta_{+, i+\frac{1}{2}}^{n+1} + \theta_{-, i+\frac{1}{2}}^{n+1} \right] \right). \end{aligned}$$

Now, due to the periodicity of the solution, we can easily show that $\sum_{i=0}^{M-1} A_i^n = 0$. Hence, we deduce (4.8) and the following estimate

$$\sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^{n+1})^2 \Delta x \leq \sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^n)^2 \Delta x \leq \sum_{i=0}^{M-1} \sum_{k \in \{+, -\}} (\theta_{k, i+\frac{1}{2}}^0)^2 \Delta x. \tag{4.19}$$

□

4 Proof of Theorem 4.4

This section is devoted to prove Theorem 4.4. The proof is outlined in three subsections.

4.1 Estimates on the periodic + linear Q^1 extension u^ϵ

Let $(x, t) \in [0, 1] \times [0, T]$. Then, there exist $i \in \{0, \dots, M-1\}$ and $n \in \{0, \dots, N_T-1\}$ such that $(x, t) \in [x_i, x_{i+1}] \times [t_n, t_{n+1}]$, where $x_i = i\Delta x$ and $t_n = n\Delta t$. For $\epsilon = (\Delta x, \Delta t)$, we define the Q^1 extension of the function defined on the grid, for any $(x, t) \in [x_i, x_{i+1}] \times [t_n, t_{n+1}]$, by

$$\begin{aligned} u_k^\epsilon(x, t) &= \left(\frac{t-t_n}{\Delta t} \right) \left\{ \left(\frac{x-x_i}{\Delta x} \right) u_{k,i+1}^{n+1} + \left(1 - \frac{x-x_i}{\Delta x} \right) u_{k,i}^{n+1} \right\} \\ &+ \left(1 - \frac{t-t_n}{\Delta t} \right) \left\{ \left(\frac{x-x_i}{\Delta x} \right) u_{k,i+1}^n + \left(1 - \frac{x-x_i}{\Delta x} \right) u_{k,i}^n \right\}. \end{aligned} \quad (4.20)$$

In particular, we can see that

$$u^\epsilon(x_i, t_n) = (u_+^\epsilon(x_i, t_n), u_-^\epsilon(x_i, t_n)) = (u_{+,i}^n, u_{-,i}^n) = u_i^n \quad \text{for } n \in \{0, \dots, N_T-1\}, i \in \{0, \dots, M-1\}.$$

The function u^ϵ can be expanded on the whole space by considering it as a 1-periodic function + linear function. Namely, we can see that

$$u^\epsilon(x+1, t) = u^\epsilon(x, t) + L_0.$$

4.1.1 Estimate on u^ϵ

From Theorem 4.2, we deduce that $u_{i+1}^{n+1}, u_i^{n+1}, u_{i+1}^n$ and u_i^n are bounded by $\zeta + L_0$, for all $i \in \{0, \dots, M-1\}$ and for all $n \in \{0, \dots, N_T-1\}$ where ζ is given by (4.6). By (4.20), we remark that u^ϵ is a convex combination of $u_{i+1}^{n+1}, u_i^{n+1}, u_{i+1}^n$ and u_i^n . Then, u^ϵ is also bounded by $\zeta + L_0$, which implies that

$$\|u_k^\epsilon\|_{L^\infty((0,1) \times (0,T))} \leq \zeta + L_0. \quad (4.21)$$

4.1.2 Estimate on $\partial_x u^\epsilon$

We have for $(x, t) \in (x_i, x_{i+1}) \times [t_n, t_{n+1}]$

$$\partial_x u^\epsilon(x, t) = \left(\frac{t-t_n}{\Delta t} \right) \theta_{i+\frac{1}{2}}^{n+1} + \left(1 - \frac{t-t_n}{\Delta t} \right) \theta_{i+\frac{1}{2}}^n. \quad (4.22)$$

We know that

$$\begin{aligned} (\partial_x u_k^\epsilon(x, t))^2 &\leq \left(\frac{t-t_n}{\Delta t} \right) (\theta_{k,i+\frac{1}{2}}^{n+1})^2 + \left(1 - \left(\frac{t-t_n}{\Delta t} \right) \right) (\theta_{k,i+\frac{1}{2}}^n)^2 \\ &\leq (\theta_{k,i+\frac{1}{2}}^{n+1})^2 + (\theta_{k,i+\frac{1}{2}}^n)^2. \end{aligned}$$

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Then, we get for $(x, t) \in (x_i, x_{i+1}) \times [t_n, t_{n+1}]$

$$\begin{aligned} \int_0^1 (\partial_x u_k^\varepsilon(x, t))^2 dx &\leq \sum_{i=0}^{M-1} \int_{x_i}^{x_{i+1}} (\partial_x u_k^\varepsilon(x, t))^2 dx \leq \sum_{i=0}^{M-1} \Delta x (\theta_{k, i+\frac{1}{2}}^{n+1})^2 + \sum_{i=0}^{M-1} \Delta x (\theta_{k, i+\frac{1}{2}}^n)^2 \\ &\leq 2 \sum_{i=0}^{M-1} \Delta x (\theta_{k, i+\frac{1}{2}}^0)^2 \leq 2 \|\partial_x \rho_0^k\|_{L^2(\mathbb{T})}^2. \end{aligned}$$

We deduce that

$$\|\partial_x u_k^\varepsilon\|_{L^\infty((0, T); L^2(\mathbb{T}))} \leq \sqrt{2} \|\partial_x \rho_0^k\|_{L^2(\mathbb{T})}. \quad (4.23)$$

4.1.3 Estimate on $\partial_t u^\varepsilon$

Let us define

$$\tau_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\Delta t}.$$

We have for $(x, t) \in [x_i, x_{i+1}] \times (t_n, t_{n+1})$

$$\partial_t u^\varepsilon(x, t) = \left(\frac{x - x_i}{\Delta x} \right) \tau_{i+1}^{n+\frac{1}{2}} + \left(1 - \frac{x - x_i}{\Delta x} \right) \tau_i^{n+\frac{1}{2}}. \quad (4.24)$$

Using the scheme (4.3), we have

$$\tau_{k, i}^{n+\frac{1}{2}} = (C_{k, i}^{n+1}[u^{n+1}])_- \theta_{k, i+\frac{1}{2}}^{n+1} - (C_{k, i}^{n+1}[u^{n+1}])_+ \theta_{k, i-\frac{1}{2}}^{n+1}. \quad (4.25)$$

We have

$$\begin{aligned} (\partial_t u_k^\varepsilon(x, t))^2 &\leq \left(\frac{x - x_i}{\Delta x} \right) (\tau_{k, i+1}^{n+\frac{1}{2}})^2 + \left(1 - \left(\frac{x - x_i}{\Delta x} \right) \right) (\tau_{k, i}^{n+\frac{1}{2}})^2 \\ &\leq (\tau_{k, i+1}^{n+\frac{1}{2}})^2 + (\tau_{k, i}^{n+\frac{1}{2}})^2 \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\int_0^1 (\partial_t u_k^\varepsilon(x, t))^2 \\ &\leq \sum_{i=0}^{M-1} \Delta x (\tau_{k, i+1}^{n+\frac{1}{2}})^2 + \sum_{i=0}^{M-1} \Delta x (\tau_{k, i}^{n+\frac{1}{2}})^2 \\ &\leq \sum_{i=0}^{M-1} \Delta x \left[(C_{k, i+1}^{n+1}[u^{n+1}])_- \theta_{k, i+\frac{3}{2}}^{n+1} - (C_{k, i+1}^{n+1}[u^{n+1}])_+ \theta_{k, i+\frac{1}{2}}^{n+1} \right]^2 \\ &\quad + \sum_{i=0}^{M-1} \Delta x \left[(C_{k, i}^{n+1}[u^{n+1}])_- \theta_{k, i+\frac{1}{2}}^{n+1} - (C_{k, i}^{n+1}[u^{n+1}])_+ \theta_{k, i-\frac{1}{2}}^{n+1} \right]^2. \end{aligned}$$

Then, using the periodicity of the gradient, we can see that

$$\begin{aligned}
 \int_0^1 (\partial_t u_k^\varepsilon(x, t))^2 &\leq 4 \sum_{i=0}^{M-1} \Delta x \left((C_{k,i+1}^{n+1}[u^{n+1}])_+ \theta_{k,i+\frac{1}{2}}^{n+1} \right)^2 + 4 \sum_{i=0}^{M-1} \Delta x \left((C_{k,i}^{n+1}[u^{n+1}])_- \theta_{k,i+\frac{1}{2}}^{n+1} \right)^2 \\
 &\leq 4\Lambda^2 \sum_{i=0}^{M-1} \Delta x (\theta_{k,i+\frac{1}{2}}^{n+1})^2 + 4\Lambda^2 \sum_{i=0}^{M-1} \Delta x (\theta_{k,i+\frac{1}{2}}^n)^2 \\
 &\leq 8\Lambda^2 \sum_{i=0}^{M-1} \Delta x (\theta_{k,i+\frac{1}{2}}^0)^2,
 \end{aligned}$$

where $\Lambda = (2(\zeta + 1) + 2L_0)(1 + |\alpha|) + \|a\|_{L^\infty(0,T)}$. Hence, we get

$$\|\partial_t u_k^\varepsilon\|_{L^\infty((0,T);L^2(\mathbb{T}))} \leq 2\sqrt{2}\Lambda \|\partial_x \rho_0^k\|_{L^2(\mathbb{T})}. \quad (4.26)$$

4.2 Extraction of a convergent subsequence of u^ε

We start by recalling the following Lemma which is necessary for the proof.

Lemma 4.1. (*Simon's Lemma [67, Corollary 4]*)

Let X , B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a uniformly bounded sequence in $L^\infty((0, T); X)$ and $(\partial_t \theta_n)_n$ is uniformly bounded in $L^r((0, T); Y)$ for $r > 1$, then $(\theta_n)_n$ is relatively compact in $C((0, T); B)$.

Combining estimates (4.21), (4.23) and (4.26), we deduce that $u_k^\varepsilon = w_k^\varepsilon + L_0 x$, where w_k^ε is 1-periodic function, is uniformly bounded in $L^\infty((0, T); H^1(0, 1))$, for any $T > 0$ and w_k^ε satisfies the following estimate

$$\|w_k^\varepsilon\|_{L^\infty(\mathbb{T} \times (0, T))} + \|\partial_x w_k^\varepsilon\|_{L^\infty((0, T); L^2(\mathbb{T}))} + \|\partial_t w_k^\varepsilon\|_{L^\infty((0, T); L^2(\mathbb{T}))} \leq \tilde{C}, \quad \text{for all } k \in \{+, -\},$$

where \tilde{C} is a constant depending only on $\alpha, L_0, \|a\|_{L^\infty(0, T)}, \|\rho_0^p\|_{(L^\infty(\mathbb{T}))^2}, \|\partial_x \rho_0^k\|_{L^2(\mathbb{T})}$ and T .

By estimates (4.21), (4.23) and (4.26) and using the compact embedding of $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, we can show that the sequence $(w_k^\varepsilon)_\varepsilon$ is relatively compact in $L^\infty(\mathbb{T} \times (0, T))$, for all $k \in \{+, -\}$ by applying Simon's Lemma for $X = H^1(\mathbb{T})$, $Y = L^2(\mathbb{T})$ and $B = L^\infty(\mathbb{T})$. Then, as ε goes to 0, we can extract a subsequence still denoted by $(w_k^{\varepsilon_\alpha})_{\varepsilon_\alpha}$ that converges strongly to some function w_k in $L^\infty(\mathbb{T} \times [0, T])$. By standard diagonalization procedure, we can extract a subsequence $(w_k^\varepsilon)_\varepsilon$ such that

$$w_k^\varepsilon \longrightarrow w_k, \quad \text{uniformly in } L^\infty(\mathbb{T} \times [0, T]), \quad \forall k \in \{+, -\}.$$

Moreover, using (4.23), we deduce that, up to extraction of a subsequence,

$$\partial_x w_k^\varepsilon \rightharpoonup \partial_x w_k \quad \text{weakly-} \star \quad \text{in } L^\infty((0, T); L^2(\mathbb{T})) \quad \text{for all } k \in \{+, -\}, \quad (4.27)$$

and using (4.26), we can see that up to extraction of a subsequence,

$$\partial_t w_k^\varepsilon \rightharpoonup \partial_t w_k \quad \text{weakly-} \star \quad \text{in } L^\infty((0, T); L^2(\mathbb{T})) \quad \text{for all } k \in \{+, -\}.$$

In particular, we see that the limit function w_k satisfies the initial data $w_k(\cdot, 0) = w_0^k(\cdot)$.

4.3 Passing to the limit in the PDE

4.3.1 Preliminaries

We show in the following Lemma a modulus of continuity satisfied by w_k^ε which is necessary for the proof of the convergence.

Lemma 4.2. (Continuity)

Let $T > 0$. Assume that $w \in L^\infty(\mathbb{T} \times (0, T))$ such that

$$\|\partial_x w\|_{L^\infty((0, T); L^2(\mathbb{T}))} + \|\partial_t w\|_{L^\infty((0, T); L^2(\mathbb{T}))} \leq C.$$

Then, for all $\delta, h \geq 0$ and all $(x, t) \in \mathbb{T} \times (0, T - \delta)$, we have:

$$|w(x + h, t + \delta) - w(x, t)| \leq \left(h^{\frac{1}{2}} + 3\delta^{\frac{1}{2}}\right) C.$$

Proof of Lemma 4.2:

For all $h > 0$ and $(x, t) \in \mathbb{T} \times (0, T - \delta)$, we have:

$$|w(x + h, t) - w(x, t)| \leq \delta \left| \int_x^{x+h} \partial_x w(y, t) dy \right| \leq h^{\frac{1}{2}} \|\partial_x w\|_{L^\infty((0, T); L^2(\mathbb{T}))} \leq h^{\frac{1}{2}} C, \quad (4.28)$$

where we have applied Hölder's inequality in the second line and in last line we have used the fact that $\partial_x w \in L^\infty((0, T); L^2(\mathbb{T}))$. Which proves the continuity in space.

Now, we prove the continuity in time, for all $\delta > 0$ and $(x, t) \in \mathbb{T} \times (0, T - \delta)$, we have:

$$\begin{aligned} & \delta |w(x, t + \delta) - w(x, t)| \\ &= \int_x^{x+\delta} |w(x, t + \delta) - w(x, t)| dy, \\ &\leq \overbrace{\int_x^{x+\delta} |w(x, t + \delta) - w(y, t + \delta)| dy}^{K_1} \\ &\quad + \overbrace{\int_x^{x+\delta} |w(y, t + \delta) - w(y, t)| dy}^{K_2} \\ &\quad + \overbrace{\int_x^{x+\delta} |w(y, t) - w(x, t)| dy}^{K_3}. \end{aligned}$$

Similarly, as in the last estimate (4.28), we can show that:

$$\begin{aligned}
 K_1 + K_3 &\leq \delta \int_x^{x+\delta} |\partial_x w(y, t + \delta)| dy + \delta \int_x^{x+\delta} |\partial_x w(y, t)| dy \\
 &\leq \delta \int_x^{x+\delta} |\partial_x w(y, t + \delta)| dy + \delta \int_x^{x+\delta} |\partial_x w(y, t)| dy \\
 &\leq \delta \delta^{\frac{1}{2}} \|\partial_x w\|_{L^\infty((0, T); L^2(\mathbb{T}))} + \delta \delta^{\frac{1}{2}} \|\partial_x w\|_{L^\infty((0, T); L^2(\mathbb{T}))} \\
 &\leq 2\delta \delta^{\frac{1}{2}} C,
 \end{aligned}$$

where we have applied Hölder's inequality in the third line and in last line we have used the fact that $\partial_x w \in L^\infty((0, T); L^2(\mathbb{T}))$.

Now, we use the fact that $\partial_t w$ is bounded in $L^\infty((0, T); L^2(\mathbb{T}))$, to get

$$\begin{aligned}
 K_2 &\leq \int_x^{x+\delta} \int_t^{t+\delta} |\partial_t w(y, s)| ds dy \\
 &\leq \delta \delta^{\frac{1}{2}} C.
 \end{aligned}$$

Collecting the estimates of K_1 , K_2 and K_3 , we obtain

$$|w(x, t + \delta) - w(x, t)| \leq 3\delta^{\frac{1}{2}} C,$$

The previous inequality joint to (4.28) implies the result. \square

Now, for $(x, t) \in (x_i, x_{i+1}) \times (t_n, t_{n+1})$, we set

$$\begin{aligned}
 G_k[u^\varepsilon](x, t) &= k \left((u_+^\varepsilon - u_-^\varepsilon)(x, t) + \alpha \int_0^1 (u_+^\varepsilon - u_-^\varepsilon)(y, t) dy + a(t) \right) = G_k[w^\varepsilon](x, t), \\
 a_x &= \frac{x - x_i}{\Delta x}, \quad b_x = 1 - a_x.
 \end{aligned}$$

Combining (4.24) and (4.25), we have

$$\begin{aligned}
 \partial_t u_k^\varepsilon(x, t) &= a_x \left[(C_{k, i+1}^{n+1}[u^{n+1}]_- - \theta_{k, i+\frac{3}{2}}^{n+1}) - (C_{k, i+1}^{n+1}[u^{n+1}]_+ + \theta_{k, i+\frac{1}{2}}^{n+1}) \right] \\
 &\quad + b_x \left[(C_{k, i}^{n+1}[u^{n+1}]_- - \theta_{k, i+\frac{1}{2}}^{n+1}) - (C_{k, i}^{n+1}[u^{n+1}]_+ + \theta_{k, i-\frac{1}{2}}^{n+1}) \right] \\
 &= (G_k[u^\varepsilon](x, t))_- \left[a_x \theta_{k, i+\frac{3}{2}}^{n+1} + b_x \theta_{k, i+\frac{1}{2}}^{n+1} \right] - (G_k[u^\varepsilon](x, t))_+ \left[a_x \theta_{k, i+\frac{1}{2}}^{n+1} + b_x \theta_{k, i-\frac{1}{2}}^{n+1} \right] + e_k^\varepsilon(x, t),
 \end{aligned} \tag{4.29}$$

where

$$\begin{aligned}
 e_k^\varepsilon(x, t) &= a_x \left[- \left[(C_{k, i+1}^{n+1}[u^{n+1}]_+ - (G_k[u^\varepsilon](x, t))_+ \right) \theta_{k, i+\frac{1}{2}}^{n+1} + \left[(C_{k, i+1}^{n+1}[u^{n+1}]_- - (G_k[u^\varepsilon](x, t))_- \right) \theta_{k, i+\frac{3}{2}}^{n+1} \right] \right. \\
 &\quad \left. + b_x \left[- \left[(C_{k, i}^{n+1}[u^{n+1}]_+ - (G_k[u^\varepsilon](x, t))_+ \right) \theta_{k, i-\frac{1}{2}}^{n+1} + \left[(C_{k, i}^{n+1}[u^{n+1}]_- - (G_k[u^\varepsilon](x, t))_- \right) \theta_{k, i+\frac{1}{2}}^{n+1} \right] \right].
 \end{aligned}$$

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In particular, for any 1-periodic test function ϕ with compact support $K = \mathbb{T} \times [0, T]$, we have

$$\begin{aligned} & \sum_{k=+,-} \left| \int_K \phi(y, \tau) e_k^\varepsilon(y, \tau) d\tau dy \right| \\ & \leq T \sum_{k=+,-} \left(\sum_{i=0}^{M-1} \Delta x (\theta_{k,i+\frac{1}{2}}^0)^2 \right)^{\frac{1}{2}} \|\phi\|_{L^\infty(\mathbb{T} \times (0, T))} \sup_{(y, \tau) \in K} \left(\sup_{|t_{n+1}-\tau| \leq \Delta t, |x_i-y| \leq \Delta x} |C_{k,i}^{n+1}[u^\varepsilon] - G_k[u^\varepsilon](y, \tau)| \right. \\ & \qquad \qquad \qquad \left. + \sup_{|t_{n+1}-\tau| \leq \Delta t, |x_{i+1}-y| \leq \Delta x} |C_{i+1}^{n+1}[u^\varepsilon] - G_k[u^\varepsilon](y, \tau)| \right). \end{aligned}$$

We know that u_k^ε can be written as

$$u_k^\varepsilon = w_k^\varepsilon + L_0 x,$$

with w_k^ε is a 1-periodic function belonging to $L^\infty(\mathbb{T} \times [0, T])$. Hence, using the uniform convergence of u^ε , the fact that $a \in C(0, T)$, (4.23), (4.26) and applying Lemma 4.2 on w_k^ε , we can deduce that

$$e_k^\varepsilon \longrightarrow 0 \quad \text{when} \quad \varepsilon = (\Delta x, \Delta t) \longrightarrow (0, 0) \quad \text{in} \quad \mathcal{D}'(\mathbb{T} \times (0, T)).$$

4.3.2 Convergence of the whole sequence

Our goal is to show that the limit $u_k = w_k + L_0 x$ verifies $\partial_t u_k + G_k[u] \partial_x u_k = 0$.

We define the function θ^ε as follows

$$\theta^\varepsilon(x, t) = \theta_{i+\frac{1}{2}}^{n+1}, \quad \text{for} \quad (x, t) \in [x_i, x_{i+1}) \times [t_n, t_{n+1}).$$

Due to the estimate (4.19), we deduce that θ^ε is uniformly bounded in $L^\infty((0, T); L^2(\mathbb{T}))$. Then, by the weakly- \star convergence, we know that there exists a limit θ , in the sense of distribution, such that for any 1-periodic test function φ (φ is smooth of compact support $K = \mathbb{T} \times [0, T]$) we have

$$\int_{\mathbb{T} \times (0, T)} \theta^\varepsilon \cdot \varphi \longrightarrow \int_{\mathbb{T} \times (0, T)} \theta \cdot \varphi. \quad (4.30)$$

From (4.29), we have

$$\begin{aligned} \partial_t u_k^\varepsilon(x, t) - e_k^\varepsilon(x, t) &= (G_k[w^\varepsilon](x, t))_- [a_x \theta_k^\varepsilon(x + \Delta x, t) + b_x \theta_k^\varepsilon(x, t)] \\ &\quad - (G_k[w^\varepsilon](x, t))_+ [a_x \theta_k^\varepsilon(x, t) + b_x \theta_k^\varepsilon(x - \Delta x, t)], \end{aligned} \quad (4.31)$$

where $a_x = \frac{x}{\Delta x} - \lfloor \frac{x}{\Delta x} \rfloor$ and $b_x = 1 - a_x$.

Let us now introduce the function A_k^ε defined by

$$A_k^\varepsilon = \int_{\mathbb{T} \times (0, T)} (\partial_t u_k^\varepsilon(x, t) - e_k^\varepsilon(x, t)) \varphi(x, t) dx dt.$$

Then, using (4.31), using the periodicity of φ, w^ε and θ^ε , we can write A_k^ε as follows

$$\begin{aligned} A_k^\varepsilon &= \int_{\mathbb{T} \times (0, T)} \theta_k^\varepsilon(x, t) [a_x(G_k[w^\varepsilon](x - \Delta x, t))_- \varphi(x - \Delta x, t) + b_x(G_k[w^\varepsilon](x, t))_- \varphi(x, t)] dx dt \\ &\quad - \int_{\mathbb{T} \times (0, T)} \theta_k^\varepsilon(x, t) [a_x(G_k[w^\varepsilon](x, t))_+ \varphi(x, t) + b_x(G_k[w^\varepsilon](x + \Delta x, t))_+ \varphi(x + \Delta x, t)] dx dt. \end{aligned}$$

Let us now define the following function

$$\begin{aligned} B_k^\varepsilon &= \int_{\mathbb{T} \times (0, T)} \theta_k^\varepsilon(x, t) [a_x(G_k[w^\varepsilon](x, t))_- \varphi(x, t) + b_x(G_k[w^\varepsilon](x, t))_- \varphi(x, t) \\ &\quad - a_x(G_k[w^\varepsilon](x, t))_+ \varphi(x, t) - b_x(G_k[w^\varepsilon](x, t))_+ \varphi(x, t)] dx dt \\ &= \int_{\mathbb{T} \times (0, T)} -G_k[w^\varepsilon](x, t) \varphi(x, t) \theta_k^\varepsilon(x, t) dx dt. \end{aligned}$$

Then, we have

$$\begin{aligned} |A_k^\varepsilon - B_k^\varepsilon| &\leq \left| \int_{\mathbb{T} \times (0, T)} a_x \theta_k^\varepsilon(x, t) [(G_k[w^\varepsilon](x - \Delta x, t))_- \varphi(x - \Delta x, t) - (G_k[w^\varepsilon](x, t))_- \varphi(x, t)] dx dt \right| \\ &\quad + \left| \int_{\mathbb{T} \times (0, T)} b_x \theta_k^\varepsilon(x, t) [-(G_k[w^\varepsilon](x + \Delta x, t))_+ \varphi(x + \Delta x, t) + (G_k[w^\varepsilon](x, t))_+ \varphi(x, t)] dx dt \right| \\ &\leq T \|\theta_k^\varepsilon\|_{L^\infty((0, T); L^2(\mathbb{T}))} \sup_{(x, t) \in K} |(G_k[w^\varepsilon](x \mp \Delta x, t))_\mp \varphi(x \mp \Delta x, t) - G_k[w^\varepsilon](x, t)_\mp \varphi(x, t)|. \end{aligned}$$

Then, using the modulus of continuity of w^ε , we can see that $|A_k^\varepsilon - B_k^\varepsilon| \rightarrow 0$ when ε goes to 0.

On the other hand, using (4.30), the weakly- \star convergence of θ^ε to θ and the strong convergence of $G_k[w^\varepsilon]$ to $G_k[w]$, we get

$$B_k^\varepsilon = \int_{\mathbb{T} \times (0, T)} -G_k[w^\varepsilon](x, t) \varphi(x, t) \theta_k^\varepsilon(x, t) dx dt \longrightarrow \int_{\mathbb{T} \times (0, T)} -G_k[w](x, t) \varphi(x, t) \theta_k(x, t) dx dt.$$

Therefore, we obtain that

$$\partial_t u_k(x, t) + G_k[u](x, t) \theta_k(x, t) = 0 \quad \text{in } \mathcal{D}'(\mathbb{T} \times (0, T)), \quad \forall k \in \{+, -\}. \quad (4.32)$$

Using (4.22), we can see that

$$\partial_x u_k^\varepsilon(x, t) = \left(\frac{t - t_n}{\Delta t} \right) \theta_k^\varepsilon(x, t) + \left(1 - \frac{t - t_n}{\Delta t} \right) \theta_k^\varepsilon(x, t - \Delta t).$$

We deduce, proceeding in a similar manner, that

$$\partial_x u_k^\varepsilon \longrightarrow \theta_k \quad \text{in } \mathcal{D}'(\mathbb{T} \times (0, T)).$$

4. Un schéma convergent pour un système de transport non-local modélisant la dynamique des densités des dislocations

Moreover, using the weak- \star convergence given by (4.27), we deduce that

$$\theta = \partial_x u. \tag{4.33}$$

Combining (4.32) with (4.33), we conclude that

$$\partial_t u_k + G_k[u] \partial_x u_k = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{T} \times (0, T)), \quad \forall k \in \{+, -\}.$$

Finally, we deduce that u_k is a solution of system (4.1).

Due to the uniqueness of the solution of the continuous problem, given by Theorem 4.1, we deduce that the whole sequence $(u_k^\varepsilon)_\varepsilon$ converges to the unique limit ρ^k . \square

5

Solution continue pour un système eikonal non-linéaire

Ce chapitre est un article accepté dans le Journal *Communications on Pure and Applied Analysis* écrit en collaboration avec Ahmad El Hajj.

Dans ce chapitre, notre étude se focalise sur un système eikonal non-linéaire décrivant l'évolution des interfaces se déplaçant avec des vitesses sans signe fortement couplées.

Ce travail constitue une généralisation de celui effectué dans le Chapitre 3. En effet, nous commençons par la régularisation du problème en lui ajoutant un terme de viscosité. Ensuite, en se basant sur une estimation entropique du gradient et d'autres estimations telle que l'estimation BV , nous montrons un résultat d'existence globale d'une solution de viscosité continue. Enfin, sous condition que le système soit quasi-monotone, nous démontrons l'unicité de la solution via un principe de comparaison.

Continuous solution for a non-linear eikonal system

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Abstract

In this work, we are dealing with a non-linear eikonal system in one dimensional space that describes the evolution of interfaces moving with non-signed strongly coupled velocities. We prove a global existence result in the framework of continuous viscosity solution. The approach is made by adding a viscosity term and passing to the limit for vanishing viscosity, relying on a new gradient entropy and BV estimates. A uniqueness result is also proved through a comparison principle property.

1 Introduction and main results

1.1 Setting of the problem

In this paper, we are interested in continuous solutions to eikonal systems in one dimension. More precisely, our work will focus on solutions $u(t, x) = (u^i(t, x))_{i=1, \dots, d}$, where d is an integer, of the following systems, for all $T > 0$

$$\partial_t u^i(t, x) = \lambda^i(t, x, u) |\partial_x u^i(t, x)| \quad \text{on } (0, T) \times \mathbb{R}, \quad \text{for } i = 1, \dots, d, \quad (5.1)$$

with the initial data:

$$u^i(0, x) = u_0^i(x), \quad x \in \mathbb{R}, \quad \text{for } i = 1, \dots, d. \quad (5.2)$$

Here we use the notations $\partial_t = \frac{\partial}{\partial t}$ and $\partial_x = \frac{\partial}{\partial x}$.

Our goal is to show that this system admits a unique continuous solution assuming that the velocity $\lambda^i : (0, T) \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}$ satisfies, for all $i \in \{1, \dots, d\}$, the following regularity:

$$(H1) \quad \left\{ \begin{array}{l} \lambda^i \in L^\infty((0, T) \times \mathbb{R} \times \mathcal{U}) \cap C((0, T) \times \mathbb{R} \times \mathcal{U}), \text{ for } i = 1, \dots, d, \\ \text{there exists } M_1 > 0 \text{ such that for } i = 1, \dots, d, \\ \sum_{i=1}^d |\lambda^i(t, x, u) - \lambda^i(t, y, v)| \leq M_1 (|u - v| + |x - y|) \text{ for all } u, v \in \mathcal{U}, x, y \in \mathbb{R} \text{ and } t \in (0, T), \end{array} \right.$$

where for real numbers $\alpha^i \leq \beta^i$, \mathcal{U} is the following box:

$$\mathcal{U} = \prod_{i=1}^d [\alpha^i, \beta^i], \quad (5.3)$$

and $|w| = \sum_{i=1, \dots, d} |w^i|$, for $w = (w^1, \dots, w^d)$.

We also assume that the Jacobian matrix of the velocity λ^i , denoted by

$$(\lambda_{,j}^i(t, x, u))_{i,j=1, \dots, d} \quad \text{where} \quad \lambda_{,j}^i = \frac{\partial \lambda^i}{\partial w^j},$$

is negative, namely

$$(H2) \quad \left| \sum_{i,j=1}^d w^i w^j \lambda_{,j}^i(t, x, u) \leq 0, \quad \forall (t, x, u) \in (0, T) \times \mathbb{R} \times \mathcal{U}, \quad \text{and} \quad \forall w = (w^1, \dots, w^d) \in \mathbb{R}^d. \right.$$

In (5.2), each component u_0^i of the initial data $u_0 = (u_0^1, \dots, u_0^d)$ is assumed to satisfy the following regularity:

$$(H3) \quad \left\{ \begin{array}{l} \alpha^i \leq u_0^i \leq \beta^i, \\ \partial_x u_0^i \in L^1(\mathbb{R}) \cap L \log L(\mathbb{R}), \end{array} \right. \quad \text{for } i = 1, \dots, d,$$

where $L \log L(\mathbb{R})$ is the following Zygmund space:

$$L \log L(\mathbb{R}) = \left\{ f \in L_{loc}^1(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} |f| \ln(e + |f|) < +\infty \right\}. \quad (5.4)$$

This space is equipped with the following norm:

$$\|f\|_{L \log L(\mathbb{R})} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}} \frac{|f|}{\mu} \ln \left(e + \frac{|f|}{\mu} \right) \leq 1 \right\}. \quad (5.5)$$

This norm is due to Luxemburg (see Adams [4], (13), Page 234).

We now present the main results.

1.2 Main results

Theorem 5.1. (Global existence and uniqueness of solution)

Assume that (H1), (H2) and (H3) are satisfied. Then, the following hold:

i) Existence of a continuous viscosity solution:

There exists a continuous viscosity solution u^i of (5.1)-(5.2), for all $i \in \{1, \dots, d\}$, where

$$u^i \in L^\infty((0, T) \times \mathbb{R}) \quad \text{and} \quad \partial_x u^i \in L^\infty((0, T); L \log L(\mathbb{R})).$$

Moreover, for all $i \in \{1, \dots, d\}$, u^i satisfies the following bounds:

$$\|u^i\|_{L^\infty((0, T) \times \mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \quad (5.6)$$

$$\|\partial_x u^i\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq \|\partial_x u_0^i\|_{L^1(\mathbb{R})} \quad (5.7)$$

and the following gradient entropy estimate:

$$\sum_{i=1}^d \|\partial_x u^i\|_{L^\infty((0,T);L \log L(\mathbb{R}))} \leq \gamma_1, \quad (5.8)$$

where $\gamma_1 \left(T, d, M_1, \Lambda, \|\partial_x u_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d} \right)$ with

$$\Lambda = \sum_{i=1}^d \Lambda^i \quad \text{and} \quad \Lambda^i = \sup_{\substack{t \in (0,T) \\ x \in \mathbb{R} \\ u \in \mathcal{U}}} |\lambda^i(t, x, u)| \quad \forall i \in \{1, \dots, d\}. \quad (5.9)$$

ii) Uniqueness of solution:

Suppose that the velocity verifies the following monotonicity assumption

$$(H4) \quad \left\{ \begin{array}{l} \lambda^j(t, x, r) - \lambda^j(t, x, s) \leq 0, \quad \text{for all vectors } r = (r^i)_{i=1, \dots, d}, s = (s^i)_{i=1, \dots, d}, \text{ such that} \\ r^j - s^j = \max_{i \in \{1, \dots, d\}} (r^i - s^i) \geq 0. \end{array} \right.$$

Then, the solution given by (i) is unique.

The key point to establish Theorem 5.1 is the gradient entropy estimate (5.8). We first consider the parabolic regularization of equation (5.1) and then we show that the smooth solution admits the L^∞ bound (5.6) and the following fundamental gradient entropy inequality:

$$\int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x u^i(t, x)|) dx - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \lambda_{i,j}^i(t, x, u) \partial_x u^i(t, x) \partial_x u^j(t, x) \leq C, \quad \text{for all } t \in (0, T) \quad (5.10)$$

where

$$0 \leq f(x) = \begin{cases} x \ln(x) + \frac{1}{e} & \text{if } x \geq 1/e, \\ 0 & \text{if } 0 \leq x \leq 1/e, \end{cases} \quad (5.11)$$

and C is a positive constant only depends on T, d, M_1 and $\|\partial_x u_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d}$.

These *a priori* estimates will allow us to pass to the limit when the regularization vanishes, using compactness arguments and stability of viscosity solutions, which leads us to prove the existence of a continuous viscosity solution. Note that, a similar gradient entropy inequality was used in [34, 35] to prove the existence and the uniqueness of solutions of (5.1) in the case of nondecreasing solutions. Also, a similar gradient entropy inequality was introduced, in the case of nondecreasing solutions, by Cannone et al. [24] to prove the global existence for a particular two-dimensional system of two coupled transport equations modeling the dynamics of dislocations densities.

In this paper, we show through Theorem 5.1 the global existence and the uniqueness of a continuous viscosity solution of system (5.1) without any monotonicity assumptions on the initial data.

1.3 A brief review of some related litterature

Many existence and uniqueness results were brightened up on similar eikonal systems. Let us mention the most known results. First, motivated by dislocation dynamics, we can point out the result done by

El Hajj and Boudjerada in [19] wherein the global existence of discontinuous viscosity BV solutions was proved for scalar one dimensional non-linear and non-local eikonal equations, including in particular the case $d = 1$ in system (5.1).

Always within the framework of dislocation dynamics, let us mention that, it was proven in El Hajj et al. [30, 33, 36] some existence and uniqueness results of continuous viscosity solution for a particular quasi-monotone (2×2) system which corresponds to the case where $(\lambda_{,j}^i)_{i,j=1,2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. The proof in [33, 36] was based on the work of Ishii, Koike [46, 47] wherein, a general theory concerning the existence and the uniqueness of viscosity solutions for quasi-monotone Hamilton-Jacobi systems, has been developed. However, in [30], the author proved a L^2 energy estimate in order to get the existence and the uniqueness. An existence result of a discontinuous BV solutions was also proved in [38] for the same (2×2) system. This result was recently extended in [37] to the case of $(d \times d)$ system under certain monotonicity condition on the velocity. In this paper we give a global existence and uniqueness result of a continuous solution in the case of general $(d \times d)$ eikonal systems without any monotonicity assumptions and sign restrictions on the velocity. This enables us to enlarge the area of applications to touch dislocation dynamics in multi-directions of propagation. In particular, system (5.1) appears naturally in the domain of dislocations densities dynamics where dislocations are moving defects presented in the crystals. In fact, in the special geometry where the dislocations assumed to be straight parallel lines (in the 3-dimensional space) and to be dependent on one variable, it is possible to describe the dynamics of these defects by system (5.1) with

$$\lambda^i(t, x, u) = \sum_{j=1}^d A_{ij} u^j$$

for some constant and non-negative symmetric matrix A (see [34, Section 5] for more modeling details).

It is worth mentioning that in the case of nondecreasing initial data, system (5.1) becomes a hyperbolic diagonal system for which results of existence and uniqueness have been shown. First, in the case of (2×2) strictly hyperbolic system with nondecreasing initial data, Lax proved in [54], the existence and uniqueness of nondecreasing smooth solutions. Also, assuming that the initial data is nondecreasing, an existence and uniqueness result of continuous solution was proved in [35] for a general $(d \times d)$ diagonal strictly hyperbolic system. The proof in [35] is based on a global existence result of a continuous nondecreasing solution established before in [34], within the framework of diagonal hyperbolic systems that are not necessarily strictly hyperbolic. In the present work, we prove an existence and uniqueness results without any monotony assumptions on the initial data, unlike what is done in [34]. In fact, we show that the gradient entropy estimate proved in [34] remains valid for a class of eikonal type systems without assuming that the initial conditions are nondecreasing. This requires obtaining a preliminary BV estimate which wasn't necessary in [34]. We also refer the reader to [16] where an interesting theory relating to the existence and uniqueness of a semi-group type solution was developed for general $(d \times d)$ strictly hyperbolic system, considering small initial data in BV norm. This approach is mainly based on a careful analysis of the vanishing viscosity approximation, giving rise to a L^1 stability estimate. Recently, it was also shown in [5] the global existence of a discontinuous viscosity solution for any $(d \times d)$ diagonal hyperbolic systems with all BV initial data.

1.4 Organization of the paper

This paper is organized as follows: In Section 2, we regularize system (5.1) by adding the viscosity term $\eta \partial_{xx}^2 u^i$, replacing the non-linear term $|\partial_x u^i|$ by $\sqrt{(\partial_x u^i)^2 + \delta^2}$ and considering regular initial data. Then, we show that this system admits a local in time regular solution, applying a fixed point argument. Section 3 is devoted to prove some (η, δ) -uniform *a priori* estimates. More precisely, we start by showing that the solution of the regularized problem $v_{\eta, \delta}^i$ satisfies a *BV* estimate and $\partial_{xx}^2 v_{\eta, \delta}^i$ verifies a L^2 preliminary estimate. Then, these estimates allow us, to obtain a gradient entropy estimate which is the key result of our paper. In Section 4, we prove some η -uniform *a priori* estimates similar to that obtained in Section 3 by passing to the limit as $\delta \rightarrow 0$. In other words, we prove, that the function v_η^i defined as the limit of $v_{\eta, \delta}^i$ when $\delta \rightarrow 0$ also satisfies a *BV* estimate and a gradient entropy estimate. After that, basing on these *a priori* estimates, we show in Section 5 that the local solution can be extended to a global one. In Section 6, we get using the stability results of continuous viscosity solutions the global existence of continuous viscosity solution with regular initial data. This leads us to show Theorem 5.1 (i) by passing to limit as $(\varepsilon, \eta) \rightarrow (0, 0)$ and using the established estimates. Finally, by comparison principle, we obtain in Section 7 the uniqueness result given by Theorem 5.1 (ii).

2 Local solution for parabolic regularized equation

We recall some preliminary local in time existence and uniqueness results for parabolic equation obtained by regularization of problem (5.1) with smoothed data $v_0 = (v_0^i)_{i=1, \dots, d}$. More precisely, we consider for $0 < \eta \leq 1$, $i = 1, \dots, d$ and $v_\eta = (v_\eta^i)_{i=1, \dots, d}$, the following problem:

$$\begin{cases} \partial_t v_\eta^i(t, x) - \eta \partial_{xx}^2 v_\eta^i(t, x) = \tilde{\lambda}^i(t, x, v_\eta) |\partial_x v_\eta^i(t, x)| & \text{on } (0, T) \times \mathbb{R}, \\ v_\eta^i(0, x) = v_0^i(x) & x \in \mathbb{R}, \end{cases} \quad (5.12)$$

where, for all $i \in \{1, \dots, d\}$, we have

$$v_0^i \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{and} \quad \partial_x v_0^i \in L^p(\mathbb{R}) \quad \text{for all } 1 \leq p \leq +\infty. \quad (5.13)$$

Moreover, $\tilde{\lambda}^i : (0, T) \times \mathbb{R} \times \mathcal{U} \rightarrow \mathbb{R}$ satisfies, for all $i \in \{1, \dots, d\}$, the following assumptions:

$$(\widetilde{H1}) \quad \left\{ \begin{array}{l} \tilde{\lambda}^i \in L^\infty((0, T) \times \mathbb{R} \times \mathcal{U}) \cap C^\infty((0, T) \times \mathbb{R} \times \mathbb{R}^d), \text{ for } i = 1, \dots, d, \\ \text{there exists } \widetilde{M}_1 > 0 \text{ such that for } i = 1, \dots, d, \\ \sum_{j=1}^d |\tilde{\lambda}^i(t, x, u) - \tilde{\lambda}^i(t, y, v)| \leq \widetilde{M}_1 (|u - v| + |x - y|) \text{ for all } u, v \in \mathcal{U}, x, y \in \mathbb{R} \text{ and } t \in (0, T), \end{array} \right.$$

and

$$(\widetilde{H2}) \quad \left| \sum_{i,j=1}^d w^i w^j \tilde{\lambda}_{,j}^i(t, x, u) \leq 0 \quad \forall (t, x, u) \in (0, T) \times \mathbb{R} \times \mathcal{U}, \quad \text{and} \quad \forall w = (w^1, \dots, w^d) \in \mathbb{R}^d. \right.$$

5. Solution continue pour un système eikonal non-linéaire

We also regularize the non-linear term $|\partial_x v_\eta^i|$ replacing it by a smooth function $\beta_\delta(\partial_x v_\eta^i)$, where β_δ is defined as follows:

$$\beta_\delta(x) = \sqrt{x^2 + \delta^2} \quad \text{for all } 0 < \delta \leq 1. \quad (5.14)$$

Then, we truncate the function β_δ by plugging the function

$$\psi_\delta(\cdot) = \phi(\sqrt{\delta} \cdot), \quad (5.15)$$

where $\phi \in C^\infty(\mathbb{R})$ is a cut-off function taking values in $[0, 1]$, supported on the interval $[-2, 2]$ and $\phi(x) \equiv 1$ on $[-1, 1]$. This brings us finally to consider for all $0 < \delta \leq \frac{1}{e}$, $i = 1, \dots, d$ and for $v_{\eta,\delta} = (v_{\eta,\delta}^i)_{i=1,\dots,d}$, the following problem:

$$\begin{cases} \partial_t v_{\eta,\delta}^i(t, x) - \eta \partial_{xx}^2 v_{\eta,\delta}^i(t, x) = \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta,\delta}^i(t, x)) & \text{on } (0, T) \times \mathbb{R}, \\ v_{\eta,\delta}^i(0, x) = v_0^i(x) & x \in \mathbb{R}. \end{cases} \quad (5.16)$$

Before communicating the result of this section, let us note

$$\tilde{\Lambda}^i = \|\tilde{\lambda}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \quad \text{and} \quad \tilde{\Lambda} = \sum_{i=1}^d \tilde{\Lambda}^i. \quad (5.17)$$

For the regularized equations (5.12) and (5.16), we have the following result.

Theorem 5.2. (Existence and uniqueness of Lipschitz continuous solution of (5.12)) *Assume that (H1) and (5.13) hold. Then, there exists $T^* > 0$ such that, we have*

i) (Existence and uniqueness of (5.16)):

System (5.16) admits a unique solution $v_{\eta,\delta} = (v_{\eta,\delta}^i)_{i=1,\dots,d} \in (C^\infty((0, T^) \times \mathbb{R}))^d \cap (W^{1,\infty}((0, T^*) \times \mathbb{R}))^d$, satisfying the following estimates:*

$$\|v_{\eta,\delta}^i\|_{L^\infty((0,T^*) \times \mathbb{R})} \leq T^* \delta \tilde{\Lambda}^i + \|v_0^i\|_{L^\infty(\mathbb{R})} \quad \text{for all } i = 1, \dots, d. \quad (5.18)$$

Moreover, there exists a constant ξ_p^δ depending on $\|v_0^i\|_{L^\infty(\mathbb{R})}$, $\|\partial_x v_0^i\|_{L^p(\mathbb{R})}$, and T^ such that*

$$\|\partial_x v_{\eta,\delta}^i\|_{L^\infty((0,T^*); L^p(\mathbb{R}))} \leq \xi_p^\delta \quad \text{for all } 1 \leq p \leq +\infty. \quad (5.19)$$

ii) (Convergence):

Up to extract a subsequence, the function $v_{\eta,\delta}^i$ converges, as δ goes to zero, to a function v_η^i locally and strongly in $C([0, T^]; W^{1,\infty}(\mathbb{R}))$. Moreover, v_η^i is the unique solution of (5.12) (in the distributional sense) belonging to $C([0, T^*]; W^{1,\infty}(\mathbb{R}))$ and satisfies, for all $0 < \eta \leq 1$ and for all $i \in \{1, \dots, d\}$, the following η -uniform estimates:*

$$\|v_\eta^i\|_{L^\infty((0,T^*) \times \mathbb{R})} \leq \|v_0^i\|_{L^\infty(\mathbb{R})}, \quad (5.20)$$

and

$$\|\partial_x v_\eta^i\|_{L^\infty((0,T^*); L^p(\mathbb{R}))} \leq \xi_p^0 \quad \text{for all } 1 \leq p \leq +\infty. \quad (5.21)$$

To prove Theorem 5.2 we need to recall the following lemma.

Lemma 5.1. (Properties of the heat kernel)

Let $G_\eta(t, x) = \frac{1}{\sqrt{4\pi\eta t}} e^{-\frac{x^2}{4\eta t}}$ be the standard heat kernel. Denoting $G_\eta(t) = G_\eta(\cdot, t)$, we have for all $t > 0$:

$$(i) \quad \|G_\eta(t)\|_{L^p(\mathbb{R})} = \begin{cases} N t^{\frac{1}{2p} - \frac{1}{2}} \text{ where } N = \left(\frac{1}{(4\pi\eta)^{\frac{p-1}{2}} \sqrt{p}} \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < +\infty \\ \frac{1}{\sqrt{4\pi\eta t}}, & \text{for } p = +\infty. \end{cases}$$

$$(ii) \quad \|\partial_x G_\eta(t)\|_{L^p(\mathbb{R})} = l_p t^{\frac{1}{2p} - 1}, \quad \text{where } l_p = \left(\frac{\Gamma(\frac{p+1}{2})}{(\sqrt{\pi})^p 2^{p-1} p^{\frac{p+1}{2}} \eta^{p-\frac{1}{2}}} \right)^{\frac{1}{p}} \text{ for all } 1 \leq p < +\infty.$$

For the proof of this lemma, we refer to Pazy [63, Theorem 5.2, Page 107].

Proof of Theorem 5.2: The proof is outlined in four steps.

Step 1 (Rewriting the equation in its integral form):

The problem (5.16) can be written in its integral form, for every $i = 1, \dots, d$, as follows:

$$v_{\eta,\delta}^i(t, x) = G_\eta(t) \star v_0^i(x) + \int_0^t G_\eta(t-s) \star (\tilde{\lambda}^i(s, \cdot, v_{\eta,\delta}(s, \cdot)) \psi_\delta(\cdot) \beta_\delta(\partial_x v_{\eta,\delta}^i(s, \cdot))) (x) ds \quad \text{for all } i \in \{1, \dots, d\}.$$

In other words, we consider the following problem:

$$\begin{cases} v_{\eta,\delta}(t, x) = (v_{\eta,\delta}^i(t, x))_{i=1,\dots,d}, & v_{\eta,\delta}(0, x) = v_0(x) = (v_0^i(x))_{i=1,\dots,d}, \\ v_{\eta,\delta}(t, x) = G_\eta(t) \star v_0(x) + B(v_{\eta,\delta})(t, x), \end{cases} \quad (5.22)$$

where, for $r(t, x) = (r^i(t, x))_{i=1,\dots,d}$, we have

$$B(r)(t, x) = \int_0^t G_\eta(t-s) \star \begin{pmatrix} \beta_\delta(\partial_x r^1(s, \cdot)) \psi_\delta(\cdot) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_\delta(\partial_x r^d(s, \cdot)) \psi_\delta(\cdot) \end{pmatrix} \begin{pmatrix} \tilde{\lambda}^1(s, \cdot, r(s, \cdot)) \\ \tilde{\lambda}^2(s, \cdot, r(s, \cdot)) \\ \vdots \\ \tilde{\lambda}^d(s, \cdot, r(s, \cdot)) \end{pmatrix} (x) ds.$$

In what follows, we show some estimates to prove in particular the contraction of the operator B and therefore the existence and uniqueness of the solution of (5.16) by applying a fixed point argument.

Step 2 (A priori estimates on B and $G_\eta(t) \star v_0$):

We introduce, for $T > 0$, the spaces

$$X = \{r = (r^i)_{i=1,\dots,d} \in (L^\infty(\mathbb{R}))^d; \partial_x r^i \in L^1(\mathbb{R})\}, \quad (5.23)$$

equipped with the following norm $\|r\|_X = \sum_{i=1}^d \|r^i\|_{L^\infty(\mathbb{R})} + \sum_{i=1}^d \|\partial_x r^i\|_{L^1(\mathbb{R})}$, and

$$X_T = \{r = (r^i)_{i=1,\dots,d} \in (L^\infty((0, T) \times \mathbb{R}))^d; \partial_x r^i \in L^\infty((0, T); L^1(\mathbb{R}))\},$$

equipped with the following norm $\|r\|_{X_T} = \sum_{i=1}^d \|r^i\|_{L^\infty((0, T) \times \mathbb{R})} + \sum_{i=1}^d \|\partial_x r^i\|_{L^\infty((0, T); L^1(\mathbb{R}))}$.

We can remark that

$$\begin{aligned}
 \|B(r)(t, \cdot)\|_X &= \sum_{i=1}^d \left\| \int_0^t G_\eta(t-s) \star \beta_\delta(\partial_x r^i(s, \cdot)) \psi_\delta(\cdot) \tilde{\lambda}^i(s, \cdot, r(s, \cdot)) ds \right\|_{L^\infty(\mathbb{R})} \\
 &+ \sum_{i=1}^d \left\| \int_0^t \partial_x G_\eta(t-s) \star \beta_\delta(\partial_x r^i(s, \cdot)) \psi_\delta(\cdot) \tilde{\lambda}^i(s, \cdot, r(s, \cdot)) ds \right\|_{L^1(\mathbb{R})}.
 \end{aligned} \tag{5.24}$$

Since

$$\begin{aligned}
 \|\psi_\delta \beta_\delta(\partial_x r^i)\|_{L^1(\mathbb{R})} &\leq \delta \|\psi_\delta\|_{L^1(\mathbb{R})} + \|\psi_\delta\|_{L^\infty(\mathbb{R})} \|\partial_x r^i\|_{L^1(\mathbb{R})} \\
 &\leq \sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|\partial_x r^i\|_{L^1(\mathbb{R})},
 \end{aligned}$$

by estimating the first line in (5.24) using Lemma 5.1 (i), with $p = +\infty$ and the second line using Lemma 5.1 (ii), with $p = 1$, we get the following estimate:

$$\begin{aligned}
 \|B(r)\|_{X_T} &\leq \sum_{i=1}^d \frac{\sqrt{T}}{\sqrt{\eta\pi}} \tilde{\Lambda}^i \left(\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|\partial_x r^i\|_{L^\infty((0,T);L^1(\mathbb{R}))} \right) \\
 &+ \sum_{i=1}^d \frac{2\sqrt{T}}{\sqrt{\eta\pi}} \tilde{\Lambda}^i \left(\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|\partial_x r^i\|_{L^\infty((0,T);L^1(\mathbb{R}))} \right) \\
 &\leq \frac{3\sqrt{T}}{\sqrt{\eta\pi}} \tilde{\Lambda} \left(\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|r\|_{X_T} \right).
 \end{aligned} \tag{5.25}$$

In addition, using Lemma 5.1 (i) for $p = 1$, we have

$$\|G_\eta(\cdot) \star v_0\|_{X_T} \leq \|v_0\|_X. \tag{5.26}$$

Step 3 (Contraction argument):

We choose to work with the Banach space $Y_T = \{r \in X_T; \|r\|_{X_T} \leq \|v_0\|_X + 1\}$. Now, we define the mapping

$$\begin{aligned}
 \mathcal{T} : Y_T &\rightarrow Y_T \\
 r &\rightarrow \mathcal{T}(r) = G_\eta(\cdot) \star v_0 + B(r).
 \end{aligned}$$

We will prove that \mathcal{T} is a well-defined contraction for T small enough. First, we prove that \mathcal{T} is well-defined. Indeed, let $r \in Y_T$, using (5.25) and (5.26), we obtain

$$\begin{aligned}
 \|\mathcal{T}(r)\|_{X_T} &\leq \|v_0\|_X + \frac{3\sqrt{T}}{\sqrt{\eta\pi}} \tilde{\Lambda} \left(\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|r\|_{X_T} \right) \\
 &\leq \|v_0\|_X + \frac{3\sqrt{T}}{\sqrt{\eta\pi}} \tilde{\Lambda} \left(\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|v_0\|_X + 1 \right).
 \end{aligned}$$

Then, taking

$$T \leq \frac{\eta\pi}{\left(3\tilde{\Lambda}(\sqrt{\delta}\|\phi\|_{L^1(\mathbb{R})} + \|v_0\|_X + 1)\right)^2} := T_1^*, \quad (5.27)$$

implies that \mathcal{T} is well-defined in $Y_{T_1^*}$.

It remains to show that \mathcal{T} is a contraction on Y_T for a certain T . Indeed, let $r_1, r_2 \in Y_T$,

$$\begin{aligned} & \|\mathcal{T}(r_1) - \mathcal{T}(r_2)\|_{X_T} \\ &= \|B(r_1) - B(r_2)\|_{X_T} \\ &\leq \left\| \int_0^t G_\eta(t-s) \star \begin{pmatrix} \beta_\delta(\partial_x r_1^1(s, \cdot))\psi_\delta(\cdot)\tilde{\lambda}^1(s, \cdot, r_1) - \beta_\delta(\partial_x r_2^1(s, \cdot))\psi_\delta(\cdot)\tilde{\lambda}^1(s, \cdot, r_2) \\ \vdots \\ \beta_\delta(\partial_x r_1^d(s, \cdot))\psi_\delta(\cdot)\tilde{\lambda}^d(s, \cdot, r_1) - \beta_\delta(\partial_x r_2^d(s, \cdot))\psi_\delta(\cdot)\tilde{\lambda}^d(s, \cdot, r_2) \end{pmatrix} ds \right\|_{X_T}. \end{aligned}$$

Then, we get as in (5.25) that

$$\begin{aligned} & \|\mathcal{T}(r_1) - \mathcal{T}(r_2)\|_{X_T} = \|B(r_1) - B(r_2)\|_{X_T} \\ &\leq \sum_{i=1}^d \int_0^t \left\| G_\eta(t-s) \star \left[\beta_\delta(\partial_x r_1^i(s, \cdot))[\tilde{\lambda}^i(t, \cdot, r_1) - \tilde{\lambda}^i(t, \cdot, r_2)] \right. \right. \\ &\quad \left. \left. + \tilde{\lambda}^i(t, \cdot, r_2)[\beta_\delta(\partial_x r_1^i(s, \cdot)) - \beta_\delta(\partial_x r_2^i(s, \cdot))] \right] \psi_\delta(\cdot) \right\|_{L^\infty(\mathbb{R})} ds \\ &+ \sum_{i=1}^d \int_0^t \left\| \partial_x G_\eta(t-s) \star \left[\beta_\delta(\partial_x r_1^i(s, \cdot))[\tilde{\lambda}^i(t, \cdot, r_1) - \tilde{\lambda}^i(t, \cdot, r_2)] \right. \right. \\ &\quad \left. \left. + \tilde{\lambda}^i(t, \cdot, r_2)[\beta_\delta(\partial_x r_1^i(s, \cdot)) - \beta_\delta(\partial_x r_2^i(s, \cdot))] \right] \psi_\delta(\cdot) \right\|_{L^1(\mathbb{R})} ds \\ &\leq \frac{3\sqrt{T}}{\sqrt{\eta\pi}} \left(\tilde{M}_1(\sqrt{\delta}\|\phi\|_{L^1(\mathbb{R})} + \|v_0\|_X + 1) + \tilde{\Lambda} \right) \|r_1 - r_2\|_{X_T}, \end{aligned}$$

Taking

$$T \leq \frac{\eta\pi}{\left(6 \left(\tilde{M}_1(\sqrt{\delta}\|\phi\|_{L^1(\mathbb{R})} + \|v_0\|_X + 1) + \tilde{\Lambda}\right)\right)^2} = T_2^*, \quad (5.28)$$

implies that \mathcal{T} is a contraction on the closed set $Y_{T_2^*}$.

Hence, by Fixed point theorem, there exists a unique solution $v_{\eta, \delta}$ to system (5.22) in Y_{T^*} , where $T^* = \min(T_1^*, T_2^*)$ satisfying in particular (5.19) for $p = 1$. This shows that $v_{\eta, \delta}$ is a solution of system (5.16) in $(0, T^*) \times \mathbb{R}$.

Step 4 (L^p -estimates, $1 \leq p \leq +\infty$):

In this step, we prove estimate (5.19) for some $T > 0$. Differentiating the integral equation in (5.22)

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with respect to x and taking the $L^p(\mathbb{R})$ norm, we get

$$\begin{aligned} \|\partial_x v_{\eta,\delta}^i(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \|G_\eta(t) \star \partial_x v_0^i\|_{L^p(\mathbb{R})} + \int_0^t \|\partial_x G_\eta(t-s) \star \tilde{\lambda}^i(s, \cdot, v_{\eta,\delta}(s, \cdot)) \psi_\delta \beta_\delta(\partial_x v_{\eta,\delta}^i(s, \cdot))\|_{L^p(\mathbb{R})} ds \\ &\leq \|G_\eta(t)\|_{L^1(\mathbb{R})} \|\partial_x v_0^i\|_{L^p(\mathbb{R})} \\ &\quad + \int_0^t \|\partial_x G_\eta(t-s)\|_{L^p(\mathbb{R})} \|\tilde{\lambda}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \|\psi_\delta \beta_\delta(\partial_x v_{\eta,\delta}^i(s, \cdot))\|_{L^1(\mathbb{R})} ds. \end{aligned}$$

We employ Lemma 5.1 (i) for $p = 1$ in the second line, and Lemma 5.1 (ii) in the third line, we obtain, for $1 \leq p < +\infty$, the following inequality

$$\begin{aligned} \|\partial_x v_{\eta,\delta}^i(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \|\partial_x v_0^i\|_{L^p(\mathbb{R})} + l_p \tilde{\Lambda}^i \|\psi_\delta \beta_\delta(\partial_x v_{\eta,\delta}^i)\|_{L^\infty((0,T); L^1(\mathbb{R}))} \int_0^t (t-s)^{\frac{1}{2p}-1} ds \\ &\leq \|\partial_x v_0^i\|_{L^p(\mathbb{R})} + 2pl_p \tilde{\Lambda}^i (\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|v_0\|_X + 1) T^{\frac{1}{2p}}. \end{aligned}$$

This leads to the following estimate

$$\left\{ \begin{array}{l} \|\partial_x v_{\eta,\delta}^i\|_{L^\infty((0,T); L^p(\mathbb{R}))} \leq \xi_p^\delta, \quad \text{for all } 1 \leq p < +\infty, \\ \text{with } \xi_p^\delta = \|\partial_x v_0\|_{(L^p(\mathbb{R}))^d} + 2pl_p \tilde{\Lambda}^i (\sqrt{\delta} \|\phi\|_{L^1(\mathbb{R})} + \|v_0\|_{(L^\infty(\mathbb{R}))^d} + \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + 1) T^{\frac{1}{2p}}. \end{array} \right. \quad (5.29)$$

Now, we use the previous estimate to obtain estimate (5.19) for $p = +\infty$. Indeed, differentiating again equation (5.22) with respect to x and taking the $L^\infty(\mathbb{R})$ norm, we obtain

$$\begin{aligned} \|\partial_x v_{\eta,\delta}^i(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|G_\eta(t) \star \partial_x v_0^i\|_{L^\infty(\mathbb{R})} + \int_0^t \|\partial_x G_\eta(t-s) \star \tilde{\lambda}^i(s, \cdot, v_{\eta,\delta}(s, \cdot)) \psi_\delta \beta_\delta(\partial_x v_{\eta,\delta}^i(s, \cdot))\|_{L^\infty(\mathbb{R})} ds \\ &\leq \|G_\eta(t)\|_{L^1(\mathbb{R})} \|\partial_x v_0^i\|_{L^\infty(\mathbb{R})} \\ &\quad + \int_0^t \|\partial_x G_\eta(t-s)\|_{L^q(\mathbb{R})} \|\tilde{\lambda}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \|\psi_\delta \beta_\delta(\partial_x v_{\eta,\delta}^i(s, \cdot))\|_{L^p(\mathbb{R})} ds, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < +\infty$. We proceed as before we deduce, using (5.29) for $1 < p < +\infty$, that

$$\begin{aligned} \|\partial_x v_{\eta,\delta}^i(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|\partial_x v_0^i\|_{L^\infty(\mathbb{R})} + l_q \tilde{\Lambda}^i (\delta \|\psi_\delta\|_{L^p(\mathbb{R})} + \|\partial_x v_{\eta,\delta}^i\|_{L^\infty((0,T); L^p(\mathbb{R}))}) \int_0^t (t-s)^{\frac{1}{2q}-1} ds \\ &\leq \|\partial_x v_0^i\|_{L^\infty(\mathbb{R})} + 2ql_q \tilde{\Lambda}^i \left(\delta^{1-\frac{1}{2p}} \|\phi\|_{L^p(\mathbb{R})} + \xi_p^\delta \right) T^{\frac{1}{2q}}, \end{aligned}$$

which leads to following estimate

$$\|\partial_x v_{\eta,\delta}^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|\partial_x v_0\|_{(L^\infty(\mathbb{R}))^d} + 2ql_q \tilde{\Lambda}^i \left(\delta^{(1-\frac{1}{2p})} \|\phi\|_{L^p(\mathbb{R})} + \xi_p^\delta \right) T^{\frac{1}{2q}} := \xi_\infty^\delta. \quad (5.30)$$

Thus (5.29) and (5.30) prove estimate (5.19) for $T = T^*$.

Step 5 (Existence and uniqueness of smooth solution, convergence):

Step 5.1 (Regularity)

The regularity of the solution is obtained by a classical bootstrap argument. Indeed, using the fact that β_δ and $\tilde{\lambda}^i$ are regular, we can remark that the right hand side of equation (5.16), namely

$f_\delta(t, x, v_{\eta, \delta}, \partial_x v_{\eta, \delta}) = \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i(t, x)) \in L^p((0, T^*) \times \mathbb{R})$ with the following δ -uniform bound:

$$\|f_\delta\|_{L^p((0, T^*) \times \mathbb{R})} \leq \tilde{\Lambda}(T^*)^{\frac{1}{p}} (\delta^{1 - \frac{1}{2p}} \|\phi\|_{L^p(\mathbb{R})} + \|\partial_x v_{\eta, \delta}^i\|_{L^\infty((0, T^*) \times \mathbb{R})}) \leq \tilde{\Lambda}(T^*)^{\frac{1}{p}} (\delta^{1 - \frac{1}{2p}} \|\phi\|_{L^p(\mathbb{R})} + \xi_p^\delta).$$

This implies via the L^p -regularity of parabolic equations (for $1 < p < +\infty$) (see [52] for more details) that $\partial_t v_{\eta, \delta}^i$ and $\partial_{xx}^2 v_{\eta, \delta}^i$ are δ -uniformly bounded in $L^p((0, T^*) \times \mathbb{R})$.

Now, we also remark that $\partial_x [f_\delta(t, x, v_{\eta, \delta}, \partial_x v_{\eta, \delta})]$ is δ -uniformly bounded in $L^p((0, T^*) \times \mathbb{R})$ with the following δ -uniform bound:

$$\begin{aligned} \|\partial_x f_\delta\|_{L^p((0, T^*) \times \mathbb{R})} &\leq \tilde{\Lambda} \|\partial_{xx}^2 v_{\eta, \delta}^i\|_{L^p((0, T^*) \times \mathbb{R})} + \widetilde{M}_1(T^*)^{\frac{1}{p}} (\delta^{1 - \frac{1}{2p}} \|\phi\|_{L^p(\mathbb{R})} + \xi_p^\delta) \\ &\quad + d \widetilde{M}_1(T^*)^{\frac{1}{p}} (\delta + \xi_\infty^\delta) \xi_p^\delta + \tilde{\Lambda}(T^*)^{\frac{1}{p}} \delta^{\frac{1}{2}} (\delta^{1 - \frac{1}{2p}} \|\phi'\|_{L^p(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})} \xi_p^\delta). \end{aligned}$$

This implies, by differentiating with respect to x equation (5.16) and using similarly the L^p -regularity, that $\partial_{tx}^2 v_{\eta, \delta}^i$ and $\partial_{xxx}^3 v_{\eta, \delta}^i$ are δ -uniformly bounded in $L^p((0, T^*) \times \mathbb{R})$.

We redifferentiate again the equation (5.16) and we repeat the same procedure, we obtain finally by using the Sobolev embedding that

$$v_{\eta, \delta} \in (C^\infty((0, T^*) \times \mathbb{R}))^d \cap (W^{1, \infty}((0, T^*) \times \mathbb{R}))^d \cap Y_{T^*}.$$

Estimate (5.18) is derived directly by applying the Maximum principle Theorem for the scalar parabolic equation (5.16) (see [52, 66]).

This completes the proof of Theorem 5.2 (i).

Step 5.2 (Convergence and proof of Theorem 5.2 (ii))

By differentiating with respect to t the equation (5.16), we get as in the previous step that $\partial_{tt}^2 v_{\eta, \delta}^i$ is δ -uniformly bounded in $L^p((0, T^*) \times \mathbb{R})$. Hence, for all $i = 1, \dots, d$, we have

$$v_{\eta, \delta}^i, \partial_x v_{\eta, \delta}^i, \partial_t v_{\eta, \delta}^i, \partial_{xx}^2 v_{\eta, \delta}^i, \partial_{tx}^2 v_{\eta, \delta}^i, \partial_{tt}^2 v_{\eta, \delta}^i \quad \text{are } \delta\text{-uniformly bounded in } L^p((0, T^*) \times \mathbb{R}),$$

and therefore we can deduce that $v_{\eta, \delta}^i$ is δ -uniformly bounded in $W^{2, p}((0, T^*) \times \mathbb{R})$. Moreover, since, for any bounded open interval K of \mathbb{R} and $p > 2$, we know that $W^{2, p}((0, T^*) \times K) \hookrightarrow W^{1, \infty}((0, T^*) \times K)$ with compact injection, we can extract a subsequence of $v_{\eta, \delta}^i$ that converges as δ goes to 0 to a function v_η^i locally and strongly in $W^{1, \infty}((0, T^*) \times \mathbb{R})$, which gives the existence of a solution of (5.12).

The estimates (5.20) and (5.21) come directly from (5.18) and (5.19) by passing to the limit as δ goes to 0.

The uniqueness is an independent result valid for Lipschitz continuous solutions. \square

3 (η, δ) -Uniform *a priori* estimates on the solution of (5.16)

In this section, we show *a priori* estimates on the solution of (5.16) obtained in Theorem 5.2 (i). The first one concerns the *BV* estimate of the equation.

Lemma 5.2. (*BV estimate*)

Suppose that $(\widetilde{H1})$ and (5.13) hold. Let $\phi \in C^\infty(\mathbb{R})$ be the cut-off function defined as in (5.15). We consider the function

$$\phi_R(\cdot) = \phi\left(\frac{\cdot}{R}\right), \quad \text{for } R > 0. \quad (5.31)$$

Let $v_{\eta,\delta} = (v_{\eta,\delta}^i)_{i=1,\dots,d} \in (C^\infty((0, T) \times \mathbb{R}))^d \cap (W^{1,\infty}((0, T) \times \mathbb{R}))^d \cap X_T$ be a solution of (5.16), that satisfies (5.18) and (5.19) for some $T > 0$. Then, for all $0 < \eta \leq 1$, we have

$$\int_{-2R}^{2R} \beta_\delta(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) dx \leq L_\delta^{R,i}, \quad \text{for } i = 1, \dots, d, \quad (5.32)$$

where $L_\delta^{R,i}$ is given explicitly in (5.40).

Proof of Lemma 5.2:

Differentiating the first equation in (5.16) with respect to x and multiplying by $\beta'_\delta(\partial_x v_{\eta,\delta}^i)$, where β_δ is defined in (5.14), we get

$$\partial_t (\beta_\delta(\partial_x v_{\eta,\delta}^i)) = \eta \partial_{xxx}^3 (\partial_x v_{\eta,\delta}^i) \beta'_\delta(\partial_x v_{\eta,\delta}^i) + \partial_x [\tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta,\delta}^i)] \beta'_\delta(\partial_x v_{\eta,\delta}^i). \quad (5.33)$$

Multiplying (5.33) by ϕ_R for $R > 0$, and integrating over the spacial variable, we get

$$\begin{aligned} & \partial_t \left[\int_{-2R}^{2R} \beta_\delta(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) dx \right] \\ &= \underbrace{\eta \int_{-2R}^{2R} \partial_x (\partial_{xx}^2 v_{\eta,\delta}^i(t, x)) \beta'_\delta(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) dx}_{I_1} \\ & \quad + \underbrace{\int_{-2R}^{2R} \partial_x (\tilde{\lambda}^i(t, x, v_{\eta,\delta}(t, x)) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta,\delta}^i(t, x))) \beta'_\delta(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) dx}_{I_2}. \end{aligned} \quad (5.34)$$

We note that these computations are verified since $v_{\eta,\delta}^i \in C^\infty((0, T) \times \mathbb{R})$. In what follows, we will show that the terms I_1 and I_2 are bounded uniformly in η and δ .

Step 1 (Estimate of I_1):

From integration by parts, we have

$$I_1 = -\eta \left[\int_{-2R}^{2R} (\partial_{xx}^2 v_{\eta,\delta}^i(t, x))^2 \beta_\delta''(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) dx + \frac{1}{R} \int_{-2R}^{2R} \partial_{xx}^2 v_{\eta,\delta}^i(t, x) \beta'_\delta(\partial_x v_{\eta,\delta}^i(t, x)) \phi'\left(\frac{x}{R}\right) dx \right].$$

Thanks to the convexity of β_δ and the positivity of ϕ_R , we know that

$$(\partial_{xx}^2 v_{\eta,\delta}^i(t, x))^2 \beta_\delta''(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) \geq 0.$$

This implies that I_1 satisfies the following inequality

$$\begin{aligned} I_1 &\leq -\frac{\eta}{R} \int_{-2R}^{2R} \partial_{xx}^2 v_{\eta,\delta}^i(t,x) \beta'_\delta(\partial_x v_{\eta,\delta}^i(t,x)) \phi' \left(\frac{x}{R} \right) dx \\ &\leq -\frac{\eta}{R} \int_{-2R}^{2R} \partial_x (\beta_\delta(\partial_x v_{\eta,\delta}^i(t,x))) \phi' \left(\frac{x}{R} \right) dx. \end{aligned}$$

Integrating again by parts the right hand side of the above inequality, we obtain

$$I_1 \leq \frac{\eta}{R^2} \int_{-2R}^{2R} \beta_\delta(\partial_x v_{\eta,\delta}^i(t,x)) \phi'' \left(\frac{x}{R} \right) dx.$$

Using (5.19), for $p = +\infty$, we get

$$I_1 \leq \frac{4\eta}{R} \beta_\delta(\xi_\infty^\delta) \|\phi''\|_{L^\infty(\mathbb{R})}. \quad (5.35)$$

Step 2 (Estimate of I_2): To estimate I_2 , we integrate by parts to get

$$\begin{aligned} I_2 &= - \underbrace{\int_{-2R}^{2R} \tilde{\lambda}^i(t,x, v_{\eta,\delta}(t,x)) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta,\delta}^i(t,x)) \beta''_\delta(\partial_x v_{\eta,\delta}^i(t,x)) \partial_{xx}^2 v_{\eta,\delta}^i(t,x) \phi_R(x) dx}_{J_1} \\ &\quad - \underbrace{\frac{1}{R} \int_{-2R}^{2R} \tilde{\lambda}^i(t,x, v_{\eta,\delta}(t,x)) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta,\delta}^i(t,x)) \beta'_\delta(\partial_x v_{\eta,\delta}^i(t,x)) \phi' \left(\frac{x}{R} \right) dx}_{J_2}. \end{aligned}$$

Furthermore, by the property of the function β_δ defined in (5.14), we can see that

$$\partial_{xx}^2 v_{\eta,\delta}^i \beta_\delta(\partial_x v_{\eta,\delta}^i) \beta''_\delta(\partial_x v_{\eta,\delta}^i) = \delta \partial_x \left(\arctan \left[\frac{\partial_x v_{\eta,\delta}^i}{\delta} \right] \right),$$

which shows that

$$J_1 = -\delta \int_{-2R}^{2R} \tilde{\lambda}^i(t,x, v_{\eta,\delta}(t,x)) \psi_\delta(x) \phi_R(x) \partial_x \left(\arctan \left[\frac{\partial_x v_{\eta,\delta}^i(t,x)}{\delta} \right] \right) dx.$$

Integrating by parts the above equality, we obtain

$$\begin{aligned} J_1 &= \delta \int_{-2R}^{2R} \frac{d}{dx} [\tilde{\lambda}^i(t,x, v_{\eta,\delta}(t,x))] \psi_\delta(x) \phi_R(x) \arctan \left[\frac{\partial_x v_{\eta,\delta}^i(t,x)}{\delta} \right] dx \\ &\quad + \frac{\delta}{R} \int_{-2R}^{2R} \tilde{\lambda}^i(t,x, v_{\eta,\delta}(t,x)) \psi_\delta(x) \phi' \left(\frac{x}{R} \right) \arctan \left[\frac{\partial_x v_{\eta,\delta}^i(t,x)}{\delta} \right] dx \\ &\quad + \delta \sqrt{\delta} \int_{-2R}^{2R} \tilde{\lambda}^i(t,x, v_{\eta,\delta}(t,x)) \phi'(\sqrt{\delta}x) \phi_R(x) \arctan \left[\frac{\partial_x v_{\eta,\delta}^i(t,x)}{\delta} \right] dx. \end{aligned}$$

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Thereafter, using Hölder inequality, we deduce that

$$J_1 \leq \delta \frac{\pi}{2} (d\widetilde{M}_1 \xi_\infty^\delta + \widetilde{M}_1) R \|\phi\|_{L^1(\mathbb{R})} + \delta \widetilde{\Lambda}^i \frac{\pi}{2} \|\phi'\|_{L^1(\mathbb{R})} + 2\pi R \delta \sqrt{\delta} \widetilde{\Lambda}^i \|\phi'\|_{L^\infty(\mathbb{R})}. \quad (5.36)$$

However, since $\beta_\delta(\partial_x v_{\eta,\delta}^i) \beta'_\delta(\partial_x v_{\eta,\delta}^i) = \partial_x v_{\eta,\delta}^i$, we can rewrite J_2 as follows

$$J_2 = -\frac{1}{R} \int_{-2R}^{2R} \widetilde{\lambda}^i(t, x, v_{\eta,\delta}(t, x)) \psi_\delta(x) \partial_x v_{\eta,\delta}^i(t, x) \phi' \left(\frac{x}{R} \right) dx.$$

Then, using Hölder inequality and (5.19) for $p = 1$, we get

$$J_2 \leq \frac{1}{R} \widetilde{\Lambda}^i \xi_1^\delta \|\phi'\|_{L^\infty(\mathbb{R})}. \quad (5.37)$$

Combining (5.36) and (5.37), we get

$$\begin{aligned} I_2 &\leq \delta \frac{\pi}{2} \left((d\widetilde{M}_1 \xi_\infty^\delta + \widetilde{M}_1) R \|\phi\|_{L^1(\mathbb{R})} + \widetilde{\Lambda}^i \|\phi'\|_{L^1(\mathbb{R})} + 4R \sqrt{\delta} \widetilde{\Lambda}^i \|\phi'\|_{L^\infty(\mathbb{R})} \right) \\ &\quad + \frac{1}{R} \widetilde{\Lambda}^i \xi_1^\delta \|\phi'\|_{L^\infty(\mathbb{R})}. \end{aligned} \quad (5.38)$$

Integrating (5.34) in time on $(0, t)$, for $0 < t < T$ and using (5.35) and (5.38), we get

$$\int_{-2R}^{2R} \beta_\delta(\partial_x v_{\eta,\delta}^i(t, x)) \phi_R(x) dx \leq \int_{-2R}^{2R} \beta_\delta(\partial_x v_0^i(x)) \phi_R(x) dx + \int_0^t (I_1 + I_2) dt \leq L_\delta^{R,i}, \quad (5.39)$$

with

$$\begin{aligned} L_\delta^{R,i} &= \int_{-2R}^{2R} \beta_\delta(\partial_x v_0^i(x)) \phi_R(x) dx + T \delta \frac{\pi}{2} \left((d\widetilde{M}_1 \xi_\infty^\delta + \widetilde{M}_1) R \|\phi\|_{L^1(\mathbb{R})} + \widetilde{\Lambda}^i \|\phi'\|_{L^1(\mathbb{R})} + 4R \sqrt{\delta} \widetilde{\Lambda}^i \|\phi'\|_{L^\infty(\mathbb{R})} \right) \\ &\quad + \frac{T}{R} \widetilde{\Lambda}^i \xi_1^\delta \|\phi'\|_{L^\infty(\mathbb{R})} + \frac{4\eta T}{R} \beta_\delta(\xi_\infty^\delta) \|\phi''\|_{L^\infty(\mathbb{R})}. \end{aligned} \quad (5.40)$$

□

Lemma 5.3. (L^2 estimate on $\partial_{xx}^2 v_{\eta,\delta}^i$)

Under the same assumptions of Lemma 5.2, we have

$$\frac{1}{2} \int_0^t \int_{\mathbb{R}} (\partial_{xx}^2 v_{\eta,\delta}^i(s, x))^2 \phi_R(x) dx ds \leq N_\delta^i,$$

where N_δ^i is a positive constant uniformly bounded on δ , that is given by

$$N_\delta^i = \frac{1}{2\eta} \int_{\mathbb{R}} (\partial_x v_0^i(x))^2 \phi_R dx + \frac{2T}{R} (\xi_\infty^\delta)^2 \|\phi''\|_{L^\infty(\mathbb{R})} + \frac{2TR}{\eta^2} (\widetilde{\Lambda}^i \beta_\delta(\xi_\infty^\delta))^2 + \frac{4T}{\eta} \widetilde{\Lambda}^i \beta_\delta(\xi_\infty^\delta) \xi_\infty^\delta \|\phi'\|_{L^\infty(\mathbb{R})},$$

with $\widetilde{\Lambda}^i$ and ξ_∞^δ are defined in (5.17) and (5.19), respectively.

Proof of Lemma 5.3:

The proof of this lemma is very classical. In fact, differentiating the first equation in (5.16) with respect

to x and multiplying then by $(\partial_x v_{\eta, \delta}^i) \phi_R$, gives

$$(\partial_{tx}^2 v_{\eta, \delta}^i)(\partial_x v_{\eta, \delta}^i) \phi_R - \eta (\partial_{xxx}^3 v_{\eta, \delta}^i)(\partial_x v_{\eta, \delta}^i) \phi_R = \partial_x [\tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i)] \partial_x v_{\eta, \delta}^i \phi_R.$$

We integrate the previous equation with respect to x , to deduce

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x v_{\eta, \delta}^i)^2 \phi_R + \eta \int_{\mathbb{R}} (\partial_{xx}^2 v_{\eta, \delta}^i)^2 \phi_R + \frac{\eta}{2R} \int_{\mathbb{R}} \partial_x [(\partial_x v_{\eta, \delta}^i)^2] \phi' \left(\frac{x}{R} \right) \\ &= - \int_{\mathbb{R}} [\tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i)] \partial_{xx}^2 v_{\eta, \delta}^i \phi_R - \frac{1}{R} \int_{\mathbb{R}} [\tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i)] \partial_x v_{\eta, \delta}^i \phi' \left(\frac{x}{R} \right), \end{aligned}$$

where we have integrated by parts the second and the third terms. We note that these computations (and the integration by parts) are justified since $v_{\eta, \delta}^i$ is a smooth function, for all $i \in \{1, \dots, d\}$.

Integrating in time on $(0, t)$, to get

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} (\partial_{xx}^2 v_{\eta, \delta}^i)^2 \phi_R &= \frac{1}{2\eta} \int_{\mathbb{R}} (\partial_x v_0^i)^2 \phi_R - \underbrace{\frac{1}{2\eta} \int_{\mathbb{R}} (\partial_x v_{\eta, \delta}^i)^2 \phi_R}_{H_1} - \underbrace{\frac{1}{2R} \int_0^t \int_{\mathbb{R}} \partial_x [(\partial_x v_{\eta, \delta}^i)^2] \phi' \left(\frac{x}{R} \right)}_{H_2} \\ &\quad - \underbrace{\frac{1}{\eta} \int_0^t \int_{\mathbb{R}} [\tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i)] \partial_{xx}^2 v_{\eta, \delta}^i \phi_R}_{H_3} \\ &\quad - \underbrace{\frac{1}{\eta R} \int_0^t \int_{\mathbb{R}} [\tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i)] \partial_x v_{\eta, \delta}^i \phi' \left(\frac{x}{R} \right)}_{H_4}. \end{aligned} \tag{5.41}$$

Since $\phi_R \geq 0$, we have

$$H_1 \leq 0. \tag{5.42}$$

In the following, we will estimate the terms H_2, H_3, H_4 . Indeed, integrating by parts in H_2 and using (5.19) for $p = +\infty$, we can see that

$$|H_2| \leq \frac{2T}{R} (\xi_\infty^\delta)^2 \|\phi''\|_{L^\infty(\mathbb{R})}. \tag{5.43}$$

Applying Young's inequality, we get from (5.19) and the fact that $0 \leq \psi_\delta, \phi_R \leq 1$, that

$$\begin{aligned} |H_3| &\leq \frac{1}{2\eta^2} \int_0^t \int_{-2R}^{2R} (\tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_\delta(x) \beta_\delta(\partial_x v_{\eta, \delta}^i))^2 + \frac{1}{2} \int_0^t \int_{-2R}^{2R} (\partial_{xx}^2 v_{\eta, \delta}^i \phi_R)^2 \\ &\leq \frac{2TR}{\eta^2} (\tilde{\Lambda}^i \beta_\delta(\xi_\infty^\delta))^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}} (\partial_{xx}^2 v_{\eta, \delta}^i)^2 \phi_R. \end{aligned} \tag{5.44}$$

Similarly, we can verify that

$$|H_4| \leq \frac{4T}{\eta} \tilde{\Lambda}^i \beta_\delta(\xi_\infty^\delta) \xi_\infty^\delta \|\phi'\|_{L^\infty(\mathbb{R})}. \tag{5.45}$$

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By injection (5.42), (5.43), (5.44) and (5.45) in (5.41), we reach the desired result. \square

The following lemma presents a preliminary gradient entropy inequality, which can be seen as an approximation of (5.10) satisfied by $v_{\eta,\delta}^i$ for all $i \in \{1, \dots, d\}$.

Lemma 5.4. (Preliminary gradient entropy estimate)

Under the same assumptions of Lemma 5.2, there exists a constant C_R^δ (that is defined explicitly in (5.55)), such that for all $0 < t < T$, we have

$$S_R^\delta(t) - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{j,i}^i(t, x, v_{\eta,\delta}) \partial_x v_{\eta,\delta}^i \partial_x v_{\eta,\delta}^j \psi_\delta \phi_R \leq C_R^\delta + S_R^\delta(0), \quad \text{for all } 0 < \frac{1}{\sqrt{\delta}} < \frac{R}{2}, \quad (5.46)$$

where $S_R^\delta(t)$ is the following entropy

$$S_R^\delta(t) = \int_{\mathbb{R}} \sum_{i=1}^d f(\beta_\delta(\partial_x v_{\eta,\delta}^i(t, x))) \phi_R(x) dx,$$

with f is defined in (5.11).

Proof of Lemma 5.4:

First, it is easy to check that $S_R^\delta(t)$ is well-defined. Now, we denote by $\theta_{\eta,\delta}^i = \partial_x v_{\eta,\delta}^i$ and compute, using (5.16),

$$\begin{aligned} \frac{d}{dt} S_R^\delta(t) &= \int_{\mathbb{R}} \sum_{i=1}^d f'(\beta_\delta(\theta_{\eta,\delta}^i)) \beta_\delta'(\theta_{\eta,\delta}^i) \partial_t \theta_{\eta,\delta}^i \phi_R \\ &= \int_{\mathbb{R}} \sum_{i=1}^d f'(\beta_\delta(\theta_{\eta,\delta}^i)) \beta_\delta'(\theta_{\eta,\delta}^i) \partial_x [\tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \beta_\delta(\theta_{\eta,\delta}^i) + \eta \partial_x \theta_{\eta,\delta}^i] \phi_R. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} \frac{d}{dt} S_R^\delta(t) &= - \overbrace{\int_{\mathbb{R}} \sum_{i=1}^d \left[f''(\beta_\delta(\theta_{\eta,\delta}^i)) (\beta_\delta'(\theta_{\eta,\delta}^i))^2 + f'(\beta_\delta(\theta_{\eta,\delta}^i)) \beta_\delta''(\theta_{\eta,\delta}^i) \right] \psi_\delta \beta_\delta(\theta_{\eta,\delta}^i) \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \partial_x \theta_{\eta,\delta}^i \phi_R}^{K_1} \\ &\quad - \overbrace{\eta \int_{\mathbb{R}} \sum_{i=1}^d \left[f''(\beta_\delta(\theta_{\eta,\delta}^i)) (\beta_\delta'(\theta_{\eta,\delta}^i))^2 + f'(\beta_\delta(\theta_{\eta,\delta}^i)) \beta_\delta''(\theta_{\eta,\delta}^i) \right] (\partial_x \theta_{\eta,\delta}^i)^2 \phi_R}^{K_2} \\ &\quad - \overbrace{\frac{1}{R} \int_{\mathbb{R}} \sum_{i=1}^d f'(\beta_\delta(\theta_{\eta,\delta}^i)) \beta_\delta'(\theta_{\eta,\delta}^i) [\tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \beta_\delta(\theta_{\eta,\delta}^i) + \eta \partial_x \theta_{\eta,\delta}^i] \phi' \left(\frac{x}{R} \right)}^{K_3}. \end{aligned}$$

We know that $K_2 \leq 0$, since the functions f, β_δ are convex and f is nondecreasing and $\phi_R \geq 0$.

Consequently,

$$\frac{d}{dt} S_R^\delta(t) \leq K_1 + K_3.$$

Step 1 (Estimate of K_1):

$$\begin{aligned}
 K_1 &= - \overbrace{\int_{\mathbb{R}} \sum_{i=1}^d [\theta_{\eta, \delta}^i f''(\beta_{\delta}(\theta_{\eta, \delta}^i)) \beta'_{\delta}(\theta_{\eta, \delta}^i) \partial_x \theta_{\eta, \delta}^i] \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi_R}_{K_{11}} \\
 &\quad - \overbrace{\delta^2 \int_{\mathbb{R}} \sum_{i=1}^d \left[\frac{f'(\beta_{\delta}(\theta_{\eta, \delta}^i))}{(\beta_{\delta}(\theta_{\eta, \delta}^i))^2} \right] \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \partial_x \theta_{\eta, \delta}^i \phi_R}_{K_{12}},
 \end{aligned}$$

where we have used the fact that $\beta_{\delta}(w)\beta'_{\delta}(w) = w$, in the first line and $\beta''_{\delta}(w)\beta_{\delta}(w) = \frac{\delta^2}{(\beta_{\delta}(w))^2}$ in the second one. From the definitions of f and β_{δ} (see (5.11) and (5.14)), we can check that, for all $0 < \delta < \frac{1}{e}$, we have

$$K_{11} = - \int_{\mathbb{R}} \sum_{i=1}^d [g'_{\delta}(\theta_{\eta, \delta}^i) \partial_x \theta_{\eta, \delta}^i] \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi_R = - \int_{\mathbb{R}} \sum_{i=1}^d \partial_x (g_{\delta}(\theta_{\eta, \delta}^i)) \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi_R,$$

where

$$g_{\delta}(x) = \begin{cases} h_{\delta}(x) - h_{\delta}(\sqrt{\frac{1}{e^2} - \delta^2}) & \text{if } x \geq \sqrt{\frac{1}{e^2} - \delta^2}, \\ 0 & \text{if } |x| \leq \sqrt{\frac{1}{e^2} - \delta^2}, \\ h_{\delta}(x) + h_{\delta}(\sqrt{\frac{1}{e^2} - \delta^2}) & \text{if } x \leq -\sqrt{\frac{1}{e^2} - \delta^2}, \end{cases} \quad \text{with } h_{\delta}(x) = x - \delta \arctan\left(\frac{x}{\delta}\right). \quad (5.47)$$

Integrating by part, we obtain

$$\begin{aligned}
 K_{11} &= \int_{\mathbb{R}} \sum_{i=1}^d g_{\delta}(\theta_{\eta, \delta}^i) \frac{d}{dx} [\tilde{\lambda}^i(t, x, v_{\eta, \delta})] \psi_{\delta} \phi_R \\
 &\quad + \frac{1}{R} \int_{\mathbb{R}} \sum_{i=1}^d g_{\delta}(\theta_{\eta, \delta}^i) \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi' \left(\frac{x}{R} \right) + \sqrt{\delta} \int_{\mathbb{R}} \sum_{i=1}^d g_{\delta}(\theta_{\eta, \delta}^i) \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \phi_R \phi'(\sqrt{\delta} x).
 \end{aligned}$$

Adding and subtracting $\theta_{\eta, \delta}^i$ in the first term, we deduce that

$$\begin{aligned}
 K_{11} &= \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_{\eta, \delta}) \theta_{\eta, \delta}^i \theta_{\eta, \delta}^j \psi_{\delta} \phi_R + \overbrace{\int_{\mathbb{R}} \sum_{i,j=1}^d (g_{\delta}(\theta_{\eta, \delta}^i) - \theta_{\eta, \delta}^i) \tilde{\lambda}_{i,j}^i(t, x, v_{\eta, \delta}) \theta_{\eta, \delta}^j \psi_{\delta} \phi_R}_{K_{11}^1} \\
 &\quad + \overbrace{\int_{\mathbb{R}} \sum_{i=1}^d \theta_{\eta, \delta}^i \partial_x \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi_R}_{K_{11}^2} + \overbrace{\int_{\mathbb{R}} \sum_{i=1}^d (g_{\delta}(\theta_{\eta, \delta}^i) - \theta_{\eta, \delta}^i) \partial_x \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi_R}_{K_{11}^3} \\
 &\quad + \overbrace{\frac{1}{R} \int_{\mathbb{R}} \sum_{i=1}^d g_{\delta}(\theta_{\eta, \delta}^i) \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \psi_{\delta} \phi' \left(\frac{x}{R} \right)}_{K_{11}^4} + \overbrace{\sqrt{\delta} \int_{\mathbb{R}} \sum_{i=1}^d g_{\delta}(\theta_{\eta, \delta}^i) \tilde{\lambda}^i(t, x, v_{\eta, \delta}) \phi_R \phi'(\sqrt{\delta} x)}_{K_{11}^5}.
 \end{aligned} \quad (5.48)$$

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By using the fact that $\|\phi\|_{L^\infty(\mathbb{R})} \leq 1$, $|g_\delta(x) - x| \leq \frac{1}{e} + \delta\pi$, for all $x \in \mathbb{R}$, ($\widetilde{H1}$) and (5.32), we get

$$|K_{11}^1| \leq \left(\frac{1}{e} + \delta\pi\right) \widetilde{M}_1 \sum_{j=1}^d \|\theta_{\eta,\delta}^j \phi_R\|_{L^1(\mathbb{R})} \leq \left(\frac{1}{e} + \delta\pi\right) \widetilde{M}_1 L_\delta^R, \quad (5.49)$$

where

$$L_\delta^R = \sum_{i=1}^d L_\delta^{R,i}. \quad (5.50)$$

Now, using (5.32), we obtain

$$|K_{11}^2| \leq \sum_{i=1}^d \|\partial_x \tilde{\lambda}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \|\theta_{\eta,\delta}^i \phi_R\|_{L^1(\mathbb{R})} \leq L_\delta^R \widetilde{M}_1.$$

Since g_δ is a function is supported in $\{x \mid |x| \geq \sqrt{\frac{1}{e^2} - \delta^2}\} = \{x \mid \beta_\delta(x) \geq \frac{1}{e}\}$ when $0 < \delta < \frac{1}{e}$, then we can see that

$$\begin{aligned} K_{11}^3 &= \int_{\mathbb{R}} \sum_{i=1}^d (g_\delta(\theta_{\eta,\delta}^i) - \theta_{\eta,\delta}^i) \partial_x \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \phi_R \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \geq \frac{1}{e}\}} \\ &\quad + \int_{\mathbb{R}} \sum_{i=1}^d (g_\delta(\theta_{\eta,\delta}^i) - \theta_{\eta,\delta}^i) \partial_x \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \phi_R \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \leq \frac{1}{e}\}} \\ &= \int_{\mathbb{R}} \sum_{i=1}^d (g_\delta(\theta_{\eta,\delta}^i) - \theta_{\eta,\delta}^i) \partial_x \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \phi_R \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \geq \frac{1}{e}\}} + \int_{\mathbb{R}} \sum_{i=1}^d (-\theta_{\eta,\delta}^i) \partial_x \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \phi_R. \end{aligned}$$

Then, we obtain using (5.32), that

$$\begin{aligned} |K_{11}^3| &\leq \left(\frac{1}{e} + \delta\pi\right) \widetilde{M}_1 \sum_{i=1}^d \int_{-2R}^{2R} \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \geq \frac{1}{e}\}} \phi_R + L_\delta^R \widetilde{M}_1 \\ &\leq e \left(\frac{1}{e} + \delta\pi\right) \widetilde{M}_1 L_\delta^R + \widetilde{M}_1 L_\delta^R \leq 2\widetilde{M}_1 L_\delta^R + \delta\pi e \widetilde{M}_1 L_\delta^R. \end{aligned}$$

Thus,

$$|K_{11}^2| + |K_{11}^3| \leq 3\widetilde{M}_1 L_\delta^R + \delta\pi e \widetilde{M}_1 L_\delta^R. \quad (5.51)$$

Since g_δ is nondecreasing function, with support in $\{x \mid \beta_\delta(x) \geq \frac{1}{e}\}$ when $0 < \delta < \frac{1}{e}$, and $0 < \frac{1}{\sqrt{\delta}} < \frac{R}{2}$, we obtain from (5.9), (5.19), (5.32) that

$$\begin{aligned} &|K_{11}^4| \\ &\leq \frac{1}{R} g_\delta(\xi_\infty^\delta) \widetilde{\Lambda} \|\phi'\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{-\frac{2}{\sqrt{\delta}}}^{\frac{2}{\sqrt{\delta}}} \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \geq \frac{1}{e}\}} \leq \frac{e}{R} g_\delta(\xi_\infty^\delta) \widetilde{\Lambda} \|\phi'\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{-R}^R \beta_\delta(\theta_{\eta,\delta}^i) \leq \frac{e}{R} g_\delta(\xi_\infty^\delta) \widetilde{\Lambda} \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R, \end{aligned}$$

where we have used the fact that $\phi_R = 1$ on $[-R, R]$, in the last line. We estimate K_{11}^5 in the same way, to arrive to the following inequality

$$|K_{11}^4| + |K_{11}^5| \leq \frac{3e}{2} \sqrt{\delta} \tilde{\Lambda} g_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R. \quad (5.52)$$

Now, collecting (5.48), (5.49), (5.51) and (5.52), we get

$$\begin{aligned} |K_{11}| &\leq \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_{\eta,\delta}) \theta_{\eta,\delta}^i \theta_{\eta,\delta}^j \psi_\delta \phi_R + \left(\frac{1}{e} + \delta\pi\right) \tilde{M}_1 L_\delta^R \\ &\quad + 3\tilde{M}_1 L_\delta^R + \delta\pi e \tilde{M}_1 L_\delta^R + \frac{3e}{2} \sqrt{\delta} \tilde{\Lambda} g_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R. \end{aligned}$$

It remains to estimate K_{12} . To do that, we integrate $|K_{12}|$ in time on $(0, t)$, for $0 < t < T$, then we use (5.19), (5.9) and the fact that f' is nondecreasing function, with support in $\{x \mid \beta_\delta(x) \geq \frac{1}{e}\}$, we deduce that

$$\int_0^t |K_{12}| \leq (\delta e)^2 \tilde{\Lambda} f'(\beta_\delta(\xi_\infty^\delta)) \int_0^t \int_{-R}^R \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \geq \frac{1}{e}\}} |\partial_x \theta_{\eta,\delta}^i|.$$

This implies, using (5.32) and Hölder's inequality and Lemma 5.3, that

$$\int_0^t |K_{12}| \leq (\delta e)^2 \tilde{\Lambda} f'(\beta_\delta(\xi_\infty^\delta)) (2e T L_\delta^R N_\delta)^{\frac{1}{2}},$$

where $N_\delta = \max_{i \in \{1, \dots, d\}} N_\delta^i$.

The previous estimate associated with the estimate of K_{11} , proved before, gives

$$\begin{aligned} \int_0^t K_1 &\leq \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_{\eta,\delta}) \theta_{\eta,\delta}^i \theta_{\eta,\delta}^j \phi_R \psi_\delta + T \left(\frac{1}{e} + \delta\pi\right) \tilde{M}_1 L_\delta^R \\ &\quad + 3T \tilde{M}_1 L_\delta^R + T \delta \pi e \tilde{M}_1 L_\delta^R + \frac{3Te}{2} \sqrt{\delta} \tilde{\Lambda} g_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R \\ &\quad + (\delta e)^2 \tilde{\Lambda} f'(\beta_\delta(\xi_\infty^\delta)) (2e T L_\delta^R N_\delta)^{\frac{1}{2}}. \end{aligned} \quad (5.53)$$

Step 2 (Estimate of K_3):

This step is devoted to bound K_3 . We begin by writing

$$K_3 = \overbrace{-\frac{1}{R} \int_{\mathbb{R}} \sum_{i=1}^d f'(\beta_\delta(\theta_{\eta,\delta}^i)) \beta'_\delta(\theta_{\eta,\delta}^i) \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \beta_\delta(\theta_{\eta,\delta}^i) \psi_\delta \phi' \left(\frac{x}{R}\right)}^{K_{31}} - \overbrace{\frac{\eta}{R} \int_{\mathbb{R}} \sum_{i=1}^d \partial_x (f(\beta_\delta(\theta_{\eta,\delta}^i))) \phi' \left(\frac{x}{R}\right)}^{K_{32}}.$$

As before, since $|\beta'_\delta(\cdot)| \leq 1$, we can show that

$$|K_{31}| \leq \frac{1}{R} f'(\beta_\delta(\xi_\infty^\delta)) \tilde{\Lambda} \beta_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} \sum_{i=1}^d \int_{-R}^R \mathbb{1}_{\{\beta_\delta(\theta_{\eta,\delta}^i) \geq \frac{1}{e}\}} \leq \frac{e}{R} f'(\beta_\delta(\xi_\infty^\delta)) \tilde{\Lambda} \beta_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R.$$

Integrating by parts in K_{32} , we can see that

$$|K_{32}| = \left| \frac{\eta}{R^2} \int_{\mathbb{R}} \sum_{i=1}^d f(\beta_\delta(\theta_{\eta,\delta}^i)) \phi''\left(\frac{x}{R}\right) \right| \leq \frac{4d\eta}{R} f(\beta_\delta(\xi_\infty^\delta)) \|\phi''\|_{L^\infty(\mathbb{R})}.$$

Bringing together the two previous inequalities, we get

$$|K_3| \leq \frac{e}{R} f'(\beta_\delta(\xi_\infty^\delta)) \tilde{\Lambda} \beta_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R + \frac{4d\eta}{R} f(\beta_\delta(\xi_\infty^\delta)) \|\phi''\|_{L^\infty(\mathbb{R})}. \quad (5.54)$$

Step 3 (Estimate on the entropy S_R^δ):

We know that

$$\frac{d}{dt} S_R^\delta(t) \leq K_1 + K_3.$$

Integrating this inequality in time on $(0, t)$, for $0 < t < T$, and using (5.53), (5.54), we obtain

$$S_R^\delta(t) - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_{\eta,\delta}) \theta_{\eta,\delta}^i \theta_{\eta,\delta}^j \psi_\delta \phi_R \leq C_R^\delta + S_R^\delta(0),$$

with

$$\begin{aligned} C_R^\delta &= T \left(\frac{1}{e} + \delta\pi \right) \tilde{M}_1 L_\delta^R + 3T \tilde{M}_1 L_\delta^R + T \delta \pi e \tilde{M}_1 L_\delta^R \\ &+ \frac{3Te}{2} \sqrt{\delta} \tilde{\Lambda} g_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R + (\delta e)^2 \tilde{\Lambda} f'(\beta_\delta(\xi_\infty^\delta)) (2eTL_\delta^R N_\delta)^{\frac{1}{2}} \\ &+ \frac{Te}{R} f'(\beta_\delta(\xi_\infty^\delta)) \tilde{\Lambda} \beta_\delta(\xi_\infty^\delta) \|\phi'\|_{L^\infty(\mathbb{R})} L_\delta^R + \frac{4Td\eta}{R} f(\beta_\delta(\xi_\infty^\delta)) \|\phi''\|_{L^\infty(\mathbb{R})}. \end{aligned} \quad (5.55)$$

Which gives the announced result. \square

Remark 5.1. We would like to draw the attention of the readers to the fact that estimate (5.19) depends on ξ_p^δ and therefore on η . Nevertheless, we can observe from (5.55) that all terms depending on the constant ξ_p^δ vanishes when $\delta \rightarrow 0$ and $R \rightarrow +\infty$. Which will lead us in the following to a η -uniform estimate.

4 η - Uniform a priori estimates on the solution of (5.12)

In this section, we show, by passing to the limit as $\delta \rightarrow 0$, three η -uniform estimates (similar to that obtained in Section 3) on the solution $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ of system (5.12) given by Theorem 5.2-(ii). The first one concerns the BV estimate, the second concerns the $L \log L$ estimate which is the key result. The third one is about the time derivative of the solution of system (5.12).

Lemma 5.5. Suppose that $(\widetilde{H}1)$ and (5.13) hold. Let $v_\eta = (v_\eta^i)_{i=1,\dots,d} \in (C([0, T]; W^{1,\infty}(\mathbb{R})))^d$ be a unique solution of (5.12), such that v_η^i is defined (as in Theorem 5.2-(ii)) by the strong limit in $C([0, T]; W_{loc}^{1,\infty}(\mathbb{R}))$, when $\delta \rightarrow 0$, of the smooth solution $v_{\eta,\delta}^i$ of (5.16). Moreover, suppose that v_η^i satisfies estimates (5.20) and (5.21) for some $T > 0$. Then, we have

$$\int_{\mathbb{R}} |\partial_x v_\eta^i(t, x)| dx \leq \int_{\mathbb{R}} |\partial_x v_0^i(x)| dx, \quad \text{for all } 0 < t < T \text{ and } i = 1, \dots, d. \quad (5.56)$$

Proof of Lemma 5.5:

By passing to the limit, δ goes to zero, in the inequality (5.39) and using the fact that $\partial_x v_{\eta,\delta}^i$ converges locally uniformly toward $\partial_x v_\eta^i$, when $\delta \rightarrow 0$, we obtain

$$\begin{aligned} & \int_{-2R}^{2R} |\partial_x v_\eta^i(t, x)| \phi_R(x) dx \\ & \leq \int_{-2R}^{2R} |\partial_x v_0^i(x)| \phi_R(x) dx + \frac{4\eta T}{R} \xi_\infty^0 \|\phi''\|_{L^\infty(\mathbb{R})} + \frac{T}{R} \tilde{\Lambda}^i \xi_1^0 \|\phi'\|_{L^\infty(\mathbb{R})} := L_0^{R,i}, \end{aligned} \quad (5.57)$$

where ξ_1^0 and ξ_∞^0 are defined respectively in (5.29) and (5.30) for $\delta = 0$. Then, passing to the limit $R \rightarrow +\infty$ leads to the desired result. \square

Lemma 5.6. (Gradient entropy inequality)

Under the same assumptions of Lemma 5.5, for any $0 < t < T$, there exists a constant $\gamma(T, \tilde{M}_1, \|\partial_x v_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d}, \tilde{\Lambda})$ (that is defined explicitly in (5.61)), such that

$$S(t) - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{,j}^i(t, x, v_\eta(t, x)) \partial_x v_\eta^i \partial_x v_\eta^j \leq \gamma, \quad \text{with} \quad S(t) = \int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x v_\eta^i(t, x)|) dx \quad (5.58)$$

where f is defined in (5.11).

For the proof of Lemma 5.6, we need the following technical lemma, which allows us to control $\|f(\cdot)\|_{L^1(\mathbb{R})}$ by $\|\cdot\|_{L \log L(\mathbb{R})}$ and $\|\cdot\|_{L^1(\mathbb{R})}$.

Lemma 5.7. (Bound of L^1 norm of f by the $L \log L$ norm)

Let f be the function defined in (5.11) and $h \in L^1(\mathbb{R})$ be a non-negative function. If $h \in L \log L(\mathbb{R})$, then $\int_{\mathbb{R}} f(h) < +\infty$ and moreover we have the following estimate:

$$\int_{\mathbb{R}} f(h) \leq 1 + \|h\|_{L \log L(\mathbb{R})} + \|h\|_{L^1(\mathbb{R})} \ln(1 + \|h\|_{L \log L(\mathbb{R})}). \quad (5.59)$$

For the proof of this lemma, we refer the reader to [32, Lemma 3.2, Page 13].

Proof of Lemma 5.6:

The announced result will be proven by passing to the limit, $\delta \rightarrow 0$ and $R \rightarrow +\infty$, successively, in the approximated gradient entropy inequality (5.46). Indeed, since the constants $N_\delta^i, L_\delta^R, \xi_\infty^\delta$ are uniformly bounded with respect to δ , and $\theta_{\eta,\delta}^i = \partial_x v_{\eta,\delta}^i$ converges locally uniformly, as δ goes to 0, toward $\partial_x v_\eta^i$ and using the continuity of $\tilde{\lambda}_{,j}^i$, we can pass to the limit in (5.46), when $\delta \rightarrow 0$, to get

$$\begin{aligned} & \int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x v_\eta^i(t, x)|) \phi_R(x) dx - \int_0^t \int_{\mathbb{R}} \sum_{i=1}^d \tilde{\lambda}_{,j}^i(t, x, v_\eta) \partial_x v_\eta^i(t, x) \partial_x v_\eta^j(t, x) \phi_R(x) dt dx \\ & \leq C_R^0 + \int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x v_0^i(x)|) \phi_R(x) dx, \end{aligned} \quad (5.60)$$

where $C_R^0 = \frac{T}{e} \tilde{M}_1 L_0^R + 3T \tilde{M}_1 L_0^R + \frac{T e}{R} f'(\xi_\infty^0) \tilde{\Lambda} \xi_\infty^0 \|\phi'\|_{L^\infty(\mathbb{R})} L_0^R + \frac{4dT\eta}{R} f(\xi_\infty^0) \|\phi''\|_{L^\infty(\mathbb{R})}$,

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where L_0^R is defined in (5.50) for $\delta = 0$, with $L_0^{R,i}$ is given in (5.57).

From (5.13), we have in particular $\partial_x v_0^i \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and therefore, it is easy to verify that $f(|\partial_x v_0^i(x)|) \in L^1(\mathbb{R})$, for all $i \in \{1, \dots, d\}$. Moreover, we know from (5.21) and (5.56) that $\partial_x v_\eta^i \in L^\infty((0, T); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. This implies that

$$f(|\partial_x v_\eta^i(t, x)|) \in L^\infty((0, T); L^1(\mathbb{R})).$$

Now, passing to the limit when $R \rightarrow +\infty$ in (5.60) and applying Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x v_\eta^i|) - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_\eta) \partial_x v_\eta^i \partial_x v_\eta^j \\ & \leq \frac{T}{e} \widetilde{M}_1 \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + 3T \widetilde{M}_1 \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + \int_{\mathbb{R}} \sum_{i=1}^d f(|\partial_x v_0^i|). \end{aligned}$$

This estimates combined with (5.59), show that

$$S(t) - \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_\eta) \partial_x v_\eta^i \partial_x v_\eta^j \leq \gamma,$$

with

$$\begin{aligned} \gamma &= \frac{T}{e} \widetilde{M}_1 \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + 3T \widetilde{M}_1 \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} \\ &+ \sum_{i=1}^d \left(1 + \|\partial_x v_0^i\|_{L \log L(\mathbb{R})} + \|\partial_x v_0^i\|_{L^1(\mathbb{R})} \ln(1 + \|\partial_x v_0^i\|_{L \log L(\mathbb{R})})\right), \end{aligned} \tag{5.61}$$

which gives the stated result. \square

The following estimate will provide the compactness in time uniformly with respect to η of the solution v_η^i , given in Theorem 5.2-(ii).

Lemma 5.8. (Estimate on the time derivative of the solution uniform on η)

Under the same assumptions of Lemma 5.5. Let $W^{-1,1}(\mathbb{R})$ be the dual space of $W^{1,\infty}(\mathbb{R})$. Then, we have

$$\|\partial_t v_\eta^i\|_{L^\infty((0,T); W^{-1,1}(\mathbb{R}))} \leq \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + \widetilde{\Lambda}^i \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d}. \tag{5.62}$$

Proof of Lemma 5.8:

Roughly speaking, the idea is to control the term $\partial_t v_{\eta,\delta}^i$ using the right hand side of (5.16), and then to pass to the limit $\delta \rightarrow 0$ and $R \rightarrow +\infty$ in order to control $\partial_t v_\eta^i$.

The proof is given by duality. We multiply the equation (5.16) by $\zeta \in L^1((0, T); W^{1,\infty}(\mathbb{R}))$ and by ϕ_R defined in (5.31). Then, integrating on $(0, T) \times \mathbb{R}$, we get

$$\int_{(0,T) \times \mathbb{R}} \zeta \partial_t v_{\eta,\delta}^i \phi_R = \eta \overbrace{\int_{(0,T) \times \mathbb{R}} \zeta \partial_{xx}^2 v_{\eta,\delta}^i \phi_R}^{O_1} + \overbrace{\int_{(0,T) \times \mathbb{R}} \zeta \tilde{\lambda}^i(t, x, v_{\eta,\delta}) \psi_\delta \beta_\delta (\partial_x v_{\eta,\delta}^i) \phi_R}^{O_2}.$$

Integrating by parts O_1 , we get

$$\begin{aligned}
 |O_1| &\leq \left| \int_{(0,T) \times \mathbb{R}} \partial_x \zeta \partial_x v_{\eta,\delta}^i \phi_R + \frac{\eta}{R} \int_{(0,T) \times \mathbb{R}} \zeta \partial_x v_{\eta,\delta}^i \phi' \left(\frac{x}{R} \right) \right| \\
 &\leq \|\partial_x \zeta\|_{L^1((0,T); L^\infty(\mathbb{R}))} \|\partial_x v_{\eta,\delta}^i \phi_R\|_{L^\infty((0,T); L^1(\mathbb{R}))} + \frac{\eta}{R} \|\zeta\|_{L^1((0,T); L^\infty(\mathbb{R}))} \|\partial_x v_{\eta,\delta}^i\|_{L^\infty(0,T); L^1(\mathbb{R})} \|\phi'\|_{L^\infty(\mathbb{R})} \\
 &\leq \|\zeta\|_{L^1((0,T); W^{1,\infty}(\mathbb{R}))} \left(L_\delta^R + \frac{\eta}{R} \xi_1^\delta \|\phi'\|_{L^\infty(\mathbb{R})} \right),
 \end{aligned} \tag{5.63}$$

where we have used inequality (5.19) and (5.32) in the second line. Similarly, using (5.32), we have

$$\begin{aligned}
 |O_2| &\leq \|\zeta\|_{L^1((0,T); L^\infty(\mathbb{R}))} \|\tilde{\Lambda}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \|\phi_R \beta_\delta(\partial_x v_{\eta,\delta}^i)\|_{L^\infty((0,T); L^1(\mathbb{R}))} \\
 &\leq \|\zeta\|_{L^1((0,T); W^{1,\infty}(\mathbb{R}))} \tilde{\Lambda}^i L_\delta^R.
 \end{aligned} \tag{5.64}$$

Collecting (5.63) and (5.64), we get

$$\int_{(0,T) \times \mathbb{R}} \zeta \partial_t v_{\eta,\delta}^i \phi_R \leq \left(L_\delta^R + \frac{\eta}{R} \xi_1^\delta \|\phi'\|_{L^\infty(\mathbb{R})} + \tilde{\Lambda}^i L_\delta^R \right) \|\zeta\|_{L^1((0,T); W^{1,\infty}(\mathbb{R}))}.$$

Hence, by duality, we get

$$\|\partial_t v_{\eta,\delta}^i\|_{L^\infty((0,T); W^{-1,1}(\mathbb{R}))} \leq L_\delta^R + \frac{\eta}{R} \xi_1^\delta \|\phi'\|_{L^\infty(\mathbb{R})} + \tilde{\Lambda}^i L_\delta^R.$$

Now, passing to the limit in the previous inequality, as δ goes to 0, and using the weakly- \star convergence in $L^\infty((0,T); W^{-1,1}(\mathbb{R}))$, we get

$$\|\partial_t v_\eta^i\|_{L^\infty((0,T); W^{-1,1}(\mathbb{R}))} \leq \liminf_{\delta \rightarrow 0} \|\partial_t v_{\eta,\delta}^i\|_{L^\infty((0,T); W^{-1,1}(\mathbb{R}))} \leq L_0^R + \frac{\eta}{R} \xi_1^0 \|\phi'\|_{L^\infty(\mathbb{R})} + \tilde{\Lambda}^i L_0^R.$$

Finally, passing to limit as R tends to $+\infty$, we obtain

$$\|\partial_t v_\eta^i\|_{L^\infty((0,T); W^{-1,1}(\mathbb{R}))} \leq L_0^\infty + \tilde{\Lambda}^i L_0^\infty = \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + \tilde{\Lambda}^i \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d}.$$

This terminates the proof. □

5 Global existence of a continuous solution of (5.12)

In this section, we are going to prove that the local in time solution obtained in Section 2 can be extended to a global one.

Theorem 5.3. (Global existence and uniqueness of continuous solution of (5.12))

Assume that (5.13) and $(\widetilde{H}1)$ hold. Then, for all $0 < \eta \leq 1$ and $T > 0$, system (5.12) admits a unique solution $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ belonging to $(C([0,T]); W^{1,\infty}(\mathbb{R}))^d$, satisfying for all $T > 0$ the following estimates

$$\|v_\eta^i\|_{L^\infty((0,T) \times \mathbb{R})} \leq \|v_0^i\|_{L^\infty(\mathbb{R})}, \tag{5.65}$$

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$$\|\partial_x v_\eta^i\|_{L^\infty((0,T);L^1(\mathbb{R}))} \leq \|\partial_x v_0^i\|_{L^1(\mathbb{R})}, \quad (5.66)$$

$$\|\partial_t v_\eta^i\|_{L^\infty((0,T);W^{-1,1}(\mathbb{R}))} \leq \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} + \tilde{\Lambda}^i \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d}. \quad (5.67)$$

Moreover, if $(\widetilde{H2})$ holds, then we have

$$\sum_{i=1}^d \|\partial_x v_\eta^i\|_{L^\infty((0,T);L \log L(\mathbb{R}))} \leq \gamma', \quad (5.68)$$

where $\gamma'(d, T, \widetilde{M}_1, \|\partial_x v_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d})$.

Proof of Theorem 5.3: The proof is outlined in two steps.

Step 1 (Global existence):

From Theorem 5.2, we have the local existence of a solution $v_\eta = (v_\eta^i)_{i=1,\dots,d}$ to system (5.12). It remains to show that this local solution can be extended to a global one.

We argue by contradiction. Assume that there exists a maximum time T_{max} such that, we have the existence of solution of the system (5.12) in the function space $X_{T_{max}}$. Similarly, using the proof of Theorem 5.2, we can obtain that $v_\eta^i \in C([0, T_{max}]; W^{1,\infty}(\mathbb{R}))$, for all $i = 1, \dots, d$. Consequently, v_η^i verifies (5.20) for $T^* = T_{max}$ and (5.21) for a certain constant

$\xi_p^0 = \xi_p^0(T_{max}, \|v_0\|_{(L^\infty(\mathbb{R}))^d}, \|\partial_x v_0\|_{(L^p(\mathbb{R}))^d})$. Thus, estimates (5.56) and (5.62) hold for $T = T_{max}$.

For every $\mu > 0$, we consider the system (5.12) with the initial conditions

$$v_0^{\mu,i}(x) = v_\eta^i(T_{max} - \mu, x), \quad \text{for all } i = 1, \dots, d.$$

We apply for the second time the same technics of the proof of Theorem 5.2 to deduce that there exists a time T_μ^* , defined as in (5.27) and (5.28) for $\delta = 1$, such that system (5.12) admits a unique solution defined until the time

$$T_0 = (T_{max} - \mu) + T_\mu^*$$

with

$$T_\mu^* = \min \left(\frac{\eta\pi}{(3\tilde{\Lambda}(\|\phi\|_{L^1(\mathbb{R})} + \|v_0^\mu\|_X + 1))^2}, \frac{\eta\pi}{\left(6 \left(\tilde{M}_1(\|\phi\|_{L^1(\mathbb{R})} + \|v_0^\mu\|_X + 1) + \tilde{\Lambda}\right)\right)^2} \right),$$

where $v_0^\mu = (v_0^{\mu,i})_{i=1,\dots,d}$ and $\tilde{\Lambda}$ is defined as in (5.17) for $T = T_{max}$.

According to (5.20) and (5.56) for $0 < t < T_{max}$, we know that

$$\begin{aligned} \|v_0^\mu\|_X &= \sum_{i=1}^d \|\partial_x v_\eta^i(T_{max} - \mu, \cdot)\|_{L^1(\mathbb{R})} + \sum_{i=1}^d \|v_\eta^i(T_{max} - \mu, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\leq \sum_{i=1}^d \|\partial_x v_\eta^i\|_{L^\infty((0,T_{max});L^1(\mathbb{R}))} + \sum_{i=1}^d \|v_\eta^i\|_{L^\infty((0,T_{max}) \times \mathbb{R})} \\ &\leq \sum_{i=1}^d \|\partial_x v_0^i\|_{L^1(\mathbb{R})} + \sum_{i=1}^d \|v_0^i\|_{L^\infty(\mathbb{R})} = \|v_0\|_X. \end{aligned}$$

Then, there exists a constant

$$C(\eta, \tilde{\Lambda}, \tilde{M}_1, \|v_0\|_{(L^\infty(\mathbb{R}))^d}, \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d}, T_{max}) > 0,$$

independent of μ such that $T_\mu^* \geq C > 0$. Passing to the limit $\mu \rightarrow 0$, we can see that $\lim_{\mu \rightarrow 0} T_\mu^* \geq C > 0$. This implies that $T_0 > T_{max}$ (for small μ). So we have contradiction and then we can construct a solution $v_\eta^i \in X_T \cap C([0, T]; W^{1,\infty}(\mathbb{R}))$, for all $T > 0$ and for all $i \in \{1, \dots, d\}$.

Step 2 (Gradient entropy estimate):

Estimates (5.65), (5.66) and (5.67) are direct consequences from (5.20), (5.56) and (5.62). To prove (5.68), we need to prove the following technical lemma, that allows us to bound $\|\cdot\|_{L \log L(\mathbb{R})}$ by $\|f(\cdot)\|_{L^1(\mathbb{R})}$ and $\|\cdot\|_{L^1(\mathbb{R})}$.

Lemma 5.9. ($L \log L$ estimate)

Let f be the function defined in (5.11) and $h \in L^1(\mathbb{R})$ be a non-negative function. If

$\int_{\mathbb{R}} f(h) < +\infty$, then $h \in L \log L(\mathbb{R})$. Moreover we have the following estimate:

$$\|h\|_{L \log L(\mathbb{R})} \leq 1 + \int_{\mathbb{R}} f(h) + \ln(1 + e^2) \|h\|_{L^1(\mathbb{R})}. \quad (5.69)$$

For the proof of this lemma, we refer the reader to [32, Lemma 4.5, Page 18]. According to $(\tilde{H}2)$, we have

$$\int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_\eta) \partial_x v_\eta^i \partial_x v_\eta^j \leq 0.$$

From this and using estimate (5.58), we can see that

$$S(t) \leq \gamma + \int_0^t \int_{\mathbb{R}} \sum_{i,j=1}^d \tilde{\lambda}_{i,j}^i(t, x, v_\eta) \partial_x v_\eta^i \partial_x v_\eta^j \leq \gamma,$$

where γ is defined in (5.61). This implies using estimates (5.56) and Lemma 5.9 that

$$\begin{aligned} & \|\partial_x v_\eta\|_{(L^\infty((0,T); L \log L(\mathbb{R})))^d} \\ & \leq \sum_{i=1}^d \left(1 + \sup_{0 \leq t \leq T} \left(\int_{\mathbb{R}} f(|\partial_x v_\eta^i|) \right) + \ln(1 + e^2) \|\partial_x v_\eta^i\|_{L^\infty((0,T); L^1(\mathbb{R}))} \right) \\ & \leq d + \gamma + \ln(1 + e^2) \|\partial_x v_0\|_{(L^1(\mathbb{R}))^d} := \gamma'. \end{aligned} \quad (5.70)$$

□

6 Existence of continuous viscosity solution

We prove in this section our existence result, cited in Theorem 5.1 (i). We proceed in two stages. First, we prove, using Theorem 5.3 and the stability results for continuous viscosity solutions, the existence and uniqueness of continuous viscosity solution of the following approximate problem, for $0 < \eta \leq 1$:

$$\begin{cases} \partial_t u_\eta^i(t, x) = \eta \partial_{xx}^2 u_\eta^i(t, x) + \lambda_\eta^i(t, x, u_\eta(t, x)) |\partial_x u_\eta^i(t, x)| & \text{in } (0, T) \times \mathbb{R}, \\ u_\eta^i(0, x) = u_{0,\eta}^i(x) & \text{in } \mathbb{R}, \end{cases} \quad (5.71)$$

where $u_{0,\eta}^i$ is the regularization of the function u_0^i , defined by

$$u_{0,\eta}^i(x) = u_0^i \star \rho_\eta^1(x), \quad (5.72)$$

and λ_η^i is the regularization of the function λ^i , given by

$$\lambda_\eta^i(t, x, w) = \hat{\lambda}^i \star \rho_\eta^{d+2}(t, x, w) \quad \forall (t, x, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, \quad (5.73)$$

with $\hat{\lambda}^i$ is the extension in \mathbb{R}^{d+2} of λ^i by 0, and ρ_η^n for $n = 1, d+2$ are the standard mollifiers defined as follows

$$\rho_\eta^n(\cdot) = \frac{1}{\eta^n} \rho^n\left(\frac{\cdot}{\eta}\right), \quad \text{such that} \quad \left\{ \begin{array}{l} \rho^n \in C_c^\infty(\mathbb{R}^n), \text{ supp } \{\rho^n\} \subseteq B(0, 1), \\ \rho^n \geq 0, \text{ and } \int_{\mathbb{R}^n} \rho^n = 1. \end{array} \right. \quad (5.74)$$

Then, we use Simon's Lemma in order to pass to the limit as $\eta \rightarrow 0$ and prove the main results cited in Theorem 5.1-(i).

For the sake of a clear presentation, we divide this section into two subsections.

6.1 Existence of continuous viscosity solution of the approximate problem

In this subsection, we prove the global existence and uniqueness result of system (5.71) considering regular initial data $u_{0,\eta} = (u_{0,\eta}^i)_{i=1,\dots,d}$. In this context, we prove that this system admits a continuous viscosity solution $u_\eta = (u_\eta^i)_{i=1,\dots,d}$, satisfying similar estimates to (5.65), (5.66), (5.67) and (5.68). Let us, first, recall the definition of the continuous viscosity solution, for systems, that was avowed by Ishii and Koike in [48]. We also refer to Barles [11], Bardi Capuzzo-dolcetta [10] and Crandall, Ishii, Lions [27] for a good introduction to viscosity solution theory in the scalar case.

Definition 5.1. (Continuous viscosity sub-solution, super-solution and solution)

For $i \in \{1, \dots, d\}$, let $H_i : (0, T) \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous Hamiltonian.

A function $w = (w^i)_{i=1,\dots,d} \in (C([0, T) \times \mathbb{R}))^d$ is a viscosity sub-solution of the following system:

$$\begin{cases} \partial_t w^i(t, x) + H_i(t, w(t, x), \partial_x w^i(t, x)) - \eta \partial_{xx}^2 w^i(t, x) = 0 & \text{in } (0, T) \times \mathbb{R}, \\ w^i(0, x) = w_0^i(x) & \text{in } \mathbb{R}, \end{cases} \quad (5.75)$$

if and only if

- $w^i(0, x) \leq w_0^i(x)$,
- for all $i \in \{1, \dots, d\}$ and for any test-function $\phi \in C^2((0, T) \times \mathbb{R})$ such that $w^i - \phi$ reaches a local maximum at a point $(t_0, x_0) \in (0, T) \times \mathbb{R}$, we have

$$\partial_t \phi(t_0, x_0) + H_i(t_0, w(t_0, x_0), \partial_x \phi(t_0, x_0)) - \eta \partial_{xx}^2 \phi(t_0, x_0) \leq 0.$$

In a similar way, a function $w = (w^i)_{i=1, \dots, d} \in (C([0, T] \times \mathbb{R}))^d$ is a viscosity super-solution of (5.75) if and only if

- $w^i(0, x) \geq w_0^i(x)$,
- for all $i \in \{1, \dots, d\}$ and for any test-function $\phi \in C^2((0, T) \times \mathbb{R})$ such that $w^i - \phi$ reaches a local minimum at point $(t_0, x_0) \in (0, T) \times \mathbb{R}$, we have

$$\partial_t \phi(t_0, x_0) + H_i(t_0, w(t_0, x_0), \partial_x \phi(t_0, x_0)) - \eta \partial_{xx}^2 \phi(t_0, x_0) \geq 0.$$

Finally, a continuous function w^i is a viscosity solution of (5.75) if and only if it is a sub- and a super-solution of (5.75).

Theorem 5.4. (Existence of continuous viscosity solution of (5.71))

Under assumptions (H1) and (H3), system (5.71) admits a unique continuous viscosity solution on $(0, T) \times \mathbb{R}$, satisfying the following L^∞ bound:

$$\|u_\eta^i\|_{L^\infty((0, T) \times \mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \quad \text{for all } i \in \{1, \dots, d\}, \quad (5.76)$$

the following BV estimate:

$$\|\partial_x u_\eta^i\|_{L^\infty((0, T); L^1(\mathbb{R}))} \leq \|\partial_x u_0^i\|_{L^1(\mathbb{R})}, \quad (5.77)$$

and the time derivative estimate:

$$\|\partial_t u_\eta^i\|_{L^\infty(0, T); W^{-1,1}(\mathbb{R})} \leq \|\partial_x u_0\|_{(L^1(\mathbb{R}))^d} + \Lambda^i \|\partial_x u_0\|_{(L^1(\mathbb{R}))^d}. \quad (5.78)$$

Moreover, if (H2) holds, we have the following gradient entropy estimate:

$$\sum_{i=1}^d \left(\|\partial_x u_\eta^i\|_{L^\infty((0, T); L \log L(\mathbb{R}))} \right) \leq \gamma_1, \quad (5.79)$$

where $\gamma_1 \left(T, d, M_1, \|\partial_x u_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d} \right)$.

For the proof of this Theorem, we need the following result.

Lemma 5.10. Let $(\rho_\eta^1)_{\eta \in (0, 1]}$ be the standard mollifier defined in (5.74). If $h \in L \log L(\mathbb{R})$, then for every $\eta \in (0, 1]$ the function $h_\eta = h \star \rho_\eta^1 \in L \log L(\mathbb{R})$ and satisfies

$$\|h_\eta\|_{L \log L(\mathbb{R})} \leq C \|h\|_{L \log L(\mathbb{R})} \quad \text{and} \quad \|h - h_\eta\|_{L \log L(\mathbb{R})} \rightarrow 0 \quad \text{as } \eta \rightarrow 0,$$

where C is a universal constant.

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For the proof of this lemma see Adams [4, Th 8.20].

Proof of Theorem 5.4:

By assumption (H3), and by classical properties of the mollifiers, we can see that, for all $\eta > 0$, the initial data $u_{0,\eta}^i$ satisfies (5.13) for all $i \in \{1, \dots, d\}$. In addition, from (H1) we can also see that λ_η^i satisfies assumption ($\widetilde{H1}$), with

$$\left\{ \begin{array}{l} \|\lambda_\eta^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \leq \Lambda^i \\ \sum_{i=1}^d \|\partial_x \lambda_\eta^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \leq M_1 \\ \sum_{i=1}^d \|\lambda_{\eta,j}^i\|_{L^\infty((0,T) \times \mathbb{R} \times \mathcal{U})} \leq M_1 \end{array} \right. \quad \text{for } i = 1, \dots, d.$$

Thus, we can apply Theorem 5.3 with $v_0^i = u_{0,\eta}^i$ and $\tilde{\lambda}^i = \lambda_\eta^i$ and prove, that system (5.71) admits a unique solution $u_\eta = (u_\eta^i)_{1 \leq i \leq d} \in (C((0,T); W^{1,\infty}(\mathbb{R})))^d$ satisfying the following estimates:

$$\begin{aligned} \|u_\eta^i\|_{L^\infty((0,T) \times \mathbb{R})} &\leq \|u_{0,\eta}^i\|_{L^\infty(\mathbb{R})} \leq \|u_0^i\|_{L^\infty(\mathbb{R})}, \\ \|\partial_x u_\eta^i\|_{L^\infty((0,T); L^1(\mathbb{R}))} &\leq \|\partial_x u_{0,\eta}^i\|_{L^1(\mathbb{R})} \leq \|\partial_x u_0^i\|_{L^1(\mathbb{R})}, \\ \|\partial_t u_\eta^i\|_{L^\infty(0,T); W^{-1,1}(\mathbb{R})} &\leq (1 + \Lambda^i) \|\partial_x u_{0,\eta}\|_{(L^1(\mathbb{R}))^d} \leq (1 + \Lambda^i) \|\partial_x u_0\|_{(L^1(\mathbb{R}))^d}. \end{aligned}$$

Moreover, from (H2) we can verify that λ_η^i satisfies ($\widetilde{H2}$) and therefore we get using (5.68) the following entropy estimate:

$$\sum_{i=1}^d (\|\partial_x u_\eta^i\|_{L^\infty((0,T); L \log L(\mathbb{R}))}) \leq d + \gamma_\eta + \ln(1 + e^2) \|\partial_x u_{0,\eta}\|_{(L^1(\mathbb{R}))^d}$$

where

$$\begin{aligned} \gamma_\eta &= \frac{TM_1}{e} \|\partial_x u_{0,\eta}\|_{(L^1(\mathbb{R}))^d} + 3TM_1 \|\partial_x u_{0,\eta}\|_{(L^1(\mathbb{R}))^d} \\ &\quad + \sum_{i=1}^d (1 + \|\partial_x u_{0,\eta}^i\|_{L \log L(\mathbb{R})} + \|\partial_x u_{0,\eta}^i\|_{L^1(\mathbb{R})} \ln(1 + \|\partial_x u_{0,\eta}^i\|_{L \log L(\mathbb{R})})). \end{aligned}$$

This gives, using Lemma 5.10 and the classical properties of the mollifiers, that there exists a constant γ_1 depending on T , d , M_1 and $\|\partial_x u_0\|_{(L^1(\mathbb{R}) \cap L \log L(\mathbb{R}))^d}$, such that (5.79) is verified with

$$\begin{aligned} \gamma_1 &= d + \ln(1 + e^2) \|\partial_x u_0\|_{(L^1(\mathbb{R}))^d} + \frac{TM_1}{e} \|\partial_x u_0\|_{(L^1(\mathbb{R}))^d} + 3TM_1 \|\partial_x u_0\|_{(L^1(\mathbb{R}))^d} \\ &\quad + \sum_{i=1}^d (1 + \mathcal{C} \|\partial_x u_0^i\|_{L \log L(\mathbb{R})} + \|\partial_x u_0^i\|_{L^1(\mathbb{R})} \ln(1 + \mathcal{C} \|\partial_x u_0^i\|_{L \log L(\mathbb{R})})). \end{aligned} \tag{5.80}$$

It is easy to verify, using the stability result of viscosity solutions (cf. Barles [11, Th 2.3]), that u_η is a continuous viscosity solution of (5.71). This comes from the fact that u_η is constructed as the uniform limit of $u_{\eta,\delta}$, that is a C^2 classical solution of the approximate problem (5.16). \square

Remark 5.2. All estimates given by Theorem 5.4 are η -uniform. In particular, we can observe from the explicit definition of the constant γ_1 in (5.80) that the gradient entropy estimate (5.79) is independent of η .

6.2 Passage to the limit as η tends to 0

6.2.1 Some useful results

First, we recall the following compactness lemmas.

Lemma 5.11. (Simon's Lemma [67, Corollary 4])

Let X , B and Y be three Banach spaces, where $X \hookrightarrow B$ with compact embedding and $B \hookrightarrow Y$ with continuous embedding. If $(\theta_n)_n$ is a uniformly bounded sequence in $L^\infty((0, T); X)$ and $(\partial_t \theta_n)_n$ is uniformly bounded in $L^r((0, T); Y)$ for $r > 1$, then $(\theta_n)_n$ is relatively compact in $C((0, T); B)$.

Lemma 5.12. (Compact embedding)

Let I be an open bounded interval of \mathbb{R} and $L \log L(I)$ be the Zygmund space, defined as (5.4) replacing \mathbb{R} by I . If we denote by

$$W^{1, L \log L}(I) = \{h \in L^1(I) \text{ such that } \partial_x h \in L \log L(I)\},$$

then the following injection:

$$W^{1, L \log L}(I) \hookrightarrow C(I),$$

is compact.

For the proof of this Lemma see R. A. Adams [4, Th 8.32].

6.2.2 Proof of Theorem 5.1-(i)

Let u_η be the continuous viscosity solution of (5.71) given by Theorem 5.4. First, we remark from estimate (5.76) that u_η^i is η -uniformly bounded in $L^\infty((0, T) \times \mathbb{R})$, for any $T > 0$. Hence, as η goes to zero, we can extract a subsequence still denoted by u_η^i , that converges weakly- \star in $L^\infty((0, T) \times \mathbb{R})$ to some limit u^i . Moreover, estimate (5.79) shows that for all open and bounded interval I of \mathbb{R} , we have

$$\sum_{i=1}^d \|\partial_x u_\eta^i\|_{L^\infty((0, T); L \log L(I))} \leq \gamma_1,$$

where γ_1 is independent of η .

From the compactness of $W^{1, L \log L}(I) \hookrightarrow L^\infty(I)$ (cf. Lemma 5.12), we can apply Simon's Lemma (i.e. Lemma 5.11), with $X = W^{1, L \log L}(I)$, $B = L^\infty(I)$ and $Y = W^{-1, 1}(I)$ to show that u_η^i is relatively compact in $L^\infty((0, T) \times I)$. Hence, as $\eta \rightarrow 0$, we can extract a subsequence, still denoted u_η^i , that satisfies the following convergences:

$$\left| \begin{array}{lll} u_\eta^i \longrightarrow u^i, & \text{strongly} & \text{in } L^\infty(I \times (0, T)), \text{ for all bounded interval } I \subset \mathbb{R}, \\ u_\eta^i \longrightarrow u^i, & \text{weakly-}\star & \text{in } L^\infty(\mathbb{R} \times (0, T)), \\ \partial_x u_\eta^i \longrightarrow \partial_x u^i, & \text{weakly-}\star & \text{in } L^\infty((0, T); L \log L(\mathbb{R})). \end{array} \right.$$

Then, using the stability result of viscosity solutions (cf. Barles [11, Th 2.3]), we can prove that u^i is a continuous viscosity solution of (5.1), which gives the existence.

Now, in order to get (5.6), it suffices to pass to the limit when $\eta \rightarrow 0$ in the estimate (5.76), using the lower semi-continuity of the L^∞ -norm with respect to the weakly- \star topology.

Note that the space E_{exp} is strictly included in the classical space L^{exp} which is the dual of $L \log L$ (see [4, 8.14, p. 236] for more details).

Likewise, since $L^\infty((0, T); L \log L(\mathbb{R}))$ is the dual of $L^1((0, T); E_{exp}(\mathbb{R}))$ (see Adams [4] for the definition of the Banach space $E_{exp}(\mathbb{R})$), we can use the lower semi-continuity of the $L^\infty((0, T); L \log L(\mathbb{R}))$ -norm with respect to the weakly- \star topology, to get

$$\|\partial_x u\|_{(L^\infty((0, T); L \log L(\mathbb{R})))^d} \leq \liminf_{\eta \rightarrow 0} \|\partial_x u_\eta\|_{(L^\infty((0, T); L \log L(\mathbb{R})))^d} \leq \gamma_1,$$

which leads to (5.8).

Finally, from (5.77) and the lower semi-continuity of the total variation, we obtain

$$TV(u^i(t, \cdot)) \leq \liminf_{\eta \rightarrow 0} TV(u_\eta^i(t, \cdot)) \leq \|\partial_x u_0^i\|_{L^1(\mathbb{R})} \quad \text{for all } t \geq 0.$$

Since $u^i(t, \cdot) \in W_{loc}^{1,1}(\mathbb{R})$, we know that the L^1 -norm of $\partial_x u^i$ coincides with the total variation of u^i which gives (5.7).

This terminates the proof of Theorem 5.1-(i). □

7 Uniqueness of the solution (Theorem 5.1 (ii))

In this section, we will prove the uniqueness of the solution of system (5.1)-(5.2) in the framework of continuous viscosity solution. To establish this result, it suffices to show that system (5.1) verifies the comparison principle, reflected by the following proposition.

Proposition 5.1. (Comparison principle)

Let $p = (p^i)_{i=1, \dots, d}$ and $q = (q^i)_{i=1, \dots, d}$ be respectively sub- and super- viscosity continuous solution of (5.1), with $p^i(0, \cdot) = p_0^i(\cdot)$ and $q^i(0, \cdot) = q_0^i(\cdot)$. Assume moreover that, there exists a constant $K > 0$ such that

$$\|p^i\|_{L^\infty((0, T) \times \mathbb{R})} \leq K \quad \text{and} \quad \|q^i\|_{L^\infty((0, T) \times \mathbb{R})} \leq K \quad \text{for all } i \in \{1, \dots, d\}. \quad (5.81)$$

Then, if $p^i(0, \cdot) \leq q^i(0, \cdot)$ in \mathbb{R} we get $p^i \leq q^i$ in $(0, T) \times \mathbb{R}$ for $i \in \{1, \dots, d\}$.

Proof of Proposition 5.1:

Let us denote by $M_{sup} = \max_{i \in \{1, \dots, d\}} \sup_{[0, T] \times \mathbb{R}} (\bar{p}^i - \bar{q}^i)$ where $\bar{p}^i(t, x) = p^i(t, x)e^{-\gamma t}$ and $\bar{q}^i(t, x) = q^i(t, x)e^{-\gamma t}$.

It is sufficient to prove that $M_{sup} \leq 0$. Let us suppose by contradiction that $M_{sup} > 0$. We duplicate the variables by considering, for all $\epsilon, \beta, \eta, \alpha$ and γ positives

$$\psi(t, s, x, y, i) = \bar{p}^i(t, x) - \bar{q}^i(s, y) - \frac{|x - y|^2}{2\epsilon} - \frac{|t - s|^2}{2\beta} - \frac{\eta}{T - t} - \alpha(|x|^2 + |y|^2),$$

We note that $\psi(\cdot, \cdot, \cdot, \cdot, i)$ is locally bounded continuous function in $[0, T]^2 \times \mathbb{R}^2$.

We can think that the maximum of ψ noted

$$M(\epsilon, \beta, \alpha, \eta, \gamma) = \sup_{\substack{(t,s) \in [0,T]^2 \\ (x,y) \in \mathbb{R}^2, i \in \{1, \dots, d\}}} \psi(t, s, x, y, i),$$

is similar with M_{sup} .

The idea is justified by the following lemma.

Lemma 5.13. *Let $(\bar{t}, \bar{s}, \bar{x}, \bar{y}, \bar{i})$ be a maximum of ψ . Then, the following properties hold:*

1. $\alpha|\bar{x}|, \alpha|\bar{y}| \leq \sqrt{2\alpha K}$.
2. $\lim_{\beta \rightarrow 0} |\bar{t} - \bar{s}| = \lim_{\epsilon \rightarrow 0} |\bar{x} - \bar{y}| = 0$.
3. $\bar{t} < T - \frac{\eta}{2K} < T$.
4. Up to extract a subsequence $\lim_{(\alpha, \eta) \rightarrow (0,0)} \lim_{\epsilon \rightarrow 0} \lim_{\beta \rightarrow 0} \bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) = M_{sup}$.
5. Up to extract a subsequence $\lim_{(\alpha, \eta) \rightarrow (0,0)} \lim_{\epsilon \rightarrow 0} \lim_{\beta \rightarrow 0} \frac{|\bar{x} - \bar{y}|^2}{2\epsilon} = 0$.
6. \bar{t}, \bar{s} are positives if ϵ, β, α and η are sufficiently small.

We first show the proof of this lemma and then we complete the proof of Proposition 5.1.

Proof of Lemma 5.13:

Using the fact that \bar{p}^i and \bar{q}^i , for $i \in \{1, \dots, d\}$, are bounded, we deduce that

$$\lim_{|x|, |y| \rightarrow \infty} \psi(t, s, x, y) = -\infty,$$

and so the function ψ reaches a maximum at a point $(\bar{t}, \bar{s}, \bar{x}, \bar{y}, \bar{i}) \in (0, T)^2 \times \mathbb{R}^2 \times \{1, \dots, d\}$.

We know that $M_{sup} = \max_{i \in \{1, \dots, d\}} \sup_{[0, T] \times \mathbb{R}} (\bar{p}^i - \bar{q}^i) > 0$, then there exists $(t^*, x^*, i^*) \in (0, T) \times \mathbb{R} \times \{1, \dots, d\}$

such that $\bar{p}^{i^*}(t^*, x^*) - \bar{q}^{i^*}(t^*, x^*) > 0$. Then, we have

$$\psi(t^*, t^*, x^*, x^*, i^*) \leq M(\epsilon, \beta, \alpha, \eta, \gamma),$$

which implies

$$0 < \bar{p}^{i^*}(t^*, x^*) - \bar{q}^{i^*}(t^*, x^*) \leq M(\epsilon, \beta, \alpha, \eta, \gamma) + \frac{\eta}{T - t^*} + 2\alpha|x^*|^2,$$

$$0 < M(\epsilon, \beta, \alpha, \eta, \gamma) + \frac{\eta}{T - t^*} + 2\alpha|x^*|^2.$$

Then, for η and α small enough, we get

$$M(\epsilon, \beta, \alpha, \eta, \gamma) > 0.$$

Thus,

$$\alpha(|\bar{x}|^2 + |\bar{y}|^2) \leq \bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) < 2K, \quad (5.82)$$

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where we have used (5.81) for the second inequality. Multiplying the previous inequality by α , yields (1) and

$$\lim_{\alpha \rightarrow 0} \alpha |\bar{x}| = \lim_{\alpha \rightarrow 0} \alpha |\bar{y}| = 0. \quad (5.83)$$

In the same way, we have

$$\frac{|\bar{x} - \bar{y}|^2}{2\epsilon} + \frac{|\bar{t} - \bar{s}|^2}{2\beta} < 2K, \quad (5.84)$$

and so

$$\lim_{\epsilon \rightarrow 0} |\bar{x} - \bar{y}|^2 = \lim_{\beta \rightarrow 0} |\bar{t} - \bar{s}|^2 = 0. \quad (5.85)$$

We also deduce that

$$\frac{\eta}{T - \bar{t}} < 2K.$$

We, then, have

$$\bar{t} < T - \frac{\eta}{2K} < T.$$

Now, let us prove (4). Indeed, for all $(t, x, i) \in (0, T) \times \mathbb{R} \times \{1, \dots, d\}$, we have

$$\psi(t, t, x, x, i) \leq \psi(\bar{t}, \bar{s}, \bar{x}, \bar{y}, \bar{i}),$$

then,

$$\begin{aligned} \bar{p}^i(t, x) - \bar{q}^i(t, x) &\leq \bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) - \frac{|\bar{x} - \bar{y}|^2}{2\epsilon} - \frac{|\bar{t} - \bar{s}|^2}{2\beta} - \frac{\eta}{T - \bar{t}} - \alpha(|\bar{x}|^2 + |\bar{y}|^2) + \frac{\eta}{T - t} + 2\alpha|x|^2 \\ &\leq \bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) + \frac{\eta}{T - t} + 2\alpha|x|^2. \end{aligned}$$

Thus,

$$\bar{p}^i(t, x) - \bar{q}^i(t, x) - 2\alpha|x|^2 - \frac{\eta}{T - t} \leq \bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}),$$

which implies that

$$\begin{aligned} \bar{p}^i(t, x) - \bar{q}^i(t, x) - 2\alpha|x|^2 - \frac{\eta}{T - t} &\leq \left(\bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{t}, \bar{x}) + \bar{q}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) \right) \\ &\leq M_{sup} + \left(\bar{q}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) \right). \end{aligned}$$

Since $\bar{t}, \bar{s}, \bar{x}, \bar{y}$ and \bar{i} are uniformly bounded with respect to β (cf. (5.82)), then using (5.85) we can extract a subsequence with respect to β , still denoted $(\bar{x}, \bar{y}, \bar{t}, \bar{s}, \bar{i})$, such that

$$\lim_{\beta \rightarrow 0} (\bar{x}, \bar{y}, \bar{t}, \bar{s}) = (\tilde{x}, \tilde{y}, \tilde{\tau}, \tilde{\tau}), \quad \text{and} \quad \lim_{\beta \rightarrow 0} \bar{i} = \tilde{i} = \tilde{i}. \quad (5.86)$$

Therefore, passing to the $\lim_{\beta \rightarrow 0}$ in the previous inequality, we get (up to extract a subsequence) that

$$\begin{aligned} \bar{p}^i(t, x) - \bar{q}^i(t, x) - 2\alpha|x|^2 - \frac{\eta}{T - t} &\leq \lim_{\beta \rightarrow 0} \left(\bar{p}^{\bar{i}}(\bar{t}, \bar{x}) - \bar{q}^{\bar{i}}(\bar{s}, \bar{y}) \right) \\ &\leq M_{sup} + \left(\bar{q}^{\bar{i}}(\tilde{\tau}, \tilde{x}) - \bar{q}^{\bar{i}}(\tilde{\tau}, \tilde{y}) \right). \end{aligned}$$

In the same way, since (\tilde{x}, \tilde{y}) verifies estimates (5.82) and (5.84), replacing (\bar{x}, \bar{y}) by (\tilde{x}, \tilde{y}) , we can extract a subsequence with respect to ϵ , still denoted $(\tilde{x}, \tilde{y}, \tilde{\tau}, \tilde{i})$, such that

$$\lim_{\epsilon \rightarrow 0} (\tilde{x}, \tilde{y}, \tilde{\tau}, \tilde{\tau}) = (\tilde{\tilde{x}}, \tilde{\tilde{x}}, \tilde{\tilde{\tau}}, \tilde{\tilde{\tau}}) \quad \text{and} \quad \lim_{\beta \rightarrow 0} \tilde{i} = \tilde{i} = \tilde{\tilde{i}}. \quad (5.87)$$

Now, passing to the $\lim_{\epsilon \rightarrow 0}$ and then $\liminf_{(\alpha, \eta) \rightarrow 0}$, $\limsup_{(\alpha, \eta) \rightarrow 0}$, in the above inequality, we obtain thanks to the continuity of \bar{q}^i that

$$\begin{aligned} \bar{p}^i(t, x) - \bar{q}^i(t, x) &\leq \liminf_{(\alpha, \eta) \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\beta \rightarrow 0} \left(\bar{p}^{\tilde{i}}(\bar{t}, \bar{x}) - \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) \right) \\ &\leq \limsup_{(\alpha, \eta) \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\beta \rightarrow 0} \left(\bar{p}^{\tilde{i}}(\bar{t}, \bar{x}) - \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) \right) \\ &\leq M_{sup}. \end{aligned}$$

This gives Lemma 5.13-(4). Similarly, we can prove that, up to extract a subsequence, we have

$$\lim_{(\alpha, \eta) \rightarrow (0,0)} \lim_{\epsilon \rightarrow 0} \lim_{\beta \rightarrow 0} \frac{|\bar{x} - \bar{y}|^2}{2\epsilon} = \lim_{(\alpha, \eta) \rightarrow (0,0)} \lim_{\epsilon \rightarrow 0} \frac{|\tilde{x} - \tilde{y}|^2}{2\epsilon} = 0. \quad (5.88)$$

Finally, for proving (5) let us suppose, for example, that $\bar{t} = 0$, then, because of $p^i(0, \cdot) \leq q^i(0, \cdot)$, we have

$$\begin{aligned} \bar{p}^{\tilde{i}}(0, \bar{x}) - \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) &\leq \bar{p}^{\tilde{i}}(0, \bar{x}) - \bar{q}^{\tilde{i}}(0, \bar{x}) + \bar{q}^{\tilde{i}}(0, \bar{x}) - \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) \\ &\leq \bar{q}^{\tilde{i}}(0, \bar{x}) - \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}). \end{aligned}$$

Then, $M_{sup} = \lim_{(\alpha, \eta) \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\beta \rightarrow 0} \bar{p}^{\tilde{i}}(0, \bar{x}) - \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) \leq 0$. But we assumed that $M_{sup} > 0$, hence the contradiction. When $\bar{s} = 0$, a similar proof could be given, which ends the proof of the Lemma 5.13. \square
Let's go back now to the proof of Proposition 5.1. Indeed, we take ϵ, β, α and η small enough such that $\bar{t} > 0$ and $\bar{s} > 0$ (see Lemma 5.13-(6)). We can remark that the function

$$(t, x) \rightarrow \bar{p}^{\tilde{i}}(t, x) - \left(\bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) + \frac{|x - \bar{y}|^2}{2\epsilon} + \frac{|t - \bar{s}|^2}{2\beta} + \frac{\eta}{T - t} + \alpha(|x|^2 + |\bar{y}|^2) \right)$$

reaches a maximum at (\bar{t}, \bar{x}) . By using the test-function

$$\phi(t, x) = \bar{q}^{\tilde{i}}(\bar{s}, \bar{y}) + \frac{|x - \bar{y}|^2}{2\epsilon} + \frac{|t - \bar{s}|^2}{2\beta} + \frac{\eta}{T - t} + \alpha(|x|^2 + |\bar{y}|^2)$$

and the fact that $p = (p^i)_{i=1, \dots, d}$ is a sub-solution of (5.1), we deduce that

$$\gamma \bar{p}^{\tilde{i}}(\bar{t}, \bar{x}) + \frac{(\bar{t} - \bar{s})}{\beta} + \frac{\eta}{(T - \bar{t})^2} - \lambda^{\tilde{i}}(\bar{t}, \bar{x}, p(\bar{t}, \bar{x})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} + 2\alpha \bar{x} \right| \leq 0. \quad (5.89)$$

In the other hand, we deduce that the function

$$(s, y) \rightarrow \bar{q}^i(s, y) - \left(\bar{p}^i(\bar{t}, \bar{x}) - \frac{|\bar{x} - y|^2}{2\epsilon} - \frac{|\bar{t} - s|^2}{2\beta} - \frac{\eta}{T - \bar{t}} - \alpha(|\bar{x}|^2 + |y|^2) \right)$$

reaches a minimum at (\bar{s}, \bar{y}) . By using the test-function

$$\phi(s, y) = \bar{p}^i(\bar{t}, \bar{x}) - \frac{|\bar{x} - y|^2}{2\epsilon} - \frac{|\bar{t} - s|^2}{2\beta} - \frac{\eta}{T - \bar{t}} - \alpha(|\bar{x}|^2 + |y|^2)$$

and the fact that $q = (q^i)_{i=1, \dots, d}$ is a super-solution of (5.1), we deduce that

$$\gamma \bar{q}^i(\bar{s}, \bar{y}) + \frac{(\bar{t} - \bar{s})}{\beta} - \lambda^i(\bar{s}, \bar{y}, q(\bar{s}, \bar{y})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right| \geq 0. \quad (5.90)$$

By subtracting (5.90) to (5.89), we deduce that

$$\gamma(\bar{p}^i(\bar{t}, \bar{x}) - \bar{q}^i(\bar{s}, \bar{y})) - \lambda^i(\bar{t}, \bar{x}, p(\bar{t}, \bar{x})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} + 2\alpha\bar{x} \right| + \lambda^i(\bar{s}, \bar{y}, q(\bar{s}, \bar{y})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right| \leq 0. \quad (5.91)$$

Sending $\beta \rightarrow 0$ in (5.91), we deduce according to (5.86) that

$$\gamma(\bar{p}^i(\bar{\tau}, \bar{x}) - \bar{q}^i(\bar{\tau}, \bar{y})) - \lambda^i(\bar{\tau}, \bar{x}, p(\bar{\tau}, \bar{x})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} + 2\alpha\bar{x} \right| + \lambda^i(\bar{\tau}, \bar{y}, q(\bar{\tau}, \bar{y})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right| \leq 0. \quad (5.92)$$

Moreover, we have

$$\psi(\bar{t}, \bar{t}, \bar{x}, \bar{y}, i) \leq \psi(\bar{t}, \bar{s}, \bar{x}, \bar{y}, i),$$

which implies

$$\bar{p}^i(\bar{t}, \bar{x}) - \bar{q}^i(\bar{t}, \bar{y}) \leq \bar{p}^i(\bar{t}, \bar{x}) - \bar{q}^i(\bar{s}, \bar{y}) > 0.$$

Passing to the limit $\beta \rightarrow 0$, we obtain

$$0 \leq \bar{p}^i(\bar{\tau}, \bar{x}) - \bar{q}^i(\bar{\tau}, \bar{y}) \geq \bar{p}^i(\bar{\tau}, \bar{x}) - \bar{q}^i(\bar{\tau}, \bar{y}).$$

Hence

$$0 \leq \bar{p}^i(\bar{\tau}, \bar{x}) - \bar{q}^i(\bar{\tau}, \bar{y}) \geq p^i(\bar{\tau}, \bar{x}) - q^i(\bar{\tau}, \bar{y})$$

By adding and subtracting the terms $\lambda^i(\bar{\tau}, \bar{y}, p(\bar{\tau}, \bar{x})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right|$, and $\lambda^i(\bar{\tau}, \bar{x}, p(\bar{\tau}, \bar{x})) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right|$ in the inequality (5.92), we deduce that

$$\begin{aligned} \gamma(\bar{p}^i(\bar{\tau}, \bar{x}) - \bar{q}^i(\bar{\tau}, \bar{y})) &+ \lambda^i(\bar{\tau}, \bar{x}, p(\bar{\tau}, \bar{x})) \left(\left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right| - \left| \frac{(\bar{x} - \bar{y})}{\epsilon} + 2\alpha\bar{x} \right| \right) \\ &+ \left(\lambda^i(\bar{\tau}, \bar{y}, p(\bar{\tau}, \bar{x})) - \lambda^i(\bar{\tau}, \bar{x}, p(\bar{\tau}, \bar{x})) \right) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right| \\ &+ \left(\lambda^i(\bar{\tau}, \bar{y}, q(\bar{\tau}, \bar{y})) - \lambda^i(\bar{\tau}, \bar{y}, p(\bar{\tau}, \bar{x})) \right) \left| \frac{(\bar{x} - \bar{y})}{\epsilon} - 2\alpha\bar{y} \right| \leq 0. \end{aligned}$$

Using (H4) in the last line, we get

$$\begin{aligned} & \gamma(\bar{p}^i(\tilde{\tau}, \tilde{x}) - \bar{q}^i(\tilde{\tau}, \tilde{y})) + \lambda^i(\tilde{\tau}, \tilde{x}, p(\tilde{\tau}, \tilde{x})) \left(\left| \frac{(\tilde{x}-\tilde{y})}{\epsilon} - 2\alpha\tilde{y} \right| - \left| \frac{(\tilde{x}-\tilde{y})}{\epsilon} + 2\alpha\tilde{x} \right| \right) \\ & + \left(\lambda^i(\tilde{\tau}, \tilde{y}, p(\tilde{\tau}, \tilde{x})) - \lambda^i(\tilde{\tau}, \tilde{x}, p(\tilde{\tau}, \tilde{x})) \right) \left| \frac{(\tilde{x}-\tilde{y})}{\epsilon} - 2\alpha\tilde{y} \right| \leq 0. \end{aligned}$$

Then, (H1) and (5.9) imply

$$\begin{aligned} & \gamma(\bar{p}^i(\tilde{\tau}, \tilde{x}) - \bar{q}^i(\tilde{\tau}, \tilde{y})) - 2\alpha(\Lambda + M_1|\tilde{y}|)|\tilde{y} - \tilde{x}| - M_1 \frac{|\tilde{y} - \tilde{x}|^2}{\epsilon} \\ & \leq \gamma(\bar{p}^i(\tilde{\tau}, \tilde{x}) - \bar{q}^i(\tilde{\tau}, \tilde{y})) - 2\Lambda\alpha|\tilde{y} - \tilde{x}| - M_1|\tilde{y} - \tilde{x}| \left| \frac{(\tilde{x} - \tilde{y})}{\epsilon} - 2\alpha\tilde{y} \right| \leq 0. \end{aligned}$$

Passing to the limit $\epsilon \rightarrow 0$ and then $(\alpha, \eta) \rightarrow (0, 0)$, we obtain by Lemma 5.13-(1)-(4), (5.87) and (5.88), that

$$\gamma M_{sup} \leq 0,$$

this contradiction ends the proof of the proposition. This result ensures the uniqueness of the solution and implies Theorem 5.1-(ii). \square

6

Convergence d'un schéma implicite pour un système diagonal hyperbolique non-conservatif

Ce chapitre est un article à paraître dans *M2AN* écrit en collaboration avec Ahmad El Hajj et Rachida Boudjerada.

Dans ce travail, nous considérons un système diagonal hyperbolique non-conservatif unidimensionnel complété par des conditions initiales Lipschitz continues. Ce système intervient naturellement dans la modélisation de la dynamique des dislocations, propageant dans plusieurs directions. Sous certaines conditions de monotonies sur la matrice Jacobienne de la vitesse, des résultats d'existence et d'unicité d'une solution Lipschitz du système, non nécessairement strictement hyperbolique, ont été établis dans [35]. Nous proposons un schéma implicite vérifiant des estimations Lipschitz discrètes similaires à celles obtenues dans [35]. Ces dernières nous permettent de montrer la convergence de la solution numérique vers la solution du problème continue sans supposer que le problème est strictement hyperbolique. Enfin, nous présentons également quelques simulations numériques.

Convergence of an implicit scheme for diagonal non-conservative hyperbolic systems

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Abstract

In this paper, we consider diagonal non-conservative hyperbolic systems in one space dimension with monotone and large Lipschitz continuous data. Under a certain nonnegativity condition on the Jacobian matrix of the velocity of the system, global existence and uniqueness results of a Lipschitz solution for this system, which is not necessarily strictly hyperbolic, was proved in [35]. We propose a natural implicit scheme satisfying a similar Lipschitz estimate at the discrete level. This property allows us to prove the convergence of the scheme without assuming it strictly hyperbolic.

AMS Classification: 35A01, 74G25, 35F20, 35F21, 70H20, 35Q74.

Key words: Implicit upwind scheme, diagonal non-conservative hyperbolic systems, transport systems, discrete gradient estimates, monotone discrete solutions, Lipschitz discrete solutions.

1 Introduction and main results

In this paper we present a convergence result for an implicit Upwind scheme considering the framework of hyperbolic systems, which are not necessarily strictly hyperbolic. Related to this work, it is worth noting that, in [59] the authors have proved a similar result for a semi-explicit scheme in the case of non-conservative strictly hyperbolic systems. Moreover, their result was only valid in the class of vanishing viscosity solutions, introduced by Bianchini and Bressan in [16]. Here, we show the convergence taking only Lipschitz continuous solutions, without any other restriction concerning the class of solutions.

Before presenting our scheme, let us, first, recall in Subsection 1.1 the setting of the continuous problem. In particular, we will restate in Subsubsection 1.1.2 the existence and uniqueness results of Lipschitz solution for the continuous problem that was shown in [35].

1.1 The continuous problem

1.1.1 Setting of the problem

In this paper, we will propose an implicit numerical scheme for hyperbolic diagonal systems in non-conservative form given by

$$\begin{cases} \partial_t v^\alpha + \lambda^\alpha(v) \partial_x v^\alpha = 0 & \text{on } (0, +\infty) \times \mathbb{R}, \text{ for } \alpha = 1, \dots, d, \\ v^\alpha(0, x) = v_0^\alpha(x) & x \in \mathbb{R}, \text{ for } \alpha = 1, \dots, d, \end{cases} \quad (6.1)$$

where $v(t, x) = (v^\alpha(t, x))_{\alpha=1, \dots, d}$, with $d \geq 1$ is an integer. Here we use the notation $\partial_t = \frac{\partial}{\partial t}$ and $\partial_x = \frac{\partial}{\partial x}$. Such systems are (sometimes) called $(d \times d)$ diagonal hyperbolic systems.

We consider a given function $(\lambda^\alpha)_{\alpha=1, \dots, d} : \mathbb{R}^d \rightarrow \mathbb{R}$, which satisfies the following regularity assumption:

$$(H1) \begin{cases} \lambda^\alpha \in C^1(\mathbb{R}^d), \text{ for } \alpha = 1, \dots, d, \\ \text{there exists } M_1 > 0 \text{ such that for } \alpha = 1, \dots, d, \\ |\lambda^\alpha(u) - \lambda^\alpha(v)| \leq M_1 |u - v| \text{ for all } u, v \in \mathbb{R}^d, \end{cases}$$

where $|w| = \sum_{\alpha=1, \dots, d} |w^\alpha|$, for $w = (w^1, \dots, w^d)$.

We denote

$$\Lambda^\alpha = \sup_{u \in \mathcal{U}} |\lambda^\alpha(u)|, \quad (6.2)$$

where \mathcal{U} is the box defined as follows

$$\mathcal{U} = \prod_{\alpha=1}^d [-M^\alpha, M^\alpha] \text{ with } M^\alpha < +\infty \text{ is a real number.}$$

In (6.1), the initial data $v_0 = (v_0^1, \dots, v_0^d)$ is assumed to satisfy the following property:

$$(H2) \left\{ \begin{array}{l} -M^\alpha \leq v_0^\alpha \leq M^\alpha, \\ v_0^\alpha \text{ is nondecreasing,} \\ \partial_x v_0^\alpha \in L^\infty(\mathbb{R}), \end{array} \right. \Bigg| \text{ for } \alpha = 1, \dots, d.$$

Now, we define the matrix

$$\lambda_\beta^\alpha(v) = \frac{\partial \lambda^\alpha}{\partial v^\beta}(v), \text{ for all } \alpha, \beta = 1, \dots, d.$$

Let us introduce various assumptions on the matrix $(\lambda_\beta^\alpha(v))_{\alpha, \beta=1, \dots, d}$ which will guarantee the existence and uniqueness of a Lipschitz solution.

(Nonnegative matrices with nonpositive off-diagonal terms)

$$(K1) \quad \begin{cases} \lambda_\beta^\alpha(v) \leq 0 & \text{for all } v \in \mathcal{U} \text{ and } \beta \neq \alpha \text{ with } \alpha, \beta \in \{1, \dots, d\}, \\ A_{\alpha, \beta} = \inf_{v \in \mathcal{U}} (\lambda_\beta^\alpha(v)) & \text{and } \sum_{\alpha, \beta=1, \dots, d} A_{\alpha, \beta} \xi_\alpha \xi_\beta \geq 0 \text{ for every } \xi = (\xi_1, \dots, \xi_d) \in [0, +\infty)^d. \end{cases}$$

(Diagonally dominant)

$$(K2) \quad \lambda_\alpha^\alpha(v) \geq \sum_{\alpha \neq \beta} (\lambda_\beta^\alpha(v))^- \text{ for all } v \in \mathcal{U} \text{ and } \alpha = 1, \dots, d,$$

where we note $x^- = \max(0, -x)$.

(Nonnegative subdiagonal matrices)

$$(K3) \quad \lambda_\beta^\alpha(v) \geq 0 \text{ for all } v \in \mathcal{U} \text{ and } \beta \geq \alpha \text{ with } \alpha, \beta \in \{1, \dots, d\}.$$

Before giving main results on the solution of the discrete problem, let us recall some existence and uniqueness results on the solution of the continuous problem.

1.1.2 Recall of useful results

Theorem 6.1. (*Existence and uniqueness of Lipschitz solution, [35, Th 1.3]*)

Assume that (H1) and (H2) are satisfied. Assume also that one of the following assumptions (K1), (K2) or (K3) holds. Then, there exists a unique function $v \in \bigcap_{T>0} [W^{1,\infty}([0, T] \times \mathbb{R})]^d$ solution of (6.1), in distributional sense, with $v(t, x) \in \mathcal{U}$ for all (t, x) .

Moreover we have for any $t \in (0, +\infty)$:

$$\sum_{\alpha=1, \dots, d} \|\partial_x v^\alpha(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sum_{\alpha=1, \dots, d} \|\partial_x v_0^\alpha\|_{L^\infty(\mathbb{R})} \text{ if (K1) holds,} \quad (6.3)$$

and

$$\max_{\alpha=1, \dots, d} \|\partial_x v^\alpha(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \max_{\alpha=1, \dots, d} \|\partial_x v_0^\alpha\|_{L^\infty(\mathbb{R})} \text{ if (K2) holds.} \quad (6.4)$$

Notice that Theorem 6.1 is based on the fact that the solution satisfies $\partial_x v^\alpha \geq 0$, for $\alpha = 1, \dots, d$, and then, we only have to bound the maximum of the gradient from one side. One of the assumptions (K1), (K2) or (K3) is a sufficient condition to control the solution of the maximum of the gradient. These *a priori* bounds are obtained considering a parabolic regularization of the system and then writing some differential inequalities satisfied in the sense of viscosity by the maximum of the gradient. The uniqueness is an independent result valid for Lipschitz solutions.

Let us mention that an application to isentropic gas dynamics was proved in [35].

In the framework of viscosity solutions and motivated by the application to dislocation densities dynamics, El Hajj, Forcadel, in [33], have shown the existence and uniqueness of Lipschitz viscosity solution in the case where $d = 2$ with a matrix $(\lambda_\beta^\alpha)_{\alpha, \beta=1, 2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. In a similar framework, in El Hajj [30], the existence and uniqueness of strong solution $W_{loc}^{1,2}([0, T] \times \mathbb{R})$ was proved. Recently, a result of global existence and uniqueness was proved in El Hajj, Oussaily [36] (cf. Chapter 3) for continuous solutions satisfying some gradient entropy estimates. We refer the readers to El Hajj, Ibrahim and Rizik [38] for

the discontinuous solutions of this system.

For a generalized (2×2) strictly hyperbolic system, in [54], Lax proved the existence and uniqueness of nondecreasing smooth solutions. In the framework of $(d \times d)$ strictly hyperbolic systems, we refer the readers to [16] where the authors have shown a global existence and uniqueness result assuming that the initial data has small total variation. Also, a result of existence and uniqueness of continuous solution was proved in [35] for a $(d \times d)$ strictly hyperbolic system.

Now, we will propose a compatible discretization with the continuous problem.

1.2 The discrete problem

1.2.1 Setting of the problem

To recover the properties of the continuous problem on the discrete level, we consider a time step $\Delta t > 0$ and a space step $\Delta x > 0$. In order to avoid the ambiguity in notations, we denote by v^α the continuous solution and $u_i^{\alpha,n}$ the discrete associated solution defined as an approximation of $v^\alpha(n\Delta t, i\Delta x)$, where $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. We propose, then, the following implicit discretization of the system

$$\forall \alpha \in \{1, \dots, d\}, \begin{cases} \frac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} + \lambda^\alpha(u_i^{n+1}) \left(\frac{u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}}{\Delta x} \right) = 0 & \text{if } \lambda^\alpha(u_i^{n+1}) \leq 0, \\ \frac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} + \lambda^\alpha(u_i^{n+1}) \left(\frac{u_i^{\alpha,n+1} - u_{i-1}^{\alpha,n+1}}{\Delta x} \right) = 0 & \text{if } \lambda^\alpha(u_i^{n+1}) \geq 0, \\ u_i^{\alpha,0} = u_0^\alpha(x_i). \end{cases}$$

It is a first-order Upwind formulation, with the velocity $\lambda^\alpha(u)$ being implicit in time.

We denote

$$\lambda_i^{\alpha,n+1} = \lambda^\alpha(u_i^{n+1}), n \in \mathbb{N}.$$

We define the positive and the negative parts $(\lambda_i^{\alpha,n+1})_+$ and $(\lambda_i^{\alpha,n+1})_-$ as follows:

$$(\lambda_i^{\alpha,n+1})_+ = \frac{1}{2} (\lambda_i^{\alpha,n+1} + |\lambda_i^{\alpha,n+1}|), \quad (\lambda_i^{\alpha,n+1})_- = \frac{1}{2} (|\lambda_i^{\alpha,n+1}| - \lambda_i^{\alpha,n+1}).$$

Both $(\lambda_i^{\alpha,n+1})_+$ and $(\lambda_i^{\alpha,n+1})_-$ are positive real numbers.

We can write the scheme in a more compact form:

$$\begin{cases} \frac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t} - (\lambda_i^{\alpha,n+1})_- \left(\frac{u_{i+1}^{\alpha,n+1} - u_i^{\alpha,n+1}}{\Delta x} \right) + (\lambda_i^{\alpha,n+1})_+ \left(\frac{u_i^{\alpha,n+1} - u_{i-1}^{\alpha,n+1}}{\Delta x} \right) = 0 \\ u_i^{\alpha,0} = u_0^\alpha(x_i). \end{cases} \quad (6.5)$$

In the sequel, we set

$$\theta_{i+\frac{1}{2}}^{\alpha,n} = \frac{u_{i+1}^{\alpha,n} - u_i^{\alpha,n}}{\Delta x}, \quad (6.6)$$

which is a discrete equivalent of $\partial_x u^\alpha$.

Remark 6.1. Here, we choose the implicit scheme since it naturally preserves the Lipschitz estimates (6.3) and (6.4) at the discrete level, which is neither the case of the explicit scheme nor that of the semi-explicit scheme.

1.3 Main results

First, we set $u_i^n = (u_i^{\alpha,n})_{\alpha=1,\dots,d}$, $u^n = (u_i^n)_{i \in \mathbb{Z}}$ and we say that $u^n \in \mathcal{U}^{\mathbb{Z}}$ if $u_i^n \in \mathcal{U}$, for all $i \in \mathbb{Z}$.

We now introduce the following CFL condition:

$$\frac{\Delta t}{\Delta x} < \min(\beta_1, \beta_2), \quad (6.7)$$

with

$$\beta_1 = \frac{1}{4\Lambda^\alpha(M^\alpha + 1)} \quad \text{and} \quad \beta_2 = \frac{1}{3(\Lambda^\alpha + M_1(M^\alpha + 1))}.$$

We present in this paper three main results. The first one is proving that the implicit scheme (6.5) has a unique bounded solution at each time-step. Then, we show a gradient estimate decay result. The third one is showing the convergence of the solution of the scheme.

Theorem 6.2. (Resolution of the implicit scheme on one time step)

Assume that assumptions (H1), (H2) and the CFL condition (6.7) are satisfied. Let $u^n \in \mathcal{U}^{\mathbb{Z}}$. Then, we get

(i) **(Existence)**

There exists a unique solution $u^{n+1} \in \mathcal{U}^{\mathbb{Z}}$ to the implicit scheme (6.5).

(ii) **(Monotonicity)**

Moreover if u^n is nondecreasing, i.e. satisfies

$$u_{i+1}^{\alpha,n} \geq u_i^{\alpha,n} \quad \text{for all } i \in \mathbb{Z}, \quad \text{and } \alpha = 1, \dots, d,$$

then u^{n+1} is also nondecreasing.

Theorem 6.3. (Discrete gradient estimates)

Suppose that assumptions (H1), (H2) are verified. If u^n is the solution of the implicit scheme (6.5) given by Theorem 6.2, then, $\theta_{i+\frac{1}{2}}^{\alpha,n}$ defined in (6.6) is nonnegative for all $n \in \mathbb{N}$ and verifies the following estimates:

$$\sum_{\alpha=1}^d \sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n} \leq \sum_{\alpha=1}^d \sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,0} \quad \text{if (K1) holds,} \quad (6.8)$$

$$\max_{\alpha=1,\dots,d} \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n} \right) \leq \max_{\alpha=1,\dots,d} \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,0} \right) \quad \text{if (K2) holds,} \quad (6.9)$$

$$\sup_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{\alpha,n} \right) \leq \mathcal{K}_\alpha(T) \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,0} \right) \quad \text{for all } n \geq n_0 \quad \text{if (K3) holds,} \quad (6.10)$$

where \mathcal{K}_α is a constant that depends on T , M_1 and $\left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\beta,0} \right)$, for all $\beta \leq \alpha - 1$ and $n_0 \in \mathbb{N}$ dependent on T .

Let us mention that estimates (6.8) and (6.9) represent the discrete case of the continuous estimates (6.3) and (6.4).

Now, we present the convergence result of our implicit scheme.

Theorem 6.4. (Convergence of the solution of the scheme)

Assume that assumptions (H1), (H2) and the CFL condition (6.7) are satisfied. We also suppose that one of the conditions (K1), (K2) or (K3) holds. Let us consider the solution $(u^n)_{n \geq 0}$ of the scheme (6.5) for the time step $\Delta t > 0$ and the space step $\Delta x > 0$ given by Theorem 6.2. Let us call $\epsilon = (\Delta t, \Delta x)$ and u^ϵ the function defined by

$$u^\epsilon(n\Delta t, i\Delta x) = u_i^n \quad \text{for } n \in \mathbb{N}, \quad i \in \mathbb{Z}.$$

Then, as ϵ goes to zero, the whole sequence $(u^\epsilon)_\epsilon$ converges to the unique Lipschitz solution v of (6.1). Moreover, for any compact $K \subset [0, +\infty) \times \mathbb{R}$, we have

$$\sup_{\substack{K \cap ((\Delta t \mathbb{N}) \times (\Delta x \mathbb{Z})) \\ \alpha=1, \dots, d}} |u^\epsilon - v| \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow (0, 0).$$

1.4 Brief literature

Numerical schemes for hyperbolic systems are mainly written for systems in conservative form which enable to recover the correct Rankine-Hugoniot shock. We refer to [55] for a review of the main classes of existing schemes. Among these schemes, convergence results are seldom found for hyperbolic systems. The Lax-Wendroff theorem [53] shows that if a consistent and conservative numerical scheme converges (in L^1 with bounded total variation), its limit is a weak solution to the hyperbolic system. However, in order to obtain convergence of the scheme, stability is needed, in general in the form of TV-stability. For the scalar Godunov scheme, convergence is obtained due to its total variation diminishing (TVD) property. This is no longer the case for systems [55]. Stability can still be proved for certain special systems of two equations, for instance in [56, 69] and in [70]. Similar results can be obtained for a class of nonlinear systems with straight-line fields [22, pp. 102-103]. Nonlinear stability can also be assessed through the use of invariant domains and entropy inequalities [18], for HLL, HLLC and kinetic solvers of Euler equations of gas dynamics.

In the case of conservative systems where the initial data has sufficiently small total variation, Glimm's random choice method [39] is probably convergent. A deterministic variant (replacing random with equidistributed sampling) has also been proved to converge under the same assumptions [58].

Beside this, for non-conservative forms, recently, Monasse and Monneau proved in [59] a convergence result of a semi-explicit scheme for diagonal non-conservative hyperbolic systems assuming that the system is strictly hyperbolic and using a discrete gradient entropy estimate which was proved in the continuous case in [35]. This result has been established in a restricted class of solutions, which is defined by the vanishing viscosity solutions, introduced by Bianchini and Bressan in [16]. In the present work, we consider the case of hyperbolic systems which are not necessarily strictly hyperbolic and we prove, basing on some Lipschitz discrete estimates, the convergence of the scheme in the framework of Lipschitz continuous solutions, without any other restriction concerning the class of solutions. Note that, the fundamental Lipschitz discrete estimates are proved, at the discrete level, assuming some nonnegative conditions on the Jacobian matrix $(\lambda_\beta^\alpha)_{\alpha, \beta}$, that ensures a L^∞ bound on the gradient of the continuous solutions. We also took an implicit scheme since it keeps continuous Lipschitz estimates (6.3)-(6.4), at the discrete level.

We should recall that in the framework of dislocation densities, it was proved in [33] a convergence result of an explicit scheme to the Lipschitz continuous solution for a particular (2×2) Hamilton-Jacobi system. We also refer to Jakobsen, Karlsen [49] and Jakobsen, Karlsen, Risebro [50] where they proved a convergence result for a weakly coupled Hamilton-Jacobi system considering a semi-discrete splitting algorithm that they proposed to approach the continuous solution.

1.5 Organization of the paper

This paper is organized as follows. In Section 2, we prove some preliminary results on the existence and monotonicity of the solution of the scheme (Theorem 6.2). Section 3 is devoted to show some discrete gradient estimates (Theorem 6.3) using conditions $(K1)$, $(K2)$ and $(K3)$ on the Jacobian matrix $(\lambda_\beta^\alpha)_{\alpha,\beta=1,\dots,d}$. Finally, based on these gradient estimates, we prove the convergence result of our scheme (Theorem 6.4) in Section 4.

2 Existence and monotonicity of the discrete solution

This section is devoted to prove the existence, the uniqueness and the monotonicity of a bounded solution for the implicit scheme (6.5). This section is divided into two subsections, in Subsection 2.1, we prove by a fixed point argument the existence and uniqueness of the discrete solution. Then, in Subsection 2.2, we show that the discrete solution is nondecreasing.

2.1 Proof of Theorem 6.2-(i):

We introduce the truncature $\bar{\lambda}^\alpha$ of λ^α by Λ^α (defined in (6.2)):

$$\bar{\lambda}^\alpha(u) = \begin{cases} \lambda^\alpha(u) & \text{if } |\lambda^\alpha(u)| \leq \Lambda^\alpha \\ \Lambda^\alpha & \text{if } |\lambda^\alpha(u)| > \Lambda^\alpha \\ -\Lambda^\alpha & \text{if } |\lambda^\alpha(u)| < -\Lambda^\alpha. \end{cases}$$

For the real number $M^\alpha < +\infty$, let us consider the box

$$\bar{\mathcal{U}} = \prod_{\alpha=1}^d [-M^\alpha - 1, M^\alpha + 1].$$

For all $i \in \mathbb{Z}$, we denote $w_i = (w_i^\alpha)_{\alpha=1,\dots,d}$ and $w = (w_i)_{i \in \mathbb{Z}}$. We define, for all $w \in \bar{\mathcal{U}}^{\mathbb{Z}}$ and for all $\alpha \in \{1, \dots, d\}$, the function $F_{w_i}^\alpha = \left(F_{w_i}^{\alpha,n} \right)_{\alpha=1,\dots,d}$ such that

$$F_{w_i}^\alpha(w) = u_i^{\alpha,n} + \frac{\Delta t}{\Delta x} \left((\bar{\lambda}^\alpha(w_i))_- (w_{i+1}^\alpha - w_i^\alpha) - (\bar{\lambda}^\alpha(w_i))_+ (w_i^\alpha - w_{i-1}^\alpha) \right). \quad (6.11)$$

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According to (6.11), we can see that the scheme (6.5) can be written as

$$\begin{cases} u_i^{\alpha, n+1} = F_{u_i^n}^\alpha(u^{n+1}), & \text{for } i \in \mathbb{Z}, n \in \mathbb{N} \text{ and } u^n \in \mathcal{U}^{\mathbb{Z}}, \\ u_i^{\alpha, 0} = u_0^\alpha(x_i). \end{cases} \quad (6.12)$$

Let us now define the mapping

$$G_{u_i^n} : \bar{\mathcal{U}}^{\mathbb{Z}} \longrightarrow \bar{\mathcal{U}}^{\mathbb{Z}}$$

$$w \longrightarrow F_{u_i^n}(w).$$

Our goal is to show that the mapping $G_{u_i^n}$ is a well-defined contraction taking $u^n \in \mathcal{U}^{\mathbb{Z}}$. This leads to prove, first, the existence and uniqueness of a fixed point of (6.12) in $\bar{\mathcal{U}}^{\mathbb{Z}}$. Then, we will show that the solution belongs to $\mathcal{U}^{\mathbb{Z}}$. To do that, we proceed in three steps.

Step 1 ($G_{u_i^n}$ is well-defined):

It is sufficient to prove that $-M^\alpha - 1 \leq F_{u_i^n}^\alpha(w) \leq M^\alpha + 1$, for all $i \in \mathbb{Z}$, if $u^n \in \mathcal{U}^{\mathbb{Z}}$. Indeed, by (6.11), we have

$$F_{u_i^n}^\alpha(w) = u_i^{\alpha, n} - \frac{\Delta t}{\Delta x} \left[(\bar{\lambda}^\alpha(w_i))_- + (\bar{\lambda}^\alpha(w_i))_+ \right] w_i^\alpha$$

$$+ \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_i))_- w_{i+1}^\alpha + \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_i))_+ w_{i-1}^\alpha.$$

Then, using the CFL condition (6.7), we obtain

$$|F_{u_i^n}^\alpha(w)| \leq M^\alpha + 4 \frac{\Delta t}{\Delta x} \Lambda^\alpha (M^\alpha + 1) \leq M^\alpha + 1,$$

which implies that the mapping $G_{u_i^n}$ is well-defined.

Step 2 ($G_{u_i^n}$ is a contraction):

We equipped $\bar{\mathcal{U}}^{\mathbb{Z}}$ with the following norm $(l^\infty)^d$:

$$\|u\|_{(l^\infty)^d} = \sum_{\alpha=1}^d \sup_{i \in \mathbb{Z}} |u_i^\alpha|.$$

We observe that, for all u and v in $\bar{\mathcal{U}}^{\mathbb{Z}}$, for all $\alpha \in \{1, \dots, d\}$,

$$\begin{aligned} & \|G_{u_i^n}(u) - G_{u_i^n}(v)\|_{(l^\infty)^d} \\ &= \sum_{\alpha=1}^d \sup_{i \in \mathbb{Z}} \left| F_{u_i^n}^\alpha(u) - F_{u_i^n}^\alpha(v) \right| \\ &= \frac{\Delta t}{\Delta x} \sum_{\alpha=1}^d \sup_{i \in \mathbb{Z}} \left| \left[(\bar{\lambda}^\alpha(u_i))_- - (\bar{\lambda}^\alpha(v_i))_- \right] (u_{i+1}^\alpha - u_i^\alpha) - \left[(\bar{\lambda}^\alpha(u_i))_+ - (\bar{\lambda}^\alpha(v_i))_+ \right] (u_i^\alpha - u_{i-1}^\alpha) \right. \\ & \quad \left. + (\bar{\lambda}^\alpha(v_i))_- [(u_{i+1}^\alpha - v_{i+1}^\alpha) - (u_i^\alpha - v_i^\alpha)] + (\bar{\lambda}^\alpha(v_i))_+ [(u_i^\alpha - v_i^\alpha) - (u_{i-1}^\alpha - v_{i-1}^\alpha)] \right|, \end{aligned}$$

where we have added and subtracted the terms $\frac{\Delta t}{\Delta x}(\bar{\lambda}^\alpha(v_i))_- u_{i+1}^\alpha$ and $\frac{\Delta t}{\Delta x}(\bar{\lambda}^\alpha(v_i))_+ u_{i-1}^\alpha$. We obtain

$$\|G_{u_i^n}(u) - G_{u_i^n}(v)\|_{(L^\infty)^d} \leq \frac{\Delta t}{\Delta x} (3\Lambda^\alpha + 3M_1(M^\alpha + 1)) \|u - v\|_{(L^\infty)^d},$$

which gives by the CFL condition (6.7) that $G_{u_i^n}$ is a contraction on $\bar{\mathcal{U}}^{\mathbb{Z}}$.

Banach fixed point theorem yields the existence and uniqueness of a solution w of (6.11) on $\bar{\mathcal{U}}^{\mathbb{Z}}$.

Step 3 (The fixed point belongs to $\mathcal{U}^{\mathbb{Z}}$):

Our goal is to prove that the fixed point w constructed in the previous steps belongs to $\mathcal{U}^{\mathbb{Z}}$, i.e., $|w_i^\alpha| \leq M^\alpha$. Indeed, let $\phi \in C^\infty(\mathbb{R})$ be a cut-off function taking values in $[0, 1]$, supported on the interval $[-2, 2]$ and $\phi(x) \equiv 1$ on $[-1, 1]$. Multiplying (6.11) by $\phi_R(x_i) = \phi(\frac{x_i}{R})$, for $R > 0$, with $x_i = i\Delta x$, we get

$$\begin{aligned} w_i^\alpha \phi_R(x_i) &= u_i^{\alpha,n} \phi_R(x_i) - \frac{\Delta t}{\Delta x} \left[(\bar{\lambda}^\alpha(w_i))_- + (\bar{\lambda}^\alpha(w_i))_+ \right] w_i^\alpha \phi_R(x_i) \\ &+ \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_i))_- w_{i+1}^\alpha (\phi_R(x_i) - \phi_R(x_{i+1})) + \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_i))_+ w_{i-1}^\alpha (\phi_R(x_i) - \phi_R(x_{i-1})) \\ &+ \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_i))_- w_{i+1}^\alpha \phi_R(x_{i+1}) + \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_i))_+ w_{i-1}^\alpha \phi_R(x_{i-1}), \end{aligned} \tag{6.13}$$

where we have added and subtracted the terms $\frac{\Delta t}{\Delta x}(\bar{\lambda}^\alpha(w_i))_- w_{i+1}^\alpha \phi_R(x_{i+1})$ and $\frac{\Delta t}{\Delta x}(\bar{\lambda}^\alpha(w_i))_+ w_{i-1}^\alpha \phi_R(x_{i-1})$. We denote $w_j^\alpha \phi_R(x_j) = \min_{i \in \mathbb{Z}} (w_i^\alpha \phi_R(x_i))$. Then, using (6.13), we obtain, for all $i \in \mathbb{Z}$,

$$\begin{aligned} w_i^\alpha \phi_R(x_i) \geq w_j^\alpha \phi_R(x_j) &\geq u_j^{\alpha,n} \phi_R(x_j) - \frac{\Delta t}{\Delta x} \left[(\bar{\lambda}^\alpha(w_j))_- + (\bar{\lambda}^\alpha(w_j))_+ \right] w_j^\alpha \phi_R(x_j) \\ &+ \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_j))_- w_{j+1}^\alpha (\phi_R(x_j) - \phi_R(x_{j+1})) + \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_j))_+ w_{j-1}^\alpha (\phi_R(x_j) - \phi_R(x_{j-1})) \\ &+ \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_j))_- w_j^\alpha \phi_R(x_j) + \frac{\Delta t}{\Delta x} (\bar{\lambda}^\alpha(w_j))_+ w_j^\alpha \phi_R(x_j) \\ &\geq u_j^{\alpha,n} \phi_R(x_j) - 2 \frac{\Delta t}{R} \Lambda^\alpha (M^\alpha + 1) \|\phi'\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where we have used the fact $w \in \bar{\mathcal{U}}^{\mathbb{Z}}$. Now, using the fact that $0 \leq \phi_R \leq 1$ and $u^n \in \mathcal{U}^{\mathbb{Z}}$, we get

$$-M^\alpha \leq -M^\alpha \phi_R(x_j) \leq u_j^{\alpha,n} \phi_R(x_j) \leq M^\alpha \phi_R(x_j) \leq M^\alpha.$$

Therefore, we obtain

$$w_i^\alpha \phi_R(x_i) \geq -M^\alpha - 2 \frac{\Delta t}{R} \Lambda^\alpha (M^\alpha + 1) \|\phi'\|_{L^\infty(\mathbb{R})}.$$

Now, by passing to the limit as $R \rightarrow +\infty$, we finally get

$$w_i^\alpha \geq -M^\alpha \quad \text{for all } i \in \mathbb{Z}.$$

We proceed in a similar manner to show that $w_i^\alpha \leq M^\alpha$, for all $i \in \mathbb{Z}$. \square

2.2 Proof of Theorem 6.2-(ii):

In what follows, we derive an evolution in time of the discrete gradient $\theta_{i+\frac{1}{2}}^{\alpha,n}$ defined by (6.6) from (6.5), which allows us to prove the monotonicity of discrete solution.

Let $u_i^{\alpha,n+1}$ be the solution of the implicit scheme (6.5). Using the definition of $\theta_{i+\frac{1}{2}}^{\alpha,n}$, we observe that

$$\theta_{i+\frac{1}{2}}^{\alpha,n+1} = \theta_{i+\frac{1}{2}}^{\alpha,n} + \frac{\Delta t}{\Delta x} \frac{u_{i+1}^{\alpha,n+1} - u_{i+1}^{\alpha,n}}{\Delta t} - \frac{\Delta t}{\Delta x} \frac{u_i^{\alpha,n+1} - u_i^{\alpha,n}}{\Delta t}.$$

Inserting (6.5) at i and $i+1$, allows us to show that $\theta_{i+\frac{1}{2}}^{\alpha,n}$ defined in (6.6) satisfies the following relation

$$\begin{aligned} \theta_{i+\frac{1}{2}}^{\alpha,n+1} &= \theta_{i+\frac{1}{2}}^{\alpha,n} - \frac{\Delta t}{\Delta x} \left[(\lambda_{i+1}^{\alpha,n+1})_+ + (\lambda_i^{\alpha,n+1})_- \right] \theta_{i+\frac{1}{2}}^{\alpha,n+1} \\ &\quad + \frac{\Delta t}{\Delta x} (\lambda_{i+1}^{\alpha,n+1})_- \theta_{i+\frac{3}{2}}^{\alpha,n+1} + \frac{\Delta t}{\Delta x} (\lambda_i^{\alpha,n+1})_+ \theta_{i-\frac{1}{2}}^{\alpha,n+1}. \end{aligned} \tag{6.14}$$

It is sufficient to show that $\theta_{i+\frac{1}{2}}^{\alpha,n+1} \geq 0$. Indeed, multiplying (6.14) by $\phi_R(x_i) = \phi(\frac{x_i}{R})$, for $R > 0$, where ϕ_R is defined in Subsection 2.1, we get

$$\begin{aligned} \theta_{i+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_i) &= \theta_{i+\frac{1}{2}}^{\alpha,n} \phi_R(x_i) - \frac{\Delta t}{\Delta x} \left[(\lambda_{i+1}^{\alpha,n+1})_+ + (\lambda_i^{\alpha,n+1})_- \right] \theta_{i+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_i) \\ &\quad + \frac{\Delta t}{\Delta x} (\lambda_{i+1}^{\alpha,n+1})_- \theta_{i+\frac{3}{2}}^{\alpha,n+1} \phi_R(x_{i+1}) + \frac{\Delta t}{\Delta x} (\lambda_i^{\alpha,n+1})_+ \theta_{i-\frac{1}{2}}^{\alpha,n+1} \phi_R(x_{i-1}) \\ &\quad + \frac{\Delta t}{\Delta x} (\lambda_{i+1}^{\alpha,n+1})_- \theta_{i+\frac{3}{2}}^{\alpha,n+1} [\phi_R(x_i) - \phi_R(x_{i+1})] + \frac{\Delta t}{\Delta x} (\lambda_i^{\alpha,n+1})_+ \theta_{i-\frac{1}{2}}^{\alpha,n+1} [\phi_R(x_i) - \phi_R(x_{i-1})], \end{aligned} \tag{6.15}$$

where we have added and subtracted the terms $\frac{\Delta t}{\Delta x} (\lambda_{i+1}^{\alpha,n+1})_- \theta_{i+\frac{3}{2}}^{\alpha,n+1} \phi_R(x_{i+1})$ and $\frac{\Delta t}{\Delta x} (\lambda_i^{\alpha,n+1})_+ \theta_{i-\frac{1}{2}}^{\alpha,n+1} \phi_R(x_{i-1})$.

Let us now denote $\theta_{k+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_k) = \min_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_i) \right)$. Then, from (6.15), we deduce that

$$\begin{aligned} \theta_{k+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_k) &\geq \theta_{k+\frac{1}{2}}^{\alpha,n} \phi_R(x_k) - \frac{\Delta t}{\Delta x} \left[\lambda_{k+1}^{\alpha,n+1} - \lambda_k^{\alpha,n+1} \right] \theta_{k+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_k) \\ &\quad + \frac{\Delta t}{\Delta x} (\lambda_{k+1}^{\alpha,n+1})_- \theta_{k+\frac{3}{2}}^{\alpha,n+1} [\phi_R(x_k) - \phi_R(x_{k+1})] + \frac{\Delta t}{\Delta x} (\lambda_k^{\alpha,n+1})_+ \theta_{k-\frac{1}{2}}^{\alpha,n+1} [\phi_R(x_k) - \phi_R(x_{k-1})]. \end{aligned}$$

Then, we obtain

$$\left(1 + \frac{\Delta t}{\Delta x} \left[\lambda_{k+1}^{\alpha,n+1} - \lambda_k^{\alpha,n+1} \right] \right) \theta_{k+\frac{1}{2}}^{\alpha,n+1} \phi_R(x_k) \geq \theta_{k+\frac{1}{2}}^{\alpha,n} \phi_R(x_k) - \frac{4}{R} \frac{\Delta t}{\Delta x} \Lambda^\alpha M^\alpha \|\phi'\|_{L^\infty(\mathbb{R})}.$$

We argue by contradiction. Assume that $\theta_{i+\frac{1}{2}}^{\alpha,n+1} < 0$ for a certain $i \in \mathbb{Z}$. Then, using the CFL condition (6.7), the fact that $\theta_{i+\frac{1}{2}}^{\alpha,n} \geq 0$ and $\phi_R \geq 0$, we obtain

$$\begin{aligned}
 \frac{1}{2}\theta_{i+\frac{1}{2}}^{\alpha,n+1}\phi_R(x_i) &\geq \left(1 + \frac{\Delta t}{\Delta x} \left[\lambda_{k+1}^{\alpha,n+1} - \lambda_k^{\alpha,n+1}\right]\right) \theta_{i+\frac{1}{2}}^{\alpha,n+1}\phi_R(x_i) \\
 &\geq \left(1 + \frac{\Delta t}{\Delta x} \left[\lambda_{k+\frac{1}{2}}^{\alpha,n+1} - \lambda_k^{\alpha,n+1}\right]\right) \theta_{k+\frac{1}{2}}^{\alpha,n+1}\phi_R(x_k) \\
 &\geq \theta_{k+\frac{1}{2}}^{\alpha,n}\phi_R(x_k) - \frac{4}{R} \frac{\Delta t}{\Delta x} \Lambda^\alpha M^\alpha \|\phi'\|_{L^\infty(\mathbb{R})} \\
 &\geq -\frac{4}{R} \frac{\Delta t}{\Delta x} \Lambda^\alpha M^\alpha \|\phi'\|_{L^\infty(\mathbb{R})}.
 \end{aligned}$$

Now, passing to the limit as $R \rightarrow +\infty$, we obtain

$$\frac{1}{2}\theta_{i+\frac{1}{2}}^{\alpha,n+1} \geq 0,$$

which gives a contradiction and then $\theta_{i+\frac{1}{2}}^{\alpha,n+1} \geq 0$. This is equivalent to $u_i^{\alpha,n+1}$ is also nondecreasing. This leads to Theorem 6.2-(ii), by applying a recursion on $n \geq 0$. \square

3 Proof of Theorem 6.3 (Discrete gradient estimates)

Our goal is to show that $\theta_{i+\frac{1}{2}}^{\alpha,n+1}$ is uniformly bounded for all $i \in \mathbb{Z}$.

We can see that $\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n+1}$ is reached at an index $j_\alpha \in \mathbb{Z}$ since $u_i^{\alpha,n}$ is bounded and nondecreasing. From (6.14), we have

$$\begin{aligned}
 \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1} = \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n+1} &= \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n} - \frac{\Delta t}{\Delta x} \left((\lambda_{j_\alpha+1}^{\alpha,n+1})_+ + (\lambda_{j_\alpha}^{\alpha,n+1})_- \right) \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1} \\
 &\quad + \frac{\Delta t}{\Delta x} (\lambda_{j_\alpha+1}^{\alpha,n+1})_- \left(\theta_{j_\alpha+\frac{3}{2}}^{\alpha,n+1} \right) + \frac{\Delta t}{\Delta x} (\lambda_{j_\alpha}^{\alpha,n+1})_+ \left(\theta_{j_\alpha-\frac{1}{2}}^{\alpha,n+1} \right).
 \end{aligned}$$

By the positivity of $\frac{\Delta t}{\Delta x} (\lambda_{j_\alpha+1}^{\alpha,n+1})_-$ and $\frac{\Delta t}{\Delta x} (\lambda_{j_\alpha}^{\alpha,n+1})_+$, we get

$$\begin{aligned}
 \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1} &\leq \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n} - \frac{\Delta t}{\Delta x} \left((\lambda_{j_\alpha+1}^{\alpha,n+1})_+ + (\lambda_{j_\alpha}^{\alpha,n+1})_- \right) \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1} \\
 &\quad + \left(\frac{\Delta t}{\Delta x} (\lambda_{j_\alpha+1}^{\alpha,n+1})_- \right) \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1} + \left(\frac{\Delta t}{\Delta x} (\lambda_{j_\alpha}^{\alpha,n+1})_+ \right) \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1},
 \end{aligned}$$

which implies that

$$\theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1} \leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n} - \frac{\Delta t}{\Delta x} \left(\lambda_{j_\alpha+1}^{\alpha,n+1} - \lambda_{j_\alpha}^{\alpha,n+1} \right) \theta_{j_\alpha+\frac{1}{2}}^{\alpha,n+1}. \quad (6.16)$$

In order to prove the discrete gradient estimates given by Theorem 6.3, we distinguish three cases. To do that, we proceed in three subsections.

3.1 Case where (K1) holds

It is sufficient to prove that the sum over $\alpha \in \{1, \dots, d\}$ of the second term in the right hand side of (6.16) is negative using assumption (K1).

We have

$$\begin{aligned}
& \frac{\Delta t}{\Delta x} \left(\lambda_{j_{\alpha}+1}^{\alpha, n+1} - \lambda_{j_{\alpha}}^{\alpha, n+1} \right) \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&= \frac{\Delta t}{\Delta x} \left[\sum_{\beta=1}^d \int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\beta}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot (u_{j_{\alpha}+1}^{\beta, n+1} - u_{j_{\alpha}}^{\beta, n+1}) d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&= \Delta t \left[\sum_{\beta=1}^d \int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\beta}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot \theta_{j_{\alpha}+\frac{1}{2}}^{\beta, n+1} d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&= \Delta t \left[\sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^d \int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\beta}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot \theta_{j_{\alpha}+\frac{1}{2}}^{\beta, n+1} d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&+ \Delta t \left[\int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\alpha}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1}.
\end{aligned}$$

Using the fact that $\lambda_{\beta}^{\alpha} \leq 0$ for $\alpha \neq \beta$, we get

$$\begin{aligned}
& \frac{\Delta t}{\Delta x} \left(\lambda_{j_{\alpha}+1}^{\alpha, n+1} - \lambda_{j_{\alpha}}^{\alpha, n+1} \right) \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&\geq \Delta t \left[\sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^d \int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\beta}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot \theta_{j_{\alpha}+\frac{1}{2}}^{\beta, n+1} d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&+ \Delta t \left[\int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\alpha}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \\
&\geq \Delta t \left[\sum_{\beta=1}^d \int_0^1 \frac{\partial \lambda^{\alpha}}{\partial u^{\beta}} (u_{j_{\alpha}}^{n+1} + \tau(u_{j_{\alpha}+1}^{n+1} - u_{j_{\alpha}}^{n+1})) \cdot \theta_{j_{\alpha}+\frac{1}{2}}^{\beta, n+1} d\tau \right] \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1}.
\end{aligned}$$

Now, using the fact that $\theta_{j_{\beta}+\frac{1}{2}}^{\beta, n+1}$ is positive and the second statement of the assumption (K1), we deduce that

$$-\frac{\Delta t}{\Delta x} \sum_{\alpha=1}^d \left(\lambda_{j_{\alpha}+1}^{\alpha, n+1} - \lambda_{j_{\alpha}}^{\alpha, n+1} \right) \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \leq -\Delta t \sum_{\alpha, \beta=1}^d A_{\alpha, \beta} \theta_{j_{\beta}+\frac{1}{2}}^{\beta, n+1} \theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1} \leq 0. \quad (6.17)$$

We deduce inequality (6.8) by combining (6.17) and the sum over $\alpha \in \{1, \dots, d\}$ of (6.16).

3.2 Case where (K2) holds

We assume that the maximum over $\alpha \in \{1, \dots, d\}$ of $\theta_{j_{\alpha}+\frac{1}{2}}^{\alpha, n+1}$ is achieved at some $\alpha_0 \in \{1, \dots, d\}$. In other words, we have

$$\max_{\alpha \in \{1, \dots, d\}} \left(\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha, n+1} \right) = \theta_{j_{\alpha_0}+\frac{1}{2}}^{\alpha_0, n+1}.$$

Then, using (6.16) at this maximum, we obtain

$$\max_{\alpha=1,\dots,d} \left[\max_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{\alpha,n+1} \right) \right] \leq \max_{\alpha=1,\dots,d} \left[\max_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{\alpha,n} \right) \right] - \frac{\Delta t}{\Delta x} \left(\lambda_{j_{\alpha_0+1}}^{\alpha_0,n+1} - \lambda_{j_{\alpha_0}}^{\alpha_0,n+1} \right) \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1}. \quad (6.18)$$

We proceed as in Subsection 3.1. Namely, it is sufficient to prove that the second term in the right hand side of inequality (6.18) is negative using assumption (K2).

Now, using the fact that $\theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \geq 0$, we get

$$\begin{aligned} & -\frac{\Delta t}{\Delta x} \left(\lambda_{j_{\alpha_0+1}}^{\alpha_0,n+1} - \lambda_{j_{\alpha_0}}^{\alpha_0,n+1} \right) \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \\ & \leq -\Delta t \left(\sum_{\substack{\beta=1 \\ \beta \neq \alpha_0}}^d \int_0^1 \frac{\partial \lambda^{\alpha_0}}{\partial u^\beta} (u_{j_{\alpha_0}}^{n+1} + \tau(u_{j_{\alpha_0+1}}^{n+1} - u_{j_{\alpha_0}}^{n+1})) \cdot \theta_{j_{\alpha_0+\frac{1}{2}}}^{\beta,n+1} d\tau \right) \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \\ & -\Delta t \int_0^1 \left(\frac{\partial \lambda^{\alpha_0}}{\partial u^{\alpha_0}} (u_{j_{\alpha_0}}^{n+1} + \tau(u_{j_{\alpha_0+1}}^{n+1} - u_{j_{\alpha_0}}^{n+1})) \cdot \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} d\tau \right) \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \\ & \leq \Delta t \left(\sum_{\substack{\beta=1 \\ \beta \neq \alpha_0}}^d \int_0^1 \left(\frac{\partial \lambda^{\alpha_0}}{\partial u^\beta} \right)^- (u_{j_{\alpha_0}}^{n+1} + \tau(u_{j_{\alpha_0+1}}^{n+1} - u_{j_{\alpha_0}}^{n+1})) \cdot \theta_{j_{\alpha_0+\frac{1}{2}}}^{\beta,n+1} d\tau \right) \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \\ & -\Delta t \int_0^1 \left(\frac{\partial \lambda^{\alpha_0}}{\partial u^{\alpha_0}} (u_{j_{\alpha_0}}^{n+1} + \tau(u_{j_{\alpha_0+1}}^{n+1} - u_{j_{\alpha_0}}^{n+1})) \cdot \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} d\tau \right) \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \\ & \leq -\Delta t \sum_{\substack{\beta=1 \\ \beta \neq \alpha_0}}^d \int_0^1 \left(\frac{\partial \lambda^{\alpha_0}}{\partial u^{\alpha_0}} (u_{j_{\alpha_0}}^{n+1} + \tau(u_{j_{\alpha_0+1}}^{n+1} - u_{j_{\alpha_0}}^{n+1})) - \left(\frac{\partial \lambda^{\alpha_0}}{\partial u^\beta} \right)^- (u_{j_{\alpha_0}}^{n+1} + \tau(u_{j_{\alpha_0+1}}^{n+1} - u_{j_{\alpha_0}}^{n+1})) \right) \left(\theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1} \right)^2 d\tau \leq 0 \end{aligned}$$

where we have used the fact that $\theta_{j_{\alpha_0+\frac{1}{2}}}^{\beta,n+1} \leq \theta_{j_{\alpha_0+\frac{1}{2}}}^{\alpha_0,n+1}$ and the assumption (K2). Hence, we get the estimate (6.9) by combining the previous inequality and (6.18).

3.3 Case where (K3) holds

We proceed using the same notations as in Subsections 3.1 and 3.2. Using (6.16) for $\alpha = 1$, we get

$$\theta_{j_1+\frac{1}{2}}^{1,n+1} \leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,n} - \frac{\Delta t}{\Delta x} \left(\lambda_{j_1+1}^{1,n+1} - \lambda_{j_1}^{1,n+1} \right) \theta_{j_1+\frac{1}{2}}^{1,n+1}. \quad (6.19)$$

Using assumption (K3) and the fact that $\theta_{j_1+\frac{1}{2}}^{\beta,n+1}$ and $\theta_{j_1+\frac{1}{2}}^{1,n+1}$ are positives, for all $\beta \in \{1, \dots, d\}$, we have

$$-\frac{\Delta t}{\Delta x} \left(\lambda_{j_1+1}^{1,n+1} - \lambda_{j_1}^{1,n+1} \right) \theta_{j_1+\frac{1}{2}}^{1,n+1} = -\Delta t \left[\sum_{\beta=1}^d \int_0^1 \frac{\partial \lambda^1}{\partial u^\beta} (u_{j_1}^{n+1} + \tau(u_{j_1+1}^{n+1} - u_{j_1}^{n+1})) \cdot \theta_{j_1+\frac{1}{2}}^{\beta,n+1} d\tau \right] \theta_{j_1+\frac{1}{2}}^{1,n+1} \leq 0. \quad (6.20)$$

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We deduce by combining (6.19) with (6.20) that

$$\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,n+1} \leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,n}.$$

We can show using a recursion on n that

$$\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,n} \leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,0}, \quad \forall n \in \mathbb{N}. \quad (6.21)$$

Now, we consider the case where $\alpha = 2$.

From (6.16), we have

$$\theta_{j_2+\frac{1}{2}}^{2,n+1} \leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,n} - \frac{\Delta t}{\Delta x} \left(\lambda_{j_2+1}^{2,n+1} - \lambda_{j_2}^{2,n+1} \right) \theta_{j_2+\frac{1}{2}}^{2,n+1}. \quad (6.22)$$

By assumption (K3) and the fact that $\theta_{j_2+\frac{1}{2}}^{\beta,n+1}$ and $\theta_{j_2+\frac{1}{2}}^{2,n+1}$ are positives, for all $\beta \in \{1, \dots, d\}$, we have

$$\begin{aligned} -\frac{\Delta t}{\Delta x} \left(\lambda_{j_2+1}^{2,n+1} - \lambda_{j_2}^{2,n+1} \right) \theta_{j_2+\frac{1}{2}}^{2,n+1} &= -\Delta t \left[\sum_{\beta=2}^d \int_0^1 \frac{\partial \lambda^2}{\partial u^\beta} (u_{j_2}^{n+1} + \tau(u_{j_2+1}^{n+1} - u_{j_2}^{n+1})) \cdot \theta_{j_2+\frac{1}{2}}^{\beta,n+1} d\tau \right] \theta_{j_2+\frac{1}{2}}^{2,n+1} \\ &\quad -\Delta t \left[\int_0^1 \frac{\partial \lambda^2}{\partial u^1} (u_{j_2}^{n+1} + \tau(u_{j_2+1}^{n+1} - u_{j_2}^{n+1})) \cdot \theta_{j_2+\frac{1}{2}}^{1,n+1} d\tau \right] \theta_{j_2+\frac{1}{2}}^{2,n+1} \\ &\leq -\Delta t \left[\int_0^1 \frac{\partial \lambda^2}{\partial u^1} (u_{j_2}^{n+1} + \tau(u_{j_2+1}^{n+1} - u_{j_2}^{n+1})) \cdot \theta_{j_2+\frac{1}{2}}^{1,n+1} d\tau \right] \theta_{j_2+\frac{1}{2}}^{2,n+1}. \end{aligned}$$

Then, we deduce using assumption (H1) that

$$-\frac{\Delta t}{\Delta x} \left(\lambda_{j_2+1}^{2,n+1} - \lambda_{j_2}^{2,n+1} \right) \theta_{j_2+\frac{1}{2}}^{2,n+1} \leq \Delta t M_1 \theta_{j_1+\frac{1}{2}}^{1,n+1} \theta_{j_2+\frac{1}{2}}^{2,n+1}. \quad (6.23)$$

Combining (6.22) and (6.23), we get using (6.21)

$$\begin{aligned} \theta_{j_2+\frac{1}{2}}^{2,n+1} &\leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,n} + \Delta t M_1 \theta_{j_1+\frac{1}{2}}^{1,n+1} \theta_{j_2+\frac{1}{2}}^{2,n+1} \\ &\leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,n} + \Delta t M_1 \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,0} \theta_{j_2+\frac{1}{2}}^{2,n+1}. \end{aligned}$$

Then, we obtain

$$\left[1 - \frac{T}{n} M_1 \left(\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,0} \right) \right] \theta_{j_2+\frac{1}{2}}^{2,n+1} \leq \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,n}.$$

Using a recursion on $n \geq n_0^2$, we can see that

$$\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,n} \leq \frac{1}{\left[1 - \frac{T}{n} M_1 \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,0} \right]^n} \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,0} \leq \mathcal{K}_2(T) \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{2,0},$$

where $\mathcal{K}_2(T) = \exp\left(2TM_1\left(\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{1,0}\right)\right)$.

Proceeding in a similar manner, for all $\beta \leq \alpha - 1$, $\alpha \geq 3$, we have

$$\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\beta,n} \leq \mathcal{K}_\beta(T) \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\beta,0}, \quad \forall \beta \leq \alpha - 1.$$

We prove that the above inequality is true for α .

Indeed, proceeding as in the case where $\alpha = 2$, we deduce that for all $n \geq n_0^\alpha$,

$$\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n+1} \leq \frac{1}{\left[1 - \frac{T}{n} M_1 \sum_{\beta=1}^{\alpha-1} \mathcal{K}_\beta(T)\right]} \max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n}.$$

Hence, using a recursion on n , we obtain

$$\max_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n} \leq \frac{1}{\left[1 - \frac{T}{n} M_1 \sum_{\beta=1}^{\alpha-1} \mathcal{K}_\beta(T)\right]^n} \max_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{\alpha,0}\right) \leq \mathcal{K}_\alpha(T) \max_{i \in \mathbb{Z}} \left(\theta_{i+\frac{1}{2}}^{\alpha,0}\right),$$

where $\mathcal{K}_\alpha(T) = \exp\left(2TM_1 \sum_{\beta=1}^{\alpha-1} \mathcal{K}_\beta(T)\right)$.

Hence, we get the estimate (6.10) for all $\alpha = 1, \dots, d$ and for all $n \geq n_0 = \max_{2 \leq \alpha \leq d} (n_0^\alpha)$. \square

4 Proof of Theorem 6.4

This section is devoted to prove Theorem 6.4. The proof is outlined in four subsections.

4.1 Estimates on the Q^1 extension u^ϵ

Let $(t, x) \in [0, T] \times \mathbb{R}$. Then, there exist $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $(t, x) \in [t_n, t_{n+1}] \times [x_i, x_{i+1}]$, where $x_i = i\Delta x$ and $t_n = n\Delta t$. For $\epsilon = (\Delta t, \Delta x)$, we define the Q^1 extension of the function defined on the grid, for any $(t, x) \in [t_n, t_{n+1}] \times [x_i, x_{i+1}]$, by

$$\begin{aligned} u^\epsilon(t, x) &= \left(\frac{t-t_n}{\Delta t}\right) \left\{ \left(\frac{x-x_i}{\Delta x}\right) u_{i+1}^{n+1} + \left(1 - \frac{x-x_i}{\Delta x}\right) u_i^{n+1} \right\} \\ &+ \left(1 - \frac{t-t_n}{\Delta t}\right) \left\{ \left(\frac{x-x_i}{\Delta x}\right) u_{i+1}^n + \left(1 - \frac{x-x_i}{\Delta x}\right) u_i^n \right\} = (u^{\epsilon,\alpha}(t, x))_{\alpha=1,\dots,d}. \end{aligned} \tag{6.24}$$

In particular, we can see that

$$u^\epsilon(t_n, x_i) = u_i^n \quad \text{for } n \in \mathbb{N}, i \in \mathbb{Z}.$$

4.1.1 Estimate on u^ϵ

From Theorem 6.2, we deduce that u_{i+1}^{n+1} , u_i^{n+1} , u_{i+1}^n and u_i^n are in \mathcal{U} , for all $i \in \mathbb{Z}$ and for all $n \in \mathbb{N}$. By (6.24), we remark that u^ϵ is a convex combination of u_{i+1}^{n+1} , u_i^{n+1} , u_{i+1}^n and u_i^n contained in a convex set \mathcal{U} . Then $u^\epsilon \in \mathcal{U}$, which implies that

$$\|u^{\epsilon,\alpha}\|_{L^\infty([0,T] \times \mathbb{R})} \leq M^\alpha. \quad (6.25)$$

4.1.2 Estimate on $\partial_x u^\epsilon$

We have for $(t, x) \in [t_n, t_{n+1}] \times (x_i, x_{i+1})$

$$\partial_x u^\epsilon(t, x) = \left(\frac{t-t_n}{\Delta t}\right) \theta_{i+\frac{1}{2}}^{n+1} + \left(1 - \frac{t-t_n}{\Delta t}\right) \theta_{i+\frac{1}{2}}^n. \quad (6.26)$$

Using the fact that $\theta_{i+\frac{1}{2}}^n$ is positive for all $n \in \mathbb{N}$ and for all $i \in \mathbb{Z}$, we deduce that

$$\begin{aligned} \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\partial_x u^{\epsilon,\alpha}(t, x)| &\leq \sup_{n \in \mathbb{N}, i \in \mathbb{Z}} \left\{ \left(\frac{t-t_n}{\Delta t}\right) \theta_{i+\frac{1}{2}}^{\alpha, n+1} + \left(1 - \frac{t-t_n}{\Delta t}\right) \theta_{i+\frac{1}{2}}^{\alpha, n} \right\} \\ &\leq \mathcal{G}(\|\partial_x u_0\|_{(L^\infty(\mathbb{R}))^d}, T), \end{aligned}$$

where we have used in the last line the estimates of Theorem 6.3 under one of the assumptions (K1), (K2) or (K3).

We deduce that

$$\|\partial_x u^{\epsilon,\alpha}\|_{L^\infty([0,T] \times \mathbb{R})} \leq \mathcal{G}. \quad (6.27)$$

4.1.3 Estimate on $\partial_t u^\epsilon$

Let us define

$$\tau_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\Delta t}.$$

We have for $(t, x) \in (t_n, t_{n+1}) \times [x_i, x_{i+1}]$

$$\partial_t u^\epsilon(t, x) = \left(\frac{x-x_i}{\Delta x}\right) \tau_{i+1}^{n+\frac{1}{2}} + \left(1 - \frac{x-x_i}{\Delta x}\right) \tau_i^{n+\frac{1}{2}}. \quad (6.28)$$

Using the scheme (6.5), we have

$$\tau_i^{\alpha, n+\frac{1}{2}} = \left(\lambda_i^{\alpha, n+1}\right)_- \theta_{i+\frac{1}{2}}^{\alpha, n+1} - \left(\lambda_i^{\alpha, n+1}\right)_+ \theta_{i-\frac{1}{2}}^{\alpha, n+1}. \quad (6.29)$$

From Theorem 6.2, we know that $u_i^{n+1} \in \mathcal{U}$. Then, we deduce that

$$|\lambda_i^{\alpha, n+1}| \leq \Lambda^\alpha.$$

Using the positivity of $\theta_{i+\frac{1}{2}}^{\alpha, n+1}$, we get

$$|\tau_i^{\alpha, n+1}| \leq 2\Lambda^\alpha \mathcal{G}.$$

Hence, we obtain

$$\|\partial_t u^{\epsilon, \alpha}\|_{L^\infty([0, T] \times \mathbb{R})} \leq 2\Lambda^\alpha \mathcal{G}, \quad (6.30)$$

where we have used in the above inequality estimates of Theorem 6.3 under one of the assumptions (K1), (K2) or (K3).

4.2 Extraction of a convergent subsequence of u^ϵ

Combining estimates (6.25), (6.27) and (6.30), we deduce that $u^{\epsilon, \alpha}$ is uniformly bounded in $W^{1, \infty}([0, T] \times \mathbb{R})$, for any $T > 0$ and satisfies the following estimate

$$\|u^{\epsilon, \alpha}\|_{L^\infty([0, T] \times \mathbb{R})} + \|\partial_x u^{\epsilon, \alpha}\|_{L^\infty([0, T] \times \mathbb{R})} + \|\partial_t u^{\epsilon, \alpha}\|_{L^\infty([0, T] \times \mathbb{R})} \leq \tilde{C}, \quad \text{for all } \alpha \in \{1, \dots, d\},$$

where \tilde{C} is a constant depending only on M^α , $\|\partial_x u_0\|_{(L^\infty(\mathbb{R}))^d}$, Λ^α and T .

Let K be a compact set in $[0, T] \times \mathbb{R}$. By estimates (6.25), (6.27) and (6.30) and using the compact embedding of $W^{1, \infty}(K) \hookrightarrow L^\infty(K)$, we can show that the sequence $(u^{\epsilon, \alpha})_\epsilon$ is relatively compact in $L^\infty(K)$, for all $\alpha \in \{1, \dots, d\}$. Then, as ϵ goes to 0, we can extract a subsequence still denoted by $(u^{\epsilon_k, \alpha})_{\epsilon_k}$ that converges strongly to some function u^α in $L^\infty(K)$. By standard diagonalization procedure, we can extract a subsequence $(u^{\epsilon, \alpha})_\epsilon$ (independent of K and α) such that

$$u^{\epsilon, \alpha} \longrightarrow u^\alpha, \quad \text{uniformly in } L^\infty(K), \quad \forall K \text{ compact and } \forall \alpha \in \{1, \dots, d\}.$$

In particular, we see that the limit function u^α satisfies the initial data $u^\alpha(0, \cdot) = u_0^\alpha(\cdot)$.

By the continuity of λ^α , we know that $\lambda^\alpha(u^\epsilon)$ converges to $\lambda^\alpha(u)$, for all $\alpha \in \{1, \dots, d\}$.

Moreover, using (6.27), we deduce that, up to extraction of a subsequence,

$$\partial_x u^{\epsilon, \alpha} \rightharpoonup \partial_x u^\alpha \quad \text{weakly-} \star \quad \text{in } L^\infty((0, T) \times \mathbb{R}), \quad \text{for all } \alpha \in \{1, \dots, d\}. \quad (6.31)$$

4.3 Passing to the limit in the PDE

4.3.1 Preliminaries

For $(t, x) \in (t_n, t_{n+1}) \times (x_i, x_{i+1})$, we set

$$\lambda^\alpha = \lambda^\alpha(u^\epsilon(t, x)), \quad a_x = \frac{x - x_i}{\Delta x}, \quad b_x = 1 - a_x.$$

Combining (6.28) and (6.29), we have

$$\begin{aligned} \partial_t u^{\epsilon, \alpha}(t, x) &= a_x \left[(\lambda_{i+1}^{\alpha, n+1})_- \theta_{i+\frac{3}{2}}^{\alpha, n+1} - (\lambda_{i+1}^{\alpha, n+1})_+ \theta_{i+\frac{1}{2}}^{\alpha, n+1} \right] + b_x \left[(\lambda_i^{\alpha, n+1})_- \theta_{i+\frac{1}{2}}^{\alpha, n+1} - (\lambda_i^{\alpha, n+1})_+ \theta_{i-\frac{1}{2}}^{\alpha, n+1} \right] \\ &= \lambda_-^\alpha \left[a_x \theta_{i+\frac{3}{2}}^{\alpha, n+1} + b_x \theta_{i+\frac{1}{2}}^{\alpha, n+1} \right] - \lambda_+^\alpha \left[a_x \theta_{i+\frac{1}{2}}^{\alpha, n+1} + b_x \theta_{i-\frac{1}{2}}^{\alpha, n+1} \right] + e^{\epsilon, \alpha}(t, x), \end{aligned} \quad (6.32)$$

6. Convergence d'un schéma implicite pour un système diagonal hyperbolique non-conservatif

where

$$e^{\epsilon,\alpha}(t,x) = a_x \left[- \left[(\lambda_{i+1}^{\alpha,n+1})_+ - \lambda_+^\alpha \right] \theta_{i+\frac{1}{2}}^{\alpha,n+1} + \left[(\lambda_{i+1}^{\alpha,n+1})_- - \lambda_-^\alpha \right] \theta_{i+\frac{3}{2}}^{\alpha,n+1} \right] \\ + b_x \left[- \left[(\lambda_i^{\alpha,n+1})_+ - \lambda_+^\alpha \right] \theta_{i-\frac{1}{2}}^{\alpha,n+1} + \left[(\lambda_i^{\alpha,n+1})_- - \lambda_-^\alpha \right] \theta_{i+\frac{1}{2}}^{\alpha,n+1} \right].$$

In particular, for any test function ζ with compact support $K = [0, T] \times [-R, R]$, where $R > 0$, we have

$$\sum_{\alpha=1}^d \left| \int_{(0,+\infty) \times \mathbb{R}} \zeta(\tau, y) e^{\epsilon,\alpha}(\tau, y) d\tau dy \right| = \sum_{\alpha=1}^d \left| \int_K \zeta(\tau, y) e^{\epsilon,\alpha}(\tau, y) d\tau dy \right| \\ \leq 4RT \sum_{\alpha=1}^d \left(\sup_{i \in \mathbb{Z}} \theta_{i+\frac{1}{2}}^{\alpha,n+1} \right) \|\zeta\|_{L^\infty((0,+\infty) \times \mathbb{R})} \sup_{(\tau,y) \in K} \left(\sup_{|t_{n+1}-\tau| \leq \Delta t, |x_i-y| \leq \Delta x} |\lambda^\alpha(u^\epsilon(t_{n+1}, x_i)) - \lambda^\alpha(u^\epsilon(\tau, y))| \right. \\ \left. + \sup_{|t_{n+1}-\tau| \leq \Delta t, |x_{i+1}-y| \leq \Delta x} |\lambda^\alpha(u^\epsilon(t_{n+1}, x_{i+1})) - \lambda^\alpha(u^\epsilon(\tau, y))| \right).$$

Then, using the continuity of u^ϵ and λ^α and the uniform convergence of u^ϵ , we deduce that

$$e^{\epsilon,\alpha} \longrightarrow 0 \quad \text{when} \quad \epsilon = (\Delta t, \Delta x) \longrightarrow (0, 0) \quad \text{in} \quad \mathcal{D}'((0, +\infty) \times \mathbb{R}).$$

4.3.2 Introduction of θ^ϵ

Our goal is to show that the limit u^α verifies $\partial_t u^\alpha + \lambda^\alpha(u) \partial_x u^\alpha = 0$.

We define the function θ^ϵ as follows

$$\theta^\epsilon(t, x) = \theta_{i+\frac{1}{2}}^{n+1}, \quad \text{for} \quad (t, x) \in [t_n, t_{n+1}) \times [x_i, x_{i+1}).$$

Due to estimates of Theorem 6.3, we deduce that θ^ϵ is uniformly bounded in $L^\infty((0, +\infty) \times \mathbb{R})$. Then, by the weakly- \star convergence, we know that there exists a limit θ , in the sense of distribution, such that for any test function φ (φ is smooth of compact support $K = [0, T] \times [-R, R] \subset [0, T] \times \mathbb{R}$, where $R > 0$) we have

$$\int_{(0,+\infty) \times \mathbb{R}} \theta^\epsilon \cdot \varphi \longrightarrow \int_{(0,+\infty) \times \mathbb{R}} \theta \cdot \varphi. \quad (6.33)$$

From (6.32), we have

$$\partial_t u^{\epsilon,\alpha}(t, x) - e^{\epsilon,\alpha}(t, x) = \lambda_-^\alpha [a_x \theta^{\epsilon,\alpha}(t, x + \Delta x) + b_x \theta^{\epsilon,\alpha}(t, x)] \\ - \lambda_+^\alpha [a_x \theta^{\epsilon,\alpha}(t, x) + b_x \theta^{\epsilon,\alpha}(t, x - \Delta x)], \quad (6.34)$$

where $a_x = \frac{x}{\Delta x} - \lfloor \frac{x}{\Delta x} \rfloor$ and $b_x = 1 - a_x$.

Let us now introduce the function $A^{\epsilon,\alpha}$ defined by

$$A^{\epsilon,\alpha} = \int_{(0,+\infty) \times \mathbb{R}} (\partial_t u^{\epsilon,\alpha}(t, x) - e^{\epsilon,\alpha}(t, x)) \varphi(t, x) dt dx.$$

Then, using (6.34), we can write $A^{\epsilon, \alpha}$ as follows

$$\begin{aligned} A^{\epsilon, \alpha} &= \int_{(0, +\infty) \times \mathbb{R}} \theta^{\epsilon, \alpha}(t, x) [a_x \lambda_-^\alpha(u^\epsilon(t, x - \Delta x)) \varphi(t, x - \Delta x) + b_x \lambda_-^\alpha(u^\epsilon(t, x)) \varphi(t, x)] dt dx \\ &\quad - \int_{(0, +\infty) \times \mathbb{R}} \theta^{\epsilon, \alpha}(t, x) [a_x \lambda_+^\alpha(u^\epsilon(t, x)) \varphi(t, x) + b_x \lambda_+^\alpha(u^\epsilon(t, x + \Delta x)) \varphi(t, x + \Delta x)] dt dx. \end{aligned}$$

Let us now define the following function

$$\begin{aligned} B^{\epsilon, \alpha} &= \int_{(0, +\infty) \times \mathbb{R}} \theta^{\epsilon, \alpha}(t, x) [a_x \lambda_-^\alpha(u^\epsilon(t, x)) \varphi(t, x) + b_x \lambda_-^\alpha(u^\epsilon(t, x)) \varphi(t, x) \\ &\quad - a_x \lambda_+^\alpha(u^\epsilon(t, x)) \varphi(t, x) - b_x \lambda_+^\alpha(u^\epsilon(t, x)) \varphi(t, x)] dt dx \\ &= \int_{(0, +\infty) \times \mathbb{R}} -\lambda^\alpha(u^\epsilon(t, x)) \varphi(t, x) \theta^{\epsilon, \alpha}(t, x) dt dx. \end{aligned}$$

Then, we have

$$\begin{aligned} |A^{\epsilon, \alpha} - B^{\epsilon, \alpha}| &\leq \left| \int_{(0, +\infty) \times \mathbb{R}} a_x \theta^{\epsilon, \alpha}(t, x) [\lambda_-^\alpha(u^\epsilon(t, x - \Delta x)) \varphi(t, x - \Delta x) - \lambda_-^\alpha(u^\epsilon(t, x)) \varphi(t, x)] dt dx \right| \\ &\quad + \left| \int_{(0, +\infty) \times \mathbb{R}} b_x \theta^{\epsilon, \alpha}(t, x) [-\lambda_+^\alpha(u^\epsilon(t, x + \Delta x)) \varphi(t, x + \Delta x) + \lambda_+^\alpha(u^\epsilon(t, x)) \varphi(t, x)] dt dx \right| \\ &\leq 2TRG \sup_{(t, x) \in K} |\lambda_{\mp}^\alpha(u^\epsilon(t, x \mp \Delta x)) \varphi(t, x \mp \Delta x) - \lambda_{\mp}^\alpha(u^\epsilon(t, x)) \varphi(t, x)|, \end{aligned}$$

which implies that $|A^{\epsilon, \alpha} - B^{\epsilon, \alpha}| \rightarrow 0$ when ϵ goes to 0.

On the other hand, using (6.33), we get

$$B^{\epsilon, \alpha} = \int_{(0, +\infty) \times \mathbb{R}} -\lambda^\alpha(u^\epsilon(t, x)) \varphi(t, x) \theta^{\epsilon, \alpha}(t, x) dt dx \rightarrow \int_{(0, +\infty) \times \mathbb{R}} -\lambda^\alpha(u(t, x)) \varphi(t, x) \theta^\alpha(t, x) dt dx.$$

Therefore, we obtain that

$$\partial_t u^\alpha + \lambda^\alpha(u) \theta^\alpha = 0 \quad \text{in } \mathcal{D}'((0, +\infty) \times \mathbb{R}), \quad \forall \alpha \in \{1, \dots, d\}. \quad (6.35)$$

Using (6.26), we can see that

$$\partial_x u^{\epsilon, \alpha}(t, x) = \left(\frac{t - t_n}{\Delta t} \right) \theta^{\epsilon, \alpha}(t, x) + \left(1 - \frac{t - t_n}{\Delta t} \right) \theta^{\epsilon, \alpha}(t - \Delta t, x).$$

We deduce, proceeding in a similar manner, that

$$\partial_x u^{\epsilon, \alpha} \rightarrow \theta^\alpha \quad \text{in } \mathcal{D}'((0, +\infty) \times \mathbb{R}).$$

Moreover, using the weak- \star convergence given by (6.31), we deduce that

$$\theta = \partial_x u. \quad (6.36)$$

Combining (6.35) with (6.36), we conclude that

$$\partial_t u^\alpha + \lambda^\alpha(u) \partial_x u^\alpha = 0 \quad \text{in } \mathcal{D}'((0, +\infty) \times \mathbb{R}), \quad \forall \alpha \in \{1, \dots, d\}.$$

Finally, we deduce that the limit u^α is a solution of system (6.1).

4.4 Convergence of the whole sequence when the limit is unique

Due to the uniqueness of the solution of the continuous problem, given by Theorem 6.1, we deduce that the whole sequence $(u^{\varepsilon, \alpha})_\varepsilon$ converges to the unique limit v^α without assuming the strict hyperbolicity of the system as in [59]. \square

5 Numerical simulations

We consider in this section a simplified one-dimensional model describing the dynamics of dislocations, where dislocations are microscopic defects present in materials, especially in metal alloys. The movement of these defects is the main explanation of plastic and viscoplastic deformations. In a particular geometry, where dislocations are assumed to be punctual defects, depending on a single variable x and moving in two fixed directions, according to the vector $(1, 0)$ (on the right) or according to the vector $-(1, 0)$ (on the left), Groma and Balogh have modeled in [40] the dynamics of the dislocations densities by a (2×2) coupled system of non-local transport equations. More precisely, it is the following system:

$$\begin{cases} \partial_t u^1(t, x) = - \left((u^1 - u^2)(t, x) + \alpha \int_0^1 (u^1 - u^2)(t, y) dy + a(t) \right) \partial_x u^1(t, x) & \text{in } (0, T) \times \mathbb{R}, \\ \partial_t u^2(t, x) = \left((u^1 - u^2)(t, x) + \alpha \int_0^1 (u^1 - u^2)(t, y) dy + a(t) \right) \partial_x u^2(t, x) & \text{in } (0, T) \times \mathbb{R}. \end{cases}$$

Here, u^1 and u^2 are two scalar valued functions, that represent respectively the dislocations moving on the right and that moving on the left. Their difference $(u^1 - u^2)$ expresses the plastic and viscoplastic deformation of the material and their spatial derivatives $\partial_x u^i$, for $i = 1, 2$, are the dislocations densities corresponding to each type. While, α is a nonnegative constant depending on the elastic coefficients of the material and $a(\cdot)$ is the exterior shear stress. We can easily see that in the particular case where $\alpha = a(t) = 0$, the model is reduced to the following system:

$$\begin{cases} \partial_t u^1(t, x) = - ((u^1 - u^2)(t, x)) \partial_x u^1(t, x) & \text{in } (0, T) \times \mathbb{R}, \\ \partial_t u^2(t, x) = ((u^1 - u^2)(t, x)) \partial_x u^2(t, x) & \text{in } (0, T) \times \mathbb{R}, \end{cases} \quad (6.37)$$

which is well in the form of (6.1).

Now, we calculate the numerical solution of system (6.37), through the Upwind scheme (6.5), choosing discretization parameters $(\Delta t, \Delta x)$ that satisfy the CFL condition (6.7) and taking the following initial data:

$$u^1(0, x) = \frac{1}{2\pi} \sin(2\pi x) + x, \quad u^2(0, x) = -\frac{1}{2\pi} \sin(2\pi x) + x, \quad \forall x \in \mathbb{R},$$

which are nondecreasing and 1-periodic + linear functions. We thus modelize a periodic distribution for the two dislocations types, with a spatial period of length 1. Note that each type of dislocations has a mean density equal to 1. In fact, the use of the periodic boundary conditions is a way of regarding what is going on in the interior of the material away from its boundary. It is enough then to see only what is happening in the interval $[0, 1]$.

We simulate below the long-time behavior of the function $(u^1 - u^2)$, which reflects the viscoplastic deformation (in the absence of shear stress). We can intuitively see from system (6.37) that this function will reach the stationary state $(u^1 - u^2) = 0$, in long time, as shown in Figure 6.1. This is consistent with physical expectations, since obviously the material will stabilize, if we stop the stress.

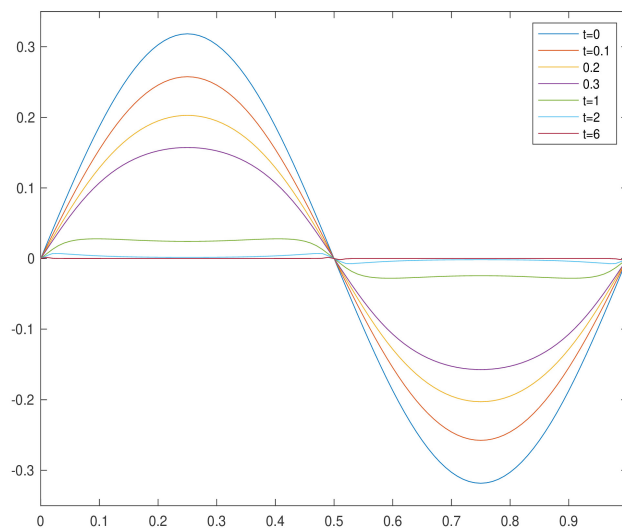


Figure 6.1: Evolution of $u^1 - u^2$ over time.

Conclusion et perspectives

Les travaux effectués dans cette thèse ont permis de répondre à plusieurs questions importantes pour la dynamique des densités des dislocations. Nous avons pu démontrer des résultats d'existence et d'unicité pour des problèmes de type Hamilton-Jacobi modélisant la dynamique des densités des dislocations se propageant dans une et plusieurs directions de glissement. Ces travaux théoriques ont été complétés par des études numériques adéquates en proposant des schémas numériques convergents. Des simulations numériques ont été aussi réalisées afin de mieux comprendre le comportement mécanique issu de la déformation plastique des matériaux.

Dans cette thèse, notre travail se focalise sur l'étude du modèle de Groma et Balogh qui ont négligé les effets aux bords et les corrélations entre dislocation-dislocation. Nous serons intéressés après cette thèse par l'étude et la compréhension d'autres modèles de la dynamique des densités des dislocations. De plus, l'objet que nous étudions après cette thèse est l'homogénéisation des systèmes hyperboliques et l'application à la dynamique des dislocations.

Du point de vue numérique, il serait intéressant de prouver une estimation d'erreur entre la solution continue et la solution discrète dans la continuité des Chapitres 4 et 6. Enfin, nous serons intéressés par l'étude numérique du problème introduit dans le Chapitre 3 dans le cadre des solutions continues $L \log L$ en proposant un schéma aux différences finies ou un schéma aux volumes finis. En effet, nous prouverons une estimation entropique discrète du gradient similaire à celle obtenue dans le cas continue (voir Chapitre 3). Cette dernière nous permettra de montrer la convergence de la solution numérique vers la solution de viscosité du problème continue en se basant sur les résultats de Barles-Souganidis [13].

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