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«Approximation diffusion pour des équations dispersives »

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TABLE OF CONTENTS

Introduction

0.1 Contexte de la thèse

Le but de cette section est de faire comprendre le lien entre les problèmes étudiés dans cette thèse et les problèmes concrets modélisés par les équations étudiées, découlant généralement de la physique. Pour cela il est important d'expliquer plus profondément les concepts manipulés dans cette thèse, comme celui d'équations dispersives, mais également de donner des outils de compréhension mathématiques, comme par exemple des notions d'intégration stochastique.

0.1.1 Equations dispersives

Les équations que nous allons retrouver tout au long de ce manuscrit permettent de modéliser des phénomènes de déplacement d'onde. Il n'y a pas d'unique définition précise décrivant exactement ce qui constitue une onde. On peut se contenter de la décrire intuitivement comme un signal reconnaissable, transmis d'un endroit d'un milieu à un autre endroit, avec une vitesse de propagation reconnaissable. Nos équations sont appelées dispersives de par les ondes qu'elles modélisent, dites elles aussi dispersives. Ces équations apparaissent notamment en optique non-linéaire, c'est-à-dire dans la description de la propagation de la lumière à travers différents milieux, ayant par exemple des indices de réfraction non-linéaires. C'est le cas par exemple pour un signal transmis à travers une fibre optique : le message est codé de façon binaire en représentant un 1 par une courte impulsion lumineuse, et un 0 par l'absence d'une telle impulsion dans le train d'ondes transmis. Il y a plusieurs problèmes de dispersion du signal dans ce procédé, ce qui soulève des enjeux importants dans les domaines des communications ou des lasers, à partir des années 1960. Par exemple on veut pouvoir augmenter le débit d'un signal, c'est-à-dire réduire la durée des impulsions ainsi que le temps entre celles-ci, et ceci tout en gardant un signal de sorti identifiable, afin de pouvoir supprimer un éventuel bruit de fond sans perte d'information.

D'un point de vue mathématique, une onde dispersive est décrite par le type de solutions du système qui la modélise plutôt que par la forme du système en soi. Un système linéaire dispersif est un système qui admet des solutions de la forme

$$\varphi = a\cos(\kappa x - \omega t), \tag{0.1.1}$$

où la fréquence ω est une fonction du nombre d'onde κ qui est déterminée par le système en lui-même. Une onde est dite dispersive si la vitesse de phase $\frac{\omega}{\kappa}$ n'est pas constante, mais dépend de κ. Ce terme s'explique par le fait qu'une solution plus générale serait un signal constitué d'une superposition de modes de la forme de (0.1.1) avec des nombres d'onde différents. Comme la vitesse de phase n'est pas constante, ces modes se propageraient à différentes vitesses, et se disperseraient. Plutôt que de regarder la vitesse de phase, on dit souvent qu'une équation est dispersive si la relation de dispersion $\omega(\kappa)$ est de dérivée seconde non nulle. Cette relation de dispersion peut s'obtenir assez facilement, comme nous le verrons sur un exemple, et fournit une bonne méthode pour intuiter du comportement de tout système linéaire dispersif. Toute cette analyse étant faite en termes de fréquences, on voit naturellement que la théorie de Fourier est très adaptée pour travailler sur ces systèmes. Ainsi, pour les problèmes linéaires, on peut utiliser cette théorie pour montrer l'existence de solutions élémentaires sous la forme de trains d'ondes sinusoïdaux, que le caractère linéaire nous permet de représenter sous forme complexe

$$\varphi(t,x) = ae^{i\kappa x - i\omega t},$$

avec a l'amplitude, en sachant qu'on peut regarder la partie réelle quand nécessaire, la vraie solution étant

$$Re(\varphi) = |a|\cos(\kappa x - \omega t + \eta), \qquad \eta = \arg(a).$$

Pour satisfaire les équations du système, κ and ω doivent satisfaire la relation de dispersion évoquée précédemment. Le caractère linéaire du système nous permet d'étudier simplement un seul mode, car des solutions plus complexes peuvent se décomposer en superpositions de modes simples. Ici $\theta = \kappa x - \omega t$ représente la phase, qui indique la position dans le cycle entre une crête, c'est-à-dire quand $Re(\varphi)$ est a son maximum, et quand $Re(\varphi)$ est à son minimum. Son gradient en espace κ est le nombre d'onde, dont l'amplitude représente le nombre moyen de crêtes par intervalle de distance de longueur 2π dans la direction de propagation de l'onde. De manière similaire, la fréquence ω donne le nombre moyen de crêtes par intervalle de temps de longueur 2π. La longueur d'onde est $\lambda = \frac{2\pi}{\kappa}$ et la période est $\tau = \frac{2\pi}{\omega}$. Pour que l'onde soit considérée comme dispersive, la vitesse de phase $c = \frac{\omega}{\kappa}$ ne peut pas être constante, car dans ce cas tous les modes se déplaceraient à la même vitesse, donc on n'observerait pas de dispersion au sein d'une superposition de modes. On donne la définition suivante pour le caractère dispersif d'une onde.

Definition 0.1.1.1. *Une onde est dite dispersive, si la relation de dispersion $\omega(\kappa)$ obtenue à partir de l'équation décrivant cette onde est d'une part à valeurs réelles, et d'autre part vérifie*

$$\omega''(\kappa) \neq 0.$$

Cette définition est plus restrictive que de demander $c'(\kappa) \neq 0$ car elle exclut le cas $\omega(\kappa) = \alpha\kappa + \beta$. Dans ce dernier cas la vitesse de groupe $\omega'(\kappa)$ serait constante et il n'y aurait pas de dispersion. On demande donc que cette vitesse de groupe dépende encore de κ, ainsi différents modes auront différentes vitesses de groupe et les paquets d'ondes auront tendance à se disperser.

Il existe une infinité d'équations dispersives, et bien que dans ce paragraphe ait été discuté seulement le cas de systèmes linéaires, il existe également des théories pour les systèmes non-linéaires. De nombreux résultats sur les deux cas peuvent être trouvés dans [Whi74]. Nous n'allons pas présenter cette théorie dans son cadre global, mais plutôt détailler quelques résultats connus sur un exemple classique d'équation dispersive : l'équation de Schrödinger.

0.1.2 L'équation de Schrödinger

L'équation de Schrödinger linéaire tient son origine de la mécanique quantique. Elle a été introduite en 1925 par Erwin Schrödinger, afin de modéliser l'évolution d'une particule au cours du temps. Cette description fait intervenir la fonction d'onde de la particule, qui donne également une interprétation probabiliste de l'état de la particule. En effet, le module au carré de la fonction d'onde définit une densité de probabilité indiquant la probabilité de présence de la particule dans une région de l'espace, lorsque l'on intègre cette densité. L'équation de Schrödinger linéaire pour une particule de masse m s'écrit originellement comme suit :

$$i\hbar\partial_t\psi(t,x) = -\frac{\hbar^2}{2m}\Delta\psi(t,x) + V(x)\psi(t,x),$$

où $\hbar = \frac{h}{2\pi}$ est la constante de Planck réduite, avec $h = 6,626.10^{-34} J.s$ la constante de Planck, constante universelle décrivant la différence d'échelle entre le caractère ondulatoire et le caractère corpusculaire de la lumière. V désigne un potentiel réel qui modélise le milieu dans lequel se trouve la particule.

Dans cette thèse nous considérons une équation de Schrödinger non-linéaire (NLS), donc on choisit de ne pas s'attarder sur l'équation linéaire, et de présenter les spécificités de notre équation. De façon plus précise, l'équation que l'on étudie est une équation de Schrödinger avec non-linéarité de type puissance. Dans ce contexte, l'équation s'obtient à partir des équations de Maxwell, et peut s'écrire sous la forme

$$i\partial_t u(t,x) = -\Delta u(t,x) + \lambda |u(t,x)|^{2\sigma} u(t,x). \tag{0.1.2}$$

Dans cette équation, le paramètre σ est un réel strictement positif, et le paramètre λ est un réel dont le signe influence le comportement de l'équation. D'un point de vue physique, l'équation de Schrödinger non-linéaire décrit la dynamique de l'enveloppe d'une onde plane quasi-monochromatique qui se propage dans un milieu faiblement dispersif, lorsqu'on néglige les effets de dissipation. On parle de non-linéarité attractive si $\lambda = -1$ et de non-linéarité répulsive si $\lambda = 1$. Cette équation est utilisée notamment dans l'étude de la propagation d'un faisceau laser dans un milieu dont l'indice de réfraction est sensible à l'amplitude de l'onde qui le traverse. Dans le cas $\lambda = -1$, l'indice de réfraction augmente avec l'amplitude de l'onde, entrainant une convergence des rayons vers la région d'amplitude la plus élevée. Cela conduit à la focalisation du faisceau, phénomène très étudié depuis des années 1960.

Lorsque l'on étudie une équation aux dérivées partielles, plusieurs questions se posent naturellement. En premier lieu on s'interroge sur la notion de solution : peut-on construire une théorie de Cauchy (locale ou globale) pour l'équation de Schrödinger non-linéaire ? Pour répondre à cette question, on essaye de trouver des lois de conservation. On identifie ainsi des invariances pour une éventuelle solution de l'équation, qui nous donnent par la suite, grâce au théorème de Noether (voir par exemple [SS99, Théorème 2.1]), les lois de conservations recherchées. Par exemple, l'équation de Schrödinger (1.1.1) est invariante par rotation : si $u(t,x)$ est solution de (1.1.1), alors $e^{i\theta}u(t,x)$, $\theta \in \mathbb{R}$, est également solution. On récupère grâce à cette invariance une propriété qu'on appelle conservation de la masse :

$$N(t) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = N(0). \tag{0.1.3}$$

On a aussi une invariance par translation en temps : si $u(t,x)$ est solution de (1.1.1), alors $u(t-t_0,x)$ est également solution. Cette invariance implique la conservation de l'énergie

$$\mathcal{E}(t)=\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u(t,x)|^2dx+\frac{\lambda}{2\sigma+2}\int_{\mathbb{R}^d}|u(t,x)|^{2\sigma+2}dx=\mathcal{E}(0). \tag{0.1.4}$$

Il existe d'autres invariances vérifiées par l'équation de Schrödinger non-linéaire, on peut par exemple les retrouver dans [SS99].

Grâce à ces lois de conservation, ainsi qu'a des théorèmes de point fixe, on arrive à conclure quant à l'existence et l'unicité des solutions de (1.1.1), en fonction du signe de λ. Dans les deux cas, on a existence et unicité locale dans l'espace $H^1(\mathbb{R}^d)$ sous réserve que $0<2\sigma<\frac{4}{d-2}$ si $d\geqslant 3$. Dans le cas défocalisant $\lambda>0$, la conservation de l'énergie (0.1.4) nous permet de borner la norme H^1 de la solution et de conclure à une existence globale. Dans le cas focalisant $\lambda<0$, l'étude est plus complexe, car l'énergie $\mathcal{E}(t)$ n'a pas de signe. Il existe néanmoins de nombreux théorèmes prouvant l'unicité et l'existence globale dans $H^1(\mathbb{R}^d)$ sous des conditions sur la puissance de la non-linéarité ou la taille de la donnée initiale. Ces résultats peuvent être retrouvés notamment dans [GV79], [GV84], [GV85] ou encore dans [Kat87]. On se restreint ici à regarder le problème dans l'espace d'énergie, mais il existe beaucoup d'autres résultats, dont des exemples peuvent être trouvés dans [Caz03]. Lorsque cette existence globale n'est pas assurée, il peut se produire une explosion de la solution en temps fini (blow up en anglais), c'est-à-dire qu'il existe un temps $T^*>0$ tel que

$$\lim_{t\to T^*}\|u(t)\|_{H^1}=+\infty.$$

Une question importante dans l'étude de l'équation de Schrödinger est la recherche de solutions stationnaires, qui gardent la même forme au cours du temps. Prenons le cas de la dimension 1, dans le cas attractif. L'équation s'écrit alors

$$i\partial_t u=-\partial_x^2u+\lambda|u|^{2\sigma}u,$$

avec $\lambda<0$. On va commencer par rechercher une solution sous la forme d'une onde localisée en espace, oscillant de manière indépendante de ce dernier :

$$u(t,x)=e^{i\omega t}\phi(x).$$

En réinjectant cette expression dans l'équation, on obtient une équation pour ϕ, qui dépend de ω, et qui nous donne finalement

$$u(t,x) = e^{i\omega t}\left(\frac{-2\lambda}{\omega(2\sigma+2)}\right)^{-\frac{1}{2\sigma}} \cosh\left(\sigma\sqrt{\omega}x\right)^{-\frac{1}{\sigma}}. \tag{0.1.5}$$

Les solutions de la forme de (0.1.5) sont appelés solitons. Ces solitons préservent leur forme au cours du temps, et se propagent à vitesse constante. Ce sont des solutions qui permettent de comprendre un peu mieux l'équation de Schrödinger, car on peut montrer que pour une large classe de données initiales tendant vers 0 à l'infini, la solution de l'équation de Schrödinger non-linéaire focalisante en dimension 1 évolue de manière asymptotique comme un ensemble fini de solitons. Il est également possible de construire des des fonctions localisées conservant leur forme en dimension supérieure, on parle d'ondes solitaires.

L'équation de Schrödinger possède bien d'autres propriétés, qui ne sont pas détaillées ici mais qui peuvent se trouver dans la littérature ([Caz03],[Kat87],[SS99],...). On s'est attardé sur l'équation avec non-linéarité puissance, car c'est celle qui va apparaître régulièrement dans les problèmes de cette thèse. Elle est notamment liée au système d'équation que l'on étudie, à savoir le système de Zakharov.

0.1.3 Le système de Zakarov

Le système de Zakharov est un système d'équations aux dérivées partielles fournissant un modèle simplifié pour modéliser des ondes de Langmuir. Les ondes de Langmuir constituent un phénomène physique prenant place dans des plasmas non ou faiblement magnétisés. Ce phénomène possède des applications aussi bien en astrophysique qu'en physique de laboratoire. Un exemple d'application est le déplacement d'une onde dans la ionosphère. Le système a été introduit par Zakharov en 1972 [Zak72] afin de décrire la dynamique couplée entre l'amplitude du champ électrique du plasma et les fluctuations à basse fréquence de sa densité ionique. En effet, on peut considérer un plasma comme un ensemble de deux fluides se pénétrant, un fluide composé d'ions et un fluide composé d'électrons. Les équations de Maxwell ainsi qu'une approche multi-échelle du problème permettent d'obtenir, sous certaines hypothèses, le système adimensionné

$$\begin{cases} i\partial_t E - \alpha\nabla\times(\nabla\times E) + \nabla(\nabla\cdot E) = \bar{n}E \\ \varepsilon^2\partial_t^2\bar{n} - \Delta\bar{n} = \Delta|E|^2 \end{cases}, \tag{0.1.6}$$

où les opérateurs $\nabla\times$ et $\nabla\cdot$ désignent respectivement le rotationnel et la divergence. Le paramètre α est assez grand dans les situations réelles, allant de 20 dans les plasmas de laboratoire jusqu'à 2.10^5 dans les gaz interstellaires. Ces équations décrivent la relation entre l'enveloppe complexe des oscillations du champ électrique proches de la fréquence du plasma et les variations plus lentes de la densité ionique moyenne du plasma. Ces variations affectent l'indice de réfraction ressenti par le champ électrique. Lorsque le terme $\varepsilon^2\partial_t^2\bar{n}$ est négligeable, la relation entre $\bar{n}$ et E conduit à une augmentation de l'indice de réfraction et donc à un phénomène d'auto-focalisation de l'onde. Dans ce cas, on est dans un régime dit de "limite subsonique", et on récupère l'équation vectorielle de Schrödinger non-linéaire :

$$i\partial_t E - \alpha\nabla\times(\nabla\times E) + \nabla(\nabla\cdot E) + |E|^2E = 0.$$

Si on s'intéresse au cas particulier $\alpha = 1$, l'opérateur linéaire dans l'équation sur E devient un Laplacien, et le système de Zakharov peut être vu comme un système de trois équations de Schrödinger scalaires pour les coordonnées de l'enveloppe du champ électrique, couplées avec une équation pour la densité ionique. Supposant qu'une seule des coordonnées du champ électrique est initialement non-nulle, que l'on note u, on obtient le système

$$\begin{cases} i\partial_t u + \Delta u = nu \\ \varepsilon^2\partial_t^2 n - \Delta n = \Delta|u|^2 \end{cases}. \tag{0.1.7}$$

Ce système est généralement appelé système de Zakharov scalaire. Il est utilisé pour modéliser de façon simple des ondes de Langmuir, lorsque le caractère vectoriel du champ électrique est négligé. À la limite subsonique, on récupère l'équation de Schrödinger cubique. Dans le cas de la dimension 1, le système de Zakharov admet aussi des solitons, tout comme l'équation de Schrödinger. En effet, comme le soliton de Schrödinger est de module constant au cours du temps, en prenant ce soliton comme donnée initiale pour u et son module au carré pour n, on obtient une solution particulière pour (0.1.7).

Ce sytème conserve également certaines quantités, notamment la masse

$$N(t) = \int_{\mathbb{R}} |u(t,x)|^2 dx = N(0),$$

ainsi que l'énergie

$$H(t) = \int_{\mathbb{R}} \left(|\nabla u(t,x)|^2 + n(t,x)|u(t,x)|^2 + \frac{1}{2}(|\varepsilon V(t,x)|^2 + |n(t,x)|^2) \right) dx = H(0),$$

où $V(t,x)$ vérifie

$$\partial_t n + \nabla \cdot V = 0.$$

Deux types de questions mathématiques ont essentiellement été traités dans la littérature. La première consiste à se demander s'il est possible de développer une théorie de Cauchy, locale ou globale, et de conclure quant à l'existence et l'unicité de solutions. La deuxième est la possibilité de rendre rigoureux le passage à la limite subsonique. Autrement dit, on se demande non seulement s'il existe une solution au problème de Cauchy pour (0.1.7), mais aussi si cette solution converge dans un certain sens vers la solution du problème de Cauchy associé à l'équation de Schrödinger cubique. Ces questions ont été résolues en 1979 par Sulem et Sulem [SS79] pour ce qui est de l'existence et l'unicité, puis par Added et Added en 1984 [AA84] et 1988 [AA88] qui ont étendu les resultats de Sulem et Sulem tout en prouvant des résultats de convergence vers l'équation de Schrödinger cubique. Plus récemment Bourgain et Colliander [BC96] ont utilisé les méthodes de décomposition selon les hautes et basses fréquences introduites par Bourgain, et Ginibre, Tsutsumi et Velo ont fait de même dans [GTV97] pour obtenir un résultat d'existence locale en dimension $d = 1, 2, 3$. D'autres résultats concernant l'existence, la limite subsonique ou même des propriétés d'explosion en temps fini sont listés dans [SS99] et apparaiseent plus récemment dans [CHT08] ou encore [MN08],[MN09].

Cette thèse étudiant des problèmes d'approximation diffusion pour le système de Zakharov et l'équation de Schrödinger non-linéaire, il est également nécessaire d'introduire des notions liées aux équations aux dérivées partielles stochastiques, et notamment des notions d'intégration stochastique dans les espaces de Hilbert.

0.1.4 Les équations aux dérivées partielles stochastiques

Les problèmes d'évolution stochastique dans des espaces de dimension infinie, tels les espace de Hilbert, sont des généralisations de la théorie d'Itô développée dans les années 1940 par Kiyoshi Itô. Les premiers résultats sur ces problèmes sont apparus dans les années 1960, et leur développement fut motivé à la fois par un soucis mathématique de généralisation d'une théorie déjà existante et par un besoin de décrire des phénomènes aléatoires faisant intervenir un très grand nombre de variables dans les domaines de la physique, de la chimie ou encore de la biologie.

Commençons d'abord par évoquer le cas de la dimension 1. Le but est de modéliser un bruit, c'est-à-dire une variable aléatoire dépendant du temps ou encore un processus stochastique, qui agit comme une perturbation sur une variable dont l'évolution est régie par une équation différentielle. Si on appelle $\dot{\xi}$ ce bruit, on est amené à considérer des équations de la forme

$$\frac{dx}{dt} = f(x,t) + g(x,t)\dot{\xi}(t).$$

Imaginons une particule évoluant dans un milieu désordonné et qui est soumise à des chocs avec d'autres particules. Supposons que ces particules soient aléatoirement réparties, que leur répartition ne dépende pas du temps et que cette perturbation n'ait pas de mémoire ; il est alors naturel que la force $\dot{\xi}$ agissant sur cette particule vérifie les hypothèses suivantes :

— La loi de $\dot{\xi}(t)$ ne dépend pas de t.

— $\dot{\xi}(t)$ et $\dot{\xi}(s)$ sont indépendants si $t \neq s$.

Un tel processus est nécessairement irrégulier. Il est plus pratique de traduire ces hypothèses en terme de sa primitive $\xi(t) = \int_0^t \dot{\xi}(s)ds$. On obtient alors :

— La loi de $(\xi(t+\tau) - \xi(\tau))_{t\geqslant 0}$ (loi du processus) ne dépend pas de $\tau \geqslant 0$.

— Pour $t, \tau \geqslant 0$, $\xi(t+\tau) - \xi(t)$ est indépendant de $(\xi(s))_{s\leqslant t}$.

Si on suppose de plus que ξ est continu par rapport à t, on peut alors montrer que ξ est nécessairement un mouvement Brownien, c'est-à-dire que l'on a en plus : pour tout $\tau > 0, t \geqslant 0$, la variable aléatoire $\xi(t+\tau) - \xi(t)$ est Gaussienne centrée de variance τ.

On va maintenant donner une définition précise du mouvement Brownien, et donner quelques unes de ses propriétés.

Soit $(\Omega, \mathcal{F}, \mathbb{P})$ un espace probabilisé et $I = [0,T]$ ou $I = \mathbb{R}^+$.

Definition 0.1.4.1. *On dit que $(\beta(t))_{t\in I}$ est un mouvement Brownien (ou processus de Wiener) sur I, à valeurs réelles, si*

1. *β est un processus à valeurs réelles et $\beta(0) = 0$.*
2. *β est continu presque sûrement sur I.*
3. *β a des accroissements indépendants, c'est-à-dire que pour tout n-uplet $(t_0, t_1, \cdots, t_n)$ avec $t_0 < t_1 < \cdots < t_n$ et $t_i \in I$ pour tout i, les variables aléatoires $\beta(t_{i+1}) - \beta(t_i)$ forment une famille indépendante.*

4. *pour tous $t, s \in I$ avec $t > s$, $\beta(t) - \beta(s)$ suit une loi normale centrée de variance $t - s$, c'est-à-dire $\beta(t) - \beta(s) \sim \mathcal{N}(0, t - s)$.*

On peut déjà remarquer que si β est un mouvement Brownien, alors pour tous $t, s \in I$ avec $t > s$, $\frac{\beta(t)-\beta(s)}{t-s}$ suit une loi Gaussienne centrée de variance $\frac{1}{t-s}$. On ne s'attend donc pas à ce que β soit dérivable. Le mouvement Brownien possède cependant d'autres propriétés. La proposition suivante en énonce quelques unes.

Proposition 0.1.4.1. *Soit β un mouvement Brownien sur $I = \mathbb{R}^+$. Alors*

1. *Homogénéité en temps : pour tout $\tau > 0$, le processus $(\beta(t + \tau) - \beta(\tau))_{t \geqslant 0}$ est un mouvement Brownien.*
2. *Symétrie : Le processus $(-\beta(t))_{t \geqslant 0}$ est un mouvement Brownien.*
3. *Scaling : Pour tout $\lambda > 0$, le processus $\left(\lambda \beta\left(\frac{t}{\lambda^2}\right)\right)_{t \geqslant 0}$ est un mouvement Brownien.*

Grâce aux propriétés stochastiques du mouvement Brownien, la théorie d'Itô permet de construire pour des processus g prévisibles et de carré intégrable en temps une notion d'intégrale contre un mouvement Brownien, à partir de limite de processus élémentaires $(g_n)_{n \in \mathbb{N}}$ de la forme $\sum_{k=0}^{\phi(n)-1} g_n(k) \mathbb{1}_{[t_k, t_{k+1}[}$ où pout tout $n \in \mathbb{N}$, $t_1 < t_2 < \cdots < t_{\phi(n)}$ représente une partition de $[0, T]$. Pour ces processus élémentaires on peut définir l'intégrale stochastique comme une combinaison linéaire d'accroissements. Ainsi

$$\begin{aligned} \int_0^T g(s) d\beta(s) &= \lim_{n \to \infty} \int_0^T g_n(s) d\beta(s) \\ &= \lim_{n \to \infty} \left(\sum_{k=0}^{\phi(n)-1} g_n(k) (\beta(t_{k+1}) - \beta(t_k)) \right). \end{aligned}$$

Cette intégration stochastique nous permet finalement de donner un sens à la notation différentielle

$$dX = f dt + g d\beta(t), \qquad t \in I,$$

qui est alors équivalente à

$$X(t) = X(0) + \int_0^t f(s) ds + \int_0^t g(s) d\beta(s), \qquad t \in I.$$

Un processus pouvant s'écrire sous cette forme est souvent appelé semi-martingale. Ceci définit une notion de différentielle stochastique d'Itô, et il est naturel de se demander s'il est possible de calculer des différentielles de fonctions composées. C'est le résultat suivant appelé formule d'Itô.

Theorem 0.1.4.1. *Soient f et g deux processus avec f adapté et g prévisible tels que*

$$\int_0^T |f(s)|ds < \infty, \ p.s., \ et \ \int_0^T |g(s)|^2 ds < \infty \ p.s.$$

Soient X vérifiant $dX = fdt + gd\beta(t)$ sur $[0,T]$, et $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ telle que $\partial_t u, \partial_x u$ et $\partial_x^2 u$ existent et sont continues sur $[0,T] \times \mathbb{R}$. Alors

$$du(t,X(t)) = \partial_t u(t,X(t))dt + \partial_x u(t,X(t))dX(t) + \frac{1}{2}\partial_x^2 u(t,X(t))g^2(t)dt. \tag{0.1.8}$$

Cette formule est le résultat fondamental permettant de construire toute une théorie des équations différentielles stochastiques. Elle permet notamment d'obtenir un résultat du type Cauchy-Lipschitz. Pour généraliser les résultats existants à la théorie des équations aux dérivées partielles stochastiques, il faut retrouver une notion d'intégration stochastique, cette fois-ci sur un espace de dimension infinie, comme par exemple un espace de Hilbert.

On considère donc maintenant H un espace de Hilbert séparable de produit scalaire $(\cdot,\cdot)_H$ et de norme $|\cdot|_H$, une base Hilbertienne $(e_k)_{k\in\mathbb{N}}$ de H et $(\beta_k)_{k\in\mathbb{N}}$ une suite de mouvements Brownien indépendants sur $(\Omega, \mathcal{F}, \mathbb{P})$. Un processus de Wiener cylindrique dans H est formellement défini par

$$W(t,x) = \sum_{k\in\mathbb{N}} \beta_k(t)e_k(x).$$

En fait, on peut voir que cette série diverge presque sûrement dans H. La dérivée de β_k est un bruit blanc en temps, et si on se place sur $[0,\pi]$, formellement on peut écrire

$$\frac{d\beta_k}{dt} = \frac{2}{\sqrt{\pi}} \sum_{j\in\mathbb{N}} Y_{jk}\cos(jt),$$

où $(Y_{jk})_{j\in\mathbb{N}}$ est une famille de variables aléatoires gaussiennes centrées réduites indépendantes. On a donc, toujours formellement,

$$\frac{dW}{dt} = \frac{2}{\sqrt{\pi}} \sum_{j,k\in\mathbb{N}} Y_{jk}\cos(jt)e_k.$$

Si par exemple $H = L^2(O)$, O ouvert de $\mathbb{R}^d$, les fonctions $\frac{2}{\sqrt{\pi}} \cos(jt)e_k(x)$ forment une base Hilbertienne de $L^2([0,\pi] \times O)$; on voit qu'on a une généralisation à plusieurs paramètres du bruit blanc, les variables t et x jouant le même rôle. Il est naturel de dire que $\frac{dW}{dt}$ est un bruit blanc en espace et en temps. La définition rigoureuse du processus de Wiener ainsi que la construction de l'intégrale stochastique nécessite les notions d'opérateurs nucléaires et Hilbert-Schmidt.

Definition 0.1.4.2. *Soit H un espace de Hilbert séparable et $\Phi : H \to H$ un opérateur linéaire. On dit que Φ est nucléaire, ou de trace finie, si pour une base Hilbertienne $(e_k)_{k\in\mathbb{N}}$ de H on a*

$$\sum_{k\in\mathbb{N}} |(\Phi e_k, e_k)_H| < \infty.$$

Ceci est alors vrai pour toute base Hilbertienne de H et

$$Tr\ \Phi = \sum_{k\in\mathbb{N}} (\Phi e_k, e_k)_H$$

ne dépend pas de la base considérée. On appelle $Tr\ \Phi$ la trace de Φ et on note $\mathcal{L}_1(H)$ l'espace des opérateurs nucléaires sur H.

Definition 0.1.4.3. *Soient H, K deux espaces de Hilbert séparables et $\Phi : H \to K$ un opérateur linéaire. On dit que Φ est Hilbert-Schmidt si pour une base $(e_k)_{k\in\mathbb{N}}$ on a*

$$\sum_{k\in\mathbb{N}} |\Phi e_k|_K^2 < \infty.$$

Ceci est alors vrai pour toute base Hilbertienne de H et si $(f_l)_{l\in\mathbb{N}}$ est une base Hilbertienne de K on a

$$Tr\ \Phi^*\Phi = \sum_{k\in\mathbb{N}} |\Phi e_k|_K^2 = \sum_{l\in\mathbb{N}} |\Phi^* f_l|_H^2 = Tr\ \Phi\Phi^*.$$

De plus cette quantité ne dépend pas des bases considérées. On note $\mathcal{L}_2(H,K)$ l'espace des opérateurs Hilbert-Schmidt de H dans K. C'est un espace de Hilbert pour le produit scalaire

$$(\Phi, \Psi)_{\mathcal{L}_2(H,K)} = Tr\ \Psi\Phi^* = Tr\Psi^*\Phi,$$

et la norme

$$|\Phi|_{\mathcal{L}_2(H,K)} = (Tr\ \Phi\Phi^*)^{\frac{1}{2}} = (Tr\ \Phi^*\Phi)^{\frac{1}{2}}.$$

Ces définitions nous permettent de donner une vraie notion de processus de Wiener cylindrique. En effet, soit cette fois H un espace de Hilbert, et $H \subset U$ un Hilbert séparable tel que l'injection soit un opérateur Hilbert-Schmidt. Alors si on définit pour $(e_k)_{k\in\mathbb{N}}$ une base Hilbertienne de H et $(\beta_k)_{k\in\mathbb{N}}$ une suite de mouvements Browniens indépendants le processus

$$W = \sum_{k\in\mathbb{N}} \beta_k e_k,$$

on peut étudier la convergence de W et on obtient

$$\mathbb{E}\left[|\sum_{k\in\mathbb{N}} \beta_k(t) e_k|_U^2 \right] = t \sum_{k\in\mathbb{N}} |e_k|_U^2 < \infty.$$

On a utilisé l'indépendance des β_k ce qui entraine $\mathbb{E}[\beta_k(t)\beta_l(t)] = 0$ si $k \neq l$. La série converge bien dans $L^2(\Omega, U)$. Remarquons qu'il est toujours possible de construire un tel espace U.

Definition 0.1.4.4. *Soit H un espace de Hilbert séparable, $(e_k)_{k\in\mathbb{N}}$ une base Hilbertienne de H et $(\beta_k)_{k\in\mathbb{N}}$ une suite de mouvements Browniens indépendants sur $I = \mathbb{R}^+$ ou $I = [0, T]$. On dit que $(W(t))_{t\in I}$ est un processus de Wiener cylindrique sur H si, pour un espace de Hilbert U tel que $H \subset U$ avec injection Hilbert-Schmidt, W est un processus Gaussien à valeurs dans U et*

$$W(t) = \sum_{k\in\mathbb{N}} \beta_k(t) e_k, \quad \mathbb{P}\ p.s., \quad t \in I.$$

On peut montrer que cette définition ne dépend pas de l'espace U. La série définissant $W(t)$ converge presque sûrement dans U car elle est définie comme somme de variables aléatoires indépendantes à distributions symétriques. Un résultat général (voir par exemple [DPZ14, Proposition 2.11]) dit que si une telle série converge en probabilité alors elle converge presque sûrement. Ici la convergence en probabilité résulte de la convergence en moyenne quadratique montrée ci-dessus. Cette définition est également indépendante de la base Hilbertienne considérée. On peut également définir des processus de Wiener plus complexes grâce aux opérateurs Hilbert-Schmidt.

Definition 0.1.4.5. *Soient deux espaces de Hilbert H et K, W un processus de Wiener cylindrique sur H et $\Phi : H \to K$ un opérateur linéaire. On dit que $\tilde{W} = \Phi W$ est un processus de Wiener de covariance $\Phi\Phi^*$ sur K.*

Bien entendu dans cette définition ΦW doit être compris comme $\sum_{k\in\mathbb{N}} \beta_k \Phi e_k$. Cette série définissant $\tilde{W}$ est convergente presque sûrement ou en moyenne quadratique dans tout espace U tel que $K \subset U$ et $\Phi : H \to U$ est Hilbert-Schmidt. On voit en prenant $H = K$ et $\Phi = Id$ qu'un processus de Wiener cylindrique est un processus de covariance Id. Dans le cas où Φ est Hilbert-Schmidt, $\tilde{W}$ est un processus Gaussien à valeurs dans K, son opérateur de covariance est en fait donné par $t\Phi\Phi^*$, qui est bien un opérateur nucléaire.

Maintenant que l'on dispose d'une notion de processus de Wiener sur des espaces de Hilbert (et donc une notion de bruit blanc), on peut définit une intégrale stochastique comme dans le cas de la dimension 1. Pour H et K deux espaces de Hilbert, de bases Hilbertiennes respectives $(e_k)_{k\in\mathbb{N}}$ et $(f_l)_{l\in\mathbb{N}}$, on définit l'intégrale d'un opérateur $\phi = (\phi(t))_{t\in[0,T]}$ prévisible de carré intégrable à valeurs dans $\mathcal{L}_2(H, K)$ l'espace des opérateurs Hilbert-Schmidt de H dans K comme

$$\int_0^T \phi(s)dW(s) = \sum_{k,l\in\mathbb{N}} (\phi(s)e_k, f_l)d\beta_k(s)f_l.$$

Une fois cette intégrale stochastique définie, on peut là encore donner un sens à la notation différentielle

$$dX = fdt + \phi dW(t), \qquad t \in [0, T],$$

qui représente

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^s \phi(s)dW(s), \qquad t \in [0, T],$$

pour ϕ prévisible tel que $\int_0^T |\phi(s)|^2_{\mathcal{L}_2(H,K)} < \infty$ $p.s.$ On peut là encore calculer des différentielles de fonctions composées, grâce à la généralisation de la formule d'Itô dans un espace de Hilbert.

Theorem 0.1.4.2. *Soit $u : [0, T] \times K \to \mathbb{R}$ une application continue telle que les dérivées $\partial_t u, \partial_x u, \partial_x^2 u$ existent et sont uniformément continues sur les bornés de $[0, T] \times K$, alors le processus $u(t, X(t))$ a une différentielle d'Itô donnée par*

$$\begin{aligned} du(t, X(t)) = &\left(\partial_t u(t, X(t)) + \partial_x u(t, X(t))f(t) + \frac{1}{2} Tr\ (\partial_x^2 u(t, X(t))\phi(t)\phi^*(t)\right) dt \\ &+ \partial_x u(t, X(t))\phi dW(t). \end{aligned} \tag{0.1.9}$$

Fort de ces concepts d'intégration et de calcul stochastique applicables aux équations aux dérivées partielles stochastiques, on peut maintenant aborder la notion d'approximation diffusion.

0.1.5 L'approximation diffusion

La notion d'approximation diffusion vient de la difficulté à modéliser des propagations d'ondes en milieu aléatoire et à analyser des problèmes multi-échelles. En effet, dans de nombreuses situations physiques, les phénomènes étudiés ont lieu dans des milieux dont on ne possède qu'une description statistique. Les équations qui les modélisent contiennent alors des termes aléatoires, et un point important de leur étude est de comprendre les différentes échelles caractéristiques du problème. Trouver le bon rapport d'échelle permet d'observer un régime asymptotique remarquable. Par exemple, l'équation

$$\partial_t Y^\varepsilon(t) = \varepsilon m(t) F(Y^\varepsilon(t)), Y(0) = y_0$$

où m est un processus stochastique, et F une fonction décrivant l'évolution observée, possède sous de bonnes hypothèses sur F et m une limite triviale $Y(t) = y_0$ pour tout $t \leqslant T$ avec T indépendant de ε. Cette équation peut par exemple décrire l'évolution d'une particule dans un champ aléatoire. On ne décèle pas sous cette échelle de temps de phénomène remarquable, donc on ne voit pas l'évolution de la particule. Cependant, sous l'échelle de temps $t \to \frac{t}{\varepsilon^2}$, l'équation devient pour $X^\varepsilon(t) = Y^\varepsilon(\frac{t}{\varepsilon^2})$:

$$\partial_t X^\varepsilon(t) = \frac{1}{\varepsilon} m\left(\frac{t}{\varepsilon^2}\right) F(X^\varepsilon(t)), \tag{0.1.10}$$

qui elle possède une limite non triviale bien connue sous de bonnes hypothèses sur le processus m. L'influence du terme aléatoire est très importante pour voir quelle échelle de temps est adéquate pour comprendre le comportement de Y^ε.

Le principe de l'approximation diffusion est dû à un résultat bien connu de la théorie des probabilités : le Théorème Centrale Limite. Celui-ci stipule que pour une suite de variables aléatoires centrées indépendantes identiquement distribuées $(M_n)_{n\in\mathbb{N}}$, la suite $\frac{1}{\sqrt{n}}\sum_{k=0}^{n} M_k$ converge en loi vers une loi normale $\mathcal{N}(0, \mathbb{E}[M_1^2])$. Ce dernier possède une version fonctionnelle, prouvant qu'une suite de fonctions aléatoires définie par $\xi_n(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]} M_j$ converge en loi dans un certain espace vers une mouvement Brownien de variance $\mathbb{E}[M_1^2]$. On peut se ramener à la notation de notre problème en définissant par exemple le processus

$$M^\varepsilon(t) = \varepsilon \int_0^{\frac{t}{\varepsilon^2}} m(s)ds$$

où m est un processus aléatoire décrivant le milieu de propagation de l'onde étudiée. Pour simplifier, supposons que ce milieu soit discret, et défini par des variables aléatoires centrées réduites $(m_n)_{n\in\mathbb{N}}$ indépendantes identiquement distribuées. Si on regarde les accroissements du processus M^ε on a pour $t_1 < t_2$

$$M^\varepsilon(t_2) - M^\varepsilon(t_1) = \varepsilon \sum_{n=[t_1/\varepsilon^2]}^{[t_2/\varepsilon^2]} m_n + \mathcal{O}(\varepsilon).$$

Le terme principal correspond au cadre du Théorème Centrale Limite énoncé, donc les accroissements du processus M^ε convergent en loi vers une variable aléatoire Gaussienne centrée de variance correspondant à la taille de l'accroissement.

$$\lim_{\varepsilon\to 0} \mathbb{E}[(M^\varepsilon(t_2) - M^\varepsilon(t_1))^2] = (t_2 - t_1)\mathbb{E}[\nu_n^2] = t_2 - t_1.$$

Comme deux accroissements disjoints de M^ε ne contiennent que des ensemble disjoints de variables indépendantes m_n, ils sont également indépendants, et cette propriété est également valable pour le processus limite. Le processus limite M possède donc des accroissements Gaussiens indépendants centrés de variance égale à la longueur de l'accroissement. On peut montrer qu'il est également continu en t avec des trajectoires continues, M est alors le mouvement Brownien. Par un changement de variable, on voit aussi que

$$M^\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t m\left(\frac{s}{\varepsilon^2}\right) ds,$$

ce qui nous indique que le processus $\frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2})$, que l'on retrouve dans l'équation (0.1.10), converge en loi vers un bruit blanc. La théorie peut être étendue sur des cas plus généraux que cet exemple, et des résultats précis peuvent être retrouvés dans [FGPS07].

Une fois qu'on peut intuiter la bonne échelle de temps grâce à ce résultat de convergence en loi, il convient de trouver l'équation limite satisfaite par le processus limite. Pour cela on a recours à la méthode de la Fonction Test Perturbée. Cette méthode est basée sur l'utilisation d'équations de Poisson et de l'alternative de Fredholm pour passer à la limite dans le problème martingale de l'équation. Pour cela on applique le générateur infinitésimal $\mathcal{L}^\varepsilon$ du processus $(X^\varepsilon, m(\frac{\cdot}{\varepsilon^2}))$ à une fonction connue φ, que l'on corrige par plusieurs (généralement 2) fonctions, afin de faire disparaitre les singularités en ε. On obtient une nouvelle fonction $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ et on est capable d'identifier une limite $\mathcal{L}^\varepsilon\varphi^\varepsilon \to \mathcal{L}\varphi$ quand ε tend vers 0. Ce générateur limite nous donne alors sous de bonnes hypothèses l'équation satisfaite par le processus limite X. Cette méthode fut introduite initialement par Papanicolaou, Strook et Varadhan [PSV77], et on peut la trouver de manière détaillée dans le livre de Fouque, Garnier, Papanicolaou et Solna [FGPS07] ou encore dans la thèse de Renaud Marty [Mar05]. De notre côté nous l'appliquerons de manière détaillée directement sur nos équations dans les différents chapitres de la thèse.

Voici donc le contexte dans lequel s'inscrit cette thèse. Nous allons travailler sur des problèmes d'approximation diffusion liés au système de Zakharov stochastique et à l'équation de Schrödinger non-linéaire stochastique.

0.2 Contenu de la thèse

Nous allons maintenant introduire les résultats obtenus au cours de cette thèse. Ces résultats seront détaillés dans les prochains chapitres.

0.2.1 Approximation diffusion pour l'équation de Schrödinger non linéaire avec potentiel aléatoire sur $\mathbb{R}^n$

Dans ce chapitre on étudie un problème d'approximation diffusion sur l'équation de Schrödinger cubique :

$$i\partial_t X^\varepsilon(t) = -\Delta X^\varepsilon(t) + \lambda|X^\varepsilon(t)|^{2\sigma}X^\varepsilon(t) + \frac{1}{\varepsilon}X(t)m\left(\frac{t}{\varepsilon^2}\right), \qquad (0.2.1)$$

sur $\mathbb{R}^n$, avec une donnée initiale régulière. Le processus m est un processus de Markov ergodique, à valeurs dans un espace $E = H^s(\mathbb{R}^n, \mathbb{R})$ pour $s > \frac{n}{2} + 3$. Dans beaucoup d'articles une hypothèse importante sur ce processus était la nécessité qu'il soit uniformément borné, afin d'obtenir des estimations a priori. Dans ce chapitre on se défait de cette hypothèse pour seulement supposer que ce processus possède suffisamment de moments. Cela introduit quelques nouvelles difficultés : par exemple lors de la méthode de la Fonction Test Perturbée, les correcteurs que l'on introduit ne peuvent plus être bornés uniformément en ε. On se place ici dans un cadre de solutions à valeurs dans l'espace de

Sobolev $H^1(\mathbb{R}^n)$, en supposant que le terme non-linéaire est sous-critique, c'est à dire que

$$0 \leqslant \sigma < \frac{2}{n-2}, n \geqslant 3, \text{ ou } 0 \leqslant \sigma, n = 1, 2.$$

Dans ce chapitre, on choisit de travailler dans des espaces de Sobolev à poids, notamment pour obtenir des résultats de tension, aussi on définit pour $0 < \zeta \leqslant 1$ l'espace

$$\Sigma^\zeta = \left\{ u \in H^1(\mathbb{R}^n, \mathbb{C}; \langle x \rangle^\zeta u \in L^2(\mathbb{R}^n, \mathbb{C}) \right\}, \tag{0.2.2}$$

muni de la norme $\|u\|_{\Sigma^\zeta} = \|u\|_{H^1} + \left\| \langle x \rangle^\zeta u \right\|_{L^2}$. Le résultat de ce chapitre est le suivant.

Theorem 0.2.1.1. *Pour tous $T > 0$ et $0 < \zeta \leqslant 1$, supposons que X_0^ε converge en loi vers X_0 dans l'espace Σ^ζ, voir (0.2.2), alors pour tout $s \in [0, 1)$ le processus X^ε à valeurs dans $C([0, T]; H^s)$, solution de (0.2.1) avec la donnée initiale X_0^ε converge en loi dans $C([0, T], H^s)$ vers X solution de*

$$idX = \Big(- \Delta X + \lambda |X|^{2\sigma} X - \frac{i}{2} F X \Big) dt - X Q^{1/2} dW \tag{0.2.3}$$

avec la donnée initiale X_0, où F, Q sont définis respectivement par (0.2.4) et (0.2.5), et où W est un processus de Wiener cylindrique sur L^2.

Pour prouver ce théorème, on se sert de la méthode de la Fonction Test Perturbée afin de passer à la limite dans le problème martingale satisfait par le processus X^ε. Dans ce chapitre on ne possède pas d'information sur l'évolution du processus m^ε. On suppose alors qu'il existe $\gamma > 6$ et une constante $\mathcal{C}$ telle que

$$\mathbb{E} \left[\sup_{t \in [0,1]} \|m(t)\|_E^\gamma \right] \leqslant \mathcal{C}.$$

On définit également le noyau de covariance

$$k(x, y) = \int_0^\infty \mathbb{E} \left[m(0)(x) m(t)(y) \right] dt \text{ pour } x, y \in \mathbb{R}^n,$$

qui nous permet de définir pour $x \in \mathbb{R}^n$

$$F(x) = k(x, x) \tag{0.2.4}$$

ainsi que l'opérateur

$$Qf(y) = Re \int_{\mathbb{R}^n} k(x,y)f(x)dx \tag{0.2.5}$$

pour $f \in H^1(\mathbb{R}^n, \mathbb{C})$. Avant de s'intéresser à la limite $\varepsilon \to 0$, on commence par montrer un résultat d'existence ainsi qu'une estimation d'énergie, car la conservation de l'énergie (0.1.4) dans le cas déterministe ne tient plus lorsque l'on ajoute une perturbation aléatoire.

Proposition 0.2.1.1. *Soit $X_0^\varepsilon \in H^1$. Alors, presque sûrement, il existe une unique solution X^ε de (0.2.1) avec la donnée initiale X_0^ε, à valeurs dans $C([0,T], H^1(\mathbb{R}^n) \cap L^r(0,T; W^{1,2\sigma+2}(\mathbb{R}^n))$, $r = \frac{4(\sigma+1)}{n\sigma}$, pour tout T. De plus, on a $\mathbb{P}-p.s.$ pour $0 \leqslant t \leqslant T$:*

$$\|X^\varepsilon(t)\|_{L^2} = \|X_0^\varepsilon\|_{L^2} \quad \textit{and} \quad \|\nabla X^\varepsilon(t)\|_{L^2} \leqslant \mathcal{C}_T e^{\mathcal{C}_{T,\varepsilon} \sup_{s\in[0,t]} \left\|m(\frac{s}{\varepsilon})\right\|_E} \left(\mathcal{E}(X_0^\varepsilon) + c_\lambda \|X_0^\varepsilon\|_{L^2}^{p_n}\right), \tag{0.2.6}$$

avec $p_n = (2\sigma + 2 - \sigma n)\frac{2}{2-\sigma n}$, $c_\lambda = -\lambda$ si $\lambda < 0$ et $c_\lambda = 0$ pour $\lambda \geqslant 0$. Enfin, si X_0^ε est $\mathcal{F}_0$-mesurable alors le processus X^ε est $(\mathcal{F}_t^\varepsilon)$-adapté.

Cette proposition nous permet, outre le fait de s'assurer que notre problème est bien posé, d'obtenir un contrôle de l'évolution des normes du processus X^ε et de son gradient, en fonction de la donnée initiale et de l'évolution du processus $m(\frac{t}{\varepsilon})$, qui est d'ordre $\varepsilon^{-\alpha}$ avec $\alpha > 0$ aussi petit que l'on souhaite. La preuve de ce résultat consiste en un principe de point fixe sur la formulation intégrale de (0.2.1) afin d'assurer le caractère bien posé et en une formule d'Itô sur l'énergie évaluée en une régularisation de X^ε, $X_\delta^\varepsilon = \rho_\delta * X^\varepsilon$ avec ρ une approximation de l'unité, couplée à un lemme de Grönwall.

On a aussi besoin de s'assurer que le processus X^ε reste bien dans l'espace Σ^ζ si la donnée initiale appartient à Σ^ζ. C'est ce qu'assure le résultat suivant, qui se démontre en appliquant la formule d'Itô à la fonctionnelle

$$\varphi_\delta(u) = \int_{\mathbb{R}^n} \langle x\rangle^{2\zeta} |u(x)|^2 e^{-\delta\langle x\rangle} dx.$$

Proposition 0.2.1.2. *Supposons que $X_0^\varepsilon \in \Sigma^\zeta$ pour $\zeta \leqslant 1$, alors $X^\varepsilon \in C([0,T], \Sigma^\zeta)$ presque sûrement pour tout $T > 0$. Plus précisément, on a*

$$\left\|X^\varepsilon(t)\langle x\rangle^\zeta\right\|_{L^2}^2 \leqslant \mathcal{C}\left(\left\|X_0^\epsilon\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t \|\nabla X^\varepsilon(s)\|_{L^2}^2 \, ds\right). \tag{0.2.7}$$

On peut ensuite commencer à mettre en place la méthode de la Fonction Test Perturbée. On note $\mathcal{M}$ le générateur infinitésimal du processus m, et $\mathcal{L}^\varepsilon$ celui du couple

$(X^\varepsilon, m(\frac{\cdot}{\varepsilon}))$. Dans cette thèse, on utilise la notion de générateur au sens où pour des fonctions test bien choisies φ le processus

$$\varphi(m_t) - \varphi(m_0) - \int_0^t \mathcal{M}\varphi(m_s)ds$$

est une martingale. Une question cruciale est de pouvoir inverser le générateur $\mathcal{M}$. Contrairement au problème sur Zakharov où on connaissait l'équation satisfaite par z^ε, et où l'on pouvait donc redéfinir par la formule d'Itô $\mathcal{M}^{-1}\varphi$, ici on doit s'assurer qu'il est bien possible de définir une telle quantité. On a donc besoin de définir une bonne classe de fonctions test sur lesquelles on pourra inverser le générateur $\mathcal{M}$.

Definition 0.2.1.1 (Bonne fonction test)**.** *On dit que $\phi : H^1_{\mathbb{C}} \times E \longrightarrow \mathbb{R}, (u,n) \mapsto \Psi(u,n)$ est une bonne fonction test si elle vérifie les propriétés suivantes :*

— *Ψ est continûment différentiable par rapport à u, la différentielle est notée $D\Psi$.*

— *Ψ est sous-polynomiale en u et n ; $\exists\, C_\Psi, \exists\, p_1 \in \mathbb{N}, \exists\, \gamma_1 < \gamma, \forall (u,n) \in H^1 \times E$,*

$$|\Psi(u,n)| \leqslant C_\Psi(1 + \|u\|_{H^1}^{p_1})(1 + \|n\|_E^{\gamma_1}).$$

— *$(u,n) \mapsto D\Psi(u,n)$ est continue de $H^1 \times E$ dans $\mathcal{L}(H^1_{\mathbb{C}}; \mathbb{C})$.*

— *$D\Psi(u,n)$ est continue par rapport à la norme H^{-1} et vérifie la borne sous-polynomiale en u suivante : $\exists\, C_\Psi, \exists\, p_2 \in \mathbb{N}, \exists\, \gamma_2 < \gamma - 1, \forall (u,h,n) \in H^1 \times H^{-1} \times E$,*

$$|D\Psi(u,n).(h)| \leqslant C_\psi(1 + \|u\|_{H^1})^{p_2}(1 + \|n\|_E^{\gamma_2}) \|h\|_{H^{-1}}.$$

— *$\forall u \in H^1_{\mathbb{C}}, \Psi(u,.) \in \mathcal{D}_{\mathcal{M}}$ où $\mathcal{D}_{\mathcal{M}}$ est le domaine de l'opérateur $\mathcal{M}$.*

— *$\mathcal{M}\Psi$ est continu et sous-polynomial en u et n ; $\exists\, C_\Psi, \exists\, p_3 \in \mathbb{N}, \exists\, \gamma_3 < \gamma, \forall (u,n) \in H^1 \times E$,*

$$|\mathcal{M}\Psi(u,n)| \leqslant C_\Psi(1 + \|u\|_{H^1}^{p_3})(1 + \|n\|_E^{\gamma_3}).$$

Pour une bonne fonction test φ, on est alors capable de calculer $\mathcal{L}^\varepsilon\varphi$:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi(u,n) = &\frac{1}{\varepsilon^2}\mathcal{M}\varphi(u,n)\\ &+ \frac{1}{\varepsilon}(D\varphi(u,n).(-iun))\\ &+ D\varphi(u,n).(i\Delta u - i\lambda|u|^{2\sigma}u).\end{aligned}$$

Cette expression fait intervenir des termes d'ordre négatifs en ε. Si on suppose que la fonction φ ne dépend que de u, le terme d'ordre -2 est alors nul. On corrige notre fonction test avec deux autres fonctions φ_1 et φ_2, qui sont aussi des bonnes fonctions test, et on obtient :

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2)(u,n) =& \frac{1}{\varepsilon^2}\mathcal{M}\varphi(u) \\ &+ \frac{1}{\varepsilon}\left(D\varphi(u).(-iun) + \mathcal{M}\varphi_1(u,n)\right) \\ &+ \mathcal{M}\varphi_2(u,n) + D\varphi_1(u,n).(-iun) + D\varphi(u).(i\Delta u - i\lambda|u|^{2\sigma}u) \\ &+ \varepsilon\left(D\varphi_2(u).(-iun) + D\varphi_1(u,n).(i\Delta u - i\lambda|u|^{2\sigma}u)\right) \\ &+ \varepsilon^2 D\varphi_2(u,n).(i\Delta u - i\lambda|u|^{2\sigma}u).\end{aligned}$$

On applique alors cette méthode à $\varphi(u) = (u,h)^\ell$ pour $h \in H^1(\mathbb{R}^n)$ et $\ell = 1, 2$. Un bon choix pour φ_1 et φ_2 nous permet de définir un générateur limite $\mathcal{L}$:

$$\mathcal{L}\varphi(u) = D\varphi(u)\cdot(i\Delta u - i\lambda|u|^{2\sigma}u) + \mathbb{E}\nu\left(D^2\varphi(u)\cdot(-iun, iu\mathcal{M}^{-1}n) + D\varphi(u)\cdot(un\mathcal{M}^{-1}n)\right), \tag{0.2.8}$$

où $\mathcal{M}^{-1}n$ est un abus de notation correspondant à $\mathcal{M}^{-1}Id(n)$. Par définition du générateur infinitésimal $\mathcal{L}^\varepsilon$, on sait que

$$\varphi^\varepsilon(X_t^\varepsilon, m_t^\varepsilon) - \varphi^\varepsilon(X_0, m_0) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X_s^\varepsilon, m_s^\varepsilon)ds$$

est une martingale. On a besoin de résultats de tension sur le processus X^ε, qui sont obtenus notamment grâce au critère d'Aldous couplé à la méthode de la Fonction Test Perturbée. Une fois ces résultats démontrés, on peut se placer dans l'espace de Skorohod et passer à la limite dans le problème martingale satisfait par X^ε et d'identifier l'opérateur $\mathcal{L}$ comme le générateur infinitésimal du processus limite. On conclut ainsi la preuve du Théorème 1.1.1.

0.2.2 Étude du système de Zakharov stochastique, et convergence vers une équation de Schrödinger non linéaire stochastique

Dans ce deuxième chapitre on étudie le système de Zakharov introduit dans la section 0.1.3, que l'on perturbe par l'ajout d'un terme stochastique. Le système devient

$$\begin{cases} i\partial_t u = -\partial_x^2 u + nu, \\ \varepsilon^2 d(\partial_t n) = \partial_x^2(n + |u|^2)dt + \phi dW(t), \end{cases} \tag{0.2.9}$$

où $W(t)$ est un processus de Wiener cylindrique et ϕ un opérateur Hilbert-Schmidt. De manière formelle, si on étudie la limite quand ε tend vers 0, on récupère

$$n = -|u|^2 - (\partial_x^2)^{-1}\phi\frac{dW}{dt},$$

et en réinjectant cette égalité dans l'équation satisfaite par u on obtient une équation de Schrödinger stochastique cubique. On s'intéresse donc au caractère bien posé du problème, ainsi qu'à la limite $\varepsilon \to 0$.

On remarque cependant rapidement qu'il est nécessaire de rajouter un amortissement à notre système, afin de récupérer la convergence. Par ailleurs, l'ajout de cet amortissement est aussi naturel pour des questions physiques, comme par exemple dans [GF06]. On introduit donc le système

$$\begin{cases} i\partial_t u = -\partial_x^2 u + nu, \\ \varepsilon^2 d(\partial_t n) + \alpha\varepsilon\partial_t n = \partial_x^2(n + |u|^2)dt + \phi dW(t) \end{cases}. \tag{0.2.10}$$

En effet, des simulations numériques sur un exemple plus simple :

$$\begin{cases} \partial_t X_t = -X_t n_t \\ \varepsilon^2 d(\partial_t n_t) + \alpha\varepsilon\partial_t n_t = -\lambda n_t dt + d\beta_t \end{cases} \tag{0.2.11}$$

nous montrent que la convergence ne se fait pas sans amortissement. Sur cet exemple simple on sait que grâce à un changement d'échelle on peut identifier l'équation limite qui est

$$dX_t = -\frac{1}{\lambda}X_t \circ d\beta_t = \frac{1}{2\lambda^2}dt - \frac{1}{\lambda}d\beta_t \tag{0.2.12}$$

dont on connait la solution explicite : $X_t = e^{-\frac{\beta_t}{\lambda}}$. Si on suppose $\alpha = 0$ et que l'on calcule

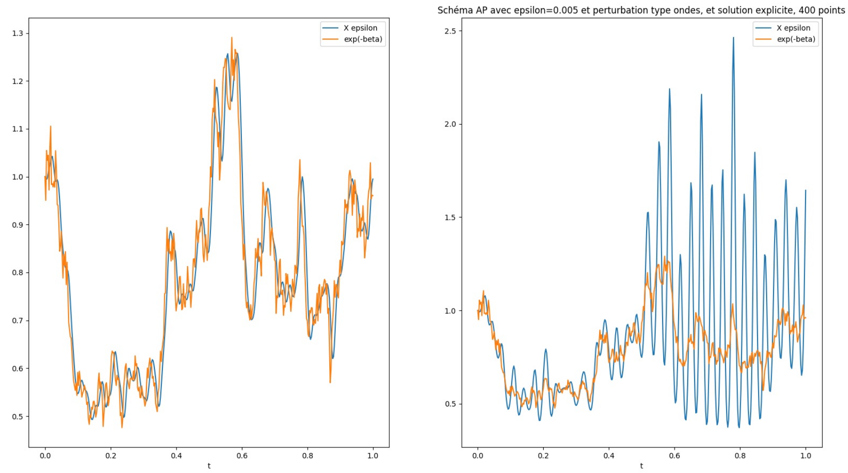

FIGURE 1 – Tracés pour $\varepsilon = 0.005$ des solutions de l'équation (0.2.11) pour $\alpha = 1$ puis $\alpha = 0$, et comparaison avec la solution de l'équation limite (0.2.12)

l'intégrale en temps de la solution n_t de (0.2.11), on obtient

$$\int_0^t n_s ds = \frac{1}{\lambda}\beta_t - \frac{1}{\lambda}\int_0^t \cos\left(\frac{\sqrt{\lambda}}{\varepsilon}(t-s)\right) d\beta_s,$$

et lorsque quon calcule la covariance on obtient

$$\mathbb{E}\left[\int_0^t \cos\left(\frac{\sqrt{\lambda}}{\varepsilon}(t-s)\right) d\beta_s \int_0^{t'} \cos\left(\frac{\sqrt{\lambda}}{\varepsilon}(t'-s)\right) d\beta_s\right] = \frac{1}{2}(t \wedge t') \cos\left(\frac{\sqrt{\lambda}}{\varepsilon}(t-t')\right)$$
$$+ \frac{\varepsilon}{4\sqrt{\lambda}}\left(\sin\left(\frac{\sqrt{\lambda}}{\varepsilon}(t-t')\right) - \sin\left(\frac{\sqrt{\lambda}}{\varepsilon}(t+t'-2(t\wedge t'))\right)\right),$$

qui oscille fortement quand ε tend vers 0. Sur la figure 1, on voit que si l'amortissement est présent la convergence semble se passer sans accroc, tandis que le comportement général de la solution de l'équation sans amortissement semble suivre la solution limite, mais avec de fortes oscillation autour de cette dernière.

Ce problème vient du fait que sans amortissement, le processus n_t ne converge pas vers un bruit blanc. En effet, en approximation diffusion le résultat clé est le fait que l'intégrale en temps de ce processus converge bien vers un mouvement Brownien, or si on trace le processus n_t avec et sans amortissement, comme illustré Figure 2 et qu'on le compare à un mouvement Brownien, on observe le même problème, qui se transmet alors à la solution de l'équation. L'influence de l'amortissement sur cette équation différentielle stochastique

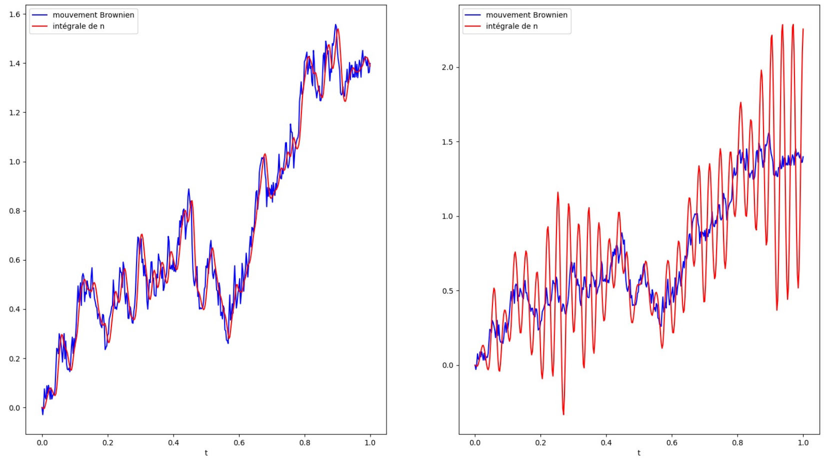

FIGURE 2 – Tracés pour $\varepsilon = 0.005$ de l'intégrale de n_t pour $\alpha = 1$ puis $\alpha = 0$, et comparaison avec le mouvement Browien limite (0.2.12)

nous donne donc une idée de ce qui peut se passer, du moins dans le cas linéaire, pour l'équation aux dérivée partielles au niveau de ses fréquences de Fourier. C'est pourquoi dans le chapitre 2 on considère directement le système (0.2.10). On montre d'abord grâce à un argument de point fixe le résultat suivant.

Theorem 0.2.2.1. *Soit* $\alpha \geqslant 0$ *et* $\phi \in \mathcal{L}^2(L^2 \cap \dot{H}^{-1})$. *Soit* $(u_0, n_0, n_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$. *Alors il existe une unique solution* $(u, n, \partial_t n)$ *de* (0.2.10) *avec des trajectoires continues à valeurs dans* $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ *telle que* $(u(0), n(0), \partial_t n(0)) = (u_0, n_0, n_1)$. *Cette solution est définie sur un intervalle aléatoire* $[0, T(\omega))$ *où* $T(\omega) > 0$ *est un temps d'arrêt tel que*

$$T(\omega) = +\infty \qquad ou \qquad \lim_{t \to T(\omega)} \max\left(\|u(t)\|_{H^2}, \|n(t)\|_{H^1}\right) = +\infty \ p.s.$$

Et en réitérant l'argument de point fixe sur les dérives en temps des équations du système (0.2.10), on montre qu'on peut gagner en régularité.

Theorem 0.2.2.2. *Soit* $\phi \in \mathcal{L}^2(H^1 \cap \dot{H}^{-1})$ *et soit* $(u, n, \partial_t n)$ *la solution de* (0.2.10) *donnée par le Théorème 0.2.2.1, et* $T(\omega)$ *le temps d'arrêt défini dans ce théorème. Alors si* $u_0 \in H^3(\mathbb{R}), n_0 \in H^2(\mathbb{R}), n_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$, *on a* $(u, n, \partial_t n)(\omega) \in C([0, T(\omega)), H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R}))$.

Les preuves de ces théorèmes suivent un schéma classique, inspiré de l'article de Kato [Kat87], et sont détaillées dans le chapitre 1 de la thèse. Elles résident notamment dans l'analyse de l'opérateur des ondes $S_\alpha(t)$ qui permet d'écrire l'équation sur n, où l'on a posé $\varepsilon = 1$ car on ne s'intéresse pas encore à la convergence, sous forme intégrale :

$$\begin{pmatrix} n(t) \\ \partial_t n(t) \end{pmatrix} = S_\alpha(t)\begin{pmatrix} n_0, n_1 \end{pmatrix} + \int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ \partial_x^2 |u(s)|^2 \end{pmatrix} ds + \int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ \phi \end{pmatrix} dW(s).$$

L'étude de cet opérateur est assez technique. C'est un multiplicateur de Fourier, ce qui nous permet de regarder facilement son comportement à l'aide de sa transformée de Fourier. Le principal problème est la dépendance en α de cet opérateur, qui nous contraint à décomposer notre étude en plusieurs domaines de fréquence.

Une fois ces résultats d'existence et d'unicité locale prouvés, on s'intéresse à récupérer un résultat plus global. On prouve alors le théorème suivant.

Theorem 0.2.2.3. *Soit $\alpha \geqslant 0$, $T > 0$ et soit $\phi \in \mathcal{L}_2(H^1 \cap \dot{H}^{-1})$. Pour toutes données initiales $u_0 \in H^3$, $n_0 \in H^2$ et $n_1 \in H^1 \cap \dot{H}^{-1}$, il existe presque sûrement une unique solution $(u, n) \in L^\infty([0, T], H^3 \times H^2)$ pour le système (0.2.10).*

Pour prouver ce théorème on décompose notre solution $n(t)$ de la façon suivante : $n(t) = m(t) + Z^\varepsilon(t)$ où les nouvelles variables sont les solutions de :

$$\varepsilon^2 \partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2\left(m + |u|^2\right),$$

$$\varepsilon^2 d(\partial_t Z^\varepsilon) + \alpha\varepsilon\partial_t Z^\varepsilon = \partial_x^2 Z^\varepsilon dt + \phi dW(t).$$

Ainsi tout le caractère stochastique de l'équation est portée par la variable Z^ε tandis que m satisfait l'équation des ondes initialement présente dans Zakharov (en ajoutant l'amortissement). Le système (0.2.10) se réécrit alors :

$$\begin{cases} i\partial_t u = -\partial_x^2 u + (m + Z^\varepsilon)u, \\ \\ \varepsilon^2\partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2(m + |u|^2). \end{cases} \tag{0.2.13}$$

On précise que la décomposition $n^\varepsilon = m^\varepsilon + Z^\varepsilon$ nous donne une décomposition des données initiales $m_0 = n_0 - Z_0^\varepsilon$, $m_1 = n_1 - Z_1^\varepsilon$. On choisit alors Z_0^ε et Z_1^ε de manière à ce que le processus Z^ε soit stationnaire, c'est-à-dire

$$(Z_0^\varepsilon, Z_1^\varepsilon) = \frac{1}{\varepsilon}\int_0^\infty S_\alpha\left(\frac{s}{\varepsilon}\right)\begin{pmatrix}0\\ \phi dW_s\end{pmatrix}.$$

Une fois cette réécriture effectuée, on essaye d'adapter les méthodes de l'article de Sulem et Sulem [SS79]. Cet article se base sur l'estimation d'énergie valable pour le système de Zakharov (0.1.7) :

$$\partial_t H(u,m) = \partial_t\left(\|\partial_x u\|_{L^2}^2 + \frac{1}{2}(\|n\|_{L^2}^2 + \|\varepsilon V\|_{L^2}^2) + \int_{\mathbb{R}} n|u|^2 dx\right) = 0, \qquad (0.2.14)$$

où V est la solution de l'équation

$$\partial_t m + \partial_x V = 0,$$

soit encore $V = -(\partial_x)^{-1}\partial_t m$, ce qui justifie l'utilisation de l'espace de Sobolev homogènes $\dot{H}^1$ présents dans les énoncés des théorèmes. Dans notre cas, cette énergie n'est plus conservée, et on commence par montrer l'égalité :

$$\partial_t H(u,m) + \int_{\mathbb{R}} Z^\varepsilon \partial_t |u|^2 dx = -\alpha\varepsilon \|V\|_{L^2}^2. \qquad (0.2.15)$$

On suit ensuite le schéma de preuve de [SS79] en gérant les termes supplémentaires dus à (0.2.15).

Les questions d'existence et d'unicité étant maintenant réglées, on s'intéresse alors à la convergence de notre système. On énonce le théorème principal de ce chapitre.

Theorem 0.2.2.4. *Soit $\alpha > 0$. Pour tout $T > 0$, et pour $u_0 \in H^3(\mathbb{R}), m_0 \in H^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}), m_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$, le processus u^ε, solution du système (0.2.13) avec $\phi \in \mathcal{L}_2(L^2(\mathbb{R}), H^3(\mathbb{R}) \cap \dot{H}^{-4}(\mathbb{R}))$, converge en loi dans $C([0,T], H^s_{loc}(\mathbb{R}))$ pour $s < 1$ vers l'unique solution u de*

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2}uF)dt - u(\partial_x^2)^{-1}\phi dW_t, \qquad (0.2.16)$$

où $F(x) = \sum_{k\in\mathbb{N}} \left((\partial_x^2)^{-1})\phi e_k\right)^2(x)$, la famille $(e_k)_{k\in\mathbb{N}}$ étant une base orthonrmée de $L^2(\mathbb{R})$.

La preuve de ce théorème constitue l'essentiel du chapitre 1. La première difficulté réside dans le fait que l'énergie H évolue de manière singulière en ε, donc on ne peut pas en tirer des estimations indépendantes de ε. Pour résoudre ce problème, on se place dans le cadre de l'approximation diffusion en changeant l'échelle de temps de notre système. En effet, on remarque que le processus Z^ε est d'ordre $\varepsilon^{-\frac{1}{2}}$ et l'équation qu'il satisfait indique qu'il dépend de t comme fonction de $\frac{t}{\varepsilon}$. On définit alors la variable $\tilde{z}^\varepsilon(t) = \varepsilon^{\frac{1}{2}} Z^\varepsilon(\varepsilon t)$. Le système (0.2.13) se réécrit alors

$$\begin{cases} i\partial_t u = -\partial_x^2 u + mu + \frac{1}{\sqrt{\varepsilon}} \tilde{z}^\varepsilon\left(\frac{t}{\varepsilon}\right) u, \\ \\ \varepsilon^2 \partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2(m + |u|^2). \end{cases} \tag{0.2.17}$$

où $\tilde{z}^\varepsilon$ est solution de l'équation aux dérivées partielles stochastique

$$\begin{cases} \partial_t \tilde{z}^\varepsilon = \tilde{\zeta}^\varepsilon, \\ d\tilde{\zeta}^\varepsilon = (-\alpha\tilde{\zeta}^\varepsilon + \partial_x^2 \tilde{z}^\varepsilon)dt + \phi d\tilde{W}_t^\varepsilon. \end{cases} \tag{0.2.18}$$

Ce processus a une loi indépendante de ε. En choisissant bien la donnée initiale de $\tilde{z}^\varepsilon$, le processus peut être pris stationnaire, avec suffisamment de moments. On est donc bien ici dans une situation classique d'approximation diffusion, pour lesquelles la méthode de la Fonction Test Perturbée est un outil puissant. Une fois ce changement d'échelle effectué on peut appliquer cette méthode à l'énergie H pour en tirer une estimation sur la quantité

$$K = \frac{1}{2}\left\|\partial_x u\right\|_{L^2}^2 + \frac{1}{4}\left\|m\right\|_{L^2}^2 + \frac{1}{2}\left\|\varepsilon V\right\|_{L^2}^2. \tag{0.2.19}$$

qui est une fonction positive. On montre ainsi que pout tout $T > 0$, il existe un temps d'arrêt τ^ε qui tend presque sûrement vers T quand ε tend vers 0, et une constante $C(T)$ indépendante de ε tels que

$$\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(K(t))^2\right] \leqslant C(T).$$

Grâce à cette estimation, on peut obtenir des résultats de tension sur nos processus en utilisant le critère d'Aldous couplé à la méthode de la Fonction Test Perturbée. Ces propriétés de tension nous permettent en passant dans l'espace de Skorohod d'obtenir des convergences faibles, en s'appuyant sur l'article [AA88] d'Added et Added, où les auteurs peuvent directement utiliser la conservation de l'énergie pour obtenir des bornes indépendantes de ε.

Avec ces résultats de convergences faibles, on peut finalement utiliser la méthode de la Fonction Test Perturbée afin d'exhiber une équation limite. On applique cette méthode à des fonctions $\varphi(u) = (u,h)^\ell$ avec $\ell = 1,2$ et h une fonction H^1 à support compact. Le cas $\ell = 1$ nous permet de montrer que

$$\varphi(u_t) - \varphi(u_0) - \int_0^t \mathcal{L}\varphi(u_s)ds,$$

où $\mathcal{L}$ est le générateur infinitésimal du processus limite u, est une martingale continue tandis que le cas $\ell = 2$ permet de calculer sa variation quadratique et de conclure grâce à un théorème de représentation martingale, que l'on peut trouver dans [DPZ14, Théorème 8.2]. Le générateur infinitésimal du couple $(u^\varepsilon, z^\varepsilon)$ appliqué à φ nous donne

$$\mathcal{L}^\varepsilon\varphi(\tilde{u}^\varepsilon) = \ell(\tilde{u}^\varepsilon,h)^{\ell-1}(i\partial_x^2\tilde{u}^\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon,h) - \frac{\ell}{\sqrt{\varepsilon}}(\tilde{u}^\varepsilon,h)^{\ell-1}(iz^\varepsilon\tilde{u}^\varepsilon,h).$$

On corrige alors φ avec deux fonctions de manière à supprimer les termes singuliers dans l'expression précédente, et on applique le générateur à $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$. On sait alors que

$$\varphi^\varepsilon(u_t^\varepsilon) - \varphi^\varepsilon(u_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(u_s^\varepsilon)ds$$

est une martingale continue. L'expression de $\mathcal{L}^\varepsilon$ nous permet d'identifier le générateur limite $\mathcal{L}$, et les résultats de tension et de convergences faibles nous permettent de passer à la limite dans le problème martingale. On a ainsi un résultat de convergence en loi du système de Zakharov stochastique avec un terme d'amortissement vers l'équation de Schrödinger stochastique cubique. On peut néanmoins s'interroger sur le lien entre le processus de Wiener dirigeant le système de Zakharov et celui dirigeant l'équation limite. Le dernier résultat du chapitre nous assure que ce sont bien les mêmes, car on peut en réalité obtenir une convergence en probabilité.

Theorem 0.2.2.5. *Soit $\alpha > 0$. Pour tout $T > 0$, et pour $(u_0, m_0, m_1) \in H^3(\mathbb{R}) \times (H^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})) \times (H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}))$, le processus u^ε solution du système (0.2.10) avec $\phi \in \mathcal{L}_2(L^2(\mathbb{R}), H^3(\mathbb{R}) \cap \dot{H}^{-4}(\mathbb{R}))$ converge en probabilité dans $C([0,T], H^s_{loc}(\mathbb{R}))$ pour $s < 1$ vers u solution de*

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2}uF)dt - u(\partial_x^2)^{-1}\phi dW_t. \tag{0.2.20}$$

où $F(x) = \sum_{k\in\mathbb{N}}\left((\partial_x^2)^{-1})\phi e_k\right)^2(x)$, et W_t est le processus de Wiener cylindrique introduit dans (0.2.10).

Pour prouver ce résultat, on se base sur un résultat de Gyöngy et Krylov [GK96], pour obtenir la convergence en probabilité de nos processus, et ainsi pouvoir passer à la limite directement dans le problème martingale. En effet, la connaissance de l'équation satisfaite par le processus z^ε nous permet de connaitre explicitement la martingale

$$\varphi^\varepsilon(u_t^\varepsilon) - \varphi^\varepsilon(u_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(u_s^\varepsilon) ds,$$

ce qui n'était pas nécessaire pour prouver la convergence en loi grâce au théorème de représentation martingale. L'argument mis en place consiste à prendre deux sous-suites quelconques du processus $(u^\varepsilon)_{\varepsilon>0}$ et de montrer que le couple constitué de ces deux sous-suites converge en loi vers une mesure supportée sur la diagonale. Le théorème de Skorohod nous permet d'obtenir ce résultat, car on a déjà montré la tension de $(u^\varepsilon)_{\varepsilon>0}$, et dans l'espace de Skorohod on peut utiliser la convergence presque sûre des copies de nos sous-suites pour conclure. Ensuite le résultat de Gyongy et Krylov nous assure la convergence en probabilité du processus $(u^\varepsilon)_{\varepsilon>0}$, ce qui permet de passer à la limite $\varepsilon \to 0$ dans l'expression explicite de la martingale et ainsi d'identifier le processus de Wiener limite.

0.2.3 Schémas numériques conservant l'asymptotique

Dans ce dernier chapitre on s'intéresse à des problèmes d'analyse numérique liés à ces problèmes d'approximation diffusion, et plus particulièrement à des schémas numériques préservant l'asymptotique ("asymptotic preserving" en anglais, ou plus simplement "AP"). On s'appuie sur l'article de Bréhier et Rakotonirina-Ricquebourg [BRR22] qui définit la notion de schéma AP pour une équation différentielle stochastique. Le but de ce chapitre est de construire un schéma numérique pour modéliser le système de Zakharov stochastique, et qui soit consistant avec la limite $\varepsilon \to 0$. Un schéma AP est une discretisation $(X_k^\varepsilon)_{k\in[\![0,N]\!]}$ de la solution $(X_t^\varepsilon)_{t\in[0,T]}$ d'une équation, qui converge quand ε tend vers 0. La propriété d'être AP consiste en le fait que le schéma pour $(X_k^\varepsilon)_{k\in[\![0,N]\!]}$ converge lui aussi quand ε tend vers 0, et que le schéma limite permet de définir une bonne discrétisation $(X_k)_{k\in[\![0,N]\!]}$ de la solution de l'équation limite $(X_t)_{t\in[0,T]}$. Cette propriété peut être illustrée par le diagramme suivant.

$$\begin{tikzcd} X_k^\varepsilon \arrow[r, "N\to\infty"] \arrow[d, "\varepsilon\to 0"'] & X_t^\varepsilon \arrow[d, "\varepsilon\to 0"] \\ X_k \arrow[r, "N\to\infty"'] & X_t \end{tikzcd}$$

De manière plus rigoureuse, on peut définir un schéma AP de la façon suivante, légèrement différente de celle donnée dans [BRR22], mais mieux adaptée aux systèmes que l'on étudie. Soit pour tout $\varepsilon > 0$ un schéma numérique $(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)_{k\in[\![0,N]\!]}$, défini par un opérateur $\Phi_{\Delta t}^\varepsilon$, c'est-à-dire que pour tout $t \in [\![0, N_1]\!]$ on a

$$(X_{k+1}^\varepsilon, n_{k+1}^\varepsilon, m_{k+1}^\varepsilon) = \Phi_{\Delta t}^\varepsilon(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon, \gamma_k),$$

où $(\gamma_k)_{k\in[\![0,N]\!]}$ est une suite de variables aléatoires Gaussiennes centrées réduites indépendantes. Ce schéma est défini en vue d'approcher la solution d'un système continu $(X_t^\varepsilon, n_t^\varepsilon, m_t^\varepsilon)_{t\in[0,T]}$, qui converge en loi vers la solution d'une équation limite $(X_t)_{t\in[0,T]}$. On fait alors les hypohèses suivantes sur le schéma numérique.

Hypothèses 0.2.3.1. *1. L'opérateur $\Phi_{\Delta t}^\varepsilon$ est défini pour tout $\varepsilon \in (0,1]$ et tout $\Delta t \in (0, \Delta t_0]$, où $\Delta t_0 > 0$ es indépendant de ε, et*

$$\sup_{\varepsilon\in(0,1]} \sup_{\Delta t\in(0,\Delta t_0]} \sup_{k\in\mathbb{N}} \frac{\mathbb{E}\left[|n_k^\varepsilon|^2 + \varepsilon^2|m_k^\varepsilon|^2\right]}{1 + \mathbb{E}\left[|n_0^\varepsilon|^2 + \varepsilon^2|m_0^\varepsilon|^2\right]} < \infty. \tag{0.2.21}$$

2. Pour tout $\varepsilon \in (0,1]$, le schéma numérique est consistant au sens faible avec le système continu : pour toute fonction continue bornée φ :

$$\mathbb{E}[\varphi(X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon)] \underset{N\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(X_T^\varepsilon, n_T^\varepsilon, m_T^\varepsilon)].$$

3. Pour tout $\Delta t \in (0, \Delta t_0]$, il existe un schéma numérique $(X_k)_{k\in[\![0,N]\!]}$ tel que pour toute fonction continue bornée φ :

$$\mathbb{E}[\varphi(X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon)] \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}[\varphi(X_N)].$$

Avec ces hypothèses, on peut définir un schéma numérique préservant l'asymptotique.

Definition 0.2.3.1. *Soit un schéma numérique* $(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)_{k\in[\![0,N]\!]}$ *satisfaisant les Hypothèses 0.2.3.1. Le schéma est dit préservant l'asymptotique, ou AP, si le schéma limite donné par la troisème hypothèse est consistant au sens faible avec l'équation limite : pour toute fonction continue bornée* φ *:*

$$\mathbb{E}[\varphi(X_N)] \underset{N\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(X_T)].$$

On commence par un modèle simple, qui nous permet de comprendre comment marche la théorie développée dans [RR21].

$$\begin{cases} \partial_t X_t^\varepsilon = \frac{\sigma(X_t^\varepsilon)n_t^\varepsilon}{\sqrt{\varepsilon}} \\ \partial_t n_t^\varepsilon = m_t^\varepsilon \\ dm_t^\varepsilon = -\frac{\alpha}{\varepsilon}m_t^\varepsilon - \frac{\lambda}{\varepsilon^2}n_t^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}}dB_t \end{cases}, \tag{0.2.22}$$

avec σ une fonction bornée. La différence avec les schémas étudiés dans [RR21] vient de l'équation sur le processus n_t^ε, qui ici est d'ordre deux, ainsi que de l'échelle en ε qui est d'ordre $-\frac{1}{2}$, afin de se rapprocher de l'étude du système de Zakharov. L'équation limite pour ce processus est

$$dX_t = \frac{\sigma(X_t)}{\lambda} \circ dB_t = \frac{1}{2\lambda^2}\sigma(X_t)\sigma'(X_t)dt + \frac{\sigma(X_t)}{\lambda}dB_t. \tag{0.2.23}$$

On approche ce système par le schéma suivant :

$$\begin{cases} \frac{n_{k+1}^\varepsilon - n_k^\varepsilon}{\Delta t} = m_{k+\theta}^\varepsilon \\ \frac{m_{k+1}^\varepsilon - m_k^\varepsilon}{\Delta t} = -\frac{\alpha}{\varepsilon}m_{k+\theta}^\varepsilon - \frac{\lambda}{\varepsilon^2}n_{k+\theta}^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}\sqrt{\Delta t}}\gamma_k \\ \frac{Y_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} = \sigma(X_k^\varepsilon)\frac{1}{\sqrt{\varepsilon}}n_{k+\theta}^\varepsilon \\ \frac{X_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} = \frac{\sigma(X_k^\varepsilon)+\sigma(Y_{k+1}^\varepsilon)}{2}\frac{1}{\sqrt{\varepsilon}}n_{k+\theta}^\varepsilon \end{cases}, \tag{0.2.24}$$

où $m_{k+\theta} = \theta m_{k+1} + (1-\theta)m_k$, $\theta \in [\frac{1}{2}, 1]$. On montre que ce schéma est bien AP au sens de [BRR22] et que le schéma limite est donné par

$$\begin{cases} Y_{k+1} = X_k + \frac{\sqrt{\Delta t}}{\lambda}\sigma(X_k)\gamma_k \\ X_{k+1} = X_k + \frac{\sqrt{\Delta t}}{\lambda}\frac{\sigma(X_k)+\sigma(Y_{k+1})}{2}\gamma_k \end{cases}. \tag{0.2.25}$$

On se rapproche ensuite un peu plus de notre problème en considérant le système :

$$\left\{\begin{array}{ll}\partial_t X_t^\varepsilon &= i\left(\mu X_t^\varepsilon + \frac{X_t^\varepsilon n_t^\varepsilon}{\sqrt{\varepsilon}}\right)\\ \partial_t n_t^\varepsilon &= m_t^\varepsilon \\ dm_t^\varepsilon &= -\frac{\alpha}{\varepsilon}m_t^\varepsilon - \frac{\lambda}{\varepsilon^2}n_t^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}}dB_t\end{array}\right. , \tag{0.2.26}$$

qui a pour équation limite

$$dX_t = i\mu X_t dt + \frac{i}{\lambda}X_t \circ dB_t. \tag{0.2.27}$$

On construit également pour ce système un schéma numérique AP :

$$\left\{\begin{array}{ll}\frac{n_{k+1}^\varepsilon - n_k^\varepsilon}{\Delta t} &= m_{k+\theta}^\varepsilon\\ \frac{m_{k+1}^\varepsilon - m_k^\varepsilon}{\Delta t} &= -\frac{\alpha}{\varepsilon}m_{k+\theta}^\varepsilon - \frac{\lambda}{\varepsilon^2}n_{k+\theta}^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}\sqrt{\Delta t}}\gamma_k\\ \frac{X_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} &= i\left(\mu X_{k+\frac{1}{2}}^\varepsilon - \frac{X_{k+\frac{1}{2}}^\varepsilon n_{k+\theta}^\varepsilon}{\sqrt{\varepsilon}}\right)\end{array}\right. . \tag{0.2.28}$$

qui converge vers le schéma limite suivant :

$$X_{k+1} = X_k + i\mu\Delta t X_{k+\frac{1}{2}} - iX_{k+\frac{1}{2}}\frac{\sqrt{\Delta t}}{\lambda}\gamma_k, \tag{0.2.29}$$

qui est bien consistant avec l'équation limite (0.2.27). On finit par construire un schéma numérique pour le système de Zakharov (0.2.17)-(0.2.18), où l'on utilise une méthode de relaxation introduite dans [Bes04] pour l'équation de Schrödinger non linéaire déterministe :

$$\left\{\begin{array}{ll}\frac{\psi_{k+1}^\varepsilon - \psi_k^\varepsilon}{2} &= |X_k^\varepsilon|^2\\ \frac{n_{k+1}^j - n_k^j}{\Delta t} &= m_{k+\frac{1}{2}}^\varepsilon\\ \frac{m_{k+1}^j - m_k^j}{\Delta t} &= -\frac{\alpha}{\varepsilon}m_{k+\frac{1}{2}}^\varepsilon + \frac{1}{\varepsilon^2}A\left(n_{k+\frac{1}{2}}^\varepsilon + \psi_{k+1}^\varepsilon\right)\\ \frac{z_{k+1}^j - z_k^j}{\Delta t} &= \zeta_{k+\frac{1}{2}}^\varepsilon\\ \frac{\zeta_{k+1}^j - \zeta_k^j}{\Delta t} &= -\frac{\alpha}{\varepsilon}\zeta_{k+\frac{1}{2}}^\varepsilon + \frac{1}{\varepsilon^2}Az_{k+\frac{1}{2}}^\varepsilon + \frac{1}{\varepsilon^{3/2}\sqrt{\Delta t}}\left(\sum_{\ell=1}^L \varphi_\ell(j)\gamma_{\ell,k}\right)_{j\in[\![1,J-1]\!]}\\ \frac{X_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} &= iAX_{k+\frac{1}{2}}^\varepsilon - iX_{k+\frac{1}{2}}^\varepsilon n_{k+\frac{1}{2}}^\varepsilon - \frac{i}{\sqrt{\varepsilon}}X_{k+\frac{1}{2}}^\varepsilon z_{k+\frac{1}{2}}^\varepsilon\end{array}\right. , \tag{0.2.30}$$

où l'on utilise une notation vectorielle pour plus de lisibilité, la matrice A encodant le Laplacien discret, et le terme $X_{k+\frac{1}{2}}^\varepsilon z_{k+\frac{1}{2}}^\varepsilon$ renvoyant au vecteur $(X_{k+\frac{1}{2}}^j z_{k+\frac{1}{2}}^j)_{i\in[\![0,J]\!]}$. Le choix de la méthode de relaxation permet de gérer l'implémentation de la non linéarité plus simplement qu'un schéma de type Crank-Nicolson par exemple. N'ayant pas de théorie pour les schémas AP en dimension infinie, on se contente d'observer le comportement asymptotique de ce schéma lorsque ε tend vers 0. On remarque alors la convergence de ce schéma vers la solution du schéma limite

$$\left\{\begin{array}{rl}\frac{\psi_{k+1}+\psi_k}{2} = & |X_k|^2 \\ \frac{X_{k+1}-X_k}{\Delta t} = & iAX_{k+\frac{1}{2}} + iX_{k+\frac{1}{2}}\psi_{K+1} \\ & -\frac{i}{\sqrt{\Delta t}}(-A)^{-1}\left(\sum_{\ell=1}^{L}\varphi_\ell(j)\gamma_{\ell,k}\right)_{j\in[\![0,J]\!]} X_{k+\frac{1}{2}}\end{array}\right. . \tag{0.2.31}$$

Des figures seront présentes tout au long de ce chapitre afin d'illustrer les différents résultats obtenus.

Chapitre 1

Diffusion approximation for the Nonlinear Schrödinger equation with random potential on $\mathbb{R}^n$

1.1 Introduction and main result

We study in this work the limit of Non-Linear Schrödinger Equation with randomness. More precisely, we consider the following problem

$$i\partial_t X(t) = -\Delta X(t) + \lambda |X(t)|^{2\sigma} X(t) + \frac{1}{\varepsilon} X(t) m(t/\varepsilon^2), \quad \varepsilon > 0, \tag{1.1.1}$$

on the domain $\mathbb{R}^n$, with regular initial data. Such equation occurs in many situations, for instance in optical fibers dynamics (see [G.A07], [UK92]). More generally, the nonlinear Schrödinger equation is an equation describing the propagation of wave in a nonhomogeneous dispersive medium and random effects often enter the description via a potential. Here we consider such a random potential which depends on time $t \geqslant 0$ and space $x \in \mathbb{R}^n$ with a scaling of the form $\frac{1}{\varepsilon} m(\frac{t}{\varepsilon^2}, x)$.

Under such a scaling, we are in the situation of diffusion approximation. The random term formally converges to a spatially dependent white noise in time and we expect to obtain a white noise driven stochastic partial differential equation at the limit. Such stochastic non-linear Schrödinger equations are used in the physics literature and has been mathematically studied, for example in a conservative version by Debussche and de Bouard ([dBD03a], [dBD05], [dBD99], ...). Barbu, Röckner and Zhang proved in [BRZ14] well posedness results in both conservative and nonconservative cases thanks to rescaling transformations, while Brzezniak and Millet studied in [BM14] the stochastic nonlinear Schrödinger equation on a two-dimensional manifold. We also cite [MRRY21] and [MRY21] where the authors study the one dimensional L^2-critical and supercritical cases for Nonlinear Schrödinger equation with spatially correlated noise and space time white noise. We believe that it is important to prove that theses equations can be limits of equations with noises appearing in physical applications, such as (1.1.1).

Our basic tool is the Perturbed Test Function method. This method provides an elegant way for diffusion approximation problems; it was first introduced by Papanicolaou, Stroock and Varadhan [PSV77] in a finite dimensional case, and one can find many applications in the book of Fouque, Garnier, Papanicolaou and Solna [FGPS07]. Generalizations in infinite dimension have been recently developped, for instance in [dBG12], [DDMV16] or [DV12]. Our aim is to develop this method in the context of the nonlinear Schrödinger equation with a random potential. The difficulty is that the fundamental object for this method is the infinitesimal generator of the Markov process associated to (1.1.1), which is a complicated object, since we work in an infinite dimensional space. Moreover the Perturbed Test Function method is based on compactness arguments, which are not trivial in the case of (1.1.1) whose space variable is in the full space $\mathbb{R}^n$. We overcome this latter problem by working in a weighted space. Note that the rescaling transformation used in [BRZ14] shows that the solution of (1.1.1) is continuous with respect to

$$\frac{1}{\varepsilon}\int_0^t m\left(\frac{s}{\varepsilon^2}\right) ds,$$

so that our result could be proved in a simpler way with this transform. However, our arguments immediately extend to more general equations, for instance, if $\frac{1}{\varepsilon}X(t)m(\frac{t}{\varepsilon^2})$ is replaced by $\frac{1}{\varepsilon}\Lambda(X(t))m(\frac{t}{\varepsilon^2})$ for a Nemitsky operator Λ with sublinear growth such that $\Lambda(z)\bar{z}$ is real valued. For the sake of clarity we decided to study the case $\Lambda(z) = z$.

The driving noise m is an ergodic Markovian process with values in a Sobolev space to be described below. In all of the articles mentioned above on diffusion approximation problem for partial differential equations, this noise is assumed to be uniformly bounded. This is important to get a priori estimates. In this article, we introduce a new argument, which allows us to replace this assumption by a much more satisfactory one: we only assume that m has sufficiently many moments. This introduces several difficulties. In particular, the correctors cannot be bounded uniformly in ε. We use a stopping time argument which, together with a control of the growth of stationary process, allows us to obtain a bound on the correctors. Another difficulty is to obtain bounds on the solutions of (1.1.1) uniformly in ε. In particular, the control of the energy is very delicate. Our idea has been used and improved in a recent work on diffusion approximation for kinetic equations (see [RR21]).

We work with solutions in the Sobolev space $H^1(\mathbb{R}^n)$ and assume that the nonlinear term is H^1-subcritical:

$$0 \leqslant \sigma < \frac{2}{n-2}, \ n \geqslant 3 \text{ or } 0 \leqslant \sigma, \ n = 1, 2. \tag{1.1.2}$$

Moreover, since we work with global solutions, we need the further assumption:

$$\sigma < \frac{2}{n}, \text{ if } \lambda < 0. \tag{1.1.3}$$

Our main result can be stated informally as follows (precise assumptions are given below).

Theorem 1.1.1. *For all $T > 0$ and $0 < \zeta \leqslant 1$, assume that X_0^ε converges in law to X_0 in the space Σ^ζ, see (1.2.1), then for all $s \in [0, 1)$ the $C([0, T]; H^s)$-valued process X^ε, solution of (1.1.1) with initial data X_0^ε converges in law in $C([0, T], H^s)$ to X solution of*

$$idX = \Big(-\Delta X + \lambda |X|^{2\sigma} X - \frac{i}{2} F X \Big) dt - X Q^{1/2} dW \tag{1.1.4}$$

with initial data X_0, where F, Q are defined respectively by (1.2.23) and (1.2.24) below, and W is a cylindrical Wiener process on L^2.

The last two terms in (1.1.4) actually correspond to the Stratonovitch noise $X_t \circ Q^{\frac{1}{2}} dW_t$. The covariance operator Q is described below and is explicit in terms of the correlation of m, see (1.2.22) and (1.2.24) below. This chapter is organized as follows. We first introduce the notations and state preliminary results on the Schrödinger equation and on the driving process m. In Section 1.3, we adapt Kato's method (see [Caz03]) to get global existence of the process X^ε. The estimates are obtained by classical manipulations of the equation and blow up when $\varepsilon \to 0$. To avoid this we adapt the perturbed test function method to our problem in Section 1.4. This enables us to get both tightness of the process and the natural generator of the limit. In Section 1.5 we use all the results proved in Section 1.3 and Skorohod Theorem (see [Bil99]) to prove the weak convergence of X^ε to X. Finally, Section 1.6 is devoted to details about an example of process m that can be considered in (1.1.1) and to proofs of technical estimates.

In this work $\mathcal{C}$, C or c denote positive constants which unless explicitely stated are independent of ε or of the smoothing parameter δ introduced below. They may eventually depend on other parameters such as T, n, σ, R or η, and if needed, we may precise the dependences by denoting them for example as $\mathcal{C}_T$.

1.2 Preliminary and Main result

1.2.1 Notations

Throughout this paper, for $p \geqslant 1$, we denote by $L^p(\mathbb{R}^n, ; \mathbb{C})$ the Lebesgue space of p integrable $\mathbb{C}$-valued functions on $\mathbb{R}^n$, endowed with the usual norm. For $p = 2$, $(\cdot, \cdot)$ is the inner product of $L^2(\mathbb{R}^n; \mathbb{C})$ given by

$$(f, g) = Re \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx.$$

For $s \in \mathbb{R}$, we use the usual Sobolev space $H^s := H^s(\mathbb{R}^n; \mathbb{C})$ of tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n; \mathbb{C})$, where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$, endowed with the usual norm

$$\|u\|_{H^s} = \|\langle \xi \rangle^s \hat{u}(\xi)\|_{L^2},$$

where $\hat{u}$ denotes the Fourier transform.

For $m \in \mathbb{N}$ and $p \geqslant 1$, we also use the standard Sobolev spaces $W^{m,p}(\mathbb{R}^n; \mathbb{C})$ consisting of functions which are in $L^p(\mathbb{R}^n; \mathbb{C})$ as well as their derivatives up to order m. It is classical that $H^m(\mathbb{R}^n; \mathbb{C}) = W^{m,2}(\mathbb{R}^n; \mathbb{C})$.

We also consider the same spaces with values in $\mathbb{R}$, they are denoted by the same symbols with $\mathbb{C}$ replaced by $\mathbb{R}$. When there is no ambiguity, we omit $\mathbb{R}^n$ and $\mathbb{C}$ or $\mathbb{R}$. For instance, we simply write H^1 for $H^1(\mathbb{R}^n; \mathbb{C})$.

In this article, the following weighted Sobolev spaces are particularly useful. We define for $0 < \zeta \leqslant 1$,

$$\Sigma^\zeta = \left\{u \in H^1(\mathbb{R}^n; \mathbb{C}); \langle x \rangle^\zeta u \in L^2\right\} \tag{1.2.1}$$

endowed with the norm $\|u\|_{\Sigma^\zeta} = \|u\|_{H^1} + \left\|\langle x \rangle^\zeta u\right\|_{L^2}$.

Finally, given a Banach space E, $C(E)$ (resp. $C^k(E)$) denotes the space of real-valued continuous (resp. C^k) functions on E, and $C_b(E)$ (resp. $C_b^k(E)$) is the space of bounded continuous functions (resp. C^k bounded as well as their derivatives up to order k).

When H, K are Hilbert spaces, we denote by $\mathcal{L}(H, K)$ the space of linear operator from H to K. If $H = K$, we simply write $\mathcal{L}(H)$. We also denote by $\mathcal{L}_2(H, K)$ the space of Hilbert-Schmidt operators from H to K.

We denote by $S(t)$ the group associated to the linear homogeneous equation and defined by $S(t) = e^{it\Delta}$. The solution of (1.1.1) is taken in the mild sense.

As already mentioned we assume that (1.1.2), (1.1.3) hold so that we are able to prove global well posedness in $H^1(\mathbb{R}^n;\mathbb{C})$ (see [Caz03]).

The energy of $u \in H^1(\mathbb{R}^n;\mathbb{C})$ is denoted by $H(u)$ and is given by

$$H(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\lambda}{2\sigma+2}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{1.2.2}$$

In order to justify the computations when getting energy estimates, we may need a regularization procedure, so we choose ρ to be a mollifier, that is

$$\rho \in C^\infty(\mathbb{R}^n,\mathbb{R}),\ \ \rho \geqslant 0,\ \int_{\mathbb{R}^n} \rho(x)dx = 1 \text{ and } \mathrm{supp}\rho \subset B(0,1). \tag{1.2.3}$$

Finally for $\delta > 0$, we define $\rho_\delta(x) = \frac{1}{\delta^n}\rho\left(\frac{x}{\delta}\right)$ and $u \star v$ stands for the convolution of u and v, when it makes sense.

1.2.2 The random process m

We assume that m is a centered, right-continuous and left-limited (càdlàg), stochastically continuous, stationary $E = H^s(\mathbb{R}^n,\mathbb{R})$-valued process, for $s > \frac{n}{2}+3$ so that $E \hookrightarrow W^{3,\infty}(\mathbb{R}^n,\mathbb{R})$, on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$. Note, in particular, that $m(t,x)$ is real-valued. We also define the rescaled process

$$m^\varepsilon(t) = m(t/\varepsilon^2), \tag{1.2.4}$$

which is centered stationary $E = H^s$-valued process $(\mathcal{F}_t^\varepsilon)_{t\in\mathbb{R}}$-adapted, where $\mathcal{F}_t^\varepsilon = \mathcal{F}_{t/\varepsilon^2}$. Below, we often use the fact that E is an algebra.

The process m is an homogeneous Markov process. We denote by $(P_t)_{t\geqslant 0}$ a transition semigroup associated to m, $\mathcal{M}$ its infinitesimal generator and $(\tilde{m}(t,n))_{t\geqslant 0, n\in E}$ a Markov process such that $\tilde{m}(t,m(s)) = m(t+s)$.

We do not need to define precisely $\mathcal{M}$ but we use it in the following sense. We assume that there exist sets $\mathcal{P}_\mathcal{M}$ and $\mathcal{D}_\mathcal{M}$ included in

$$C_P = \{\psi \in C(E), \exists\, C > 0,\ p \geqslant 0 : \forall\, n \in E, |\psi(n)| \leqslant C(1+\|n\|_E)^p\}, \tag{1.2.5}$$

such that for $\psi \in \mathcal{P}_{\mathcal{M}}$ there exists $\mathcal{M}^{-1}\psi \in \mathcal{D}_{\mathcal{M}}$ such that for $n \in E$

$$\mathcal{M}^{-1}\psi(\tilde{m}(t,n)) - \int_0^t \psi(\tilde{m}(s,n))ds \tag{1.2.6}$$

is an integrable martingale. We write $\mathcal{M}\mathcal{M}^{-1}\psi = \psi$. This is an implicit assumption of ergodicity on m. We assume that $\tilde{m}$ has a unique invariant measure ν, which is the invariant law of m. Clearly, our setting requires that

$$\mathbb{E}_\nu[\psi] = \int_E \psi(n)d\nu(n) = 0, \qquad \psi \in \mathcal{P}_{\mathcal{M}}. \tag{1.2.7}$$

We need that $\mathcal{P}_M$ contains sufficiently many functions. Let us define for u, $v \in H^1$:

$$\Psi_1^{u,v}(n) = (un, v).$$

We assume that for each u, $v \in H^1$, $\Psi_1^{u,v} \in \mathcal{P}_M$ and there exists $L_1 : E \to E$ continuous such that

$$\mathcal{M}^{-1}\Psi_1^{u,v}(n) = (uL_1(n), v).$$

Informally, this says: $\mathcal{M}^{-1}n = L_1(n)$.

Then we define for u, $v \in H^1$:

$$\Psi_2^{u,v}(n) = (unL_1(n), v) - \mathbb{E}_\nu(unL_1(n), v),$$
$$\Psi_3^{u,v}(n) = (u\nabla n, v\nabla L_1(n)) - \mathbb{E}_\nu(u\nabla n, v\nabla L_1(n)),$$

and assume that for each u, $v \in H^1$, $i = 2$, 3, $\Psi_i^{u,v} \in \mathcal{P}_M$ and there exists $L_2 : E \to E$ and $L_3 = E \to H^{s-1}$ continuous such that

$$\mathcal{M}^{-1}\Psi_i^{u,v}(n) = (uL_i(n), v).$$

Again, informally this may be written as:

$$\mathcal{M}^{-1}(nL_1(n) - \mathbb{E}_\nu(nL_1(n))) = \mathcal{M}^{-1}(n\mathcal{M}^{-1}n - \mathbb{E}_\nu(n\mathcal{M}^{-1}n)) = L_2(n)$$

and

$$\mathcal{M}^{-1}(\nabla n \cdot \nabla L_1(n) - \mathbb{E}_\nu(\nabla n \cdot \nabla L_1(n))) = \mathcal{M}^{-1}(\nabla n \cdot \nabla \mathcal{M}^{-1} n - \mathbb{E}_\nu(\nabla n \cdot \nabla \mathcal{M}^{-1} n)) \\ = L_3(n).$$

We need to invert further a function of n. For u, $v \in H^1$, we define

$$\Psi_4^{u,v}(n) = (un, v)(uL_1(n), v) - \mathbb{E}_\nu(un, v)(uL_1(n), v)$$

and assume that for each u, $v \in H^1$, $\Psi_4^{u,v} \in \mathcal{P}_M$ and there exists $L_4 : E \to L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ continuous such that

$$\mathcal{M}^{-1}\Psi_4^{u,v}(n) = Re \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)u(y)L_4(n)(x,y)\bar{v}(x)\bar{v}(y)dxdy.$$

Informally:

$$L_4(n)(x,y) = \mathcal{M}^{-1}(n(x)L_1(n)(y) - \mathbb{E}_\nu n(x)L_1(n)(y)) \text{ and } L_2(n)(x) = L_4(n)(x,x). \tag{1.2.8}$$

The boundedness assumption made in previous articles on diffusion approximation for PDEs is replaced here by the weaker assumption: there exists $\gamma > 6$ and a constant $\mathcal{C}$, such that for all $k \in \mathbb{N}$, the following estimate holds

$$\mathbb{E}\left(\sup_{t\in[k,k+1]} \|m(t)\|_E^\gamma\right) \leqslant \mathcal{C}. \tag{1.2.9}$$

In particular, this implies:

$$\int_E \|n\|_E^\gamma d\nu < \infty$$

and

$$\sup_{t\in[0,T]} \|m(t)\|_E < \infty, \ a.s.,$$

for any $T \geqslant 0$. Notice that, by stationarity of m, it is sufficient to assume that (1.2.9) holds for $k = 0$.

We need some control on the growth of the functions $L_i, i : 1, cdots, 4$, introduced above. We assume that there exists $\eta < \gamma/2 - 1$ and $p \in \mathbb{N}$ such that:

$$\begin{aligned} \|L_i(n)\|_E &\leqslant c(1 + \|n\|_E)^\eta,\ n \in E,\ i = 1, 2, \\ \|L_3(n)\|_{H^{s-1}} &\leqslant c(1 + \|n\|_E)^\eta, n \in E, \\ \|L_4(n)\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} &\leqslant c(1 + \|n\|_E)^\eta,\ n \in E. \end{aligned} \tag{1.2.10}$$

In many situations, $\mathcal{M}^{-1}\psi$ is given by:

$$\mathcal{M}^{-1}\psi(n) = -\int_0^\infty P_t\psi(n)dt,$$

and, in particular, for each $n \in E$, $P_t\psi(n)$ is an integrable function: $t \mapsto P_t f(n) \in L^1(0, \infty)$. Here we do not need this but we simply use the assumption that there exists $\lambda \in L^1(0, \infty)$ such that:

$$\|\mathbb{E}(m(t, n))\|_E \leqslant \lambda(t)(1 + \|n\|_E)^{\gamma-1} \tag{1.2.11}$$

and that L_1 is given by

$$(w, L_1(n)) = -\int_0^\infty \mathbb{E}\left[(w, m(t, n))\right] dt, \quad w \in E'. \tag{1.2.12}$$

The following Lemma is similar to [DPZ14, Lemma 15.4.4] and is fundamental to avoid the uniform boundedness assumption on the process m.

Lemma 1.2.2.1. *For all $T > 0$, and $\varepsilon > 0$, and $\alpha > \frac{2}{\gamma}$*

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|m^\varepsilon(t)\|_E \geqslant \varepsilon^{-\alpha}\right) \to 0,$$

when ε goes to 0.

Proof. For $k \in \mathbb{N}$, we denote by η_k the process

$$\eta_k := \sup_{t\in[k,k+1]} \|m(t)\|_E\,,$$

then by Markov inequality and (1.2.9) we have for all δ and k

$$\mathbb{P}(\eta_k \geqslant k^\delta) \leqslant \frac{\mathcal{C}}{k^{\delta\gamma}},$$

and choosing δ such that $\delta\gamma > 1$, we have

$$\sum_{k\geqslant 0} \mathbb{P}(\eta_k \geqslant k^\delta) < \infty,$$

so we get by Borel-Cantelli lemma the existence of random variables Z_1, Z_2 such that

$$\mathbb{P} - a.s., \forall t \in \mathbb{R}^+, \|m(t)\|_E \leqslant Z_1 + |t|^\delta Z_2.$$

Finally, since

$$\left\{ \sup_{t\in[0,T]} \|m^\varepsilon(t)\|_E \geqslant \varepsilon^{-\alpha} \right\} \subset \left\{ Z_1 + \left|\frac{T}{\varepsilon^2}\right|^\delta Z_2 \geqslant \varepsilon^{-\alpha} \right\}$$

and the probability of the right-hand side event goes to 0, when $\varepsilon \to 0$, under the condition $\alpha > 2\delta$, this ends the proof. □

We naturally define the $(\mathcal{F}_t^\varepsilon)$-stopping time τ^ε by

$$\tau^\varepsilon = \inf\{t \in [0,T], \|m^\varepsilon(t)\|_E \geqslant \varepsilon^{-\alpha}\}. \tag{1.2.13}$$

We will use many times that $\mathbb{P}(\tau^\varepsilon < T) \leqslant \mathbb{P}(\sup_{t\in[0,T]} \|m^\varepsilon(t)\|_E \geqslant \varepsilon^{-\alpha})$, which together with Lemma 1.2.2.1 yields

$$\mathbb{P}(\tau^\varepsilon < T) \to_{\varepsilon\to 0} 0. \tag{1.2.14}$$

1.2.3 An example

An example of concrete assumptions on the process m that are sufficient to satisfy all of the above hypotheses is the following:

1. For every $n \in E$, $(\tilde{m}(t,n))_{t\geqslant 0}$ is a stochastically continuous Markov process associated to the semigroup $(P_t)_{t\geqslant 0}$.
2. $(P_t)_{t\geqslant 0}$ is Feller.
3. For every $k \in \mathbb{N}$ and $n \in E$, $\mathbb{E}[\|\tilde{m}(t,n)\|_E^k]$ is finite and there exist $C_k > 0, \ell_k \in \mathbb{N}$

such that for every $t \in [0,T]$

$$\mathbb{E}[\|\tilde{m}(t,n)\|_E^k] \leqslant C_k(1+\|n\|_E)^{\ell_k}, \qquad n \in E. \tag{1.2.15}$$

4. There exist $\kappa > 0, \gamma > 0, k_0 \in \mathbb{N}$ such that for any $n_1, n_2 \in E$ we can construct a coupling $(m^1(t), m^2(t))$ of $(\tilde{m}(t,n_1), \tilde{m}(t,n_2))_{t\geqslant 0}$ such that

$$\mathbb{P}\left(m^1(t) \neq m^2(t)\right) \leqslant \kappa\left(1+\|n_1\|_E+\|n_2\|_E\right)^{k_0} e^{-\gamma t}. \tag{1.2.16}$$

It follows classically from this last assumption that $(P_t)_{t\geqslant 0}$ has a unique invariant measure ν, which is exponentially mixing. We finally need

5. $$\int_E n\nu(dn) = 0,$$

   and for any $k \in \mathbb{N}$

$$\int_E \|n\|_E^k \, \nu(dn) < \infty. \tag{1.2.17}$$

We then take m as a stationary process with law ν such that $\tilde{m}(t, m(s)) = m(t+s)$. For $\psi \in C_P$ such that $\int_E \psi(n) d\nu(n) = 0$, we may define

$$\mathcal{M}^{-1}\psi(n) = -\int_0^\infty P_t\psi(n)dt. \tag{1.2.18}$$

Our assumptions imply that $\mathcal{M}^{-1}\psi \in C_P$. We choose

$$\mathcal{D}_{\mathcal{M}} = \left\{\mathcal{M}^{-1}\psi, \psi \in C_P\right\}, \qquad \mathcal{P}_{\mathcal{M}} = \left\{\psi \in C_P, \int_E \psi(n)d\nu(n) = 0\right\}.$$

Clearly for any $\mathcal{M}^{-1}\psi \in \mathcal{D}_{\mathcal{M}}$, we may take $\mathcal{M}\mathcal{M}^{-1}\psi = \psi$ and $L_1, \cdots, L_4$ can be constructed and follow our assumptions.

Let us now construct a process satisfying 1 to 5. We consider the following stochastic equation in a Hilbert space H: for $t > 0, x \in H$

$$\begin{cases} dX_t = (AX_t + F(X_t))dt + \sigma dW_t, \\ X_0 = x, \end{cases} \tag{1.2.19}$$

where A is an unbounded operator on $D(A)$ dense in H, invertible with a compact inverse, which generates an analytic semigroup $(e^{tA})_{t\geqslant 0}$ and such that

$$\langle Ax, x\rangle \leqslant -\lambda_1 \|x\|_H^2, \qquad x \in D(A), \tag{1.2.20}$$

and

$$\left\|e^{tA}\right\|_{\mathcal{L}(H)} \leqslant Me^{-\eta t}, \qquad t > 0,$$

for some $k > 0, M > 0, \eta > 0$. Assume moreover that, for all $t > 0$, e^{tA} is Hilbert-Schmidt on H and

$$\left\|e^{tA}\right\|_{\mathcal{L}_2(H)} \leqslant Lt^{-\gamma}, \qquad t > 0,$$

with $L > 0, \gamma \in [0, \frac{1}{2})$.

Remark that if $A = -(-\Delta)^r$ where $-\Delta$ is the Laplace operator on a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^d, d \in \mathbb{N}^*$ with Dirichlet boundary conditions, then A satisfies our assumptions provided that $r > \frac{d}{2}$.

The nonlinear term F can be chosen in various ways, for simplicity we assume that it is Lipschitz bounded. The covariance of the noise σ is an invertible operator on H. It has been proved in [DHT11] that (1.2.19) defines a Markov process in H. We consider a continuous linear invertible operator $\Lambda : H \to E$ and take

$$m(t, n) = \Lambda\left(X(t, \Lambda^{-1}n + \bar{x}) - \bar{x}\right), \tag{1.2.21}$$

where $\bar{x} = \displaystyle\int_H x\nu(dx)$, ν being the invariant measure of X_t, and, for $x \in H$, $X(t, x)$ being the unique solution of (1.2.19). The assumprions 1, 2 and 4 follow from corresponding properties on X_t proved in [DHT11]. Concerning 3 and 5, they may be proved by classical computations based on the change of unknown

$$Y(t) = X(t) - \int_0^t e^{A(t-s)}\sigma dW_s$$

in (1.2.19) and energy estimates. In Section 1.6.1 we give details and prove that the process $m(t, n)$ satisfies conditions 1 to 5 and that these conditions are indeed sufficient to construct $L_1, \cdots, L_4$ as above.

Remark 1.2.3.1. *We could also build an example based on a Markov chain in E as in [DRV21].*

1.2.4 The covariance operator

We note that assumptions (1.2.9) and (1.2.11) imply that $t \mapsto \mathbb{E}(m(0)(x)(m(t)(y))$ is integrable over $[0, \infty)$, so that we can define:

$$k(x,y) = \int_0^\infty \mathbb{E}(m(0)(x)m(t)(y) + m(t)(x)m(0)(y))dt \text{ for } x, y \in \mathbb{R}^n. \tag{1.2.22}$$

It is not difficult to check that $k \in W^{3,\infty}(\mathbb{R}^n \times \mathbb{R}^n) \cap H^1(\mathbb{R}^n \times \mathbb{R}^n)$. We assume that $k(\cdot,\cdot) \in L^1(\mathbb{R}^n)$, and that $\Delta_1 k(\cdot,\cdot) \in L^1(\mathbb{R}^n)$, with Δ_1 denoting the Laplacian operator with respect to the first variable of k.

We also define $F \in E$ by

$$F(x) = k(x,x) \tag{1.2.23}$$

and Q the linear operator on $H^1_{\mathbb{C}}$ by

$$Qf(y) = Re \int_{\mathbb{R}^n} k(x,y)f(x)dx \text{ for } f \in H^1_{\mathbb{C}}. \tag{1.2.24}$$

The following lemma, whose proof can be found in [D.M60], is useful to prove that Q is non-negative.

Lemma 1.2.4.1 (Wiener-Kintchine). *Let $\big(x(t)\big)_{t\in\mathbb{R}}$ be a real-valued stationary and centered process, we set $C(t) = \mathbb{E}(x(t)x(0))$ and assume C is integrable on $\mathbb{R}$. Defining*

$$\hat{x}_T(\nu) = \int_{-T}^{T} x(t)e^{-i\nu t}dt,$$

we have

$$\int_{\mathbb{R}} C(\tau)e^{i\tau\nu}d\tau = \lim_{T\to+\infty} \frac{1}{2T}\mathbb{E}\Big|\hat{x}_T(\nu)\Big|^2.$$

We now use the last assumptions on m and $\mathcal{P}_{\mathcal{M}}$ to show the following properties:

Proposition 1.2.1. *The operator Q has finite trace on L^2, is self-adjoint and non-negative: $(Qf,f) \geqslant 0$ for all $f \in L^2$. Moreover, the following identities hold*

$$k(x,y) = \mathbb{E}\int_{\mathbb{R}} m(t)(x)m(0)(y)dt, \tag{1.2.25}$$

$$\int_0^\infty \mathbb{E}(m(t)(x)m(0)(y))dt = -\int_E n(y)L_1(n)(x)d\nu(n), \tag{1.2.26}$$

where ν is the law of $m(0)$.

Proof. We know that $k(\cdot,\cdot) \in L^1(\mathbb{R}^n)$, so $Tr(Q) < \infty$, and k is obviously symmetric. Moreover, the stationarity of m yields

$$\int_0^\infty \mathbb{E}\Big(m(0)(y)m(t)(x)\Big)dt = \int_0^\infty \mathbb{E}\Big(m(-t)(x)m(0)(y)\Big)dt$$
$$= \int_{-\infty}^0 \mathbb{E}\Big(m(0)(x)m(t)(y)\Big)dt,$$

and we easily get (1.2.25). Let us now prove the positivity of Q. We define $x(t) = (m(t), f)$, which is a centered stationary process, and denote by $C(t) = \mathbb{E}(x(t)x(0))$ its correlation function. Then definitions (1.2.24) and (1.2.25) of k and Q and Lemma 1.2.4.1 yield

$$(Qf, f) = \int_{\mathbb{R}} C(\tau)d\tau = \lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\Big|\hat{x}_T(0)\Big|^2 \geqslant 0.$$

Finally, applying (1.2.12) with $w = \delta_x \in E'$, we write:

$$-\int_E n(y)L_1(n)(x)d\nu(n) = -\mathbb{E}\Big(m(0)(y)L_1(m(0))(x)\Big)$$
$$= \mathbb{E}\Big(m(0)(y)\int_0^\infty \mathbb{E}\left(\tilde{m}(t, m(0))(x)|\mathcal{F}_0\right)dt\Big)$$
$$= \int_0^\infty \mathbb{E}(m(t)(x)m(0)(y))dt,$$

this proves (1.2.26) □

Thanks to Proposition 1.2.1, we may define the operator $Q^{1/2}$, which is Hilbert-Schmidt on L^2. Let us denote by q its kernel. Then, we have

$$k(x, y) = \int_{\mathbb{R}^n} q(x, z)q(z, y)dz.$$

We need a little more smoothness on this operator. We make an extra assumption:

$$F \in W^{1,1} \cap W^{1,\infty}. \tag{1.2.27}$$

This implies that $Q^{1/2}$ is Hilbert-Schmidt from L^2 to H^1 since:

$$\|Q^{1/2}\|^2_{\mathcal{L}_2(L^2,H^1)} \leqslant \|k(\cdot,\cdot)\|_{L^1(\mathbb{R}^n)} + \|\Delta_1 k(\cdot,\cdot)\|_{L^1(\mathbb{R}^n)}.$$

We assume that it is also γ-radonifying from L^2 to $W^{1,p}$ for any $p \geqslant 2$. It is shown in [BL10] that this amounts to assuming that $q \in W^{1,p}(\mathbb{R}^n; L^2(\mathbb{R}^n))$.

Under this assumption it is shown in [dBD03a] that (1.1.4) has a unique global solution in H^1.

1.2.5 Main result

We state here our main result, where we recall all the assumptions needed on the process m.

Theorem 1.2.1. *Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $0 \leqslant s < 1$, and let $\sigma \in \mathbb{R}$ such that*

$$0 \leqslant \sigma < \frac{2}{n-2}, n \geqslant 3 \text{ or } 0 \leqslant \sigma, n = 1, 2$$

and

$$\sigma < \frac{2}{n} \text{ if } \lambda < 0.$$

Consider the Nonlinear Schrödinger equation

$$i\partial_t X^\varepsilon(t) = -\Delta X^\varepsilon(t) + \lambda |X^\varepsilon(t)|^{2\sigma} X^\varepsilon(t) + \frac{1}{\varepsilon} X^\varepsilon(t) m(t/\varepsilon^2),$$

where m is a centered stationary $E = H^s(\mathbb{R}^n)$-valued homogeneous Markov process, $(\mathcal{F}_t)$-adapted, for which there exist $\gamma > 6$ and $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,1]} \|m(t)\|_E^\gamma\right] \leqslant C.$$

Assume furthermore that (1.2.10),(1.2.11) and (1.2.12) are satisfied. Finally, consider F and Q respectively defined in (1.2.23), (1.2.24), with k defined in (1.2.22), which can be written as

$$k(x,y) = \int_{\mathbb{R}^n} q(x,z)q(z,y)dz,$$

where q is the kernel of the operator $Q^{\frac{1}{2}}$. Assume that $F \in W^{1,1}(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ and that $q \in W^{1,p}(\mathbb{R}^n, L^2(\mathbb{R}^n))$ for any $p \geqslant 1$. Then for all $T > 0$ and $0 < \zeta \leqslant 1$, assuming that X_0^ε converges in law to X_0 in Σ^ζ defined in (1.2.1), the $C([0,T], H^s)$-valued process X^ε solution of the above Schrödinger equation with initial data X_0^ε converges in law in

$C([0,T],H^s(\mathbb{R}^n))$ to X solution of

$$\begin{cases} idX_t = \Big(-\Delta X_t + \lambda|X_t|^{2\sigma}X_t - \frac{i}{2}FX_t\Big)dt - XQ^{1/2}dW_t \\ X(0) = X_0 \end{cases},$$

where W_t is a cylindrical Wiener process on $H^s(\mathbb{R}^n)$.

The proof of this theorem is detailled in Section 1.5. We first study the equation (1.1.1) and the generator of the couple (X^ϵ, m^ϵ). This allows us to introduce the perturbed test function method, which is the key tool to obtain tightness and prove the main result.

1.3 The equation

1.3.1 The Cauchy problem

This subsection is devoted to proving the following proposition. It states the existence of X^ϵ and provides a bound in H^1. This bound is obtained by standard arguments but is not uniform in ε. A uniform bound is obtained with more sophisticated tools below.

Proposition 1.3.1. *For any $\varepsilon > 0$, given $X_0^\varepsilon \in H^1$, there exists almost surely a unique mild solution X^ε of (1.1.1) with initial data X_0^ε which lies in $C([0,T],H^1(\mathbb{R}^n) \cap L^r(0,T;W^{1,2\sigma+2}(\mathbb{R}^n))$, $r = \frac{4(\sigma+1)}{n\sigma}$, for any T. Moreover, we have $\mathbb{P}-a.s.$ for $0 \leqslant t \leqslant T$:*

$$\begin{aligned} &\|X^\varepsilon(t)\|_{L^2} = \|X_0^\varepsilon\|_{L^2}\,, \\ &\|\nabla X^\varepsilon(t)\|_{L^2} \leqslant \mathcal{C}_T e^{\mathcal{C}_{T,\varepsilon}\sup_{s\in[0,t]}\|m^\varepsilon(s)\|_E}\left(H(X_0^\varepsilon) + c_\lambda\|X_0^\varepsilon\|_{L^2}^{p_n}\right), \end{aligned} \tag{1.3.1}$$

with $p_n = (2\sigma+2-\sigma n)\frac{2}{2-\sigma n}$, $c_\lambda = -\lambda$ if $\lambda < 0$ and $c_\lambda = 0$ for $\lambda \geqslant 0$. Finally, if X_0^ε is $\mathcal{F}_0$-measurable, then the process X^ε is $(\mathcal{F}_t^\varepsilon)$-adapted.

Proof. We solve (1.1.1) pathwise and consider the equation

$$i\partial_t u^\varepsilon = -\Delta u^\varepsilon + \lambda|u^\varepsilon|^{2\sigma}u^\varepsilon + \frac{1}{\varepsilon}m^\varepsilon(t,\omega)u^\varepsilon.$$

We use here Kato's method (see [Caz03, Theorem 4.4.1]) to get existence and uniqueness of $X^\varepsilon(\omega) = u^\varepsilon$ for a small enough T_0. We recall that X^ε is the fixed-point of ϕ defined for

$u \in \mathcal{E}_T$ by:

$$\phi(u)(t) = S(t)X_0^\varepsilon(\omega) - i\lambda \int_0^t S(t-s)|u(s)|^{2\sigma}u(s)ds - \frac{i}{\varepsilon}\int_0^t S(t-s)u(s)m^\varepsilon(s)ds \quad (1.3.2)$$

in the space

$$\begin{aligned}\mathcal{E}_T =& \{u \in L^\infty([0,T_0];H^1(\mathbb{R}^n)) \cap L^r([0,T_0];W^{1,2\sigma+2}(\mathbb{R}^n));\\ & \|u\|_{L^\infty([0,T_0];H^1(\mathbb{R}^n))} \leqslant M, \|u\|_{L^r([0,T_0];W^{1,2\sigma+2}(\mathbb{R}^n))} \leqslant M\}\end{aligned}$$

endowed with the distance

$$d(u,v) = \|u-v\|_{L^\infty([0,T_0];L^2(\mathbb{R}^n))} + \|u-v\|_{L^r(0,T_0;L^{2\sigma+2}(\mathbb{R}^n))}\,.$$

Kato's method proves that ϕ is a contraction for well chosen M, T_0, and allows us to get the continuity in time of the solution.

Since ϕ given by (1.3.2) maps $(\mathcal{F}_t^\varepsilon)$-adapted processes onto $(\mathcal{F}_t^\varepsilon)$-adapted processes, and that X^ε is obtained by iterating ϕ, this gives that X^ε is $(\mathcal{F}_t^\varepsilon)$-adapted.

Moreover, it is easy to verify that $\|X^\varepsilon(t,\omega)\|_{L^2} = \|X_0^\varepsilon\|_{L^2}$. To prove the bound on the gradient, we use the energy, and a regularization is necessary to justify the computations. We denote now by $X_\delta^\varepsilon = \rho_\delta \star X^\varepsilon(\omega) \in H^k$ for all $k \in \mathbb{N}$. Then we compute $H(\rho_\delta \star X^\varepsilon(t))$:

$$\begin{aligned}&\frac{dH(\rho_\delta \star X^\varepsilon(t))}{dt} = dH(X_\delta^\varepsilon(t)).(\rho_\delta \star \partial_t X^\varepsilon)\\ &= Re\left(\int_{\mathbb{R}^n} (-\Delta X_\delta^\varepsilon + \lambda|X_\delta^\varepsilon|^{2\sigma}X_\delta^\varepsilon)\overline{\partial_t X_\delta^\varepsilon}\right)\\ &= Re\left(\int_{\mathbb{R}^n} (-\Delta X_\delta^\varepsilon + \lambda|X_\delta^\varepsilon|^{2\sigma}X_\delta^\varepsilon)(-i\Delta \bar{X}_\delta^\varepsilon + i\lambda\rho_\delta \star (|X^\varepsilon|^{2\sigma}\bar{X}^\varepsilon) + \frac{i}{\varepsilon}\rho_\delta \star (\bar{X}^\varepsilon m^\varepsilon))\right)\\ &= Re\left(\int_{\mathbb{R}^n} (-\Delta X_\delta^\varepsilon, i\lambda\rho_\delta \star (|X^\varepsilon|^{2\sigma}\bar{X}^\varepsilon) + \frac{i}{\varepsilon}\rho_\delta \star (\bar{X}^\varepsilon m^\varepsilon)\right)\\ &\quad + Re\left(\int_{\mathbb{R}^n} (\lambda|X_\delta^\varepsilon|^{2\sigma}X_\delta^\varepsilon)(-i\Delta \bar{X}_\delta^\varepsilon + i\lambda\rho_\delta \star (|X^\varepsilon|^{2\sigma}\bar{X}^\varepsilon) + \frac{i}{\varepsilon}\rho_\delta \star (\bar{X}^\varepsilon m^\varepsilon))\right).\end{aligned}$$

Since $\nabla X^\varepsilon \in L^\infty([0,T_0];L^2) \cap L^r([0,T_0];L^{2\sigma+2}))$ we get that $\nabla X_\delta^\varepsilon \to \nabla X^\varepsilon$ in $L^r(0,T_0;L^{2\sigma+2})$ and similarly $\rho_\delta \star (|X^\varepsilon|^{2\sigma}X^\varepsilon)$ converges to $|X^\varepsilon|^{2\sigma}X^\varepsilon$ in $L^{r'}(0,T_0;W^{1,\frac{2\sigma+2}{2\sigma+1}})$ as δ goes to 0. The other terms are treated similarly and after integration in time, and $\delta \to 0$ we obtain

$$H(X^\varepsilon(t)) = H(X_0^\varepsilon) - \frac{1}{\varepsilon}\int_0^t Im \int_{\mathbb{R}^n} \nabla X^\varepsilon(s).\nabla m^\varepsilon(s,\omega)\bar{X}^\varepsilon(s)ds$$

which leads to, using Cauchy-Schwarz and Young inequality

$$H(X^\varepsilon(t)) \leqslant H(X_0^\varepsilon) + \frac{1}{\varepsilon}\int_0^t \|m^\varepsilon(s)\|_E \left(\frac{\|\nabla X^\varepsilon(s)\|_{L^2}^2}{2} + \frac{\|X_0^\varepsilon\|_{L^2}^2}{2}\right) ds \tag{1.3.3}$$

This implies (1.3.1) when $\lambda \geqslant 0$. For $\lambda < 0$, we use Gagliardo-Nirenberg's inequality and Young inequality and have for $u \in H^1$ and $\sigma < 2/n$

$$\frac{1}{2\sigma+2}\|u\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant \mathcal{C}\,\|u\|_{L^2}^{2\sigma+2-\sigma n}\,\|\nabla u\|_{L^2}^{\sigma n} \leqslant \frac{1}{4}\|\nabla u\|_{L^2}^2 + \mathcal{C}\,\|u\|_{L^2}^{p_n}.$$

Thus the condition (1.1.3) implies that the energy provides a control on the L^2 norm of the gradient. In fact, the energy bound (1.3.3), Gronwall lemma and the conservation of the L^2 norm yield the result. Global existence follows from (1.3.1) and the conservation of the L^2 norm.

□

1.3.2 Persistence in the spaces Σ^ζ

The result given in this subsection is a slight improvement of [Caz03, Theorem 6.5.1]. Contrary to the estimate in H^1, standard arguments suffice to give a uniform bound of the weighted L^2 norm.

Proposition 1.3.2. *Assume $X_0^\varepsilon \in \Sigma^\zeta$ for some $\zeta \leqslant 1$. Then $X^\varepsilon \in C([0,T],\Sigma^\zeta)$ almost surely for any $T > 0$. More precisely, there exists $\mathcal{C} > 0$ a deterministic constant such that*

$$\left\|X^\varepsilon(t)\langle x\rangle^\zeta\right\|_{L^2}^2 \leqslant \mathcal{C}\left(\left\|X_0^\epsilon\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t \|\nabla X^\varepsilon(s)\|_{L^2}^2\, ds\right). \tag{1.3.4}$$

Proof. Recall that we denote by $\langle x\rangle = \sqrt{1+|x|^2}$. Given $\delta > 0$, denote by φ_δ the function

$$\varphi_\delta(u) = \int_{\mathbb{R}^n} \langle x\rangle^{2\zeta}|u(x)|^2 e^{-\delta\langle x\rangle}dx.$$

We have

$$\begin{aligned}\frac{d}{dt}\varphi_\delta(X^\varepsilon(t)) &= 2(X^\varepsilon(t), \partial_t X^\varepsilon(t)\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}) \\ &= 2(X^\varepsilon(t), i\Delta X^\varepsilon(t)\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}) \\ &= -2\left(X^\varepsilon(t), i\nabla X^\varepsilon(t)\nabla(\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle})\right) \\ &= 2Im\left(\int_{\mathbb{R}^n}\bar{X}^\varepsilon(t)\nabla X^\varepsilon(t).\nabla\left(\langle x\rangle^{2\zeta}e^{-\delta\langle x\rangle}\right)\right).\end{aligned}$$

Note that X^ϵ is sufficiently smooth and no regularization argument is necessary here.

Since $\nabla(\langle x\rangle^{2\zeta}) = 2\zeta\langle x\rangle^{2\zeta-2}x$, we get

$$\begin{aligned}\varphi_\delta(X^\varepsilon(t)) =& \varphi_\delta(u_0^\varepsilon) \\ &+ 2Im\left(\int_0^t\int_{\mathbb{R}^n}\bar{X}^\varepsilon(t)\nabla X^\varepsilon(t).\left((2\zeta\langle x\rangle^{2\zeta-2} - 2\zeta\delta\langle x\rangle^{4\zeta-2})e^{-\delta\langle x\rangle}x\right)\right)dx,\end{aligned} \tag{1.3.5}$$

using then $|x| \leqslant \langle x\rangle$ yields

$$\begin{aligned}\varphi_\delta(X^\varepsilon(t)) &\leqslant \varphi_\delta(u_0^\varepsilon) \\ &+4\zeta\left(\int_0^t\int_{\mathbb{R}^n}\left|X^\varepsilon(t)\langle x\rangle^\zeta e^{-\frac{\delta}{2}\langle x\rangle^{2\zeta}}\right|\left|\nabla X^\varepsilon(t).\left((\langle x\rangle^{\zeta-2} - \delta\langle x\rangle^{3\zeta-2})e^{-\frac{\delta}{2}\langle x\rangle}\right)\langle x\rangle\right|\right)dx\end{aligned}$$

and since $t(t^{\zeta-2} - \delta t^{3\zeta-2})e^{-\frac{\delta}{2}t}$ is bounded for $t \geqslant 1$ and $\delta \leqslant 1$, and $\varphi_\delta(u_0^\varepsilon) \leqslant \left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2$, we deduce

$$\varphi_\delta(X^\varepsilon(t)) \leqslant \left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \mathcal{C}\int_0^t\|\nabla X^\varepsilon(s)\|_{L^2}\sqrt{\varphi_\delta(X^\varepsilon(s))}ds,$$

which, thanks to Young inequality and Gronwall lemma, leads to

$$\varphi_\delta(X^\varepsilon(t)) \leqslant \mathcal{C}\left(\left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t\|\nabla X^\varepsilon(s)\|_{L^2}^2 ds\right).$$

Finally, by Fatou's lemma, letting $\delta \downarrow 0$:

$$\left\|X^\varepsilon(t)\langle x\rangle^\zeta\right\|_{L^2}^2 \leqslant \mathcal{C}\left(\left\|u_0\langle x\rangle^\zeta\right\|_{L^2}^2 + \int_0^t\|\nabla X^\varepsilon(s)\|_{L^2}^2 ds\right)$$

which is exactly (1.3.4). Then by (1.3.1), we see that the right-hand side of (1.3.4) is bounded, which implies $X^\varepsilon \in C_w([0,T], \Sigma^\zeta)$, where the subscript w indicates weak continuity in time. Letting $\delta \downarrow 0$ in (1.3.5) we see that

$$\left\| X^\varepsilon(t)\langle x\rangle^\zeta \right\|_{L^2}^2 = \left\| u_0\langle x\rangle^\zeta \right\|_{L^2}^2 + 4\zeta Im \int_0^t \left(\int_{\mathbb{R}^n} X^\varepsilon(s)\langle x\rangle^{2\zeta-2}\nabla X^\varepsilon(s).xdx \right) ds$$

and since the right-hand side is continuous, we get the strong continuity: $X^\varepsilon \in C([0,T],\Sigma^\zeta)$. ☐

1.3.3 Generator of $(X^\varepsilon, m^\varepsilon)$

In order to determine the law of the limiting process X, we need to identify the generator of X^ε. Clearly, X^ε is not a Markov process, because its increments depend on m^ε, but the couple $(X^\varepsilon, m^\varepsilon)$ is a Markov process, since m is.

We compute now the generator $\mathcal{L}^\varepsilon$ of the process $(X^\varepsilon, m^\varepsilon)$. We are able to compute this generator acting on functions such as in the next definition.

Definition 1.3.3.1 (Good test function)**.** *We say that $\Psi : H^1_{\mathbb{C}} \times E \longrightarrow \mathbb{R}, (u,n) \mapsto \Psi(u,n)$ is a good test function if the following holds:*

— *Ψ is continuously differentiable with respect to u, the differential is denoted by $D\Psi$.*

— *Ψ is subpolynomial in u and n; $\exists\, C_\Psi, \exists\, p_1 \in \mathbb{N}, \exists\, \gamma_1 < \gamma, \forall (u,n) \in H^1 \times E$,*

$$|\Psi(u,n)| \leqslant C_\Psi(1 + \|u\|_{H^1}^{p_1})(1 + \|n\|_E^{\gamma_1}).$$

— *$(u,n) \mapsto D\Psi(u,n)$ is continuous from $H^1 \times E$ to $\mathcal{L}(H^1_{\mathbb{C}};\mathbb{C})$.*

— *$D\Psi(u,n)$ is continuous with respect to the H^{-1} norm and satisfies the following subpolynomial bound in u: $\exists\, C_\Psi, \exists\, p_2 \in \mathbb{N}, \exists\, \gamma_2 < \gamma - 1, \forall (u,h,n) \in H^1 \times H^{-1} \times E$,*

$$|D\Psi(u,n).(h)| \leqslant C_\psi(1 + \|u\|_{H^1})^{p_2}(1 + \|n\|_E^{\gamma_2}) \|h\|_{H^{-1}}.$$

— *$\forall u \in H^1_{\mathbb{C}}, \Psi(u,.) \in \mathcal{D}_\mathcal{M}$*

— *$\mathcal{M}\Psi$ is continuous and subpolynomial in u and n; $\exists\, C_\Psi, \exists\, p_3 \in \mathbb{N}, \exists\, \gamma_3 < \gamma, \forall (u,n) \in H^1 \times E$,*

$$|\mathcal{M}\Psi(u,n)| \leqslant C_\Psi(1 + \|u\|_{H^1}^{p_3})(1 + \|n\|_E^{\gamma_3}).$$

Remark 1.3.3.1. *We have not defined the infinitesimal generator of $\mathcal{M}$. The fifth assumption in Definition 1.3.3.1 should be understood as: for each $u \in H^1_{\mathbb{C}}$, $\psi(u,\cdot) \in \mathcal{D}_\mathcal{M}$*

and there exists $\varphi(u,\cdot) \in \mathcal{P}_{\mathcal{M}}$ *such that* $\psi(u,\cdot) = \mathcal{M}^{-1}\varphi(u,\cdot)$*. Then we write* $\varphi(u,\cdot) = \mathcal{M}\psi(u,\cdot)$*. We also need to consider functions* $\psi(u)$ *which do not depend on* n*, in this case we set* $\mathcal{M}\psi(u) = 0$*.*

With these good test functions, we may identify the generator. For the definition of the predictable quadratic variation of a martingale, we refer to [JS03] and recall that it coincides with the quadratic variation when the martingale is continuous.

Proposition 1.3.3. *For* φ *a good test function, the infinitesimal generator* $\mathcal{L}^\varepsilon$ *of* $(X^\varepsilon, m^\varepsilon)$ *is given by the formula:*

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi(u,n) =& \frac{1}{\varepsilon^2}\mathcal{M}\varphi(u,n)\\ &+ \frac{1}{\varepsilon}(D\varphi(u,n).(-iun))\\ &+ D\varphi(u,n).(i\Delta u - i\lambda|u|^{2\sigma}u)\end{aligned}$$

for $u \in H^1_{\mathbb{C}}$*,* $n \in E$*. More precisely, if* $\mathbb{E}\,\|X_0^\varepsilon\|_{H^1}^p < \infty$ *for all* $p > 0$*:*

$$M_\varphi(t) = \varphi(X^\varepsilon(t), m^\varepsilon(t)) - \varphi(X_0^\varepsilon, m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi(X^\varepsilon(s), m^\varepsilon(s))ds$$

is a càdlàg and integrable $(\mathcal{F}_t^\varepsilon)$ *zero-mean martingale. If furthermore* γ_1*,* γ_2+1*,* $\gamma_3 < \gamma/2$ *and* φ^2 *is also a good test function, its continuous quadratic variation is given by*

$$\langle M_\varphi, M_\varphi\rangle_t = \int_0^t (\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s), m^\varepsilon(s))ds. \tag{1.3.6}$$

Proof. Let $t \geqslant s \geqslant 0$ and $s = t_0 < \dots < t_n = t$ be a subdivision of $[s,t]$ with $sup_i|t_{i+1}-t_i| = \delta$. Given g a $\mathcal{F}_s^\varepsilon$-measurable and bounded function, we have

$$\mathbb{E}\left[(\varphi(X^\varepsilon(t), m^\varepsilon(t)) - \varphi(X_0^\varepsilon, m^\varepsilon(0)))\, g\right] = \mathbb{E}\left[\left(\int_s^t \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right) g\right] + I + II,$$

where

$$
\begin{aligned}
I =& \sum_{i=0}^{n-1} \mathbb{E}\left[(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_{i+1}))\right. \\
& - \int_{t_i}^{t_{i+1}} D\varphi(X^\varepsilon(s), m^\varepsilon(s)).(i\Delta X^\varepsilon(s) - i\lambda |X^\varepsilon(s)|^{2\sigma} X^\varepsilon(s) - \frac{i}{\varepsilon} X^\varepsilon(s) m^\varepsilon(s)) ds \ )\left. g\right] \\
=& \mathbb{E}\left[\left(\int_0^t i_\delta(s) ds\right) g\right]
\end{aligned}
$$

with

$$
i_\delta(s) = -\sum_{i=0}^{n-1} \mathbf{1}_{[t_i, t_{i+1}]}(s) \left(D\varphi(X^\varepsilon(s), m^\varepsilon(t_{i+1})) - D\varphi(X^\varepsilon(s), m^\varepsilon(s))\right) . \frac{dX^\varepsilon}{dt}(s),
$$

and

$$
\begin{aligned}
II =& \sum_{i=0}^{n-1} \mathbb{E}\left[(\varphi(X^\varepsilon(t_i), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))\right. \\
& - \frac{1}{\varepsilon^2} \int_{t_i}^{t_{i+1}} \mathcal{M}\varphi(X^\varepsilon(s), m^\varepsilon(s)) ds \ )\left. g\right] \\
=& \mathbb{E}\left[\left(\int_0^t ii_\delta(s) ds\right) g\right].
\end{aligned}
$$

To treat i_δ, we write $\frac{dX^\varepsilon}{dt}(\sigma) = i\Delta X^\varepsilon(s) - i|X^\varepsilon)(s)|^{2\sigma} X^\varepsilon(s) - i\frac{1}{\varepsilon} X^\varepsilon(s) m^\varepsilon(s)$, and since $D\varphi$ is regularizing and subpolynomial, we have

$$
\left| D\varphi(X^\varepsilon(s), m^\varepsilon(s)) \cdot \left(i\Delta X^\varepsilon + \frac{X^\varepsilon m^\varepsilon}{\varepsilon}(s) \right) \right| \leqslant C_{\epsilon,\varphi}(1 + \|X^\varepsilon(s)\|_{H^1}^{p_2+1})(1 + \|m^\varepsilon(s)\|_E^{\gamma_2+1}). \tag{1.3.7}
$$

Also, since $\sigma < \frac{2}{n-2}$, we know that H^1 is embedded in $L^{2\sigma+2}$ and by duality $L^{\frac{2\sigma+2}{2\sigma+1}}$ is embedded in H^{-1}. We deduce:

$$
\begin{aligned}
|D\varphi(X^\varepsilon, n) \cdot (|X^\varepsilon|^{2\sigma} X^\varepsilon)| &\leqslant C_{\epsilon,\varphi}(1 + \|X^\varepsilon(s)\|_{H^1}^{p_2})(1 + \|m^\varepsilon(s)\|_E^{\gamma_2}) \, \||X^\varepsilon|^{2\sigma} X^\varepsilon\|_{L^{\frac{2\sigma+2}{2\sigma+1}}} \\
&\leqslant C_{\epsilon,\varphi}(1 + \|X^\varepsilon(s)\|_{H^1}^{p_2})(1 + \|m^\varepsilon(s)\|_E^{\gamma_2}) \, \|X^\varepsilon\|_{H^1}^{2\sigma+1}
\end{aligned}
$$

which gives, thanks to (1.3.1) and (1.2.9), the uniform (in (s, ω)) integrability of i_δ Moreover, since X^ε is almost surely continuous and m^ε is stochastically continous, $D\varphi(X^\varepsilon(s), m^\varepsilon(t_{i+1})) - D\varphi(X^\varepsilon(s), m^\varepsilon(s))$ converges to 0 when $\delta \to 0$ in probability, so I converges to 0, as δ tends to 0.

We claim that

$$ii_\delta(s) = \frac{1}{\varepsilon^2}\sum_{i=0}^{n-1}\mathbf{1}_{[t_i,t_{i+1}]}\left(\mathcal{M}\varphi(X^\varepsilon(t_i),m^\varepsilon(s)) - \mathcal{M}\varphi(X^\varepsilon(s),m^\varepsilon(s))\right).$$

Indeed, for $u \in H^1$, $\varphi(u,\cdot) \in \mathcal{D}_{\mathcal{M}}$ (see Definition 1.3.3.1) and

$$M^\varepsilon(u,t) = \varphi(u,m^\varepsilon(t)) - \varphi(u,m(0)) - \int_0^t \frac{1}{\varepsilon^2}\mathcal{M}\varphi(u,m^\varepsilon(s))ds$$

is a $(\mathcal{F}_t^\varepsilon)$-martingale. As $X^\varepsilon(t_i)$ and g are $\mathcal{F}_{t_i}^\varepsilon$-measurable, we get

$$\begin{aligned}\mathbb{E}\left([\varphi(X^\varepsilon(t_i),m^\varepsilon(t_{i+1}))-\right.&\left.\varphi(X^\varepsilon(t_i),m^\varepsilon(t_i))]\,g\right)\\ &= \frac{1}{\varepsilon^2}\mathbb{E}\left(\left[\int_{t_i}^{t_{i+1}}\mathcal{M}\varphi(X^\varepsilon(t_i),m^\varepsilon(s))ds\right]g\right).\end{aligned}$$

Since $\mathcal{M}\varphi$ satisfies the polynomial bound in Definition 1.3.3.1, we get the uniform integrability of ii_δ. The convergence in probability to 0 comes from the continuity of $\mathcal{M}\varphi$.

The quadratic variation $\langle M,M\rangle$ is characterized by the property that $M^2(t) - \int_0^t\langle M,M\rangle(s)ds$ is a martingale. Let $s < t$ and $(t_i)_i$ be a subdivision of $[s,t]$ with $\sup_i|t_{i+1}-t_i| = \delta$. Recall the following sequence of identity:

$$\begin{aligned}&\mathbb{E}(M^2(t) - M^2(s) - \int_0^t(\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s),m^\varepsilon(s))ds\\ &= \sum_i\mathbb{E}(M^2(t_{i+1}) - M^2(t_i) - \int_{t_i}^{t_{i+1}}(\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s),m^\varepsilon(s))ds\\ &= \sum_i\mathbb{E}((M(t_{i+1}) - M(t_i))^2 - \int_{t_i}^{t_{i+1}}(\mathcal{L}^\varepsilon\varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s),m^\varepsilon(s))ds,\end{aligned} \tag{1.3.8}$$

where the last equality follows from

$$\mathbb{E}((M(t_{i+1}) - M(t_i))^2) = \mathbb{E}(M^2(t_{i+1}) - 2M(t_{i+1})M(t_i) + M^2(t_i))$$

and

$$\mathbb{E}(M(t_{i+1})M(t_i)) = \mathbb{E}(\mathbb{E}(M(t_{i+1})M(t_i)|\mathcal{F}_{t_i})) = \mathbb{E}(M^2(t_i)).$$

Hence, it suffices to show that the right hand side of (1.3.8) goes to zero as $\delta \to 0$.

Let us write:

$$
\begin{aligned}
(M(t_{i+1}) - M(t_i))^2 = &\Big(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i)) \\
&- \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\Big)^2 \\
=&\varphi^2(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) \\
&- 2\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1}))\varphi(X^\varepsilon(t_i), m^\varepsilon(t_i)) + \varphi^2(X^\varepsilon(t_i), m^\varepsilon(t_i)) \\
&- 2(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))) \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma \\
&+ \left(\int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right)^2.
\end{aligned}
$$

Applying the above to φ^2 implies that the process

$$
\tilde{M}(t) = \varphi^2(X^\varepsilon(t), m^\varepsilon(t)) - \varphi^2(X^\varepsilon(0), m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi^2(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma
$$

is a martingale for the filtration generated by m^ε. We can now write

$$
\begin{aligned}
\sum_{i=0}^{n-1}(M(t_{i+1}) - M(t_i))^2 = &\sum_{i=0}^{n-1} \tilde{M}(t_{i+1}) - \tilde{M}(t_i) + \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi^2(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma \\
&- 2\varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))\Big(\varphi\big(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))\big)\Big) \\
&- 2(\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))) \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma \\
&+ \left(\int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right)^2
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_i \mathbb{E}((M(t_{i+1}) - M(t_i))^2 - \int_{t_i}^{t_{i+1}} (\mathcal{L}^\varepsilon \varphi^2 - 2\varphi\mathcal{L}^\varepsilon\varphi)(X^\varepsilon(s), m^\varepsilon(s))ds \\
&= 2\sum_i \mathbb{E}\Bigg(\int_{t_i}^{t_{i+1}} \varphi\mathcal{L}^\varepsilon\varphi(X^\varepsilon(s), m^\varepsilon(s))ds - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i)) \\
&\qquad \times (\varphi\left(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})\right) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i)))\Bigg) \\
&-2\sum_i \mathbb{E}\left((\varphi(X^\varepsilon(t_{i+1}), m^\varepsilon(t_{i+1})) - \varphi(X^\varepsilon(t_i), m^\varepsilon(t_i))) \int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right) \\
&+\sum_i \mathbb{E}\left(\left(\int_{t_i}^{t_{i+1}} \mathcal{L}^\varepsilon\varphi(X^\varepsilon(\sigma), m^\varepsilon(\sigma))d\sigma\right)^2\right).
\end{aligned}$$

Again, the inequalities (1.2.9) and (1.3.1) and uniform integrability can be used to prove that the three terms of the right hand side go to zero under the extra assumption that $\gamma_1,\ \gamma_2 + 1,\ \gamma_3 < \gamma/2$.

□

Remark 1.3.3.2. *This proof is not completely rigourous. Indeed, we have differentiated* $\varphi(X^\varepsilon(t))$ *with respect to* t *but we do not know whether* X^ϵ *is* C^1 *with values in* H^1*. This is easily overcome by a regularization argument as in the proof of Proposition 1.3.1: we replace* φ *by* $\varphi_\delta = \varphi(\rho_\delta * \cdot)$ *and let* $\delta \to 0$ *at the end of the proof.*

1.4 The perturbed test function method

1.4.1 Correctors

From the expression of $\mathcal{L}^\varepsilon\varphi(u, n)$ in Proposition 1.3.3, we see that the negative powers of ε are present. The term of order -2 cancels if φ does not depend on n. Since we are interested only in the behaviour of X^ε when $\varepsilon \to 0$, it is natural to consider such functions. To treat the -1 order term we need to add correctors to φ. Assuming $\varphi_1,\ \varphi_2$ are good test functions we have

$$
\begin{aligned}
\mathcal{L}^\varepsilon(\varphi+\varepsilon\varphi_1+\varepsilon^2\varphi_2)(u,n) = &\frac{1}{\varepsilon^2}\mathcal{M}\varphi(u) \\
&+\frac{1}{\varepsilon}\left(D\varphi(u).(-iun)+\mathcal{M}\varphi_1(u,n)\right) \\
&+\mathcal{M}\varphi_2(u,n)+D\varphi_1(u,n).(-iun)+D\varphi(u).(i\Delta u-i\lambda|u|^{2\sigma}u) \\
&+\varepsilon\left(D\varphi_2(u).(-iun)+D\varphi_1(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u)\right) \\
&+\varepsilon^2 D\varphi_2(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u),
\end{aligned}
$$

where the notation $D\varphi$ denotes the derivative with respect to u. Let us compute formally the correctors. As already mentionned, the -2 order term vanishes because we chose a function φ depending only on u. The first corrector is chosen so that the -1 order term cancels. It is formally given by:

$$
\varphi_1(u,n)=\mathcal{M}^{-1}D\varphi(u)\cdot(iun)=D\varphi(u)\cdot(iu\mathcal{M}^{-1}n)=D\varphi(u)\cdot(iuL_1(n)).
$$

The second corrector enables to identify the limit generator. The average of the third line is given by

$$
\begin{aligned}
\mathcal{L}\varphi(u)=&D\varphi(u)\cdot(i\Delta u-i\lambda|u|^{2\sigma}u) \\
&+\mathbb{E}_\nu\left(D^2\varphi(u)\cdot(-iun,iu\mathcal{M}^{-1}n)+D\varphi(u)\cdot(un\mathcal{M}^{-1}n)\right) \\
=&D\varphi(u)\cdot(i\Delta u-i\lambda|u|^{2\sigma}u) \\
&+\mathbb{E}_\nu\left(D^2\varphi(u)\cdot(-iun,iuL_1(n))+D\varphi(u)\cdot(unL_1(n))\right)
\end{aligned}
\tag{1.4.1}
$$

and we choose φ_2 such that:

$$
\mathcal{M}\varphi_2(u,n)+D\varphi_1(u,n).(-iun)+D\varphi(u).(i\Delta u-i\lambda|u|^{2\sigma}u)-\mathcal{L}\varphi(u)=0.
$$

In this way, we formally get $\mathcal{L}^\varepsilon(\varphi+\varepsilon\varphi_1+\varepsilon^2\varphi_2)(u,n)\to\mathcal{L}\varphi(u)$ and we indeed identify the limit generator.

We do not need to justify rigorously the above computation for many functions. As we shall see below, for our purpose, it is sufficient to consider test functions of the form: $\varphi(u)=(u,h)^\ell$ for $h\in H^1$, $\ell=1,2$. It is clearly a good test function and satisfies all assumptions of Proposition 1.3.3.

Proposition 1.4.1 (First corrector)**.** *Let $\varphi(u)=(u,h)^\ell$ with $h\in H^1$, $\ell=1,2$. Then there* *exists φ_1 a good test function such that:*

$$D\varphi(u).(-iun) + \mathcal{M}\varphi_1(u,n) = 0 \quad \forall u,n \in H^1_{\mathbb{C}} \times E$$

Moreover, $\varphi_1 = \ell(u,h)^{\ell-1}(iuL_1(n),h)$ and satisfies all assumptions of Proposition 1.3.3.

Proof of Proposition 1.4.1. For any $k \in H^1$, $D\varphi(u) \cdot k = \ell(u,h)^{\ell-1}(k,h)$. Therefore:

$$\begin{aligned}\int_E D\varphi(u).(-iun)d\nu(n) =& \ell(u,h)^{\ell-1}\int_E (-iun,h)d\nu(n) = \ell(u,h)^{\ell-1}(-iu\int_E nd\nu(n),h)\\ =& 0,\end{aligned}$$

where ν is the law of $m(t)$, which is centered. Thanks to our assumptions, φ_1 is given by

$$\varphi_1(u,n) = \ell(u,h)^{\ell-1}(iuL_1(n),h).$$

By (1.2.10), we easily see that this is a good test function and Proposition 1.3.3 applies. $\square$

Note that thanks to (1.2.10), we have:

$$|\varphi_1(u,n)| \leqslant C(1+\|n\|_E)^\eta \|u\|^\ell_{H^1}\|h\|^\ell_{H^1}.$$

Moreover, for $k \in H^1$,

$$D\varphi_1(u,n)\cdot k = \ell(u,h)^{\ell-1}(ikL_1(n),h) + \ell(\ell-1)(k,h)(iuL_1(n),h).$$

We now compute the second corrector. For the test function $\varphi(u) = (u,h)^\ell$, we have

$$\begin{aligned}\mathcal{L}\varphi(u) =& \ell(u,h)^{\ell-1}(i\Delta u - i\lambda|u|^{2\sigma}u,h)\\ &+ \mathbb{E}_\nu\left(\ell(\ell-1)(-iun,h)(iuL_1(n),h) + \ell(u,h)^{\ell-1}(unL_1(n),h)\right).\end{aligned}$$

The equation for φ_2 then writes:

$$\begin{aligned}&\mathcal{M}\varphi_2(u,n) - \ell(\ell-1)(iun,h)(iuL_1(n),h) + \ell(u,h)^{\ell-1}(unL_1(n),h))\\ &-\mathbb{E}_\nu\left(\ell(\ell-1)(-iun,h)(iuL_1(n),h) + \ell(u,h)^{\ell-1}(unL_1(n),h)\right) = 0.\end{aligned}$$

The following proposition is again a straigthforward application of our assumptions.

Proposition 1.4.2 (Second corrector). *Let $\varphi(u) = (u,h)^\ell$ with $h \in H^1$, $\ell = 1,2$. Then there exists φ_2 a good test function such that:*

$$\mathcal{M}\varphi_2(u,n) - \ell(\ell-1)(iun,h)(iuL_1(n),h) + \ell(u,h)^{\ell-1}(unL_1(n),h))$$
$$-\mathbb{E}_\nu\left(\ell(\ell-1)(-iun,h)(iuL_1(n),h) + \ell(u,h)^{\ell-1}(unL_1(n),h)\right) = 0.$$

Moreover, φ_2 satisfies all assumptions of Proposition 1.3.3.

Clearly, φ_2 is given by:

$$\varphi_2(u,n) = -(uL_2(n),h)$$

if $\ell = 1$ and

$$\varphi_2(u,n) = -2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)u(y)L_4(n)(x,y)\bar{h}(x)\bar{h}(y)dxdy - 2(u,h)(uL_2(n),h)$$

for $\ell = 2$. Thanks to (1.2.10), we have in both cases:

$$|\varphi_2(u,n)| \leqslant C(1+\|n\|_E)^\eta \|u\|_{H^1}^\ell \|h\|_{H^1}^\ell.$$

Lemma 1.4.1.1 (Perturbed test-function method). *Let $\varphi(u) = (u,h)^\ell$, where $h \in H^1$, $\ell = 1,2$, and φ_1, φ_2 given by Propostions 1.4.1 and 1.4.2. For $\varepsilon \in (0,1)$, we define $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$. Then φ^ε verifies for $u \in H^1$, $n \in E$:*

1. $$|\varphi^\varepsilon(u,n) - \varphi(u)| \leqslant C\varepsilon\|u\|_{H^1}^\ell\|h\|_{H^1}^\ell(1+\|n\|_E)^\eta.$$

2. $$|\mathcal{L}^\varepsilon\varphi^\varepsilon(u,n) - \mathcal{L}\varphi(u)| \leqslant C\varepsilon(1+\|u\|_{H^1}^{2\sigma+\ell})\|h\|_{H^1}^\ell(1+\|n\|_E)^{\eta+1}. \tag{1.4.2}$$

3. *The process*

$$M_{\varphi^\varepsilon}(t) = \varphi^\varepsilon(X^\varepsilon(t), m^\varepsilon(t)) - \varphi^\varepsilon(X_0^\varepsilon, m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s), m^\varepsilon(s))ds$$

is a càdlàg and integrable $(\mathcal{F}_t)$ zero-mean martingale.

Proof of Lemma 1.4.1.1. We treat the case $\ell = 1$. The case $\ell = 2$ is similar but lengthier. The first assertion clearly follows from the bound we have written above on φ_1 and φ_2.

By definition of $\mathcal{L}\varphi$:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(u,n)-\mathcal{L}\varphi(u)=&\varepsilon\left(D\varphi_1(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u)+\varepsilon D\varphi_2(u,n).(-iun)\right)\\&+\varepsilon^2 D\varphi_2(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u).\end{aligned}$$

Recalling the expressions of φ_1, φ_2, we have

$$\begin{aligned}&\left|D\varphi_1(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u)+\varepsilon D\varphi_2(u,n).(-iun)\right|\\&\leqslant C|h|_{H^1}(|L_1(n)|_E\|i\Delta u-i\lambda|u|^{2\sigma}u\|_{H^{-1}}+\varepsilon|u|_{H^1}\|nL_2(n)\|_E)\end{aligned}$$

and

$$\begin{aligned}&\left|D\varphi_2(u,n).(i\Delta u-i\lambda|u|^{2\sigma}u)\right|\\&\leqslant C|h|_{H^1}|L_2(n)|_E\|i\Delta u-i\lambda|u|^{2\sigma}u\|_{H^{-1}}.\end{aligned}$$

We have seen in the proof of Proposition 1.3.3 that

$$\|i\Delta u-i\lambda|u|^{2\sigma}u\|_{H^{-1}}\leqslant C(1+\|u\|_{H^1})^{2\sigma+1}.$$

The estimate of the second assertion follows easily. By Proposition 1.4.1, Proposition 1.4.2 we know that φ^ε is a good test function and satisfies the assumptions of Proposition 1.3.3 so that the third point is also clear. □

1.4.2 Tightness of the process X^ε

In this subsection, we aim to obtain tightness of the family of stopped processes $(X^{\varepsilon,\tau^\varepsilon})_\varepsilon$, where $X^{\varepsilon,\tau^\varepsilon}(t)=X^\varepsilon(t\wedge\tau^\varepsilon)$. The definition (1.2.13) of τ^ε depends on α. We choose $\alpha<2/\gamma$ such that $\alpha(\eta+1)<1$.

The crucial ingredient used in previous works on diffusion approximation in infinite dimension is an assumption on uniform boundedness of the driving process in the adequate functional space, which would be $L^\infty(0,T;E)$ in our case, w.r.t. ε (see [DV12] and [dBG12]). Under our weaker assumptions, the result remains true provided we use the stopping time τ^ε. We will see that this is sufficient to conclude.

Proposition 1.4.3. *Assuming $X_0^\varepsilon\to X_0,\mathbb{P}-a.s.$ in the space Σ^ζ for some $0<\zeta\leqslant 1$, then the family of processes $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon>0}$ is tight in $C([0,T],H^s)$ for $s<1$.*

This result strongly relies on the following a priori estimate.

Lemma 1.4.2.1. *Assume that* $\mathbb{E}\left(\|X_0^\varepsilon\|_{L^2} + H(X_0^\varepsilon)\right)^2 \leqslant \mathcal{C}$ *(independently of* ε*), and let* X^ε *be the solution of* (1.1.1) *with initial data* X_0^ε *and* τ^ε *the stopping time introduced in* (1.2.13)*. Then for any stopping time* $\tau \leqslant T$*, there exists a constant* $\mathcal{C}(T)$ *depending on* T *but not on* ε *such that for* $t \in [0,T]$*:*

$$\mathbb{E}\left[\left\|X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{H^1}\right] \leqslant \mathcal{C}(T). \tag{1.4.3}$$

The proofs of Lemma 1.4.2.1 and Proposition 1.4.3 are technical and are postponed to Section 1.6.

We remark that this lemma, together with (1.3.4) and assuming $\mathbb{E}\left\|X_0^\varepsilon\langle x\rangle^\zeta\right\|_{L^2}^2 \leqslant \mathcal{C}$ show that the following inequality holds

$$\mathbb{E}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{\Sigma^\zeta}^p\right) \leqslant \mathcal{C}. \tag{1.4.4}$$

1.5 Proof of Theorem 1.1.1

The proof of Theorem 1.1.1 is divided into 3 steps. First we identify the SPDE associated to $\mathcal{L}$, then we prove the weak convergence of X^ε to X solution of (1.1.4), linked to $\mathcal{L}$, and finally, we conclude using the uniqueness of the solution.

Proof of Theorem 1.1.1. Step 1: Identification of the limiting generator

For $h \in H^{2-s}$ with $s < 1$, we use the functionals $\varphi(u) = (u,h)$ and φ^2, which are clearly good test functions. We now compute $\mathcal{L}\varphi$ and $\mathcal{L}\varphi^2$,

$$\begin{aligned}
\mathcal{L}\varphi(u) &= D\varphi(u).(i\Delta u - i\lambda|u|^{2\sigma}u) + \mathbb{E}_\nu\left(\underbrace{D^2\varphi(-iun, iuL_1(n))}_{0} + D\varphi(u).(unL_1(n))\right)\\
&= \left(i\Delta u - i\lambda|u|^{2\sigma}u + \mathbb{E}_\nu unL_1(n), h\right)\\
&= \left(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF, h\right),
\end{aligned}$$

where F is given by (1.2.23) (see (1.2.26)). In the case of φ^2, we again have

$$D\varphi^2(u).(i\Delta u - i\lambda|u|^{2\sigma}u) + \mathbb{E}_\nu\left(D\varphi^2(u).(un\mathcal{M}^{-1}n)\right) = D\varphi^2(u).(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF)$$

but now the term $\mathbb{E}_\nu D^2\varphi(-iun, iu\mathcal{M}^{-1}n)$ does not vanish,

$$\begin{aligned}\mathbb{E}_\nu D^2\varphi^2(-iun, iuL_1(n)) &= 2\int_E (-iun, h)(iuL_1(n), h)d\nu(n)\\ &= 2\mathbb{E}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m(0)(x)Re(iu(x)\bar{h}(x))\big(\int_0^\infty m(t)(y)dt\big)Re(iu(y)\bar{h}(y))dxdy\Big)\\ &\quad \text{since } m(t) \text{ is real-valued}\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x,y)Re(iu(x)\bar{h}(x))Re(iu(y)\bar{h}(y))dxdy.\end{aligned}$$

Let us denote now by q the kernel of $Q^{1/2}$, where Q is given by (1.2.24). Then we have

$$k(x,y) = \int_{\mathbb{R}^n} q(x,z)q(z,y)dz$$

and

$$\begin{aligned}&\mathbb{E}_\nu D^2\varphi^2(-iun, iuL_1(n))\\ &\qquad= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} q(x,z)q(z,y)Re(iu(x)\bar{h}(x))Re(iu(y)\bar{h}(y))dzdxdy\\ &\qquad= \int_{\mathbb{R}^n}\big(iuq(.,z),h\big)\big(iuq(.,z),h\big)dz\\ &\qquad= \frac{1}{2}Tr\big(D^2\varphi^2.(iuQ^{1/2}, iuQ^{1/2})\big).\end{aligned}$$

Finally, we have

$$\mathcal{L}\varphi(u) = D\varphi(u).(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF) \tag{1.5.1}$$

and

$$\mathcal{L}\varphi^2(u) = D\varphi^2(u).(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF) + \frac{1}{2}Tr\big(D^2\varphi^2.(iuQ^{1/2}, iuQ^{1/2})\big). \tag{1.5.2}$$

Step 2: Convergence

Given $0 < \zeta \leqslant 1$, by Propostion 1.4.3, we have a subsequence of $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon>0}$, still denoted by $(X^{\varepsilon,\tau^\varepsilon})$, of law P^ε and a probability measure P on $C([0,T],H^s)$ such that

$$P^\varepsilon \to P \text{ weakly on } C([0,T],H^s).$$

Since $[0,T]$ is compact and H^s is separable, $C([0,T],H^s)$ is also separable, and by Skohorod theorem (see [Bil99]) there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $\tilde{X}^\varepsilon, \tilde{X}$ random variables on $\tilde{\Omega}$ with values in $C([0,T], H^s)$ such that

$$\tilde{X}^\varepsilon \to \tilde{X} \text{ in } C([0,T], H^s), \quad \tilde{\mathbb{P}} \text{ a.s. as } \varepsilon \to 0$$

and $\mathcal{L}(\tilde{X}^\varepsilon) = P^\varepsilon$ and $\mathcal{L}(\tilde{X}) = P$. For $h \in H^{2-s}$, we use the test function $\varphi(u) = (u, h)$ and φ_1, φ_2 the correctors given by Propositions 1.4.1 and 1.4.2. We define $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$, then by Lemma 1.4.1.1, the process

$$M(t) = \varphi^\varepsilon(X^\varepsilon(t), m^\varepsilon(t)) - \varphi^\varepsilon(X^\varepsilon(0), m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s), m^\varepsilon(s))ds$$

is a martingale for the filtration $(\mathcal{F}_t^\varepsilon)_t$, and the stopped process $M(t \wedge \tau^\varepsilon)$ is also a martingale, that is, for all $0 \leqslant s_1 \leqslant \ldots \leqslant s_n \leqslant s \leqslant t$ and $g \in C_b\big((H^1)^n\big)$

$$\begin{aligned}\mathbb{E}\Big(\big(&\varphi^\varepsilon(X^\varepsilon(t\wedge\tau^\varepsilon), m^\varepsilon(t\wedge\tau^\varepsilon)) - \varphi^\varepsilon(X^\varepsilon(s\wedge\tau^\varepsilon), m^\varepsilon(s\wedge\tau^\varepsilon)) \\ &- \int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s'), m^\varepsilon(s'))ds'\big) g(X^\varepsilon(s_1\wedge\tau^\varepsilon), \ldots, X^\varepsilon(s_n\wedge\tau^\varepsilon))\Big) = 0.\end{aligned}$$

Moreover, we easily have

$$\begin{aligned}&\int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s'), m^\varepsilon(s'))ds'\big) g(X^\varepsilon(s_1\wedge\tau^\varepsilon), \ldots, X^\varepsilon(s_n\wedge\tau^\varepsilon)) \\ &\quad = \int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(s'), m^{\varepsilon\wedge\tau^\varepsilon}(s))ds'\big) g(X^{\varepsilon,\tau^\varepsilon}(s_1), \ldots, X^{\varepsilon,\tau^\varepsilon}(s_n)).\end{aligned}$$

Then we get:

$$\begin{aligned}
&\mathbb{E}\Big(\big(\varphi(X^{\varepsilon,\tau^\varepsilon}(t))-\varphi(X^{\varepsilon,\tau^\varepsilon}(s))-\int_s^t \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'\big)g(X^{\varepsilon,\tau^\varepsilon}(s_1),...,X^{\varepsilon,\tau^\varepsilon}(s_n))\Big)\\
&\quad=\mathbb{E}\Big[\Big(-\varepsilon\big(\varphi_1(X^{\varepsilon,\tau^\varepsilon}(t),m^\varepsilon(t\wedge\tau^\varepsilon))-\varphi_1(X^{\varepsilon,\tau^\varepsilon}(s),m^\varepsilon(s\wedge\tau^\varepsilon))\big)\\
&\quad-\varepsilon^2\big(\varphi_2(X^{\varepsilon,\tau^\varepsilon}(t),m^\varepsilon(t\wedge\tau^\varepsilon))-\varphi_2(X^{\varepsilon,\tau^\varepsilon}(s),m^\varepsilon(s\wedge\tau^\varepsilon))\big)\\
&\quad-\Big(\int_{s\wedge\tau^\varepsilon}^s \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'-\int_t^{t\wedge\tau^\varepsilon}\mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'\\
&\quad-\int_{s\wedge\tau^\varepsilon}^{t\wedge\tau^\varepsilon}(\mathcal{L}^\varepsilon\varphi^\varepsilon-\mathcal{L}\varphi)(X^\varepsilon(s'),m^\varepsilon(s'\wedge\tau^\varepsilon))ds'\Big)g(X^{\varepsilon,\tau^\varepsilon}(s_1),...,X^{\varepsilon,\tau^\varepsilon}(s_n))\Big]\\
&\\
&\quad=\mathbb{E}[T_1+T_2+T_3+T_4].
\end{aligned}$$

We now use Lemma 1.4.1.1 and Lemma 1.4.2.1 to get the bound:

$$\begin{aligned}
\mathbb{E}[T_1+T_2]&\leqslant C\varepsilon\mathbb{E}\left(\sup_{s'\in[0,T]}\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{H^1}\|h\|_{H^1}\left(1+\sup_{s'\in[0,T]}\|m^\varepsilon(s'\wedge\tau^\varepsilon)\|_E\right)^\eta\right)\\
&\leqslant C\varepsilon^{1-\alpha\eta}.
\end{aligned}$$

Similarly

$$\mathbb{E}[T_4]\leqslant C\varepsilon^{1-\alpha(\eta+1)}.$$

By the embedding $H^1\subset L^{2\sigma+2}$ (see (1.1.2)) and Hölder's inequality, we have

$$\begin{aligned}
\left|\mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))\right|\leqslant&\|\nabla h\|_{L^2}\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{L^2}+\|h\|_{L^{2\sigma+2}}\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{L^{2\sigma+2}}^{2\sigma+1}\\
&+\|F\|_{L^\infty}\|h\|_{L^2}\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{L^2}\\
\leqslant&\mathcal{C}(1+\left\|X^{\varepsilon,\tau^\varepsilon}(s')\right\|_{H^1})^{2\sigma+1}
\end{aligned}$$

and Lemma 1.4.2.1 yields

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\Big|\int_{s\wedge\tau^\varepsilon}^s\mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds'\Big)g(X^\varepsilon(s_1),...,X^\varepsilon(s_n))\Big|\Big)\Big]\\
&\quad\leqslant\mathcal{C}\mathbb{E}\left[(s-s\wedge\tau^\varepsilon)\left(1+\mathcal{C}(T)\right)^{2\sigma+1}\right]\\
&\quad\leqslant\mathcal{C}\left(\mathbb{E}\left[(s-s\wedge\tau^\varepsilon)^2\right]\right)^{1/2}:=r_1(\varepsilon),
\end{aligned}$$

with $r_1(\varepsilon) \to 0$ when $\varepsilon \to 0$ by Lemma 1.2.2.1 and the uniform integrability. Similarly, we have

$$\mathbb{E}\left|\int_t^{t\wedge\tau^\varepsilon} \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s'))ds' g(X^\varepsilon(s_1),...,X^\varepsilon(s_n))\right| \leqslant r_2(\varepsilon),$$

with $r_2(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Finally, we obtain

$$\left|\mathbb{E}\Big(\big(\varphi(X^{\varepsilon,\tau^\varepsilon}(t)) - \varphi(X^{\varepsilon,\tau^\varepsilon}(s)) - \textstyle\int_s^t \mathcal{L}\varphi(X^{\varepsilon,\tau^\varepsilon}(s')ds'\big) g(X^{\varepsilon,\tau^\varepsilon}(s_1),...,X^{\varepsilon,\tau^\varepsilon}(s_n))\Big)\right|$$
$$\leqslant C\varepsilon^{1-(\eta+1)\alpha} + r_1(\varepsilon) + r_2(\varepsilon), \tag{1.5.3}$$

where C does not depend on ε. Moreover, as $X^{\varepsilon,\tau^\varepsilon}$ and $\tilde{X}^\varepsilon$ have the same law, then (1.5.3) is also true by replacing $X^{\varepsilon,\tau^\varepsilon}$ by $\tilde{X}^\varepsilon$, and $\mathbb{P}$ by $\tilde{\mathbb{P}}$. Since φ, $\mathcal{L}\varphi$ and g are continuous from H^s to $\mathbb{R}$ (the continuity of $\mathcal{L}\varphi$ requires the continuity of the nonlinearity which is given by the embedding $H^s \subset L^{2\sigma+1}$), taking the limit $\varepsilon \to 0$, we get

$$\tilde{\mathbb{E}}\Big(\big(\varphi(\tilde{X}(t)) - \varphi(\tilde{X}(s)) - \int_s^t \mathcal{L}\varphi(\tilde{X}(s'))ds'\big) g(\tilde{X}(s_1),...,\tilde{X}(s_n))\Big) = 0. \tag{1.5.4}$$

Then the process

$$\tilde{M}(t) = \varphi(\tilde{X}(t)) - \varphi(\tilde{X}(0)) - \int_0^t \mathcal{L}\varphi(\tilde{X}(s))ds$$

is a martingale with respect to the filtration $\mathcal{G}_s$ generated by $\tilde{X}(s)$. Note that this martingale is continuous.

Similarly, we can pass to the limit $\varepsilon \to 0$ in the definition of the quadratic variation and obtain that the quadratic variation of $\tilde{M}$ is given by:

$$\langle\tilde{M},\tilde{M}\rangle(t) = \int_0^t \big(\mathcal{L}(\varphi)^2 - 2\varphi\mathcal{L}\varphi\big)(\tilde{X}(s))ds.$$

Note that this step requires the use of the perturbed test function method applied to $\varphi^2(u) = (u,h)^2$.

From (1.5.1) and (1.5.2), we deduce

$$\begin{aligned}\big(\mathcal{L}(\varphi)^2 - 2\varphi\mathcal{L}\varphi\big)(u) =& 2(u,h)(i\Delta u - i\lambda|u|^{2\sigma}u - \frac{1}{2}uF, h) + Tr\big((iuQ^{1/2}, h)^2\big) \\ & - 2(u,h)\Big((i\Delta u - \frac{i}{2}uF, h)\Big) \\ =& Tr\big((iuQ^{1/2}, h)^2\big)\end{aligned}$$

The continuous H^{-1}-valued martingale

$$\mathbf{M}(t) = \tilde{X}(t) - \tilde{X}(0) - \int_0^t i\Delta\tilde{X}(s) - i|\tilde{X}(s)|^{2\sigma}\tilde{X}(s) - \frac{1}{2}\tilde{X}(s)F$$

has the quadratic variation

$$\int_0^t \left(i\tilde{X}(s)Q^{1/2}\right)\left(i\tilde{X}(s)Q^{1/2}\right)^* ds$$

then, using the martingale representation theorem (see [DPZ14]) and up to enlarging the probability space, there exists a cylindrical Wiener process W such that

$$\mathbf{M}(t) = \int_0^t i\tilde{X}(s)Q^{1/2}dW(s).$$

Step 3: Uniqueness of the limit
Note that $\tilde{X}^\varepsilon$ also satisifies (1.4.3). Letting $\varepsilon \to 0$, we deduce that $\tilde{X} \in L^p(\tilde{\Omega}; L^\infty([0,T]; H^1(\mathbb{R}^n)))$ for any $p \geqslant 1$ is a martingale solution of (1.1.4). In particular, it has paths in $L^\infty([0,T]; H^1(\mathbb{R}^n))$. We know from [BRZ14] that under our assumptions that (1.1.4) has a unique solution with paths in $L^\infty([0,T]; H^1(\mathbb{R}^n))$. This implies uniqueness in law for martingale solutions.

As P is the law of $\tilde{X}$, we deduce this is the law of the solution of (1.1.4). By uniqueness of the limit, we conclude that the whole sequence $(X^{\varepsilon,\tau^\varepsilon})$ converges in law to X, in the space of probability measure of $C([0,T], H^s)$.

Finally, we obviously have for $\delta > 0$,

$$\left\{\sup_{0\leqslant t\leqslant T}\left\|X^\varepsilon(t) - X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{H^1} > \delta\right\} \subset \{\tau^\varepsilon < T\},$$

and together with Lemma 1.2.2.1 yields the convergence in probability of $X^\varepsilon - X^{\varepsilon,\tau^\varepsilon}$ to

0.

Using finally [Bil99, Theorem 4.1], we obtain the weak convergence of X^ε to X in $C([0,T],H^s)$.

□

1.6 Appendix

1.6.1 Details about the example in Section 1.2.3

Consider X_t solution of (1.2.19):

$$\begin{cases} dX_t = (AX_t + F(X_t))dt + \sigma dW_t \\ X_0 = x \end{cases}, \tag{1.6.1}$$

where $t > 0$, and $x \in H$ with H a Hilbert space. Then we define

$$m(t,n) = \Lambda\left(X(t,\Lambda^{-1}n + \bar{x}) - \bar{x}\right),$$

with $\Lambda : H \to E$ a continuous invertible operator and $\bar{x} = \int_H x\nu(dx)$ where ν is the invariant measure of X_t. In order to show that m satisfies the assumptions in E, it is sufficient to show that these assumptions are satisfied by X in H. It is straightforward that X_t is stochastically continuous, and it is proved in [DHT11] that X_t satisfies the coupling assumption 4. We first prove that the moment of order 2 of X_t is bounded. Let us define

$$Z_t = \int_0^t e^{A(t-s)}\sigma dW_s, \qquad Y_t = X_t - Z_t, \quad t \geqslant 0. \tag{1.6.2}$$

Thanks to the assumptions made on A and σ we know that there exists $\alpha_0 > 0$ such that for any $k \in \mathbb{N}$,

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|Z_t\|^k_{D((-A)^{\alpha_0})}\right] \leqslant C, \tag{1.6.3}$$

where $D((-A)^\alpha)$ is the domain of the operator $(-A)^\alpha$. Besides Y_t is solution of

$$\begin{cases} dY_t = (AY_t + F(Y_t + Z_t))dt \\ Y_0 = x \end{cases}. \tag{1.6.4}$$

Lemma 1.6.1.1. *Let $x \in H$, and X_t the solution of equation (1.2.19) with initial data x.*

Then for any $k \in \mathbb{N}$ *there exists a constant* C_k *depending on* T *and* x *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^k\right] \leqslant C_k.$$

Proof. The proof is based on a classical energy estimate. We start from the equation satisfied by Y_t:

$$\partial_t Y_t = AY_t + F(Y_t + Z_t),$$

and take the scalar product with Y_t. We use the assumptions on A and the boundedness of F to obtain the inequality

$$\frac{1}{2}\partial_t \|Y_t\|_H^2 + \|Y_t\|^2_{D((-A)^{\frac{1}{2}})} \leqslant M \|Y_t\|_H$$

which, using (1.2.20) for the right-hand side of the inequality, gives:

$$\frac{1}{2}\partial_t \|Y_t\|_H^2 + \frac{1}{2}\|Y_t\|^2_{D((-A)^{\frac{1}{2}})} \leqslant \frac{M^2}{2\lambda_1}, \tag{1.6.5}$$

where λ_1 is the first eigenvalue of $-A$. Finally using (1.2.20) again:

$$\partial_t \|Y_t\|_H^2 + \lambda_1 \|Y_t\|_H^2 \leqslant \frac{M^2}{\lambda_1}.$$

Now multiply this inequality by $\|Y_t\|_H^{k-2}$ for $k \in \mathbb{N}$:

$$\begin{aligned}\frac{k}{2}\partial_t \|Y_t\|_H^k + \lambda_1 \|Y_t\|_H^k &\leqslant C \|Y_t\|_H^{k-2} \\ &\leqslant C_k + \frac{\lambda_1}{2}\|Y_t\|_H^k.\end{aligned}$$

A Grönwall argument allows us to conclude that

$$\|Y_t\|_H^k \leqslant \frac{2C_k}{\lambda_1} + \|x\|_H^k e^{-\frac{\lambda_1}{2}t}. \tag{1.6.6}$$

Then, coming back to the process X_t, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^k\right] \leqslant C_k \mathbb{E}\left[\sup_{t\in[0,T]} \left(\|Z_t\|_H^k + \|Y_t\|_H^k\right)\right] \leqslant C_k(x),$$

because of the bound that we just proved for $\|Y_t\|_H^2$ and because all moments of Z_t are bounded. $\square$

Now we can start proving that X_t satisfies the different assumptions 1 to 5. The following lemma ensures that X_t verifies 2.

Lemma 1.6.1.2. *Let X_t the process which is solution of (1.2.19), and denote by P_t its transition semigroup. Then $(P_t)_{t\geqslant 0}$ is Feller.*

Proof. Let $x, y \in H$, denote by X_t^x and X_t^y the solutions of (1.2.19) with initial data x and y. Then we have

$$d(X_t^x - X_t^y) = (A(X_t^x - X_t^y) + F(X_t^x) - F(X_t^y))\, dt.$$

By classical energy estimate, using the assumptions on A and the Lipschitz assumption on F, we get by a Grönwall argument that

$$\|X_t^x - X_t^y\|_H \leqslant Ce^{Ct} \|x - y\|_H \,.$$

Thus for any $x \in H$, for any $t \geqslant 0$, the process X_t^y converges almost surely to X_t^x if y converges to x. Now let $\varphi \in C(H)$ with polynomial growth, and $(x_n)_{n\in\mathbb{N}}$ a sequence in H which converges to $x \in H$ as n goes to $+\infty$. The almost sure convergence proved above implies that $\varphi(X_t^{x_n}) - \varphi(X_t^x)$ converges almost surely to 0 in H as n goes to $+\infty$, and Lemma 1.6.1.1 ensures uniform integrability, so that finally

$$\lim_{n\to+\infty} \|P_t\varphi(x_n) - P_t\varphi(x)\|_H = 0.$$

$\square$

Now we are interested in the existence and uniqueness of an invariant measure for the process X_t.

Lemma 1.6.1.3. *Let $x \in H$. Then P_t the transition semigroup of the process X_t solution of (1.2.19) with initial data x has a unique invariant measure denoted by ν.*

Proof. The existence of an invariant measure is based on the Krylov-Bogoliubov theorem. Here again we use the processes Y_t and Z_t defined in (1.6.2), and start from equation (1.6.5):

$$\frac{1}{2}\partial_t \|Y_t\|_H^2 + \frac{1}{2} \|Y_t\|^2_{D((-A)^{\frac{1}{2}})} \leqslant \frac{M^2}{2\lambda_1}.$$

If we integrate this inequality, we get for $t > 0$:

$$\frac{1}{t}\left\|Y_t\right\|_H^2 + \frac{1}{t}\int_0^t \left\|Y_s\right\|^2_{D((-A)^{\frac{1}{2}})} ds \leqslant C + \frac{1}{t}\left\|Y_0\right\|_H^2,$$

so that, using (1.6.3) there exists a random variable κ with all moments finite such that

$$\frac{1}{t}\int_0^t \left\|(-A)^{\alpha_0} X_s\right\|_H ds \leqslant \kappa.$$

If we denote by $\mu_t = \frac{1}{t}\int_0^t \mathcal{L}(X_s)ds$ with $\mathcal{L}(X)$ the law of X solution of (1.6.1), we have

$$\mu_t\left(B^c_{D((-A)^{\alpha_0})}(0,R)\right) \leqslant \frac{C}{R}.$$

Thus the sequence $(\mu_t)_{t>0}$ is tight and by Krylov-Bogoliubov theorem there exists an invariant measure ν.

Now we prove the uniqueness of this invariant measure. It results from the coupling assumption 4 proved in [DHT11] which gives for φ a continuous bounded function and $x, y \in H$:

$$\begin{aligned} |P_t\varphi(x) - P_t\varphi(y)| &= \left|\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]\right| \\ &= \left|\mathbb{E}\left[(\varphi(X_t^x) - \varphi(X_t^y))\,\mathbb{1}_{X_t^x \neq X_t^y}\right]\right. \\ &\leqslant 2\left\|\varphi\right\|_0 (1 + \left\|x\right\|_H^2 + \left\|y\right\|_H^2)e^{-\gamma t}, \end{aligned} \tag{1.6.7}$$

where we used Cauchy-Schwarz inequality in the last step of the computation and denoted by $\left\|\varphi\right\|_0$ the supremum of φ. Let ν_1, ν_2 be two invariant measures. Let $A \subset H$ and $\varphi = \mathbb{1}_{A\cap B_H(0,R)}$ for $R > 0$. The fact that ν_1 and ν_2 are both invariant measures gives

$$|\nu_1(A\cap B_H(0,R)) - \nu_2(A\cap B_H(0,R))| = \left|\int_H P_t\varphi(x)\nu_1(dx) - \int_H P_t\varphi(y)\nu_2(dy)\right|$$

and according to the inequality coming from the coupling assumption we get:

$$|\nu_1(A\cap B_H(0,R)) - \nu_2(A\cap B_H(0,R))| \leqslant 2(1+R^2)e^{-\gamma t},$$

which goes to 0 as t goes to $+\infty$. However, the left-hand side of this inequality does not depend on t, thus it is equal to 0, for any $R > 0$. Finally, we get that for any $A \subset H$, $\nu_1(A) = \nu_2(A)$. $\square$

It remains to prove that X_t satisfies assumptions 3 and 5. For this we decompose X_t and we have for any $\varepsilon > 0$

$$\|X_t\|_H^k \leqslant (\|Z_t\|_H + \|Y_t\|_H)^k \leqslant C_{k,\varepsilon} \|Z_t\|_H^k + (1+\varepsilon) \|Y\|_H^k ,$$

which gives us that the assumption 3 is satisfied. Indeed, we recall that $Y_0 = x$ and that all the moments of Z_t are bounded, so that using (1.6.6)

$$\mathbb{E}\left[\|X_t\|_H^k\right] \leqslant C(1 + \|x\|_H)^k.$$

Now assuming that $X_0 \sim \nu$, we have

$$\int_H \|x\|_H^k \, \nu(dx) = \int_H \mathbb{E}\left[\|X_t\|_H^k\right] \nu(dx)$$

because ν is the invariant measure. Then

$$\begin{aligned}\int_H \|x\|_H^k \, \nu(dx) &\leqslant \int_H C_{k,\varepsilon} \mathbb{E}\left[\|Z_t\|_H^k\right] + (1+\varepsilon)\mathbb{E}\left[\|Y_t\|_H^k\right] \nu(dx) \\ &\leqslant C_{k,\varepsilon} + (1+\varepsilon)e^{-\frac{\lambda_1}{2}t} \int_H \mathbb{E}\left[\|Y_0\|_H^k\right] \nu(dx)\end{aligned}$$

because $\mathbb{E}\left[\|Z_t\|_H^k\right] \leqslant C$. Besides, $X_0 = Y_0$ and ν is the invariant measure:

$$\int_H \|x\|_H^k \, \nu(dx) \leqslant C_{k,\varepsilon} + (1+\varepsilon)e^{-\frac{\lambda_1}{2}t} \int_H \|x\|_H^k \, \nu(dx),$$

and assuming that $(1+\varepsilon)e^{-\frac{\lambda_1}{2}t} < 1$, we proved that for every $k \in \mathbb{N}$:

$$\int_H \|x\|_H^k \, \nu(dx) < \infty.$$

Finally we proved that X_t satisfies assumptions 1 to 5 (except the zero-mean property), and so does $m(t,n)$ according to its definition in (1.2.21).

Now we can work on the construction of the functionnals $L_1, \cdots, L_4$. We state the following result, which ensures the existence of L_1.

Lemma 1.6.1.4. *Let* $u, v \in H^1(\mathbb{R}^n)$*, and define* $\varphi^{u,v}(n) = (un, v)$*. Then if we define*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = -\int_0^\infty P_t\varphi^{u,v}(n)dt,$$

there exists $L_1 : E \to E$ *such that*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_1(n), v), \quad \textit{and} \quad \|L_1(n)\|_E \leqslant C(1 + \|n\|_E)^2.$$

Proof. The process $\varphi^{u,v}(n)$ is centered, thus we can write

$$\begin{aligned}|P_t\varphi^{u,v}(n)| &= |P_t\varphi^{u,v}(n) - \int_E P_t\varphi^{u,v}(\bar{n})\nu(d\bar{n})| \\ &\leqslant \int_E |P_t\varphi^{u,v}(n) - P_t\varphi^{u,v}(\bar{n})|\nu(d\bar{n})\end{aligned}$$

because ν is the invariant measure. Thus denoting by $(m^1_{n,\bar{n}}, m^2_{n,\bar{n}})$ a coupling of $(m(t,n), m(t,\bar{n}))$ satisfying (1.2.16), we have

$$\begin{aligned}|P_t\varphi^{u,v}(n)| &\leqslant \int_E |\mathbb{E}\left[\varphi^{u,v}(m(t,n)) - \varphi^{u,v}(m(t,\bar{n}))\right]|\nu(d\bar{n}) \\ &\leqslant \int_E |\mathbb{E}\left[\left(\varphi^{u,v}(m^1_{n,\bar{n}}(t)) - \varphi^{u,v}(m^2_{n,\bar{n}}(t))\right)\mathbb{1}_{m^1\neq m^2}\right]|\nu(d\bar{n}) \\ &\leqslant \int_E \mathbb{E}\left[\varphi^{u,v}(m^1_{n,\bar{n}}(t))^2 + \varphi^{u,v}(m^2_{n,\bar{n}}(t))^2\right]^{\frac{1}{2}}\mathbb{P}(m^1_{n,\bar{n}} \neq m^2_{n,\bar{n}})^{\frac{1}{2}}\nu(d\bar{n}).\end{aligned}$$

Denote by E' the dual space of E, then:

$$\begin{aligned}|P_t\varphi^{u,v}(n)| \leqslant C\|uv\|_{E'}\int_E &\left(\mathbb{E}\left[\|m(t,n)\|_E^2\right]^{\frac{1}{2}} + \mathbb{E}\left[\|m(t,\bar{n}\|_E^2\right]^{\frac{1}{2}}\right) \\ &\times (1 + \|n\|_E^2 + \|\bar{n}\|_E^2)^{\frac{1}{2}}e^{-\frac{\gamma}{2}t}\nu(d\bar{n}).\end{aligned}$$

Besides we have

$$\begin{aligned}\mathbb{E}\left[\|m(t,n)\|_E^2\right] &= \mathbb{E}\left[\left\|\Lambda X(t, \Lambda^{-1}n + \bar{x}) - \bar{x}\right\|_E^2\right] \\ &\leqslant C\left(1 + \mathbb{E}\left[\left\|X(t, \Lambda^{-1}n + \bar{x})\right\|_H^2\right]\right) \\ &\leqslant C\left(1 + \|n\|_E\right)^2\end{aligned}$$

according to assumption 3. Finally we get

$$|P_t\varphi^{u,v}(n)| \leqslant C\left\|uv\right\|_{E'}(1+\left\|n\right\|_E)^2 e^{-\frac{\gamma}{2}t}.$$

Thus $\mathcal{M}^{-1}\varphi^{u,v}(n)$ is well-defined and

$$\mathcal{M}^{-1}\varphi^{u,v}(n) \leqslant C\left\|uv\right\|_{E'}(1+\left\|n\right\|_E)^2. \tag{1.6.8}$$

We want now to show that we may choose $\mathcal{M}\mathcal{M}^{-1}\varphi^{u,v} = \varphi^{u,v}$. In other words, we want

$$\mathcal{M}^{-1}\varphi^{u,v}(m(t,n)) - \int_0^t \varphi^{u,v}(m(s,n))ds$$

to be a martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. Let $r \leqslant t$. On one hand wa have

$$\begin{aligned}
&\mathbb{E}\left[-\int_0^\infty P_\tau\varphi^{u,v}(m(t,n))d\tau - \int_0^t \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right] \\
&= -\mathbb{E}\left[\int_0^\infty \mathbb{E}\left[\varphi^{u,v}(m(t+\tau,n)) \mid \mathcal{F}_t\right]d\tau + \int_0^t \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right] \\
&= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(t+\tau,n))d\tau + \int_0^t \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right] \\
&= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right].
\end{aligned}$$

On the other hand we use the fact that $\int_0^r \varphi^{u,v}(m(s,n))ds \in \mathcal{F}_r$ to compute

$$\begin{aligned}
&\mathcal{M}^{-1}\varphi^{u,v}(m(r,n)) - \int_0^r \varphi^{u,v}(m(s,n))ds \\
&= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(r+\tau,n))d\tau + \int_0^r \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right] \\
&= -\mathbb{E}\left[\int_0^\infty \varphi^{u,v}(m(s,n))ds \mid \mathcal{F}_r\right],
\end{aligned}$$

thus the martingale property is satisfied.

Finally, for every $n \in E$ we have constructed a linear form on E': $uv \mapsto \mathcal{M}^{-1}\varphi^{u,v}(n)$. It is continuous, since

$$\left\|\mathcal{M}^{-1}\varphi^{u,v}(n)\right\|_E \leqslant C(1+\left\|n\right\|_E)^2\left\|uv\right\|_{E'}.$$

Thus, since E is a reflexive space, there exists $L_1(n) \in E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (L_1(n), uv) = (uL_1(n), v) \quad \text{and} \quad \|L_1(n)\|_E \leqslant C(1 + \|n\|_E)^2.$$

□

So far we have proved that it is possible to construct $L_1(n)$ which satisfies our assumptions. Let us now construct the other functionals.

Lemma 1.6.1.5. *Let $u, v \in H^1(\mathbb{R}^n)$, and define $\varphi^{u,v}(n) = (u(nL_1(n) - \mathbb{E}_\nu [nL_1(n)]), v)$. Then if we define*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = -\int_0^\infty P_t\varphi^{u,v}(n)dt,$$

there exists $L_2 : E \to E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_2(n), v) \quad \textit{and} \quad \|L_2(n)\|_E \leqslant C(1 + \|n\|_E)^3.$$

Proof. We proceed as above and get

$$\begin{aligned}
&|P_t\varphi^{u,v}(n)| \\
&\leqslant C \|uv\|_{E'} \int_E \mathbb{E}\left[\left\|m(t,n)L_1(m(t,n)) - \mathbb{E}_\nu[nL^1(n)]\right\|_E^2\right]^{\frac{1}{2}} (1 + \|n\|^2 + \|\bar{n}\|_E^2)^{\frac{1}{2}} e^{-\frac{\gamma}{2}t} \\
&\quad + \mathbb{E}\left[\left\|m(t,\bar{n})L_1(m(t,\bar{n})) - \mathbb{E}_\nu[nL^1(n)]\right\|_E^2\right]^{\frac{1}{2}} (1 + \|n\|^2 + \|\bar{n}\|_E^2)^{\frac{1}{2}} e^{-\frac{\gamma}{2}t}\nu(d\bar{n}).
\end{aligned}$$

We use the fact that E is an algebra to estimate

$$\begin{aligned}
\mathbb{E}\left[\left\|m(t,n)L_1(m(t,n)) - \mathbb{E}_\nu[nL^1(n)]\right\|_E^2\right] &\leqslant C\left(1 + \mathbb{E}\left[\|m(t,n)\|_E^2 \|L_1(m(t,n))\|_E^2\right]\right) \\
&\leqslant C\left(1 + \|n\|_E\right)^4.
\end{aligned}$$

Finally, we get

$$|P_t\varphi^{u,v}(n)| \leqslant C \|uv\|_{E'} \left(1 + \|n\|_E\right)^3 e^{-\frac{\gamma}{2}t},$$

and we can continue the proof exactly as for the construction on $L_1(n)$. Thus for any $n \in E$ there exists $L_2(n) \in E$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_2(n), v) \quad \text{and} \quad \|L_2(n)\|_E \leqslant C(1 + \|n\|_E)^3.$$

□

We have the same result for L_3.

Lemma 1.6.1.6. *Let $u, v \in H^1(\mathbb{R}^n)$, and define $\varphi^{u,v}(n) = (u(\nabla n \nabla L_1(n) - \mathbb{E}_\nu [\nabla n \nabla L_1(n)]), v)$. Then if we set*

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = -\int_0^\infty P_t \varphi^{u,v}(n) dt$$

there exists $L_3 : E \to H^{s-1}$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_3(n), v) \quad \textit{and} \quad \|L_3(n)\|_{H^{s-1}} \leqslant C(1 + \|n\|_E)^3.$$

Proof. We follow the same steps as for the two previous proofs, except that when we bound $\varphi^{u,v}$ we have

$$|\varphi^{u,v}(n)| \leqslant \|uv\|_{H^{1-s}} \|\nabla n \nabla L_1(n) - \mathbb{E}_\nu [\nabla n \nabla L_1(n)]\|_{H^{s-1}},$$

and we use the fact that $H^{s-1}(\mathbb{R}^n)$ is an algebra because $s > \frac{d}{2} + 3$ to write

$$|\varphi^{u,v}(n)| \leqslant C \|uv\|_{H^{1-s}} (\|n\|_E \|L_1(n)\|_E + C).$$

Thus the proof gives us for any $n \in E$ the existence of $L_3(n) \in H^{s-1}$ such that

$$\mathcal{M}^{-1}\varphi^{u,v}(n) = (uL_3(n), v) \quad \text{and} \quad \|L_3(n)\|_{H^{s-1}} \leqslant C(1 + \|n\|_E)^3.$$

□

Finally, we need to construct the last functional L_4, which is slightly different from the previous ones.

Lemma 1.6.1.7. *Let $w \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, and define*

$$\varphi^w(n) = \int_{\mathbb{R}^{2n}} w(x, y) (n(x)L_1(n)(y) - \mathbb{E}_\nu [n(x)L_1(n)(y)]) dxdy.$$

Then if we set

$$\mathcal{M}^{-1}\varphi^w(n) = -\int_0^\infty P_t \varphi^w(n) dt$$

there exists $L_4 : E \to L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *such that*

$$\mathcal{M}^{-1}\varphi^w(n) = \int_{\mathbb{R}^{2n}} w(x,y)L_4(n)(x,y)dxdy \quad \textit{and} \quad \|L_4(n)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C(1+\|n\|_E)^3.$$

Proof. The proof of this lemma is very similar to the proofs of the previous lemmas, but this time

$$|\varphi^w(n)| \leqslant \|w\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} \|n(x)L_1(n)(y) - \mathbb{E}_\nu [n(x)L_1(n)(y)]\|_{E\times E}$$

because $E \hookrightarrow L^\infty(\mathbb{R}^n)$. Thus we get

$$|P_t\varphi^w(n)| \leqslant C \|w\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)} (1+\|n\|_E)^3 e^{-\frac{\gamma}{2}t},$$

and using the same procedure as for the previous proofs for any $n \in E$ there exists $L_4(n) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\mathcal{M}^{-1}\varphi^w(n) = \int_{\mathbb{R}^{2n}} w(x,y)L_4(n)(x,y)dxdy \quad \text{and} \quad \|L_4(n)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \leqslant C(1+\|n\|_E)^3.$$

□

We have constructed the last functional in a slightly different way than in (1.2.8), but taking $w(x,y) = u(x)u(y)\bar{v}(x)\bar{v}(y)$ we recover the same expression for $\mathcal{M}^{-1}\varphi^w(n)$.

1.6.2 Proof of Lemma 1.4.2.1

As seen in Section 1.3, a straight application of standard energy arguments produce unsatisfactory dependence on ε. The idea is to use the perturbed test functions method. This mimics the Itô formula which is used to get a priori estimates for the limit equation with white noise (see [dBD03a]).

If one tries to use similar arguments as for Lemma 1.4.1.1 with the functionnal H, defined by (1.2.2), in place of linear or quadratic functional, this requires a lot of smoothness on H and the useless assumption ($\sigma \geqslant 1/2$). We proceed slightly differently. We first smooth the functional and take advantage of the various cancelations before constructing the correctors.

Proof of Lemma 1.4.2.1. We consider the functionnal $\varphi_\delta(u) = H(\rho_\delta \star u)$. We claim that it is a good test function. Indeed, we have for $u, h \in H^1$

$$D\varphi_\delta(u) \cdot h = \left(-\Delta\rho_\delta \star u + \lambda|\rho_\delta \star u|^{2\sigma}(\rho_\delta \star u), \rho_\delta \star h\right), \tag{1.6.9}$$

and there is no diffulty to verify that φ_δ matches Definition 1.3.3.1. By Proposition 1.3.3, we know that the process

$$M_\delta^\varepsilon(t) = \varphi_\delta(X^\varepsilon(t)) - \varphi_\delta(X_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon\varphi_\delta(X^\varepsilon(s), m^\varepsilon(s))ds,$$

is a martingale. It can be seen that $\varphi_\delta(u) \to H(u)$ when $\delta \to 0$ and $u \in H^1$. Moreover, we have

$$\begin{aligned}
&\mathcal{L}^\varepsilon\varphi_\delta(X^\varepsilon(s), m^\varepsilon(s)) \\
&\quad= \frac{1}{\varepsilon}\left(-\Delta\rho_\delta \star X^\varepsilon(s) + \lambda|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}(\rho_\delta \star X^\varepsilon), -i\rho_\delta \star (X^\varepsilon(s)m^\varepsilon(s))\right) \\
&\quad+ \left(-\Delta\rho_\delta \star X^\varepsilon(s) + \lambda|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}(\rho_\delta \star X^\varepsilon), i\rho_\delta \star \left(\Delta X^\varepsilon(s) - \lambda|X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s)\right)\right),
\end{aligned}$$

and we proved in Proposition 1.3.1 that the second term of the right-hand side converges to 0, when $\delta \to 0$. Similarly we have when δ tends to 0

$$|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s) \to |X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s) \text{ in } L^{\frac{2\sigma+2}{2\sigma+1}},$$

$$\rho_\delta \star X^\varepsilon(s)m^\varepsilon(s) \to X^\varepsilon(s)m^\varepsilon(s) \text{ in } L^{2\sigma+2},$$

so that

$$\begin{aligned}
\left(|\rho_\delta \star X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s), i\rho_\delta \star X^\varepsilon(s)m^\varepsilon(s)\right) &\to \left(|X^\varepsilon(s)|^{2\sigma}X^\varepsilon(s), iX^\varepsilon(s)m^\varepsilon(s)\right) \\
&= Re\left(\int_{\mathbb{R}^n} i|X^\varepsilon(s)|^{2\sigma+2}m^\varepsilon(s)dx\right) = 0,
\end{aligned}$$

since m is real valued. In the same way, we have

$$(-\Delta\rho_\delta \star X^\varepsilon(s), i\rho_\delta \star X^\varepsilon(s)m^\varepsilon(s)) \to (\nabla X^\varepsilon(s), iX^\varepsilon(s)\nabla m^\varepsilon(s)).$$

Finally, as δ converges to 0, $M_\delta(t)$ converges almost surely to

$$M^\varepsilon(t) := H(X^\varepsilon(t)) - H(X_0^\varepsilon) - \int_0^t \frac{1}{\varepsilon}(\nabla X^\varepsilon(s), iX^\varepsilon(s)\nabla m^\varepsilon(s))\,ds,$$

and since $M_\delta(t)$ is a $(\mathcal{F}_t^\varepsilon)$-martingale, from (1.3.1) and the dominated convergence theorem, we deduce immediately that $M^\varepsilon(t)$ is also a $(\mathcal{F}_t^\varepsilon)$-martingale.

We define then the first corrector

$$\varphi_1(u,n) = (\nabla u, -iu\nabla L_1(n)) . \tag{1.6.10}$$

This is not a good test function. For $\delta > 0$, we set $\varphi_{1,\delta}(u,n) = \varphi_1(\rho_\delta \star u, n)$. There is no difficulty to see that $\varphi_{1,\delta}$ is a good test function and then, we obtain by Proposition 1.3.3 that the process

$$\begin{aligned} M_{1,\delta}^\varepsilon(t) :=& \varphi_{1,\delta}(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_{1,\delta}(X^\varepsilon(0), m^\varepsilon(0)) \\ &- \int_0^t \mathcal{L}^\varepsilon \varphi_{1,\delta}(X^\varepsilon(s), m^\varepsilon(s))ds, \end{aligned}$$

is a martingale. After some computations, we have

$$\begin{aligned} \mathcal{L}^\varepsilon \varphi_{1,\delta}(u,n) =& \frac{1}{\varepsilon^2} (\nabla\rho_\delta \star u, -i\rho_\delta \star u\nabla n) \\ &+ \frac{1}{\varepsilon} \left[(\nabla\rho_\delta \star (un), \nabla L_1(n)\rho_\delta \star u) - (\nabla\rho_\delta \star u, n\nabla L_1(n)\rho_\delta \star u) \right] \\ &+ 2 (\Delta\rho_\delta \star u, \nabla L_1(n)\nabla\rho_\delta \star u) + (\Delta\rho_\delta \star u, \rho_\delta \star u\Delta L_1(n)) \\ &- 2\lambda \left(\rho_\delta \star (|u|^{2\sigma}u), \nabla\rho_\delta \star u\nabla L_1(n) \right) - \lambda \left(\rho_\delta \star (|u|^{2\sigma}u), \rho_\delta \star u\Delta L_1(n) \right), \end{aligned}$$

and after integration by parts, we get

$$(\Delta\rho_\delta \star u, \rho_\delta \star u\Delta L_1(n)) = -(\nabla\rho_\delta \star u, \nabla\rho_\delta \star u\Delta L_1(n)) - (\nabla\rho_\delta \star u, \rho_\delta \star u\nabla\Delta L_1(n)),$$

$$(\Delta\rho_\delta \star u, \nabla L_1(n)\nabla\rho_\delta \star u) = -\frac{1}{2} (\nabla\rho_\delta \star u, \nabla\rho_\delta \star u\Delta L_1(n)),$$

and

$$\left(\nabla\rho_\delta \star (|u|^{2\sigma}u), \rho_\delta \star u\nabla L_1(n) \right) = -\left(\rho_\delta \star (|u|^{2\sigma}u), \nabla(\rho_\delta \star u)\nabla L_1(n) + \rho_\delta \star u\Delta L_1(n) \right).$$

Finally, taking the limit $\delta \to 0$ and using

$$(2\sigma+2) \left(|u|^{2\sigma}u, \nabla u\nabla L_1(n) \right) = \left(\nabla|u|^{2\sigma+2}, \nabla L_1(n) \right) = -\left(|u|^{2\sigma+2}, \Delta L_1(n) \right),$$

we get that the process $M_1^\varepsilon(t)$ given by

$$\begin{aligned}
M_1^\varepsilon(t) :=& \varphi_1(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_1(X^\varepsilon(0), m^\varepsilon(0)) \\
&- \int_0^t \frac{1}{\varepsilon^2} \left(\nabla X^\varepsilon(s), -iX^\varepsilon(s)\nabla m^\varepsilon(s)\right) \\
&- \frac{1}{\varepsilon} \left[(X^\varepsilon(s)\nabla m^\varepsilon(s), X^\varepsilon \nabla L_1(m^\varepsilon(s))) \right] \\
&+ 2 \left(\nabla X^\varepsilon(s), \nabla X^\varepsilon(s)\Delta L_1(m^\varepsilon(s))\right) + \left(\nabla X^\varepsilon(s), X^\varepsilon(s)\nabla \Delta L_1(m^\varepsilon(s))\right) \\
&+ \lambda \frac{\sigma}{\sigma+1} \left(|X^\varepsilon(s)|^{2\sigma+2}, \Delta L_1(m^\varepsilon(s)) \right) ds
\end{aligned}$$

is a martingale.

We finally consider the C_P^1 test function

$$\varphi_2(u,n) := \mathcal{M}^{-1}\left((u\nabla n, u\nabla L_1(n) - \mathbb{E}_\nu\left(u\nabla n, u\nabla L_1(n)\right)) \right) = (uL_3(n), u).$$

Using Proposition 1.3.3, we know that

$$M_2^\varepsilon(t) := \varphi_2(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_2(X^\varepsilon(0), m^\varepsilon(0)) - \int_0^t \mathcal{L}^\varepsilon \varphi_2(X^\varepsilon(s), m^\varepsilon(s)) ds \qquad (1.6.11)$$

is a $(\mathcal{F}_t^\varepsilon)$-martingale. After computation we obtain $\mathcal{L}^\varepsilon\varphi_2$:

$$\begin{aligned}
\mathcal{L}^\varepsilon \varphi_2(u,n) =& \frac{1}{\varepsilon^2}\left(u(\nabla n \nabla L_1(n) - \mathbb{E}_\nu \nabla n \nabla L_1(n)), u\right) \\
&- 2\left(i\nabla u . \nabla L_3(n), u\right)
\end{aligned} \qquad (1.6.12)$$

where we strongly used that n is real-valued.

Consequently, we know that the following process $\mathbf{M}^\varepsilon(t)$ is a martingale, with respect to the filtration $(\mathcal{F}_t^\varepsilon)$.

$$\begin{aligned}
\mathbf{M}^\varepsilon(t) :=& M^\varepsilon(t) + \varepsilon M_1^\varepsilon(t) + \varepsilon^2 M_2^\varepsilon(t) \\
=& H(X^\varepsilon(t)) - H(X_0^\varepsilon) + \varepsilon\left(\varphi_1(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_1(X_0^\varepsilon, m^\varepsilon(0))\right) \\
&+ \varepsilon^2\left(\varphi_2(X^\varepsilon(t), m^\varepsilon(t)) - \varphi_2(X_0^\varepsilon, m^\varepsilon(0))\right) \\
&- \int_0^t (X^\varepsilon(s)(\mathbb{E}_\nu \nabla n \nabla L_1(m^\varepsilon(s)), X^\varepsilon(s)) \\
&\quad + 2\varepsilon\left(\nabla X^\varepsilon(s), \nabla X^\varepsilon(s)\Delta L_1(m^\varepsilon(s))\right) + \varepsilon\left(\nabla X^\varepsilon(s), X^\varepsilon(s)\nabla\Delta L_1(m^\varepsilon(s))\right) \\
&\quad + \varepsilon\frac{\lambda\sigma}{\sigma+1}\left(|X^\varepsilon(s)|^{2\sigma+2}, \Delta L_1(m^\varepsilon(s))\right) \\
&\quad - 2\varepsilon^2\left(i\nabla X^\varepsilon(s)\nabla L_3(m^\varepsilon(s), X^\varepsilon(s))\right) ds.
\end{aligned}$$

Hence, since τ^ε, given by (1.2.13), is a bounded $(\mathcal{F}_t^\varepsilon)$-stopping time, the process $\mathbf{M}(t\wedge\tau^\varepsilon)$ is a martingale. From the identity above, the L^2 conservation (1.6.10) and (1.6.12) we have

$$\begin{aligned}
\mathbb{E}[H(X^{\varepsilon,\tau^\varepsilon}(t)] \leqslant \mathbb{E}\Big[&|H(X_0^\varepsilon)| + \mathcal{C}\varepsilon \left\|L_1(m^\varepsilon)\right\|_E \left\|X_0^\varepsilon\right\|_{L^2}\left(\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{L^2} + \left\|\nabla X_0^\varepsilon\right\|_{L^2}\right) \\
&+ \mathcal{C}\varepsilon^2 \left\|L_3(m^\varepsilon)\right\|_{H^\kappa} \left\|X_0^\varepsilon\right\|_{L^2}^2 + \mathbf{M}(t\wedge\tau^\varepsilon) \\
&+ \mathcal{C}\int_0^{t\wedge\tau^\varepsilon} \mathbb{E}_\nu\left[\nabla n \nabla L_1(n)\right] \left\|X_0^\varepsilon\right\|_{L^2}^2 + \varepsilon\left\|L^1(m^\varepsilon)\right\|_E \left\|X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^{2\sigma+2}}^{2\sigma+1} \\
&\quad + \varepsilon\left\|L_1(m^\varepsilon)\right\|_E\left(\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2 + \left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}\left\|X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}\right) \\
&\quad + \varepsilon^2\left(\left\|L_3(m^\varepsilon)\right\|_{H^\kappa}^2 + \left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2\right)\left\|X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2} ds\Big]
\end{aligned}$$

Gagliardo-Nirenberg's inequality yields

$$\frac{1}{2\sigma+2}\left\|u\right\|_{L^{2\sigma+2}}^{2\sigma+2} \leqslant \mathcal{C}\left\|u\right\|_{L^2}^{2\sigma+2-\sigma n}\left\|\nabla u\right\|_{L^2}^{\sigma n} \leqslant \frac{1}{4}\left\|\nabla u\right\|_{L^2}^2 + \mathcal{C}\left\|u\right\|_{L^2}^{q_n}, \tag{1.6.13}$$

with $q_n = (2\sigma+2-\sigma n)\frac{2}{2-\sigma n}$. We recall that $\|X^\varepsilon(t)\|_{L^2} = \|X_0^\varepsilon\|_{L^2}$ and that

$\|m^\varepsilon(t\wedge\tau^\varepsilon)\|_{L^2}\leqslant\varepsilon^{-\alpha}$ according to Lemma 1.2.2.1. Hence using (1.2.10)

$$\begin{aligned}
&\mathbb{E}\Big[\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{L^2}^2\Big]\\
&\leqslant 2\mathbb{E}\Big[|H(X_0^\varepsilon)|+\mathcal{C}\varepsilon^{1-\alpha\eta}(\|X_0^\varepsilon\|_{L^2}\|\nabla X_0^\varepsilon\|_{L^2}+\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{L^2}^2+\|X_0^\varepsilon\|_{L^2}^2)\\
&+\mathcal{C}\|X_0\|_{L^2}^{q_n}+\mathcal{C}\varepsilon^{2(1-\alpha\eta)}\left\|X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{L^2}^2\\
&+\mathcal{C}\int_0^{t\wedge\tau^\varepsilon}\mathbb{E}_\nu\left[\nabla n\nabla L_1(n)\right]\|X_0^\varepsilon\|_{L^2}^2+\varepsilon^{1-\alpha\eta}(\|X_0^\varepsilon\|_{L^2}^2+\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2)\\
&+\varepsilon^{1-\alpha\eta}(\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2+\|X_0^\varepsilon\|_{L^2}^{q_n})+\varepsilon^2(1+\|X_0^\varepsilon\|_{L^2}^2+\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2)ds\Big],
\end{aligned}$$

then for ε small enough we get

$$\mathbb{E}\left[\left\|\nabla X^{\varepsilon,\tau^\varepsilon}\right\|_{L^2}^2\right]\leqslant\mathcal{C}(T,\|X_0^\varepsilon\|_{L^2}^{q_n},|H(X_0^\varepsilon)|)+\mathcal{C}\varepsilon^{1-\alpha}\int_0^{t\wedge\tau^\varepsilon}\mathbb{E}\left[\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2\right]ds. \tag{1.6.14}$$

Gronwall's lemma and the conservation of the L^2 norm give us that for any $t\in[0,T]$

$$\mathbb{E}\left[\left\|X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{H^1}^2\right]\leqslant\mathcal{C}(T). \tag{1.6.15}$$

Then, taking τ a stopping time such that $\tau<T$, we can do the same computations as before and get a similar bound as (1.6.14):

$$\begin{aligned}
\mathbb{E}\left[\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(\tau)\right\|_{L^2}^2\right]&\leqslant\mathcal{C}(T,\|X_0^\varepsilon\|_{L^2}^{q_n},|H(X_0^\varepsilon)|)+\mathcal{C}\varepsilon^{1-\alpha}\mathbb{E}\left[\int_0^{\tau\wedge\tau^\varepsilon}\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2ds\right]\\
&\leqslant C(T,\|X_0^\varepsilon\|_{L^2}^{q_n},|H(X_0^\varepsilon)|)+\mathcal{C}\int_0^T\mathbb{E}\left[\left\|\nabla X^{\varepsilon,\tau^\varepsilon}(s)\right\|_{L^2}^2\right]ds
\end{aligned}$$

and using the bound (1.6.15) we conclude the proof of the lemma. □

1.6.3 Proof of Proposition 1.4.3

Proof of Proposition 1.4.3. To prove the tightness of the sequence $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon\to0}$ we use Aldous criterion, which can be found in [Bil99, Theorem 16.10] in the case of finite dimension. In the case of an infinite dimensional separable space H, the hypothesis (16.22) in [Bil99] has to be replaced. Let us state the criterion we use.

Proposition 1.6.1. *Let H be a complete separable space, and $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of process on $[0,T]$ such that, for any $t \in [0,T]$, $u^\varepsilon(t)$ is H valued. Assume that*

1. *For every $\eta > 0$, for every $t \geqslant 0$ there exists a compact set $\Gamma_{\eta,t} \subset H$, such that*
$$\inf_{\varepsilon>0} \mathbb{P}(u^\varepsilon(t) \in \Gamma_{\eta,t}) \geqslant 1-\eta, \tag{1.6.16}$$

2. *For every $\lambda, \eta > 0$, there exist δ_0, ε_0 such that for $\delta < \delta_0$, and $\varepsilon < \varepsilon_0$, if τ is a stopping time then*
$$\mathbb{P}\left(\|u^\varepsilon(\tau+\delta) - u^\varepsilon(\tau)\|_H > \lambda\right) \leqslant \eta. \tag{1.6.17}$$

Then the sequence $(u^\varepsilon)_{\varepsilon>0}$ is tight in $C([0,T],H)$.

We can prove this result using the same proof as for [Bil99, Theorem 16.10], but instead of the hypothesis (16.22) we assume (a) in [EK86, Theorem 7.9]. Here, Lemma 1.4.2.1 and Proposition 1.3.2 ensure that $(X^{\varepsilon,\tau^\varepsilon})_{\varepsilon>0}$ satisfies (1.6.16) for $H = H^s(\mathbb{R}^n)$, $0 \leqslant s < 1$. Indeed, according to these two lemmas, we have

$$\mathbb{E}\left[\left\|X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{\Sigma^\zeta}\right] \leqslant C(T).$$

Thus for any $\delta > 0$ and $t \in [0,T]$:

$$\mathbb{P}\left(\left\|X^{\varepsilon,\tau^\varepsilon}(t)\right\|_{H^1\cap\Sigma^\zeta} > R\right) \leqslant \frac{C(T)}{R} \leqslant \delta$$

for $R > 0$ large enough. Thus, $X^{\varepsilon,\tau^\varepsilon}(t)$ lives in a bounded set of $H^1(\mathbb{R}^n) \cap \Sigma^\zeta$ with probability larger than $1-\delta$. It remains to prove that the embedding $H^1(\mathbb{R}^n) \cap \Sigma^\zeta \hookrightarrow H^s(\mathbb{R}^n)$ is compact. Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^n) \cap \Sigma^\zeta$. Then by compact embedding and diagonal extraction, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, which converges to $u \in L^2(B(0,R))$ for every $R > 0$. Now we compute:

$$\begin{aligned}\|u_{n_k} - u\|^2_{L^2(\mathbb{R}^n)} &\leqslant \|u_{n_k} - u\|^2_{L^2(B(0,R))} + \int_{B(0,R)^c} |u_{n_k}(x) - u(x)|^2 dx \\ &\leqslant \|u_{n_k} - u\|^2_{L^2(B(0,R))} + (1+R)^{-\zeta} \int_{B(0,R)^c} \langle x \rangle^{\zeta} \left(|u_{n_k}(x)|^2 + |u(x)|^2\right) dx\end{aligned}$$

for any $R > 0$, thus $(u_{n_k})_{k\in\mathbb{N}}$ converges to u in $L^2(\mathbb{R}^n)$, and is bounded in $H^1(\mathbb{R}^n)$, so that by interpolation the subsequence also converges to u in $H^s(\mathbb{R}^n), 0 \leqslant s < 1$.

We also need to prove that $X^{\varepsilon,\tau^\varepsilon}$ satisfies the second condition of Proposition 1.6.1. We first prove that (1.6.17) holds with H replaced by $H^{-\frac{1}{2}}(\mathbb{R}^n, \mathbb{C})$. For this we use the Perturbed Test Function method, and the equality:

$$\|u(\tau+\delta) - u(\tau)\|^2_{H^{-\frac{1}{2}}} = \|u(\tau+\delta)\|^2_{H^{-\frac{1}{2}}} - \|u(\tau)\|^2_{H^{-\frac{1}{2}}} - 2\langle u(\tau+\delta) - u(\tau), u(\tau)\rangle_{H^{-\frac{1}{2}}}.$$

First we apply the Perturbed Test Function method to $\varphi(u) = \|u\|^2_{H^{-\frac{1}{2}}}$. We compute $\mathcal{L}^\varepsilon\varphi$:

$$\mathcal{L}^\varepsilon \varphi(X^\varepsilon) = 2\langle (1-\Delta)^{-\frac{1}{2}} X^\varepsilon, i\Delta X^\varepsilon - i\lambda |X^\varepsilon|^{2\sigma} X^\varepsilon\rangle - \frac{2}{\varepsilon}\langle (1-\Delta)^{-\frac{1}{2}} X^\varepsilon, iX^\varepsilon m(\frac{t}{\varepsilon^2})\rangle.$$

We choose the first corrector φ_1 in order to cancel the term of negative order in ε, and recalling that $L_1(n) = \mathcal{M}^{-1} n$ we get

$$\varphi_1(u,n) = 2\langle (1-\Delta)^{-\frac{1}{2}} u, iuL_1(n)\rangle, \tag{1.6.18}$$

for which we have a bound using the hypothesis (1.2.10):

$$|\varphi_1(u,n)| \leqslant 2\|u\|_{H^{-1}} \|u\|_{L^2} \|L_1(n)\|_E \leqslant 2\|u\|^2_{L^2} \|L_1(n)\|_E. \tag{1.6.19}$$

Now we apply the infinitesimal generator $\mathcal{L}^\varepsilon$ to $\varphi + \varepsilon\varphi_1$:

$$\begin{aligned}\mathcal{L}^\varepsilon (\varphi + \varepsilon\varphi_1)(X^\varepsilon, m^\varepsilon) =& 2\langle (1-\Delta)^{-\frac{1}{2}} X^\varepsilon, i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma} X^\varepsilon\rangle \\ &+ 2\langle (1-\Delta)^{-\frac{1}{2}} X^\varepsilon, X^\varepsilon m^\varepsilon L_1(m^\varepsilon)\rangle \\ &+ 2\langle (1-\Delta)^{-\frac{1}{2}} (X^\varepsilon m^\varepsilon), X^\varepsilon L_1(m^\varepsilon)\rangle \\ &+ \varepsilon D_u\varphi_1(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma} X^\varepsilon).\end{aligned}$$

Thus, treating the operator $(1-\Delta)^{-\frac{1}{2}}$ as a kernel operator with kernel k_Δ, we set

$$\begin{aligned}\varphi_2(u,n) =& Re\int_{\mathbb{R}^{2n}} \bar{u}(x)k_\Delta(x,y)u(y)L_4(n)(y,x)dxdy \\ &- 2\langle(1-\Delta)^{-\frac{1}{2}}u, uL_2(n)\rangle,\end{aligned} \tag{1.6.20}$$

where we recall that formally $L_2(n) = \mathcal{M}^{-1}\left(nL_1(n) - \mathbb{E}_\nu[nL_1(n)]\right)$. Finally, defining $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$, we obtain that:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon, m^\varepsilon) =& 2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon, i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon\rangle \\ &+ 2\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon, X^\varepsilon\mathbb{E}_\nu\left[nL_1(n)\right]\rangle \\ &- 2\mathbb{E}_\nu\left[\langle(1-\Delta)^{-\frac{1}{2}}\left(X^\varepsilon n\right), X^\varepsilon L_1(n)\rangle\right] \\ &+ \varepsilon\left(D_u\varphi_1(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon) + D_u\varphi_2(-iX^\varepsilon m^\varepsilon)\right) \\ &+ \varepsilon^2 D_u\varphi_2(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon).\end{aligned} \tag{1.6.21}$$

First we get an estimate of $\varphi_2(u,n)$ in (1.6.20). We have

$$\langle(1-\Delta)^{-\frac{1}{2}}u, uL_2(n)\rangle \leqslant \|u\|_{H^{-1}}\|u\|_{L^2}\|L_2(n)\|_{L^\infty},$$

and for the other term in the right-hand side of (1.6.20) we use the assumptions made on L_4 to get the bound:

$$Re\int_{\mathbb{R}^{2n}} \bar{u}(x)k_\Delta(x,y)u(y)L_4(n)(y,x)dxdy \leqslant \|u\|_{H^{-1}}\|u\|_{L^2}\|L_4(n)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}.$$

Finally we obtain the following bound on φ_2:

$$\begin{aligned}|\varphi_2(u,n)| \leqslant& 2\|u\|_{H^{-1}}\|u\|_{L^2}\left(\|L_2(n)\|_{L^\infty} + \|L_4(n)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\right) \\ \leqslant& 2\|u\|_{L^2}^2\left(\|L_2(n)\|_{L^\infty} + \|L_4(n)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\right),\end{aligned} \tag{1.6.22}$$

with L_2 and L_4 defined in (1.2.8).

Now we work on the different terms of the expression of $\mathcal{L}^\varepsilon\varphi^\varepsilon$ in (1.6.21). First we deal with the terms of order 0 in ε. Using integrations by parts and (1.6.13) we get

$$\langle(1-\Delta)^{-\frac{1}{2}}u, i\Delta u - i\lambda|u|^{2\sigma}u\rangle \leqslant \|u\|_{L^2}\|u\|_{H^1} + \|u\|_{L^2}\left(\|u\|_{L^2}^{q_n} + \|u\|_{H^1}^2\right). \tag{1.6.23}$$

Then we have

$$\begin{aligned}
&\langle (1-\Delta)^{-\frac{1}{2}}u, u\mathbb{E}_\nu\left[nL_1(n)\right]\rangle \\
&= -Re\int_{\mathbb{R}^n}(1-\Delta)^{-\frac{1}{2}}u(x)\bar{u}(x)\mathbb{E}\int_0^\infty m(0,x)m(s,x)dsdx \\
&= \frac{1}{2}Re\int_{\mathbb{R}^n}(1-\Delta)^{-\frac{1}{2}}u(x)\bar{u}(x)k(x,x)dxdydz,
\end{aligned}$$

where $k \in W^{3,\infty}(\mathbb{R}^n\times\mathbb{R}^n)\cap H^1(\mathbb{R}^n\times\mathbb{R}^n)$ is defined in (1.2.22). We get:

$$\langle (1-\Delta)^{-\frac{1}{2}}u, u\mathbb{E}_\nu\left[nL_1(n)\right]\rangle \leqslant \|k\|^2_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n))}\|u\|^2_{L^2}. \tag{1.6.24}$$

We get the same estimate for the last term of order 0:

$$\mathbb{E}_\nu\left[\langle (1-\Delta)^{-\frac{1}{2}}(un), uL_1(n)\rangle\right] \leqslant \|k\|^2_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n))}\|u\|^2_{L^2}. \tag{1.6.25}$$

Now we focus on the terms of order 1 in ε in (1.6.21). We recall the expression of the first corrector $\varphi_1(u,n) = 2\langle (1-\Delta)^{-\frac{1}{2}}u, iuL_1(n)\rangle$. Thus

$$\begin{aligned}
D_u\varphi_1(i\Delta u - i\lambda|u|^{2\sigma}u) =& 2\langle (1-\Delta)^{-\frac{1}{2}}u, (\lambda|u|^{2\sigma}u-\Delta u)L_1(n)\rangle \\
&+ 2\langle (1-\Delta)^{-\frac{1}{2}}(\Delta u - \lambda|u|^{2\sigma}u), uL_1(n)\rangle.
\end{aligned}$$

We use integrations by parts to deal with the terms involving Δu and we get that

$$\langle (1-\Delta)^{-\frac{1}{2}}u, \Delta uL_1(n)\rangle + \langle (1-\Delta)^{-\frac{1}{2}}\Delta u, uL_1(n)\rangle \leqslant C\|u\|_{L^2}\|u\|_{H^1}\|L_1(n)\|_E.$$

For the other terms, we use (1.6.13) to get

$$\begin{aligned}
\langle (1-\Delta)^{-\frac{1}{2}}u, |u|^{2\sigma}uL_1(n)\rangle +& \langle (1-\Delta)^{-\frac{1}{2}}(|u|^{2\sigma}u), uL_1(n)\rangle \\
&\leqslant \|u\|_{L^2}\left(\|u\|^{q_n}_{L^2}+\|u\|^2_{H^1}\right)\|L_1(n)\|_{L^\infty}.
\end{aligned}$$

Finally

$$|D_u\varphi_1(i\Delta u - i\lambda|u|^{2\sigma}u)| \leqslant C\|u\|_{L^2}\|u\|_{H^1}\|L_1(n)\|_E + \|u\|_{L^2}(\|u\|^{q_n}_{L^2}+\|u\|_{H^1})\|L_1(n)\|_{L^\infty}. \tag{1.6.26}$$

It remains to control the terms coming from the introduction of the second corrector φ_2

written in (1.6.20). We have

$$\begin{aligned}\frac{1}{2}D_u\varphi_2(h) =& -\langle(1-\Delta)^{-\frac{1}{2}}h, uL_2(n)\rangle - \langle(1-\Delta)^{-\frac{1}{2}}u, hL_2(n)\rangle \\ =& Re\int_{\mathbb{R}^{2n}} h(x)k_\Delta(x,y)\bar{u}(y)\left(L_4(n)(x,y)+L_4(n)(y,x)\right)dxdy\end{aligned} \tag{1.6.27}$$

where k_Δ denotes the kernel of the operator $(1-\Delta)^{\frac{1}{2}}$ and $L_4(n)$ is defined in (1.2.8). We start by the estimate of $D_u\varphi_2(-iX^\varepsilon m^\varepsilon)$. We can easily bound the terms involving $L_2(n)$:

$$\begin{aligned}\langle(1-\Delta)^{-\frac{1}{2}}(iX^\varepsilon m^\varepsilon), uL_2(m^\varepsilon)\rangle + &\langle(1-\Delta)^{-\frac{1}{2}}X^\varepsilon, iX^\varepsilon m^\varepsilon L_2(m^\varepsilon)\rangle \\ &\leqslant C\left\|X^\varepsilon\right\|_{L^2}^2\left\|m^\varepsilon\right\|_{L^\infty}\left\|L_2(m^\varepsilon)\right\|_{L^\infty}.\end{aligned}$$

For the other term, compute:

$$\begin{aligned}Re\int_{\mathbb{R}^{2n}} h(x)k_\Delta(x,y)\bar{u}(y)(L_4(n)(x,y)+&L_4(n)(y,x))dxdy \\ &\leqslant 2\left\|h\right\|_{L^2}\left\|u\right\|_{H^{-1}}\left\|L_4(n)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}.\end{aligned} \tag{1.6.28}$$

We can use this bound to control the terms which do not involve $L_2(m^\varepsilon)$ in $D_u\varphi(-iX^\varepsilon m^\varepsilon)$:

$$\begin{aligned}Re\int_{\mathbb{R}^{2n}} & iX^\varepsilon(x)m^\varepsilon(x)k_\Delta(x,y)\bar{X}^\varepsilon(y)\left(L_4(m^\varepsilon)(x,y)+L_4(m^\varepsilon)(y,x)\right)dxdy \\ &\leqslant 2\left\|L_4(m^\varepsilon)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} Re\int_{\mathbb{R}^n} i(1-\Delta)^{-\frac{1}{2}}\bar{X}^\varepsilon(x)X^\varepsilon(x)m^\varepsilon(x)dx \\ &\leqslant 2\left\|L_4(m^\varepsilon)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\left\|X^\varepsilon\right\|_{L^2}^2\left\|m^\varepsilon\right\|_{L^\infty},\end{aligned}$$

Finally, we have

$$|D_u\varphi_2(-iX^\varepsilon m^\varepsilon)| \leqslant C\left\|X^\varepsilon\right\|_{L^2}^2\left\|m^\varepsilon\right\|_{L^\infty}\left(\left\|L_2(m^\varepsilon)\right\|_{L^\infty}+\left\|L_4(m^\varepsilon)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}\right) \tag{1.6.29}$$

where L_2 and L_4 are defined in (1.2.8).

It remains to control $D_u\varphi_2(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)$. We use the same computations as for $D_u\varphi_1(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)$ and the bounds (1.6.13) and (1.6.28) to get:

$$\begin{aligned}|D_u\varphi_2(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)| \leqslant & C\left\|X^\varepsilon\right\|_{L^2}\left\|L_2(m^\varepsilon)\right\|_{L^\infty}\left(\|X^\varepsilon\|_{L^2}^{p_n}+\|X^\varepsilon\|_{H^1}^2\right)\\ &+C\left\|X^\varepsilon\right\|_{L^2}\left\|X^\varepsilon\right\|_{H^1}\left\|L_2(m^\varepsilon)\right\|_E\\ +C\left\|L_4(m^\varepsilon)\right\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)}&\left(\|X^\varepsilon\|_{L^2}\|X^\varepsilon\|_{H^1}+\|X^\varepsilon\|_{L^2}\left(\|X^\varepsilon\|_{L^2}^{q_n}+\|X^\varepsilon\|_{H^1}^2\right)\right).\end{aligned} \tag{1.6.30}$$

Finally, estimates (1.6.22), (1.6.26), (1.6.29), (1.6.30) coupled with Lemma 1.2.2.1, conservation of the L^2-norm and Lemma 1.4.2.1 give us for τ^ε the stopping time defined in (1.2.13):

$$\begin{aligned}\mathbb{E}\left[|\varphi(X^{\varepsilon,\tau^\varepsilon})-\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon},m^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(\varepsilon^{1-\alpha}+\varepsilon^{2-2\alpha}),\\ \mathbb{E}\left[|\mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon},m^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(1+\varepsilon^{1-\alpha}+\varepsilon^{2-2\alpha}),\end{aligned}$$

where $X^{\tau^\varepsilon}, m^{\tau^\varepsilon}$ denote the stopped process $X^{\tau^\varepsilon}(t)=X(t\wedge\tau^\varepsilon), m^{\tau^\varepsilon}(t)=m(t\wedge\tau^\varepsilon)$.

We use this last estimate and the fact that

$$\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(t))-\varphi^\varepsilon(X^\varepsilon)-\int_0^t\mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(s))ds$$

is a martingale to compute for any stopping time τ:

$$\begin{aligned}&\mathbb{E}\Big[\left\|X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2.-\left\|X^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2\Big]\\ &\quad=\mathbb{E}\left[\varphi(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon))-\varphi(X^\varepsilon(\tau\wedge\tau^\varepsilon))\right]\\ &\quad\leqslant\mathbb{E}\left[\varphi^\varepsilon(X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon))-\varphi^\varepsilon(X^\varepsilon(\tau\wedge\tau^\varepsilon))\right]+C\varepsilon^{1-\alpha}+C\varepsilon^{2-2\alpha}\\ &\quad\leqslant\mathbb{E}\left[\int_{\tau\wedge\tau^\varepsilon}^{(\tau+\delta)\wedge\tau^\varepsilon}\mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}(s),m^{\varepsilon,\tau^\varepsilon}(s))ds\right]+C\varepsilon^{1-\alpha}+C\varepsilon^{2-2\alpha}\\ &\quad\leqslant C(T)\delta\left(1+\varepsilon^{1-\alpha}+\varepsilon^{2-2\alpha}\right)+C\varepsilon^{1-\alpha}+C\varepsilon^{2-2\alpha}.\end{aligned}$$

Finally, for δ and ε small enough we get:

$$\mathbb{E}\left[\left\|X^\varepsilon((\tau+\delta)\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2-\left\|X^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}^2\right]\leqslant\eta. \tag{1.6.31}$$

Then we apply the Pertubed Test Function method on $\varphi(u)=\langle u,h\rangle_{H^{-\frac12}}$ for a fixed function $h \in L^2(\mathbb{R}^n)$. Computations lead us to choose two correctors:

$$\begin{aligned}\varphi_1(u,n) =&\langle iu\mathcal{M}^{-1}n, (1-\Delta)^{-\frac{1}{2}}h\rangle\\ =&\langle iuL_1(n), (1-\Delta)^{-\frac{1}{2}}h\rangle,\\ \varphi_2(u,n) =&\langle u\mathcal{M}^{-1}\left(\mathbb{E}_\nu\left[nL_1(n)\right] - nL_1(n)\right), (1-\Delta)^{-\frac{1}{2}}h\rangle\\ =&\langle uL_2(n), (1-\Delta)^{-\frac{1}{2}}h\rangle,\end{aligned}$$

and the infinitesimal generator applied to $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ is

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon, m^\varepsilon) &= \langle i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon\rangle + \langle X^\varepsilon\mathbb{E}_\nu\left[nL_1(n)\right], (1-\Delta)^{-\frac{1}{2}}h\rangle\\ &+ \varepsilon\Big(\langle(\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon - \Delta X^\varepsilon)L_1(m^\varepsilon), (1-\Delta)^{-\frac{1}{2}}h\rangle - \langle iX^\varepsilon m^\varepsilon L_2(m^\varepsilon), (1-\Delta)^{-\frac{1}{2}}h\rangle\Big)\\ &+ \varepsilon^2\langle(i\Delta X^\varepsilon - i\lambda|X^\varepsilon|^{2\sigma}X^\varepsilon)L_2(m^\varepsilon, (1-\Delta)^{-\frac{1}{2}}h\rangle.\end{aligned} \tag{1.6.32}$$

Here again there are several quantities which need to be bounded. We use similar computations as previously done for $\|u\|^2_{H^{-\frac{1}{2}}}$ and obtain:

$$\begin{aligned}\mathbb{E}\left[|\varphi(X^{\varepsilon,\tau^\varepsilon}) - \varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}, m^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(\varepsilon^{1-\alpha} + \varepsilon^{2-2\alpha}),\\ \mathbb{E}\left[|\mathcal{L}^\varepsilon\varphi^\varepsilon(X^{\varepsilon,\tau^\varepsilon}, m^{\varepsilon,\tau^\varepsilon})|\right] &\leqslant C(T)(1 + \varepsilon^{1-\alpha} + \varepsilon^{2-2\alpha}),\end{aligned}$$

where $C(T)$ also depends on the L^2-norm of h. Now taking τ as a stopping time and $h = X^\varepsilon(\tau \wedge \tau^\varepsilon)$, knowing that conditioning by $\mathcal{F}_{\tau\wedge\tau^\varepsilon}$

$$M_t := \varphi^\varepsilon(X^\varepsilon(t)) - \varphi^\varepsilon(X_0) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(X^\varepsilon(s))ds$$

is a martingale, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\big\langle X^{\varepsilon}((\tau+\delta)\wedge\tau^{\varepsilon}) - X^{\varepsilon}(\tau\wedge\tau^{\varepsilon}), X^{\varepsilon}(\tau\wedge\tau^{\varepsilon})\big\rangle_{H^{-\frac{1}{2}}}\Big] \\
&\quad = \mathbb{E}\left[\varphi(X^{\varepsilon}((\tau+\delta)\wedge\tau^{\varepsilon})) - \varphi(X^{\varepsilon}(\tau\wedge\tau^{\varepsilon}))\right] \\
&\quad \leqslant \mathbb{E}\left[\varphi^{\varepsilon}(X^{\varepsilon}((\tau+\delta)\wedge\tau^{\varepsilon})) - \varphi^{\varepsilon}(X^{\varepsilon}(\tau\wedge\tau^{\varepsilon}))\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha} \\
&\quad \leqslant \mathbb{E}\left[\mathbb{E}\left[\varphi^{\varepsilon}(X^{\varepsilon}((\tau+\delta)\wedge\tau^{\varepsilon})) - \varphi^{\varepsilon}(X^{\varepsilon}(\tau\wedge\tau^{\varepsilon})) \mid \mathcal{F}_{\tau\wedge\tau^{\varepsilon}}\right]\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha} \\
&\quad \leqslant \mathbb{E}\left[\mathbb{E}\left[\int_{\tau\wedge\tau^{\varepsilon}}^{(\tau+\delta)\wedge\tau^{\varepsilon}} \mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(X^{\varepsilon}(s),)ds \mid \mathcal{F}_{\tau\wedge\tau^{\varepsilon}}\right]\right] \\
&\qquad + \mathbb{E}\left[\mathbb{E}\left[M_{(\tau+\delta)\wedge\tau^{\varepsilon}} - M_{\tau\wedge\tau^{\varepsilon}} \mid \mathcal{F}_{\tau\wedge\tau^{\varepsilon}}\right]\right] + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha} \\
&\quad \leqslant C(1+\varepsilon^{1-\delta}+\varepsilon^{2-2\delta})\delta + C\varepsilon^{1-\alpha} + C\varepsilon^{2-2\alpha}.
\end{aligned}$$

Thus for ε, δ small enough, we have

$$\mathbb{E}\left[\big\langle X^{\varepsilon}((\tau+\delta)\wedge\tau^{\varepsilon}) - X^{\varepsilon}(\tau\wedge\tau^{\varepsilon}), X^{\varepsilon}(\tau\wedge\tau^{\varepsilon})\big\rangle_{H^{-\frac{1}{2}}}\right] \leqslant \eta. \tag{1.6.33}$$

Finally, gathering (1.6.31),(1.6.33) and using Markov inequality, we get that for any stopping time τ and $R, \eta > 0$, for ε, δ small enough:

$$\mathbb{P}\left(\left\|X^{\varepsilon,\tau^{\varepsilon}}(\tau+\delta) - X^{\varepsilon,\tau^{\varepsilon}}(\tau)\right\|_{H^{-\frac{1}{2}}} > R\right) \leqslant \eta.$$

So far we have already proved that $(X^{\varepsilon,\tau^{\varepsilon}})_{\varepsilon}$ satisfies condition (1.6.16), and condition (1.6.17) for $H = H^{-\frac{1}{2}}(\mathbb{R}^n)$. Using this last condition and Lemma 1.4.2.1, we can prove by interpolation that $(X^{\varepsilon,\tau^{\varepsilon}})_{\varepsilon}$ also satisfies condition (1.6.17) with $H = H^s(\mathbb{R}^n)$, for any $s < 1$. Thus Aldous criterion is satisfied, the process $(X^{\varepsilon,\tau^{\varepsilon}})_{\varepsilon}$ is tight in $\mathcal{C}([0,T], H^s(\mathbb{R}^n))$. $\square$

Chapter 2

The Stochastic Zakharov system, convergence to a Nonlinear Stochastic Schrödinger equation

Introduction

The Zakharov system is a simplified model for the description of long wave Langmuir turbulence in an ionized plasma. Langmuir waves are rapid oscillations of the electron density. The system was introduced in [Zak72] as:

$$\begin{cases} i\partial_t u = -\Delta u + nu, \\ \varepsilon^2 \partial_t^2 n = \Delta(n + |u|^2). \end{cases} \tag{2.0.1}$$

The function u represents the envelope of the electric field, and n denotes the deviation of the ion density from its mean mode. The term ε^2 refers to the inverse of the ion speed. If we formally study this system and more precisely its limit as ε goes to 0, we obtain a cubic Nonlinear Schrödinger equation:

$$i\partial_t u + \Delta u + |u|^2 u = 0. \tag{2.0.2}$$

This system was first studied by Sulem and Sulem in [SS79] where they proved global existence of weak solutions in the energy space in dimension $d = 1, 2, 3$ using a Galerkin method, and global well-posedness in dimension 1 for initial data

$$(u_0, n_0, n_1) \in H^m(\mathbb{R}) \times H^{m-1}(\mathbb{R}) \times (H^{m-2}(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})), \qquad \text{for any } m \geqslant 3.$$

Added and Added in [AA84] improved the results of [SS79] by showing global well-posedness in dimension 2 for small initial data, using Brezis-Gallouet inequality. A local well-posedness result was proved by Schochet and Weinstein in [SW86] with an existence of a time interval independent of ε, which allowed them to show the convergence as ε goes to 0 to a solution of (2.0.2). In [AA88] Added and Added detailled the rate of convergence for small amplitude solutions, depending on the compatibility of initial data, which can involve initial layer effects, and the optimal rates for this convergence were obtained in [OT91]. This limit was also studied for instance in [MN08], where the authors were interested by the convergence of the Klein-Gordon-Zakharov system to the Nonlinear Schrödinger equation.

Other results about local well-posedness came from [OT92] for data in $H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, $1 \leqslant n \leqslant 3$, and from [Col97] in the energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times (L^2(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3))$. Bourgain and Colliander also used a low-high frequency decomposition method in [BC96], which was used again by Ginibre, Tsutsumi and Velo in [GTV97] to get some refinements for the local well-posedness in dimension $d = 1, 2, 3$. More results about local and global well-posedness can be found for instance in [Col98, CHT08, MN09, Pec01] for different dimensions or different nonlinearities.

Here we are interested in a stochastic version of the Zakharov system restricted to the dimension 1, where we add a damping term. It corresponds to the following problem:

$$\begin{cases} i\partial_t u = -\partial_x^2 u + nu, \\ \varepsilon^2 d(\partial_t n) + \alpha\varepsilon\partial_t n = \partial_x^2(n + |u|^2)dt + \phi dW(t), \end{cases} \tag{2.0.3}$$

where $\alpha > 0$. This problem is complemented with inital data $(u(0), n(0), \partial_t n(0)) = (u_0, n_0, n_1)$, which belong to some Sobolev spaces that we will specify in the different sections of this work. The operator ϕ is a Hilbert-Schmidt operator, and the random process W_t is a cylindrical Wiener process on $L^2(\mathbb{R})$. This stochastic perturbation corresponds to a spatially correlated white noise in time, and can be physically understood as random external fluctuations in the studied system, as for example thermal fluctuations. The damping is introduced to compensate the input of energy due to the noise. The Zakharov system with a damping term is studied even without any stochastic perturbation, for example in [GF06], where the authors provide a numerical analysis of Langmuir turbulence on incoherently scattered spectra.

As in the deterministic case, we expect that the solution of (2.0.3) converges in some sense to a stochastic Nonlinear Schrödinger equation. The conservative version of this equation with a multiplicative noise has been studied by Debussche and de Bouard in [dBD99] and [dBD03b], while Barbu, Röckner and Zhang in [BRZ14] used a different approach using rescaling transformations to prove well-posedness results in both conservative and nonconservative cases. We also cite Brzezniak and Millet [BM14], who studied the stochastic Nonlinear Schrödinger equation on a two-dimensional compact Riemannian manifold.

The aim of this chapter is to prove some local and global well-posedness results for the system (2.0.3), and to prove a weak convergence as ε goes to 0 to a stochastic partial differential equation. The main difficulty comes from the fact that the energy of our system

$$H = \|\partial_x u\|_{L^2}^2 + \frac{1}{2}\left(\|n\|_{L^2}^2 + \left\|\varepsilon\partial_x^{-1}\partial_t n\right\|_{L^2}^2\right) + \int_{\mathbb{R}} n|u|^2 dx \tag{2.0.4}$$

is no longer preserved, and more importantly it has a singular evolution as ε goes to 0. Indeed, one can apply Itô formula to H, and terms which are not controlled by the energy or terms of order ε^{-2} will appear. In order to deal with this singular evolution and to be able to pass to the limit in ε, we use a predictor method usually called the Perturbed Test Function method. We use this method in the case of infinite dimensions, which allows us to pass to the limit in a martingale problem to identify the limit equation. This method was first introduced in a finite dimensional setting in [PSV77], and many examples of applications can be found in the book of Garnier, Fouque, Papanicolaou and Solna [FGPS07]. It was generalized to the infinite dimensional case for instance in [dBG12],[DV12] and used in [DDMV16].

In this chapter we prove almost sure local and global well-posedness of the system (2.0.3), based on [SS79], and we prove that it converges in a weak sense to a stochastic Nonlinear Schrödinger equation:

$$idu_t = (-\partial_x^2 u_t - |u_t|^2 u_t - \frac{i}{2}u_t F)dt - u_t(\partial_x^2)^{-1}\phi dW_t.$$

The last two terms in this equation correspond to the Stratonovitch noise $u_t \circ (\partial_x^2)^{-1}\phi dW_t$.

The perturbed test function method usually allows to prove convergence in law of solutions. However, due to the special form of the noise considered here, we obtain convergence in probability. Note that the well-posedness results that we proved are still true without the addition of the damping term, namely for $\alpha = 0$, but the convergence of the system to the stochastic Nonlinear Schrödinger equation requires $\alpha > 0$. It is not clear at all whether (2.0.3) has a limit $\varepsilon \to 0$ in the case $\alpha = 0$.

In Section 2.1, we state some preliminary results which will be needed in the other sections. In Section 2.2 we prove that the system is almost surely locally well-posed in

$L^\infty([0,T],H^3(\mathbb{R})\times H^2(\mathbb{R})\times H^1(\mathbb{R}))$ and almost surely globally well-posed in the space $L^\infty([0,T],H^3(\mathbb{R})\times(H^2(\mathbb{R})\cap\dot H^{-1}(\mathbb{R})\times(H^1(\mathbb{R})\cap\dot H^{-1}(\mathbb{R})))$. In Section 2.3 we state our main result on the convergence of the system (2.0.3) to a stochastic Nonlinear Schrödinger equation, and we state some results, which will be useful in order to prove this theorem. Finally we prove Theorem 2.3.0.1 using weak convergence results based on the work of [AA88] in the deterministic case applied to a martingale problem.

2.1 Preliminary and notations

Throughout this work, for $p\geqslant 1$, we denote by $L^p(\mathbb{R},\mathbb{C})$ the Lebesgue space of p integrable $\mathbb{C}$-valued functions on $\mathbb{R}$, endowed with its usual norm. For $p=2$, $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{R},\mathbb{C})$ given by

$$(f,g)=Re\int_{\mathbb{R}}f(x)\bar g(x)dx,$$

where Re denotes the real part of the integral. Sometimes we will just write $L^p:=L^p(\mathbb{R})$ without any additional precision of the underlying space. In fact, the only $\mathbb{C}$-valued function in our problem is u, and all the others functions are $\mathbb{R}$-valued. For $s\in\mathbb{R}$, we use the Sobolev space $H^s:=H^s(\mathbb{R})$ of tempered distributions $f\in\mathcal{S}'(\mathbb{R})$ such that $(1+\xi^2)^{\frac{s}{2}}\hat f(\xi)\in L^2(\mathbb{R})$, $\hat f$ being the Fourier transform of f, endowed with its usual norm. Besides we will need the homogeneous Sobolev space $\dot H^s:=\dot H^s(\mathbb{R})$ of tempered distributions $f\in\mathcal{S}'(\mathbb{R})$ such that $|\xi|^s\hat f(\xi)\in L^2(\mathbb{R})$. We also denote by $H^s_{loc}(\mathbb{R})=H^s_{loc}$ the space of distributions f such that for every $\ell>0$, $f\in H^s([-\ell,\ell])$. Finally, if H,K are Hilbert spaces, $\mathcal{L}_2(H,K)$ is the space of Hilbert-Schmidt operators from H to K (when $H=K$ we write $\mathcal{L}_2(H)$).

To rigorously introduce the system (2.0.3), we set $(\Omega,\mathcal{F},\mathbb{P})$ a probability space, endowed with a filtration $(\mathcal{F}_t)_{t>0}$. We also introduce a sequence $(\beta_k)_{k\in\mathbb{N}}$ of real-valued Brownian motions associated to this filtration. We denote by $(e_k)_{k\in\mathbb{N}}$ a Hilbert basis of $L^2(\mathbb{R})$ and

$$W(t,x,\omega)=\sum_{k\in\mathbb{N}}\beta_k(t,\omega)e_k(x)$$

so the quantity ϕdW can be interpreted as

$$\phi dW(t,x,\omega)=\sum_{k\in\mathbb{N}}\phi e_k(x)d\beta_k(t,\omega).$$

If $\phi \in \mathcal{L}_2(L^2(\mathbb{R}), H)$ with H a Hilbert space, this series converges for any $t \geqslant 0$ in $L^2(\Omega, H)$. Indeed, according to Ito's isometry we have

$$\begin{aligned}
\mathbb{E}\left[\left\|\sum_{k\in\mathbb{N}} \phi e_k(x)\beta_k(t)\right\|_H^2\right] &= t\sum_{k\in\mathbb{N}} \|\phi e_k\|_H^2 \\
&= t\,\|\phi\|^2_{\mathcal{L}_2(L^2,H)} < +\infty.
\end{aligned}$$

In all this section we fix $\varepsilon = 1$. We can rewrite the system in its integral form:

$$\begin{cases}
u(t) = & U(t)u_0 - i\int_0^t U(t-s)n(s)u(s)ds, \\
\begin{pmatrix} n(t) \\ m(t) \end{pmatrix} = & S_\alpha(t)\begin{pmatrix} n_0 \\ n_1 \end{pmatrix} + \displaystyle\int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ \partial_x^2|u|^2(s) \end{pmatrix} ds \\
& + \displaystyle\int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ \phi \end{pmatrix} dW(s),
\end{cases} \tag{2.1.1}$$

where $U(t)$ is the Schrödinger group which appears in the solution of $i\partial_t\psi = -\partial_x^2\psi$ with initial data $\psi(0) = \psi_0$, namely $\psi(t,x) = U(t)\psi_0(x)$, and $S_\alpha(t)$ is the semigroup associated to the wave equation with damping:

$$\begin{cases} \partial_t n = m \\ \partial_t m + \alpha m = \partial_x^2 n \end{cases},$$

so that the solution of this equation with initial data (n_0, n_1) can be written as $(n(t), \partial_t n(t)) = S_\alpha(t)(n_0, n_1)$. We denote respectively the right-hand sides of the system by $\Gamma_1(u,n)$ and $\Gamma_2(u,n,m)$. For any bounded measurable function φ we define the Fourier multiplier $\varphi((-\partial_x^2)^{\frac{1}{2}}) = \mathcal{F}^{-1}\varphi(|\xi|)\mathcal{F}$ as an operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Thus, we can define $S_\alpha(t)$ writing $S_\alpha(t)\phi = \mathcal{F}^{-1}(m_\alpha(t,\xi)\hat{\phi}(\xi))$ where:

$$m_\alpha(t,\xi) = e^{-\frac{\alpha}{2}t}\begin{pmatrix} m_\alpha^{1,1}(t,\xi) & m_\alpha^{1,2}(t,\xi) \\ m_\alpha^{2,1}(t,\xi) & m_\alpha^{2,2}(t,\xi) \end{pmatrix} \tag{2.1.2}$$

with

$$\begin{aligned}
m_\alpha^{1,1}(t,\xi) &= \cosh(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t) + \frac{\alpha}{\sqrt{\alpha^2-4\xi^2}}\sinh(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t),\\
m_\alpha^{1,2}(t,\xi) &= \frac{2}{\sqrt{\alpha^2-4\xi^2}}\sinh(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t),\\
m_\alpha^{2,1}(t,\xi) &= \left(\frac{\sqrt{\alpha^2-4\xi^2}}{2} - \frac{\alpha^2}{2\sqrt{\alpha^2-4\xi^2}}\right)\sinh(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t)\\
m_\alpha^{2,2}(t,\xi) &= \cosh(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t) - \frac{\alpha}{\sqrt{\alpha^2-4\xi^2}}\sinh(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t)
\end{aligned} \tag{2.1.3}$$

if $|\xi| < \frac{\alpha}{2}$, and

$$\begin{aligned}
m_\alpha^{1,1}(t,\xi) &= \cos(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t) + \frac{\alpha}{\sqrt{4\xi^2-\alpha^2}}\sin(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t),\\
m_\alpha^{1,2}(t,\xi) &= \frac{2}{\sqrt{4\xi^2-\alpha^2}}\sin(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t),\\
m_\alpha^{2,1}(t,\xi) &= \left(\frac{\sqrt{4\xi^2-\alpha^2}}{2} - \frac{\alpha^2}{2\sqrt{4\xi^2-\alpha^2}}\right)\sin(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t)\\
m_\alpha^{2,2}(t,\xi) &= \cos(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t) - \frac{\alpha}{\sqrt{4\xi^2-\alpha^2}}\sin(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t)
\end{aligned} \tag{2.1.4}$$

if $|\xi| > \frac{\alpha}{2}$.

In order to prove local well-posedness of equation (2.0.3), we adapt the method of [Kat87]. For this we introduce the spaces:

$$\begin{aligned}
Z &= \{u \in L^\infty(I, L^2(\mathbb{R})), \partial_x^2 u \in L^\infty(I, L^2(\mathbb{R})), \partial_t u \in L^\infty(I, L^2(\mathbb{R}))\},\\
&= \{u \in L^\infty(I, H^2(\mathbb{R})), \partial_t u \in L^\infty(I, L^2(\mathbb{R}))\},\\
Y &= \{n \in L^\infty(I, H^1(\mathbb{R})), \partial_t n \in L^\infty(I, L^2(\mathbb{R}))\},\\
H &= L^\infty(I, L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})),
\end{aligned} \tag{2.1.5}$$

where I is a time interval of the form $[0,T]$ for some $T > 0$. We also introduce the set

$$Z' = \{f \in L^\infty(I, L^2(\mathbb{R})), \partial_t f \in L^1(I, L^2(\mathbb{R}))\}.$$

The following lemmas are well-known and can be found in [Kat87].

Lemma 2.1.0.1. *The operator $U(\cdot)$ is a bounded operator from $L^2(\mathbb{R})$ to*

$L^\infty([0,T],L^2(\mathbb{R}))$ and from $H^1(\mathbb{R})$ to $L^\infty([0,T],H^1(\mathbb{R}))$. The associated norms are independent of T.

Lemma 2.1.0.2. *$U(\cdot)$ is a bounded operator from $H^2(\mathbb{R})$ to Z, and $v \mapsto \int_0^t U(t-s)v(s)ds$ is a bounded operator from Z' to Z. Besides there exists $\gamma > 0$ such that:*

$$\|U(t)v\|_Z \leqslant \gamma \, \|v\|_{H^2},$$

$$\left\| \int_0^t U(t-s)v(s)ds \right\|_Z \leqslant (2\gamma+1) \, \|v\|_{Z'}.$$

Because of the damping term in the second equation of (2.0.3), we have to take care of different problems in the estimates of S_α, depending on the frequencies we are looking at. Indeed, a singularity at the frequency $\xi = \alpha^2$ appears in the expression of S_α. It is not a problem when we are interested in the local well-posedness, but it involves additional hypothesis on the initial conditions and on the operator ϕ when we want to prove global well-posedness and when we study the limit as $\varepsilon \to 0$.

Lemma 2.1.0.3. *$S_\alpha(\cdot)$ is a bounded operator from $H^1(\mathbb{R}) \times L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ to $Y \times H$ space.The associated norm is independent of T, and we also denote by γ its supremum.*

Proof. Here we work with $\alpha \geqslant 0$. The scheme of this proof will be used again for all the estimates that we will need on the wave operator. We denote by $(S_\alpha(t)(n,m))_i$the i-th component of $S_\alpha(t)(n,m)$, for $i = 1, 2$. Thanks to equations (2.1.3) and (2.1.4), we have:

$$\begin{aligned}
&\|(S_\alpha(t)(n,m))_1\|_{H^1}^2 \leqslant C \int_{|\xi| \leqslant \frac{\alpha}{2}} (1+|\xi|^2) e^{-\alpha t} \left(\left(\cosh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2} t \right) \right.\right. \\
&+ \left.\left. \frac{\alpha^2}{\alpha^2-4\xi^2} \sinh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2} t \right) \right) \hat{n}^2(\xi) + \frac{4}{\alpha^2-4\xi^2} \sinh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2} t \right) \hat{m}^2(\xi) \right) d\xi \\
&+ C \int_{|\xi| \geqslant \frac{\alpha}{2}} (1+|\xi|^2) e^{-\alpha t} \left(\left(\cos^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2} t \right) \right.\right. \\
&+ \left.\left. \frac{\alpha^2}{4\xi^2-\alpha^2} \sin^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2} t \right) \right) \hat{n}^2(\xi) + \frac{4}{4\xi^2-\alpha^2} \sin^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2} t \right) \hat{m}^2(\xi) \right) d\xi.
\end{aligned}$$

We decompose $\mathbb{R}$ in four frequency domains.

<u>First domain</u>:$|\xi| \leqslant \frac{\alpha}{4}$

Here we can bound the integral on the domain $|\xi| \leqslant \frac{\alpha}{4}$ by

$$C\int_{|\xi|\leqslant\frac{\alpha}{4}} (1+|\xi|^2)\left(\left(1+\frac{\alpha^2}{\alpha^2-4\xi^2}\right)\hat{n}^2+\frac{4}{\alpha^2-4\xi^2}\hat{m}^2\right)d\xi$$
$$\leqslant C_\alpha\left(\|n\|_{L^2}^2+\|m\|_{L^2}^2\right).$$

Since $|\xi|$ is bounded on this domain.

<u>Second domain</u>: $\frac{\alpha}{4} \leqslant |\xi| \leqslant \frac{\alpha}{2}$

Here we have a singularity at $|\xi| = \frac{\alpha}{2}$. We deal with it thanks to the following computation:

$$\begin{aligned}
&C\int_{\frac{\alpha}{4}\leqslant|\xi|\leqslant\frac{\alpha}{2}} (1+|\xi|^2)e^{-\alpha t}\left(\frac{\alpha^2}{\alpha^2-4\xi^2}\sinh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\hat{n}^2\right.\\
&\left.+\cosh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\hat{n}^2+\frac{4}{\alpha^2-4\xi^2}\sinh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\hat{m}^2\right)d\xi\\
&\leqslant C\int_{\frac{\alpha}{4}\leqslant|\xi|\leqslant\frac{\alpha}{2}} (1+|\xi|^2)e^{-\alpha t}\left((1+\frac{\alpha^2t^2}{4})\cosh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\hat{n}^2\right.\\
&\qquad\left.+t^2\cosh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\hat{m}^2\right)d\xi.
\end{aligned}$$

Since we can bound $t \mapsto t^2e^{-\alpha t}\cosh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)$ independently of time because $\frac{\alpha}{4} \leqslant |\xi|$, the bound becomes

$$C_\alpha\int_{\frac{\alpha}{4}\leqslant|\xi|\leqslant\frac{\alpha}{2}} (1+|\xi|^2)(\hat{n}^2+\hat{m}^2)d\xi \leqslant C_\alpha(\|n\|_{L^2}^2+\|m\|_{L^2}^2).$$

<u>Third domain</u>: $\frac{\alpha}{2} \leqslant |\xi| \leqslant \alpha$

Here we have to deal with the same singularity as in the above domain. Thus we have:

$$\begin{aligned}
C\int_{\frac{\alpha}{2}\leqslant|\xi|\leqslant\alpha}&(1+|\xi|^2)e^{-\alpha t}\left(\left(\cos^2(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t)+\frac{\alpha^2}{4\xi^2-\alpha^2}\sin^2(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t)\right)\hat{n}^2\right.\\
&\left.+\frac{4}{4\xi^2-\alpha^2}\sin^2(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t)\hat{m}^2\right)d\xi\\
&\leqslant\int_{\frac{\alpha}{2}\leqslant|\xi|\leqslant\alpha}(1+|\xi|^2)e^{-\alpha t}\left((1+\frac{\alpha^2t^2}{4})\hat{n}^2+t^2\hat{m}^2\right)d\xi\\
&\leqslant C_\alpha(\|n\|_{L^2}^2+\|m\|_{L^2}^2),
\end{aligned}$$

where we used that $(1+\xi^2)\leqslant(1+\alpha^2)$.

<u>Fourth domain</u>: $|\xi|\geqslant\alpha$

Here we do not have any singularity, and the bound becomes:

$$\begin{aligned}
C\int_{|\xi|\geqslant\alpha}&(1+|\xi|^2)e^{-\alpha t}\left(\left(\cos^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right)+\frac{\alpha^2}{4\xi^2-\alpha^2}\sin^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right)\right)\hat{n}^2\right.\\
&\left.+\frac{4}{4\xi^2-\alpha^2}\sin^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right)\hat{m}^2\right)d\xi\\
&\leqslant C\int_{|\xi|\geqslant\alpha}(1+|\xi|^2)\left((1+\frac{\alpha^2}{4\xi^2-\alpha^2})\hat{n}^2+\frac{4}{4\xi^2-\alpha^2}\hat{m}^2\right)d\xi\\
&\leqslant C_\alpha(\|n\|_{H^1}^2+\|m\|_{L^2}^2).
\end{aligned}$$

The same procedure gives us the bound:

$$\|(S_\alpha(t)(n,m))_2\|_{L^2\cap\dot{H}^{-1}}^2\leqslant C\left(\|n\|_{H^1}^2+\|m\|_{L^2\cap\dot{H}^{-1}}^2\right).$$

and we remark that $(\partial_tS_\alpha(t)(n,m))_1=(S_\alpha(t)(n,m))_2$, which concludes the proof. □

Remark 2.1.0.1. *In the above proof, all the bounds that we obtained depend on α. However, one can verify that the result is still true for $\alpha=0$. Besides, note that $S_\alpha(t)$ is a contraction semigroup in the space $H^1(\mathbb{R})\times L^2(\mathbb{R})\cap\dot{H}^{-1}(\mathbb{R})$.*

From Lemma 2.1.0.3 we get another result, which will be usefull in order to estimate the second term of the integral form of equation (2.0.3).

Lemma 2.1.0.4. *The operator*

$$G(t) : \begin{pmatrix} n \\ m \end{pmatrix} \mapsto \int_0^t S_\alpha(t-s) \begin{pmatrix} n(s) \\ m(s) \end{pmatrix} ds$$

is a bounded operator from $Y \times H$ into itself.

Proof. Thanks to Lemma 2.1.0.3 we easily get that

$$\|(G(t)(n,m))_1\|_{L^\infty([0,T],H^1)} \leqslant \gamma T(\|n\|_{L^\infty([0,T],H^1)} + \|m\|_{L^\infty([0,T],L^2)})$$

and

$$\|(G(t)(n,m))_2\|_{L^\infty([0,T],L^2\cap\dot{H}^{-1})} \leqslant \gamma T(\|n\|_{L^\infty([0,T],H^1)} + \|m\|_{L^\infty([0,T],L^2\cap\dot{H}^{-1})}).$$

Then, we remark that

$$(\partial_t G(t)(n,m))_1 = n(t) + \int_0^t (S_\alpha(t-s)(n(s),m(s)))_2\, ds$$

and we get

$$\begin{aligned}\|(\partial_t G(t)(n,m))_1\|_{L^\infty([0,T],L^2)} &\leqslant \|n\|_{L^\infty([0,T],L^2)} + \gamma T(\|n\|_{L^\infty([0,T],H^1)} + \|m\|_{L^\infty([0,T],L^2)}) \\ &\leqslant (1+\gamma T)(\|n\|_Y + \|m\|_H).\end{aligned}$$

□

We will also need a result about the stochastic convolution.

Lemma 2.1.0.5. *For $\phi \in \mathcal{L}_2(L^2 \cap \dot{H}^{-1})$*

$$\int_0^t S_\alpha(t-s) \begin{pmatrix} 0 \\ \phi \end{pmatrix} dW(s) \in Y \times H \quad a.s.$$

Proof. We apply [DPZ14, Theorem 6.10] to $S_\alpha(t)$, which is a contraction semigroup, and we get

$$\mathbb{E}\left[\sup_{[0,T]}\left\|\int_0^t S_\alpha(t-s)\begin{pmatrix}0\\ \phi\end{pmatrix}dW(s)\right\|^2_{H^1\times L^2\cap\dot{H}^{-1}}\right]\leqslant C\mathbb{E}\left[\int_0^t\|\phi\|^2_{\mathcal{L}_2(L^2\cap\dot{H}^{-1})}\,ds\right]$$
$$\leqslant CT\,\|\phi\|^2_{\mathcal{L}_2(L^2\cap\dot{H}^{-1})}.$$

The fact that we can control the derivative in time of the first component with the second component as in the proof of Lemma 2.1.0.4 concludes the proof. □

Now we state the last lemma, which will provide important estimates for the application of the Perturbed Test Function method in the last section. It is specifically in this lemma that the need of the homogeneous Sobolev spaces appears, because of the integration in time in the low frequency domain.

Lemma 2.1.0.6. *For $k,\ell\in\mathbb{N}$, there exists a constant $C>0$ (depending on α) such that*

$$\int_0^\infty\|(S_\alpha(t)(n,m))_1\|^2_{H^k\cap\dot{H}^{-\ell}}\,dt\leqslant C\left(\|n\|^2_{H^k\cap\dot{H}^{-(\ell+1)}}+\|m\|^2_{H^{k-1}\cap\dot{H}^{-(\ell+1)}}\right),\qquad(2.1.6)$$

$$\int_0^\infty\|(S_\alpha(t)(n,m))_2\|^2_{H^k\cap\dot{H}^{-\ell}}\,dt\leqslant C\left(\|n\|^2_{H^{k+1}\cap\dot{H}^{-(\ell+1)}}+\|m\|^2_{H^k\cap\dot{H}^{-(\ell+1)}}\right),\qquad(2.1.7)$$

and for $i=1,2$

$$\int_0^\infty\int_t^\infty\|(S_\alpha(s)(0,\phi e_k))_i\|^2_{H^k\cap\dot{H}^{-\ell}}\,dsdt\leqslant C\,\|\phi e_k\|^2_{H^k\cap\dot{H}^{-(\ell+2)}}.\qquad(2.1.8)$$

Proof. These estimates are proved using the same tools, so we choose to prove only the first one in details, and we focus on the H^k norm because the procedure for the $\dot{H}^{-\ell}$ norm is similar. Thanks to the explicit form of S_α, given in (2.1.3) and (2.1.4), we can bound the expression $\int_0^\infty \|(S_\alpha(t)(n,m))_1\|_{H^k}^2 \, dt$ by:

$$\begin{aligned}
C\int_0^\infty \int_{|\xi|\leqslant\frac{\alpha}{2}} (1+|\xi|^2)^k e^{-\alpha t} \Bigg[\frac{\alpha^2}{\alpha^2-4\xi^2} \sinh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right) \hat{n}^2(\xi) \\
+ \cosh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right) \hat{n}^2 + \frac{4}{\alpha^2-4\xi^2} \sinh^2\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right) \hat{m}^2(\xi) \Bigg] d\xi dt \\
+ C\int_0^\infty \int_{|\xi|\geqslant\frac{\alpha}{2}} (1+|\xi|^2)^k e^{-\alpha t} \Bigg[\frac{\alpha^2}{4\xi^2-\alpha^2} \sin^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right) \hat{n}^2(\xi) \\
+ \cos^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right) \hat{n}^2 + \frac{4}{4\xi^2-\alpha^2} \sin^2\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right) \hat{m}^2(\xi) \Bigg] d\xi dt.
\end{aligned}$$

Now we decompose the frequency space as in Lemma 2.1.0.3.

<u>First domain:</u> $|\xi| \leqslant \frac{\alpha}{4}$.

Here we integrate in time and get the bound:

$$\begin{aligned}
C\int_{|\xi|\leqslant\frac{\alpha}{4}} (1+|\xi|^2)^k \left(\frac{1}{\alpha-\sqrt{\alpha^2-4\xi^2}} + \frac{1}{\alpha+\sqrt{\alpha^2-4\xi^2}} \right) \\
\times \left(\left(1+\frac{\alpha^2}{\alpha^2-4\xi^2}\right)\hat{n}^2 + \frac{4}{\alpha^2-4\xi^2}\hat{m}^2 \right) d\xi \\
= C\int_{|\xi|\leqslant\frac{\alpha}{4}} 2\alpha\frac{(1+|\xi|^2)^k}{4\xi^2} \left(\left(1+\frac{\alpha^2}{\alpha^2-4\xi^2}\right)\hat{n}^2 + \frac{4}{\alpha^2-4\xi^2}\hat{m}^2 \right) d\xi
\end{aligned}$$

and on this domain $(1+|\xi|^2)^k$ and $\frac{\alpha^2}{\alpha^2-4\xi^2}$ are bounded, thus we can bound this expression by

$$C(\|n\|_{\dot{H}^{-1}}^2 + \|m\|_{\dot{H}^{-1}}^2).$$

Note that the $\dot{H}^{-1}$ regularity is needed because of the integration in time in this low frequency domain.

<u>Second domain:</u> $\frac{\alpha}{4} \leqslant |\xi| \leqslant \frac{\alpha}{2}$.

Here we use the inequality $\frac{\sinh(x)}{x} \leqslant \cosh(x)$ and we integrate by parts in time to get the bound

$$C\int_{\frac{\alpha}{4}\leqslant|\xi|\leqslant\frac{\alpha}{2}}(1+|\xi|^2)^k\frac{\alpha}{2\xi^2}\hat{n}^2+\left(\frac{\left(\alpha+\sqrt{\alpha^2-4\xi^2}\right)^3}{(4\xi^2)^3}(\hat{n}^2+\hat{m}^2)\right.$$
$$\left.+\frac{\left(\alpha-\sqrt{\alpha^2-4\xi^2}\right)^3}{(4\xi^2)^3}(\hat{n}^2+\hat{m}^2)\right)d\xi$$
$$\leqslant C(\|n\|_{L^2}^2+\|m\|_{L^2}^2),$$

since on this domain we have upper and lower bounds on ξ.
Third domain: $\frac{\alpha}{2}\leqslant\xi\leqslant\alpha$
Here we use the inequality $\frac{\sin(x)}{x}\leqslant 1$ and we integrate by parts in time to get the bound

$$C(\|n\|_{L^2}^2+\|m\|_{L^2}^2)$$

by the same procedure as for the second domain.
Fourth domain: $|\xi|\geqslant\alpha$
Here we integrate in time to bound our quantity by

$$C\int_{|\xi|\geqslant\alpha}\frac{1}{\alpha}(1+|\xi|^2)^k\left(\left(1+\frac{\alpha^2}{4\xi^2-\alpha^2}\right)\hat{n}^2+\frac{4}{4\xi^2-\alpha^2}\hat{m}^2\right)d\xi$$
$$\leqslant C(\|n\|_{H^k}^2+\|m\|_{H^{k-1}}^2),$$

because on this domain $\frac{1}{4\xi^2-\alpha^2}$ is bounded.

Gathering all these estimates, we obtain:

$$\int_0^\infty\|(S_\alpha(t)(n,m))_1\|_{H^k}^2\,dt\leqslant C\left(\|n\|_{H^k\cap\dot{H}^{-1}}^2+\|m\|_{H^{k-1}\cap\dot{H}^{-1}}^2\right)$$

so that the first estimate is proved. The computations are the same for the other estimates, except that for the last one, where we have to integrate in time twice, thus $\dot{H}^{-2}$ regularity is needed. □

2.2 Well-Posedness of the problem

2.2.1 Local Well-Posedness

In this section, we prove local well-posedness of the problem (2.0.3). Here, ε is fixed so we do not loose generality by writing all the computations with $\varepsilon = 1$. We start with a first result, which is proved according to the method used by Kato in [Kat87] for the Schrödinger equation, and contraction arguments. Then we state a second result, which shows that we can recover more regularity on our solutions. This kind of regularity will be needed in order to show afterwards that the problem (2.0.3) is actually globally well-posed, based on [SS79].

Existence of solutions in $H^2 \times H^1$

In this section we prove the following theorem.

Theorem 2.2.1.1. *Let $\alpha \geqslant 0$ and $\phi \in \mathcal{L}^2(L^2 \cap \dot{H}^{-1})$. Let $(u_0, n_0, n_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$. Then there exists a unique solution $(u, n, \partial_t n)$ to (2.0.3) with continuous valued paths in $H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$ such that $(u(0), n(0), \partial_t n(0)) = (u_0, n_0, n_1)$. This solution is defined on a random interval $[0, T(\omega))$, where $T(\omega) > 0$ is a stopping time such that*

$$T(\omega) = +\infty \qquad \textit{or} \qquad \lim_{t \to T(\omega)} \max\left(\|u(t)\|_{H^2}, \|n(t)\|_{H^1}\right) = +\infty.$$

To prove this theorem we set up a fixed-point scheme. We adapt Kato's arguments from [Kat87] on the Schrödinger equation. We start the proof by showing a contraction argument for the integral form of the first equation of the system, for fixed $n \in Y$. Thus, to any $n \in Y$ we can associate a solution u to the first equation of (2.0.3). Then we will show that the difference between two solutions u_1 and u_2 of this equation can be controlled by the difference between n_1 and n_2, the associated functions, which will allow us to prove that the second equation of (2.0.3) also verifies a contraction argument. First we define

$$C(\omega) = \left\| \int_0^t S(t-s) \begin{pmatrix} 0 \\ \phi \end{pmatrix} dW(s) \right\|_{Y_0 \times H_0}, \tag{2.2.1}$$

$$Y_0 = \{n \in L^\infty([0,T_0], H^1(\mathbb{R})), \partial_t n \in L^\infty([0,T_0], L^2(\mathbb{R}))\},$$
$$H_0 = L^\infty([0,T_0], L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})),$$

where T_0 is a deterministic time, fixed in advance and arbitrarily large. We are interested by the first equation of the system. Let us set

$$F : (u,n) \mapsto un.$$

We state the following lemma, which gathers some results on F.

Lemma 2.2.1.1. — *F is continuous from $H^2 \times H^1$ to L^2, and maps bounded sets into bounded sets.*

— *We have for $s,t \in [0,T]$*

$$\|F(u(t),n(t)) - F(u(s),n(s))\|_{L^2} \leqslant C|t-s| \, \|u\|_Z \, \|n\|_Y \,, u \in Z, n \in Y. \tag{2.2.2}$$

— *F maps a bounded set of $Z \times Y$ into a bounded set of Z' with*

$$\begin{aligned} \|F(u,n) - F(u_0,n_0)\|_{Z'} \leqslant CT \, \|u\|_Z \, \|n\|_Y \,, \\ u \in Z, n \in Y, u_0 \in H^2; n_0 \in H^1. \end{aligned} \tag{2.2.3}$$

Proof. The proof of this lemma is straightforward and is an adaptation of Lemmas 3.1 and 3.4 in [Kat87]. □

Now let $n \in \mathcal{B}_{R_0}(Y)$ be fixed. We now have to show that some ball $\mathcal{B}_{R_1}(Z)$ is invariant by $\Gamma_1(\cdot, n)$, where Γ_1 denotes the right-hand side of the first equation of (2.1.1), and that $\Gamma_1(\cdot, n)$ is a contraction for a well-chosen norm.
Fix $R_1 > 2\gamma \|u_0\|_{H^2} + 2(2\gamma+1) \|u_0\|_{H^2} \|n_0\|_{H^1}$, γ given by Lemma 2.1.0.2, we define the following set:

$$E := \{u \in Z, \|u\|_Z \leqslant R_1\}. \tag{2.2.4}$$

Lemma 2.2.1.2. *If $u \in E, n \in \mathcal{B}_{R_0}(Y)$, then $\|\Gamma_1(u,n)\|_Z \leqslant R_1$ for $T > 0$ small enough depending on R_0.*

Proof. Thanks to the previous lemma, we have

$$\begin{aligned} \|\Gamma_1(u,n)\|_Z &\leqslant \gamma \|u_0\|_{H^2} + (2\gamma+1) \|nu\|_{Z'} \\ &\leqslant \gamma \|u_0\|_{H^2} + (2\gamma+1)(\|u_0 n_0\|_{Z'} + \|nu - n_0 u_0\|_{Z'}), \end{aligned}$$

where $n_0u_0 \in Z'$ is seen as a constant function in time, so that its Z' norm is $\|n_0u_0\|_{L^2}$. Finally,

$$\|\Gamma_1(u,n)\|_Z \leqslant \gamma \|u_0\|_{H^2} + (2\gamma+1)(\|u_0\|_{H^2} \|n_0\|_{H^1} + 4R_0T \|u\|_Z)$$

and thus, for $u \in E$, and $T > 0$ small enough so that $4(2\gamma+1)R_0T < \frac{R_1}{2}$, we get the result. □

Lemma 2.2.1.3. *For all $n \in \mathcal{B}_{R_0}(Y)$, $\Gamma_1(\cdot,n)$ is a contraction in the norm $\|\cdot\|_{L^\infty([0,T],H^1)}$, for $u \in E$ and $T > 0$ small enough depending on $R_0 > 0$.*

Proof. Let $u_1, u_2 \in E$ and $n \in \mathcal{B}_{R_0}(Y)$, we have using the integral form of the first equation of (2.0.3):

$$\|\Gamma_1(u_1,n) - \Gamma_1(u_2,n)\|_{L^\infty([0,T],H^1)} \leqslant \gamma T \|n\|_{L^\infty([0,T],H^1)} \|u_1 - u_2\|_{L^\infty([0,T],H^1)}$$

thus, for T small enough (so that $\gamma T R_0 < 1$) we get the result. □

Note that the time T depends on R_0 but not on R_1. Now we focus on the wave equation. Let us fix $\omega \in \Omega$. According to Lemma 2.2.1.3, for any $R_0 > 0$, for all $n \in \mathcal{B}_{R_0}(Y)$, $\Gamma_1(\cdot,n)$ is a contraction in the norm $\|\cdot\|_{L^\infty([0,T],H^1)}$, thus by the Picard fixed point theorem there exists a unique solution $u \in Z$ to the first equation of (2.0.3) up to T_0. We write $u = \mathcal{T}(n)$. Now, let

$$R_2 > 2\gamma \|(n_0,n_1)\|_{H^1\times L^2} + 2C(\omega)$$

with $C(\omega)$ defined in (2.2.1). We also fix $R > \max(R_1,R_2)$. Thus, if we choose $n \in Y, \|n\|_Y \leqslant R$, namely $n \in \mathcal{B}_R(Y)$, we get a solution $u = \mathcal{T}(n)$. This solution will satisfy $\|u\|_Z \leqslant R$ according to Lemma 2.2.1.2. First we show that for $n_1, n_2 \in \mathcal{B}_R(Y)$ we can estimate the difference between two solutions $u_1 = \mathcal{T}(n_1), u_2 = \mathcal{T}(n_2)$.

Proposition 2.2.1.1. *Let $n_1, n_2 \in \mathcal{B}_R(Y)$ and associated solutions $u_1 = \mathcal{T}(n_1), u_2 = \mathcal{T}(n_2)$. Then for T small enough, depending only on R, there exists $C > 0$ such that*

$$\|u_1 - u_2\|_{L^\infty H^2} \leqslant (CR+2) \|n_1 - n_2\|_Y .$$

To prove this result we need a lemma.

Lemma 2.2.1.4. *Let $n_1, n_2 \in \mathcal{B}_R(Y)$ and associated solutions $u_1 = \mathcal{T}(n_1), u_2 = \mathcal{T}(n_2)$. There exists $C > 0$, and there exists $T > 0$ small enough such that $\gamma RCT < 1$ and:*

$$\|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)} \leqslant \frac{\gamma RCT}{1-\gamma RCT}(\|n_1 - n_2\|_Y + \|u_1 - u_2\|_{L^\infty([0,T],H^1)}).$$

Proof. We need to estimate $\|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)}$. For $k = 1, 2$, we denote by u_k the solution of the equation

$$i\partial_t u_k = -\partial_x^2 u_k + n_k u_k$$

with initial data $u_k(0) = u_0$. Thus $\partial_t u_k$ is solution of

$$i\partial_t(\partial_t u_k) = -\partial_x^2(\partial_t u_k) + n_k\partial_t u_k + u_k\partial_t n_k$$

with initial data $v_0 = i\partial_x^2 u_0 - in_0u_0$, which can be rewritten as:

$$\partial_t u_k(t) = U(t)v_0 - i\int_0^t U(t-s)(n_k(s)\partial_t u_k(s) + u_k(s)\partial_t n_k(s))ds.$$

Then it comes:

$$\begin{aligned}\|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)} \leqslant & \gamma T \|(n_1\partial_t u_1 + u_1\partial_t n_1) - (n_2\partial_t u_2 + u_2\partial_t n_2)\|_{L^\infty([0,T],L^2)} \\ \leqslant & \gamma CT(\|\partial_t n_1\|_{L^\infty([0,T],L^2)} \|u_1 - u_2\|_{L^\infty([0,T],H^1)} \\ & + \|u_2\|_{L^\infty([0,T],H^1)} \|\partial_t n_1 - \partial_t n_2\|_{L^\infty([0,T],L^2)} \\ & + \|\partial_t u_1\|_{L^\infty([0,T],L^2)} \|n_1 - n_2\|_{L^\infty([0,T],H^1)} \\ & + \|n_2\|_{L^\infty([0,T],H^1)} \|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)}).\end{aligned}$$

Since $\|u_k\|_Z \leqslant R$, $\|n_k\|_Y \leqslant R$ we obtain choosing T small enough:

$$\|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)} \leqslant \frac{\gamma RCT}{1-\gamma RCT}(\|n_1 - n_2\|_Y + \|u_1 - u_2\|_{L^\infty([0,T],H^1)}).$$

□

Now we can prove Proposition 2.2.1.1.

Proof of Proposition 2.2.1.1. We start by a first computation, recalling that for $k = 1, 2$

we have $u_k = \Gamma_1(u_k, n_k)$:

$$\begin{aligned}
\|u_1 - u_2\|_{L^\infty([0,T],H^1)} &= \|\Gamma_1(u_1,n_1) - \Gamma_1(u_2,n_2)\|_{L^\infty([0,T],H^1)} \\
&= \left\| \int_0^t U(t-s)(n_1u_1 - n_2u_2)(s)ds \right\|_{L^\infty([0,T],H^1)} \\
&\leqslant \gamma T(\|n_1\|_{L^\infty([0,T],H^1)} \|u_1 - u_2\|_{L^\infty([0,T],H^1)} \\
&\quad + \|u_2\|_{L^\infty([0,T],H^1)} \|n_1 - n_2\|_{L^\infty([0,T],H^1)}) \\
&\leqslant \gamma RT(\|u_1 - u_2\|_{L^\infty([0,T],H^1)} + \|n_1 - n_2\|_{L^\infty([0,T],H^1)}) \\
&\leqslant \frac{\gamma RT}{1-\gamma RT} \|n_1 - n_2\|_{L^\infty([0,T],H^1)},
\end{aligned}$$

if T is chosen such that $\gamma RCT < 1$. Besides we have thanks to the first equation of (2.0.3)

$$\begin{aligned}
\|u_1 - u_2\|_{L^\infty H^2} \leqslant\ & \|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)} + \|u_1n_1 - u_2n_2\|_{L^\infty([0,T],L^2)} \\
& + \|u_1 - u_2\|_{L^\infty([0,T],L^2)} \\
\leqslant\ & \|\partial_t u_1 - \partial_t u_2\|_{L^\infty([0,T],L^2)} + CR(\|n_1 - n_2\|_Y + \|u_1 - u_2\|_{L^\infty([0,T],H^1)}) \\
& + \|u_1 - u_2\|_{L^\infty([0,T],H^1)},
\end{aligned}$$

and using Lemma 2.2.1.4:

$$\begin{aligned}
\|u_1 - u_2\|_{L^\infty H^2} &\leqslant \left(\frac{\gamma RCT}{1-\gamma RCT} + CR + 1\right)(\|n_1 - n_2\|_Y + \|u_1 - u_2\|_{L^\infty([0,T],H^1)}) \\
&\leqslant \left(\frac{\gamma RCT}{1-\gamma RCT} + CR + \frac{\gamma RT}{1-\gamma RT}\right) \|n_1 - n_2\|_Y \\
&\leqslant (CR+1) \|n_1 - n_2\|_Y
\end{aligned}$$

because we can assume that $\frac{\gamma RT}{1-\gamma RT} < \frac{1}{2}$ and $\frac{\gamma RCT}{1-\gamma RCT} < \frac{1}{2}$ by choosing T small enough. □

Now we can set up the fixed point argument for:

$$(n, \partial_t n) = \Gamma_2(\mathcal{T}(n), n, \partial_t n) =: \Gamma_3(n, \partial_t n)$$

for $n \in \mathcal{B}_R(Y)$, where Γ_2 is defined by the right-hand side of the second equation of (2.1.1).

Lemma 2.2.1.5. *For $T > 0$ small enough depending on $R > 0$, Γ_3 is a contraction on $\mathcal{B}_R(Y \times H)$ for the norm $\|\cdot\|_{Y\times H}$.*

Proof. Let $(n_1, m_1), (n_2, m_2) \in \mathcal{B}_R(Y \times H)$. For $k = 1, 2$, we denote $u_k = \mathcal{T}(n_k)$. Thus

$$\Gamma_3(n_1, m_1) - \Gamma_3(n_2, m_2) = G(t)(0, \partial_x^2 |u_1|^2 - \partial_x^2 |u_2|^2),$$

with $G(t)$ defined in Lemma 2.1.0.4. According to Lemma 2.1.0.4 we have:

$$\begin{aligned}
\|\Gamma_3(n_1, m_1) - \Gamma_3(n_2, m_2)\|_{Y \times H} &\leqslant \gamma T \left\| \partial_x^2 |u_1|^2 - \partial_x^2 |u_2|^2 \right\|_{L^\infty([0,T],L^2)} \\
&\leqslant 2\gamma T (CR \|u_1 - u_2\|_{L^\infty H^2} \\
&+ \left\| |\partial_x u_1|^2 - |\partial_x u_2|^2 \right\|_{L^\infty([0,T],L^2)}) \\
&\leqslant 2\gamma T (CR \|u_1 - u_2\|_{L^\infty H^2} + 2R \|u_1 - u_2\|_{L^\infty([0,T],H^1)}) \\
&\leqslant \tilde{C} T \|n_1 - n_2\|_Y
\end{aligned}$$

according to Proposition 2.2.1.1. Thus for $\tilde{C}T < 1$ the map Γ_3 is a contraction. $\square$

It remains to prove that the ball $\mathcal{B}_R(Y \times H)$ is invariant under the mapping Γ_3.

Lemma 2.2.1.6. *For $T > 0$ small enough depending on $R > 0$, and $(n, m) \in \mathcal{B}_R(Y \times H)$, we have $\|\Gamma_3(n, m)\|_{Y \times H} \leqslant R$.*

Proof.

$$\begin{aligned}
\|\Gamma_3(n, m)\|_{Y \times H} \leqslant& \left\| S_\alpha(t) \begin{pmatrix} n_0 \\ n_1 \end{pmatrix} \right\|_{Y \times H} + \left\| G(t) \begin{pmatrix} 0 \\ \partial_x^2 |u|^2(s) \end{pmatrix} \right\|_{Y \times H} \\
&+ \left\| \int_0^t S_\alpha(t - s) \begin{pmatrix} 0 \\ \phi \end{pmatrix} dW(s) \right\|_{Y \times H} \\
\leqslant& \gamma \|(n_0, n_1)\|_{H^1 \times L^2} + \gamma T \left\| \partial_x^2 |u|^2 \right\|_{L^\infty([0,T],L^2)} + C(\omega) \\
\leqslant& \gamma \|(n_0, n_1)\|_{H^1 \times L^2} + 2\gamma T (CR \|u\|_{L^\infty H^2} + \left\| |\partial_x u|^2 \right\|_{L^\infty([0,T],L^2)}) + C(\omega) \\
\leqslant& \gamma \|(n_0, n_1)\|_{H^1 \times L^2} + C(\omega) + 2\gamma T (CR \|u\|_{L^\infty H^2} + \|\partial_x u\|^2_{L^\infty([0,T],H^1)}),
\end{aligned}$$

where $u = \mathcal{T}(n)$, $G(t)$ is defined in Lemma 2.1.0.4, and $C(\omega)$ in (2.2.1). Since $\|u\|_Z \leqslant R$, we have

$$\begin{aligned}
\|\Gamma_3(n, m)\|_{Y \times H} &\leqslant \gamma \|(n_0, n_1)\|_{H^1 \times L^2} + C(\omega) + 4\gamma C R^2 T \\
&\leqslant \frac{R_2}{2} + 4\gamma C R^2 T.
\end{aligned}$$

Thus for T small enough such that $4\gamma CRT < \frac{1}{2}$, we get the result, because $R > R_2$. $\square$

Thanks to all these lemmas, we obtain the existence of a unique solution $(u, n, \partial_t n) \in \mathcal{B}_R(H^2 \times H^1 \times L^2)$ using Picard fixed point theorem. We can extend this solution while $(u, n, \partial_t n)$ is bounded in $H^2 \times H^1 \times L^2$ using classical arguments, since the existence time only depends on the initial data in their $H^2 \times H^1 \times L^2$ norm.

Propagation of regularity

In this subsection we show that if the initial data is more regular, then the solution of the system keeps the same regularity as long as it exists in the sense of Theorem 2.2.1.1.

Theorem 2.2.1.2. *Let $\phi \in \mathcal{L}^2(H^1 \cap \dot{H}^{-1})$ and let $(u, n, \partial_t n)$ be the solution of (2.0.3) given by Theorem 2.2.1.1, and $T(\omega)$ be the stopping time defined in this theorem. Then if $u_0 \in H^3(\mathbb{R}), n_0 \in H^2(\mathbb{R}), n_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$, we actually have $(u, n, \partial_t n)(\omega) \in C([0, T(\omega)), H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^1(\mathbb{R}))$.*

To prove this theorem we will differentiate the system (2.0.3) and set up a similar fixed-point scheme as the one used to prove Theorem 2.2.1.1.
Given $(u_0, n_0, n_1) \in H^3 \times H^2 \times H^1$, a fixed $\omega \in \Omega$, a random time $T^*(\omega)$, we have a solution to the system (2.0.3) $(u, n, \partial_t n) \in Z^* \times Y^* \times H^*$ given by Theorem 2.2.1.1, where

$$
\begin{aligned}
Z^* &= \{u \in L^\infty([0, T^*], H^2(\mathbb{R})), \partial_t u \in L^\infty([0, T^*], L^2(\mathbb{R}))\}, \\
Y^* &= \{n \in L^\infty([0, T^*], H^1(\mathbb{R})), \partial_t n \in L^\infty([0, T^*], L^2(\mathbb{R}))\}, \\
H^* &= L^\infty([0, T^*], L^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})).
\end{aligned}
$$

Furthermore, if we differentiate the system (2.0.3) in space, defining $u_x = \partial_x u, n_x = \partial_x n$ and $m_x = \partial_x m$ we get:

$$
\begin{cases}
i\partial_t u_x = -\partial_x^2 u_x + n_x u + n u_x, \\
\partial_t n_x = m_x, \\
dm_x = -\alpha m_x + \partial_x^2(n_x + \partial_x |u|^2)dt + \partial_x \phi dW(t).
\end{cases}
\tag{2.2.5}
$$

One can already notice in the above system that we will need more regularity on ϕ, if we want to apply the same procedure as in the previous subsection. We need to stress the dependence in u_x in the last equation of the system (2.2.5). Writing

$$
\partial_x^3 |u|^2 = 6Re(\bar{u_x}\partial_x^2 u) + 2Re(\bar{u}\partial_x^2 u_x)
$$

allows us to consider the following system:

$$\begin{cases} v(t) = & U(t)v_0 + \int_0^t U(t-s)(nv+uh)(s)ds, \\ \begin{pmatrix} h(t) \\ k(t) \end{pmatrix} = & S_\alpha(t)\begin{pmatrix} h_0 \\ k_0 \end{pmatrix} + \displaystyle\int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ 6Re(\bar{v}\partial_x^2 u) + 2Re(\bar{u}\partial_x^2 v) \end{pmatrix}(s)ds \\ & + \displaystyle\int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ \partial_x \phi \end{pmatrix} dW(s), \end{cases} \tag{2.2.6}$$

where $v_0 = \partial_x u_0$, $h_0 = \partial_x n_0$, $k_0 = \partial_x n_1$. We denote respectively by $\Lambda_1(v,h)$ and $\Lambda_2(v,h,k)$ the right-hand sides of (2.2.6). These two functions also depend on u and n. We will use the same method as in the previous subsection. We define the function

$$\Psi_{u,n} : (v,h) \mapsto nv + uh.$$

It is clear that this function depends on n and u. We state the following result which gathers some immediate properties of $\Psi_{u,n}$, whose proofs are left to the reader.

Lemma 2.2.1.7. — *$\Psi_{u,n} : H^2 \times H^1 \to L^2$ continuously and boundedly (it maps a bounded set into a bounded set).*

— *$\Psi_{u,n}$ maps $Z \times Y$ into Z' boundedly, these spaces are defined in (2.1.5), with $T \leqslant T^*$ arbitrary. Furthermore,*

$$\|\Psi_{u,n}(v_1,h_1) - \Psi_{u,n}(v_2,h_2)\|_{Z'} \leqslant CMT(\|v_1 - v_2\|_Z + \|h_1 - h_2\|_Y) \tag{2.2.7}$$

with $M = \max(\|u\|_{Z^*}, \|n\|_{Y^*})$.

Now we state the following result:

Proposition 2.2.1.2. *For all $h \in Y$, there exists a unique solution $v \in Z$ to the first equation of the system (2.2.5) with initial data v_0. The time interval $[0,T]$, on which the solution is defined is actually $[0,T^*]$.*

To prove this result we start by proving a lemma:

Lemma 2.2.1.8. *For all $h \in Y$, the operator $\Lambda_1(\cdot,h)$ maps Z into Z and is a contraction for the norm $\|\cdot\|_{L^\infty([0,T],H^1)}$, for $T \leqslant T^*$ such that $\gamma T \|n\|_{Y*} < 1$.*

Proof. Let $v \in Z$. Lemma 2.2.1.2 implies

$$\begin{aligned}\|\Lambda_1(v,h)\|_Z &\leqslant \|U(t)v_0\|_Z + \left\|\int_0^t U(t-s)(nv+uh)(s)ds\right\|_Z \\ &\leqslant \gamma\|v_0\|_{H^2} + (2\gamma+1)\|nv+uh\|_{Z'} \\ &\leqslant \gamma\|v_0\|_{H^2} + (2\gamma+1)(1+2T)\left(\|n\|_{Y*}\|v\|_Z + \|u\|_{Z*}\|h\|_Y\right) < \infty\end{aligned}$$

thus we have $\Lambda_1(\cdot,h) : Z \to Z$. For the contraction argument, we have the estimate:

$$\|\Lambda_1(v_1,h)-\Lambda_1(v_2,h)\|_{L^\infty([0,T],H^1)} \leqslant \gamma T(\|n\|_{L^\infty([0,T],H^1)}\|v_1-v_2\|_{L^\infty([0,T],H^1)}) \tag{2.2.8}$$

and thus for T small enough (namely $\gamma T\|n\|_{Y*} < 1$) the result is proved. □

Thanks to this lemma, we can establish the existence of a solution v on a time interval $[0,T_1]$, with $T_1 \leqslant T^*$. Besides, since T_1 only depends on $\|n\|_{Y*}$, the solution may be extended to $[0,T^*]$ by iterations, in a classical way.

Now, as in the previous subsection, we can focus on the wave equation. Let us fix $\omega \in \Omega$. Proposition 2.2.1.2 states that for all $h \in Y$, there exists a unique solution $v \in Z$ to the first equation of the system (2.2.5). We denote $v = \tilde{\mathcal{T}}(h)$.

Proposition 2.2.1.3. *Let $h_1,h_2 \in Y$ and let $v_1 = \tilde{\mathcal{T}}(h_1), v_2 = \tilde{\mathcal{T}}(h_2)$ be the associated solutions. Then there exists $C>0$ such that if $T>0$ is small enough so that*

$$\gamma T\|n\|_{Y*} < 1 \quad \text{and} \quad \frac{\gamma MCT}{1-\gamma MCT} < 1$$

with $M = \max(\|u\|_{Z*}, \|n\|_{Y*})$, we have

$$\|v_1-v_2\|_{L^\infty H^2} \leqslant C\|h_1-h_2\|_Y.$$

To prove this result we need the following lemma:

Lemma 2.2.1.9. *Let $h_1,h_2 \in Y$ and let $v_1 = \tilde{\mathcal{T}}(h_1), v_2 = \tilde{\mathcal{T}}(h_2)$ be the associated solutions. There exists $C>0$ such that if $T>0$ is small enough so that $\gamma MCT < 1$:*

$$\|\partial_t v_1 - \partial_t v_2\|_{L^\infty([0,T],L^2)} \leqslant \frac{\gamma MCT}{1-\gamma MCT}(\|h_1-h_2\|_Y + \|v_1-v_2\|_{L^\infty([0,T],H^1)})$$

with $M = \max(\|u\|_{Z*}, \|n\|_{Y*})$.

Proof. For $k = 1, 2$, v_k is solution of the equation

$$i\partial_t v_k = -\partial_x^2 v_k + nv_k + uh_k \tag{2.2.9}$$

with initial data $v_k(0) = v_0$. Thus $\partial_t v_k$ is solution of

$$i\partial_t(\partial_t v_k) = -\partial_x^2(\partial_t v_k) + n\partial_t v_k + v_k\partial_t n + u\partial_t h_k + h_k\partial_t u.$$

Then we conclude by the same procedure as in Lemma 2.2.1.4 since $\|u\|_{Z*} \leqslant M$, $\|n\|_{Y*} \leqslant M$. □

Now we can prove Proposition 2.2.1.3. Equation (2.2.9) gives us

$$\begin{aligned}
\|v_1 - v_2\|_{H^2} &\leqslant \|\partial_t v_1 - \partial_t v_2\|_{L^2} + \|n(v_1 - v_2) + u(h_1 - h_2)\|_{L^2} + \|v_1 - v_2\|_{L^2} \\
&\leqslant \|\partial_t v_1 - \partial_t v_2\|_{L^2} + CM(\|h_1 - h_2\|_Y + \|v_1 - v_2\|_{L^\infty([0,T],H^1)}) \\
&\quad + \|v_1 - v_2\|_{L^2}.
\end{aligned}$$

Using the expression of Λ_1 in (2.2.6), we obtain

$$\begin{aligned}
\|v_1 - v_2\|_{L^\infty([0,T],H^1)} &= \|\Lambda_1(v_1, h_1) - \Lambda_1(v_2, h_2)\|_{L^\infty([0,T],H^1)} \\
&\leqslant \gamma T\left(\|n\|_{Y*}\|v_1 - v_2\|_{L^\infty([0,T],H^1)} + \|u\|_{Z*}\|h_1 - h_2\|_{L^\infty([0,T],H^1)}\right).
\end{aligned}$$

Then, according to Lemma 2.2.1.9 and for $T > 0$ satisfying the assumptions of Proposition 2.2.1.1, we get the conclusion.

Now we can set up the fixed point argument. We start by giving a result on the stochastic convolution:

Proposition 2.2.1.4. *Let $\phi \in \mathcal{L}_2(L^2, H^1 \cap \dot{H}^{-1})$, then*

$$\int_0^t S_\alpha(t-s)\begin{pmatrix} 0 \\ \partial_x\phi \end{pmatrix} dW(s) \in Y \times H \quad a.s.$$

Proof. As in the proof of Lemma 2.1.0.5, we use [DPZ14, Theorem 6.10] to get

$$\mathbb{E}\left[\sup_{[0,T]}\left\|\int_0^t S_\alpha(t-s)\begin{pmatrix}0\\ \partial_x\phi\end{pmatrix}dW(s)\right\|^2_{H^1\times L^2\cap\dot{H}^{-1}}\right]\leqslant C\gamma\mathbb{E}\left[\int_0^t\|\partial_x\phi\|^2_{\mathcal{L}_2(L^2\cap\dot{H}^{-1})}ds\right]$$
$$\leqslant C\gamma T\|\phi\|^2_{\mathcal{L}_2(L^2,H^1\cap\dot{H}^{-1})},$$

and the fact that we can control the derivative in time of the first component with the second component concludes the proof. □

Propositions 2.2.1.3 and 2.2.1.4 allow us to prove the following result.

Proposition 2.2.1.5. *For $\phi\in\mathcal{L}_2(L^2,H^1)$, there exists a unique solution $h\in H$ to the second equation of the system (2.2.6) with initial data $(\partial_x n_0,\partial_x n_1)$ and $v=\tilde{\mathcal{T}}(h)$. The time interval $[0,T]$, on which the solution is defined is actually $[0,T^*]$.*

Lemma 2.2.1.10. *$\Lambda_3(h,\partial_t h)=\Lambda_2(\tilde{\mathcal{T}}(h),h,\partial_t h)$ takes values in $Y\times H$ if $(h,\partial_t h)\in Y\times H$. Moreover, there exists a constant $C>0$ depending on $\|u\|_{Z^*}$ and $\|n\|_{Y^*}$ such that if $T>0$ is small enough so that $CT<1$, Λ_3 is a contraction on $Y\times H$ with the norm $\|\cdot\|_{Y\times H}$.*

Proof. Let $(h,m)\in Y\times H$, setting $v=\tilde{\mathcal{T}}(h)$ we have according to (2.2.6)

$$\|\Lambda_3(h,m)\|_{Y\times H}\leqslant\gamma\|(\partial_x n_0,\partial_x n_1)\|_{H^1\times L^2}+\gamma T\left\|6Re(\bar{v}\partial_x^2 u)+2Re(\bar{u}\partial_x^2 v)\right\|_{L^2}+\tilde{C}(\omega)$$
$$\leqslant\gamma\|(\partial_x n_0,\partial_x n_1)\|_{H^1\times L^2}+\gamma CMT\|v\|_Z+\tilde{C}(\omega)$$

where $\tilde{C}(\omega)$ is the constant which bounds the stochastic convolution in $Y\times H$. For the contraction argument, let $(h_1,m_1),(h_2,m_2)\in Y\times H$. Then

$$\|\Lambda_3(h_1,m_1)-\Lambda_3(h_2,m_2)\|_{Y\times H}\leqslant\gamma T(6\|u\|_{H^2}\|v_1-v_2\|_{H^1}+2\|u\|_{H^1}\|v_1-v_2\|_{H^2})$$
$$\leqslant CT\|h_1-h_2\|_Y,$$

using Proposition 2.2.1.3 and Lemma 2.2.1.9, where C depends only on the norms $\|u\|_{Z^*}$ and $\|n\|_{Y^*}$. For $T>0$ such that $CT<1$ we get the result. □

Now we can prove Theorem 2.2.1.2. Indeed, according to Propositions 2.2.1.2 and 2.2.1.5, we have proved the existence of a unique solution to the system (2.2.5) with initial data $(\partial_x u_0,\partial_x n_0,\partial_x n_1)$, on a time interval $[0,T(\omega)]$, with $T(\omega)$, which is the random time of Theorem 2.2.1.1.

Furthermore, $(\partial_x u, \partial_x n, \partial_x \partial_t n)$ is also a solution of this system, with the same initial data, so we can conclude by uniqueness that $(u, n, \partial_t n) \in L^\infty([0, T(\omega)), H^3 \times H^2 \times H^1)$.

2.2.2 Global Well-Posedness

In this section, we follow the article [SS79] by Sulem and Sulem to get a priori estimates in order to get a global result. We are still considering the system

$$\begin{cases} i\partial_t u = -\partial_x^2 u + nu \\ \partial_t n = \mu \\ \varepsilon^2 d\mu + \alpha\varepsilon\mu = \partial_x^2(n + |u|^2)dt + \phi dW(t) \end{cases} \tag{2.2.10}$$

but now we keep track of ε in the estimates. We will see that they are not uniform in ε.

We prove below the following theorem.

Theorem 2.2.2.1. *Let $\alpha \geqslant 0$, $T > 0$ and let $\phi \in \mathcal{L}^2(H^1 \cap \dot{H}^{-1})$. For initial data $u_0 \in H^3$, $n_0 \in H^2$ and $n_1 \in H^1 \cap \dot{H}^{-1}$, there exists almost surely a unique solution $(u, n) \in L^\infty([0, T], H^3 \times H^2)$ for the system (2.0.3).*

Modification of the system

To prove our theorem, we decompose $n = m + Z^\varepsilon$ where $Z^\varepsilon \in L^\infty([0, T], H^2 \cap \dot{H}^{-1})$ a.s according to Lemmas 2.1.0.5 and 2.2.1.4, and satisfies

$$\begin{aligned} \varepsilon^2 d(\partial_t Z^\varepsilon) &= (-\alpha\varepsilon\partial_t Z^\varepsilon + \partial_x^2 Z^\varepsilon)dt + \phi dW_t, \\ Z^\varepsilon(0) &= \frac{1}{\varepsilon}\int_{-\infty}^0 S_\alpha(-s)\begin{pmatrix} 0 \\ \phi \end{pmatrix} dW_s. \end{aligned} \tag{2.2.11}$$

Note that Z^ε is a stationary process. Thus (u, n) is a solution of the system (2.2.10) if and only if (u, m) is a solution of the system:

$$\begin{cases} i\partial_t u = -\partial_x^2 u + (m + Z^\varepsilon)u, \\ \\ \varepsilon^2\partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2(m + |u|^2). \end{cases} \tag{2.2.12}$$

In order to prove our theorem, we need to introduce

$$V = -\partial_x^{-1}\partial_t m, \tag{2.2.13}$$

for which we have to assume that $\partial_t m \in \dot{H}^{-1}$. This is actually true thanks to the hypothesis on the initial data since

$$\partial_t m(t) = (S_\alpha(t)(n_0, n_1))_2 + \int_0^t \left(S_\alpha(t-s)(0, \partial_x^2 |u|^2)\right)_2 ds$$

and a computation with the same procedure as in Lemma 2.1.0.3 shows that $\partial_t m \in \dot{H}^{-1}$ under the assumptions of Theorem 2.2.2.1. The equation satisfied by m gives us that

$$\varepsilon^2 \partial_t V + \alpha\varepsilon V + \partial_x(m + |u|^2) = 0. \tag{2.2.14}$$

Proof of the global Well-Posedness

If $\alpha = 0$ and $\phi = 0$, one can prove by taking the scalar product of the first equation of (2.2.12) with $\bar{u}$ and $\partial_t \bar{u}$ and the second equation of (2.2.12) with $(\partial_x^2)^{-1}\partial_t m$ that the deterministic system preserves the mass and the energy:

$$N(u) = \|u\|_{L^2}^2, \quad \text{and} \quad H(u,m) = \|\partial_x u\|_{L^2}^2 + \frac{1}{2}(\|m\|_{L^2}^2 + \|\varepsilon V\|_{L^2}^2) + \int_{\mathbb{R}} m|u|^2 dx. \tag{2.2.15}$$

In our case, we still have conservation of the mass N, and we have:

Proposition 2.2.2.1. *For $(u_0, m_0, m_1) \in H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, the local solution (u, m) of the system (2.2.12) satisfies:*

$$\partial_t N = 0, \qquad \partial_t H(u,m) + \int_{\mathbb{R}} Z^\varepsilon \partial_t |u|^2 dx = -\alpha\varepsilon \|V\|_{L^2}^2. \tag{2.2.16}$$

Proof. The following computations are justified because we assumed enough regularity on our solutions u, m, Z^ε. We recall that m, Z^ε are $\mathbb{R}$-valued, and u is the only $\mathbb{C}$-valued function. For the conservation of mass we take the scalar product of the first equation of the system (2.2.12) with iu. For the energy we start from the system (2.2.12) and take the scalar product of the first equation with $i\partial_t u$, then we take the scalar product of the second equation with $(\partial_x^2)^{-1}\partial_t m$. We get by integration by parts:

$$\partial_t \|\partial_x u\|_{L^2}^2 + \int_{\mathbb{R}} m\partial_t |u|^2 + Z^\varepsilon \partial_t |u|^2 dx = 0,$$

$$-\frac{1}{2}\partial_t \|\varepsilon V\|_{L^2}^2 = \int_{\mathbb{R}} \partial_t m(m + |u|^2) dx + \alpha\varepsilon \|V\|_{L^2}^2.$$

Taking the difference between the first equation and the second equation we deduce:

$$\partial_t \left(\|\partial_x u\|_{L^2}^2 + \frac{1}{2}\left(\|m\|_{L^2}^2 + \|\varepsilon V\|_{L^2}^2\right) + \int_{\mathbb{R}} m|u|^2 dx \right) + \int_{\mathbb{R}} Z^\varepsilon \partial_t |u|^2 dx = -\alpha\varepsilon \|V\|_{L^2}^2 .$$

□

This result allows us to obtain the following bound (we work with $\omega \in \Omega$ fixed so that $Z^\varepsilon \in H^2$ a.s. and there exists a constant such that $\|\partial_x Z^\varepsilon\|_{L^\infty([0,T],H^1)}^2 \leqslant C(\omega)$ a.s.):

Proposition 2.2.2.2. *For $(u_0, m_0, m_1) \in H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, the local solution (u, m) of the system (2.2.12) satisfies*

$$\frac{1}{2}\|\partial_x u\|_{L^2}^2 + \frac{1}{4}\|m\|_{L^2}^2 + \frac{1}{2}\|\varepsilon V\|_{L^2}^2 \leqslant (C + \|u_0\|_{L^2}^2 \|\partial_x Z^\varepsilon\|_{L^\infty([0,T],H^1)}^2 t)e^t \tag{2.2.17}$$

with $C = |H_0| + 2N$ where $H_0 = H(u_0, m_0)$.

Proof. Thanks to the previous Proposition:

$$\begin{aligned}\|\partial_x u\|_{L^2}^2 + \frac{1}{2}\left(\|m\|_{L^2}^2 + \left\|\varepsilon\partial_x^{-1}\partial_t m\right\|_{L^2}^2\right) \leqslant& |H_0| + \left|\int_{\mathbb{R}} m|u|^2 dx\right| \\ &+ \left|\int_0^t \int_{\mathbb{R}} Z^\varepsilon \partial_t |u|^2 dx ds\right| - \alpha\varepsilon \|V\|_{L^2([0,t],L^2)}^2 .\end{aligned}$$

Then we neglect the negative term, we use the first equation (2.2.12) to develop $\partial_t |u|^2$, and by integration by parts we get:

$$\begin{aligned}\|\partial_x u\|_{L^2}^2 + \frac{1}{2}\left(\|m\|_{L^2}^2 + \left\|\varepsilon\partial_x^{-1}\partial_t m\right\|_{L^2}^2\right) \leqslant& |H_0| + \left|\int_{\mathbb{R}} m|u|^2 dx\right| \\ &+ 2\left|\int_0^t \int_{\mathbb{R}} \partial_x Z^\varepsilon Im(\bar{u}\partial_x u) dx ds\right| \\ \leqslant& |H_0| + \left|\int_{\mathbb{R}} m|u|^2 dx\right| \\ &+ \int_0^t \|\partial_x Z^\varepsilon\|_{L^\infty}^2 \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 ds.\end{aligned}$$

For the second term of the right-hand side of the inequality, we use [SS79, Lemma 2] which states that

$$\left|\int_{\mathbb{R}} m|u|^2 dx\right| \leqslant \frac{1}{4}\|m\|_{L^2}^2 + 4N^{\frac{3}{2}} \|\partial_x u\|_{L^2}$$

to finally get:

$$\begin{aligned}\|\partial_x u\|_{L^2}^2 + \frac{1}{4}\|m\|_{L^2}^2 + \frac{1}{2}\left\|\varepsilon\partial_x^{-1}\partial_t m\right\|_{L^2}^2 \leqslant & |H_0| + CN^{\frac{3}{2}}\|\partial_x u\|_{L^2} \\ & + \int_0^t \|\partial_x Z^\varepsilon\|_{L^\infty}^2 \|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2\, ds.\end{aligned} \tag{2.2.18}$$

Now we can apply Grönwall's Lemma and use mass conservation to get the desired estimate. □

Then we have the following lemma which gathers the equalities stated in [SS79]. The difference in our case comes from the damping and stochastic terms:

Lemma 2.2.2.1. *For* $(u_0, m_0, m_1) \in H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$*, the local solution* (u, m) *of the system* (2.2.12) *satisfies*

$$\partial_t\left(\left\|\varepsilon(\partial_x)^{-1}\partial_t^2 m\right\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2\right) + 2\int_{\mathbb{R}} \partial_t^2 m\partial_t|u|^2 dx + 2\alpha\varepsilon\|\partial_t V\|_{L^2}^2 = 0, \tag{2.2.19}$$

$$\partial_t\|\partial_t u\|_{L^2}^2 = 2Im\int_{\mathbb{R}} u\partial_t\bar{u}(\partial_t m + \partial_t Z^\varepsilon)dx, \tag{2.2.20}$$

$$-\partial_t\|\partial_t u\|_{L^2}^2 = \int_{\mathbb{R}}(m + Z^\varepsilon)\partial_t|\partial_t u|^2 dx + \int_{\mathbb{R}} \partial_t(m + Z^\varepsilon)\left(\partial_t^2|u|^2 - 2|\partial_t u|^2\right)dx. \tag{2.2.21}$$

Proof. First equality:
Applying the equation (2.2.14) satisfied by $V = -\partial_x^{-1}\partial_t m$ we have, using integration by parts:

$$\begin{aligned}\partial_t\left(\left\|\varepsilon(\partial_x)^{-1}\partial_t^2 m\right\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2\right) &= 2\int_{\mathbb{R}} \varepsilon^2\partial_t V\partial_t^2 V + \partial_t m\partial_t^2 m dx \\ &= 2\int_{\mathbb{R}} \partial_t V\left(-\alpha\varepsilon\partial_t V - \partial_x\partial_t(m + |u|^2)\right) + \partial_t m\partial_t^2 m dx \\ &= 2\int_{\mathbb{R}} -\alpha\varepsilon|\partial_t V|^2 - \partial_t m\partial_t^2 m - \partial_t^2 m\partial_t|u|^2 + \partial_t m\partial_t^2 m dx \\ &= -2\int_{\mathbb{R}} \partial_t^2 m\partial_t|u|^2 dx - 2\alpha\varepsilon\|\partial_t V\|_{L^2}^2.\end{aligned}$$

Second equality:
We start from the equation

$$i\partial_t u = -\partial_x^2 u + (m + Z^\varepsilon)u.$$

Then we differentiate this equation in time:

$$\partial_t^2 u = i\partial_t\partial_x^2 u - i\partial_t(mu) - i\partial_t(Z^\varepsilon u).$$

We take the scalar product with $\partial_t \bar{u}$ and get

$$\begin{aligned}\partial_t \|\partial_t u\|_{L^2}^2 &= 2Re\int_{\mathbb{R}} i\partial_t\partial_x^2 u\partial_t\bar{u} - i\partial_t(mu)\partial_t\bar{u} - i\partial_t(Z^\varepsilon u)\partial_t\bar{u}dx \\ &= 2Im\int_{\mathbb{R}} u\partial_t\bar{u}\partial_t(m+Z^\varepsilon)dx,\end{aligned}$$

where we used integrations by parts.
<u>Third equality:</u>
We start from the first equation, and write $\partial_x^2 u = nu - i\partial_t u$. Then

$$\partial_t\partial_x^2 u = \partial_t((m+Z^\varepsilon)u) - i\partial_t^2 u,$$

and taking the scalar product with $\partial_t^2 u$ we get

$$Re\int_{\mathbb{R}} \partial_t\partial_x^2 u\partial_t^2\bar{u}dx = Re\int_{\mathbb{R}} \partial_t((m+Z^\varepsilon)u)\partial_t^2\bar{u} - i|\partial_t^2 u|^2 dx,$$

so that

$$\begin{aligned}-\frac{1}{2}\partial_t \|\partial_x\partial_t u\|_{L^2}^2 &= Re\int_{\mathbb{R}} \partial_t(m+Z^\varepsilon)u\partial_t^2\bar{u} + \frac{1}{2}(m+Z^\varepsilon)\partial_t|\partial_t u|^2 dx \\ &= \frac{1}{2}\int_{\mathbb{R}}(m+Z^\varepsilon)\partial_t|\partial_t u|^2 dx + \frac{1}{2}\int_{\mathbb{R}}\partial_t(m+Z^\varepsilon)\left(\partial_t^2|u|^2 - 2|\partial_t u|^2\right)dx.\end{aligned}$$

□

By gathering (2.2.19)-(2.2.21) and integrating in time we obtain:

$$\begin{aligned}
2\|\partial_t u\|_{H^1}^2 + \|\varepsilon\partial_t V\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2 + 2\int_{\mathbb{R}} \partial_t m \partial_t |u|^2 + m|\partial_t u|^2 dx& \\
= C + \int_0^t \int_{\mathbb{R}} 6\partial_t m|\partial_t u|^2 + 4Im(u\partial_t \bar{u}\partial_t m) dxds& \\
+ 4Im \int_0^t \int_{\mathbb{R}} u\partial_t \bar{u}\partial_t Z^\varepsilon dxds - 2\int_{\mathbb{R}} Z^\varepsilon |\partial_t u|^2 dx& \\
- 2\int_0^t \int_{\mathbb{R}} \partial_t Z^\varepsilon \left(\partial_t^2 |u|^2 - 3|\partial_t u|^2\right) dxds& \\
- 2\alpha\varepsilon \|V\|_{L^2([0,t],L^2)}^2&
\end{aligned} \tag{2.2.22}$$

where C is a constant depending on the initial conditions coming from the integration in time which is finite according to the assumed regularities of u_0, m_0, m_1. Note that we also used:

$$\int_0^t \int_{\mathbb{R}} Z^\varepsilon \partial_t |\partial_t u|^2 dx = \int_{\mathbb{R}} Z^\varepsilon |\partial_t u|^2 dx - \int_0^t \int_{\mathbb{R}} \partial_t Z^\varepsilon |\partial_t u|^2 dxds.$$

The aim here is to get an estimate on $\|\partial_t u\|_{H^1}^2 + \|\varepsilon\partial_t V\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2$ via the Grönwall's argument. We will need several lemmas.

Lemma 2.2.2.2. *For $(u_0, m_0, m_1) \in H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, there exists a constant $C > 0$ depending on the initial data such that the local solution (u, m) of the system (2.2.12) satisfies*

$$\begin{aligned}
\left|\int_{\mathbb{R}} 6\partial_t m|\partial_t u|^2 + 4Im(u\partial_t \bar{u}\partial_t m) dxds\right| \leqslant & \frac{2(C + C(\omega)t)}{\varepsilon} e^t (1 + \|\partial_t u\|_{H^1}^2) \\
& + C\left(\|\partial_t u\|_{H^1}^2 + \|\partial_t m\|_{L^2}^2\right).
\end{aligned} \tag{2.2.23}$$

Proof. Integrating by parts, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}} 6\partial_t m|\partial_t u|^2 + 4Im(u\partial_t \bar{u}\partial_t m) dx\right| &= \left|\int_{\mathbb{R}} -6\partial_x V|\partial_t u|^2 + 4Im(u\partial_t \bar{u}\partial_t m) dx\right| \\
&\leqslant C\left(\|V\|_{L^2} \left\|\partial_x |\partial_t u|^2\right\|_{L^2} + \|u_0\|_{L^2} \|\partial_t u\|_{L^\infty} \|\partial_t m\|_{L^2}\right),
\end{aligned}$$

where we used that $\|u\|_{L^2}$ is preserved and that $\|V\|_{L^2}$ is controlled by the bound (2.2.17).

Using again (2.2.17), we get

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}} 6\partial_t m|\partial_t u|^2 + 4Im(u\partial_t\bar{u}\partial_t m)dx\right| \leqslant & \frac{\sqrt{C+C(\omega)t}}{\varepsilon}e^{t/2}\left\|2Re(\partial_x u\partial_x\partial_t u)\right\|_{L^2}\\ &+C\left(\|\partial_t u\|_{H^1}^2+\|\partial_t m\|_{L^2}^2\right)\\ \leqslant & \frac{2(C+C(\omega)t)}{\varepsilon}e^t\|\partial_t u\|_{H^1}^2\\ &+C\left(\|\partial_t u\|_{H^1}^2+\|\partial_t m\|_{L^2}^2\right).\end{aligned}$$

□

Lemma 2.2.2.3. *For $(u_0,m_0,m_1)\in H^3\times H^2\times(H^1\cap\dot{H}^{-1})$, the local solution (u,m) of the system (2.2.12) satisfies*

$$\begin{aligned}2\left|\int_{\mathbb{R}}\partial_t m\partial_t|u|^2+m|\partial_t u|^2dx\right| \leqslant & 2\left(\frac{8(C+C(\omega)t)}{\varepsilon}e^t\right)^2+\frac{1}{2}\|\partial_t u\|_{H^1}^2\\ &+8(C+C(\omega)t)e^t\|\partial_t u\|_{L^2}^2.\end{aligned}\tag{2.2.24}$$

Proof. We integrate by parts to get

$$\begin{aligned}2\left|\int_{\mathbb{R}}\partial_t m\partial_t|u|^2+m|\partial_t u|^2dx\right| \leqslant & 4\|V\|_{L^2}\left(\|\partial_x u\|_{L^2}\|\partial_t u\|_{H^1}+\|u\|_{H^1}\|\partial_x\partial_t u\|_{L^2}\right)\\ &+2\|m\|_{L^2}\|\partial_t u\|_{L^4}^2\\ \leqslant & 8\|V\|_{L^2}\|u\|_{H^1}\|\partial_t u\|_{H^1}+2\|m\|_{L^2}\|\partial_t u\|_{L^2}\|\partial_t u\|_{H^1}\end{aligned}$$

where we used Gagliardo-Nirenberg inequality for the second term of the right-hand side of the inequality. Using the fact that $\|V\|_{L^2}$, $\|u\|_{H^1}$ and $\|m\|_{L^2}$ are controlled by H, it comes:

$$\begin{aligned}2\left|\int_{\mathbb{R}}\partial_t m\partial_t|u|^2+m|\partial_t u|^2dx\right| \leqslant & \frac{8(C+C(\omega)t)}{\varepsilon}e^t\|\partial_t u\|_{H^1}\\ &+2(C+C(\omega)t)^{\frac{1}{2}}e^{t/2}\left(\frac{1}{\delta}\|\partial_t u\|_{L^2}^2+\delta\|\partial_t u\|_{H^1}^2\right)\\ \leqslant & 2\left(\frac{8(C+C(\omega)t)}{\varepsilon}e^t\right)^2+\frac{1}{4}\|\partial_t u\|_{H^1}^2\\ &+2(C+C(\omega)t)^{\frac{1}{2}}e^{t/2}\left(\frac{1}{\delta}\|\partial_t u\|_{L^2}^2+\delta\|\partial_t u\|_{H^1}^2\right)\end{aligned}$$

and choosing δ to have $2(C + C(\omega)t)^{\frac{1}{2}} e^{t/2}\delta = \frac{1}{4}$, the result is proved. $\square$

Lemma 2.2.2.4. *For $(u_0, m_0, m_1) \in H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, the local solution (u, m) of the system (2.2.12) satisfies*

$$\begin{aligned} \|\partial_t u\|_{L^2}^2 \leqslant & C + \int_0^t (C + C(\omega)s)e^s \left(\|\partial_t u\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2\right) ds \\ & + \int_0^t (C + C(\omega)s)e^s \left(\|\partial_t Z^\varepsilon\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2\right) ds. \end{aligned} \tag{2.2.25}$$

Proof. We use the identity (2.2.20) from Lemma 2.2.2.1:

$$\begin{aligned} \partial_t \|\partial_t u\|_{L^2}^2 &= 2Im \int_{\mathbb{R}} u \partial_t \bar{u} (\partial_t m + \partial_t Z^\varepsilon) dx \\ &\leqslant C \|u\|_{L^\infty} \|\partial_t u\|_{L^2} \|\partial_t m\|_{L^2} + \|u\|_{L^\infty} \|\partial_t Z^\varepsilon\|_{L^2} \|\partial_t u\|_{L^2}. \end{aligned}$$

Then we integrate in time and we control $\|u\|_{H^1}$ by the energy:

$$\begin{aligned} \|\partial_t u\|_{L^2}^2 \leqslant & C + \int_0^t (C + C(\omega)s)e^s \left(\|\partial_t u\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2\right) ds \\ & + \int_0^t (C + C(\omega)s)e^s \left(\|\partial_t Z^\varepsilon\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2\right) ds. \end{aligned}$$

$\square$

Now we have to estimate the terms in equation (2.2.22) involving the stochastic convolution Z^ε, namely,

$$4Im \int_0^t \int_{\mathbb{R}} u \partial_t \bar{u} \partial_t Z^\varepsilon dx ds + 2 \int_{\mathbb{R}} Z^\varepsilon |\partial_t u|^2 dx + 2 \int_0^t \int_{\mathbb{R}} \partial_t Z^\varepsilon \left(\partial_t^2 |u|^2 - 3|\partial_t u|^2\right) dx ds. \tag{2.2.26}$$

We denote by I, II and III the three integrals. We know from the definition of Z^ε that thanks to the assumption on ϕ and to Lemma 2.1.0.5, Proposition 2.2.1.4, almost surely $(Z^\varepsilon, \partial_t Z^\varepsilon) \in L^\infty([0,T], H^2) \times L^\infty([0,T], H^1)$ and $(\partial_x Z^\varepsilon, \partial_t \partial_x Z^\varepsilon) \in L^\infty([0,T], H^1) \times L^\infty([0,T], L^2)$. We prove the following results:

Lemma 2.2.2.5. *For $(u_0, m_0, m_1) \in H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, the local solution (u, m) of the system (2.2.12) satisfies for I, II and III the three terms of (2.2.26):*

$$|I| \leqslant \int_0^t (C + C(\omega)s)e^s \|\partial_t Z^\varepsilon\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2 ds, \tag{2.2.27}$$

$$|II| \leqslant \frac{1}{2}\left(\|Z^\varepsilon\|_{L^2}^2 \|\partial_t u\|_{L^2}^2 + \|\partial_t u\|_{H^1}^2\right), \tag{2.2.28}$$

$$|III| \leqslant 4\int_0^t (C + C(\omega)s)e^s \|\partial_t u\|_{H^1} \|\partial_t Z^\varepsilon\|_{H^1} ds + 3\int_0^t \|\partial_t Z^\varepsilon\|_{L^2} \|\partial_t u\|_{H^1}^2 ds. \tag{2.2.29}$$

In these inequalities $C > 0$ denotes a constant depending on the initial data (u_0, m_0, m_1) and $C(\omega)$ is the constant bounding the stochastic convolution in the $L^\infty([0,T], H^2) \times L^\infty([0,T], L^2)$-norm.

Proof. For the first estimate:

$$\left|Im \int_0^t \int_{\mathbb{R}} u\partial_t \bar{u}\partial_t Z^\varepsilon dxds\right| \leqslant C\int_0^t \|u\|_{L^\infty} \|\partial_t Z^\varepsilon\|_{L^2} \|\partial_t u\|_{L^2} ds$$

and we use the fact that $\|u\|_{H^1}$ is controlled by the energy to get the result.
For the second estimate:

$$\begin{aligned}\left|\int_{\mathbb{R}} Z^\varepsilon |\partial_t u|^2 dx\right| &\leqslant \|Z^\varepsilon\|_{L^2} \|\partial_t u\|_{L^4}^2 \\ &\leqslant \frac{1}{2}\left(\|Z^\varepsilon\|_{L^2}^2 \|\partial_t u\|_{L^2}^2 + \|\partial_t u\|_{H^1}^2\right)\end{aligned}$$

using the Gagliardo-Nirenberg inequality.
Finally for the third estimate, we note first that:

$$\partial_t |u|^2 = -2Im(\bar{u}\partial_x^2 u).$$

Thus,

$$\partial_t^2 |u|^2 = -2Im(\partial_t \bar{u}\partial_x^2 u + \bar{u}\partial_x^2 \partial_t u),$$

so that, integrating the first term by parts,

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} \partial_t Z^\varepsilon \left(\partial_t^2 |u|^2 \;\; -3|\partial_t u|^2\right) dxds\right| \\
&\qquad = \left| - \int_0^t \int_{\mathbb{R}} \partial_t Z^\varepsilon \left(2Im(\partial_t \bar{u} \partial_x^2 u + \bar{u} \partial_x^2 \partial_t u) + 3|\partial_t u|^2\right) dxds \right| \\
&\qquad \leqslant 2 \left| \int_0^t \int_{\mathbb{R}} Im(\partial_x \partial_t \bar{u} \partial_x u \partial_t Z^\varepsilon + \partial_t \bar{u} \partial_x \partial_t Z^\varepsilon \partial_x u \right. \\
&\qquad\qquad \left. + \partial_x \bar{u} \partial_t Z^\varepsilon \partial_x \partial_t u + \bar{u} \partial_x \partial_t Z^\varepsilon \partial_x \partial_t u) dxds \right| \\
&\qquad\qquad + 3 \int_0^t \|\partial_t Z^\varepsilon\|_{L^2} \|\partial_t u\|_{H^1}^2 \, ds \\
&\qquad \leqslant \int_0^t 4 \|\partial_t u\|_{H^1} \|\partial_x u\|_{L^2} \|\partial_t Z^\varepsilon\|_{H^1} \\
&\qquad\qquad + \|u\|_{H^1} \|\partial_t Z^\varepsilon\|_{H^1} \|\partial_t u\|_{H^1} \, dxds \\
&\qquad\qquad + 3 \int_0^t \|\partial_t Z^\varepsilon\|_{L^2} \|\partial_t u\|_{H^1}^2 \, ds.
\end{aligned}$$

Then, using Propositions 2.2.2.1 and 2.2.2.2, we get:

$$|III| \leqslant 4 \int_0^t (C + C(\omega)s)e^s \|\partial_t u\|_{H^1} \|\partial_t Z^\varepsilon\|_{H^1} \, ds + 3 \int_0^t \|\partial_t Z^\varepsilon\|_{L^2} \|\partial_t u\|_{H^1}^2 \, ds.$$

□

According to (2.2.22) and Lemmas 2.2.2.2, 2.2.2.3 and 2.2.2.5, for $t \in [0, T]$ we get the estimate:

$$\begin{aligned}
2 \|\partial_t u\|_{H^1}^2 + \|\varepsilon \partial_t V\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2 \leqslant & C(T, \varepsilon, \omega) \left(1 + \int_0^t \|\partial_t u(s)\|_{H^1}^2 + \|\partial_t m(s)\|_{L^2}^2 \, ds\right) \\
& + \frac{1}{2} \|\partial_t u\|_{H^1}^2 + C(T, \varepsilon, \omega) \|\partial_t u\|_{L^2}^2 .
\end{aligned}$$

We use Lemma 2.2.2.4 to estimate the term $\|\partial_t u\|_{L^2}^2$. Finally, we obtain

$$\|\partial_t u\|_{H^1}^2 + \|\varepsilon \partial_t V\|_{L^2}^2 + \|\partial_t m\|_{L^2}^2 \leqslant C(T, \varepsilon, \omega) \left(1 + \int_0^t \|\partial_t u(s)\|_{H^1}^2 + \|\partial_t m(s)\|_{L^2}^2 \, ds\right), \tag{2.2.30}$$

and, thanks to the Grönwall's Lemma, the quantities $\|\partial_t u\|_{H^1}$, $\|\partial_t m\|_{L^2}$ and $\|\varepsilon \partial_t V\|_{L^2}$ are bounded for $t \in [0, T]$. Thus, we can prove the following lemma:

Lemma 2.2.2.6. *Let $T > 0$, and $(u_0, m_0, m_1) \in H^3 \times H^2 \times H^1 \cap \dot{H}^{-1}$. Then the quantities $\|m\|_{H^1}$ and $\|u\|_{H^3}$ are bounded for $t \in [0, T]$.*

Proof. From equation (2.2.14) we get

$$\varepsilon^2 \partial_t V + \alpha\varepsilon V + \partial_x(m + |u|^2) = 0,$$

so that

$$\begin{aligned} \|\partial_x m\|_{L^2} &\leqslant \varepsilon^2 \|\partial_t V\|_{L^2} + \left\|\partial_x |u|^2\right\|_{L^2} + \alpha\varepsilon \|V\|_{L^2} \\ &\leqslant \varepsilon^2 \|\partial_t V\|_{L^2} + \alpha\varepsilon \|V\|_{L^2} + \|u\|_{H^1}^2 , \end{aligned}$$

thus the result for $\|m\|_{H^1}$ is proved. Then we look at the first equation of the system (2.2.12), which gives us:

$$\begin{aligned} \left\|\partial_x^2 u\right\|_{H^1} &\leqslant \|\partial_t u\|_{H^1} + \|(m + Z^\varepsilon)u\|_{H^1} \\ &\leqslant \|\partial_t u\|_{H^1} + \|m\|_{H^1} \|u\|_{H^1} + \|Z^\varepsilon\|_{H^1} \|u\|_{H^1} , \end{aligned}$$

and we get the desired result. $\square$

It remains to get a H^2 bound on m, and a H^1 bound on $\partial_t m$. For this purpose we note that thanks to the equation satisfied by m:

$$\begin{aligned} \partial_t \left(\left\|\partial_x^2 m\right\|_{L^2}^2 + \|\varepsilon\partial_x\partial_t m\|_{L^2}^2 \right) &\leqslant 2 \int_{\mathbb{R}} \partial_x \partial_t m \partial_x^3 |u|^2 dx \\ &\leqslant \left\|\partial_x^3 |u|^2\right\|_{L^2}^2 + \|\partial_x \partial_t m\|_{L^2}^2 \\ &\leqslant M(t) + \frac{1}{\varepsilon^2} \left(\left\|\partial_x^2 m\right\|_{L^2}^2 + \|\varepsilon\partial_x\partial_t m\|_{L^2}^2 \right) \end{aligned}$$

since the norm $\|\partial_x^3 |u|^2\|_{L^2}^2$ is bounded on $[0, T]$ thanks to Lemma 2.2.2.6. Hence, the Theorem is proved.

2.3 Convergence of the stochastic Zakharov system to the stochastic Schrödinger equation

In this section we prove that the system converges to a stochastic Nonlinear Schrödinger equation when ε goes to 0. We start from the system (2.2.12) that we recall here:

$$\begin{cases} i\partial_t u = -\partial_x^2 u + (m + Z^\varepsilon)u, \\ \\ \varepsilon^2\partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2(m + |u|^2), \end{cases} \tag{2.3.1}$$

with Z^ε solution of

$$\varepsilon^2 d(\partial_t Z^\varepsilon) + \alpha\varepsilon\partial_t Z^\varepsilon = \partial_x^2 Z^\varepsilon dt + \phi dW_t. \tag{2.3.2}$$

Let us state the first main result of this chapter.

Theorem 2.3.0.1. *Let $\alpha > 0$. For any $T > 0$, and for $u_0 \in H^3(\mathbb{R}), m_0 \in H^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}), m_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$, the process $(u^\varepsilon, m^\varepsilon)$ solution of the system (2.3.1) with $u(0) = u_0, m(0) = m_0, \partial_t m(0) = m_1$ and $\phi \in \mathcal{L}_2(L^2(\mathbb{R}), H^3(\mathbb{R}) \cap \dot{H}^{-4}(\mathbb{R}))$ converges in law in $C([0,T], H^s_{loc}(\mathbb{R})) \times C([0,T], H^{s-1}_{loc}(\mathbb{R}))$ for $s < 1$ to $(u, -|u|^2)$ solution of*

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2}uF)dt - u(\partial_x^2)^{-1}\phi dW_t, \tag{2.3.3}$$

where $F(x) = \sum_{k\in\mathbb{N}} \left((\partial_x^2)^{-1})\phi e_k\right)^2(x)$.

In order to understand the limit behavior, we first note that Z^ε is of order $\varepsilon^{-1/2}$, as can be seen by computing $\mathbb{E}(\|Z^\epsilon(t)\|^2_{L^2})$ for instance. Also, the scaling of (2.3.2) indicates that it depends on t as a function of $\frac{t}{\varepsilon}$. It is thus natural to define $z^\varepsilon(t) = \varepsilon^{1/2}Z^\varepsilon(\varepsilon t)$. This is done in the next subsection and we see that we are exactly in a situation where the Perturbed Test Function method works.

We then adapt the proof of [AA88] by Added and Added, for which we need a uniform bound in ε on $\|\partial_x u\|_{L^2}, \|m\|_{L^2}$ and $\|\varepsilon V\|_{L^2}$ (where $V = -\partial_x^{-1}\partial_t m$). In order to get such a bound, we apply the Perturbed Test Function method to the energy H defined in equation (2.2.15). Moreover, we will need to gather this bound with some tightness results to be able to pass to the limit $\varepsilon \to 0$ as in [AA88], this will be done in the fourth subsection. Finally, the last subsection states the weak convergence results obtained, and gives the limit $\varepsilon \to 0$ in the martingale problem in order to prove Theorem 2.3.0.1. Since we are in a situation, where the driving process is an Ornstein-Uhlenbeck process and all correctors can be computed explicitly, we are able to have explicit martingales when studying the martingale problem. We take advantage of this to obtain the convergence in probability of the system (2.3.1) to equation (2.3.3). Note that the convergence in probability in a similar but a simpler problem, where the Ornstein-Uhlenbeck solves an equation of the order one and the setting is finite dimensional, has been obtained in [GL22].

In the first three subsections, in order to lighten the notations we do not specify the dependence on ε for u, m and V. We start to keep track of this dependence in the fourth subsection, where we prove the tightness.

2.3.1 Rescaling of the system

We already know that (u, n) is a solution of the system (2.2.10) if (u, m) is a solution of (2.3.1). We can rewrite the equation on Z^ε in its integral form:

$$\varepsilon^2 Z^\varepsilon(t) = -\alpha \int_0^t \int_0^s \varepsilon \partial_t Z^\varepsilon(\sigma) d\sigma ds + \int_0^t \int_0^s \partial_x^2 Z^\varepsilon(\sigma) d\sigma ds + \int_0^t \int_0^s \phi dW_\sigma ds.$$

In order to understand the orders of the different terms in (2.3.1), we write Z^ε as

$$Z^\varepsilon(t) = \varepsilon^{-\frac{1}{2}} \tilde{z}^\varepsilon(\varepsilon^{-1} t) \tag{2.3.4}$$

and rewrite all the integrals in the rescaled time. For the first term, we have:

$$\begin{aligned}\int_0^t \int_0^s \varepsilon \partial_t Z^\varepsilon(\sigma) d\sigma ds &= \int_0^t \varepsilon Z^\varepsilon(s) ds \\ &= \int_0^{\varepsilon^{-1} t} \int_0^s \varepsilon^{\frac{3}{2}} \partial_t \tilde{z}^\varepsilon(\sigma) d\sigma ds.\end{aligned}$$

Then

$$\begin{aligned}\int_0^t \int_0^s \partial_x^2 Z^\varepsilon(\sigma) d\sigma ds &= \int_0^t \int_0^s \varepsilon^{-\frac{1}{2}} \partial_x^2 \tilde{z}^\varepsilon(\varepsilon^{-1}\sigma) d\sigma ds \\ &= \int_0^{\varepsilon^{-1} t} \int_0^s \varepsilon^{\frac{3}{2}} \partial_x^2 \tilde{z}^\varepsilon(\sigma) d\sigma ds.\end{aligned}$$

Finally,

$$\begin{aligned}\int_0^t \phi W(s)ds &= \int_0^t \varepsilon^{\frac{1}{2}}\phi\tilde{W}^\varepsilon(\varepsilon^{-1}s)ds\\ &= \int_0^{\varepsilon^{-1}t} \varepsilon^{\frac{3}{2}}\phi\tilde{W}^\varepsilon(s)ds,\end{aligned}$$

where $\tilde{W}^\varepsilon(t) = \varepsilon^{-\frac{1}{2}}W(\varepsilon t)$ is still a cylindrical Wiener process. Now by replacing $\varepsilon^{-1}t$ by t and deriving twice, we obtain an equation which does not depend on ε:

$$d(\partial_t \tilde{z}^\varepsilon) + \alpha\partial_t\tilde{z}^\varepsilon = \partial_x^2\tilde{z}^\varepsilon dt + \phi d\tilde{W}_t^\varepsilon. \tag{2.3.5}$$

Now we can rewrite the system as:

$$\begin{cases} i\partial_t u = -\partial_x^2 u + mu + \frac{1}{\sqrt{\varepsilon}}\tilde{z}^\varepsilon\left(\frac{t}{\varepsilon}\right)u,\\ \partial_t m = \mu\\ \varepsilon^2\partial_t\mu + \alpha\varepsilon\mu = \partial_x^2(m+|u|^2), \end{cases} \tag{2.3.6}$$

with $\tilde{z}^\varepsilon$ solution of the following equation

$$\begin{cases} \partial_t\tilde{z}^\varepsilon = \tilde{\zeta}^\varepsilon,\\ d\tilde{\zeta}^\varepsilon = (-\alpha\tilde{\zeta}^\varepsilon + \partial_x^2\tilde{z}^\varepsilon)dt + \phi d\tilde{W}_t^\varepsilon. \end{cases} \tag{2.3.7}$$

Clearly, the process $(\tilde{z}^\varepsilon, \tilde{\zeta}^\varepsilon)$ has a law independent of ε and we see that (2.3.6) corresponds to the classical situation of diffusion-approximation, for which the Perturbed test Function method is a very efficient tool.

In the sequel, we apply this method but in order to work with a constant Wiener process, we do not use $(\tilde{z}^\varepsilon, \tilde{\zeta}^\varepsilon)$ anymore. We use the process: $(z^\varepsilon(t), \zeta^\varepsilon(t)) = (\varepsilon^{1/2}Z^\varepsilon(t), \varepsilon^{3/2}\partial_t Z^\varepsilon(t))$, which is of order 1 and is, in fact, the standard process used in this method.

2.3.2 The driving process $(z^\varepsilon, \zeta^\varepsilon)$ and its generator

In this subsection we state some results linked to the process $(z^\varepsilon, \zeta^\varepsilon)$ and its infinitesimal generator. It satisfies the equation:

$$\begin{cases} \partial_t z^\varepsilon = \dfrac{1}{\varepsilon}\zeta^\varepsilon, \\ d\zeta^\varepsilon = \dfrac{1}{\varepsilon}(-\alpha\zeta^\varepsilon + \partial_x^2 z^\varepsilon)dt + \dfrac{1}{\varepsilon^{1/2}}\phi dW_t. \end{cases} \tag{2.3.8}$$

Its infinitesimal generator is $\frac{1}{\varepsilon}\mathcal{M}$, where $\mathcal{M}$ is the infinitesimal generator of $(\tilde{z}^\varepsilon, \tilde{\zeta}^\varepsilon)$ or of the ε independent process (z, ζ) satisfying:

$$\begin{cases} \partial_t z = \zeta, \\ d\zeta = (-\alpha\zeta + \partial_x^2 z)dt + \phi dW_t. \end{cases} \tag{2.3.9}$$

There exists a unique Gaussian invariant measure for the transition semigroup associated with the equation satisfied by $(z^\varepsilon, \zeta^\varepsilon)$ (see [DPZ14, Chapter 11]), uniqueness follows from the decay of S_α for $\alpha > 0$. This invariant measure does not depend on ε and we denote it by ν. The existence of this measure allows us to take the process $(z^\varepsilon, \zeta^\varepsilon)$ stationary by choosing the initial condition of the system (2.3.7):

$$\begin{pmatrix} z_0^\varepsilon \\ \zeta_0^\varepsilon \end{pmatrix} = \frac{1}{\varepsilon^{1/2}} \int_0^\infty S_\alpha\left(\frac{s}{\varepsilon}\right) \begin{pmatrix} 0 \\ \phi dW_s \end{pmatrix}. \tag{2.3.10}$$

We also choose an initial condition (z_0, ζ_0) such that (z, ζ) satisfying (2.3.9) is a stationary process. This implies that the initial data of equation (2.0.3) is decomposed as $n_0 = m_0 + \frac{1}{\sqrt{\varepsilon}} z_0, n_1 = m_1 + \frac{1}{\sqrt{\varepsilon}}\zeta_0$. The regularity of (z_0, ζ_0) depends on the regularity of ϕ as a Hilbert-Schmidt operator, as seen in Lemma 2.1.0.5. Thus the regularity of (n_0, n_1) only depends on the regularity of (m_0, m_1) and the regularity of ϕ as a Hilbert-Schmidt operator

Then for any continuous bounded function φ on $H^2 \times H^1 \cap \dot{H}^{-1}$, we have an explicit expression for the expectation with respect to the invariant measure ν:

$$\begin{aligned}\mathbb{E}_\nu[\varphi(z,\zeta)] &= \mathbb{E}\left[\varphi(z_0^\varepsilon,\zeta_0^\varepsilon)\right]\\ &= \mathbb{E}\left[\varphi\left(\frac{1}{\varepsilon^{1/2}}\int_0^\infty S_\alpha(\frac{s}{\varepsilon})\begin{pmatrix}0\\ \phi dW_s\end{pmatrix}_1, \frac{1}{\varepsilon^{1/2}}\int_0^\infty S_\alpha(\frac{s}{\varepsilon})\begin{pmatrix}0\\ \phi dW_s\end{pmatrix}_2\right)\right]\\ &= \mathbb{E}\left[\varphi(z_0,\zeta_0)\right] = \mathbb{E}\left[\varphi\left(\int_0^\infty S_\alpha(s)\begin{pmatrix}0\\ \phi dW_s\end{pmatrix}_1, \int_0^\infty S_\alpha(s)\begin{pmatrix}0\\ \phi dW_s\end{pmatrix}_2\right)\right].\end{aligned}$$

In order to justify the computations of this paper, we will need to take $\phi \in \mathcal{L}_2(H^3 \cap \dot{H}^{-4})$, which guarantees that $(z^\varepsilon,\zeta^\varepsilon) \in C([0,T],H^3 \cap \dot{H}^{-3})^2$ almost surely, for any $T > 0$ (thanks to similar arguments as for Lemmas 2.1.0.3 and 2.1.0.5, because the operator $S_\alpha(t)$ commutes with the operator ∂_x). The stationarity allows us to control the growth of $(z^\varepsilon,\zeta^\varepsilon)$ thanks to the following Proposition.

Proposition 2.3.2.1. *For any $T > 0$ and any $\delta > 0$:*

$$\mathbb{P}(\sup_{[0,T]}(\|z^\varepsilon(t)\|_{H^3\cap\dot{H}^{-3}}, \|\zeta^\varepsilon(t)\|_{H^3\cap\dot{H}^{-3}}) \geqslant \varepsilon^{-\delta}) \to 0$$

as ε goes to 0.

Proof. For $k \in \mathbb{N}$, we denote by η_k the process

$$\eta_k = \sup_{t\in[k,k+1]}\left(\|z(t)\|_{H^3\cap\dot{H}^{-3}} + \|\zeta(t)\|_{H^3\cap\dot{H}^{-3}}\right).$$

Since the process (z,ζ) is Gaussian, it has finite moments and since it is stationary $\mathbb{E}[\eta_k^\gamma]$ is finite and does not depend on k for all $\gamma > 0$. Thus, by Markov inequality for any $\delta > 0$, choosing $\gamma > 1/\delta$ we have $\sum_{k\in\mathbb{N}}\mathbb{P}(\eta_k > k^\delta) < \infty$. And by Borel-Cantelli lemma, we are able to prove that there exist two random variables Z_1, Z_2 such that almost surely

$$\|z(t)\|_{H^3\cap\dot{H}^{-3}} + \|\zeta(t)\|_{H^3\cap\dot{H}^{-3}} \leqslant Z_1 + |t|^\delta Z_2. \tag{2.3.11}$$

This estimate implies the result for $(z(\frac{t}{\varepsilon}),\zeta(\frac{t}{\varepsilon}))$ and thus for $(z^\varepsilon,\zeta^\varepsilon)$ since it has the same law. □

According to Proposition 2.3.2.1, if we introduce

$$\tau_\delta^\varepsilon = \inf\left\{t : \|z^\varepsilon(t)\|_{H^3\cap\dot{H}^{-3}} \geqslant \varepsilon^{-\delta}, \|\zeta^\varepsilon(t)\|_{H^3\cap\dot{H}^{-3}} \geqslant \varepsilon^{-\delta}\right\}, \tag{2.3.12}$$

we know that τ_δ^ε goes to ∞ in probability as ε goes to 0.

Now we focus on the infinitesimal generator $\mathcal{M}$ of (z,ζ). Indeed in the following sections we use the Perturbed Test Function method, for which we have to find the inverse of the generator applied to several quantities. We do not wish to specify the domain of the generator. For a Borel function φ, we write $\mathcal{M}\varphi$ when $\varphi(z(t),\zeta(t)) - \int_0^t \mathcal{M}\varphi(z(s),\zeta(s))ds$ is a well defined quantity and is an integrable martingale. Similarly, we write $\mathcal{M}^{-1}\varphi$ when $\mathcal{M}^{-1}\varphi(z(t),\zeta(t)) - \int_0^t \varphi(z(s),\zeta(s))ds$ is a well defined quantity and is an integrable martingale.

For a function f, we try to find a function φ such that $\mathcal{M}\varphi = f$ (we write $\varphi = \mathcal{M}^{-1}f$). Clearly, we need $\mathbb{E}_\nu[f(z,\zeta)] = 0$, recall that ν is the invariant measure of (z,ζ). Given $\eta = (z,\zeta)$, we denote by $(z(t,\eta),\zeta(t,\eta))$ the solution of (2.3.9) with initial data η at time 0. Then, since the process has all moments, it is classical to see that for any φ measurable with polynomial growth, we have:

$$\mathcal{M}^{-1}\varphi(\eta) = -\mathbb{E}\int_0^\infty \varphi(z(t,\eta),\zeta(t,\eta))dt.$$

The following results give properties on $\mathcal{M}\varphi$ for some test functions φ used in the Perturbed Test Function method.

Lemma 2.3.2.1. *For $(z,\zeta) \in (H^3 \cap \dot{H}^{-2})^2$ and $\phi \in \mathcal{L}_2(H^1 \cap \dot{H}^{-2})$ we have*

$$\mathcal{M}^{-1}z = (\partial_x^2)^{-1}\zeta + \alpha(\partial_x^2)^{-1}z, \tag{2.3.13}$$

where $\mathcal{M}^{-1}z$ is a notation for $\mathcal{M}^{-1}\pi_1(z,\zeta)$, π_1 being the projection on the first coordinate.

Proof. By equation (2.3.9), we have

$$d\left((\partial_x^2)^{-1}\zeta + \alpha(\partial_x^2)^{-1}z\right) = zdt + (\partial_x^2)^{-1}\phi dW_t,$$

and thus, $(\partial_x^2)^{-1}\zeta(t) + \alpha(\partial_x^2)^{-1}z(t) - \int_0^t z(s)ds$ is a martingale. Note that the zero-mean condition is satisfied because ν is a centered Gaussian measure. □

Proposition 2.3.2.2. *For $(z,\zeta) \in (H^1 \cap \dot{H}^{-3}) \times (L^2 \cap \dot{H}^{-3})$ and $\phi \in \mathcal{L}_2(H^1 \cap \dot{H}^{-4})$ we have*

$$\left\| \mathcal{M}^{-1} \left(z\mathcal{M}^{-1}z - \mathbb{E}_\nu \left[z\mathcal{M}^{-1}z \right] \right) \right\|_{L^\infty} \leqslant C \left(C_\phi + \|z\|^2_{H^1 \cap \dot{H}^{-3}} + \|\zeta\|^2_{L^2 \cap \dot{H}^{-3}} \right), \tag{2.3.14}$$

where $\mathbb{E}_\nu$ denotes the expectation under the invariant measure and C_ϕ is a constant depending on the Hilbert-Schmidt norm of ϕ in the space $H^1 \cap \dot{H}^{-4}$.

We also have for $(z,\zeta) \in (H^2 \cap \dot{H}^{-3}) \times (H^1 \cap \dot{H}^{-3})$ and $\phi \in \mathcal{L}_2(H^2 \cap \dot{H}^{-4})$

$$\left\| \mathcal{M}^{-1} \left(\partial_x z \partial_x \mathcal{M}^{-1}z - \mathbb{E}_\nu \left[\partial_x z \partial_x \mathcal{M}^{-1}z \right] \right) \right\|_{L^\infty} \leqslant C \left(C_\phi + \|z\|^2_{H^2 \cap \dot{H}^{-2}} + \|\zeta\|^2_{H^1 \cap \dot{H}^{-2}} \right), \tag{2.3.15}$$

where this time C_ϕ is a constant depending on the Hilbert-Schmidt norm of ϕ in the space $H^2 \cap \dot{H}^{-4}$.

In order to prove Proposition 2.3.2.2, we state the following lemma:

Lemma 2.3.2.2.

$$\begin{aligned}
\mathcal{M}^{-1} \left(z\mathcal{M}^{-1}z - \mathbb{E}_\nu \left[z\mathcal{M}^{-1}z \right] \right) = - \int_0^\infty \Big(& \left(S_\alpha(t)(z,\zeta)\right)_1 (\partial_x^2)^{-1} \left(S_\alpha(t)(z,\zeta)\right)_2 \\
& - \sum_{k\in\mathbb{N}} \int_t^\infty \left(S_\alpha(s)(0,\phi e_k)\right)_1 (\partial_x^2)^{-1} \left(S_\alpha(s)(0,\phi e_k)\right)_2 ds \\
& + \alpha \left(S_\alpha(t)(z,\zeta)\right)_1 (\partial_x^2)^{-1} \left(S_\alpha(t)(z,\zeta)\right)_1 \\
& - \alpha \sum_{k\in\mathbb{N}} \int_t^\infty \left(S_\alpha(s)(0,\phi e_k)\right)_1 (\partial_x^2)^{-1} \left(S_\alpha(s)(0,\phi e_k)\right)_1 ds \Big) dt
\end{aligned} \tag{2.3.16}$$

where S_α is the semigroup defined in (2.1.3) and (2.1.4). Besides we denote by $(S_\alpha(t)(z,\zeta))_i$ the i-th component of $S_\alpha(t)(z,\zeta)$.

Proof. Thanks to Lemma 2.3.2.1, we know that

$$\begin{aligned}
\mathcal{M}^{-1} \left(z\mathcal{M}^{-1}z - \mathbb{E}_\nu \left[z\mathcal{M}^{-1}z \right] \right) = & \mathcal{M}^{-1} \left(z(\partial_x^2)^{-1}\zeta - \mathbb{E}_\nu \left[z(\partial_x^2)^{-1}\zeta \right] \right) \\
& + \alpha \mathcal{M}^{-1} \left(z(\partial_x^2)^{-1}z - \mathbb{E}_\nu \left[z(\partial_x^2)^{-1}z \right] \right).
\end{aligned}$$

For the first term, we write:

$$\begin{aligned}\mathbb{E}_\nu\left[z(\partial_x^2)^{-1}\zeta\right] &= \int z(\partial_x^2)^{-1}\zeta d\nu(z,\zeta) \\ &= \mathbb{E}\left[\left(\int_{-\infty}^0 S_\alpha(-s)(0,\phi dW_s)\right)_1 \left(\int_{-\infty}^0 (\partial_x^2)^{-1} S_\alpha(-s)(0,\phi dW_s)\right)_2\right] \\ &= \sum_{k\in\mathbb{N}} \int_0^\infty \left(S_\alpha(s)(0,\phi e_k)\right)_1 \left((\partial_x^2)^{-1} S_\alpha(s)(0,\phi e_k)\right)_2 ds,\end{aligned} \tag{2.3.17}$$

thanks to Itô isometry. It follows:

$$\begin{aligned}&\mathcal{M}^{-1}\left(z(\partial_x^2)^{-1}\zeta - \mathbb{E}_\nu\left[z(\partial_x^2)^{-1}\zeta\right]\right) \\ &\quad = -\int_0^\infty \mathbb{E}\Bigg[\left(S_\alpha(t)(z,\zeta) + \int_0^t S_\alpha(t-s)(0,\phi dWs)\right)_1 \\ &\quad \times (\partial_x^2)^{-1}\left(S_\alpha(t)(z,\zeta) + \int_0^t S_\alpha(t-s)(0,\phi dWs)\right)_2 - \mathbb{E}_\nu\left[z(\partial_x^2)^{-1}\zeta\right]\Bigg] dt \\ &\quad = -\int_0^\infty \left(S_\alpha(t)(z,\zeta)\right)_1 (\partial_x^2)^{-1}\left(S_\alpha(t)(z,\zeta)\right)_2 \\ &\qquad - \sum_{k\in\mathbb{N}} \int_t^\infty \left(S_\alpha(s)(0,\phi e_k)\right)_1 (\partial_x^2)^{-1}\left(S_\alpha(s)(0,\phi e_k)\right)_2 ds \quad dt.\end{aligned}$$

The argument is the same for the second term. □

Proof of Proposition 2.3.2.2. Thanks to Lemma 2.3.2.2, we just have to use the embedding $H^1 \hookrightarrow L^\infty$ and bound the products of terms using:

$$\begin{aligned}\int_0^\infty \left\|\left(S_\alpha(t)(z,\zeta)\right)_1\right\|_{H^1} \left\|(\partial_x^2)^{-1}\left(S_\alpha(t)(z,\zeta)\right)_2\right\|_{H^1} dt \leqslant \int_0^\infty &\left\|\left(S_\alpha(t)(z,\zeta)\right)_1\right\|_{H^1}^2 \\ &+ \left\|\left(S_\alpha(t)(z,\zeta)\right)_2\right\|_{L^2\cap\dot{H}^{-2}}^2 dt\end{aligned}$$

and we conclude thanks to Lemma 2.1.0.6, which allows us to bound all these quantities. The scheme of proof is the same for the second estimate, except that the ∂_x operator requires H^2 regularity. □

2.3.3 The Perturbed Test Function method

In this section we detail the Perturbed Test Function method applied to our system of equations. Recall that $\mathcal{M}$ is the infinitesimal generator of (z,ζ). We denote by $\mathcal{L}^\varepsilon$ the infinitesimal generator of $(u, m, \mu, z^\varepsilon, \zeta^\varepsilon)$. Provided φ is a smooth function, the Itô formula gives:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi(u, m, \varepsilon^2\mu, z^\varepsilon, \zeta^\varepsilon) =& D_u\varphi(i\partial_x^2 u - imu) - \frac{1}{\sqrt{\varepsilon}} D_u\varphi(iz^\varepsilon u) + D_m\varphi(\mu) \\ &+ D_\mu\varphi(\partial_x^2(m + |u|^2) - \alpha\varepsilon\mu) + \frac{1}{\varepsilon}\mathcal{M}\varphi,\end{aligned}$$

where

$$\mathcal{M}\varphi = D_z\varphi(\zeta) + D_\zeta\varphi(\partial_x^2 z - \alpha\zeta) + \frac{1}{2}Tr(\phi^* D_\zeta^2\varphi\phi). \tag{2.3.18}$$

The aim is to compute the generator $\mathcal{L}^\varepsilon$ applied to a real-valued function $\varphi(u, m, \varepsilon^2\mu)$ corrected by two functions φ_1 and φ_2 in order to deal with the singularity appearing above.

In this case φ does not depend on (z, ζ), thus, $\mathcal{M}\varphi = 0$. In order to deal with the term of order $-\frac{1}{2}$ in ε, it seems natural to take as the first corrector

$$\begin{aligned}\varphi_1(u, z, \zeta) &= D_u\varphi(i\mathcal{M}^{-1}zu) \\ &= D_u\varphi\left(iu(\partial_x^2)^{-1}(\zeta + \alpha z)\right)\end{aligned} \tag{2.3.19}$$

according to Lemma 2.3.2.1. The addition of this term will give us new terms to control in $\mathcal{L}^\varepsilon(\varphi + \sqrt{\varepsilon}\varphi_1)$, because of the growth of $(z^\varepsilon, \zeta^\varepsilon)$ as a negative power of ε (see Proposition 2.3.2.1). Thus, we also consider a second corrector

$$\begin{aligned}\varphi_2(u, z, \zeta) &= \mathcal{M}^{-1}\left(D_u\varphi_1(iuz) - \mathbb{E}_\nu\left[D_u\varphi_1(iuz)\right]\right) \\ &= \mathcal{M}^{-1}\left(D_u\left(D_u\varphi(iu\mathcal{M}^{-1}z)\right)(iuz) - \mathbb{E}_\nu\left[D_u\left(D_u\varphi(iu\mathcal{M}^{-1}z)\right)(iuz)\right]\right),\end{aligned} \tag{2.3.20}$$

where ν is the invariant measure of z^ε, which is the same as the invariant measure of z. Finally, we apply the generator $\mathcal{L}^\varepsilon$ to $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$, which is

$$\begin{aligned}\varphi^\varepsilon(u, m, \mu, z^\varepsilon, \zeta^\varepsilon) =& \varphi(u, m, \mu) + \sqrt{\varepsilon} D_u\varphi\left(iu(\partial_x^2)^{-1}(\zeta^\varepsilon + \alpha z^\varepsilon)\right) \\ &+ \varepsilon\mathcal{M}^{-1}\left(D_u\varphi_1(iuz^\varepsilon) - \mathbb{E}_\nu\left[D_u\varphi_1(iuz)\right]\right).\end{aligned} \tag{2.3.21}$$

Now we know how to correct any function $\varphi(u, m, \varepsilon^2\mu)$ in order to avoid any singularity in ε, thanks to the control of $(z^\varepsilon, \zeta^\varepsilon)$ given by Proposition 2.3.2.1.

2.3.4 Energy estimate

In this section we consider initial data $u_0 \in H^3(\mathbb{R}), m_0 \in H^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}), m_1 \in H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})$, and $(z_0, \zeta_0) \in (H^3(\mathbb{R}) \cap \dot{H}^{-3}(\mathbb{R}))^2$. We use the energy

$$H(u(t), m(t), V(t)) = \|\partial_x u(t)\|_{L^2}^2 + \frac{1}{2}(\|m(t)\|_{L^2}^2 + \|\varepsilon V(t)\|_{L^2}^2) + \int_{\mathbb{R}} m(t)|u(t)|^2 dx$$

where $V = -\partial_x^{-1}\mu$. This functional is well-defined under the assumptions of Theorem 2.2.2.1 on the initial data. Proposition 2.2.2.1 gives a bound on the energy H. However, since Z^ε behaves like $\varepsilon^{-1/2}$, this bound is useless for the limit as $\varepsilon \to 0$.

Let us also introduce

$$K_t = K(u(t), m(t), V(t)) = \frac{1}{2}\|\partial_x u(t)\|_{L^2}^2 + \frac{1}{4}\|m(t)\|_{L^2}^2 + \frac{1}{2}\|\varepsilon V(t)\|_{L^2}^2 \tag{2.3.22}$$

which is a positive functional. It can be shown that

$$K - 2N^3 \leqslant H \leqslant 3K + 2N^3 \tag{2.3.23}$$

where N is defined in (2.2.15) (see [SS79, Lemma 2]) and we know from Proposition 2.2.2.1 that the mass $N = \|u\|_{L^2}^2$ is conserved by the first equation of (2.3.1).
The quantity K will be useful since it is positive, and since we have bounds between K and H. We will use the Perturbed Test Function method to get bounds on H, which will give us bounds on K.

Our aim here is to apply the Perturbed Test Function method in order to get a bound on H independent of ε. We want to apply the generator $\mathcal{L}^\varepsilon$ of $(u, m, \mu, z^\varepsilon, \zeta^\varepsilon)$, the solution of (2.3.6) and (2.3.7), to the energy H, defined in equation (2.2.15). According to the Itô formula, we have

$$\begin{aligned}\mathcal{L}^\varepsilon H =& D_u H(i\partial_x^2 u - imu) - \frac{1}{\sqrt{\varepsilon}} D_u H(iz^\varepsilon u) \\ &+ D_m H(\mu) + \frac{1}{\varepsilon^2} D_\mu H(\partial_x^2(m + |u|^2) - \alpha\varepsilon\mu) + \frac{1}{\varepsilon}\mathcal{M}H\end{aligned}$$

and $\mathcal{M}H = 0$ because H does not depend on (z, ζ). Thus we have:

$$\begin{aligned}
\mathcal{L}^\varepsilon H =& 2Re\int_{\mathbb{R}} \partial_x \bar{u}(i\partial_x^3 u - i\partial_x(mu)) + m\bar{u}(i\partial_x^2 u - imu)dx \\
&- \frac{2}{\sqrt{\varepsilon}} Re\int_{\mathbb{R}} \partial_x \bar{u}\partial_x(iz^\varepsilon u) + im|u|^2 z^\varepsilon dx \\
&+ \int_{\mathbb{R}} \mu(m+|u|^2)dx + \int_{\mathbb{R}} \partial_x^{-1}\mu\partial_x(m+|u|^2)dx - \alpha\varepsilon \left\|\partial_x^{-1}\mu\right\|_{L^2}^2 \\
=& 2Re\int_{\mathbb{R}} -i\left[|\partial_x^2 u|^2 + \partial_x\bar{u}\partial_x(mu) + \partial_x u\partial_x(m\bar{u}) + m^2|u|^2\right]dx \\
&- \frac{2}{\sqrt{\varepsilon}} Re\int_{\mathbb{R}} i\partial_x(z^\varepsilon u)\partial_x\bar{u}dx - \alpha\varepsilon\left\|\partial_x^{-1}\mu\right\|_{L^2}^2 \\
=& -\alpha\varepsilon\left\|\partial_x^{-1}\mu\right\|_{L^2}^2 - \frac{2}{\sqrt{\varepsilon}} Re\int_{\mathbb{R}} iu\partial_x\bar{u}\partial_x z^\varepsilon dx.
\end{aligned}$$

According to section 2.3.3 we already know the expression of the two correctors $H_1(u, z, \zeta)$ and $H_2(u, z, \zeta)$. The first corrector can be computed with the help of Lemma 2.3.2.1:

$$\begin{aligned}
H_1 &= 2Re\int_{\mathbb{R}} iu\partial_x\bar{u}\partial_x\mathcal{M}^{-1}z^\varepsilon dx \\
&= 2Re\int_{\mathbb{R}} iu\partial_x\bar{u}\left(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon\right)dx.
\end{aligned} \tag{2.3.24}$$

We have

$$\begin{aligned}
\sqrt{\varepsilon}D_u H_1(i\partial_x^2 u - imu - \frac{i}{\sqrt{\varepsilon}}z^\varepsilon u) =& 4\sqrt{\varepsilon}\int_{\mathbb{R}} |\partial_x u|^2(\zeta^\varepsilon + \alpha z^\varepsilon)dx \\
&+ \sqrt{\varepsilon}\int_{\mathbb{R}} \partial_x|u|^2(\partial_x\zeta^\varepsilon + \alpha\partial_x z^\varepsilon)dx \\
&- 2\sqrt{\varepsilon}\int_{\mathbb{R}} |u|^2\partial_x m(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon)dx \\
&- 2\int_{\mathbb{R}} |u|^2\partial_x z^\varepsilon(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon)dx
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{\varepsilon}(D_\zeta H_1(d\zeta^\varepsilon) + D_z H_1(dz^\varepsilon)) =& \left(\frac{2}{\sqrt{\varepsilon}} Re\int_{\mathbb{R}} iu\partial_x\bar{u}\partial_x z^\varepsilon dx\right)dt \\
&+ 2Re\int_{\mathbb{R}} iu\partial_x\bar{u}\sum_{k\in\mathbb{N}}\partial_x^{-1}(\phi e_k)d\beta_k(t)dx.
\end{aligned}$$

It implies that

$$\begin{aligned} d(H+\sqrt{\varepsilon}H_1) = &\Big(-\alpha\varepsilon \left\|\partial_x^{-1}\mu\right\|_{L^2}^2 - 2\int_{\mathbb{R}} |u|^2\partial_x z^\varepsilon(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon)dx \\ &+ 4\sqrt{\varepsilon}\int_{\mathbb{R}} |\partial_x u|^2(\zeta^\varepsilon + \alpha z^\varepsilon)dx + \sqrt{\varepsilon}\int_{\mathbb{R}} \partial_x|u|^2(\partial_x\zeta^\varepsilon + \alpha\partial_x z^\varepsilon)dx \\ &-2\sqrt{\varepsilon}\int_{\mathbb{R}} |u|^2\partial_x m(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon)dx\Big)\,dt \\ &+ 2Re\int_{\mathbb{R}} iu\partial_x\bar{u}\sum_{k\in\mathbb{N}}\partial_x^{-1}(\phi e_k)d\beta_k(t)dx \\ =&\mathcal{L}^\varepsilon(H+\sqrt{\varepsilon}H_1)dt + dX_t, \end{aligned} \tag{2.3.25}$$

where

$$X_t = 2Re\int_{\mathbb{R}}\int_0^t iu\partial_x\bar{u}\sum_{k\in\mathbb{N}}\partial_x^{-1}(\phi e_k)d\beta_k(s)dx \tag{2.3.26}$$

is a martingale.

First, we can see that

$$\begin{aligned} |H_1| &\leqslant 2\left\|u\right\|_{L^2}\left\|\partial_x u\right\|_{L^2}(\left\|\partial_x^{-1}\zeta^\varepsilon\right\|_{L^\infty}(t,\omega) + \alpha\left\|\partial_x^{-1}z^\varepsilon\right\|_{L^\infty}(t,\omega)) \\ &\leqslant C\left\|u\right\|_{L^2}(\left\|\partial_x^{-1}\zeta^\varepsilon\right\|_{L^\infty}(t,\omega) + \alpha\left\|\partial_x^{-1}z^\varepsilon\right\|_{L^\infty}(t,\omega))K^{\frac{1}{2}}. \end{aligned} \tag{2.3.27}$$

As explained in Section 2.3.3, we need a second corrector in order to get a uniform bound as ε goes to 0. We choose H_2 such that

$$\begin{aligned} \mathcal{M}H_2 &= 2\int_{\mathbb{R}} |u|^2\partial_x z^\varepsilon(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon)dx - 2\mathbb{E}_\nu\left[\int_{\mathbb{R}} |u|^2\partial_x z(\partial_x^{-1}\zeta + \alpha\partial_x^{-1}z)dx\right] \\ &= 2\int_{\mathbb{R}} |u|^2\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon dx - 2\mathbb{E}_\nu\left[\int_{\mathbb{R}} |u|^2\partial_x z\partial_x\mathcal{M}^{-1}z dx\right]. \end{aligned} \tag{2.3.28}$$

In other words we set

$$H_2 = 2\int_{\mathbb{R}} |u|^2\mathcal{M}^{-1}\left(\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]\right)dx. \tag{2.3.29}$$

Defining then $H_t^\varepsilon = H(u(t), m(t), V(t)) + \sqrt{\varepsilon}H_1(u(t), z^\varepsilon(t), \zeta^\varepsilon(t)) + \varepsilon H_2(u(t), z^\varepsilon(t), \zeta^\varepsilon(t))$

and $V = -\partial_x^{-1}\mu$, we obtain:

$$\begin{aligned}\mathcal{L}^\varepsilon(H_t^\varepsilon) =& -2\int_{\mathbb{R}} |u|^2\mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]dx\\&+\sqrt{\varepsilon}\left(4\int_{\mathbb{R}}|\partial_x u|^2(\zeta^\varepsilon+\alpha z^\varepsilon)dx+\int_{\mathbb{R}}\partial_x|u|^2(\partial_x\zeta^\varepsilon+\alpha\partial_x z^\varepsilon)dx\right.\\&\left.-2\int_{\mathbb{R}}|u|^2\partial_x m(\partial_x^{-1}\zeta^\varepsilon+\alpha\partial_x^{-1}z^\varepsilon)dx\right)\\&-\alpha\left\|\sqrt{\varepsilon}V\right\|_{L^2}^2+4\varepsilon Re\int_{\mathbb{R}} i\bar{u}\partial_x^2 u\mathcal{M}^{-1}\left(\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon-\mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]\right)dx.\end{aligned} \tag{2.3.30}$$

Remark 2.3.4.1. *The second corrector also contributes to the martingale part, which can be computed thanks to Itô formula and Lemma 2.3.2.2 adding*

$$\begin{aligned}Y_T =&\varepsilon\int_{s=0}^T\int_{\mathbb{R}}|u|^2\int_{t=0}^\infty \partial_x\left(S_\alpha(t)(z^\varepsilon,\phi dW_s)\right)_1\\&\times\left(\partial_x^{-1}\left(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon)\right)_2+\alpha\partial_x^{-1}\left(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon)\right)_1\right)\\&+\partial_x\left(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon)\right)_1\left(\partial_x^{-1}\left(S_\alpha(t)(z^\varepsilon,\phi dW_s)\right)_2+\alpha\partial_x^{-1}\left(S_\alpha(t)(z^\varepsilon,\phi dW_s)\right)_1\right)dtdx.\end{aligned} \tag{2.3.31}$$

If we write all the computations with Itô formula, it is equivalent to write $dH^\varepsilon = \mathcal{L}^\varepsilon H^\varepsilon dt + d(X_t + Y_t)$ with X_t defined in (2.3.26) and Y_t in (2.3.31) . We have many quantities that we need to estimate. We start by the estimation of H_2:

$$\begin{aligned}|H_2| &\leqslant C\|u\|_{L^2}^2\left\|\mathcal{M}^{-1}\left(\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon-\mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]\right)\right\|_{L^\infty}\\&\leqslant C\left(C_\phi+\|z^\varepsilon\|_{H^2\cap\dot{H}^{-2}}^2+\|\zeta^\varepsilon\|_{H^2\cap\dot{H}^{-2}}^2\right)\end{aligned} \tag{2.3.32}$$

according to Proposition 2.3.2.2.

From now on we will work for t in an interval $[0,\tau_\delta^\varepsilon)$, where τ_δ^ε is defined in the previous subsection in equation (2.3.12). We choose $\delta \leqslant \frac{1}{6}$ (in this subsection $\frac{1}{4}$ is sufficient but we will need a smaller δ for example in section 2.3.5).
Now, for $t < \tau_\delta^\varepsilon$, we can get bounds between $H^\varepsilon = H + \sqrt{\varepsilon}H_1 + \varepsilon H_2$ and K:

Lemma 2.3.4.1. *There exist positive constants $\tilde{C}$ and C_l depending on the mass N and the Hilbert-Schmidt norm of ϕ in the space $H^2 \cap \dot{H}^{-4}$ such that*

$$\frac{1}{2}K - C_l \leqslant H^\varepsilon \leqslant 4K + \tilde{C}. \tag{2.3.33}$$

Proof. From the bounds (2.3.23), (2.3.27), and (2.3.32) we have on H, H_1 and H_2, we get directly that $H^\varepsilon \leqslant 4K + \tilde{C}$, and that

$$H^\varepsilon \geqslant K - 2N^3 - C(C_\phi + 2) - CK^{\frac{1}{2}} \geqslant \frac{1}{2}K - C_l.$$

□

We can also bound $\mathcal{L}^\varepsilon(H^\varepsilon)$, which will lead us to get an energy estimate.

Proposition 2.3.4.1. *There exist B and C two positive constants depending on the mass N and the Hilbert-Schmidt norm of ϕ in the space $H^2 \cap \dot{H}^{-4}$ such that for any ε with $0 < \varepsilon \leqslant 1$, on $[0, \tau_\delta^\varepsilon]$*

$$\mathcal{L}^\varepsilon(H_t^\varepsilon) + \alpha \left\| \sqrt{\varepsilon} V(t) \right\|_{L^2}^2 \leqslant \varepsilon K_t^2 + BK_t + C. \tag{2.3.34}$$

Proof. We start from equation (2.3.30). First

$$\left| 2 \int_{\mathbb{R}} |u|^2 \mathbb{E}_\nu \left[\partial_x z \partial_x \mathcal{M}^{-1} z \right] dx \right| \leqslant C_\phi \left\| u \right\|_{L^2}^2$$

because we know the expression of $\mathcal{M}^{-1} z$ thanks to Lemma 2.3.2.1 and we can compute explicitly $\mathbb{E}_\nu \left[\partial_x z \partial_x \mathcal{M}^{-1} z \right]$. Besides

$$\begin{aligned} \left| 4\sqrt{\varepsilon} \int_{\mathbb{R}} |\partial_x u|^2 (\zeta^\varepsilon + \alpha z^\varepsilon) dx \right| &\leqslant C\sqrt{\varepsilon} \left\| \partial_x u \right\|_{L^2}^2 \left(\left\| \zeta^\varepsilon \right\|_{H^1} + \alpha \left\| z^\varepsilon \right\|_{H^1} \right) \\ &\leqslant C\varepsilon^{\frac{1}{2} - \delta} K, \end{aligned}$$

and

$$\begin{aligned} \left| \sqrt{\varepsilon} \int_{\mathbb{R}} \partial_x |u|^2 \left(\partial_x \zeta^\varepsilon + \alpha \partial_x z^\varepsilon \right) dx \right| &\leqslant C\sqrt{\varepsilon} \left\| u \right\|_{L^2} \left\| \partial_x u \right\|_{L^2} \left(\left\| \partial_x \zeta^\varepsilon \right\|_{L^\infty} + \alpha \left\| \partial_x z^\varepsilon \right\|_{L^\infty} \right) \\ &\leqslant C\sqrt{\varepsilon} \left\| \partial_x u \right\|_{L^2} \left(\left\| \zeta^\varepsilon \right\|_{H^2} + \alpha \left\| z^\varepsilon \right\|_{H^2} \right) \\ &\leqslant C\varepsilon^{\frac{1}{2} - \delta} K^{\frac{1}{2}} \\ &\leqslant C + K. \end{aligned}$$

It remains to estimate two terms in the right-hand side of (2.3.30) . First, by integration by parts:

$$\sqrt{\varepsilon}\left|\int_{\mathbb{R}} |u|^2\partial_x m\left(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon\right)dx\right| \leqslant \sqrt{\varepsilon}\int_{\mathbb{R}}\partial_x|u|^2 m\left(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon\right)dx + \sqrt{\varepsilon}\int_{\mathbb{R}}|u|^2 m\left(\zeta^\varepsilon + \alpha z^\varepsilon\right)dx.$$

We handle the second integral using [SS79, Lemma 2]:

$$\sqrt{\varepsilon}\int_{\mathbb{R}}|u|^2 m\left(\zeta^\varepsilon + \alpha z^\varepsilon\right)dx \leqslant \sqrt{\varepsilon}\left(\|\zeta^\varepsilon\|_{L^\infty} + \alpha\|z^\varepsilon\|_{L^\infty}\right)(K+C) \leqslant C\varepsilon^{\frac{1}{2}-\delta}(K+C)$$

and for the first integral:

$$\begin{aligned}
&\sqrt{\varepsilon}\left|\int_{\mathbb{R}}\partial_x|u|^2 m\left(\partial_x^{-1}\zeta^\varepsilon + \alpha\partial_x^{-1}z^\varepsilon\right)dx\right| \\
&\qquad\leqslant 2\sqrt{\varepsilon}\left(\left\|\partial_x^{-1}\zeta^\varepsilon\right\|_{L^\infty} + \alpha\left\|\partial_x^{-1}z^\varepsilon\right\|_{L^\infty}\right)\|u\|_{L^\infty}\|\partial_x u\|_{L^2}\|m\|_{L^2} \\
&\qquad\leqslant C\varepsilon^{\frac{1}{2}-\delta}\|u\|_{L^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}\|\partial_x u\|_{L^2}\|m\|_{L^2} \\
&\qquad\leqslant C\varepsilon^{\frac{1}{2}-\delta}(N^{\frac{1}{2}}K + N^{\frac{1}{4}}K^{\frac{5}{4}}) \\
&\qquad\leqslant C\varepsilon^{\frac{1}{2}-\delta}(K + K^{\frac{5}{4}}).
\end{aligned}$$

Then, for the last term we have according to Lemma 2.3.2.2, also by integration by parts:

$$\begin{aligned}
&\left|4\varepsilon Re\int_{\mathbb{R}} i\bar{u}\partial_x^2 u\ \mathcal{M}^{-1}\left(\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]\right)dx\right| \\
&\qquad\leqslant \left|4\varepsilon Re\int_{\mathbb{R}} i\bar{u}\partial_x u\partial_x\mathcal{M}^{-1}\left(\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]\right)dx\right| \\
&\qquad\leqslant C\varepsilon\|u\|_{L^2}\|\partial_x u\|_{L^2}\left\|\partial_x\mathcal{M}^{-1}\left(\partial_x z^\varepsilon\partial_x\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu\left[\partial_x z\partial_x\mathcal{M}^{-1}z\right]\right)\right\|_{L^\infty} \\
&\qquad\leqslant C\varepsilon K^{\frac{1}{2}}(C_\phi + \|z^\varepsilon\|^2_{H^3\cap\dot{H}^{-2}} + \|\zeta^\varepsilon\|^2_{H^3\cap\dot{H}^{-2}}) \\
&\qquad\leqslant CK^{\frac{1}{2}}\varepsilon(1+\varepsilon^{-2\delta}).
\end{aligned}$$

Gathering all these estimates we finally get

$$\begin{aligned}
\mathcal{L}^\varepsilon(H^\varepsilon) + \alpha\left\|\sqrt{\varepsilon}V\right\|_{L^2}^2 \leqslant& C_\phi + C\varepsilon^{\frac{1}{2}-\delta}K + C + K + C\varepsilon^{\frac{1}{2}-\delta}(K+C) + C\varepsilon^{\frac{1}{2}-\delta}(K+K^{\frac{5}{4}}) \\
&+ CK^{\frac{1}{2}}\varepsilon(1+\varepsilon^{-2\delta}) \\
\leqslant& \varepsilon K^2 + BK + C,
\end{aligned}$$

since $\delta \leqslant \frac{1}{6}$. □

Proposition 2.3.4.1 allows us to state the two following results which will be used as energy estimates.

Proposition 2.3.4.2. *Let the initial conditions (u_0, m_0, m_1) be in $H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, (z_0, ζ_0) defined in (2.3.10) and $\phi \in \mathcal{L}_2(H^3 \cap \dot{H}^{-3})$. Then, for any $T > 0$, there exists a constant $C(T) > 0$, which is independent of ε such that the solution (u, m) of the system (2.3.6) satisfies*

$$\mathbb{E}[\sup_{t \leqslant \tau^\varepsilon} (K(t))^2] \leqslant C(T),$$

where $K(t)$ is defined in equation (2.3.22) with $u = u(t), m = m(t), \mu = \partial_t m(t)$ solution of (2.3.1) , and τ^ε is a stopping time going to T as ε goes to 0.

Proposition 2.3.4.2 will be very useful for the following sections because it implies weak convergences on the stopped processes. In fact, we can get a bound on the following quantity, which will also be necessary.

Proposition 2.3.4.3. *Let the initial conditions (u_0, m_0, m_1) be in $H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1})$, (z_0, ζ_0) defined in (2.3.10) and $\phi \in \mathcal{L}_2(H^3 \cap \dot{H}^{-3})$.Then, for any $T > 0$, there exists a constant $C(T) > 0$, which is independent of ε, such that the solution (u, m) of the system (2.3.6) satisfies*

$$\mathbb{E}\left[\left\|\sqrt{\varepsilon} V\right\|^2_{L^2([0,\tau^\varepsilon), L^2)}\right] \leqslant C(T),$$

where $V = -\partial_x^{-1}\partial_t m$.

The proofs of these two Propositions are postponed to the Appendix.

2.3.5 Tightness of the processes

Let us fix $T > 0$. Thanks to the bound obtained in Proposition 2.3.4.2, we are able to prove the tightness of our processes, which will allow us according to Skorohod Theorem to get almost sure convergence results at the cost of a change in the probability space.

Let $(u_0, m_0, m_1) \in H^3 \times (H^2 \cap \dot{H}^{-1}) \times (H^1 \cap \dot{H}^{-1})$ and $(z_0, \zeta_0) \in (H^3 \cap \dot{H}^{-2})^2$ be the initial conditions. Then for $\varepsilon > 0$, denote by $(u^\varepsilon, m^\varepsilon)$ the solution to the system (2.3.6) given by Theorem 2.2.2.1. We will use Aldous criterion (see [Bil99, Theorem 16.10]), for which we need a control of the module of continuity. It leads us to consider the following integrals (we recall that $V^\varepsilon = -\partial_x^{-1}\partial_t m^\varepsilon$):

$$M^\varepsilon(t) = \int_0^t m^\varepsilon(s)ds, \qquad \mathcal{V}^\varepsilon(t) = \int_0^t V^\varepsilon(s)ds. \tag{2.3.35}$$

We also fix $\delta \leqslant \frac{1}{6}$ and introduce the space

$$E = \left\{ f \in C(\mathbb{R}, H^3 \cap \dot{H}^{-3}), \quad \sup_{t\in\mathbb{R}} \frac{\|f(t)\|_{H^3\cap\dot{H}^{-3}}}{1+|t|^\delta} < \infty \right\}. \tag{2.3.36}$$

According to Proposition 2.3.2.1 and more precisely to (2.3.11) we know that the process (z,ζ) solution to (2.3.7) takes values in $E\times E$ almost surely. We need the Skorohod copies of (z,ζ) to belong to this space in order to recover weak convergences in the Skorohod space. Our aim is to prove the following Proposition.

Proposition 2.3.5.1. *Let $(u_0, m_0, m_1) \in H^3 \times (H^2 \cap \dot{H}^{-1}) \times (H^1 \cap \dot{H}^{-1})$ and (z_0, ζ_0) be the initial conditions defined in (2.3.10), and $\phi \in \mathcal{L}_2(H^3 \cap \dot{H}^{-4})$. Denote by $(u^\varepsilon, m^\varepsilon)$ the solution of (2.3.6), and (z,ζ) the solution of (2.3.7). Then the sequence $(u^\varepsilon, M^\varepsilon, \sqrt{\varepsilon}\mathcal{V}^\varepsilon, z, \zeta)$ where $M^\varepsilon, \mathcal{V}^\varepsilon$ are defined in (2.3.35) is tight in the space $\mathcal{C}^0([0,T], H^s_{loc}) \times \mathcal{C}^0([0,T], H^{-\sigma}_{loc}) \times H^s([0,T], H^{-\sigma}_{loc}) \times E \times E$ for any $s < 1, \sigma > 0$.*

In order to prove this result, we first study the process u^ε, for which we want to apply Aldous criterion that can be found in a finite dimensional case in [Bil99, Thorem 16.10]. We use it in an infinite dimensional case, and one can check that the proof in [Bil99, Thorem 16.10] is still valid. Here again we will need to use the Perturbed Test function method to show that u^ε satisfies this criterion.

Aldous criterion for u^ε

In this section we show that the process $(u^\varepsilon)_\varepsilon$ is tight in the space $\mathcal{C}^0([0,T], H^s_{loc})$. For this purpose, we use the Aldous criterion which can be found in [Bil99]. In the first step we need to verify two statements, which are Lemma 2.3.5.1 and Proposition 2.3.5.2.

Lemma 2.3.5.1. *Let $\varepsilon > 0$ and denote by $(u^\varepsilon, m^\varepsilon)$ the solution of (2.3.6) given by Theorem 2.2.2.1. Then*

$$\lim_{R\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{H^1} > R \right) = 0. \tag{2.3.37}$$

Proof. We know thanks to the definition of K in (2.3.22) and Proposition 2.3.4.2 that

$$\mathbb{P}\left(\sup_{t\in[0,\tau^\varepsilon]}\|u^\varepsilon(t)\|_{H^1}^2 > R\right) \leqslant \frac{C(T)}{R}$$

with C independent of ε, and where τ^ε is defined in Proposition 2.3.4.2.
Thus, writing

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]}\|u^\varepsilon(t)\|_{H^1}^2 > R\right) \leqslant & \mathbb{P}\left(\sup_{t\in[0,\tau^\varepsilon]}\|u^\varepsilon(t)\|_{H^1}^2 > R\right) + \mathbb{P}\left(\tau^\varepsilon < T\right) \\ \leqslant & \frac{C(T)}{R} + \mathbb{P}\left(\tau^\varepsilon < T\right).\end{aligned} \tag{2.3.38}$$

We can see that passing to the lim sup in ε makes the second term of the right-hand-side of the equation vanish according to Lemma 2.4.0.6, while it does not impact the first term. Finally, taking the limit when R goes to ∞, we get (2.3.37). $\square$

Proposition 2.3.5.2. *Let $\varepsilon > 0$ and denote by $(u^\varepsilon, m^\varepsilon)$ the solution of (2.3.6) given by Theorem 2.2.2.1. For every λ, η, there exist δ_0, ε_0 such that for $\bar{\delta} < \delta_0$ and $\varepsilon < \varepsilon_0$, if τ is a stopping time with $\tau + \delta_0 \leqslant T$ a.s. , then we have*

$$\mathbb{P}\left(\left\|u^\varepsilon(\tau+\bar{\delta}) - u^\varepsilon(\tau)\right\|_{H^{-\frac{1}{2}}} > \lambda\right) \leqslant \eta. \tag{2.3.39}$$

In order to prove Proposition 2.3.5.2, we will need two lemmas, whose proofs will use the Perturbed Test Function method.

Lemma 2.3.5.2. *For any $\eta > 0$, there exist δ_0, ε_0 such that for any bounded stopping time τ with $\tau + \delta_0 \leqslant T$, for $\bar{\delta} < \delta_0$ and $\varepsilon < \varepsilon_0$, we have*

$$\mathbb{E}\left[\left\|u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)\right\|_{H^{-\frac{1}{2}}}^2 - \|u^\varepsilon(\tau\wedge\tau^\varepsilon)\|_{H^{-\frac{1}{2}}}^2\right] \leqslant \eta. \tag{2.3.40}$$

Proof. We apply the perturbed test function method to $\varphi(u) = \|u\|_{H^{-\frac{1}{2}}}^2$, for any $u \in H^3$. We compute the infinitesimal generator $\mathcal{L}^\varepsilon$ applied to φ and we get

$$\mathcal{L}^\varepsilon\varphi(u) = 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, i\partial_x^2 u - im^\varepsilon u\rangle + \frac{2}{\sqrt{\varepsilon}}\langle(1-\partial_x^2)^{-\frac{1}{2}}u, -iz^\varepsilon u\rangle. \tag{2.3.41}$$

To deal with the term of order $-\frac{1}{2}$ in ε, we choose the first corrector

$$\begin{aligned}\varphi_1(u) &= 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, iu\mathcal{M}^{-1}z^\varepsilon\rangle \\ &= 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, iu((\partial_x^2)^{-1}\zeta^\varepsilon + \alpha(\partial_x^2)^{-1}z^\varepsilon)\rangle.\end{aligned} \tag{2.3.42}$$

We can already bound φ_1 on $[0,\tau^\varepsilon)$: for all $t<\tau^\varepsilon$

$$\begin{aligned}\varphi_1(u(t)) &\leqslant C\|u(t)\|_{H^{-1}}\|u(t)\|_{L^2}\left(\|\zeta^\varepsilon(t)\|_{\dot H^{-2}\cap\dot H^{-1}} + \alpha\|z^\varepsilon(t)\|_{\dot H^{-2}\cap\dot H^{-1}}\right) \\ &\leqslant C(T)\|u(t)\|_{L^2}^2\varepsilon^{-\delta} = C(T)\varepsilon^{-\delta}.\end{aligned} \tag{2.3.43}$$

If we apply the infinitesimal generator $\mathcal{L}^\varepsilon$ to $\varphi(u)+\sqrt{\varepsilon}\varphi_1(u)$ we get

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi+\sqrt{\varepsilon}\varphi_1)(u) =& 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, i\partial_x^2u - im^\varepsilon u\rangle \\ &+ 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, uz^\varepsilon\mathcal{M}^{-1}z^\varepsilon\rangle - 2\langle i(1-\partial_x^2)^{-\frac{1}{2}}(z^\varepsilon u), iu\mathcal{M}^{-1}z^\varepsilon\rangle \\ &+ 2\sqrt{\varepsilon}\Big[\langle(1-\partial_x^2)^{-\frac{1}{2}}u, (m^\varepsilon u - \partial_x^2u)\mathcal{M}^{-1}z^\varepsilon\rangle \\ &+\langle i(1-\partial_x^2)^{-\frac{1}{2}}(\partial_x^2u - m^\varepsilon u), iu\mathcal{M}^{-1}z^\varepsilon\rangle\Big].\end{aligned}$$

Here again, we have to deal with a term which seems to be of order 0 in ε, but the fact that z^ε grows at the speed $\varepsilon^{-\delta}$ makes us introduce a second corrector φ_2 such that

$$\begin{aligned}\mathcal{M}\varphi_2(u) =& 2\langle(1-\partial_x^2)^{-\frac{1}{2}}(z^\varepsilon u), u\mathcal{M}^{-1}z^\varepsilon\rangle - 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, uz^\varepsilon\mathcal{M}^{-1}z^\varepsilon\rangle \\ &- \mathbb{E}_\nu\Big[2\langle(1-\partial_x^2)^{-\frac{1}{2}}(zu), u\mathcal{M}^{-1}z\rangle - 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, uz\mathcal{M}^{-1}z\rangle\Big].\end{aligned} \tag{2.3.44}$$

We need to get an estimate on φ_2. Thus we write it using Lemma 2.3.2.2, which allows us to have an explicit expression for $\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right)$:

$$\begin{aligned}\varphi_2(u) =& -2\langle(1-\partial_x^2)^{-\frac{1}{2}}u, u\mathcal{M}^{-1}(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu\left[z^\varepsilon\mathcal{M}^{-1}z^\varepsilon\right])\rangle \\ &- \int_0^\infty\int_{\mathbb{R}}\bar u(x)(1-\partial_x^2)^{-\frac{1}{2}}\left(u(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon))_1\right) \\ &\times\left[(\partial_x^2)^{-1}(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon))_2 + \alpha(\partial_x^2)^{-1}(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon))_1\right](x) \\ &-\bar u(x)\sum_{k\in\mathbb{N}}\int_t^\infty(1-\partial_x^2)^{-\frac{1}{2}}\left(u(S_\alpha(t)(0,\phi e_k))_1\right) \\ &\times\left[(\partial_x^2)^{-1}(S_\alpha(t)(0,\phi e_k))_2 + \alpha(\partial_x^2)^{-1}(S_\alpha(t)(0,\phi e_k))_1\right](x)dsdxdt.\end{aligned}$$

The first term of the right-hand side of the above equation is bounded for $t\in[0,\tau^\varepsilon)$

thanks to Proposition 2.3.2.2 by

$$\begin{aligned}C(T)\left\|u\right\|_{H^{-1}}\left\|u\right\|_{L^2}&\left\|\mathcal{M}^{-1}(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon-\mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right])\right\|_{L^\infty}\\&\leqslant C(T)\left\|u\right\|_{L^2}^2C_\phi(1\left\|z^\varepsilon\right\|_{H^1\cap\dot{H}^{-3}}^2+\left\|\zeta^\varepsilon\right\|_{H^1\cap\dot{H}^{-3}}^2)\\&\leqslant C(T)\varepsilon^{-2\delta}\end{aligned}$$

while if we denote by II the integral in the right-hand side above, we have

$$\begin{aligned}II\leqslant\left\|u\right\|_{L^2}^2\Bigg[\int_0^\infty&\left\|S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon)_1\right\|_{L^\infty}\left(\left\|(\partial_x^2)^{-1}S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon)_2\right\|_{L^\infty}\right.\\&\left.+\alpha\left\|(\partial_x^2)^{-1}(S_\alpha(t)(z^\varepsilon,\zeta^\varepsilon))_1\right\|_{L^\infty}\right)\\-\sum_{k\in\mathbb{N}}\int_t^\infty&\left\|S_\alpha(t)(0,\phi e_k)_1\right\|_{L^\infty}\left(\left\|(\partial_x^2)^{-1}S_\alpha(t)(0,\phi e_k)_2\right\|_{L^\infty}\right.\\&\left.+\alpha\left\|(\partial_x^2)^{-1}S_\alpha(t)(0,\phi e_k)_1\right\|_{L^\infty}\right)dsdt\Bigg].\end{aligned}$$

Then, using the Sobolev embedding $H^1\hookrightarrow L^\infty$ and Lemma 2.1.0.6, we get

$$\begin{aligned}II&\leqslant C(T)\left\|u\right\|_{L^2}^2C_\phi(1+\left\|z^\varepsilon\right\|_{H^1\cap\dot{H}^{-3}}^2+\left\|\zeta^\varepsilon\right\|_{H^1\cap\dot{H}^{-3}}^2)\\&\leqslant C(T)\varepsilon^{-2\delta},\end{aligned}$$

where C_ϕ depends on the Hilbert-Schmidt norm of ϕ in the space $H^3\cap\dot{H}^{-4}$. Finally we have an estimate on φ_2 for $t\in[0,\tau^\varepsilon)$:

$$\varphi_2(u)\leqslant C(T)\varepsilon^{-2\delta},\tag{2.3.45}$$

where C depends on the L^2 norm of u and the Hilbert-Schmidt norm of ϕ.

Moreover, we have

$$\begin{aligned}D_u\varphi_2(u)\cdot h=&2\Big[\langle h,\mathcal{M}^{-1}\left(z^\varepsilon(1-\partial_x^2)^{-\frac12}(u\mathcal{M}^{-1}z^\varepsilon)-\mathbb{E}_\nu[z(1-\partial_x^2)^{-\frac12}(u^\varepsilon\mathcal{M}^{-1}z)]\right)\rangle\\&+\langle\mathcal{M}^{-1}\left((1-\partial_x^2)^{-\frac12}(z^\varepsilon u)\mathcal{M}^{-1}z^\varepsilon-\mathbb{E}_\nu[(1-\partial_x^2)^{-\frac12}(z^\varepsilon u)\mathcal{M}^{-1}z]\right),h\rangle\\&-\langle(1-\partial_x^2)^{-\frac12}h,u\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon-\mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right)\rangle\\&-\langle(1-\partial_x^2)^{-\frac12}u,h\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon-\mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right)\rangle\Big]\quad.\end{aligned}\tag{2.3.46}$$

Defining $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$, we know that

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi^\varepsilon)(u^\varepsilon) =& \frac{1}{\sqrt{\varepsilon}}\left(D_u\varphi(-iz^\varepsilon u^\varepsilon) + \mathcal{M}\varphi_1(u^\varepsilon)\right)\Big(D_u\varphi(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon) \\ &+ D_u\varphi_1(-iz^\varepsilon u^\varepsilon) + \mathcal{M}\varphi_2(u^\varepsilon)\Big) \\ &+ \sqrt{\varepsilon}\left(D_u\varphi_1(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon) + D_u\varphi_2(-iz^\varepsilon u^\varepsilon)\right) \\ &+ \varepsilon D_u\varphi_2(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon)\end{aligned}$$

which gives us

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi^\varepsilon)(u^\varepsilon) =& 2\langle(1-\partial_x^2)^{-\frac{1}{2}}u^\varepsilon, i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon\rangle \\ &+ 2\mathbb{E}_\nu\Big[\langle(1-\partial_x^2)^{-\frac{1}{2}}u^\varepsilon, u^\varepsilon z\mathcal{M}^{-1}z\rangle - 2\langle(1-\partial_x^2)^{-\frac{1}{2}}(z^\varepsilon u^\varepsilon), u^\varepsilon\mathcal{M}^{-1}z\rangle\Big] \\ &+ 2\sqrt{\varepsilon}\Big[\langle(1-\partial_x^2)^{-\frac{1}{2}}u^\varepsilon, (m^\varepsilon u^\varepsilon - \partial_x^2 u^\varepsilon)\mathcal{M}^{-1}z^\varepsilon\rangle \\ &+ \langle i(1-\partial_x^2)^{-\frac{1}{2}}(\partial_x^2 u^\varepsilon - m^\varepsilon u^\varepsilon), iu^\varepsilon\mathcal{M}^{-1}z^\varepsilon\rangle\Big] \\ &+ \sqrt{\varepsilon}D_u\varphi_2(-iz^\varepsilon u^\varepsilon) + \varepsilon D_u\varphi_2(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon).\end{aligned}$$

Thanks to the Sobolev embedding $H^1 \hookrightarrow L^\infty$ and Lemma 2.1.0.6 we can state that

$$\begin{aligned}\sqrt{\varepsilon}|D_u\varphi_2(-iz^\varepsilon u^\varepsilon)| &\leqslant C(T)\sqrt{\varepsilon}\,\|u^\varepsilon\|_{L^2}^2\,\|z^\varepsilon\|_{H^1}\,C_\phi(1+\|z^\varepsilon\|^2_{H^1\cap\dot{H}^{-3}} + \|\zeta^\varepsilon\|^2_{H^1\cap\dot{H}^{-3}}) \\ &\leqslant C(T)\varepsilon^{\frac{1}{2}-3\delta}.\end{aligned}$$

For the term $\varepsilon D_u\varphi_2(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon)$, we will have to deal with terms such as

$$\langle(1-\partial_x^2)^{-\frac{1}{2}}(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon), u^\varepsilon\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right)\rangle.$$

We bound them by:

$$(\|u^\varepsilon\|_{H^1}\|u^\varepsilon\|_{L^2} + \|u^\varepsilon\|_{H^1}\|m^\varepsilon\|_{L^2}\|u^\varepsilon\|_{L^2})\left\|\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right)\right\|_{L^\infty}$$

in order to have (using the conservation of the L^2 norm of u^ε)

$$\begin{aligned}\varepsilon D_u\varphi_2(i\partial_x^2 u^\varepsilon - im^\varepsilon u^\varepsilon) &\leqslant C(T)\varepsilon(\|u^\varepsilon\|^2_{H^1} + \|m^\varepsilon\|^2_{L^2})C_\phi(1+\|z^\varepsilon\|^2_{H^1\cap\dot{H}^{-3}} + \|\zeta^\varepsilon\|^2_{H^1\cap\dot{H}^{-3}}) \\ &\leqslant C(T)(\|u^\varepsilon\|^2_{H^1} + \|m^\varepsilon\|^2_{L^2})\varepsilon^{1-2\delta}.\end{aligned}$$

We get:

$$\begin{aligned}|\mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon)| \leqslant & C\|u^\varepsilon\|_{H^1} + \|m^\varepsilon\|_{L^2}\|u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^{-1}} + C_\phi\|u^\varepsilon\|_{L^2}\|u^\varepsilon\|_{H^{-1}} \\ & + C(T)\sqrt{\varepsilon}(\|u^\varepsilon\|_{L^2}\|u^\varepsilon\|_{H^1} + \|m^\varepsilon\|_{L^2}\|u^\varepsilon\|_{L^\infty}\|u^\varepsilon\|_{H^{-1}})(\|\zeta^\varepsilon\|_{\dot H^{-2}} + \|z^\varepsilon\|_{\dot H^{-2}}) \\ & + C(T)\varepsilon^{\frac12-3\delta} + C(\|u^\varepsilon\|^2_{H^1} + \|m^\varepsilon\|^2_{L^2})\varepsilon^{1-2\delta}\end{aligned}$$

and taking the supremum on $[0,\tau^\varepsilon)$:

$$|\mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon)| \leqslant C(T)\left(1 + \|u^\varepsilon\|^2_{L^\infty([0,\tau^\varepsilon),H^1)} + \|m^\varepsilon\|^2_{L^\infty([0,\tau^\varepsilon),L^2)}\right)\left(1 + \varepsilon^{\frac12-3\delta} + \varepsilon^{1-2\delta}\right). \tag{2.3.47}$$

Now we use the bound $|\varphi^\varepsilon(u^\varepsilon)| \leqslant \varphi(u^\varepsilon) + C\varepsilon^{\frac12-\delta} + C\varepsilon^{1-2\delta}$ (where C depends on T, on the L^2 norm of u^ε and the Hilbert-Schmidt norm of ϕ in the space $H^3\cap\dot H^{-4}$) and the fact that

$$\varphi^\varepsilon(u^\varepsilon(t)) - \varphi^\varepsilon(u_0) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon(s))ds$$

is a martingale to compute:

$$\begin{aligned}&\mathbb{E}\left[\left\|u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)\right\|^2_{H^{-\frac12}} - \|u^\varepsilon(\tau\wedge\tau^\varepsilon)\|^2_{H^{-\frac12}}\right] \\ &\qquad = \mathbb{E}\left[\varphi(u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)) - \varphi(u^\varepsilon(\tau\wedge\tau^\varepsilon))\right] \\ &\qquad \leqslant \mathbb{E}\left[\varphi^\varepsilon(u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)) - \varphi^\varepsilon(u^\varepsilon(\tau\wedge\tau^\varepsilon))\right] + C\varepsilon^{\frac12-\delta} + C\varepsilon^{1-2\delta} \\ &\qquad \leqslant C\mathbb{E}\left[\left(1 + \|u^\varepsilon\|^2_{L^\infty([0,\tau^\varepsilon),H^1)} + \|m^\varepsilon\|^2_{L^\infty([0,\tau^\varepsilon),L^2)}\right)\right]\bar\delta\left(1+\varepsilon^{\frac12-3\delta}+\varepsilon^{1-2\delta}\right) \\ &\qquad\qquad + C\varepsilon^{\frac12-\delta} + C\varepsilon^{1-2\delta} \\ &\qquad \leqslant C\bar\delta\left(1+\varepsilon^{\frac12-3\delta}+\varepsilon^{1-2\delta}\right) + C\varepsilon^{\frac12-\delta} + C\varepsilon^{1-2\delta},\end{aligned}$$

where we have bounded $\mathbb{E}\left[\left(1 + \|u^\varepsilon\|^2_{L^\infty([0,\tau^\varepsilon),H^1)} + \|m^\varepsilon\|^2_{L^\infty([0,\tau^\varepsilon),L^2)}\right)\right]$ applying the Proposition 2.3.4.2. Finally, for $\bar\delta$ and ε small enough we get:

$$\mathbb{E}\left[\left\|u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)\right\|^2_{H^{-\frac12}} - \|u^\varepsilon(\tau\wedge\tau^\varepsilon)\|^2_{H^{-\frac12}}\right] \leqslant \eta.$$

□

The second lemma states a similar result.

Lemma 2.3.5.3. *Let $T>0$. For any $\eta>0$, and for any bounded stopping time τ, there* *exist* δ_0, ε_0 *such that for* $\bar{\delta} < \delta_0$ *and* $\varepsilon < \varepsilon_0$

$$\mathbb{E}\left[\left\langle u^\varepsilon((\tau + \bar{\delta}) \wedge \tau^\varepsilon) - u^\varepsilon(\tau \wedge \tau^\varepsilon), u^\varepsilon(\tau \wedge \tau^\varepsilon)\right\rangle_{H^{-\frac{1}{2}}}\right] \leqslant \eta. \tag{2.3.48}$$

Proof. We apply the Pertubed Test Function method to $\varphi(u) = \langle u, h\rangle_{H^{-\frac{1}{2}}}$ for a fixed function $h \in H^{-\frac{1}{2}}$.
We have

$$\mathcal{L}^\varepsilon\varphi(u) = \langle i(1-\partial_x^2)^{-\frac{1}{2}}\partial_x^2 u, h\rangle_{L^2} - \langle im^\varepsilon u, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} - \frac{1}{\sqrt{\varepsilon}}\langle iz^\varepsilon u, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2},$$

so we choose a first corrector φ_1 such that

$$\begin{aligned}\varphi_1(u) &= \langle iu\mathcal{M}^{-1}z^\varepsilon, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \\ &= \langle iu((\partial_x^2)^{-1}\zeta^\varepsilon + \alpha(\partial_x^2)^{-1}z^\varepsilon), (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2}.\end{aligned} \tag{2.3.49}$$

We have the bound

$$|\varphi_1(u)| \leqslant C(T)\left\|u\right\|_{L^2}\left\|h\right\|_{L^2}\left(\|\zeta^\varepsilon\|_{L^2\cap\dot{H}^{-2}} + \|z^\varepsilon\|_{L^2\cap\dot{H}^{-2}}\right).$$

Then we get

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi + \sqrt{\varepsilon}\varphi_1)(u) =&\langle i(1-\partial_x^2)^{-\frac{1}{2}}\partial_x^2 u, h\rangle_{L^2} - \langle im^\varepsilon u, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \\ &+\langle uz^\varepsilon\mathcal{M}^{-1}z^\varepsilon, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \\ &+\sqrt{\varepsilon}\langle(m^\varepsilon u - \partial_x^2 u)\mathcal{M}^{-1}z^\varepsilon, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2},\end{aligned}$$

and we choose a second corrector φ_2 such that

$$\varphi_2(u) = -\langle u\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right), (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2}, \tag{2.3.50}$$

which we can bound on $[0, \tau^\varepsilon)$ according to Lemma 2.1.0.6:

$$\begin{aligned}|\varphi_2(u^\varepsilon)| &\leqslant \|u^\varepsilon\|_{H^{-1}}\left\|h\right\|_{L^2}\left\|\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right)\right\|_{L^\infty} \\ &\leqslant \|u^\varepsilon\|_{L^2}\left\|h\right\|_{L^2}C_\phi(1 + \|z^\varepsilon\|^2_{H^1\cap\dot{H}^{-3}} + \|\zeta^\varepsilon\|^2_{H^1\cap\dot{H}^{-3}}) \\ &\leqslant C(T)\left\|h\right\|_{L^2}\varepsilon^{-2\delta}.\end{aligned}$$

Finally, if we define $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$, we get

$$\begin{aligned}
\mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon) =& \langle i(1-\partial_x^2)^{-\frac{1}{2}}\partial_x^2 u^\varepsilon, h\rangle_{L^2} - \langle im^\varepsilon u^\varepsilon, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \\
&+ \langle u^\varepsilon \mathbb{E}_\nu[z\mathcal{M}^{-1}z], (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \\
&+ \sqrt{\varepsilon}\Big[\langle (m^\varepsilon u^\varepsilon - \partial_x^2 u^\varepsilon)\mathcal{M}^{-1}z^\varepsilon, (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \\
&+ \langle iu^\varepsilon z^\varepsilon \mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right), (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2}\Big] \\
&+ \varepsilon\langle (im^\varepsilon u^\varepsilon - i\partial_x^2 u^\varepsilon)\mathcal{M}^{-1}\left(z^\varepsilon\mathcal{M}^{-1}z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z]\right), (1-\partial_x^2)^{-\frac{1}{2}}h\rangle_{L^2} \quad .
\end{aligned} \tag{2.3.51}$$

We can bound this quantity on $[0,\tau^\varepsilon)$ thanks to Lemma 2.1.0.6 and Proposition 2.3.2.1:

$$\begin{aligned}
|\mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon)| \leqslant& C\left\|u^\varepsilon\right\|_{H^1}\left\|h\right\|_{L^2} + C\left\|u^\varepsilon\right\|_{H^1}\left\|m^\varepsilon\right\|_{L^2}\left\|h\right\|_{L^2} \\
&+ C_\phi\left\|u^\varepsilon\right\|_{L^2}\left\|h\right\|_{L^2} \\
&+ C\sqrt{\varepsilon}\left\|u^\varepsilon\right\|_{H^1}(1+\left\|m^\varepsilon\right\|_{L^2})\left\|h\right\|_{L^2}(\left\|\zeta^\varepsilon\right\|_{L^2\cap\dot{H}^{-2}} + \left\|z^\varepsilon\right\|_{L^2\cap\dot{H}^{-2}}) \\
&+ C_\phi\sqrt{\varepsilon}\left\|u^\varepsilon\right\|_{L^2}\left\|h\right\|_{L^2}\left\|z^\varepsilon\right\|_{H^1}(1+\left\|z^\varepsilon\right\|^2_{H^1\cap\dot{H}^{-3}} + \left\|\zeta^\varepsilon\right\|^2_{H^1\cap\dot{H}^{-3}}) \\
&+ C_\phi\varepsilon\left\|u^\varepsilon\right\|_{H^1}(1+\left\|m^\varepsilon\right\|_{L^2})\left\|h\right\|_{L^2}(1+\left\|z^\varepsilon\right\|^2_{H^1\cap\dot{H}^{-3}} + \left\|\zeta^\varepsilon\right\|^2_{H^1\cap\dot{H}^{-3}}) \\
\\
\leqslant& C\left\|u^\varepsilon\right\|_{H^1}\left\|h\right\|_{L^2} + C\left\|u^\varepsilon\right\|_{H^1}\left\|m^\varepsilon\right\|_{L^2}\left\|h\right\|_{L^2} + C_\phi\left\|u^\varepsilon\right\|_{L^2}\left\|h\right\|_{L^2} \\
&+ C\left\|u^\varepsilon\right\|_{H^1}(1+\left\|m^\varepsilon\right\|_{L^2})\left\|h\right\|_{L^2}\varepsilon^{\frac{1}{2}-\delta} \\
&+ C_\phi\left\|u^\varepsilon\right\|_{L^2}\left\|h\right\|_{L^2}\varepsilon^{\frac{1}{2}-3\delta} + C_\phi\left\|u^\varepsilon\right\|_{H^1}(1+\left\|m^\varepsilon\right\|_{L^2})\left\|h\right\|_{L^2}\varepsilon^{1-2\delta},
\end{aligned}$$

and taking the supremum on $[0,\tau^\varepsilon)$, we have thanks to Proposition 2.3.4.2:

$$\mathbb{E}\left[|\mathcal{L}^\varepsilon\varphi^\varepsilon(u)|\right] \leqslant C(1+\varepsilon^{\frac{1}{2}-3\delta} + \varepsilon^{1-2\delta}) \tag{2.3.52}$$

where C is a constant depending on T, on the initial conditions and on the Hilbert-Schmidt norm of ϕ. In this last estimate the dependence on $\left\|h\right\|_{L^2}$ is not written because we will use this estimate with $h = u^\varepsilon(\tau\wedge\tau^\varepsilon)$ and the conservation of the mass allows us to bound this term by C.

Now taking $h = u^\varepsilon(\tau \wedge \tau^\varepsilon)$, we have

$$\begin{aligned}|\varphi^\varepsilon(u^\varepsilon)| &\leqslant |\varphi(u^\varepsilon)| + C\varepsilon^{\frac{1}{2}-\delta}\|u^\varepsilon\|_{L^2}^2 + C_\phi\varepsilon^{1-2\delta}\|u^\varepsilon\|_{L^2}^2\\ &\leqslant |\varphi(u^\varepsilon)| + C\varepsilon^{\frac{1}{2}-\delta} + C_\phi\varepsilon^{1-2\delta}\end{aligned}$$

and we know that

$$X_t := \varphi^\varepsilon(u^\varepsilon(t)) - \varphi^\varepsilon(u_0) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon(s))ds$$

is a martingale.
Thus

$$\begin{aligned}\mathbb{E}\left[\langle u^\varepsilon((\tau+\bar\delta)\ \wedge\tau^\varepsilon) - u^\varepsilon(\tau\wedge\tau^\varepsilon), u^\varepsilon(\tau\wedge\tau^\varepsilon)\rangle_{H^{-\frac{1}{2}}}\right] &=\\ &\mathbb{E}\left[\varphi(u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)) - \varphi(u^\varepsilon(\tau\wedge\tau^\varepsilon))\right]\\ &\leqslant \mathbb{E}\left[\varphi^\varepsilon(u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)) - \varphi^\varepsilon(u^\varepsilon(\tau\wedge\tau^\varepsilon))\right] + C\varepsilon^{\frac{1}{2}-\delta} + C_\phi\varepsilon^{1-2\delta}\\ &\leqslant \mathbb{E}\left[\mathbb{E}\left[\varphi^\varepsilon(u^\varepsilon((\tau+\bar\delta)\wedge\tau^\varepsilon)) - \varphi^\varepsilon(u^\varepsilon(\tau\wedge\tau^\varepsilon)) \mid \mathcal{F}_{\tau\wedge\tau^\varepsilon}\right]\right]\\ &+ C\varepsilon^{\frac{1}{2}-\delta} + C_\phi\varepsilon^{1-2\delta}\\ &\leqslant \mathbb{E}\left[\mathbb{E}\left[\int_{\tau\wedge\tau^\varepsilon}^{(\tau+\bar\delta)\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon(s))ds \mid \mathcal{F}_{\tau\wedge\tau^\varepsilon}\right]\right]\\ &+ \mathbb{E}\left[\mathbb{E}\left[X_{(\tau+\bar\delta)\wedge\tau^\varepsilon} - X_{\tau\wedge\tau^\varepsilon} \mid \mathcal{F}_{\tau\wedge\tau^\varepsilon}\right]\right] + C\varepsilon^{\frac{1}{2}-\delta} + C_\phi\varepsilon^{1-2\delta}\\ &\leqslant C(1+\varepsilon^{\frac{1}{2}-3\delta} + \varepsilon^{1-2\delta})\bar\delta + C\varepsilon^{\frac{1}{2}-\delta} + C_\phi\varepsilon^{1-2\delta}\\ &\leqslant \eta\end{aligned}$$

for $\varepsilon, \bar\delta$ small enough. □

Thanks to Lemmas 2.3.5.2 and 2.3.5.3, we are now able to prove Proposition 2.3.5.2.

Proof of Proposition 2.3.5.2. Let λ and η be two positive numbers, and let τ be a stopping time such that $\tau < T - \delta$. Then:

$$\begin{aligned}
\mathbb{P}\left(\left\|u^\varepsilon(\tau+\bar{\delta})-u^\varepsilon(\tau)\right\|_{H^{-\frac12}}>\lambda\right) &= \mathbb{P}\left(\left\{\left\|u^\varepsilon(\tau+\bar{\delta})-u^\varepsilon(\tau)\right\|_{H^{-\frac12}}>\lambda\right\}\cap\{\tau^\varepsilon\geqslant T\}\right)\\
&\quad+\mathbb{P}\left(\left\{\left\|u^\varepsilon(\tau+\bar{\delta})-u^\varepsilon(\tau)\right\|_{H^{-\frac12}}>\lambda\right\}\cap\{\tau^\varepsilon< T\}\right)\\
&\leqslant \mathbb{P}\left(\left\|u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)-u^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}>\lambda\right)\\
&\quad+\mathbb{P}(\tau^\varepsilon<T)\\
&\leqslant \mathbb{P}\left(\left\|u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)-u^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|_{H^{-\frac12}}>\lambda\right)+\frac{\eta}{2}
\end{aligned}$$

thanks to Lemma 2.4.0.6 for ε small enough. Then we use the following identity:

$$\begin{aligned}
&\mathbb{E}\left[\left\|u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)-u^\varepsilon(\tau\wedge\tau^\varepsilon)\right\|^2_{H^{-\frac12}}\right]\\
&\quad=\mathbb{E}\left[\left\|u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)\right\|^2_{H^{-\frac12}}-\|u^\varepsilon(\tau\wedge\tau^\varepsilon)\|^2_{H^{-\frac12}}\right]\\
&\quad-2\mathbb{E}\left[\langle u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)-u^\varepsilon(\tau\wedge\tau^\varepsilon),u^\varepsilon(\tau\wedge\tau^\varepsilon)\rangle_{H^{-\frac12}}\right].
\end{aligned}$$

Thus thanks to Markov inequality we get

$$\begin{aligned}
&\mathbb{P}\Big(\left\|u^\varepsilon(\tau+\bar{\delta})-u^\varepsilon(\tau)\right\|^2_{H^{-\frac12}}>\lambda^2\Big)\\
&\qquad\leqslant\frac{1}{\lambda^2}\left(\mathbb{E}\left[\left\|u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)\right\|^2_{H^{-\frac12}}-\|u^\varepsilon(\tau\wedge\tau^\varepsilon)\|^2_{H^{-\frac12}}\right]\right.\\
&\qquad\left.+2\mathbb{E}\left[\langle u^\varepsilon((\tau+\bar{\delta})\wedge\tau^\varepsilon)-u^\varepsilon(\tau\wedge\tau^\varepsilon),u^\varepsilon(\tau\wedge\tau^\varepsilon)\rangle_{H^{-\frac12}}\right]\right)+\frac{\eta}{2}\\
&\qquad\leqslant\eta
\end{aligned}$$

applying Lemmas 2.3.5.2 and 2.3.5.3, with $\varepsilon, \bar{\delta}$ small enough. □

Finally, we can prove the following result.

Proposition 2.3.5.3. *The sequence (u^ε) is tight in $\mathcal{C}^0([0,T],H^s_{loc})$ for $s<1$.*

Proof. Lemma 2.3.5.1 and Proposition 2.3.5.2 allow us to prove by interpolation that (u^ε) satisfies (2.3.37) and (2.3.39) in any space H^s_{loc} with $s<1$, hence, according to Aldous criterion the result is proved. □

The other processes

Now that we have explained how to get the tightness of the process $(u^\varepsilon)_\varepsilon$, we want to briefly explain how to get the tightness for the other quantities. For the sequence $(M^\varepsilon)_\varepsilon$ defined in (2.3.35) we use again Aldous criterion and Proposition 2.3.4.2 coupled with the compact embedding $L^2 \hookrightarrow H^{-\sigma}_{loc}$ to get the tightness in $\mathcal{C}^0([0,T], H^{-\sigma}_{loc})$. For the sequence $(\sqrt{\varepsilon}\mathcal{V}^\varepsilon)_\varepsilon$ defined in (2.3.35), we also use Propostion 2.3.4.2 and the compact embedding $H^1([0,T], L^2) \hookrightarrow H^s([0,T], H^{-\sigma}_{loc})$ to conclude. Finally, z and ζ are constant random processes, which both belong almost surely to E defined in (2.3.36), according to the proof of Proposition 2.3.2.1 thus the tightness is established.

2.3.6 Convergence to the Stochastic Schrödinger equation

In this subsection, we prove the main result of this paper, which is Theorem 2.3.0.1. In order to prove this theorem, we have to prove some weak convergence results, which are similar to those that we can find in [AA88]. Then we will be able to pass to the limit $\varepsilon \to 0$ in the martingale problem. We know that $(u^\varepsilon, n^\varepsilon)$ is a solution of the system (2.0.3) if and only if $(u^\varepsilon, m^\varepsilon)$, where $m^\varepsilon = n^\varepsilon - Z^\varepsilon$, with Z^ε solution of (2.2.11), is a solution of the system (2.3.6). We will work with this latter system. The regularities on the initial data and on ϕ assumed in Theorem 2.3.0.1 ensure that the process $(z,\zeta) \in (H^3 \cap \dot{H}^{-3})$ and that the computations made below are justified.

Weak convergences

The previous section has proved that the sequence $(u^\varepsilon, M^\varepsilon, \sqrt{\varepsilon}\mathcal{V}^\varepsilon, z, \zeta)$ is tight in $C([0,T], H^s_{loc}) \times C([0,T], H^{-\sigma}_{loc}) \times H^s([0,T], H^{-\sigma}_{loc}) \times E \times E$ for $s < 1, \sigma < 0$. Then, according to Skorohod Theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of random variables $(\tilde{u}^\varepsilon, \tilde{M}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon, \tilde{z}^\varepsilon, \tilde{\zeta}^\varepsilon)$ on $\tilde{\Omega}$ which converges almost surely in the same space to $(\tilde{u}, \tilde{M}, \tilde{\mathcal{V}}, \tilde{z}, \tilde{\zeta})$ when ε goes to 0. Furthermore, the theorem ensures that for all ε the new random variables have the same laws as the original random variables. Now we define

$$\tilde{m}^\varepsilon = \partial_t \tilde{M}^\varepsilon, \qquad \tilde{m} = \partial_t \tilde{M}, \qquad \tilde{V}^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \partial_t \tilde{\mathcal{V}}^\varepsilon, \tag{2.3.53}$$

where the derivatives are taken in the sense of distributions. We also define the process

$$\left(\tilde{\mathbf{z}}^\varepsilon(t), \tilde{\boldsymbol{\zeta}}^\varepsilon(t)\right) = \left(\tilde{z}^\varepsilon\left(\frac{t}{\varepsilon}\right), \tilde{\zeta}^\varepsilon\left(\frac{t}{\varepsilon}\right)\right). \tag{2.3.54}$$

The equality in law between $\tilde{z}^\varepsilon$ and z implies that $\tilde{\mathbf{z}}^\varepsilon$ has the same law as z^ε. Since the copies of the processes have the same laws as the original processes, they satisfy the same equations, with the same regularities thanks to Proposition 2.2.2.2 and bound (2.2.30). We also define

$$\tilde{K}_t^\varepsilon = \frac{1}{4}\|\tilde{u}^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\tilde{m}^\varepsilon\|_{L^2}^2 + \frac{1}{4}\left\|\varepsilon\tilde{V}^\varepsilon\right\|_{L^2}^2. \tag{2.3.55}$$

Now we need to obtain weak convergence results in order to adapt the proof of [AA88] and identify the limit of the processes u^ε and m^ε. This is possible according to the next Proposition, which gives us a bound on the new random variables which is similar to the bounds obtained in Lemmas 2.3.4.2 and 2.3.4.3. Note that the set E plays here an important role.

Proposition 2.3.6.1. *For $\tilde{K}^\varepsilon$ and $\tilde{V}^\varepsilon$ defined in equations (2.3.55) and (2.3.53), there exists a stopping time $\tilde{\tau}^\varepsilon$ and a constant $C(T) > 0$ independent of ε such that:*

$$\mathbb{E}\left[\sup_{t\in[0,\tilde{\tau}^\varepsilon]}(\tilde{K}_t^\varepsilon)^2 + \alpha\left\|\sqrt{\varepsilon}\tilde{V}^\varepsilon\right\|_{L^2([0,\tilde{\tau}^\varepsilon],L^2)}^2\right] \leqslant C(T). \tag{2.3.56}$$

Moreover, this stopping time $\tilde{\tau}^\varepsilon$ goes almost surely to T when ε goes to 0.

Proof. The only difference between the system satisfied by $(u^\varepsilon, m^\varepsilon)$ and the one satisfied by $(\tilde{u}^\varepsilon, \tilde{m}^\varepsilon)$ is that the process z is replaced by a process $\tilde{z}^\varepsilon$, which a priori depends on ε. If it was not the case, we could do the same computations as in Section 2.3.4 and easily recover the bound claimed above in this Proposition. The only problem appears when we introduce the stopping time τ_δ^ε thanks to Proposition 2.3.2.1. Indeed, we would like to introduce in the same way the stopping time

$$\tilde{\tau}_\delta^\varepsilon = \inf\left\{t : \|\tilde{\mathbf{z}}^\varepsilon\|_{H^3\cap\dot{H}^{-3}} \geqslant \varepsilon^{-\delta}, \left\|\tilde{\boldsymbol{\zeta}}^\varepsilon\right\|_{H^3\cap\dot{H}^{-3}} \geqslant \varepsilon^{-\delta}\right\} \tag{2.3.57}$$

but we do not know a priori its behaviour when ε goes to 0. Actually it is not a problem, because the Skorohod Theorem ensures that $\tilde{z}^\varepsilon$ converges almost surely to $\tilde{z}$ in E. Thus, for almost every $\omega \in \tilde{\Omega}$, there exists a constant $C(\omega)$ such that for ε small enough

$$\|\tilde{z}^\varepsilon(t)\|_{H^3\cap\dot{H}^{-3}} \leqslant \|\tilde{z}(t)\|_{H^3\cap\dot{H}^{-3}} + C(\omega)(1+|t|^\delta).$$

Besides, $\tilde{z}$ and z have the same law, so there exist almost surely two random variables Z_1 and Z_2 such that:

$$\|\tilde{z}(t)\|_{H^3\cap\dot{H}^{-3}} \leqslant Z_1 + |t|^\delta Z_2.$$

Finally, for every $\varepsilon > 0$, almost surely there exists two random variables $\tilde{Z}_1, \tilde{Z}_2$ independent of ε such that:

$$\|\tilde{z}^\varepsilon(t)\|_{H^3 \cap \dot{H}^{-3}} \leqslant \tilde{Z}_1 + |t|^\delta \tilde{Z}_2.$$

Thus the stopping time $\tilde{\tau}_\delta^\varepsilon$ goes almost surely to infinity when ε goes to 0. It also allows us to control the growth of $(\tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon)$. □

According to Proposition 2.3.6.1 we have, in addition to the almost sure convergences given by Skorohod Theorem, weak convergences (by extracting subsequences) for the stopped processes. The sequence $(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}, \tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}, \sqrt{\varepsilon}\tilde{V}^\varepsilon_{\tilde{\tau}^\varepsilon})$ converges in the space $L^4(\tilde{\Omega}, L^\infty([0,T], H^1)) \times L^4(\tilde{\Omega}, L^\infty([0,T], L^2)) \times L^2(\tilde{\Omega}, [0,T], \mathbb{R})$ to $(\tilde{u}, \tilde{m}, \tilde{V})$ weak star. Besides the weak limits are the same as the almost sure limits.

Now we state two lemmas, which are adapted from [AA88].

Lemma 2.3.6.1. *The weak limit $\tilde{m}$ of the stopped process $\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}$ is equal to $-|\tilde{u}|^2$.*

Proof. First we can easily show that $|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}|^2$ converges almost surely to $|\tilde{u}|^2$ in $L^\infty([0,T], H^s_{loc})$. Now, let φ be a test function in $L^2(\tilde{\Omega}, \mathcal{D}((0,T), H^{-\sigma}_{loc}))$. The equation on $\tilde{V}^\varepsilon_{\tilde{\tau}^\varepsilon}$ gives us:

$$\mathbb{E}\left[\langle \varepsilon^2 \partial_t \tilde{V}^\varepsilon_{\tilde{\tau}^\varepsilon}, \varphi\rangle + \alpha\langle \varepsilon \tilde{V}^\varepsilon_{\tilde{\tau}^\varepsilon}, \varphi\rangle\right] = -\mathbb{E}\left[\langle \partial_x \left(\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon} + |\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}|^2\right), \varphi\rangle\right].$$

On the other hand

$$\mathbb{E}\left[\langle \varepsilon^2 \partial_t \tilde{V}^\varepsilon_{\tilde{\tau}^\varepsilon}, \varphi\rangle\right] = -\varepsilon^{\frac{3}{2}}\mathbb{E}\left[\langle \sqrt{\varepsilon}\tilde{V}^\varepsilon_{\tilde{\tau}^\varepsilon}, \partial_t \varphi\rangle\right]$$

which goes to 0 as ε goes to 0 according to the weak convergence in $L^2(\tilde{\Omega}, [0,T], \mathbb{R})$ ensured by Proposition 2.3.6.1. The second term in the left-hand side of the equation goes to 0 as ε goes to 0 with the same procedure. From this we recover that:

$$\partial_x(\tilde{m} + |\tilde{u}|^2) = 0 \text{ a.s. in } L^\infty([0,T], H^{-\sigma}_{loc}).$$

Since $\tilde{m}, |\tilde{u}|^2 \in L^2(\tilde{\Omega}, L^\infty([0,T], L^2))$, it follows that $\tilde{m} + |\tilde{u}|^2 = 0$. □

Lemma 2.3.6.2. *The process $\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}$ converges weakly to $-|\tilde{u}|^2\tilde{u}$ in $L^2(\Omega, L^2([0,T], H^{-1}))$.*

Proof. Let $\phi \in L^2(\Omega, L^2([0,T], H^1))$ with compact support in space.

$$\begin{aligned}\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \left(\tilde{m}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} \tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} + |\tilde{u}|^2 \tilde{u}\right) \phi dx dt\right] =& \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \tilde{m}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} \left(\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} - \tilde{u}\right) \phi dx dt\right] \\ &+ \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \tilde{u} \left(\tilde{m}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} + |\tilde{u}|^2\right) \phi dx dt\right] \\ =& I + II\end{aligned}$$

On one hand, we know that there exists $R > 0$ such that supp$\phi \subset [0,T] \times [-R,R]$, thus

$$\begin{aligned} I \leqslant& \mathbb{E}\left[\|\tilde{m}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}\|_{L^\infty([0,T],L^2)} \|\phi\|_{L^2([0,T],H^1)} \|\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} - \tilde{u}\|_{L^\infty([0,T],L^2([-R,R]))}\right] T^{\frac{1}{2}} \\ \leqslant& \|\tilde{m}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}\|_{L^4(\tilde{\Omega},L^\infty([0,T],L^2))} \|\phi\|_{L^2(\tilde{\Omega},L^2([0,T],H^1))} \|\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} - \tilde{u}\|_{L^4(\tilde{\Omega},L^\infty([0,T],L^2([-R,R]))} T^{\frac{1}{2}}.\end{aligned}$$

Since ϕ has compact support in space, we know from the almost sure convergence of $(\tilde{u}^{\varepsilon})$ in $C([0,T], H^s_{loc})$ with $s < 1$ that we have convergence of $\|\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}} - \tilde{u}\|_{L^\infty([0,T],L^2([-R,R]))}$ to 0 for almost every $\omega \in \tilde{\Omega}$. This result coupled with the uniform integrability coming from the conservation of the mass ensures that I goes to 0 as ε goes to 0.

On the other hand, we remark that $\tilde{u} \in L^\infty(\tilde{\Omega}, L^\infty([0,T], L^2))$ thanks to the conservation of mass. Thus, $\phi\tilde{u} \in L^2(\Omega, L^1([0,T], L^2))$ and we use the weak convergence of $\tilde{m}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}$ to $-|\tilde{u}|^2$ to conclude that II goes to 0 when ε goes to 0. □

These lemmas will be used when we pass to the limit in ε in the martingale problem.

Martingale Problem

In this subsection, we prove Theorem 2.3.0.1. We use the Perturbed Test Function method to identify the limit generator when ε goes to 0 and we pass to this limit in the martingale problem.

We define the function $\varphi : u \mapsto (u,h)^\ell$ for $h \in H^1$ a fixed function with compact support, and $\ell = 1, 2$. In fact it is sufficient to test our generator only on these functions. The case $\ell = 1$ allows us to state that

$$\varphi(\tilde{u}_t) - \varphi(u_0) - \int_0^t \mathcal{L}\varphi(\tilde{u}_s) ds,$$

where $\mathcal{L}$ is the limit generator, is a continuous martingale, while the case $\ell = 2$ allows us to compute its quadratic variation, and thus, conclude thanks to a martingale representation theorem which can be found in [DPZ14, Theorem 8.2]. As in the previous sections, we compute the generator $\mathcal{L}^\varepsilon\varphi$ and then correct it with two correctors. This can be made by using the Ito formula thanks to the regularities of $\tilde{u}^\varepsilon, \tilde{m}^\varepsilon, \tilde{Z}^\varepsilon, \partial_t\tilde{Z}^\varepsilon$ and the regularities of φ and its correctors that one can compute using the approach in Section 2.3.3. First we compute:

$$\mathcal{L}^\varepsilon\varphi(\tilde{u}^\varepsilon) = \ell(\tilde{u}^\varepsilon, h)^{\ell-1}(i\partial_x^2\tilde{u}^\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon, h) - \frac{\ell}{\sqrt{\varepsilon}}(\tilde{u}^\varepsilon, h)^{\ell-1}(i\tilde{\mathbf{z}}^\varepsilon\tilde{u}^\varepsilon, h).$$

We define the first corrector using the analysis made in Section 2.3.3:

$$\begin{aligned}\varphi_1(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon) &= \ell(\tilde{u}^\varepsilon, h)^{\ell-1}(i\tilde{u}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon, h)\\ &= \ell(\tilde{u}^\varepsilon, h)^{\ell-1}(i\tilde{u}^\varepsilon(\partial_x^2)^{-1}(\tilde{\boldsymbol{\zeta}}^\varepsilon + \alpha\tilde{\mathbf{z}}^\varepsilon), h).\end{aligned} \tag{2.3.58}$$

We can easily bound this first corrector:

$$|\varphi_1(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon)| \leqslant C\left\|\tilde{u}^\varepsilon\right\|_{L^2}^\ell \left\|h\right\|_{L^2}^\ell \left(\left\|(\partial_x^2)^{-1}\tilde{\boldsymbol{\zeta}}^\varepsilon\right\|_{L^\infty} + \alpha\left\|(\partial_x^2)^{-1}\tilde{\mathbf{z}}^\varepsilon\right\|_{L^\infty}\right). \tag{2.3.59}$$

The addition of this corrector gives us new terms in the generator:

$$\begin{aligned}\mathcal{L}^\varepsilon(\varphi + \sqrt{\varepsilon}\varphi_1)(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon) =&\ell(\tilde{u}^\varepsilon, h)^{\ell-1}(i\partial_x^2\tilde{u}^\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon, h) + \ell(\tilde{u}^\varepsilon, h)^{\ell-1}(\tilde{u}^\varepsilon\tilde{\mathbf{z}}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon, h)\\ &-\ell(\ell-1)(i\tilde{\mathbf{z}}^\varepsilon\tilde{u}^\varepsilon, h)(i\tilde{u}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon, h) + \sqrt{\varepsilon}D_u\varphi_1(i\partial_x^2\tilde{u}^\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon).\end{aligned}$$

The terms depending on $\tilde{\mathbf{z}}^\varepsilon$ lead us to introducing the second corrector φ_2 such that

$$\begin{aligned}\mathcal{M}\varphi_2(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon) =& -\ell(\tilde{u}^\varepsilon, h)^{\ell-1}(\tilde{u}^\varepsilon\tilde{\mathbf{z}}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon, h) + \mathbb{E}_\nu\left[\ell(\tilde{u}^\varepsilon, h)^{\ell-1}(\tilde{u}^\varepsilon z\mathcal{M}^{-1}z, h)\right]\\ &+\ell(\ell-1)(i\tilde{\mathbf{z}}^\varepsilon\tilde{u}^\varepsilon, h)(i\tilde{u}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon, h) - \mathbb{E}_\nu\left[\ell(\ell-1)(iz\tilde{u}^\varepsilon, h)(i\tilde{u}^\varepsilon\mathcal{M}^{-1}z, h)\right].\end{aligned}$$

In other words, we define φ_2 as:

$$\begin{aligned}\varphi_2(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon) =& -\ell(\tilde{u}^\varepsilon, h)^{\ell-1}\left(\tilde{u}\mathcal{M}^{-1}\left(\tilde{\mathbf{z}}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon - \mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right]\right), h\right)\\ &+\ell(\ell-1)\mathcal{M}^{-1}\left((i\tilde{\mathbf{z}}^\varepsilon\tilde{u}^\varepsilon, h)(i\tilde{u}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon, h) - \mathbb{E}_\nu\left[(iz\tilde{u}^\varepsilon, h)(i\tilde{u}^\varepsilon\mathcal{M}^{-1}z, h)\right]\right).\end{aligned} \tag{2.3.60}$$

Here again we can bound this corrector according to Lemma 2.1.0.6:

$$\begin{aligned}|\varphi_2(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon)| &\leqslant C\left\|\tilde{u}^\varepsilon\right\|_{L^2}^\ell \left\|h\right\|_{L^2}^\ell \left\|\mathcal{M}^{-1}\left(\tilde{\mathbf{z}}^\varepsilon\mathcal{M}^{-1}\tilde{\mathbf{z}}^\varepsilon - \mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right]\right)\right\|_{L^\infty(\mathbb{R})}\\ &\leqslant C_\phi\left\|\tilde{u}^\varepsilon\right\|_{L^2}^\ell \left\|h\right\|_{L^2}^\ell \left(1 + \left\|\tilde{\mathbf{z}}^\varepsilon\right\|_{H^1\cap\dot{H}^{-3}}^2 + \left\|\tilde{\boldsymbol{\zeta}}^\varepsilon\right\|_{H^1\cap\dot{H}^{-3}}^2\right).\end{aligned} \tag{2.3.61}$$

Finally, defining $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$, we get:

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(\tilde{u}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon, \tilde{\boldsymbol{\zeta}}^\varepsilon) &= \ell(\tilde{u}^\varepsilon, h)^{\ell-1}(i\partial_x^2\tilde{u}^\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon, h) + \ell(\tilde{u}^\varepsilon, h)^{\ell-1}(\tilde{u}^\varepsilon\mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right], h)\\ &- \ell(\ell-1)\mathbb{E}_\nu\left[(iz\tilde{u}^\varepsilon, h)(i\tilde{u}^\varepsilon\mathcal{M}^{-1}z, h)\right]\\ &+ \sqrt{\varepsilon}D_u\varphi_1(i\partial_x^2\tilde{u}\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon)\\ &+ \sqrt{\varepsilon}D_u\varphi_2(-i\tilde{\mathbf{z}}^\varepsilon\tilde{u}^\varepsilon) + \varepsilon D_u\varphi_2(i\partial_x^2\tilde{u}^\varepsilon - i\tilde{m}^\varepsilon\tilde{u}^\varepsilon).\end{aligned} \tag{2.3.62}$$

This expression helps us define the following quantity, which will be the limit generator of our equation.

Definition 2.3.6.1. *For $\tilde{u} \in H^3$ and $\varphi(\cdot) = (\cdot, h)^\ell$, $\ell = 1, 2$, where $h \in H^1$, we define the operator $\mathcal{L}\varphi$ as*

$$\begin{aligned}\mathcal{L}\varphi(\tilde{u}) =&\ell(\tilde{u}, h)^{\ell-1}(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u}, h) + \ell(\tilde{u}, h)^{\ell-1}(\tilde{u}\mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right], h)\\ &- \ell(\ell-1)\mathbb{E}_\nu\left[(i\tilde{u}\mathcal{M}^{-1}z, h)(i\tilde{u}z, h)\right].\end{aligned} \tag{2.3.63}$$

Now that we have the expression of $\mathcal{L}^\varepsilon\varphi^\varepsilon$ and a candidate for the limit generator, we state the following result, which gathers the main arguments of the proof of Theorem 2.3.0.1.

Proposition 2.3.6.2. *Let $\varphi(u) = (u, h)^\ell$, $\ell = 1, 2$, where $h \in H^1$ has a compact support. Let φ_1, φ_2 be the two correctors defined in equations (2.3.58) and (2.3.60). We define $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$. Then:*

1. *$|\varphi^\varepsilon(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}, \tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon}, \tilde{\zeta}^\varepsilon_{\tilde{\tau}^\varepsilon}) - \varphi(\tilde{u})| \leqslant C_\phi\sqrt{\varepsilon}\left\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\right\|^\ell_{L^2}\|h\|^\ell_{L^2}\left(1 + \|\tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon}\|^2_E + \left\|\tilde{\boldsymbol{\zeta}}^\varepsilon_{\tilde{\tau}^\varepsilon}\right\|^2_E\right)$, where E is defined in Equation (2.3.36).*
2. *$\lim_{\varepsilon\to 0}\int_0^T \mathbb{E}\left[|\mathcal{L}^\varepsilon\varphi^\varepsilon(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}, \tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon}, \tilde{\boldsymbol{\zeta}}^\varepsilon_{\tilde{\tau}^\varepsilon}) - \mathcal{L}\varphi(\tilde{u})|\right](t)dt = 0.$*
3. *The quantity M^ε_t defined by*

$$(M^\varepsilon_t, h) = \varphi^\varepsilon(\tilde{u}^\varepsilon_t, \tilde{\mathbf{z}}^\varepsilon_t, \tilde{\boldsymbol{\zeta}}^\varepsilon_t) - \varphi^\varepsilon(\tilde{u}_0, \tilde{\mathbf{z}}_0, \tilde{\boldsymbol{\zeta}}_0) - \int_0^t \mathcal{L}^\varepsilon\varphi^\varepsilon(\tilde{u}^\varepsilon_s, \tilde{\mathbf{z}}^\varepsilon_s, \partial_t\tilde{\boldsymbol{\zeta}}^\varepsilon_s)ds \tag{2.3.64}$$

   is a martingale for the filtration $\tilde{\mathcal{F}}^\varepsilon$ generated by the process $(\tilde{u}^\varepsilon)$.

Proof. The first point of the Proposition is clear according to the bounds (2.3.59) and (2.3.61).

For the second point, we have

$$\begin{aligned}\mathcal{L}^\varepsilon\varphi^\varepsilon(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon},\tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon},\tilde{\boldsymbol{\zeta}}^\varepsilon_{\tilde{\tau}^\varepsilon})-\mathcal{L}\varphi(\tilde{u}) &= \ell(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon},h)^{\ell-1}\left(i\partial_x^2\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}+\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right],h\right)\\ &-\ell(\tilde{u},h)^{\ell-1}\left(i\partial_x^2\tilde{u}+i|\tilde{u}|^2\tilde{u}+\tilde{u}\mathbb{E}_\nu\left[z\mathcal{M}^{-1}z\right],h\right)\\ &+\ell(\ell-1)\mathbb{E}_\nu\left[\left(i\tilde{u}\mathcal{M}^{-1}z,h\right)\left(i\tilde{u}z,h\right)-\left(i\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\mathcal{M}^{-1}z,h\right)\left(i\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}z,h\right)\right]\\ &+\sqrt{\varepsilon}D_u\varphi_1(i\partial_x^2\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon})+\sqrt{\varepsilon}D_u\varphi_2(-i\tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon})\\ &+\varepsilon D_u\varphi_2(i\partial_x^2\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}).\end{aligned}$$

We start with the terms of order $\frac{1}{2}$ and 1 in ε, deducing

$$\sqrt{\varepsilon}D_u\varphi_1(i\partial_x^2\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon})\leqslant C\varepsilon^{\frac{1}{2}-\delta}\left(\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{H^1}+\|\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}\right)\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}^{\ell-1}\|h\|_{H^1}^\ell,$$

$$\sqrt{\varepsilon}D_u\varphi_2(-i\tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon})\leqslant C_\phi\sqrt{\varepsilon}\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}^\ell\|h\|_{L^2}^\ell\varepsilon^{-\delta}(1+\varepsilon^{2\delta}),$$

using Propositions 2.3.2.1 and 2.3.2.2. Furthermore,

$$\varepsilon D_u\varphi_2(i\partial_x^2\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon})\leqslant C_\phi\varepsilon\left(\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{H^1}+\|\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}\right)\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2}^{\ell-1}\|h\|_{H^1}^\ell(1+\varepsilon^{-2\delta})$$

Thus, we can conclude thanks to Proposition 2.3.6.1 that the expectation of these three terms goes to 0 when ε goes to 0. Furthermore, the conservation of mass, the weak convergence of $\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}$ given by Proposition 2.3.6.1 and the weak convergence of $\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}$ proved in Lemma 2.3.6.2 give us that the expectation of all the terms of order 0 goes to 0 as ε goes to 0. Indeed, if we look for example at the following term

$$(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon},h)(-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon},h)-(\tilde{u},h)(i|\tilde{u}|^2\tilde{u},h),$$

we can rewrite it as

$$(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-\tilde{u},h)(-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon},h)+(\tilde{u},h)(-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-i|\tilde{u}|^2\tilde{u},h).$$

Then the expectation of the first term gives

$$\begin{aligned}&\mathbb{E}\left[(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-\tilde{u},h)\ (-i\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon},h)\right]\\ &\qquad\leqslant\|u_0\|_{L^2}\|\tilde{m}^\varepsilon_{\tilde{\tau}^\varepsilon}\|_{L^2(\tilde{\Omega},L^\infty([0,T],L^2))}\|h\|_{H^1}^2\|\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}-\tilde{u}\|_{L^2(\tilde{\Omega},L^\infty([0,T],L^2_{loc}))}\end{aligned}$$

since h has a compact support, and thanks to the conservation of mass and the convergence of $\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}$ in $\mathcal{C}^0([0,T], H^s_{loc})$ we know that this expectation goes to 0 as ε goes to 0. The other term is handled by using the conservation of mass and Lemma 2.3.6.2 thanks to the integral in time.

The last point of the Proposition is quite obvious because the variables $\tilde{u}^{\varepsilon}, \tilde{m}^{\varepsilon}, \tilde{z}^{\varepsilon}, \tilde{\zeta}^{\varepsilon}$ are sufficiently smooth to ensure that we can apply Itô formula to φ^{ε}, so that M^{ε}_t is a martingale. □

Now we are now in position to prove the Theorem 2.3.0.1.
First step: Identification of the limit generator.

We define $k(x,y) = \mathbb{E}_{\nu}\left[z(x)\mathcal{M}^{-1}z(y) + z(y)\mathcal{M}^{-1}z(x)\right]$, and $F(x) = k(x,x)$. The next result gives the link between k and the Hilbert-Schmidt operator ϕ.

Lemma 2.3.6.3. *For $x, y \in \mathbb{R}$ and (z,ζ) solution of equation (2.3.7), we have:*

$$-k(x,y) = \sum_{k\in\mathbb{N}} (\partial_x^2)^{-1}\phi e_k(x)(\partial_x^2)^{-1}\phi e_k(y). \tag{2.3.65}$$

The proof of Lemma 2.3.6.3 is technical and is postponed to the Appendix of this chapter. Now, for $h \in H^1$ with compact support, we use the functionals $\varphi(u) = (u,h)^{\ell}$ for $\ell = 1,2$ which have been studied in Proposition 2.3.6.2. If we look at $\mathcal{L}\varphi$ with $\ell = 1$ defined in Definition 2.3.6.1, we can write thanks to Definition 2.3.6.1 and (2.3.65):

$$\begin{aligned}\mathcal{L}\varphi(\tilde{u}) &= \left(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u} + \frac{1}{2}\tilde{u}F(x), h\right)\\ &= D_u\varphi\left(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u} - \frac{1}{2}\tilde{u}\sum_{k\in\mathbb{N}}\left((\partial_x^2)^{-1}\phi e_k(x)\right)^2\right).\end{aligned}$$

Furthermore, if we study $\mathcal{L}\varphi$ for $\ell = 2$, we get

$$\begin{aligned}\mathcal{L}\varphi(\tilde{u}) =& 2(\tilde{u},h)\left(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u} + \tilde{u}\mathbb{E}_{\nu}[z\mathcal{M}^{-1}z], h\right)\\ &- 2\int_{\mathbb{R}}\int_{\mathbb{R}} Re\left(i\tilde{u}(x)\bar{h}(x)\right) Re\left(i\tilde{u}(y)\bar{h}(y)\right)\mathbb{E}_{\nu}\left[z(x)\mathcal{M}^{-1}z(y)\right]dxdy\\ =& D_u\varphi\left(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u} - \frac{1}{2}\tilde{u}\sum_{k\in\mathbb{N}}\left((\partial_x^2)^{-1}\phi e_k(x)\right)^2\right)\\ &- \int_{\mathbb{R}}\int_{\mathbb{R}} Re\left(i\tilde{u}(x)\bar{h}(x)\right) Re\left(i\tilde{u}(y)\bar{h}(y)\right)k(x,y)dxdy.\end{aligned}$$

We focus on the second term of the right-hand-side of the equation and use Lemma 2.3.6.3:

$$
\begin{aligned}
-\int_{\mathbb{R}}\int_{\mathbb{R}} Re\left(i\tilde{u}(x)\bar{h}(x)\right) Re\left(i\tilde{u}(y)\bar{h}(y)\right) k(x,y)dxdy =& \sum_{k\in\mathbb{N}} \left(iu(\partial_x^2)^{-1}\phi e_k, h\right)^2 \\
=& \frac{1}{2} Tr(iu(\partial_x^2)^{-1}\phi)^* D_u^2\varphi(iu(\partial_x^2)^{-1}\phi).
\end{aligned}
$$

Finally, we have for $\ell = 1$

$$
\mathcal{L}\varphi(\tilde{u}) = D_u\varphi\left(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u} - \frac{1}{2}\tilde{u}\sum_{k\in\mathbb{N}}\left((\partial_x^2)^{-1}\phi e_k(x)\right)^2\right), \tag{2.3.66}
$$

and for $\ell = 2$

$$
\begin{aligned}
\mathcal{L}\varphi(\tilde{u}) =& D_u\varphi\left(i\partial_x^2\tilde{u} + i|\tilde{u}|^2\tilde{u} - \frac{1}{2}\tilde{u}\sum_{k\in\mathbb{N}}\left((\partial_x^2)^{-1}\phi e_k(x)\right)^2\right) \\
&+ \frac{1}{2} Tr(iu(\partial_x^2)^{-1}\phi)^* D_u^2\varphi(iu(\partial_x^2)^{-1}\phi).
\end{aligned} \tag{2.3.67}
$$

<u>Second step:</u> Convergence.

We work here with $\ell = 1$. We know according to Proposition 2.3.6.2 that $M^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}$, where M^{ε} is defined in (2.3.64), is a stopped martingale which means that for every $h \in H^1$ with compact support, for every $g \in \mathcal{C}_b\left((H^s_{loc})^n\right)$ and for every $\sigma = s_1 < s_2 < \cdots < s_n = t$, we have

$$
\begin{aligned}
\mathbb{E}\Big[\Big(&\varphi^{\varepsilon}(\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}, \tilde{\mathbf{z}}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}, \tilde{\boldsymbol{\zeta}}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}})(t) - \varphi^{\varepsilon}(\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}, \tilde{\mathbf{z}}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}, \tilde{\boldsymbol{\zeta}}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}})(\sigma) \\
&- \int_{\sigma\wedge\tilde{\tau}^{\varepsilon}}^{t\wedge\tilde{\tau}^{\varepsilon}} \mathcal{L}^{\varepsilon}\varphi^{\varepsilon}(\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}, \tilde{\mathbf{z}}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}, \tilde{\boldsymbol{\zeta}}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}})(r)dr\Big) g\left(\tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}(s_1), \cdots, \tilde{u}^{\varepsilon}_{\tilde{\tau}^{\varepsilon}}(s_n)\right)\Big] = 0.
\end{aligned} \tag{2.3.68}
$$

We want to show that the limit process is still a martingale. This is the following statement:

Lemma 2.3.6.4. *Let $\varphi(u) = (u, h)$ where $h \in H^1$ with compact support. Then the process M_t defined by:*

$$
(M_t, h) = \varphi(\tilde{u}_t) - \varphi(\tilde{u}_0) - \int_0^t \mathcal{L}\varphi(\tilde{u}_s)ds \tag{2.3.69}
$$

is a martingale for the filtration $\mathcal{G}_s$ generated by $\tilde{u}_s$.

Proof. Let $g \in \mathcal{C}_b\left((H^s_{loc})^n\right)$, and $\sigma = s_1 < s_2 < \cdots < s_n = t$. Then

$$\begin{aligned}
&\mathbb{E}\left[\left(\varphi(\tilde{u}_t) - \varphi(\tilde{u}_\sigma) - \int_\sigma^t \mathcal{L}\varphi(\tilde{u}_r)dr\right) g\left(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_1), \cdots, \tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_n)\right)\right] \\
&= \mathbb{E}\Bigg[\Bigg(\left(\varphi(\tilde{u}_t) - \varphi(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(t))\right) - \left(\varphi(\tilde{u}_\sigma) - \varphi(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(\sigma))\right) \\
&- \sqrt{\varepsilon}\left(\varphi_1(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(t)) - \varphi_1(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(\sigma))\right) - \varepsilon\left(\varphi_2(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(t)) - \varphi_2(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(\sigma))\right) \\
&- \left(\int_\sigma^{\sigma\wedge\tilde{\tau}^\varepsilon} \mathcal{L}\varphi(\tilde{u}_r)dr - \int_t^{t\wedge\tilde{\tau}^\varepsilon} \mathcal{L}\varphi(\tilde{u}_r)dr\right) \\
&- \int_{\sigma\wedge\tilde{\tau}^\varepsilon}^{t\wedge\tilde{\tau}^\varepsilon} \mathcal{L}\varphi(\tilde{u}_r) - \mathcal{L}^\varepsilon\varphi^\varepsilon(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(r), \tilde{\mathbf{z}}^\varepsilon_{\tilde{\tau}^\varepsilon}(r), \tilde{\boldsymbol{\zeta}}^\varepsilon_{\tilde{\tau}^\varepsilon}(r))dr\Bigg) g(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_1), \cdots, \tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_n))\Bigg] \\
&= \mathbb{E}\left[T_1 + T_2 + T_3 + T_4\right].
\end{aligned}$$

Note that φ_1 and φ_2 depend also on $\tilde{m}^\varepsilon, \tilde{\mathbf{z}}^\varepsilon$ and $\tilde{\boldsymbol{\zeta}}^\varepsilon$ but we just write the dependance in $\tilde{u}^\varepsilon$ for convenience. According to Proposition 2.3.6.2 and the fact that g is bounded, we know that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[T_1 + T_2\right] = 0.$$

The same proposition ensures that $\mathbb{E}[T_4]$ goes to 0 as ε goes to 0. Finally for T_3:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_\sigma^{\sigma\wedge\tilde{\tau}^\varepsilon} \mathcal{L}\varphi(\tilde{u}_r)dr\right) g(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_1), \cdots, \tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_n))\right] &\leqslant \mathbb{E}\left[\int_\sigma^{\sigma\wedge\tilde{\tau}^\varepsilon} C_\phi \left\|\tilde{u}\right\|_{H^1} \left\|h\right\|_{H^1} dr\right] \\
&\leqslant \mathbb{E}\left[(\sigma\wedge\tilde{\tau}^\varepsilon - \sigma) C_\phi \left\|\tilde{u}\right\|_{L^\infty([0,T],H^1)} \left\|h\right\|_{H^1}\right] \\
&\leqslant C\mathbb{E}\left[(\sigma\wedge\tilde{\tau}^\varepsilon - \sigma)^2\right]^{\frac{1}{2}}
\end{aligned}$$

because $\tilde{u} \in L^4(\tilde{\Omega}, L^\infty([0,T], H^1))$ according to Proposition 2.3.6.1. We obviously have the same bound for the other term of T_3. Thus, $\mathbb{E}[T_3]$ goes to 0 as ε goes to 0, since the stopping time $\tilde{\tau}^\varepsilon$ goes almost surely to T as ε goes to 0.

We have proved that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\left(\varphi(\tilde{u}_t) - \varphi(\tilde{u}_\sigma) - \int_\sigma^t \mathcal{L}\varphi(\tilde{u}_r)dr\right) g\left(\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_1), \cdots, \tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}(s_n)\right)\right] = 0, \tag{2.3.70}$$

and we use the continuity of g and the almost sure convergence of $\tilde{u}^\varepsilon_{\tilde{\tau}^\varepsilon}$ to get:

$$\mathbb{E}\left[\left(\varphi(\tilde{u}_t) - \varphi(\tilde{u}_\sigma) - \int_\sigma^t \mathcal{L}\varphi(\tilde{u}_r)dr\right) g\left(\tilde{u}(s_1), \cdots, \tilde{u}(s_n)\right)\right] = 0 \tag{2.3.71}$$

so that the process M_t is a martingale for the filtration $\mathcal{G}_s$ generated by $\tilde{u}_s$. □

Similarly to Lemma 2.3.6.4, we can pass to the limit $\varepsilon \to 0$ in the definition of the quadratic variation of M^ε defined in (2.3.64) to get that

$$(\langle M, M\rangle_t h, h) = \int_0^t \left(\mathcal{L}(\varphi^2) - 2\varphi\mathcal{L}\varphi\right)(\tilde{u}_s)ds. \tag{2.3.72}$$

Note that this step requires the use of the Perturbed Test Function method on φ for $\ell = 2$, which has been described in the first step in (2.3.67), and the use of Proposition 2.3.6.2. We know from equations (2.3.66) and (2.3.67) that

$$\left(\mathcal{L}(\varphi^2) - 2\varphi\mathcal{L}\varphi\right)(\tilde{u}) = \frac{1}{2}Tr(iu(\partial_x^2)^{-1}\phi)^* D_u^2\varphi^2(iu(\partial_x^2)^{-1}\phi).$$

Thus, the martingale

$$M_t = \tilde{u}_t - u_0 - \int_0^t i\partial_x^2\tilde{u}_s + i|\tilde{u}_s|^2\tilde{u}_s - \frac{1}{2}\tilde{u}_s\sum_{k\in\mathbb{N}}\left((\partial_x^2)^{-1}\phi e_k\right)^2 ds$$

has the quadratic variation:

$$\int_0^t Tr\left(i\tilde{u}_s(\partial_x^2)^{-1}\phi\right)\left(i\tilde{u}_s(\partial_x^2)^{-1}\phi\right)^* ds.$$

Then, using the martingale representation theorem and up to enlarging the probability space, there exists a cylindrical Wiener process $\bar{W}$ such that

$$M_t = \int_0^t i\tilde{u}_s(\partial_x^2)^{-1}\phi d\bar{W}_s$$

so that $\tilde{u}$ is a martingale solution of (2.3.3).

Third step: Uniqueness

So far we have proved the convergence to equation (2.3.3) for a subsequence, we need to conclude with the convergence on the whole sequence. We know from Proposition 2.3.6.1 that $\tilde{u} \in L^4(\tilde{\Omega}, L^\infty([0,T], H^1))$. In particular, it has paths in $L^\infty([0,T], H^1)$. We know from [dBD03b] that equation (2.3.3) has a unique solution with paths in $L^\infty([0,T], H^1)$ a.s. This leads to the conclusion of the proof of Theorem 2.3.0.1.

2.3.7 Convergence in probability

The aim of this section is to prove that we can actually get a better result than the convergence in law of the solution u^ε of (2.0.3) to the solution u of (2.3.3). It is not usually the case in approximation diffusion problems, and it is due here to the particular form of the process $(z^\varepsilon, \zeta^\varepsilon)$.

Theorem 2.3.7.1. *Let $\alpha > 0$. For any $T > 0$, and for $(u_0, m_0, m_1) \in H^3(\mathbb{R}) \times (H^2(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R})) \times (H^1(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}))$, the process $(u^\varepsilon, m^\varepsilon)$ solution of the system (2.3.1) with $\phi \in \mathcal{L}_2(L^2(\mathbb{R}), H^3(\mathbb{R}) \cap \dot{H}^{-4}(\mathbb{R}))$ converges in probability in $C([0,T], H^s_{loc}(\mathbb{R}))$ for $s < 1$ to u solution of*

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2} uF)dt - u(\partial_x^2)^{-1}\phi dW_t. \tag{2.3.73}$$

where $F(x) = \sum_{k\in\mathbb{N}} \left((\partial_x^2)^{-1})\phi e_k\right)^2(x)$, and W_t is the cylindrical Wiener process introduced in (2.0.3).

In the previous sections, it was sufficient to show that u^ε was solution of the martingale problem (2.3.64). Actually the correctors φ_1 and φ_2 introduced in (2.3.58) and (2.3.60) give us an explicit expression of the martingale M_t^ε defined in (2.3.64). Indeed, the process $(z^\varepsilon, \zeta^\varepsilon)$ that we study is solution of (2.3.9) where W_t is the cylindrical Wiener process introduced in the system (2.0.3). When we apply the Perturbed Test Function method to $\varphi(u) = (u, h)$ for $h \in H^1$ with compact support, the first corrector $\sqrt{\varepsilon}\varphi_1(u^\varepsilon) = \sqrt{\varepsilon}(iu^\varepsilon \mathcal{M}^{-1} z^\varepsilon, h)$ contributes to the martingale part according to Equation (2.3.9) satisfied by z^ε, adding

$$X_t^\varepsilon = \int_0^t \left(iu^\varepsilon (\partial_x^2)^{-1}\phi dW_s, h\right). \tag{2.3.74}$$

In the same way, the martingale part coming from the second corrector $\varepsilon\varphi_2(u^\varepsilon) = \varepsilon(u^\varepsilon \mathcal{M}^{-1}(z^\varepsilon \mathcal{M}^{-1} z^\varepsilon - \mathbb{E}_\nu[z\mathcal{M}^{-1}z], h)$ is

$$\begin{aligned} Y_T^\varepsilon =& \varepsilon Re \int_{s=0}^T \int_{\mathbb{R}} u^\varepsilon(s,x)\bar{h}(x) \\ &\times \int_{t=0}^\infty (S_\alpha(t)(z^\varepsilon, \phi dW_s))_1 \left((\partial_x^2)^{-1} (S_\alpha(t)(z^\varepsilon, \zeta^\varepsilon))_2 + \alpha(\partial_x^2)^{-1} (S_\alpha(t)(z^\varepsilon, \zeta^\varepsilon))_1\right) \\ &+ (S_\alpha(t)(z^\varepsilon, \zeta^\varepsilon))_1 \left((\partial_x^2)^{-1} (S_\alpha(t)(z^\varepsilon, \phi dW_s))_2 + \alpha(\partial_x^2)^{-1} (S_\alpha(t)(z^\varepsilon, \phi dW_s))_1\right) dtdx. \end{aligned} \tag{2.3.75}$$

Thus the previous section and more precisely the third item of Proposition 2.3.6.2 ensures that

$$\varphi^\varepsilon(u_t^\varepsilon) - \varphi^\varepsilon(u_0) - \int_0^t \mathcal{L}^\varepsilon \varphi^\varepsilon(u_s^\varepsilon)ds = X_t^\varepsilon + Y_t^\varepsilon, \tag{2.3.76}$$

with $\varphi^\varepsilon = \varphi + \sqrt{\varepsilon}\varphi_1 + \varepsilon\varphi_2$. We can use Lemma 2.1.0.6 to estimate the quadratic variation of Y_t^ε to show that

$$d\langle Y^\varepsilon, Y^\varepsilon\rangle_{t\wedge\tau^\varepsilon} \leqslant C_\phi \varepsilon \left\|u^\varepsilon\right\|_{L^2}^2 \left\|h\right\|_{H^1}^2 \left(1+\varepsilon^{-4\delta}\right)$$

where τ^ε is the stopping time defined in (2.4.8). Thus we know that

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{t\in[0,\tau^\varepsilon]} Y_t^\varepsilon\right] = 0. \tag{2.3.77}$$

To prove Theorem 2.3.7.1, we will use a result which is claimed in [GK96] in the finite dimensional case, and used in [MR05] in the infinite dimensional case. We will also need to use [DGHT11, Lemma 2.1] which allows us to pass to the limit $\varepsilon \to 0$ directly in the martingale part of (2.3.76).

Proof of Theorem 2.3.7.1. Let $(u^{\varepsilon_k})_k$ and $(u^{\eta_k})_k$ be two subsequences of the sequence $(u^\varepsilon)_\varepsilon$, where for any $\varepsilon > 0$, u^ε is a solution of (2.0.3). Then the proof of Proposition 2.3.5.1 ensures that the sequence

$$\left(u^{\varepsilon_k}, u^{\eta_k}, M^{\varepsilon_k}, M^{\eta_k}, \sqrt{\varepsilon_k}\mathcal{V}^{\varepsilon_k}, \sqrt{\eta_k}\mathcal{V}^{\eta_k}, z, \zeta, (\beta_n)_{n\in\mathbb{N}}\right)$$

is tight in the space $(\mathcal{C}^0([0,T], H^s_{loc}))^2 \times (\mathcal{C}^0([0,T], H^{-\sigma}_{loc}))^2 \times (H^s([0,T], H^{-\sigma}_{loc}))^2 \times E \times E \times \mathcal{C}([0,T])^{\mathbb{N}}$ for $s < 1, \sigma > 0$. According to Skorohod Theorem, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of random variables on $\tilde{\Omega}$, with the same laws, which converges almost surely in this space as ε goes to 0. We denote respectively by $\tilde{u}$ and $\tilde{v}$ the almost sure limits of the new subsequences $\tilde{u}^{\varepsilon_k}$ and $\tilde{u}^{\eta_k}$. For any $n \in \mathbb{N}$ we also denote by $\tilde{\beta}_n$ the limit of the subsequence $(\tilde{\beta}_n^k)$, and we define:

$$\tilde{W}_t^k(x) = \sum_{n\in\mathbb{N}} e_n(x)\tilde{\beta}_n^k(t), \qquad \tilde{W}_t(x) = \sum_{n\in\mathbb{N}} e_n(x)\tilde{\beta}_n(t).$$

According to Section 2.3.6, the stopped process $\tilde{u}^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}$, where $\tilde{\tau}^\varepsilon$ is the stopping time, which appears in Proposition 2.3.6.1, satisfies for $\varphi(u) = (u, h)$, where $h \in H^1(\mathbb{R})$ with compact support,

$$\varphi^{\varepsilon_k}(\tilde{u}^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}(t)) - \varphi^{\varepsilon_k}(\tilde{u}_0) - \int_0^{t\wedge\tilde{\tau}^{\varepsilon_k}} \mathcal{L}^{\varepsilon_k}\varphi^{\varepsilon_k}(\tilde{u}^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}(s))ds = \int_0^{t\wedge\tilde{\tau}^{\varepsilon_k}} \big(i\tilde{u}^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}(s)(\partial_x^2)^{-1}\phi d\tilde{W}^k_s, h\big) + \tilde{Y}^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}, \tag{2.3.78}$$

where $\tilde{Y}^\varepsilon$ is defined as Y^ε in (2.3.75) but using the new random variables. The stopped process $\tilde{u}^{\eta_k}_{\tilde{\tau}^{\eta_k}}$ satisfies the same equation with ε_k replaced by η_k. If we look at the left-hand side of this equation, the section 2.3.6 and Proposition 2.3.6.2 ensure that it converges in probability to

$$(\tilde{u}_t, h) - (\tilde{u}_0, h) - \int_0^t \mathcal{L}\varphi(\tilde{u}_s)ds,$$

where $\mathcal{L}$ is the limit generator defined in Definition 2.3.6.1. We also know by (2.3.77) that $Y^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}$ converges in probability to 0. It remains to deal with the integral in the right-hand side of the above equation. With this aim in mind we apply [DGHT11, Lemma 2.1] to the sequences $(\tilde{W}^k)_k$ and $(\tilde{u}^{\varepsilon_k}_{\tilde{\tau}^{\varepsilon_k}}(\partial_x^2)^{-1}\phi)_k$. Thus the integral converges in probability to

$$\int_0^t \big(i\tilde{u}(s)(\partial_x^2)^{-1}\phi d\tilde{W}_s, h\big)$$

as ε goes to 0. The procedure is the same for $\tilde{u}^{\eta_k}_{\tilde{\tau}^{\eta_k}}$, and finally the limits $\tilde{u}$ and $\tilde{v}$ are both solutions of

$$(u_t, h) - (u_0, h) - \int_0^t \left(i\partial_x^2 u_s + i|u_s|^2 u_s + \frac{1}{2}u_s F, h\right) ds = \int_0^t \big(iu_s(\partial_x^2)^{-1}\phi d\tilde{W}_s, h\big)\,,$$

with $F(x) = \sum_{k\in\mathbb{N}} \left((\partial_x^2)^{-1}\phi e_k(x)\right)^2$. We conclude thanks to the uniqueness of solutions to the stochastic Schrödinger equation that $\tilde{u} = \tilde{v}$.

Thus, thanks to Skorohod theorem (which states the equality in law between u^ε and its copy $\tilde{u}^\varepsilon$, we have proved that for all subsequences $(u^{\varepsilon_k})_k$, $(u^{\eta_k})_k$, the couple $(u^{\varepsilon_k}, u^{\eta_k})_k$ converges in law when ε goes to 0 to a limit (u, u). Thanks to a result from Gyongy-Krylov (see [GK96]), the sequence $(u^\varepsilon)_\varepsilon$ converges in probability as ε goes to 0 to a limit u, that we will identify. Indeed, we know from Section 2.3.6 that for $\varphi(u) = (u, h)$ where $h \in H^1(\mathbb{R})$ with compact support we have

$$\varphi^\varepsilon(u^\varepsilon_{\tau^\varepsilon}(t)) - \varphi^\varepsilon(u_0) - \int_0^{t\wedge\tau^\varepsilon} \mathcal{L}^\varepsilon\varphi^\varepsilon(u^\varepsilon_{\tau^\varepsilon}(s))ds = \int_0^{t\wedge\tau^\varepsilon} \big(iu^\varepsilon_{\tau^\varepsilon}(s)(\partial_x^2)^{-1}\phi dW_s, h\big) + Y^\varepsilon_{\tau^\varepsilon}, \tag{2.3.79}$$

with τ^ε defined in (2.4.8) and Y_t^ε in (2.3.75). Now using the convergence in probability of $\tilde{u}^\varepsilon_{\tau^\varepsilon}$ and [DGHT11, Lemma 2.1], we conclude that the limit u is a weak solution of

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2} uF)dt - u(\partial_x^2)^{-1}\phi dW_t, \tag{2.3.80}$$

where W_t is the cylindrical Wiener process of (2.0.3). Actually we can prove that u is a mild solution of equation (2.3.80). Indeed, Fatou's Lemma gives us that $u \in L^2(\Omega, L^\infty([0,T], H^1(\mathbb{R})))$, and the embedding $H^1(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ ensures that the nonlinearity $(t,x) \mapsto |u(t,x)|^2 u(t,x)$ belongs to $L^\infty([0,T], L^{\frac{4}{3}}(\mathbb{R}))$. Thus $|u|^2 u \in L^1(0,T,H^{-1}(\mathbb{R}) = L^1([0,T], D((-A)^{-\frac{1}{2}}))$ where $D((-A)^{-\frac{1}{2}})$ denotes the domain of the operator $(-A)^{-\frac{1}{2}}$ with $A = -i\partial_x^2$.

We would like to insist on the fact that here we used the convergence in probability to pass to the limit directly in the martingale term in (2.3.78), since $\lim \tilde{W}^k = \tilde{W}$, whereas in Section 2.3.6 we just used the martingale representation theorem to ensure the existence of some Wiener process, which allowed us to give an explicit expression of the martingale term. In this section we prove that this Wiener process is the one that is introduced in a first place in (2.0.3). $\square$

2.4 Appendix

This appendix gathers several proofs which although technical are not essential to the understanding of the chapter.

We start with the following deterministic lemma. The proof allows us to understand why we set up such a procedure to get our final bound on the energy H defined in (2.2.15), which is stated in Proposition 2.3.4.2.

Lemma 2.4.0.1. *Let $x(t)$ a positive function satisfying the differential inequality:*

$$x'(t) \leqslant \varepsilon x(t)^2 + bx(t) + c, \tag{2.4.1}$$

where $\varepsilon > 0, b, c > 0$, with the initial data $x_0 \geqslant 0$. Then there exists τ^ε such that for all $t \in [0, \tau^\varepsilon]$:

$$x(t) \leqslant 4(x_0 + \frac{2c}{b})e^{bt}.$$

Furthermore, τ^ε goes to infinity as ε goes to 0.

Proof of Lemma 2.4.0.1. As we are interested in the case where ε is "small", we can assume that $\Delta = b^2 - 4\varepsilon c > 0$, therefore

$$\varepsilon x(t)^2 + bx(t) + c = \varepsilon(x - r_1)(x - r_2)$$

with

$$r_1 = \frac{-b + \sqrt{\Delta}}{2\varepsilon}, \qquad r_2 = \frac{-b - \sqrt{\Delta}}{2\varepsilon}, \qquad r_1 > r_2, \qquad r_1, r_2 < 0.$$

Under the assumptions of Lemma 2.4.0.1 we have:

$$x'(t)\left(\frac{1}{(x(t) - r_1)(x(t) - r_2)}\right) \leqslant \varepsilon \text{ which is equivalent to } \frac{d}{dt}\left(\frac{1}{\sqrt{\Delta}} \ln \frac{x(t) - r_1}{x(t) - r_2}\right) \leqslant 1$$

and we get the bound

$$\frac{x(t) - r_1}{x(t) - r_2} \leqslant \frac{x_0 - r_1}{x_0 - r_2} e^{\sqrt{\Delta} t}. \tag{2.4.2}$$

In order to derive a bound on $x(t)$ we need

$$\frac{x_0 - r_1}{x_0 - r_2} e^{\sqrt{\Delta} t} < 1.$$

In other words

$$t < t_\varepsilon^0 = \frac{1}{\sqrt{\Delta}} \ln\left(\frac{x_0 - r_2}{x_0 - r_1}\right) \xrightarrow[\varepsilon \to 0]{} +\infty.$$

For $0 \leqslant t < t_\varepsilon^0$, we thus have

$$\begin{aligned} x(t) &\leqslant \frac{-r_2(x_0 - r_1)e^{\sqrt{\Delta} t} + r_1(x_0 - r_2)}{x_0 - r_2 - (x_0 - r_1)e^{\sqrt{\Delta} t}} \\ &\leqslant \frac{2b(x_0 + \frac{2c}{b})e^{bt}}{b - 2\varepsilon(x_0 + \frac{2c}{b})e^{bt}}, \end{aligned}$$

since $-r_1 \leqslant \frac{2c}{b}$. It follows, for $t < \tau^\varepsilon = \frac{1}{b} \ln\left(\frac{b^2}{2\varepsilon(2x_0 b + 4c)}\right)$,

$$\begin{aligned} x(t) &\leqslant \frac{2b(x_0 + \frac{2c}{b})e^{bt}}{b - 2\varepsilon(x_0 + \frac{2c}{b})\frac{b^2}{2\varepsilon(2x_0 b + 4c)}} \\ &\leqslant 4(x_0 + \frac{2c}{b})e^{bt}. \end{aligned}$$

We can easily verify that $\tau^\varepsilon < t^0_\varepsilon$ and that $\tau^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} +\infty$. $\square$

Now we detail the procedure to get a proof of Propositions 2.3.4.2 and 2.3.4.3. Proposition 2.3.4.1 and the scheme of proof of Lemma 2.4.0.1 lead us to introducing the function θ, which is the primitive function of

$$\frac{1}{\varepsilon x^2 + \tilde{M}x + \tilde{N}},$$

where $\tilde{M}$ and $\tilde{N}$ are positive constants that we will choose later. We will apply this function to $(H^\varepsilon)^2$, so we define $\tilde{\theta}(x) = \theta(x^2)$. From the proof of Lemma 2.4.0.1, we know that

$$\tilde{\theta}(x) = \frac{1}{\sqrt{\Delta}} \ln\left(\frac{x^2 - r_1}{x^2 - r_2}\right), \tag{2.4.3}$$

where $\Delta = \tilde{M}^2 - 4\varepsilon\tilde{N}$ and r_1, r_2 are the roots of $\varepsilon x^2 + \tilde{M}x + \tilde{N} = 0$, $0 > r_1 > r_2$. The aim is then to use the Itô formula in order to estimate $\tilde{\theta}(H^\varepsilon)$. First we need some results on the martingale term $X_t + Y_t$ introduced in equations (2.3.26) and (2.3.31).

Lemma 2.4.0.2. *For $t \leqslant \tau^\varepsilon_\delta$, where τ^ε_δ is defined in equation (2.3.12), the martingales X_t and Y_t defined in equations (2.3.26) and (2.3.31) satisfy:*

$$d\langle X, X\rangle_t \leqslant C_\phi K dt, \qquad d\langle Y, Y\rangle_t \leqslant C_\phi(1+K)dt, \qquad d\langle X, Y\rangle_t \leqslant C_\phi(1+K)dt. \tag{2.4.4}$$

The constant C_ϕ depends on the mass $\|u\|^2_{L^2}$ and the Hilbert-Schmidt norm of ϕ in the space $H^3 \cap \dot{H}^{-4}$.

Proof. First we have for $t \leqslant \tau^\varepsilon_\delta$:

$$\begin{aligned} d\langle X, X\rangle_t &= \sum_{k\in\mathbb{N}} \left(Im \int_{\mathbb{R}} u\partial_x \bar{u} \partial_x^{-1}(\phi e_k) dx \right)^2 dt \\ &\leqslant 4\,\|u\|_{L^2} \sum_{k\in\mathbb{N}} \left\|\partial_x^{-1}(\phi e_k)\right\|^2_{L^\infty} K dt \\ &\leqslant C_\phi K dt. \end{aligned}$$

Then we use Lemma 2.1.0.6 to estimate the quadratic variation of Y_t:

$$d\langle Y, Y\rangle_t \leqslant \varepsilon\,\|u\|^4_{L^4}\, C_\phi(1 + \|z^\varepsilon\|^4_{H^2\cap\dot{H}^{-2}} + \|\zeta^\varepsilon\|^4_{H^2\cap\dot{H}^{-2}})dt$$

and using the Gagliardo-Nirenberg inequality and Proposition 2.3.2.1 we get (assuming $\varepsilon < 1$ and δ small enough):

$$d\langle Y, Y\rangle_t \leqslant C_\phi(1+K)dt.$$

The argument of proof is exactly the same for the last quadratic variation. □

Lemma 2.4.0.3. *There exists a constant $\Gamma > 0$ and a martingale M_t such that for $t < \tau_\delta^\varepsilon$:*

$$\tilde{\theta}(H_t^\varepsilon) \leqslant \tilde{\theta}(H_0^\varepsilon) + \Gamma t + M_t \tag{2.4.5}$$

where we recall that $H^\varepsilon = H + \sqrt{\varepsilon}H_1 + \varepsilon H_2$, with H_1 defined in equation (2.3.24) and H_2 in equation (2.3.29).

We will state some results on the martingale M_t introduced in Lemma 2.4.0.3, but first we focus on the proof of this lemma and on another result which will give us a bound on K.

Proof. The Itô formula gives us:

$$\begin{aligned}
d\tilde{\theta}(H^\varepsilon) =& \tilde{\theta}'(H_t^\varepsilon)dH_t^\varepsilon + \frac{1}{2}\tilde{\theta}''(H_t^\varepsilon)d\langle H^\varepsilon, H^\varepsilon\rangle_t \\
=& \tilde{\theta}'(H^\varepsilon)\mathcal{L}^\varepsilon(H^\varepsilon)dt + \frac{1}{2}\tilde{\theta}''(H^\varepsilon)d\langle X+Y, X+Y\rangle_t + \tilde{\theta}'(H^\varepsilon)d\left(X_t + Y_t\right) \\
=& (I + II)dt + dM_t
\end{aligned}$$

where

$$M_t = \int_0^t \tilde{\theta}'(H^\varepsilon)d\left(X_t + Y_t\right) \tag{2.4.6}$$

is a martingale.

We want to estimate I and II. First we remark that

— $\tilde{\theta}'(x) = \frac{2x}{\varepsilon x^4 + \tilde{M}x^2 + \tilde{N}}$,

— $\tilde{\theta}''(x) = \frac{2(\varepsilon x^4 + \tilde{M}x^2 + \tilde{N}) - 8\varepsilon x^4 - 4\tilde{M}x^2}{(\varepsilon x^4 + \tilde{M}x^2 + \tilde{N})^2} \leqslant \frac{2}{\tilde{M}x^2 + \tilde{N}}$.

Let us start with the first term, using Proposition 2.3.4.1 and Lemma 2.3.4.1:

$$I \leqslant \frac{2H^\varepsilon(\varepsilon K^2 + BK + C)}{\varepsilon(H^\varepsilon)^4 + \tilde{M}(H^\varepsilon)^2 + \tilde{N}} \leqslant \frac{(8K + 2\tilde{C})(\varepsilon K^2 + BK + C)}{\varepsilon(H^\varepsilon)^4 + \tilde{M}(H^\varepsilon)^2 + \tilde{N}}.$$

Using Young inequality, there exist constants $\tilde{\alpha}$ and $\tilde{\beta}$ such that

$$I \leqslant \frac{\varepsilon K^4 + \tilde{\alpha} K^2 + \tilde{\beta}}{\varepsilon (H^\varepsilon)^4 + \tilde{M}(H^\varepsilon)^2 + \tilde{N}}.$$

Next, we use the bound $H^\varepsilon \geqslant \frac{1}{2}K - C_l$ stated in (2.3.4.1) and we distinguish two cases. If $K \leqslant 4C_l$ then $I \leqslant \frac{4^4 C_l^4 + 16\tilde{\alpha} C_l^2 + \tilde{\beta}}{\tilde{N}}$ which is a constant. Otherwise $(H^\varepsilon)^2 \geqslant (\frac{1}{2}K - C_l)^2 \geqslant (\frac{1}{4}K)^2$ and

$$I \leqslant \frac{\varepsilon K^4 + \tilde{\alpha} K^2 + \tilde{\beta}}{\varepsilon (\frac{1}{4}K)^4 + \tilde{M}(\frac{1}{4}K)^2 + \tilde{N}}.$$

Thus, choosing $\tilde{M}$ large enough yields that, I is bounded. Next we estimate II, using Lemma 2.4.0.2. If $K \leqslant 4C_l$, then

$$II \leqslant \frac{4C_\phi(1+K)}{\tilde{M}(H^\varepsilon)^2 + \tilde{N}} \leqslant \frac{C_\phi(1+C_l)}{\tilde{N}}.$$

Otherwise $H^\varepsilon \geqslant \frac{1}{4}K$ and

$$II \leqslant \frac{4C_\phi(1+K)}{\frac{\tilde{M}}{16}K^2 + \tilde{N}},$$

which is bounded. Finally, for $t \leqslant \tau_\delta^\varepsilon$, I and II are bounded if we choose $\tilde{M}, \tilde{N}$ large enough, and

$$d\tilde{\theta}(H^\varepsilon) \leqslant \Gamma + dM_t \quad \Rightarrow \quad \tilde{\theta}(H^\varepsilon) \leqslant \tilde{\theta}(H_0^\varepsilon) + \Gamma t + M_t$$

which concludes the proof. □

Our aim is now to use the previous result to get a bound on K, following the scheme in the proof of Lemma 2.4.0.1.

Lemma 2.4.0.4. *There exists a stopping time τ^ε and constants $C, \Gamma > 0$ such that for all $t \in [0, \tau^\varepsilon]$:*

$$K(t)^2 \leqslant C e^{\tilde{M}(\Gamma t + M_t)}, \tag{2.4.7}$$

where M_t is the martingale defined in equation (2.4.6).

Proof. The proof follows the same steps as the proof of deterministic Lemma 2.4.0.1. We start from the inequality given by the Lemma 2.4.0.3

$$\tilde{\theta}(H^\varepsilon) \leqslant \tilde{\theta}(H_0^\varepsilon) + \Gamma t + M_t, \qquad \text{for } t < \tau_\delta^\varepsilon,$$

and we define the stopping time

$$\tau_1^\varepsilon = \inf\left\{t : \Gamma t + M_t > \frac{1}{2\sqrt{\Delta}} \ln\left(\frac{(H_0^\varepsilon)^2 - r_2}{(H_0^\varepsilon)^2 - r_1}\right)\right\} \wedge \tau_\delta^\varepsilon,$$

where r_1, r_2, and Δ are defined at the beginning of this subsection. Then, as in the proof of Lemma 2.4.0.1 , for $t < \tau_1^\varepsilon$ we have, replacing x by $(H^\varepsilon)^2$ and x_0 by $(H_0^\varepsilon)^2$:

$$(H_t^\varepsilon)^2 \leqslant \frac{-r_2\left((H_0^\varepsilon)^2 - r_1\right)e^{\sqrt{\Delta}(\Gamma t + M_t)} + r_1\left(H_0^\varepsilon)^2 - r_2\right)}{(H_0^\varepsilon)^2 - r_2 - \left((H_0^t)^2 - r_1\right)e^{\sqrt{\Delta}(\Gamma t + M_t)}}$$

$$\leqslant \frac{2\tilde{M}\left((H_0^\varepsilon)^2 + 2\frac{\tilde{N}}{\tilde{M}}\right)e^{\tilde{M}(\Gamma t + M_t)}}{\tilde{M} - 2\varepsilon^2\left((H_0^\varepsilon)^2 + 2\frac{N}{M}\right)e^{\tilde{M}(\Gamma t + M_t)}}.$$

Now, for

$$t \leqslant \tau^\varepsilon = \inf\left\{t : \Gamma t + M_t > \frac{1}{\tilde{M}} \ln\left(\frac{\tilde{M}}{2\varepsilon\left(2(H_0^\varepsilon)^2\tilde{M} + 4\frac{\tilde{N}}{\tilde{M}}\right)}\right)\right\} \wedge \tau_\delta^\varepsilon \wedge T, \tag{2.4.8}$$

we have

$$(H_t^\varepsilon)^2 \leqslant 4\left((H_0^\varepsilon)^2 + 2\frac{\tilde{N}}{\tilde{M}}\right)e^{\tilde{M}(\Gamma t + M_t)}.$$

We can easily verify as in Lemma 2.4.0.1 that $\tau^\varepsilon \leqslant \tau_1^\varepsilon$. This bound on H^ε will help us get a bound on K. Here again we have two cases. If $K \leqslant 4C_l$ then K^2 is obviously bounded. Otherwise $H^\varepsilon \geqslant \frac{1}{4}K$ and thus

$$K^2 \leqslant 64\left((H_0^\varepsilon)^2 + 2\frac{\tilde{N}}{\tilde{M}}\right)e^{\tilde{M}(\Gamma t + M_t)}.$$

□

We remark that the bound we obtained on K involves the martingale M_t. In order to get a bound on the supremum over $t \leqslant \tau^\varepsilon$ of K we need to control the quadratic variation of M_t. This is done in the following result.

Lemma 2.4.0.5. *There exists a constant C such that*

$$d\langle M, M\rangle_t \leqslant C dt,$$

where M_t is the martingale defined in equation (2.4.6).

Proof. Our aim is to bound the quadratic variation of

$$M_t = \tilde{\theta}(H_t^\varepsilon) - \tilde{\theta}(H_0^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \tilde{\theta}(H_s^\varepsilon) ds.$$

We recall that we explicitly know the expression of M_t:

$$M_t = \int_0^t \tilde{\theta}'(H_s^\varepsilon) d\left(X_s + Y_s\right)$$

where X_t and Y_t are respectively defined in (2.3.26) and (2.3.31). Thus we know that

$$\begin{aligned} d\langle M, M\rangle_t &= \left(\tilde{\theta}'(H^\varepsilon)\right)^2 d\langle X+Y, X+Y\rangle_t \\ &\leqslant \frac{4(H^\varepsilon)^2}{\left(\varepsilon(H^\varepsilon)^4 + \tilde{M}(H^\varepsilon)^2 + \tilde{N}\right)^2} C_\phi(1+K)dt \end{aligned} \tag{2.4.9}$$

according to Lemma 2.4.0.2. We use the bound from Lemma 2.3.4.1 to conclude that $d\langle M, M\rangle_t \leqslant C dt$. □

Now, this bound on the quadratic variation gives us a new result about the stopping time τ^ε.

Lemma 2.4.0.6. *Let the initial conditions (u_0, m_0, m_1) of equation (2.3.6) be in $H^1 \times L^2 \times \dot{H}^{-1} \times (H^2 \cap \dot{H}^{-1})^2$ and (z_0, ζ_0) defined in (2.3.10). The stopping time τ^ε defined in (2.4.8) goes a.s. to T as ε goes to zero.*

Proof. We denote by $F(\varepsilon)$ the quantity

$$\frac{1}{\tilde{M}} \ln\left(\frac{\tilde{M}}{2\varepsilon\left(2(H_0^\varepsilon)^2\tilde{M} + 4\frac{\tilde{N}}{\tilde{M}}\right)}\right)$$

and we define, for $T > 0$,

$$\eta_k = \sup_{t\in[k,k+1]\wedge T} (\Gamma t + M_t).$$

We have used a martingale inequality (see [DPZ14, Theorem 3.14])

$$\mathbb{E}[\eta_k] \leqslant \Gamma T + 3\mathbb{E}[\langle M, M\rangle_{(k+1)\wedge T}^{\frac{1}{2}}] \leqslant CT$$

thanks to Lemma 2.4.0.5. Then for $\delta > 1$

$$\sum_{k\in\mathbb{N}} \mathbb{P}(\eta_k > k^\delta) \leqslant CT \sum_{k\in\mathbb{N}} \frac{1}{k^\delta} < +\infty$$

and thanks to Borel-Cantelli Lemma there exist two random variables X_1 and X_2 such that almost surely, for every $t \in [0,T]$, $\Gamma t + M_t \leqslant X_1 + |t|^\delta X_2$. Thus almost surely $\tau^\varepsilon \geqslant \tilde{\tau}_\varepsilon$ where

$$\tilde{\tau}_\varepsilon = \inf_{t\in[0,T]} \left\{ X_1 + t^\delta X_2 > F(\varepsilon) \right\}$$

which goes a.s. to T as ε goes to 0 because $\lim_{\varepsilon\to 0} F(\varepsilon) = +\infty$. □

Now we can estimate the bound on the supremum of K_t on the time interval $[0, \tau^\varepsilon]$, and prove Proposition 2.3.4.2.

Proof of Proposition 2.3.4.2. From equation (2.4.7), we deduce

$$\sup_{t\leqslant\tau^\varepsilon} (K_t)^2 \leqslant Ce^{\tilde{M}\Gamma\tau^\varepsilon} \sup_{t\leqslant\tau^\varepsilon} e^{\tilde{M}M_t}.$$

On the other hand, we have by Ito's formula

$$e^{\tilde{M}M_t} = 1 + \int_0^t \tilde{M}e^{\tilde{M}M_s} dM_s + \frac{\tilde{M}^2}{2} \int_0^t e^{\tilde{M}M_s} d\langle M, M\rangle_s$$

where we have already shown that $d\langle M, M\rangle_t \leqslant Cdt$. Thus thanks to a martingale inequality (see [DPZ14, Theorem 3.14]), we get

$$\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon} (K_t)^2\right] \leqslant C(T) \left(1 + \mathbb{E}\left[\langle \int_0^T e^{\tilde{M}M_s} dM_s \rangle^{\frac{1}{2}}\right] + \mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon} \int_0^t e^{\tilde{M}M_s} d\langle M, M\rangle_s\right]\right)$$

and using Cauchy-Schwarz inequality on the last expectation, we have

$$\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon} (K_t)^2\right] \leqslant C(T) \left(1 + \mathbb{E}\left[\left(\int_0^T e^{2\tilde{M}M_s} d\langle M, M\rangle_s\right)\right]^{\frac{1}{2}}\right). \tag{2.4.10}$$

We want to control the expectation, and using Girsanov formula:

$$\begin{aligned}\mathbb{E}\left[\int_0^T e^{2\tilde{M}M_s} ds\right] &= \int_0^T \mathbb{E}\left[e^{2\tilde{M}M_s - \langle 2\tilde{M}M_s\rangle + \langle 2\tilde{M}M_s\rangle}\right] ds \\ &\leqslant \int_0^T \mathbb{E}\left[e^{4\tilde{M}M_s - 2\langle 2\tilde{M}M_s\rangle}\right]^{\frac{1}{2}} \mathbb{E}\left[e^{2\langle 2\tilde{M}M_s\rangle}\right]^{\frac{1}{2}} ds \\ &\leqslant \int_0^T e^{4\tilde{M}^2 C s} ds\end{aligned}$$

hence finally, we get

$$\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(K_t)^2\right] \leqslant C(T).$$

□

Proof of Proposition 2.3.4.3. According to Propostion 2.3.4.1, we have

$$\mathcal{L}^\varepsilon(H^\varepsilon) + \alpha \left\|\sqrt{\varepsilon}V\right\|_{L^2}^2 \leqslant \varepsilon K^2 + BK + C. \tag{2.4.11}$$

Now, we know that fot $t \leqslant \tau^\varepsilon$

$$H_t^\varepsilon + \alpha \int_0^t \left\|\sqrt{\varepsilon}V\right\|_{L^2}^2 ds = \int_0^t \mathcal{L}^\varepsilon(H_s^\varepsilon) + \left\|\sqrt{\varepsilon}V\right\|_{L^2}^2 ds + (X_t + Y_t) + H_0^\varepsilon$$

where X_t and Y_t are the martingales defined in equation (2.3.26) and (2.3.31). We also have from Lemma 2.3.4.1 that $\frac{1}{2}K - C_l \leqslant H^\varepsilon$, and thanks to the previous equation we get:

$$\frac{1}{2}K(t) + \alpha \int_0^t \left\|\sqrt{\varepsilon}V\right\|_{L^2}^2 ds \leqslant \int_0^t (\varepsilon K^2 + BK + C) ds + (X_t + Y_t) + C.$$

It implies that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon} K(t) + \alpha \int_0^{\tau^\varepsilon} \left\|\sqrt{\varepsilon}V\right\|_{L^2}^2 ds\right] \\ &\qquad \leqslant C + C\int_0^T \mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(K(t))^2\right] ds + \mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(X_t + Y_t)\right],\end{aligned}$$

and using Proposition 2.3.4.2 and a martingale inequality, which can be found in [DPZ14, Theorem 3.14], we obtain:

$$\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon} K(t) + \alpha \left\|\sqrt{\varepsilon}V\right\|^2_{L^2([0,\tau^\varepsilon),L^2)} ds\right] \leqslant C(T) + 3\mathbb{E}\left[\langle X+Y, X+Y\rangle^{\frac{1}{2}}\right]. \tag{2.4.12}$$

According to Lemma 2.4.0.2 and Proposition 2.3.4.2 we get

$$\begin{aligned}\mathbb{E}\left[\frac{1}{2}\sup_{t\leqslant\tau^\varepsilon} K(t) + \alpha \left\|\sqrt{\varepsilon}V\right\|^2_{L^2([0,\tau^\varepsilon),L^2)} ds\right] &\leqslant C(T) + C_\phi \mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(1+K)\right] \\ &\leqslant C(T),\end{aligned}$$

which concludes the proof. □

We would like to propose another proof of Proposition 2.3.4.2, with another stopping time that we will still denote by τ^ε. This second proof is simpler, but we think it is interesting to write these two methods because the stopping time introduced in the first proof is more explicit than the one in the second proof.

Second proof of Proposition 2.3.4.2. First we prove that there exists $C(T) > 0$ and a stopping time τ^ε going to T in probability as ε goes to 0 such that

$$\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(H_t^\varepsilon)^2\right] \leqslant C(T).$$

We start from Propostion 2.3.4.1, which states that

$$\mathcal{L}^\varepsilon(H_t^\varepsilon) \leqslant \varepsilon K_t^2 + BK_t + C,$$

where K is defined in (2.3.22). It gives us that

$$dH_t^\varepsilon \leqslant (\varepsilon K^2 + BK + C)dt + dN_t$$

with N_t the martingale defined by $N_t = X_t + Y_t$, where X_t and Y_t are introduced respectively in (2.3.26) and (2.3.31). We are interested in an estimate for $(H^\varepsilon)^2$, so we use

Lemma 2.3.4.1 to get with the Itô formula:

$$\begin{aligned} d(H_t^\varepsilon)^2 =& 2H_t^\varepsilon dH_t^\varepsilon + H_t^\varepsilon d\langle H^\varepsilon, H^\varepsilon\rangle_t \\ \leqslant& 2H_t^\varepsilon \left(4\varepsilon(H_t^\varepsilon)^2 + BH_t^\varepsilon + C\right) dt + 2H_t^\varepsilon dN_t + H_t^\varepsilon d\langle N, N\rangle_t. \end{aligned}$$

Then we use Lemma 2.4.0.2 and Young's inequality to obtain

$$d(H_t^\varepsilon)^2 \leqslant \left(\varepsilon^2(H_t^\varepsilon)^4 + B(H_t^\varepsilon)^2 + C\right) dt + 2H_t^\varepsilon dN_t,$$

where the constants B and C might be slightly different, but still independent of ε. Now we introduce the stopping time

$$\tau^\varepsilon = \inf\left\{t \in [0,T]; \varepsilon^2(H_t^\varepsilon)^2 \geqslant 1\right\} \wedge \tau_\delta^\varepsilon, \tag{2.4.13}$$

with τ_δ^ε the stopping time defined in (2.3.12). For $t \leqslant \tau^\varepsilon$ we get

$$d(H_t^\varepsilon)^2 \leqslant \left((1+B)(H_t^\varepsilon)^2 + C\right) dt + 2H_t^\varepsilon dN_t,$$

which can be rewritten under its integral form:

$$(H_t^\varepsilon)^2 \leqslant (H_0)^2 + Ct + \int_0^t (1+B)(H_s^\varepsilon)^2 ds + 2\int_0^t H_s^\varepsilon dN_s.$$

Taking the expectation of this last inequality, we use Grönwall's Lemma and obtain the estimate:

$$\mathbb{E}\left[(H_{t\wedge\tau^\varepsilon}^\varepsilon)^2\right] \leqslant \left(C + \mathbb{E}\left[(H_0)^2\right]\right) e^{(1+B)t\wedge\tau^\varepsilon}, \tag{2.4.14}$$

Finally, we can estimate the supremum over $[0,\tau^\varepsilon]$:

$$\begin{aligned} \mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(H_t^\varepsilon)^2\right] \leqslant& (H_0)^2 + CT + (1+B)\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2 ds\right] + \mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}\int_0^t 2H_s^\varepsilon dN_s\right] \\ \leqslant& (H_0)^2 + CT + (1+B)\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2 ds\right] + \mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2 d\langle N, N\rangle_s^{\frac{1}{2}}\right] \end{aligned}$$

where we used a martingale inequality which can be found for example in [DPZ14, Theorem 3.14]. Now using Lemma 2.4.0.2, we can write

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(H_t^\varepsilon)^2\right] \leqslant&(H_0)^2+CT+(1+C_\phi+B)\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2ds\right]\\&+C_\phi\mathbb{E}\left[\left(\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^3ds\right)^{\frac{1}{2}}\right]\\ \leqslant&(H_0)^2+CT+C\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2ds\right]\\&+C_\phi\mathbb{E}\left[\left(\sup_{t\leqslant\tau^\varepsilon}(H_t^\varepsilon)^2\right)^{\frac{1}{2}}\left(\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2ds\right)^{\frac{1}{2}}\right],\end{aligned}$$

and finally,

$$\frac{1}{2}\mathbb{E}\left[\sup_{t\leqslant\tau^\varepsilon}(H_t^\varepsilon)^2\right] \leqslant(H_0)^2+CT+C\mathbb{E}\left[\int_0^{T\wedge\tau^\varepsilon}(H_s^\varepsilon)^2ds\right],$$

which allows us to conclude thanks to equation (2.4.14). It remains to prove the behavior of τ^ε as ε goes to 0. We simply use Markov's inequality:

$$\begin{aligned}\mathbb{P}\left(\tau^\varepsilon<T\right)=&\mathbb{P}\left(\sup_{t\leqslant\tau^\varepsilon}(H_t^\varepsilon)^2\geqslant\frac{1}{\varepsilon^2}\right)\\ \leqslant&\varepsilon^2C(T),\end{aligned}$$

which concludes the proof of Proposition 2.3.4.2, since K is bounded by H^ε thanks to Lemma 2.3.4.1. □

Proof of Lemma 2.3.6.3. We want to compute $k(x,y) = \mathbb{E}_\nu\left[z(x)\mathcal{M}^{-1}z(y)+z(y)\mathcal{M}^{-1}z(x)\right]$. We know from Lemma 2.3.2.1 that $\mathcal{M}^{-1}z = (\partial_x^2)^{-1}\zeta+\alpha(\partial_x^2)^{-1}z$. We start by computing the term $\mathbb{E}_\nu\left[z(x)(\partial_x^2)^{-1}z(y)\right]$. According to equation (2.3.17), this term is equal to

$$\sum_{k\in\mathbb{N}}\int_0^\infty(S_\alpha(t)(0,\phi e_k))_1(x)(\partial_x^2)^{-1}(S_\alpha(t)(0,\phi e_k))_1(y)dt$$

with

$$
\begin{aligned}
(S_\alpha(t)(0,\phi e_k))_1(x) =& 2e^{-\frac{\alpha}{2}t}\int_{|\xi|\leqslant\frac{\alpha}{2}} \sinh\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\frac{\hat{\phi e_k}(\xi)}{\sqrt{\alpha^2-4\xi^2}}e^{ix\xi}d\xi \\
&+ 2e^{-\frac{\alpha}{2}t}\int_{|\xi|\geqslant\frac{\alpha}{2}} \sin\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right)\frac{\hat{\phi e_k}(\xi)}{\sqrt{4\xi^2-\alpha^2}}e^{ix\xi}d\xi,
\end{aligned}
$$

where we have used (2.1.3) and (2.1.4). Thus we compute

$$
\begin{aligned}
&4\sum_{k\in\mathbb{N}}\int_0^\infty e^{-\alpha t}\left[\int_{|\xi|\leqslant\frac{\alpha}{2}} \sinh\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t\right)\frac{\hat{\phi e_k}(\xi)}{\sqrt{\alpha^2-4\xi^2}}e^{ix\xi}d\xi\right. \\
&\qquad\qquad \left.+\int_{|\xi|\geqslant\frac{\alpha}{2}} \sin\left(\frac{\sqrt{4\xi^2-\alpha^2}}{2}t\right)\frac{\hat{\phi e_k}(\xi)}{\sqrt{4\xi^2-\alpha^2}}e^{ix\xi}d\xi\right] \\
&\qquad\left[\int_{|\eta|\leqslant\frac{\alpha}{2}} \sinh\left(\frac{\sqrt{\alpha^2-4\eta^2}}{2}t\right)\frac{\widehat{(\partial_x^2)^{-1}\phi} e_k(\eta)}{\sqrt{\alpha^2-4\eta^2}}e^{iy\eta}d\eta\right. \\
&\qquad\qquad \left.+\int_{|\eta|\geqslant\frac{\alpha}{2}} \sin\left(\frac{\sqrt{4\eta^2-\alpha^2}}{2}t\right)\frac{\widehat{(\partial_x^2)^{-1}\phi} e_k(\eta)}{\sqrt{4\eta^2-\alpha^2}}e^{iy\eta}d\eta\right]dt \\
&= I + II + III + IV.
\end{aligned}
$$

We focus on the computation of the low frequency product:

$$
\begin{aligned}
I = 2\sum_{k\in\mathbb{N}}\int_{|\xi|\leqslant\frac{\alpha}{2}}\int_{|\eta|\leqslant\frac{\alpha}{2}} &\frac{\hat{\phi e_k}(\xi)}{\sqrt{\alpha^2-4\xi^2}}\frac{\widehat{(\partial_x^2)^{-1}\phi} e_k(\eta)}{\sqrt{\alpha^2-4\eta^2}}e^{i(x\xi+y\eta)} \\
&\int_0^\infty e^{-\alpha t}\left[\cosh\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t+\frac{\sqrt{\alpha^2-4\eta^2}}{2}t\right)\right. \\
&\qquad\left.-\cosh\left(\frac{\sqrt{\alpha^2-4\xi^2}}{2}t-\frac{\sqrt{\alpha^2-4\eta^2}}{2}t\right)\right]dtd\eta d\xi.
\end{aligned}
$$

The term in the time integral is equal to

$$
\begin{aligned}
&\frac{1}{2}\left[\frac{1}{\alpha-\frac{1}{2}\sqrt{\alpha^2-4\xi^2}-\frac{1}{2}\sqrt{\alpha^2-4\eta^2}}+\frac{1}{\alpha+\frac{1}{2}\sqrt{\alpha^2-4\xi^2}+\frac{1}{2}\sqrt{\alpha^2-4\eta^2}}\right. \\
&\quad\left.-\frac{1}{\alpha-\frac{1}{2}\sqrt{\alpha^2-4\xi^2}+\frac{1}{2}\sqrt{\alpha^2-4\eta^2}}-\frac{1}{\alpha+\frac{1}{2}\sqrt{\alpha^2-4\xi^2}-\frac{1}{2}\sqrt{\alpha^2-4\eta^2}}\right] \\
&=\frac{\alpha\sqrt{\alpha^2-4\xi^2}\sqrt{\alpha^2-4\eta^2}}{(\xi^2-\eta^2)^2+2\alpha^2(\xi^2+\eta^2)},
\end{aligned}
$$

so that

$$I = 2\alpha \sum_{k\in\mathbb{N}} \int_{|\xi|\leqslant\frac{\alpha}{2}} \int_{|\eta|\leqslant\frac{\alpha}{2}} \frac{\hat{\phi e_k}(\xi)\widehat{(\partial_x^2)^{-1}\phi} e_k(\eta)}{(\xi^2-\eta^2)^2+2\alpha^2(\xi^2+\eta^2)} e^{i(x\xi+y\eta)} d\eta d\xi.$$

We can do similar computations for the other products, and we obtain

$$\mathbb{E}_\nu\left[z(x)(\partial_x^2)^{-1}z(y)\right] = \sum_{k\in\mathbb{N}} \int_{\mathbb{R}}\int_{\mathbb{R}} K_1(\xi,\eta)\hat{\phi e_k}(\xi)\hat{\phi e_k}(\eta)e^{i(x\xi+y\eta)}d\eta d\xi, \tag{2.4.15}$$

with

$$K_1(\xi,\eta) = -\frac{1}{\eta^2}\times\frac{2\alpha}{(\xi^2-\eta^2)^2+2\alpha^2(\xi^2+\eta^2)}. \tag{2.4.16}$$

The computation of $\mathbb{E}_\nu\left[z(x)(\partial_x^2)^{-1}\zeta(y)\right]$ is very similar so that we do not write it in details. We obtain

$$\begin{aligned}\mathbb{E}_\nu\left[z(x)(\partial_x^2)^{-1}\zeta(y)\right] = &-\frac{\alpha}{2}\mathbb{E}_\nu\left[z(x)(\partial_x^2)^{-1}z(y)\right]\\ &+\sum_{k\in\mathbb{N}}\int_{\mathbb{R}}\int_{\mathbb{R}} K_2(\xi,\eta)\hat{\phi e_k}(\xi)\hat{\phi e_k}(\eta)e^{i(x\xi+y\eta)}d\eta d\xi,\end{aligned} \tag{2.4.17}$$

with

$$K_2(\xi,\eta) = -\frac{1}{\eta^2}\times\frac{\alpha^2+\xi^2-\eta^2}{(\xi^2-\eta^2)^2+2\alpha^2(\xi^2+\eta^2)}. \tag{2.4.18}$$

Finally, defining $K(\xi,\eta)=\frac{\alpha}{2}K_1(\xi,\eta)+K_2(\xi,\eta)$ we obtain

$$\begin{aligned}\mathbb{E}_\nu\big[z(x)\mathcal{M}^{-1}z(y) &+z(y)\mathcal{M}^{-1}z(x)\big]\\ &=\sum_{k\in\mathbb{N}}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(K(\xi,\eta)+K(\eta,\xi)\right)\hat{\phi e_k}(\xi)\hat{\phi e_k}(\eta)e^{i(x\xi+y\eta)}d\eta d\xi\\ &=-\sum_{k\in\mathbb{N}}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\xi^2\eta^2}\hat{\phi e_k}(\xi)\hat{\phi e_k}(\eta)e^{i(x\xi+y\eta)}d\eta d\xi\end{aligned} \tag{2.4.19}$$

which gives us the result of Lemma 2.3.6.3. □

CHAPTER 3

ASYMPTOTIC PRESERVING SCHEMES

Introduction

This chapter is devoted to numerical results obtained in our study of the stochastic Zakharov system

$$\begin{cases} i\partial_t X_t^\varepsilon = -\partial_x^2 X_t^\varepsilon + n_t^\varepsilon X_t^\varepsilon \\ \varepsilon^2 d(\partial_t n_t^\varepsilon) + \alpha\varepsilon\partial_t n_t^\varepsilon = \partial_x^2\left(n_t^\varepsilon + |X_t^\varepsilon|^2\right) dt + \phi dW_t \end{cases}. \tag{3.0.1}$$

More precisely we try to construct a numerical scheme which approximates the stochastic Zakarov system, and which converges to a numerical scheme approximating the stochastic Nonlinear Schrödinger equation, limit of Zakharov system as ε goes to 0:

$$idX_t = \left(-\partial_x^2 X_t - |X_t|^2 X_t - iX_t F\right) dt - X_t(\partial_x^2)^{-1}\phi dW_t, \tag{3.0.2}$$

the last two terms corresponding to the Stratonovitch noise $X_t \circ (\partial_x^2)^{-1}\phi dW_t$. For this purpose we study the methods of Asymptotic Preserving schemes. First notions of the AP schemes were developed in [Jin99] for multiscale partial differential equations, and similar studies can be found in [HJL17], [Jin12],[Pup19]. In the domain of stochastic partial differential equations, we can cite numerical results as for examples [BSDDM05] and [DDM02], where the authors study the focusing Nonlinear Schrödinger equation in dimension 1 and 2. Cohen and Dujardin also introduce in [CD17] an explicit exponential integrator to present numerical experiments on Nonlinear Schrödinger equations with white noise dispersion, and develop this notion for the stochastic Manakov equation in [BCD20]. Asymptotic Preserving properties and numerical schemes were then studied for stochastic equations in [AF19], [DM16] and [Mar06]. This kind of numerical schemes allows us to approximate simultaneously the original multiscale model and its limit. Indeed, there exist numerical schemes, which are consistent for any fixed value of $\varepsilon > 0$, but which limiting schemes are not consistent with the good interpretation of the noise, namely the Stratonovich interpretation. We can refer to [FG18] or [LAE08] for examples of such schemes. The definition of AP schemes for stochastic differential equations is detailed in [BRR22]. In the deterministic setting, some numerical schemes have been studied for the Zakharov system, and its behavior as ε gets close to 0. One refers, for example, to [Su18], where the author presents three different schemes based on an exponential wave integrator in phase for the wave equation in (3.0.1) and a time-splitting technique for the Shrödinger equation in (3.0.1), in order to compare their convergence rates in time in the subsonic limit regime $\varepsilon << 1$.

We begin with simple models of stochastic differential equations in order to expose in a simplified context the theory of Asymptotic Preserving schemes, and then we try to get closer to a stochastic differential system which could correspond to the equations satisfied by the Fourier modes of the solution of (3.0.1) without the nonlinearity. Finally we work on the infinite-dimensional case, where the Asymptotic Preserving schemes theory is no longer available. In this case we provide numerical simulations which seem to show the convergence to the solution of a numerical scheme encoding the solution of (3.0.2).

3.1 Definition of AP scheme and first example

We are interested in the following equation: for $t \geqslant 0$

$$\begin{cases} \partial_t X_t^\varepsilon = \frac{\sigma(X_t^\varepsilon) n_t^\varepsilon}{\sqrt{\varepsilon}} \\ \partial_t n_t^\varepsilon = m_t^\varepsilon \\ dm_t^\varepsilon = \left(-\frac{\alpha}{\varepsilon} m_t^\varepsilon - \frac{\lambda}{\varepsilon^2} n_t^\varepsilon\right) dt + \frac{1}{\varepsilon^{\frac{3}{2}}} dB_t \end{cases} . \tag{3.1.1}$$

Here, X_t^ε and m_t^ε are $\mathbb{R}$-valued processes, σ is a $\mathcal{C}_b^2(\mathbb{R})$ function, B_t is a real Brownian motion, α and λ are two positive constants. It can be shown that this equation converges as ε goes to 0, to the following stochastic differential equation:

$$dX_t = \frac{\sigma(X_t)}{\lambda} \circ dB_t = \frac{1}{2\lambda^2} \sigma(X_t)\sigma'(X_t) dt + \frac{\sigma(X_t)}{\lambda} dB_t, \tag{3.1.2}$$

where $\circ dB_t$ denotes the Stratonovitch convention. The aim here is to create an AP scheme for this problem. Let $T > 0$, $N \in \mathbb{N}$ and let $\Delta t = \frac{T}{N}$ denote the time-step size. Let $(\gamma_k)_{0 \leqslant k \leqslant N-1}$ be a family of independent standard $\mathbb{R}$-valued Gaussian random variables. We assume that the initial conditions $X_0 \in \mathbb{R}$ and $(n_0^\varepsilon, m_0^\varepsilon) \in \mathbb{R}^2$ are deterministic quantities satisfying

$$X_0^\varepsilon = x_0^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} x_0, \qquad \sup_{\varepsilon\in(0,1]} |n_0^\varepsilon| < \infty, \qquad \sup_{\varepsilon\in(0,1]} |m_0^\varepsilon| < \infty. \tag{3.1.3}$$

If we denote by $\mathcal{L}^\varepsilon$ the infinitesimal generator of $(X^\varepsilon, m^\varepsilon, n^\varepsilon)$ we have

$$\mathcal{L}^\varepsilon\varphi(x,n,m) = \frac{1}{\sqrt{\varepsilon}} n\sigma(x)\partial_x\varphi + m\partial_n\varphi - \left(\frac{\alpha}{\varepsilon}m + \frac{\lambda}{\varepsilon^2}n\right)\partial_m\varphi + \frac{1}{2\varepsilon^3}\partial_m^2\varphi. \tag{3.1.4}$$

Now we can define our numerical scheme based on the scheme (1.6) introduced in [BRR22] but adapted to the second order equation in time in (3.0.1):

$$\begin{cases} \frac{n_{k+1}^\varepsilon - n_k^\varepsilon}{\Delta t} = m_{k+\frac{1}{2}}^\varepsilon \\ \frac{m_{k+1}^\varepsilon - m_k^\varepsilon}{\Delta t} = -\frac{\alpha}{\varepsilon} m_{k+\frac{1}{2}}^\varepsilon - \frac{\lambda}{\varepsilon^2} n_{k+\frac{1}{2}}^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}\sqrt{\Delta t}}\gamma_k \\ \frac{Y_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} = \sigma(X_k^\varepsilon)\frac{1}{\sqrt{\varepsilon}} n_{k+\frac{1}{2}}^\varepsilon \\ \frac{X_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} = \frac{\sigma(X_k^\varepsilon)+\sigma(Y_{k+1}^\varepsilon)}{2}\frac{1}{\sqrt{\varepsilon}} n_{k+\frac{1}{2}}^\varepsilon \end{cases}, \tag{3.1.5}$$

where $m_{k+\frac{1}{2}} = \frac{m_{k+1}+m_k}{2}$. Note that we can define this scheme using an operator denoted by $\Phi_{\Delta t}^\varepsilon$:

$$\forall n \in \{0,1,\cdots,N-1\}, \quad (X_{k+1}^\varepsilon, n_{k+1}^\varepsilon, m_{k+1}^\varepsilon) = \Phi_{\Delta t}^\varepsilon(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon, \gamma_k). \tag{3.1.6}$$

We are interested by the asymptotic preserving property of our scheme. Thus we need to introduce a limiting scheme, which approximates the limiting equation (3.1.2). We denote this numerical scheme by $(X_k)_{k\in[\![0,N]\!]}$, and we define it in the same way as for the scheme (3.1.5) using an operator denoted by $\Phi_{\Delta t}$:

$$\forall n \in \{0,1,\cdots,N-1\}, \quad X_{k+1} = \Phi_{\Delta t}(X_k, \gamma_k). \tag{3.1.7}$$

We recall the definition of an asymptotic preserving numerical scheme, which can be found in [RR21]. We first state three assumptions.

Assumptions 3.1.0.1. *1. The operator $\Phi_{\Delta t}^\varepsilon$ is defined for all $\varepsilon \in (0,1]$ and all $\Delta t \in (0, \Delta t_0]$, where $\Delta t_0 > 0$ is independent of ε, and*

$$\sup_{\varepsilon\in(0,1]} \sup_{\Delta t\in(0,\Delta t_0]} \sup_{k\in\mathbb{N}} \frac{\mathbb{E}\left[|n_k^\varepsilon|^2 + \varepsilon^2|m_k^\varepsilon|^2\right]}{1+\mathbb{E}\left[|n_0^\varepsilon|^2 + \varepsilon^2|m_0^\varepsilon|^2\right]} < \infty. \tag{3.1.8}$$

2. For all $\varepsilon \in (0,1]$, the numerical scheme (3.1.5) is consistent in the weak sense with *the system* (3.1.1)*: for all bounded continuous function* φ*:*

$$\mathbb{E}[\varphi(X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon)] \underset{N\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(X_T^\varepsilon, n_T^\varepsilon, m_T^\varepsilon)].$$

3. *For every* $\Delta t \in (0, \Delta t_0]$*, there exists a limiting scheme* $(X_k)_k$*, given in* (3.1.7)*, such that for every bounded continuous function* φ*:*

$$\mathbb{E}[\varphi(X_N^\varepsilon)] \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}[\varphi(X_N)].$$

With these assumptions, we can give the definition of an asymptotic preserving scheme.

Definition 3.1.0.1. *Let Assumptions 3.1.0.1 be satisfied. The scheme* (3.1.5) *is said to be asymptotic preserving if the limiting scheme given in the third assumption is consistent in the weak sense with the limiting equation* (3.1.2)*: for all bounded continuous function* φ*:*

$$\mathbb{E}[\varphi(X_N)] \underset{N\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(X_T)]$$

Note that the definition that we gave is similar to [BRR20, Definition 3.4]. The only differences is that we adapted [BRR20, Property 3.1] to our problem, and that the third assumption in Assumptions 3.1.0.1 replaces Property 3.3 in [BRR20]. In [BRR20] it is only assumed that the weak convergence is true for one time step, but without the uniform convergence property, it is not possible to recover our assumption, which is necessary for the definition of an asymptotic preserving scheme.

We also would like to give details about a remark mentionned in [BRR20], which gives a way to prove the weak consistency of the scheme with $\varepsilon > 0$ fixed and of the limiting scheme.

Lemma 3.1.0.1. *The second point of Assumptions 3.1.0.1, namely the weak consistency of the scheme* (3.1.5)*, can be verified using the following sufficient criterion: for all* $\varphi \in \mathcal{C}_b^4(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$*, when* Δt *goes to 0:*

$$\mathbb{E}\left[\varphi\left(\Phi_{\Delta t}^\varepsilon(x,n,m,\gamma)\right)\right] = \varphi(x,n,m) + \Delta t\mathcal{L}^\varepsilon\varphi(x,n,m) + R(\Delta t,\varepsilon,\varphi,x,n,m), \tag{3.1.9}$$

where the last term satisfies $|R(\Delta t,\varepsilon,\varphi,x,n,m)| \leqslant C(\varphi,\varepsilon)\left(1+|n|^2+|m|^2\right)\Delta t^{\frac{3}{2}}$ *for some constant* $C(\varphi,\varepsilon) > 0$.

Proof. Let $\varphi \in \mathcal{C}_b^4(\mathbb{R}^3)$, and $\varepsilon > 0$ fixed. Then

$$\begin{aligned}\mathbb{E}\left[\varphi\left(X^\varepsilon(T,x), n^\varepsilon(T,n), m^\varepsilon(T,m)\right)\right] - \mathbb{E}\left[\varphi\left(X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon\right)\right] \\ = u(T,x,n,m) - \mathbb{E}[u(0, X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon)],\end{aligned}$$

where u is the solution of

$$\begin{cases} \partial_t u(t,x,n,m) = \mathbb{E}\left[\mathcal{L}^\varepsilon \varphi\left(X^\varepsilon(t,x), n^\varepsilon(t,n), m^\varepsilon(t,m)\right)\right] \\ u(0,x,n,m) = \varphi(x,n,m) \end{cases}.$$

We compute

$$\begin{aligned} u(T,x,n,m) - &\mathbb{E}[u(0, X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon)] \\ &= -\sum_{k=0}^{N-1} \mathbb{E}[u(T-t_{k+1}, X_{k+1}^\varepsilon, n_{k+1}^\varepsilon, m_{k+1}^\varepsilon) - u(T-t_k, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)] \\ &= -\sum_{k=0}^{N-1} \mathbb{E}\big[\big(u(T-t_{k+1}, X_{k+1}^\varepsilon, n_{k+1}^\varepsilon, m_{k+1}^\varepsilon) - u(T-t_{k+1}, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)\big) \\ &+ (u(T-t_{k+1}, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon) - u(T-t_k, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon))\big]. \end{aligned}$$

Now, conditioning by $(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)$ and using (3.1.9), we get

$$\begin{aligned} u(T, x,& n, m) - \mathbb{E}[u(0, X_N^\varepsilon, n_N^\varepsilon, m_N^\varepsilon)] \\ &= -\sum_{k=0}^{N-1} \Delta t \mathbb{E}\left[\mathcal{L}^\varepsilon \varphi\left(X^\varepsilon(T-t_{k+1}, X_k^\varepsilon), n^\varepsilon(T-t_{k+1}, n_k^\varepsilon), m^\varepsilon(T-t_{k+1}, m_k^\varepsilon)\right)\right] \\ &+ \mathbb{E}[R(\Delta t, \varepsilon, \varphi, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)] \\ &- \mathbb{E}\int_{t_k}^{t_{k+1}} \mathcal{L}^\varepsilon \varphi\left(X^\varepsilon(T-s, X_k^\varepsilon), n^\varepsilon(T-s, n_k^\varepsilon), m^\varepsilon(T-s, m_k^\varepsilon)\right) ds. \end{aligned}$$

Thanks to the assumption made on R and using (3.1.8) in Assumptions 3.1.0.1, we know that

$$\begin{aligned} \sum_{k=0}^{N-1} R(\Delta t, \varepsilon, \varphi, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon) &\leqslant C(\varphi, \varepsilon) \sum_{k=0}^{N-1} (1 + |n_k^\varepsilon|^2 + |m_k^\varepsilon|^2) \Delta t^{\frac{3}{2}} \\ &\leqslant C(\varphi, \varepsilon)(1 + \frac{1}{\varepsilon^2}) \mathbb{E}[|n_0^\varepsilon|^2 + \varepsilon^2 |m_0^\varepsilon|^2] \sqrt{\Delta t}. \end{aligned}$$

It remains to estimate the following quantity:

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Delta t\mathcal{L}^\varepsilon\varphi(X^\varepsilon(T-t_{k+1},X_k^\varepsilon),n^\varepsilon(T-t_{k+1},n_k^\varepsilon),m^\varepsilon(T-t_{k+1},m_k^\varepsilon))\\
&\qquad\qquad -\int_{t_k}^{t_{k+1}}\mathcal{L}^\varepsilon\varphi\left(X^\varepsilon(T-s,X_k^\varepsilon),n^\varepsilon(T-s,n_k^\varepsilon),m^\varepsilon(T-s,m_k^\varepsilon)\right)ds\Bigg]\\
&=\mathbb{E}\Bigg[\int_{t_k}^{t_{k+1}}\mathcal{L}^\varepsilon\varphi(X^\varepsilon(T-t_{k+1},X_k^\varepsilon),n^\varepsilon(T-t_{k+1},n_k^\varepsilon),m^\varepsilon(T-t_{k+1},m_k^\varepsilon))\\
&\qquad\qquad -\mathcal{L}^\varepsilon\varphi\left(X^\varepsilon(T-s,X_k^\varepsilon),n^\varepsilon(T-s,n_k^\varepsilon),m^\varepsilon(T-s,m_k^\varepsilon)\right)ds\Bigg].
\end{aligned}
$$

Denoting by v the function $\mathcal{L}^\varepsilon\varphi(X^\varepsilon(\cdot,X_k^\varepsilon),n^\varepsilon(\cdot,n_k^\varepsilon),m^\varepsilon(\cdot,m_k^\varepsilon))$, we have by Itô formula

$$
\begin{aligned}
&v(T-t_{k+1})-v(T-s)\\
&\quad=\int_s^{t_{k+1}}D_x\mathcal{L}^\varepsilon\varphi(X^\varepsilon(T-\tau,X_k^\varepsilon),n^\varepsilon(T-\tau,n_k^\varepsilon),m^\varepsilon(T-\tau,m_k^\varepsilon))dX_\tau^\varepsilon\\
&\quad\quad+\int_s^{t_{k+1}}D_n\mathcal{L}^\varepsilon\varphi(X^\varepsilon(T-\tau,X_k^\varepsilon),n^\varepsilon(T-\tau,n_k^\varepsilon),m^\varepsilon(T-\tau,m_k^\varepsilon))dn_\tau^\varepsilon\\
&\quad\quad+\int_s^{t_{k+1}}D_m\mathcal{L}^\varepsilon\varphi(X^\varepsilon(T-\tau,X_k^\varepsilon),n^\varepsilon(T-\tau,n_k^\varepsilon),m^\varepsilon(T-\tau,m_k^\varepsilon))dm_\tau^\varepsilon\\
&\quad\quad+\frac{1}{2}\int_s^{t_{k+1}}D_m^2\mathcal{L}^\varepsilon\varphi(X^\varepsilon(T-\tau,X_k^\varepsilon),n^\varepsilon(T-\tau,n_k^\varepsilon),m^\varepsilon(T-\tau,m_k^\varepsilon))d\langle m^\varepsilon,m^\varepsilon\rangle_\tau.
\end{aligned}
$$

Then, using the boundedness of the functions φ and σ in (3.1.5), and (3.1.8) in Assumptions 3.1.0.1, we can bound all these quantities and we get that

$$
\mathbb{E}\left[v(T-t_{k+1})-v(T-s)\right]\leqslant C(\varphi,\sigma,\varepsilon)\mathbb{E}[|n_0^\varepsilon|^2+\varepsilon^2|m_0^\varepsilon|^2]\Delta t,
$$

and as we integrate this term between t_k and t_{k+1}, we obtain:

$$
\begin{aligned}
&|\mathbb{E}\left[\varphi\left(X^\varepsilon(T,x),n^\varepsilon(T,n),m^\varepsilon(T,m)\right)\right]-\mathbb{E}\left[\varphi\left(X_N^\varepsilon,n_N^\varepsilon,m_N^\varepsilon\right)\right]|\\
&\quad\leqslant\sum_{k=0}^{N-1}C(\varphi,\varepsilon)\mathbb{E}[|n_0^\varepsilon|^2+\varepsilon^2|m_0^\varepsilon|^2]\Delta t^2\\
&\quad\leqslant C(\varphi,\varepsilon)\mathbb{E}[|n_0^\varepsilon|^2+\varepsilon^2|m_0^\varepsilon|^2]\Delta t,
\end{aligned}
$$

and thus, we have the weak consistency stated in the second point of Assumptions 3.1.0.1.

$\square$

We prove the following result.

Proposition 3.1.0.1. *Under assumption* (3.1.3)*, the numerical scheme* (3.1.5) *is asymptotic preserving in the sense of Definition 3.1.0.1. Moreover, the limiting scheme is given by*

$$\begin{cases} Y_{k+1} = X_k + \frac{\sqrt{\Delta t}}{\lambda}\sigma(X_k)\gamma_k \\ X_{k+1} = X_k + \frac{\sqrt{\Delta t}}{\lambda}\frac{\sigma(X_k)+\sigma(Y_{k+1})}{2}\gamma_k \end{cases}. \tag{3.1.10}$$

Proof of Proposition 3.1.0.1. First we can prove that the process $(n_k^\varepsilon, m_k^\varepsilon)_k$ satisfies (3.1.8). This is done in Appendix with $n^\varepsilon_{k+\frac{1}{2}}$ and $m^\varepsilon_{k+\frac{1}{2}}$ replaced by n^ε_{k+1} and m^ε_{k+1} in order to minimize the technicalities of the proof, but the arguments are the same for the scheme (3.1.5). The bound (3.1.8) in Assumptions 3.1.0.1 ensures that $\varepsilon m_k^\varepsilon$ and n_k^ε are bounded in $L^1(\Omega)$. According to (3.1.5) we have

$$\varepsilon^{\frac{3}{2}}\frac{m^\varepsilon_{k+1}-m^\varepsilon_k}{\Delta t} = -\alpha\sqrt{\varepsilon}\frac{n^\varepsilon_{k+1}-n^\varepsilon_k}{\Delta t} - \frac{\lambda}{\sqrt{\varepsilon}}n^\varepsilon_{k+\frac{1}{2}} + \frac{1}{\sqrt{\Delta t}}\gamma_k,$$

thus in $L^1(\Omega)$:

$$\lim_{\varepsilon\to 0}\frac{\lambda}{\sqrt{\varepsilon}}n^\varepsilon_{k+\frac{1}{2}} = \frac{1}{\sqrt{\Delta t}}\gamma_k. \tag{3.1.11}$$

According to this convergence, we can prove by a recursion argument that for every $k \in [\![0, N]\!]$, X_k^ε converges in probability to X_k and Y_k^ε converges in probability to Y_k as ε goes to 0, with X_k and Y_k defined by the limiting scheme (3.1.10). Indeed, it is true for $k = 0$ because of Assumption (3.1.3). Besides for $\eta, \delta > 0$ we have

$$\begin{aligned}
&\mathbb{P}(|Y^\varepsilon_{k+1} - Y_{k+1}| \geqslant \delta) \\
=&\mathbb{P}(|X_k^\varepsilon - X_k + \Delta t\sigma(X_k^\varepsilon)(\frac{1}{\sqrt{\varepsilon}}n^\varepsilon_{k+\frac{1}{2}} - \frac{1}{\lambda\sqrt{\Delta t}}\gamma_k) + \frac{\sqrt{\Delta t}}{\lambda}(\sigma(X_k^\varepsilon) - \sigma(X_k))\gamma_k| \geqslant \delta) \\
\leqslant&\mathbb{P}(|X_k^\varepsilon - X_k| \geqslant \frac{\delta}{3}) + \mathbb{P}(\Delta t M|\frac{1}{\sqrt{\varepsilon}}n^\varepsilon_{k+\frac{1}{2}} - \frac{1}{\lambda\sqrt{\Delta t}}\gamma_k| \geqslant \frac{\delta}{3}) \\
&+ \mathbb{P}(\frac{L\sqrt{\Delta t}}{\lambda}|\gamma_k||X_k^\varepsilon - X_k| \geqslant \frac{\delta}{3}),
\end{aligned}$$

where $\sigma(x) \leqslant M$ and L is the Lipschitz constant of σ. We deal with the first two terms of the right-hand side of the inequality using the recursion assumption and the convergence in $L^1(\Omega)$ of $\frac{1}{\sqrt{\varepsilon}}n_{k+\frac{1}{2}}$. For the last term, we have

$$\mathbb{P}(\frac{L\sqrt{\Delta t}}{\lambda}|\gamma_k||X_k^\varepsilon - X_k| \geqslant \frac{\delta}{3}) \leqslant \mathbb{P}(|\gamma_k| \geqslant R) + \mathbb{P}(|X_k^\varepsilon - X_k| \geqslant \frac{\delta\lambda}{3RL\sqrt{\Delta t}}).$$

Thus for R large enough and ε small enough, we get that $\mathbb{P}(|Y_{k+1}^\varepsilon - Y_{k+1}| \geqslant \delta) \leqslant \eta$. The procedure is the same for X_k^ε and X_k. Thanks to this convergence in probability, we obtain that for every bounded continuous function φ

$$\mathbb{E}[\varphi(X_N^\varepsilon)] \underset{\varepsilon\to 0}{\longrightarrow} \mathbb{E}[\varphi(X_N)],$$

where $(X_k)_k$ is defined by the limiting scheme (3.1.10). We have proved that the numerical scheme (3.1.5) converges to the limiting scheme (3.1.10), which corresponds to the third item of Assumption 3.1.0.1. It remains to prove that the schemes (3.1.5) and (3.1.10) are consistent with (3.1.1) and (3.1.2), respectively.

On one hand let $\varepsilon > 0$ be fixed. To prove the consistency of (3.1.5) with (3.1.1) it is sufficient to prove that, for every $\varphi \in \mathcal{C}_b^2(\mathbb{R}\times\mathbb{R}\times\mathbb{R})$, when Δt goes to 0,

$$\begin{aligned}\mathbb{E}\left[\varphi(X_{k+1}^\varepsilon, n_{k+1}^\varepsilon, m_{k+1}^\varepsilon)\right] =& \mathbb{E}\left[\varphi(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)\right] + \Delta t\mathbb{E}\left[\mathcal{L}^\varepsilon\varphi(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)\right] \\ &+ R(\Delta t, \varepsilon, \varphi, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon),\end{aligned} \tag{3.1.12}$$

with R a function satisfying the same assumption as the function R in (3.1.9). We first prove that

$$(X_{k+1}^\varepsilon, Y_{k+1}^\varepsilon, n_{k+1}^\varepsilon, m_{k+1}^\varepsilon) = (X_k^\varepsilon, X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon) + \kappa^\varepsilon\left(\frac{1}{2}\right),$$

where κ^ε denotes any random function such that

$$\mathbb{E}[|\kappa^\varepsilon(p)|] \leqslant C(\varphi, \varepsilon, n_0^\varepsilon, m_0^\varepsilon)\Delta t^p. \tag{3.1.13}$$

For example the equation on n_{k+1}^ε gives us $n_{k+1}^\varepsilon = n_k^\varepsilon + \kappa^\varepsilon(1)$ with $\kappa^\varepsilon(1) = \Delta t m_{k+\frac{1}{2}}^\varepsilon$ which is bounded in $L^1(\Omega)$ according to (3.1.8). The boundedness of σ and the estimate (3.1.8) gives us the result. Indeed, we have

$$\begin{aligned}\mathbb{E}\left[|Y_{k+1}^\varepsilon - X_k^\varepsilon|\right] &= \frac{\Delta t}{\sqrt{\varepsilon}}\mathbb{E}\left[|\sigma(X_k^\varepsilon)n_{k+\frac{1}{2}}^\varepsilon|\right] \\ &\leqslant C\frac{\Delta t}{\sqrt{\varepsilon}}\mathbb{E}\left[|n_{k+\frac{1}{2}}^\varepsilon|\right],\end{aligned}$$

which proves the result for Y^ε_{k+1} using (3.1.8). The procedure is the same for X^ε_{k+1}. Hence, using again (3.1.5):

$$\begin{aligned}
n^\varepsilon_{k+1} &= n^\varepsilon_k + \Delta t m^\varepsilon_k + \kappa^\varepsilon\left(\frac{3}{2}\right),\\
m^\varepsilon_{k+1} &= m^\varepsilon_k - \frac{\alpha\Delta t}{\varepsilon}m^\varepsilon_k - \frac{\lambda\Delta t}{\varepsilon^2}n^\varepsilon_k + \frac{\sqrt{\Delta t}}{\varepsilon^{\frac{3}{2}}}\gamma_k + \kappa^\varepsilon\left(\frac{3}{2}\right),\\
X^\varepsilon_{k+1} &= X^\varepsilon_k + \frac{\Delta t}{\sqrt{\varepsilon}}\sigma(X^\varepsilon_k)n^\varepsilon_k + \kappa^\varepsilon(2).
\end{aligned}$$

Thanks to these estimates we can prove (3.1.12):

$$\begin{aligned}
&\mathbb{E}\left[\varphi(X^\varepsilon_{k+1}, n^\varepsilon_{k+1}, m^\varepsilon_{k+1})\right]\\
&= \mathbb{E}\left[\varphi(X^\varepsilon_k, n^\varepsilon_{k+1}, m^\varepsilon_{k+1}) + \frac{\Delta t}{\sqrt{\varepsilon}}\sigma(X^\varepsilon_k)n^\varepsilon_k\partial_x\varphi(X^\varepsilon_k, n^\varepsilon_{k+1}, m^\varepsilon_{k+1}) + \kappa^\varepsilon(2)\right]\\
&= \mathbb{E}\Bigg[\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1}) + \Delta t m^\varepsilon_k\partial_n\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1})\\
&+ \frac{\Delta t}{\sqrt{\varepsilon}}\sigma(X^\varepsilon_k)n^\varepsilon_k\partial_x\varphi(X^\varepsilon_k, n^\varepsilon_{k+1}, m^\varepsilon_{k+1}) + \kappa^\varepsilon\left(\frac{3}{2}\right)\Bigg].
\end{aligned}$$

Since $\partial_n\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1}) = \partial_n\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k) + \kappa^\varepsilon\left(\frac{1}{2}\right)$ and $\partial_x\varphi(X^\varepsilon_k, n^\varepsilon_{k+1}, m^\varepsilon_{k+1}) = \partial_x\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k) + \kappa^\varepsilon\left(\frac{1}{2}\right)$, we get

$$\begin{aligned}
&\mathbb{E}\Bigg[\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1}) + \Delta t m^\varepsilon_k\partial_n\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1})\\
&\qquad + \frac{\Delta t}{\sqrt{\varepsilon}}\sigma(X^\varepsilon_k)n^\varepsilon_k\partial_x\varphi(X^\varepsilon_k, n^\varepsilon_{k+1}, m^\varepsilon_{k+1}) + \kappa^\varepsilon\left(\frac{3}{2}\right)\Bigg]\\
&= \mathbb{E}\Bigg[\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1}) + \Delta t m^\varepsilon_k\partial_n\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k)\\
&\qquad + \frac{\Delta t}{\sqrt{\varepsilon}}\sigma(X^\varepsilon_k)n^\varepsilon_k\partial_x\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k) + \kappa^\varepsilon\left(\frac{3}{2}\right)\Bigg].
\end{aligned} \tag{3.1.14}$$

Besides, the estimate on m^ε_{k+1} found using (3.1.5) gives us

$$\begin{aligned}
\mathbb{E}\left[\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_{k+1})\right] =& \mathbb{E}\Bigg[\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k) - \Delta t\left(\frac{\alpha}{\varepsilon}m^\varepsilon_k + \frac{\lambda}{\varepsilon^2}n^\varepsilon_k\right)\partial_m\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k)\\
&+ \frac{\sqrt{\Delta t}}{\varepsilon^{\frac{3}{2}}}\gamma_k\partial_m\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k) + \frac{\Delta t}{2\varepsilon^3}\gamma_k^2\partial_m^2\varphi(X^\varepsilon_k, n^\varepsilon_k, m^\varepsilon_k) + \kappa^\varepsilon\left(\frac{3}{2}\right)\Bigg].
\end{aligned}$$

The expectation of the term of order $\frac{1}{2}$ in Δt vanishes using the independence between $(X_k^\varepsilon, n_k^\varepsilon, m_k^\varepsilon)$ and the standard Gaussian random variable γ_k. Then if we replace the expression of $\mathbb{E}\left[\varphi(X_k^\varepsilon, n_k^\varepsilon, m_{k+1}^\varepsilon)\right]$ in (3.1.14), we can observe that the infinitesimal generator $\mathcal{L}^\varepsilon\varphi$ is obtained by grouping all the terms of order 1 in Δt. Hence, (3.1.12) follows with the function $R(\Delta t, \varepsilon, \varphi, x, n, m)$ given by $\mathbb{E}\left[\kappa^\varepsilon\left(\frac{3}{2}\right)\right]$, where $\kappa^\varepsilon\left(\frac{3}{2}\right)$ appears in the last equation, and obviously satisfies the conditions of Lemma 3.1.0.1.

On the other hand, recalling that (X_k) is given by the numerical scheme defined by (3.1.10), it remains to prove that for every function $\varphi \in \mathcal{C}_b^2(\mathbb{R})$, when Δt goes to 0,

$$\mathbb{E}\left[\varphi(X_{k+1})\right] = \mathbb{E}\left[\varphi(X_k)\right] + \Delta t \mathbb{E}\left[\mathcal{L}\varphi(X_k)\right] + \tilde{R}(\varphi, \Delta t, X_k), \tag{3.1.15}$$

where $\mathcal{L}$ is the infinitesimal generator of the process X_t solution of (3.1.2), given by

$$\mathcal{L}\varphi(x) = \frac{1}{2\lambda^2}\sigma'(x)\sigma(x)\varphi'(x) + \frac{1}{2\lambda^2}\sigma^2(x)\varphi''(x),$$

and $\tilde{R}$ is a function such that $|\tilde{R}(\varphi, \Delta t, x)| \leqslant C(\varphi)\Delta t^{\frac{3}{2}}$. Performing an expansion at order $\sqrt{\Delta t}$, we get

$$\frac{\sigma(X_k) + \sigma(Y_{k+1})}{2} = \sigma(X_k) + \frac{\sqrt{\Delta t}}{2\lambda}\sigma(X_k)\sigma'(X_k)\gamma_k + \kappa(1),$$

with κ denoting any random function such that $\mathbb{E}[\kappa(p)] \leqslant C(\varphi)\Delta t^p$. It allows us to obtain the following expansion for X_{k+1}:

$$X_{k+1} = X_k + \frac{\sqrt{\Delta t}}{\lambda}\sigma(X_k)\gamma_k + \frac{\Delta t}{2\lambda^2}\sigma(X_k)\sigma'(X_k)\gamma_k^2 + \kappa(\frac{3}{2}).$$

Since γ_k is centered and γ_k and X_k are independant random variables, we obtain (3.1.15) using similar computations as those used to obtain (3.1.12). This proves Proposition 3.1.0.1. □

Now we compare the scheme (3.1.5) for $\varepsilon = 0.01$ and different values of Δt with a Milstein scheme for the limit equation. In our implementations, we take $\sigma(x) = \cos(2\pi x)$. Figure 3.1 shows us that in the case of small ε, for the smallest value of Δt, the AP scheme behaves almost as the Milstein scheme, which is defined by

$$X_{k+1} = X_k + \frac{\sqrt{\Delta t}}{\lambda}\sigma(X_k)\gamma_k + \frac{\Delta t}{2\lambda^2}\sigma(X_k)\sigma'(X_k)\gamma_k^2. \tag{3.1.16}$$

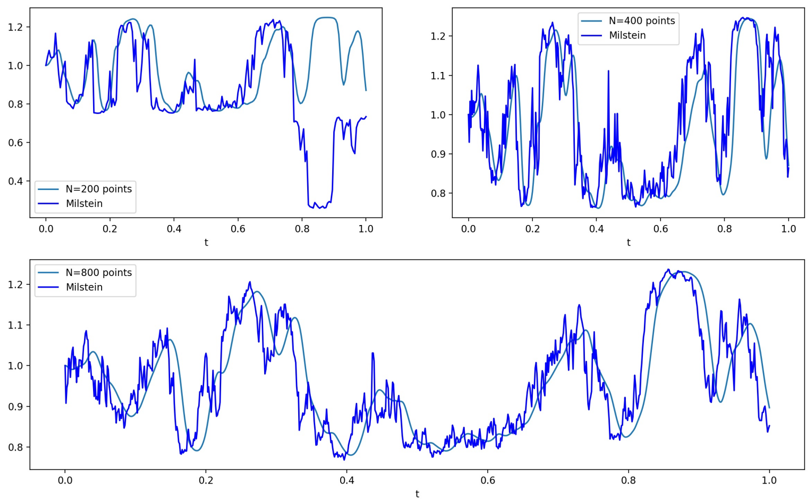

Figure 3.1 – AP scheme for different values of Δt, with $\sigma(x) = \cos(2\pi x)$, $\varepsilon = 0.01$, and comparison with Milstein scheme.

We also give an illustration of the convergence as ε goes to zero. Figure 3.2 represents the AP scheme (3.1.5) for three different values of ε, and compares it with the Milstein scheme (3.1.16) encoding the limit equation (3.1.2). For the smallest value of ε, the AP scheme is almost indistinguishable of the Milstein scheme.

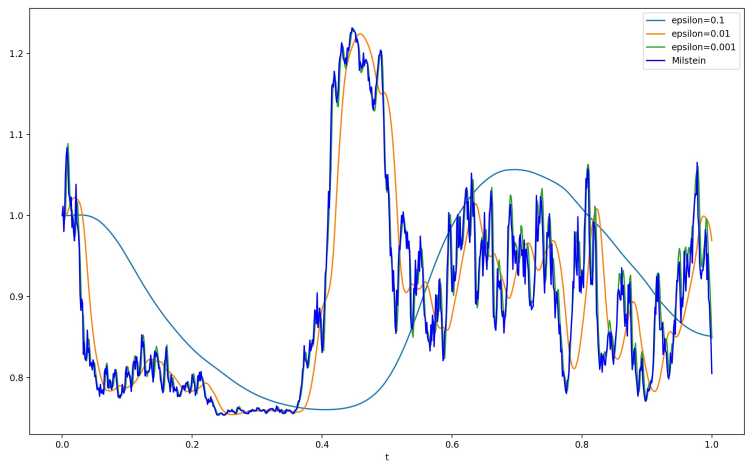

Figure 3.2 – AP scheme for different values of ε, with $\sigma(x) = \cos(2\pi x)$, $\Delta t = 1.10^{-3}$, and comparison with Milstein scheme.

3.2 ODE of Schrödinger type

Now we consider a slightly different equation, which is for $t \geqslant 0$

$$\begin{cases} \partial_t X_t^\varepsilon = i\left(\mu X_t^\varepsilon + \frac{1}{\sqrt{\varepsilon}} X_t^\varepsilon n_t^\varepsilon\right) \\ \partial_t n_t^\varepsilon = m_t^\varepsilon \\ dm_t^\varepsilon = -\frac{\alpha}{\varepsilon} m_t^\varepsilon - \frac{\lambda}{\varepsilon^2} n_t^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}} dB_t \end{cases}, \tag{3.2.1}$$

with $\alpha, \lambda, \mu \geqslant 0$. Here the process n^ε satisfies the same equation as in Equation (3.1.1) but the process X^ε satisfies an equation which is more similar to a Schrödinger-type equation, and thus is $\mathbb{C}$-valued. It can be proved that the solutions of (3.2.1) converge in law to the solutions of the following equation:

$$dX_t = i\mu X_t dt + \frac{i}{\lambda} X_t \circ dB_t. \tag{3.2.2}$$

The difference with the previous problem is that the equation on X^ε can be written as

$$\partial_t X_t^\varepsilon = i\left(b(X_t^\varepsilon) + \frac{1}{\sqrt{\varepsilon}} \sigma(X_t^\varepsilon) n_t^\varepsilon\right),$$

where b and σ are now two unbounded functions. Thus we cannot use a prediction correction scheme, as in (3.1.5), for this equation, because it is not a conservative scheme in this case. Indeed, the boundedness assumption on σ was useful to prove that

$$(Y_{k+1}^\varepsilon, X_{k+1}^\varepsilon) = (X_k^\varepsilon, X_k^\varepsilon) + \kappa^\varepsilon\left(\frac{1}{2}\right),$$

with κ^ε a function satisfying (3.1.13), and we cannot expect to prove a similar result without the conservation of the quantity $|X_k^\varepsilon|^2$. Thus we use a different scheme, which is the following:

$$\begin{cases} \frac{n_{k+1}^\varepsilon - n_k^\varepsilon}{\Delta t} = m_{k+\frac{1}{2}}^\varepsilon \\ \frac{m_{k+1}^\varepsilon - m_k^\varepsilon}{\Delta t} = -\frac{\alpha}{\varepsilon} m_{k+\frac{1}{2}}^\varepsilon - \frac{\lambda}{\varepsilon^2} n_{k+\frac{1}{2}}^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}\sqrt{\Delta t}} \gamma_k \\ \frac{X_{k+1}^\varepsilon - X_k^\varepsilon}{\Delta t} = i\left(\mu X_{k+\frac{1}{2}}^\varepsilon - \frac{1}{\sqrt{\varepsilon}} X_{k+\frac{1}{2}}^\varepsilon n_{k+\frac{1}{2}}^\varepsilon\right) \end{cases}. \tag{3.2.3}$$

In this scheme it is obvious that for every $\varepsilon > 0$ and all $k \in \mathbb{N}$ we have

$$|X_k^\varepsilon|^2 = |X_0^\varepsilon|^2 \tag{3.2.4}$$

since n_k^ε is $\mathbb{R}$-valued. We state the following result on this scheme:

Proposition 3.2.0.1. *Under the assumption* (3.1.3)*, the numerical scheme* (3.2.3) *is asymptotic preserving in the sense of Definition 3.1.0.1. Moreover, the limiting scheme is given by*

$$X_{k+1} = X_k + i\mu\Delta t X_{k+\frac{1}{2}} - iX_{k+\frac{1}{2}}\frac{\sqrt{\Delta t}}{\lambda}\gamma_k. \tag{3.2.5}$$

Proof. Here (3.1.8) is still satisfied because we did not change the equations on n_k^ε and m_k^ε. Thus we have the same convergence in $L^1(\Omega)$ of $\frac{\lambda}{\sqrt{\varepsilon}}n_{k+\frac{1}{2}}^\varepsilon$ to $\frac{1}{\sqrt{\Delta t}}\gamma_k$ as ε goes to 0. The only result that we have to verify is that we still have

$$X_{k+1}^\varepsilon = X_k^\varepsilon + \kappa^\varepsilon\left(\frac{1}{2}\right).$$

For that we compute using Young's inequality

$$\begin{aligned}\mathbb{E}\left[|X_{k+1}^\varepsilon - X_k^\varepsilon|\right] &= \mathbb{E}\left[|i\mu\Delta t X_{k+\frac{1}{2}}^\varepsilon - iX_{k+\frac{1}{2}}^\varepsilon\frac{\Delta t}{\sqrt{\varepsilon}}n_{k+\frac{1}{2}}^\varepsilon|\right] \\ &\leqslant \Delta t\mathbb{E}\left[|i\mu X_{k+\frac{1}{2}}^\varepsilon|\right] + \frac{\Delta t^{\frac{1}{2}}}{2}\mathbb{E}\left[|X_{k+\frac{1}{2}}|^2\right] + \frac{\Delta t^{\frac{3}{2}}}{2\varepsilon}\mathbb{E}\left[|n_{k+\frac{1}{2}}^\varepsilon|^2\right],\end{aligned}$$

and we conclude thanks to (3.1.8) and (3.2.4). The rest of the proof is straightforward using the same procedure as the proof of Proposition 3.1.0.1. □

Here again we have some illustrations of the convergence as ε goes to 0. Figure 3.3 represents the modulus of the solution of the AP scheme (3.2.3) and of the limiting scheme (3.2.5), for $\varepsilon = 0.01$ and $\Delta t = 2.10^{-3}$. Observe that the behavior of the AP scheme is close to the one of the limiting scheme. We can also observe separately the real and imaginary parts of the two schemes, as represented in Figure 3.4, and note the same behavior.

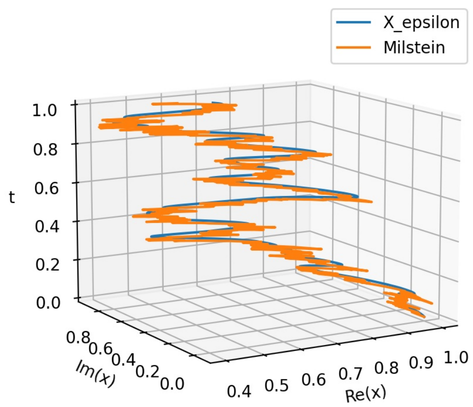

Figure 3.3 – Evolution of the AP scheme (3.2.3) for $\Delta t = 2.10^{-3}$, $\varepsilon = 0.01$ and comparison with the evolution of the limiting scheme (3.2.5).

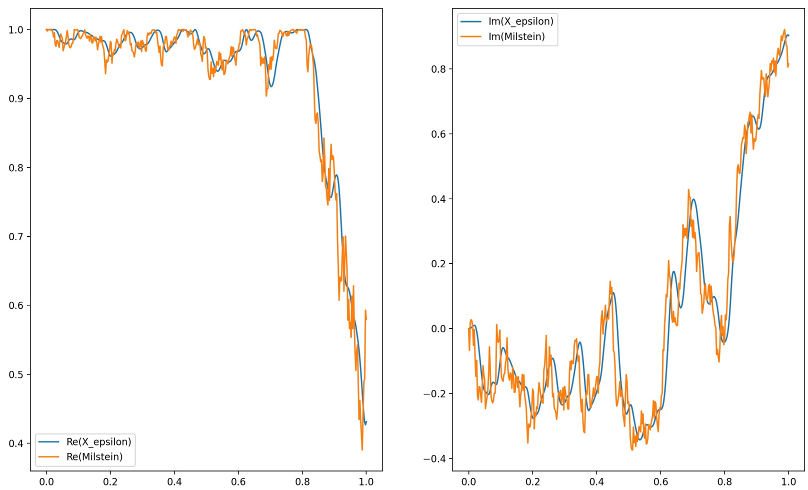

Figure 3.4 – Real and imaginary parts of the AP scheme (3.2.3) for $\Delta t = 2.10^{-3}$, $\varepsilon = 0.01$ and comparison with the real and imaginary parts of the limiting scheme (3.2.5).

3.3 Numerical scheme for the stochastic Zakharov system

We finally study the Zakharov system

$$\left\{\begin{array}{l} \partial_t X^\varepsilon = i\partial_x^2 X^\varepsilon - iX^\varepsilon n^\varepsilon \\ \partial_t n^\varepsilon = m^\varepsilon \\ dm^\varepsilon = \left(-\frac{\alpha}{\varepsilon} m^\varepsilon + \frac{1}{\varepsilon^2}\partial_x^2\left(n^\varepsilon + |X^\varepsilon|^2\right)\right)dt + \frac{1}{\varepsilon^2}\phi dW_t \end{array}\right. . \tag{3.3.1}$$

We try to find a numerical scheme which is a good approximation of (3.3.1), and which converges to a good numerical scheme for the limiting Schrödinger equation. We consider Dirichlet conditions on the boundary of the domain $[-R, R]$ for $R > 0$. Let $J \in \mathbb{N}$ and let $\Delta x = \frac{2R}{J}$ denote the space-step size. For $j \in [\![0, J]\!]$, we denote by X_k^j the quantity given by the numerical scheme which approximates the solution $X(k\Delta t, -R + j\Delta x)$. We also define the matrix $A \in \mathcal{M}_{J-1}(\mathbb{R})$, which encodes the discrete Laplacian, as

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}. \tag{3.3.2}$$

Finally, for $L \in \mathbb{N}$, we define for $\ell \in [\![1, L]\!]$ the function φ_ℓ as

$$\varphi_\ell(j) = \sin\left(\frac{\pi \ell j \Delta x}{R}\right), \qquad j \in [\![0, J]\!], \tag{3.3.3}$$

and the family $(\gamma_{\ell,k})$ of independent standard $\mathbb{R}$-valued Gaussian random variables. We know according to (2.3.6) and (2.3.8) in Chapter II that the system (3.3.1) can be written as

$$\left\{\begin{array}{l} \partial_t X^\varepsilon(t) = i\partial_x^2 X^\varepsilon(t) - iX^\varepsilon(t) n^\varepsilon(t) - \frac{i}{\sqrt{\varepsilon}} X^\varepsilon(t) z^\varepsilon(t) \\ \partial_t n^\varepsilon(t) = m^\varepsilon(t) \\ \partial_t m^\varepsilon(t) = -\frac{\alpha}{\varepsilon} m^\varepsilon(t) + \frac{1}{\varepsilon^2}\partial_x^2\left(n^\varepsilon(t) + |X^\varepsilon(t)|^2\right) \\ \partial_t z^\varepsilon(t) = \zeta^\varepsilon(t) \\ d\zeta^\varepsilon(t) = \left(-\frac{\alpha}{\varepsilon}\zeta^\varepsilon(t) + \frac{1}{\varepsilon^2}\partial_x^2 z^\varepsilon(t)\right)dt + \frac{1}{\varepsilon^{\frac{3}{2}}}\phi dW_t \end{array}\right. . \tag{3.3.4}$$

We denote by $(X_0, n_0, m_0, z_0, \zeta_0)$ the initial data. Our first candidate is the following scheme, which has good conservatives properties.

Lemma 3.3.0.1. *Consider the numerical scheme*

$$\begin{cases} \frac{n^\varepsilon_{k+1}-n^\varepsilon_k}{\Delta t} = m^\varepsilon_{k+\frac{1}{2}} \\ \frac{m^\varepsilon_{k+1}-m^\varepsilon_k}{\Delta t} = -\frac{\alpha}{\varepsilon} m^\varepsilon_{k+\frac{1}{2}} + \frac{1}{\varepsilon^2} A \left(n^\varepsilon_{k+\frac{1}{2}} + \frac{|X^\varepsilon_{k+1}|^2+|X^\varepsilon_k|^2}{2} \right) \\ \frac{z^\varepsilon_{k+1}-z^\varepsilon_k}{\Delta t} = \zeta^\varepsilon_{k+\frac{1}{2}} \\ \frac{\zeta^\varepsilon_{k+1}-\zeta^\varepsilon_k}{\Delta t} = -\frac{\alpha}{\varepsilon}\zeta^\varepsilon_{k+\frac{1}{2}} + \frac{1}{\varepsilon^2} A z^\varepsilon_{k+\frac{1}{2}} + \frac{1}{\varepsilon^{3/2}\sqrt{\Delta t}} \left(\sum_{\ell=1}^L \varphi_\ell(j)\gamma_{\ell,k} \right)_{j\in[\![1,J-1]\!]} \\ \frac{X^\varepsilon_{k+1}-X^\varepsilon_k}{\Delta t} = iAX^\varepsilon_{k+\frac{1}{2}} - iX^\varepsilon_{k+\frac{1}{2}} n^\varepsilon_{k+\frac{1}{2}} - \frac{i}{\sqrt{\varepsilon}} X^\varepsilon_{k+\frac{1}{2}} z^\varepsilon_{k+\frac{1}{2}} \end{cases}, \tag{3.3.5}$$

with the initial data and the boundary conditions

$$\begin{aligned} X^0_k = n^0_k = m^0_k = z^0_k = \zeta^0_k &= 0, & k &\in [\![0, N]\!], \\ X^J_k = n^J_k = m^J_k = z^J_k = \zeta^J_k &= 0, & k &\in [\![0, N]\!], \\ (X^j_0, n^j_0, m^j_0, z^j_0, \zeta^j_0) &= (X_0, n_0, m_0, z_0, \zeta_0)(-R + j\Delta x), & j &\in [\![1, J-1]\!]. \end{aligned}$$

where A is defined in (3.3.2), and the product $X^\varepsilon_{k+\frac{1}{2}} n^\varepsilon_{k+\frac{1}{2}}$ denotes the vector $(X^j_{k+\frac{1}{2}} n^j_{k+\frac{1}{2}})_{j\in[\![1,J-1]\!]}$. Define

$$\begin{aligned} H^\varepsilon_k = {} & \left\| (-A)^{\frac{1}{2}} X^\varepsilon_k \right\|^2_{\ell^2([\![1,J-1]\!])} + \frac{1}{2} \left(\|n^\varepsilon_k\|^2_{\ell^2([\![0,J]\!])} + \left\| \varepsilon(-A)^{-\frac{1}{2}} m^\varepsilon_k \right\|^2_{\ell^2([\![0,J]\!])} \right) \\ & + \langle n^\varepsilon_k, |X^\varepsilon_k|^2 \rangle, \end{aligned} \tag{3.3.6}$$

where $\|\cdot\|_{\ell^2([\![1,J-1]\!])}$ and $\langle \cdot, \cdot \rangle$ are respectively the notations for

$$\|u\|_{\ell^2([\![1,J-1]\!])} = \left(\Delta x \sum_{j=1}^{J-1} (u^j)^2 \right)^{\frac{1}{2}}, \text{ and } \langle u, v \rangle = \Delta x \sum_{j=1}^{J-1} u^j v^j.$$

Then the numerical scheme preserves the following quantities for every $k \in [\![0, N]\!]$:

$$\begin{gathered} \|X^\varepsilon_k\|^2_{\ell^2([\![1,J-1]\!])}, \\ H^\varepsilon_k + \alpha \sum_{l=0}^{k-1} \left\| \sqrt{\varepsilon}(-A)^{-\frac{1}{2}} m^\varepsilon_{l+\frac{1}{2}} \right\|^2_{\ell^2[\![1,J-1]\!]} + \frac{1}{\sqrt{\varepsilon}} \sum_{l=0}^{k-1} \langle z^\varepsilon_{l+\frac{1}{2}}, \frac{|X^\varepsilon_{l+1}|^2 - |X^\varepsilon_l|^2}{\Delta t} \rangle. \end{gathered} \tag{3.3.7}$$

Proof. First we take the scalar product between the last equation of (3.3.5) and $i\frac{\bar{X}^\varepsilon_{k+1}-\bar{X}^\varepsilon_k}{\Delta t}$, and then we take the real part. We have

$$
\begin{aligned}
0 =& Re\left(-\langle AX^\varepsilon_{k+\frac12}, \frac{\bar X^\varepsilon_{k+1}-\bar X^\varepsilon_k}{\Delta t}\rangle + \langle X^\varepsilon_{k+\frac12} n^\varepsilon_{k+\frac12}, \frac{\bar X^\varepsilon_{k+1}-\bar X^\varepsilon_k}{\Delta t}\rangle\right)\\
&+ Re\left(\frac{1}{\sqrt{\varepsilon}}\langle X^\varepsilon_{k+\frac12} z^\varepsilon_{k+\frac12}, \frac{\bar X^\varepsilon_{k+1}-\bar X^\varepsilon_k}{\Delta t}\rangle\right)
\end{aligned}
$$

which gives us

$$
\begin{aligned}
0 =& \frac{1}{\Delta t}\left(\left\|(-A)^{\frac12}X^\varepsilon_{k+1}\right\|^2_{\ell^2([\![1,J-1]\!])} - \left\|(-A)^{\frac12}X^\varepsilon_{k}\right\|^2_{\ell^2([\![1,J-1]\!])}\right)\\
&+ \frac{1}{\Delta t}\langle n^\varepsilon_{k+\frac12}, |X^\varepsilon_{k+1}|^2 - |X^\varepsilon_k|^2\rangle + \frac{1}{\sqrt{\varepsilon}\Delta t}\langle z^\varepsilon_{k+\frac12}, |X^\varepsilon_{k+1}|^2 - |X^\varepsilon_k|^2\rangle
\end{aligned}
\tag{3.3.8}
$$

Then we take the scalar product between the second equation of (3.3.5) and $A^{-1}m^\varepsilon_{k+\frac12}$:

$$
\begin{aligned}
\langle \frac{m^\varepsilon_{k+1}-m^\varepsilon_k}{\Delta t}, A^{-1}m^\varepsilon_{k+\frac12}\rangle =& \frac{1}{\varepsilon^2}\langle A\left(n^\varepsilon_{k+\frac12} + \frac{|X^\varepsilon_{k+1}|^2+|X^\varepsilon_k|^2}{2}\right), A^{-1}m^\varepsilon_{k+\frac12}\rangle\\
&- \frac{\alpha}{\varepsilon}\langle m^\varepsilon_{k+\frac12}, A^{-1}m^\varepsilon_{k+\frac12}\rangle.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
&-\frac{1}{2\Delta t}\left(\left\|(-A)^{-\frac12}m^\varepsilon_{k+1}\right\|^2_{\ell^2([\![1,J-1]\!])} - \left\|(-A)^{-\frac12}m^\varepsilon_{k}\right\|^2_{\ell^2([\![1,J-1]\!])}\right)\\
&\quad = \frac{1}{\varepsilon^2}\langle n^\varepsilon_{k+\frac12} + \frac{|X^\varepsilon_{k+1}|^2+|X^\varepsilon_k|^2}{2}, m^\varepsilon_{k+\frac12}\rangle + \frac{\alpha}{\varepsilon}\left\|(-A)^{-\frac12}m^\varepsilon_{k+\frac12}\right\|^2_{\ell^2[\![1,J-1]\!]}.
\end{aligned}
\tag{3.3.9}
$$

Now we add (3.3.8) with (3.3.9) multiplied by $-\varepsilon^2$. We get

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\left\|(-A)^{\frac{1}{2}}X_{k+1}^\varepsilon\right\|^2_{\ell^2(\llbracket 1,J-1\rrbracket)}-\left\|(-A)^{\frac{1}{2}}X_{k}^\varepsilon\right\|^2_{\ell^2(\llbracket 1,J-1\rrbracket)}\right)\\
&\quad+\frac{\varepsilon^2}{2\Delta t}\left(\left\|(-A)^{-\frac{1}{2}}m_{k+1}^\varepsilon\right\|^2_{\ell^2(\llbracket 0,J\rrbracket)}-\left\|(-A)^{-\frac{1}{2}}m_{k}^\varepsilon\right\|^2_{\ell^2(\llbracket 0,J\rrbracket)}\right)\\
&\quad+\frac{1}{\Delta t}\langle n_{k+\frac{1}{2}}^\varepsilon,|X_{k+1}^\varepsilon|^2-|X_k^\varepsilon|^2\rangle\\
&=-\langle n_{k+\frac{1}{2}}^\varepsilon,m_{k+\frac{1}{2}}^\varepsilon\rangle-\langle\frac{|X_{k+1}^\varepsilon|^2+|X_k^\varepsilon|^2}{2},m_{k+\frac{1}{2}}^\varepsilon\rangle-\alpha\varepsilon\left\|(-A)^{-\frac{1}{2}}m_{k+\frac{1}{2}}^\varepsilon\right\|^2_{\ell^2\llbracket 1,J-1\rrbracket}\\
&\quad-\frac{1}{\sqrt{\varepsilon}\Delta t}\langle z_{k+\frac{1}{2}}^\varepsilon,|X_{k+1}^\varepsilon|^2-|X_k^\varepsilon|^2\rangle\\
&=-\frac{1}{2\Delta t}\left(\left\|n_{k+1}^\varepsilon\right\|^2_{\ell^2(\llbracket 1,J-1\rrbracket)}-\|n_k^\varepsilon\|^2_{\ell^2(\llbracket 1,J-1\rrbracket)}\right)-\langle\frac{|X_{k+1}^\varepsilon|^2+|X_k^\varepsilon|^2}{2},\frac{n_{k+1}^\varepsilon-n_k^\varepsilon}{\Delta t}\rangle\\
&\quad-\alpha\varepsilon\left\|(-A)^{-\frac{1}{2}}m_{k+\frac{1}{2}}^\varepsilon\right\|^2_{\ell^2\llbracket 1,J-1\rrbracket}-\frac{1}{\sqrt{\varepsilon}\Delta t}\langle z_{k+\frac{1}{2}}^\varepsilon,|X_{k+1}^\varepsilon|^2-|X_k^\varepsilon|^2\rangle
\end{aligned}$$

using the first equation of (3.3.5). We conclude the proof noticing that

$$\begin{aligned}
\frac{1}{\Delta t}\langle n_{k+\frac{1}{2}}^\varepsilon,|X_{k+1}^\varepsilon|^2-|X_k^\varepsilon|^2\rangle+\frac{1}{2\Delta t}\langle|X_{k+1}^\varepsilon|^2+|X_k^\varepsilon|^2,n_{k+1}^\varepsilon-n_k^\varepsilon\rangle\\
=\frac{1}{\Delta t}\langle n_{k+1}^\varepsilon,|X_{k+1}^\varepsilon|^2\rangle-\langle n_k^\varepsilon,|X_k^\varepsilon|^2\rangle
\end{aligned}$$

□

It is difficult to implement the previous scheme using a finite difference approach because of the nonlinearity, so we choose to use a relaxation method, which is detailled for example in [Bes04] for the nonlinear deterministic Schrödinger equation. We introduce a new variable ψ_k^ε, and the scheme becomes

$$\left\{\begin{aligned}
\frac{\psi_{k+1}^\varepsilon+\psi_k^\varepsilon}{2}&=|X_k^\varepsilon|^2\\
\frac{n_{k+1}^\varepsilon-n_k^\varepsilon}{\Delta t}&=m_{k+\frac{1}{2}}^\varepsilon\\
\frac{m_{k+1}^j-m_k^j}{\Delta t}&=-\frac{\alpha}{\varepsilon}m_{k+\frac{1}{2}}^\varepsilon+\frac{1}{\varepsilon^2}A\left(n_{k+\frac{1}{2}}^\varepsilon+\psi_{k+1}^\varepsilon\right)\\
\frac{z_{k+1}^\varepsilon-z_k^\varepsilon}{\Delta t}&=\zeta_{k+\frac{1}{2}}^\varepsilon\\
\frac{\zeta_{k+1}^\varepsilon-\zeta_k^\varepsilon}{\Delta t}&=-\frac{\alpha}{\varepsilon}\zeta_{k+\frac{1}{2}}^\varepsilon+\frac{1}{\varepsilon^2}Az_{k+\frac{1}{2}}^\varepsilon+\frac{1}{\varepsilon^{3/2}\sqrt{\Delta t}}\left(\sum_{\ell=1}^L\varphi_\ell(j)\gamma_{\ell,k}\right)_{j\in\llbracket 1,J-1\rrbracket}\\
\frac{X_{k+1}^\varepsilon-X_k^\varepsilon}{\Delta t}&=iAX_{k+\frac{1}{2}}^\varepsilon-iX_{k+\frac{1}{2}}^\varepsilon n_{k+\frac{1}{2}}^\varepsilon-\frac{i}{\sqrt{\varepsilon}}X_{k+\frac{1}{2}}^\varepsilon z_{k+\frac{1}{2}}^\varepsilon
\end{aligned}\right.\tag{3.3.10}$$

with $\psi_0^\varepsilon=|X_0^\varepsilon|^2$. The advantage of this scheme is that it is simpler to implement since it is only linearly implicit. However, except for the ℓ^2-norm of X_k^ε, there is no obvious conservation for this scheme, contrary to the scheme (3.3.5) (see Lemma 3.3.0.1). We can assume formally that the numerical scheme (3.3.10) converges to the following scheme, which is close to the numerical scheme studied in [Bes04] but has an additional term coming from the stochastic perturbation:

$$\left\{\begin{array}{rcl} \frac{\psi_{k+1}+\psi_k}{2} & = & |X_k|^2 \\ \frac{X_{k+1}-X_k}{\Delta t} & = & iAX_{k+\frac{1}{2}} + iX_{k+\frac{1}{2}}\psi_{k+1} \\ & & -\frac{i}{\sqrt{\Delta t}}(-A)^{-1}\left(\sum_{\ell=1}^{L}\varphi_\ell(j)\gamma_{\ell,k}\right)_{j\in[\![1,J-1]\!]} X_{k+\frac{1}{2}} \end{array}\right. . \tag{3.3.11}$$

We compare the scheme (3.3.10) with another scheme, introduced in [Gla92]. In our version, we add a term corresponding to the damping, and a term describing the stochastic perturbation:

$$\left\{\begin{array}{l} i\frac{X^j_{k+1}-X^j_k}{\Delta t} + AX^j_{k+\frac{1}{2}} = n^j_{k+\frac{1}{2}}X^j_{k+\frac{1}{2}} + \frac{1}{\sqrt{\varepsilon}}z^j_{k+\frac{1}{2}}X^j_{k+\frac{1}{2}} \\ \\ \frac{n^j_{k+1}-2n^j_k+n^j_{k-1}}{\Delta t^2} + \frac{\alpha}{\varepsilon}\frac{n^j_{k+1}-n^j_{k-1}}{2\Delta t} - \frac{1}{2\varepsilon^2}A(n^j_{k+1}+n^j_{k-1}) = \frac{1}{\varepsilon^2}A(|X^j_k|^2) \\ \\ \frac{z^j_{k+1}-2z^j_k+z^j_{k-1}}{\Delta t^2} + \frac{\alpha}{\varepsilon}\frac{z^j_{k+1}-z^j_{k-1}}{2\Delta t} - \frac{1}{2\varepsilon^2}A(z^j_{k+1}+z^j_{k-1}) = \frac{1}{\varepsilon^{3/2}\sqrt{\Delta t}}\sum_{\ell=1}^{L}\varphi_\ell(j)\gamma_{\ell,k} \end{array}\right. . \tag{3.3.12}$$

In [Gla92], it is shown in Theorem 1 that the original scheme preserves a discrete energy. We give the equivalent of [Gla92, Theorem 1] for our scheme (3.3.12).

Proposition 3.3.0.1. *Define a new variable $u_k^\varepsilon = (u_k^j)_{j\in[\![1,J-1]\!]}$ satisfying*

$$Au_k^\varepsilon = \frac{n_{k+1}^\varepsilon - n_k^\varepsilon}{\Delta t}. \tag{3.3.13}$$

Then the quantity H_k^ε defined for $k \in [\![0, N-1]\!]$ by

$$\begin{aligned} H_k^\varepsilon = {} & \left\|(-A)^{\frac{1}{2}}X_{k+1}^\varepsilon\right\|^2_{\ell^2([\![1,J-1]\!])} + \frac{1}{2}\left\|\varepsilon(-A)^{\frac{1}{2}}u_k^\varepsilon\right\|^2_{\ell^2([\![1,J-1]\!])} \\ & + \frac{1}{2}\left\|n^\varepsilon_{k+\frac{1}{2}}\right\|^2_{\ell^2([\![1,J]\!])} + \langle n^\varepsilon_{k+\frac{1}{2}}, |X^\varepsilon_{k+1}|^2\rangle \end{aligned} \tag{3.3.14}$$

satisfies for any $k \in [\![0, N-1]\!]$*:*

$$
\begin{aligned}
H_k^\varepsilon = H_0^\varepsilon + \frac{\alpha\varepsilon\Delta t}{4}\sum_{\ell=0}^{k}\langle A(u_\ell^\varepsilon + u_{\ell-1}^\varepsilon), u_\ell^\varepsilon + u_{\ell-1}^\varepsilon\rangle \\
- \frac{1}{2\sqrt{\varepsilon}}\sum_{\ell=1}^{k}\langle z_\ell^\varepsilon + z_{\ell+1}^\varepsilon, |X_{\ell+1}^\varepsilon|^2 - |X_\ell^\varepsilon|^2\rangle,
\end{aligned}
\tag{3.3.15}
$$

with A *defined in* (3.3.2).

Proof. The proof of this result follows exactly the same steps as [Gla92, Theorem 1], and the two new quantities which appear in the right-hand side of (3.3.15) come from the terms that we have added in our scheme in order to describe the damping and the stochastic perturbation. □

We run a few tests on the schemes (3.3.10) and (3.3.12). First we fix $\varepsilon > 0$, and we study the behavior of these two schemes as Δt goes to 0. We choose to take as initial data for X^ε a Gaussian function $X_0^\varepsilon = e^{-\frac{x^2}{2}}$ while $z_0^\varepsilon = \zeta_0^\varepsilon = 0$. The initial conditions are taken such that $n_0^\varepsilon = -|X_0^\varepsilon|^2$. We should choose z_0^ε and ζ_0^ε as in Chapter II, Section 2.3 in order to correspond to the stationary processes, but the convergence to the equilibrium is exponential with rate $\frac{t}{\varepsilon}$. It creates an initial layer of size ε, but since we do not study the solutions for small values of t, it will not have a big influence on our resluts. Figure 3.5 shows for both of the schemes the relative error

$$
\frac{\left\|X_{N_i}^\varepsilon - X_N^{\varepsilon,ref}\right\|_{\ell^2([\![0,J]\!])}}{\left\|X_N^{\varepsilon,ref}\right\|_{\ell^2([\![0,J]\!])}}
$$

for several values Δt_i such that $N_i\Delta t_i = T$, and where $X^{\varepsilon,ref}$ denotes the solution of the numerical scheme with the smallest Δt, namely $\Delta t = 0.001$ in Figure 3.5.

We also represent the relative error

$$
\frac{\left\|X_N^\varepsilon - X_N^{\varepsilon,ref}\right\|_{\ell^2([\![0,J_i]\!]}}{\left\|X_N^{\varepsilon,ref}\right\|_{\ell^2([\![0,J]\!]}}
$$

for several values Δx_i such that $J_i\Delta x_i = 2R$, where this time $X^{\varepsilon,ref}$ denotes the solution of the numerical scheme with the smallest Δx, namely $\Delta x = 2.10^{-3}$. Figure 3.6 shows that the schemes are more sensitive to the variations of Δx, but also that the error is

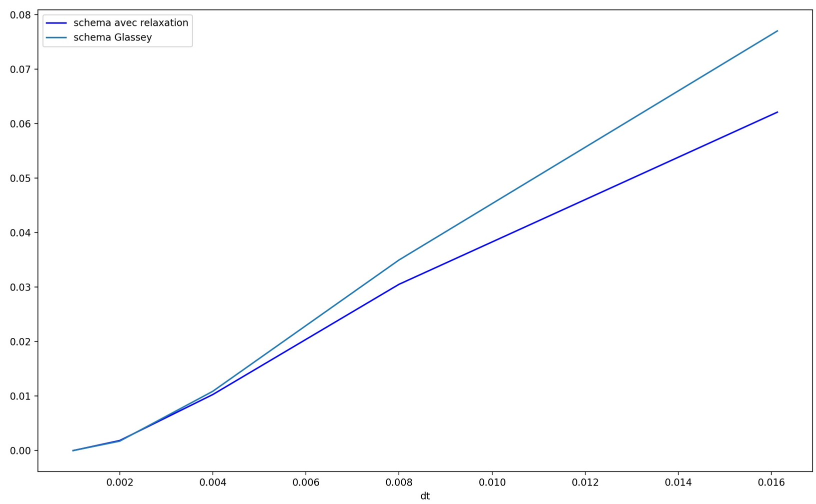

Figure 3.5 – Relative error in ℓ^2-norm for schemes (3.3.10) and (3.3.12) with different values of Δt, for $\varepsilon = 0.01$, and $\Delta x = 0.1$.

relatively small as Δx is small.

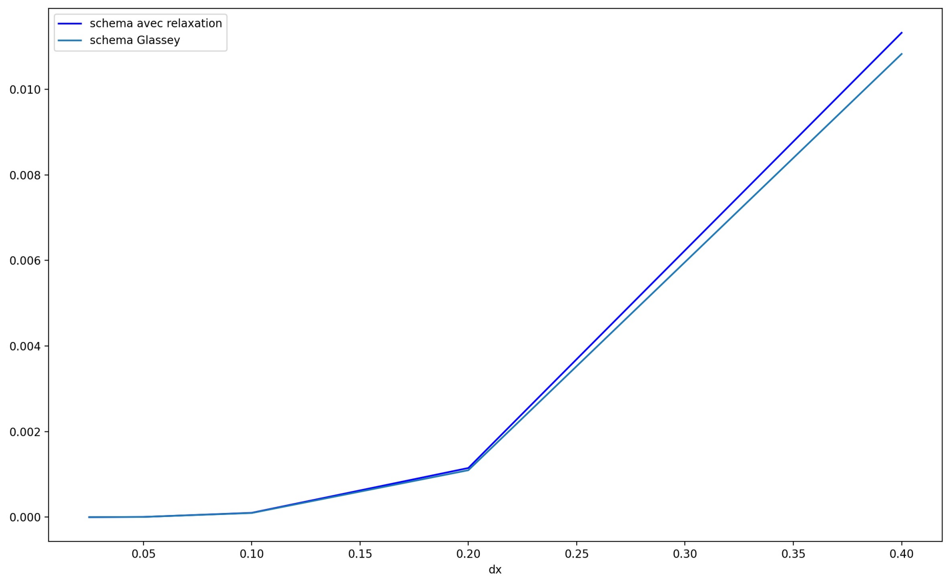

Figure 3.6 – Relative error in ℓ^2-norm for schemes (3.3.10) and (3.3.12) with different values of Δx, for $\varepsilon = 0.01$, and $\Delta t = 1 \times 10^{-3}$.

Then, we study the convergence as ε goes to 0. It is quite obvious that the numerical scheme (3.3.10) formally converges to the numerical scheme (3.3.11). Hence we can compare the solution of (3.3.11) with the solution of (3.3.10) for several values of ε to see if we can indeed observe some convergence as ε goes to 0. Thus we represent the relative error

$$\frac{\|X_N - X_N^\varepsilon\|_{\ell^2([\![0,J]\!])}}{\|X_N\|_{\ell^2([\![0,J]\!])}}$$

where X is the solution of (3.3.11) and X^ε the solution of (3.3.10). In Figure 3.7, we represent this error for initial data such that $X_0 = X_0^\varepsilon = e^{-\frac{x^2}{2}}$ and $n_0^\varepsilon = -|X_0^\varepsilon|^2$.

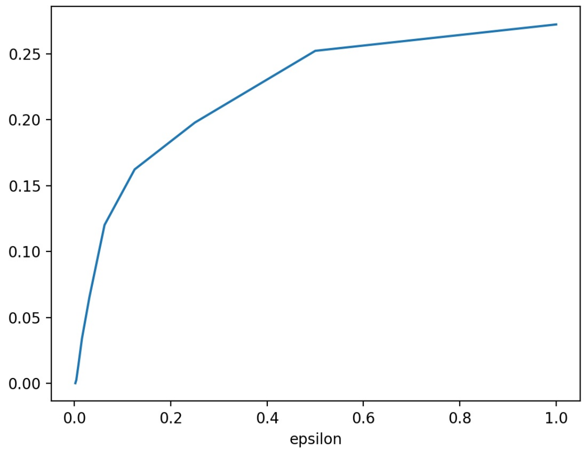

Figure 3.7 – Relative error in ℓ^2-norm between the limiting scheme (3.3.11) and the scheme (3.3.10) for different values of ε, with initial data $X_0 = X_0^\varepsilon = e^{-\frac{x^2}{2}}$, $\Delta t = 0.01$, $\Delta x = 0.1$.

We can also represent this relative error for $X_0 = X_0^\varepsilon = \sqrt{2}\cosh(x)^{-1}$ the soliton and $n_0^\varepsilon = -|X_0^\varepsilon|^2$, which is a standing wave solution for deterministic nonlinear Schrödinger equation and Zakharov system. This is done in Figure 3.8. In both cases, we observe the convergence of the solution of the scheme (3.3.10) to the solution of (3.3.11), but the convergence seems to be stronger with the soliton as initial data than with the Gaussian function.

The situation is different for the numerical scheme (3.3.12), because we can not define a limiting scheme. Indeed, even for the deterministic case, letting ε going to 0 in the equation on m_{k+1}^ε in (3.3.12) gives us for example

$$n_{k+1}^\varepsilon + n_{k-1}^\varepsilon = -2|X_k^\varepsilon|^2, \tag{3.3.16}$$

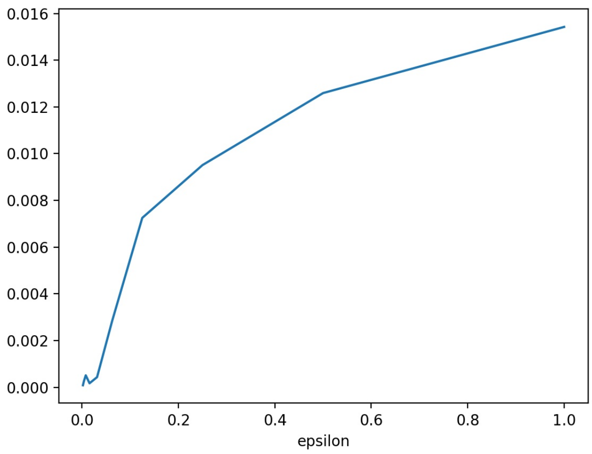

Figure 3.8 – Relative error in ℓ^2-norm between the limiting scheme (3.3.11) and the scheme (3.3.10) for different values of ε, with initial data $X_0 = X_0^\varepsilon = \sqrt{2}\cosh(x)^{-1}$, $\Delta t = 0.01$, $\Delta x = 0.1$.

but the equation on X_{k+1}^ε requires n_k^ε and not n_{k-1}^ε. This scheme needs initial conditions for X, but also for n, which is not supposed to appear to describe the nonlinear Schrödinger equation. More importantly, one can check that the discretization of the noise in (3.3.12) introduces correlations between the increments of the Wiener process, thus (3.3.12) cannot be a good scheme for small values of ε since the discretization does not converge to a Stratonovich noise. Figures 3.9 and 3.10 represent the error $\left\|X_N^{\varepsilon,r} - X_N^{\varepsilon,G}\right\|_{\ell^2[\![0,J]\!]}$, with $X^{\varepsilon,r}$ solution of (3.3.10) and $X^{\varepsilon,G}$ solution of (3.3.12) for several values of ε. In both cases we choose initial data such that $n_0^\varepsilon = -|X_0^\varepsilon|^2$, but in Figure 3.9 we take $X_0^{\varepsilon,r} = X_0^{\varepsilon,G} = e^{-\frac{x^2}{2}}$ whereas in Figure 3.10 we take $X_0^{\varepsilon,r} = X_0^{\varepsilon,G} = \sqrt{2}\cosh(x)^{-1}$. In both cases, we observe that even if the two solutions are close, the difference $\left\|X_N^{\varepsilon,r} - X_N^{\varepsilon,G}\right\|_{\ell^2[\![0,J]\!]}$ strongly increases as ε goes to 0, which seems to confirm our intuition concerning the behavior of the scheme (3.3.12) for small values of ε.

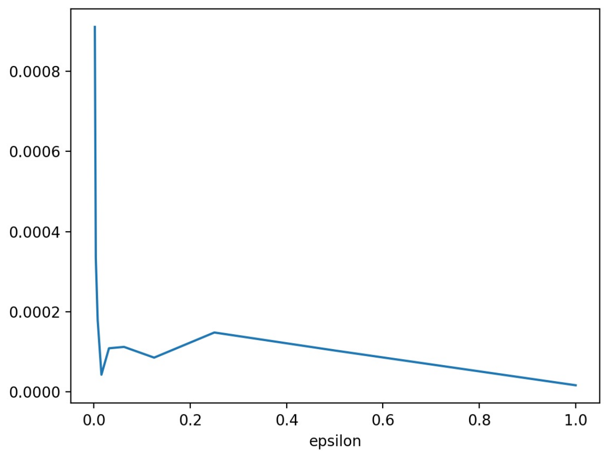

Figure 3.9 – Error in ℓ^2-norm between the schemes (3.3.12) and (3.3.10) for different values of ε, with initial data $X_0^{\varepsilon,r} = X_0^{\varepsilon,G} = e^{-\frac{x^2}{2}}$, $\Delta t = 0.01$, $\Delta x = 0.1$.

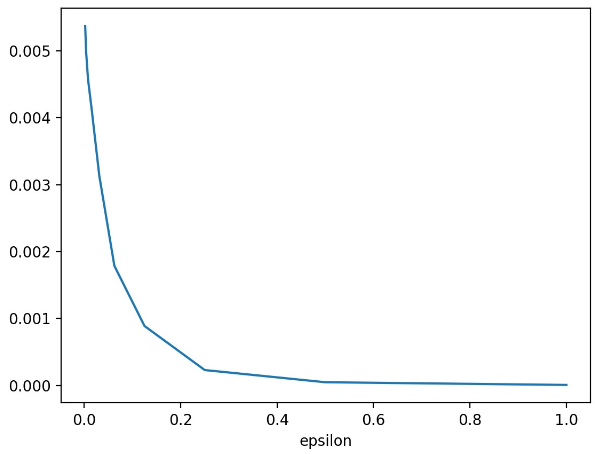

Figure 3.10 – Error in ℓ^2-norm between the schemes (3.3.12) and (3.3.10) for different values of ε, with initial data $X_0^{\varepsilon,r} = X_0^{\varepsilon,G} = \sqrt{2}\cosh(x)^{-1}$, $\Delta t = 0.01$, $\Delta x = 0.1$.

3.4 Appendix

We prove that the process $(n_k^\varepsilon, m_k^\varepsilon)_k$ defined in (3.1.5) satisfies (3.1.8) in Assumptions 3.1.0.1. We recall the numerical scheme:

$$\begin{cases} \frac{n_{k+1}^\varepsilon - n_k^\varepsilon}{\Delta t} = m_{k+\frac{1}{2}}^\varepsilon \\ \frac{m_{k+1}^\varepsilon - m_k^\varepsilon}{\Delta t} = -\frac{\alpha}{\varepsilon} m_{k+\frac{1}{2}}^\varepsilon - \frac{\lambda}{\varepsilon^2} n_{k+\frac{1}{2}}^\varepsilon + \frac{1}{\varepsilon^{\frac{3}{2}}\sqrt{\Delta t}}\gamma_k \end{cases} . \tag{3.4.1}$$

We prove that the scheme satisfies (3.1.8) replacing the quantities $n^\varepsilon_{k+\frac{1}{2}}$ and $m^\varepsilon_{k+\frac{1}{2}}$ by n^ε_{k+1} and m^ε_{k+1} in order to lighten the computations. The scheme becomes:

$$\begin{pmatrix} n^\varepsilon_{k+1} \\ m^\varepsilon_{k+1} \end{pmatrix} = A^{-1} \begin{pmatrix} n^\varepsilon_{k} \\ m^\varepsilon_{k} \end{pmatrix} + A^{-1} \begin{pmatrix} 0 \\ \frac{\sqrt{\Delta t}}{\varepsilon^{\frac{3}{2}}}\gamma_k \end{pmatrix} \tag{3.4.2}$$

with

$$A = \begin{pmatrix} 1 & -\Delta t \\ \frac{\lambda \Delta t}{\varepsilon^2} & 1 + \frac{\alpha \Delta t}{\varepsilon} \end{pmatrix}.$$

The eigenvalues of the matrix A are

$$\begin{cases} r_\pm = 1 + \frac{\Delta t}{2\varepsilon}\left(\alpha \pm \sqrt{\alpha^2 - 4\lambda}\right) & \text{if } \alpha^2 - 4\lambda > 0, \\ r = 1 + \frac{\alpha \Delta t}{2\varepsilon} & \text{if } \alpha^2 - 4\lambda = 0, \\ r_\pm = 1 + \frac{\Delta t}{2\varepsilon}\left(\alpha \pm i\sqrt{4\lambda - \alpha^2}\right) & \text{if } \alpha^2 - 4\lambda < 0. \end{cases}$$

Note that we will often be in the third case with complex eigenvalues, because formally λ represents an eigenvalue the Laplace operator while α stands for the damping term. We place ourselves in this case: there exists a matrix $P \in GL_2(\mathbb{C})$ composed of eigenvectors of A such that

$$P^{-1}\begin{pmatrix} n^\varepsilon_{k+1} \\ m^\varepsilon_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{r_+} & 0 \\ 0 & \frac{1}{r_-} \end{pmatrix} P^{-1} \begin{pmatrix} n^\varepsilon_{k} \\ m^\varepsilon_{k} \end{pmatrix} + \begin{pmatrix} \frac{1}{r_+} & 0 \\ 0 & \frac{1}{r_-} \end{pmatrix} P^{-1} \begin{pmatrix} 0 \\ \frac{\sqrt{\Delta t}}{\varepsilon^{\frac{3}{2}}}\gamma_k \end{pmatrix} \tag{3.4.3}$$

A computation gives us:

$$P^{-1} = \frac{i\varepsilon}{\sqrt{4\lambda - \alpha^2}} \begin{pmatrix} \frac{1}{2\varepsilon}\left(\alpha - i\sqrt{4\lambda - \alpha^2}\right) & -1 \\ -\frac{1}{2\varepsilon}\left(\alpha + i\sqrt{4\lambda - \alpha^2}\right) & 1 \end{pmatrix}.$$

Then if we define

$$\begin{pmatrix} a^\varepsilon_{k} \\ b^\varepsilon_{k} \end{pmatrix} = P^{-1} \begin{pmatrix} n^\varepsilon_{k} \\ m^\varepsilon_{k} \end{pmatrix}, \tag{3.4.4}$$

we get, according to (3.4.3) and because γ_k is a standard Gaussian random variable:

$$\begin{aligned}\mathbb{E}\left[|a_{k+1}^\varepsilon|^2+|b_{k+1}^\varepsilon|^2\right] \leqslant \mathbb{E}\Bigg[&\left|\frac{1}{r_+}\right|^2\left(a_k^\varepsilon-\frac{i\sqrt{\Delta t}}{\sqrt{\varepsilon}\sqrt{4\lambda-\alpha^2}}\gamma_k\right)\left(\bar{a_k^\varepsilon}+\frac{i\sqrt{\Delta t}}{\sqrt{\varepsilon}\sqrt{4\lambda-\alpha^2}}\gamma_k\right)\\ &+\left|\frac{1}{r_-}\right|^2\left(b_k^\varepsilon+\frac{i\sqrt{\Delta t}}{\sqrt{\varepsilon}\sqrt{4\lambda-\alpha^2}}\gamma_k\right)\left(\bar{b_k^\varepsilon}-\frac{i\sqrt{\Delta t}}{\sqrt{\varepsilon}\sqrt{4\lambda-\alpha^2}}\gamma_k\right)\Bigg].\end{aligned}$$

Now, using the independence between $(a_k^\varepsilon, b_k^\varepsilon)$ and γ_k, and the fact that γ_k is a standard Gaussian random variable, to deal with the products $\mathbb{E}[(a_k^\varepsilon-\bar{a_k^\varepsilon})\gamma_k]$ and $\mathbb{E}[(\bar{b_k^\varepsilon}-b_k^\varepsilon)\gamma_k]$, we get

$$\begin{aligned}\mathbb{E}\left[|a_{k+1}^\varepsilon|^2+|b_{k+1}^\varepsilon|^2\right] \leqslant \mathbb{E}\Bigg[&\left|\frac{1}{r_+}\right|^2\left(|a_k^\varepsilon|^2+\frac{1}{4\lambda-\alpha^2}\frac{\Delta t}{\varepsilon}\gamma_k^2\right)\\ &+\left|\frac{1}{r_-}\right|^2\left(|b_k^\varepsilon|^2+\frac{1}{4\lambda-\alpha^2}\frac{\Delta t}{\varepsilon}\gamma_k^2\right)\Bigg].\end{aligned}$$

Since $|r_+|>1$ and $|r_-|>1$, a straightforward recursion argument gives us:

$$\begin{aligned}\mathbb{E}\left[|a_{k+1}^\varepsilon|^2+|b_{k+1}^\varepsilon|^2\right] &\leqslant \mathbb{E}\left[|a_0^\varepsilon|^2+|b_0^\varepsilon|^2\right]+\frac{1}{4\lambda-\alpha^2}\frac{\Delta t}{\varepsilon}\sum_{\ell=1}^{k+1}\left|\frac{1}{r_+}\right|^{2\ell}+\left|\frac{1}{r_-}\right|^{2\ell}\\ &\leqslant \mathbb{E}\left[|a_0^\varepsilon|^2+|b_0^\varepsilon|^2\right]+\frac{1}{4\lambda-\alpha^2}\frac{\Delta t}{\varepsilon}\left(\frac{1}{|r_+|^2-1}+\frac{1}{|r_-|^2-1}\right).\end{aligned}$$

We detail the computation for the term involving r_+, the procedure is the same for the other term.

$$\begin{aligned}\frac{1}{4\lambda-\alpha^2}\frac{\Delta t}{\varepsilon}\frac{1}{|r_+|^2-1} &= \frac{1}{4\lambda-\alpha^2}\frac{\Delta t}{\varepsilon}\frac{1}{\frac{\alpha\Delta t}{\varepsilon}+\frac{\alpha^2\Delta t^2}{4\varepsilon^2}}\\ &\leqslant \frac{1}{4\lambda-\alpha^2}\frac{1}{\alpha}\end{aligned}$$

since $4\lambda-\alpha^2>0$. We conclude that

$$\sup_{\varepsilon\in(0,1]}\sup_{\Delta t\in(0,\Delta t_0]}\sup_{k\in\mathbb{N}}\frac{\mathbb{E}\left[|a_k^\varepsilon|^2+|b_k^\varepsilon|^2\right]}{1+\mathbb{E}\left[|a_0^\varepsilon|^2+|b_0^\varepsilon|^2\right]}<\infty, \tag{3.4.5}$$

and from this result we get (3.1.8) for n_k^ε and m_k^ε using the relation (3.4.4).

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DOCTORAT
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école normale supérieure rennes

Titre : Approximation diffusion pour des équations dispersives

Mot clés : Approximation diffusion, équation de Schrödinger non linéaire, système de Zakharov, schémas AP

Résumé : Cette thèse porte sur des problèmes d'approximation diffusion. On s'intéresse plus particulièrement à l'équation de Schrödinger avec une non linéarité de type puissance, perturbée par un potentiel aléatoire. On montre la convergence en loi de la solution de cette équation vers celle d'une équation de Schrödinger stochastique, dirigée par un processus de Wiener. On étudie également le système de Zakharov stochastique, et l'on se sert de la méthode de la Fonction Test Perturbée pour prouver une convergence vers une équation de Schrödinger cubique stochastique. Enfin, on s'intéresse à la notion de schémas numériques préservant l'asymptotique. On propose alors un schéma numérique approchant la solution du système de Zakharov stochastique, qui converge vers un schéma numérique approchant la solution de l'équation de Schrödinger stochastique limite.

Title: Approximation diffusion for dispersive equations

Keywords: Approximation diffusion, Nonlinear Schrödinger equation, Zakharov system, AP schemes

Abstract: This thesis deals with approximation diffusion problems. More precisely we study the Nonlinear Schrödinger equation with a random potential. We prove that the solution of this equation converges in distribution to the solution of a stochastic Nonlinear Schrödinger equation, driven by a Wiener process. We also study the stochastic Zakharov system, and we use the Perturbed test Function method to prove the convergence of the solution of this system to the solution of a cubic stochastic Schrödinger equation. Finally we study the theory of Asymptotic Preserving numerical schemes. We present a numerical scheme which approximates the solution of the stochastic Zakharov system, and which converges to a numerical scheme approximating the soluton of the limiting stochastic Schrödinger equation.