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# Géométrie, dynamique et rigidité de groupes d'automorphismes de produits libres

Yassine Guerch

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Géométrie, dynamique et rigidité de  
groupes d'automorphismes de produits  
libres  
*Geometry, dynamics and rigidity of automorphism groups of free  
products*

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**Thèse soutenue à Paris-Saclay, le 4 juillet 2022, par**

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**FMJH**  
Jacques Hadamard



*À toi Papa, Bouchaïb, premier docteur Guerch.  
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voir leur beauté intrinsèque. Tu seras toujours dans mes  
pensées et dans mon cœur.  
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*“[Une thèse] avance comme un train, tu comprends, comme un train dans la nuit. [...] Les choses s'accrochent... comme des wagons, l'histoire avance sur ses rails, le public-voyageur ne quitte pas [la thèse], il se laisse véhiculer du point de départ au terminus et il traverse des paysages qui sont des émotions.”*

Citation adaptée de l'œuvre de François Truffaut  
*La nuit américaine* (1973)



# Chapitre 1

## Introduction

Dans cette thèse, nous nous intéressons à des problèmes de théorie géométrique des groupes et leur lien avec la géométrie à courbure négative et les systèmes dynamiques, en particulier la dynamique mesurable des groupes. Nous nous intéressons notamment à des phénomènes de rigidité de structures géométriques associés aux groupes considérés. Nous étudions plus particulièrement le groupe des automorphismes de certains produits libres de groupes de type fini. Soit  $n \in \mathbb{N}^*$  et soit  $F$  un groupe cyclique d'ordre 2. Nous considérons dans ce manuscrit les groupes des automorphismes de  $\mathbb{F}_n$ , groupe libre (non abélien si  $n \geq 2$ ) de rang  $n$ , et de  $W_n$ , groupe de Coxeter universel de rang  $n$ , produit libre de  $n$  copies de  $F$ . Si  $G$  est un groupe, nous noterons  $\text{Aut}(G)$  le groupe des automorphismes de  $G$  et  $\text{Int}(G)$  le groupe des *automorphismes intérieurs* de  $G$ , sous-groupe distingué de  $\text{Aut}(G)$  constitué des conjugaisons globales par des éléments de  $G$ . Soit  $\text{Out}(G) = \text{Aut}(G)/\text{Int}(G)$  le groupe des automorphismes extérieurs de  $G$ . Les groupes d'intérêt dans ce manuscrit sont les groupes  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$ .

Dans tout ce travail, nous nous efforcerons de comparer les résultats obtenus avec ceux déjà connus du groupe modulaire  $\text{Mod}(S) = \pi_0(\text{Homeo}^+(S))$  d'une surface  $S$  compacte, connexe, orientable de genre  $g \geq 2$  (voir par exemple l'étude comparative de Bestvina [Bes2], Vogtmann [Vog], et une exposition de Paulin [Pau3]). Cette comparaison s'avère fructueuse car un certain nombre de résultats sur  $\text{Out}(\mathbb{F}_n)$  ont été obtenus en s'inspirant de démonstrations de résultats analogues pour  $\text{Mod}(S)$ . Par ailleurs, si  $S$  est une surface à exactement une composante connexe de bord, le groupe fondamental de  $S$  est isomorphe à un groupe libre non abélien de rang  $2g$ , et le théorème de Dehn-Nielsen-Baer (voir par exemple [FarM, Theorem 8.8]) affirme que le groupe  $\text{Mod}(S)$  s'identifie à un sous-groupe de  $\text{Out}(F_{2g})$ .

Nous étudierons plus particulièrement dans cette thèse des résultats de *rigidité* des groupes  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$ . Les résultats de rigidité auxquels nous nous intéressons peuvent être classés en deux catégories. La première catégorie regroupe des résultats sur la structure des symétries du groupe : nous parlerons de phénomènes de rigidité lorsque le groupe ne contient pas d'autres symétries que celles naturelles. Les symétries peuvent ici prendre la forme des automorphismes du groupe ou des automorphismes entre sous-groupes d'indice fini du groupe considéré. Cette notion de rigidité est à rapprocher du

théorème de rigidité de Mostow-Prasad-Margulis [Mos1, Mos2] affirmant qu'un isomorphisme entre les groupes fondamentaux de deux variétés riemanniennes hyperboliques de dimension au moins 3, connexes, de volume fini est induit par une isométrie entre les variétés riemanniennes (voir la partie 1.1 pour des développements). Ainsi, la géométrie d'une telle variété hyperbolique est entièrement déterminée par sa topologie. En ce sens, les résultats de rigidité que nous allons considérer dans cette thèse permettent de caractériser les groupes considérés avec peu d'informations, puisqu'il suffira par exemple de considérer uniquement les symétries du groupe pour le reconstruire.

Ces rigidités de nature algébrique sont à mettre en parallèle avec des rigidités de nature géométrique. En effet, les groupes considérés agissent par isométries sur certains espaces et cette action induit un isomorphisme entre le groupe considéré et le groupe des isométries de l'espace. Cette rigidité géométrique se rapproche des résultats de Tits [Tit2] affirmant que le groupe des automorphismes simpliciaux d'un immeuble sphérique associé à un groupe algébrique simple, connexe de rang au moins 2 est isomorphe au groupe algébrique lui-même (voir également la partie 1.1). Ces résultats de rigidité géométrique ont, comme nous le verrons dans la suite de cette introduction, de grandes connexions avec les résultats de rigidité algébrique susmentionnés, puisque nous démontrerons des résultats de rigidité algébrique en utilisant des résultats de rigidité géométrique.

La deuxième catégorie de rigidité étudiée concerne l'existence d'éléments *génériques*, c'est-à-dire d'éléments qui portent en eux toute l'information en cours d'étude du groupe ; cette information peut par exemple être de nature dynamique. La construction d'éléments génériques dans les cas qui nous intéresseront reposera sur des arguments dynamiques, avec la construction d'éléments du groupe considéré dont la dynamique est riche. Nous pouvons rapprocher cette catégorie de rigidité avec des résultats d'alternative de Tits [Tit1] décrivant les sous-groupes d'un groupe linéaire : tout sous-groupe  $H$  non résoluble d'un groupe linéaire contient un groupe libre non abélien. En effet, la démonstration de Tits, sur laquelle nous reviendrons dans la partie 1.3, utilise des arguments dynamiques similaires à ceux que nous utiliserons pour construire des éléments génériques.

L'étude de ces questions de rigidité repose traditionnellement sur la construction d'espaces géométriques sur lesquels les groupes considérés admettent de "jolies" actions, c'est-à-dire propres ou cocompactes (voir la partie 1.2 pour des développements). L'idée générale est de construire un dictionnaire entre les propriétés algébriques du groupe et les propriétés géométriques de l'action, afin que les propriétés algébriques se déduisent des propriétés géométriques et réciproquement. L'enjeu est alors en grande partie de construire l'espace géométrique adapté à la question algébrique posée.

## 1.1 Actions sur des espaces de déformation et premiers résultats de rigidité

Puisque nous considérons dans cette thèse des groupes d'automorphismes extérieurs  $\text{Out}(G)$  de certains groupes  $G$  de type fini, il est naturel de construire des espaces géométriques sur lesquels agit  $\text{Out}(G)$  à partir d'espaces géométriques sur lesquels agit  $G$ . Par exemple, si  $G$  est de type fini, le groupe  $G$  agit naturellement par multiplication



à gauche sur le graphe de Cayley de  $G$  associé à une partie génératrice de  $G$ . En particulier, dans le cas du groupe  $\mathbb{F}_n$ , si la partie génératrice est une base libre  $\mathfrak{B}$  de  $\mathbb{F}_n$ , alors le graphe de Cayley  $T_{\mathfrak{B}}$  de  $\mathbb{F}_n$  associé à  $\mathfrak{B}$  est un arbre simplicial et le groupe  $\mathbb{F}_n$  agit librement sur  $T_{\mathfrak{B}}$ . Plus généralement, d'après [Ser1, Théorème 4], les groupes libres  $\mathbb{F}_n$  pour  $n \in \mathbb{N}$  sont les seuls groupes de type fini admettant une action libre sur un arbre simplicial. Les arbres fournissent ainsi une large collection d'objets géométriques sur lesquels  $\mathbb{F}_n$  admet une jolie action. Nous supposons dans la suite une certaine familiarité du lecteur ou de la lectrice avec les actions de groupes sur les arbres (voir par exemple [Ser1, CM]).

Il n'existe cependant pas d'arbre simplicial non trivial *canonique* sur lequel agisse  $\mathbb{F}_n$  librement et, de ce fait, il n'existe pas d'arbre simplicial non trivial sur lequel  $\mathbb{F}_n$  agit et tel que l'action de  $\mathbb{F}_n$  s'étende à  $\text{Aut}(\mathbb{F}_n)$  tout entier. Il convient alors de créer un espace géométrique sur lequel agit  $\text{Out}(\mathbb{F}_n)$  et qui contient l'information de toutes les actions libres de  $\mathbb{F}_n$  sur des arbres simpliciaux. Cet espace est l'*Outre-espace* de  $\mathbb{F}_n$ , noté  $CV_n$ , introduit par Culler et Vogtmann [CV], qui est défini comme suit. Un point de  $CV_n$  est une classe d'homothétie  $\mathbb{F}_n$ -équivariante d'arbres simpliciaux métriques admettant une action par isométrie de  $\mathbb{F}_n$  libre et minimale. Ici, une action est *minimale* s'il n'existe pas de sous-arbre propre invariant par l'action du groupe. L'ensemble  $CV_n$  est muni de la topologie de Gromov-Hausdorff équivariante [Pau1]. Muni de cette topologie, l'espace  $CV_n$  peut être équipé d'une structure de complexe simplicial ayant des faces manquantes. Le groupe  $\text{Aut}(\mathbb{F}_n)$  agit à droite par précomposition de l'action et l'action de  $\text{Aut}(\mathbb{F}_n)$  passe au quotient en une action à droite proprement discontinue de  $\text{Out}(\mathbb{F}_n)$ . L'Outre-espace est un *espace de déformation*, analogue de l'espace de Teichmüller d'une surface hyperbolique compacte, connexe, orientable ou de l'espace symétrique associé à un groupe de Lie semi-simple (voir par exemple les travaux de Forester [For] et de Guirardel et Levitt [GuL2]). Le quotient  $CV_n/\text{Out}(\mathbb{F}_n)$  est alors un *espace de modules*, analogue au quotient de l'espace de Teichmüller par le groupe modulaire de la surface considérée. L'Outre-espace est un espace contractile comme démontré dans [CV], et donc l'action de  $\text{Out}(\mathbb{F}_n)$  sur  $CV_n$  permet par exemple de montrer que la dimension cohomologique virtuelle de  $\text{Out}(\mathbb{F}_n)$  est finie et égale à  $2n - 3$ . Par ailleurs, l'espace  $CV_n$  se rétracte par déformation forte de manière  $\text{Out}(\mathbb{F}_n)$ -équivariante sur un complexe simplicial appelé l'*épine de l'Outre-espace* de  $\mathbb{F}_n$ , dont le 1-squelette est noté  $K(\mathbb{F}_n)$ . Les sommets de  $K(\mathbb{F}_n)$  sont les classes d'homéomorphisme  $\mathbb{F}_n$ -équivariants d'arbres simpliciaux munis d'une action libre et minimale de  $\mathbb{F}_n$ , deux sommets  $\mathcal{X}$  et  $\mathcal{X}'$  de  $K(\mathbb{F}_n)$  étant reliés par une arête si et seulement s'il existe  $X \in \mathcal{X}$  et  $X' \in \mathcal{X}'$  tels que  $X$  s'écrase sur  $X'$ , c'est-à-dire que  $X'$  est obtenu à partir de  $X$  en contractant des orbites d'arêtes, ou réciproquement. Le groupe  $\text{Out}(\mathbb{F}_n)$  agit alors sur  $K(\mathbb{F}_n)$  par automorphisme de graphes et l'action de  $\text{Out}(\mathbb{F}_n)$  sur  $K(\mathbb{F}_n)$  est cocompacte.

Les espaces de modules d'actions de groupes sur des arbres interviennent par ailleurs naturellement dans la compactification de certains espaces de représentations, comme démontré par Morgan et Shalen ([MS2], voir également les travaux de Bestvina [Bes1] et de Paulin [Pau1]). En effet, soit  $\Gamma$  un groupe de type fini n'ayant pas de sous-groupes d'indice fini contenant un sous-groupe distingué abélien infini et  $G$  un groupe

algébrique linéaire simple sur  $\mathbb{R}$  de rang réel 1. Soit  $\text{Hom}_{fd}(\Gamma, G)$  l'ensemble des représentations fidèles et discrètes de  $\Gamma$  dans  $G$  muni de la topologie compacte-ouverte, et soit  $\text{Hom}_{fd}(\Gamma, G)/G$  l'espace quotient des représentations fidèles et discrètes de  $\Gamma$  dans  $G$  modulo l'action propre par conjugaison par un élément de  $G$  (voir par exemple [Par1]). Soit  $\text{SLF}(\Gamma)$  l'ensemble des actions de  $\Gamma$  sur des arbres (possiblement réels), sans point fixe global, sans sous-arbre invariant propre, à stabilisateurs d'arc virtuellement nilpotents, modulo isométrie  $\Gamma$ -équivariante, muni de la topologie de Gromov-Hausdorff équivariante et soit  $\mathbb{P}\text{SLF}(\Gamma)$  le projectifié de  $\text{SLF}(\Gamma)$ . Comme interprété par Parreau [Par1, Par2], Morgan et Shalen ont montré que l'on pouvait compactifier l'espace  $\text{Hom}_{fd}(\Gamma, G)/G$  par un fermé de  $\mathbb{P}\text{SLF}(\Gamma)$ . De tels résultats ont par exemple permis de comprendre les représentations du groupe fondamental d'une surface hyperbolique dans le groupe des isométries du plan hyperbolique réel (voir par exemple [Ota, Chapitre 2]).

Les espaces de modules d'actions de groupes sur des arbres s'avérant être des espaces géométriques privilégiés dans l'étude d'actions de groupes, la construction d'espaces d'actions sur des arbres s'est ainsi notamment propagée dans l'étude des produits libres de groupes de type fini. Soit  $G$  un groupe de type fini et soit

$$G = G_1 * \dots * G_k * B$$

une décomposition de  $G$  en produits libres de groupes. Notons  $\mathcal{F} = \{[G_1], \dots, [G_k]\}$  l'ensemble des classes de conjugaison dans  $G$  des sous-groupes  $G_1, \dots, G_k$ . L'ensemble  $\mathcal{F}$  forme un *système de facteurs libres de  $G$* , c'est-à-dire qu'il existe un sous-groupe  $B$  de  $G$  tel que  $G = G_1 * \dots * G_k * B$ . L'étude de telles décompositions de groupes, et donc l'étude de groupes *relativement à des ensembles de classes de conjugaison de sous-groupes* est à mettre en parallèle avec l'étude des groupes kleinien (c'est-à-dire des sous-groupes discrets de  $\text{PSL}(2, \mathbb{C})$ ) et de leur action par isométries sur l'espace hyperbolique réel  $\mathbb{H}_{\mathbb{R}}^3$ . En effet, les groupes kleinien sont généralement étudiés relativement à leurs sous-groupes *paraboliques*, c'est-à-dire les sous-groupes maximaux du groupe kleinien considéré fixant exactement un point à l'infini de  $\mathbb{H}_{\mathbb{R}}^3$ .

Guirardel et Levitt [GuL1] ont construit un analogue de l'Outre-espace de  $\mathbb{F}_n$  pour des produits libres de groupes de type fini. L'*Outre-espace de  $G$  relativement à  $\mathcal{F}$* , noté  $\mathcal{O}(G, \mathcal{F})$ , est l'ensemble des classes d'isométries  $G$ -équivariantes d'arbres simpliciaux métriques munis d'une action minimale par isométrie de  $G$ , à stabilisateur d'arêtes triviaux et dont les stabilisateurs de sommets sont des conjugués des  $G_i$  avec  $i \in \{1, \dots, k\}$ . L'ensemble  $\mathcal{O}(G, \mathcal{F})$  est muni de la topologie de Gromov-Hausdorff équivariante. Rappelons qu'un groupe  $G'$  est *librement indécomposable* s'il n'est pas isomorphe à  $\mathbb{Z}$  et s'il n'existe pas de sous-groupes non triviaux  $A'$  et  $B'$  de  $G'$  tels que  $G' = A' * B'$ . Lorsque la décomposition de  $G$  relativement à  $\mathcal{F}$  coïncide avec une *décomposition de Grushko* de  $G$ , c'est-à-dire que, pour tout  $i \in \{1, \dots, k\}$ , le groupe  $G_i$  est librement indécomposable et que  $B$  est libre, nous parlerons simplement de l'Outre-espace de  $G$  et le noterons  $\mathcal{O}(G)$ . En particulier, dans le cas de  $W_n$ , les stabilisateurs de sommets des arbres considérés dans  $\mathcal{O}(W_n)$  sont finis, soit triviaux soit isomorphes à  $F$ . L'espace  $\mathcal{O}(W_n)$  est contractile comme démontré dans [GuL1]. Comme dans le cas de  $CV_n$ , le groupe  $\text{Aut}(W_n)$  agit à droite sur  $\mathcal{O}(W_n)$  par précomposition de l'action, et cette action passe au quotient en

une action proprement discontinue de  $\text{Out}(W_n)$ . Par ailleurs, l'espace  $\mathcal{O}(W_n)$  se rétracte par déformation forte  $\text{Out}(W_n)$ -équivariante sur un complexe simplicial appelé *l'épine de l'Outre-espace de  $W_n$*  dont le 1-squelette est noté  $K(W_n)$ . Les sommets de  $K(W_n)$  sont les classes d'homéomorphismes  $W_n$ -équivariants d'arbres simpliciaux munis d'une action de  $W_n$  minimale, à stabilisateur d'arêtes triviaux et à stabilisateurs de sommets finis, deux sommets  $\mathcal{X}$  et  $\mathcal{X}'$  de  $K(W_n)$  étant reliés par une arête si et seulement s'il existe  $X \in \mathcal{X}$  et  $X' \in \mathcal{X}'$  tel que  $X$  s'écrase sur  $X'$  ou réciproquement. Le groupe  $\text{Out}(W_n)$  agit alors sur  $K(W_n)$  par automorphisme de graphes et l'action de  $\text{Out}(W_n)$  sur  $K(W_n)$  est cocompacte. Nous démontrons le théorème suivant.

**Théorème 1.1.1** (Theorem 3.1.1). *Soit  $n \geq 4$ . Le morphisme naturel*

$$\text{Out}(W_n) \rightarrow \text{Aut}(K(W_n))$$

*est un isomorphisme.*

Nous obtenons ainsi un résultat de rigidité au sens où les seules symétries de l'Outre-espace de  $W_n$  sont celles naturelles induites par l'action. Le théorème 1.1.1 donne une classification complète des symétries de l'épine de l'Outre-espace pour tout  $n$ . En effet, pour  $n = 2$ , l'épine de l'Outre-espace de  $W_2$  est réduite à un point. Lorsque  $n = 3$ , l'épine de l'Outre-espace de  $W_3$  est un arbre trivalent, isomorphe à l'épine de l'Outre-espace (réduit) de  $F_2$  ; de ce fait, son groupe d'automorphismes de graphes est non dénombrable. Le théorème 1.1.1 montre que l'épine de l'Outre-espace de  $W_n$  est un *modèle géométrique rigide pour  $\text{Out}(W_n)$* , c'est-à-dire que l'on peut considérer le groupe  $\text{Out}(W_n)$  comme étant exactement le groupe des automorphismes du graphe  $K(W_n)$ .

De tels modèles géométriques rigides ont été démontrés l'être dans d'autres contextes. Par exemple, dans le cadre des groupes algébriques, Tits a démontré que, si le rang de l'immeuble sphérique associé à un groupe algébrique simple connexe est au moins 2, le groupe des automorphismes simpliciaux de l'immeuble est exactement le groupe algébrique [Tit2]. Un tel résultat de rigidité géométrique est également vrai pour les immeubles affines de dimension au moins 3. La démonstration repose sur le fait que l'immeuble à l'infini d'un tel immeuble affine est sphérique de rang au moins 2. De ce fait, la description du bord à l'infini permet de mieux comprendre les symétries de l'immeuble lui-même.

Dans le cas du groupe modulaire d'une surface compacte, connexe, orientable  $S$  de genre au moins 3, Royden [Roy] a démontré que le groupe des isométries de l'espace de Teichmüller (pour la distance de Teichmüller) est exactement  $\text{Mod}^\pm(S)$ , le groupe modulaire étendu de la surface. Un autre modèle géométrique rigide pour  $\text{Mod}^\pm(S)$  est le *graphe des courbes*. Introduit par Harvey ([Har], voir également [Kla, MasM]), le graphe des courbes est le graphe dont les sommets sont les classes d'homotopie de courbes fermées simples et essentielles, deux sommets  $[c_1]$  et  $[c_2]$  étant reliés par une arête s'il existe  $c'_1 \in [c_1]$  et  $c'_2 \in [c_2]$  tels que  $c'_1 \cap c'_2 = \emptyset$ . Le groupe  $\text{Mod}(S)$  agit naturellement sur le graphe des courbes et Ivanov [Iva2] a démontré que le graphe des courbes d'une surface compacte, connexe, orientable de genre au moins 3 constituait un modèle géométrique rigide pour  $\text{Mod}^\pm(S)$ . Enfin, lorsque  $n \geq 3$ , l'épine de l'Outre-espace

de  $\mathbb{F}_n$  constitue un modèle géométrique rigide pour  $\text{Out}(\mathbb{F}_n)$  par des résultats de Bridson et Vogtmann [BV2]. Le fait que l'épine de  $CV_n$  constitue un modèle géométrique rigide pour  $\text{Out}(\mathbb{F}_n)$  n'implique cependant pas que le graphe  $K(W_n)$  soit un modèle géométrique rigide pour  $\text{Out}(W_n)$ , comme nous le verrons dans la partie 1.5. D'autres modèles géométriques rigides avaient déjà été construits pour  $\text{Out}(W_n)$ . Par exemple l'espace de McCullough-Miller pour  $W_n$ , espace homotopiquement équivalent et de manière  $\text{Out}(W_n)$ -équivariante à l'Outre-espace de  $W_n$  [MM], est un modèle géométrique rigide pour  $\text{Out}(W_n)$  par un résultat de Piggott [Pig]. Néanmoins, il n'existe pas, à notre connaissance, d'isomorphisme  $\text{Out}(W_n)$ -équivariant entre le 1-squelette de l'espace de McCullough-Miller et  $K(W_n)$  et, de ce fait, nous ne pouvons pas déduire directement la rigidité géométrique de  $K(W_n)$  à partir de celle de l'espace de McCullough-Miller.

Il s'avère par ailleurs que la démonstration du théorème 1.1.1 repose sur le fait que certains sous-graphes de  $K(W_n)$  sont également des modèles géométriques rigides pour  $\text{Out}(W_n)$ . Un graphe de première importance est le *graphe des  $\{0\}$ -étoiles et des  $F$ -étoiles* noté  $L(W_n)$ . Une  $\{0\}$ -étoile est une classe d'équivalence  $\mathcal{X}$  dans  $K(W_n)$  dont un représentant  $X \in \mathcal{X}$  est tel que le quotient  $W_n \setminus X$  soit un graphe de groupes dont le graphe sous-jacent est un arbre ayant  $n$  feuilles et  $n + 1$  sommets. Une  $F$ -étoile est une classe d'équivalence  $\mathcal{X}$  dans  $K(W_n)$  dont un représentant  $X \in \mathcal{X}$  est tel que le quotient  $W_n \setminus X$  soit un graphe de groupes dont le graphe sous-jacent est un arbre ayant  $n - 1$  feuilles et  $n$  sommets. Les  $\{0\}$ -étoiles et les  $F$ -étoiles jouent le rôle des *roses* de  $CV_n$ , c'est-à-dire des classes d'équivalence de graphes de Cayley pour les parties génératrices libres de  $\mathbb{F}_n$ . Le graphe  $L(W_n)$  est alors le sous-graphe de  $K(W_n)$  dont les sommets sont les  $\{0\}$ -étoiles et les  $F$ -étoiles, et où deux sommets de  $L(W_n)$  sont adjacents si, et seulement si, ils sont adjacents dans  $K(W_n)$ . Le groupe  $\text{Out}(W_n)$  agit à droite sur  $L(W_n)$  par précomposition de l'action et nous démontrons le théorème de rigidité suivant.

**Théorème 1.1.2** (Corollary 3.3.2). *Soit  $n \geq 5$ . Le morphisme naturel*

$$\text{Out}(W_n) \rightarrow \text{Aut}(L(W_n))$$

*est un isomorphisme.*

La démonstration du théorème 1.1.2 repose sur une étude locale du graphe  $L(W_n)$ . Nous étudions en particulier les cycles dans les boules de rayon 4 de  $L(W_n)$ . Ceci nous permet de décrire localement le graphe  $L(W_n)$  et cette description locale est suffisante pour obtenir une description globale des symétries de  $L(W_n)$ . Ce passage de la rigidité locale à la rigidité globale est à mettre en parallèle avec la démonstration de la rigidité géométrique des immeubles sphériques associés à des groupes algébriques faite par Tits [Tit2] qui reposait sur l'étude des cycles dans des boules de rayon 2 dans l'immeuble.

L'importance du théorème 1.1.2 provient du fait qu'il permet une traduction d'une question de rigidité algébrique de  $\text{Out}(W_n)$  en une question de rigidité géométrique. En effet, les stabilisateurs dans  $L(W_n)$  d'une  $\{0\}$ -étoile et celui d'une  $F$ -étoile adjacente engendrent  $\text{Out}(W_n)$ . En montrant d'une part que tout automorphisme de  $\text{Out}(W_n)$  préserve l'ensemble des stabilisateurs de  $\{0\}$ -étoiles et de  $F$ -étoiles, et d'autre part que tout automorphisme de  $\text{Out}(W_n)$  préserve l'ensemble des stabilisateurs d'arêtes de

$L(W_n)$ , nous voyons que tout automorphisme de  $\text{Out}(W_n)$  induit un automorphisme du graphe  $L(W_n)$ . En utilisant la rigidité géométrique, nous déduisons le théorème de rigidité algébrique suivant.

**Théorème 1.1.3** (Théorème 2.1.1). (1) *Si  $n \geq 5$ , alors  $\text{Out}(\text{Out}(W_n)) = \{1\}$ .*

(2) *Si  $n = 4$ , alors  $\text{Out}(\text{Out}(W_n))$  est isomorphe à  $\mathbb{Z}/2\mathbb{Z}$ .*

L'automorphisme extérieur nontrivial de  $\text{Out}(W_4)$  donné par le théorème 1.1.3 (2) a une interprétation très simple en terme géométrique puisqu'il permute le stabilisateur d'une  $\{0\}$ -étoile avec celui d'une  $F$ -étoile adjacente. Nous obtenons par ailleurs une description de cet automorphisme sur un système de générateurs de  $\text{Out}(W_n)$ . Le théorème 1.1.3 donne une classification complète des automorphismes de  $\text{Out}(W_n)$  pour  $n \in \mathbb{N}^*$ . En effet, le groupe  $\text{Out}(\text{Out}(W_2))$  est isomorphe à un groupe cyclique d'ordre 2 [Tho, Lemmas 1.4.2, 1.4.3]. Par ailleurs, le groupe  $\text{Out}(W_3)$  est isomorphe à un groupe projectif linéaire de rang 2 à coefficients dans  $\mathbb{Z}$  [Gue1, Proposition 2.2], donc son groupe d'automorphismes extérieurs est trivial par un résultat de Hua et Reiner [HR]. Notons également que, si la démonstration du théorème 1.1.3 donnée dans [Gue1] ne fait pas directement intervenir le graphe  $L(W_n)$ , la démonstration montre effectivement que tout automorphisme de  $\text{Out}(W_n)$  préserve l'ensemble des stabilisateurs de  $\{0\}$ -étoiles et de  $F$ -étoiles. Le théorème 1.1.3 est inspiré de résultats analogues dans différents contextes. En effet, Mostow [Mos2] a démontré que le groupe des automorphismes extérieurs de réseaux irréductibles uniformes de groupes de Lie réels, connexes, semi-simples et non localement isomorphes à  $\text{SL}_2(\mathbb{R})$  est fini. De même, Bridson et Vogtmann [BV1] ont démontré que tout automorphisme du groupe des automorphismes extérieurs d'un groupe libre de rang  $N$  (avec  $N \geq 3$ ) est une conjugaison. Le résultat de Bridson et Vogtmann ne permet cependant pas de déduire le théorème 1.1.3 comme nous l'expliquerons dans la partie 1.5. Enfin, Ivanov [Iva2] a démontré un résultat similaire dans le cas du groupe modulaire étendu d'une surface compacte, connexe, orientable de genre  $g \geq 2$ . La méthode de démonstration évoquée pour le théorème 1.1.3 est d'ailleurs similaire à la démonstration d'Ivanov. En effet, Ivanov utilise tout d'abord le fait que le graphe des courbes de la surface est un modèle géométrique rigide pour  $\text{Mod}^\pm(S)$  et donne ensuite une caractérisation algébrique des stabilisateurs de sommets du graphe des courbes grâce aux twists de Dehn de la surface. Ivanov démontre en fait un résultat sur les isomorphismes entre sous-groupes d'indice fini de  $\text{Mod}^\pm(S)$ . Comme nous l'expliquerons dans la partie suivante, nous pouvons également obtenir un résultat sur les isomorphismes entre sous-groupes d'indice fini de  $\text{Out}(W_n)$ . Ce résultat repose sur l'action de  $\text{Out}(W_n)$  sur certains espaces hyperboliques au sens de Gromov, comme nous l'expliquons aussi dans le début de la prochaine partie.

## 1.2 Action sur des espaces hyperboliques au sens de Gromov et rigidité du commensurateur abstrait

L'étude de  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$  repose également sur la construction d'espaces *hyperboliques au sens de Gromov* (voir [Gro1] pour tout prérequis et également [GdlH]) sur lesquels  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$  agissent par isométries. Par exemple, l'utilisation des espaces hyperboliques pour comprendre certains groupes agissant par isométries sur ces derniers est un argument central dans la démonstration du théorème de rigidité de Mostow en rang 1 [Mos1], avec l'usage des propriétés de l'espace hyperbolique réel  $\mathbb{H}_{\mathbb{R}}^3$  ainsi que son bord à l'infini.

Un autre exemple d'espace hyperbolique au sens de Gromov est le suivant. Masur et Minsky [MasM] ont démontré que le graphe des courbes d'une surface compacte, connexe, orientable de genre au moins 2, tout en n'étant pas localement fini, est hyperbolique au sens de Gromov. L'hyperbolicité du graphe des courbes est alors devenu un outil fondamental dans l'étude de  $\text{Mod}(S)$ . Mentionnons par exemple que Dahmani, Guirardel et Osin [DGO] ont démontré, en utilisant l'action de  $\text{Mod}(S)$  sur le graphe des courbes, que tout sous-groupe  $H$  de  $\text{Mod}(S)$  qui n'est pas virtuellement abélien est *SQ universel*, c'est-à-dire que tout groupe dénombrable se plonge dans un quotient de  $H$ .

Dans le contexte de  $\mathbb{F}_n$  et de  $W_n$ , la construction d'espaces hyperboliques au sens de Gromov sur lesquels  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$  agissent par isométries repose sur leurs actions sur des arbres. Soient  $G$  un groupe de type fini et  $\mathcal{F}$  un système de facteurs libres de  $G$ . Un *scindement de  $G$*  est une action de  $G$  sur un arbre simplicial, cette action étant supposée minimale et sans point fixe global. Un scindement de  $G$  est *relatif à  $\mathcal{F}$*  si, pour tout sous-groupe  $H$  de  $G$  tel que  $[H] \in \mathcal{F}$ , le groupe  $H$  fixe un sommet de  $G$ . Un *scindement libre de  $G$  relatif à  $\mathcal{F}$*  est un scindement de  $G$  relatif à  $\mathcal{F}$  à stabilisateurs d'arêtes triviaux. Notons que les arbres considérés dans l'Outre-espace  $\mathcal{O}(G, \mathcal{F})$  sont des cas particuliers de scindements libres. Un élément  $g$  de  $G$  est dit  *$\mathcal{F}$ -périphérique*, ou *périphérique* s'il n'y a pas d'ambiguïté, s'il existe un sous-groupe  $H$  de  $G$  tel que  $g \in H$  et  $[H] \in \mathcal{F}$ . Sinon,  $g$  est  *$\mathcal{F}$ -non-périphérique*. Si  $S$  et  $S'$  sont deux scindements de  $G$  relatifs à  $\mathcal{F}$ , on dit que  $S$  est un *raffinement* de  $S'$  si  $S'$  est obtenu à partir de  $S$  en écrasant des orbites d'arêtes. Deux scindements  $S$  et  $S'$  sont *compatibles* s'ils ont un raffinement commun.

Le *graphe des facteurs libres de  $G$  relativement à  $\mathcal{F}$* , noté  $\text{FF}(G, \mathcal{F})$ , est le graphe dont les sommets sont les classes d'homéomorphisme  $G$ -équivariant de scindements libres de  $G$  relatifs à  $\mathcal{F}$ , deux classes d'équivalence  $\mathcal{S}$  et  $\mathcal{S}'$  étant reliés par une arête s'il existe  $S \in \mathcal{S}$  et  $S' \in \mathcal{S}'$  tels que  $S$  et  $S'$  sont compatibles ou s'il existe un élément  $g \in G$  qui est  $\mathcal{F}$ -non-périphérique tel que  $g$  fixe un sommet dans  $S$  et dans  $S'$ . L'hyperbolicité du graphe des facteurs libres avait été démontrée dans le cas où  $G = \mathbb{F}_n$  et  $\mathcal{F} = \emptyset$  par Bestvina et Feighn [BF2] (voir également une démonstration de Kapovich et Rafi [KR]) et dans le cas relatif par Handel et Mosher [HaM3] (toujours pour  $G = \mathbb{F}_n$ ). Guirardel et Horbez [GuH2] ont démontré que, en dehors de certains systèmes de facteurs libres *sporadiques* (voir loc. cit.), le graphe  $\text{FF}(G, \mathcal{F})$  est hyperbolique au sens de Gromov.

Le groupe  $\text{Out}(G, \mathcal{F})$  des automorphismes extérieurs de  $G$  préservant  $\mathcal{F}$  agit à droite

par isométries sur  $\text{FF}(G, \mathcal{F})$  par précomposition de l'action. Les éléments loxodromiques de  $\text{Out}(G, \mathcal{F})$  sont les *automorphismes extérieurs complètement irréductibles relativement à  $\mathcal{F}$* , c'est-à-dire les éléments de  $\text{Out}(G, \mathcal{F})$  dont aucune puissance non triviale ne préserve de systèmes de facteurs libres propres contenant  $\mathcal{F}$ .

Nous donnons à présent une autre justification du cadre relatif avec une comparaison avec le groupe modulaire d'une surface. En effet, Ivanov [Iva1] a démontré une classification des sous-groupes de  $\text{Mod}(S)$ , où  $S$  est une surface compacte, connexe et hyperbolique. Cette classification affirme qu'à tout sous-groupe  $H$  de  $\text{Mod}(S)$  nous pouvons associer une décomposition maximale de  $S$  en sous-surfaces essentielles invariantes à isotopie près par un sous-groupe d'indice fini de  $H$ . La démonstration de ce résultat s'effectue en deux étapes. La première étape est de montrer, en utilisant l'action de  $H$  sur le graphe des courbes, que soit  $H$  possède un sous-groupe d'indice fini qui fixe la classe d'isotopie d'une courbe fermée simple et essentielle, soit  $H$  contient un élément pseudo-Anosov. Si  $H$  contient un pseudo-Anosov, alors  $S$  ne contient pas de sous-surface essentielle propre invariante par un sous-groupe d'indice fini de  $H$ . Sinon, quitte à passer à un sous-groupe d'indice fini, on peut supposer que  $H$  fixe la classe d'isotopie  $[c]$  d'une courbe fermée, simple et essentielle  $c$  de  $S$ . Le groupe  $H$  préserve alors la classe d'isotopie de  $\overline{S - c}$  et le résultat se démontre par récurrence sur la complexité topologique de chaque composante connexe de  $\overline{S - c}$ .

Dans le cadre de  $\text{Out}(\mathbb{F}_n)$ , l'introduction du cadre relatif est nécessaire. En effet, un possible analogue des éléments pseudo-Anosov sont les automorphismes complètement irréductibles mentionnés ci-dessus. L'analogue des sous-surfaces dans le théorème d'Ivanov serait alors les facteurs libres de  $\mathbb{F}_n$ . La première étape de la démonstration est analogue. En effet, un résultat dû à Handel et Mosher [HaM1] pour les sous-groupes de type fini puis généralisé par Horbez [Hor1] à tous les sous-groupes de  $\text{Out}(\mathbb{F}_n)$  affirme que tout sous-groupe de  $\text{Out}(\mathbb{F}_n)$  soit contient un automorphisme extérieur complètement irréductible soit contient un sous-groupe d'indice fini qui fixe la classe de conjugaison d'un facteur libre. La démonstration de Horbez repose sur l'action de  $\text{Out}(\mathbb{F}_n)$  sur le complexe des facteurs libres. Cependant, si  $A$  est un facteur libre de  $\mathbb{F}_n$ , il n'existe pas de facteur libre *naturel*  $B$  de  $\mathbb{F}_n$  tel que  $\mathbb{F}_n = A * B$ . De ce fait, un sous-groupe de  $\text{Out}(\mathbb{F}_n)$  fixant la classe de conjugaison de  $A$  ne fixe pas nécessairement de facteur libre complémentaire et un argument de récurrence à la Ivanov ne fonctionne pas.

À la place, il convient d'utiliser les graphes des facteurs libres *relatifs*. En effet, si  $H$  est un sous-groupe de  $\text{Out}(\mathbb{F}_n)$  qui fixe un système de facteurs libres  $\mathcal{F}$ , alors  $H$  agit par isométries sur  $\text{FF}(\mathbb{F}_n, \mathcal{F})$ . L'alternative de Handel et Mosher se généralise alors au cadre relatif (voir [HaM4] pour le cas où  $H$  est de type fini, et [GuH2] pour le cas général).

Nous mentionnons à présent l'existence d'un autre graphe hyperbolique sur lequel  $\text{Out}(G, \mathcal{F})$  agit par isométries. Il s'agit du *graphe des scindements libres de  $G$  relatifs à  $\mathcal{F}$* , noté  $\overline{K}(G, \mathcal{F})$ . Le graphe  $\overline{K}(G, \mathcal{F})$  est le graphe dont les sommets sont les classes d'homéomorphismes  $G$ -équivalents de scindements libres de  $G$  relativement à  $\mathcal{F}$ , deux sommets  $\mathcal{S}$  et  $\mathcal{S}'$  étant reliés par une arête s'il existe  $S \in \mathcal{S}$  et  $S' \in \mathcal{S}'$  tels que  $S$  s'écrase sur  $S'$  ou réciproquement. Le graphe  $\overline{K}(G, \mathcal{F})$  a l'avantage d'avoir des stabilisateurs beaucoup plus rigides que ceux de  $\text{FF}(G, \mathcal{F})$ . En effet, les stabilisateurs de sommets

de  $\text{FF}(G, \mathcal{F})$  sont isomorphes à  $\text{Out}(G, \mathcal{F}')$ , où  $\mathcal{F}'$  est un système de facteurs libres non trivial de  $G$  contenant proprement  $\mathcal{F}$ , alors que les stabilisateurs de sommets de  $\overline{K}(G, \mathcal{F})$  ont une description précise donnée par Levitt [Lev1]. Notons par ailleurs que  $\overline{K}(G, \mathcal{F})$  peut être considéré comme une complétion simpliciale de l'Outre-espace de  $G$  relativement à  $\mathcal{F}$  (rappelons que l'Outre-espace de  $G$  relativement à  $\mathcal{F}$  possède une structure de complexe simplicial auquel il manque des faces), d'où la notation  $\overline{K}(G, \mathcal{F})$ . De ce fait, les questions de rigidité concernant  $\overline{K}(G, \mathcal{F})$  sont plus simples à traiter que les questions de rigidité concernant  $\text{FF}(G, \mathcal{F})$  (mentionnons tout de même que Bestvina et Bridson ont annoncé avoir démontré que le graphe des facteurs libres de  $\mathbb{F}_n$  est un modèle géométrique rigide pour  $\text{Out}(\mathbb{F}_n)$ ). Handel et Mosher [HaM2] ont démontré que  $\overline{K}(\mathbb{F}_n)$  est hyperbolique, Horbez [Hor2] l'a démontré dans le cas où  $G$  est général. Le groupe  $\text{Out}(G, \mathcal{F})$  admet une action naturelle par isométries sur  $\overline{K}(G, \mathcal{F})$  par précomposition de l'action. Dans le cas de  $\mathbb{F}_n$ , Aramayona et Souto [AS] ont démontré que le graphe des scindements libres de  $\mathbb{F}_n$  est un modèle géométrique rigide pour  $\text{Out}(\mathbb{F}_n)$ . Dans le cas de  $W_n$ , le graphe  $\overline{K}(W_n)$  constitue également un modèle géométrique rigide pour  $\text{Out}(W_n)$ .

**Théorème 1.2.1** (Theorem 3.1.2). *Soit  $n \geq 4$ . Le morphisme naturel*

$$\text{Out}(W_n) \rightarrow \text{Aut}(\overline{K}(W_n))$$

*est un isomorphisme.*

Les actions de  $\text{Out}(W_n)$  et  $\text{Out}(\mathbb{F}_n)$  sur ces espaces hyperboliques ne sont cependant pas proprement discontinues. En effet, les stabilisateurs de sommets de  $\overline{K}(W_n)$  sont en général infinis. Levitt [Lev1, Proposition 4.2] en a donné une description complète. Les stabilisateurs de sommets contiennent en particulier un sous-groupe distingué appelé *groupe de twists*, que nous décrivons maintenant dans le cas d'un exemple simple. Soient  $A$  et  $B$  deux sous-groupes infinis de  $W_n$  tels que  $W_n = A * B$  et soit  $S$  le scindement libre tel que le graphe sous-jacent à  $W_n \setminus S$  contient exactement une arête et tel que les stabilisateurs de sommets de  $S$  sont exactement les conjugués de  $A$  et les conjugués de  $B$ . Soit  $\phi \in \text{Out}(W_n)$  tel que  $\phi$  préserve la classe d'homéomorphisme  $W_n$ -équivariant  $\mathcal{S}$  de  $S$ . Quitte à considérer une puissance de  $\phi$ , nous pouvons supposer que  $\phi$  préserve la classe de conjugaison de  $A$  et la classe de conjugaison de  $B$ . Soit  $\Phi \in \phi$  tel que  $\Phi(A) = A$ . L'automorphisme  $\Phi$  induit un automorphisme extérieur  $[\Phi|_A] \in \text{Out}(A)$  et cet automorphisme extérieur ne dépend pas du choix de  $\Phi$  car  $A$  est un sous-groupe *malnormal* de  $\mathbb{F}_n$  : pour tout élément  $g \in \mathbb{F}_n - A$ , nous avons  $gAg^{-1} \cap A = \{e\}$ . Soit  $\text{Stab}_{\text{Out}(W_n)}^0(\mathcal{S})$  le sous-groupe d'indice 2 de  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{S})$  qui consiste en les automorphismes extérieurs fixant les classes de conjugaison de  $A$  et de  $B$ . Nous avons alors un morphisme

$$\text{Stab}_{\text{Out}(W_n)}^0(\mathcal{S}) \rightarrow \text{Out}(A) \times \text{Out}(B).$$

Le noyau de ce morphisme est le *groupe de twists de  $\mathcal{S}$* . Un twist de  $\mathcal{S}$  possède alors un représentant qui préserve  $A$  et  $B$  et qui agit comme une conjugaison globale sur  $A$  (resp.



$B$ ) par un élément de  $A$  (resp.  $B$ ). De ce fait, le groupe de twists de  $\mathcal{S}$  est isomorphe à  $A \times B$ .

Plus généralement, les groupes de twists d'un scindement libre de  $W_n$  sont isomorphes à des produits directs de groupes isomorphes à des groupes de Coxeter universels. Dans l'étude des questions de rigidité, l'existence de tels produits directs revêt une importance particulière puisque l'action par isométrie de produits directs sur un espace hyperbolique au sens de Gromov est facilement compréhensible et contraint l'existence de points fixes dans l'espace hyperbolique ou dans son bord (voir par exemple [HW2, Proposition 4.2]). Par exemple, un groupe isomorphe à  $\mathbb{Z}^2$  agissant par isométries sur un espace hyperbolique propre devra nécessairement fixer un point dans la compactification de l'espace hyperbolique. Ceci implique notamment que les groupes hyperboliques ne contiennent pas de sous-groupes isomorphes à  $\mathbb{Z}^2$ . De ce fait, l'action de  $\text{Out}(W_n)$  sur le graphe des facteurs libres permet de caractériser algébriquement certains groupes de twists et plus généralement certains stabilisateurs de sommets du graphe des scindements libres. Cette caractérisation algébrique des stabilisateurs est l'argument central dans la démonstration du théorème de rigidité algébrique suivant.

**Théorème 1.2.2** (Theorem 4.8.2). *Soit  $n \geq 5$  et soit  $F: H_1 \rightarrow H_2$  un isomorphisme entre deux sous-groupes d'indice fini de  $\text{Out}(W_n)$ . Alors il existe un unique  $f \in \text{Out}(W_n)$  tel que  $F$  soit la restriction de la conjugaison globale par  $f$ .*

Nous montrons en fait la rigidité du *commensurateur abstrait* de  $\text{Out}(W_n)$ . Soit  $G$  un groupe. Le commensurateur abstrait de  $G$  est le groupe  $\text{Comm}(G)$  des classes d'équivalence d'isomorphismes entre deux sous-groupes d'indice fini de  $G$ . Deux isomorphismes sont dans la même classe d'équivalence s'ils sont égaux en restriction à un sous-groupe d'indice fini commun de leur domaine de définition. Si  $f_1: G_1 \rightarrow H_1$  et  $f_2: G_2 \rightarrow H_2$  sont deux isomorphismes entre sous-groupes d'indice fini de  $G$ , la loi de composition de  $\text{Comm}(G)$  est :

$$[f_1: G_1 \rightarrow H_1] \circ [f_2: G_2 \rightarrow H_2] = [f_1 \circ f_2: f_2^{-1}(H_2 \cap G_1) \rightarrow f_1(G_1 \cap H_2)].$$

Le commensurateur abstrait a été popularisé par Serre dans le cadre de l'étude des groupes arithmétiques et le problème de congruences de sous-groupes d'indice fini dans  $\text{SL}(n, \mathbb{Z})$ . Notons que tout automorphisme de  $G$  induit un élément de  $\text{Comm}(G)$ , et donc l'action  $G$  sur lui-même par conjugaison induit un morphisme

$$G \rightarrow \text{Comm}(G).$$

Le commensurateur abstrait est une notion de symétrie du groupe  $G$  qui est plus faible que celle de son groupe d'automorphismes. Par exemple, le commensurateur abstrait du groupe  $\mathbb{Z}^m$  est isomorphe à  $\text{GL}(m, \mathbb{Q})$  alors que son groupe d'automorphismes est  $\text{GL}(m, \mathbb{Z})$ . De même, le commensurateur abstrait d'un groupe libre non abélien n'est pas de type fini [BB]. Le résultat de rigidité suivant, qui est une conséquence du théorème 1.2.2, montre que, dans le cas de  $\text{Out}(W_n)$ , les deux notions de symétrie que sont le commensurateur abstrait et le groupe des automorphismes coïncident.

**Corollaire 1.2.3** (Theorem 4.8.2). *Soit  $n \geq 5$ . Le morphisme*

$$\text{Out}(W_n) \rightarrow \text{Comm}(\text{Out}(W_n))$$

*est un isomorphisme.*

Le théorème 1.2.2 et le corollaire 1.2.3 sont inspirés de résultats analogues dans différents contextes. En effet, le théorème de Mostow-Prasad-Margulis et le théorème d'arithmécité de Margulis (voir par exemple [Zim]) impliquent par exemple que, si  $\Gamma$  est un réseau dans un groupe de Lie  $G$  simple, connexe, non compact, de centre trivial et tel que  $G \neq \text{PSL}(2, \mathbb{R})$ , alors  $\Gamma$  est d'indice fini dans  $\text{Comm}(\Gamma)$  si, et seulement si,  $\Gamma$  n'est pas arithmétique ; sinon  $\text{Comm}(\Gamma)$  est dense dans  $G$ . De plus, dans les deux cas,  $\text{Comm}(\Gamma)$  est isomorphe au *commensurateur de  $\Gamma$  dans  $G$* , c'est-à-dire à l'ensemble des éléments  $g$  de  $G$  tels que  $\Gamma \cap g\Gamma g^{-1}$  est d'indice fini dans  $\Gamma$  et dans  $g\Gamma g^{-1}$ . De même, Ivanov [Iva2] a démontré que, si  $S$  est une surface connexe, compacte, orientable de genre au moins 3, alors le morphisme naturel  $\text{Mod}^\pm(S) \rightarrow \text{Comm}(\text{Mod}^\pm(S))$  est un isomorphisme. Dans le cas de  $\text{Out}(\mathbb{F}_n)$ , la rigidité du commensurateur abstrait de  $\text{Out}(\mathbb{F}_n)$  a été démontrée par Farb et Handel [FarH] pour  $n \geq 4$ , puis a été étendue au cas  $n = 3$  par Horbez et Wade [HW2] par des méthodes géométriques qui ont inspiré la démonstration du théorème 1.2.2 et du corollaire 1.2.3. Enfin, de tels résultats de rigidité ont été étendus à d'autres groupes, tels les groupes modulaires des corps en anses [Hen] ou les groupes modulaires de grosses surfaces [BDR].

### 1.3 Dynamique sur des espaces compacts et croissance dans les groupes hyperboliques

Comme nous l'avons évoqué, la démonstration du théorème 1.2.2 repose sur l'étude de l'action de  $\text{Out}(W_n)$  sur un espace hyperbolique, le graphe des facteurs libres, et la recherche de points fixes dans une bordification naturelle. Il convient de souligner de manière approfondie (et nous en avons déjà aperçu deux occurrences dans la partie 1.2) l'importance de l'étude de l'action par homéomorphismes d'un groupe d'isométries d'un espace hyperbolique au sens de Gromov  $X$  sur le bord à l'infini  $\partial_\infty X$  de  $X$  (voir par exemple [Gro1, GdlH, BriHae]). Notons par exemple que l'étude des groupes kleiniens s'appuie principalement sur leur action par isométrie sur l'espace hyperbolique réel  $\mathbb{H}_{\mathbb{R}}^3$  de dimension 3 et leur action par homéomorphismes (en fait conformes) sur leur ensemble limite. Le bord à l'infini de  $X$  a une structure riche, et peut être considéré à la fois de manière topologique, dynamique, métrique ou analytique. La classification des isométries d'un espace hyperbolique au sens de Gromov (voir par exemple [GdlH, Chapitre 8]) repose essentiellement sur l'étude de l'action des isométries sur le bord de Gromov et les points fixes associés. Cette classification illustre par ailleurs la dynamique des isométries de  $X$  sur la bordification  $X \cup \partial_\infty X$ . Par exemple, si  $\phi$  est une isométrie loxodromique de  $X$ , alors  $\phi$  fixe exactement deux points  $a_\phi^{+\infty}, a_\phi^{-\infty} \in \partial_\infty X$  et, pour tout  $x \in X \cup \partial_\infty X - \{a_\phi^{\pm\infty}\}$ , la suite  $(\phi^{\pm m}(x))_{m \in \mathbb{N}}$  converge vers  $a_\phi^{\pm\infty}$ .

Ce type de dynamique est un exemple de *dynamique Nord-Sud*. Soit  $X$  un espace topologique compact, métrisable et  $\phi$  un homéomorphisme de  $X$ . Alors  $\phi$  agit avec une dynamique Nord-Sud sur  $X$  s'il existe deux compacts disjoints propres  $\Delta_{\pm}(\phi)$  de  $X$  invariants par  $\phi$  et tels que tout point de  $X - \Delta_{\pm}(\phi)$  converge vers un point  $p \in \Delta_{\mp}(\phi)$  sous l'itération de  $\phi^{\mp 1}$ . De tels compacts  $\Delta_{\pm}(\phi)$  sont appelés des compacts *attractifs* et *répulsifs* associés à  $\phi$ . Les résultats de dynamique Nord-Sud sont des outils privilégiés pour obtenir des résultats de classification des groupes considérés. Par exemple, Tits [Tit1] (voir également [dlH1]), en utilisant des arguments de dynamique Nord-Sud et de tennis de table (voir par exemple [dlH2, BriHae]), a démontré une alternative pour les groupes linéaires : tout sous-groupe d'un groupe linéaire soit contient un sous-groupe libre non abélien de rang 2 soit est résoluble.

Ainsi, puisque le graphe des courbes d'une surface  $S$  compacte, connexe, orientable de genre  $g \geq 2$  est hyperbolique au sens de Gromov, les éléments pseudo-Anosov, qui sont précisément les éléments loxodromiques du graphe des courbes par des résultats de Masur et Minsky [MasM], agissent avec une dynamique Nord-Sud sur la réunion du graphe des courbes et de son bord à l'infini. De même les automorphismes extérieurs complètement irréductibles d'un produit libre  $G$  agissent avec une dynamique Nord-Sud sur la réunion du graphe des facteurs libres avec son bord à l'infini, et on peut en déduire une alternative de Tits pour ces groupes [GuH2].

Des résultats de dynamique Nord-Sud existent également dans le cadre d'espaces non hyperboliques. Par exemple, le résultat de Tits [Tit1] suscit  utilise de la dynamique projective et notamment le résultat de dynamique Nord-Sud suivant. Soit  $\phi$  une matrice carr e r elle de rang  $n \geq 2$  diagonalisable dans  $\mathbb{R}$  et telle qu'il existe une valeur propre  $\lambda > 1$  de  $\phi$  telle que, pour toute valeur propre  $\lambda'$  de  $\phi$  distincte de  $\lambda$ , nous ayons  $|\lambda| > |\lambda'|$ . Alors la classe projective d'un vecteur propre de  $\phi$  associ     $\lambda$  est un point attractif dans  $\mathbb{RP}^n$ . De plus, la classe projective du sous-espace vectoriel de  $\mathbb{R}^n$  engendr  par les vecteurs propres de  $\phi$  associ s   des valeurs propres distinctes de  $\lambda$  est le compact r pulsif de  $\phi$  dans  $\mathbb{RP}^n$ . Notons que, dans ce cas,  $\Delta_{-}(\phi)$  n'est pas r duit   un point si  $n \geq 3$ . Par ailleurs, Thurston [Thu] a d montr  que les hom omorphismes pseudo-Anosov d'une surface connexe, compacte, orientable de genre au moins 2 agissent avec une dynamique Nord-Sud sur l'espace des feuilletages mesur s projectifs et sur le compactifi  de Thurston de l'espace de Teichm ller. En utilisant cette dynamique Nord-Sud, Ivanov [Iva1] (voir  galement les travaux de McCarthy [McC]) a alors d montr  que tout sous-groupe de  $\text{Mod}(S)$  contenant un  l ment pseudo-Anosov est soit virtuellement cyclique, soit contient un sous-groupe libre non ab lien compos  uniquement d' l ments pseudo-Anosov. De m me Levitt et Lustig [LL] ont d montr  que les automorphismes ext rieurs compl tement irr ductibles de  $\mathbb{F}_n$  agissent avec une dynamique Nord-Sud sur le compactifi  de l'Outre-espace de  $\mathbb{F}_n$ .

Dans cette th se, nous  tudions des propri t s dynamiques des  l ments de  $\text{Out}(\mathbb{F}_n)$ . Nous construisons un dictionnaire entre l'action d'un  l ment  $\phi \in \text{Out}(\mathbb{F}_n)$  sur un espace topologique compact appropri   $X$  et la *croissance de la longueur de classes de conjugaisons de  $\mathbb{F}_n$  par it ration de  $\phi$* . L' tude de la dynamique sur l'espace  $X$  repose alors sur une traduction en une dynamique sur le groupe  $\mathbb{F}_n$ , et inversement. Les r sultats de

dynamiques étudiés seront ainsi intrinsèquement liés avec la *croissance* dans les groupes. Initiée par Švarc, Milnor et Wolf, et particulièrement développée par Guivarc'h, Gromov et Grigorchuk, les problèmes de croissance dans un groupe forment un domaine majeur en théorie géométrique des groupes (voir par exemple [LS, Man1, Hel]). Ces problèmes de croissance sont reliés aux alternatives sur les sous-groupes que nous évoquions plus tôt. En effet, l'alternative que Tits a démontrée pour les groupes linéaires implique par un résultat de Milnor et Wolf que, si  $G$  est un sous-groupe d'un groupe linéaire, alors le volume d'une boule dans un graphe de Cayley de  $G$  croît de manière exponentielle ou polynomiale.

Dans ce manuscrit, nous nous intéressons aux problèmes de croissance d'éléments du groupe sous l'itération d'automorphismes du groupe. Soient  $G$  un groupe de type fini,  $S$  une partie génératrice finie de  $G$  et  $d$  la distance des mots sur  $G$  associée à  $S$ . Soient  $\Phi$  un automorphisme de  $G$  et  $g \in G$ . La *croissance de  $g$  sous itération de  $\Phi$*  correspond au comportement asymptotique de la suite

$$(\ell_S(\Phi^m(g)))_{m \in \mathbb{N}} = (d(\Phi^m(g), e))_{m \in \mathbb{N}}.$$

De même, si  $\phi$  est un automorphisme extérieur de  $G$ , la *croissance de (la classe de conjugaison de)  $g$  sous itération de  $\phi$*  correspond au comportement asymptotique de la suite

$$(\ell_S(\phi^m([g])))_{m \in \mathbb{N}} = \left( \min_{g' \in [g], \Phi \in \phi} \ell_S(\Phi^m(g')) \right)_{m \in \mathbb{N}}.$$

Nous dirons que  $g$  est à *croissance polynomiale relativement à  $\phi$*  s'il existe un polynôme  $P \in \mathbb{Z}[X]$  tel que, pour tout  $m \in \mathbb{N}^*$ , nous ayons :

$$\ell_S(\phi^m([g])) \leq P(m),$$

et nous dirons que  $g$  est à *croissance exponentielle par itération de  $\phi$*  s'il existe  $C, \lambda > 0$  tels que, pour tout  $m \in \mathbb{N}^*$ , nous ayons :

$$\ell_S(\phi^m([g])) \geq C e^{\lambda m}.$$

Pour une large famille de groupes comprenant les groupes libres abéliens et les groupes hyperboliques sans torsion, un élément du groupe est soit à croissance exponentielle soit à croissance polynomiale relativement à un automorphisme extérieur (voir par exemple [Cou, Theorem 1.1]). De plus, ce fait ne dépend pas du choix de la partie génératrice finie du groupe. Notons cependant que cette alternative n'est pas vérifiée pour tous les groupes et que Coulon [Cou] a construit des groupes ayant des éléments à croissance intermédiaire sous itération d'automorphismes.

Nous donnons à présent des exemples d'automorphismes d'un groupe libre non abélien  $\mathbb{F}_n$  de rang  $n$  dont les éléments ont différents types de croissance.

(1) Un exemple d'automorphisme pour lequel tous les éléments sont à croissance polynomiale est le suivant. Soit  $F_3 = \langle a, b, c \rangle$  et soit  $\phi$  l'automorphisme fixant  $a$ , envoyant  $b$  sur  $ba$  et envoyant  $c$  sur  $cb$ . Alors  $b$  est à croissance linéaire relativement à  $\phi$  et  $c$  est à croissance quadratique relativement à  $\phi$ .

(2) Un exemple d'automorphisme pour lequel tous les éléments sont à croissance exponentielle est le suivant. Soit  $M \in \mathrm{GL}(3, \mathbb{Z})$  une matrice primitive telle qu'il existe une unique valeur propre  $\lambda$  de  $M$ , comptée avec multiplicité, telle que  $|\lambda| > 1$  et telle que pour toute valeur propre  $\mu$  de  $M$  distincte de  $\lambda$ , nous ayons  $|\mu| < 1$ . Soit  $\phi_M \in \mathrm{Out}(F_3)$  tel que l'image de  $\phi_M$  par le morphisme d'abélianisation  $\Theta: \mathrm{Out}(F_3) \rightarrow \mathrm{GL}(3, \mathbb{Z})$  soit égale à  $M$ . Alors un résultat de Gersten et Stallings [GS] implique que tous les éléments de  $\mathbb{F}_n$  sont à croissance exponentielle par itération de  $\phi_M$ .

(3) Un exemple d'automorphisme possédant des éléments à croissance exponentielle et des éléments à croissance polynomiale est le suivant. Soit  $\mathbb{F}_3 = \langle a, b, c \rangle$  et soit  $\phi$  l'automorphisme envoyant  $a$  sur  $b$ ,  $b$  sur  $ab$  et  $c$  sur  $caba^{-1}b^{-1}$ . Alors, si  $g \in \mathbb{F}_3 - \langle aba^{-1}b^{-1}, c \rangle$ , l'élément  $g$  est à croissance exponentielle par itération de  $\phi$ . L'élément  $aba^{-1}b^{-1}$  est fixé par  $\phi^2$  et l'élément  $c$  est à croissance linéaire par itération de  $\phi$ .

Ces questions de croissance ont par ailleurs des conséquences sur la structure algébrique des groupes. En effet, Brinkmann [Bri] (voir également les travaux de Bestvina et Feighn [BF1]) a démontré que si  $\phi \in \mathrm{Out}(\mathbb{F}_n)$ , alors la suspension  $\mathbb{F}_n \rtimes_{\phi} \mathbb{Z}$  est un groupe hyperbolique au sens de Gromov si et seulement si  $\phi \in \mathrm{Out}(\mathbb{F}_n)$  est un élément *atoroïdal*, c'est-à-dire si tous les éléments de  $\mathbb{F}_n$  sont à croissance exponentielle par itération de  $\phi$ . Notons qu'une conséquence d'un résultat de Levitt [Lev2, Corollary 1.6] est qu'un automorphisme  $\phi \in \mathrm{Out}(\mathbb{F}_n)$  est atoroïdal si et seulement si pour tout  $k \in \mathbb{N}^*$ , l'automorphisme extérieur  $\phi^k$  ne fixe pas la classe de conjugaison d'un élément non trivial de  $\mathbb{F}_n$ . Ce résultat est à comparer avec un résultat de Thurston affirmant que la suspension d'une surface compacte, connexe, orientable de genre  $g \geq 2$  par un élément pseudo-Anosov est hyperbolique (voir par exemple la démonstration d'Otal [Ota] qui utilise des actions de groupes sur des arbres).

Dans cette thèse, nous considérons ces questions de croissance du point de vue des sous-groupes de  $\mathrm{Out}(\mathbb{F}_n)$ . Si  $\phi \in \mathrm{Out}(\mathbb{F}_n)$ , nous noterons  $\mathrm{Poly}(\phi)$  l'ensemble des éléments de  $\mathbb{F}_n$  à croissance polynomiale par itération de  $\phi$ . Si  $H$  est un sous-groupe de  $\mathrm{Out}(\mathbb{F}_n)$ , nous noterons  $\mathrm{Poly}(H) = \bigcap_{\phi \in H} \mathrm{Poly}(\phi)$ , c'est-à-dire l'ensemble des éléments de  $\mathbb{F}_n$  à croissance polynomiale par itération de chaque élément de  $H$ . Nous démontrons le résultat suivant, qui donne l'existence d'éléments dynamiquement génériques dans les sous-groupes de  $\mathrm{Out}(\mathbb{F}_n)$ .

**Théorème 1.3.1** (Theorem 7.1.1). *Soit  $n \in \mathbb{N}^*$  et soit  $H$  un sous-groupe de  $\mathrm{Out}(\mathbb{F}_n)$ . Il existe  $\phi \in H$  tel que  $\mathrm{Poly}(\phi) = \mathrm{Poly}(H)$ .*

Le théorème 1.3.1 montre ainsi que, si  $H$  est un sous-groupe de  $\mathrm{Out}(\mathbb{F}_n)$ , tout élément de  $\mathbb{F}_n$  à croissance exponentielle par itération d'un élément de  $H$  est à croissance exponentielle par itération de tout élément générique de  $H$  donné par le théorème 1.3.1. Un élément générique ainsi construit encode donc toute la croissance exponentielle de  $H$ .

Le théorème 1.3.1 possède un analogue dans le contexte du groupe modulaire d'une surface  $S$  fermée, connexe, orientable, munie d'une structure hyperbolique. En effet, une conséquence de la classification de Nielsen-Thurston (voir par exemple [FarM, The-

orem 13.2]) et de l'étude de la croissance d'éléments pseudo-Anosov (voir par exemple [FarM, Theorem 14.23]) est que la croissance sous les itérés d'un élément de  $\text{Mod}(S)$  de la longueur hyperbolique d'un représentant géodésique de la classe d'homotopie d'une courbe fermée et essentielle est soit exponentielle, soit linéaire. Par ailleurs, la croissance linéaire provient de twists de Dehn tandis que la croissance exponentielle provient d'éléments pseudo-Anosov de sous-surfaces de  $S$ . Comme nous l'avons vu précédemment, Ivanov [Iva1] a démontré que, pour tout sous-groupe  $H$  de  $\text{Mod}(S)$ , quitte à passer à un sous-groupe d'indice fini de  $H$ , il existe un nombre fini de classes d'homotopies de courbes fermées, simples et essentielles deux à deux disjointes  $C_1, \dots, C_k$  invariantes par  $H$  et telles que, pour toute composante connexe  $S'$  de  $S - \bigcup_{i=1}^k C_i$ , la restriction  $H|_{S'} \subseteq \text{Mod}(S')$  soit est triviale, soit contient un pseudo-Anosov. Par ailleurs, il existe un élément  $f \in H$  tel que, pour toute composante connexe  $S'$  de  $S - \bigcup_{i=1}^k C_i$  telle que la restriction  $H|_{S'} \subseteq \text{Mod}(S')$  n'est pas triviale, l'élément  $f|_{S'} \in \text{Mod}(S')$  est un pseudo-Anosov. L'élément  $f$  peut se construire de la manière suivante. Pour toute sous-surface  $S'$  de  $S - \bigcup_{i=1}^k C_i$  telle que la restriction  $H|_{S'} \subseteq \text{Mod}(S')$  n'est pas triviale, le groupe  $H|_{S'}$  contient un élément loxodromique du graphe des courbes de  $S'$ . Par un résultat de Clay et Uyanik [CU1, Theorem 5.1], il existe  $f \in H$  tel que, pour toute sous-surface  $S'$  de  $S - \bigcup_{i=1}^k C_i$  telle que la restriction  $H|_{S'} \subseteq \text{Mod}(S')$  n'est pas triviale, l'élément  $f|_{S'}$  est loxodromique dans le graphe des courbes de  $S'$ , donc est un élément pseudo-Anosov de  $\text{Mod}(S')$ .

Mentionnons que le théorème 1.3.1 a été démontré dans le cas où  $\text{Poly}(H) = \{1\}$  par Clay et Uyanik [CU2]. En effet, ces derniers démontrent que, dans ce cas, le groupe  $H$  contient un élément atoroïdal. Nous généralisons ainsi le théorème de Clay et Uyanik en considérant les automorphismes atoroïdaux comme étant les automorphismes extérieurs  $\phi$  de  $\text{Out}(\mathbb{F}_n)$  tels que, pour tout élément  $g$  de  $\text{Out}(\mathbb{F}_n)$  non trivial,  $g$  est à croissance exponentielle relativement à  $\phi$ . Lorsque nous considérons les éléments atoroïdaux comme étant les éléments de  $\text{Out}(\mathbb{F}_n)$  tels qu'aucune puissance ne fixe de classe de conjugaison d'éléments non trivial de  $\mathbb{F}_n$ , nous pouvons nous demander si une autre généralisation du théorème de Clay et Uyanik est possible. En effet, nous pouvons nous poser la question suivante. Soit  $H$  un sous-groupe de  $\text{Out}(\mathbb{F}_n)$  tel que  $H$  fixe virtuellement la classe de conjugaison d'un sous-groupe  $A$  de  $\mathbb{F}_n$ . Est-il vrai que soit  $H$  fixe virtuellement la classe de conjugaison d'un élément  $g$  de  $\mathbb{F}_n$  qui n'est contenu dans aucun conjugué de  $A$ , soit il existe  $\phi \in H$  tel que les seuls éléments de  $\mathbb{F}_n$  dont les classes de conjugaison sont fixées par  $\phi$  soient contenus dans des conjugués de  $A$  ?

Malheureusement, un tel résultat n'est pas valable. En effet, soit  $F_3 = \langle a, b, c \rangle$  un groupe libre non abélien de rang 3. Soit  $\phi_a$  (resp.  $\phi_b$ ) l'automorphisme de  $F_3$  qui fixe  $a$  et  $b$  et qui envoie  $c$  sur  $ca$  (resp.  $c$  sur  $cb$ ). Soit  $H = \langle [\phi_a], [\phi_b] \rangle$ . Alors  $H$  fixe la classe de conjugaison de tous les éléments de  $\langle a, b \rangle$ . Par ailleurs, pour tout élément  $\psi \in H$ , il existe un unique représentant  $\Psi$  de  $\psi$  et un élément  $g_\psi$  de  $\langle a, b \rangle$  tel que  $\Psi(\langle a, b \rangle) = \langle a, b \rangle$ ,  $\Psi|_{\langle a, b \rangle} = \text{id}_{\langle a, b \rangle}$  et  $\Psi(c) = cg_\psi$ . Ainsi,  $\psi$  fixe la classe de conjugaison du sous-groupe engendré par  $g_\psi$  et  $cg_\psi c^{-1}$ . Cependant, pour tout  $\psi \in H$ , si  $\phi \in H$  est tel que  $g_\phi$  ne centralise pas  $g_\psi$ , alors  $\phi$  ne fixe la classe de conjugaison d'aucun élément de  $\langle g_\psi, cg_\psi c^{-1} \rangle$  autre que les puissances de  $g_\psi$  et de  $cg_\psi c^{-1}$ .

Ainsi, le théorème de Clay et Uyanik ne se généralise pas à tous les sous-groupes de  $\mathbb{F}_n$  lorsque l'on considère les classes de conjugaison fixées. Néanmoins, dans l'exemple construit, le rang du sous-groupe de  $\mathbb{F}_n$  dont la classe de conjugaison est fixée par  $H$  est égal à 2. Ce rang est en fait minimal comme le montre le résultat suivant, conséquence du théorème 1.3.1 et du théorème de Kolchin pour les sous-groupes de  $\text{Out}(\mathbb{F}_n)$  dû à Bestvina, Feighn et Handel [BFH3], analogue du théorème de Kolchin pour les sous-groupes de  $\text{GL}(n, \mathbb{Z})$  (voir par exemple [Ser2]). Si  $\phi \in \text{Out}(\mathbb{F}_n)$ , nous notons  $\text{Fix}(\phi)$  l'ensemble des classes de conjugaison d'éléments de  $\mathbb{F}_n$  fixées par  $\phi$ .

**Théorème 1.3.2** (Corollary 7.5.3). *Soit  $n \geq 3$ , soit  $H$  un sous-groupe de  $\text{Out}(\mathbb{F}_n)$  et soient  $g_1, \dots, g_k$  des éléments non triviaux de  $\mathbb{F}_n$ . Supposons que, pour tout  $i \in \{1, \dots, k\}$ , le groupe  $H$  fixe virtuellement la classe de conjugaison de  $g_i$ . Alors l'une des assertions suivantes (mutuellement exclusives) est vérifiée.*

- (1) *Il existe un élément non trivial  $g_{k+1} \in \mathbb{F}_n$  qui n'est conjugué à aucune puissance d'un  $g_i$  pour  $i \in \{1, \dots, k\}$  et dont la classe de conjugaison est virtuellement fixée par  $H$ .*
- (2) *Il existe  $\phi \in H$  tel que  $\text{Fix}(\phi) = \{[\langle g_1 \rangle], \dots, [\langle g_k \rangle]\}$ .*

Le résultat suivant montre que la deuxième alternative du théorème 1.3.2 intervient en particulier lorsque le sous-groupe  $H$  de  $\text{Out}(\mathbb{F}_n)$  considéré est géométrique (au sens de [HaM4, Theorem J]).

**Proposition 1.3.3** (Corollary 7.5.4). *Soit  $n \geq 3$  et soit  $H$  un sous-groupe de  $\text{Out}(\mathbb{F}_n)$ . Supposons qu'il existe un unique sous-groupe cyclique maximal non trivial  $\langle g \rangle$  de  $\mathbb{F}_n$  dont la classe de conjugaison est virtuellement fixée par  $H$ . Supposons que  $g$  ne soit contenu dans aucun facteur libre propre de  $\mathbb{F}_n$ . Alors il existe une surface compacte, connexe  $S$  et une identification de  $\pi_1(S)$  avec  $\mathbb{F}_n$  tel que  $H$  soit un sous-groupe de  $\text{Mod}(S)$  contenant un élément pseudo-Anosov.*

La démonstration du théorème 1.3.1 repose sur des arguments de dynamique Nord-Sud et de tennis de table comme présentés précédemment. Nous cherchons donc à construire un espace topologique  $X$  compact, métrisable sur lequel  $H$  agit par homéomorphismes et tel qu'il existe un élément  $\phi \in H$  tel que  $\phi$  agisse sur  $X$  avec une dynamique Nord-Sud. Un élément  $\phi \in H$  sera en fait dynamiquement générique si et seulement si  $\phi$  agit sur  $X$  avec une dynamique Nord-Sud. Nous créons ainsi un dictionnaire entre des propriétés dynamiques de l'action de  $H$  sur  $X$  (la dynamique Nord-Sud de certains éléments de  $H$ ) et la croissance d'éléments de  $\mathbb{F}_n$  par itérations d'éléments de  $H$ .

L'espace  $X$  construit sera un *espace de courants*, espace que nous présentons dans la prochaine section. Cet espace a l'avantage, comme nous le verrons, qu'il existe une application ensembliste  $\Theta$  injective et  $H$ -équivariante des classes de conjugaison d'éléments de  $\mathbb{F}_n - \text{Poly}(H)$  vers  $X$ , dont l'image est dense. Ceci facilitera la construction du dictionnaire évoqué puisque la dynamique de l'action de  $H$  sur  $X$  dépendra complètement de l'étude de la dynamique de  $H$  sur l'image de  $\Theta$ .

## 1.4 Action sur des espaces de courants et construction d'éléments génériques

Dans la section précédente, nous avons présenté le bord à l'infini d'un espace hyperbolique au sens de Gromov d'un point de vue topologique et dynamique. Dans cette section, nous adoptons un point de vue analytique dans nos considérations du bord à l'infini. Nous considérons ainsi des mesures de Radon sur le bord à l'infini de certains groupes hyperboliques. Ceci est à rapprocher des travaux de Furstenberg [Fur1, Fur2, Fur3] et de l'étude d'un groupe  $\Gamma$  par l'utilisation des bords de Poisson-Furstenberg, qui sont des espaces mesurés munis d'une action de  $\Gamma$  qui quasi-préserve la mesure. Le bord de Poisson-Furstenberg intervient notamment dans la considération de problèmes de rigidité proches de ceux du théorème de superrigidité de Margulis (voir par exemple l'exposition de Bader et Furman [BaFur, Theorem 4.8]). Dans le cas où  $\Gamma$  est un groupe hyperbolique muni d'une mesure de probabilité  $\mu$  de premier moment fini et tel que le groupe engendré par le support de  $\mu$  est non élémentaire, le bord de Poisson-Furstenberg est isomorphe au bord à l'infini du groupe  $\Gamma$  par des résultats de Kaimanovich [Kai].

Dans [Pat, Sul], Patterson et Sullivan ont défini des mesures de Radon quasi invariantes  $\mu_\Gamma$  sur le bord à l'infini d'un espace hyperbolique réel  $\mathbb{H}_{\mathbb{R}}^n$  pour  $n \geq 2$  associés à des groupes convexes cocompacts  $\Gamma$  d'isométries de  $\mathbb{H}_{\mathbb{R}}^n$ . La mesure  $\mu_\Gamma$  ainsi construite a pour support l'ensemble limite  $\Lambda\Gamma$  de  $\Gamma$ . Cette mesure  $\mu_\Gamma$  permet par exemple d'obtenir des informations sur les propriétés ergodiques de l'action de  $\Gamma$  sur  $\Lambda\Gamma$  ou sur la croissance des orbites de  $\Gamma$  dans  $\mathbb{H}_{\mathbb{R}}^n$ . Ces mesures de Radon quasi invariantes ont par la suite été élargies aux espaces hyperboliques au sens de Gromov par Coornaert [Coor].

Pour obtenir des mesures de Radon invariantes et non plus quasi invariantes, une méthode générale est de considérer le *double bord* de l'espace hyperbolique au sens de Gromov. Soit  $X$  un espace hyperbolique au sens de Gromov et soit  $\partial_\infty X$  le bord à l'infini de  $X$ . Le *double bord de  $X$*  est l'espace topologique quotient

$$\partial^2 X = (\partial_\infty X \times \partial_\infty X \setminus \Delta) / \sim,$$

où  $\sim$  est la relation d'équivalence engendrée par la relation  $(x, y) \sim (y, x)$  et  $\Delta$  est la diagonale. Si  $\Gamma$  est un sous-groupe discret du groupe des isométries de  $X$  agissant de manière proprement discontinue et cocompacte sur  $X$ , un *courant de  $\Gamma$*  est une mesure de Radon positive ou nulle  $\Gamma$ -invariante sur le double bord de  $X$ . L'ensemble des courants  $\text{Curr}(X, \Gamma)$  est muni de la topologie vague (ou faible-étoile), où une suite de courants  $(\mu_n)_{n \in \mathbb{N}} \in (\text{Curr}(X, \Gamma))^{\mathbb{N}}$  converge vers un courant  $\mu \in \text{Curr}(X, \Gamma)$  si, et seulement si, pour tout borélien  $B$  de  $\partial^2 X$  tel que  $\mu(\partial B) = 0$ , la suite  $(\mu_n(B))_{n \in \mathbb{N}}$  converge vers  $\mu(B)$ . On définit également l'espace des courants projectifs

$$\mathbb{P}\text{Curr}(X, \Gamma) = (\text{Curr}(X, \Gamma) - \{0\}) / \sim,$$

où deux courants  $\mu, \nu \in \text{Curr}(X, \Gamma)$  sont équivalents s'il existe  $\lambda \in \mathbb{R}_+^*$  tel que  $\mu = \lambda\nu$ . Les courants ont été introduits par Ruelle et Sullivan [RS] et largement utilisés par Bonahon ([Bon1, Chapitre 4], voir également [Bon3]) dans son étude des surfaces hyperboliques



compactes, connexes et orientables. En effet, si  $S$  est une telle surface, son groupe fondamental agit sur son revêtement universel  $\tilde{S}$ , identifié par une isométrie  $\pi_1(S)$ -équivariante avec le plan hyperbolique réel  $\mathbb{H}_{\mathbb{R}}^2$ . De ce fait, nous pouvons définir l'espace des courants de  $S$ , noté  $\text{Curr}(S)$ , comme étant l'espace  $\text{Curr}(S) = \text{Curr}(\mathbb{H}_{\mathbb{R}}^2, \pi_1(S))$ . Bonahon [Bon3] a démontré que cette définition ne dépendait pas du choix de la structure hyperbolique de  $S$  (à unique homéomorphisme équivariant près). Nous pouvons de même définir l'espace des courants projectifs de  $S$  comme étant l'espace  $\mathbb{P}\text{Curr}(S) = \mathbb{P}\text{Curr}(\mathbb{H}_{\mathbb{R}}^2, \pi_1(S))$ . L'espace  $\text{Curr}(S)$  est un espace métrisable. De plus, l'espace  $\mathbb{P}\text{Curr}(S)$  est un espace métrisable compact. L'espace  $\text{Curr}(S)$  admet une action naturelle de  $\text{Mod}(S)$  par homéomorphismes puisque tout élément de  $\text{Mod}(S)$  induit un homéomorphisme de  $\partial_{\infty}\mathbb{H}_{\mathbb{R}}^2$ . Ainsi, si  $f \in \text{Mod}(S)$  et si  $\mu \in \text{Curr}(S)$ , alors, pour tout borélien  $B$  de  $\partial^2\mathbb{H}_{\mathbb{R}}^2$  on pose

$$f(\mu)(B) = \mu(f^{-1}(B)).$$

L'action de  $\text{Mod}(S)$  sur  $\text{Curr}(S)$  passe au quotient en une action de  $\text{Mod}(S)$  sur  $\mathbb{P}\text{Curr}(S)$ , l'espace des courants projectifs de  $S$ .

Les classes d'homotopies de courbes fermées essentielles de  $S$  définissent naturellement des courants que nous présentons. Soit  $[\gamma]$  une telle classe d'homotopie et  $\gamma$  son représentant géodésique. Soit  $\tilde{\gamma}$  une géodésique infinie de  $\tilde{S}$  se projetant sur  $\gamma$ . Alors  $\tilde{\gamma}$  définit deux points distincts à l'infini  $\tilde{\gamma}^{\pm\infty}$ , donc un élément  $\{\tilde{\gamma}^{\pm\infty}\}$  de  $\partial^2\tilde{S}$ . Le courant associé à  $[\gamma]$ , noté  $\eta_{[\gamma]}$  est

$$\eta_{[\gamma]} = \sum_{g \in \pi_1(S)/C([\gamma])} \delta_{g\{\tilde{\gamma}^{\pm\infty}\}},$$

où  $\delta_{g\{\tilde{\gamma}^{\pm\infty}\}}$  est la mesure de Dirac en  $g\{\tilde{\gamma}^{\pm\infty}\}$  et  $C([\gamma])$  le centralisateur de  $[\gamma]$  dans  $\pi_1(S)$ . Nous obtenons donc une application de l'ensemble des classes d'homotopie de courbes fermées essentielles de  $S$  dans l'espace des courants projectifs de  $S$ , et l'image de cette application est dense dans  $\mathbb{P}\text{Curr}(S)$  [Bon3]. L'espace  $\mathbb{P}\text{Curr}(S)$  est donc un espace naturel d'action de  $\text{Mod}(S)$  afin de comprendre l'action de  $\text{Mod}(S)$  sur les classes d'homotopie de courbes fermées essentielles. Par ailleurs, les éléments pseudo-Anosov de  $\text{Mod}(S)$  agissent avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(S)$  par des résultats de Thurston ([Thu], voir également les travaux d'Uyanik [Uya1]) et, de ce fait,  $\mathbb{P}\text{Curr}(S)$  est un objet privilégié d'arguments de tennis de table. Par ailleurs il existe un plongement naturel du compactifié de Thurston de l'espace de Teichmüller de  $S$  dans  $\mathbb{P}\text{Curr}(S)$  [Bon3].

La construction d'espaces de courants associés à des actions de groupes fondamentaux de surfaces par isométries sur le revêtement universel fut généralisée par Bonahon [Bon3] à tous les groupes hyperboliques au sens de Gromov. L'espace des courants d'un groupe libre non abélien  $\mathbb{F}_n$ , noté  $\text{Curr}(\mathbb{F}_n)$  et son espace projectif associé  $\mathbb{P}\text{Curr}(\mathbb{F}_n)$  furent particulièrement étudiés ([Mar, Kap, KapL, CHL, Uya2]). Soit  $T$  le graphe de Cayley de  $\mathbb{F}_n$  associé à une base libre de  $\mathbb{F}_n$ . L'espace des courants de  $\mathbb{F}_n$  est

$$\text{Curr}(\mathbb{F}_n) = \text{Curr}(T, \mathbb{F}_n),$$

l'espace  $\text{Curr}(\mathbb{F}_n)$  ne dépend pas du choix de  $T$  à unique homéomorphisme équivariant près. L'espace  $\text{Curr}(\mathbb{F}_n)$  est métrisable et  $\mathbb{P}\text{Curr}(\mathbb{F}_n)$  est métrisable et compact. Tout

comme dans le cas de  $\text{Mod}(S)$ , le groupe  $\text{Aut}(\mathbb{F}_n)$  agit sur  $\text{Curr}(\mathbb{F}_n)$  et sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n)$  par précomposition. Les mesures considérées étant  $\mathbb{F}_n$ -invariantes, l'action de  $\text{Aut}(\mathbb{F}_n)$  passe au quotient en une action de  $\text{Out}(\mathbb{F}_n)$  par homéomorphismes. Comme dans le cas des surfaces où nous pouvons considérer  $\mathbb{P}\text{Curr}(S)$  comme une complétion "topologique" des courants associés aux classes d'homotopies de courbes fermées simples et essentielles, nous pouvons considérer  $\mathbb{P}\text{Curr}(\mathbb{F}_n)$  comme une complétion "topologique" des courants associés aux classes de conjugaison d'éléments non triviaux de  $\mathbb{F}_n$ . Soit  $[g]$  une classe de conjugaison d'éléments non triviaux de  $\mathbb{F}_n$ . Soit  $\text{Ax}(g)$  l'axe de translation de  $g$  dans  $T$ . Alors  $\text{Ax}(g)$  définit deux points distincts à l'infini  $g^{\pm\infty}$  de  $\partial_\infty\mathbb{F}_n$  et un point  $\{g^{\pm\infty}\}$  du double bord de  $\mathbb{F}_n$ . Alors  $[g]$  définit un courant  $\eta_{[g]}$  par

$$\eta_{[g]} = \sum_{h \in \mathbb{F}_n / C(g)} \delta_{h\{g^{\pm\infty}\}},$$

où  $C(g)$  est le centralisateur de  $g$ . Bonahon [Bon3] (voir aussi [Mar]) démontre que l'ensemble  $\{\eta_{[g]}\}_{g \in \mathbb{F}_n - \{e\}}$  est dense dans  $\mathbb{P}\text{Curr}(\mathbb{F}_n)$ .

L'espace des courants a également un accouplement avec l'Outre-espace de  $\mathbb{F}_n$ . Soit  $cv_n$  l'Outre-espace non projectifié de  $\mathbb{F}_n$ , c'est-à-dire l'ensemble des classes d'isométrie  $\mathbb{F}_n$ -équivariante d'arbres simpliciaux métriques admettant une action par isométries de  $\mathbb{F}_n$  libre et minimale. L'ensemble  $cv_n$  est muni de la topologie de Gromov-Hausdorff équivariante. Soit  $\mathcal{T} \in cv_n$ , soient  $T$  un représentant de  $\mathcal{T}$  et  $g \in \mathbb{F}_n$  un élément non trivial. On note  $i(\mathcal{T}, \eta_{[g]})$  la longueur de translation de  $g$  dans  $T$ . Cette longueur de translation ne dépend ni du choix du représentant de la classe de conjugaison de  $g$  ni du choix du représentant de  $\mathcal{T}$ . Un résultat de Kapovich et Lustig ([KapL], voir également [Kap]) implique que la fonction  $i$  s'étend continûment en une fonction  $\text{Out}(\mathbb{F}_n)$ -équivariante par rapport à l'action diagonale

$$i: \overline{cv}_n \times \text{Curr}(S) \rightarrow \mathbb{R}_+.$$

Cet accouplement a ainsi été une motivation pour obtenir des propriétés dynamiques de l'action de  $\text{Out}(\mathbb{F}_n)$  sur  $\text{Curr}(\mathbb{F}_n)$  similaires à celles de l'action de  $\text{Out}(\mathbb{F}_n)$  sur  $CV_n$ . Par exemple, les éléments complètement irréductibles de  $\text{Out}(\mathbb{F}_n)$  agissent avec une dynamique Nord-Sud sur l'espace des courants projectifs [Mar]. Les éléments atorôïdaux de  $\text{Out}(\mathbb{F}_n)$  agissent également avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n)$  par des résultats de Lustig et Uyanik [LU2, Uya2].

Les espaces de courants sont ainsi des espaces privilégiés afin de construire des espaces topologiques compacts, métrisables et tels que l'ensemble des courants associés aux classes de conjugaison des éléments considérés soit dense. Rappelons que, pour un automorphisme extérieur  $\phi \in \text{Out}(\mathbb{F}_n)$  fixé, nous cherchons à construire un espace topologique  $X$  permettant de traduire les propriétés de croissance exponentielle d'éléments de  $\mathbb{F}_n$  par itération de  $\phi$  en des propriétés dynamiques de l'action de  $\phi$  sur  $X$ , et inversement. Pour cela, nous construisons une notion de courants *relativement* à  $\text{Poly}(\phi)$ . Intuitivement, les courants seront des mesures de Radon positives ou nulles  $\mathbb{F}_n$ -invariantes sur le double bord à l'infini de  $\mathbb{F}_n$  privé du double bord à l'infini de  $\text{Poly}(\phi)$ , donc des mesures de Radon sur une notion *double bord de  $\mathbb{F}_n$  relativement* à  $\text{Poly}(\phi)$ . Il convient de ce fait

de préciser la notion de double bord relatif. Par ailleurs, il est nécessaire de vérifier que le double bord relatif est un espace topologique “convenable” afin que l’espace des courants relatifs conserve les propriétés topologiques de compacité et de métrisabilité. Nous allons en fait introduire la notion de courant de  $\mathbb{F}_n$  *relativement à des systèmes de sous-groupes malnormaux*. La notion de systèmes de sous-groupes malnormaux était déjà présente dans les travaux d’Handel et Mosher [HaM4]. Nous introduisons dans cette thèse la notion de courants associés à de tels systèmes de sous-groupes malnormaux.

Un *système de sous-groupes de  $\mathbb{F}_n$*  est un ensemble fini (éventuellement vide) de classes de conjugaison de sous-groupes de type fini de  $\mathbb{F}_n$ . Rappelons qu’un sous-groupe  $A$  de  $\mathbb{F}_n$  est malnormal si, pour tout élément  $g \in \mathbb{F}_n - A$ , nous avons  $A \cap gAg^{-1} = \{1\}$ . Un *système de sous-groupes malnormal* est un système de sous-groupes  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  de  $\mathbb{F}_n$  tel que, pour tout  $i \in \{1, \dots, r\}$ , le groupe  $A_i$  est malnormal et, pour tous les sous-groupes distincts  $A, B$  de  $\mathbb{F}_n$  tels que  $[A], [B] \in \mathcal{A}$ , nous avons  $A \cap B = \{e\}$ .

Par exemple, les systèmes de facteurs libres de  $\mathbb{F}_n$  sont des exemples de systèmes de sous-groupes malnormaux. Un autre exemple de systèmes de sous-groupes malnormaux est le suivant. Soit  $T$  un arbre réel muni d’une action minimale de  $\mathbb{F}_n$  par isométries et tel que les stabilisateurs d’arcs soient triviaux. Par les travaux de Gaboriau et Levitt [GaL], le rang des stabilisateurs de sommets est au plus égal à  $n$  et le nombre de classes de conjugaison de stabilisateurs de sommets est fini. Soit  $\mathcal{A}_T$  le système de sous-groupes de  $\mathbb{F}_n$  consistant en les classes de conjugaison des stabilisateurs de sommets de  $T$ . Alors  $\mathcal{A}_T$  est un système de sous-groupes malnormal par un résultat de Handel et Mosher ([HaM4, Lemma II.3.1]) et est appelé le *système de sous-groupes de sommets de  $T$* . Par ailleurs, tous les systèmes de facteurs libres sont des systèmes de sous-groupes de sommets d’un certain arbre  $T$  simplicial. Cependant, il existe des systèmes de sous-groupes de sommets qui ne sont pas des systèmes de facteurs libres. Par exemple, soit  $S$  une surface compacte, connexe, orientable, hyperbolique avec exactement une composante connexe de bord, que l’on suppose totalement géodésique. Alors le groupe fondamental de  $S$  est isomorphe à un groupe libre non abélien  $\mathbb{F}_n$ . Soit  $\Lambda$  une lamination géodésique mesurée de  $S$  sans feuille compacte. Soit  $\tilde{\Lambda}$  la préimage de  $\Lambda$  dans le revêtement universel  $\tilde{S}$  de  $S$  et soit  $T$  l’arbre dual (voir par exemple [MS1]) de  $\tilde{\Lambda}$ . Alors une identification de  $\pi_1(S)$  avec  $\mathbb{F}_n$  induit une action de  $\mathbb{F}_n$  sur  $T$  à stabilisateurs d’arcs triviaux. Par ailleurs, le groupe fondamental de la composante connexe de  $S - \Lambda$  contenant la composante de bord est le stabilisateur d’un sommet de  $T$ , et ce dernier n’est contenu dans aucun facteur libre de  $\mathbb{F}_n$ .

Un autre exemple de systèmes de sous-groupes malnormaux de  $\mathbb{F}_n$  est le suivant. Soit  $\phi \in \text{Out}(\mathbb{F}_n)$  tel qu’il existe  $g \in \mathbb{F}_n$  à croissance exponentielle par itération de  $\phi$ . Un *sous-groupe à croissance polynomiale par itération de  $\phi$*  est un sous-groupe  $P$  de  $\mathbb{F}_n$  tel qu’il existe  $\Phi \in \phi$  et  $k \in \mathbb{N}^*$  tel que  $\Phi^k(P) = P$  et tout élément de  $P$  est à croissance polynomiale par itération de  $\Phi^k$ . Un résultat de Levitt [Lev2, Proposition 1.4] montre qu’il existe un nombre fini  $[H_1], \dots, [H_k]$  de sous-groupes maximaux à croissance polynomiale par itération de  $\phi$  et que l’ensemble  $\{[H_1], \dots, [H_k]\}$  est un système de sous-groupes malnormal. De ce fait, puisque  $\text{Poly}(\phi) = \bigcup_{i=1}^r \bigcup_{g \in \mathbb{F}_n} gH_i g^{-1}$ , il existe un système de sous-groupes malnormal canoniquement associé à  $\text{Poly}(\phi)$ , noté  $\mathcal{A}(\phi)$ .

Par exemple, dans l'exemple (1) de la partie 1.3, nous avons  $\mathcal{A}(\phi) = \{[\mathbb{F}_n]\}$ . Dans l'exemple (2), nous avons  $\mathcal{A}(\phi) = \emptyset$ . Enfin, dans l'exemple (3), nous avons  $\mathcal{A}(\phi) = \{[\langle aba^{-1}b^{-1}, c \rangle]\}$ . Nous ne savons pas si tout système de sous-groupes malnormaux  $\mathcal{A}(\phi)$  pour  $\phi \in \text{Out}(\mathbb{F}_n)$  est un système de sous-groupes de sommets. Néanmoins, les travaux récents de Mutanguha [Mut] laissent à penser que ce n'est pas le cas. Le théorème 1.3.1 implique également le fait suivant.

**Corollaire 1.4.1** (Corollary 7.5.2). *Soit  $n \geq 3$  et  $H$  un sous-groupe de  $\text{Out}(\mathbb{F}_n)$ . Si  $\text{Poly}(H) \neq \{1\}$ , il existe des sous-groupes maximaux non triviaux  $A_1, \dots, A_r$  de  $\mathbb{F}_n$  tels que*

$$\text{Poly}(H) = \bigcup_{i=1}^k \bigcup_{g \in F_n} gA_i g^{-1}$$

et  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  est un système de sous-groupes malnormal.

L'intérêt d'introduire les systèmes de sous-groupes malnormaux dans nos considérations à l'infini provient des faits suivants. D'une part, tout sous-groupe de type fini  $A$  de  $\mathbb{F}_n$  est quasi-convexe. De ce fait, si  $A$  est non trivial, il existe un unique plongement topologique  $A$ -équivariant  $\partial_\infty A \hookrightarrow \partial_\infty \mathbb{F}_n$ , et nous identifions  $\partial_\infty A$  avec son image dans  $\partial_\infty \mathbb{F}_n$  et  $\partial^2 A$  avec son image dans  $\partial^2 \mathbb{F}_n$ . D'autre part, par la définition des systèmes de sous-groupes malnormaux, pour tout système de sous-groupes malnormal  $\mathcal{A}$  de  $\mathbb{F}_n$  et pour tous les sous-groupes distincts  $A, B$  de  $\mathbb{F}_n$  tels que  $[A], [B] \in \mathcal{A}$ , nous avons  $\partial_\infty A \cap \partial_\infty B = \{e\}$  (voir par exemple [HaM4, Fact I.1.2]). De ce fait, les doubles bords  $\partial^2 A$  et  $\partial^2 B$  sont également disjoints.

Soit  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  un système de sous-groupes malnormal de  $\mathbb{F}_n$ . Nous définissons le *double bord de  $\mathcal{A}$*  par

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^r \bigcup_{g \in \mathbb{F}_n} \partial^2 (gA_i g^{-1})$$

et le double bord de  $\mathbb{F}_n$  relativement à  $\mathcal{A}$ , par

$$\partial^2(\mathbb{F}_n, \mathcal{A}) = \partial^2 \mathbb{F}_n - \partial^2 \mathcal{A}.$$

L'ensemble  $\partial^2(\mathbb{F}_n, \mathcal{A})$  est un ouvert de  $\partial^2 \mathbb{F}_n$ , stable sous l'action de  $\mathbb{F}_n$  par translation à gauche (voir [Gue4, Lemma 2.5]). Muni de la topologie induite par celle de  $\partial^2 \mathbb{F}_n$ , il est de ce fait localement compact et admet une orbite dense sous l'action de  $\mathbb{F}_n$ .

La construction du double bord de  $\mathbb{F}_n$  relativement à un système de sous-groupes malnormal est à rapprocher de la notion de bord à l'infini de *groupes relativement hyperboliques*. Soit  $G$  un groupe de type fini et soit  $\{[P_1], \dots, [P_r]\}$  un ensemble fini de classes de conjugaison de groupes de type fini non triviaux de  $G$ . Le groupe  $G$  est *hyperbolique relativement* à  $\{[P_1], \dots, [P_r]\}$  s'il existe un espace hyperbolique au sens de Gromov  $X$  tel que  $G$  agisse de manière proprement discontinue sur  $X$  et tel que les sous-groupes maximaux fixant exactement un point de  $\partial_\infty X$  soient précisément les conjugués des  $P_i$  avec  $i \in \{1, \dots, r\}$ . La notion d'hyperbolicité relative fut introduite par Gromov [Gro1]

(voir également les travaux de Farb [Far] et Bowditch [Bow]). Par exemple, le groupe fondamental d'une variété hyperbolique  $M$  connexe, de volume fini est hyperbolique relativement aux classes de conjugaison des sous-groupes paraboliques de  $\pi_1(M)$  associés aux pointes de  $M$ . Par des résultats de Bowditch [Bow, Theorem 7.12] et Hruska [Hru, Theorem 5.1], pour tout système de sous-groupe malnormal  $\mathcal{A}$  de  $\mathbb{F}_n$ , le groupe  $\mathbb{F}_n$  est hyperbolique relativement à  $\mathcal{A}$ . L'hyperbolicité relative intervient ainsi naturellement lorsque l'on considère des problèmes de croissance sous itération d'automorphismes. Par exemple, un cas particulier d'un théorème de Dahmani et Krishna [DS] implique que, pour tout automorphisme extérieur d'ordre infini  $\phi \in \text{Out}(\mathbb{F}_n)$  et tout représentant  $\Phi$  de  $\phi$ , l'extension  $\mathbb{F}_n \rtimes_{\Phi} \mathbb{Z}$  est hyperbolique relativement à la suspension de  $\mathcal{A}(\phi)$ .

À tout système de sous-groupes malnormal  $\mathcal{A}$  de  $\mathbb{F}_n$ , nous pouvons ainsi associer le bord  $\partial_{\infty}X(\mathcal{A})$  de l'espace Gromov-hyperbolique  $X(\mathcal{A})$  intervenant dans la définition d'hyperbolicité relative (la classe d'homéomorphisme  $\mathbb{F}_n$ -équivariante de  $\partial_{\infty}X(\mathcal{A})$  ne dépend pas du choix de  $X(\mathcal{A})$  par un résultat de Bowditch [Bow, Theorem 9.4]). Nous pouvons ainsi construire deux bords à l'infini  $\partial^2X(\mathcal{A})$  et  $\partial^2(\mathbb{F}_n, \mathcal{A})$  naturellement associés à  $\mathcal{A}$ . Dans cette thèse, nous préférons travailler avec  $\partial^2(\mathbb{F}_n, \mathcal{A}(\phi))$  puisqu'il a l'avantage d'être un ouvert de  $\partial^2\mathbb{F}_n$  alors qu'il n'existe pas, à notre connaissance, de plongement topologique naturel de  $\partial^2X(\mathcal{A})$  dans  $\partial^2\mathbb{F}_n$ .

Soit  $\mathcal{A}$  un système de sous-groupes malnormal de  $\mathbb{F}_n$ . Nous définissons les *courants de  $\mathbb{F}_n$  relativement à  $\mathcal{A}$*  comme étant les mesures de Radon positives ou nulles  $\mathbb{F}_n$ -invariantes sur le double bord relatif  $\partial^2(\mathbb{F}_n, \mathcal{A})$ . L'ensemble  $\text{Curr}(\mathbb{F}_n, \mathcal{A})$  des courants de  $\mathbb{F}_n$  relativement à  $\mathcal{A}$  est muni de la topologie faible-étoile. C'est un espace métrisable et l'espace quotient  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A})$  des courants projectifs est compact et métrisable. Soit  $\text{Out}(\mathbb{F}_n, \mathcal{A})$  le sous-groupe de  $\text{Out}(\mathbb{F}_n)$  préservant  $\mathcal{A}$ . Alors  $\text{Out}(\mathbb{F}_n, \mathcal{A})$  agit sur  $\text{Curr}(\mathbb{F}_n, \mathcal{A})$  et  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A})$  par homéomorphismes par précomposition.

Notons que, lorsque  $\mathcal{A}$  est un système de facteurs libres, les courants de  $\mathbb{F}_n$  relativement à des systèmes de facteurs libres avaient déjà été étudiés par Gupta [Gup1, Gup2] et par Guirardel et Horbez [GuH1] dans le cadre plus général des produits libres de groupes de type fini. Mentionnons par ailleurs que Guirardel et Horbez ont démontré que le résultat d'accouplement décrit précédemment pour  $\text{Curr}(\mathbb{F}_n)$  et  $CV_n$  n'est plus valable si l'on remplace  $\text{Curr}(\mathbb{F}_n)$  par  $\text{Curr}(\mathbb{F}_n, \mathcal{A})$  et  $cv_n$  par  $\mathcal{O}(\mathbb{F}_n, \mathcal{A})$  où  $\mathcal{A}$  est un système de facteurs libres.

Soit  $g \in \mathbb{F}_n$  un élément  *$\mathcal{A}$ -non-périphérique*, c'est-à-dire tel qu'il n'existe pas de sous-groupe  $A$  de  $\mathbb{F}_n$  tel que  $[A] \in \mathcal{A}$  et  $g \in A$ . Alors le courant  $\eta_{[g]} \in \text{Curr}(\mathbb{F}_n)$  induit par restriction un courant non nul dans  $\text{Curr}(\mathbb{F}_n, \mathcal{A})$ , que nous notons encore  $\eta_{[g]}$ . Nous montrons le résultat suivant.

**Théorème 1.4.2** (Proposition 5.4.1). *Soit  $n \geq 3$  et soit  $\mathcal{A}$  un système de sous-groupes malnormal de  $\mathbb{F}_n$ . Soit  $\text{NP}(\mathcal{A})$  l'ensemble des éléments  $\mathcal{A}$ -non-périphériques de  $\mathbb{F}_n$ . Alors l'ensemble  $\{\eta_{[g]}\}_{g \in \text{NP}(\mathcal{A})}$  est dense dans  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A})$ .*

Ainsi, lorsque  $\phi \in \text{Out}(\mathbb{F}_n)$  et  $\mathcal{A} = \mathcal{A}(\phi)$ , le théorème 1.4.2 implique que l'ensemble des courants relatifs projectifs associés à des éléments à croissance exponentielle par itération de  $\phi$  est dense dans  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(\phi))$ . Cette propriété facilite la compréhension

des propriétés dynamiques de l'action de  $\phi$  sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(\phi))$ . En effet, la densité de l'ensemble  $\{[\eta_{[g]}]\}_{g \in \text{NP}(\mathcal{A})}$  dans  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A})$  est un argument important dans la démonstration du théorème suivant.

**Théorème 1.4.3** (Theorem 6.5.1). *Soit  $n \geq 3$  et soit  $\phi \in \text{Out}(\mathbb{F}_n)$  un élément tel que  $\text{Poly}(\phi) \neq \mathbb{F}_n$ . Alors  $\phi$  agit avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(\phi))$ .*

La démonstration du théorème 1.4.3 repose également sur l'utilisation de représentants topologiques adaptés des éléments de  $\text{Out}(\mathbb{F}_n)$  appelés *réseaux ferroviaires complètement scindés*. Ces réseaux ferroviaires ont été introduits par Feighn et Handel [FH], généralisant des réseaux ferroviaires introduits par Bestvina et Handel [BH] (voir également [BFH1]). Ils permettent de contrôler efficacement les problèmes de croissance lorsque l'on itère un automorphisme extérieur et sont, de ce fait, bien adaptés aux questions étudiées.

Nous pouvons maintenant esquisser la démonstration du théorème 1.3.1. Soit  $H$  un sous-groupe de  $\text{Out}(\mathbb{F}_n)$ . Un argument de récurrence montre que, pour tout facteur libre propre (possiblement trivial) maximal  $A$  de  $\mathbb{F}_n$  dont la classe de conjugaison est  $H$ -invariante, on peut trouver un élément  $\phi_A \in H$  tel que  $\text{Poly}(\phi_A) \cap A = \text{Poly}(H) \cap A$ . Soit  $H|_A$  l'image de  $H$  dans  $\text{Out}(A)$  et soit  $\Phi_A \in \phi_A$  tel que  $\Phi_A(A) = A$ . Le théorème 1.4.3 montre alors que l'élément  $[\Phi_A|_A] \in \text{Out}(A)$  agit avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(A, \mathcal{A}([\Phi_A|_A]))$ . Par hypothèse de récurrence, le groupe  $H|_A$  agit également par homéomorphismes sur  $\mathbb{P}\text{Curr}(A, \mathcal{A}([\Phi_A|_A]))$  et sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}([\Phi_A|_A]))$ . Un argument de tennis de table faisant intervenir les espaces  $\mathbb{P}\text{Curr}(A, \mathcal{A}([\Phi_A|_A]))$  et  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}([\Phi_A|_A]))$  permet alors de construire un automorphisme extérieur  $\psi \in H$  tel que  $\text{Poly}(\psi) = \text{Poly}(H)$ .

Nous résumons le lien entre l'existence d'éléments génériques et l'action sur un espace de courants relatifs dans le théorème suivant. Si  $H$  est un sous-groupe de  $\text{Out}(\mathbb{F}_n)$  et  $X$  un sous-ensemble de  $\text{Curr}(\mathbb{F}_n, \mathcal{A}(H))$ , nous noterons  $\text{Vect}(X)$  l'espace vectoriel engendré par  $X$ . Nous noterons également

$$p: \text{Curr}(\mathbb{F}_n, \mathcal{A}(H)) - \{0\} \rightarrow \mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(H))$$

la projection canonique.

**Théorème 1.4.4.** *Soit  $n \geq 2$ , soit  $H$  un sous-groupe de  $\text{Out}(\mathbb{F}_n)$  et soit  $\phi \in H$ . L'élément  $\phi$  est dynamiquement générique si, et seulement si,  $\phi$  agit avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(H))$ , tel que  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  soit de dimension finie et tel que l'action de  $\phi$  sur  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  soit diagonalisable sans valeur propre égale à 1.*

**Démonstration.** Si  $\phi \in H$  est un élément dynamiquement générique, alors il agit avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(H)) = \mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(\phi))$  par le théorème 1.4.3. Par le lemme 6.4.7, l'espace vectoriel  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  est de dimension finie. De plus, par la proposition 6.4.12, l'action de  $\phi$  sur  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  est diagonalisable sans valeur propre égale à 1.

Réciproquement, supposons que  $\phi \in H$  ne soit pas dynamiquement générique et qu'il agisse avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(H))$ , tel que l'espace vectoriel  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  soit de dimension finie et tel que l'action de  $\phi$  sur  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  soit diagonalisable. Montrons que l'une des valeurs propres de l'action de  $\phi$  sur  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  est égale à 1. Pour cela, nous allons tout d'abord montrer que  $\Delta_+(\phi) \cup \Delta_-(\phi)$  contient l'ensemble des courants associés aux éléments  $g \in \text{Poly}(\phi) - \text{Poly}(H)$ . Soit  $g \in \text{Poly}(\phi) - \text{Poly}(H)$  tel que  $\eta_{[g]} \notin \Delta_+(\phi) \cup \Delta_-(\phi)$ . Soit  $f: G \rightarrow G$  un réseau ferroviaire complètement scindé de  $\phi$  (voir Proposition 6.2.5) et soit  $\|\cdot\|: \text{Curr}(\mathbb{F}_n, \mathcal{A}(H)) \rightarrow \mathbb{R}$  la fonction longueur simpliciale associée à  $G$  (voir Section 6.2.4). Si  $w \in \mathbb{F}_n$ , alors

$$\|\eta_{[w]}\| = \ell(\gamma_w),$$

où  $\ell(\gamma_w)$  est la longueur dans  $G$  du cycle  $\gamma_w$  de  $G$  associé à la classe de conjugaison de  $w$ . Soit  $N$  tel que le chemin réduit  $[f^N(\gamma_w)]$  associé à  $f^N(\gamma_w)$  soit complètement scindé et tel que, pour toute unité de scindement  $\sigma$  de  $G$ , nous ayons  $\ell([f^N(\sigma)]) \geq \sigma$ . Ainsi, pour tout  $n \geq N$ , nous avons

$$\ell([f^{N(n+1)}(\gamma_w)]) \geq \ell([f^{nN}(\gamma_w)]).$$

De ce fait, quitte à remplacer  $\gamma_w$  par  $[f^N(\gamma_w)]$  et  $\phi$  par  $\phi^N$ , nous pouvons supposer que, pour tout  $n \in \mathbb{N}$ , nous avons

$$\|\eta_{\phi^{n+1}([g])}\| \geq \|\eta_{\phi^n([g])}\|.$$

Puisque  $\phi$  agit avec une dynamique Nord-Sud sur  $\mathbb{P}\text{Curr}(\mathbb{F}_n, \mathcal{A}(H))$ , en utilisant par exemple [Kap, Lemma 3.5] il existe  $[\mu_+] \in \Delta_+(\phi)$  tel que

$$\lim_{n \rightarrow \infty} \frac{\eta_{[\phi^n(g)]}}{\|\eta_{\phi^n([g])}\|} = \mu_+.$$

La classe d'homothétie de  $\mu_+$  étant invariante par  $\phi$ , il existe  $\lambda > 0$  tel que  $\phi(\mu_+) = \lambda\mu_+$ . Soit  $B$  un borélien de  $\partial^2(\mathbb{F}_n, \mathcal{A}(H))$  tel que  $\mu_+(\partial B) = 0$  et  $\mu_+(B) > 0$ . Nous avons donc

$$\lim_{n \rightarrow \infty} \frac{\eta_{[\phi^n(g)]}}{\|\eta_{\phi^n([g])}\|}(B) = \mu_+(B)$$

et

$$\lim_{n \rightarrow \infty} \frac{\phi(\eta_{[\phi^n(g)]})}{\|\eta_{\phi^n([g])}\|}(B) = \lim_{n \rightarrow \infty} \frac{\eta_{[\phi^{n+1}(g)]}}{\|\eta_{\phi^n([g])}\|}(B) = \lambda\mu_+(B).$$

Ainsi, nous avons

$$\lim_{n \rightarrow \infty} \frac{\eta_{\phi^n([g])}(B)}{\eta_{\phi^{n+1}([g])}(B)} = \lambda.$$

Comme

$$\lim_{n \rightarrow \infty} \frac{\eta_{[\phi^n(g)]}}{\|\eta_{\phi^n([g])}\|}(B) = \lim_{n \rightarrow \infty} \frac{\eta_{[\phi^{n+1}(g)]}}{\|\eta_{\phi^{n+1}([g])}\|}(B) = \mu_+(B),$$

nous avons :

$$\lim_{n \rightarrow \infty} \frac{\|\eta_{\phi^{n+1}([g])}\|}{\|\eta_{\phi^n([g])}\|} = \lambda.$$

La suite  $\{\|\eta_{\phi^n([g])}\|\}_{n \in \mathbb{N}}$  étant croissante, nous avons  $\lambda \geq 1$ . Puisque  $g \in \text{Poly}(\phi)$ , la suite  $\{\|\eta_{\phi^n([g])}\|\}_{n \in \mathbb{N}}$  croît au plus à vitesse polynomiale. De ce fait, nous avons  $\lambda = 1$ .

De même, soit  $[\mu_-] \in \Delta_-(\phi)$  un courant projectif limite de  $\{\phi^{-n}([\eta_g])\}_{n \in \mathbb{N}}$ . Alors  $\mu_-$  est également fixé par  $\phi$  et donc, pour tout  $t \in [0, 1]$ , le courant projectif  $[t\mu_+ + (1-t)\mu_-]$  est fixé par  $\phi$ . Puisque  $\phi$  agit avec une dynamique Nord-Sud, pour tout  $t \in [0, 1]$ , nous avons  $[t\mu_+ + (1-t)\mu_-] \in \Delta_+(\phi) \cup \Delta_-(\phi)$ . Or, nous avons  $\mu_+ \in \Delta_+(\phi)$  et  $\mu_- \in \Delta_-(\phi)$ . Ceci contredit le fait que  $\Delta_+(\phi)$  et  $\Delta_-(\phi)$  soient deux compacts disjoints.

Ainsi, l'espace vectoriel de dimension finie  $\text{Vect}(\Delta_+(\phi) \cup \Delta_-(\phi))$  contient l'ensemble  $X$  des courants associés aux éléments  $g \in \text{Poly}(\phi) - \text{Poly}(H)$ . Puisque  $X$  est invariant par  $\phi$  et que l'action de  $\phi$  sur  $\text{Vect}(p^{-1}(\Delta_+(\phi)) \cup p^{-1}(\Delta_-(\phi)))$  est diagonalisable, il existe  $g \in \text{Poly}(\phi) - \text{Poly}(H)$  et  $\lambda > 0$  tel que  $\phi(\eta_{[g]}) = \eta_{\phi([g])} = \lambda\eta_{[g]}$ . Puisque  $g$  est à croissance polynomiale sous itération de  $\phi$ , nous avons nécessairement  $\lambda = 1$ , ce qui conclut.  $\square$

## 1.5 Comparaison entre $\text{Out}(\mathbb{F}_n)$ et $\text{Out}(W_n)$

Nous terminons cette introduction par une section comparative (qui n'est pas reprise dans les chapitres 2 à 7) entre les deux groupes d'intérêt de cette thèse que sont  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$ . Il existe tout d'abord un morphisme canonique de  $\text{Aut}(W_n)$  vers  $\text{Aut}(\mathbb{F}_{n-1})$ . Soit  $x_1, \dots, x_n$  une partie génératrice standard de  $W_n$  et soit  $\Phi: W_n \rightarrow \mathbb{Z}/2\mathbb{Z}$  le morphisme qui, pour tout  $i \in \{1, \dots, k\}$ , envoie  $x_i$  sur l'élément non trivial de  $\mathbb{Z}/2\mathbb{Z}$ . Par des résultats de Mühlherr [Müh], le noyau  $B_n = \ker(\Phi)$  de ce morphisme est un sous-groupe caractéristique de  $W_n$  isomorphe à un groupe libre non abélien de rang  $n - 1$ , engendré par  $x_1x_2, x_2x_3, \dots, x_{n-1}x_n$ . Par ailleurs, le morphisme induit  $\text{Aut}(W_n) \rightarrow \text{Aut}(\mathbb{F}_{n-1})$  est injectif et c'est un isomorphisme lorsque  $n = 3$ . Le groupe  $\text{Inn}(W_n)$  est envoyé sur un sous-groupe de  $\text{Inn}(\mathbb{F}_{n-1}) \rtimes \langle r \rangle$  où  $r$  est l'automorphisme de  $\mathbb{F}_{n-1}$  qui, pour une base fixée  $e_1, \dots, e_{n-1}$  de  $\mathbb{F}_{n-1}$  associe, pour tout  $i \in \{1, \dots, n - 1\}$ , l'élément  $e_i$  à l'élément  $e_i^{-1}$  (voir par exemple [Hea, Lemma 2.6]). Le groupe  $\text{Out}(W_n)$  s'injecte alors dans le groupe  $\text{Out}(\mathbb{F}_{n-1}) / \langle\langle r \rangle\rangle$  ([Hea, Lemma 2.7]).

Ces relations algébriques entre  $\text{Out}(W_n)$  et  $\text{Out}(\mathbb{F}_{n-1})$  ont également une traduction en termes géométriques sur leurs Outre-espaces associés. Il existe en effet un plongement  $\text{Out}(B_n)$ -équivariant  $K(W_n) \hookrightarrow K(\mathbb{F}_{n-1})$  que nous décrivons maintenant. Soit  $\mathcal{X}$  un sommet de  $K(W_n)$  et soit  $X$  un représentant de  $\mathcal{X}$ . Puisque les stabilisateurs de  $X$  pour l'action de  $W_n$  sont finis, le groupe  $B_n$  agit librement sur  $X$ . La restriction de l'action sur  $X$  à  $B_n$  reste minimale et définit alors un morphisme  $\text{Out}(B_n)$ -équivariant  $K(W_n) \rightarrow K(\mathbb{F}_{n-1})$ . Ce morphisme est injectif puisque  $B_n$  est un sous-groupe d'indice fini de  $W_n$ , donc détermine entièrement l'action de  $W_n$  sur un arbre  $X$ . Nous pouvons ainsi voir  $K(W_n)$  comme un sous-graphe de  $K(\mathbb{F}_{n-1})$ . Le plongement  $K(W_n) \hookrightarrow K(\mathbb{F}_{n-1})$  n'est en général pas surjectif, excepté dans le cas où  $n = 3$  ([Gue2, Proposition 2.6]).



Ces relations algébriques et géométriques entre les groupes  $\text{Out}(W_n)$  et  $\text{Out}(\mathbb{F}_{n-1})$  justifient l'intuition que les groupes  $\text{Out}(W_n)$  et  $\text{Out}(\mathbb{F}_{n-1})$  partagent de nombreuses propriétés de rigidité algébrique et géométrique, comme nous l'avons évoqué dans les premières sections de cette introduction. Les résultats de rigidité de  $\text{Out}(W_n)$  ne se déduisent cependant pas de celles de  $\text{Out}(\mathbb{F}_{n-1})$ . Par exemple, il n'existe pas de manière naturelle d'associer à un automorphisme de  $\text{Out}(W_n)$  un automorphisme de  $\text{Out}(\mathbb{F}_n)$ . De ce fait le théorème 1.1.3 ne se déduit pas du résultat similaire pour  $\text{Out}(\mathbb{F}_n)$ . De même, les techniques utilisées pour démontrer des résultats de rigidité pour  $\text{Out}(\mathbb{F}_n)$  reposent en général sur l'existence *d'extensions HNN*, c'est-à-dire de morphismes surjectif  $\mathbb{F}_n \rightarrow \mathbb{Z}$ , alors qu'il n'en existe pas pour le groupe  $W_n$ .

Par ailleurs, certaines des propriétés de rigidité de  $\text{Out}(\mathbb{F}_n)$  ne sont pas connues pour  $\text{Out}(W_n)$ . Par exemple,  $\text{Out}(\mathbb{F}_n)$  vérifie une propriété de rigidité pour *l'équivalence mesurée*. Deux groupes dénombrables  $\Gamma_1$  et  $\Gamma_2$  sont *mesurablement équivalents* s'il existe un espace mesuré standard  $(\Sigma, \mu)$  muni d'une action de  $\Gamma_1 \times \Gamma_2$  par automorphismes boréliens préservant la mesure et tel que, pour tout  $i \in \{1, 2\}$ , le groupe  $\Gamma_i$  agit librement sur  $\Sigma$  et il existe un borélien  $B_i$  de mesure finie tel que l'union des translatés de  $B_i$  par  $\Gamma_i$  recouvrent  $\Sigma$  et tel que l'intersection de deux translatés distincts de  $B_i$  par  $\Gamma_i$  soit de mesure nulle. La notion d'équivalence mesurée a été introduite par Gromov [Gro2] et est un équivalent mesurable à la notion de quasi-isométries entre groupes de type fini. Guirardel et Horbez [GuH3] ont démontré que, pour  $n \geq 3$ , tout groupe  $G$  mesurablement équivalent à  $\text{Out}(\mathbb{F}_n)$  est virtuellement isomorphe à  $\text{Out}(\mathbb{F}_n)$  : il existe des sous-groupes d'indice fini  $G_1$  de  $G$  et  $G_2$  de  $\text{Out}(\mathbb{F}_n)$  et des sous-groupes distingués finis  $N_1$  de  $G_1$  et  $N_2$  de  $G_2$  tels que les groupes  $G_1/N_1$  et  $G_2/N_2$  soient isomorphes. De tels résultats de rigidité étaient déjà connus dans le cas de réseaux dans des groupes de Lie simples de rang supérieur par Furman [Furm] et dans le cas du groupe modulaire d'une surface fermée, connexe, orientable de genre au moins 2 par Kida [Kid].

*Question.* Le groupe  $\text{Out}(W_n)$  vérifie-t-il une propriété de rigidité pour l'équivalence mesurée pour  $n \geq 4$  ?

Les groupes  $\text{Out}(\mathbb{F}_n)$  et  $\text{Out}(W_n)$  diffèrent cependant significativement sur certaines propriétés. Par exemple, Varghese [Var] a démontré que le groupe  $\text{Out}(W_n)$  n'a pas la propriété  $(T)$  de Kazhdan pour  $n \geq 2$ , alors que Kaluba, Kielak et Nowak [KaKN] ont montré que le groupe  $\text{Out}(\mathbb{F}_n)$  a la propriété  $(T)$  pour  $n \geq 5$  (voir également les travaux de Kaluba, Nowak et Ozawa [KNO] pour le cas  $n = 5$ ) et étendu à  $n = 4$  par Nitsche [Nit].

Nous terminons cette introduction par une analyse, dans le cas de  $\text{Out}(W_n)$ , des questions de croissance dans le groupe comme étudiées dans les sections 1.3 et 1.4 de cette introduction pour  $\text{Out}(\mathbb{F}_n)$ . Tout comme dans le cas de  $\text{Out}(\mathbb{F}_n)$ , la croissance d'éléments de  $W_n$  sous itération d'éléments de  $\text{Out}(W_n)$  est toujours soit exponentielle soit polynomiale. Un exemple d'automorphisme  $\phi \in \text{Out}(W_n)$  tel que  $\text{Poly}(\phi) = W_n$  est le suivant (nous remercions Nicholas Touikan pour l'avoir porté à notre connaissance). Soit  $W_4 = \langle a, b, c, d \rangle$ , soit  $\Phi$  l'automorphisme envoyant  $a$  sur  $a$ ,  $b$  sur  $aba$ ,  $c$  sur  $bc b$  et  $d$  sur  $cdc$  et soit  $\phi$  la classe d'automorphismes extérieurs de  $\Phi$ . Alors  $ac$  est à croissance linéaire sous itération de  $\phi$  et  $ad$  est à croissance quadratique sous itération de  $\phi$ .

Notons qu'il existe un ordre partiel des systèmes de facteurs libres d'un groupe  $G$ , où  $\mathcal{F}_1 \leq \mathcal{F}_2$  si pour tout sous-groupe  $A$  de  $G$  tel que  $[A] \in \mathcal{F}_1$ , il existe un sous-groupe  $B$  de  $G$  contenant  $A$  tel que  $[B] \in \mathcal{F}_2$ . De tels suites de systèmes de facteurs libres sont particulièrement adaptées à l'étude des sous-groupes  $H \in \text{Out}(\mathbb{F}_n)$  tels que  $\text{Poly}(\phi) = F_n$ . En effet, soit  $H$  un tel sous-groupe, que l'on suppose de type fini. En utilisant le théorème de Kolchin pour  $\text{Out}(\mathbb{F}_n)$  [BFH3], on montre qu'il existe un graphe  $G'$ , de groupe fondamental isomorphe à  $\mathbb{F}_n$ , contenant exactement  $k$  arêtes  $e_1, \dots, e_k$  tel que, pour tout  $\phi \in H$ , il existe une équivalence d'homotopie  $f: G' \rightarrow G'$  représentant  $\phi$  telle que, pour tout  $i \in \{1, \dots, k\}$ , l'application  $f$  préserve le sous-graphe  $H_i$  de  $G'$  constitué des arêtes  $e_1, \dots, e_i$ . Les sous-graphes  $H_i$  induisent alors une suite

$$\mathcal{F}_1 \leq \dots \leq \mathcal{F}_k = \{[F_n]\}$$

de systèmes de facteurs libres de  $\mathbb{F}_n$  invariante par  $H$ . Les décompositions en systèmes de facteurs libres sont également particulièrement adaptées à l'étude des sous-groupes  $H$  de  $\text{Out}(W_n)$  tels que  $\text{Poly}(\phi) = W_n$ , comme le montre le résultat suivant. Rappelons qu'une suite  $\mathcal{F}_1 \leq \mathcal{F}_2$  de systèmes de facteurs libres de  $W_n$  est *sporadique* s'il existe deux sous-groupes  $A, B$  de  $W_n$  tels que  $[A], [B] \in \mathcal{F}_1$  et  $\mathcal{F}_2 = (\mathcal{F}_1 - \{[A], [B]\}) \cup \{[A * B]\}$ .

**Proposition 1.5.1.** *Soit  $n \geq 1$  et soit  $H \subseteq \text{Out}(W_n)$  un sous-groupe de type fini tel que  $\text{Poly}(H) = W_n$ . Il existe un sous-groupe d'indice fini  $H'$  de  $H$  ainsi qu'une suite  $\mathcal{F}_1 \leq \dots \leq \mathcal{F}_k = \{[W_n]\}$  de systèmes de facteurs libres de  $W_n$   $H'$ -invariante tels que, pour tout  $i \in \{1, \dots, k\}$ , l'extension  $\mathcal{F}_i \leq \mathcal{F}_{i+1}$  soit sporadique.*

**Démonstration.** Nous démontrons le résultat par récurrence sur  $n$ . Lorsque  $n = 1$ , le résultat est immédiat car  $\text{Out}(W_1) = \{1\}$ . Soit  $n \geq 2$  et soit  $\mathcal{F}$  un système de facteurs libres de  $W_n$  propre,  $H$ -périodique et maximal. Quitte à passer à un sous-groupe d'indice fini de  $H$ , nous pouvons supposer que  $\mathcal{F}$  est  $H$ -invariant. Supposons par l'absurde que  $\mathcal{F}$  ne soit pas sporadique. Alors, par [GuH2, Theorem 7.1], le sous-groupe  $H$  contient un élément  $\phi \in H$  complètement irréductible relativement à  $\mathcal{F}$ . Par [GuH2, Theorem 4.1], l'élément  $\phi$  est un élément loxodromique du graphe (hyperbolique au sens de Gromov par un résultat de Guirardel et Horbez [GuH2, Proposition 2.11] car  $\mathcal{F}$  n'est pas sporadique)  $\text{FF}(W_n, \mathcal{F})$  des facteurs libres de  $W_n$  relativement à  $\mathcal{F}$ . Donc  $\phi$  fixe un point au bord à l'infini de  $\text{FF}(W_n, \mathcal{F})$ . Par [GuH2, Theorem 3.4],  $\phi$  fixe la classe d'homothétie  $W_n$ -équivariante d'un arbre réel  $T$  non trivial muni d'une action minimale de  $W_n$  dans le bord de l'Outre-espace  $\mathcal{O}(W_n, \mathcal{F})$  de  $W_n$  relativement à  $\mathcal{F}$ . De ce fait, il existe une homothétie  $I: T \rightarrow T$  telle que pour tous les  $x \in T$  et  $g \in \mathbb{F}_n$ , nous ayons  $I(gx) = \Phi(g)I(x)$ . Par ailleurs, puisque  $\phi$  est complètement irréductible, le coefficient de dilatation  $\lambda$  de  $I$  est différent de 1 ([GuH2, Corollary 6.7]). Pour tout  $g \in W_n$ , soit  $\ell_T(g) = \inf_{x \in T} d(x, gx)$ . Cette borne inférieure est atteinte pour un certain  $x_g \in T$ . Alors, pour tout  $k \in \mathbb{N}^*$ , nous avons  $\lambda^k \ell_T(g) = \ell_T(\Phi^k(g))$  et ce dernier terme est borné par un polynôme en la variable  $k$  puisque  $\text{Poly}(\phi) = W_n$  par des résultats de Culler et Morgan [CM, Propositions 1.5, 1.8]. Donc pour tout  $g \in W_n$ , nous avons  $\ell_T(g) = 0$  et  $g$  fixe un point de  $T$ . Puisque  $W_n$  est de type fini, il fixe un point de  $T$  (voir par exemple [CM, Section 3]). Ceci contredit la non trivialité de l'action.

Donc il existe  $k \in \mathbb{N}^*$ , des sous-groupes  $A$  et  $B$  de  $W_n$  tel que  $\mathcal{F} = \{[W_k], [W_{n-k}]\}$  avec  $A$  isomorphe à  $W_k$  et  $B$  isomorphe à  $W_{n-k}$ . Nous avons ainsi deux morphismes  $H \rightarrow \text{Out}(A)$  et  $H \rightarrow \text{Out}(B)$  et l'hypothèse de récurrence appliquée aux images de ces morphismes permet de conclure.  $\square$

La proposition 1.5.1 indique que les techniques d'étude de la croissance polynomiale de  $\text{Out}(W_n)$  peuvent se rapprocher de celles utilisées pour étudier la croissance polynomiale de  $\text{Out}(\mathbb{F}_n)$ . Nous pouvons par ailleurs imaginer que ces questions de croissance aient un impact sur le calcul de la fonction de Dehn de  $\text{Out}(W_n)$ . En effet, Bridson et Vogtmann [BV3] ont démontré que la fonction de Dehn du groupe  $\text{Out}(\mathbb{F}_n)$  était équivalente à une fonction exponentielle. La démonstration exploite l'existence de sous-groupes  $H$  de  $\text{Out}(\mathbb{F}_n)$  qui sont des *sous-groupes de Kolchin*, c'est-à-dire tels que  $\text{Poly}(H) = \mathbb{F}_n$ . La question naturelle à se poser est alors la suivante.

*Question. La fonction de Dehn de  $\text{Out}(W_n)$  est-elle exponentielle ?*

Nous présentons à présent la structure du manuscrit. Les chapitres 2, 3 et 4 concernent les résultats de rigidité dans  $\text{Out}(W_n)$ . Nous présentons dans le deuxième chapitre la démonstration du théorème 1.1.3 concernant les automorphismes de  $\text{Out}(W_n)$ . Il correspond à l'article [Gue1]. Le troisième chapitre est consacré à la rigidité géométrique de  $\text{Out}(W_n)$  et la construction de différents modèles géométriques rigides pour  $\text{Out}(W_n)$ , il correspond à l'article [Gue2]. Enfin dans le quatrième chapitre, nous démontrons le théorème 1.2.2 et le corollaire 1.2.3 sur la rigidité du commensurateur abstrait de  $\text{Out}(W_n)$ . Il correspond à l'article [Gue3].

Nous consacrons les trois derniers chapitres aux résultats sur la croissance dans  $\text{Out}(\mathbb{F}_n)$  et les courants relatifs. Le chapitre 5 est dédié à la construction des courants relatifs à un système de sous-groupes malnormal et la démonstration du théorème 1.4.2. Il correspond à l'article [Gue4]. Dans le chapitre 6, nous démontrons les résultats de dynamique Nord-Sud sur les courants relatifs et le théorème 1.4.3. Il correspond à l'article [Gue5]. Enfin, dans le dernier chapitre, correspondant à l'article [Gue6], nous terminons par la démonstration du théorème 1.3.1, et nous en donnons quelques applications.

## Chapitre 2

# Automorphismes du groupe des automorphismes d'un groupe de Coxeter universel

### 2.1 Introduction

Soit  $n$  un entier plus grand que 2. On note  $F = \mathbb{Z}/2\mathbb{Z}$  le groupe cyclique d'ordre 2 et  $W_n = *_n F$  le groupe de Coxeter universel de rang  $n$ , produit libre de  $n$  copies de  $F$ . Si  $G$  est un groupe, on note  $\text{Out}(G) = \text{Aut}(G)/\text{Int}(G)$  son groupe d'automorphismes extérieurs. Nous démontrons dans cet article les résultats suivants.

**Théorème 2.1.1.** *Si  $n \geq 5$ , alors  $\text{Out}(\text{Out}(W_n)) = \{1\}$ . Si  $n = 4$ , alors  $\text{Out}(\text{Out}(W_n))$  est isomorphe à  $\mathbb{Z}/2\mathbb{Z}$ .*

**Théorème 2.1.2.** *Si  $n \geq 4$ , alors  $\text{Out}(\text{Aut}(W_n)) = \{1\}$ .*

De tels résultats sont déjà connus dans le cas où  $n = 2$  (cf. [Tho, Lemma 1.4.2, Lemma 1.4.3]) où tous les automorphismes de  $\text{Out}(W_2)$  sont intérieurs et où  $\text{Out}(\text{Aut}(W_2))$  est un groupe cyclique d'ordre 2. Dans le cas où  $n = 3$ , les groupes  $\text{Aut}(W_3)$  et  $\text{Out}(W_3)$  sont isomorphes à  $\text{Aut}(\text{FF}_2)$  et  $\mathbb{PGL}(2, \mathbb{Z})$  respectivement, avec  $\text{FF}_2$  un groupe libre de rang 2 (cf. [Var, Lemma 2.3]). Nous obtenons donc une description de  $\text{Out}(\text{Out}(W_n))$  pour tout entier  $n$  plus grand que 2.

De telles questions de rigidité algébrique ont déjà été résolues dans des cas similaires. En effet, Mostow [Mos2] a démontré que le groupe des automorphismes extérieurs de réseaux irréductibles uniformes de groupes de Lie réels, connexes, semi-simples et non localement isomorphes à  $\text{SL}_2(\mathbb{R})$  est fini. De même, Ivanov [Iva2, Theorem 2] a démontré un résultat similaire dans le cas du groupe modulaire d'une surface compacte, connexe, orientable de genre  $g \geq 2$ . Enfin, Bridson et Vogtmann [BV1] ont démontré que tout automorphisme du groupe des automorphismes extérieurs d'un groupe libre de rang  $N$  (avec  $N \geq 3$ ) est une conjugaison. Ce dernier cas a motivé l'étude de la rigidité algébrique

de  $\text{Out}(W_n)$  d'une part à cause de la propriété d'universalité pour les groupes engendrés par des éléments d'ordre 2 de  $W_n$ , d'autre part car, si  $n \geq 3$ , le groupe  $\text{Aut}(W_n)$  s'injecte dans  $\text{Aut}(F_{n-1})$  (cf. [Müh, Theorem A]).

Pour démontrer les théorèmes 2.1.1 et 2.1.2, nous étudions l'action de  $W_n$  sur un complexe simplicial de drapeaux introduit par Guirardel et Levitt. Plus précisément, nous cherchons à comprendre les stabilisateurs de certains sommets de ce complexe. En effet, les stabilisateurs de ces sommets formant une partie génératrice de  $\text{Aut}(W_n)$  et  $\text{Out}(W_n)$ , comprendre l'image de ces stabilisateurs par des automorphismes de  $\text{Aut}(W_n)$  et  $\text{Out}(W_n)$  nous permettra de faciliter l'étude de ces derniers. L'étude de l'action de  $W_n$  sur un complexe simplicial se justifie également par la démonstration des théorèmes similaires dans les cas des réseaux des groupes de Lie semi-simples, du groupe modulaire d'une surface de type fini et du groupe des automorphismes d'un groupe libre qui passait également par l'étude de l'action du groupe étudié sur un espace géométrique adapté. En particulier, dans le cas du groupe des automorphismes extérieurs d'un groupe libre de rang  $N$ , cet objet géométrique était *l'outre-espace de Culler-Vogtmann*  $\text{CV}_N$ , qui fut introduit par Culler et Vogtmann dans [CV].

Dans le cas de  $W_n$ , Guirardel et Levitt [GuL1] ont introduit un espace topologique analogue à l'outre-espace de Culler et Vogtmann, appelé *l'outre-espace d'un produit libre*. Dans le cas d'un produit libre de copies de  $F$ , cet espace sera noté  $\mathbb{PO}(W_n)$ . Ce dernier est défini comme un ensemble de classes d'homothétie de graphes de groupes métriques marqués de groupe fondamental isomorphe à  $W_n$ . Muni de la topologie dite *faible*, l'espace  $\mathbb{PO}(W_n)$  se rétracte par déformation forte sur un complexe simplicial de drapeaux, appelé *l'épine de  $\mathbb{PO}(W_n)$* . Le groupe  $\text{Out}(W_n)$  agit naturellement sur  $\mathbb{PO}(W_n)$  et sur son épine par précomposition du marquage. Le groupe  $\text{Aut}(W_n)$  agit quant à lui sur *l'autre espace de  $W_n$* , noté  $\mathbb{PA}(W_n)$ . Nous renvoyons à la partie 2 pour des précisions.

La démonstration du théorème 2.1.1 est inspirée de celle de Bridson et Vogtmann dans le cas d'un groupe libre [BV1], mais des complications structurelles apparaissent. Nous présentons la démonstration dans le cas de  $\text{Out}(W_n)$ , le cas de  $\text{Aut}(W_n)$  étant similaire. Son plan, très simplifié, est le suivant. L'épine de l'outre-espace  $\mathbb{PO}(W_n)$  contient, à la différence de celle de l'outre-espace de Culler-Vogtmann qui n'en contient qu'un, deux types de sommets distingués, à savoir les *{0}-étoiles* et les *F-étoiles*, voir la partie 2 et la figure 2.1.

Nous étudions tout d'abord les stabilisateurs des *{0}-étoiles* et des *F-étoiles* sous l'action de  $\text{Out}(W_n)$ . Nous montrons dans la partie 3 que les sous-groupes de  $\text{Out}(W_n)$  isomorphes à  $\mathfrak{S}_n$  sont les stabilisateurs de *{0}-étoiles* et les sous-groupes de  $\text{Out}(W_n)$  isomorphes au produit semi-direct  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$  sont les stabilisateurs de *F-étoiles*. Ces derniers représentent un cas nouveau en comparaison de la preuve de [BV1] dans le cas d'un groupe libre. De ce fait, tout automorphisme  $\alpha$  de  $\text{Out}(W_n)$  préserve l'ensemble des stabilisateurs de *{0}-étoiles* et l'ensemble des stabilisateurs de *F-étoiles*. Fixons  $\alpha \in \text{Aut}(\text{Out}(W_n))$ . Le groupe  $\text{Out}(W_n)$  agissant transitivement sur l'ensemble des *{0}-étoiles*, nous pouvons supposer que  $\alpha$  induit un automorphisme du stabilisateur d'une

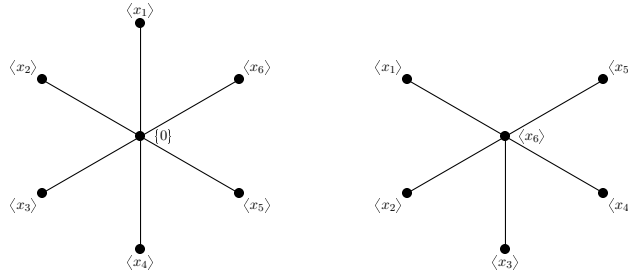


Figure 2.1: Exemples de graphes de groupes dont les classes d'équivalence sont respectivement une  $\{0\}$ -étoile et une  $F$ -étoile (cas  $n = 6$ ). Les arêtes ont des groupes associés triviaux. L'ensemble  $\{x_1, \dots, x_6\}$  est une partie génératrice standard de  $W_6$ .

$\{0\}$ -étoile  $\mathcal{X}$ . Les stabilisateurs de  $\{0\}$ -étoiles étant isomorphes à  $\mathfrak{S}_n$ , si  $n \geq 5$  et  $n \neq 6$ , nous pouvons supposer que la restriction de  $\alpha$  au stabilisateur de  $\mathcal{X}$  est égale à l'identité. Nous montrons alors qu'un tel  $\alpha$  préserve le stabilisateur d'une  $F$ -étoile  $\mathcal{Y}$  adjacente à  $\mathcal{X}$ , et que la restriction de  $\alpha$  au stabilisateur de  $\mathcal{Y}$  est en fait l'identité. Le groupe  $\text{Out}(W_n)$  étant engendré par l'union des stabilisateurs d'une  $\{0\}$ -étoile et d'une  $F$ -étoile adjacente, ceci conclut la démonstration si  $n \geq 5$ . Le cas  $n = 4$ , qui présente un automorphisme extérieur exceptionnel, est traité dans la partie 4.

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## 2.2 Préliminaires

Nous rappelons tout d'abord la définition de l'outre-espace  $\mathbb{PO}(W_n)$  introduit par Guirardel et Levitt dans [GuL1]. Un point de  $\mathbb{PO}(W_n)$  est une classe d'homothétie de graphes de groupes métriques  $X$  de groupe fondamental  $W_n$  munis d'un isomorphisme de groupes appelé *marquage*  $\rho: W_n \rightarrow \pi_1(X)$  (pour un choix indifférent de point base) vérifiant :

- (1) le graphe sous-jacent à  $X$  est un arbre fini ;
- (2) tous les groupes d'arêtes sont triviaux ;
- (3) il y a exactement  $n$  sommets de groupes associés isomorphes à  $F$  ;
- (4) tous les autres sommets ont un groupe associé trivial ;
- (5) toute feuille de l'arbre sous-jacent a un groupe associé non trivial ;
- (6) si  $v$  est un sommet de groupe associé trivial, alors  $\deg(v) \geq 3$ .

Deux graphes métriques marqués  $(X, \rho)$  et  $(X', \rho')$  sont dans la même classe d'homothétie s'il existe une homothétie  $f: X \rightarrow X'$  (i.e. un homéomorphisme multipliant toutes les

longueurs des arêtes par un même scalaire strictement positif) telle que  $f_* \circ \rho = \rho'$ . On note  $[X, \rho]$  la classe d'homothétie d'un tel graphe de groupes métrique marqué  $(X, \rho)$ . Si le marquage est sous-entendu, on notera  $\mathcal{X}$  la classe d'homothétie. Le groupe  $\text{Aut}(W_n)$  agit par précomposition du marquage. Étant donné que pour tout  $\alpha \in \text{Inn}(W_n)$ , et pour tout  $\mathcal{X} \in \mathbb{PO}(W_n)$ , nous avons  $\alpha(\mathcal{X}) = \mathcal{X}$ , l'action de  $\text{Aut}(W_n)$  sur  $\mathbb{PO}(W_n)$  induit une action de  $\text{Out}(W_n)$  sur  $\mathbb{PO}(W_n)$ .

La définition de *l'autre espace de  $W_n$* , noté  $\mathbb{PA}(W_n)$ , est identique à celle de  $\mathbb{PO}(W_n)$  à ceci près que chaque graphe de groupes métrique considéré est muni d'un point base  $v$ . Le *marquage* est alors un isomorphisme de groupes  $\rho: W_n \rightarrow \pi_1(X, v)$ . Les homothéties considérées préservent les points bases. Le groupe  $\text{Aut}(W_n)$  agit par précomposition du marquage.

L'ensemble  $\mathbb{PO}(W_n)$  (resp.  $\mathbb{PA}(W_n)$ ) est muni d'une topologie. Pour tout élément  $[X, \rho] \in \mathbb{PO}(W_n)$ , soit  $(X, \rho)$  un représentant de cette classe d'équivalence tel que la somme des longueurs des arêtes du graphe sous-jacent soit égale à 1. Le graphe de groupes  $(X, \rho)$  définit alors un simplexe ouvert obtenu en faisant varier les longueurs des arêtes du graphe sous-jacent à  $(X, \rho)$ , de manière à ce que la somme des longueurs des arêtes soit toujours égale à 1. Une classe d'équivalence  $[X', \rho'] \in \mathbb{PO}(W_n)$  définit une face de codimension 1 du simplexe associé à  $(X, \rho)$  si l'on peut obtenir  $(X', \rho')$  à partir de  $(X, \rho)$  en écrasant une arête du graphe sous-jacent à  $X$ . La *topologie faible* sur  $\mathbb{PO}(W_n)$  est alors définie de la manière suivante : un ensemble est ouvert si, et seulement si, son intersection avec chaque simplexe ouvert est ouverte.

Nous rappelons à présent la définition d'un rétract par déformation forte  $\text{Out}(W_n)$ -équivariant de  $\mathbb{PO}(W_n)$ , appelé *l'épine de l'outre-espace*. L'épine de  $\mathbb{PO}(W_n)$  est le complexe simplicial de drapeaux dont les sommets sont les simplexes ouverts associés à chaque classe d'équivalence  $[X, \rho]$ , et où deux sommets correspondant à des classes d'équivalence de graphes de groupes marqués  $[X, \rho]$  et  $[X', \rho']$  sont reliés par une arête si  $[X, \rho]$  définit une face du simplexe associé à  $[X', \rho']$  ou réciproquement. L'épine de  $\mathbb{PA}(W_n)$  est définie de manière similaire. Il existe un plongement de l'épine de  $\mathbb{PO}(W_n)$  dans  $\mathbb{PO}(W_n)$  ayant pour image l'épine barycentrique de  $\mathbb{PO}(W_n)$ . Par la suite, nous identifierons l'épine de  $\mathbb{PO}(W_n)$  avec son image par ce plongement. De même, il existe un plongement de l'épine de  $\mathbb{PA}(W_n)$  dans  $\mathbb{PA}(W_n)$  ayant pour image l'épine barycentrique de  $\mathbb{PA}(W_n)$ .

Si  $X$  est un graphe de groupes, on note  $\text{Aut}_{gr}(X)$  le groupe des automorphismes du graphe sous-jacent à  $X$ . Si  $X$  est un graphe de groupes pointé, la notation  $\text{Aut}_{gr}(X)$  désigne le groupe des automorphismes du graphe pointé sous-jacent à  $X$ . Nous appellerons  $\{0\}$ -*étoile* la classe d'équivalence dans  $\mathbb{PO}(W_n)$  d'un graphe de groupes marqué dont le graphe sous-jacent est un arbre ayant  $n$  feuilles et  $n + 1$  sommets et de longueur d'arêtes constante. Nous appellerons  $F$ -*étoile* la classe d'équivalence dans  $\mathbb{PO}(W_n)$  d'un graphe de groupes marqué dont le graphe sous-jacent est un arbre ayant  $n - 1$  feuilles et  $n$  sommets et de longueur d'arêtes constante. Les sommets correspondants dans l'épine de  $\mathbb{PO}(W_n)$  sont encore appelés  $\{0\}$ -*étoiles* et  $F$ -*étoiles*. Dans le cas de  $\mathbb{PA}(W_n)$ , les

définitions des  $\{0\}$ -étoiles et des  $F$ -étoiles sont identiques à ceci près que l'on suppose également que le point base est le centre (l'unique sommet qui n'est pas une feuille) du graphe.

On fixe désormais une partie génératrice standard  $\{x_1, \dots, x_n\}$  de  $W_n$ .

Le groupe  $\text{Aut}(W_n)$  (et donc  $\text{Out}(W_n)$ ) est de type fini. Nous décrivons maintenant une partie génératrice finie. Pour tout  $i \in \{1, \dots, n-1\}$ , on note  $\tau_i: W_n \rightarrow W_n$  l'automorphisme envoyant  $x_i$  sur  $x_{i+1}$ ,  $x_{i+1}$  sur  $x_i$  et qui fixe tous les autres générateurs. Pour tous les  $i, j \in \{1, \dots, n\}$  tels que  $i \neq j$ , on note  $\sigma_{i,j}: W_n \rightarrow W_n$  l'automorphisme qui envoie  $x_i$  sur  $x_j x_i x_j$  et qui fixe tous les autres générateurs. La proposition suivante est due à Mühlherr.

**Proposition 2.2.1.** *[Müh, Theorem B] Le groupe  $\text{Aut}(W_n)$  est engendré par  $\tau_1, \dots, \tau_{n-1}$  et par  $\sigma_{1,2}$ .*

Si  $\alpha$  est un élément de  $\text{Aut}(W_n)$ , sa classe d'automorphismes extérieurs sera notée  $[\alpha]$ . Soit  $p: \text{Aut}(W_n) \rightarrow \text{Out}(W_n)$  la projection canonique. On note  $\tilde{A}_n = \langle \tau_1, \dots, \tau_{n-1} \rangle$  et  $A_n = p(\tilde{A}_n)$ . Les groupes  $\tilde{A}_n$  et  $A_n$  sont isomorphes au groupe symétrique  $\mathfrak{S}_n$ . On note  $\tilde{U}_n = \langle \tau_1, \dots, \tau_{n-2}, \sigma_{1,n} \rangle$  et  $U_n = p(\tilde{U}_n)$ . On voit que  $\tilde{U}_n$  est isomorphe au produit semi-direct  $F^{n-1} \rtimes \mathfrak{S}_{n-1}$ , alors que  $U_n$  est isomorphe au produit semi-direct  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ , où  $\mathfrak{S}_{n-1}$  agit dans les deux cas par permutation des facteurs, en considérant  $F^{n-2}$  comme le quotient de  $F^{n-1}$  par le sous-groupe  $F$  diagonal. Soient  $\tilde{B}_n = \langle \tau_1, \dots, \tau_{n-2} \rangle$  et  $B_n = p(\tilde{B}_n)$ . Les groupes  $\tilde{B}_n$  et  $B_n$  sont isomorphes à  $\mathfrak{S}_{n-1}$ .

Nous traitons à présent le cas où  $n = 3$ . Soit  $\epsilon: W_3 \rightarrow \mathbb{Z}/2\mathbb{Z}$  le morphisme envoyant, pour tout  $i \in \{1, 2, 3\}$ , l'élément  $x_i$  sur 1. Mühlherr ([Müh, Theorem A]) a montré que  $\ker(\epsilon)$  est un sous-groupe caractéristique de  $W_3$ . De plus,  $\ker(\epsilon)$  est un groupe libre à 2 générateurs, librement engendré par  $x_1 x_2$  et  $x_2 x_3$ . Ceci induit un morphisme  $\rho: \text{Aut}(W_3) \rightarrow \text{Aut}(\text{FF}_2)$ , qui est en fait un isomorphisme (c.f. [Var, Lemma 2.3]).

**Proposition 2.2.2.** *Le morphisme  $\rho: \text{Aut}(W_3) \rightarrow \text{Aut}(\text{FF}_2)$  induit un isomorphisme entre  $\text{Out}(W_3)$  et  $\mathbb{PGL}(2, \mathbb{Z})$ .*

**Démonstration.** Soient  $a$  et  $b$  les générateurs de  $\text{FF}_2$ . On remarque tout d'abord que  $\text{Int}(\text{FF}_2) \subseteq \rho(\text{Int}(W_3))$ . Donc le noyau du morphisme surjectif  $\text{Aut}(W_3) \rightarrow \text{Out}(\text{FF}_2)$  est inclus dans  $\text{Int}(W_3)$ . Pour tout  $i \in \{1, 2, 3\}$ , soit  $ad_{x_i} \in \text{Aut}(W_3)$  la conjugaison globale par  $x_i$ . Un calcul immédiat montre que, pour tout  $i \in \{1, 2, 3\}$ ,  $\rho(ad_{x_i})$  est dans la classe d'automorphisme extérieur du morphisme  $\iota: \text{FF}_2 \rightarrow \text{FF}_2$  envoyant  $a$  sur  $a^{-1}$  et  $b$  sur  $b^{-1}$ . De ce fait, puisque le sous-groupe  $\langle [\iota] \rangle$  est distingué dans  $\text{Out}(\text{FF}_2)$ , le morphisme  $\rho$  induit un isomorphisme entre  $\text{Out}(W_3)$  et  $\text{Out}(\text{FF}_2)/\langle [\iota] \rangle$ . Comme  $\iota$  est envoyé par le morphisme d'abélianisation sur  $-\text{Id} \in \text{GL}(2, \mathbb{Z})$ , on voit que  $\text{Out}(W_3)$  est isomorphe à  $\mathbb{PGL}(2, \mathbb{Z})$ .  $\square$

Nous allons démontrer les théorèmes 2.1.1 et 2.1.2 en étudiant les stabilisateurs des  $\{0\}$ -étoiles et des  $F$ -étoiles sous l'action de  $\text{Out}(W_n)$  et  $\text{Aut}(W_n)$ . Pour cela, nous utiliserons les résultats suivants, dus respectivement à Hensel et Kielak et à Guirardel et Levitt, qui donnent des informations sur les points fixes de sous-groupes de  $\text{Out}(W_n)$ .



**Proposition 2.2.3.** [HK, Corollary 6.1.] Soient  $n \geq 1$  un entier et  $H$  un sous-groupe fini de  $\text{Out}(W_n)$ . Alors  $H$  fixe un point de  $\mathbb{PO}(W_n)$ .

**Corollaire 2.2.4.** Soient  $n \geq 1$  un entier et  $H$  un sous-groupe fini de  $\text{Aut}(W_n)$ . Alors  $H$  fixe un point de  $\mathbb{PA}(W_n)$ .

**Démonstration.** Soit  $p: \text{Aut}(W_n) \rightarrow \text{Out}(W_n)$  la projection canonique. Alors  $p(H)$  est un sous-groupe fini de  $\text{Out}(W_n)$ , donc par la proposition 2.2.3,  $p(H)$  fixe un point  $\mathcal{X}$  de l'outre-espace. Soient  $X$  un représentant de  $\mathcal{X}$  et  $\bar{X}$  son graphe sous-jacent. Comme tout automorphisme intérieur agit sur  $X$ , et que  $p(H)$  agit également sur  $X$ , on en déduit que  $H$  agit sur  $X$ . Étant donné que  $H$  est fini et que  $\bar{X}$  est un arbre, on voit que  $H$  fixe un point  $v$  de  $\bar{X}$ . Donc la classe d'homothétie du graphe de groupes métrique marqué pointé  $(X, v)$  est fixée par  $H$ .  $\square$

**Proposition 2.2.5.** [GuL2, Theorem 8.3.] Soit  $n \geq 2$  un entier. Si  $H$  est un sous-groupe de type fini de  $\text{Out}(W_n)$  (resp.  $\text{Aut}(W_n)$ ) fixant un point de  $\mathbb{PO}(W_n)$  (resp.  $\mathbb{PA}(W_n)$ ), alors l'ensemble des points fixes de  $H$  est contractile pour la topologie faible.  $\square$

On note  $\text{Fix}_{\mathbb{PO}(W_n)}(G)$  l'ensemble des points fixes d'un sous-groupe  $G$  de  $\text{Out}(W_n)$  dans  $\mathbb{PO}(W_n)$  (ou  $\text{Fix}(G)$  s'il n'y a pas d'ambiguïté). On note de plus  $\text{Fix}_{K_n}(G)$  l'ensemble des points fixes de  $G$  contenus dans l'épine de  $\mathbb{PO}(W_n)$ . Puisque l'épine de  $\mathbb{PO}(W_n)$  est un rétract par déformation forte  $\text{Out}(W_n)$ -équivariant de  $\mathbb{PO}(W_n)$ , nous déduisons de la proposition 2.2.5 le résultat suivant.

**Corollaire 2.2.6.** Soit  $n \geq 2$  un entier. Si  $H$  est un sous-groupe de type fini de  $\text{Out}(W_n)$  fixant un point de l'épine de  $\mathbb{PO}(W_n)$ , alors l'ensemble  $\text{Fix}(H)$  des points fixes de  $H$  dans l'épine de  $\mathbb{PO}(W_n)$  est connexe pour la topologie faible.

Soit  $\mathcal{X}$  un point de l'épine de  $\mathbb{PO}(W_n)$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $T$  l'arbre de Bass-Serre associé à  $X$ . Nous définissons à présent un morphisme de groupes

$$\Phi: \text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X).$$

Soient  $[\alpha] \in \text{Stab}_{\text{Out}(W_n)}(\mathcal{X})$ , et  $\alpha \in \text{Aut}(W_n)$  un représentant de  $[\alpha]$ . Il existe un automorphisme  $\tilde{H}_\alpha \in \text{Aut}(T)$  tel que pour tout  $x \in T$ , et pour tout  $g \in W_n$  on ait  $\alpha(g)\tilde{H}_\alpha(x) = \tilde{H}_\alpha(gx)$ . L'automorphisme  $\tilde{H}_\alpha$  induit un automorphisme  $H_\alpha \in \text{Aut}_{gr}(X)$ , et l'application  $\alpha \mapsto H_\alpha$  passe au quotient pour donner un morphisme

$$\Phi: \text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X).$$

Nous pouvons à présent démontrer un résultat identique au corollaire 2.2.6 dans le cas de  $\mathbb{PA}(W_n)$ .

**Corollaire 2.2.7.** Soit  $n \geq 2$  un entier. Si  $H$  est un sous-groupe fini de  $\text{Aut}(W_n)$  fixant un point de l'épine de  $\mathbb{PA}(W_n)$ , alors l'ensemble  $\text{Fix}(H)$  des points fixes de  $H$  dans l'épine de  $\mathbb{PA}(W_n)$  est connexe pour la topologie faible.

**Démonstration.** Soient  $\mathcal{X}$  et  $\mathcal{Y}$  deux points de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  fixés par  $H$ . Soit  $p_1: \mathbb{P}\mathcal{A}(W_n) \rightarrow \mathbb{P}\mathcal{O}(W_n)$  le morphisme canonique d'oubli du point base. On rappelle que  $p: \text{Aut}(W_n) \rightarrow \text{Out}(W_n)$  est la projection canonique. Alors  $p(H)$  fixe  $p_1(\mathcal{X})$  et  $p_1(\mathcal{Y})$ , donc par le corollaire 2.2.6 il existe dans  $\text{Fix}_{K_n}(p(H))$  un chemin continu  $P$  de  $p_1(\mathcal{X})$  vers  $p_1(\mathcal{Y})$ . Soient  $\mathcal{X}_1, \dots, \mathcal{X}_n$  les sommets de  $K_n$  consécutifs dans  $P$  (on suppose  $p_1(\mathcal{X}) = \mathcal{X}_1$  et  $\mathcal{X}_n = p_1(\mathcal{Y})$ ) tels que, pour tout  $i \in \{1, \dots, n-1\}$ ,  $X_i$  et  $X_{i+1}$  sont reliés par une arête dans  $K_n$ . Soit  $X_1$  un représentant de  $\mathcal{X}_1$  et pour tout  $i \in \{2, \dots, n\}$ , soit  $X_i$  un représentant de  $\mathcal{X}_i$  obtenu en écrasant ou en éclatant une forêt de  $X_{i-1}$ . Pour tout  $i \in \{1, \dots, n\}$ , comme tout automorphisme intérieur agit trivialement sur  $X_i$ , et puisque  $p(H)$  agit également sur  $X_i$ , on en déduit que  $H$  agit sur  $X_i$ . De plus, étant donné que  $H$  est fini et que le graphe sous-jacent  $\overline{X}_i$  de  $X_i$  est un arbre, on voit que  $H$  fixe un point  $v_i$  de  $\overline{X}_i$ . Pour tout  $i$ , soit  $\tilde{\mathcal{X}}_i$  la classe d'équivalence du graphe métrique marqué pointé  $(X_i, v_i)$  (on suppose que  $\tilde{\mathcal{X}}_1 = \mathcal{X}$  et  $\tilde{\mathcal{X}}_n = \mathcal{Y}$ ). Alors  $\tilde{\mathcal{X}}_i$  est fixé par  $H$ .

Nous construisons à présent pour tout  $i \in \{1, \dots, n-1\}$ , un chemin continu inclus dans l'ensemble des points fixes de  $H$  dans l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  entre  $\tilde{\mathcal{X}}_i$  et  $\tilde{\mathcal{X}}_{i+1}$ , ce qui conclura. La construction étant symétrique, nous pouvons supposer, quitte à changer les représentants  $X_i$  et  $X_{i+1}$ , que  $X_{i+1}$  est obtenu à partir de  $X_i$  en écrasant une forêt  $\mathcal{F}$ . Soient  $\Delta$  le simplexe ouvert dans  $\mathbb{P}\mathcal{A}(W_n)$  associé à  $(X_i, v_i)$  et  $e$  l'arête de l'épine barycentrique de  $\mathbb{P}\mathcal{A}(W_n)$  reliant  $\tilde{\mathcal{X}}_i$  et  $\tilde{\mathcal{X}}_{i+1}$ . Pour toute arête  $f$  de  $\mathcal{F}$ , soit  $\ell_f$  la longueur de  $f$ . Pour tout  $t \in [0, 1]$ , soient  $X_i^t$  le graphe de groupes métrique obtenu à partir de  $X_i$  en donnant à toute arête  $f \in \mathcal{F}$  la longueur  $(1-t)\ell_f$ , et  $pr_t: X_i \rightarrow X_i^t$  la projection canonique. On observe que  $X_i^0 = X_i$  et que  $X_i^1 = X_{i+1}$ .

Puisque  $H$  stabilise  $X_i$  et  $X_{i+1}$ , on voit que  $H$  stabilise la forêt  $F$ . Donc, pour tout  $t \in [0, 1]$ , le groupe  $H$  stabilise  $X_i^t$ . Puisque  $H$  fixe le sommet  $v_i$  de  $\overline{X}_i$ , il fixe également, pour tout  $t \in [0, 1]$ , le sommet  $pr_t(v_i)$ . Ceci induit un chemin continu de  $\tilde{\mathcal{X}}_i$  vers la classe d'équivalence dans  $K_n$  de  $(X_{i+1}, pr_1(v_i))$ . Si  $pr_1(v_i) \neq v_{i+1}$ , alors, puisque le graphe sous-jacent à  $X_{i+1}$  est un arbre,  $H$  fixe l'unique arc dans  $\overline{X}_{i+1}$  reliant  $pr_1(v_i)$  et  $v_{i+1}$ . Ceci induit alors un chemin continu contenu dans l'ensemble des points fixes de  $H$  dans l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  entre la classe d'équivalence dans  $K_n$  de  $(X_{i+1}, pr_1(v_i))$  et  $\tilde{\mathcal{X}}_{i+1}$ , ce qui conclut.  $\square$

Soient  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  et  $X$  un représentant de  $\mathcal{X}$ . On note  $\Phi: \text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X)$  le morphisme naturel. Nous donnons maintenant une description de  $\ker(\Phi)$ . Soit  $[X, \rho]$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ . On note  $(X, \rho)$  un représentant de  $[X, \rho]$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . Soit  $e$  une arête de  $\overline{X}$  reliant le sommet  $v = o(e)$  au sommet  $w = t(e)$ . Soit  $z \in G_v$  un élément du groupe associé au sommet  $v$ , et  $\bar{z}$  son antécédent par  $\rho$ . Nous définissons à présent le *twist par  $z$  autour de  $e$* . Soit  $G_u$  le groupe associé à un sommet  $u$ . Le twist par  $z$  autour de  $e$ , noté  $D_z$ , est l'automorphisme de  $W_n$ , bien défini modulo conjugaison, qui est égal à l'identité sur  $\rho^{-1}(G_u)$  si  $u$  est dans la même composante connexe de  $\overline{X}$  privé de l'intérieur de  $e$  que  $v$ , et qui à  $x \in \rho^{-1}(G_u)$  associe  $\bar{z}x\bar{z}^{-1}$  si  $u$  n'est pas dans la même composante connexe que  $v$ . Nous avons le résultat suivant, dû à Levitt.

**Proposition 2.2.8.** *[Lev1, Proposition 2.2 and 3.1] Soit  $n \geq 2$  un entier. Soient  $\mathcal{X}$  un*

point de l'épine de l'outre-espace  $\mathbb{P}\mathcal{O}(W_n)$  et  $X$  un représentant de  $\mathcal{X}$ . Soient  $v_1, \dots, v_n$  les sommets du graphe sous-jacent de  $X$  de groupe associé isomorphe à  $F$  et soit  $n_i$  le degré de  $v_i$  pour  $i = 1, \dots, n$ . Le noyau du morphisme  $\Phi: \text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X)$  (noté  $\text{Out}_0(W_n)$  dans [Lev1]) est isomorphe à  $\prod_{i=1}^n F^{n_i-1}$ , et il est engendré par les twists autour des arêtes dont l'origine appartient à  $\{v_1, \dots, v_n\}$  et n'est pas une feuille.

**Remarque 2.2.9.** Dans le cas où  $\mathcal{X} \in \mathbb{P}\mathcal{A}(W_n)$ , le noyau est engendré par les twists autour des arêtes  $e$  dont l'origine  $o(e)$  appartient à  $\{v_1, \dots, v_n\}$  et n'est pas une feuille, et telles que, si  $o(e)$  est distinct du point base  $v_*$ , ces arêtes ne soient pas contenues dans l'unique chemin reliant  $o(e)$  à  $v_*$ . En particulier, si le groupe associé à  $v_*$  est trivial et si  $n_i$  est le degré de  $v_i$  pour  $i = 1, \dots, n$ , alors le noyau est isomorphe à  $\prod_{i=1}^n F^{n_i-1}$ . Si le groupe associé à  $v_*$  est non trivial, et si on suppose  $v_* = v_n$ , alors le noyau est isomorphe à  $\left(\prod_{i=1}^{n-1} F^{n_i-1}\right) \times F^{n_n}$ .

### 2.3 Stabilisateurs des $\{0\}$ -étoiles et des $F$ -étoiles

Nous étudions tout d'abord les stabilisateurs des  $\{0\}$ -étoiles.

**Lemme 2.3.1.** Soit  $n \geq 4$  un entier. Soient  $G$  un sous-groupe fini de  $\text{Out}(W_n)$  isomorphe à  $\mathfrak{S}_n$ , et  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  fixé par  $G$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\bar{X}$  le graphe sous-jacent à  $X$ . Si le nombre de feuilles de  $\bar{X}$  est  $n$ , alors  $\mathcal{X}$  est une  $\{0\}$ -étoile.

**Démonstration.** Soit  $v$  un sommet de  $\bar{X}$  qui n'est pas une feuille et qui soit à distance maximale du centre<sup>1</sup> de  $\bar{X}$ .

**Affirmation.** Si  $m = \deg(v)$ , alors  $v$  est adjacent à au moins  $m - 1$  feuilles de  $\bar{X}$ .

**Démonstration.** L'hypothèse de maximalité sur  $v$  implique qu'il y a au plus un sommet adjacent à  $v$  qui n'est pas une feuille, car sinon nous pourrions trouver un sommet  $w$  adjacent à  $v$  qui ne serait pas une feuille et qui serait à distance strictement plus grande du centre que  $v$ .  $\square$

Maintenant, le groupe associé à  $v$  est trivial car  $\bar{X}$  possède exactement  $n$  sommets de groupes associés non triviaux, et ces sommets sont tous des feuilles car  $\bar{X}$  possède  $n$  feuilles. De ce fait,  $\deg(v) \geq 3$  et  $v$  est adjacent à au moins deux feuilles, notées  $v_1$  et  $v_2$ .

Soient  $L$  l'ensemble des feuilles de  $\bar{X}$ , et  $w$  une feuille de  $\bar{X}$  distincte de  $v_1$  et  $v_2$ . Puisque les seuls sommets de  $\bar{X}$  dont les groupes associés sont non triviaux sont des feuilles, la proposition 2.2.8 montre que le morphisme naturel  $G \rightarrow \text{Aut}_{gr}(X)$  est injectif. Ainsi, étant donné que le groupe  $G$  est isomorphe à  $\mathfrak{S}_n$ , et que  $\bar{X}$  possède  $n$  feuilles,

<sup>1</sup>Rappelons que le centre d'un arbre métrique compact non vide est l'unique milieu d'un segment de longueur maximale.

le morphisme naturel  $\text{Aut}_{gr}(X) \hookrightarrow \text{Bij}(L)$  est un isomorphisme. Donc il existe un automorphisme de  $\overline{X}$  envoyant  $v_1$  sur  $w$  et fixant  $v_2$ . De ce fait,  $w$  est adjacent à  $v$ . Ainsi,  $v$  est adjacent à toutes les feuilles de  $\overline{X}$ . Puisque le groupe  $\text{Aut}_{gr}(X)$  est isomorphe à  $\text{Bij}(L)$ , toutes les arêtes de  $\overline{X}$  ont même longueur. De ce fait,  $\mathcal{X}$  est une  $\{0\}$ -étoile.  $\square$

**Remarque 2.3.2.** Le résultat est identique dans le cas de  $\mathbb{P}\mathcal{A}(W_n)$ . En effet, soit  $G$  un sous-groupe fini de  $\text{Aut}(W_n)$  isomorphe à  $\mathfrak{S}_n$ , et  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  fixé par  $G$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . Supposons que  $\overline{X}$  possède  $n$  feuilles. Alors la remarque 2.2.9 donne que le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est un sous-groupe distingué de  $G$  d'ordre au plus 2. Comme  $G$  est isomorphe à  $\mathfrak{S}_n$ , et que  $n \geq 4$ , le morphisme est injectif. La même démonstration que le lemme 2.3.1 montre alors que  $X$  possède  $n$  feuilles et  $n + 1$  sommets. Il reste à montrer que le point base est le centre de  $\overline{X}$ . Mais ceci provient du fait que le groupe  $G$  est isomorphe à  $\text{Aut}_{gr}(X)$  qui lui-même est isomorphe à  $\text{Bij}(L)$ . Ainsi, nécessairement, le point base est le centre de  $\overline{X}$ . Donc  $\mathcal{X}$  est une  $\{0\}$ -étoile.

**Proposition 2.3.3.** Soient  $n \geq 5$  un entier et  $G$  un sous-groupe de  $\text{Out}(W_n)$  isomorphe à  $\mathfrak{S}_n$ . Alors  $G$  est le stabilisateur dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  d'une unique  $\{0\}$ -étoile.

**Démonstration.** Puisque  $G$  est fini, d'après la proposition 2.2.3, il existe un point  $\mathcal{X}$  de l'épine de l'outre-espace qui est fixé par  $G$ . Soit  $X$  un représentant de  $\mathcal{X}$ . D'après la proposition 2.2.8, il existe un entier  $k$  tel que le noyau de l'application naturelle  $G \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k \cap G$ .

Or  $F^k \cap G$  est un 2-sous-groupe distingué de  $G \simeq \mathfrak{S}_n$ . Donc, comme  $n \geq 5$ , un tel sous-groupe est trivial. De ce fait,  $G$  s'injecte dans  $\text{Aut}_{gr}(X)$ . Or tout automorphisme d'un arbre est entièrement déterminé par la permutation qu'il induit sur l'ensemble des feuilles. Ainsi, si  $\overline{X}$  est le graphe sous-jacent à  $X$  et si  $L$  est l'ensemble des feuilles de  $\overline{X}$ ,

$$G \hookrightarrow \text{Aut}_{gr}(X) \hookrightarrow \text{Bij}(L).$$

Or les représentants des éléments de  $\mathbb{P}\mathcal{O}(W_n)$  possèdent au plus  $n$  sommets de groupes non triviaux et toutes les feuilles possèdent des groupes associés non triviaux. Donc  $|L| \leq n$ . Donc, comme  $G$  s'injecte dans  $\text{Bij}(L)$  et que  $G$  est isomorphe à  $\mathfrak{S}_n$ , on voit que  $G$  est isomorphe à  $\text{Aut}_{gr}(X)$  et que  $\text{Aut}_{gr}(X)$  est isomorphe à  $\text{Bij}(L)$ . De ce fait,  $\overline{X}$  possède  $n$  feuilles. Par le lemme 2.3.1,  $\mathcal{X}$  est une  $\{0\}$ -étoile.

Montrons maintenant l'unicité. Puisque l'ensemble des  $\{0\}$ -étoiles est discret dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ , par le corollaire 2.2.6, on conclut que  $G$  fixe une unique  $\{0\}$ -étoile dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ .  $\square$

**Remarque 2.3.4.** Dans le cas de  $\mathbb{P}\mathcal{A}(W_n)$ , le résultat de la proposition 2.3.3 est vrai pour  $n \geq 4$ . En effet, dans le cas où  $n \geq 5$ , la démonstration est identique à celle de la proposition 2.3.3 en utilisant cette fois la remarque 2.3.2.

Dans le cas où  $n = 4$ , soit  $\mathcal{X} \in \mathbb{P}\mathcal{A}(W_4)$  un point fixé par un sous-groupe  $G$  de  $\text{Aut}(W_n)$  isomorphe à  $\mathfrak{S}_n$ . On note  $X$  un représentant de  $\mathcal{X}$ ,  $\overline{X}$  le graphe sous-jacent à

$X$  et  $v_*$  le point base de  $\overline{X}$ . Soit  $H$  le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$ . Supposons par l'absurde que  $H$  ne soit pas trivial. Alors, par la remarque 2.2.9, le groupe  $H$  est un 2-groupe. Comme le seul 2-sous-groupe distingué de  $\mathfrak{S}_4$  est le groupe de Klein, le groupe  $H$  est isomorphe à  $F^2$ . Nous distinguons différents cas, selon le fait que le groupe associé à  $v_*$  soit trivial ou non et selon le nombre de sommets qui ne sont pas des feuilles et qui ont un groupe associé non trivial. On remarque immédiatement que, puisque tout arbre possède au moins 2 feuilles, le nombre de sommets qui ne sont pas des feuilles et de groupes associés non triviaux est au plus 2.

Supposons que  $\overline{X}$  contienne deux sommets qui ne soient pas des feuilles et dont les groupes associés sont isomorphes à  $F$  et que le groupe associé à  $v_*$  soit trivial.

Soient  $w_1$  et  $w_2$  ces deux sommets. Alors  $\deg(v_*) \geq 3$ . Comme chaque composante connexe de  $\overline{X} - \{v_*\}$  contient au moins une feuille,  $\overline{X}$  contiendrait 5 sommets de groupes associés non triviaux. Ceci contredit le fait qu'il y a exactement 4 sommets dans le graphe de groupes associés non triviaux.

Supposons que  $\overline{X}$  contienne deux sommets qui ne sont pas des feuilles et dont les groupes associés sont isomorphes à  $F$  et que le groupe associé à  $v_*$  ne soit pas trivial.

Alors la description du noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  donné dans la remarque 2.2.9 donne que le cardinal du noyau est au moins 8, ce qui contredit le fait que  $H$  est de cardinal 4.

Supposons que  $\overline{X}$  contienne un seul sommet, noté  $w$ , de groupe associé non trivial et qui ne soit pas une feuille et que le groupe associé à  $v_*$  soit trivial. Alors  $\deg(v_*) \geq 3$ . Comme chaque composante connexe de  $\overline{X} - \{v_*\}$  contient au moins une feuille, et qu'il existe un sommet de groupe associé non trivial et qui ne soit pas une feuille,  $\deg(v_*) = 3$ . De plus, puisqu'il y a exactement 4 sommets dans le graphe de groupes associés non triviaux, chaque composante connexe de  $\overline{X} - \{v_*\}$  contient exactement une feuille. Donc  $v_*$  est relié à exactement 2 feuilles et  $w$  est relié à une seule feuille et à  $v_*$ . Or le cardinal du groupe des automorphismes d'un tel graphe est égal à 2. Comme le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal 4, ceci contredit le fait que  $G$  est isomorphe à  $\mathfrak{S}_4$ .

Supposons que  $\overline{X}$  contienne un seul sommet, noté  $w$ , de groupe associé non trivial et qui ne soit pas une feuille. Si  $v_*$  est une feuille, alors le graphe possède exactement 3 feuilles, dont l'une est le point base. De ce fait, comme tout automorphisme de  $\overline{X}$  est induit par son action sur les feuilles, le groupe des automorphismes d'un tel graphe pointé est de cardinal 2. Comme le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal 4, ceci contredit le fait que  $G$  est isomorphe à  $\mathfrak{S}_4$ .

Supposons alors que le point base  $v_*$  ne soit pas une feuille. Par les cas précédents,  $v_* = w$ . Comme le nombre de sommets de groupes non trivial est exactement 4, et que tout sommet de groupe associé trivial est de degré au moins 3, le graphe  $\overline{X}$  contient au plus un sommet de groupe associé trivial. Le cas où le nombre de sommets de groupe associé trivial est égal à 1 n'est pas possible car alors le cardinal du groupe

des automorphismes d'un tel graphe est égal à 2, contredisant le fait que le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal 4 et que  $G$  est isomorphe à  $\mathfrak{S}_4$ .

Dans le cas où le nombre de sommets de groupe associé trivial est nul, on voit que  $\mathcal{X}$  est une  $F$ -étoile. Or, par la remarque 2.2.9, le cardinal du noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est égal à 8, d'où une contradiction.

En conclusion, le morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est également injectif dans le cas où  $\mathcal{X}$  appartient à  $\mathbb{P}\mathcal{A}(W_4)$  et  $n = 4$ . La suite de la démonstration est alors identique à la proposition 2.3.3.  $\square$

Nous démontrons à présent un résultat similaire pour les  $F$ -étoiles. Pour cela, nous avons besoin du lemme suivant.

**Lemme 2.3.5.** *Soient  $n \geq 4$  un entier et  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . Soit  $k$  l'entier tel que le noyau du morphisme naturel  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k$ . Alors  $k \leq n - 2$ . Par ailleurs,  $k = n - 2$  si, et seulement si, l'ensemble  $V\overline{X}$  des sommets de  $\overline{X}$  est de cardinal  $n$ .*

**Démonstration.** Supposons que  $|V\overline{X}| > n$ . Soient  $v$  un sommet de groupe associé trivial et  $e$  une arête de  $X$  reliant  $v$  à un sommet  $w$ . Une telle arête existe car  $\overline{X}$  est connexe et le nombre de sommets de  $\overline{X}$  de groupe non trivial est égal à  $n$ .

**Affirmation.** Soient  $Y$  le graphe de groupes marqué obtenu à partir de  $X$  en contractant l'arête  $e$  et  $\mathcal{Y}$  sa classe d'équivalence dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ . Alors le noyau du morphisme naturel  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{Y}) \rightarrow \text{Aut}_{gr}(Y)$  est isomorphe à  $F^l$ , avec  $l = k$  si le groupe associé à  $w$  est trivial, et  $l \geq k + 1$  sinon.

**Démonstration.** Si le groupe associé à  $w$  est trivial, alors contracter l'arête  $e$  ne modifie pas le degré des sommets dont le groupe associé est non trivial. Donc, dans ce cas,  $k = l$ . Supposons maintenant que le groupe associé à  $w$  ne soit pas trivial. Notons  $\overline{v\overline{w}}$  le sommet obtenu en contractant  $e$ . Le groupe associé à  $\overline{v\overline{w}}$  est non trivial. Alors, puisque, par hypothèse,  $\text{deg}(v) \geq 3$ , nous avons :

$$\text{deg}(\overline{v\overline{w}}) = \text{deg}(v) + \text{deg}(w) - 2 \geq \text{deg}(w) + 1.$$

Ainsi, dans ce cas,  $l \geq k + 1$ .  $\square$

De ce fait, si  $|V\overline{X}| > n$ , il existe une arête reliant un sommet de groupe associé trivial et un sommet de groupe associé non trivial. Par l'affirmation précédente, l'entier  $k$  associé au morphisme  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X)$  n'est pas maximal.

Ainsi, pour calculer la borne maximale de  $k$ , nous pouvons supposer que  $\overline{X}$  possède  $n$  sommets, tous de groupe associé non trivial. Donc,

$$\sum_{v \in V\overline{X}} \text{deg}(v) = 2|E\overline{X}| = 2n - 2,$$

la dernière égalité provenant du fait que  $\overline{X}$  soit un arbre. Ainsi,

$$k = \sum_{v \in V\overline{X}} (\deg(v) - 1) = \sum_{v \in V\overline{X}} \deg(v) - n = 2n - 2 - n = n - 2.$$

Donc,  $k \leq n - 2$ , et si  $|V\overline{X}| = n$ , alors  $k = n - 2$ .

Supposons maintenant que  $k = n - 2$ . Par l'affirmation précédente, la procédure de contraction présentée fait croître strictement  $k$  lorsque l'on contracte une arête reliant un sommet de groupe associé trivial et un sommet de groupe associé non trivial. Donc  $\overline{X}$  ne peut pas contenir de sommets ayant un groupe associé trivial. Donc le cardinal de  $V\overline{X}$  est égal à  $n$ .  $\square$

**Remarque 2.3.6.** Dans le cas de  $\mathbb{P}\mathcal{A}(W_n)$ , soit  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . Soit  $k$  l'entier tel que le noyau du morphisme naturel  $\text{Stab}_{\text{Aut}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k$ . Alors une démonstration identique au lemme 2.3.5 montre que  $k \leq n - 1$  avec égalité si, et seulement si,  $|V\overline{X}| = n$ .

Nous pouvons maintenant montrer le résultat suivant concernant les stabilisateurs de  $F$ -étoiles dans  $\text{Out}(W_n)$ .

**Proposition 2.3.7.** (1) Soit  $n \geq 4$  un entier. Le cardinal maximal d'un sous-groupe fini de  $\text{Out}(W_n)$  est  $2^{n-2}(n-1)!$ .

(2) Supposons  $n \geq 5$ . Soient  $G$  un sous-groupe de  $\text{Out}(W_n)$ , et  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  fixé par  $G$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . Si  $\overline{X}$  possède  $n$  feuilles, alors  $|G| < 2^{n-2}(n-1)!$ .

(3) Supposons  $n \geq 4$ . Soient  $G$  un sous-groupe de  $\text{Out}(W_n)$  isomorphe à  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ , et  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  fixé par  $G$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . Si le nombre de feuilles de  $\overline{X}$  est  $n - 1$ , alors  $\mathcal{X}$  est une  $F$ -étoile.

(4) Supposons  $n \geq 5$ . Soit  $G$  un sous-groupe de  $\text{Out}(W_n)$  isomorphe à  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ . Alors  $G$  est le stabilisateur d'une unique  $F$ -étoile.

**Démonstration.** Si  $\mathcal{X}$  est un élément de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ , nous noterons  $X$  un représentant de  $\mathcal{X}$ . Nous noterons également  $\overline{X}$  le graphe sous-jacent à  $X$  et  $L$  l'ensemble des feuilles de  $\overline{X}$ . Puisque  $\overline{X}$  est un arbre, tout automorphisme de  $\overline{X}$  est entièrement déterminé par son action sur les feuilles. Donc le morphisme de restriction de  $\text{Aut}_{gr}(X)$  dans  $\text{Bij}(L)$  est injectif.

Montrons l'assertion (1). Puisque tout sous-groupe fini de  $\text{Out}(W_n)$  fixe un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  par la proposition 2.2.3, il suffit de montrer que, pour  $\mathcal{X}$  un point de l'épine de l'outre-espace,  $|\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})| \leq 2^{n-2}(n-1)!$ . D'après la proposition 2.2.8, il existe un entier  $k$  tel que le noyau du morphisme naturel  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{X}) \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k$ . De ce fait,  $|\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})| \leq 2^k |\text{Aut}_{gr}(X)|$ .

Nous distinguons deux cas, selon le cardinal de  $L$ .

- Supposons que  $|L| \leq n - 1$ . Alors  $\text{Aut}_{gr}(X)$ , qui s'injecte dans  $\text{Bij}(L)$ , s'injecte dans  $\mathfrak{S}_{n-1}$ . Ainsi,

$$|\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})| \leq 2^k |\text{Aut}_{gr}(X)| \leq 2^k (n-1)! \leq 2^{n-2} (n-1)!,$$

où la dernière inégalité découle du lemme 2.3.5.

- Supposons que  $|L| = n$ . Alors tous les sommets ayant des groupes associés non triviaux sont des feuilles. Ainsi,  $k = 0$  par la proposition 2.2.8. Puisque  $\text{Bij}(L)$  est isomorphe à  $\mathfrak{S}_n$ , nous avons

$$|\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})| \leq |\text{Aut}_{gr}(X)| \leq n!.$$

Or puisque  $n \geq 4$ , nous avons  $n \leq 2^{n-2}$ , donc  $n! \leq 2^{n-2} (n-1)!$ , ce qui conclut.

Donc, pour tout sous-groupe fini  $G$  de  $\text{Out}(W_n)$ , l'ordre de  $G$  est au plus  $2^{n-2} (n-1)!$ . Cette borne est atteinte par le groupe  $U_n = \langle [\tau_1], \dots, [\tau_{n-2}], [\sigma_{1,n}] \rangle$ , qui est isomorphe au produit semi-direct  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ .

Soient  $n \geq 5$  et  $G$ ,  $\mathcal{X}$  et  $X$  comme dans l'énoncé de l'assertion (2). Par la proposition 2.2.8, il existe un entier  $k$  tel que le noyau du morphisme naturel  $G \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k \cap G$ . Puisque  $\overline{X}$  possède  $n$  feuilles, par la proposition 2.2.8, l'entier  $k$  est nul. De ce fait, le groupe  $G$  s'injecte dans  $\text{Aut}_{gr}(X)$ , qui s'injecte dans  $\text{Bij}(L)$ . Donc  $|G| \leq n!$ . Or  $2^{n-2} (n-1)! \leq n!$  implique que  $n \leq 4$ . D'où  $|G| < 2^{n-2} (n-1)!$ .

Soient  $n \geq 4$  et  $G$ ,  $\mathcal{X}$  et  $X$  comme dans l'énoncé de (3). Comme  $G$  est de cardinal maximal parmi les sous-groupes finis de  $\text{Out}(W_n)$ , nous avons  $G = \text{Stab}_{\text{Out}(W_n)}(\mathcal{X})$ . Donc, par la proposition 2.2.8, il existe un entier  $k$  tel que le noyau du morphisme naturel  $G \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k$ . Ainsi, puisque  $\text{Aut}_{gr}(X)$  s'injecte dans  $\text{Bij}(L)$  et que ce dernier est isomorphe à  $\mathfrak{S}_{n-1}$ , on voit que  $|G| \leq 2^k (n-1)!$ . Comme  $k \leq n-2$  par le lemme 2.3.5, et puisque  $|G| = 2^{n-2} (n-1)!$ , on a nécessairement  $k = n-2$ . Le lemme 2.3.5 donne alors que  $\overline{X}$  possède exactement  $n$  sommets. De ce fait,  $\overline{X}$  possède  $n-1$  feuilles et  $n$  sommets. Par ailleurs, on voit également que  $\text{Aut}_{gr}(X)$  est isomorphe à  $\text{Bij}(L)$ . De ce fait, toutes les arêtes de  $\overline{X}$  ont la même longueur. Donc  $\mathcal{X}$  est une  $F$ -étoile.

Supposons enfin que  $n \geq 5$  et que  $G$  soit un sous-groupe de  $\text{Out}(W_n)$  isomorphe à  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ . Par la proposition 2.2.3, le groupe  $G$  fixe un point  $\mathcal{X}$  de l'épine de l'outre-espace. Comme  $G$  est de cardinal maximal parmi les sous-groupes finis de  $\text{Out}(W_n)$ , nous avons  $G = \text{Stab}_{\text{Out}(W_n)}(\mathcal{X})$ . Donc, par la proposition 2.2.8, il existe un entier  $k$  tel que le noyau du morphisme naturel  $G \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k$ .

**Affirmation.** L'arbre  $\overline{X}$  possède exactement  $n-1$  feuilles.

**Démonstration.** L'assertion (2) dit que  $\overline{X}$  possède au plus  $n-1$  feuilles. Nous avons

$$|G| = 2^{n-2} (n-1)! \leq 2^k |\text{Aut}_{gr}(X)| \leq 2^{n-2} |\text{Aut}_{gr}(X)|;$$



où la dernière égalité provient du lemme 2.3.5. Donc  $|\text{Aut}_{gr}(X)| \geq (n-1)!$ . Ainsi, puisque  $\overline{X}$  possède au plus  $n-1$  feuilles, le groupe  $\text{Bij}(L)$ , dans lequel s'injecte  $\text{Aut}_{gr}(X)$ , est isomorphe à  $\mathfrak{S}_{n-1}$ . Donc le cardinal de  $L$  est  $n-1$ .  $\square$

De ce fait,  $\overline{X}$  possède  $n-1$  feuilles. Par l'assertion (3),  $\mathcal{X}$  est une  $F$ -étoile dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ . Par le corollaire 2.2.6, l'ensemble des points fixes de  $G$  est connexe. Puisque l'ensemble des  $F$ -étoiles est discret dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ , on conclut que  $G$  fixe une unique  $F$ -étoile dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ .  $\square$

**Remarque 2.3.8.** Dans le cas de  $\text{Aut}(W_n)$ , soient  $G$  un sous groupe fini de  $\text{Aut}(W_n)$  et  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  fixé par  $G$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe pointé sous-jacent à  $X$ .

(1) *Si  $n \geq 4$ , le cardinal de  $G$  est plus petit que  $2^{n-1}(n-1)!$ .*

La démonstration pour le cas où le nombre de feuilles de  $\overline{X}$  est plus petit que  $n-1$  est identique à celle de la proposition 2.3.7 (1) en utilisant cette fois la remarque 2.3.6. Dans le cas où le nombre de feuilles est égal à  $n$ , le noyau du morphisme naturel  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal plus petit que 2 par la remarque 2.2.9, donc  $|G| \leq 2n! \leq 2^{n-1}(n-1)!$  car  $n \geq 4$ .

(2) *Si  $n \geq 5$  et si  $\overline{X}$  possède  $n$  feuilles, alors  $|G| < 2^{n-1}(n-1)!$ .*

En effet, par la remarque 2.2.9, le cardinal du noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est plus petit que 2, donc  $|G| \leq 2n! < 2^{n-1}(n-1)!$  car  $n \geq 5$ .

(3) *Si  $n \geq 4$ , si  $G$  est isomorphe à  $F^{n-1} \rtimes \mathfrak{S}_{n-1}$  et si  $\overline{X}$  possède au plus  $n-1$  feuilles, alors  $\mathcal{X}$  est une  $F$ -étoile.*

En effet, une démonstration identique à celle de la proposition 2.3.7 (3) montre que  $\overline{X}$  possède  $n-1$  feuilles et  $n$  sommets. Montrons alors que le point base est le centre de  $X$ . Ceci découle du fait que le groupe des automorphismes de  $\overline{X}$  est isomorphe à  $\mathfrak{S}_{n-1}$  car le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est isomorphe à  $F^{n-1}$  et que  $G$  est isomorphe à  $F^{n-1} \rtimes \mathfrak{S}_{n-1}$ .

(4) *Si  $n \geq 4$  et si  $G$  est isomorphe à  $F^{n-1} \rtimes \mathfrak{S}_{n-1}$ , tout point de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  fixé par  $G$  est une  $F$ -étoile.*

En effet, l'existence d'une  $F$ -étoile fixée par  $G$  lorsque  $n \geq 5$  se déduit des faits précédents.

Dans le cas où  $n = 4$ , soit  $\mathcal{X}$  un point de l'épine de l'outre-espace fixé par  $G$ . Soient  $X$  un représentant de  $\mathcal{X}$  et  $\overline{X}$  le graphe sous-jacent à  $X$ . On note  $L$  l'ensemble des feuilles de  $\overline{X}$ . Si  $\overline{X}$  possède au plus  $n-1$  feuilles, alors, par le fait précédent,  $\mathcal{X}$  est une  $F$ -étoile. Supposons que  $\overline{X}$  possède exactement  $n$  feuilles. Alors la remarque 2.2.9 montre que le noyau du morphisme naturel  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal au plus 2. Il ne peut pas être injectif car le cardinal de  $G$  est égal à 48 alors que le groupe  $\text{Aut}_{gr}(X)$  s'injecte dans  $\text{Bij}(L)$  de cardinal égal à 24. Donc le noyau du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal égal à 2. Ainsi, le point base de  $\overline{X}$  est une feuille. Or, puisque  $\text{Aut}_{gr}(X)$  s'injecte dans  $\text{Bij}(L)$  et que l'image du morphisme  $G \rightarrow \text{Aut}_{gr}(X)$  est de cardinal égal à 24, on voit que  $\text{Aut}_{gr}(X)$  est isomorphe à  $\text{Bij}(L)$ . Ceci contredit le fait que le point base de  $\overline{X}$  est une feuille. En conclusion,  $\overline{X}$  possède au plus  $n-1$  feuilles. Donc  $\mathcal{X}$  est

une  $F$ -étoile. La démonstration de l'unicité de la  $F$ -étoile fixée par  $G$  est alors identique à celle de la démonstration de la proposition 2.3.7 (4).

**Lemme 2.3.9.** *Soit  $n$  un entier.*

- (1) *Supposons que  $n \geq 5$ . Soit  $G$  un sous-groupe de  $\mathfrak{S}_n$  isomorphe à  $\mathfrak{S}_{n-1}$ . Il existe un automorphisme de  $\mathfrak{S}_n$  envoyant  $G$  sur  $\{f \in \text{Bij}(\{1, \dots, n\}) : f(n) = n\}$ .*
- (2) *Si  $n \geq 4$  et  $n \neq 6$  et si  $G$  est un sous-groupe de  $\text{Bij}(\{1, \dots, n\})$  isomorphe à  $\mathfrak{S}_{n-1}$ , alors il existe un entier  $i \in \{1, \dots, n\}$  tel que  $G = \{f \in \text{Bij}(\{1, \dots, n\}) : f(i) = i\}$ .*

**Démonstration.** (1) L'action de  $\mathfrak{S}_n$  sur  $\mathfrak{S}_n/G$  par multiplication à gauche est un morphisme de groupes  $\phi: \mathfrak{S}_n \rightarrow \text{Bij}(\mathfrak{S}_n/G)$ . Le noyau de ce morphisme est un sous-groupe distingué de  $\mathfrak{S}_n$  inclus dans  $G$ . Or,  $G$  est d'indice  $n$ . Donc, étant donné que  $n \geq 5$ , le noyau de ce morphisme est trivial. Donc, puisque les groupes  $\mathfrak{S}_n$  et  $\text{Bij}(\mathfrak{S}_n/G)$  ont même cardinal fini, le morphisme  $\phi$  est un isomorphisme. Soit  $\tilde{\psi}: \mathfrak{S}_n/G \rightarrow \{1, \dots, n\}$  une bijection envoyant  $\{G\}$  sur  $n$ , et  $\psi: \text{Bij}(\mathfrak{S}_n/G) \rightarrow \mathfrak{S}_n$  l'isomorphisme induit par  $\tilde{\psi}$ . Alors  $\psi \circ \phi$  est un automorphisme de  $\mathfrak{S}_n$  envoyant  $G$  sur le sous-groupe de  $\mathfrak{S}_n$  fixant  $n$ .

(2) Nous commençons par traiter le cas où  $n = 4$ . Il découle d'une inspection des sous-groupes de  $\mathfrak{S}_4$  isomorphes à  $\mathfrak{S}_3$ . En effet,  $\mathfrak{S}_4$  possède exactement 4 sous-groupes isomorphes à  $\mathfrak{S}_3$ . Donc, il existe un entier  $i \in \{1, 2, 3, 4\}$  tel que  $G = \{f \in \text{Bij}(\{1, \dots, n\}) : f(i) = i\}$ .

Supposons maintenant que  $n \geq 5$  et que  $n \neq 6$ . Par le premier point du lemme, il existe un automorphisme  $\phi$  de  $\mathfrak{S}_n$  envoyant  $G$  sur  $\{f \in \text{Bij}(\{1, \dots, n\}) : f(n) = n\}$ . Or, si  $n \neq 6$ , tout automorphisme de  $\mathfrak{S}_n$  est intérieur. Comme les automorphismes intérieurs préservent le fait d'être le stabilisateur d'un entier, il existe un entier  $i \in \{1, \dots, n\}$  tel que  $G = \{f \in \text{Bij}(\{1, \dots, n\}) : f(i) = i\}$ .  $\square$

Étudions les points fixes du groupe  $B_n$  dans l'épine de l'outre-espace de  $W_n$ .

**Proposition 2.3.10.** *Soient  $n \geq 4$  et  $B_n = \langle [\tau_1], \dots, [\tau_{n-2}] \rangle$ .*

- (1) *Les seuls sommets fixés par  $B_n$  dans l'épine de l'outre-espace de  $W_n$  sont des  $\{0\}$ -étoiles et des  $F$ -étoiles.*
- (2) *Le groupe  $B_n$  fixe une unique  $F$ -étoile et une unique  $\{0\}$ -étoile.*

**Remarque.** La proposition 2.3.10 diffère des propositions 2.3.3 et 2.3.7 car elle porte uniquement sur un sous-groupe *particulier* de  $\text{Out}(W_n)$ . Nous ne savons pas si le résultat reste vrai pour un sous-groupe de  $\text{Out}(W_n)$  isomorphe à  $\mathfrak{S}_{n-1}$  quelconque.

**Démonstration.** (1) Soient  $\mathcal{X}$  un sommet de l'épine de  $\mathbb{PO}(W_n)$  fixé par  $B_n$  et  $X$  un représentant de  $\mathcal{X}$ . Soient  $\overline{X}$  le graphe sous-jacent à  $X$ ,  $L$  l'ensemble des feuilles de  $\overline{X}$  et  $v_1, \dots, v_n$  les sommets de  $\overline{X}$  dont les groupes associés sont non triviaux. Par la proposition 2.2.8, il existe un entier  $k$  tel que le noyau du morphisme naturel  $B_n \rightarrow \text{Aut}_{gr}(X)$  soit isomorphe à  $F^k \cap B_n$ . Or, ce noyau est un sous-groupe de  $F^k$ , et ce dernier est

engendré par des twists. Pour tout  $i \in \{1, \dots, n\}$ , soit  $y_i$  l'antécédent par le marquage de  $X$  du générateur du groupe associé à  $v_i$ . Pour tout  $i \in \{1, \dots, n\}$ , les compositions de twists contenues dans  $F^k \cap B_n$  préservent la classe de conjugaison dans  $W_n$  de  $y_i$  alors que les permutations du groupe engendré par  $\{[\tau_1], \dots, [\tau_{n-2}]\}$  ne préservent pas ces dernières. De ce fait, nous avons  $F^k \cap B_n = \{1\}$ .

Le groupe  $\text{Aut}_{gr}(X)$  s'injecte dans  $\text{Bij}(L)$ . Par ailleurs, étant donné que le morphisme  $\phi: B_n \rightarrow \text{Aut}_{gr}(X)$  est injectif, et que  $B_n$  est isomorphe à  $\mathfrak{S}_{n-1}$ , nous avons  $|L| \geq n - 1$ . De plus, chaque feuille ayant un groupe associé non trivial, nous avons  $|L| \leq n$ . Donc  $L \in \{n - 1, n\}$ . Examinons les deux cas possibles.

Si  $|L| = n - 1$ , alors  $\text{Aut}_{gr}(X)$  est isomorphe à  $\text{Bij}(L)$ . Montrons que  $\mathcal{X}$  est une  $F$ -étoile. Soit  $v$  un sommet qui n'est pas une feuille à distance maximale du centre de  $\overline{X}$ . L'hypothèse de maximalité sur  $v$  implique qu'il y a au plus un sommet adjacent à  $v$  qui n'est pas une feuille, car sinon nous pourrions trouver un sommet  $w$  adjacent à  $v$  qui ne serait pas une feuille et qui serait à distance strictement plus grande du centre que  $v$ . De ce fait,  $v$  est adjacent à au moins  $\deg(v) - 1$  feuilles.

Si le groupe associé à  $v$  est non trivial, alors  $v$  est fixé par  $B_n$  car c'est le seul sommet de  $\overline{X}$  qui soit de groupe associé non trivial et qui ne soit pas une feuille. Donc puisque  $B_n$  est isomorphe à  $\text{Aut}_{gr}(X)$ , le sommet  $v$  est fixé par  $\text{Aut}_{gr}(X)$ . Enfin, puisque tout élément de  $\text{Bij}(L)$  est induit par un élément de  $\text{Aut}_{gr}(X)$ , le sommet  $v$  est adjacent à toutes les feuilles et  $\mathcal{X}$  est une  $F$ -étoile.

Si  $v$  est un sommet de groupe trivial, alors, par hypothèse,  $\deg(v) \geq 3$ . De ce fait,  $v$  est adjacent à au moins deux feuilles, notées  $v_1$  et  $v_2$ . Soit  $w$  une feuille de  $\overline{X}$  distincte de  $v_1$  et  $v_2$ . Puisqu'il existe un automorphisme de  $\overline{X}$  envoyant  $v_1$  sur  $w$  et fixant  $v_2$ , alors, nécessairement,  $w$  est adjacent à  $v$ . Donc  $v$  est adjacent à toutes les feuilles. Ceci n'est pas possible car alors  $X$  contiendrait uniquement  $n - 1$  sommets de groupe associé non trivial. Donc  $v$  est nécessairement un sommet de groupe associé non trivial et  $\mathcal{X}$  est une  $F$ -étoile.

Supposons que  $|L| = n$ . Montrons alors que  $\mathcal{X}$  est une  $\{0\}$ -étoile. Le groupe  $\text{Aut}_{gr}(X)$  s'injecte dans  $\text{Bij}(L)$  qui est isomorphe à  $\mathfrak{S}_n$ . Par ailleurs, puisque  $B_n$  s'injecte dans  $\text{Aut}_{gr}(X)$ , l'image de  $\text{Aut}_{gr}(X)$  dans  $\text{Bij}(L)$  contient un sous-groupe de  $\text{Bij}(L)$  isomorphe à  $\mathfrak{S}_{n-1}$ .

Soit  $H$  l'image de  $B_n$  dans  $\text{Aut}_{gr}(X)$ . Par le lemme 2.3.9 (2), si  $n \neq 6$ , il existe une feuille  $v_1$  de  $\overline{X}$  telle que l'image de  $H$  dans  $\text{Bij}(L)$  soit égale à  $\text{Stab}_{\text{Bij}(L)}(v_1)$ . Soit  $v$  le sommet adjacent à  $v_1$ . Puisque  $v$  n'est pas une feuille,  $\deg(v) \geq 3$ . Ou bien  $v$  est adjacent à une autre feuille distincte de  $v_1$ , ou bien  $v$  est adjacent à une unique feuille.

Si  $v$  est adjacent à une unique feuille, il existe dans  $\overline{X}$  des feuilles de  $L - \{v_1\}$  à distance au moins 4. Soient  $w_1$  et  $w_2$  deux telles feuilles distinctes de  $v_1$ , telles que  $w_1$  soit à distance maximale du centre et que  $w_2$  soit une feuille distincte de  $v_1$  à distance maximale de  $w_1$ . Puisque la valence de tout sommet de groupe associé trivial est au moins 3, il existe une feuille  $w_3$  à distance 2 de  $w_2$ . Or l'image de  $H$  dans  $\text{Bij}(L)$  est égale à  $\text{Stab}_{\text{Bij}(L)}(v_1)$ . Donc il existe un automorphisme de  $\overline{X}$  fixant  $w_3$  et envoyant  $w_2$  sur  $w_1$ , ce qui n'est pas possible par hypothèse sur  $w_1$  et  $w_2$ .

Donc  $v$  est adjacent à une feuille distincte de  $v_1$ , que l'on note  $v_2$ . Soit  $w$  une feuille de  $\overline{X}$  distincte de  $v_1$  et  $v_2$ . Étant donné qu'il existe un automorphisme de  $\overline{X}$  envoyant  $v_2$  sur  $w$  et fixant  $v_1$ , le sommet  $w$  est à distance 2 de  $v_2$ . En particulier,  $\mathcal{X}$  est une  $\{0\}$ -étoile.

Traisons maintenant le cas où  $n = 6$ . On numérote de 1 à 6 les feuilles. Une construction explicite d'un représentant de l'unique automorphisme extérieur non trivial de  $\mathfrak{S}_6$  (cf. [Mil]) donne que l'unique (à conjugaison près) sous-groupe de  $\text{Bij}(L)$  isomorphe à  $\mathfrak{S}_5$  et qui ne soit pas un stabilisateur de feuille est le groupe

$$H = \langle (12)(34)(56), (16)(24)(35), (14)(23)(56), (16)(25)(34) \rangle.$$

Supposons alors que  $H$  soit inclus dans l'image de  $\text{Aut}_{gr}(X)$  dans  $\text{Bij}(L)$ . Le groupe  $H$  agit transitivement sur les feuilles de  $\overline{X}$ . De ce fait, tous les sommets reliés à des feuilles sont adjacents à un même nombre  $k$  de feuilles. Les seules valeurs possibles pour  $k$  sont  $k \in \{1, 2, 3, 6\}$ . Le cas où  $k = 1$  n'est pas possible car tout sommet qui n'est pas une feuille est de degré au moins 3 (tous les sommets dont les groupes associés sont non triviaux sont des feuilles). De plus,  $k \neq 3$  car le groupe des automorphismes d'un tel graphe ne pourrait contenir simultanément les permutations  $(12)(34)(56)$ ,  $(16)(24)(35)$  et  $(14)(23)(56)$ . Enfin,  $k \neq 2$  car alors  $\overline{X}$  posséderait 3 sommets adjacents à 2 feuilles. Cependant le groupe des automorphismes d'un tel graphe ne pourrait contenir simultanément les permutations  $(12)(34)(56)$ ,  $(16)(24)(35)$  et  $(16)(25)(34)$ . Donc  $k = 6$  et  $X$  est une  $\{0\}$ -étoile.

Ainsi,  $B_n$  fixe uniquement des  $\{0\}$ -étoiles et des  $F$ -étoiles.

(2) Montrons maintenant que  $B_n$  fixe une unique  $F$ -étoile. Soit  $X$  le graphe de groupes marqué dont le graphe sous-jacent possède  $n$  sommets, notés  $v_1, \dots, v_n$ , tel que les feuilles du graphe sous-jacent soient  $v_1, \dots, v_{n-1}$ , et tel que pour tout  $i \in \{1, \dots, n\}$ , l'image réciproque par le marquage du générateur du groupe associé à  $v_i$  soit  $x_i$ . Soit  $\mathcal{X}$  la classe d'équivalence de  $X$ . Alors  $\mathcal{X}$  est une  $F$ -étoile et le stabilisateur de  $\mathcal{X}$  est  $U_n$ . Puisque  $B_n \subseteq U_n$ , ceci montre l'existence.

Montrons maintenant l'unicité. Soit  $\mathcal{Y}$  une autre  $F$ -étoile fixée par  $B_n$ . On note  $Y$  un représentant de  $\mathcal{Y}$ . Par le corollaire 2.2.6, il existe dans  $\text{Fix}_{K_n}(B_n)$  un chemin continu de  $\mathcal{X}$  vers  $\mathcal{Y}$ . Puisque deux  $F$ -étoiles distinctes ne sont pas reliées par une arête dans l'épine de  $\mathbb{PO}(W_n)$ , et puisque tout sommet de  $\text{Fix}_{K_n}(B_n)$  est une  $\{0\}$ -étoile ou une  $F$ -étoile, ce chemin passe par une  $\{0\}$ -étoile adjacente à  $\mathcal{X}$ .

**Affirmation.** Soient  $\mathcal{Z}$  une  $\{0\}$ -étoile adjacente à  $\mathcal{X}$  et  $Z$  un représentant de  $\mathcal{Z}$ . On note  $\overline{Z}$  le graphe sous-jacent à  $Z$  et  $v_1, \dots, v_n$  les sommets de  $\overline{Z}$  dont les groupes associés sont non triviaux. Alors l'image réciproque par le marquage de  $Z$  des générateurs des groupes associés aux sommets  $v_1, \dots, v_n$  est, à conjugaison près :

$$\{x_n^{\alpha_1} x_1 x_n^{\alpha_1}, \dots, x_n^{\alpha_{n-1}} x_{n-1} x_n^{\alpha_{n-1}}, x_n\},$$

avec  $\alpha_i \in \{0, 1\}$  pour tout  $i \in \{1, \dots, n-1\}$ .

**Démonstration.** Pour tout  $i \in \{1, \dots, n\}$ , soit  $y_i$  le générateur du groupe associé à  $v_i$ . Puisque  $\mathcal{Z}$  est adjacente à  $\mathcal{X}$ , il existe une arête  $e$  de  $\overline{Z}$  telle que le graphe de groupes marqué  $Z'$  dont le graphe sous-jacent est obtenu à partir de  $\overline{Z}$  en contractant  $e$  soit dans la classe  $\mathcal{X}$ . Quitte à renuméroter, on peut supposer que l'un des sommets de  $e$  est  $v_n$ . Soient  $T_X$  et  $T_{Z'}$  les arbres de Bass-Serre associés à  $X$  et  $Z'$ . Les graphes de groupes  $X$  et  $Z'$  étant équivalents, il existe un homéomorphisme  $W_n$ -équivariant  $f: T_X \rightarrow T_{Z'}$ . Soit  $v$  le sommet de  $T_X$  de stabilisateur  $\langle x_n \rangle$ . Alors  $f(v)$  a pour stabilisateur  $\langle x_n \rangle$ . Par ailleurs, étant donné que les sommets adjacents à  $v$  ont pour stabilisateurs  $\langle x_1 \rangle, \dots, \langle x_{n-1} \rangle, \langle x_n x_1 x_n \rangle, \dots, \langle x_n x_{n-1} x_n \rangle$ , les sommets adjacents à  $f(v)$  ont pour stabilisateurs  $\langle x_1 \rangle, \dots, \langle x_{n-1} \rangle, \langle x_n x_1 x_n \rangle, \dots, \langle x_n x_{n-1} x_n \rangle$ . Donc, tout sous-graphe fini et connexe de  $T_{Z'}$  ayant  $n$  sommets et  $n - 1$  feuilles et de centre  $f(v)$  est tel que les stabilisateurs des feuilles sont

$$\langle x_n^{\alpha_1} x_1 x_n^{\alpha_1} \rangle, \dots, \langle x_n^{\alpha_{n-1}} x_{n-1} x_n^{\alpha_{n-1}} \rangle,$$

avec  $\alpha_i \in \{0, 1\}$  pour tout  $i \in \{1, \dots, n-1\}$ . Ainsi, l'image réciproque par le marquage de  $Z$  des générateurs des groupes associés aux sommets  $v_1, \dots, v_n$  est, à conjugaison près :

$$\langle x_n^{\alpha_1} x_1 x_n^{\alpha_1} \rangle, \dots, \langle x_n^{\alpha_{n-1}} x_{n-1} x_n^{\alpha_{n-1}} \rangle,$$

avec  $\alpha_i \in \{0, 1\}$  pour tout  $i \in \{1, \dots, n-1\}$ . □

Ainsi, au vu de la description des  $\{0\}$ -étoiles adjacentes à  $\mathcal{X}$ , le groupe  $B_n$  fixe une unique  $\{0\}$ -étoile adjacente à  $\mathcal{X}$  : la  $\{0\}$ -étoile  $Z$  telle que les antécédents par le marquage des générateurs des groupes de sommets non triviaux soient, à conjugaison près,  $x_1, \dots, x_n$ . On note  $\mathcal{Z}$  la classe d'équivalence de  $Z$  et  $\overline{Z}$  le graphe sous-jacent à  $Z$ .

Soit  $\mathcal{Y}'$  une  $F$ -étoile adjacente à  $\mathcal{Z}$ . Notons  $Y'$  un représentant de  $\mathcal{Y}'$  et  $\overline{Y}'$  le graphe sous-jacent à  $Y'$ . Il existe une arête  $e$  de  $\overline{Z}$  telle que le graphe de groupes  $Z'$  obtenu en contractant  $e$  soit dans  $\mathcal{Y}'$ . Les antécédents par le marquage de  $Y'$  des générateurs des groupes de sommets sont donc, à conjugaison près,  $x_1, \dots, x_n$ .

Ainsi, puisque  $B_n$  permute les sommets de tout point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  dont l'image réciproque par le marquage des groupes associés sont  $\langle x_1 \rangle, \dots, \langle x_{n-1} \rangle$ , on voit que l'unique  $F$ -étoile adjacente à  $\mathcal{Z}$  fixée par  $B_n$  est  $\mathcal{X}$ . Donc,  $B_n$  fixe une unique  $F$ -étoile dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ .

Montrons enfin que  $B_n$  fixe une unique  $\{0\}$ -étoile. Soit  $Z$  le graphe de groupes marqué dont le graphe sous-jacent possède  $n + 1$  sommets,  $n$  feuilles, notées  $w_1, \dots, w_n$ , et tel que pour tout  $i \in \{1, \dots, n\}$ , l'image réciproque par le marquage du générateur du groupe associé à  $w_i$  soit  $x_i$ . Soit  $\mathcal{Z}$  la classe d'équivalence de  $Z$ . Alors  $\mathcal{Z}$  est une  $\{0\}$ -étoile et le stabilisateur de  $\mathcal{Z}$  est  $A_n$ . Puisque  $B_n \subseteq A_n$ , ceci montre l'existence.

Montrons l'unicité. Soit  $\mathcal{Y}$  une autre  $\{0\}$ -étoile fixée par  $B_n$ . Par le corollaire 2.2.6, il existe un chemin continu dans  $\text{Fix}_{K_n}(B_n)$  de  $\mathcal{Z}$  vers  $\mathcal{Y}$ . Au vu de l'assertion (1) de la proposition, ce chemin passe uniquement par des  $\{0\}$ -étoiles et des  $F$ -étoiles. Or,  $B_n$

fixe une unique  $F$ -étoile  $\mathcal{X}$ , et par la dernière affirmation, l'unique  $\{0\}$ -étoile adjacente à  $\mathcal{X}$  et fixée par  $B_n$  est  $\mathcal{Z}$ . Donc  $B_n$  fixe une unique  $\{0\}$ -étoile dans l'épine de  $\mathbb{P}\mathcal{O}(W_n)$ .  $\square$

**Remarque 2.3.11.** Soit  $n \geq 4$ . Dans le cas de  $\text{Aut}(W_n)$ , soit  $\tilde{B}_n = \langle \tau_1, \dots, \tau_{n-2} \rangle$ , qui est encore isomorphe à  $\mathfrak{S}_{n-1}$ . Soit  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  fixé par  $\tilde{B}_n$ . On note  $X$  un représentant de  $\mathcal{X}$  et  $\bar{X}$  le graphe sous-jacent à  $X$ .

(1) *Soit  $\mathcal{X}$  est une  $F$ -étoile, soit  $\bar{X}$  possède  $n$  feuilles et  $n + 1$  sommets.*

En effet, une démonstration identique à celle de la proposition 2.3.10 (1) montre que le morphisme  $\tilde{B}_n \rightarrow \text{Aut}_{gr}(X)$  est injectif, et que le nombre de feuilles de  $\bar{X}$  est soit égal à  $n - 1$ , soit égal à  $n$ . S'il est égal à  $n - 1$ , une démonstration identique à celle de la proposition 2.3.10 (1) montre que  $\bar{X}$  possède  $n$  sommets et  $n - 1$  feuilles. Comme le groupe  $\text{Aut}_{gr}(X)$  contient un sous-groupe isomorphe à  $\mathfrak{S}_{n-1}$  et que  $\bar{X}$  possède  $n - 1$  feuilles, on voit que, nécessairement, le point base de  $X$  est son centre. Donc  $\mathcal{X}$  est une  $F$ -étoile. Si le nombre de feuilles de  $\bar{X}$  est égal à  $n$ , une démonstration identique à celle de la proposition 2.3.10 (1) montre que  $\bar{X}$  possède  $n + 1$  sommets et  $n$  feuilles.

(2) *Le groupe  $\tilde{B}_n$  fixe une unique  $F$ -étoile.*

En effet, il fixe une  $F$ -étoile car  $\tilde{B}_n$  est un sous-groupe de  $\tilde{U}_n = \langle \tau_1, \dots, \tau_{n-2}, \sigma_{1,n} \rangle$  et ce dernier est isomorphe à  $F^{n-1} \rtimes \mathfrak{S}_{n-1}$ . De ce fait, la remarque 2.3.8 (4) permet de conclure. Nous appellerons  $\mathcal{X}$  l'unique  $F$ -étoile fixée par  $\tilde{U}_n$ .

Pour l'unicité, soit  $\mathcal{Y}$  une autre  $F$ -étoile fixée par  $\tilde{B}_n$ . Puisque l'ensemble des  $F$ -étoiles dans l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  n'est pas connexe, tout chemin continu entre  $\mathcal{X}$  et  $\mathcal{Y}$  et contenu dans l'ensemble des points fixes de  $\tilde{B}_n$  pour l'action de  $\text{Aut}(W_n)$  sur l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  passe par un point  $\mathcal{Z}$  ayant un représentant  $Z$  de graphe sous-jacent possédant  $n$  feuilles et  $n + 1$  sommets. Soit  $\bar{Z}$  le graphe sous-jacent à  $Z$ , et  $v_1, \dots, v_n$  les feuilles de  $\bar{Z}$ . Une démonstration identique à celle de la première affirmation de la démonstration de la proposition 2.3.10 (2) montre que l'image réciproque par le marquage de  $Z$  des générateurs des groupes associés aux sommets  $v_1, \dots, v_n$  est respectivement ou bien  $x_1, \dots, x_{n-1}, x_n$  ou bien  $x_n x_1 x_n, \dots, x_n x_{n-1} x_n, x_n$ . De plus, la description de  $\tilde{B}_n$  montre que le point base de  $Z$  est contenu dans l'arête reliant le centre de  $\bar{Z}$  et  $v_n$ .

Soit maintenant  $\mathcal{Z}'$  un sommet de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  fixé par  $\tilde{B}_n$ , adjacent à  $\mathcal{Z}$  et qui n'est pas une  $F$ -étoile. Puisque  $\mathcal{Z}'$  possède  $n$  feuilles et  $n + 1$  sommets par le premier point de la remarque, un représentant  $Z'$  de  $\mathcal{Z}'$  est obtenu à partir de  $Z$  en déplaçant le point base dans l'arête reliant le centre de  $\bar{Z}$  et  $v_n$ . De ce fait, l'image réciproque par le marquage des générateurs des groupes associés aux feuilles de  $\bar{Z}'$  sont les mêmes que pour  $\mathcal{Z}$ .

Donc, pour conclure sur l'unicité de la  $F$ -étoile fixée par  $\tilde{B}_n$ , il suffit d'étudier les  $F$ -étoiles fixées par  $\tilde{B}_n$  est adjacente à  $\mathcal{Z}$ . Soit  $\mathcal{Y}'$  une  $F$ -étoile adjacente à  $\mathcal{Z}$ . Notons  $Y'$  un représentant de  $\mathcal{Y}'$  et  $\bar{Y}'$  le graphe sous-jacent à  $Y'$ . Il existe une arête  $e$  de  $\bar{Z}$  telle que le graphe de groupes  $Z'$  obtenu en contractant  $e$  soit dans  $\mathcal{Y}'$ . Les antécédents par le marquage de  $Y'$  des générateurs des groupes de sommets sont donc, à conjugaison près,  $x_1, \dots, x_n$ . Ainsi, puisque  $\tilde{B}_n$  permute transitivement les sommets de tout point de l'épine de  $\mathbb{P}\mathcal{A}(W_n)$  dont l'image réciproque par le marquage des groupes associés sont

$\langle x_1 \rangle, \dots, \langle x_{n-1} \rangle$ , on voit que l'unique  $F$ -étoile adjacente à  $\mathcal{Z}$  fixée par  $\tilde{B}_n$  est  $\mathcal{X}$ . Donc,  $\tilde{B}_n$  fixe une unique  $F$ -étoile dans l'épine de  $\mathbb{P}\mathcal{A}(W_n)$ .

## 2.4 Rigidité des automorphismes extérieurs d'un groupe de Coxeter universel

Le but de cette partie est de démontrer le théorème 2.1.1. Nous distinguons différents cas, selon la valeur de  $n$ . Soit  $\alpha \in \text{Aut}(\text{Out}(W_n))$ .

### 2.4.1 Démonstration dans le cas $n \geq 5$ et $n \neq 6$

Soit  $\mathcal{X}_1$  la  $\{0\}$ -étoile fixée par le sous-groupe fini  $A_n$  de  $\text{Out}(W_n)$  (l'unicité provient de la proposition 2.3.3). Alors, d'après la proposition 2.3.3,  $\alpha(A_n)$  est le stabilisateur d'une unique  $\{0\}$ -étoile  $\mathcal{X}_2$ . Or  $\text{Out}(W_n)$  agit transitivement sur l'ensemble des  $\{0\}$ -étoiles, donc il existe  $\psi \in \text{Out}(W_n)$  tel que  $\psi(\mathcal{X}_1) = \mathcal{X}_2$ . Posons  $\alpha_0 = \text{ad}(\psi) \circ \alpha$ , alors  $\alpha_0(A_n) = \text{ad}(\psi) \circ \alpha(A_n) = A_n$ .

Puisque  $\alpha_0|_{A_n}$  est un automorphisme de  $A_n$ , que  $A_n$  est isomorphe à  $\mathfrak{S}_n$  et que, pour  $n \neq 6$ , le groupe  $\text{Out}(\mathfrak{S}_n)$  est trivial, quitte à changer  $\alpha_0$  dans sa classe d'automorphisme extérieur, on peut supposer que  $\alpha_0|_{A_n} = \text{id}_{A_n}$ .

Maintenant, étant donné que  $B_n \subseteq U_n$ , nous avons  $\alpha_0(B_n) = B_n \subseteq \alpha_0(U_n)$ . Or par la proposition 2.3.10 (2),  $B_n$  fixe une unique  $F$ -étoile. Par ailleurs, le stabilisateur de cette  $F$ -étoile est  $U_n$ . Donc, puisque  $\alpha_0(U_n)$  est également le stabilisateur d'une unique  $F$ -étoile par la proposition 2.3.7 (4), on obtient que  $\alpha_0(U_n) = U_n$ .

Or  $U_n$  est isomorphe au produit semi-direct  $F^{n-2} \rtimes B_n$ , et  $B_n$  agit sur  $F^{n-2}$  (vu comme le quotient de  $F^{n-1}$  par son sous-groupe diagonal  $F$ ) par permutation des facteurs. Soit  $\sigma \in B_n$ . On note  $\text{fix}(\sigma)$  l'ensemble des points fixes de  $\sigma$  agissant par conjugaison dans  $F^{n-2}$ . Puisque, pour tout  $\sigma \in B_n$ ,  $\alpha_0(\sigma) = \sigma$ , on voit que, pour tout  $\sigma \in \{0\} \rtimes B_n$  et pour tout  $g \in F^{n-2} \times \{1\}$ ,  $\alpha_0(\sigma g \sigma^{-1}) = \sigma \alpha_0(g) \sigma^{-1}$ ; en particulier, si  $g \in \text{fix}(\sigma)$ , alors  $\alpha_0(g) \in \text{fix}(\sigma)$ .

Soit maintenant  $\sigma = (2 \dots n-1) \in B_n$ . Alors  $\text{fix}(\sigma) = \{0, [\sigma_{1,n}]\}$ . Donc, puisque  $\alpha_0([\sigma_{1,n}]) \in \text{fix}(\sigma)$ , on a  $\alpha_0([\sigma_{1,n}]) = [\sigma_{1,n}]$ . De même, pour tout  $i \in \{1, \dots, n-1\}$ ,  $\alpha_0([\sigma_{i,n}]) = [\sigma_{i,n}]$ . Ainsi,  $\alpha_0|_{F^{n-2}} = \text{id}_{F^{n-2}}$ . Puisque, par ailleurs,  $\alpha_0$  est l'identité sur  $B_n$ , on voit que  $\alpha_0|_{U_n} = \text{id}_{U_n}$ . De ce fait, étant donné que  $\alpha_0|_{A_n} = \text{id}_{A_n}$  et que  $A_n$  et  $U_n$  engendrent  $\text{Out}(W_n)$  par la proposition 2.2.1, on voit que  $\alpha_0 = \text{id}$  et le résultat s'en déduit.

### 2.4.2 Démonstration dans le cas $n = 6$

Dans le cas où  $n = 6$ , la proposition 2.3.3 s'appliquant encore, soit  $\alpha_0$  un représentant de la classe d'automorphismes extérieurs de  $\alpha$  tel que  $\alpha_0(A_n) = A_n$ . Supposons que la classe d'automorphisme extérieur de  $\alpha_0|_{A_n}$  soit non triviale. Alors une description explicite d'un automorphisme engendrant l'unique classe d'automorphismes extérieurs de  $\mathfrak{S}_6$  (cf. [Mil]) donne, en identifiant  $A_n$  et  $\mathfrak{S}_6$  par l'unique isomorphisme envoyant  $\tau_i$

sur la permutation  $(i \ i + 1)$  pour  $1 \leq i \leq 5$ , que

$$\alpha_0(B_n) = \langle [(12)(34)(56)], [(16)(24)(35)], [(14)(23)(56)], [(16)(25)(34)] \rangle.$$

Ainsi,  $\alpha_0(B_n)$  agit transitivement sur les classes de conjugaison de  $\{x_1, \dots, x_n\}$ . Alors, puisque  $\alpha_0(B_n) \subseteq \alpha_0(U_n)$ , par le quatrième point de la proposition 2.3.7,  $\alpha_0(B_n)$  fixe une  $F$ -étoile  $\mathcal{X}$ . Soit  $X$  un représentant de  $\mathcal{X}$ . Par la proposition 2.2.8, le noyau du morphisme naturel  $\alpha_0(B_n) \rightarrow \text{Aut}_{gr}(X)$  est isomorphe à  $F^{n-2} \cap \alpha_0(B_n)$ .

Or  $F^{n-2} \cap \alpha_0(B_n)$  est un 2-sous-groupe distingué de  $\alpha_0(B_n)$ . Comme  $\alpha_0(B_n)$  est isomorphe à  $\mathfrak{S}_{n-1}$  et que  $n = 6$ , nous avons  $F^{n-2} \cap \alpha_0(B_n) = \{1\}$ . Donc  $\alpha_0(B_n)$  est isomorphe à  $\text{Aut}_{gr}(X)$  car  $\text{Aut}_{gr}(X)$  est isomorphe à  $\mathfrak{S}_{n-1}$ . Soient maintenant  $\bar{X}$  le graphe sous-jacent à  $X$ ,  $v_1, \dots, v_{n-1}$  les feuilles de  $\bar{X}$ , et  $v_n$  le centre de  $\bar{X}$ . Pour  $j \in \{1, \dots, n\}$ , soit  $\langle y_j \rangle$  l'image réciproque par le marquage du groupe associé à  $v_j$ . Le groupe  $\text{Aut}_{gr}(X)$ , et donc  $\alpha_0(B_n)$ , s'identifie à l'ensemble des bijections de  $\{v_1, \dots, v_n\}$  fixant  $v_n$ . Or, par la proposition 2.2.1, il existe  $\pi \in \text{Bij}(\{x_1, \dots, x_n\})$  telle que pour tout  $i \in \{1, \dots, n\}$ , il existe  $z_i \in W_n$  vérifiant :

$$y_i = z_i x_{\pi(i)} z_i^{-1}.$$

Ceci contredit le fait que  $\alpha_0(B_n)$  s'identifie à l'ensemble des bijections de  $\{v_1, \dots, v_n\}$  fixant  $v_n$  car le groupe  $\alpha_0(B_n)$  agit transitivement sur l'ensemble des classes de conjugaison de  $\{x_1, \dots, x_n\}$ . Donc la classe d'automorphisme extérieur de  $\alpha_0|_{A_n}$  est triviale et on conclut comme dans 2.4.1.

### 2.4.3 Démonstration dans le cas $n = 4$

Dans le cas où  $n = 4$ , la proposition 2.3.3 et le quatrième point de la proposition 2.3.7 ne sont plus valables car alors tout sous-groupe de  $\text{Out}(W_n)$  isomorphe à  $\mathfrak{S}_4$  est isomorphe au produit semi-direct  $V \rtimes \mathfrak{S}_3$ , où  $V$  est le groupe de Klein. Nous avons cependant la proposition suivante.

**Proposition 2.4.1.** *Soient  $n = 4$  et  $G$  un sous-groupe de  $\text{Out}(W_n)$  isomorphe au produit semi-direct  $F^{n-2} \rtimes \mathfrak{S}_{n-1}$ . Alors  $G$  est soit le stabilisateur d'une unique  $F$ -étoile, soit le stabilisateur d'une unique  $\{0\}$ -étoile. Les deux cas sont mutuellement exclusifs.*

**Démonstration.** Soient  $\mathcal{X}$  un point de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  fixé par  $G$  (qui existe par la proposition 2.2.3), et  $X$  un représentant de  $\mathcal{X}$ . Soient  $\bar{X}$  le graphe sous-jacent à  $X$  et  $L$  l'ensemble des feuilles de  $\bar{X}$ . La proposition 2.4.1 se démontre de manière identique à la proposition 2.3.7 (3), à ceci près que l'on ne peut pas exclure le cas où  $\bar{X}$  possède  $n$  feuilles. Il faut alors distinguer le cas où  $|L| = n - 1$  et  $|L| = n$ . Si  $\bar{X}$  possède  $n$  feuilles, le lemme 2.3.1 donne que  $\mathcal{X}$  est une  $\{0\}$ -étoile. Si  $\bar{X}$  possède  $n - 1$  feuilles, alors la proposition 2.3.7 (3) donne que  $\mathcal{X}$  est une  $F$ -étoile.

Montrons maintenant que  $G$  ne peut fixer à la fois une  $\{0\}$ -étoile et une  $F$ -étoile. Par la proposition 2.3.7 (1),  $G$  est le stabilisateur de tout point fixé par  $G$ .



Supposons que  $G$  soit le stabilisateur d'une  $\{0\}$ -étoile  $\mathcal{X}$ . Soient  $X$  un représentant de  $\mathcal{X}$ , et  $\bar{X}$  le graphe sous-jacent à  $X$ . Soient  $v_1, \dots, v_n$  les sommets de  $\bar{X}$  dont les groupes associés sont non triviaux et, pour tout  $i \in \{1, \dots, n\}$ , soit  $y_i$  l'image réciproque par le marquage du générateur du groupe associé à  $v_i$ . Alors le groupe  $G$  est le groupe engendré par les permutations de  $\{y_1, \dots, y_n\}$ .

Soit  $\mathcal{Y}$  une  $F$ -étoile dans l'épine de  $\mathbb{PO}(W_n)$  fixée par  $G$ . Par le corollaire 2.2.6,  $\text{Fix}_{K_n}(G)$  est connexe. Il existe donc un chemin continu dans  $\text{Fix}_{K_n}(G)$  de  $\mathcal{X}$  vers  $\mathcal{Y}$ . Les sommets par lesquels passe ce chemin sont uniquement des  $\{0\}$ -étoiles et des  $F$ -étoiles au vu des points stabilisés par  $G$ . Or le groupe engendré par les permutations de  $\{y_1, \dots, y_n\}$  ne fixe aucune  $F$ -étoile adjacente à  $X$ . En effet, le groupe  $G$  contiendrait un élément permutant le centre de la  $F$ -étoile avec une feuille, ce qui n'est pas possible. Donc  $G$  ne fixe aucune  $F$ -étoile.

Enfin, l'unicité du point fixe provient du fait que l'ensemble des  $\{0\}$ -étoiles et l'ensemble des  $F$ -étoiles sont discrets dans l'épine de  $\mathbb{PO}(W_n)$  alors que l'ensemble des points fixes de  $G$  est connexe par le corollaire 2.2.6.  $\square$

Nous pouvons maintenant montrer le théorème 2.1.1 dans le cas  $n = 4$ .

Soit  $\alpha \in \text{Aut}(\text{Out}(W_n))$ . Soit  $\mathcal{X}_1$  la  $\{0\}$ -étoile fixée par le sous-groupe fini  $A_n \simeq \mathfrak{S}_4$  de  $\text{Out}(W_n)$ . Par la proposition 2.4.1,  $\alpha(A_n)$  fixe soit une  $\{0\}$ -étoile, soit une  $F$ -étoile.

Si  $\alpha(A_n)$  fixe une  $\{0\}$ -étoile, alors la même démonstration que pour le cas où  $n \neq 6$  dans la partie 2.4.1 montre que quitte à changer  $\alpha$  dans sa classe d'automorphisme extérieurs, nous avons  $\alpha|_{A_n} = \text{id}_{A_n}$ . Par la proposition 2.4.1, le groupe  $U_n \simeq F^2 \rtimes \mathfrak{S}_3$  fixe soit une  $\{0\}$ -étoile, soit une  $F$ -étoile. Étant donné que  $B_n \subseteq U_n$  fixe une unique  $\{0\}$ -étoile  $\rho$  et une unique  $F$ -étoile  $\rho'$  et que  $\alpha|_{B_n} = \text{id}_{B_n}$ , on voit que  $\alpha(U_n)$  est soit le stabilisateur de  $\rho$ , soit le stabilisateur de  $\rho'$ . Cependant, puisque le stabilisateur de  $\rho$  est  $A_n$  et que  $\alpha|_{A_n} = \text{id}_{A_n}$ , on voit que  $\alpha(U_n)$  est le stabilisateur de  $\rho'$ . Donc  $\alpha(U_n) = U_n$ . Le reste de la démonstration est alors identique à celle du cas où  $n \neq 6$  dans la partie 2.4.1.

Supposons que  $\alpha(A_n)$  fixe une unique  $F$ -étoile. Construisons à présent un représentant de la classe d'automorphismes extérieurs de  $\alpha$ . Puisque  $\text{Out}(W_n)$  agit transitivement sur les  $F$ -étoiles, quitte à changer  $\alpha$  dans sa classe d'automorphismes extérieurs, on peut supposer que  $\alpha(A_n) = U_n$ . Soit  $V$  le groupe de Klein contenu dans  $A_n$ . Alors  $\alpha(V)$  est l'unique 2-sous-groupe distingué non trivial de  $U_n$ . Donc

$$\alpha(V) = \langle [\sigma_{1,4}], [\sigma_{2,4}], [\sigma_{3,4}] \rangle.$$

Ainsi, puisque  $B_n \cap V = \{\text{id}\}$ , on voit que  $\alpha(B_n) \cap \alpha(V) = \{\text{id}\}$ . Par ailleurs,  $A_n = B_n V$ , donc  $U_n = \alpha(B_n) \alpha(V)$ . De ce fait,  $\alpha(B_n)$  est un sous-groupe de  $U_n$  d'ordre 6. Or, il existe une unique classe de conjugaison de sous-groupes d'ordre 6 dans  $U_n$ . Donc, quitte à changer  $\alpha$  dans sa classe d'automorphismes extérieurs, on peut supposer que  $\alpha(B_n) = B_n$ . De même, puisque  $B_n$  est isomorphe à  $\mathfrak{S}_3$ , quitte à changer  $\alpha$  dans sa classe d'automorphisme extérieur, on peut supposer que  $\alpha|_{B_n} = \text{id}_{B_n}$ .

Déterminons à présent l'image de  $[\tau_3]$  et  $[\sigma_{3,4}]$  par  $\alpha$ . Puisque  $[\tau_1][\tau_3] \in V$ , on voit que  $\alpha([\tau_1][\tau_3]) \in \{[\sigma_{1,4}], [\sigma_{2,4}], [\sigma_{3,4}]\}$ . Or,  $[\tau_1]$  commute avec  $[\tau_1][\tau_3]$ , donc  $\alpha([\tau_1][\tau_3])$  doit également commuter avec  $[\tau_1]$ . De ce fait,  $\alpha([\tau_1][\tau_3]) = [\sigma_{3,4}]$  et  $\alpha([\tau_3]) = [\tau_1][\sigma_{3,4}]$ .

Déterminons l'image de  $[\sigma_{3,4}]$  par  $\alpha$ . Puisque  $\alpha(B_n) = B_n$ , le groupe  $\alpha(U_n)$  est le stabilisateur d'un point fixe de  $B_n$ . Par la proposition 2.3.10,  $B_n$  fixe uniquement deux sommets de l'épine de  $\mathbb{P}\mathcal{O}(W_n)$  : la  $\{0\}$ -étoile stabilisée par  $A_n$  et la  $F$ -étoile stabilisée par  $U_n$ . Comme  $\alpha(A_n) = U_n$ , on a nécessairement  $\alpha(U_n) = A_n$ . Donc  $\alpha([\sigma_{3,4}]) \in V$ . Puisque  $[\sigma_{3,4}]$  commute avec  $[\tau_1]$ , on obtient que  $\alpha([\sigma_{3,4}]) = [\tau_1][\tau_3]$ .

Donc  $\alpha$  se restreint en l'identité sur  $B_n$ , envoie  $[\tau_3]$  sur  $[\tau_1][\sigma_{3,4}]$  et  $[\sigma_{3,4}]$  sur  $[\tau_1][\tau_3]$ . Comme  $B_n$ ,  $[\tau_3]$  et  $[\sigma_{3,4}]$  engendrent  $\text{Out}(W_4)$ , ceci montre qu'un tel automorphisme  $\alpha$ , s'il existe, est unique modulo automorphisme intérieur.

Réciproquement, montrons que l'application  $\alpha$  de  $B_n \cup \{[\tau_3], [\sigma_{3,4}]\}$  dans  $\text{Out}(W_4)$  définie par  $\alpha|_{B_n} = \text{id}_{B_n}$ ,  $\alpha([\tau_3]) = [\tau_1][\sigma_{3,4}]$  et  $\alpha([\sigma_{3,4}]) = [\tau_1][\tau_3]$  s'étend de manière unique en un morphisme de groupes de  $\text{Out}(W_4)$ . Comme  $[\tau_1]$  commute avec  $[\tau_3]$  et  $[\sigma_{3,4}]$ , ceci montre que  $\alpha$  est involutif, donc un automorphisme de  $\text{Out}(W_4)$ . Sa classe dans  $\text{Out}(\text{Out}(W_4))$  est non triviale (car son action sur l'épine de  $\mathbb{P}\mathcal{O}(W_4)$  est non triviale), ce qui montre le théorème 2.1.1 lorsque  $n = 4$ .

Pour simplifier les notations, nous notons  $[i j]$  la classe d'automorphismes extérieurs de la transposition permutant  $x_i$  et  $x_j$ . Notons

$$S = \{[i j] \mid 1 \leq i, j \leq 4\} \cup \{[\sigma_{i,j}] \mid 1 \leq i \neq j \leq 4\},$$

qui est une partie génératrice de  $\text{Out}(W_4)$  par la proposition 2.2.1. Un petit calcul élémentaire montre que, si  $i = 1, 2$ , alors

$$[i 4] = [i 3][3 4][i 3], [\sigma_{i,4}] = [i 3][\sigma_{3,4}][i 3],$$

$$\alpha([i 3])\alpha([3 4])\alpha([i 3]) = [j k][\sigma_{i,4}] \text{ et } \alpha([i 3])\alpha([\sigma_{3,4}])\alpha([i 3]) = [j k][i 4],$$

où  $\{j, k\} = \{1, 2, 3\} - \{i\}$ . Considérons l'application  $\tilde{\alpha}$  de  $S$  dans  $\text{Out}(W_4)$  étendant  $\alpha$  sur  $S \cap (B_n \cup \{[3 4], [\sigma_{3,4}]\})$  et telle que, si  $i = 1, 2$ ,

$$\tilde{\alpha}([i 4]) = [j k][\sigma_{i,4}] \text{ et } \tilde{\alpha}([\sigma_{i,4}]) = [j k][i 4],$$

où  $\{j, k\} = \{1, 2, 3\} - \{i\}$ . Des calculs élémentaires pour lesquels nous renvoyons à l'appendice 2.5 montrent que cette application préserve, quand  $n = 4$ , la présentation de  $\text{Out}(W_n)$  donnée par [Gil, Theorem 2.20], ce qui conclut.

#### 2.4.4 Démonstration de la rigidité de $\text{Aut}(W_n)$

Nous démontrons à présent le théorème 2.1.2. Soient  $n \geq 4$  et  $\alpha \in \text{Aut}(\text{Aut}(W_n))$ . Soient  $\tilde{A}_n = \langle \tau_1, \dots, \tau_{n-1} \rangle$ ,  $\tilde{B}_n = \langle \tau_1, \dots, \tau_{n-2} \rangle$  et  $\tilde{U}_n = \langle \tau_1, \dots, \tau_{n-2}, \sigma_{1,n} \rangle$ . En utilisant les remarques 2.3.4, 2.3.8 (4) et 2.3.11 (2), et en effectuant une démonstration identique à celle du théorème 2.1.1 dans les cas où  $n \geq 5$ , on voit que, quitte à changer  $\alpha$  dans sa classe d'automorphismes extérieurs,  $\alpha|_{\tilde{A}_n} = \text{id}_{\tilde{A}_n}$  et que  $\alpha(\tilde{U}_n) = \tilde{U}_n$ .

Or  $\tilde{U}_n$  est isomorphe à  $F^{n-1} \rtimes \tilde{B}_n$ . Soit  $\sigma \in \tilde{B}_n$ . On note  $\text{fix}(\sigma)$  l'ensemble des points fixes de  $\sigma$  agissant par conjugaison dans  $F^{n-1}$ . On voit que pour tout  $\sigma \in \{0\} \rtimes \tilde{B}_n$  et

pour tout  $g \in F^{n-1} \rtimes \{1\}$ ,  $\alpha(\sigma g \sigma^{-1}) = \sigma \alpha(g) \sigma^{-1}$  ; en particulier, si  $g \in \text{fix}(\sigma)$ , alors  $\alpha(g) \in \text{fix}(\sigma)$ .

Soit maintenant  $\sigma = (2 \dots n - 1) \in B_n$ . Alors  $\text{fix}(\sigma) = \{0, \sigma_{1,n}, \prod_{i \neq 1,n} \sigma_{i,n}, \prod_{i=1}^{n-1} \sigma_{i,n}\}$ .

Donc  $\alpha(\sigma_{1,n}) \in \{\sigma_{1,n}, \prod_{i \neq 1,n} \sigma_{i,n}, \prod_{i=1}^{n-1} \sigma_{i,n}\}$ . Comme  $\prod_{i=1}^{n-1} \sigma_{i,n}$  est l'unique élément non trivial

dans le centre de  $\tilde{U}_n$ , on voit que  $\alpha(\sigma_{1,n}) \neq \prod_{i=1}^{n-1} \sigma_{i,n}$ .

Supposons par l'absurde que  $\alpha(\sigma_{1,n}) = \prod_{i \neq 1,n} \sigma_{i,n}$ . Pour  $j \in \{1, \dots, n-1\}$ , notons  $(1 j)$  la transposition de  $\tilde{B}_n$  permutant  $x_1$  et  $x_j$ . Alors, on voit que, pour tout  $j \in \{1, \dots, n-1\}$ ,  $\alpha(\sigma_{j,n}) = \alpha((1 j)\sigma_{1,n}(1 j)) = \prod_{i \neq j,n} \sigma_{i,n}$ .

Un calcul immédiat montre alors que, pour tout  $j \neq k, n$ , et  $k < n$ ,

$$\alpha(\sigma_{k,j}) = \alpha((j n)\sigma_{k,n}(j n)) = \prod_{i \neq j,k} \sigma_{i,j}.$$

Or  $\sigma_{1,2}\sigma_{3,4} = \sigma_{3,4}\sigma_{1,2}$ , alors que

$$\begin{aligned} \alpha(\sigma_{1,2})\alpha(\sigma_{3,4})(x_1) &= \prod_{i \neq 1,2} \sigma_{i,2} \prod_{i \neq 3,4} \sigma_{i,4}(x_1) = x_2 x_4 x_2 x_1 x_2 x_4 x_2 \quad \text{et que} \\ \alpha(\sigma_{3,4})\alpha(\sigma_{1,2})(x_1) &= \prod_{i \neq 3,4} \sigma_{i,4} \prod_{i \neq 1,2} \sigma_{i,2}(x_1) = x_4 x_1 x_4. \end{aligned}$$

Donc  $\alpha(\sigma_{1,2})\alpha(\sigma_{3,4}) \neq \alpha(\sigma_{3,4})\alpha(\sigma_{1,2})$ . Ceci contredit le fait que  $\alpha$  est un morphisme de groupes. Ainsi,  $\alpha(\sigma_{1,n}) = \sigma_{1,n}$ . Par la proposition 2.2.1, nous avons  $\alpha = \text{id}$ . Ceci conclut la démonstration du théorème 2.1.2.

## 2.5 Présentation du groupe $\text{Out}(W_4)$

Soit  $n = 4$ . Pour simplifier les notations, nous notons  $[i j]$  la classe d'automorphismes extérieurs de la transposition permutant  $x_i$  et  $x_j$ . Nous rappelons que l'application ensembliste  $\alpha: B_n \cup \{[\tau_3], [\sigma_{3,4}]\} \rightarrow \text{Out}(W_n)$  est définie par :

$$\begin{aligned} \alpha|_B &= \text{id}_{B_n} ; \\ \alpha([3 4]) &= [1 2][\sigma_{3,4}] ; \\ \alpha([\sigma_{3,4}]) &= [1 2][3 4]. \end{aligned}$$

Nous montrons dans cette appendice que l'application  $\alpha$  s'étend de manière unique en un morphisme de groupes de  $\text{Out}(W_4)$  dans lui-même. Nous montrons pour cela qu'il préserve l'ensemble des relations d'une présentation de  $\text{Out}(W_4)$ . La présentation suivante est due à Gilbert.

**Proposition 2.5.1.** [Gil, Theorem 2.20] Soit  $n$  un entier plus grand que 2. Une présentation de  $\text{Out}(W_n)$  est donnée par :

- (1) la partie génératrice  $S$  constituée de l'ensemble des permutations  $[i j]$  pour les entiers distincts  $i, j \in \{1, \dots, n\}$  ainsi que l'ensemble des éléments  $[\sigma_{i,j}]$  pour les entiers distincts  $i, j \in \{1, \dots, n\}$  ;
- (2) les relations suivantes :
  - (a) pour tout  $i \in \{1, \dots, n\}$ ,  $\prod_{j \neq i} [\sigma_{j,i}] = 1$  ;
  - (b) pour tous les  $i, j, k, \ell \in \{1, \dots, n\}$  avec  $i \neq j$  et  $k \neq \ell$ , si on pose  $\tau = (i j)$ , alors  $[i j][k \ell] = [\tau(k) \tau(\ell)][i j]$  ;
  - (c) pour tout  $j \in \{1, \dots, n\}$ , pour tous les  $i, k \in \{1, \dots, n\} - \{j\}$  distincts,  $[\sigma_{i,j}][\sigma_{k,j}] = [\sigma_{k,j}][\sigma_{i,j}]$  ;
  - (d) pour tous les  $i, j \in \{1, \dots, n\}$  distincts,  $[\sigma_{i,j}][\sigma_{i,j}] = 1$  ;
  - (e) pour tous les  $i, j, k, \ell \in \{1, \dots, n\}$  deux à deux distincts,  $[\sigma_{i,j}][\sigma_{k,\ell}] = [\sigma_{k,\ell}][\sigma_{i,j}]$  ;
  - (f) pour tous les  $i, j, k, \ell \in \{1, \dots, n\}$ , tels que  $k \neq \ell$ , si  $\tau = (i j)$ ,  $[i j][\sigma_{k,\ell}] = [\sigma_{\tau(k),\tau(\ell)}][i j]$  ;
  - (g) pour tous les  $i, j, k \in \{1, \dots, n\}$  deux à deux distincts,  $[\sigma_{j,i}][\sigma_{i,k}][\sigma_{j,k}] = [\sigma_{j,k}][\sigma_{i,k}][\sigma_{j,i}]$ .

Nous remarquons que, dans le cas où  $n = 4$ , la relation (g) se déduit des relations (a), (d) et (e).

**Proposition 2.5.2.** L'application  $\alpha$  se prolonge de manière unique en un morphisme de groupes de  $\text{Out}(W_4)$  dans lui-même.

**Démonstration.** Nous définissons tout d'abord une application prolongeant  $\alpha$  sur la partie génératrice  $S$  de  $\text{Out}(W_4)$  définie dans la proposition 2.5.1. Un petit calcul élémentaire montre que, si  $i = 1, 2$ ,  $[i 4] = [i 3][3 4][i 3]$ ,  $[\sigma_{i,4}] = [i 3][\sigma_{3,4}][i 3]$ ,  $\alpha([i 3])\alpha([3 4])\alpha([i 3]) = [j k][\sigma_{i,4}]$ ,  $\alpha([i 3])\alpha([\sigma_{3,4}])\alpha([i 3]) = [j k][i 4]$ , où  $\{j, k\} = \{1, 2, 3\} - \{i\}$ .

Nous considérons à présent l'application  $\tilde{\alpha}$  de  $S$  dans  $\text{Out}(W_4)$  étendant  $\alpha$  sur  $S \cap (B_n \cup \{[3 4], [\sigma_{3,4}]\})$  et telle que, si  $i = 1, 2$ ,  $\tilde{\alpha}([i 4]) = [j k][\sigma_{i,4}]$ ,  $\tilde{\alpha}([\sigma_{i,4}]) = [j k][i 4]$ , où  $\{j, k\} = \{1, 2, 3\} - \{i\}$  et, si  $i$  et  $j$  sont distincts et si  $j \neq 4$ ,  $\tilde{\alpha}([\sigma_{i,j}]) = [\sigma_{j,4}][i j][k \ell][\sigma_{j,4}]$ , où  $\{k, \ell\} = \{1, 2, 3, 4\} - \{i, j\}$ .

Vérifions maintenant que  $\tilde{\alpha}$  préserve la présentation de  $\text{Aut}(W_n)$ . Ceci montrera que  $\tilde{\alpha}$  se prolonge en un morphisme de groupes de  $\text{Out}(W_4)$  dans lui-même. De plus, étant donné que  $B \cup \{[3 4], [\sigma_{3,4}]\}$  est une partie génératrice de  $\text{Out}(W_4)$  (cf. [Müh, Theorem B]), au vu de la définition de  $\tilde{\alpha}$ , ceci conclura la démonstration de la proposition. Nous écrivons pour chaque cas la relation vérifiée en préalable à la démonstration.

- (1) Pour tout  $i$ , pour tous les  $j, k, \ell \in \{1, 2, 3, 4\} - \{i\}$  deux à deux distincts,  $[\sigma_{j,i}][\sigma_{k,i}][\sigma_{\ell,i}] = 1$ .

$$\tilde{\alpha}([\sigma_{1,4}])\tilde{\alpha}([\sigma_{2,4}])\tilde{\alpha}([\sigma_{3,4}]) = [2\ 3][1\ 4][1\ 3][2\ 4][1\ 2][3\ 4] = 1.$$

Si  $j \neq 4$ , et si  $i, k, \ell \in \{1, 2, 3, 4\} - \{j\}$  sont deux à deux distincts,

$$\begin{aligned}\tilde{\alpha}([\sigma_{i,j}])\tilde{\alpha}([\sigma_{k,j}])\tilde{\alpha}([\sigma_{\ell,j}]) &= [\sigma_{j,4}][i\ j][k\ \ell][\sigma_{j,4}][\sigma_{j,4}][j\ k][i\ \ell][\sigma_{j,4}][\sigma_{j,4}][j\ \ell][i\ k][\sigma_{j,4}] \\ &= [\sigma_{j,4}][i\ j][k\ \ell][j\ k][i\ \ell][j\ \ell][i\ k][\sigma_{j,4}] = [\sigma_{j,4}][\sigma_{j,4}] = 1.\end{aligned}$$

(2) Pour tous les  $i, j, k, \ell$  vérifiant  $i \neq j$  et  $k \neq \ell$ , si on pose  $\sigma = (i\ j)$ , alors  $[i\ j][k\ \ell] = [\sigma(k)\ \sigma(\ell)][i\ j]$ .

Puisque  $\tilde{\alpha}$  est l'identité sur  $B_n$ , cette relation est vérifiée si  $i, j, k, \ell \in \{1, 2, 3\}$ . Vérifions les autres cas. Soient  $i, j \in \{1, 2, 3\}$  distincts et  $k \in \{1, 2, 3\} - \{i, j\}$ .

$$\begin{aligned}\tilde{\alpha}([i\ 4])\tilde{\alpha}([j\ 4]) &= [j\ k][\sigma_{i,4}][i\ k][\sigma_{j,4}] = [j\ i][j\ k][\sigma_{k,4}][\sigma_{j,4}] \\ &= [j\ i][j\ k][\sigma_{i,4}] = \tilde{\alpha}([j\ i])\tilde{\alpha}([i\ 4]).\end{aligned}$$

Maintenant, si  $i, j, k \in \{1, 2, 3\}$  sont deux à deux distincts,

$$\tilde{\alpha}([i\ j])\tilde{\alpha}([k\ 4]) = [i\ j][i\ j][\sigma_{k,4}] = [\sigma_{k,4}] = \tilde{\alpha}([k\ 4])\tilde{\alpha}([i\ j]).$$

(3) Pour tout  $j$ , pour tous les  $i, k \in \{1, 2, 3, 4\} - \{j\}$  distincts, nous avons  $[\sigma_{i,j}][\sigma_{k,j}] = [\sigma_{k,j}][\sigma_{i,j}]$ .

On note  $\ell$  l'élément distinct de  $i, j$  et  $k$ .

Supposons que  $j \neq 4$ . Alors

$$\begin{aligned}\tilde{\alpha}([\sigma_{i,j}])\tilde{\alpha}([\sigma_{k,j}]) &= [\sigma_{j,4}][j\ i][k\ \ell][j\ k][i\ \ell][\sigma_{j,4}] \\ &= [\sigma_{j,4}][j\ \ell][i\ k][\sigma_{j,4}] = \tilde{\alpha}([\sigma_{\ell,j}]) \\ &= \tilde{\alpha}([\sigma_{k,j}])\tilde{\alpha}([\sigma_{i,j}]).\end{aligned}$$

Dans le cas où  $j = 4$ ,

$$\tilde{\alpha}([\sigma_{i,4}])\tilde{\alpha}([\sigma_{k,4}]) = [k\ \ell][i\ 4][i\ \ell][k\ 4] = [i\ \ell][k\ 4][\ell\ k][i\ 4] = \tilde{\alpha}([\sigma_{k,4}])\tilde{\alpha}([\sigma_{i,4}]).$$

(4) Pour tout  $i \neq j$ , nous avons  $[\sigma_{i,j}][\sigma_{i,j}] = 1$ .

On note  $\ell$  et  $k$  les deux éléments distincts de  $i$  et  $j$ .

Supposons que  $j \neq 4$ . Alors

$$\tilde{\alpha}([\sigma_{i,j}])\tilde{\alpha}([\sigma_{i,j}]) = [\sigma_{j,4}][j\ i][k\ \ell][j\ i][k\ \ell][\sigma_{j,4}] = 1.$$

Si  $j = 4$ , alors

$$\tilde{\alpha}([\sigma_{i,4}])\tilde{\alpha}([\sigma_{i,4}]) = [k\ \ell][i\ 4][k\ \ell][i\ 4] = 1.$$

(5) Si  $i, j, k, \ell$  sont deux à deux distincts, alors  $[\sigma_{i,j}][\sigma_{k,\ell}] = [\sigma_{k,\ell}][\sigma_{i,j}]$ .

Nous traitons tout d'abord le cas où  $j = 4$  ou  $\ell = 4$ . Par symétrie, nous pouvons supposer que  $j = 4$ .

$$\begin{aligned}
\tilde{\alpha}([\sigma_{i,4}])\tilde{\alpha}([\sigma_{k,\ell}]) &= [k \ell][i 4][\sigma_{\ell,4}][k \ell][i 4][\sigma_{\ell,4}] \\
&= [k \ell][\sigma_{\ell,i}][k \ell][i 4][\sigma_{\ell,i}][i 4] \\
&= [\sigma_{k,i}][k \ell][i 4][\sigma_{k,i}][k \ell][i 4] \\
&= [\sigma_{\ell,4}][\sigma_{k,i}][\sigma_{\ell,4}][k \ell][i 4][\sigma_{\ell,4}][\sigma_{k,i}][\sigma_{\ell,4}][k \ell][i 4] \\
&= [\sigma_{\ell,4}][k \ell][\sigma_{\ell,i}][\sigma_{k,4}][\sigma_{\ell,i}][\sigma_{k,4}][i 4][\sigma_{\ell,4}][k \ell][i 4] \\
&= [\sigma_{\ell,4}][k \ell][i 4][\sigma_{\ell,4}][k \ell][i 4] = \tilde{\alpha}([\sigma_{k,\ell}])\tilde{\alpha}([\sigma_{i,4}]).
\end{aligned}$$

Nous effectuons maintenant le cas où  $i = 4$  ou  $k = 4$ . Par symétrie, nous pouvons supposer  $i = 4$ .

$$\begin{aligned}
\tilde{\alpha}([\sigma_{4,j}])\tilde{\alpha}([\sigma_{k,\ell}]) &= [\sigma_{j,4}][j 4][k \ell][\sigma_{j,4}][\sigma_{\ell,4}][k \ell][j 4][\sigma_{\ell,4}] \\
&= [\sigma_{j,4}][j 4][k \ell][\sigma_{k,4}][k \ell][j 4][\sigma_{\ell,4}] \\
&= [\sigma_{j,4}][\sigma_{\ell,j}][\sigma_{\ell,4}] ; \\
\tilde{\alpha}([\sigma_{k,\ell}])\tilde{\alpha}([\sigma_{4,j}]) &= [\sigma_{\ell,4}][j 4][k \ell][\sigma_{\ell,4}][\sigma_{j,4}][k \ell][j 4][\sigma_{j,4}] \\
&= [\sigma_{\ell,4}][j 4][k \ell][\sigma_{k,4}][k \ell][j 4][\sigma_{j,4}] \\
&= [\sigma_{\ell,4}][\sigma_{\ell,j}][\sigma_{j,4}] \\
&= [\sigma_{j,4}][\sigma_{\ell,4}][\sigma_{j,4}][\sigma_{\ell,j}][\sigma_{\ell,4}][\sigma_{j,4}][\sigma_{\ell,4}] \\
&= [\sigma_{j,4}][\sigma_{k,4}][\sigma_{\ell,j}][\sigma_{k,4}][\sigma_{\ell,4}] \\
&= [\sigma_{j,4}][\sigma_{\ell,j}][\sigma_{\ell,4}] = \tilde{\alpha}([\sigma_{4,j}])\tilde{\alpha}([\sigma_{k,\ell}]).
\end{aligned}$$

(6) Pour tous les  $i, j, k, \ell$  tels que  $k \neq \ell$ , si  $\tau = (i j)$ , alors  $[i j][\sigma_{k,\ell}] = [\sigma_{\tau(k),\tau(\ell)}][i j]$ .

On note  $a$  et  $b$  les éléments vérifiant  $\{a, b\} = \{1, 2, 3, 4\} - \{k, \ell\}$ .

Nous supposons tout d'abord que  $i, j \in \{1, 2, 3\}$ . Supposons également que  $\ell \neq 4$ . Si  $\{i, j\} \cap \{k, \ell\} = \emptyset$ , alors  $\{a, b\} = \{i, j\}$  et  $\tau(k) = k$  et  $\tau(\ell) = \ell$ . Donc

$$\tilde{\alpha}([i j])\tilde{\alpha}([\sigma_{k,\ell}]) = [i j][\sigma_{\ell,4}][k \ell][i j][\sigma_{\ell,4}] = [\sigma_{\ell,4}][k \ell][i j][\sigma_{\ell,4}][i j] = \tilde{\alpha}([\sigma_{k,\ell}])\tilde{\alpha}([i j]).$$

Si  $\{i, j\} \cap \{k, \ell\} = \{k\} = \{i\}$ , alors  $\{a, b\} = \{j, 4\}$  et  $\tau(k) = j$  et  $\tau(\ell) = \ell$ . Donc

$$\tilde{\alpha}([i j])\tilde{\alpha}([\sigma_{i,\ell}]) = [i j][\sigma_{\ell,4}][i \ell][j 4][\sigma_{\ell,4}] = [\sigma_{\ell,4}][j \ell][i 4][\sigma_{\ell,4}][i j] = \tilde{\alpha}([\sigma_{j,\ell}])\tilde{\alpha}([i j]).$$

Si  $\{i, j\} \cap \{k, \ell\} = \{\ell\} = \{i\}$ , alors  $\{a, b\} = \{j, b\}$  avec  $b \notin \{i, k\}$  et  $\tau(k) = k$  et  $\tau(\ell) = j$ . Donc

$$\tilde{\alpha}([i j])\tilde{\alpha}([\sigma_{k,i}]) = [i j][\sigma_{i,4}][k i][j b][\sigma_{i,4}] = [\sigma_{j,4}][k j][i b][\sigma_{j,4}][i j] = \tilde{\alpha}([\sigma_{k,j}])\tilde{\alpha}([i j]).$$

Si  $\{i, j\} \cap \{k, \ell\} = \{i, j\}$ , alors  $\{i, j\} \cap \{a, b\} = \emptyset$ . De plus, puisque  $i$  et  $j$  jouent un rôle symétrique, nous pouvons supposer que  $\tau(k) = j$  et  $\tau(\ell) = i$ . Donc

$$\tilde{\alpha}([i j])\tilde{\alpha}([\sigma_{i,j}]) = [i j][\sigma_{j,4}][i j][a b][\sigma_{j,4}] = [\sigma_{i,4}][i j][a b][\sigma_{i,4}][i j] = \tilde{\alpha}([\sigma_{j,i}])\tilde{\alpha}([i j]).$$

Supposons maintenant que  $\ell = 4$ .

Si  $\{i, j\} \cap \{k, \ell\} = \emptyset$ , alors  $\{a, b\} = \{i, j\}$  et  $\tau(k) = k$  et  $\tau(\ell) = \ell$ . Donc

$$\tilde{\alpha}([i j])\tilde{\alpha}([\sigma_{k,4}]) = [i j][i j][k 4] = [i j][k 4][i j] = \tilde{\alpha}([\sigma_{k,4}])\tilde{\alpha}([i j]).$$

Si  $\{i, j\} \cap \{k, \ell\} = \{k\} = \{i\}$ , alors  $\{a, b\} = \{j, b\}$ , avec  $b \neq i$  et  $b \neq 4$  et  $\tau(k) = j$  et  $\tau(\ell) = \ell$ . Donc

$$\tilde{\alpha}([i j])\tilde{\alpha}([\sigma_{i,4}]) = [i j][j b][i 4] = [i b][j 4][i j] = \tilde{\alpha}([\sigma_{j,4}])\tilde{\alpha}([i j]).$$

Supposons maintenant que  $j = 4$ . Supposons également que  $\ell \neq 4$ . Puisque  $j = 4$ , le cas où  $\ell = 4$  est symétrique au cas où  $k = 4$ .

Si  $\{i, 4\} \cap \{k, \ell\} = \emptyset$ , alors  $\{a, b\} = \{i, 4\}$  et  $\tau(k) = k$  et  $\tau(\ell) = \ell$ . Donc

$$\begin{aligned} \tilde{\alpha}([i 4])\tilde{\alpha}([\sigma_{k,\ell}]) &= [k \ell][\sigma_{i,4}][\sigma_{\ell,4}][k \ell][i 4][\sigma_{\ell,4}] \\ &= [\sigma_{i,4}][\sigma_{k,4}][k \ell][k \ell][i 4][\sigma_{\ell,4}] \\ &= [\sigma_{\ell,4}][k \ell][i 4][\sigma_{k,4}][k \ell] \\ &= [\sigma_{\ell,4}][k \ell][i 4][\sigma_{i,4}][\sigma_{k,4}][\sigma_{i,4}][k \ell] = \\ &= [\sigma_{\ell,4}][k \ell][i 4][\sigma_{\ell,4}][\sigma_{i,4}][k \ell] = \tilde{\alpha}([\sigma_{k,\ell}])\tilde{\alpha}([i 4]). \end{aligned}$$

Si  $\{i, 4\} \cap \{k, \ell\} = \{k\} = \{i\}$ , alors  $\{a, b\} = \{a, 4\}$ , avec  $a \neq i$  et  $a \neq \ell$  et  $\tau(k) = 4$  et  $\tau(\ell) = \ell$ . Donc

$$\begin{aligned} \tilde{\alpha}([i 4])\tilde{\alpha}([\sigma_{i,\ell}]) &= [a \ell][\sigma_{i,4}][\sigma_{\ell,4}][i \ell][a 4][\sigma_{\ell,4}] \\ &= [\sigma_{i,4}][\sigma_{a,4}][a \ell][i \ell][a 4][\sigma_{\ell,4}] \\ &= [\sigma_{\ell,4}][a \ell][i \ell][a 4][\sigma_{\ell,4}] \\ &= [\sigma_{\ell,4}][i a][\ell 4][\sigma_{a,4}][a \ell] \\ &= [\sigma_{\ell,4}][i a][\ell 4][\sigma_{i,4}][\sigma_{a,4}][\sigma_{i,4}][a \ell] \\ &= [\sigma_{\ell,4}][i a][\ell 4][\sigma_{\ell,4}][\sigma_{i,4}][a \ell] = \tilde{\alpha}([\sigma_{4,\ell}])\tilde{\alpha}([i 4]). \end{aligned}$$

Si  $\{i, 4\} \cap \{k, \ell\} = \{k\} = \{4\}$ , alors  $\{a, b\} = \{a, i\}$ , avec  $a \neq 4$  et  $a \neq \ell$  et  $\tau(k) = i$  et  $\tau(\ell) = \ell$ . Donc

$$\begin{aligned} \tilde{\alpha}([i 4])\tilde{\alpha}([\sigma_{4,\ell}]) &= [a \ell][\sigma_{i,4}][\sigma_{\ell,4}][4 \ell][a i][\sigma_{\ell,4}] \\ &= [\sigma_{i,4}][\sigma_{a,4}][a \ell][4 \ell][a i][\sigma_{\ell,4}] \\ &= [\sigma_{\ell,4}][a \ell][4 \ell][a i][\sigma_{\ell,4}] \\ &= [\sigma_{\ell,4}][4 a][\ell i][\sigma_{a,4}][a \ell] \\ &= [\sigma_{\ell,4}][4 a][\ell i][\sigma_{i,4}][\sigma_{a,4}][\sigma_{i,4}][a \ell] \\ &= [\sigma_{\ell,4}][4 a][\ell i][\sigma_{\ell,4}][\sigma_{i,4}][a \ell] = \tilde{\alpha}([\sigma_{i,\ell}])\tilde{\alpha}([i 4]). \end{aligned}$$

Si  $\{i, 4\} \cap \{k, \ell\} = \{\ell\} = \{i\}$ , alors  $\{a, b\} = \{a, 4\}$ , avec  $a \neq k$  et  $a \neq i$  et  $\tau(k) = k$  et  $\tau(\ell) = 4$ . Donc

$$\begin{aligned}
\tilde{\alpha}([i\ 4])\tilde{\alpha}([\sigma_{k,i}]) &= [a\ k][\sigma_{i,4}][\sigma_{i,4}][k\ i][a\ 4][\sigma_{i,4}] \\
&= [a\ k][k\ i][a\ 4][\sigma_{i,4}] \\
&= [a\ i][k\ 4][a\ k][\sigma_{i,4}] = \tilde{\alpha}([\sigma_{k,4}])\tilde{\alpha}([i\ 4]).
\end{aligned}$$

Si  $\{i, 4\} \cap \{k, \ell\} = \{i, 4\}$ , alors  $k = 4$  et  $\{a, b\} \cap \{i, 4\} = \emptyset$  et  $\tau(k) = i$  et  $\tau(\ell) = 4$ .  
Donc

$$\begin{aligned}
\tilde{\alpha}([i\ 4])\tilde{\alpha}([\sigma_{4,i}]) &= [a\ b][\sigma_{i,4}][\sigma_{i,4}][4\ i][a\ b][\sigma_{i,4}] \\
&= [a\ b][4\ i][a\ b][\sigma_{i,4}] \\
&= [4\ i][a\ b][a\ b][\sigma_{i,4}] = \tilde{\alpha}([\sigma_{i,4}])\tilde{\alpha}([i\ 4]).
\end{aligned}$$

Donc  $\tilde{\alpha}$  préserve toutes les relations données dans la proposition 2.5.1. De ce fait,  $\tilde{\alpha}$  se prolonge en un morphisme de groupes de  $\text{Out}(W_4)$  dans lui-même. Ceci conclut la démonstration.  $\square$



## Chapitre 3

# The symmetries of the Outer space of a universal Coxeter group

### 3.1 Introduction

Let  $n$  be an integer greater than 1. Let  $F = \mathbb{Z}/2\mathbb{Z}$  be a cyclic group of order 2 and  $W_n = *_n F$  be a universal Coxeter group, which is a free product of  $n$  copies of  $F$ . The geometric study of automorphisms groups of free products is currently in strong expansion, see for instance [MM, GuL1, Pig, HaM3, Gup2, GuH2, DL]. This paper proves a major geometric rigidity result of the outer automorphism group  $\text{Out}(W_n)$  of  $W_n$ .

The study of  $\text{Out}(W_n)$  benefits from analogies with algebraic groups, with  $\text{Out}(F_N)$ , the outer automorphism group of a free group of rank  $N$ , and with the mapping class group of a connected compact surface. As usual in geometric group theory, the understanding of the group  $\text{Out}(W_n)$  is related to the construction of geometric spaces on which it acts nicely (properly or cocompactly). Such constructions appear in the study of  $\text{Out}(F_N)$ , which involves the study of its action on the spine of the Outer space introduced by Culler and Vogtmann in [CV]. Similarly, the study of the mapping class group of a connected compact surface involves the study of its action on the Teichmüller space and on the curve graph of the surface, while the study of algebraic groups implies the study of their actions on buildings.

The spaces introduced in these cases are *rigid geometric models* in the following sense: the symmetries of these spaces are induced by elements of the group itself. Indeed, for algebraic groups, Tits showed that, if the rank of a spherical building associated with a simple connected algebraic group is at least 2, then the full group of simplicial automorphisms of the building is isomorphic to the algebraic group itself ([Tit2]). In the context of a connected orientable compact surface of genus at least 3, Royden proved that the group of isometries of the Teichmüller space with respect to the Teichmüller metric coincides with the extended mapping class group of the surface ([Roy]). Moreover, Ivanov ([Iva2, Theorem 1]) showed that the group of simplicial automorphisms of the graph of curves is isomorphic to the extended mapping class group. In the context of  $\text{Out}(F_N)$ ,

Bridson and Vogtmann proved that, if  $N \geq 3$ , the group of simplicial automorphisms of the spine of Outer space is isomorphic to  $\text{Out}(F_N)$  ([BV2]).

In the case of  $\text{Out}(W_n)$ , spaces on which  $\text{Out}(W_n)$  acts properly or cocompactly include the McCullough-Miller space [MM] or  $\mathbb{P}\mathcal{O}(W_n)$ , the outer space of  $W_n$  introduced by Guirardel and Levitt in [GuL1]. These two spaces are  $\text{Out}(W_n)$ -equivariantly homotopy equivalent (see [MM, Theorem 8.5.]). Moreover, it was proved by Piggott ([Pig, Theorem 1.1]) that, for  $n \geq 4$ , the McCullough-Miller space is a rigid geometric model for  $\text{Out}(W_n)$ : the group of simplicial automorphisms of the McMullough-Miller space is isomorphic to  $\text{Out}(W_n)$ .

In this article, we study the action of  $\text{Out}(W_n)$  on a simplicial flag complex on which  $\mathbb{P}\mathcal{O}(W_n)$  retracts  $\text{Out}(W_n)$ -equivariantly, called *the spine of  $\mathbb{P}\mathcal{O}(W_n)$*  and denoted by  $K_n$ . Vertices of  $K_n$  are homothety classes of marked graphs of groups whose fundamental group is isomorphic to  $W_n$ . Two homothety classes  $\mathcal{X}$  and  $\mathcal{Y}$  of marked graphs of groups are adjacent in  $K_n$  if they have representatives  $X$  and  $Y$  such that one can obtain  $Y$  from  $X$  by collapsing a forest in the underlying graph of  $X$ , or conversely. The group  $\text{Out}(W_n)$  naturally acts on  $K_n$  by precomposing the marking. The aim of this article is to prove that  $K_n$  is a rigid geometric model for  $\text{Out}(W_n)$  in the following sense. Here we denote by  $\text{Aut}(K_n)$  the group of simplicial automorphisms of  $K_n$ .

**Theorem 3.1.1.** *Let  $n \geq 4$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(K_n)$$

*is an isomorphism.*

Theorem 3.1.1 gives a complete classification of the simplicial automorphisms of  $K_n$  for every  $n$ . Indeed, the graph  $K_2$  is reduced to a point and  $K_3$  is a tree (in fact isomorphic to the spine of the Outer space of a nonabelian free group of rank 2, see Proposition 3.2.6). The question of the study of simplicial automorphisms of  $K_n$  is first motivated by the aforementioned examples, but also by algebraic results on  $\text{Out}(W_n)$ . Indeed, for instance in the case of the mapping class group of a connected orientable compact surface of genus at least 3, the fact that the curve complex is a rigid geometric model for the extended mapping class group is used by Ivanov in order to prove that any automorphism of the extended mapping class group is in fact a conjugation (see [Iva2, Theorem 2]). Similarly, the fact that the spine of Outer space is a rigid geometric model for  $\text{Out}(F_N)$  with  $N \geq 3$  is related to the fact that any automorphism of  $\text{Out}(F_N)$  is a conjugation ([BV1]). As, for  $n \geq 4$ , any automorphism of  $\text{Out}(W_n)$  is a conjugation (see [Gue1, Théorème 1.1]) and as the proof relies on the study of the action of  $\text{Out}(W_n)$  on  $K_n$ , it was natural to expect that  $K_n$  is a rigid geometric model for  $\text{Out}(W_n)$ . Even though the McCullough-Miller space and  $\mathbb{P}\mathcal{O}(W_n)$  are  $\text{Out}(W_n)$ -equivariantly homotopy equivalent, the author does not know how to deduce the rigidity of  $K_n$  out of the rigidity of the McCullough-Muller space. Indeed, there is no canonical graph isomorphism between  $K_n$  and the McCullough-Miller space, and corresponding vertices in the McCullough-Miller space and in  $K_n$  do not share the same properties of minimality. For instance the negative link of a  $\{0\}$ -star (see Sections 3.2.3 and 3.4 for

precise definitions) is nontrivial in  $K_n$ , whereas it is trivial in the McCullough-Miller space.

The proof of Theorem 3.1.1 relies on the study of the action of  $\text{Out}(W_n)$  on a subgraph of  $K_n$  called *the graph of  $\{0\}$ -stars and  $F$ -stars* and denoted by  $L_n$ . Vertices of  $L_n$  are  $\{0\}$ -stars and  $F$ -stars (see Section 3.2.3 and Figure 3.1). Two vertices of  $L_n$  are adjacent if and only if they are adjacent in  $K_n$ . We first prove that  $L_n$  is a rigid geometric model for  $W_n$  (see Theorem 3.3.1). This relies on studying systoles of  $L_n$ , that is, embedded cycles of minimal length. For this, we introduce (see Section 3.3) a new complexity associated with an edge of  $L_n$ , and a relative complexity associated with pairs of  $\{0\}$ -stars. For  $n = 3$ , the same study is not possible as the  $\{0\}$ -stars are no longer the vertices with minimal degree in  $L_n$ . We do not know whether Theorem 3.3.1 holds for  $n = 3$ .

The rest of the proof consists in showing that there exists a homomorphism from  $\text{Aut}(K_n)$  to  $\text{Aut}(L_n)$  defined by restriction which turns out to be injective. We note that the characterization of the vertices of  $L_n$  in  $K_n$  is only based on the study of the possible decompositions of the link of the vertices of  $K_n$ . This differs from the proof of the similar result by Bridson and Vogtmann in the case of  $\text{Out}(F_n)$  since they used homological arguments in order to characterize some vertices of the spine of Outer space. Another major difference is that the strictly local rigidity properties of  $L_n$  are much weaker than the ones of the spine of Outer space, and we need to explore the combinatorial balls of radius 4 in  $L_n$  in order to acquire a sufficient rigidity. Note that in the case of algebraic groups, Tits only needed to explore the combinatorial balls of radius 2.

In Section 3.5, we study the simplicial completion of  $K_n$ , denoted by  $\overline{K}_n$ . The simplicial complex  $\overline{K}_n$  is also known as *the free splitting complex of  $W_n$*  (see [AS, HaM2] and Section 3.5). This complex has an analogue in the case of a free group of rank  $N$ , called *the free splitting complex of  $F_N$* . It was proved by Aramayona and Souto that the free splitting complex of  $F_N$  is also a rigid geometric model for  $\text{Out}(F_N)$  when  $N \geq 3$  (see [AS, Theorem 1]). In Section 3.5, we prove the following theorem:

**Theorem 3.1.2.** *Let  $n \geq 4$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(\overline{K}_n)$$

*is an isomorphism.*

Theorem 3.1.2 can be deduced from Theorem 3.1.1 as follows. The spine  $K_n$  has a natural embedding into  $\overline{K}_n$ . We first show that any automorphism of  $\overline{K}_n$  preserves the image of  $K_n$ . This gives a homomorphism  $\text{Aut}(\overline{K}_n) \rightarrow \text{Aut}(K_n)$  and the main point, using techniques of Scott-Swarup and Horbez-Wade, is to prove its injectivity. We then conclude using Theorem 3.1.1.

**Acknowledgments.** I warmly thank my advisors, Camille Horbez and Frédéric Paulin, for their precious advices and for carefully reading the different versions of this article. I thank the anonymous referee for suggesting a proof of the description of the symmetries of  $K_3$ .

## 3.2 Preliminaries

### 3.2.1 Background on the outer space of $W_n$

Let  $n$  be an integer greater than 1. Let  $F = \mathbb{Z}/2\mathbb{Z}$  be a cyclic group of order 2 and  $W_n = \ast_n F$  be the universal Coxeter group of order  $n$ . We recall the definition of the outer space  $\mathbb{PO}(W_n)$  introduced by Guirardel and Levitt in [GuL1]. A point in  $\mathbb{PO}(W_n)$  is a homothety class of metric graph of groups  $X$  whose fundamental group is  $W_n$ , equipped with a group isomorphism  $\rho: W_n \rightarrow \pi_1(X)$  called a *marking*, which satisfies :

- (1) the underlying graph of  $X$  is a finite tree ;
- (2) every edge group is trivial ;
- (3) there are exactly  $n$  vertices whose associated group is isomorphic to  $F$  ;
- (4) all the other vertices have trivial associated group ;
- (5) if  $v$  is a vertex whose associated group is trivial, then  $\deg(v) \geq 3$ .

Two metric graphs of groups  $(X, \rho)$  and  $(X', \rho')$  are in the same homothety class if there exists a homothety  $f: X \rightarrow X'$  (meaning an application multiplying all edge lengths by the same scalar) and such that  $f_* \circ \rho = \rho'$ . We denote by  $[X, \rho]$  the homothety class of such a metric graph of groups  $(X, \rho)$ . If the marking is implicit, we denote by  $\mathcal{X}$  the homothety class. The group  $\text{Aut}(W_n)$  acts by precomposing the marking. As, for any  $\alpha \in \text{Inn}(W_n)$ , and for any  $\mathcal{X} \in \mathbb{PO}(W_n)$ , we have  $\alpha(\mathcal{X}) = \mathcal{X}$ , the action of  $\text{Aut}(W_n)$  induces an action of  $\text{Out}(W_n)$ .

The set  $\mathbb{PO}(W_n)$  is equipped with a topology which we recall now. Suppose that  $[X, \rho] \in \mathbb{PO}(W_n)$  and let  $(X, \rho)$  be the representative of  $[X, \rho]$  such that the sum of the edge lengths is equal to 1. To  $(X, \rho)$  we associate a simplex by varying the lengths of the edges, so that the sum of the edge lengths is still equal to 1. A homothety class  $[X', \rho'] \in \mathbb{PO}(W_n)$  defines a codimension 1 face of the simplex associated with  $(X, \rho)$  if we can obtain  $(X', \rho')$  from  $(X, \rho)$  by contracting an edge of the underlying graph of  $X$ . The *weak topology* is then defined in the following way: a set is open if and only if its intersection with every open simplex is open.

We now recall the definition of a deformation retract of  $\mathbb{PO}(W_n)$  known as *the spine of  $\mathbb{PO}(W_n)$*  and denoted by  $K_n$ . It is a flag complex whose vertices are the open simplices associated with each homothety class  $[X, \rho] \in \mathbb{PO}(W_n)$ . Two vertices corresponding to two homothety classes  $[X, \rho]$  and  $[X', \rho']$  are adjacent if  $[X, \rho]$  defines a face of the simplex associated with  $[X', \rho']$  and conversely. There is an embedding  $F: K_n \hookrightarrow \mathbb{PO}(W_n)$  whose image is the barycentric spine of  $\mathbb{PO}(W_n)$ . We will from then on identify  $K_n$  with  $F(K_n)$ .

We now give a description of the stabilizer of a point in  $K_n$  due to Levitt. If  $\mathcal{X} \in VK_n$ , we denote by  $\text{Stab}(\mathcal{X})$  the stabilizer of  $\mathcal{X}$  under the action of  $\text{Out}(W_n)$ . Let  $X$  be

a representative of  $\mathcal{X}$ . We denote by  $\text{Stab}^0(\mathcal{X})$  the subgroup of  $\text{Stab}(\mathcal{X})$  made of all elements  $F \in \text{Out}(W_n)$  such that the automorphism induced by  $F$  on  $X$  is the identity. We write the next proposition in a more general context where the nontrivial vertex groups are not necessarily isomorphic to  $F$  (see Section 3.5).

**Proposition 3.2.1.** *[Lev1, Proposition 4.2] Let  $n \geq 4$  and  $\mathcal{X} \in VK_n$ . Let  $X$  be a representative of  $\mathcal{X}$  and let  $v_1, \dots, v_k$  be the vertices of  $X$  with nontrivial associated groups. For  $i \in \{1, \dots, k\}$ , let  $G_i$  be the group associated with  $v_i$ . Then  $\text{Stab}^0(\mathcal{X})$  is isomorphic to*

$$\prod_{i=1}^k G_i^{\deg(v_i)-1} \rtimes \text{Aut}(G_i) ,$$

where  $\text{Aut}(G_i)$  acts on  $G_i^{\deg(v_i)-1}$  diagonally.

**Remark 3.2.2.** *More generally, Proposition 3.2.1 remains true for every  $n \geq 3$  and every free splitting of  $W_n$  (see the definitions at the beginning of Section 3.5).*

### 3.2.2 The symmetries of $K_3$

In this section, we describe the spine of  $\mathbb{PO}(W_n)$  when  $n = 3$ . Let  $\{x_1, \dots, x_n\}$  be a standard generating set of  $W_n$ . Let  $\Theta: W_n \rightarrow \mathbb{Z}/2\mathbb{Z}$  be the homomorphism which sends, for every  $i \in \{1, \dots, n\}$ , the element  $x_i$  to the nontrivial element of  $\mathbb{Z}/2\mathbb{Z}$ . By a result of Mühlherr [Müh, Theorem A], the group  $B_n = \ker(\theta)$  is a nonabelian free group of rank  $n - 1$  and, when  $n = 3$ , the group  $B_3$  is generated by  $a = x_1x_2$  and  $b = x_2x_3$ . Let  $\phi \in \text{Out}(B_3)$  be the outer class of the automorphism which sends  $a$  to  $a^{-1}$  and  $b$  to  $b^{-1}$ . Then  $\phi$  belongs to the center of  $\text{Out}(B_3)$  and has order 2. Let  $G = B_3 \rtimes_{\phi} \mathbb{Z}/2\mathbb{Z}$ .

**Lemma 3.2.3.** *The group  $G$  is isomorphic to  $W_3$ .*

**Proof.** The group  $G$  is generated by  $(a, \phi)$ ,  $(1, \phi)$  and  $(b, \phi)$ , and all these elements have order 2. Hence there exists a surjective homomorphism  $\Psi: W_3 \rightarrow G$  which sends  $x_1$  to  $(a, \phi)$ ,  $x_2$  to  $(1, \phi)$  and  $x_3$  to  $(b, \phi)$ . We claim that  $\Psi$  is an isomorphism. Since  $B_3$  has index 2 in  $W_3$  and since every nontrivial normal subgroup of  $W_3$  is infinite, it suffices to show that  $\ker(\Psi) \cap B_3 = \{1\}$ . Note that  $x_1x_2$  is sent by  $\Psi$  to  $(a, 1)$  and  $x_3x_2$  is sent by  $\Psi$  to  $(b, 1)$ . Hence the homomorphism  $\Psi$  restricts to a surjective homomorphism  $\Psi|_{B_3}: B_3 \rightarrow B_3$ . Since  $B_3$  is a nonabelian free group of rank 2, it is Hopfian. Hence  $\Psi|_{B_3}$  is an isomorphism. This concludes the proof.  $\square$

Let  $CV_2$  be the *Outer space* of a nonabelian free group  $F_2$  of rank 2, that is the space whose elements are the  $F_2$ -equivariant isometry classes of metric trees equipped with a minimal, free action of  $F_2$ , where  $CV_2$  is equipped with the weak topology as defined in the case of  $\mathbb{PO}(W_n)$ . For a graph  $G$ , an edge  $e$  of  $G$  is separating if the graph  $\overline{G - e}$  has two connected components. The space  $CV_2$   $\text{Out}(F_2)$ -equivariantly retracts onto the *reduced Outer space*  $CV_2^r$ , where a point  $\mathcal{X}$  in  $CV_2^r$  is such that, for every representative  $X$  of  $\mathcal{X}$ , the quotient  $F_2 \backslash X$  does not have a separating edge. As in the case of  $\mathbb{PO}(W_n)$ , the space  $CV_2^r$   $\text{Out}(F_2)$ -equivariantly retracts onto the *spine of the reduced outer space*,

denoted by  $K'_2$ . It is the graph whose vertices are the  $F_2$ -equivariant homeomorphism classes of simplicial trees  $X$  equipped with a minimal, free action of  $F_2$  such that  $F_2 \backslash X$  does not have a separating edge and there exists an edge between two equivalence classes  $\mathcal{X}$  and  $\mathcal{X}'$  if there exists  $X \in \mathcal{X}$  and  $X' \in \mathcal{X}'$  such that  $X$  collapses onto  $X'$  or conversely. The graph  $K'_2$  is isomorphic to a trivalent tree (see [CV, Section 1.4]). The group  $\text{Out}(F_2)$  acts on  $K'_2$  by precomposition of the action. The element  $\phi \in \text{Out}(B_3) \simeq \text{Out}(F_2)$  is in the center of  $\text{Out}(B_3)$ . Hence  $\phi$  acts as the identity on  $K'_2$  and, if  $X$  is a representative of a vertex  $\mathcal{X}$  of  $K'_2$ , the action of  $B_3$  on  $X$  extends to an action of  $G$  on  $X$ . By Lemma 3.2.3, the group  $G$  is isomorphic to  $W_3$ .

**Lemma 3.2.4.** *Let  $\mathcal{X}$  be a vertex of  $K'_2$  and let  $X$  be a representative of  $\mathcal{X}$ . Then  $\phi$  fixes a unique vertex in  $X$ .*

**Proof.** Note that  $B_3 \backslash X$  is either a rose or a theta graph. Let  $T$  be a maximal tree in  $B_3 \backslash X$ . Let  $c, d \in B_3$  be such that the two edges in  $\overline{B_3 \backslash X}$  are labeled by  $c$  and  $d$ . Then  $\{c, d\}$  is a free basis of  $B_3$ . Let  $\Phi$  be a representative of  $\phi$  such that  $\Phi(c) = c^{-1}$  and  $\Phi(d) = d^{-1}$ . Then  $\Phi$  preserves the axes  $\text{Ax}(c)$  and  $\text{Ax}(d)$  of  $c$  and  $d$ . Moreover,  $\Phi$  has a unique fixed point  $x_c$  in  $\text{Ax}(c)$  and a unique fixed point  $x_d$  in  $\text{Ax}(d)$ . We claim that  $x_c = x_d$ . Indeed, otherwise  $\Phi$  would fix pointwise the unique geodesic path  $\tau$  between  $x_c$  and  $x_d$ . Note that the elements  $c$  and  $d$  are chosen so that  $\text{Ax}(c) \cap \text{Ax}(d) \neq \emptyset$ . Thus the path  $\tau$  is contained in  $\text{Ax}(c) \cup \text{Ax}(d)$ . Since  $x_c$  is the unique element in  $\text{Ax}(c)$  fixed by  $\Phi$  and since  $x_d$  is the unique element in  $\text{Ax}(d)$  fixed by  $\Phi$ , we have  $x_c = x_d$ . Let  $y$  be a point in  $X$  fixed by  $\Phi$ . We claim that  $y = x_c$ . Indeed, as above, the element  $\Phi$  fixes pointwise the geodesic path between  $x_c$  and  $y$ . This path must contain an edge in  $\text{Ax}(c) \cup \text{Ax}(d)$  since  $\text{Ax}(c) \cup \text{Ax}(d)$  covers a fundamental domain of  $X$  for the action of  $B_3$ . Since  $x_c$  is the unique point in  $\text{Ax}(c) \cup \text{Ax}(d)$  fixed by  $\Phi$ , we have  $x_c = y$ . This concludes the proof.  $\square$

**Remark 3.2.5.** *Let  $\mathcal{X}$  be an equivalence class of metric trees in  $CV_2$  and let  $X$  be a representative of  $\mathcal{X}$ . Suppose that  $B_3 \backslash X$  contains a separating edge. Then  $\phi$  fixes an edge in  $X$ , which corresponds to a lift of the separating edge of  $B_3 \backslash X$ .*

Let  $\mathcal{X}$  be a vertex of  $K'_2$  and let  $X$  be a representative of  $\mathcal{X}$ . By Lemma 3.2.4, the edge stabilizers of  $X$  for the action of  $G$  are trivial. Moreover, the vertex stabilizers are either trivial or isomorphic to  $F$ . Hence we have a natural simplicial map

$$\Theta': K'_2 \rightarrow K_3$$

which is  $\text{Out}(G)$ -equivariant.

**Proposition 3.2.6.** *The natural application*

$$\Theta': K'_2 \rightarrow K_3.$$

*is an isomorphism.*

**Proof.** It suffices to construct an inverse for  $\Theta'$ . Let  $\mathcal{X}$  be a vertex in  $K_3$  and let  $X$  be a representative of  $\mathcal{X}$ . The action of  $W_3$  on  $X$  restricts to an action of  $B_3$  on  $X$ . Since the vertex stabilizers of  $X$  are trivial or isomorphic to  $F$ , the action of  $B_3$  on  $X$  is free. Since  $B_3$  has index 2 in  $W_3$ , the action of  $B_3$  on  $X$  is minimal. Moreover, by Remark 3.2.5, since  $X$  does not have an edge with nontrivial stabilizer for the action of  $G$ , the quotient  $B_3 \backslash X$  does not have a separating edge. Hence it induces a simplicial map  $K_3 \rightarrow K'_2$  which is the inverse of  $\Theta'$ .  $\square$

In particular, since the simplicial automorphism group of a trivalent tree is uncountable, the group  $\text{Aut}(K_3)$  is not isomorphic to  $\text{Out}(W_3)$ .

### 3.2.3 The graph of $\{0\}$ -stars and $F$ -stars.

In order to prove Theorem 3.1.1, we introduce a graph included in the spine  $K_n$  called *the graph of  $\{0\}$ -stars and  $F$ -stars*.

**Definition 3.2.7.** (1) A  $\{0\}$ -star is the equivalence class in  $K_n$  of a metric graph of groups whose underlying graph has  $n + 1$  vertices and  $n$  leaves.

(2) A  $F$ -star is the equivalence class in  $K_n$  of a metric graph of groups whose underlying graph has  $n$  vertices and  $n - 1$  leaves.

(3) The *graph of  $\{0\}$ -stars and  $F$ -stars*, denoted by  $L_n$ , is the full subgraph of  $K_n$  whose vertices are exactly the  $\{0\}$ -stars and the  $F$ -stars. There is an edge between two vertices of  $L_n$  if and only if there is an edge between the corresponding vertices in  $K_n$ .

As  $\text{Aut}(W_n)$  acts on  $K_n$  by precomposition of the action, the graph  $L_n$  is invariant by  $\text{Out}(W_n)$ .

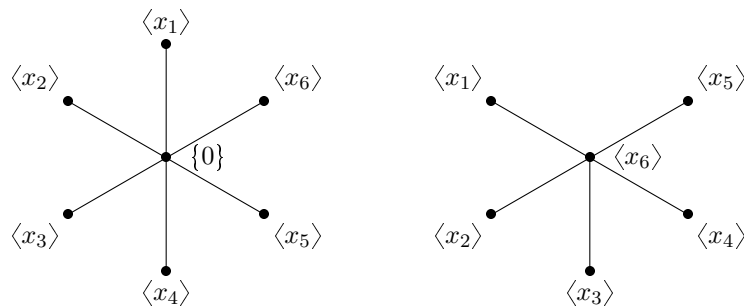


Figure 3.1: A  $\{0\}$ -star (left) and an  $F$ -star (right).

Since any two  $\{0\}$ -stars are at distance at least 2 in  $K_n$ , the neighbors of a  $\{0\}$ -star in  $L_n$  are  $F$ -stars. Conversely, since any two  $F$ -stars are at distance at least 2 in  $K_n$ , the neighbors of an  $F$ -star in  $L_n$  are  $\{0\}$ -stars. The number of neighbors in  $L_n$  of a  $\{0\}$ -star is equal to  $n$ . They correspond to collapsing exactly one edge of the underlying graph. The number of neighbors in  $L_n$  of an  $F$ -star is equal to  $2^{n-2}$ . They correspond to blowing-up the central vertex of the underlying graph while applying a partial conjugation by the

generator of the preimage by the marking of the group associated with the center. As  $\text{Aut}(W_n)$  acts transitively on the set of free bases of  $W_n$ , we see that  $\text{Aut}(W_n)$  acts transitively on the set of  $\{0\}$ -stars. Thus, as partial conjugations and permutations generate  $\text{Aut}(W_n)$  by [Müh, Theorem B], it follows that the graph  $L_n$  is connected.

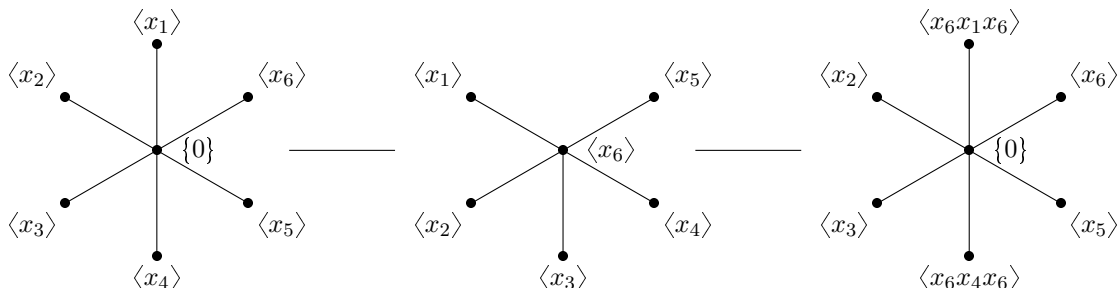


Figure 3.2: Examples of two neighbours of an  $F$ -star in  $L_6$ .

### 3.3 Rigidity of the graph of $\{0\}$ -stars and $F$ -stars

In this section, we prove the following theorem.

**Theorem 3.3.1.** *Let  $n \geq 4$ . Let  $f$  be an automorphism of  $L_n$  preserving  $O_n$  and  $F_n$ . Then  $f$  is induced by the action of a unique element  $\gamma$  of  $\text{Out}(W_n)$ .*

For  $n \geq 5$ , any  $F$ -star has  $2^{n-2}$  neighbours in  $L_n$  and any  $\{0\}$ -star has  $n$  neighbours in  $L_n$ . As  $2^{n-2} > n$  precisely when  $n \geq 5$ , we see that every automorphism of  $L_n$  preserves the set of  $\{0\}$ -stars and the set of  $F$ -stars. We thus have the following corollary.

**Corollary 3.3.2.** *Let  $n \geq 5$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$$

*is an isomorphism.* □

Before proving Theorem 3.3.1, we first prove a lemma which characterises the number of paths in a ball of radius 4 centered at a  $\{0\}$ -star.

Let  $\mathcal{X}$  be a  $\{0\}$ -star, and  $(X, \rho)$  a representative of  $\mathcal{X}$ . Let  $v_1, \dots, v_n$  be the  $n$  leaves of the underlying graph of  $X$ . For  $i \in \{1, \dots, n\}$ , let  $x_i$  be the preimage by  $\rho$  of the generator of the group associated with  $v_i$ , and let  $\mathcal{Y}_i$  be the  $F$ -star adjacent to  $\mathcal{X}$  such that a representative of  $\mathcal{Y}_i$  is obtained from  $X$  by contracting the edge adjacent to  $v_i$ . For distinct  $i, j \in \{1, \dots, n\}$ , let  $\sigma_{j,i}: W_n \rightarrow W_n$  be the automorphism sending  $x_j$  to  $x_i x_j x_i$  and, for  $k \neq j$ , fixing  $x_k$ . In this context we will call  $x_i$  the *twistor* of  $\sigma_{j,i}$ . For distinct  $i, j \in \{1, \dots, n\}$ , let  $(i j)$  be the automorphism of  $W_n$  switching  $x_i$  and  $x_j$  and, for  $k \neq i, j$ , fixing  $x_k$ . A theorem of Mühlherr (c.f. [Müh, Theorem B]) implies that



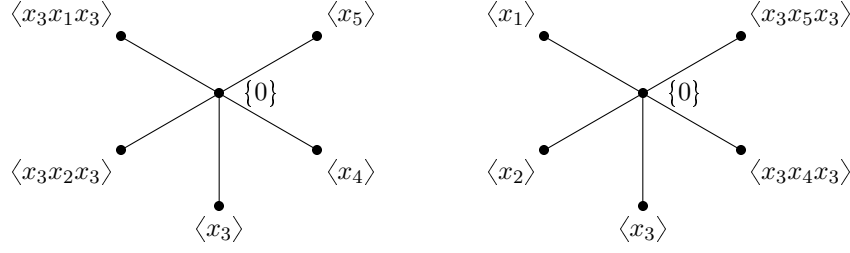


Figure 3.3: Two representatives of the same homothety class  $\mathcal{X}'$  realizing  $k_{\mathcal{X},i}(\mathcal{X}')$ .

$\{\sigma_{i,j} \mid i \neq j\} \cup \{(i j) \mid i \neq j\}$  is a generating set of  $\text{Aut}(W_n)$ . Note that, for every integers  $i, j, k, \ell$ , there exist  $p, q$  such that  $(i j)\sigma_{k,\ell}(i j) = \sigma_{p,q}$ .

We now fix  $i \in \{1, \dots, n\}$ . Let  $\mathcal{X}'$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_i$  and distinct from  $\mathcal{X}$ . Let  $(X', \rho')$  be a representative of  $\mathcal{X}'$ . Let  $w_1, \dots, w_n$  be the leaves of the underlying graph of  $X'$ , and, for  $j \in \{1, \dots, n\}$ , let  $y_j$  be the preimage by  $\rho'$  of the generator of the group associated with  $w_j$ . Up to composition by an inner automorphism and reordering, either  $y_j = x_j$  or  $y_j = x_i y_j x_i$  (see Figure 3.2 with  $i = n = 6$ ). Thus, there exist  $k \in \{1, \dots, n-1\}$  and  $i_1, \dots, i_k \in \{1, \dots, \hat{i}, \dots, n\}$  pairwise distinct such that, for all  $j \in \{1, \dots, n\}$ ,

$$\left( \prod_{l=1}^k \sigma_{i_l, i} \right) (x_j) = y_j.$$

Let  $\text{Inn}^\#(W_n) = \langle \text{Inn}(W_n), \{\sigma_{i,j} \mid i \neq j\} \rangle$ . We define the *first term complexity* of  $\mathcal{X}'$  by

$$k_{\mathcal{X},i}(\mathcal{X}') = \min \left\{ k \mid \begin{array}{l} \exists i_1, \dots, i_k \in \{1, \dots, \hat{i}, \dots, n\}, I \in \text{Inn}^\#(W_n) \text{ such that} \\ \forall j \in \{1, \dots, n\}, I \circ \left( \prod_{l \in \{1, \dots, k\}} \sigma_{i_l, i} \right) (x_j) = y_j \end{array} \right\}.$$

This definition does not depend on the choice of a representative of  $\mathcal{X}'$ . Note that the sequence  $i_1, \dots, i_k$  realizing the minimum is not necessarily unique (see Figure 3.3 with  $n = 5$  and  $i = 3$ ). However, if  $k_{\mathcal{X},i}(\mathcal{X}') \neq n - k_{\mathcal{X},i}(\mathcal{X}') - 1$ , such a sequence is unique.

We now define a notion of relative complexity in  $O_n$ , the set of  $\{0\}$ -stars in  $L_n$ . Let  $\mathcal{Z}$  be a  $\{0\}$ -star in  $L_n$  distinct from  $\mathcal{X}$  and let  $(Z, \psi)$  be a representative of  $\mathcal{Z}$ . Let  $w_1, \dots, w_n$  be the leaves of the underlying graph of  $Z$ , and, for  $j \in \{1, \dots, n\}$ , let  $z_j$  be the preimage by  $\psi$  of the generator of the group associated with  $w_j$ . As  $\{\sigma_{i,j} \mid i \neq j\} \cup \{(i j) \mid i \neq j\}$  is a generating set of  $\text{Aut}(W_n)$  (c.f. [Müh, Theorem B]), we see that, up to composition by an inner automorphism and reordering,

$$\begin{aligned} \exists k \in \mathbb{N}, \exists (i_1, j_1), \dots, (i_k, j_k) \in \{1, \dots, n\}^2 - \{(x, x) \mid x \in \{1, \dots, n\}\}, \\ \forall j \in \{1, \dots, n\}, \left( \prod_{m=1}^k \sigma_{i_m, j_m} \right) (x_j) = z_j. \end{aligned}$$

We now define the *second term complexity* of  $\mathcal{Z}$  by

$$\ell_{\mathcal{X}}(\mathcal{Z}) = \min \left\{ \ell \left| \begin{array}{l} \exists a_1, \dots, a_\ell \in \{1, \dots, n\} \text{ pairwise distinct,} \\ \exists I \in \text{Inn}^\#(W_n) \text{ such that, } \exists k \in \mathbb{N} \\ \exists (i_1, j_1), \dots, (i_k, j_k) \in \{1, \dots, n\} \times \{a_1, \dots, a_\ell\} - \{(x, x) \mid x \in \{a_1, \dots, a_\ell\}\}, \\ \forall j \in \{1, \dots, n\}, I \circ \left( \prod_{m=1}^k \sigma_{i_m, j_m} \right) (x_j) = z_j \end{array} \right. \right\}.$$

The intuition behind the second term complexity is the following. We want to count the minimal number  $N$  of elements of  $\{x_1, \dots, x_n\}$  such that, for all  $j \in \{1, \dots, n\}$ , the generator  $z_j$  can be obtained from  $x_j$  using partial conjugations by at most  $N$  twistors.

Note that, in the definition of  $\ell_{\mathcal{X}}(\mathcal{Z})$ , if  $\left( \prod_{m=1}^k \sigma_{i_m, j_m} \right) (x_j) = z_j$ , and if  $m_1, m_2 \in \{1, \dots, k\}$  are distinct, we do not require that  $j_{m_1} \neq j_{m_2}$ , so that the same twistor can appear in distinct partial conjugations. Note also that  $\ell_{\mathcal{X}}(\mathcal{Z})$  does not depend on the choice of a representative of  $\mathcal{Z}$ .

**Lemma 3.3.3.** *Let  $\mathcal{X}$ ,  $(X, \rho)$ ,  $v_1, \dots, v_n$  and  $(\mathcal{Y}_i)_{i=1, \dots, n}$  be as above.*

(1) *Fix  $i \in \{1, \dots, n\}$  and let  $\mathcal{X}'$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_i$  and distinct from  $\mathcal{X}$ . Then  $\ell_{\mathcal{X}}(\mathcal{X}') = 1$  and a set  $\{a_1, \dots, a_\ell\}$  realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}')$  is  $\{i\}$ .*

(2) *Let  $B(\mathcal{X}, r)$  be the closed ball in  $L_n$  of radius  $r$  centered at  $\mathcal{X}$ . Let  $\mathcal{Z} \in B(\mathcal{X}, 4) \cap O_n$ . Then  $\ell_{\mathcal{X}}(\mathcal{Z}) \leq 2$ . Moreover, the set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  is unique.*

**Proof.** Let  $(X', \rho')$  be a representative of  $\mathcal{X}'$ . Let  $y_1, \dots, y_n$  be the preimage by  $\rho'$  of the generators of the nontrivial vertex groups of  $X'$ . Then, up to composing by an inner automorphism and reordering, for all  $j \in \{1, \dots, n\}$ , either  $y_j = x_j$  or  $y_j = x_i x_j x_i$ . Thus, for all  $j \in \{1, \dots, n\}$ , the only twistor that we need in order to obtain  $y_j$  from  $x_j$  using partial conjugations is  $x_i$ . Since  $\mathcal{X}' \neq \mathcal{X}$ , it follows that  $\ell_{\mathcal{X}}(\mathcal{X}') = 1$  and that a set realizing the minimum defining it is  $\{i\}$ .

For the second assertion, let  $Z$  be a representative of  $\mathcal{Z}$ , and let  $z_1, \dots, z_n$  be the preimage by the marking of the generators of the vertex groups. Then, there exist  $j, k \in \{1, \dots, n\}$  such that, for all  $m \in \{1, \dots, n\}$ , one of the following holds:

- (1)  $z_m = x_m$ ,
- (2)  $z_m = x_j x_m x_j$ ,
- (3)  $z_m = x_k x_m x_k$ ,
- (4)  $z_m = x_k x_j x_k x_m x_k x_j x_k$ ,
- (5)  $z_m = x_k x_j x_m x_j x_k$ .

Thus, for all  $m \in \{1, \dots, n\}$ , as we only need  $x_j$  and  $x_k$  as twistors to obtain  $z_m$  from  $x_m$ , we see that  $\ell_{\mathcal{X}}(\mathcal{Z}) \leq 2$ .

Moreover, the twistors  $x_j$  and  $x_k$  are the unique elements of  $\{x_1, \dots, x_n\}$  such that, for all  $i \in \{1, \dots, n\}$ , the generator  $z_i$  is obtained from  $x_i$  by partial conjugations using  $x_j$  and  $x_k$  as twistors. Thus, for all  $\mathcal{Z} \in B(\mathcal{X}, 4) \cap O_n$ , the set  $\{a_1, \dots, a_\ell\}$  realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  is unique.  $\square$

We isolate here a technical argument that will appear frequently in the proof of Lemma 3.3.5.

**Lemma 3.3.4.** *Let  $\mathcal{X}$ ,  $(X, \rho)$ ,  $v_1, \dots, v_n$  and  $(\mathcal{Y}_i)_{i=1, \dots, n}$  be as above.*

*Fix  $i \in \{1, \dots, n\}$  and let  $\mathcal{X}'$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_i$  and distinct from  $\mathcal{X}$ . Let  $k, \ell \in \{1, \dots, n\} - \{i\}$  be distinct. Let  $\mathcal{X}_k^{(2)}$  be a  $\{0\}$ -star such that:*

- $d(\mathcal{X}', \mathcal{X}_k^{(2)}) = 2$ ,
- $\ell_{\mathcal{X}}(\mathcal{X}_k^{(2)}) = 2$  and a set realizing the minimum defining it is  $\{i, k\}$ .

*Let  $\mathcal{X}_k^{(3)}$  be a  $\{0\}$ -star at distance 2 of  $\mathcal{X}_k^{(2)}$  and such that any set realizing  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)})$  contains  $\ell$ . Then  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)}) \geq 3$ .*

**Proof.** Let  $(X', \rho')$  be a representative of  $\mathcal{X}'$ . Let  $w_1, \dots, w_n$  be the leaves of the underlying graph of  $X'$ , and, for  $m \in \{1, \dots, n\}$ , let  $y_m$  be the preimage by  $\rho'$  of the generator of the group associated with  $w_m$ . For  $j \in \{2, 3\}$ , let  $(X_k^{(j)}, \psi^{(j)})$  be a representative of  $\mathcal{X}_k^{(j)}$ , let  $w_1^{(j)}, \dots, w_n^{(j)}$  be the  $n$  leaves of the underlying graph of  $X_k^{(j)}$  and, for  $m \in \{1, \dots, n\}$ , let  $y_m^{(j)}$  be the preimage by  $\psi^{(j)}$  of the generator of the group associated with  $w_m^{(j)}$ . Note that, up to composition by an inner automorphism and reordering, for all  $m \in \{1, \dots, n\}$ ,

$$y_m^{(2)} = x_i^{\gamma_m} x_k^{\beta_m} x_i^{\alpha_m} x_m x_i^{\alpha_m} x_k^{\beta_m} x_i^{\gamma_m}, \quad \alpha_m, \beta_m, \gamma_m \in \{0, 1\}.$$

Note also that  $\gamma_m = 1$  precisely when  $y_k = x_i x_k x_i$  and  $\beta_m = 1$ . Thus, for all  $m \in \{1, \dots, n\}$ , the element  $y_m^{(2)}$  is obtained from  $x_m$  using partial conjugations with twistors  $x_i$  and  $x_k$ . Moreover, as  $k \neq i$ , and as  $\mathcal{X}' \neq \mathcal{X}$ , there exists  $n_1 \in \{1, \dots, n\}$  such that  $\alpha_{n_1} \neq 0$  or  $\gamma_{n_1} \neq 0$ . Since  $\mathcal{X}_k^{(2)} \neq \mathcal{X}'$ , there exists  $n_2$  such that  $\beta_{n_2} \neq 0$ .

As  $\ell$  is contained in any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)})$ , there exists  $p \in \mathbb{N}$  and  $m_1 \in \{1, \dots, n\}$  such that

$$\left( \prod_{m=1}^p \sigma_{i_m, j_m} \right) (x_{m_1}) = y_{m_1}^{(3)},$$

and there exists  $m$  such that  $j_m = \ell$ .

**Claim.** The elements  $x_k$  and  $x_i$  are twistors of any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)})$ .

**Proof.** As  $\ell$  is contained in any set realizing  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)})$ , a representative of  $\mathcal{X}_k^{(3)}$  is obtained from  $X_k^{(2)}$  by contracting the edge adjacent to  $w_\ell^{(2)}$  and then blowing-up an edge at the central vertex. We then distinguish different cases according to the value of  $y_\ell^{(2)}$ .

If  $y_\ell^{(2)} = x_\ell$ , then for all  $m \in \{1, \dots, n\}$ ,

$$y_m^{(3)} = x_\ell^{\delta_m} x_i^{\gamma_m} x_k^{\beta_m} x_i^{\alpha_m} x_m x_i^{\alpha_m} x_k^{\beta_m} x_i^{\gamma_m} x_\ell^{\delta_m}, \quad \alpha_m, \beta_m, \gamma_m, \delta_m \in \{0, 1\}.$$

Since  $n_1$  and  $n_2$  are such that  $\alpha_{n_1} \neq 0$  or  $\gamma_{n_1} \neq 0$ , and  $\beta_{n_2} \neq 0$ , the claim follows.

If  $y_\ell^{(2)} = x_i x_\ell x_i$ , then for all  $m \in \{1, \dots, n\}$ , we have

$$y_m^{(3)} = x_i^{\delta_m} x_\ell^{\delta_m} x_i^{\gamma_m + \delta_m} x_k^{\beta_m} x_i^{\alpha_m} x_m x_i^{\alpha_m} x_k^{\beta_m} x_i^{\gamma_m + \delta_m} x_\ell^{\delta_m} x_i^{\delta_m}, \quad \alpha_m, \beta_m, \gamma_m, \delta_m \in \{0, 1\}.$$

Since  $n_2$  is such that  $\beta_{n_2} \neq 0$  and  $n_1$  is such that  $\gamma_{n_1} + \delta_{n_1} \neq 0$ , or  $\alpha_{n_1} \neq 0$  or  $\delta_{n_1} \neq 0$ , the claim follows.

If  $y_\ell^{(2)} = x_k x_\ell x_k$ , then for all  $m \in \{1, \dots, n\}$ , we have

$$y_m^{(3)} = x_k^{\delta_m} x_\ell^{\delta_m} x_k^{\delta_m} x_i^{\gamma_m} x_k^{\beta_m} x_i^{\alpha_m} x_m x_i^{\alpha_m} x_k^{\beta_m} x_i^{\gamma_m} x_k^{\delta_m} x_\ell^{\delta_m} x_k^{\delta_m}, \quad \alpha_m, \beta_m, \gamma_m, \delta_m \in \{0, 1\}.$$

Since  $n_1$  and  $n_2$  are such that  $\alpha_{n_1} \neq 0$  or  $\gamma_{n_1} \neq 0$ , and  $\beta_{n_2} \neq 0$ , the result follows.

Finally, if  $y_\ell^{(2)} = x_k x_i x_\ell x_i x_k$ , or if  $y_\ell^{(2)} = x_i x_k x_\ell x_k x_i$ , or if  $y_\ell^{(2)} = x_i x_k x_i x_\ell x_i x_k x_i$ , then  $y_{m_1}$  is obtained from  $x_{m_1}$  using  $x_i$ ,  $x_k$  and  $x_\ell$  as twistors. This concludes the proof of the claim.  $\square$

Thus,  $i$ ,  $k$  and  $\ell$  are contained in any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)})$ , and this implies that  $\ell_{\mathcal{X}}(\mathcal{X}_k^{(3)}) \geq 3$ .  $\square$

We are now ready to prove a lemma concerning the number of embedded paths in  $B(\mathcal{X}, 4)$ .

**Lemma 3.3.5.** *Let  $\mathcal{X}$ ,  $(X, \rho)$ ,  $v_1, \dots, v_n$  and  $(\mathcal{Y}_i)_{i=1, \dots, n}$  be as above.*

*Fix  $i \in \{1, \dots, n\}$  and let  $\mathcal{X}'$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_i$  and distinct from  $\mathcal{X}$ . Let  $(X', \rho')$  be a representative of  $\mathcal{X}'$  and let  $\overline{X}'$  be the underlying graph of  $X'$ . Let*

$$\{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')}\} \subseteq \{1, \dots, \hat{i}, \dots, n\}$$

*be a set realizing the minimum defining  $k_{\mathcal{X}, i}(\mathcal{X}')$ , and  $j \in \{1, \dots, \hat{i}, \dots, n\}$ . Let  $x'_1, \dots, x'_n$  be the preimages by  $\rho'$  of the generators of the nontrivial vertex groups. Up to reordering, suppose that, for all  $k \in \{1, \dots, n\}$ ,  $x'_k$  is obtained from  $x_k$  by a conjugation.*

- (1) *If  $x'_j = x_j$ , the number of distinct injective edge paths in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  of length at most 5 between  $\mathcal{X}'$  and  $\mathcal{Y}_j$  is equal to  $2^{n - k_{\mathcal{X}, i}(\mathcal{X}') - 2} - 1$ .*
- (2) *If  $x'_j = x_i x_j x_i$ , the number of distinct injective edge paths in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  of length at most 5 between  $\mathcal{X}'$  and  $\mathcal{Y}_j$  is equal to  $2^{k_{\mathcal{X}, i}(\mathcal{X}') - 1} - 1$ .*
- (3) *Let  $\mathcal{Z}$  be a  $\{0\}$ -star distinct from  $\mathcal{X}$  and adjacent to  $\mathcal{Y}_j$  and such that  $k_{\mathcal{X}, j}(\mathcal{Z}) = 1$ . Let  $\{t\}$  be a set realizing the minimum defining  $k_{\mathcal{X}, j}(\mathcal{Z})$ . Suppose that  $x'_j = x_j$ .  
If  $t \in \{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')}\}$ , then there is no path between  $\mathcal{X}'$  and  $\mathcal{Z}$  of length at most 4 in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$ . If  $t \notin \{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')}\}$ , then there is at least one path between  $\mathcal{X}'$  and  $\mathcal{Z}$  of length at most 4 in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$ .*

**Proof.** We prove the case  $x'_j = x_j$ . The proof of the case  $x'_j = x_i x_j x_i$  is similar. The proof consists in showing that the possible arcs  $P$  are as represented in Figure 3.4.

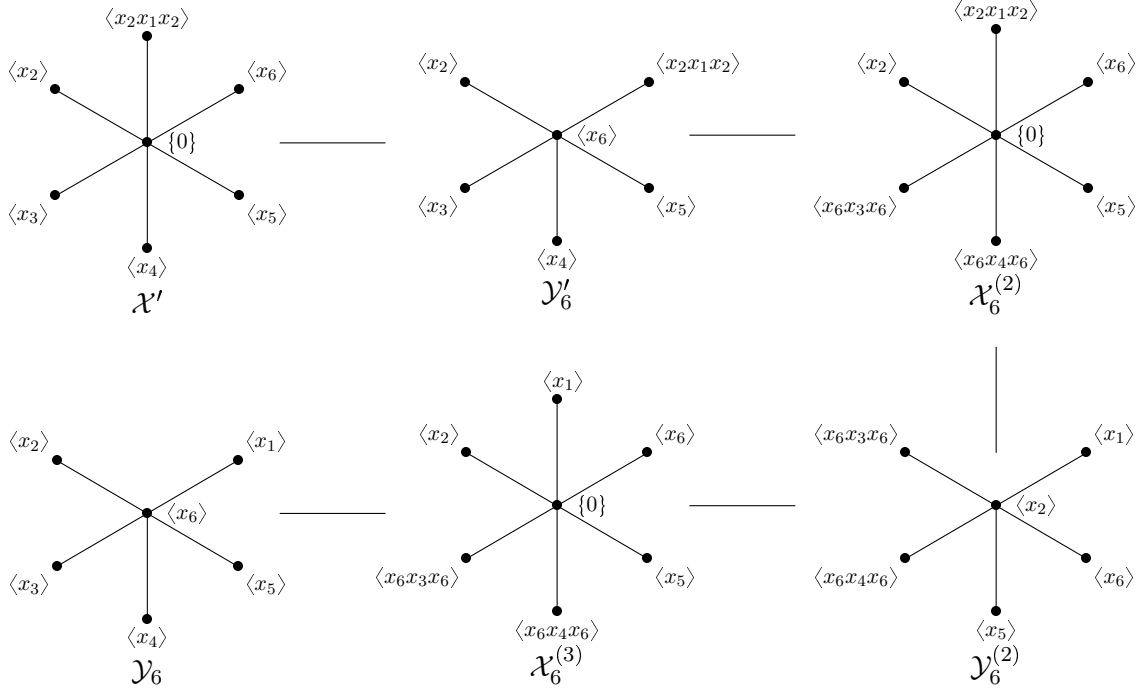


Figure 3.4: Example of a path in Lemma 3.3.5 between  $\mathcal{X}'$  (adjacent to  $\mathcal{Y}_i$  with  $i = 2$ ) and  $\mathcal{Y}_j$  with  $j = 6$ .

Let  $P$  be an arc (that is, an injective edge path), in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  between  $\mathcal{X}'$  and  $\mathcal{Y}_j$  of length at most 5. Let  $w_1, \dots, w_n$  be the leaves of  $\overline{X}'$ , and, for  $k \in \{1, \dots, n\}$ , let  $y_k$  be the preimage by  $\rho'$  of the generator of the group associated with  $w_k$ . Note that, up to reordering and composing by an inner automorphism, if  $k \notin \{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')} \}$ , then  $y_k = x_k$ , and, if  $k \in \{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')} \}$ , then  $y_k = x_i x_k x_i$ . For  $k \neq i$ , let  $\mathcal{Y}'_k$  be the  $F$ -star such that a representative of  $\mathcal{Y}'_k$  is obtained from  $X'$  by contracting the edge adjacent to  $w_k$ .

**Claim.** If  $k \notin \{i, j\}$ , the path  $P$  cannot contain  $\mathcal{Y}'_k$ .

**Proof.** Suppose towards a contradiction that  $\mathcal{Y}'_k \in P$ , with  $k \notin \{i, j\}$ . Since  $P$  is an arc, there exists in  $P$  a  $\{0\}$ -star  $\mathcal{X}_k^{(2)}$  adjacent to  $\mathcal{Y}'_k$  and distinct from  $\mathcal{X}'$ . By Lemma 3.3.3 (2), we see that  $l_{\mathcal{X}}(\mathcal{X}_k^{(2)}) \leq 2$ . We claim that  $l_{\mathcal{X}}(\mathcal{X}_k^{(2)}) = 2$ . Indeed, let  $(X_k^{(2)}, \psi)$  be a representative of  $\mathcal{X}_k^{(2)}$ , let  $w_1^{(2)}, \dots, w_n^{(2)}$  be the  $n$  leaves of the underlying graph of  $X_k^{(2)}$  and, for  $m \in \{1, \dots, n\}$ , let  $y_m^{(2)}$  be the preimage by  $\psi$  of the generator of the group associated with  $w_m^{(2)}$ . Note that, up to composition by an inner automorphism and reordering, for all  $m \in \{1, \dots, n\}$ ,

$$y_m^{(2)} = x_i^{\gamma_m} x_k^{\beta_m} x_i^{\alpha_m} x_m x_i^{\alpha_m} x_k^{\beta_m} x_i^{\gamma_m}, \quad \alpha_m, \beta_m, \gamma_m \in \{0, 1\}.$$

Note also that  $\gamma_m = 1$  precisely when  $y_k = x_i x_k x_i$  and  $\beta_m = 1$ . Thus, for all  $m \in \{1, \dots, n\}$ , the element  $y_m^{(2)}$  is obtained from  $x_m$  using partial conjugations with twistors  $x_i$  and  $x_k$ . Moreover, as  $k \neq i$ , and as  $\mathcal{X}' \neq \mathcal{X}$ , there exists  $n_1$  such that either  $\alpha_{n_1} \neq 0$  or  $\gamma_{n_1} \neq 0$ . As  $\mathcal{X}_k^{(2)} \neq \mathcal{X}'$ , there exists  $n_2$  such that  $\beta_{n_2} \neq 0$ . It implies that  $l_{\mathcal{X}}(\mathcal{X}_k^{(2)}) = 2$ .

Therefore, by Lemma 3.3.3 (1), the  $\{0\}$ -star  $\mathcal{X}_k^{(2)}$  is not adjacent to  $\mathcal{Y}_j$  since any  $\{0\}$ -star  $\mathcal{Z}$  adjacent to  $\mathcal{Y}_j$  is such that  $l_{\mathcal{X}}(\mathcal{Z}) = 1$ .

So  $P$  contains an  $F$ -star  $\mathcal{Y}_k^{(2)}$  adjacent to  $\mathcal{X}_k^{(2)}$  and distinct from  $\mathcal{Y}'_k$  (see Figure 3.4 with  $k = 6$ ). We claim that a representative of  $\mathcal{Y}_k^{(2)}$  is obtained from  $X_k^{(2)}$  by contracting the edge adjacent to  $w_i^{(2)}$ . Indeed, if it is not the case, one of the following two possibilities holds.

(i) A representative of  $\mathcal{Y}_k^{(2)}$  is obtained from  $X_k^{(2)}$  by contracting the edge adjacent to  $w_k^{(2)}$ . But then we go back to  $\mathcal{Y}'_k$ , which contradicts the fact that  $P$  is an arc.

(ii) A representative of  $\mathcal{Y}_k^{(2)}$  is obtained from  $X_k^{(2)}$  by contracting the edge adjacent to  $w_\ell^{(2)}$ , with  $\ell \neq i, k$ . Let  $\mathcal{X}_k^{(3)}$  be the  $\{0\}$ -star in  $P$  adjacent to  $\mathcal{Y}_k^{(2)}$  and distinct from  $\mathcal{X}_k^{(2)}$ , and let  $X_k^{(3)}$  be a representative of  $\mathcal{X}_k^{(3)}$ . Then, there exist  $p \in \mathbb{N}$ ,  $m_1 \in \{1, \dots, n\}$  and  $i_1, \dots, i_p, j_1, \dots, j_p \in \{1, \dots, n\}$  such that, if  $y_{m_1}$  is the preimage by the marking of a nontrivial vertex group of  $X_k^{(3)}$ , we have

$$\left( \prod_{m=1}^p \sigma_{i_m, j_m} \right) (x_{m_1}) = y_{m_1},$$

and there exists  $m$  such that  $j_m = \ell$ . Therefore, by Lemma 3.3.4, we see that  $l_{\mathcal{X}}(\mathcal{X}_k^{(3)}) \geq 3$ . But, by Lemma 3.3.3 (2), we have  $\mathcal{X}_k^{(3)} \notin B(\mathcal{X}, 4)$  and this contradicts the fact that  $P \subseteq B(\mathcal{X}, 4)$ .

Therefore a representative of  $\mathcal{Y}_k^{(2)}$  is obtained from  $X_k^{(2)}$  by contracting the edge adjacent to  $w_i^{(2)}$  (see Figure 3.4).

But then, for every  $\{0\}$ -star  $\mathcal{Z}$  adjacent to  $\mathcal{Y}_k^{(2)}$ , the set realizing  $l_{\mathcal{X}}(\mathcal{Z})$  must contain  $k$ . Indeed, let  $Z$  be a representative of  $\mathcal{Z}$  and let  $z_1, \dots, z_n$  be the preimages by the marking of the generators of the nontrivial vertex groups. Up to composition by an inner automorphism and reordering,  $z_i = y_i^{(2)}$  and there exists  $m \in \{1, \dots, n\}$  such that  $z_m \neq y_m^{(2)}$ . Thus, if  $y_i^{(2)}$  is obtained from  $x_i$  using partial conjugations such that one of the twistors is  $x_k$ , then a set realizing the minimum defining  $l_{\mathcal{X}}(\mathcal{Z})$  contains  $k$ . Moreover, if  $y_i^{(2)} = x_i$ , then, since  $n_2$  is such that  $\beta_{n_2} \neq 0$ , we see that  $z_{n_2}$  is obtained from  $x_{n_2}$  using partial conjugations such that one of the twistors is  $x_k$ . In any case, the set realizing  $l_{\mathcal{X}}(\mathcal{Z})$  must contain  $k$ . On the other hand, by Lemma 3.3.3 (1), if  $\mathcal{Z}'$  is a  $\{0\}$ -star adjacent to  $\mathcal{Y}_j$ , then the set realizing  $l_{\mathcal{X}}(\mathcal{Z}')$  only contains  $j$ . Since a set realizing the second term complexity is unique by Lemma 3.3.3 (2), we see that  $\mathcal{Z}$  and

$\mathcal{Y}_j$  are not adjacent in  $L_n$ . This leads to a contradiction since we suppose that the length of  $P$  is at most 5.  $\square$

So the above claim implies that the path  $P$  either contains  $\mathcal{Y}_i$  or  $\mathcal{Y}'_j$  (note that  $\mathcal{Y}_i = \mathcal{Y}'_i$ ). The case  $\mathcal{Y}_i \in P$  cannot occur by the following claim.

**Claim.** There does not exist a  $\{0\}$ -star adjacent to  $\mathcal{Y}_i$  and distinct from  $\mathcal{X}$  at distance 3 from  $\mathcal{Y}_j$  in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$ .

**Proof.** Let  $\mathcal{X}_i^{(2)}$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_i$  and distinct from  $\mathcal{X}$  and  $\mathcal{X}'$ . Let  $X_i^{(2)}$  be a representative of  $\mathcal{X}_i^{(2)}$ . Let  $v_1^{(2)}, \dots, v_n^{(2)}$  be the leaves of the underlying graph of  $X_i^{(2)}$  and, for  $m \in \{1, \dots, n\}$ , let  $a_m^{(2)}$  be the preimage by the marking of the generator of the groups associated with  $v_m^{(2)}$ . Up to reordering and composing by an inner automorphism, we can suppose that, for all  $m \in \{1, \dots, n\}$ , either  $a_m^{(2)} = x_m$  or  $a_m^{(2)} = x_i x_m x_i$  (this is possible by Lemma 3.3.3 (1)). Let  $\mathcal{X}_i^{(3)}$  be a  $\{0\}$ -star distinct from  $\mathcal{X}$  and at distance 2 of  $\mathcal{X}_i^{(2)}$ , let  $X_i^{(3)}$  be a representative of  $\mathcal{X}_i^{(3)}$  and let  $a_1^{(3)}, \dots, a_n^{(3)}$  be the preimages by the marking of the generators of the nontrivial vertex groups. Then one of the following holds:

(a) The  $\{0\}$ -star  $\mathcal{X}_i^{(3)}$  is adjacent to  $\mathcal{Y}_i$ . By Lemma 3.3.3 (1), a set realizing  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$  is equal to  $\{i\}$ . On the other hand, the set realizing the minimum defining the second term complexity of every  $\{0\}$ -star adjacent to  $\mathcal{Y}_j$  contains  $j$ . As  $i \neq j$ , we see that  $\mathcal{X}^{(3)}$  cannot be adjacent to  $\mathcal{Y}_j$ .

(b) There exist  $p \in \mathbb{N}$ ,  $k \in \{1, \dots, n\} - \{i\}$ ,  $\ell, i_1, \dots, i_p, j_1, \dots, j_p \in \{1, \dots, n\}$  and  $s \in \{1, \dots, p\}$  such that

$$\left( \prod_{m=1}^p \sigma_{i_m, j_m} \right) (x_\ell) = a_\ell^{(3)}$$

and  $j_s = k$ . Thus  $k$  is contained in any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$ . Moreover, we claim that  $i$  is contained in any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$ . Indeed, as  $k$  is contained in a set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$ , a representative of  $\mathcal{X}_i^{(3)}$  is obtained from  $X_i^{(2)}$  as follows. We first contract the edge adjacent to the vertex  $v_k^{(2)}$ . This gives an  $F$ -star denoted by  $Y_i^{(3)}$ . Then, a representative of  $\mathcal{X}_i^{(3)}$  is obtained from  $Y_i^{(3)}$  by blowing-up an edge. If  $a_k^{(2)} = x_k$ , then, as  $\mathcal{X}_i^{(2)}$  is adjacent to  $\mathcal{Y}_i$ , a set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(2)})$  is equal to  $\{i\}$  by Lemma 3.3.3 (1). As  $a_k^{(2)} = x_k$ , we see that either  $a_m^{(3)} = a_m^{(2)}$  or  $a_m^{(3)} = x_k a_m^{(2)} x_k$ . Thus, as  $i \neq k$ , we see that  $i$  is contained in a set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$ . If  $a_k^{(2)} = x_i x_k x_i$ , then  $a_k^{(3)} = a_k^{(2)} = x_i x_k x_i$  and any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$  must contain  $i$ . Therefore, in any case, we have that  $\{i, k\}$  is contained in any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)})$ . This shows that  $\ell_{\mathcal{X}}(\mathcal{X}_i^{(3)}) \geq 2$ . However, since the  $\{0\}$ -stars adjacent to  $\mathcal{Y}_j$  have second term complexity equal to 1 by Lemma 3.3.3 (1), we see that  $\mathcal{X}_i^{(2)}$  cannot be such that  $d_{B(\mathcal{X}, 4) - \{\mathcal{X}\}}(\mathcal{X}_i^{(2)}, \mathcal{Y}_j) = 3$ .  $\square$

Thus,  $P$  contains  $\mathcal{Y}'_j$ . As any two distinct  $F$ -stars are at distance at least 2 in  $L_n$ , the path  $P$  contains a  $\{0\}$ -star  $\mathcal{X}_j^{(2)}$  adjacent to  $\mathcal{Y}'_j$  and distinct from  $\mathcal{X}'$  (see Figure 3.4). Let  $(X_j^{(2)}, \psi)$  be a representative of  $\mathcal{X}_j^{(2)}$ , let  $w_1^{(2)}, \dots, w_n^{(2)}$  be the  $n$  leaves of the underlying graph of  $X_j^{(2)}$  and, for  $m \in \{1, \dots, n\}$ , let  $y_m^{(2)}$  be the preimage by  $\psi$  of the generator of the group associated with  $w_m^{(2)}$ . Note that, up to composition by an inner automorphism and reordering, for all  $m \in \{1, \dots, n\}$ ,

$$y_m^{(2)} = x_j^{\beta_m} x_i^{\alpha_m} x_m x_i^{\alpha_m} x_j^{\beta_m}, \quad \alpha_m, \beta_m \in \{0, 1\}.$$

As  $\mathcal{X}_j^{(2)} \neq \mathcal{X}'$ , there exist  $k, l \in \{1, \dots, n\}$  such that  $\alpha_k \neq 0$  and  $\beta_l \neq 0$ . Thus,  $\ell_{\mathcal{X}}(\mathcal{X}_j^{(2)}) = 2$  and a set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_j^{(2)})$  is  $\{i, j\}$ . This also implies that the  $\{0\}$ -star  $\mathcal{X}_j^{(2)}$  is not adjacent to  $\mathcal{Y}_j$  by Lemma 3.3.3 (1). So  $P$  contains an  $F$ -star  $\mathcal{Y}_j^{(2)}$  adjacent to  $\mathcal{X}_j^{(2)}$  and distinct from  $\mathcal{Y}_j$  and  $\mathcal{Y}'_j$  (see Figure 3.4). We claim that a representative of  $\mathcal{Y}_j^{(2)}$  is obtained from  $X_j^{(2)}$  by contracting the edge that contains  $w_i^{(2)}$ . Indeed, if it is not the case, then one of the following holds.

(i) A representative of  $\mathcal{Y}_j^{(2)}$  is obtained from  $X_j^{(2)}$  by contracting the edge that contains  $w_j^{(2)}$ . Then  $\mathcal{Y}_j^{(2)} = \mathcal{Y}'_j$ , and this contradicts the fact that  $P$  is an arc.

(ii) A representative of  $\mathcal{Y}_j^{(2)}$  is obtained from  $X_j^{(2)}$  by contracting the edge that contains  $w_\ell^{(2)}$ , with  $\ell \neq i, j$ . Let  $\mathcal{X}_\ell^{(3)}$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_j^{(2)}$  and distinct from  $\mathcal{X}_j^{(2)}$ , and let  $X_\ell^{(3)}$  be a representative of  $\mathcal{X}_\ell^{(3)}$ . As  $\mathcal{X}_\ell^{(3)} \neq \mathcal{X}_j^{(2)}$  any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{X}_\ell^{(3)})$  must contain  $\ell$ . Accordingly, since  $\ell \neq i, j$  we see by Lemma 3.3.4 that  $\ell_{\mathcal{X}}(\mathcal{X}_\ell^{(3)}) \geq 3$ . This contradicts the fact that  $P \subseteq B(\mathcal{X}, 4) - \{\mathcal{X}\}$  by Lemma 3.3.3 (2).

Therefore, a representative of  $\mathcal{Y}_j^{(2)}$  is obtained from  $X_j^{(2)}$  by contracting the edge adjacent to  $w_i^{(2)}$ . We now distinguish two cases, according to the value of  $\beta_i$ .

- Claim.** (1) If  $\beta_i = 0$ , then, for all  $m \in \{1, \dots, \hat{i}, \dots, n\}$ , we have  $(\alpha_m, \beta_m) \neq (1, 1)$ .  
(2) If  $\beta_i = 1$ , then, for all  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\alpha_m = 1$ , the pair  $(\alpha_m, \beta_m)$  equals  $(1, 1)$ .

**Proof.** Let  $\mathcal{Z}$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_j^{(2)}$ , let  $Z$  be a representative of  $\mathcal{Z}$ , and let  $z_1, \dots, z_n$  be the preimages by the marking of the generators of the nontrivial vertex groups of  $Z$ .

- (1) Suppose that  $\beta_i = 0$  and that there exists  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $(\alpha_m, \beta_m) = (1, 1)$ . Then any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  must contain  $i$  because, as  $y_m^{(2)} = x_j x_i x_m x_i x_j$ , and as  $y_i^{(2)} = x_i$ , we see that, up to composing by an inner automorphism and reordering, we have that  $z_i = x_i$  and either  $z_m = x_j x_i x_m x_i x_j$  or  $z_m = x_i x_j x_i x_m x_i x_j$ .



(2) Suppose now that  $\beta_i = 1$  and that there exists  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\alpha_m = 1$  and such that the pair  $(\alpha_m, \beta_m)$  equals  $(1, 0)$ . Then any set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  must contain  $i$  because, as  $y_m^{(2)} = x_i x_m x_i$ , and as  $y_i^{(2)} = x_j x_i x_j$ , we see that, up to composing by an inner automorphism and reordering, we have that  $z_i = x_j x_i x_j$  and either  $z_m = x_i x_m x_i$  or  $z_m = x_j x_i x_j x_i x_m x_i x_j x_i x_j$ .

So, in both cases, for every  $\{0\}$ -star  $\mathcal{Z}$  adjacent to  $\mathcal{Y}_j^{(2)}$ , the set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  must contain  $i$ .

Let  $\mathcal{Z}$  be the  $\{0\}$ -star in  $P$  adjacent to  $\mathcal{Y}^{(2)}$  and distinct from  $\mathcal{X}^{(2)}$ . Then the set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  must contain  $i$  by the above. Since the length of  $P$  is at most 5, the  $\{0\}$ -star  $\mathcal{Z}$  is adjacent to  $\mathcal{Y}_j$ . But then, by Lemma 3.3.3 (1), the set realizing the minimum defining  $\ell_{\mathcal{X}}(\mathcal{Z})$  is equal to  $j$ . Since a set realizing the second term complexity is unique by Lemma 3.3.3 (2), we see that  $\mathcal{Z}$  and  $\mathcal{Y}_j$  cannot be adjacent and this leads to a contradiction.  $\square$

So if  $\beta_i = 0$ , then, for all  $m \in \{1, \dots, \hat{i}, \dots, n\}$ , the pair  $(\alpha_m, \beta_m) \neq (1, 1)$  and if  $\beta_i = 1$ , for all  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\alpha_m = 1$ , the pair  $(\alpha_m, \beta_m)$  equals  $(1, 1)$ .

We now claim that there are exactly  $2^{n-k_{\mathcal{X},i}(\mathcal{X}')-2} - 1$  possible values for the sequence  $(\beta_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_n)$ .

First, if  $\beta_i = 1$ , then by the above claim, for all  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\alpha_m = 1$ , we have  $\beta_m = 1$ . Using a global conjugation by  $x_j$ , it then follows that every marked graph of groups whose associated sequence  $(\beta_1, \dots, \beta_n)$  satisfies the above claim and is such that  $\beta_i = 1$  is equivalent to a marked graph of groups whose associated sequence  $(\beta'_1, \dots, \beta'_n)$  satisfies the above claim and is such that  $\beta'_i = 0$ . Thus we can suppose that, for such a sequence  $(\beta_1, \dots, \beta_n)$ , we have  $\beta_i = 0$ .

Moreover, by the above claim, all the pairs  $(\alpha_m, \beta_m)$  such that  $\alpha_m = 1$  have the same value for  $\beta_m$ . Thus the sequence  $(\beta_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_n)$  is determined by the pairs  $(\alpha_m, \beta_m)$  such that  $\alpha_m = 0$  and the choice of  $\beta_m$ . By hypothesis, there are exactly  $k_{\mathcal{X},i}(\mathcal{X}')$  values of  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\alpha_m = 1$  since  $\alpha_m = 1$  if and only if  $x'_m = x_i x_m x_i$ . It then suffices to choose whether  $\beta_m = 0$  or  $\beta_m = 1$ . Furthermore, let  $(\beta_1, \dots, \beta_n)$  and  $(\beta'_1, \dots, \beta'_n)$  be two distinct sequences satisfying the above claim and such that  $\beta_i = \beta'_i = 0$ . Then there exists  $m \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\beta_m = 1$  and  $\beta'_m = 0$ . Thus, since  $\beta_i = \beta'_i = 0$ , the associated marked graph of groups are not equivalent and the two sequences give rise to two distinct equivalence classes of marked graph of groups. Finally, since  $\mathcal{X}^{(2)} \neq \mathcal{X}'$ , there exists  $k \in \{1, \dots, \hat{i}, \dots, n\}$  such that  $\beta_k = 1$ . Hence there are  $2^{n-k_{\mathcal{X},i}(\mathcal{X}')-2} - 1$  possible values for the sequence  $(\beta_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_n)$ .

Let  $\mathcal{Z}$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}_j^{(2)}$  and distinct from  $\mathcal{X}_j^{(2)}$  and let  $Z$  be a representative of  $\mathcal{Z}$ . Let  $z_1, \dots, z_n$  be the preimage by the marking of the nontrivial associated groups. Then, for every sequence  $(\beta_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_n)$  satisfying the above claim, there exists exactly one such  $\mathcal{Z}$  such that, up to composing by an inner automorphism and reordering, for all  $\ell \in \{1, \dots, n\}$ , we have either  $z_\ell = x_j x_\ell x_j$  or  $z_\ell = x_\ell$ . Such a  $\{0\}$ -star is adjacent to both  $\mathcal{Y}_j^{(2)}$  and  $\mathcal{Y}_j$ . We call this  $\{0\}$ -star  $\mathcal{X}_j^{(3)}$ .

Thus, there exists a unique  $\{0\}$ -star  $\mathcal{X}_j^{(3)}$  adjacent to both  $\mathcal{Y}_j^{(2)}$  and  $\mathcal{Y}_j$ . Since  $P$  is an arc of length at most 5, it must contain  $\mathcal{X}_j^{(3)}$ . Thus an arc in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  with length at most 5 between  $\mathcal{X}'$  and  $\mathcal{Y}_j$  is completely determined by a sequence  $(\beta_1, \dots, \hat{\beta}_i, \dots, \hat{\beta}_j, \dots, \beta_n)$  satisfying the above claim. This concludes the proof of Lemma 3.3.5 (1).

In order to prove the third assertion of the lemma, let  $P$  be an arc between  $\mathcal{X}'$  and  $\mathcal{Z}$  of length at most 4. Then there exists an arc  $P'$  between  $\mathcal{X}'$  and  $\mathcal{Y}_j$  of length at most 5 which contains  $P$ . Thus,  $P$  is contained in one of the paths constructed in the proof of the first assertion of the lemma. Therefore, using the notations of the proof of Lemma 3.3.5 (1), we see that  $\mathcal{Z} = \mathcal{X}_j^{(3)}$ . Let  $X_j^{(3)}$  be a representative of  $\mathcal{X}_j^{(3)}$  and let  $y_1^{(3)}, \dots, y_n^{(3)}$  be the preimages by the marking of the generators of the nontrivial vertex groups. Then for all  $m \in \{1, \dots, n\}$ , if  $\alpha_m = 0$ , then  $y_m^{(3)} = x_j^{\beta_m} x_m x_j^{\beta_m}$ , and if  $\alpha_m = 1$ , then either  $y_m^{(3)} = x_m$  or  $y_m^{(3)} = x_j x_m x_j$ . Moreover, by construction, we know that there exists  $m$  such that  $\alpha_m = 0$  and  $\beta_m \neq 0$ . As  $k_{\mathcal{X}, j}(\mathcal{Z}) = 1$ , and as  $\alpha_t = 0$  if and only if  $t \notin \{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')}\}$ , we see that there is an arc between  $\mathcal{X}'$  and  $\mathcal{Z}$  of length at most 4 if and only if  $t \notin \{i_1, \dots, i_{k_{\mathcal{X}, i}(\mathcal{X}')}\}$ . This concludes the proof.  $\square$

**Proposition 3.3.6.** *Let  $n \geq 4$ . Let  $\mathcal{X} \in O_n$ . Let  $f \in \text{Aut}(L_n)$  be such that  $f$  restricted to the star of  $\mathcal{X}$  is the identity. Then  $f = \text{id}_{L_n}$ .*

**Proof.** In order to prove Proposition 3.3.6, we prove that  $f$  fixes the star of all  $\{0\}$ -stars at distance 2 from  $\mathcal{X}$ . This concludes by propagation since  $L_n$  is connected.

First, we prove that  $f$  fixes  $B(\mathcal{X}, 2) \cap O_n - \{\mathcal{X}\}$ . Let  $\mathcal{X}_1, \mathcal{X}_2 \in B(\mathcal{X}, 2) \cap O_n$  be distinct  $\{0\}$ -stars. If there exist distinct  $i, j \in \{1, \dots, n\}$  such that  $\mathcal{X}_1$  is adjacent to  $\mathcal{Y}_i$  and  $\mathcal{X}_2$  is adjacent to  $\mathcal{Y}_j$ , then  $f(\mathcal{X}_1) \neq \mathcal{X}_2$  because  $f(\mathcal{Y}_i) = \mathcal{Y}_i$ ,  $f(\mathcal{Y}_j) = \mathcal{Y}_j$  and there is no  $\{0\}$ -star adjacent to both  $\mathcal{Y}_i$  and  $\mathcal{Y}_j$  apart from  $\mathcal{X}$ .

Suppose that there exists  $i$  such that  $\mathcal{Y}_i$  is adjacent to both  $\mathcal{X}_1$  and  $\mathcal{X}_2$ . For  $\alpha \in \{1, 2\}$ , let  $X_\alpha$  be a representative of  $\mathcal{X}_\alpha$  and let  $y_1^\alpha, \dots, y_n^\alpha$  be the preimages by the marking of the generators of the nontrivial vertex groups of  $X_\alpha$ . Since  $\mathcal{X}_1 \neq \mathcal{X}_2$ , we see that, up to reordering and composing by an inner automorphism, there exist  $j, k \in \{1, \dots, n\}$  such that  $y_i^1 = y_i^2 = x_i$ , such that  $y_j^1 = x_j$  and  $y_j^2 = x_i x_j x_i$  and such that  $y_k^1 = y_k^2$ . By Lemma 3.3.5 (1), if  $k_{\mathcal{X}, i}(\mathcal{X}_1) \neq k_{\mathcal{X}, i}(\mathcal{X}_2)$ , then the number of arcs of length at most 5 in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  between  $\mathcal{X}_1$  and  $\mathcal{Y}_k$  is distinct from the number of arcs of length at most 5 in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  between  $\mathcal{X}_2$  and  $\mathcal{Y}_k$ . Suppose that  $k_{\mathcal{X}, i}(\mathcal{X}_1) + k_{\mathcal{X}, i}(\mathcal{X}_2) \neq n - 1$ . In particular, we have that  $n - k_{\mathcal{X}, i}(\mathcal{X}_1) - 2 \neq k_{\mathcal{X}, i}(\mathcal{X}_2) - 1$ . Therefore, by Lemma 3.3.5 (1) and (2), the number of arcs of length at most 5 in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  between  $\mathcal{X}_1$  and  $\mathcal{Y}_j$  is distinct from the number of arcs of length at most 5 in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  between  $\mathcal{X}_2$  and  $\mathcal{Y}_j$ . Thus  $f(\mathcal{X}_1) \neq \mathcal{X}_2$  since  $f$  restricted to the star of  $\mathcal{X}$  is the identity. In particular, since  $n \geq 4$ , if  $\mathcal{X}' \in B(\mathcal{X}, 2) - \{\mathcal{X}\}$  is such that  $\mathcal{X}'$  is adjacent to  $\mathcal{Y}_i$  and that  $k_{\mathcal{X}, i}(\mathcal{X}') = 1$ , then  $f(\mathcal{X}') = \mathcal{X}'$ .

It remains the case where  $k_{\mathcal{X}, i}(\mathcal{X}_1) = k_{\mathcal{X}, i}(\mathcal{X}_2) = \frac{n-1}{2}$ . Let  $\mathcal{X}^k$  be the  $\{0\}$ -star adjacent to  $\mathcal{Y}_k$  such that  $k_{\mathcal{X}, i}(\mathcal{X}^k) = 1$  and such that the set realizing  $k_{\mathcal{X}, i}(\mathcal{X}^k)$  is  $\{j\}$ .

As  $k_{\mathcal{X},i}(\mathcal{X}^k) = 1$ , we have that  $f(\mathcal{X}^k) = \mathcal{X}^k$ . Moreover, as  $y_j^1 = x_j$  and as  $y_j^2 = x_i x_j x_i$ , Lemma 3.3.5 (3) implies that there is no path of length at most 4 between  $\mathcal{X}_2$  and  $\mathcal{X}^k$  in  $B(\mathcal{X}, 4) - \{\mathcal{X}\}$  while there is one such path between  $\mathcal{X}_1$  and  $\mathcal{X}^k$ . Thus  $f(\mathcal{X}_1) = \mathcal{X}_1$ . Hence  $f$  fixes  $B(\mathcal{X}, 2) \cap O_n - \{\mathcal{X}\}$ .

Now let  $\mathcal{X}' \in B(\mathcal{X}, 2) \cap O_n - \{\mathcal{X}\}$  and let  $\mathcal{Y}$  be the  $F$ -star adjacent to both  $\mathcal{X}$  and  $\mathcal{X}'$  (the uniqueness of this  $F$ -star follows from the uniqueness of the set realizing the minimum defining  $\ell_{\mathcal{X}}$ , see Lemma 3.3.3 (2)). Let  $X'$  be a representative of  $\mathcal{X}'$  and let  $v'_1, \dots, v'_n$  be the leaves of the underlying graph of  $X'$  and, for  $i \in \{1, \dots, n\}$ , let  $x'_i$  be the preimage by the marking of  $X'$  of the generators of the group associated with  $v'_i$ . Then, up to reordering, we can suppose that a representative of  $\mathcal{Y}$  is obtained from  $X'$  by contracting the edge adjacent to  $v'_n$ . Let  $\mathcal{Y}^1$  and  $\mathcal{Y}^2$  be two distinct  $F$ -stars adjacent to  $\mathcal{X}'$  and distinct from  $\mathcal{Y}$ . We prove that  $f(\mathcal{Y}^1) \neq \mathcal{Y}^2$ . Up to reordering, we can suppose that, for  $\alpha \in \{1, 2\}$ , a representative of  $\mathcal{Y}^\alpha$  is obtained from  $X'$  by contracting the edge adjacent to  $v'_\alpha$ . Let  $\mathcal{Z}$  be a  $\{0\}$ -star adjacent to  $\mathcal{Y}$  such that :

- (1)  $k_{\mathcal{X}',n}(\mathcal{Z}) = 1$  ;
- (2) a set realizing the minimum defining  $k_{\mathcal{X}',n}(\mathcal{Z})$  is  $\{1\}$ .

Then Lemma 3.3.5 (1) and (2) tells us that the number of paths of length at most 5 in  $B(\mathcal{X}', 4) - \{\mathcal{X}'\}$  between  $\mathcal{Z}$  and  $\mathcal{Y}^1$  is equal to  $2^{k_{\mathcal{X}',n}(\mathcal{Z})-1} - 1$  while the number of paths of length at most 5 in  $B(\mathcal{X}', 4) - \{\mathcal{X}'\}$  between  $\mathcal{Z}$  and  $\mathcal{Y}^2$  is equal to  $2^{n-k_{\mathcal{X}',n}(\mathcal{Z})-2} - 1$ . Since  $k_{\mathcal{X}',n}(\mathcal{Z}) = 1$ , since  $n \geq 4$  and since  $f$  restricted to the star of  $\mathcal{Y}$  is the identity, we see that  $f(\mathcal{Y}^1) \neq \mathcal{Y}^2$  and the proposition follows.  $\square$

**Proof of Theorem 3.3.1.** The uniqueness of  $\gamma$  is immediate since no automorphism of  $W_n$  fixes the conjugacy class of each element appearing in every free generating set of  $W_n$ . It thus suffices to prove that every automorphism preserving  $O_n$  and  $F_n$  is induced by an element of  $\text{Out}(W_n)$ . Let  $f$  be an automorphism of  $L_n$  preserving  $O_n$  and  $F_n$ . Since  $\text{Out}(W_n)$  acts transitively on  $O_n$ , we can suppose, up to composing by an element of  $\text{Out}(W_n)$ , that  $f$  fixes a  $\{0\}$ -star  $\mathcal{X}$ . Now  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})$  is isomorphic to  $\mathfrak{S}_n$  and every element of  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})$  acts on the underlying graph of a representative  $X$  of  $\mathcal{X}$  by permuting the leaves. As a representative of any  $F$ -star adjacent to  $\mathcal{X}$  is obtained from  $X$  by contracting the edge adjacent to a leaf, we see that  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{X})$  acts transitively on the link of  $\mathcal{X}$ . Thus, we can suppose, up to composing by an element of  $\text{Out}(W_n)$ , that  $f$  fixes the star of  $\mathcal{X}$ . Proposition 3.3.6 then implies that  $f$  is the identity. This concludes the proof of Theorem 3.3.1.  $\square$

### 3.4 Rigidity of the outer space of $W_n$

The aim of this section is to prove Theorem 3.1.1, by constructing an injective homomorphism  $\text{Aut}(K_n) \hookrightarrow \text{Aut}(L_n)$ . We first give a characterization of the  $\{0\}$ -stars and the  $F$ -stars which is preserved under automorphisms of  $K_n$ . This characterization relies on a study of the link of the vertices of  $K_n$ . We begin with some definitions.

**Definition 3.4.1.** Let  $X$  be a graph. A *join* of  $X$  is a decomposition of  $X$  into two nontrivial subgraphs  $A$  and  $B$  such that  $VA \cap VB = \emptyset$  and, for all  $a \in VA$  and  $b \in VB$ , the vertices  $a$  and  $b$  are adjacent in  $X$ . We then write  $X = A * B$ .

The fact of being decomposed as a join is preserved under automorphisms of graphs. In the case of a vertex  $x \in VK_n$ , there is a natural decomposition of the link  $\text{lk}(x)$  of  $x$  in  $K_n$ .

**Definition 3.4.2.** Let  $x = \mathcal{X} \in VK_n$ . Let  $X$  be a representative of  $\mathcal{X}$ .

(1) The *positive link* of  $x$ , denoted by  $\text{lk}_+(x)$ , is the maximal subgraph of  $\text{lk}(x)$  whose set of vertices consists in the homothety classes which have a representative that collapses onto  $X$ .

(2) The *negative link* of  $x$ , denoted by  $\text{lk}_-(x)$ , is the maximal subgraph of  $\text{lk}(x)$  whose set of vertices consists of homothety classes which have a representative  $Y$  such that  $X$  collapses onto  $Y$ .

For all vertices  $x$  of  $K_n$ , by definition of the adjacency in  $K_n$ , we have

$$\text{lk}(x) = \text{lk}_+(x) * \text{lk}_-(x).$$

It is in fact, as we will prove in Proposition 3.4.7 below, the only decomposition of  $\text{lk}(x)$  as a join.

**Lemma 3.4.3.** Let  $n \geq 4$ . Let  $x = \mathcal{X} \in VK_n$  be such that  $\text{lk}_+(x) \neq \emptyset$ . Let  $X$  be a representative of  $\mathcal{X}$  and let  $\overline{X}$  be its underlying graph.

(1) If  $\text{lk}_+(x)$  is nontrivial and has no edge, then  $2 \leq |\text{lk}_+(x)| \leq 3$ . Moreover,  $|\text{lk}_+(x)| = 3$  if and only if the underlying graph of any representative of  $x$  has  $n$  leaves.

(2) Let  $\text{lk}_+^1(x)$  be the set of vertices of  $K_n$  such that any element of  $\text{lk}_+^1(x)$  has a representative that can be obtained from  $X$  by blowing-up exactly one edge. Then  $|\text{lk}_+^1(x)| \geq 2$ .

**Proof.** Suppose that  $\text{lk}_+(x)$  is nontrivial and has no edge. Then the graph  $\overline{X}$  has at least  $n - 1$  leaves. Otherwise, one can blow-up two distinct edges at two distinct vertices of  $\overline{X}$  with nontrivial vertex groups which are not leaves. This gives rise to two vertices in the positive link of  $x$  that are linked by an edge. This contradicts the fact that  $\text{lk}_+(x)$  has no edge.

Moreover, if  $\overline{X}$  has exactly  $n - 1$  leaves, then all vertices of  $\overline{X}$  with trivial associated groups have valence 3 since otherwise one can blow-up an edge at a non-leaf vertex of  $\overline{X}$  with nontrivial vertex group and another edge at a valence-four vertex of  $\overline{X}$  with trivial vertex group. This gives rise to two vertices in the positive link of  $x$  that are linked by an edge. Moreover, the only non-leaf vertex with nontrivial associated group has valence equal to 2 since otherwise one can blow-up two edges at this vertex, giving rise to two vertices in the positive link of  $x$  that are linked by an edge.

If  $\overline{X}$  has  $n$  leaves, then at most one vertex of  $\overline{X}$  has degree at least 4 since otherwise one can blow-up two edges at two distinct vertices of  $\overline{X}$ . This gives rise to two vertices

in the positive link of  $x$  that are linked by an edge. Thus  $\overline{X}$  has at most one vertex  $v$  with degree at least 4. The degree of  $v$  is in fact equal to 4 since otherwise one can blow up a two-edge graph at  $v$ , which gives rise to two vertices in the positive link of  $x$  that are linked by an edge.

Thus, there are two possibilities for  $\overline{X}$ .

(i) The graph  $\overline{X}$  has  $n$  leaves. Moreover, there are exactly one vertex  $v$  of valence 4 and  $|V\overline{X}| - (n + 1)$  vertices of valence 3. In this case, the number of possible vertices in  $\text{lk}_+(x)$  corresponds to partitioning the set of edges adjacent to  $v$  into two subsets of order 2. This shows that  $|\text{lk}_+(x)| = 3$ .

(ii) The graph  $\overline{X}$  has  $n - 1$  leaves. Moreover, there are exactly one vertex  $v$  of valence 2 and  $|V\overline{X}| - n$  vertices of valence 3. In this case, the group associated with  $v$  is nontrivial and it is the only vertex of  $\overline{X}$  that has nontrivial associated group and is not a leaf. In that case, the number of possible vertices in  $\text{lk}_+(x)$  corresponds to blowing-up an edge  $e$  at  $v$  so that one of the endpoint of  $e$  is a leaf. Since  $v$  has valence 2, Proposition 3.2.1 implies that  $\text{Stab}^0(x)$  is isomorphic to  $F$ . Thus, there are two possibilities for blowing-up the edge  $e$  (either blowing it up while applying the nontrivial element of  $\text{Stab}^0(x)$  or blowing it up such that the preimages by the marking of the generators of the nontrivial vertex groups of the new graph of groups are the same as the preimages by the marking of  $X$  of the generators of the nontrivial vertex groups). This shows that  $|\text{lk}_+(x)| = 2$ .

We now prove the second part of the lemma. Suppose that  $\text{lk}_+(x)$  is nontrivial (it might have edges). Suppose first that  $\overline{X}$  has at most  $n - 2$  leaves. Let  $v_1$  and  $v_2$  be two vertices of  $\overline{X}$  with nontrivial associated groups that are not leaves. Then one can find two elements of  $\text{lk}_+^1(x)$  by blowing up an edge at either  $v_1$  or  $v_2$ . Thus,  $|\text{lk}_+^1(x)| \geq 2$ .

Finally, if  $\overline{X}$  has at least  $n - 1$  leaves, then the constructions of distinct elements of  $\text{lk}_+^1(x)$  are similar to the case where  $\text{lk}_+(x)$  is nontrivial and has no edge.  $\square$

**Lemma 3.4.4.** *Let  $n \geq 4$ . Suppose that  $x = \mathcal{X} \in VK_n$  is such that  $\text{lk}_-(x)$  is nontrivial and has no edge. Let  $X$  be a representative of  $\mathcal{X}$  and  $\overline{X}$  be its underlying graph.*

- (1) *There exists a unique vertex in  $\overline{X}$  with trivial associated group.*
- (2) *The negative link satisfies  $3 \leq |\text{lk}_-(x)| \leq n$ . Moreover,  $|\text{lk}_-(x)| = n$  if and only if  $x$  is a  $\{0\}$ -star.*

**Proof.** (1) The graph  $\overline{X}$  contains at least one vertex with trivial associated group since otherwise there would not exist an element  $\mathcal{Y} \in VK_n$  such that a representative of  $\mathcal{Y}$  is obtained from  $X$  by collapsing a forest. This would contradict the fact that  $\text{lk}_-(x)$  is nontrivial. Thus  $\overline{X}$  contains at least one vertex with trivial associated group.

Suppose towards a contradiction that  $\overline{X}$  contains two vertices with trivial associated groups. Then, since the degree of any vertex of  $\overline{X}$  with trivial associated group is at least 3, there exists two distinct edges  $e_1$  and  $e_2$  in  $\overline{X}$  that can be simultaneously collapsed to get a new element in  $VK_n$ . Moreover, if  $i \in \{1, 2\}$ , and if  $\mathcal{Y}_i$  is the homothety classes of the marked graph of groups obtained from  $X$  by collapsing  $e_i$ , then  $\mathcal{Y}_1, \mathcal{Y}_2 \in \text{lk}_-(x)$  and

$\mathcal{Y}_1$  and  $\mathcal{Y}_2$  are adjacent in  $\text{lk}_-(x)$  and distinct. This contradicts the fact that  $\text{lk}_-(x)$  has no edge. Thus, there exists a unique vertex in  $\overline{X}$  with trivial associated group.

(2) Let  $v$  be the unique vertex in  $\overline{X}$  with trivial associated group guaranteed by the first assertion. It follows that  $\deg(v) \geq 3$ . Thus  $|\text{lk}_-(x)| \geq 3$ . Since  $\overline{X}$  contains exactly  $n$  vertices with nontrivial associated group,  $\deg(v) \leq n$ . Thus  $|\text{lk}_-(x)| \leq n$ .

Now, if  $|\text{lk}_-(x)| = n$ , then  $\deg(v) = n$ . Thus  $\overline{X}$  contains exactly  $n$  leaves and  $n + 1$  vertices and  $\mathcal{X}$  is a  $\{0\}$ -star. Conversely, if  $\mathcal{X}$  is a  $\{0\}$ -star, then there exists exactly one vertex in  $\overline{X}$  with trivial associated group. Moreover, its degree is equal to  $n$ . Thus  $|\text{lk}_-(x)| = n$ .  $\square$

**Lemma 3.4.5.** *Let  $n \geq 4$ . Let  $x = \mathcal{X} \in VK_n$  be such that  $\text{lk}_-(x)$  is nontrivial. Let  $X$  be a representative of  $\mathcal{X}$  and let  $\overline{X}$  be its underlying graph. Let  $v_1, \dots, v_n$  be the vertices of  $\overline{X}$  with nontrivial associated group. Let  $e \in E\overline{X}$  and let  $\{v_{i_1}, \dots, v_{i_k}\} \amalg \{v_{j_1}, \dots, v_{j_l}\}$  be the partition of  $\{v_1, \dots, v_n\}$  obtained by considering the vertices contained in each connected component of  $\overline{X} - \overset{\circ}{e}$ .*

(1) *Let  $F_0 \subseteq \overline{X}$  be a forest (that may be empty) such that the homothety class of the marked graph of groups  $Y$  obtained from  $X$  by collapsing  $F_0$  is a vertex of  $K_n$ . Let  $p: \overline{X} \rightarrow \overline{Y}$  be the canonical projection.*

*Then, if  $p(e)$  is not a vertex, it is the unique edge  $f$  of  $\overline{Y}$  such that the partition of  $\{p(v_1), \dots, p(v_n)\}$  induced by  $\overline{Y} - \overset{\circ}{f}$  is  $\{p(v_{i_1}), \dots, p(v_{i_k})\} \amalg \{p(v_{j_1}), \dots, p(v_{j_l})\}$ .*

(2) *Let  $y, z \in \text{lk}_-(x)$  be distinct vertices. Let  $Y$  and  $Z$  be representatives of  $y$  and  $z$  respectively, and let  $\overline{Y}$  and  $\overline{Z}$  be their underlying graphs. Let  $p_y: \overline{X} \rightarrow \overline{Y}$  and  $p_z: \overline{X} \rightarrow \overline{Z}$  be the natural projections.*

*If one can obtain  $Z$  from  $Y$  by collapsing a forest of  $\overline{Y}$ , and if  $p_z(e)$  is not a point, there exists a unique edge  $\widetilde{p_z(e)} \in E\overline{Y}$  such that the partition of  $\{p_y(v_1), \dots, p_y(v_n)\}$  induced by  $\widetilde{p_z(e)}$  is*

$$\{p_y(v_{i_1}), \dots, p_y(v_{i_k})\} \amalg \{p_y(v_{j_1}), \dots, p_y(v_{j_l})\}.$$

**Remark 3.4.6.** Let  $X, Y$  and  $Z$  be as in the above statement. Let  $G$  be the forest of  $Y$  such that  $Z$  is obtained from  $Y$  by collapsing  $G$ .

(1) The statements of the lemmas can be reinterpreted in terms of decompositions in free factors of  $W_n$ . Indeed, a partition of the vertices with nontrivial associated groups  $\{v_1, \dots, v_n\} = A \amalg B$  induced by an edge of  $\overline{X}$  gives rise to a decomposition of  $W_n$  as  $W_n = W_k^A * W_{n-k}^B$  well-defined up to global conjugation. In this case,  $W_k^A$  is generated by the groups associated with the vertices in  $A$ , and  $W_{n-k}^B$  is generated by the groups associated with the vertices in  $B$ . In particular, Lemma 3.4.5 (1) can be stated as follows.

*If  $X$  is a graph of groups whose fundamental group is  $W_n$ , and if  $e$  and  $f$  are distinct edges of the underlying graph of  $X$ , then  $e$  and  $f$  induce distinct free factor decompositions.*

Moreover, if  $Y$  is a graph of groups obtained from  $X$  by collapsing a forest, and if  $g$  is an edge of the underlying graph of  $Y$ , then there exists a unique edge  $\tilde{g}$  in the underlying graph of  $X$  which induces the same free factor decomposition as  $g$ .

(2) Let  $H$  be a forest in  $\bar{Z}$ . The second statement of the lemma gives a unique minimal forest  $\tilde{H}$  in  $\bar{Y}$  that lifts  $H$ . Indeed, if  $h \in EH$ , let  $\tilde{h}$  be the unique edge of  $\bar{Y}$  given by Lemma 3.4.5 (2). Then  $\{\tilde{h}\}_{h \in H}$  is a lift of  $H$ . This lift has the property that  $\tilde{H} \cap G$  is contained in the leaves of  $G$  and that every vertex of  $\tilde{H}$  is adjacent to an edge in  $\tilde{H}$ . We call it *the canonical lift* of  $H$ .

**Proof.** For the first statement, we only need to prove the uniqueness result. Let  $f$  be an edge of  $\bar{Y}$  distinct from  $p(e)$ . Let  $A_1 \amalg A_2$  be the partition of  $\{p(v_1), \dots, p(v_n)\}$  induced by  $p(e)$ , and let  $B_1 \amalg B_2$  be the partition of  $\{p(v_1), \dots, p(v_n)\}$  induced by  $f$ . We prove that there exist two vertices  $v$  and  $w$  of  $\bar{Y}$  with nontrivial associated groups such that  $v$  and  $w$  are in the same connected component of  $\bar{Y} - \overset{\circ}{f}$  while they are not in the same connected component of  $\bar{Y} - \overset{\circ}{p(e)}$ , or conversely. This will imply that there exists  $\alpha \in \{1, 2\}$  such that  $B_\alpha \cap A_1 \neq \emptyset$  and that  $B_\alpha \cap A_2 \neq \emptyset$ , or conversely. This will conclude the proof. There are two cases to distinguish, according to the endpoints of  $p(e)$ .

If both of the endpoints of  $p(e)$  have nontrivial associated groups, then, since  $\bar{Y}$  is a tree,  $p(e)$  is necessarily the unique edge of  $\bar{Y}$  such that the endpoints of  $p(e)$  are in distinct connected components of  $\bar{Y} - \overset{\circ}{p(e)}$ .

Suppose that one of the endpoints of  $p(e)$ , denoted by  $v_0$ , has trivial associated group. Then there exists an arc  $P$  between two distinct leaves of  $\bar{Y}$ , say  $p(v_i)$  and  $p(v_j)$ , such that  $p(e)$  and  $f$  are (up to replacing them by their opposite edges) contained in this path and in this order. Since  $v_0$  has trivial associated group,  $\deg(v_0) \geq 3$ . Thus, up to exchanging the roles of  $p(v_i)$  and  $p(v_j)$ , there exists a path  $P'$  between  $p(v_i)$  and a leaf of  $\bar{Y}$ , say  $p(v_k)$ , distinct from both  $p(v_i)$  and  $p(v_j)$ , such that  $P'$  contains  $v_0$  (see Figure 3.5).

So if  $P'$  contains  $p(e)$  (see Figure 3.5, Case 1), then  $p(v_i)$  and  $p(v_k)$  are not contained in the same connected component of  $\bar{X} - \overset{\circ}{p(e)}$  while they are contained in the same connected component of  $\bar{X} - \overset{\circ}{f}$ . If  $P'$  does not contain  $p(e)$ , then there are two cases to distinguish.

Let  $v_1$  be the other endpoint of  $p(e)$ . If there exists  $\ell \in \{1, \dots, n\}$  such that we have  $v_1 = p(v_\ell)$  (see Figure 3.5, Case 2), then  $p(v_j)$  and  $p(v_\ell)$  are contained in the same connected component of  $\bar{X} - \overset{\circ}{p(e)}$  while they are not contained in the same connected component of  $\bar{X} - \overset{\circ}{f}$ .

If  $v_1$  has trivial associated group (see Figure 3.5, Case 3), then  $\deg(v_1) \geq 3$ . So there exists  $\ell \in \{1, \dots, n\}$  such that  $v_1$  is contained in the arc  $P^{(2)}$  between  $p(v_j)$  and  $p(v_\ell)$  and such that  $p(e)$  is not contained in  $P^{(2)}$ . Thus  $p(v_j)$  and  $p(v_\ell)$  are contained in the same connected component of  $\bar{X} - \overset{\circ}{p(e)}$  while they are not contained in the same connected component of  $\bar{X} - \overset{\circ}{f}$ . In any case,  $p(e)$  and  $f$  do not generate the same partition of  $\{p(v_1), \dots, p(v_n)\}$ .

Let  $Y$  and  $Z$  be as in the second statement of the lemma. By the first state-

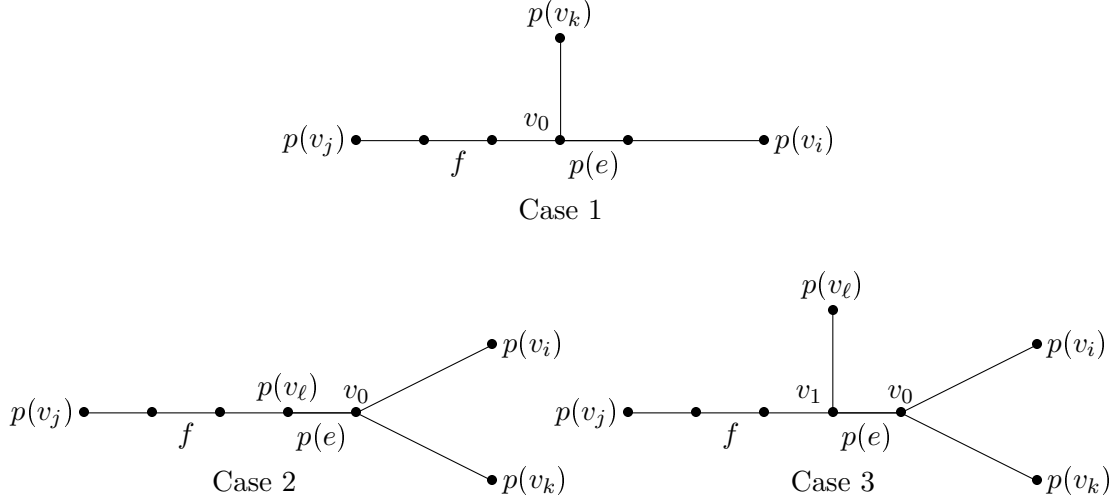


Figure 3.5: The arcs constructed in Lemma 3.4.5.

ment of the lemma, there exists a unique edge  $\widetilde{p_z(e)} \in E\overline{Y}$  such that the partition of  $\{p_y(v_1), \dots, p_y(v_n)\}$  induced by  $\widetilde{p_z(e)}$  is  $\{p_y(v_{i_1}), \dots, p_y(v_{i_k})\} \amalg \{p_y(v_{j_1}), \dots, p_y(v_{j_l})\}$  (namely it is  $p_y(e)$ ), and we take this edge to be our lift.  $\square$

**Proposition 3.4.7.** *Let  $n \geq 4$ , and  $x = \mathcal{X} \in VK_n$ . Suppose that both  $\text{lk}_-(x)$  and  $\text{lk}_+(x)$  are nontrivial. The only nontrivial decomposition of  $\text{lk}(x)$  as a join is  $\text{lk}(x) = \text{lk}_+(x) * \text{lk}_-(x)$ .*

**Proof.** Let  $X$  be a representative of  $\mathcal{X}$  and let  $\overline{X}$  be its underlying graph. Let  $\text{lk}(x) = A * B$  be a nontrivial decomposition as a join of  $\text{lk}(x)$  such that  $A \neq \text{lk}_+(x), \text{lk}_-(x)$ . Then there exist  $x_1, x_2 \in \text{lk}_+(x)$  or  $x_1, x_2 \in \text{lk}_-(x)$  such that  $x_1 \in A$  and  $x_2 \in B$ . For  $i \in \{1, 2\}$ , let  $\mathcal{X}_i$  be the homothety class corresponding to  $x_i$  and let  $X_i$  be a representative. Let  $\overline{X}_i$  be the underlying graph of  $X_i$ . Since  $x_1$  and  $x_2$  are joined by an edge, up to renumbering and changing the representatives, there exists a forest  $F_0$  in  $\overline{X}_1$  such that  $\overline{X}_2$  is obtained from  $\overline{X}_1$  by collapsing  $F_0$ . We now investigate both cases.

Suppose first that  $x_1, x_2 \in \text{lk}_+(x)$ . We are going to construct two other vertices  $z_1$  and  $z_2$  such that  $z_1 \in A$ ,  $z_2 \in B$  and  $z_1$  and  $z_2$  are not linked by an edge, which will lead to a contradiction (see Figure 3.6).

Since  $x_2 \in \text{lk}_+(x)$ , up to changing the representative  $X$  of  $\mathcal{X}$ , there exists a forest  $G$  in  $\overline{X}_2$  such that  $\overline{X}$  is obtained from  $\overline{X}_2$  by collapsing  $G$ . Let  $\widetilde{G}$  be the canonical lift of  $G$  in  $\overline{X}_1$ .

Let  $f \in EF_0$ . Let  $\mathcal{Z}_0$  be the homothety class of the marked graph of groups  $Z_0$  obtained from  $X_1$  by collapsing  $f$  and let  $z_0$  be the corresponding vertex in  $K_n$ . Since a representative of  $\mathcal{Z}_0$  is obtained from  $X_1$  by collapsing an edge, we see that  $x_1 \in \text{lk}_+(z_0)$ . Moreover, since  $f \in EF_0$ , we see that  $z_0 \in \text{lk}_+(x_2)$  and  $z_0 \in \text{lk}_+(x)$ . Lemma 3.4.3 (2) applied to  $z_0$  then implies that there exists  $z_1 = \mathcal{Z}_1 \in \text{lk}_+(z_0)$  distinct from  $x_1$  such that the underlying graph of any representative of  $z_1$  has the same number of edges as  $\overline{X}_1$ .



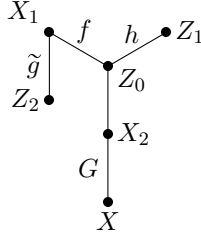


Figure 3.6: The adjacency of the homothety classes constructed in the first case of Lemma 3.4.7.

Since  $z_1 \in \text{lk}_+(z_0)$  and  $z_0 \in \text{lk}_+(x)$ , we have  $z_1 \in \text{lk}_+(x)$ . As  $z_1$  has a representative that has the same number of edges as  $X_1$ , and as  $x_1 \neq z_1$ , we see that  $z_1 \notin \text{lk}(x_1)$ . Therefore we have  $z_1 \in A$ .

In order to construct  $z_2$ , let  $\tilde{g}$  be an edge in  $\tilde{G}$ . Let  $Z_2$  be the homothety class of the marked graph of group  $Z_2$  obtained from  $X_1$  by collapsing  $\tilde{g}$ . Let  $z_2$  be the corresponding vertex in  $K_n$ . Then, since  $\tilde{g} \in \tilde{G}$ , we see that  $z_2 \in \text{lk}_+(x)$ . As  $\tilde{g} \in E\tilde{G}$ , and as two distinct edges induce distinct free factor decompositions by Remark 3.4.6 (1), there exists an edge  $g \in E\bar{X}_2$  (namely the edge whose lift in  $\bar{X}_1$  is  $\tilde{g}$ ) such that the free factor decomposition induced by  $g$  is distinct from the free factor decomposition induced by any edge of the underlying graph of  $Z_2$ . Thus we see that  $x_2$  and  $z_2$  cannot be adjacent. Indeed, if it was the case, then as  $Z_2$  is obtained from  $X_1$  by collapsing exactly one edge, either  $|EX_2| = |EZ_2|$  or there would exist a representative  $Z'_2$  of  $Z_2$  such that  $X_2$  is obtained from  $Z'_2$  by collapsing a forest. Both cases would contradict Remark 3.4.6 (1) because the edge  $g$  of  $\bar{X}_2$  induces a free factor decomposition that is not induced by any edge of the underlying graph of  $Z'_2$ . This implies that  $z_2 \notin \text{lk}(x_2)$  and that  $z_2 \in B$ .

**Claim.** The vertices  $z_1$  and  $z_2$  are not adjacent in  $\text{lk}(v)$ .

**Proof.** Suppose towards a contradiction that  $z_1$  and  $z_2$  are adjacent. Let  $Z_1$  be a representative of the homothety class corresponding to  $z_1$ , and, for  $i \in \{1, 2\}$ , let  $\bar{Z}_i$  be the underlying graph of  $Z_i$ . As  $|E\bar{Z}_1| = |E\bar{Z}_2| + 1$ , up to changing the representatives  $Z_1$  and  $Z_2$ , we can suppose that  $Z_2$  is obtained from  $Z_1$  by collapsing an edge  $e \in E\bar{Z}_1$ . Let  $h$  be the edge in  $\bar{Z}_1$  such that the marked graph of groups obtained from  $Z_1$  by collapsing  $h$  is in  $z_0$ . As  $Z_1$  is distinct from  $X_1$ , the edge  $h$  is such that the free factor decomposition of  $W_n$  induced by  $h$  is distinct from the one induced by any edge of  $\bar{X}_1$ . Thus, by Lemma 3.4.5 (2), the free factor decomposition of  $W_n$  induced by  $h$  is distinct from the one induced by any edge of  $\bar{Z}_2$ . Therefore, by Remark 3.4.6 (1), the marked graph of groups  $Z_2$  is obtained from  $Z_1$  by collapsing  $h$ . This implies that  $z_2 = z_0$  by the choice of  $h$ . However,  $\tilde{G} \cap F_0$  does not contain any edge by the properties of the canonical lift of  $G$  (see Remark 3.4.6 (2)). Thus there exists an edge in  $\bar{Z}_0$  which induces the same free factor decomposition of  $W_n$  as  $\tilde{g}$ . As  $Z_2$  is obtained from  $X_1$  by collapsing  $\tilde{g}$ , Lemma 3.4.5 (1) implies that there is no edge in  $\bar{Z}_2$  that induces the same free factor decomposition as  $\tilde{g}$ . Thus  $z_0 \neq z_2$ , and this leads to a contradiction.  $\square$

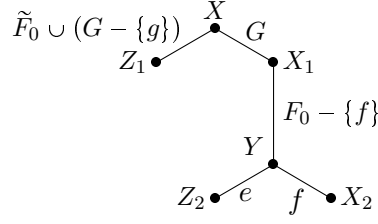


Figure 3.7: The adjacency of the homothety classes constructed in the second case of Lemma 3.4.7.

Therefore  $z_1$  and  $z_2$  are not adjacent in  $\text{lk}(x)$ . However,  $z_1 \in A$  and  $z_2 \in B$ . This contradicts the fact that  $\text{lk}(x) = A * B$  is a join decomposition.

Now suppose that  $x_1, x_2 \in \text{lk}_-(x)$ . We use the same strategy as when  $x_1, x_2 \in \text{lk}_+(x)$  (see Figure 3.7).

Since  $x_1 \in \text{lk}_-(x)$ , up to changing the representative  $X$  of  $\mathcal{X}$ , there exists a forest  $G$  in  $\bar{X}$  such that  $\bar{X}_1$  is obtained from  $\bar{X}$  by collapsing  $G$ . Let  $g \in EG$ . Let  $\tilde{F}_0$  be the canonical lift of  $F_0$  in  $\bar{X}$ . Remark that  $\tilde{F}_0 \cap G$  does not contain any edge. Let  $\mathcal{Z}_1$  be the homothety class of the marked graph of groups  $Z_1$  obtained from  $X$  by collapsing  $\tilde{F}_0 \cup (G - \{g\})$  and let  $z_1$  be the corresponding vertex. Let  $p_{x_1}: \bar{X} \rightarrow \bar{X}_1$  and  $p_{z_1}: \bar{X} \rightarrow \bar{Z}_1$  be the natural projections. We claim that  $x_1$  and  $z_1$  are not adjacent. Indeed, suppose that  $x_1$  and  $z_1$  are adjacent. As  $x_1$  and  $z_1$  are distinct, we see that  $|E\bar{X}_1| \neq |E\bar{Z}_1|$ . Therefore, as  $|E(\tilde{F}_0 \cup (G - \{g\}))| \geq |EG|$ , we see that  $|E\bar{X}_1| > |E\bar{Z}_1|$ , and a representative of  $\mathcal{Z}_1$  is obtained from a representative of  $\mathcal{X}_1$  by collapsing a forest. Let  $X'_1$  be a representative of  $\mathcal{X}_1$  obtained from  $Z_1$  by blowing-up a forest. As  $p_{z_1}(g)$  is an edge in  $\bar{Z}_1$ , Remark 3.4.6 (1) implies that there exists a unique edge  $\tilde{g}$  in  $\bar{X}_1$  such that  $\tilde{g}$  induces the same free factor decomposition as  $p_{z_1}(g)$  and  $g$ . But since  $p_{x_1}(g)$  is a point, Remark 3.4.6 (1) implies that there is no edge in  $\bar{X}'_1$  which induces the same free factor decomposition as  $g$ . So  $x_1$  and  $z_1$  are not adjacent and  $z_1 \in A$ .

In order to construct  $z_2$ , let  $f \in EF_0$ . Let  $Y$  be the marked graph of groups obtained from  $X_1$  by collapsing  $F_0 - \{f\}$ . Let  $p_0: \bar{X}_1 \rightarrow \bar{Y}$  be the natural projection. Let  $a_1$  and  $a_2$  be the endpoints of  $p_0(f)$ . Let  $X_1/F_0$  be the marked graph of groups obtained from  $X_1$  by collapsing  $F_0$ . Since the homothety class of  $X_1/F_0$  is an element of  $K_n$  (namely it is  $\mathcal{X}_2$ ), one of the endpoints of  $p_0(f)$  has trivial associated group. Suppose without loss of generality that  $a_1$  has trivial associated group. In particular,  $\deg(a_1) \geq 3$ . Let  $a_3$  and  $a_4$  be two distinct vertices adjacent to  $a_1$  other than  $a_2$  and let  $e$  be the edge between  $a_1$  and  $a_3$ . Finally let  $\mathcal{Z}_2$  be the homothety class of the marked graph of groups  $Z_2$  obtained from  $Y$  by collapsing  $\{e\}$ . Let  $z_2$  be the corresponding vertex in  $K_n$ . Then, since  $|EX_2| = |EZ_2|$  and since  $X_2$  and  $Z_2$  are obtained from  $X$  by collapsing two distinct forests, we see that  $z_2$  and  $x_2$  are not adjacent in  $K_n$ . So  $z_2 \in B$ .

Let us prove that  $z_1$  and  $z_2$  are not adjacent in  $\text{lk}(x)$ . Suppose towards a contradiction that  $z_1$  and  $z_2$  are adjacent. As  $G$  contains at least one edge, we have that  $Z_1$  is obtained from  $X$  by collapsing  $|F_0| + |G| - 1$  edges. Moreover,  $Z_2$  is obtained from  $X$

by collapsing  $|F_0| + |G|$  edges. This implies that the number of edges of a representative of  $z_1$  is greater than the number of edges of a representative of  $z_2$ . Thus, there exists a representative of  $z_1$  that collapses onto a representative of  $z_2$ . Let  $p_{z_2}: \overline{X} \rightarrow \overline{Z}_2$  be the natural projection. Let  $\tilde{f} \in \tilde{F}_0$  be the canonical lift of  $f$  in  $\overline{X}$ . Since  $p_{z_2}(\tilde{f})$  is an edge in  $\overline{Z}_2$ , Remark 3.4.6 (1) implies that there exists an edge in  $\overline{Z}_2$  which induces the same free factor decomposition as  $\tilde{f}$ . But, as  $p_{z_1}(\tilde{f})$  is a point in  $\overline{Z}_1$ , Remark 3.4.6 (1) shows that there is no edge in  $\overline{Z}_2$  that induces the same free factor decomposition as  $\tilde{f}$ . Thus,  $z_1$  and  $z_2$  are not adjacent.

This contradicts the fact that  $\text{lk}(x) = A * B$  is a join decomposition. This concludes the proof of the proposition.  $\square$

**Corollary 3.4.8.** *Let  $n \geq 4$  and  $f \in \text{Aut}(K_n)$ . Then  $f$  preserves the set of  $\{0\}$ -stars and the set of  $F$ -stars.*

**Proof.** Let  $\rho$  be a  $\{0\}$ -star. Since  $\text{lk}_-(\rho)$  has no edge and is of cardinal equal to  $n$ , Proposition 3.4.7 tells us that either  $\text{lk}_+(f(\rho))$  has no edge and its cardinal is equal to  $n$ , or  $\text{lk}_-(f(\rho))$  has no edge and its cardinal is equal to  $n$ . Since  $n \geq 4$ , Lemma 3.4.3 (1) tells us that the first case is not possible. So  $\text{lk}_-(f(\rho))$  has no edge and its cardinal is equal to  $n$ . Then Lemma 3.4.4 (2) shows that  $f(\rho)$  is a  $\{0\}$ -star.

Let  $\rho'$  be an  $F$ -star. Then there exists a  $\{0\}$ -star  $\rho$  such that  $\rho' \in \text{lk}_-(\rho)$ . Therefore,  $f(\rho') \in \text{lk}_-(f(\rho))$ . As  $f(\rho)$  is a  $\{0\}$ -star and since the negative link of a  $\{0\}$ -star is composed of  $F$ -stars, we see that  $f(\rho')$  is an  $F$ -star.  $\square$

Thus, there exists a homomorphism  $\text{Aut}(K_n) \rightarrow \text{Aut}(L_n)$  defined by restriction. We now prove that this homomorphism is in fact injective.

**Lemma 3.4.9.** *Let  $n \geq 4$ . Let  $f \in \text{Aut}(K_n)$  be such that  $f|_{O_n} = \text{id}_{O_n}$  and  $f|_{F_n} = \text{id}_{F_n}$ . Let  $y = \mathcal{Y} \in VK_n$  be such that  $\text{lk}_-(v)$  is trivial. Then  $f(y) = y$ .*

**Proof.** In order to prove Lemma 3.4.9, we prove the following claim.

**Claim.** Let  $0 \leq k \leq n - 3$ . Let  $\mathcal{X}$  and  $\mathcal{Y}$  be vertices of  $K_n$ . Let  $X$  and  $Y$  be representatives of  $\mathcal{X}$  and  $\mathcal{Y}$ . We write  $\overline{X}$  and  $\overline{Y}$  for their underlying graphs. Suppose that  $\mathcal{X}$  has a nontrivial negative link with no edge and that  $\mathcal{Y}$  has a trivial negative link. If  $\overline{X}$  has  $k$  vertices with nontrivial associated group that are not leaves, and if  $\overline{Y}$  has  $k + 1$  vertices with nontrivial associated group that are not leaves, then  $f(\mathcal{X}) = \mathcal{X}$  and  $f(\mathcal{Y}) = \mathcal{Y}$ .

Lemma 3.4.9 then follows from the claim because for every vertex  $y \in VK_n$  with trivial negative link, there exists  $k \in \{0, \dots, n - 3\}$  such that  $y$  has a representative  $Y$  whose underlying graph has exactly  $k + 1$  vertices with nontrivial associated group that are not leaves.

We prove the claim by induction on  $k$ . When  $k = 0$ ,  $\overline{X}$  has  $n$  leaves, so by Lemma 3.4.4 (2), we have that  $|\text{lk}_-(v)| = n$ . Thus, by Lemma 3.4.4 (2), we see that  $\mathcal{X}$  is a  $\{0\}$ -star. Moreover,  $\overline{Y}$  has  $n - 1$  leaves and  $n$  vertices, so  $\mathcal{Y}$  is an  $F$ -star. Thus, when  $k = 0$ , the claim is a restatement of the fact that  $f$  fixes the  $\{0\}$ -stars and the  $F$ -stars.

Now suppose that the claim is true for some  $0 \leq k \leq n - 4$ . Let  $\mathcal{X}$  and  $\mathcal{Y}$  be such that  $\mathcal{X}$  has a nontrivial negative link with no edge and that  $\mathcal{Y}$  has a trivial negative link. Let  $X$  and  $Y$  be representatives of  $\mathcal{X}$  and  $\mathcal{Y}$ , and let  $\overline{X}$  and  $\overline{Y}$  their underlying graphs. Suppose that  $\overline{X}$  has  $k + 1$  vertices with nontrivial associated group that are not leaves, and that  $\overline{Y}$  has  $k + 2$  vertices with nontrivial associated group that are not leaves.

We start by showing that  $f(\mathcal{X}) = \mathcal{X}$ . First, by Proposition 3.4.7, the homothety class  $f(\mathcal{X})$  has either a nontrivial negative link with no edge or a nontrivial positive link with no edge.

**Claim.** The homothety class  $\mathcal{X}$  cannot be sent by  $f$  to a homothety class  $z = \mathcal{Z}$  such that  $\text{lk}_+(z)$  has no edge.

**Proof.** Suppose towards a contradiction that it is the case. By Lemma 3.4.4,  $|\text{lk}_-(\mathcal{X})| \geq 3$ , while by Lemma 3.4.3 (1),  $|\text{lk}_+(z)| \leq 3$ . Thus,  $|\text{lk}_-(\mathcal{X})| = |\text{lk}_+(z)| = 3$ . But then, Lemma 3.4.3 (1) implies that the underlying graph of any representative of  $\mathcal{Z}$  has  $n$  leaves. However, such a vertex  $z$  is adjacent to  $n$   $F$ -stars whereas  $\mathcal{X}$  is adjacent to at most one  $F$ -star. Indeed if  $k + 1 = 1$ , the homothety class  $\mathcal{X}$  is adjacent to exactly one  $F$ -star obtained from  $X$  by collapsing the unique edge between the vertex with trivial associated group (the uniqueness of this vertex follows from Lemma 3.4.4 (1)) and the non-leaf vertex with nontrivial associated group. If  $k + 1 \geq 2$ , then  $\mathcal{X}$  is not adjacent to an  $F$ -star because  $\overline{X}$  has at least two vertices with nontrivial associated group that are not leaves, whereas any  $F$ -star has exactly one such vertex. As the set of  $F$ -stars is fixed by  $f$ , we get a contradiction.  $\square$

So  $f(\mathcal{X})$  has a nontrivial negative link with no edge. Let  $v$  be the unique vertex of  $\overline{X}$  with trivial associated group given by Lemma 3.4.4 (1).

**Claim.** The underlying graph of any representative of  $f(\mathcal{X})$  has exactly  $n - k - 1$  leaves.

**Proof.** By the induction hypothesis, the automorphism  $f$  fixes all vertices of  $K_n$  whose negative link is nontrivial and has no edges and such that the underlying graph of any representative has at least  $n - k$  leaves. Thus, the underlying graph of any representative of  $f(\mathcal{X})$  has at most  $n - k - 1$  leaves.

Now, suppose that  $\mathcal{Z}$  is the homothety class of a marked graph of groups  $Z$  whose underlying graph has at most  $n - k - 2$  leaves and such that  $\text{lk}_-(\mathcal{Z})$  is nontrivial and has no edge. Then  $\text{lk}_-(\mathcal{Z})$  does not contain any homothety class of marked graphs of groups whose underlying graph has  $n - k - 1$  leaves. But  $\text{lk}_-(\mathcal{X})$  contains one such homothety class, namely the homothety class of a marked graph of groups obtained from  $X$  by collapsing an edge between  $v$  and a vertex that is not a leaf. As  $f$  fixes all vertices of  $K_n$  with trivial negative link and such that the underlying graph of any representative has at least  $n - k - 1$  leaves and as  $f(\text{lk}_-(\mathcal{X})) = \text{lk}_-(f(\mathcal{X}))$ , it follows that  $f(\mathcal{X}) \neq \mathcal{Z}$ . Thus, the underlying graph of any representative of  $f(\mathcal{X})$  has at least  $n - k - 1$  leaves. Therefore the underlying graph of any representative of  $f(\mathcal{X})$  has exactly  $n - k - 1$  leaves.  $\square$

To prove that, in fact,  $f(\mathcal{X}) = \mathcal{X}$ , we distinguish between two cases, according to the vertices adjacent to  $v$ . Note that, as  $\overline{X}$  is connected, the vertex  $v$  is adjacent to at least one vertex that is not a leaf.

**Case 1.** Suppose that  $v$  is adjacent to at least two vertices  $w_1$  and  $w_2$  that are not leaves.

For  $i \in \{1, 2\}$ , let  $e_i$  be the edge between  $v$  and  $w_i$ , and let  $\mathcal{Y}_i$  be the homothety class of the marked graph of groups  $Y_i$  obtained from  $X$  by collapsing  $e_i$ . Then  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  are homothety classes of marked graphs of groups with trivial negative link and such that the underlying graphs of  $Y_1$  and  $Y_2$  have  $k + 1$  vertices with nontrivial associated group that are not leaves. By induction hypothesis,  $f(\mathcal{Y}_1) = \mathcal{Y}_1$  and  $f(\mathcal{Y}_2) = \mathcal{Y}_2$ . Let  $p_1: X \rightarrow Y_1$  and  $p_2: X \rightarrow Y_2$  be the natural projections. In Case 1, the fact that  $f(\mathcal{X}) = \mathcal{X}$  is a consequence of the following claim.

**Claim.** The homothety class  $\mathcal{X}$  is the only vertex in  $\text{lk}(\mathcal{Y}_1) \cap \text{lk}(\mathcal{Y}_2)$  whose negative link is nontrivial and has no edge.

**Proof.** Let  $\mathcal{Z} \in \text{lk}(\mathcal{Y}_1) \cap \text{lk}(\mathcal{Y}_2)$  be such that  $\text{lk}_-(\mathcal{Z})$  is nontrivial and has no edge. Assume towards a contradiction that  $\mathcal{Z} \neq \mathcal{X}$ . As  $\mathcal{Y}_1$  has trivial negative link, for all  $\mathcal{Z}' \in VK_n$  such that  $\mathcal{Z}' \in \text{lk}(\mathcal{Y}_1)$ , we have in fact  $\mathcal{Z}' \in \text{lk}_+(\mathcal{Y}_1)$ . Thus, there exists a representative  $Z$  of  $\mathcal{Z}$  such that  $Z$  is obtained from  $Y_1$  by blowing-up a forest  $F_0$ . Let  $\overline{Z}$  be the underlying graph of  $Z$ , and let  $p_1^Z: \overline{Z} \rightarrow \overline{Y}_1$  be the natural projection.

We claim that there exists a unique edge in  $F_0$ . Indeed, otherwise there would exist two vertices in  $\overline{Z}$  with trivial associated groups. As  $\text{lk}_-(\mathcal{Z})$  has no edge, this would contradict Lemma 3.4.4 (1). Thus, there exists a unique edge  $f \in EF_0$ .

Since  $\mathcal{Z} \in \text{lk}(\mathcal{Y}_2)$  and since  $\text{lk}_-(\mathcal{Z})$  is nontrivial and has no edge, Lemma 3.4.4 (1) implies that there exists an edge  $g$  such that the homothety class of the marked graph of groups  $Z/\{g\}$  obtained from  $Z$  by collapsing  $g$  is  $\mathcal{Y}_2$ . Let  $p_2^Z: \overline{Z} \rightarrow \overline{Z}/\{g\}$  be the natural projection. By Remark 3.4.6 (1), there exists a unique edge  $h \in E\overline{Z}$  such that  $p_2^Z(h)$  induces the same free factor decomposition of  $W_n$  as  $p_2(e_1)$ . But since  $Z$  is a blow-up of  $Y_1$  by an edge, and since  $Y_1$  is obtained from  $X$  by collapsing  $e_1$ , Lemma 3.4.5 (2) implies that  $p_1^Z(h)$  is reduced to a point. Therefore  $f = h$  and  $\overline{Z}$  is obtained from  $Y_1$  by blowing-up the edge  $e_1$ . It follows that the graph  $\overline{Z}$  is isomorphic to the graph  $\overline{X}$ . Thus, we can suppose that  $\overline{X} = \overline{Z}$ . We can also suppose, by Lemma 3.4.5 (2), that  $g = e_2$ . As  $v$  has trivial associated group,  $\deg(v) \geq 3$ . If  $\mathcal{X} \neq \mathcal{Z}$ , since both  $X$  and  $Z$  are obtained from  $Y_1$  by blowing-up the edge  $e_1$ , there exist an integer  $\ell \in \{0, 1\}$  and a vertex  $w_3 \in V\overline{X}$  distinct from  $w_1$  and  $w_2$  and adjacent to  $v$  such that:

- (1) For  $i \in \{1, 2, 3\}$ , the preimage by the marking of  $X$  of the generator of the group associated with  $w_i$  is  $x_i$  ;
- (2) The preimage by the marking of  $Z$  of the generator of the group associated with  $w_2$  is  $x_1^\ell x_2 x_1^\ell$  and the preimage by the marking of  $Z$  of the generator of the group associated with  $w_3$  is  $x_1^{\ell+1} x_3 x_1^{\ell+1}$ .

As  $p_2(w_2)$  and  $p_2(w_3)$  are in the same connected component of  $p_2(\overline{X}) - \{p_2(w_1)\}$ , it follows that  $p(e_1)$  and  $p_2(e_1)$  induces distinct free factor decompositions of  $W_n$ . This contradicts the fact that  $Z/\{e_1\} \in \mathcal{Y}_2$  by Remark 3.4.6 (1). The claim follows.  $\square$

**Case 2.** Suppose that  $v$  is adjacent to only one vertex  $w$  that is not a leaf.

Let  $e$  be the edge between  $v$  and  $w$  and let  $\mathcal{Y}'$  be the homothety class of the marked graph of groups  $Y'$  obtained from  $X$  by collapsing  $e$ . Let  $\overline{Y}'$  be the underlying graph of  $Y'$ . Let  $p_X: \overline{X} \rightarrow \overline{Y}'$  be the natural projection. Then, as  $\text{lk}_-(\mathcal{Y}')$  is trivial and as  $\mathcal{Y}'$  has a representative whose underlying graph has  $n - k - 1$  leaves, by induction hypothesis, we see that  $f(\mathcal{Y}') = \mathcal{Y}'$ . So  $f(\mathcal{X}) \in \text{lk}(\mathcal{Y}')$ . Thus a representative  $Z$  of  $f(\mathcal{X})$  is obtained from  $Y'$  by blowing-up a forest  $F_0 \subseteq E\overline{Z}$ . As  $\text{lk}_-(f(\mathcal{X}))$  has no edge, the forest  $F_0$  contains a unique edge  $e'$ . Let  $\overline{Z}$  be the underlying graph of  $Z$ , and  $p_Z: \overline{Z} \rightarrow \overline{Y}'$  be the canonical projection.

Suppose towards a contradiction that  $f(\mathcal{X}) \neq \mathcal{X}$ . By the claim above Case 1, the underlying graph of any representative of  $f(\mathcal{X})$  has exactly  $n - k - 1$  leaves. Therefore none of the two endpoints of  $e'$  is a leaf. Thus, as one of the endpoints of  $e'$  has trivial associated group, there exists a vertex  $a \in V\overline{Y}'$  such that  $\deg(a) \geq 3$  and such that  $e'$  collapses onto  $a$ . As  $\mathcal{Y}'$  has trivial negative link, we see that the group associated with  $a$  is nontrivial. Let  $y_i$  be the preimage by the marking of  $Y'$  of the generator of the group associated with  $a$ . Let  $\tilde{a}$  be the lift of  $a$  in  $\overline{Z}$  such that  $\tilde{a}$  has nontrivial associated group. Then  $y_i$  is the preimage by the marking of  $Z$  of the generator of the group associated with  $\tilde{a}$ . Let  $\tilde{b}$  be the endpoint of  $e'$  distinct from  $\tilde{a}$  (see Figure 3.8). As  $Z/\{e'\} \in \mathcal{Y}'$ , the vertex  $\tilde{b}$  has trivial associated group and  $\deg(\tilde{b}) \geq 3$ . Moreover, by the previous case, the vertex  $\tilde{b}$  cannot be adjacent to two vertices that are not leaves.

Suppose first that  $Z$  is not a blow-up of  $Y'$  at  $p_X(e)$ . This implies that  $p_X^{-1}(a)$  is a vertex.

As  $\tilde{b}$  is not adjacent to two vertices that are not leaves, there exist two distinct leaves  $w_1$  and  $w_2$  of  $\overline{Y}'$  adjacent to  $a$  such that  $w_1$  and  $w_2$  have lifts  $\tilde{w}_1$  and  $\tilde{w}_2$  in  $\overline{Z}$  with nontrivial associated group that are adjacent to  $\tilde{b}$ . Let  $y_j$  and  $y_k$  be the preimages by the marking of  $Y'$  of the groups associated with  $w_1$  and  $w_2$ . Then there exist  $\alpha_j, \alpha_k \in \{0, 1\}$  such that  $y_i^{\alpha_j} y_j y_i^{\alpha_j}$  and  $y_i^{\alpha_k} y_k y_i^{\alpha_k}$  are the preimages by the marking of  $Z$  of the groups associated with  $\tilde{w}_1$  and  $\tilde{w}_2$ .

Let  $Z'_1$  be the homothety class of the marked graph of groups  $Z'_1$  defined as follows (see Figure 3.8):

- The underlying graph of  $Z'_1$  is obtained from  $\overline{Y}'$  by pulling-up an edge  $h$  at  $a$  so that one of the two endpoints of  $h$  is a leaf. Let  $p': \overline{Z}'_1 \rightarrow \overline{Y}'$  be the projection. Let  $x$  be a vertex of the underlying graph of  $Z'_1$ . Remark that, as  $w_2$  is a leaf,  $p'^{-1}(w_2)$  is a leaf.
- If  $x$  is distinct from  $p'^{-1}(w_2)$  and is such that  $p'(x) \neq a$ , then the group associated with  $x$  in  $Z'_1$  is the same one as the group associated with  $p'(x)$ .
- If  $p'(x) = a$  and if  $x$  is a leaf, then the group associated with  $x$  is the same one as the group associated with  $a$ .
- If  $p'(x) = a$  and if  $x$  is not a leaf, then  $x$  has trivial associated group.

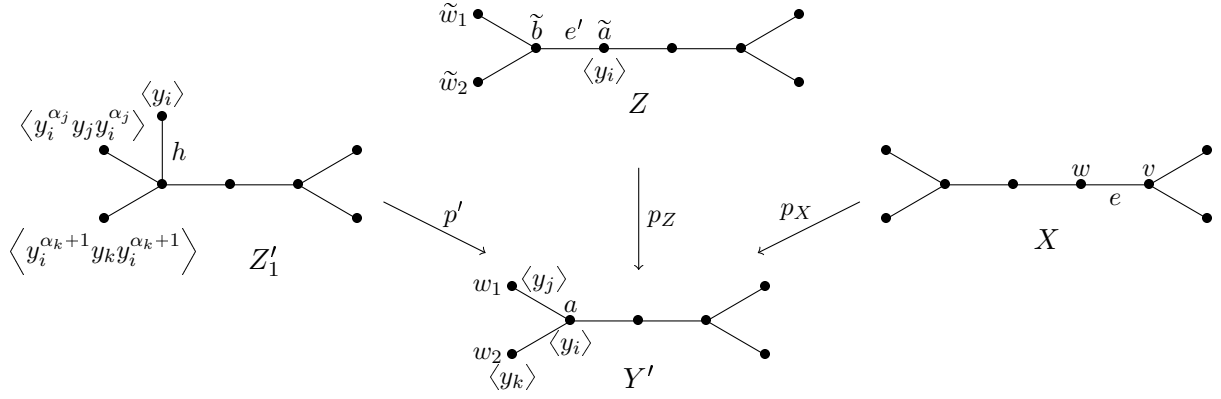


Figure 3.8: The construction of  $Z'_1$  in Lemma 3.4.9 when  $p_X(e) \neq a$ .

• Finally, the preimage by the marking of  $Z'_1$  of the generator of the group associated with  $p'^{-1}(w_2)$  is  $y_i^{\alpha_k+1} y_k y_i^{\alpha_k+1}$ . By the induction hypothesis,  $Z'_1$  is fixed by  $f$ . What is more,  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, Z'_1) = 2$ . Indeed, a common refinement of  $X$  and  $Z'_1$  is obtained from  $X$  by pulling-up the edge  $h$  at  $p_X^{-1}(a)$  (this is possible since  $p_X(e) \neq a$ ).

**Claim.** In  $\text{lk}(\mathcal{Y}')$ , we have  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, Z'_1) > 2$ .

**Proof.** Since both  $\mathcal{Z}$  and  $Z'_1$  have nontrivial negative link with no edge, we see by Lemma 3.4.4 (1) that  $|V\overline{\mathcal{Z}}| = |V\overline{Z'_1}|$ . As both  $\overline{\mathcal{Z}}$  and  $\overline{Z'_1}$  are trees, we have  $|E\overline{\mathcal{Z}}| = |E\overline{Z'_1}|$ . Thus, as  $\mathcal{Z} \neq Z'_1$ , we have  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, Z'_1) > 1$ . As  $\mathcal{Y}'$  has trivial negative link, the only way  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, Z'_1) = 2$  is that  $Z$  and  $Z'_1$  have a common refinement. Let  $z$  be the leaf of  $\overline{Z'_1}$  such that  $p'(z) = a$ . Then  $p'^{-1}(w_1)$  and  $p'^{-1}(w_2)$  are in the same connected component of  $\overline{Z'_1} - \{z\}$ . Let  $Z_1^{(2)}$  be a refinement of  $Z_1$ , and let  $y_1^{(2)}, \dots, y_n^{(2)}$  be the preimages by the marking of  $Z_1^{(2)}$  of the generators of the nontrivial vertex groups of  $Z_1^{(2)}$ . Suppose that, for all  $m \in \{1, \dots, n\}$ , there exists  $\alpha_m \in \{0, 1\}$  such that  $y_m^{(2)} = y_i^{\alpha_m} y_m y_i^{\alpha_m}$  and that there exist  $m_0$  and  $m_1$  such that  $\alpha_{m_0} = 0$  and  $\alpha_{m_1} = 1$ . Since the preimage by the marking of the generator of the group associated with  $z$  is  $y_i$ , we see that  $Z_1^{(2)}$  is obtained from  $Z'_1$  by blowing-up a forest and applying a twist at an edge whose terminal point is  $z$ . As a consequence, since  $p'^{-1}(w_1)$  and  $p'^{-1}(w_2)$  are in the same connected component of  $\overline{Z'_1} - \{z\}$ , there does not exist a refinement of  $Z'_1$  such that the preimages by the marking of the generator of the group associated with lifts of  $p'^{-1}(w_1)$  and  $p'^{-1}(w_2)$  are respectively  $y_i^{\alpha_j} y_j y_i^{\alpha_j}$  and  $y_i^{\alpha_k} y_k y_i^{\alpha_k}$ . Thus,  $Z$  and  $Z'_1$  do not have any common refinement and  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, Z'_1) > 2$ .  $\square$

Since  $d_{\text{lk}(f(\mathcal{Y}'))}(f(\mathcal{X}), f(Z'_1)) = d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, Z'_1) = 2$ , the last claim implies that  $f(\mathcal{X}) = \mathcal{X}$  when  $p_X(e) \neq a$ .

Suppose now that  $p_X(e) = a$ . Then, as  $|\text{lk}_-(\mathcal{X})| = |\text{lk}_-(f(\mathcal{X}))|$ , and as  $\overline{X}$  and  $\overline{Z}$  both have a unique vertex with trivial associated group by Lemma 3.4.4 (1) (namely  $v$

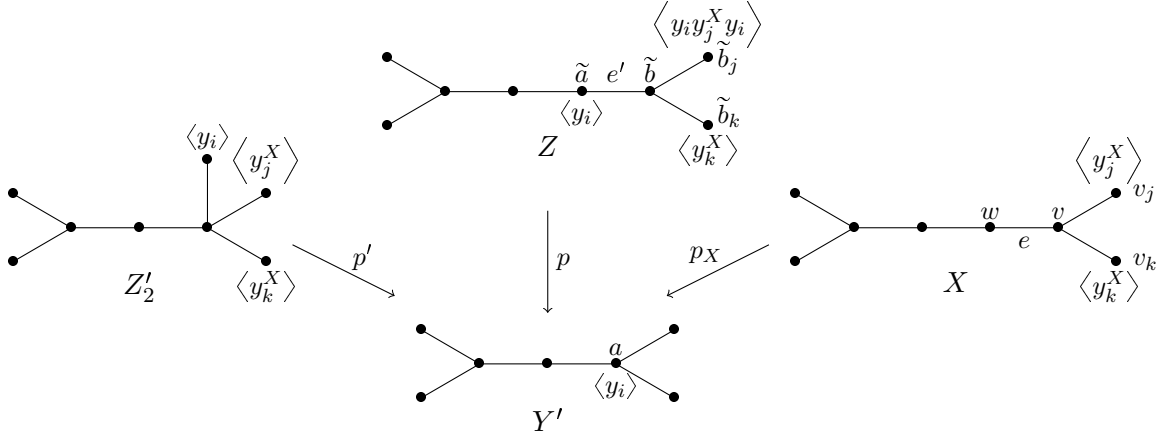


Figure 3.9: The construction of  $Z'_2$  in Lemma 3.4.9 in Case (i).

and  $\tilde{b}$ ), we see that  $\deg(v) = \deg(\tilde{b}) = m$ . Moreover, both  $v$  and  $\tilde{b}$  are adjacent to a unique vertex that is not a leaf (namely  $w$  and  $\tilde{a}$ ). Thus, both  $v$  and  $\tilde{b}$  are adjacent to exactly  $m - 1$  leaves. Note that, as  $v$  and  $\tilde{b}$  have trivial associated group,  $m - 1 \geq 2$ . Let  $v_1, \dots, v_{m-1}$  be the leaves of  $\bar{X}$  adjacent to  $v$ , and let  $\tilde{b}_1, \dots, \tilde{b}_{m-1}$  be the leaves of  $\bar{Z}$  adjacent to  $\tilde{b}$ . For  $j \in \{1, \dots, m - 1\}$ , let  $y_j^X$  be the preimage by the marking of the generator of the group associated with  $v_j$  and let  $y_j^Z$  be the preimage by the marking of the generator of the group associated with  $\tilde{b}_j$ . As we suppose that  $\mathcal{X} \neq \mathcal{Z}$ , up to reordering and composing by an inner automorphism, one of the following holds.

(i) There exist  $j, k \in \{1, \dots, m - 1\}$  distinct such that  $y_j^X = y_i y_j^Z y_i$  and  $y_k^X = y_k^Z$  (see Figure 3.9).

(ii) There exist  $j, k \in \{1, \dots, m - 1\}$  distinct and a leaf  $\tilde{a}_k$  adjacent to  $\tilde{a}$  such that  $y_j^X = y_i y_j^Z y_i$  and such that the preimage by the marking of the generator of the group associated with  $\tilde{a}_k$  is  $y_k^X$ . Moreover, there exists a leaf  $w_0$  in  $\bar{X}$  adjacent to  $w$  such that the preimage by the marking of the generator of the group associated with  $w_0$  is  $y_k^Z$  (see Figure 3.10).

(iii) There exist  $j, \ell \in \{1, \dots, m - 1\}$  distinct and a leaf  $\tilde{a}_j$  adjacent to  $\tilde{a}$  such that  $y_\ell^X = y_\ell^Z$  and such that the preimage by the marking of the generator of the group associated with  $\tilde{a}_j$  is  $y_j^X$  (see Figure 3.11).

(iv) For all  $j \in \{1, \dots, m - 1\}$ , there exists a leaf  $\tilde{a}_j$  adjacent to  $\tilde{a}$  such that the preimage by the marking of the generator of the group associated with  $\tilde{a}_j$  is  $y_j^X$ .

We then distinguish two cases, according to whether  $y_j^X = y_i y_j^Z y_i$  or not.

Suppose first that  $y_j^Z = y_i y_j^X y_i$  (Cases (i) and (ii)). Let  $Z'_2$  be the homothety class of the marked graph of groups  $Z'_2$  defined as follows (see Figures 3.9 and 3.10):

- The underlying graph  $\bar{Z}'_2$  of  $Z'_2$  is obtained from  $\bar{Y}'$  by blowing-up an edge  $h$  at  $a$  so that one of the two endpoints of  $h$  is a leaf. Let  $p': \bar{Z}'_2 \rightarrow \bar{Y}'$  be the projection. Let  $x$



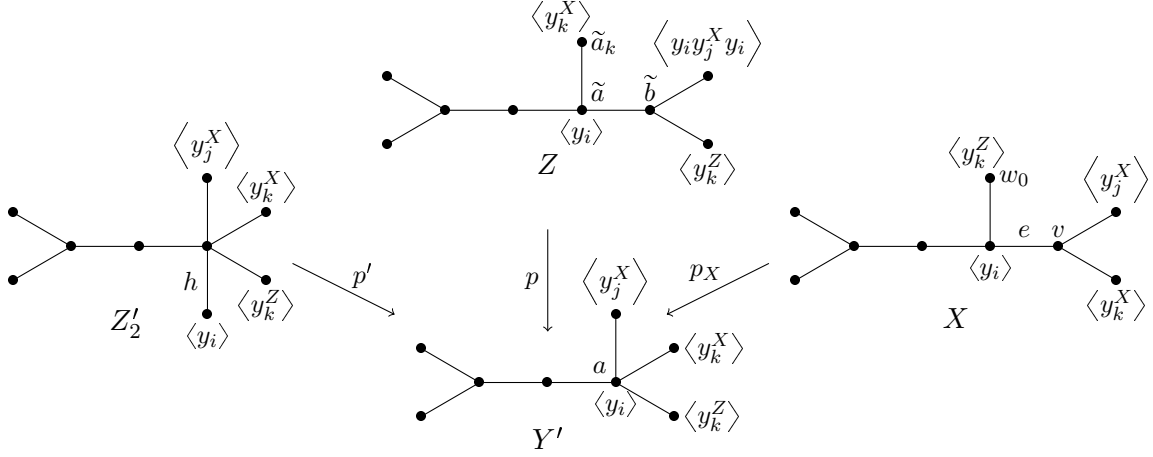


Figure 3.10: The construction of  $Z'_2$  in Lemma 3.4.9 in Case (ii).

be a vertex of the underlying graph of  $Z'_2$ . Let  $y$  be the preimage by the marking of the generator of the group associated with  $p'(x)$ .

- If  $p'(x) \neq a$  or if  $p'(x) = a$  and  $x$  is a leaf, then the preimage by the marking of  $Z'_2$  of the generator of the group associated with  $x$  is  $y$ .
- If  $p'(x) = a$  and  $x$  is not a leaf, then  $x$  has trivial associated group.

By the induction hypothesis, as  $\overline{Z}'_2$  has  $k$  vertices with nontrivial associated vertex groups that are not leaves, the homothety class  $\mathcal{Z}'_2$  is fixed by  $f$ . What is more,  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, \mathcal{Z}'_2) = 2$ . Indeed, a common refinement of  $Z$  and  $Z'_2$  is obtained from  $X$  by blowing-up the edge  $h$  at the vertex  $w$  of  $p_X^{-1}(a)$  with nontrivial associated group.

**Claim.** In  $\text{lk}(\mathcal{Y}')$ , we have  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, \mathcal{Z}'_2) > 2$ .

**Proof.** Since both  $\mathcal{Z}$  and  $\mathcal{Z}'_2$  have nontrivial negative link with no edge, we see by Lemma 3.4.4 (1) that  $|V\overline{\mathcal{Z}}| = |V\overline{\mathcal{Z}'_2}|$ . As both  $\overline{\mathcal{Z}}$  and  $\overline{\mathcal{Z}'_2}$  are trees, we have  $|E\overline{\mathcal{Z}}| = |E\overline{\mathcal{Z}'_2}|$ . Thus, as  $\mathcal{Z} \neq \mathcal{Z}'_2$ , we have  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, \mathcal{Z}'_2) > 1$ . As  $\mathcal{Y}'$  has trivial negative link, the only way  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, \mathcal{Z}'_2) = 2$  is that  $Z$  and  $Z'_2$  have a common refinement. Let  $z$  be the leaf of  $\overline{\mathcal{Z}'_2}$  such that  $p'(z) = a$ . Then the preimage by the marking of the group associated with  $z$  is  $\langle y_i \rangle$ .

Let  $Z_2^{(2)}$  be a refinement of  $Z'_2$ , let  $\overline{Z}_2^{(2)}$  be its underlying graph and let  $y_1^{(2)}, \dots, y_n^{(2)}$  be the preimages by the marking of  $Z_2^{(2)}$  of the generators of the nontrivial vertex groups of  $Z_2^{(2)}$ . Since both  $Z$  and  $Z'_2$  are obtained from  $Y'$  by blowing-up an edge at  $a$  while applying a twist around an edge adjacent to  $a$ , a potential common refinement of  $Z$  and  $Z'_2$  is obtained from  $Y'$  by blowing-up a forest while applying a twist around an edge adjacent to  $a$ . Thus, we may assume that, for all  $m \in \{1, \dots, n\}$ , there exists  $\alpha_m \in \{0, 1\}$  such that  $y_m^{(2)} = y_i^{\alpha_m} y_m y_i^{\alpha_m}$ .

Suppose first that  $y_k^X = y_k^Z$  (Case (i)). Let  $\tilde{v}_j$  and  $\tilde{v}_k$  be the lifts in  $\overline{Z}_2^{(2)}$  of respectively  $p'^{-1}(p_X(v_j))$  and  $p'^{-1}(p_X(v_k))$  with nontrivial associated group. Since  $p'^{-1}(p_X(v_j))$  and

$p'^{-1}(p_X(v_k))$  are contained in the same connected component of  $\overline{Z}'_2 - \{z\}$ , there exists  $\alpha \in \{0, 1\}$  such that the preimages by the marking of  $Z_2^{(2)}$  of the generators of the groups associated with  $\tilde{v}_j$  and  $\tilde{v}_k$  are respectively  $y_i^\alpha y_j^X y_i^\alpha$  and  $y_i^\alpha y_k^X y_i^\alpha$ . As a consequence, there does not exist a refinement of  $Z'_2$  such that the preimages by the marking of the generators of the groups associated with the lifts of  $p'^{-1}(p_X(v_j))$  and  $p'^{-1}(p_X(v_k))$  with nontrivial associated groups are respectively  $y_i y_j^X y_i$  and  $y_k^X$ . Thus,  $Z$  and  $Z'_2$  do not have any common refinement and  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, \mathcal{Z}'_2) > 2$ .

Suppose now that there exists a leaf  $w_0$  in  $\overline{X}$  adjacent to  $w$  such that the preimage by the marking of the generator of the group associated with  $w_0$  is  $y_k^Z$  (Case (ii)). Let  $\tilde{v}_j, \tilde{v}_k$  and  $\tilde{w}_0$  be the lifts in  $\overline{Z}_2^{(2)}$  of respectively  $p'^{-1}(p_X(v_j)), p'^{-1}(p_X(v_k))$  and  $p'^{-1}(p_X(w_0))$  with nontrivial associated group. Since  $p'^{-1}(p_X(v_j)), p'^{-1}(p_X(v_k))$  and  $p'^{-1}(p_X(w_0))$  are contained in the same connected component of  $\overline{Z}'_2 - \{z\}$ , there exists  $\alpha \in \{0, 1\}$  such that the preimages by the marking of  $Z_2^{(2)}$  of the generators of the groups associated with  $\tilde{v}_j$  and  $\tilde{v}_k$  are respectively  $y_i^\alpha y_j^X y_i^\alpha, y_i^\alpha y_k^X y_i^\alpha$  and  $y_i^\alpha y_k^Z y_i^\alpha$ . As a consequence, there does not exist a refinement of  $Z'_2$  such that the preimages by the marking of the generators of the groups associated with the lifts of  $p'^{-1}(p_X(v_j)), p'^{-1}(p_X(v_k))$  and  $p'^{-1}(p_X(w_0))$  with nontrivial associated groups are respectively  $y_i y_j^X y_i, y_k^X$  and  $y_k^Z$ . Thus,  $Z$  and  $Z'_2$  do not have any common refinement and  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, \mathcal{Z}'_2) > 2$ .  $\square$

Suppose now that  $y_j^Z \neq y_i y_j^X y_i$  (Cases (iii) and (iv)). Then there exists a leaf  $\tilde{a}_j$  adjacent to  $\tilde{a}$  such that the preimage by the marking of the generator of the group associated with  $\tilde{a}_j$  is  $y_j^X$ . Let  $a_j = p(\tilde{a}_j)$ . Moreover, as  $\mathcal{Z} \neq \mathcal{X}$ , either there exists  $\ell \in \{1, \dots, m-1\}$  such that either  $y_\ell^Z = y_i y_\ell^X y_i$  (Case (ii)) or  $y_\ell^Z = y_\ell^X$  (Case (iii)) or there exist  $\ell \in \{1, \dots, m-1\} - \{j\}$  and a leaf  $\tilde{a}_\ell$  of  $\overline{Z}$  adjacent to  $\tilde{a}$  such that the preimage by the marking of the generator of the group associated with  $\tilde{a}_\ell$  is  $y_\ell^X$  (Case (iv)). By the claim above (see Case (ii)), we can suppose that  $y_\ell^Z \neq y_i y_\ell^X y_i$ .

Let  $\mathcal{Z}'_3$  be the homothety class of the marked graph of groups  $Z'_3$  defined as follows (see Figure 3.11 with  $y_\ell^X = y_\ell^Z$ ):

- The underlying graph  $\overline{Z}'_3$  of  $Z'_3$  is obtained from  $\overline{Y}'$  by blowing-up an edge  $h$  at  $a$  so that one of the two endpoints of  $h$  is a leaf. Let  $p': \overline{Z}'_3 \rightarrow \overline{Y}'$  be the projection. Let  $x$  be a vertex of the underlying graph of  $Z'_3$ . Let  $y$  be the preimage by the marking of the generator of the group associated with  $p'(x)$ .
  - If  $p'(x) \neq a, a_j$  or if  $p'(x) = a$  and  $x$  is a leaf, then the preimage by the marking of  $Z'_3$  of the generator of the group associated with  $x$  is  $y$ .
  - If  $p'(x) = a$  and  $x$  is not a leaf, then  $x$  has trivial associated group.
  - If  $p'(x) = a_j$ , then  $x$  is a leaf. Moreover, the preimage by the marking of  $Z'_3$  of the generator of the group associated with  $x$  is  $y_i y_j^X y_i$ .

Since  $Z'_3$  has one less vertex with nontrivial associated group which is not a leaf, by the induction hypothesis,  $\mathcal{Z}'_3$  is fixed by  $f$ . What is more,  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{Z}, \mathcal{Z}'_3) = 2$ . Indeed, a common refinement of  $Z$  and  $Z'_3$  is obtained from  $Z$  by blowing-up the edge  $h$  at the vertex of  $p^{-1}(a)$  with nontrivial associated group.

**Claim.** In  $\text{lk}(\mathcal{Y}')$ , we have  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, \mathcal{Z}'_3) > 2$ .

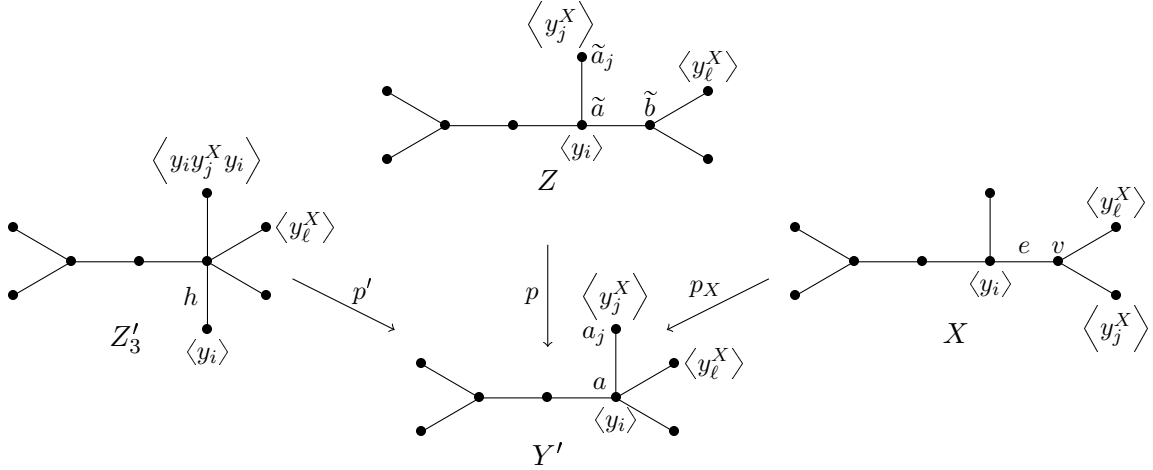


Figure 3.11: The construction of  $Z'_3$  in Lemma 3.4.9 in the case  $y_l^X = y_l^Z$ .

**Proof.** The proof is identical for Cases (iii) and (iv). Since both  $\mathcal{X}$  and  $Z'_3$  have nontrivial negative link with no edge we see by Lemma 3.4.4 (1) that  $|V\bar{\mathcal{X}}| = |V\bar{Z}'_3|$ . As both  $\bar{\mathcal{X}}$  and  $\bar{Z}'_3$  are trees, we have  $|E\bar{\mathcal{X}}| = |E\bar{Z}'_3|$ . Thus, as  $\mathcal{X} \neq Z'_3$ , we have  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, Z'_3) > 1$ . As  $\mathcal{Y}'$  has trivial negative link, the only way that  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, Z'_3) = 2$  is that  $X$  and  $Z'_3$  have a common refinement. Let  $z$  be the leaf of  $\bar{Z}'_3$  such that  $p'(z) = a$ . Then the preimage by the marking of the group associated with  $z$  is  $\langle y_i \rangle$ .

Let  $Z_3^{(2)}$  be a refinement of  $Z'_3$ , let  $\bar{Z}_3^{(2)}$  be its underlying graph and let  $y_1^{(2)}, \dots, y_n^{(2)}$  be the preimages by the marking of  $Z_3^{(2)}$  of the generators of the nontrivial vertex groups of  $Z_3^{(2)}$ . Since both  $Z$  and  $Z'_3$  are obtained from  $Y'$  by blowing-up an edge at  $a$  while applying a twist around an edge adjacent to  $a$ , a potential common refinement of  $Z$  and  $Z'_3$  is obtained from  $Y'$  by blowing-up a forest while applying a twist around an edge adjacent to  $a$ . Thus, we may assume that, for all  $m \in \{1, \dots, n\}$ , there exists  $\alpha_m \in \{0, 1\}$  such that  $y_m^{(2)} = y_i^{\alpha_m} y_m y_i^{\alpha_m}$ . Let  $\tilde{v}_j$  and  $\tilde{v}_l$  be the lifts in  $\bar{Z}_3^{(2)}$  of respectively  $p'^{-1}(p_X(v_j))$  and  $p'^{-1}(p_X(v_l))$  with nontrivial associated group. Since  $p'^{-1}(p_X(v_j))$  and  $p'^{-1}(p_X(v_l))$  are contained in the same connected component of  $\bar{Z}'_3 - \{z\}$ , there exists  $\alpha \in \{0, 1\}$  such that the preimages by the marking of  $Z_3^{(2)}$  of the generators of the groups associated with  $\tilde{v}_j$  and  $\tilde{v}_l$  are respectively  $y_i^{\alpha+1} y_j^X y_i^{\alpha+1}$  and  $y_i^\alpha y_l^X y_i^\alpha$ . As a consequence, there does not exist a refinement  $Z_0$  of  $Z'_3$  such that the preimages by the marking of the generators of the groups associated with the lifts of  $p'^{-1}(p_X(v_j))$  and  $p'^{-1}(p_X(v_l))$  with nontrivial associated groups are respectively  $y_j^X$  and  $y_l^X$ . Thus,  $X$  and  $Z'_3$  do not have any common refinement and  $d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, Z'_3) > 2$ .  $\square$

Since  $d_{\text{lk}(f(\mathcal{Y}'))}(f(\mathcal{X}), f(Z'_3)) = d_{\text{lk}(\mathcal{Y}')}(\mathcal{X}, Z'_3) = 2$ , the two claims imply that  $f(\mathcal{X}) = \mathcal{X}$ .

We now prove that  $f(\mathcal{Y}) = \mathcal{Y}$ . Let  $v_1$  be a vertex of  $\bar{\mathcal{Y}}$  that is adjacent to at least

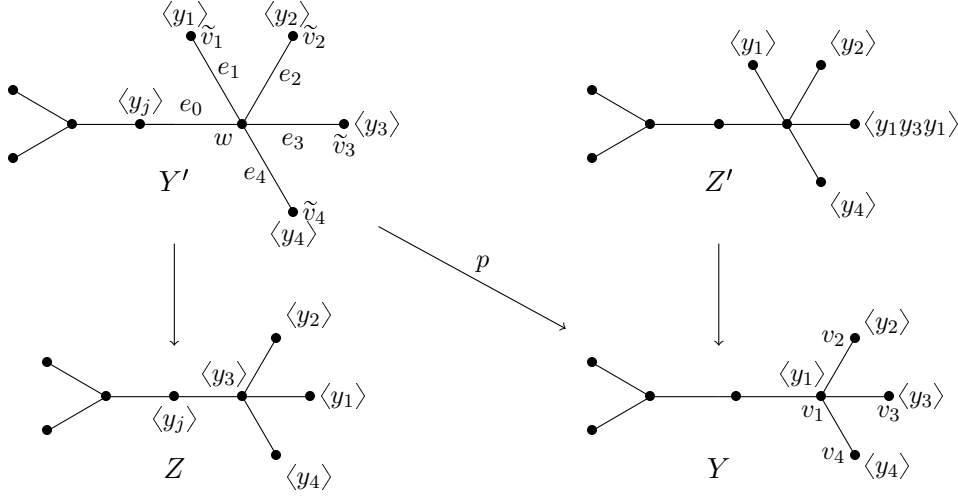


Figure 3.12: The constructions of  $Y'$  and  $Z'$  in Lemma 3.4.9.

one leaf. Let  $v_2, \dots, v_\ell$  be the leaves adjacent to  $v_1$  and, for  $i \in \{1, \dots, \ell\}$ , let  $y_i$  be the preimage by the marking of the generator of the group associated with  $v_i$ . Let  $\mathcal{Y}'$  be the equivalence class of the marked graph of groups  $Y'$  defined as follows (see Figure 3.12):

- The underlying graph  $\bar{Y}'$  of  $Y'$  is obtained from  $Y$  by blowing-up an edge  $e_1$  at  $v_1$  such that one of the endpoint of  $e_1$  is a leaf. Let  $p: Y' \rightarrow Y$  be the natural projection. Let  $x$  be a vertex of  $\bar{Y}'$ , and let  $y$  be the preimage by the marking of the generator of the group associated with  $p(x)$ .

- If  $p(x) \neq v_1$  or if  $p(x) = v_1$  and  $x$  is a leaf, then the preimage by the marking of  $Y'$  of the generator of the group associated with  $x$  is  $y$ .

- If  $p(x) = v_1$  and  $x$  is not a leaf, then  $v_1$  has trivial associated group.

By the previous step, as  $Y'$  has  $k$  vertices with nontrivial associated groups that are not leaves, and as  $\text{lk}_-(\mathcal{Y}')$  is nontrivial and has no edge, we see that  $f(\mathcal{Y}') = \mathcal{Y}'$ . By the second claim in the proof of Lemma 3.4.9, the negative link of  $f(\mathcal{Y}')$  is nontrivial and has no edge. Therefore,  $f$  preserves the negative link of  $\mathcal{Y}'$  and  $f(\mathcal{Y}) \in \text{lk}_-(\mathcal{Y}')$ . Let  $w$  be the endpoint of  $e_1$  with trivial associated group. For  $i \in \{1, \dots, \ell\}$ , let  $\tilde{v}_i$  be the leaf of  $\bar{Y}'$  which lifts  $v_i$ . Let  $e_2, \dots, e_\ell$  be the edges of  $\bar{Y}'$  such that for all  $i \in \{2, \dots, \ell\}$ , the endpoints of  $e_i$  are  $\tilde{v}_i$  and  $w$ .

We claim that there exists a unique  $i \in \{1, \dots, \ell\}$  such that a representative of  $f(\mathcal{Y})$  is obtained from  $Y'$  by contracting  $e_i$ . Indeed, as  $\text{lk}_-(\mathcal{Y}')$  is nontrivial and has no edge, Lemma 3.4.4 (1) implies that  $\bar{Y}'$  has exactly one vertex with trivial associated group, namely  $w$ . Therefore, a representative of  $f(\mathcal{Y})$  is obtained from  $Y'$  by contracting a unique edge adjacent to  $w$ .

Suppose towards a contradiction that there exists an edge  $e_0$  in  $\bar{Y}'$  between  $w$  and a vertex  $w'$  with nontrivial associated group that is not a leaf and such that a representative of  $f(\mathcal{Y})$  is obtained from  $Y'$  by collapsing  $e_0$ . Let  $y_j$  be the preimage by the marking of  $Y'$  of the generator of the group associated with  $w'$ . Let  $\mathcal{Z}_0$  be the homothety class of the

marked graph of groups obtained from  $Y'$  by contracting  $e_0$ . By induction hypothesis,  $f(\mathcal{Z}_0) = \mathcal{Z}_0$ . Thus  $f(\mathcal{Y}) \neq \mathcal{Z}_0$ .

Thus, there exists a unique  $i \in \{1, \dots, \ell\}$  such that a representative of  $f(\mathcal{Y})$  is obtained from  $Y'$  by contracting  $e_i$ . We claim that  $i = 1$ . Indeed, for  $i \neq 1$ , let  $\mathcal{Z}$  be the equivalence class of the marked graph of groups  $Z$  obtained from  $Y'$  by collapsing  $e_i$ . Let  $\mathcal{Z}'$  be the equivalence class of the marked graph of groups  $Z'$  whose underlying graph is  $\overline{Y}'$  and such that the preimage by the markings of the generators of the groups associated with  $\tilde{v}_1, \dots, \tilde{v}_\ell$  are  $y_1, \dots, y_{i-1}, y_1 y_i y_1, y_{i+1}, \dots, y_\ell$ . Then  $\mathcal{Z}' \in \text{lk}_+(\mathcal{Y})$  because  $Z'$  is obtained from  $Y$  first by precomposing the marking of  $Y$  by the automorphism which sends  $y_i$  to  $y_1 y_i y_1$  and fixes all the other  $y_i$  and then blowing-up an edge at  $v_1$  such that one of the endpoints of this edge is a leaf and then. However,  $\mathcal{Z}' \notin \text{lk}(\mathcal{Z})$  because the vertex of  $\overline{Z}$  whose preimage by the marking of the associated group is  $\langle y_i \rangle$  is a leaf. Therefore there is no refinement of  $Z$  such that there exist two vertex groups of the refinement such that the preimage by the marking of the generators of the vertex groups are respectively  $y_1 y_i y_1$  and  $y_k$ . As  $f(\mathcal{Z}') = \mathcal{Z}'$  by the previous step, we have  $f(\mathcal{Y}) \in \text{lk}(\mathcal{Z}')$ . Therefore,  $f(\mathcal{Y}) \neq \mathcal{Z}$  and  $f(\mathcal{Y}) = \mathcal{Y}$ .  $\square$

We can now show the injectivity of the homomorphism  $\text{Aut}(K_n) \rightarrow \text{Aut}(L_n)$ .

**Proposition 3.4.10.** *Let  $n \geq 4$ . Let  $f \in \text{Aut}(K_n)$  such that  $f|_{O_n} = \text{id}_{O_n}$  and  $f|_{F_n} = \text{id}_{F_n}$ . Then  $f = \text{id}_{K_n}$ .*

**Proof.** Let  $k \in \mathbb{N}$  and let  $\mathcal{X} \in VK_n$  be such that the underlying graph  $\overline{X}$  of a representative  $X$  of  $\mathcal{X}$  has exactly  $k$  vertices with trivial associated group. We prove by induction on  $k$  that  $f(\mathcal{X}) = \mathcal{X}$ . If  $k = 0$ , then  $\mathcal{X}$  has trivial negative link. Thus, by Lemma 3.4.9, we have  $f(\mathcal{X}) = \mathcal{X}$ .

Suppose now that  $k \geq 1$ . Then, as any representative of an element of  $\text{lk}_-(\mathcal{X})$  is obtained from  $X$  by collapsing at least an edge, by the induction hypothesis, we have  $f|_{\text{lk}_-(\mathcal{X})} = \text{id}|_{\text{lk}_-(\mathcal{X})}$  and  $\text{lk}_-(f(\mathcal{X})) = \text{lk}_-(\mathcal{X})$ .

Suppose towards a contradiction that  $f(\mathcal{X}) \neq \mathcal{X}$ . Let  $Y$  be a representative of  $f(\mathcal{X})$ , and let  $\overline{Y}$  be the underlying graph of  $Y$ . By the induction hypothesis,  $\overline{Y}$  has at least  $k$  vertices with trivial associated group. Since  $\mathcal{X} \neq f(\mathcal{X})$ , there exists an edge  $e \in E\overline{Y}$  such that the free factor decomposition of  $W_n$  induced by  $e$  is distinct from the free factor decomposition induced by any edge of  $\overline{X}$ . Let  $\mathcal{Z} \in \text{lk}_-(\mathcal{X})$ . Let  $Z$  be a representative of  $\mathcal{Z}$  obtained from  $X$  by collapsing a forest, and let  $\overline{Z}$  be the underlying graph of  $Z$ . By Remark 3.4.6 (1), for any edge  $f \in E\overline{Z}$ , there exists an edge  $\tilde{f} \in E\overline{X}$  such that the free factor decomposition induced by  $f$  is the same one as the free factor decomposition induced by  $\tilde{f}$ . Thus, there does not exist any edge of  $\overline{Z}$  which induces, up to global conjugation, the same free factor decomposition of  $W_n$  as  $e$ . But, for any edge  $f$  of  $\overline{Y}$ , there exists  $\mathcal{Z}' \in \text{lk}_-(\mathcal{Y})$ , a representative  $Z'$  of  $\mathcal{Z}'$  with underlying graph  $\overline{Z}'$  and an edge  $g$  of  $\overline{Z}'$  such that the free factor decomposition induced by  $e$  is the same as the one induced by  $g$  ( $Z$  is obtained from  $Y$  by contracting an edge distinct from  $f$ ). This contradicts the fact that  $\text{lk}_-(f(\mathcal{X})) = \text{lk}_-(\mathcal{X})$ . Thus  $f(\mathcal{X}) = \mathcal{X}$  and  $f = \text{id}_{K_n}$ .  $\square$

**Proof of Theorem 3.1.1.** Let  $n \geq 4$ . The injectivity is immediate since the homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$  is injective by Theorem 3.3.1 and since  $L_n$  is a subgraph of  $K_n$ . We now prove surjectivity. Let  $f \in \text{Aut}(K_n)$ . By Proposition 3.4.8, the automorphism  $f$  induces an automorphism  $\tilde{f} \in \text{Aut}(L_n)$ . By Theorem 3.3.1,  $\tilde{f}$  is induced by an element  $\gamma \in \text{Out}(W_n)$ . Since the homomorphism  $\text{Aut}(K_n) \rightarrow \text{Aut}(L_n)$  is injective by Proposition 3.4.10,  $f$  is induced by  $\gamma$ . This concludes the proof.  $\square$

### 3.5 Rigidity of the simplicial completion of $K_n$

Let  $n \geq 4$ . A *splitting* of  $W_n$  is a minimal, simplicial  $W_n$ -action on a simplicial tree  $S$  and such that:

- (1) The finite graph  $W_n \backslash S$  is not empty and not reduced to a point.
- (2) Vertices of  $S$  with trivial stabilizer have degree at least 3.

Here *minimal* means that  $W_n$  does not preserve any proper subtree of  $S$ . A splitting  $S$  of  $W_n$  is *free* if all edge stabilizers are trivial. A splitting  $S'$  is a *blow-up*, or equivalently a *refinement*, of a splitting  $S$  if  $S$  is obtained from  $S'$  by collapsing some edge orbits in  $S'$ . Two splittings are *compatible* if they have a common refinement. If  $k \geq 1$  is an integer, a free splitting  $S$  is a *k-edge free splitting* if  $W_n \backslash S$  has exactly  $k$  edges. An *F-one-edge free splitting* is a one-edge free splitting  $S$  such that one of the vertex groups of  $W_n \backslash S$  is isomorphic to  $W_{n-1}$  while the other vertex group is isomorphic to  $F$ .

The simplicial completion of  $K_n$ , denoted by  $\overline{K}_n$ , is the flag complex such that:

- The vertices of  $\overline{K}_n$  are the equivalence classes of free splittings of  $W_n$ , where two free splittings  $S$  and  $S'$  are equivalent if there exists a  $W_n$ -equivariant homeomorphism between them.
- Two equivalence classes of free splittings  $\mathcal{S}$  and  $\mathcal{S}'$  are adjacent in  $\overline{K}_n$  if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  refines  $S'$  or conversely.

In the literature, this complex is also called the *free splitting complex*. The free splitting complex appears as well in the study of the outer automorphism group of a free group of finite rank and more generally in the study of the outer automorphism group of a free product of groups (see [AS, HaM3, HaM2]). In particular, Handel-Mosher ([HaM3, HaM2]) in the case of  $\text{Out}(F_N)$  and Handel-Mosher and Horbez ([HaM3, Hor2]) in the case of the outer automorphism group of a free product of groups proved that this complex is Gromov hyperbolic.

We have a canonical injective homomorphism  $K_n \hookrightarrow \overline{K}_n$  defined as follows. Let  $\mathcal{X} \in VK_n$  be the equivalence class of a marked graph of groups, and let  $X$  be a representative of  $\mathcal{X}$ . Let  $S$  be a Bass-Serre tree corresponding to  $X$ , and let  $\mathcal{S}$  be the equivalence class of  $S$ . Then the map

$$\begin{aligned} \Phi & : K_n \rightarrow \overline{K}_n \\ & \mathcal{X} \mapsto \mathcal{S} \end{aligned}$$

is a well-defined injective homomorphism. From now on, we identify  $K_n$  with its image in  $\overline{K}_n$ .

The group  $\text{Aut}(W_n)$  acts on  $\overline{K}_n$  by precomposition of the action. For any  $\alpha \in \text{Inn}(W_n)$  and for any  $S \in \overline{K}_n$ , we have  $\alpha(S) = S$ . Therefore the action of  $\text{Aut}(W_n)$  induces an action of  $\text{Out}(W_n)$ .

In this section, we prove Theorem 3.1.2. In order to do so, we first show that any automorphism of  $\overline{K}_n$  preserves  $K_n$ . Thus, we have a restriction homomorphism  $\text{Aut}(\overline{K}_n) \rightarrow \text{Aut}(K_n)$  which, as we will see, turns out to be injective. Theorem 3.1.2 then follows from Theorem 3.1.1.

We first characterise the vertices of  $K_n$  in  $\overline{K}_n$ .

**Proposition 3.5.1.** *Let  $n \geq 4$ . Let  $\mathcal{S} \in V\overline{K}_n$ . If  $\mathcal{S} \in VK_n$ , then  $\mathcal{S}$  has finite valence in  $\overline{K}_n$ . If  $\mathcal{S} \in V\overline{K}_n - VK_n$ , then  $\mathcal{S}$  has infinite valence in  $\overline{K}_n$ .*

**Proof.** Suppose that  $\mathcal{S} \in VK_n$ , and let  $\mathcal{S}' \in \text{lk}(\mathcal{S})$ . Let  $S$  and  $S'$  be representatives of  $\mathcal{S}$  and  $\mathcal{S}'$ . If  $S$  refines  $S'$ , then  $W_n \setminus S'$  is obtained from  $W_n \setminus S$  by collapsing a forest. Since  $W_n \setminus S$  is a finite tree, there are only finitely many possibilities for  $W_n \setminus S'$ , hence finitely many possibilities for  $S'$ . If  $S'$  refines  $S$ , then, since  $\mathcal{S} \in K_n$ , the equivalence class  $\mathcal{S}'$  also belongs to  $K_n$ . Thus, we have  $\mathcal{S}' \in \text{lk}_+^{K_n}(\mathcal{S})$  where  $\text{lk}_+^{K_n}(\mathcal{S})$  is the positive link of  $\mathcal{S}$  in  $K_n$ . Since  $\text{lk}_+^{K_n}(\mathcal{S})$  is finite, there are only finitely many possibilities for  $\mathcal{S}'$ . Hence  $\text{lk}(\mathcal{S})$  is finite.

Now suppose that  $\mathcal{S} \in V\overline{K}_n - VK_n$ . Let  $S$  be a representative of  $\mathcal{S}$ . Since we have  $\mathcal{S} \in V\overline{K}_n - VK_n$ , there exists a vertex of  $S$  whose stabilizer contains a subgroup  $G$  of  $W_n$  isomorphic to  $W_2$ . Since  $\text{Aut}(W_2)$  is isomorphic to  $W_2$  (see e.g. [Tho, Lemma 1.4.2]), we see that  $\text{Stab}(S)$  is infinite by Proposition 3.2.1. Moreover, we claim that there exists  $\mathcal{S}' \in \text{lk}(\mathcal{S}) \cap VK_n$ . Indeed, let  $v$  be a vertex of  $W_n \setminus S$  whose associated group is isomorphic to  $W_i$  with  $i \geq 2$ . then one can construct an element  $T \in K_i$  and then blow-up  $T$  at  $v$ . The equivalence class of the result is an element in  $\text{lk}(\mathcal{S})$ . Applying the process to every vertex of  $W_n \setminus S$  with infinite associated vertex group gives an element  $\mathcal{S}' \in \text{lk}(\mathcal{S}) \cap VK_n$ . As  $\text{Stab}(\mathcal{S})$  acts on  $\text{lk}(\mathcal{S})$ , the orbit of  $\mathcal{S}'$  under the action of  $\text{Stab}(\mathcal{S})$  is infinite (recall that  $\text{Stab}(\mathcal{S}')$  is finite by Proposition 3.2.1). Thus  $\text{lk}(\mathcal{S})$  is infinite.  $\square$

Thus, Proposition 3.5.1 tells us that any automorphism of  $\overline{K}_n$  preserves  $K_n$ . This gives a restriction homomorphism

$$\text{Aut}(\overline{K}_n) \rightarrow \text{Aut}(K_n).$$

In the rest of the section, we prove that this homomorphism is injective. In order to show this, we first prove that any automorphism of  $\overline{K}_n$  which fixes  $K_n$  pointwise also fixes the set of one-edge free splittings pointwise. We will then conclude by the following proposition, due to Scott and Swarup.

**Theorem 3.5.2.** [SS, Theorem 2.5] *Let  $n \geq 4$ . Any set  $\{S_1, \dots, S_k\}$  of pairwise distinct, pairwise compatible, one-edge free splittings of  $W_n$  has a unique refinement  $S$  such that  $W_n \setminus S$  has exactly  $k$  edges. If  $S$  is a free splitting such that  $W_n \setminus S$  has exactly  $k$  edges, then  $S$  refines exactly  $k$  distinct one-edge free splittings.*

The next lemma is inspired by [HW1, Lemma 2.3] due to Horbez and Wade.

**Lemma 3.5.3.** *Let  $n \geq 4$ . For all  $\mathcal{S} \in V\overline{K}_n$ , the following assertions are equivalent.*

- (1) *There exists  $S \in \mathcal{S}$  such that  $S$  is an  $F$ -one-edge free splitting.*
- (2) *The equivalence class  $\mathcal{S}$  satisfies the following properties.*
  - (a) *The link of  $\mathcal{S}$  is infinite.*
  - (b) *There exists a  $\{0\}$ -star  $\mathcal{X}$  such that  $\mathcal{X} \in \text{lk}(\mathcal{S})$ .*
  - (c) *There exist  $\mathcal{S}_1, \mathcal{S}_2 \in \text{lk}(\mathcal{S})$  such that  $d_{\overline{K}_n}(\mathcal{S}_1, \mathcal{S}_2) = 2$  and such that  $\mathcal{S}_1 - \mathcal{S} - \mathcal{S}_2$  is the unique path of length 2 joining  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .*

**Proof.** We first prove that (1) implies (2). Let  $S \in \mathcal{S}$  be an  $F$ -one-edge free splitting. Then  $\mathcal{S} \notin V\overline{K}_n$  and Proposition 3.5.1 implies that  $\text{lk}(\mathcal{S})$  is infinite, which proves Property (a).

In order to prove Property (b), let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S$ . Let  $X$  be the  $\{0\}$ -star such that, if  $w_1, \dots, w_n$  are the leaves of  $\overline{X}$ , and if  $i \in \{1, \dots, n\}$ , then the stabilizer of  $w_i$  is  $\langle x_i \rangle$ . Let  $\mathcal{X}$  be the equivalence class of  $X$ . Then  $\mathcal{X} \in \text{lk}(\mathcal{S})$ .

In order to prove Property (c), let  $S_1$  be the 2-edge free splitting induced by the decomposition

$$W_n = \langle x_1, x_2 \rangle * \langle x_3, \dots, x_{n-1} \rangle * \langle x_n \rangle,$$

where the preimage by the marking of the group associated with the central vertex of  $W_n \setminus S_1$  is  $\langle x_3 \dots, x_{n-1} \rangle$ . Let  $S_2$  be the 2-edge free splitting induced by the decomposition

$$W_n = \langle x_1, x_3 \rangle * \langle x_2, \widehat{x}_3, \dots, x_{n-1} \rangle * \langle x_n \rangle,$$

where the preimage by the marking of the group associated with the central vertex of  $W_n \setminus S_2$  is  $\langle x_2, \widehat{x}_3, \dots, x_{n-1} \rangle$ . For  $i \in \{1, 2\}$ , let  $\mathcal{S}_i$  be the equivalence class of  $S_i$ . Then  $\mathcal{S}_1, \mathcal{S}_2 \in \text{lk}(\mathcal{S})$ . Moreover, the equivalence classes  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are not adjacent in  $\overline{K}_n$  since both  $S_1$  and  $S_2$  are 2-edge free splittings, thus, there does not exist  $i \in \{1, 2\}$ ,  $j \in \{1, 2\} - \{i\}$  such that  $S_i$  collapses onto  $S_j$ . So  $d_{\overline{K}_n}(\mathcal{S}_1, \mathcal{S}_2) = 2$ .

**Claim.** Let  $\mathcal{T} \in V\overline{K}_n$  be such that  $\mathcal{S}_1 - \mathcal{T} - \mathcal{S}_2$  is a path of length 2 joining  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Then  $\mathcal{S} = \mathcal{T}$ .

**Proof.** Suppose towards a contradiction that there exists a representative  $T$  of  $\mathcal{T}$  such that  $T$  is a common refinement of  $S_1$  and  $S_2$ . For  $i \in \{1, \dots, n\}$ , let  $v_i$  be the only vertex of  $T$  fixed by  $x_i$ . Note that, for  $i \neq j$ , the vertices  $v_i$  and  $v_j$  may not be distinct. Since  $T$  refines  $S_1$ , for every edge  $e \in ET$ , one of the following holds:

- the vertices  $v_1, v_2$  and  $v_3$  belong to the same connected component of  $T - \{\hat{e}\}$ ,
- the vertices  $v_1$  and  $v_2$  belong to a connected component of  $T - \{\hat{e}\}$  distinct from the one that contains  $v_3$ ,



- there exist  $i \in \{1, 2\}$  and  $j \in \{1, 2\} - \{i\}$  such that  $v_i$  is in a connected component of  $T - \{\hat{e}\}$  distinct from the one containing  $v_j, v_3$  and  $v_n$ .

But if  $T$  refines  $S_2$ , there exists  $e \in ET$  such that  $v_1$  and  $v_3$  belong to a connected component of  $T - \{\hat{e}\}$  distinct from the one that contains  $v_2$  and  $v_n$ . This leads to a contradiction.

Thus, there exists a representative  $T$  of  $\mathcal{T}$  such that both  $S_1$  and  $S_2$  collapse to  $T$ . As  $S$  is the only such one-edge free splitting, the claim follows.  $\square$

We now prove that (2) implies (1). Suppose that  $\mathcal{S}$  satisfies the properties of Assertion (2) of the lemma.

**Claim.** Property (c) implies that  $\mathcal{S}$  has a representative  $S$  that is either a one-edge free splitting or is such that there is no free splitting of  $W_n$  that properly refines  $S$ .

**Proof.** Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be as in Property (c), and, for  $i \in \{1, 2\}$ , let  $S_i$  be a representative of  $\mathcal{S}_i$ . Let  $S$  be a representative of  $\mathcal{S}$ . There are three cases to distinguish.

- If  $S$  refines  $S_1$  and if  $S_2$  refines  $S$ , then  $S_2$  refines  $S_1$ , so that  $d_{\overline{K}_n}(S_1, S_2) \leq 1$ . This leads to a contradiction.
- If  $S$  refines both  $S_1$  and  $S_2$ , then there does not exist any proper refinement of  $S$  as this would contradict the uniqueness of the path of length 2 between  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .
- If  $S$  is refined by both  $S_1$  and  $S_2$ , then  $S$  is a one-edge free splitting as otherwise there would exist a splitting  $S'$  that is properly refined by  $S$ . This would contradict the uniqueness of the path.

The claim follows.  $\square$

Since a free splitting which has no proper refinement is in  $K_n$ , the above claim, Property (a) and Proposition 3.5.1 imply that  $S$  is a one-edge free splitting. Property (b) implies in fact that  $S$  is an  $F$ -one-edge free splitting as the  $F$ -one-edge-free splittings are the only one-edge free splittings that are adjacent to a  $\{0\}$ -star. The lemma follows.  $\square$

**Lemma 3.5.4.** *Let  $n \geq 4$ . Let  $f \in \text{Aut}(\overline{K}_n)$  be such that  $f|_{L_n} = \text{id}_{L_n}$ . Let  $\mathcal{S}$  be the equivalence class of an  $F$ -one-edge free splitting  $S$ . Then  $f(\mathcal{S}) = \mathcal{S}$ .*

**Proof.** As  $f \in \text{Aut}(\overline{K}_n)$ , Corollary 3.4.8 and Lemmas 3.5.1 and 3.5.3 imply that  $f(\mathcal{S})$  is the equivalence class of an  $F$ -one-edge free splitting  $S'$ . Let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S$ . Let  $\mathcal{X}$  be the equivalence class of the  $F$ -star  $X$  represented in Figure 3.13 on the left.

Since  $f(\mathcal{X}) = \mathcal{X}$ , the free splitting  $S'$  is an  $F$ -one-edge free splitting obtained from  $X$  by collapsing  $n - 1$  edges. But if  $T$  is an  $F$ -one-edge free splitting obtained from

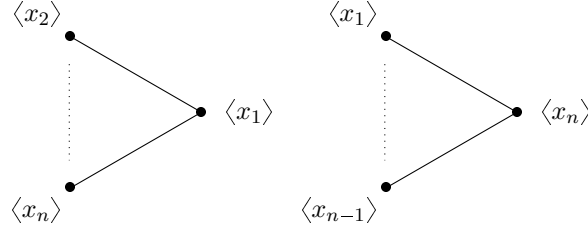


Figure 3.13: The  $F$ -stars  $X$  and  $X'$  of the proof of Lemma 3.5.4.

$X$  by collapsing  $n - 1$  edges, then there exists  $i \in \{1, \dots, n\}$  such that the free factor decomposition of  $W_n$  induced by  $T$  is

$$W_n = \langle x_1, \dots, \hat{x}_i, \dots, x_n \rangle * \langle x_i \rangle.$$

For  $i \in \{1, \dots, n\}$ , we will denote by  $T_i$  the  $F$ -one-edge free splitting with associated free factor decomposition  $\langle x_1, \dots, \hat{x}_i, \dots, x_n \rangle * \langle x_i \rangle$ , and by  $\mathcal{T}_i$  its equivalence class. For  $i \neq n$ , the free splitting  $T_i$  is a collapse of the  $F$ -star  $X'$  depicted in Figure 3.13 on the right, whereas  $S$  is not a collapse of  $X'$ .

Let  $\mathcal{X}'$  be the equivalence class of  $X'$ . Since  $f(\mathcal{X}') = \mathcal{X}'$ , we have that  $f(\mathcal{S})$  is not adjacent to  $\mathcal{X}'$ . But, for all  $i \neq n$ , the equivalence class  $\mathcal{T}_i$  is adjacent to  $\mathcal{X}'$ . Thus, for all  $i \neq n$ , we have  $f(\mathcal{S}) \neq \mathcal{T}_i$ . Therefore, as  $\mathcal{S} = \mathcal{T}_n$ , we conclude that  $f(\mathcal{S}) = \mathcal{S}$ .  $\square$

**Proof of Theorem 3.1.2.** By Proposition 3.5.1, there exists a homomorphism

$$\text{Aut}(\overline{K}_n) \rightarrow \text{Aut}(K_n)$$

induced by the restriction to  $K_n$ . In order to prove Theorem 3.1.2, it suffices to prove that this homomorphism is injective. Let  $f \in \text{Aut}(\overline{K}_n)$  be such that  $f|_{K_n} = \text{id}_{K_n}$ . Let us prove that  $f = \text{id}$ . By Theorem 3.5.2, it suffices to prove that, for any equivalence class  $\mathcal{S}$  of a one-edge free splitting  $S$ , we have  $f(\mathcal{S}) = \mathcal{S}$ . Indeed, let  $\mathcal{S}$  be the equivalence class of a free splitting. Then, by Theorem 3.5.2, there exist  $k$  one-edge free splittings  $\mathcal{S}_1, \dots, \mathcal{S}_k$  such that  $\mathcal{S}$  is the unique vertex of  $\overline{K}_n$  such that, for all  $i \in \{1, \dots, k\}$ ,  $\mathcal{S}$  is adjacent to  $\mathcal{S}_i$ . Thus, if, for any equivalence class  $\mathcal{S}$  of a one-edge free splitting  $S$ , we have  $f(\mathcal{S}) = \mathcal{S}$ , then  $f = \text{id}$ .

Suppose that  $\mathcal{S}$  is the equivalence class of a one-edge free splitting  $S$ . The case where  $S$  is an  $F$ -one-edge free splitting was proved in Lemma 3.5.4. If  $S$  is not an  $F$ -one-edge free splitting, let  $W_n = \langle x_1, \dots, x_k \rangle * \langle x_{k+1}, \dots, x_n \rangle$  be the free factor decomposition of  $W_n$  induced by  $S$ , with  $2 \leq k \leq n - 2$ . Let  $X$  be the free splitting of  $W_n$  depicted in Figure 3.14, and let  $\mathcal{X}$  be its equivalence class.

Then  $\mathcal{X} \in VK_n$ , so  $f(\mathcal{X}) = \mathcal{X}$ . As  $\mathcal{S} \in \text{lk}(\mathcal{X})$ , we also have that  $f(\mathcal{S}) \in \text{lk}(\mathcal{X})$ . Moreover,  $f(\mathcal{S}) \notin VK_n$  by Proposition 3.5.1. Thus, a representative of  $f(\mathcal{S})$  is obtained from  $X$  by collapsing a forest  $F$ .

**Claim.** Any splitting  $S'$  distinct from  $S$  and obtained from  $X$  by collapsing a forest is either an  $F$ -one-edge free splitting or is adjacent to an  $F$ -one-edge free splitting.

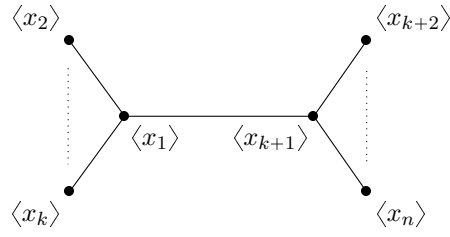


Figure 3.14: The free splitting  $X$  of the proof of Theorem 3.1.2.

**Proof.** If  $S' \neq S$  is obtained from  $X$  by collapsing a forest, and if  $S'$  is not an  $F$ -one-edge free splitting, there exists an edge  $e \in V(W_n \setminus S')$  such that  $e$  is adjacent to a leaf. This edge determines an  $F$ -one-edge free splitting adjacent to  $S'$ .  $\square$

Thus, by Lemma 3.5.4, any equivalence class  $\mathcal{S}' \in \text{lk}_-(\mathcal{X})$  is determined by the equivalence classes of  $F$ -one edge free splittings that are adjacent to  $\mathcal{S}'$ . Therefore we have  $f(\mathcal{S}) = \mathcal{S}$  and the equivalence class of any one-edge free splitting is fixed by  $f$ . Theorem 3.5.2 then implies that  $f = \text{id}$ . This concludes the proof of Theorem 3.1.2.  $\square$

## Chapitre 4

# Commensurations of the outer automorphism group of a universal Coxeter group

### 4.1 Introduction

Given a group  $G$ , the *abstract commensurator* of  $G$ , denoted by  $\text{Comm}(G)$ , is the group of equivalence classes of isomorphisms between finite index subgroups of  $G$ . Two such isomorphisms are equivalent if they agree on some common finite index subgroup of their domain. Note that every automorphism of  $G$  induces an element of  $\text{Comm}(G)$ , and in particular the action of  $G$  on itself by global conjugation gives a homomorphism  $G \rightarrow \text{Comm}(G)$ .

The abstract commensurator of  $G$  captures a notion of symmetry for the group that is weaker than its group of automorphisms. For instance, the abstract commensurator of  $\mathbb{Z}^m$  is isomorphic to  $\text{GL}(m, \mathbb{Q})$  while the abstract commensurator of a nonabelian free group is not finitely generated (see [BB]). However, some groups satisfy strong rigidity properties and the group  $\text{Comm}(G)$  is then not much larger than  $\text{Aut}(G)$  or  $G$  itself. For instance, the Mostow-Prasad-Margulis rigidity theorem and Margulis arithmeticity theorem (see for instance [Zim]) imply that if  $\Gamma$  is a lattice in a connected noncompact simple Lie group  $G$  with trivial center, and if  $G \neq \text{PSL}(2, \mathbb{R})$ , then  $\Gamma$  is a finite index subgroup of  $\text{Comm}(\Gamma)$  if and only if  $\Gamma$  is not arithmetic, otherwise  $\text{Comm}(\Gamma)$  is dense in  $G$ . In the case of the extended mapping class group of a connected orientable closed surface  $S_g$  of genus  $g$  at least 3, we have an even stronger result due to Ivanov [Iva2] since the natural homomorphism  $\text{Mod}^\pm(S_g) \rightarrow \text{Comm}(\text{Mod}^\pm(S_g))$  is an isomorphism. This result also extends to the case of the mapping class group of a connected orientable surface with genus equal to 2 and with at least two boundary components. In the context of the outer automorphism group of a free group  $F_N$  of rank  $N$ , Farb and Handel ([FarH]) proved that, for  $N \geq 4$ , the natural map from  $\text{Out}(F_N)$  to  $\text{Comm}(\text{Out}(F_N))$  is an isomorphism and that every isomorphism between two finite index subgroups of  $\text{Out}(F_N)$  extends

to an inner automorphism of  $\text{Out}(F_N)$ . This result was later extended by Horbez and Wade ([HW2]) to the case  $N = 3$  using a more geometric approach. Their techniques also enabled them to compute the abstract commensurator of many interesting subgroups of  $\text{Out}(F_N)$ , like its Torelli subgroup. These rigidity results have been extended to other groups, such as handlebody groups ([Hen]) and big mapping class groups ([BDR]).

In this article, we are interested in the outer automorphism group of a universal Coxeter group. Let  $n$  be an integer greater than 1. Let  $F = \mathbb{Z}/2\mathbb{Z}$  be a cyclic group of order 2 and  $W_n = \ast_n F$  be a universal Coxeter group of rank  $n$ , that is a free product of  $n$  copies of  $F$ . We prove the following theorem.

**Theorem 4.1.1.** *Let  $n \geq 5$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Comm}(\text{Out}(W_n))$$

*is an isomorphism.*

The group  $\text{Out}(W_2)$  is finite and the group  $\text{Out}(W_3)$  is isomorphic to  $\text{PGL}(2, \mathbb{Z})$ . This gives an almost complete classification except for  $n = 4$ , where our proof for  $n \geq 5$  cannot be immediately adapted to this case as  $\text{Out}(W_4)$  does not contain any direct product of two nonabelian free groups. Hence the case  $n = 4$  remains open. One step towards the understanding of  $\text{Out}(W_4)$  is given in [Gue1], where we proved that  $\text{Out}(W_4)$  has a nontrivial outer automorphism. The conclusion of Theorem 4.1.1 will therefore not be true if one can prove that this outer automorphism remains not trivial for every finite index subgroup of  $\text{Out}(W_n)$ . Theorem 4.1.1 is a major improvement of [Gue1, Théorème 1.1] which states that, for  $n \geq 5$ , the only automorphisms of  $\text{Out}(W_n)$  are the global conjugations. In turn, Theorem 4.1.1 implies that every isomorphism between two finite index subgroups of  $\text{Out}(W_n)$  is given by a conjugation by an element of  $\text{Out}(W_n)$ . The proof of the present Theorem 4.1.1 significantly differs from the one of [Gue1, Théorème 1.1] since the proof of [Gue1, Théorème 1.1] is based on the study of torsion subgroups of  $\text{Out}(W_n)$ , whereas  $\text{Out}(W_n)$  is virtually torsion free (see [GuL1, Corollary 5.5]).

Our proof of Theorem 4.1.1 is inspired by the proof of the similar result in the context of  $\text{Out}(F_N)$  given by Horbez and Wade ([HW2]). However, new ideas are required in this situation. Indeed, to our knowledge, there is no way to compute the abstract commensurator of  $\text{Out}(W_n)$  by identifying it with a subgroup of  $\text{Out}(F_N)$ . Moreover, the study of the restriction of automorphisms of  $W_n$  to some finite index nonabelian free subgroup of  $W_n$  is not sufficient to understand the abstract commensurator of  $\text{Out}(W_n)$ , as it does not give informations about finite index subgroups of  $\text{Out}(W_n)$ . Finally, the proof of Horbez and Wade relies extensively on the possibility of writing a free group as an HNN extension, which is not possible in a universal Coxeter group. Instead, we use the fact that  $W_n$  can be written as a free product  $W_n = A * B$ , where  $B$  is a finite abelian subgroup of  $W_n$ .

We now sketch our proof of Theorem 4.1.1. Following a strategy that dates back to Ivanov's work ([Iva2]), we study the action of  $\text{Out}(W_n)$  on various graphs which are *rigid*, that is, every graph automorphism is induced by an element of  $\text{Out}(W_n)$ . These

graphs include the spine  $K_n$  of the Outer space of  $W_n$  as defined by Guirardel and Levitt in [GuL1], generalizing Culler and Vogtmann's Outer space of a free group ([CV]), or the free splitting graph  $\overline{K}_n$  of  $W_n$  (see [Gue2, Theorem 1.1 and 1.2] and Section 4.2.2 for definitions). The proof of Theorem 4.1.1 relies on the action of  $\text{Out}(W_n)$  on a subset of the vertices of  $\overline{K}_n$ , called *the set of  $W_k$ -stars*. Let  $k \in \{0, \dots, n-1\}$ . A  $W_k$ -star is a free splitting  $S$  of  $W_n$  such that the underlying graph of the induced graph of groups  $W_n \setminus S$  is a tree with  $n-k$  edges, such that the degree of one of the vertices, called the *center*, is equal to  $n-k$ , and such that the group associated with the center is isomorphic to  $W_k$  and the groups associated with the leaves are all isomorphic to  $F$ . The  $W_k$ -stars are the analogue for  $W_n$  of the roses in the Outer space of a free group. They play a significant role in the proof of other rigidity results for  $\text{Out}(W_n)$  (see [Gue1, Gue2]).

This allows us to introduce a graph called the *graph of one-edge compatible  $W_{n-2}$ -stars*, and denoted by  $X_n$ . It is defined as follows: vertices are  $W_n$ -equivariant homeomorphism classes of  $W_{n-2}$ -stars, where two vertices  $\mathcal{S}$  and  $\mathcal{S}'$  are adjacent if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  have both a common refinement and a common collapse. We prove the following result.

**Theorem 4.1.2.** *Let  $n \geq 5$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(X_n)$$

*is an isomorphism.*

Our proof of Theorem 4.1.2 requires the rigidity of another graph, called the *graph of  $W_*$ -stars*, and denoted by  $X'_n$ . It is the graph whose vertices are the  $W_n$ -equivariant homeomorphism classes of  $W_k$ -stars with  $k$  varying in  $\{0, \dots, n-2\}$ , where two vertices  $\mathcal{S}$  and  $\mathcal{S}'$  are adjacent if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  refines  $S'$  or conversely. We first show that every graph automorphism of  $X_n$  induces a graph automorphism of  $X'_n$  and that the induced map  $\text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$  is injective. Using the rigidity of  $X'_n$  (see Theorem 4.3.4), we show that any graph automorphism of  $X_n$  is induced by an element of  $\text{Out}(W_n)$ .

We then show that every commensuration  $f$  of  $\text{Out}(W_n)$  induces a graph automorphism of  $X_n$ . Once we have that result, a general argument (see Proposition 4.2.1) gives the isomorphism between  $\text{Out}(W_n)$  and  $\text{Comm}(\text{Out}(W_n))$ . In order to construct such a homomorphism  $\text{Comm}(\text{Out}(W_n)) \rightarrow \text{Aut}(X_n)$ , we first give an algebraic characterisation of the stabilizers of equivalence classes of  $W_{n-2}$ -stars. The characterization relies on the examination of maximal abelian subgroups of  $\text{Out}(W_n)$  and of direct products of nonabelian free groups in  $\text{Out}(W_n)$ . In particular, we prove (see Theorem 4.5.1), using the action of  $\text{Out}(W_n)$  on a simplicial complex called the *free factor complex of  $W_n$* , the following result.

**Theorem 4.1.3.** *Let  $n \geq 4$ . The maximal number of factors in a direct product of non-abelian free groups contained in  $\text{Out}(W_n)$  is equal to  $n-3$ .*

One example of such a maximal direct product of nonabelian free subgroups of  $\text{Out}(W_n)$  is the following one. Let  $W_n = \langle x_1, \dots, x_n \rangle$  be a standard generating set

for  $W_n$  and let  $W = \langle x_1, x_2, x_3 \rangle$ . For every  $i \geq 4$  and every  $w \in W$ , let  $F_{i,w}$  be the automorphism which fixes  $x_j$  for every  $j \neq i$  and which sends  $x_i$  to  $wx_iw^{-1}$ . Let  $[F_{i,w}]$  be the outer automorphism class of  $F_{i,w}$  and let  $H_i = \langle [F_{i,w}]_{w \in W} \rangle$ . Then the group  $\langle H_i \rangle_{i \geq 4}$  is a subgroup of  $\text{Out}(W_n)$  isomorphic to a direct product of  $n - 3$  groups isomorphic to  $W_3$ .

The complete characterisation of stabilizers of equivalence classes of  $W_{n-2}$ -stars being quite technical, we do not give the complete statement in the introduction (see Propositions 4.6.10 and 4.7.9). However, we remark that this characterisation relies on the following key points: the fact that stabilizers of equivalence classes of  $W_{n-2}$ -stars contain a maximal free abelian subgroup and the fact that it contains a direct product of  $n - 3$  nonabelian free groups. The characterisation also features a study of the group of twists of a  $W_{n-2}$ -star, which is a direct product of two virtually nonabelian free groups by a result of Levitt ([Lev1]) and such that each of which has finite index in the centralizer in  $\text{Out}(W_n)$  of the other.

This characterisation being preserved by commensurations of  $\text{Out}(W_n)$ , it induces a homomorphism from  $\text{Comm}(\text{Out}(W_n))$  to the group  $\text{Bij}(VX_n)$  of bijections of the set of vertices of  $X_n$ . In order to show that this map extends to the edge set of  $X_n$ , we also present an algebraic characterisation of compatibility of  $W_{n-2}$ -stars, which is essentially based on the fact that if the intersection of stabilizers of equivalence classes of  $W_{n-1}$ -stars contains a maximal abelian subgroup of  $\text{Out}(W_n)$ , then the  $W_{n-1}$ -stars are pairwise compatible (see Propositions 4.6.12 and 4.8.1). We deduce that the map  $\text{Comm}(\text{Out}(W_n)) \rightarrow \text{Bij}(VX_n)$  extends to a map  $\text{Comm}(\text{Out}(W_n)) \rightarrow \text{Aut}(X_n)$ , which completes our proof.

Finally, we prove in the appendix the rigidity of another natural graph endowed with an  $\text{Out}(W_n)$ -action, called the *graph of  $W_{n-1}$ -stars*. It is the graph whose vertices are  $W_n$ -equivariant homeomorphism classes of  $W_{n-1}$ -stars, where two vertices  $\mathcal{S}$  and  $\mathcal{S}'$  are adjacent if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  have a common refinement. This graph arises naturally in the study of  $\text{Out}(W_n)$  and its action on the free splitting graph  $\bar{K}_n$  as it is isomorphic to the full subgraph of  $\bar{K}_n$  whose vertices are the equivalence classes of  $W_k$ -stars, with  $k$  varying in  $\{0, \dots, n - 1\}$ . This gives another geometric rigid model for  $\text{Out}(W_n)$  (see Theorem 4.9.1).

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## 4.2 Preliminaries

### 4.2.1 Commensurations

Let  $G$  be a group. The *abstract commensurator* of  $G$ , denoted by  $\text{Comm}(G)$ , is the group whose elements are the equivalence classes of isomorphisms between finite index subgroups of  $G$  for the following equivalence relation. Two isomorphisms between finite

index subgroups  $f: H_1 \rightarrow H_2$  and  $f': H'_1 \rightarrow H'_2$  are equivalent if they agree on some common finite index subgroup  $H$  of their domains. If  $f$  is an isomorphism between finite index subgroups, we denote by  $[f]$  the equivalence class of  $f$ . The identity of  $\text{Comm}(G)$  is the equivalence class of the identity map on  $G$ . Let  $[f], [f'] \in \text{Comm}(G)$ , and let  $f: H_1 \rightarrow H_2$  and  $f': H'_1 \rightarrow H'_2$  be representatives. The composition law  $[f] \cdot [f']$  is given by  $[f] \cdot [f'] = [f \circ f'|_{f'^{-1}(H_1) \cap H'_1}]$ . Note that if  $H$  is a finite index subgroup of  $G$ , then the natural map  $\text{Comm}(G) \rightarrow \text{Comm}(H)$  obtained by restriction is an isomorphism.

Two subgroups  $G_1$  and  $G_2$  in  $G$  are commensurable if  $G_1 \cap G_2$  has finite index in both  $G_1$  and  $G_2$ . Being commensurable is an equivalence relation. If  $H$  is a subgroup of  $G$ , we will denote by  $[H]$  its commensurability class in  $G$ . The group  $\text{Comm}(G)$  acts on the set of all commensurability classes as follows. Let  $[H]$  be the commensurability class of a subgroup  $H$ . Let  $[f] \in \text{Comm}(G)$  and let  $f: H_1 \rightarrow H_2$  be a representative of  $[f]$ . Then we define  $[f] \cdot [H]$  by setting  $[f] \cdot [H] = [f(H \cap H_1)]$ .

The next result, due to Horbez and Wade, gives a sufficient condition for  $\text{Comm}(G)$  to be rigid. It comes from ideas due to Ivanov when studying mapping class groups (see [Iva2]). It requires the existence of a graph on which  $G$  acts by graph automorphisms.

**Proposition 4.2.1.** [HW2, Proposition 1.1] *Let  $G$  be a group. Let  $X$  be simplicial graph such that  $G$  acts on  $X$  by graph automorphisms. Let  $\text{Aut}(X)$  be the group of graph automorphisms of  $X$ . Assume that:*

- (1) *the natural homomorphism  $G \rightarrow \text{Aut}(X)$  is an isomorphism,*
- (2) *given two distinct vertices  $v$  and  $w$  of  $X$ , the groups  $\text{Stab}_G(v)$  and  $\text{Stab}_G(w)$  are not commensurable in  $G$ ,*
- (3) *the sets  $\mathcal{I} = \{[\text{Stab}_G(v)] \mid v \in VX\}$  and  $\mathcal{J} = \{([\text{Stab}_G(v)], [\text{Stab}_G(w)]) \mid vw \in EX\}$  are  $\text{Comm}(G)$ -invariant (in the latter case with respect to the diagonal action).*

*Then any isomorphism  $f: H_1 \rightarrow H_2$  between finite index subgroups of  $G$  is given by the conjugation by an element of  $G$  and the natural map  $G \rightarrow \text{Comm}(G)$  is an isomorphism.  $\square$*

### 4.2.2 Free splittings and free factor systems of $W_n$

Let  $n$  be an integer greater than 1. Let  $F = \mathbb{Z}/2\mathbb{Z}$  be a cyclic group of order 2 and  $W_n = *_n F$  be a universal Coxeter group of rank  $n$ . A *splitting* of  $W_n$  is a minimal, simplicial  $W_n$ -action on a simplicial tree  $S$  such that:

- (1) The finite graph  $W_n \backslash S$  is not empty and not reduced to a point.
- (2) Vertices of  $S$  with trivial stabilizer have degree at least 3.

Here *minimal* means that  $W_n$  does not preserve any proper subtree of  $S$ . A splitting  $S$  of  $W_n$  is *free* if all edge stabilizers are trivial. A splitting  $S'$  is a *blow-up*, or equivalently a *refinement*, of a splitting  $S$  if  $S$  is obtained from  $S'$  by collapsing some edge orbits in  $S'$ . Two splittings are *compatible* if they have a common refinement. We define an



equivalence class in the set of free splittings, where two splittings  $S$  and  $S'$  are equivalent if there exists a  $W_n$ -equivariant homeomorphism between them.

A *free factor system* of  $W_n$  is a set  $\mathcal{F}$  of conjugacy classes of subgroups of  $W_n$  which arises as the set of all conjugacy classes of nontrivial point stabilizers in some (nontrivial) free splitting of  $W_n$ . Equivalently, there exist  $k \in \mathbb{N} - \{0, 1\}$  and  $[A_1], \dots, [A_k]$  conjugacy classes of nontrivial, proper subgroups of  $W_n$  such that  $W_n = A_1 * \dots * A_k$  and  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$ . The free factor system is *sporadic* if  $k = 2$ , and *nonsporadic* otherwise. The set of all free factor systems of  $W_n$  has a natural partial order, where  $\mathcal{F} \leq \mathcal{F}'$  if every factor of  $\mathcal{F}$  is conjugate into one of the factors of  $\mathcal{F}'$ . Remark that if  $\{x_1, \dots, x_n\}$  is a standard generating set of  $W_n$ , then for every free factor system  $\mathcal{F}$  of  $W_n$  and every  $i \in \{1, \dots, n\}$ , there exists  $[A] \in \mathcal{F}$  such that  $x_i$  is conjugate into  $A$ . In other words, the free factor system  $\{[x_1], \dots, [x_n]\}$  is a minimum for the partial order on the set of free factor systems of  $W_n$ .

Let  $\mathcal{F}$  be a free factor system of  $W_n$ . We denote by  $\text{Out}(W_n, \mathcal{F})$  the subgroup of  $\text{Out}(W_n)$  consisting of all outer automorphisms that preserve all the conjugacy classes of subgroups in  $\mathcal{F}$ . If  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$ , we denote by  $\text{Out}(W_n, \mathcal{F}^{(t)})$  the subgroup of  $\text{Out}(W_n, \mathcal{F})$  consisting of all outer automorphisms which have a representative whose restriction to each  $A_i$  with  $i \in \{1, \dots, k\}$  is a global conjugation by some  $g_i \in W_n$ .

A  $(W_n, \mathcal{F})$ -*tree* is an  $\mathbb{R}$ -tree equipped with a  $W_n$ -action by isometries and such that every subgroup of  $W_n$  whose conjugacy class belongs to  $\mathcal{F}$  is elliptic. A *free splitting of  $W_n$  relative to  $\mathcal{F}$*  is a free splitting of  $W_n$  such that every free factor in  $\mathcal{F}$  is elliptic. A *free factor of  $(W_n, \mathcal{F})$*  is a subgroup of  $W_n$  which arises as a point stabilizer in a free splitting of  $W_n$  relative to  $\mathcal{F}$ . A free factor of  $(W_n, \mathcal{F})$  is *proper* if it is nontrivial, not equal to  $W_n$  and not conjugate to an element of  $\mathcal{F}$ . An element  $g \in W_n$  is  $\mathcal{F}$ -*peripheral* (or simply *peripheral* if there is no ambiguity) if it is conjugate into one of the subgroups of  $\mathcal{F}$ , and  $\mathcal{F}$ -*nonperipheral* otherwise. In particular, for every free factor system  $\mathcal{F}$  of  $W_n$ , and every element  $x \in W_n$  appearing in a standard generating set of  $W_n$ , we see that  $x$  is  $\mathcal{F}$ -peripheral.

### 4.2.3 The Outer space of $(W_n, \mathcal{F})$

We recall the definition of the *unprojectivised Outer space of  $(W_n, \mathcal{F})$* , denoted by  $\mathcal{O}(W_n, \mathcal{F})$  and introduced by Guirardel and Levitt in [GuL1]. It is the set of all  $(W_n, \mathcal{F})$ -equivariant isometry classes  $\mathcal{S}$  of metric simplicial trees with a nontrivial action of  $W_n$ , with trivial arc stabilizers and such that a subgroup is elliptic if and only if it is peripheral. The set  $\mathcal{O}(W_n, \mathcal{F})$  is equipped with the Gromov-Hausdorff equivariant topology introduced in [Paul]. The *projectivised Outer space of  $(W_n, \mathcal{F})$* , denoted by  $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ , is defined as the space of homothety classes of trees in  $\mathcal{O}(W_n, \mathcal{F})$ . The spaces  $\mathcal{O}(W_n, \mathcal{F})$  and  $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$  come equipped with a right action of  $\text{Out}(W_n, \mathcal{F})$  given by precomposition of the actions.

The space  $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$  has a natural structure of a simplicial complex with missing faces. Indeed, every element  $\mathcal{S} \in \mathbb{P}\mathcal{O}(W_n, \mathcal{F})$  defines an open simplex as follows. Let  $S$

be a representative of  $\mathcal{S}$  such that the sum of the edge lengths of  $W_n \setminus S$  is equal to 1. We associate an open simplex by varying the lengths of the edges, so that the sum of the edge lengths is still equal to 1. A homothety class  $\mathcal{S}' \in \mathbb{P}\mathcal{O}(W_n, \mathcal{F})$  of a splitting  $S'$  defines a codimension 1 face of the simplex associated with  $\mathcal{S}$  if we can obtain  $S'$  from some representative  $S$  of  $\mathcal{S}$  by contracting one orbit of edges in  $S$ .

The closure  $\overline{\mathcal{O}(W_n, \mathcal{F})}$  of Outer space in the space of all isometry classes of minimal nontrivial  $W_n$ -actions on  $\mathbb{R}$ -trees, equipped with the Gromov-Hausdorff equivariant topology, was identified in [Hor3] with the space of all *very small*  $(W_n, \mathcal{F})$ -trees, which are the  $(W_n, \mathcal{F})$ -trees whose arc stabilizers are either trivial, or cyclic, root-closed and nonperipheral, and whose tripod stabilizers are trivial. The space  $\overline{\mathbb{P}\mathcal{O}(W_n, \mathcal{F})}$  equipped with the quotient topology is compact (see [Hor3, Theorem 1]).

We recall the definition of a simplicial complex on which the space  $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$  retracts  $\text{Out}(W_n, \mathcal{F})$ -equivariantly, called the *spine of Outer space of*  $(W_n, \mathcal{F})$  and denoted by  $K(W_n, \mathcal{F})$ . It is the flag complex whose vertices are the  $W_n$ -equivariant homeomorphism classes  $\mathcal{S}$  of free splittings relative to  $\mathcal{F}$  with the property that, if  $S \in \mathcal{S}$ , then all elliptic subgroups in  $S$  are peripheral. Two vertices  $\mathcal{S}$  and  $\mathcal{S}'$  in  $K(W_n, \mathcal{F})$  are linked by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  refines  $S'$  or conversely. There is an embedding  $F: K(W_n, \mathcal{F}) \hookrightarrow \mathbb{P}\mathcal{O}(W_n, \mathcal{F})$  whose image is the barycentric spine of  $\mathbb{P}\mathcal{O}(W_n, \mathcal{F})$ . We will from now on identify  $K(W_n, \mathcal{F})$  with  $F(K(W_n, \mathcal{F}))$ . If  $\mathcal{F}$  consists of exactly  $n$  copies of  $F$ , we simply write  $K_n$  for  $K(W_n, \mathcal{F})$ . In this case the dimension of the simplicial complex  $K_n$  is  $n - 2$ . Indeed, if  $\mathcal{S}$  is an equivalence class of a free splitting  $S$  in  $K_n$  such that the number of edges of  $W_n \setminus S$  is minimal, then, the number of edges in  $W_n \setminus S$  is equal to  $n - 1$ . If  $\mathcal{S}$  is an equivalence class of a free splitting  $S$  in  $K_n$  such that the number of edges of  $W_n \setminus S$  is maximal, then  $W_n \setminus S$  has  $n$  leaves and every vertex of  $W_n \setminus S$  that is not a leaf has degree equal to 3. As  $S$  is a tree, this shows that the number of edges in  $W_n \setminus S$  is equal to  $2n - 3$ . Since, every splitting  $S$  of  $K_n$  collapses onto a splitting  $S'$  such that  $W_n \setminus S'$  has  $n - 1$  edges, we see that the dimension of  $K_n$  is equal to  $2n - 3 - (n - 1) = n - 2$ .

The *free splitting graph of*  $W_n$ , denoted by  $\overline{K}_n$ , is the following graph. The vertices of  $\overline{K}_n$  are the  $W_n$ -equivariant homeomorphism classes of free splittings. Two distinct equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  are joined by an edge in  $\overline{K}_n$  if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  refines  $S'$  or conversely. The free splitting graph of  $W_n$  is the 1-skeleton of the closure of  $K_n$  in the space of free splittings of  $W_n$ . The group  $\text{Aut}(W_n)$  acts on  $\overline{K}_n$  on the right by precomposition of the action. As  $\text{Inn}(W_n)$  acts trivially on  $\overline{K}_n$ , the action of  $\text{Aut}(W_n)$  induces an action of  $\text{Out}(W_n)$  on  $\overline{K}_n$ .

#### 4.2.4 The free factor graph of $(W_n, \mathcal{F})$

Let  $\mathcal{F}$  be a free factor system of  $W_n$ . We now define a Gromov hyperbolic graph on which  $\text{Out}(W_n, \mathcal{F})$  acts by isometries. The *free factor graph relative to*  $\mathcal{F}$ , denoted by  $\text{FF}(W_n, \mathcal{F})$ , is the following graph. Its vertices are the  $W_n$ -equivariant homeomorphism classes of free splittings of  $W_n$  relative to  $\mathcal{F}$ . Two equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  are joined by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  are compatible or share a common nonperipheral elliptic element. The free factor graph is always hyperbolic (see

[BF2, HaM3, GuH2]). The next proposition is due to Guirardel and Horbez. Here, if  $H$  is a subgroup of  $\text{Out}(W_n)$  and if  $\mathcal{F}$  is a free factor system of  $W_n$ , we say that  $\mathcal{F}$  is *H-periodic* if there exists a finite index subgroup  $H'$  of  $H$  such that  $H'(\mathcal{F}) = \mathcal{F}$ .

**Proposition 4.2.2.** [GuH2, Theorem 5.1] *Let  $n \geq 3$  and let  $\mathcal{F}$  be a nonsporadic free factor system of  $W_n$ . Let  $H$  be a subgroup of  $\text{Out}(W_n, \mathcal{F})$  which acts on  $\text{FF}(W_n, \mathcal{F})$  with bounded orbits. Then there exists an  $H$ -periodic free factor system  $\mathcal{F}'$  such that  $\mathcal{F} \leq \mathcal{F}'$  and  $\mathcal{F} \neq \mathcal{F}'$ .  $\square$*

The Gromov boundary of  $\text{FF}(W_n, \mathcal{F})$  has been described in terms of *relatively arational trees* (see the work of Reynolds [Rey] for the definition of an arational tree in the context of a free group, the work of Bestvina-Reynolds and Hamenstädt ([BR, Ham] for the description of the boundary in the case of a free group, and the work of Guirardel-Horbez [GuH2] in the case of a free product). A  $(W_n, \mathcal{F})$ -tree  $T$  is *arational* if no proper  $(W_n, \mathcal{F})$ -free factor acts elliptically on  $T$  and, for every proper  $(W_n, \mathcal{F})$ -free factor  $A$ , the  $A$ -minimal invariant subtree of  $T$  (that is the union of the axes of the loxodromic elements of  $A$  for the action of  $W_n$  on  $T$ , see [CM, Proposition 3.1]) is a simplicial  $A$ -tree in which every nontrivial point stabilizer can be conjugated into one of the subgroups of  $\mathcal{F}$ . We equip each arational  $(W_n, \mathcal{F})$ -tree with the *observers' topology*: this is the topology on a tree  $T$  such that a basis of open sets is given by the connected components of the complements of points in  $T$ . We equip the set of arational  $(W_n, \mathcal{F})$ -trees with an equivalence relation, where two arational  $(W_n, \mathcal{F})$ -trees are equivalent if they are  $W_n$ -equivariantly homeomorphic with the observers' topology.

**Theorem 4.2.3.** [GuH2, Theorem 3.4] *Let  $n \geq 3$ . Let  $\mathcal{F}$  be a nonsporadic free factor system of  $W_n$ . The Gromov boundary of  $\text{FF}(W_n, \mathcal{F})$  is  $\text{Out}(W_n, \mathcal{F})$ -equivariantly homeomorphic to the space of all equivalence classes of arational  $(W_n, \mathcal{F})$ -trees.  $\square$*

**Lemma 4.2.4.** [GuH1, Proposition 13.5] *Let  $n \geq 3$ . Let  $\mathcal{F}$  be a nonsporadic free factor system of  $W_n$ , and let  $H$  be a subgroup of  $\text{Out}(W_n, \mathcal{F})$ . If  $H$  fixes a point in  $\partial_\infty \text{FF}(W_n, \mathcal{F})$ , then  $H$  has a finite-index subgroup that fixes the homothety class of an arational  $(W_n, \mathcal{F})$ -tree.  $\square$*

#### 4.2.5 Groups of twists

Let  $S$  be a splitting of  $W_n$ , let  $v \in VS$ , let  $e$  be an edge with origin  $v$ , and let  $z$  be an element of the centralizer  $C_{G_v}(G_e)$  of  $G_v$  in  $G_e$ . We define the *twist by  $z$  around  $e$*  to be the automorphism  $D_{e,z}$  of  $W_n$  defined as follows (see [Lev1]). Let  $\bar{S}$  be the splitting obtained from  $S$  by collapsing all the edges of  $S$  outside of the orbit of  $e$ . Then  $\bar{S}$  is a tree. Let  $\bar{e}$  be the image of  $e$  in  $\bar{S}$  and let  $\bar{v}$  be the image of  $v$  in  $\bar{S}$ . Let  $\bar{w}$  be the endpoint of  $\bar{e}$  distinct from  $\bar{v}$ . The automorphism  $D_{e,z}$  is defined to be the unique automorphism that acts as the identity on  $G_{\bar{v}}$  and as conjugation by  $z$  on  $G_{\bar{w}}$ . The element  $z$  is called the *twistor of  $D_{e,z}$* . It is well-defined up to composing on the right by an element of  $C_{W_n}(G_{\bar{w}}) \cap C_{G_v}(G_e)$ . The *group of twists of  $S$*  is the subgroup of  $\text{Out}(W_n)$  generated by all twists around oriented edges of  $S$ .

We now give a description of the stabilizer of a point in  $\overline{K}_n$  due to Levitt. If  $\mathcal{S} \in V\overline{K}_n$ , we denote by  $\text{Stab}(\mathcal{S})$  the stabilizer of  $\mathcal{S}$  under the action of  $\text{Out}(W_n)$ . Let  $S$  be a representative of  $\mathcal{S}$ . We denote by  $\text{Stab}^0(\mathcal{S})$  the subgroup of  $\text{Stab}(\mathcal{S})$  consisting of all elements  $F \in \text{Out}(W_n)$  such that the graph automorphism induced by  $F$  on  $W_n \setminus S$  is the identity.

**Proposition 4.2.5.** [Lev1, Propositions 2.2, 3.1 and 4.2] *Let  $n \geq 4$  and  $\mathcal{S} \in V\overline{K}_n$ . Let  $S$  be a representative of  $\mathcal{S}$  and let  $v_1, \dots, v_k$  be the vertices of  $W_n \setminus S$  with nontrivial associated groups. For  $i \in \{1, \dots, k\}$ , let  $G_i$  be the group associated with  $v_i$ .*

(1) *The group  $\text{Stab}^0(\mathcal{S})$  fits in an exact sequence*

$$1 \rightarrow \mathcal{T} \rightarrow \text{Stab}^0(\mathcal{S}) \rightarrow \prod_{i=1}^k \text{Out}(G_i) \rightarrow 1,$$

where  $\mathcal{T}$  is the group of twists of  $S$ .

(2) *The group  $\text{Stab}^0(\mathcal{S})$  is isomorphic to*

$$\prod_{i=1}^k G_i^{\deg(v_i)-1} \rtimes \text{Aut}(G_i),$$

where  $\text{Aut}(G_i)$  acts on  $G_i^{\deg(v_i)-1}$  diagonally.

(3) *The group of twists  $\mathcal{T}$  of  $S$  is isomorphic to*

$$\mathcal{T} \simeq \bigoplus_{i=1}^k G_i^{\deg(v_i)} / Z(G_i),$$

where the center  $Z(G_i)$  of  $G_i$  is embedded diagonally in  $G_i^{\deg(v_i)}$ . □

**Remark 4.2.6.** In [Lev1, Proposition 2.2], Levitt shows that the kernel of the natural homomorphism  $\text{Stab}^0(\mathcal{S}) \rightarrow \prod_{i=1}^k \text{Out}(G_i)$  given by the action on the vertex groups is generated by bitwists. Since edge stabilizers are trivial, the group of bitwists is equal to the group of twists. More generally (see [Lev1, Proposition 2.3]), if the outer automorphism group of every edge stabilizer is finite (in particular, if edge stabilizers are isomorphic to  $\mathbb{Z}$  or to  $F$ ) then the group of twists is a finite index subgroup of the group of bitwists.

Finally, if the centralizer in  $W_n$  of an edge stabilizer is trivial, then the group of bitwists about this edge is trivial. Therefore, if the edge stabilizer is not cyclic, then the group of bitwists about this edge is trivial. In all cases, we see that, for every equivalence class  $\mathcal{S}$  of a splitting  $S$  of  $W_n$ , the group of twists of  $\mathcal{S}$  is a finite index subgroup of the group of bitwists of  $W_n$ .

We establish one last fact about twists about edges whose centralizer is cyclic (see [CL2, Lemma 5.3] for a similar statement in the context of the outer automorphism group of a nonabelian free group).

**Lemma 4.2.7.** *Let  $n \geq 3$  and let  $\mathcal{S}$  be the equivalence class of a splitting  $S$ . Suppose that there exists an edge  $e$  of  $S$  with cyclic stabilizer and let  $D$  be the outer automorphism class of a twist about  $e$ . Let  $H_{\mathcal{S}}$  be the subgroup of  $\text{Stab}^0(\mathcal{S})$  which induces the identity on the edge stabilizer  $G_e$  of  $e$ . Then  $D$  is central in  $H_{\mathcal{S}}$*

*In particular,  $\text{Stab}^0(\mathcal{S})$  has a finite index subgroup  $H_{\mathcal{S}}$  such that  $D$  is central in  $H_{\mathcal{S}}$ .*

**Proof.** Let  $U$  be a splitting onto which  $S$  collapses (or  $S$  itself if  $S$  does not have a nontrivial collapse), and let  $\mathcal{U}$  be its equivalence class. Then  $\text{Stab}^0(\mathcal{S}) \subseteq \text{Stab}^0(\mathcal{U})$ . Thus, we may suppose, up to collapsing all orbits of edges of  $S$  except the one containing  $e$ , that  $S$  has exactly one orbit of edges. Let  $v$  and  $w$  be the two endpoints of  $e$  and let  $G_v$  and  $G_w$  be their edge stabilizers. Let  $f \in H_{\mathcal{S}}$  and let  $F$  be a representative of  $f$  such that  $F(G_v) = G_v$ ,  $F(G_w) = G_w$  and  $F|_{G_e} = \text{id}_{G_e}$  (this representative exists since  $f \in H_{\mathcal{S}}$ ). Let  $z \in C_{G_v}(G_e)$  be such that  $D_{e,z}$  is a representative of  $D$ . Then, since  $F(z) = z$ , for every  $x \in W_n$ , we have  $D_{e,z} \circ F \circ D_{e,z}^{-1}(x) = F(x)$ . Hence  $f$  and  $D$  commutes and  $D$  is central in  $H_{\mathcal{S}}$ . Since the outer automorphism group of a cyclic group is finite, we see that  $H_{\mathcal{S}}$  is a finite index subgroup of  $\text{Stab}^0(\mathcal{S})$ . This concludes the proof.  $\square$

### 4.3 Geometric rigidity in the graph of $W_k$ -stars

We start by defining  $W_k$ -stars, which are the main splittings of interest in this article.

**Definition 4.3.1.** Let  $n \geq 3$ , and let  $k \geq 1$  be an integer.

- (1) A free splitting  $S$  is a  $k$ -edge free splitting if  $W_n \setminus S$  has exactly  $k$  edges.
- (2) Suppose that  $0 \leq k \leq n - 2$ . A  $W_k$ -star is an  $(n - k)$ -edge free splitting such that:
  - the underlying graph of  $W_n \setminus S$  has  $n - k + 1$  vertices and one of them, called the *center of  $W_n \setminus S$* , has degree exactly  $n - k$ ,
  - the group associated with the center of  $W_n \setminus S$  is isomorphic to  $W_k$  (we use the convention that  $W_0 = \{1\}$  and that  $W_1 = F$ ),
  - the group associated with any leaf of  $W_n \setminus S$  is isomorphic to  $F$ .
- (3) A  $W_{n-1}$ -star is a one-edge free splitting  $S$  such that one of the vertex groups of  $W_n \setminus S$  is isomorphic to  $W_{n-1}$  while the other vertex group is isomorphic to  $F$ .

Note that, in [Gue2], a  $W_{n-1}$ -star is called an  $F$ -one-edge free splitting. Using Proposition 4.2.5 (2), we see that, if  $k \in \{0, \dots, n - 2\}$ , and if  $\mathcal{S}$  is the equivalence class of a  $W_k$ -star, then the group  $\text{Stab}^0(\mathcal{S})$  is isomorphic to  $W_k^{n-k-1} \rtimes \text{Aut}(W_k)$ .

Note that, if  $S$  is a  $W_k$ -star with  $k \in \{0, \dots, n - 2\}$  and  $S'$  is a splitting on which  $S$  collapses, then there exists  $\ell \in \{k, \dots, n - 1\}$  such that  $S'$  is a  $W_{\ell}$ -star. In particular, for every  $k \in \{0, \dots, n - 2\}$ , if  $S$  is a  $W_k$ -star, then every one-edge free splitting on which  $S$  collapses is a  $W_{n-1}$ -star. A similar statement is also true for refinements of  $W_k$ -stars (see Lemma 4.3.8).

### 4.3.1 Rigidity of the graph of $W_*$ -stars

We introduce in this section a graph, *the graph of one-edge compatible  $W_{n-2}$ -stars*, on which  $\text{Out}(W_n)$  acts by simplicial automorphisms. We prove that this graph is a rigid geometric model for  $\text{Out}(W_n)$ . The proof relies on the study of the rigidity of an additional graph on which  $\text{Out}(W_n)$  acts, the *graph of  $W_*$ -stars*, to be defined after Theorem 4.3.3.

**Definition 4.3.2.** (1) The *graph of  $W_{n-2}$ -stars*, denoted by  $\tilde{X}_n$ , is the graph whose vertices are the  $W_n$ -equivariant homeomorphism classes of  $W_{n-2}$ -stars, where two equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  are joined by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  are compatible.

(2) The *graph of one-edge compatible  $W_{n-2}$ -stars*, denoted by  $X_n$ , is the graph whose vertices are the  $W_n$ -equivariant homeomorphism classes of  $W_{n-2}$ -stars where two equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  are joined by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  have a common refinement which is a  $W_{n-3}$ -star.

Note that the adjacency in the graph  $X_n$  is equivalent to having both a common collapse (which is a  $W_{n-1}$ -star) and a common refinement. The graph  $X_n$  is a subgraph of  $\tilde{X}_n$ . The group  $\text{Aut}(W_n)$  acts on  $\tilde{X}_n$  and  $X_n$  by precomposition of the action. As  $\text{Inn}(W_n)$  acts trivially on  $X_n$ , the action of  $\text{Aut}(W_n)$  induces an action of  $\text{Out}(W_n)$ . We denote by  $\text{Aut}(X_n)$  the group of graph automorphisms of  $X_n$ . In Section 4.3.2, we prove the following theorem.

**Theorem 4.3.3.** *Let  $n \geq 5$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(X_n)$$

*is an isomorphism.*

In order to prove this theorem, we take advantage of the action of  $\text{Out}(W_n)$  on another graph, namely the *graph of  $W_*$ -stars*, denoted by  $X'_n$ . The vertices of this graph are the  $W_n$ -equivariant homeomorphism classes of  $W_k$ -stars, with  $k$  varying in  $\{0, \dots, n-2\}$ . Two equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  are joined by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  refines  $S'$  or conversely. Note that we have a natural embedding  $X'_n \hookrightarrow \bar{K}_n$ . We identify from now on  $X'_n$  with its image in  $\bar{K}_n$ . In this section, we prove the following theorem.

**Theorem 4.3.4.** *Let  $n \geq 5$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(X'_n)$$

*is an isomorphism.*

Theorem 4.3.4 relies on the fact that  $X'_n$  contains a rigid subgraph, namely the graph of  $\{0\}$ -stars and  $F$ -stars, and denoted by  $L_n$ . The vertices of this graph are the  $W_n$ -equivariant homeomorphism classes of  $\{0\}$ -stars and  $F$ -stars. Two equivalence classes  $\mathcal{S}$

and  $\mathcal{S}'$  are joined by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  refines  $S'$  or conversely.

We recall the following theorem.

**Theorem 4.3.5.** [Gue2, Theorem 3.1, Corollary 3.2] *Let  $n \geq 4$ . Let  $f$  be an automorphism of  $L_n$  preserving the set of  $\{0\}$ -stars and the set of  $F$ -stars. Then  $f$  is induced by the action of a unique element  $\gamma$  of  $\text{Out}(W_n)$ . In particular, for every  $n \geq 5$ , the natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$$

*is an isomorphism.* □

The strategy in order to prove Theorem 4.3.4 is to show that every automorphism of  $X'_n$  preserves  $L_n$  and that the natural map  $\text{Aut}(X'_n) \rightarrow \text{Aut}(L_n)$  is injective.

**Remark 4.3.6.** *Using the same techniques, we may prove that the graph of  $W_{n-1}$ -stars is rigid. This is done in the appendix (see Theorem 4.9.1).*

First we recall a theorem due to Scott and Swarup.

**Theorem 4.3.7.** [SS, Theorem 2.5] *Let  $n \geq 4$ . Any set  $\{S_1, \dots, S_k\}$  of pairwise nonequivalent, pairwise compatible, one-edge free splittings of  $W_n$  has a unique refinement  $S$  such that  $W_n \setminus S$  has exactly  $k$  edges. Moreover, the equivalence class of  $S$  only depends on the equivalence classes of  $S_1, \dots, S_k$ . If  $S$  is a free splitting such that  $W_n \setminus S$  has exactly  $k$  edges, then  $S$  refines exactly  $k$  pairwise nonequivalent one-edge free splittings.* □

We also need the following lemma concerning refinements of  $W_k$ -stars.

**Lemma 4.3.8.** *Let  $k, \ell \in \{0, \dots, n-1\}$  and let  $S$  and  $S'$  be respectively a  $W_k$ -star and a  $W_\ell$ -star. If  $S$  and  $S'$  have a common refinement, then there exists  $j \in \{0, \dots, n-2\}$  and a  $W_j$ -star  $S''$  which refines both  $S$  and  $S'$ . Moreover,  $S''$  can be chosen such that  $S''$  is a refinement of  $S$  and  $S'$  with the minimal number of orbits of edges.*

**Proof.** Let  $S_1, \dots, S_{n-k}$  be  $n-k$   $W_{n-1}$ -stars onto which  $S$  collapses and let  $S'_1, \dots, S'_{n-\ell}$  be  $n-\ell$   $W_{n-1}$ -stars onto which  $S'$  collapses. Then the set  $\{S_1, \dots, S_{n-k}, S'_1, \dots, S'_{n-\ell}\}$  is a set of pairwise compatible  $W_{n-1}$ -stars. For every  $s \in \{1, \dots, n-k\}$  and every  $t \in \{1, \dots, n-\ell\}$ , let  $\mathcal{S}_s$  be the equivalence class of  $S_s$  and  $\mathcal{S}'_t$  be the equivalence class of  $S'_t$ . Let  $n-j = |\{\mathcal{S}_1, \dots, \mathcal{S}_{n-k}, \mathcal{S}'_1, \dots, \mathcal{S}'_{n-\ell}\}|$ . By Theorem 4.3.7, there exists a free splitting  $S''$  with  $n-j$  edges which refines every  $W_{n-1}$ -star of the set  $\{S_1, \dots, S_{n-k}, S'_1, \dots, S'_{n-\ell}\}$ . But, as  $F$  is freely indecomposable, a common refinement of two  $W_{n-1}$ -stars  $U$  and  $U'$  is obtained from  $U$  by blowing-up an edge at the vertex of  $W_n \setminus U$  whose associated group is isomorphic to  $W_{n-1}$ . Since  $U'$  is also a  $W_{n-1}$ -star, this common refinement has two orbits of edges and the two corresponding leaves have a stabilizer isomorphic to  $F$ , hence it is a  $W_{n-2}$ -star. The same argument shows that, if  $U_0$  is a  $W_{n-1}$ -star and if  $U_1$  is a  $W_k$ -star with  $k \in \{1, \dots, n-1\}$  compatible with  $U_0$ , then a common refinement of  $U_0$  and  $U_1$  with a minimal number of orbits of edges is either a  $W_k$ -star (if the equivalence classes of  $U_0$  and  $U_1$  are adjacent in  $\overline{K}_n$ ) or a  $W_{k-1}$ -star. Therefore, by induction on

$i \in \{1, \dots, n - \ell\}$ , we see that a common refinement of  $\{S_1, \dots, S_{n-k}, S'_1, \dots, S'_{n-\ell}\}$  with the minimal number of orbits of edges is a  $W_j$ -star. This shows that  $S''$  is a  $W_j$ -star. This concludes the proof.  $\square$

Lemma 4.3.8 implies that the set of  $W_k$ -stars with  $k$  varying in  $\{0, \dots, n - 1\}$  is closed under taking collapse and taking refinement with a minimal number of orbits of edges.

**Lemma 4.3.9.** *Let  $n \geq 5$ . For every  $f \in \text{Aut}(X'_n)$ , we have  $f(L_n) = L_n$ . Moreover, if  $f|_{L_n} = \text{id}_{L_n}$ , then  $f = \text{id}_{X'_n}$ .*

**Proof.** Let  $f \in \text{Aut}(X'_n)$ . The fact that  $f(L_n) = L_n$  follows from the fact that vertices of  $K_n \cap X'_n$  in  $X'_n$  are characterized by the fact that they are the vertices with finite valence. The proof is identical to the proof of [Gue2, Proposition 5.1].

Now suppose that  $f|_{L_n} = \text{id}_{L_n}$  and let  $\mathcal{S}$  be the equivalence class of a  $W_{n-2}$ -star  $S$ . Let us prove that  $f(\mathcal{S}) = \mathcal{S}$ . Let  $\{x_1, \dots, x_n\}$  be a standard generating set of  $W_n$  such that the free factor decomposition of  $W_n$  induced by  $S$  is

$$W_n = \langle x_1 \rangle * \langle x_2, \dots, x_{n-1} \rangle * \langle x_n \rangle.$$

Let  $\mathcal{X}$  be the equivalence class of the  $F$ -star  $X$  depicted in Figure 4.1.

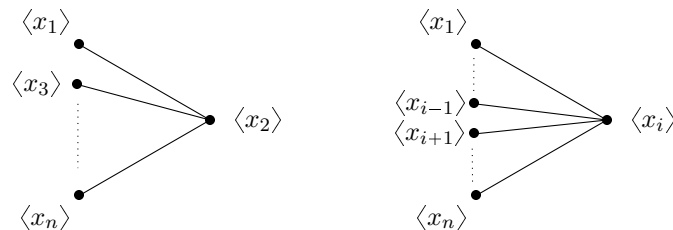


Figure 4.1: The  $F$ -stars  $X$  (on the left) and  $X'$  (on the right) of the proof of Lemma 4.3.9.

We see that  $\mathcal{S}$  and  $\mathcal{X}$  are adjacent in  $X'_n$ . Therefore, as  $f(\mathcal{X}) = \mathcal{X}$ , we see that  $f(\mathcal{S})$  and  $\mathcal{X}$  are adjacent in  $X'_n$ .

Let  $\mathcal{S}'$  be the equivalence class of a  $W_{n-2}$ -star adjacent to  $\mathcal{X}$  and distinct from  $\mathcal{S}$ . Then, as  $\mathcal{X}$  and  $\mathcal{S}'$  are adjacent, there exist distinct  $i, j \in \{1, \dots, n\}$  with  $i, j \neq 2$  and a representative  $S'$  of  $\mathcal{S}'$  such that the free factor decomposition of  $W_n$  induced by  $S'$  is

$$W_n = \langle x_i \rangle * \langle x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n \rangle * \langle x_j \rangle.$$

Since  $\mathcal{S} \neq \mathcal{S}'$ , we may suppose that  $i \notin \{1, n\}$ . But then  $\mathcal{S}$  is adjacent to the equivalence class  $\mathcal{X}'$  of the  $F$ -star  $X'$  depicted in Figure 4.1 whereas  $\mathcal{S}'$  is not adjacent to  $\mathcal{X}'$ . Since  $f(\mathcal{X}') = \mathcal{X}'$ , this shows that  $f(\mathcal{S}) \neq \mathcal{S}'$ .

Finally, let  $k \in \{2, \dots, n - 3\}$  and let  $\mathcal{S}^{(2)}$  be the equivalence class of a  $W_k$ -star  $S^{(2)}$  which is adjacent to  $\mathcal{X}$ . We prove that  $f(\mathcal{S}) \neq \mathcal{S}^{(2)}$ . Since  $k \leq n - 3$ , the underlying graph of  $W_n \setminus S^{(2)}$  has at least 3 edges. Therefore, there exists  $i \notin \{1, n\}$  and a leaf  $v$  of the underlying graph of  $W_n \setminus S^{(2)}$  such that the preimage by the marking of  $W_n \setminus S^{(2)}$  of the generator of the group associated with  $v$  is  $x_i$ . But then the equivalence class  $\mathcal{S}^{(2)}$



is not adjacent to the equivalence class  $\mathcal{X}'$  of the  $F$ -star  $X'$  depicted in Figure 4.1. As  $\mathcal{S}$  is adjacent to  $\mathcal{X}'$  and as  $f(\mathcal{X}') = \mathcal{X}'$ , we see that  $f(\mathcal{S}) \neq \mathcal{S}^{(2)}$ . Therefore,  $f(\mathcal{S}) = \mathcal{S}$ .

The above paragraphs show that  $f$  fixes pointwise the set of equivalence classes of  $W_{n-2}$ -stars. Let  $k \in \{2, \dots, n-3\}$  and let  $\mathcal{T}$  be the equivalence class of a  $W_k$ -star  $T$ . By Theorem 4.3.7, the equivalence class  $\mathcal{T}$  is uniquely determined by the set of  $W_{n-1}$ -stars on which  $T$  collapses. Since two distinct equivalence classes of  $W_{n-2}$ -stars are adjacent in  $\overline{K}_n$  to distinct pairs of equivalence classes of  $W_{n-1}$ -stars, the equivalence class  $\mathcal{T}$  is uniquely determined by the set of  $W_{n-2}$ -stars on which it collapses. Since  $f$  fixes pointwise the set of equivalence classes of  $W_{n-2}$ -stars, we see that  $f(\mathcal{T}) = \mathcal{T}$ . Hence  $f = \text{id}_{X'_n}$ . This concludes the proof.  $\square$

**Proof of Theorem 4.3.4.** Let  $n \geq 5$ . We first prove the injectivity. The homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$  is injective by Theorem 4.3.5. Moreover, the homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$  factors through  $\text{Out}(W_n) \rightarrow \text{Aut}(X'_n) \rightarrow \text{Aut}(L_n)$ . We therefore deduce the injectivity of  $\text{Out}(W_n) \rightarrow \text{Aut}(X'_n)$ . We now prove the surjectivity. Let  $f \in \text{Aut}(X'_n)$ . By Lemma 4.3.9, we have a homomorphism  $\Phi: \text{Aut}(X'_n) \rightarrow \text{Aut}(L_n)$  defined by restriction. By Theorem 4.3.5, the automorphism  $\Phi(f)$  is induced by an element  $\gamma \in \text{Out}(W_n)$ . Since the homomorphism  $\text{Aut}(X'_n) \rightarrow \text{Aut}(L_n)$  is injective by Lemma 4.3.9,  $f$  is induced by  $\gamma$ . This concludes the proof.  $\square$

### 4.3.2 Rigidity of the graph of one-edge compatible $W_{n-2}$ -stars

In this section, we prove Theorem 4.3.3. In order to do so, we construct an injective homomorphism  $\text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ . First, we need to show some technical results concerning the graph  $X_n$ . Indeed, let  $\Delta$  be a triangle (that is, a cycle of length 3) in  $X_n$ , and let  $\mathcal{S}_1, \mathcal{S}_2$  and  $\mathcal{S}_3$  be the vertices of this triangle. By Theorem 4.3.7, for every  $i \in \{1, 2, 3\}$ , there exists  $S_i \in \mathcal{S}_i$  such that  $S_1, S_2$  and  $S_3$  have a common refinement  $S$ , and we suppose that  $S$  has the minimal number of orbits of edges among the common refinements of  $S_1, S_2$  and  $S_3$ . Since  $S_1, S_2$  and  $S_3$  are  $W_{n-2}$ -stars, there exists  $k \in \{0, \dots, n-3\}$  such that  $S$  is a  $W_k$ -star. By definition of the adjacency in  $X_n$ , the splitting  $S$  is either a  $W_{n-3}$ -star or a  $W_{n-4}$ -star (see Figure 4.2). Our first result shows that we can distinguish these two types of triangles.

**Lemma 4.3.10.** *Let  $n \geq 5$ . Let  $\mathcal{S}_1, \mathcal{S}_2$  and  $\mathcal{S}_3$  be three equivalence classes of  $W_{n-2}$ -stars which are pairwise adjacent in  $X_n$ . Let  $S_1, S_2$  and  $S_3$  be representatives of  $\mathcal{S}_1, \mathcal{S}_2$  and  $\mathcal{S}_3$  which have a common refinement  $S$ . Suppose that  $S$  is the refinement of  $S_1, S_2$  and  $S_3$  which has the minimal number of orbit of edges. Then  $S$  is a  $W_{n-4}$ -star if and only if there exists an equivalence class  $\mathcal{S}_4$  of a  $W_{n-2}$ -star  $S_4$  distinct from  $\mathcal{S}_1, \mathcal{S}_2$  and  $\mathcal{S}_3$  such that, for every  $i \in \{1, 2, 3\}$ , the equivalence classes  $\mathcal{S}_i$  and  $\mathcal{S}_4$  are adjacent in  $X_n$ .*

**Proof.** Suppose first that  $S$  is a  $W_{n-4}$ -star. Let  $\{x_1, \dots, x_n\}$  be a standard generating set of  $W_n$  such that the free factor decomposition of  $W_n$  induced by  $S$  is

$$W_n = \langle x_1 \rangle * \langle x_2 \rangle * \langle x_3 \rangle * \langle x_4 \rangle * \langle x_5, \dots, x_n \rangle.$$

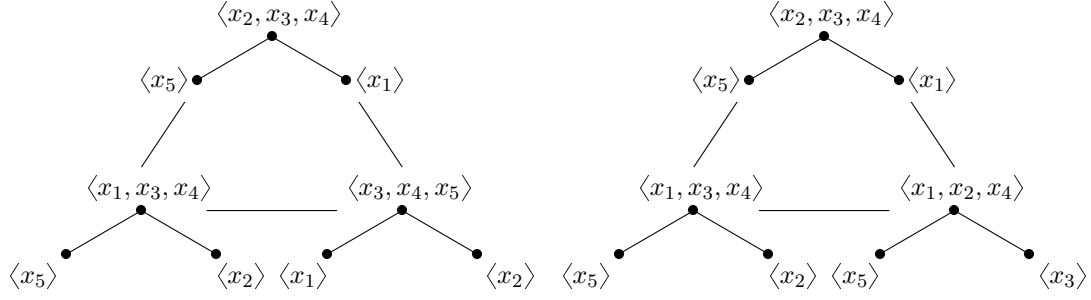


Figure 4.2: Two triangles in  $X_n$ , one corresponding to a  $W_{n-3}$ -star (on the left) and one corresponding to a  $W_{n-4}$ -star (on the right).

Since being adjacent in  $X_n$  is equivalent to having a common refinement which is a  $W_{n-3}$ -star and having a common collapse which is a  $W_{n-1}$ -star, the  $W_{n-2}$ -stars  $S_1$  and  $S_2$  share a common collapse  $S'$  which is a  $W_{n-1}$ -star. Let  $\mathcal{S}'$  be the equivalence class of  $S'$ . We claim that there exists an orbit of edges  $E$  in  $S_3$  such that the splitting obtained from  $S_3$  by collapsing every orbit of edges of  $S_3$  except  $E$  is in  $\mathcal{S}'$ . Indeed, suppose towards a contradiction that this is not the case. Then, as for every  $i \in \{1, 2\}$ , the equivalence classes  $\mathcal{S}_i$  and  $S_3$  are adjacent in  $X_n$ , we see that, for every  $i \in \{1, 2\}$ , the splittings  $S_i$  and  $S_3$  share a common collapse onto a  $W_{n-1}$ -star  $S'_i$ . Recall that we supposed that there does not exist an orbit of edges  $E$  in  $S_3$  such that the splitting obtained from  $S_3$  by collapsing every orbit of edges of  $S_3$  except  $E$  is in  $\mathcal{S}'$ . This implies that for every  $i \in \{1, 2\}$ , the equivalence class  $\mathcal{S}'_i$  of  $S'_i$  is distinct from  $\mathcal{S}'$ . Since  $S_1$  and  $S_2$  are  $W_{n-2}$ -stars, they collapse onto exactly 2 distinct  $W_{n-1}$ -stars. Therefore, for every  $i \in \{1, 2\}$ , the equivalence classes  $\mathcal{S}'$  and  $\mathcal{S}'_i$  are the two equivalence classes of  $W_{n-1}$ -stars onto which  $S_i$  collapses. It follows that a common refinement of  $S'_1$ ,  $S'_2$  and  $\mathcal{S}'$  is also a common refinement of  $S_1$ ,  $S_2$  and  $S_3$ . But a common refinement of  $S'_1$ ,  $S'_2$  and  $\mathcal{S}'$  is a  $W_{n-3}$ -star. This contradicts the fact that  $S$  has the minimal number of edges among common refinements of  $S_1$ ,  $S_2$  and  $S_3$ . Thus  $S_3$  collapses onto a  $W_{n-1}$ -star in the equivalence class  $\mathcal{S}'$ . Let  $j \in \{1, \dots, 4\}$  be such that the free factor decomposition of  $W_n$  induced by  $\mathcal{S}'$  is:

$$W_n = \langle x_j \rangle * \langle x_1, \dots, \widehat{x}_j, \dots, x_n \rangle.$$

Let  $\mathcal{S}_4$  be the equivalence class of the  $W_{n-2}$ -star  $S_4$  whose induced free factor decomposition is:

$$W_n = \langle x_j \rangle * \langle x_1, \dots, \widehat{x}_5, \dots, \widehat{x}_j, \dots, x_n \rangle * \langle x_5 \rangle.$$

Then, for every  $i \in \{1, 2, 3\}$ , the equivalence classes  $\mathcal{S}_4$  and  $\mathcal{S}_i$  are adjacent in  $X_n$ .

Conversely, suppose that  $S$  is a  $W_{n-3}$ -star. Let  $\{x_1, \dots, x_n\}$  be a standard generating set of  $W_n$  such that the free factor decomposition of  $W_n$  induced by  $S$  is

$$W_n = \langle x_1 \rangle * \langle x_2 \rangle * \langle x_3 \rangle * \langle x_4, \dots, x_n \rangle.$$

Then, up to reordering, we may suppose that, for every  $i \in \{1, 2, 3\}$  the free factor

decomposition of  $W_n$  induced by  $S_i$  is:

$$W_n = \langle x_i \rangle * \langle x_{i+1} \rangle * \langle x_1, \dots, \widehat{x_i}, \widehat{x_{i+1}}, \dots, x_n \rangle,$$

where, for  $i = 3$ , the index  $i + 1$  is taken modulo 3. Let  $\mathcal{S}'$  be the equivalence class of a  $W_{n-2}$ -star  $S'$  adjacent to  $\mathcal{S}_1$  in  $X_n$  and distinct from  $\mathcal{S}_2$  and  $\mathcal{S}_3$ . Then, up to changing the representative  $S'$ , there exists  $j \in \{1, 2\}$  such that  $S'$  collapses onto the  $W_{n-1}$ -star whose associated free factor decomposition is:

$$W_n = \langle x_j \rangle * \langle x_1, \dots, \widehat{x_j}, \dots, x_n \rangle.$$

If  $j = 1$ , then, as  $\mathcal{S}'$  is distinct from  $\mathcal{S}_1$  and  $\mathcal{S}_3$ , we see that  $\mathcal{S}'$  is not adjacent to  $\mathcal{S}_2$  in  $X_n$ . If  $j = 2$ , then, as  $\mathcal{S}'$  is distinct from  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , we see that  $\mathcal{S}'$  is not adjacent to  $\mathcal{S}_3$  in  $X_n$ . In both cases, we see that there exists  $i \in \{2, 3\}$  such that  $\mathcal{S}'$  is not adjacent to  $\mathcal{S}_i$ . This concludes the proof.  $\square$

**Corollary 4.3.11.** *Let  $n \geq 5$ . Let  $k \geq 4$  and let  $\mathcal{S}_1, \dots, \mathcal{S}_k$  be  $k$  equivalence classes of  $W_{n-2}$ -stars which are pairwise adjacent in  $X_n$ . For  $i \in \{1, \dots, k\}$ , let  $S_i$  be a representative of  $\mathcal{S}_i$ . Let  $S$  be a refinement of  $S_1, \dots, S_k$  whose number of orbits of edges is minimal. Then  $S$  is a  $W_{n-k-1}$ -star.*

**Proof.** For every distinct  $i, j \in \{1, \dots, k\}$ , the equivalence classes  $\mathcal{S}_i$  and  $\mathcal{S}_j$  are adjacent in  $X_n$ . Hence, for every distinct  $i, j \in \{1, \dots, k\}$ , there exists a common refinement of  $S_i$  and  $S_j$  which is a  $W_{n-3}$ -star. This implies that, for every  $p \in \{1, \dots, k\}$  and for every  $i_1, \dots, i_p \in \{1, \dots, k\}$ , a common refinement of  $S_{i_1}, \dots, S_{i_p}$  is obtained from a common refinement of  $S_{i_1}, \dots, S_{i_{p-1}}$  whose number of orbits of edges is minimal by adding at most one orbit of edges. We claim that a common refinement of  $S_{i_1}, \dots, S_{i_p}$  whose number of orbits of edges is minimal has exactly  $p + 1$  orbits of edges. Indeed, otherwise there would exist  $i, j, \ell \in \{1, \dots, k\}$  pairwise distinct such that a  $W_{n-3}$ -star which refines both  $S_i$  and  $S_j$  also refines  $S_\ell$ . This is not possible by Lemma 4.3.10 since  $k \geq 4$ . This proves the claim. Taking  $p = k$  concludes the proof of the lemma.  $\square$

**Proposition 4.3.12.** *Let  $n \geq 5$ . There exists a  $\text{Out}(W_n)$ -equivariant injective homomorphism  $\tilde{\Phi}: \text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ .*

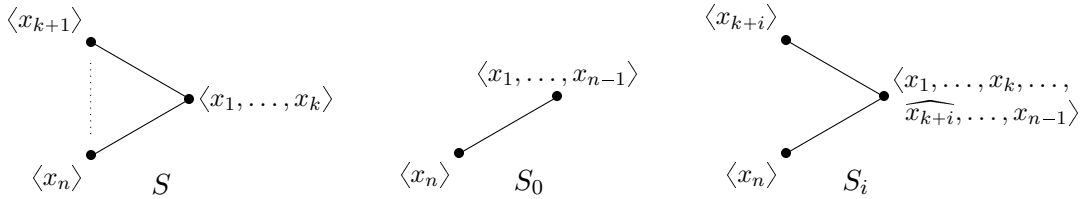


Figure 4.3: The construction of the map  $\text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ .

**Proof.** We first exhibit a map  $\Phi: \text{Aut}(X_n) \rightarrow \text{Bij}(VX'_n)$ . Let  $f \in \text{Aut}(X_n)$ . Let  $k \in \{0, \dots, n - 2\}$  and let  $\mathcal{S}$  be the equivalence class of a  $W_k$ -star  $S$ . If  $k = n - 2$ ,

then we set  $\Phi(f)(\mathcal{S}) = f(\mathcal{S})$ . If  $k \leq n - 3$ , let  $S_0$  be a  $W_{n-1}$ -star refined by  $S$ . Let  $S_1, \dots, S_{n-k-1}$  be the  $W_{n-2}$ -stars such that, for every  $i \in \{1, \dots, n - k - 1\}$ ,  $S$  refines  $S_i$  and  $S_i$  refines  $S_0$  (see Figure 4.3). For every  $i \in \{1, \dots, n - k - 1\}$ , let  $\mathcal{S}_i$  be the equivalence class of  $S_i$ , and let  $T_i$  be a representative of  $f(\mathcal{S}_i)$ . By Corollary 4.3.11, if  $n - k - 1 \geq 4$ , the  $W_{n-2}$ -stars  $T_1, \dots, T_{n-k-1}$  are refined by a  $W_k$ -star  $T'$ . This  $W_k$ -star is unique up to  $W_n$ -equivariant homeomorphism by Theorem 4.3.7. In the case where  $k = n - 3$ , we have  $n - k - 1 = 2$  and, since  $f(\mathcal{S}_1)$  and  $f(\mathcal{S}_2)$  are adjacent in  $X_n$ , the splittings  $T_1$  and  $T_2$  are refined by a  $W_{n-3}$ -star  $T'$  and it is unique up to  $W_n$ -equivariant homeomorphism by Theorem 4.3.7. Finally, when  $k = n - 4$ , Lemma 4.3.10 implies that a common refinement of  $T_1, T_2$  and  $T_3$  with the minimal number of orbits of edges is a  $W_{n-4}$ -star  $T'$ , and it is unique up to  $W_n$ -equivariant homeomorphism by Theorem 4.3.7. In all cases, let  $\mathcal{T}'$  be the equivalence class of  $T'$ . We set  $\Phi(f)(\mathcal{S}) = \mathcal{T}'$ .

We now prove that  $\Phi$  is well-defined. Let  $k \in \{0, \dots, n - 2\}$  and let  $\mathcal{S}$  be the equivalence class of a  $W_k$ -star  $S$ . Let  $S_0$  and  $S'_0$  be two distinct  $W_{n-1}$ -stars onto which  $S$  collapses and let  $\mathcal{S}_0$  and  $\mathcal{S}'_0$  be their equivalence classes. Let  $S_1, \dots, S_{n-k-1}$  be the  $W_{n-2}$ -stars such that, for every  $i \in \{1, \dots, n - k - 1\}$ ,  $S$  refines  $S_i$  and  $S_i$  refines  $S_0$  and let  $S'_1, \dots, S'_{n-k-1}$  be the  $W_{n-2}$ -stars such that, for every  $i \in \{1, \dots, n - k - 1\}$ ,  $S$  refines  $S'_i$  and  $S'_i$  refines  $S'_0$ . For  $i \in \{1, \dots, n - k - 1\}$ , let  $\mathcal{S}_i$  be the equivalence class of  $S_i$  and let  $\mathcal{S}'_i$  be the equivalence class of  $S'_i$ . For every  $i \in \{1, \dots, n - k - 1\}$ , let  $T_i$  be a representative of  $f(\mathcal{S}_i)$  and let  $T'_i$  be a representative of  $f(\mathcal{S}'_i)$ . Let  $T$  be a  $W_k$ -star which refines  $T_1, \dots, T_{n-k-1}$  and let  $T'$  be a  $W_k$ -star which refines  $T'_1, \dots, T'_{n-k-1}$ . Finally, let  $\mathcal{T}$  be the equivalence class of  $T$  and let  $\mathcal{T}'$  be the equivalence class of  $T'$ . We claim that  $\mathcal{T} = \mathcal{T}'$ . Indeed, we first remark that there exist  $i, j \in \{1, \dots, n - k - 1\}$  such that  $\mathcal{S}_i = \mathcal{S}'_j$ : it is the equivalence class of the  $W_{n-2}$ -star which refines both  $S_0$  and  $S'_0$ . Up to reordering, we may suppose that  $i = j = 1$ , that  $S_1 = S'_1$  and that  $T_1 = T'_1$ . Therefore, both  $T$  and  $T'$  collapse onto  $T_1$ .

Let  $U_2, \dots, U_{n-k-1}$  be the  $W_{n-3}$ -stars such that, for every  $j \in \{2, \dots, n - k - 1\}$ , the  $W_{n-3}$ -star  $U_j$  refines  $S_1$  and  $U_j$  is refined by  $S$ . For every  $j \in \{2, \dots, n - k - 1\}$  there exist  $\ell, \ell' \in \{2, \dots, n - k - 1\}$  such that  $S_\ell$  and  $S'_{\ell'}$  are refined by  $U_j$ . Therefore, the map  $g: \{2, \dots, n - k - 1\} \rightarrow \{2, \dots, n - k - 1\}$  sending  $\ell$  to  $\ell'$  is a bijection. Thus, we may suppose that  $g$  is the identity, that is, we may suppose that  $j = \ell = \ell'$ . It follows that for every  $j \in \{2, \dots, n - k - 1\}$ , the equivalence class of the  $W_{n-3}$ -star which refines  $S_1$  and  $S_j$  is the same one as the equivalence class of the  $W_{n-3}$ -star which refines  $S_1$  and  $S'_j$ . Therefore, for every  $i \in \{2, \dots, n - k - 1\}$ , the set  $\{S_1, \mathcal{S}_i, \mathcal{S}'_i\}$  defines a triangle in  $X_n$  which corresponds to the equivalence class of a  $W_{n-3}$ -star. By Lemma 4.3.10, for every  $i \in \{2, \dots, n - k - 1\}$ , the set  $\{f(\mathcal{S}_1), f(\mathcal{S}_i), f(\mathcal{S}'_i)\}$  defines a triangle in  $X_n$  which corresponds to the equivalence class of a  $W_{n-3}$ -star. Thus, up to changing the representative  $T'_i$ , for every  $i \in \{1, \dots, n - k - 1\}$ , the  $W_{n-3}$ -star which refines  $T_1$  and  $T_i$  is the same one as the  $W_{n-3}$ -star which refines  $T_1$  and  $T'_i$ . As  $\mathcal{T}$  and  $\mathcal{T}'$  are characterized by the set of equivalence classes of  $W_{n-3}$ -stars which collapses onto  $T_1$  and on which  $T$  and  $T'$  collapse, we see that  $\mathcal{T} = \mathcal{T}'$ . Therefore, the map  $\Phi(f): VX'_n \rightarrow VX'_n$  is well-defined. As  $\Phi(f) \circ \Phi(f^{-1}) = \Phi(f \circ f^{-1}) = \text{id}$ , we see that  $\Phi(f)$  is a bijection.

We now prove that the map  $\Phi: \text{Aut}(X_n) \rightarrow \text{Bij}(VX'_n)$  induces a monomorphism

$\tilde{\Phi}: \text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ . Let  $f \in \text{Aut}(X_n)$  and let us prove that  $\Phi(f)$  preserves  $EX'_n$ . Let  $\mathcal{S}, \mathcal{S}'$  be adjacent vertices in  $X'_n$ . Up to exchanging the roles of  $\mathcal{S}$  and  $\mathcal{S}'$ , we may suppose that there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S'$  collapses onto  $S$ . Let  $k, \ell \in \{1, \dots, n-2\}$  be such that  $S$  is a  $W_k$ -star and  $S'$  is a  $W_{k-\ell}$ -star. Let  $S_0$  be a  $W_{n-1}$ -star such that  $S$  refines  $S_0$ . Let  $S_1, \dots, S_{n-k-1}$  be the  $W_{n-2}$ -stars such that, for every  $i \in \{1, \dots, n-k-1\}$ ,  $S$  refines  $S_i$  and  $S_i$  refines  $S_0$ . As  $S'$  refines  $S$ , there exist  $\ell$   $W_{n-2}$ -stars  $S_{n-k}, \dots, S_{n-k+\ell-1}$  such that the  $W_{n-2}$ -stars  $S_1, \dots, S_{n-k+\ell-1}$  are the  $n-k+\ell-1$   $W_{n-2}$ -stars which collapse onto  $S_0$  and which are refined by  $S'$ . For every  $i \in \{1, \dots, n-k+\ell-1\}$ , let  $\mathcal{S}_i$  be the equivalence class of  $S_i$ . By definition of  $\Phi(f)$ , there exist a representative  $T$  of  $\Phi(f)(\mathcal{S})$  and representatives  $T_1, \dots, T_{n-k-1}$  of  $f(\mathcal{S}_1), \dots, f(\mathcal{S}_{n-k-1})$  such that  $T$  is a common refinement of  $T_1, \dots, T_{n-k-1}$ . Moreover, there exist a representative  $T'$  of  $\Phi(f)(\mathcal{S}')$  and representatives  $T_{n-k}, \dots, T_{n-k+\ell-1}$  of  $f(\mathcal{S}_{n-k}), \dots, f(\mathcal{S}_{n-k+\ell-1})$  such that  $T'$  is a common refinement of  $f(\mathcal{S}_1), \dots, f(\mathcal{S}_{n-k+\ell-1})$ . As  $\{f(\mathcal{S}_1), \dots, f(\mathcal{S}_{n-k-1})\}$  is a subset of  $\{f(\mathcal{S}_1), \dots, f(\mathcal{S}_{n-k+\ell-1})\}$ , we see that  $f(\mathcal{S})$  and  $f(\mathcal{S}')$  are adjacent. This shows that the application  $\Phi(f): VX_n \rightarrow VX'_n$  induces a homomorphism  $\tilde{\Phi}: \text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ . Finally, the facts that  $\tilde{\Phi}$  is injective and is  $\text{Out}(W_n)$ -equivariant follow from the fact that, for every equivalence class  $\mathcal{S}$  of  $W_{n-2}$ -stars, we have  $f(\mathcal{S}) = \Phi(f)(\mathcal{S})$ . This concludes the proof.  $\square$

**Proof of Theorem 4.3.3.** Let  $n \geq 5$ . We first prove the injectivity. The homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(X'_n)$  is injective by Theorem 4.3.4. Moreover, the homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(X'_n)$  factors through  $\text{Out}(W_n) \rightarrow \text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ . We therefore deduce the injectivity of  $\text{Out}(W_n) \rightarrow \text{Aut}(X_n)$ . We now prove the surjectivity. Let  $f \in \text{Aut}(X_n)$ . By Proposition 4.3.12, we have a homomorphism  $\tilde{\Phi}: \text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$ . By Theorem 4.3.4, the automorphism  $\tilde{\Phi}(f)$  is induced by an element  $\gamma \in \text{Out}(W_n)$ . Since the homomorphism  $\text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$  is injective by Proposition 4.3.12,  $f$  is induced by  $\gamma$ . This concludes the proof.  $\square$

## 4.4 The group of twists of a $W_{n-1}$ -star

In this section, we study the centralizers in  $\text{Out}(W_n)$  of twists about a  $W_{n-1}$ -star. We first show that to a free factor of  $W_n$  isomorphic to  $W_{n-1}$ , one can associate a canonical equivalence class of  $W_{n-1}$ -star (see Lemma 4.4.4). We then show that, for an outer automorphism  $f$  in the stabilizer of the equivalence class  $\mathcal{S}$  of a  $W_{n-1}$ -star, there exists a canonical representative  $F$  of  $f$  such that  $f$  commutes with a twist  $T$  of the  $\mathcal{S}$  if and only if  $F$  fixes the twistor of  $T$  (see Lemma 4.4.11) We first need some preliminary results about stabilizers of free factors of  $W_n$  isomorphic to  $W_{n-1}$ .

Let  $\{x_1, \dots, x_n\}$  be a standard generating set of  $W_n$ . For distinct  $i, j \in \{1, \dots, n\}$ , let  $\sigma_{j,i}: W_n \rightarrow W_n$  be the automorphism sending  $x_j$  to  $x_i x_j x_i$  and, for  $k \neq j$ , fixing  $x_k$ . For distinct  $i, j \in \{1, \dots, n\}$ , let  $(i j)$  be the automorphism of  $W_n$  switching  $x_i$  and  $x_j$  and, for  $k \neq i, j$ , fixing  $x_k$ . The following theorem is due to Mühlherr.

**Theorem 4.4.1.** [Müh, Theorem B] *Let  $n \geq 2$ . The set  $\{\sigma_{i,j} \mid i \neq j\} \cup \{(i j) \mid i \neq j\}$  is a generating set of  $\text{Aut}(W_n)$ .*

We now introduce a finite index subgroup of  $\text{Out}(W_n)$  which will be used throughout the remainder of this paper. For every  $i, j \in \{1, \dots, n\}$  distinct, both  $\sigma_{i,j}$  and  $(i\ j)$  preserve the set of conjugacy classes  $\{[x_1], \dots, [x_n]\}$ . Since  $\{\sigma_{i,j} \mid i \neq j\} \cup \{(i\ j) \mid i \neq j\}$  generates  $\text{Aut}(W_n)$  by Theorem 4.4.1, we see that we have a well-defined homomorphism  $\text{Out}(W_n) \rightarrow \text{Bij}(\{[x_1], \dots, [x_n]\})$ . Let  $\text{Out}^0(W_n)$  be the kernel of this homomorphism. The group  $\text{Out}^0(W_n)$  has finite index in  $\text{Out}(W_n)$ . We will mostly work in  $\text{Out}^0(W_n)$  from now on because of the following lemma.

**Lemma 4.4.2.** *Let  $n \geq 3$  and let  $f \in \text{Out}^0(W_n)$ . Suppose that  $f$  fixes the equivalence class  $\mathcal{S}$  of a free splitting  $S$ . Then the graph automorphism of the underlying graph of  $W_n \setminus S$  induced by  $f$  is the identity. Therefore we have  $\text{Stab}_{\text{Out}^0(W_n)}(\mathcal{S}) = \text{Stab}_{\text{Out}^0(W_n)}^0(\mathcal{S})$ .*

**Proof.** The underlying graph  $\overline{W_n \setminus S}$  of  $W_n \setminus S$  is a tree. Moreover, since  $S$  is a free splitting, if  $L$  is the set of leaves of  $\overline{W_n \setminus S}$ , then the set  $\{[G_v]\}_{v \in L}$  is a free factor system of  $W_n$ . Note that, as  $\{[x_1], \dots, [x_n]\}$  is a free factor system of  $W_n$  which is minimal for inclusion, for every  $i \in \{1, \dots, n\}$ , there exists one  $v \in VS$  such that  $x_i \in G_v$ . Since  $S$  is a free splitting, for every  $i \in \{1, \dots, n\}$ , the element  $x_i$  is contained in a unique vertex group. Moreover, for every  $v \in L$ , there exist  $k \in \{0, \dots, n-1\}$  and  $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$  such that  $G_v$  is isomorphic to  $W_k$  and  $\{[x_{i_1}] \cap G_v, \dots, [x_{i_k}] \cap G_v\}$  is a free factor system of  $G_v$ . As  $f \in \text{Out}^0(W_n)$ , and as  $f$  fixes  $\mathcal{S}$ , it follows that, for every  $v \in L$ , we have  $f([G_v]) = [G_v]$ . Hence the graph automorphism  $\hat{f}$  of  $\overline{W_n \setminus S}$  induced by  $f$  acts as the identity on  $L$ . As any graph automorphism of a finite tree is determined by its action on the set of leaves, it follows that  $\hat{f} = \text{id}$ . This concludes the proof.  $\square$

**Remark 4.4.3.** *The subgroup  $\text{Out}^0(W_n)$  of  $\text{Out}(W_n)$  is our (weak) analogue of the subgroup  $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$  of  $\text{Out}(F_N)$ , which is defined as the kernel of the natural homomorphism  $\text{Out}(F_N) \rightarrow \text{GL}(N, \mathbb{Z}/3\mathbb{Z})$ . Indeed, the group  $\text{IA}_N(\mathbb{Z}/3\mathbb{Z})$  satisfies a statement similar to Lemma 4.4.2, but it has the additional property that if  $\phi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})$  has a periodic orbit in the free splitting graph of  $F_N$ , then the cardinality of this orbit is equal to 1. In the context of  $\text{Out}^0(W_n)$ , we do not know if  $\text{Out}^0(W_n)$  contains a torsion free finite index subgroup which satisfies this property.*

The next lemma relates the stabilizer of a free factor of  $W_n$  isomorphic to  $W_{n-1}$  and the stabilizer of a  $W_{n-1}$ -star.

**Lemma 4.4.4.** *Let  $n \geq 3$ . Let  $A$  be a free factor of  $W_n$  isomorphic to  $W_{n-1}$ . Then, up to  $W_n$ -equivariant homeomorphism, there exists a unique free splitting  $S$  in which  $A$  is elliptic. In particular, if  $f \in \text{Out}(W_n)$  is such that  $f([A]) = [A]$ , then  $f$  fixes the equivalence class of  $S$ .*

**Proof.** By definition of a free factor, there exists a free splitting  $S$  of  $W_n$  such that  $A$  is elliptic in  $S$ . This proves the existence. We now prove the uniqueness statement. We may assume that  $\{x_1, \dots, x_{n-1}\}$  is a standard generating set of  $A$  and  $x_n \in W_n$  is such that

$$W_n = A * \langle x_n \rangle.$$

Then, the free factor system  $\mathcal{F} = \{[A], [\langle x_n \rangle]\}$  is a sporadic free factor system which contains  $[A]$ . Let  $\mathcal{F}'$  be a free factor system of  $W_n$  which contains  $[A]$ . Since the free factor system  $\{[\langle x_1 \rangle], \dots, [\langle x_n \rangle]\}$  is the minimal element of the set of free factor systems of  $W_n$ , we see that there exists  $[B] \in \mathcal{F}'$  such that  $x_n \in B$ . As  $\mathcal{F}'$  contains  $[A]$  and as  $W_n = A * \langle x_n \rangle$ , it follows that  $W_n = A * B$  and that  $B \subseteq \langle x_n \rangle$ . Therefore  $[B] = [\langle x_n \rangle]$  and  $\mathcal{F}' = \{[A], [\langle x_n \rangle]\}$ . We deduce that  $\mathcal{F}$  is the unique nontrivial free factor system which contains  $[A]$ . But the spine  $K(W_n, \mathcal{F})$  of the Outer space relative to  $\mathcal{F}$  is reduced to a point, i.e. it is reduced to a unique equivalence class of free splittings. This proves the uniqueness statement.  $\square$

**Remark 4.4.5.** *In the context of  $\text{Out}(F_N)$ , the analogue of the splitting given by Lemma 4.4.4 is the following one. Let  $[A]$  be the conjugacy class of a free factor of  $F_N$  isomorphic to  $F_{N-1}$ . Then the canonical splitting associated with  $A$  is the splitting corresponding to the HNN extension  $F_N = A*$  over the trivial group. However, there does not exist a natural choice (up to conjugacy) of an element  $g \in F_N$  such that  $\{[A], [g]\}$  is a free factor system of  $F_N$ .*

Let  $S$  be a splitting with exactly one orbit of edges, whose stabilizer is root-closed and isomorphic to  $\mathbb{Z}$ . Then the group of twists of  $S$  is isomorphic to  $\mathbb{Z}$  by a result of Levitt (see [Lev1, Proposition 3.1]). The next proposition is similar to a result in the case of the outer automorphism group of a free group (see [CL1] and [HW2, Lemma 2.7]). Recall that an element  $w \in W_n$  is *root-closed* if there does not exist  $w_0 \in W_n$  and an integer  $n \geq 2$  such that  $w = w_0^n$ .

**Lemma 4.4.6.** *Let  $n \geq 3$ . Let  $A$  be a free factor of  $W_n$  isomorphic to  $W_{n-1}$  and let  $w \in A$  be a root-closed element of infinite order. Let  $x \in W_n$  be such that  $W_n = A * \langle x \rangle$ . Let  $S$  be the equivalence class of a splitting  $S$  whose associated amalgamated decomposition of  $W_n$  is the following:*

$$W_n = A *_{\langle w \rangle} (\langle w \rangle * \langle x \rangle).$$

*Let  $D$  be a nontrivial twist about  $S$ . Let  $\mathcal{R}$  be the equivalence class of a free splitting  $R$  of  $W_n$  such that  $D(\mathcal{R}) = \mathcal{R}$ . Let  $R'$  and  $S'$  be metric representatives of  $R$  and  $S$ , let  $\mathcal{R}'$  and  $\mathcal{S}'$  be their  $W_n$ -equivariant isometry classes and let  $[\mathcal{R}']$  and  $[\mathcal{S}']$  be their homothety classes.*

(1) *In  $\overline{\mathbb{P}\mathcal{O}(W_n)}$ , there exists an increasing function  $\psi: \mathbb{N} \rightarrow \mathbb{N}$  such that*

$$\lim_{n \rightarrow \infty} D^{\psi(n)}([\mathcal{R}']) = [\mathcal{S}'].$$

(2) *The splittings  $S$  and  $R$  are compatible.*

**Proof.** We prove the first part. As  $\overline{\mathbb{P}\mathcal{O}(W_n)}$  is compact, up to passing to a subsequence, there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$  and a  $W_n$ -equivariant isometry class  $\mathcal{T}$  of an  $\mathbb{R}$ -tree  $T$  such that

$$\lim_{n \rightarrow \infty} \lambda_n D^n(\mathcal{R}') = \mathcal{T}.$$

Since translation length functions are continuous for the Gromov-Hausdorff topology (see [Paul1]), for every  $g \in W_n$ , we have:

$$\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \|g\|_{\mathcal{T}},$$

where  $\|g\|_{\mathcal{T}}$  is the translation length of  $g$  in  $T$ . Hence, for every  $g \in W_n$ , the limit  $\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')}$  is finite. But as  $D$  has infinite order, we have  $\lim_{n \rightarrow \infty} \lambda_n = 0$ . As there exists a representative  $\phi \in \text{Aut}(W_n)$  of  $D$  such that  $\phi_A = \text{id}_A$ , for every  $g \in A$ , we have:

$$\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \lim_{n \rightarrow \infty} \lambda_n \|g\|_{\mathcal{R}'} = 0.$$

Hence every element of  $A$  fixes a point in  $T$ . As  $A$  is finitely generated, this implies that  $A$  fixes a point in  $T$  (see for instance [CM, Section 3]). Similarly, we see that  $\langle w \rangle * \langle x \rangle$  fixes a point in  $T$ . As  $W_n = A * \langle x \rangle$ , we see that  $A$  and  $\langle w \rangle * \langle x \rangle$  cannot fix the same point in  $T$ . Let  $U$  be the free splitting of  $W_n$  associated with the free factor decomposition  $W_n = A * \langle x \rangle$ . Let  $v_0$  be the vertex of  $U$  fixed by  $A$ , let  $v_1$  be the vertex fixed by  $x$  and let  $v_2$  be the vertex fixed by  $w x w^{-1}$ . Let  $e_1$  be the edge between  $v_0$  and  $v_1$  and  $e_2$  be the edge between  $v_0$  and  $v_2$ . The arguments above show that we have a canonical  $W_n$ -equivariant morphism from  $U$  to  $T$ . This morphism is obtained by a fold of the edges  $e_1$  and  $e_2$  of  $U$  and this fold is extended  $W_n$ -equivariantly. Since  $w$  is root-closed, there is no other edge of  $U$  that can be folded as otherwise the stabilizer of an edge of  $T$  would not be cyclic. Therefore the  $\mathbb{R}$ -tree  $T$  is simplicial and the decomposition of  $W_n$  associated with  $W_n \setminus T$  is

$$W_n = A *_{\langle w \rangle} (\langle w \rangle * \langle x \rangle).$$

Hence  $\mathcal{T} = \mathcal{S}'$  and the first statement follows.

Let us prove the second statement. For every  $n \in \mathbb{N}$ , the equivalence classes  $\lambda_n D^n(\mathcal{R})$  and  $\mathcal{R}$  have compatible representatives. But as  $\lim_{n \rightarrow \infty} \lambda_n D^n(\mathcal{R}) = \mathcal{S}$ , it follows from [GuL5, Corollary A.12] that, in the limit, the splittings  $S$  and  $R$  are compatible.  $\square$

**Lemma 4.4.7.** *Let  $n \geq 3$  and let  $\mathcal{S}$  be the equivalence class of a  $W_{n-1}$ -star  $S$ . Let  $T$  be the group of twists of  $\mathcal{S}$  and let  $f \in T$  be an element of infinite order. Let  $\mathcal{R}$  be the equivalence class of a  $W_{n-1}$ -star  $R$  such that  $f(\mathcal{R}) = \mathcal{R}$ . Then  $S$  and  $R$  are compatible.*

**Proof.** Let

$$W_n = A * \langle x_n \rangle$$

be a free factor decomposition of  $W_n$  associated with  $S$  and let  $z_f \in A$  be the twistor of  $f$ . Let  $z$  be a root-closed element of  $A$  such that there exists  $m \geq 1$  with  $z^m = z_f$ . Let  $h \in T$  be the twist about  $z$ . We see that  $h^m = f$ . Let  $\mathcal{S}'$  be the splitting associated with the following amalgamated decomposition of  $W_n$ :

$$W_n = A *_{\langle z \rangle} (\langle x_n \rangle * \langle z \rangle).$$

Let  $\mathcal{S}'$  be the equivalence class of  $\mathcal{S}'$ . Let  $T'$  be the group of twists of  $\mathcal{S}'$ . Since  $A$  is isomorphic to  $W_{n-1}$  and since  $z$  is root-closed, we see that  $C_A(z) = \langle z \rangle$ . Therefore  $T'$



is isomorphic to  $\mathbb{Z}$  and a generator of  $T'$  is  $h$ . As  $f(\mathcal{R}) = \mathcal{R}$ , Lemma 4.4.6 implies that  $S'$  and  $R$  are compatible. Let  $U$  be a common refinement of  $S'$  and  $R$  whose number of orbits of edges is minimal. Since both  $S'$  and  $R$  are one-edge splittings and are different, the splitting  $U$  has 2 orbits of edges. It follows that  $W_n \setminus U$  is obtained from  $W_n \setminus S'$  by blowing-up an edge at one of the two vertices of  $W_n \setminus S'$ . Let  $\tilde{v}$  be the vertex of  $S'$  whose stabilizer is  $A$  and let  $v$  be its image in  $W_n \setminus S'$ . Let  $\tilde{w}$  be the vertex of  $S'$  fixed by  $\langle x_n \rangle * \langle z \rangle$  and let  $w$  be its image in  $W_n \setminus S'$ .

**Claim.** Either  $\mathcal{S} = \mathcal{R}$  or the splitting  $W_n \setminus U$  is obtained from  $W_n \setminus S'$  by blowing-up an edge at  $v$ .

**Proof.** Suppose that  $W_n \setminus U$  is obtained from  $W_n \setminus S'$  by blowing-up an edge at  $w$ . Then, since the group  $G_w$  associated with  $w$  is  $\langle x_n \rangle * \langle z \rangle$  and since  $z$  must fix an edge of  $U$ , we see that a free splitting of  $G_w$  such that  $z$  fixes a vertex is a  $(G_w, \{\langle z \rangle, \langle x_n \rangle\})$ -free splitting. But  $(G_w, \{\langle z \rangle, \langle x_n \rangle\})$  has exactly one such equivalence class of one-edge free splitting: the one with vertex stabilizers conjugated with  $\langle z \rangle$  and  $\langle x_n \rangle$ . This implies that  $\mathcal{R} = \mathcal{S}$ . The claim follows.  $\square$

Suppose that  $\mathcal{R} \neq \mathcal{S}$ . The claim implies that the amalgamated decomposition of  $W_n$  associated with  $U$  is

$$W_n = B * C *_{\langle z \rangle} (\langle z \rangle * \langle x_n \rangle),$$

where  $B$  and  $C$  are free factors of  $W_n$  such that  $A = B * C$  and  $z \in C$ . Let  $U'$  be a refinement of  $U$  whose associated amalgamated decomposition of  $W_n$  is:

$$W_n = B * C *_{\langle z \rangle} \langle z \rangle * \langle x_n \rangle,$$

that is,  $z$  and  $x_n$  fix distinct points in  $U'$ . Then, since  $A = B * C$ , the splitting  $U'$  is a refinement of  $S$ . This concludes the proof.  $\square$

**Proposition 4.4.8.** *Let  $n \geq 3$ . Let  $S$  be a  $W_{n-1}$ -star and let  $f \in \text{Out}(W_n)$  be a twist about the unique edge of  $W_n \setminus S$ . Let  $g \in \text{Out}^0(W_n)$  be such that  $g \in C_{\text{Out}(W_n)}(f)$ . Then  $g(\mathcal{S}) = \mathcal{S}$ .*

**Proof.** Let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition associated with  $S$  and let  $\mathcal{S}$  be the equivalence class of  $S$ . By Lemma 4.4.4, in order to prove that  $g(\mathcal{S}) = \mathcal{S}$ , it suffices to show that  $g$  preserves the conjugacy class of  $A = \langle x_1, \dots, x_{n-1} \rangle$ . Let  $\tilde{f}$  be a representative of  $f$  such that  $\tilde{f}|_A = \text{id}_A$ . Let  $\tilde{g}$  be a representative of  $g$ . Suppose towards a contradiction that  $\tilde{g}$  does not preserve the conjugacy class of  $A$ . By hypothesis, there exists  $I \in \text{Inn}(W_n)$  such that  $\tilde{f} \circ \tilde{g} = I \circ \tilde{g} \circ \tilde{f}$ . Thus,

$$\tilde{f} \circ \tilde{g}(A) = I \circ \tilde{g} \circ \tilde{f}(A) = I \circ \tilde{g}(A).$$

Therefore,  $f$  preserves the conjugacy class of  $\tilde{g}(A)$ . By Lemma 4.4.4,  $f$  fixes the unique equivalence class  $\mathcal{R}$  of the  $W_{n-1}$ -star  $R$  associated with  $\tilde{g}(A)$ . By Lemma 4.4.7, the

splittings  $S$  and  $R$  are compatible. Since we suppose that  $\tilde{g}(A) \notin [A]$ , there exists a common refinement  $S'$  of  $S$  and  $R$  which is a  $W_{n-2}$ -star. Thus, there exists  $y_n \in W_n$  such that the free factor decomposition associated with  $S'$  is

$$W_n = \langle x_n \rangle * B * \langle y_n \rangle,$$

where  $B$  is such that  $A = B * \langle y_n \rangle$  and  $B * \langle x_n \rangle$  is a conjugate of  $\tilde{g}(A)$ . Up to changing the representative  $\tilde{g}(A)$ , we may suppose that  $\tilde{g}(A) = B * \langle x_n \rangle$ . This implies that  $x_n \in \tilde{g}(A)$ , that is  $\tilde{g}^{-1}(x_n) \in A$ . But, since  $A = \langle x_1, \dots, x_{n-1} \rangle$ , we see that  $[\tilde{g}^{-1}(x_n)] \in \{[x_1], \dots, [x_{n-1}]\}$ . This contradicts the fact that  $g \in \text{Out}^0(W_n)$ .  $\square$

Combining Lemma 4.4.7 and Proposition 4.4.8, we have the following corollary.

**Corollary 4.4.9.** *Let  $n \geq 3$ . Let  $\mathcal{S}$  and  $\mathcal{R}$  be two distinct  $W_n$ -equivariant homeomorphism classes of two  $W_{n-1}$ -stars  $S$  and  $R$ . Let  $f$  and  $g$  be twists about respectively  $S$  and  $R$  such that  $f$  and  $g$  commute. Then  $S$  and  $R$  are compatible.*

**Proof.** Let  $k \geq 1$  be such that  $g^k \in \text{Out}^0(W_n)$ . By Proposition 4.4.8, since  $g^k$  and  $f$  commute, we have  $g^k(\mathcal{S}) = \mathcal{S}$ . Since  $g^k$  is a twist about  $\mathcal{R}$ , by Lemma 4.4.7, we have that  $S$  and  $R$  are compatible.  $\square$

Let  $\mathcal{S}$  be the equivalence class of a  $W_{n-1}$ -star  $S$  and let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  associated with  $S$ . Let  $A = \langle x_1, \dots, x_{n-1} \rangle$ . Let  $f \in \text{Stab}_{\text{Out}(W_n)}(\mathcal{S})$ . Then any representative of  $f$  sends  $A$  to a conjugate of itself. Let  $\tilde{f}'$  be a representative of  $f$  such that  $\tilde{f}'(A) = A$ . Since the vertices in  $S$  fixed by  $A$  and  $x_n$  are adjacent, and since the stabilizer of every vertex in  $S$  adjacent to the vertex fixed by  $A$  is a conjugate of  $\langle x_n \rangle$  by an element of  $A$ , we see that  $\tilde{f}'(x_n) = xx_nx^{-1}$  with  $x \in A$ . Therefore, there exists a representative  $\tilde{f}$  of  $f$  such that  $\tilde{f}(A) = A$  and  $\tilde{f}(x_n) = x_n$ . The automorphism  $\tilde{f}$  is the unique representative of  $f$  such that  $\tilde{f}(A) = A$  and  $\tilde{f}(x_n) = x_n$ .

We have a similar result for  $W_{n-2}$ -stars. Indeed, let  $\mathcal{S}'$  be the equivalence class of a  $W_{n-2}$ -star  $S'$  and let

$$W_n = \langle x_1 \rangle * \langle x_2, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  associated with  $S'$  and let  $B = \langle x_2, \dots, x_{n-1} \rangle$ . Let  $f \in \text{Stab}_{\text{Out}^0(W_n)}(\mathcal{S}')$ . A similar argument as in the case of a  $W_{n-1}$ -star shows that there exists a representative  $\tilde{f}$  of  $f$  such that  $\tilde{f}(B) = B$  and  $\tilde{f}(x_n) = x_n$ .

**Lemma 4.4.10.** *Let  $n \geq 4$ . Let  $\mathcal{S}$  be the  $W_n$ -equivariant homeomorphism class of a  $W_{n-1}$ -star  $S$ . Let  $T$  be the group of twists of  $\mathcal{S}$ . Let  $\mathcal{S}'$  be the  $W_n$ -equivariant homeomorphism class of a  $W_{n-2}$ -star  $S'$  which refines  $S$ . Let  $e$  be the edge of  $W_n \setminus S'$  such that a representative of  $\mathcal{S}$  is obtained from  $W_n \setminus S'$  by collapsing the edge distinct from  $e$ . Let  $T'$  be the group of twists of  $S'$  about the edge  $e$ . Then  $T \cap \text{Stab}_{\text{Out}^0(W_n)}(\mathcal{S}') \subseteq T'$ .*

**Proof.** Let

$$W_n = \langle x_1 \rangle * \langle x_2, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S'$  and let  $A = \langle x_2, \dots, x_{n-1} \rangle$ . Let

$$W_n = B * \langle y_n \rangle$$

be the free factor decomposition associated with  $S$ . Up to changing the representative  $S$ , we may suppose that  $B = \langle x_1, \dots, x_{n-1} \rangle$  and that  $y_n = x_n$ . Let  $f \in T \cap \text{Stab}_{\text{Out}^0(W_n)}(S')$ . Let  $\tilde{f}$  be the representative of  $f$  such that  $\tilde{f}(B) = B$  and  $\tilde{f}(x_n) = x_n$  which exists since  $f \in \text{Stab}_{\text{Out}(W_n)}(S)$ . Since  $f \in T$ , there exists  $g \in B$  such that  $\tilde{f}|_B$  is the global conjugation by  $g$ . Let  $\tilde{f}'$  be a representative of  $f$  such that  $\tilde{f}'(A) = A$  and  $\tilde{f}'(x_n) = x_n$ , which exists since  $f \in \text{Stab}_{\text{Out}^0(W_n)}(S')$ . Since the centralizer in  $W_n$  of  $x_n$  is  $\langle x_n \rangle$  and since  $A$  is malnormal in  $W_n$ , we see that  $\tilde{f} = \tilde{f}'$ . Hence  $\tilde{f}(A) = A$ , and, since  $A$  is malnormal, we see that  $g \in A$ . Therefore,  $f \in T'$ , which concludes the proof.  $\square$

**Lemma 4.4.11.** *Let  $n \geq 3$ . Let  $\mathcal{S}$  be the equivalence class of a  $W_{n-1}$ -star  $S$  and let*

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

*be the free factor decomposition associated with  $S$ . Let  $A = \langle x_1, \dots, x_{n-1} \rangle$ . Let  $T$  be the group of twist of  $S$ . For  $f \in T$ , let  $z_f \in A$  be the twistor of  $f$ . Let  $g \in \text{Stab}(\mathcal{S})$  and let  $\tilde{g}$  be a representative of  $g$  such that  $\tilde{g}(A) = A$  and  $\tilde{g}(x_n) = x_n$ . Then  $g \in C_{\text{Out}(W_n)}(\langle f \rangle)$  if and only if  $\tilde{g}(z_f) = z_f$ .*

**Proof.** By Proposition 4.2.5 (2), the group  $\text{Stab}(\mathcal{S})$  is isomorphic to  $\text{Aut}(A)$ . The isomorphism  $\text{Stab}(\mathcal{S}) \rightarrow \text{Aut}(A)$  is defined by sending  $f \in \text{Stab}(\mathcal{S})$  to its representative  $\tilde{f}$  such that  $\tilde{f}(A) = A$  and  $\tilde{f}(x_n) = x_n$ . In particular, for every  $h_1, h_2 \in \text{Out}(W_n) \cap \text{Stab}(\mathcal{S})$ , we see that  $h_1$  and  $h_2$  commute if and only if there exist representatives  $\tilde{h}_1$  and  $\tilde{h}_2$  of  $h_1$  and  $h_2$  respectively such that  $\tilde{h}_1(A) = A$ ,  $\tilde{h}_2(A) = A$ ,  $\tilde{h}_1(x_n) = \tilde{h}_2(x_n) = x_n$  and  $\tilde{h}_1 \circ \tilde{h}_2 = \tilde{h}_2 \circ \tilde{h}_1$ . Moreover, Proposition 4.2.5 (2) identifies the group of twists  $T$  with the group  $\text{Inn}(A)$ . For  $a \in A$ , let  $\text{ad}_a$  be the inner automorphism of  $A$  induced by  $a$ . Since, for every  $h \in \text{Aut}(A)$  and every  $a \in A$ , we have  $h \text{ad}_a h^{-1} = \text{ad}_{h(a)}$ , we see that  $h$  commutes with  $\text{ad}_a$  if and only if  $h(a) = a$ . Hence  $g \in C_{\text{Out}(W_n)}(\langle f \rangle)$  if and only if  $\tilde{g}(z_f) = z_f$ .  $\square$

## 4.5 Direct products of nonabelian free groups in $\text{Out}(W_n)$

Following [HW2, Section 6], we define the *product rank* of a group  $H$ , denoted by  $\text{rk}_{\text{prod}}(H)$ , to be the maximal integer  $k$  such that a direct product of  $k$  nonabelian free groups embeds in  $H$ . Note that, if  $H'$  is a finite index subgroup of  $H$ , then  $\text{rk}_{\text{prod}}(H') = \text{rk}_{\text{prod}}(H)$ . Moreover, if  $\phi: H \rightarrow \mathbb{Z}$  is a homomorphism, then  $\text{rk}_{\text{prod}}(\ker(\phi)) = \text{rk}_{\text{prod}}(H)$ . The aim of this section is to prove the following theorem.

**Theorem 4.5.1.** (1) For every  $n \geq 3$ , we have  $\text{rk}_{\text{prod}}(\text{Aut}(W_n)) = n - 2$ .

(2) For every  $n \geq 4$ , we have  $\text{rk}_{\text{prod}}(\text{Out}(W_n)) = n - 3$ .

(3) Suppose that  $n \geq 5$ . If  $H$  is a subgroup of  $\text{Out}(W_n)$  isomorphic to a direct product of  $n - 3$  nonabelian free groups, then  $H$  has a subgroup  $H'$  isomorphic to a direct product of  $n - 3$  nonabelian free groups which virtually fixes the  $W_n$ -equivariant homeomorphism class of a  $W_{n-1}$ -star. In addition,  $H$  does not virtually fix the  $W_n$ -equivariant homeomorphism class of any one-edge free splitting that is not a  $W_{n-1}$ -star.

We first recall an estimate regarding product ranks and group extensions due to Horbez and Wade.

**Lemma 4.5.2.** [HW2, Lemma 6.3] Let  $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$  be a short exact sequence of groups. Then  $\text{rk}_{\text{prod}}(G) \leq \text{rk}_{\text{prod}}(N) + \text{rk}_{\text{prod}}(Q)$ .  $\square$

In order to compute the product rank of  $\text{Out}(W_n)$ , we take advantage of its action on the Gromov hyperbolic free factor complex. We recall a general result concerning actions of direct products on a hyperbolic space.

**Lemma 4.5.3.** [HW2, Proposition 4.2, Lemma 4.4] Let  $X$  be a Gromov hyperbolic space, and let  $H$  be a group acting by isometries on  $X$ . Assume that  $H$  contains a normal subgroup  $K$  isomorphic to a direct product  $K = \prod_{i=1}^k K_i$ .

If there exists  $j \in \{1, \dots, k\}$  such that  $K_j$  contains a loxodromic element, then  $\prod_{i \neq j} K_i$  has a finite orbit in  $\partial_\infty X$ .

If there exist two distinct  $i, j \in \{1, \dots, k\}$  such that both  $K_i$  and  $K_j$  contain a loxodromic element, then  $H$  has a finite orbit in  $\partial_\infty X$ .

If, for every  $j \in \{1, \dots, k\}$ , the group  $K_j$  does not contain a loxodromic element, then either  $K$  has a finite orbit in  $\partial_\infty X$  or  $H$  has bounded orbits in  $X$ .  $\square$

Let  $\mathcal{F}$  be a free factor system of  $W_n$ . Recall that  $\mathcal{O}(W_n, \mathcal{F})$  is the outer space of  $W_n$  relative to  $\mathcal{F}$ . Given  $T \in \overline{\mathcal{O}(W_n, \mathcal{F})}$ , let  $[T]$  be the homothety class of  $T$ . The homothetic stabilizer  $\text{Stab}([T])$  is the stabilizer of  $[T]$  for the action of  $\text{Out}(W_n, \mathcal{F})$  on  $\mathbb{P}\overline{\mathcal{O}(W_n, \mathcal{F})}$ . Equivalently,  $\Phi \in \text{Out}(W_n, \mathcal{F})$  lies in  $\text{Stab}([T])$  if there exists a lift  $\tilde{\Phi} \in \text{Aut}(W_n, \mathcal{F})$  of  $\Phi$  and a homothety  $I_{\tilde{\Phi}}: T \rightarrow T$  such that, for all  $g \in W_n$  and  $x \in T$ , we have  $I_{\tilde{\Phi}}(gx) = \tilde{\Phi}(g)I_{\tilde{\Phi}}(x)$ . The scaling factor of  $I_{\tilde{\Phi}}$  does not depend on the choice of a representative of  $\Phi$ , and we denote it by  $\lambda_T(\Phi)$ . This gives a homomorphism

$$\begin{aligned} \text{Stab}([T]) &\rightarrow \mathbb{R}_+^* \\ \Phi &\mapsto \lambda_T(\Phi). \end{aligned}$$

The kernel of this morphism is called the *isometric stabilizer of  $T$*  and is denoted by  $\text{Stab}^{\text{is}}(T)$ . It is the stabilizer of  $T$  for the action of  $\text{Out}(W_n, \mathcal{F})$  on  $\overline{\mathcal{O}(W_n, \mathcal{F})}$ .

**Lemma 4.5.4.** [GuH2, Lemma 6.1] Let  $n \geq 3$ . Let  $\mathcal{F}$  be a nonsporadic free factor system of  $W_n$ . For every  $T \in \overline{\mathcal{O}(W_n, \mathcal{F})}$ , the image of the morphism  $\lambda_T$  is a cyclic (maybe trivial) subgroup of  $\mathbb{R}_+^*$ .  $\square$

We will also use a theorem due to Guirardel and Horbez which assigns to every nonempty collection of free splittings whose elementwise stabilizer is infinite a canonical (not necessarily free) splitting.

**Theorem 4.5.5.** [GuH3, Theorem 6.12] *Let  $n \geq 3$ . There exists an  $\text{Out}(W_n)$ -equivariant map which assigns to every nonempty collection  $\mathcal{C}$  of free splittings of  $W_n$  whose elementwise  $\text{Out}(W_n)$ -stabilizer is infinite, a nontrivial splitting  $U_{\mathcal{C}}$  of  $W_n$  whose set of vertices  $VU_{\mathcal{C}}$  has a  $W_n$ -invariant partition  $VU_{\mathcal{C}} = V_1 \sqcup V_2$  with the following properties:*

- (1) *For every vertex  $v \in V_1$ , the following holds:*
  - (a) *either some edge incident on  $v$  has trivial stabilizer, or the set of stabilizers of edges incident on  $v$  induces a nontrivial free factor system of the vertex stabilizer  $G_v$ ,*
  - (b) *there exists a finite index subgroup  $H_0$  of the elementwise stabilizer of the collection  $\mathcal{C}$  such that every outer automorphism in  $H_0$  has a representative in  $\text{Aut}(W_n)$  which restricts to the identity on  $G_v$ .*
- (2) *The collection of all conjugacy classes of stabilizers of vertices in  $V_2$  is a free factor system of  $W_n$ .  $\square$*

Finally, we state a proposition due to Guirardel and Horbez concerning the isometric stabilizer of an arational tree.

**Proposition 4.5.6.** [GuH2, Proposition 6.5] *Let  $n \geq 3$ . Let  $\mathcal{F}$  be a nonsporadic free factor system of  $W_n$ , and let  $T$  be an arational  $(W_n, \mathcal{F})$ -tree. Let  $H$  be a subgroup of  $\text{Out}(W_n, \mathcal{F})$  which is virtually contained in  $\text{Stab}^{\text{is}}(T)$ . Then  $H$  has a finite index subgroup  $H'$  which fixes infinitely many  $(W_n, \mathcal{F})$ -free splittings, and in particular  $H$  fixes the conjugacy class of a proper  $(W_n, \mathcal{F})$ -free factor.  $\square$*

Note that the statement of Proposition 4.5.6 in [GuH2] only mentions that  $H'$  fixes one  $(W_n, \mathcal{F})$ -free splitting, but the proof uses an arbitrary free splitting of  $W_n$ , so that one can construct infinitely many pairwise distinct free splittings fixed by  $H'$  by varying the chosen free splitting of  $W_n$ .

**Proof of Theorem 4.5.1.** The proof is inspired by [HW2, Theorem 6.1] due to Horbez and Wade and [HHW, Theorem 4.3] due to Hensel, Horbez and Wade.

We first prove that if  $n \geq 4$ , then  $\text{rk}_{\text{prod}}(\text{Out}(W_n)) \geq n - 3$  and that, if  $n \geq 3$ , then  $\text{rk}_{\text{prod}}(\text{Aut}(W_n)) \geq n - 2$ . Pick a standard generating set  $\{x_1, \dots, x_n\}$  of  $W_n$ . Then the group  $H$  generated by  $\{x_1x_2, x_2x_3\}$  is a nonabelian free group (see [Müh, Theorem A]).

Suppose first that  $n \geq 4$ . For  $i \in \{4, \dots, n\}$  and  $h \in H$ , let  $F_{i,h}$  be the automorphism sending  $x_i$  to  $hx_ih^{-1}$  and, for  $j \neq i$ , fixing  $x_j$ . Then, for every distinct  $i, j \in \{4, \dots, n\}$  and for every  $g, h \in H$ , the automorphisms  $F_{i,g}$  and  $F_{j,h}$  commute, giving a direct product of  $n - 3$  nonabelian free groups in  $\text{Out}(W_n)$ . Moreover, for every  $g, h \in H$ , and every  $i \in \{4, \dots, n\}$ , the inner automorphism  $\text{ad}_g$  commutes with  $F_{i,h}$ , which yields a direct

product of  $n - 2$  nonabelian free groups in  $\text{Aut}(W_n)$ . In the case where  $n = 3$ , the group  $\text{Aut}(W_3)$  contains the subgroup  $\langle \text{ad}_h \rangle_{h \in H}$ , which is a nonabelian free group.

We now prove that, if  $n \geq 3$ , then  $\text{rk}_{\text{prod}}(\text{Aut}(W_n)) \leq n - 2$ , if  $n = 3$ , then  $\text{rk}_{\text{prod}}(\text{Out}(W_n)) = 1$  and if  $n \geq 4$ , then  $\text{rk}_{\text{prod}}(\text{Out}(W_n)) \leq n - 3$ . The proof is by induction on  $n$ . The base case where  $n = 3$  follows from the fact that the group  $\text{Aut}(W_3)$  is isomorphic to  $\text{Aut}(F_2)$  (see [Var, Lemma 2.3]) and the fact that the group  $\text{Aut}(F_2)$  does not contain a direct product of two nonabelian free groups (see [HW2, Lemma 6.2]). Moreover, by [Gue1, Proposition 2.2], the group  $\text{Out}(W_3)$  is isomorphic to  $\text{PGL}(2, \mathbb{Z})$  which is virtually free.

Let  $k \geq \max\{n - 3, 2\}$  and let  $H = H_1 \times H_1 \times \dots \times H_k$  be a subgroup of  $\text{Out}(W_n)$  isomorphic to a direct product of  $k$  nonabelian free groups. Note that  $k = n - 3$  if  $n \geq 5$  and  $k = 2$  if  $n = 4$ . We prove that there exists a subgroup  $K$  of  $H$  isomorphic to a direct product of  $k$  nonabelian free groups which virtually fixes a one-edge free splitting of  $W_n$ . Let  $\mathcal{F}$  be a maximal  $H$ -periodic free factor system. If  $\mathcal{F}$  is sporadic, then  $H$  virtually fixes a one-edge free splitting, so we are done. Therefore, we may suppose that  $\mathcal{F}$  is nonsporadic. As  $\mathcal{F}$  is supposed to be maximal, by Proposition 4.2.2, the group  $H$  acts on  $\text{FF}(W_n, \mathcal{F})$  with unbounded orbits. Lemma 4.5.3 implies that, after possibly reordering the factors, the group  $H' = H_1 \times H_2 \times \dots \times H_{k-1}$  has a finite orbit in  $\partial_\infty \text{FF}(W_n, \mathcal{F})$ . By Lemma 4.2.4, the group  $H'$  virtually fixes the homothety class  $[T]$  of an arational  $(W_n, \mathcal{F})$ -tree  $T$ .

Let  $H_0$  be a normal subgroup of finite index in  $H'$  that is contained in  $\text{Stab}([T])$ .

**Claim.** The group  $H$  contains a subgroup isomorphic to a direct product of  $k$  nonabelian free groups, which fixes the equivalence class of a one-edge free splitting.

**Proof.** By Lemma 4.5.4, the homomorphism  $\lambda_T|_{H_0}$  from  $H_0$  to  $\mathbb{R}_+^*$  given by the scaling factor has cyclic image. As  $H_0$  contains a direct product of  $k - 1$  nonabelian free groups, so does  $P = \ker(\lambda_T|_{H_0})$  (see the beginning of Section 4.5). In particular, the intersection of  $P$  with every direct factor  $H_i$  of  $H'$  is a nonabelian free group. As  $P$  is contained in the isometric stabilizer of  $T$ , Proposition 4.5.6 implies that  $P$  contains a finite index subgroup  $P_0$  which fixes infinitely many  $(W_n, \mathcal{F})$ -free splittings.

Let  $\mathcal{C}$  be the (nonempty) collection of all  $(W_n, \mathcal{F})$ -free splittings fixed by the infinite group  $P_0$ , let  $U_{\mathcal{C}}$  be the splitting provided by Theorem 4.5.5, and let  $\mathcal{U}_{\mathcal{C}}$  be its equivalence class. Since  $P_0$  commutes with  $H_k$ , the equivalence class  $\mathcal{U}_{\mathcal{C}}$  is  $(P_0 \times H_k)$ -invariant.

Suppose first that the splitting  $U_{\mathcal{C}}$  contains an edge  $e \in EU_{\mathcal{C}}$  with trivial stabilizer. Let  $U'$  be the splitting obtained from  $U_{\mathcal{C}}$  by collapsing every edge of  $U_{\mathcal{C}}$  that is not contained in the orbit of  $e$ , and let  $\mathcal{U}'$  be its equivalence class. Then  $\mathcal{U}'$  is the equivalence class of a one-edge free splitting virtually fixed by  $P_0 \times H_k$ . Since  $P_0$  contains a direct product of  $k - 1$  nonabelian free groups, the claim follows.

Thus, we can suppose that all edge stabilizers of  $U_{\mathcal{C}}$  are nontrivial. We show that this leads to a contradiction. Let  $VU_{\mathcal{C}} = V_1 \amalg V_2$  be the partition of  $VU_{\mathcal{C}}$  given by Theorem 4.5.5. Let  $P'$  be a finite index subgroup of  $P_0$  which acts trivially on the quotient  $W_n \setminus U_{\mathcal{C}}$ . We claim that the intersection of  $P'$  with the group of twists of  $U_{\mathcal{C}}$  is trivial. Indeed, let  $e$  be an oriented edge of  $U_{\mathcal{C}}$ . As every subgroup of  $W_n$  with nontrivial centralizer is cyclic, if the edge stabilizer  $G_e$  of  $e$  is not cyclic, the group of

twists around this edge is trivial. Thus, oriented edges with nontrivial group of twists have cyclic stabilizers. But twists about edges with cyclic stabilizers are central in a finite index subgroup of  $\text{Stab}^0(U_C)$  by Lemma 4.2.7. Let  $P''$  is a finite index subgroup of  $P'$ . Then the intersection of  $P''$  with every direct factor  $H_i$  of  $H'$  is a nonabelian free group. Therefore every element of  $P''$  is contained in a nonabelian free subgroup of  $P''$ . In particular, the center of every finite index subgroup of  $P'$  is trivial. Thus we see that the intersection of  $P'$  with the group of twists is trivial. By Remark 4.2.6, up to passing to a further finite index subgroup of  $P'$ , we may suppose that the intersection of  $P'$  with the group of bitwists is trivial.

By Proposition 4.2.5 (1) and Remark 4.2.6, the fact that the intersection of  $P'$  with the group of bitwists is trivial implies that we have an injective homomorphism

$$P' \rightarrow \prod_{v \in W_n \setminus VU_C} \text{Out}(G_v).$$

By Theorem 4.5.5 (1)(b), for every vertex  $v \in V_1$ , the homomorphism  $P' \rightarrow \text{Out}(G_v)$  has finite image. Therefore, up to passing to a finite index subgroup of  $P'$ , we have an injective map

$$P' \rightarrow \prod_{v \in W_n \setminus V_2} \text{Out}(G_v).$$

By Theorem 4.5.5 (2), for every  $v \in V_2$ , the vertex stabilizer  $G_v$  is an element of a free factor system of  $W_n$ . Therefore, there exists  $k$  such that  $G_v$  is isomorphic to  $W_k$ . By Lemma 4.5.2, we have:

$$n - 4 \leq k - 1 = \text{rk}_{\text{prod}}(P') \leq \sum_{v \in W_n \setminus V_2} \text{rk}_{\text{prod}}(\text{Out}(G_v)).$$

By induction, we see that, if  $|W_n \setminus V_2| \geq 2$ , then

$$\sum_{v \in W_n \setminus V_2} \text{rk}_{\text{prod}}(\text{Out}(G_v)) \leq n - 6,$$

which leads to a contradiction. Thus  $|W_n \setminus V_2| = 1$ . Let  $v \in W_n \setminus V_2$ . Then there exists  $\ell \in \{1, \dots, n-1\}$  such that  $G_v$  is isomorphic to  $W_\ell$ . If  $\ell \leq n-2$ , then

$$\text{rk}_{\text{prod}}(\text{Out}(G_v)) \leq n - 5,$$

which leads to a contradiction. If  $\ell = n-1$ , then the free factor system  $\mathcal{F}$  contains a free factor isomorphic to  $W_{n-1}$  and is therefore a sporadic free factor system, which leads to a contradiction.  $\square$

Therefore, we see that there exists a subgroup  $K$  of  $H$  isomorphic to a direct product of  $k$  nonabelian free groups such that  $K$  fixes the  $W_n$ -equivariant homeomorphism class of a one-edge-free splitting  $\mathcal{S}$ . We now prove that  $\mathcal{S}$  is the equivalence class of a  $W_{n-1}$ -star. Let  $S$  be a representative of  $\mathcal{S}$ , let  $v_1$  and  $v_2$  be the vertices of the underlying graph of  $W_n \setminus \mathcal{S}$  and, for  $i \in \{1, 2\}$ , let  $k_i$  be such that  $W_{k_i}$  is isomorphic to  $G_{v_i}$ . Let  $K_0$  be the

finite index subgroup of  $K$  which acts as the identity on  $W_n \setminus S$ . Then  $K_0 \subseteq \text{Stab}^0(\mathcal{S})$ . By Proposition 4.2.5 (2), the group  $\text{Stab}^0(\mathcal{S})$  is isomorphic to  $\text{Aut}(W_{k_1}) \times \text{Aut}(W_{k_2})$ . Suppose towards a contradiction that, for every  $i \in \{1, 2\}$ , we have that  $k_i \neq 1$ . Suppose first that, for every  $i \in \{1, 2\}$ , we have  $k_i \geq 3$ . Then, by Lemma 4.5.2, we see that:

$$k = \text{rk}_{\text{prod}}(K_0) \leq \text{rk}_{\text{prod}}(\text{Aut}(W_{k_1})) + \text{rk}_{\text{prod}}(\text{Aut}(W_{k_2})) \leq k_1 - 2 + k_2 - 2 = n - 4,$$

where the second inequality comes from the induction hypothesis. If there exists  $i \in \{1, 2\}$  such that  $k_i = 2$ , then, as  $\text{Aut}(W_2)$  is virtually cyclic (it is isomorphic to  $W_2$  by [Tho, Lemma 1.4.2]), we see that:

$$k = \text{rk}_{\text{prod}}(K_0) \leq \text{rk}_{\text{prod}}(\text{Aut}(W_{k_1})) + \text{rk}_{\text{prod}}(\text{Aut}(W_{k_2})) \leq k_1 - 2 \leq n - 4.$$

In both cases, we have a contradiction as  $k \geq n - 3$  when  $k \geq 5$  and  $k = n - 2$  when  $n = 4$ . Thus, there exists  $i \in \{1, 2\}$  such that  $k_i = 1$ . This shows that  $S$  is a  $W_{n-1}$ -star. In particular, when  $k = n - 3$ , that is, when  $n \geq 5$ , this proves Theorem 4.5.1 (3).

Since  $K_0 \subseteq \text{Stab}^0(\mathcal{S})$ , Proposition 4.2.5 (2) implies that

$$k = \text{rk}_{\text{prod}}(K_0) \leq \text{rk}_{\text{prod}}(\text{Aut}(W_{n-1})) = n - 1 - 2 = n - 3.$$

When  $n = 4$ , then  $k = 2 = n - 2$ . Therefore, we have a contradiction in this case. This shows that, for all  $n \geq 4$ , the product rank of  $\text{Out}(W_n)$  is equal to  $n - 3$ . This concludes the proof of Theorem 4.5.1 (2).

It remains to prove that, if  $n \geq 4$ , we have  $\text{rk}_{\text{prod}}(\text{Aut}(W_n)) \leq n - 2$ . We have the following short exact sequence

$$1 \rightarrow W_n \rightarrow \text{Aut}(W_n) \rightarrow \text{Out}(W_n) \rightarrow 1.$$

By Lemma 4.5.2, as  $W_n$  is virtually free, we see that

$$\text{rk}_{\text{prod}}(\text{Aut}(W_n)) \leq \text{rk}_{\text{prod}}(W_n) + \text{rk}_{\text{prod}}(\text{Out}(W_n)) = 1 + n - 3 = n - 2.$$

This concludes the proof of Theorem 4.5.1 (1).  $\square$

## 4.6 Subgroups of stabilizers of $W_{n-1}$ -stars

In the next two sections, we prove an algebraic characterisation of stabilizers of equivalence classes of  $W_{n-2}$ -stars. In this section, we take advantage of properties satisfied by stabilizers of equivalence classes of  $W_{n-2}$ -stars which are sufficiently rigid to show that a subgroup  $H$  of  $\text{Out}(W_n)$  which satisfies these properties virtually fixes a  $W_{n-1}$ -star. In the next section, we will take advantage of the fact that stabilizers of equivalence classes of compatible  $W_{n-2}$ -stars have large intersections to give a characterisation of stabilizers of equivalence classes of  $W_{n-2}$ -stars.

Let  $\Gamma$  be a finite index subgroup of the group  $\text{Out}^0(W_n)$  (defined after Theorem 4.4.1). We introduce the following algebraic property for a subgroup  $H \subseteq \Gamma$ .

( $P_{W_{n-2}}$ ) The group  $H$  satisfies the following three properties:



- (1) The group  $H$  contains a normal subgroup isomorphic to a direct product  $K_1 \times K_2$  of two normal subgroups such that each one contains a nonabelian finitely generated normal free subgroup of finite index and such that for every  $i \in \{1, 2\}$ , for every nontrivial normal subgroup  $P$  of a finite index subgroup  $K'_i$  of  $K_i$ , and for every finite index subgroup  $P'$  of  $P$ , the group  $C_{\text{Out}^0(W_n)}(P')$  contains  $K_{i+1}$  as a finite index subgroup (where indices are taken modulo 2).
- (2) The group  $H$  contains a direct product of  $n - 3$  nonabelian free groups.
- (3) The group  $H$  contains a subgroup isomorphic to  $\mathbb{Z}^{n-2}$ .

**Remark 4.6.1.** (1) Notice that property  $(P_{W_{n-2}})$  is closed under taking finite index subgroups.

(2) Hypothesis  $(P_{W_{n-2}})$  (1) implies that, if for every  $i \in \{1, 2\}$ , the group  $P_i$  is a finite index subgroup of a nontrivial normal subgroup of a finite index subgroup of  $K_i$ , the centralizer in  $\text{Out}^0(W_n)$  of  $P_1 \times P_2$  is finite.

We first prove that the stabilizer in  $\Gamma$  of the equivalence class of a  $W_{n-2}$ -star satisfies  $(P_{W_{n-2}})$ . We then show that a group satisfying  $(P_{W_{n-2}})$  virtually fixes the equivalence class of a  $W_{n-1}$ -star.

#### 4.6.1 Properties of $\mathcal{Z}_{RC}$ -factors

In order to prove that the stabilizer in  $\Gamma$  of the equivalence class of a  $W_{n-2}$ -star satisfies  $(P_{W_{n-2}})$ , we first need some background concerning  $\mathcal{Z}_{RC}$ -splittings. Let  $G$  be a finitely generated group. A  $\mathcal{Z}_{RC}$ -splitting of  $G$  is a splitting of  $G$  such that every edge stabilizer is either trivial or isomorphic to  $\mathbb{Z}$  and root-closed. A  $\mathcal{Z}_{RC}$ -factor of  $G$  is a subgroup of  $G$  which arises as a vertex stabilizer of a  $\mathcal{Z}_{RC}$ -splitting of  $G$ . Note that since edge stabilizers are root-closed, so are the vertex stabilizers.

We now describe a finite index subgroup of  $W_n$  that we will use in the proof of Proposition 4.6.3. Let  $\mathbb{F}$  be the kernel of the homomorphism  $W_n \rightarrow F$  which sends every generator of a standard generating set of  $W_n$  to the nontrivial element of  $F$ . Remark that  $\mathbb{F}$  does not depend on the choice of the basis. Indeed, if  $\{x_1, \dots, x_n\}$  is a standard generating set of  $W_n$ , and if  $x$  is an element of  $W_n$  of order 2, there exists  $i \in \{1, \dots, n\}$  and  $g \in W_n$  such that  $x = gx_i g^{-1}$ . We have the following result due to Mühlherr.

**Lemma 4.6.2.** [Müh, Theorem A] *The group  $\mathbb{F}$  is a nonabelian free group of rank  $n - 1$  which is a characteristic subgroup of  $W_n$ . Moreover, the natural restriction homomorphism*

$$\text{Aut}(W_n) \rightarrow \text{Aut}(\mathbb{F})$$

*is injective.*

We now outline here some properties of  $\mathcal{Z}_{RC}$ -factors (see e.g. [HW2, Proposition 7.3]).

**Proposition 4.6.3.** *Let  $n \geq 3$ . The  $\mathcal{Z}_{RC}$ -factors of  $W_n$  satisfy the following properties.*

- (1) *Let  $H$  be a finitely generated subgroup of  $W_n$  which is not virtually cyclic. There exists  $g \in H$  which is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $H$ .*
- (2) *There exists  $C \in \mathbb{N}^*$  such that, for every strictly ascending chain  $G_1 \subsetneq \dots \subsetneq G_k$  of  $\mathcal{Z}_{RC}$ -factors of  $W_n$ , one has  $k \leq C$ .*
- (3) *If a subgroup  $K$  of  $W_n$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$  and if  $P$  is either a finite index subgroup of  $K$  or a nontrivial normal subgroup of  $K$ , then  $P$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ .*
- (4) *A subgroup  $K$  of  $W_n$  is contained in a proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$  if and only if every element of  $K$  is contained in a proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ .*

**Proof.** The first assertion is a consequence of [GeH, Lemma 4.3] due to Genevois and Horbez.

For the second assertion, let  $G_1 \subsetneq \dots \subsetneq G_k$  be a sequence of strictly ascending  $\mathcal{Z}_{RC}$ -factors. Then, since  $\mathcal{Z}_{RC}$ -factors are root-closed, for every  $i \geq 3$  the group  $G_i$  is not cyclic. Thus, as we want an upper bound on the number of subgroups of such a sequence, we may suppose that for every  $i \in \{1, \dots, n\}$ , the group  $G_i$  is not cyclic. We claim that, for every  $i \in \{1, \dots, k\}$ , there exists  $\phi_i \in \text{Aut}(W_n)$  such that  $\text{Fix}(\phi_i) = G_i$ . Indeed, let  $S_i$  be a  $\mathcal{Z}_{RC}$ -splitting of  $W_n$  such that there exists  $v \in VS_i$  whose stabilizer is equal to  $G_i$ . Up to collapsing edges, we may suppose that every vertex of  $S_i$  has nontrivial stabilizer. Let  $e_1, \dots, e_\ell$  be the edges with origin  $v$  which are in pairwise distinct orbits. Let  $F_0 \subseteq \{e_1, \dots, e_\ell\}$  be the subset made of all edges with nontrivial stabilizer. By the definition of a  $\mathcal{Z}_{RC}$ -splitting, for every  $e_s \in F_0$ , the group  $G_{e_s}$  is cyclic. For every  $e_s \in F_0$ , let  $z_s$  be a generator of  $G_{e_s}$ . For every  $e_{s'} \in \{e_1, \dots, e_\ell\} - F_0$ , let  $z_{s'} \in G_i$  be such that, if  $w_{s'}$  is the endpoint of  $e_{s'}$  distinct from  $v$ , we have  $z_{s'} G_{w_{s'}} z_{s'}^{-1} \neq G_{w_{s'}}$ . Let  $\phi_i = D_{e_1, z_1} \circ \dots \circ D_{e_\ell, z_\ell}$  be a multitwist about every edge with origin  $v$ . Then, as the centralizer of an infinite cyclic subgroup of  $W_n$  is infinite cyclic, we have  $\text{Fix}(\phi_i) = G_i$ . Therefore, in order to prove the second assertion, it suffices to prove that there exists  $C \in \mathbb{N}^*$  such that for every strictly ascending chain  $\text{Fix}(\phi_1) \subsetneq \dots \subsetneq \text{Fix}(\phi_k)$  of fixed points sets of automorphisms of  $W_n$ , one has  $k \leq C$ .

Let  $\mathbb{F}$  be the characteristic subgroup of  $W_n$  given by Lemma 4.6.2 and let

$$\Phi: \text{Aut}(W_n) \rightarrow \text{Aut}(\mathbb{F})$$

be the natural injective homomorphism given by restriction. Then

$$\text{Fix}(\Phi(\phi_1)) \subseteq \dots \subseteq \text{Fix}(\Phi(\phi_k))$$

is an ascending chain of fixed points sets.

**Claim.** For every  $i \in \{2, \dots, k-1\}$ , the set  $\{\text{Fix}(\Phi(\phi_{i-1})), \text{Fix}(\Phi(\phi_i)), \text{Fix}(\Phi(\phi_{i+1}))\}$  contains at least 2 elements.

**Proof.** Suppose towards a contradiction that

$$|\{\text{Fix}(\Phi(\phi_{i-1})), \text{Fix}(\Phi(\phi_i)), \text{Fix}(\Phi(\phi_{i+1}))\}| = 1.$$

As  $\text{Fix}(\phi_{i-1}) \subsetneq \text{Fix}(\phi_i)$  and  $\text{Fix}(\Phi(\phi_{i-1})) = \text{Fix}(\Phi(\phi_i))$ , there exists  $a \in W_n - \mathbb{F}$  such that  $\phi_i(a) = a$  and  $\phi_{i-1}(a) \neq a$ . Since the index of  $\mathbb{F}$  is equal to 2, we see that  $\phi_{i-1}(a^2) = a^2$ . Therefore,  $\phi_{i-1}(a)^2 = a^2$  and  $\phi_{i-1}(a)$  is a square root of  $a^2$ . If  $a^2$  has infinite order, its only square root is  $a$ . This implies that  $\phi_{i-1}(a) = a$ , a contradiction. Thus we can assume that  $a$  has order 2 and, up to changing the basis  $\{x_1, \dots, x_n\}$ , we may suppose that  $a = x_1$ .

As the index of  $\mathbb{F}$  is equal to 2, we have  $W_n = \mathbb{F} \amalg x_1\mathbb{F}$ . Let  $x \in \text{Fix}(\phi_{i+1}) - \mathbb{F}$ . Then there exists  $y \in \mathbb{F}$  such that  $x = x_1y$ . As  $x_1 \in \text{Fix}(\phi_i)$  and  $\text{Fix}(\phi_i) \subsetneq \text{Fix}(\phi_{i+1})$ , we have that  $\phi_{i+1}(x_1) = x_1$ . Hence  $\phi_{i+1}(y) = y$ . As  $y \in \mathbb{F}$  and  $\text{Fix}(\Phi(\phi_i)) = \text{Fix}(\Phi(\phi_{i+1}))$ , we see that  $\phi_i(y) = y$  and  $\phi_i(x) = \phi_i(x_1y) = x_1y = x$ . Therefore we have that  $\text{Fix}(\phi_i) = \text{Fix}(\phi_{i+1})$ , which is a contradiction. The claim follows.  $\square$

The claim implies that the length of the strictly ascending chain associated with  $\text{Fix}(\Phi(\phi_1)) \subseteq \dots \subseteq \text{Fix}(\Phi(\phi_k))$  is at least equal to  $\frac{k}{2}$ . But any strictly ascending chain of fixed subgroups in a free group on  $n-1$  generators has length at most  $2(n-1)$  (see [MV, Theorem 4.1]). Therefore, there exists  $C$  which depends only on  $n$  such that  $k \leq C$ . The second assertion of Proposition 4.6.3 follows.

We now prove the third assertion. Let  $P$  and  $K$  be as in Proposition 4.6.3 (3). If  $K$  is a virtually infinite cyclic group, then  $K$  is either isomorphic to  $\mathbb{Z}$  or to  $W_2$ . Let  $a$  be a generator of the subgroup of  $K$  isomorphic to  $\mathbb{Z}$  and root-closed in  $K$ . Since  $\langle a \rangle$  is a finite index subgroup of  $K$  and since  $K$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ , then neither is  $a$ . Remark that any nontrivial normal subgroup of  $K$  intersects the subgroup  $\langle a \rangle$  non trivially. Therefore, if  $P$  is contained in a proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ , then  $a$  is elliptic in a  $\mathcal{Z}_{RC}$ -splitting. This contradicts the fact that  $a$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ .

So we can assume that  $K$  is not virtually cyclic. As every finite index subgroup contains a nontrivial normal subgroup of  $K$ , we may assume that  $P$  is a nontrivial normal subgroup of  $K$ . Notice that  $P$  is necessarily noncyclic. Suppose towards a contradiction that  $P$  is contained in a  $\mathcal{Z}_{RC}$ -factor. Then there exists a  $\mathcal{Z}_{RC}$ -splitting  $S$  of  $W_n$  such that  $P$  is elliptic in  $S$ . Since edge stabilizers are cyclic, the group  $P$  fixes a unique vertex  $x$  of  $S$ . But, as  $P$  is normal in  $K$ , for every  $k \in K$ , we have that  $kx$  is also fixed by  $P$ , hence we have  $kx = x$ . Therefore,  $x$  is fixed by  $K$ , which contradicts the fact that  $K$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor.

We finally prove Proposition 4.6.3 (4). Suppose that  $K$  is contained in a proper  $\mathcal{Z}_{RC}$ -factor. Then it is clear that every element of  $K$  is contained in a proper  $\mathcal{Z}_{RC}$ -factor.

Conversely, assume that  $K$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ . Let us prove that there exists  $g \in K$  such that  $g$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor. By Proposition 4.6.3 (2), there exists a bound on the length of an increasing chain of  $\mathcal{Z}_{RC}$ -factors of  $W_n$ . Therefore, the group  $K$  contains a finitely generated subgroup  $K'$  which is not contained in any proper  $\mathcal{Z}_{RC}$ -factor. By Proposition 4.6.3 (1), there exists  $g \in K'$  such that  $g$  is not contained in a proper  $\mathcal{Z}_{RC}$ -factor of  $K'$ . Let  $S$  be a  $\mathcal{Z}_{RC}$ -splitting of  $W_n$ . As  $K'$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ , the group  $K'$  has a well-defined, nontrivial minimal subtree  $S_{K'}$  with respect to the action of  $K'$  on  $S$ . As  $S$  is a  $\mathcal{Z}_{RC}$ -splitting of  $W_n$ , the splitting  $S_{K'}$  is a  $\mathcal{Z}_{RC}$ -splitting of  $K'$ . Since  $g$  is

not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $K'$ , it follows that  $g$  is a hyperbolic isometry of  $S_{K'}$  and is not elliptic in  $S$ . As  $S$  is arbitrary, it follows that  $g$  is not contained in any  $\mathcal{Z}_{RC}$ -factor of  $W_n$ .  $\square$

Proper  $\mathcal{Z}_{RC}$ -factors appear naturally when studying stabilizers of conjugacy classes of elements as shown by the following theorem. Recall that, if  $\mathcal{H} = \{H_1, \dots, H_k\}$  is a finite family of finitely generated subgroups of  $W_n$ , the group  $\text{Out}(W_n, \mathcal{H}^{(t)})$  is the subgroup of  $\text{Out}(W_n)$  consisting of all outer automorphisms  $\phi \in \text{Out}(W_n)$  such that, for every  $i \in \{1, \dots, k\}$ , there exists a representative  $\tilde{\phi}_i \in \text{Aut}(W_n)$  of  $\phi$  such that  $\tilde{\phi}_i(H_i) = H_i$  and  $\tilde{\phi}_i|_{H_i} = \text{id}_{H_i}$ .

**Theorem 4.6.4.** [GuL4, Theorem 7.14] *Let  $n \geq 3$  and let  $g \in W_n$ . Then the subgroup  $\text{Out}(W_n, \langle g \rangle)$  of outer automorphisms which preserve  $\langle g \rangle$  up to conjugacy is infinite if and only if  $g$  is contained in a proper  $\mathcal{Z}_{RC}$ -factor of  $W_n$ .*

*More generally, Let  $G$  be a finitely generated Gromov hyperbolic group. If  $\mathcal{H}$  is a finite family of finitely generated subgroups of  $G$ , then the group  $\text{Out}(G, \mathcal{H}^{(t)})$  is infinite if and only if there exists a nontrivial  $\mathcal{Z}_{RC}$ -splitting  $S$  of  $G$  such that every subgroup of  $\mathcal{H}$  fixes a vertex of  $S$ .*  $\square$

#### 4.6.2 Stabilizers of $W_{n-2}$ -stars satisfy $(P_{W_{n-2}})$

**Lemma 4.6.5.** *Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . Let  $\mathcal{S}$  be the equivalence class of a  $W_{n-2}$ -star  $S$ . Let  $e_1$  and  $e_2$  be the two edges of  $W_n \setminus S$  and, for  $i \in \{1, 2\}$ , let  $T'_i$  be the group of twists about  $e_i$  in  $\text{Stab}_\Gamma(\mathcal{S})$ . Let  $i \in \{1, 2\}$ , let  $T_i$  be a finite index subgroup of  $T'_i$  and let  $P'$  be a finite index subgroup of a nontrivial normal subgroup of  $T_i$ . Then for every finite index subgroup  $P_0$  of  $P'$ , the group  $P_0$  fixes exactly one equivalence class of  $W_{n-2}$ -stars.*

**Proof.** Let

$$W_n = \langle x_1 \rangle * \langle x_3, \dots, x_n \rangle * \langle x_2 \rangle$$

be a free factor decomposition associated with  $W_n \setminus S$  and  $A = \langle x_3, \dots, x_n \rangle$ . Up to exchanging the roles of  $e_1$  and  $e_2$ , we may suppose that  $P'$  is contained in the group of twists of the equivalence class of the  $W_{n-1}$ -star  $\mathcal{S}_1$  whose associated free factor decomposition of  $W_n$  is, up to global conjugation:

$$W_n = \langle x_1 \rangle * \langle x_2, x_3, \dots, x_n \rangle.$$

Let  $B = \langle x_2, x_3, \dots, x_n \rangle$  and let  $\mathcal{S}_1$  be the equivalence class of  $S_1$ . Finally, let  $\mathcal{S}_2$  be the equivalence class of the  $W_{n-1}$ -star  $S_2$  whose associated free factor decomposition of  $W_n$  is, up to global conjugation:

$$W_n = \langle x_2 \rangle * \langle x_1, x_3, \dots, x_n \rangle.$$

Let  $C = \langle x_1, x_3, \dots, x_n \rangle = A * \langle x_1 \rangle$ .

We claim that the only equivalence classes of  $W_{n-1}$ -stars fixed by any finite index subgroup of  $P'$  are  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Indeed, fix  $i \in \{1, 2\}$ . The group  $T_i$  is isomorphic to a

finite index subgroup  $N$  of  $W_{n-2}$ . By Proposition 4.6.3 (3) applied with  $K = W_{n-2}$  and  $P = N$ , as  $n \geq 5$ , the group  $N$  is not contained in any proper  $\mathcal{Z}_{RC}$ -free factor of  $W_{n-2}$ . By Proposition 4.6.3 (4), there exists  $g \in N$  such that  $W_{n-2}$  is freely indecomposable relative to  $g$ . Hence there exists  $g \in A$  such that  $A$  is freely indecomposable relative to  $g$  and  $P'$  contains the twist about  $e_1$  whose twistor is  $g$ . Note that this twist can be seen as a twist about the  $W_{n-1}$ -star  $S_1$ . Let  $\mathcal{S}'_1$  be the equivalence class of the one-edge cyclic splitting  $S'_1$  whose associated amalgamated decomposition of  $W_n$  is, up to global conjugation:

$$W_n = (\langle x_1 \rangle * \langle g \rangle) *_{\langle g \rangle} B.$$

Let  $\mathcal{S}_3$  be the equivalence class of a  $W_{n-1}$ -star  $S_3$  fixed by some finite index subgroup of  $P'$  and distinct from  $\mathcal{S}_1$ . Let

$$W_n = \langle y \rangle * D$$

be the free factor decomposition associated with  $S_3$ . We claim that  $\mathcal{S}_3 = \mathcal{S}_2$ . As  $P'$  contains the twist about  $g$ , by Lemma 4.4.7, the splitting  $S_3$  is compatible with  $S'_1$ . Let  $U$  be a two-edge refinement of  $S'_1$  and  $S_3$ . Then  $U$  is obtained from  $S_3$  by blowing-up an edge at vertices whose stabilizers are conjugate to  $D$ . Moreover,  $U$  is obtained from  $S'_1$  by blowing-up an edge at vertices whose stabilizers are conjugate to  $B$  or by blowing-up an edge at the vertices whose stabilizers are conjugate to  $\langle x_1 \rangle * \langle g \rangle$ . But, the second case can only occur when  $\mathcal{S}_3 = \mathcal{S}_1$  (see the claim in the proof of Lemma 4.4.7). Therefore, we may suppose that  $U$  is obtained from  $S'_1$  by blowing up an edge at vertices whose stabilizers are conjugate to  $B$ . Thus, up to applying a global conjugation, we may assume that  $\langle x_1 \rangle * \langle g \rangle \subseteq D$ . But, as  $g$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $A$  and as  $A \cap D$  is a free factor of  $A$ , we see that  $A \cap D = A$ . Hence  $A * \langle x_1 \rangle \subseteq D$ , and, as  $A * \langle x_1 \rangle$  is isomorphic to  $W_{n-1}$ , we have in fact  $A * \langle x_1 \rangle = D$ . It follows that  $C = D$  and, by Lemma 4.4.4, we see that  $\mathcal{S}_2 = \mathcal{S}_3$ . Thus the only equivalence classes of  $W_{n-1}$ -stars fixed by finite index subgroups of  $P'$  are  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

Therefore the only equivalence classes of  $W_{n-2}$ -stars fixed by finite index subgroups of  $P'$  are the equivalence classes of the  $W_{n-2}$ -stars which refine  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . As  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are refined by a unique (up to  $W_n$ -equivariant homeomorphism)  $W_{n-2}$ -star by Theorem 4.3.7, we conclude that  $\mathcal{S}$  is the only equivalence class of  $W_{n-2}$ -star fixed by finite index subgroups of  $P'$ . This completes the proof.  $\square$

**Proposition 4.6.6.** *Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . Let  $\mathcal{S}$  be the equivalence class of a  $W_{n-2}$ -star  $S$ . Then  $\text{Stab}_\Gamma(\mathcal{S})$  satisfies  $(P_{W_{n-2}})$ . Moreover, we can choose for the subgroup  $K_1 \times K_2$  of Property  $(P_{W_{n-2}})$  (1) the direct product of the groups of twists of  $\mathcal{S}$  about the two edges of  $S$ .*

**Proof.** The fact that  $\text{Stab}_\Gamma(\mathcal{S})$  satisfies  $(P_{W_{n-2}})$  (2) follows from the fact that  $\text{Stab}_\Gamma(\mathcal{S})$  contains the stabilizer in  $\Gamma$  of the equivalence class of a  $W_3$ -star obtained from  $S$  by blowing-up  $n - 5$  edges at the center of  $W_n \setminus S$ . Indeed, Proposition 4.2.5 (3) ensures that the group of twists of a  $W_3$ -star is isomorphic to a direct product of  $n - 3$  copies of  $W_3$ .

The fact that  $\text{Stab}_\Gamma(\mathcal{S})$  satisfies  $(P_{W_{n-2}})$  (3) follows from the fact that  $\text{Stab}_\Gamma(\mathcal{S})$  contains the stabilizer in  $\Gamma$  of the equivalence class of a  $W_2$ -star obtained from  $S$  by

blowing-up  $n - 4$  edges at the center of  $W_n \setminus S$ . Indeed the group of twists of a  $W_2$ -star is isomorphic to a direct product of  $n - 2$  copies of  $W_2$  by Proposition 4.2.5 (3).

Let us now prove that  $\text{Stab}_\Gamma(\mathcal{S})$  satisfies  $(P_{W_{n-2}})$  (1). Let  $T'$  be the group of twists of  $\mathcal{S}$  and let  $T = T' \cap \Gamma$ . The group  $T$  is normal in  $\text{Stab}_\Gamma(\mathcal{S})$  since  $\Gamma \subseteq \text{Out}^0(W_n)$ . By Proposition 4.2.5 (3), the group  $T'$  is isomorphic to  $T'_1 \times T'_2$ , where, for  $i \in \{1, 2\}$ ,  $T'_i$  is the group of twists in  $\text{Out}(W_n)$  about one edge of  $W_n \setminus S$ . For  $i \in \{1, 2\}$ , let  $T_i = T'_i \cap \Gamma$ . For every  $i \in \{1, 2\}$ , the group  $T_i$  is a normal subgroup of  $\text{Stab}_\Gamma(\mathcal{S})$  and the group  $T_1 \times T_2$  is a normal subgroup of  $\text{Stab}_\Gamma(\mathcal{S})$ . Let  $T_1^{(2)}$  be a finite index subgroup of  $T_1$  and let  $P'$  be a finite index subgroup of a nontrivial normal subgroup of  $T_1^{(2)}$ . We prove that the centralizer of  $P'$  in  $\Gamma$  contains  $T_2$  as a finite index subgroup. This will conclude the proof of the proposition by symmetry of  $T_1$  and  $T_2$ . By Lemma 4.6.5, the equivalence class  $\mathcal{S}$  is the only equivalence class of  $W_{n-2}$ -star fixed by every finite index subgroup of  $P'$ . Hence  $C_\Gamma(P')$  fixes  $\mathcal{S}$ .

Let  $H$  be a finite index subgroup of  $C_\Gamma(P')$  which fixes  $\mathcal{S}$ . Let

$$W_n = \langle x_1 \rangle * \langle x_3, \dots, x_n \rangle * \langle x_2 \rangle$$

be a free factor decomposition associated with  $W_n \setminus S$  and  $A = \langle x_3, \dots, x_n \rangle$ . By Proposition 4.2.5 (1), the kernel of the natural homomorphism  $H \rightarrow \text{Out}(A)$  is isomorphic to  $H \cap T$ . We claim that the image of  $H$  in  $\text{Out}(A)$  is finite. Indeed, as  $P'$  is a finite index subgroup of a nontrivial normal subgroup of a finite index subgroup of  $T_1$  and as  $T_1$  is isomorphic to a finite index subgroup of  $W_{n-2}$ , we see that  $P'$  is isomorphic to a finite index subgroup  $N$  of a nontrivial normal subgroup of a finite index subgroup of  $W_{n-2}$ . By Proposition 4.6.3 (3),  $N$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_{n-2}$ . By Proposition 4.6.3 (4), there exists  $g \in N$  such that  $g$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $W_{n-2}$ . Thus, there exists  $g \in A$  such that  $g$  is not contained in any proper  $\mathcal{Z}_{RC}$ -factor of  $A$  and the twist about  $g$  is contained in  $P'$ . As  $H$  commutes with the twist about  $g$ , Lemma 4.4.11 implies that  $H$  preserves the conjugacy class of  $g$ . Hence, by Theorem 4.6.4, the image of  $H$  in  $\text{Out}(A)$  is finite.

Thus,  $H \cap T$  has finite index in  $H$  and in  $C_\Gamma(P')$ . But, as  $H$  commutes with  $P' \subseteq T_1$ , and as  $T_1$  is virtually a nonabelian free group, the intersection  $H \cap T_2$  has finite index in  $H \cap T$ , hence has finite index in  $C_\Gamma(P')$ . This completes the proof.  $\square$

### 4.6.3 Groups satisfying $(P_{W_{n-2}})$ and stabilizers of $W_{n-1}$ -stars

We prove in this section that if  $H$  is a subgroup of  $\text{Out}(W_n)$  which satisfies  $(P_{W_{n-2}})$ , then  $H$  virtually fixes the equivalence class of a  $W_{n-1}$ -star. We first recall a general lemma.

**Lemma 4.6.7.** *Let  $G$  be a group and let  $N$  be a finitely generated normal subgroup of  $G$ . Let  $n \in \mathbb{N}^*$ .*

- (1) *There exist only finitely many subgroups of  $N$  of index equal to  $n$ .*
- (2) *For every finite index subgroup  $N'$  of  $N$  there exists a finite index subgroup  $G'$  of  $G$  such that  $N'$  is a normal subgroup of  $G'$ .*

**Proof.** Assertion (1) is well known, we only prove assertion (2). Let  $N'$  be a subgroup of  $N$  of index  $n$  and let  $g \in G$ . As  $N$  is a normal subgroup of  $G$ , the automorphism  $\text{ad}_g: G \rightarrow G$  induces an automorphism  $\text{ad}_g|_N: N \rightarrow N$  by restriction. Therefore,  $\text{ad}_g$  permutes the subgroups of index  $n$  in  $N$ . Since there exists a finite number of subgroups of index  $n$  in  $N$  by the first assertion, we see that there exists a finite index subgroup  $G'$  of  $G$  such that, for every  $g \in G'$ , we have  $\text{ad}_g(N') = N'$ . Therefore  $N'$  is a normal subgroup of  $G'$ . This concludes the proof.  $\square$

**Lemma 4.6.8.** *Let  $n \geq 5$ . Let  $H$  be a subgroup of  $\text{Out}^0(W_n)$  satisfying  $(P_{W_{n-2}})$ . Let  $K_1 \times K_2$  be a normal subgroup of  $H$  given by  $(P_{W_{n-2}})$  (1). Then one of the following holds.*

(1) *For every  $i \in \{1, 2\}$ , the group  $K_i$  does not virtually fix the equivalence class of a free splitting.*

(2) *The group  $H$  virtually fixes the equivalence class of a one-edge free splitting.*

**Proof.** Suppose that there exists  $i \in \{1, 2\}$  such that  $K_i$  virtually fixes the equivalence class of a free splitting. Up to reordering, we may assume that  $i = 1$ . Let  $K'_1$  be a finite index subgroup of  $K_1$  which fixes the equivalence class of a free splitting, and let  $\mathcal{C}$  be the set of all equivalence classes of free splittings fixed by  $K'_1$ . Since  $K_1$  is a finitely generated normal subgroup of  $H$ , by Lemma 4.6.7 (2), there exists a finite index subgroup  $H_0$  of  $H$  such that  $K'_1$  is a normal subgroup of  $H_0$ . In particular, the set  $\mathcal{C}$  is preserved by  $H_0$ .

Suppose first that the set  $\mathcal{C}$  is finite. Then the set  $\mathcal{C}$  is virtually fixed pointwise by  $H_0$ . Hence the group  $H$  virtually fixes the equivalence class of a free splitting.

So we may assume that the set  $\mathcal{C}$  is infinite. Let  $U_{\mathcal{C}}$  be the splitting provided by Theorem 4.5.5, and let  $\mathcal{U}_{\mathcal{C}}$  be its equivalence class. By the equivariance property in Theorem 4.5.5 the equivalence class  $\mathcal{U}_{\mathcal{C}}$  is  $H_0$ -invariant. Suppose first that the splitting  $U_{\mathcal{C}}$  contains an edge  $e \in EU_{\mathcal{C}}$  with trivial stabilizer. Let  $U'$  be the splitting obtained from  $U_{\mathcal{C}}$  by collapsing every edge of  $U_{\mathcal{C}}$  that are not contained in the orbit of  $e$ , and let  $\mathcal{U}'$  be its equivalence class. Then  $\mathcal{U}'$  is the equivalence class of a one-edge free splitting virtually fixed by  $H$ .

Thus, we may assume that all edge stabilizers of  $U_{\mathcal{C}}$  are nontrivial. We show that this leads to a contradiction. Let  $H'$  be the subgroup of finite index in  $H_0$  which acts trivially on  $W_n \setminus U_{\mathcal{C}}$ . We claim that the intersection of  $H'$  with the group of twists of  $U_{\mathcal{C}}$  is finite. Indeed, let  $e$  be an oriented edge of  $U_{\mathcal{C}}$ . As  $W_n$  is virtually free, if the edge stabilizer  $G_e$  of  $e$  is not cyclic, the group of twists about this edge is trivial. Thus, as we suppose that all edge stabilizers are nontrivial, oriented edges with nontrivial group of twists have cyclic stabilizers. But by Lemma 4.2.7 twists about edges with cyclic stabilizers are central in a finite index subgroup of  $\text{Stab}^0(U_{\mathcal{C}})$ . Note that Remark 4.6.1 (2) implies that the center of every finite index subgroup of  $H'$  is finite. Therefore the intersection of  $H'$  with the group of twists is finite. By Remark 4.2.6, the intersection of  $H'$  with the group of bitwists is finite. Thus, up to passing to a finite index subgroup, we may suppose that the map

$$H' \rightarrow \prod_{v \in V(W_n \setminus U_{\mathcal{C}})} \text{Out}(G_v)$$

given by the action on the vertex groups is injective.

Let  $VU_C = V_1 \amalg V_2$  be the partition of  $VU_C$  given by Theorem 4.5.5 and, for every  $i \in \{1, 2\}$ , let  $H_i$  be the subgroup of  $H'$  made of all automorphisms whose image in  $\prod_{v \in W_n \setminus V_i} \text{Out}(G_v)$  is trivial. Then  $H_1$  and  $H_2$  centralize each other and, by Theorem 4.5.5 (1)(b), the group  $H_1 \cap K'_1$  is a finite index subgroup of  $K'_1$ . Thus  $H_2$  centralizes a finite index subgroup of  $K'_1$ . We prove that  $\text{rk}_{\text{prod}}(H_2) \geq 2$ , which will contradict the fact that the centralizer of every finite index subgroup of  $K'_1$  is virtually free.

By Theorem 4.5.5 (2), the set of all conjugacy classes of groups  $G_v$  with  $v \in V_2$  is a free factor system of  $W_n$ . In particular, for every  $v \in V_2$ , there exists  $k_v \in \{0, \dots, n-1\}$  such that  $G_v$  is isomorphic to  $W_{k_v}$ . Suppose first that  $|W_n \setminus V_2| \geq 3$ . In this case, by Theorem 4.5.1 (2) and since  $\text{rk}_{\text{prod}}(\text{Out}(W_3)) = 1$  and  $\text{rk}_{\text{prod}}(\text{Out}(W_2)) = 0$ , for all  $v \in V_2$ , we have  $\text{rk}_{\text{prod}}(\text{Out}(W_{k_v})) \leq k_v - 2$ . Hence

$$\text{rk}_{\text{prod}} \left( \prod_{v \in W_n \setminus V_2} \text{Out}(G_v) \right) \leq n - 6.$$

Since  $\text{rk}_{\text{prod}}(H') = n - 3$ , using Lemma 4.5.2, we see that  $\text{rk}_{\text{prod}}(H_2) \geq 3$ . This leads to a contradiction. Suppose now that  $|W_n \setminus V_2| = 2$  and let  $v_1, v_2 \in W_n \setminus V_2$  be distinct. Then for every  $i \in \{1, 2\}$  there exists  $k_i \in \{1, \dots, n-1\}$  such that  $G_{v_i}$  is isomorphic to  $W_{k_i}$ . If  $W_n = W_{k_1} * W_{k_2}$ , then the group  $H'$  virtually fixes the equivalence class of the one-edge free splitting determined by this free factor decomposition of  $W_n$ . So we may assume that  $W_n \neq W_{k_1} * W_{k_2}$ . This implies that  $k_1 + k_2 \leq n - 1$ . Hence

$$\text{rk}_{\text{prod}} \left( \prod_{v \in W_n \setminus V_2} \text{Out}(G_v) \right) \leq n - 5.$$

Since  $\text{rk}_{\text{prod}}(H') = n - 3$ , using Lemma 4.5.2, we see that  $\text{rk}_{\text{prod}}(H_2) \geq 2$ . This leads to a contradiction. Suppose now that  $|W_n \setminus V_2| = 1$ , and let  $v \in W_n \setminus V_2$ . Then there exists  $k \in \{1, \dots, n-1\}$  such that  $G_v$  is isomorphic to  $W_k$ . Suppose first that  $k \leq n - 2$ . Then by Theorem 4.5.1 (2), and since  $\text{rk}_{\text{prod}}(\text{Out}(W_3)) = 1$ ,  $\text{rk}_{\text{prod}}(\text{Out}(W_1)) = 0$  and  $\text{rk}_{\text{prod}}(\text{Out}(W_2)) = 0$ , if  $n \neq 5$ , we have

$$\text{rk}_{\text{prod}}(\text{Out}(W_k)) \leq n - 5.$$

Thus, by Lemma 4.5.2, we see that  $\text{rk}_{\text{prod}}(H_2) \geq 2$ . When  $n = 5$ , the case where  $k = 3$  and  $\text{rk}_{\text{prod}}(\text{Out}(W_k)) = 1 = n - 4$  can occur. But by Property  $(P_{W_{n-2}})$  (3), the group  $H'$  contains a subgroup isomorphic to  $\mathbb{Z}^3$ . Since  $\text{Out}(W_3)$  is virtually free, the group  $H_2$  contains a subgroup isomorphic to  $\mathbb{Z}^2$ . This contradicts the fact that the centralizer of every finite index subgroup of  $K'_1$  is virtually nonabelian free. Hence we have  $k = n - 1$ . But then, by Lemma 4.4.4, the group  $H'$  (and hence the group  $H$ ) virtually fixes the equivalence class of a  $W_{n-1}$ -star. This concludes the proof.  $\square$



**Lemma 4.6.9.** *Let  $n \geq 5$ . Let  $\mathcal{F}$  be a nonsporadic free factor system. Let  $H$  be a subgroup of  $\text{Out}^0(W_n) \cap \text{Out}(W_n, \mathcal{F})$  containing a direct product of  $n - 3$  nonabelian free groups. Then  $H$  cannot contain a finite index subgroup which fixes the homothety class of a  $(W_n, \mathcal{F})$ -arational tree.*

**Proof.** Suppose towards a contradiction that  $H$  has a finite index subgroup which fixes the equivalence class of a  $(W_n, \mathcal{F})$ -arational tree. Up to passing to a finite index subgroup, we may suppose that  $H$  itself fixes the homothety class of a  $(W_n, \mathcal{F})$ -arational tree. By Lemma 4.5.4, there exists a homomorphism from  $H$  to  $\mathbb{Z}$  whose kernel  $K'$  is exactly the isometric stabilizer of a  $(W_n, \mathcal{F})$ -arational tree. Note that  $K'$  contains a direct product of  $n - 3$  nonabelian free groups as it is the kernel of a homomorphism from  $H$  to  $\mathbb{Z}$ . By Proposition 4.5.6, there exists a finite index subgroup  $K$  of  $K'$  such that  $K$  fixes infinitely many equivalence classes of free splittings. Let  $\mathcal{C}$  be the collection of all equivalence classes of free splittings fixed by  $K$ .

We claim that  $\mathcal{C}$  is in fact finite, which will lead to a contradiction. Since  $K \subseteq \text{Out}^0(W_n)$ , Lemma 4.4.2 implies that if  $\mathcal{S}$  is the equivalence class of a free splitting  $S$  fixed by  $K$ , then the group  $K$  fixes the equivalence class of every one-edge free splitting onto which  $S$  collapses. By Theorem 4.3.7, if  $\mathcal{S}$  is the equivalence class of a free splitting  $S$ , then  $\mathcal{S}$  is determined by the finite set of equivalence classes of one-edge free splittings onto which  $S$  collapses. Therefore, it suffices to show that  $K$  can only fix finitely many equivalence classes of one-edge free splittings. Let  $\mathcal{S}$  be the equivalence class of a one-edge free splitting fixed by  $K$ . Since  $K$  contains a direct product of  $n - 3$  nonabelian free groups, Theorem 4.5.1 (3) implies that  $\mathcal{S}$  is a  $W_{n-1}$ -star. Let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be a free factor decomposition associated with  $\mathcal{S}$  and let  $A = \langle x_1, \dots, x_{n-1} \rangle$ . By Proposition 4.2.5 (1), the kernel of the natural homomorphism  $K \rightarrow \text{Out}(A)$  is the intersection of  $K$  with the group of twists  $T$  of  $\mathcal{S}$ . By Theorem 4.5.1 (2), the product rank of  $\text{Out}(A)$  is equal to  $n - 4$ . Since  $K$  contains a direct product of  $n - 3$  nonabelian free groups, we see that  $K \cap T$  is infinite. Therefore, for every equivalence class  $\mathcal{S}$  of a  $W_{n-1}$ -star  $S$  fixed by  $K$ , the group  $K$  contains an infinite twist about  $\mathcal{S}$ .

Let  $\mathcal{S}$  and  $\mathcal{S}'$  be two distinct equivalence classes of  $W_{n-1}$ -stars fixed by  $K$ . Let  $S$  be a representative of  $\mathcal{S}$  and let  $S'$  be a representative of  $\mathcal{S}'$ . We claim that  $S$  and  $S'$  are compatible. Indeed, by the above, there exists  $f \in K$  of infinite order such that  $f$  is a twist about  $\mathcal{S}$ . Since  $f$  fixes  $\mathcal{S}'$ , Lemma 4.4.7 implies that  $S$  and  $S'$  are compatible. Therefore, for every distinct equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  of one-edge free splittings fixed by  $K$ , there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  are compatible. By Theorem 4.3.7, this is only possible when  $\mathcal{C}$  is finite. This leads to a contradiction since  $K$  must fix infinitely many equivalence classes of free splittings. This concludes the proof.  $\square$

**Proposition 4.6.10.** *Let  $n \geq 5$ . Let  $H$  be a subgroup of  $\text{Out}^0(W_n)$  satisfying  $(P_{W_{n-2}})$ . Then  $H$  virtually fixes the equivalence class of a  $W_{n-1}$ -star.*

**Proof.** The proof is inspired by [HW2, Proposition 8.2] and [HHW, Proposition 6.5]. We prove that  $H$  virtually fixes the equivalence class of a one-edge free splitting. Since  $H$  contains a direct product of  $n - 3$  nonabelian free groups, we will then conclude by Theorem 4.5.1 (3). Suppose towards a contradiction that  $H$  does not virtually fix the equivalence class of a one-edge free splitting. Let  $\mathcal{F}$  be a maximal  $H$ -periodic free factor system. We can assume that  $\mathcal{F}$  is nonsporadic otherwise  $H$  virtually fixes the equivalence class of a one-edge free splitting and we are done. As  $\mathcal{F}$  is maximal, by Proposition 4.2.2, the group  $H$  acts with unbounded orbits on  $\text{FF}(W_n, \mathcal{F})$ .

Let  $K_1 \times K_2$  be a normal subgroup of  $H$  given by  $(P_{W_{n-2}})$  (1). Suppose first that neither  $K_1$  nor  $K_2$  contains a loxodromic element on  $\text{FF}(W_n, \mathcal{F})$ . As  $H$  has unbounded orbits on  $\text{FF}(W_n, \mathcal{F})$ , Lemma 4.5.3 implies that  $K_1 \times K_2$  has a finite orbit in  $\partial_\infty \text{FF}(W_n, \mathcal{F})$ .

By Lemma 4.2.4, there exists a finite index subgroup  $K'_1 \times K'_2$  of  $K_1 \times K_2$  such that  $K'_1 \times K'_2$  fixes the homothety class of an arational  $(W_n, \mathcal{F})$ -tree  $T$ . Since  $K_1 \times K_2$  does not contain a loxodromic element,  $K'_1 \times K'_2$  fixes  $T$  up to isometry, not just homothety (see e.g. [GuH2, Proposition 6.2]). By Proposition 4.5.6, the group  $K'_1 \times K'_2$  virtually fixes infinitely many equivalence classes of  $(W_n, \mathcal{F})$ -free splittings. By Lemma 4.6.8, the group  $H$  virtually fixes the equivalence class of a one-edge free splitting of  $W_n$ .

So we may suppose that there exists a loxodromic element  $\Phi \in K_1 \times K_2$ . First suppose that there exists a unique  $i \in \{1, 2\}$  such that the group  $K_i$  contains a loxodromic element  $\Phi_i$ . We may assume, up to reordering, that only  $K_2$  contains a loxodromic element  $\Phi$ . Therefore by Lemma 4.5.3, the group  $K_1$  virtually fixes a point in  $\partial_\infty \text{FF}(W_n, \mathcal{F})$ . By Lemma 4.2.4, the group  $K_1$  virtually fixes the homothety class an arational  $(W_n, \mathcal{F})$ -tree  $T$ . Let  $K'_1$  be a normal subgroup of  $K_1$  of finite index that is contained in  $\text{Stab}([T])$ . As  $K'_1$  does not contain any loxodromic element, as in the above step,  $K'_1$  fixes  $T$  up to isometry. By Proposition 4.5.6, the group  $K'_1$  fixes the equivalence class of a free splitting relative to  $\mathcal{F}$ . By Lemma 4.6.8, the group  $H$  virtually fixes the equivalence class of a one-edge free splitting of  $W_n$ .

Now suppose that for every  $i \in \{1, 2\}$ , the group  $K_i$  contains a loxodromic element. By Lemma 4.5.3, the whole group  $H$  virtually fixes a point in  $\partial_\infty \text{FF}(W_n, \mathcal{F})$ . By Lemma 4.2.4, the group  $H$  virtually fixes the homothety class of an arational tree. This contradicts Lemma 4.6.9.

Therefore, in all cases, the group  $H$  virtually fixes the equivalence class  $\mathcal{S}$  of a one-edge free splitting  $S$ . By Theorem 4.5.1 (3), since  $H$  contains a direct product of  $n - 3$  nonabelian free groups, the group  $H$  virtually fixes the equivalence class of a  $W_{n-1}$ -star.  $\square$

We now prove a proposition which gives a sufficient condition for equivalence classes of  $W_{n-1}$ -stars provided by Proposition 4.6.10 to be compatible. We first need the following result due to Krstić and Vogtmann.

**Proposition 4.6.11.** *[KV, Corollary 10.2] Let  $n \geq 3$ . The virtual cohomological dimension of  $\text{Out}(W_n)$  is equal to  $n - 2$ . In particular, the maximal rank of a free abelian subgroup of  $\text{Out}(W_n)$  is equal to  $n - 2$ .*

**Proposition 4.6.12.** *Let  $n \geq 5$  and let  $\Gamma$  be a subgroup of  $\text{Out}^0(W_n)$  of finite index. Let  $k \in \mathbb{N}^*$  and let  $H_1, \dots, H_k$  be subgroups of  $\Gamma$  which satisfy  $(P_{W_{n-2}})$  and such that the intersection  $\bigcap_{i=1}^k H_i$  contains a subgroup  $H$  isomorphic to  $\mathbb{Z}^{n-2}$ . For  $i \in \{1, \dots, k\}$ , let  $\mathcal{S}_i$  be the equivalence class of a  $W_{n-1}$ -star  $S_i$  which is virtually fixed by  $H_i$ . Then, for every  $i, j \in \{1, \dots, k\}$ , the  $W_{n-1}$ -stars  $S_i$  and  $S_j$  are compatible.*

**Proof.** Let  $i, j \in \{1, \dots, k\}$  be distinct integers. Let  $H'$  be a finite index subgroup of  $H$  contained in  $\text{Stab}_\Gamma(\mathcal{S}_i) \cap \text{Stab}_\Gamma(\mathcal{S}_j)$ . Let  $A_i$  and  $A_j$  be the vertex groups isomorphic to  $W_{n-1}$  of respectively  $W_n \setminus \mathcal{S}_i$  and  $W_n \setminus \mathcal{S}_j$  (well defined up to conjugation). By Proposition 4.6.11, the rank of a maximal abelian subgroup of  $\text{Out}(W_{n-1})$  is equal to  $n - 3$ . Therefore, the kernel of the homomorphisms  $H' \rightarrow \text{Out}(A_i)$  and  $H' \rightarrow \text{Out}(A_j)$  given by the action on the vertex group contains an element of infinite order. Let  $f_i \in \ker(H' \rightarrow \text{Out}(A_i))$  and  $f_j \in \ker(H' \rightarrow \text{Out}(A_j))$  be infinite order elements. By Proposition 4.2.5 (1),  $f_i$  and  $f_j$  are twists about respectively  $S_i$  and  $S_j$ . As  $f_i$  and  $f_j$  commute, by Corollary 4.4.9,  $S_i$  and  $S_j$  are compatible. This concludes the proof.  $\square$

## 4.7 Algebraic characterization of stabilizers of $W_{n-2}$ -stars

In this section, we give an algebraic characterization of stabilizers of  $W_{n-2}$ -stars. By the previous section, we know that groups which satisfy  $(P_{W_{n-2}})$  virtually stabilize equivalence classes of  $W_{n-1}$ -stars, and we have given an algebraic criterion to show that these  $W_{n-1}$ -stars are compatible. In order to prove that a group  $H$  which satisfies  $(P_{W_{n-2}})$  virtually stabilizes the equivalence class of a  $W_{n-2}$ -star, we study the intersection of a normal subgroup  $K_1 \times K_2$  of  $H$  given by  $(P_{W_{n-2}})$  (1) with the group of twists of the equivalence class of a  $W_{n-1}$ -star virtually fixed by  $H$ .

### 4.7.1 Groups of twists in groups satisfying $(P_{W_{n-2}})$

We start this section with a lemma which gives a sufficient condition for a group  $H$  satisfying  $(P_{W_{n-2}})$  to be the stabilizer of a  $W_{n-2}$ -star.

**Lemma 4.7.1.** *Let  $n \geq 5$  and let  $\Gamma$  be a subgroup of finite index of  $\text{Out}^0(W_n)$ . Let  $H$  be a subgroup of  $\Gamma$  which satisfies  $(P_{W_{n-2}})$  and let  $K_1 \times K_2$  be a normal subgroup of  $H$  given by  $(P_{W_{n-2}})$  (1). Let  $\mathcal{S}_1$  be the equivalence class of a  $W_{n-1}$ -star  $S_1$  virtually fixed by  $H$  and let  $T_1$  be the group of twists of  $\mathcal{S}_1$ .*

*Suppose that  $T_1 \cap K_1$  is infinite and that there exists an equivalence class  $\mathcal{S}_2$  of a  $W_{n-1}$ -star  $S_2$  such that the intersection of  $K_2$  with the group of twists  $T_2$  of  $\mathcal{S}_2$  is infinite. Then  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are compatible and  $H$  virtually fixes the equivalence class  $\mathcal{S}$  of the  $W_{n-2}$ -star which refines  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Moreover,  $\mathcal{S}$  is the unique equivalence class of a  $W_{n-2}$ -star virtually fixed by  $H$ . Finally, the groups  $T_1 \cap \text{Stab}_\Gamma(\mathcal{S})$  and  $K_1$  (resp.  $T_2 \cap \text{Stab}_\Gamma(\mathcal{S})$  and  $K_2$ ) are commensurable.*

**Proof.** For  $i \in \{1, 2\}$ , let  $f_i \in T_i \cap K_i$  be of infinite order. First remark that, as  $f_1$  and  $f_2$  generate a free abelian group of order 2, we have  $T_1 \neq T_2$  because the group of twists

of a  $W_{n-1}$ -star is virtually a nonabelian free group. Hence we have  $\mathcal{S}_1 \neq \mathcal{S}_2$ . As  $K_1$  commutes with  $f_2$ , Proposition 4.4.8 shows that  $K_1$  fixes  $\mathcal{S}_2$ . As  $K_1$  contains a twist of  $\mathcal{S}_1$ , Lemma 4.4.7 shows that  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are compatible.

Let  $S$  be a  $W_{n-2}$ -star which refines  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , let  $\mathcal{S}$  be its equivalence class and let  $T$  be the group of twists of  $\mathcal{S}$  in  $\Gamma$ . Then  $T$  contains a finite index normal subgroup isomorphic to  $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$ , where  $K_1^{\mathcal{S}_1}$  and  $K_2^{\mathcal{S}_2}$  are virtually nonabelian free groups. By Proposition 4.6.6, we can choose  $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$  such that  $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$  is a group satisfying Property  $(P_{W_{n-2}})$  (1). Moreover, up to reordering,  $K_1^{\mathcal{S}_1} \subseteq T_1$  and  $K_2^{\mathcal{S}_2} \subseteq T_2$ . Since  $K_1$  fixes both  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , we see that  $K_1$  fixes  $\mathcal{S}$ . Therefore, by Proposition 4.2.5 (1), we have a homomorphism  $\Phi: K_1 \rightarrow \text{Out}(W_{n-2})$  whose kernel is exactly  $K_1 \cap T$ . By Lemma 4.4.10, we see that  $T_1 \cap \text{Stab}_\Gamma(\mathcal{S}) \cap K_1^{\mathcal{S}_1}$  is a finite index subgroup of  $T_1 \cap \text{Stab}_\Gamma(\mathcal{S})$ . As  $K_1 \cap T_1$  is infinite, so is  $K_1 \cap K_1^{\mathcal{S}_1}$ . Let  $P = \ker(\Phi) \cap K_1^{\mathcal{S}_1} = K_1 \cap K_1^{\mathcal{S}_1}$ . Then, since  $K_1 \subseteq \text{Out}^0(W_n)$ , the group  $K_1^{\mathcal{S}_1} \cap K_1$  is a normal subgroup of  $K_1$ . Therefore  $P$  is a nontrivial normal subgroup of  $K_1$ . By Property  $(P_{W_{n-2}})$  (1), we see that  $K_2$  is a finite index subgroup of  $C_\Gamma(P)$ . But  $P$  is centralized by  $K_2^{\mathcal{S}_2}$  since  $P \subseteq K_1^{\mathcal{S}_1}$ . Hence  $K_2^{\mathcal{S}_2} \cap K_2$  is a finite index subgroup of  $K_2^{\mathcal{S}_2}$ . As  $K_1^{\mathcal{S}_1}$  is a finite index subgroup of the centralizer of  $K_2^{\mathcal{S}_2}$  by Property  $(P_{W_{n-2}})$  (1), and as  $K_1$  is a finite index subgroup of the centralizer of  $K_2$ , we see that  $K_1^{\mathcal{S}_1} \cap K_1$  has finite index in  $K_1$  and therefore  $P$  has finite index in  $K_1$ . Let

$$W_n = \langle x_1 \rangle * \langle x_3, \dots, x_n \rangle * \langle x_2 \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S$  and let  $A = \langle x_3, \dots, x_n \rangle$ . Then, up to reordering, for every  $f \in P$ , there exists  $z_f \in A$  and a representative  $F$  of  $f$  such that  $F$  sends  $x_1$  to  $z_f x_1 z_f^{-1}$ , and, for every  $i \neq 1$ , fixes  $x_i$ .

**Claim.** The only equivalence classes of  $W_{n-1}$ -stars which are virtually fixed by  $K_1$  are  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

**Proof.** Let  $\mathcal{S}_3$  be the equivalence class of a  $W_{n-1}$ -star  $S_3$  virtually fixed by  $K_1$ . Suppose towards a contradiction that  $\mathcal{S}_3$  is distinct from both  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Let  $K'_1 = K_1 \cap \text{Stab}_\Gamma(\mathcal{S}_3)$  and  $P' = P \cap \text{Stab}_\Gamma(\mathcal{S}_3)$ . Then, as  $P$  is an infinite subgroup of the group of twists of  $\mathcal{S}_1$ , and as  $P'$  is a finite index subgroup of  $P$ , we see that  $P'$  is an infinite subgroup of the group of twists of  $\mathcal{S}_1$ . By Lemma 4.4.7, we see that  $\mathcal{S}_1$  and  $\mathcal{S}_3$  are compatible. Let  $S'$  be a  $W_{n-2}$ -star that refines  $\mathcal{S}_1$  and  $\mathcal{S}_3$  and let  $\mathcal{S}'$  be its equivalence class. Let

$$W_n = \langle y_1 \rangle * \langle y_3, \dots, y_n \rangle * \langle y_2 \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S'$  and let  $B = \langle y_3, \dots, y_n \rangle$ . Since  $S$  is a refinement of  $\mathcal{S}_1$ , we may suppose that  $B * \langle y_2 \rangle = A * \langle x_2 \rangle$  and that  $y_1$  is a conjugate of  $x_1$  by an element of  $B * \langle y_2 \rangle$ . Up to applying a global conjugation, we may also suppose that  $y_1 = x_1$  and that  $B * \langle y_2 \rangle = A * \langle x_2 \rangle$ .

Let  $T'$  be the group of twists of  $\mathcal{S}'$ . Then  $T'$  contains a finite index normal subgroup isomorphic to  $P'_1 \times P'_2$ , where both  $P'_1$  and  $P'_2$  are virtually nonabelian free subgroups of  $T'$  which correspond to the groups of twists about the two edges of  $W_n \setminus S'$ . Then, as  $P'$  is a group of twists of  $\mathcal{S}_1$ , and as  $P'$  fixes  $\mathcal{S}'$ , by Lemma 4.4.10, up to reordering, the group  $P'$  is contained in  $P'_1$ .

Let  $f' \in P'_1$ , let  $F'$  be the representative of  $f'$  which acts as the identity on  $B * \langle y_2 \rangle$  and let  $z_{f'} \in B$  be the twistor of  $F'$ . Then  $F'$  acts as the identity on  $A * \langle x_2 \rangle$  and  $F'(x_1) = z_{f'} x_1 z_{f'}^{-1}$ . Recall that for every  $\psi \in P'$ , there exists a unique  $z_\psi \in A$  and a unique representative  $\Psi$  of  $\psi$  such that  $\Psi$  sends  $x_1$  to  $z_\psi x_1 z_\psi^{-1}$ , and, for every  $i \neq 1$ , fixes  $x_i$ . Thus, a necessary condition for  $f'$  to be in  $P'$  is that  $z_{f'} \in A \cap B$ .

But as  $A$  and  $B$  are free factors of  $W_n$ , the group  $A \cap B$  is a free factor of  $B$ . To see this, let  $U$  be a free splitting of  $W_n$  such that  $A$  is a vertex stabilizer of  $U$  and let  $U_B$  be the minimal subtree of  $B$  in  $U$ . Then, as  $U$  is a free splitting of  $W_n$ , we see that  $U_B$  is a free splitting of  $B$ . But then, as  $A$  is a vertex stabilizer in  $U$ , we see that  $A \cap B$  is a vertex stabilizer in  $U_B$ . Therefore,  $A \cap B$  is a free factor of  $B$ . Thus one can find a  $W_{n-3}$ -star  $S^{(2)}$  which refines  $S'$  and such that, for every  $f' \in P'$ , the twistor  $z_{f'}$  fixes a vertex of  $S^{(2)}$ . Indeed, one can equivariantly blow-up an edge  $e$  at the vertex of  $S'$  whose stabilizer is  $B$  such that the stabilizer of one of the endpoints of  $e$  is a subgroup  $C$  isomorphic to  $W_{n-3}$  with  $A \cap B \subseteq C$ . Therefore we may also assume that  $S^{(2)}$  is a  $W_{n-3}$ -star. Let  $\mathcal{S}^{(2)}$  be the equivalence class of  $S^{(2)}$ . By Proposition 4.2.5 (3), the group of twists of  $\mathcal{S}^{(2)}$  is isomorphic to a direct product  $W_{n-3}^3$  of three infinite groups, where each factor is a group of twists about an edge of  $W_n \setminus \mathcal{S}^{(2)}$ . This implies that  $P'$  is contained in exactly one of the three factors isomorphic to  $W_{n-3}$ . It follows that the centralizer of  $P'$  contains two elements which generates a free abelian group of order 2. This contradicts the fact that the centralizer of  $P'$  is virtually a nonabelian free group by  $(P_{W_{n-2}})$  (1). The claim follows.  $\square$

The claim above then implies, as  $K_1$  is a normal subgroup of  $H$ , that  $H$  virtually fixes  $\mathcal{S}_2$ . As  $H$  virtually fixes  $\mathcal{S}_1$ , we see that  $H$  virtually fixes the equivalence class  $\mathcal{S}$ . Moreover, the above claim shows that  $\mathcal{S}$  is the unique equivalence class of a  $W_{n-2}$ -star virtually fixed by  $K_1$ , and hence virtually fixed by  $H$ .

We finally prove that  $K_1$  and  $T_1 \cap \text{Stab}_\Gamma(\mathcal{S})$  (resp.  $K_2$  and  $T_2 \cap \text{Stab}_\Gamma(\mathcal{S})$ ) are commensurable. By Lemma 4.4.10, for every  $i \in \{1, 2\}$  we see that  $K_i^{\mathcal{S}_i} \cap T_i \cap \text{Stab}_\Gamma(\mathcal{S})$  is a finite index subgroup of  $T_i \cap \text{Stab}_\Gamma(\mathcal{S})$ . Moreover, for every  $i \in \{1, 2\}$  and every  $f \in K_i^{\mathcal{S}_i}$ , the twist  $f$  of  $\mathcal{S}$  is also a twist of  $\mathcal{S}_i$ . Hence we have  $K_i^{\mathcal{S}_i} \subseteq T_i \cap \text{Stab}_\Gamma(\mathcal{S})$ . Therefore, for every  $i \in \{1, 2\}$ , the groups  $K_i^{\mathcal{S}_i}$  and  $T_i \cap \text{Stab}_\Gamma(\mathcal{S})$  are commensurable. Hence it suffices to show that, for every  $i \in \{1, 2\}$ , the groups  $K_i$  and  $K_i^{\mathcal{S}_i}$  are commensurable.

Recall that  $K_2^{\mathcal{S}_2} \cap K_2$  is a finite index subgroup of  $K_2^{\mathcal{S}_2}$  and that  $K_1^{\mathcal{S}_1} \cap K_1$  has finite index in  $K_1$ . Since  $H$  virtually fixes  $\mathcal{S}$ , and since  $K_2^{\mathcal{S}_2}$  is a normal subgroup of  $\text{Stab}_\Gamma(\mathcal{S})$ , we see that  $K_2^{\mathcal{S}_2} \cap K_2$  is a normal subgroup of a finite index subgroup of  $K_2$ . We know that  $K_2^{\mathcal{S}_2} \cap K_2$  commutes with  $K_1^{\mathcal{S}_1}$  because  $K_1^{\mathcal{S}_1}$  and  $K_2^{\mathcal{S}_2}$  commute with each other. Thus, by Property  $(P_{W_{n-2}})$  (1) applied to  $K_1 \times K_2$ , the centralizer of  $K_2^{\mathcal{S}_2} \cap K_2$  contains  $K_1$  as a finite index subgroup. This shows that  $K_1 \cap K_1^{\mathcal{S}_1}$  is a finite index subgroup of  $K_1^{\mathcal{S}_1}$ . Hence  $K_1$  and  $K_1^{\mathcal{S}_1}$  are commensurable. By Property  $(P_{W_{n-2}})$  (1) applied to  $K_1^{\mathcal{S}_1} \times K_2^{\mathcal{S}_2}$ , the centralizer of a finite index subgroup of  $K_1^{\mathcal{S}_1}$  contains  $K_2^{\mathcal{S}_2}$  as a finite index subgroup. Moreover, the centralizer of a finite index subgroup of  $K_1$  contains  $K_2$  as a finite index subgroup. Hence the centralizer of  $K_1 \cap K_1^{\mathcal{S}_1}$  contains both  $K_2$  and  $K_2^{\mathcal{S}_2}$  as finite index subgroups. Thus  $K_2$  and  $K_2^{\mathcal{S}_2}$  are commensurable. This completes the

proof of Lemma 4.7.1. □

Lemma 4.7.1 suggests that in order to show that a group  $H$  which satisfies  $(P_{W_{n-2}})$  is in fact virtually the stabilizer of the equivalence class of a  $W_{n-2}$ -star, it suffices to study the intersection of  $H$  with groups of twists. A first step towards such a result is the following lemma.

**Lemma 4.7.2.** *Let  $n \geq 5$  and let  $\Gamma$  be a subgroup of  $\text{Out}^0(W_n)$  of finite index. Let  $H$  be a subgroup of  $\Gamma$  satisfying  $(P_{W_{n-2}})$  and let  $K_1 \times K_2$  be a normal subgroup of  $H$  given by  $(P_{W_{n-2}})$  (1). Let  $\mathcal{S}$  be the equivalence class of a  $W_{n-1}$ -star  $S$  virtually fixed by  $H$  and let  $T$  be the group of twists of  $\mathcal{S}$  contained in  $\Gamma$ .*

*There exists a unique  $i \in \{1, 2\}$  such that  $K_i \cap T$  is infinite. Moreover,  $H \cap T \cap K_i$  has finite index in  $H \cap T$ .*

**Proof.** Up to passing to a finite index subgroup of  $H$ , we may suppose that  $H$  fixes  $\mathcal{S}$ . The uniqueness assertion follows from the fact that  $T$  is virtually a nonabelian free group and that  $K_1 \times K_2$  is a direct product. Therefore, up to reordering, we may suppose that  $K_1 \cap T$  is finite. Since  $\text{Out}(W_n)$  is virtually torsion free by [GuL1, Corollary 5.3], there exists a finite index subgroup  $K'_1$  of  $K_1$  such that  $K'_1 \cap T$  is trivial. Since  $K_1$  is a finitely generated normal subgroup of  $H$ , Lemma 4.6.7 implies that there exists a finite index subgroup  $H'$  of  $H$  such that  $K'_1$  is a normal subgroup of  $H'$ . Therefore, we may suppose that  $K_1 \cap T$  is trivial. By Proposition 4.2.5 (1), the natural homomorphism  $K_1 \rightarrow \text{Out}(W_{n-1})$  given by the action on the vertex groups is injective.

We claim that  $H \cap T$  is infinite. Indeed, consider the natural homomorphism  $\Phi: H \rightarrow \text{Out}(W_{n-1})$ . By Proposition 4.6.11, the rank of a maximal free abelian subgroup of  $\text{Out}(W_{n-1})$  is equal to  $n - 3$ . As  $H$  contains a subgroup isomorphic to  $\mathbb{Z}^{n-2}$  by  $(P_{W_{n-2}})$  (3), the kernel of  $H \rightarrow \text{Out}(W_{n-1})$  is infinite. But, by Proposition 4.2.5 (1), this is precisely  $H \cap T$ . Therefore,  $H \cap T$  is infinite.

We now prove that  $H \cap T \cap K_2$  has finite index in  $H \cap T$ . This will conclude the proof as  $H \cap T$  is infinite. Let  $K = \Phi^{-1}(\Phi(K_2))$ . Note that  $H \cap T \subseteq K$ . Then, as  $K_2$  is normal in  $H$ , we see that  $K$  is a normal subgroup of  $H$  which contains  $H \cap T$  and  $K_2$ . We claim that  $K \cap K_1$  is finite. Indeed, suppose towards a contradiction that there exists  $f \in K \cap K_1$  of infinite order. Then, as the homomorphism

$$\Phi|_{K_1}: K_1 \rightarrow \text{Out}(W_{n-1})$$

is injective, the element  $\Phi(f)$  has infinite order. By definition of  $K$ , we see that  $\Phi(f) \in \Phi(K_1) \cap \Phi(K_2)$ . But, as the homomorphism  $\Phi|_{K_1}: K_1 \rightarrow \text{Out}(W_{n-1})$  is injective, and as  $K_1$  is virtually a nonabelian free group, there exists  $g \in K_1$  of infinite order such that  $\Phi(g)$  does not commute with  $\Phi(f)$ . Since  $\Phi(f) \in \Phi(K_2)$  this contradicts the fact that  $K_1$  and  $K_2$  commute with each other. Hence  $K \cap K_1$  is finite.

The groups  $K$  and  $K_1$  are two normal subgroups of  $H$  with finite intersection. Let  $K_1^{(2)}$  be a finite index normal subgroup of  $K_1$  such that  $K \cap K_1^{(2)} = \{1\}$ . Since  $K_1$  is finitely generated, by Lemma 4.6.7 (2), there exists a finite index subgroup  $H^{(2)}$  of  $H$  such that  $K_1^{(2)}$  is a normal subgroup of  $H^{(2)}$ . Hence  $K_1^{(2)}$  and  $K \cap H^{(2)}$  are normal

subgroups of  $H^{(2)}$  with trivial intersection. Therefore,  $K \cap H^{(2)} \subseteq C_\Gamma(K_1^{(2)})$ . But, Property  $(P_{W_{n-2}})$  (1) implies that  $K$  and  $K_2$  are commensurable. Since  $K$  contains  $H \cap T$ , we see that  $K_2 \cap T$  and  $H \cap T$  are commensurable. This concludes the proof.  $\square$

#### 4.7.2 Groups satisfying $(P_{W_{n-2}})$ and stabilizers of $W_{n-2}$ -stars

In this section we prove that a subgroup of  $\text{Out}^0(W_n)$  which satisfies  $(P_{W_{n-2}})$  virtually fixes the equivalence class of a  $W_{n-2}$ -star. We first prove a series of properties for elements of  $\text{Out}(W_n)$ .

**Lemma 4.7.3.** *Let  $n \geq 3$ . Let  $w \in W_n$  be a root-closed element of infinite order. Let  $\mathcal{S}$  be the equivalence class of a splitting  $S$  whose associated amalgamated decomposition of  $W_n$  is:*

$$W_n = A *_{\langle w \rangle} B,$$

where  $A$  and  $B$  are subgroups of  $W_n$  containing  $w$ . Let  $D$  be a nontrivial twist about  $S$ . Let  $h \in W_n$ . Then  $D$  preserves the conjugacy class of  $h$  if and only if there exists  $h' \in W_n$  such that  $h' \in [h]$  and  $h' \in A \cup B$ .

**Proof.** It is clear that  $D$  preserves the conjugacy classes of elements in  $A$  and  $B$ . Conversely, let  $h \in W_n$  be such that  $D([h]) = [h]$ . Let  $R$  be a Grushko splitting of  $W_n$ . Let  $R'$  and  $S'$  be metric representatives of  $R$  and  $S$ , let  $\mathcal{R}'$  and  $\mathcal{S}'$  be their  $W_n$ -equivariant isometry classes and let  $[\mathcal{R}']$  and  $[\mathcal{S}']$  be their homothety classes. As  $\mathbb{P}\overline{\mathcal{O}(W_n)}$  is compact, up to passing to a subsequence, there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$  and a  $W_n$ -equivariant isometry class  $\mathcal{T}$  of an  $\mathbb{R}$ -tree  $T$  such that

$$\lim_{n \rightarrow \infty} \lambda_n D^n(\mathcal{R}') = \mathcal{T}.$$

Since translation length functions are continuous for the Gromov-Hausdorff topology (see [Paul]), for every  $g \in W_n$ , we have:

$$\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \|g\|_{\mathcal{T}},$$

where  $\|g\|_{\mathcal{T}}$  is the translation length of  $g$  in  $T$ . Hence, for every  $g \in W_n$ , the limit  $\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')}$  is finite. But as there exists  $g' \in W_n$  such that  $\|g'\|_{D^n(\mathcal{R}')}$  tends to infinity as  $n$  goes to infinity, we have  $\lim_{n \rightarrow \infty} \lambda_n = 0$ . As there exists a representative  $\phi \in \text{Aut}(W_n)$  of  $D$  such that  $\phi_A = \text{id}_A$ , for every  $g \in A$ , we have:

$$\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \lim_{n \rightarrow \infty} \lambda_n \|g\|_{\mathcal{R}'} = 0.$$

Hence every element of  $A$  fixes a point in  $T$ . As  $A$  is finitely generated, this implies that  $A$  fixes a point in  $T$  (see for instance [CM, Section 3]). Similarly, we see that the groups  $B$  and  $\langle h \rangle$  fix points in  $T$ . As  $W_n = \langle A, B \rangle$ , we see that  $A$  and  $B$  cannot fix the same point in  $T$ . Thus, there exists a natural  $W_n$ -equivariant application  $\Psi: S' \rightarrow T$ . Let us prove that  $\Psi$  is an isometry. It suffices to prove that  $\Psi$  is a local isometry, that is, it suffices to prove that the application  $\Psi$  does not fold edges. By  $W_n$ -equivariance

and symmetry, it suffices to prove that, If  $e$  and  $e'$  are two distinct edges of  $S'$  whose origin is the vertex fixed by  $A$ , then  $\Psi(e) \neq \Psi(e')$ . Suppose towards a contradiction that  $\Psi(e) = \Psi(e')$ . Then  $\langle G_e, G_{e'} \rangle$  fixes  $\Psi(e)$ . Note that  $G_e$  and  $G_{e'}$  are isomorphic to  $\mathbb{Z}$ . Moreover, since  $w$  is root-closed, we neither have  $G_e \subseteq G_{e'}$  nor  $G_{e'} \subseteq G_e$ . Since  $G_e$  is a malnormal subgroup of  $W_n$  and since  $G_{e'}$  is a nontrivial conjugate of  $G_e$ , we see that  $G_e \cap G_{e'} = \{1\}$ . Hence  $\langle G_e, G_{e'} \rangle$  is a nonabelian free group which fixes an arc in  $T$ . But arc stabilizers in  $T$  are cyclic, a contradiction. Hence  $\Psi$  is an isometric embedding and, by minimality of  $T$ , the application  $\Psi$  is a  $W_n$ -equivariant isometry. Therefore, as  $h$  fixes a point in  $T$ , it also fixes a point in  $S'$ . Therefore,  $h$  is contained in a conjugate of  $A$  or  $B$ .  $\square$

For the next proposition, recall the definition of the subgroup  $\mathbb{F}$  of  $W_n$  from Lemma 4.6.2.

**Proposition 4.7.4.** *Let  $n \geq 3$ . Let  $(H_N)_{N \in \mathbb{N}^*}$  be an increasing sequence of subgroups of  $\mathbb{F}$ . There exists an integer  $n_0$  such that for every  $N \geq n_0$ , we have*

$$\text{Out}(W_n, H_N^{(t)}) = \text{Out}(W_n, H_{n_0}^{(t)}).$$

**Proof.** We show that the result is a consequence of a similar result in the context of the automorphism group of a nonabelian free group due to Martino and Ventura ([MV, Corollary 4.2]). Since  $\mathbb{F}$  is a nonabelian free group, we may suppose that, for every  $N \in \mathbb{N}^*$ , the group  $H_N$  is a nonabelian free group. Hence for every  $N \in \mathbb{N}^*$ , we have  $C_{W_n}(H_N) = \{1\}$ . Therefore, for every  $N \in \mathbb{N}^*$  and every  $\phi \in \text{Out}(W_n, H_N^{(t)})$ , there exists a unique representative  $\Phi \in \text{Aut}(W_n)$  of  $\phi$  such that  $\Phi(H_N) = H_N$  and  $\Phi|_{H_N} = \text{id}_{H_N}$ . This implies that, for every  $N \in \mathbb{N}^*$ , we have an injective homomorphism  $\text{Out}(W_n, H_N^{(t)}) \hookrightarrow \text{Aut}(W_n, H_N)$ , where  $\text{Aut}(W_n, H_N)$  is the group of automorphisms of  $W_n$  which fix every element of  $H_N$ . Therefore, it suffices to prove the result for  $\text{Aut}(W_n, H_N)$ . Since there exists an injective homomorphism  $\text{Aut}(W_n) \rightarrow \text{Aut}(\mathbb{F})$  and since, for every  $N \in \mathbb{N}^*$ , we have  $H_N \subseteq \mathbb{F}$ , it suffices to prove that there exists  $n_0 \in \mathbb{N}^*$  such that, for every  $N \geq n_0$ , we have  $\text{Aut}(\mathbb{F}, H_N) = \text{Aut}(\mathbb{F}, H_{n_0})$ . We then conclude using [MV, Corollary 4.2].  $\square$

We now recall a theorem due to Guirardel and Levitt which provides a canonical splitting for a relative one-ended hyperbolic group (recall that a group  $G$  is *one-ended relative to a family of subgroups*  $\mathcal{H}$  if  $G$  does not have a one-edge splitting with finite edge stabilizers such that every subgroup of  $\mathcal{H}$  fixes a point).

**Theorem 4.7.5.** [GuL5, Theorem 9.14] *Let  $G$  be a hyperbolic group and let  $\mathcal{H}$  be a family of subgroups such that  $G$  is one-ended relative to  $\mathcal{H}$ . There exists a splitting  $S$  of  $G$  such that:*

- (1) *Every edge stabilizer is virtually infinite cyclic.*
- (2) *For every  $H \in \mathcal{H}$ , the group  $H$  is elliptic in  $S$ .*
- (3) *The tree  $S$  is invariant under all automorphisms of  $G$  preserving  $\mathcal{H}$ . Moreover,  $S$  is compatible with every splitting  $S'$  with virtually cyclic edge stabilizers and such that for every  $H \in \mathcal{H}$ , the group  $H$  is elliptic in  $S'$ .*



(4) Let  $H \in \mathcal{H}$  be such that  $H$  is virtually a (possibly not finitely generated) nonabelian free group, and let  $v$  be the vertex of  $S$  fixed by  $H$ . Let  $\text{Inc}_v$  be the finite set of representatives of all conjugacy classes of groups associated with edges in  $S$  which are incident to  $v$ . Then the group  $\text{Out}(G_v, \{H, \text{Inc}_v\}^{(t)})$  is finite.

Assertion (4) is a bit stronger than what is stated in [GuL5], hence we add some explanations.

*Proof of Assertion (4) of Theorem 4.7.5.* Let  $S$ ,  $H$  and  $v$  be as in Assertion (4). By for instance [Pau2, Proposition 2.5], the set  $\text{Inc}_v$  is finite. Note that every group in  $\text{Inc}_v$  is virtually cyclic by Assertion (1). Thus, the set  $\text{Inc}_v$  is a finite set of finitely generated groups. Up to taking finite index subgroups, we may suppose that  $H \subseteq \mathbb{F}$  and that for every subgroup  $H'$  in  $\text{Inc}_v$ , the group  $H'$  is contained in  $\mathbb{F}$ . Suppose towards a contradiction that  $\text{Out}(G_v, \{H, \text{Inc}_v\}^{(t)})$  is infinite. Suppose first, following the terminology of [GuL5], that the vertex  $v$  is *rigid*. By Proposition 4.7.4, there exists a finitely generated subgroup  $K$  of  $H$  such that  $\text{Out}(W_n, H^{(t)}) = \text{Out}(W_n, K^{(t)})$ . By [GuL4, Theorem 7.14], there exists a one-edge splitting  $U$  of  $G_v$  whose edge stabilizer is isomorphic to  $\mathbb{Z}$  such that  $K$  and every group in  $\text{Inc}_v$  are elliptic in  $U$ . Since  $v$  is a rigid vertex, there exists  $h \in H$  such that  $h$  acts loxodromically on  $U$ . Since every group in  $\text{Inc}_v$  fixes a point in  $U$ , one can blow up the splitting  $U$  at the vertex  $v$  of  $S$ . This gives a refinement  $S'$  of  $S$ . Let  $D'$  be a nontrivial infinite twist of  $U$ . Then  $D'$  induces a twist  $D$  of  $S'$ . By Lemma 4.7.3, the element  $D$  fixes the conjugacy class of  $K$  but does not fix the conjugacy class of  $h$ . This contradicts  $\text{Out}(W_n, H^{(t)}) = \text{Out}(W_n, K^{(t)})$ .

So we may suppose, following the terminology of [GuL5], that the vertex  $v$  is *flexible*. By [GuL5, Theorem 9.14 (2)], as  $H$  is virtually a nonabelian free group, the vertex  $v$  is a *QH vertex* (see [GuL5, Definition 5.13]). But the definition of a QH vertex implies, as  $H$  is contained in  $\mathcal{H}$ , that the group  $H$  must be virtually contained in a boundary subgroup of the fundamental group of the orbifold associated with  $G_v$ . Thus the group  $H$  must be virtually cyclic, a contradiction.  $\square$

We also need some results about splittings over virtually cyclic groups, whose generalization to virtually free groups is due to Cashen.

**Theorem 4.7.6.** [Cas, Theorem 1.2] *Let  $G_1$  and  $G_2$  be finitely generated virtually nonabelian free groups, and let  $C$  be a virtually cyclic group which is a proper subgroup of both  $G_1$  and  $G_2$ . Then  $G_1 *_C G_2$  is virtually a nonabelian free group if and only if there exists  $i \in \{1, 2\}$  such that  $G_i$  has a splitting with finite edge stabilizers such that  $C$  is a vertex stabilizer.*

**Corollary 4.7.7.** *Let  $n \geq 3$  and let  $G_1, G_2$  be subgroups of  $W_n$  such that  $W_n = G_1 *_C G_2$  is a nontrivial amalgamated product of  $W_n$ , where  $C$  is isomorphic to  $W_2$  and  $G_1$  and  $G_2$  are not virtually cyclic.*

(1) *There exists  $i \in \{1, 2\}$  such that  $C$  is a free factor of  $G_i$ . Moreover, if  $j \in \{1, 2\} - \{i\}$ , then  $G_j$  is a free factor of  $W_n$ .*

(2) *There exist  $3 \leq k_1, k_2 \leq n - 1$  such that  $k_1 + k_2 = n + 2$  and, for every  $i \in \{1, 2\}$ , the group  $G_i$  is isomorphic to  $W_{k_i}$ . In particular,  $n \geq 4$ .*

**Proof.** (1) By Lemma 4.6.2, the subgroup  $\mathbb{F}$  of  $W_n$  is a nonabelian free group of finite index. Since both  $G_1$  and  $G_2$  are not virtually cyclic, the intersections  $G_1 \cap \mathbb{F}$  and  $G_2 \cap \mathbb{F}$  are finite index subgroups of  $G_1$  and  $G_2$  which are nonabelian free groups. Hence  $G_1$  and  $G_2$  are virtually nonabelian free groups. Moreover, since  $W_n$  and  $C$  are finitely generated, so are  $G_1$  and  $G_2$ . By Theorem 4.7.6, up to exchanging the roles of  $G_1$  and  $G_2$ , we may suppose that  $G_1$  has a splitting  $S$  such that every edge stabilizer is finite and  $C$  is the stabilizer of a vertex  $v \in VS$ . Note that, since the finite subgroups of  $W_n$  are all isomorphic to  $F$ , every edge stabilizer of  $S$  is either trivial or isomorphic to  $F$ . Since every element of  $W_n$  of order 2 is a conjugate of an element in a standard generating set of  $W_n$ , every nontrivial edge stabilizer is a free factor of both of its endpoint stabilizers. Let  $V_1$  be the set of vertices of  $S$  distinct from  $v$  and fixed by a subgroup of  $C$  isomorphic to  $F$ . Therefore, for every  $w \in V_1$ , there exists a subgroup  $A_w$  of  $G_w$  and an element  $x_w \in C$  of order 2 such that  $G_w = A_w * \langle x_w \rangle$ . Let  $S_0$  be a splitting of  $W_n$  obtained from  $S$  by blowing-up, at every vertex  $w \in V_1$ , the free splitting  $A_w * \langle x_w \rangle$  and by attaching the edge fixed by  $x_w$  to its corresponding fixed point. Let  $S'$  be the splitting of  $W_n$  obtained from  $S_0$  by collapsing every edge with nontrivial stabilizer. Then the stabilizer in  $G_1$  of every edge of  $S'$  adjacent to the vertex fixed by  $C$  has trivial stabilizer. Thus,  $C$  is a free factor of  $G_1$  and there exists  $H_1 \subset G_1$  such that  $G_1 = H_1 * C$ . This proves the first assertion of (1). The second assertion of (1) follows from the fact that

$$W_n = G_1 *_C G_2 = (H_1 * C) *_C G_2 = H_1 * G_2.$$

Hence  $H_1$  and  $G_2$  are free factors of  $W_n$ .

(2) Therefore, there exist  $h_1, k_2 \in \{1, \dots, n - 2\}$  with  $h_1 + k_2 = n$  such that  $H_1$  is isomorphic to  $W_{h_1}$  and  $G_2$  is isomorphic to  $W_{k_2}$ . Thus  $G_1$  is isomorphic to  $W_{h_1+2}$ . Set  $k_1 = h_1 + 2$ . Since the amalgamated product is nontrivial and since  $G_1$  and  $G_2$  are not virtually cyclic, we have  $3 \leq k_1, k_2 \leq n - 1$ . This proves (2).  $\square$

**Lemma 4.7.8.** *Let  $n \geq 4$  and let  $S$  be a splitting of  $W_n$ . Let  $\mathcal{S}$  be its equivalence class. Let  $v_1$  and  $v_2$  be adjacent vertices of  $S$  and let  $e$  be the edge between  $v_1$  and  $v_2$ . Suppose that  $G_e$  is isomorphic to  $W_2$ . Let  $f \in \text{Stab}_{\text{Out}(W_n)}(\mathcal{S})$  be such that:*

- (1) *the graph automorphism of  $W_n \setminus S$  induced by  $f$  is trivial;*
- (2) *the natural homomorphisms  $\langle f \rangle \rightarrow \text{Out}(G_{v_1}, G_e)$  and  $\langle f \rangle \rightarrow \text{Out}(G_{v_2}, G_e)$  are trivial.*

*Then  $f$  has a representative which acts as the identity on  $\langle G_{v_1}, G_{v_2} \rangle$ .*

**Proof.** By (2), the outer automorphism  $f$  has two representatives  $F_1$  and  $F_2$  such that for every  $i \in \{1, 2\}$ , we have  $F_i(G_{v_i}) = G_{v_i}$  and  $F_i|_{G_{v_i}} = \text{id}_{G_{v_i}}$ . Note that  $G_{v_1} \cap G_{v_2} = G_e$ . Hence  $F_1$  and  $F_2$  acts as the identity on  $G_e$ . Therefore,  $F_1$  and  $F_2$  differ by an inner automorphism  $\text{ad}_z$  with  $z \in C_{W_n}(G_e)$ . However, since  $G_e$  is isomorphic to  $W_2$ , we have  $C_{W_n}(G_e) = \{e\}$ . Hence  $F_1 = F_2$ . This concludes the proof.  $\square$

**Proposition 4.7.9.** *Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . Let  $H$  be a subgroup of  $\Gamma$  which satisfies  $(P_{W_{n-2}})$ . Then  $H$  virtually stabilizes the equivalence class of a  $W_{n-2}$ -star. Moreover, this equivalence class is unique.*

**Proof.** By Proposition 4.6.10, the group  $H$  virtually fixes the equivalence class  $\mathcal{S}$  of a  $W_{n-1}$ -star  $S$ . Let

$$W_n = A * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S$ . Up to passing to a finite index subgroup, we may suppose that  $H$  fixes  $\mathcal{S}$ . Let  $T$  be the group of twists of  $\mathcal{S}$  contained in  $\Gamma$ . By Proposition 4.2.5 (2), the group  $\text{Stab}(\mathcal{S})$  is isomorphic to  $\text{Aut}(A)$  and the group of twists of  $\mathcal{S}$  is identified with the inner automorphism group of  $A$ .

Let  $K_1 \times K_2$  be a normal subgroup of  $H$  given by Property  $(P_{W_{n-2}})$  (1). By Lemma 4.7.2, up to exchanging the roles of  $K_1$  and  $K_2$ , we may assume that  $K_1 \cap T$  is infinite, that  $H \cap T \cap K_1$  is a finite index subgroup of  $H \cap T$  and that  $K_2 \cap T$  is finite. Up to passing to a finite index subgroup of  $H$ , we may assume that  $K_2 \cap T = \{1\}$ . In particular, the natural homomorphism  $\phi: K_2 \rightarrow \text{Out}(A)$  is injective. Let  $K \subseteq A$  be the group of twistors associated with twists contained in  $K_1$ . Note that to every splitting  $S_0$  of  $A$  such that  $K$  fixes a unique vertex of  $S_0$ , one can deduce a splitting  $S'_0$  of  $W_n$  such that  $K$  fixes a point of  $S'_0$ . Indeed, by blowing-up the splitting  $S_0$  at the vertex  $v$  of  $S$  whose associated group is  $A$ , and by attaching the edges of  $S$  adjacent to  $v$  to the vertex fixed by  $K$ , we obtain a splitting  $S'_0$  of  $W_n$  such that  $K$  fixes a point of  $S'_0$ . Let  $\mathcal{S}'_0$  be the equivalence class of  $S'_0$ . We claim that the group  $K_1 \cap T$  fixes  $\mathcal{S}'_0$ . Indeed, let  $e_0$  be the edge of  $S'_0$  adjacent to the vertex  $v_0$  fixed by  $K$  and the vertex fixed by  $\langle x_n \rangle$ . Since the stabilizer of  $e_0$  is trivial, Proposition 4.2.5 implies that the group of twists about  $e_0$  at the vertex  $v_0$  contains all the twists whose twistor is an element of  $K$ . Hence  $K_1 \cap T$  fixes  $\mathcal{S}'_0$ .

We now construct a one-edge free splitting  $S_0$  of  $A$  such that  $K$  fixes a vertex of  $S_0$ . By the above discussion, this will give a two-edge free splitting of  $W_n$  such that  $K$  fixes a vertex of this splitting which is not a leaf and whose equivalence class is fixed by  $K_1 \cap T$ . We distinguish between three cases, according to whether  $A$  is one-ended relative to  $K$  and according to the edge stabilizers of a splitting of  $A$  relative to  $K$ .

**Case 1.** There exists a free splitting  $S_0$  of  $A$  such that  $K$  fixes a vertex of  $S_0$ .

In particular, the corresponding splitting  $S'_0$  of  $W_n$  constructed above is a free splitting of  $W_n$ . We claim that the splitting  $S'_0$  has two orbits of edges. Indeed, suppose that  $S'_0$  has  $k$  orbits of edges, with  $k \geq 3$ . Then,  $S'_0$  is obtained from  $S$  by blowing-up at least two orbits of edges at  $v$ . Therefore, the group of twistors  $K$  is contained in a free factor  $B$  of  $W_n$  isomorphic to  $W_{n-3}$ . Let  $B'$  be a free factor of  $W_n$  isomorphic to  $W_2$  such that

$$W_n = \langle x_n \rangle * B * B'$$

and let  $R$  be the free splitting associated with this decomposition. Then the equivalence class  $\mathcal{R}$  of  $R$  is a free splitting of  $W_n$  fixed by  $K_1 \cap T$ . But by Proposition 4.2.5 (3), the group of twists of  $\mathcal{R}$  is isomorphic to  $B \times B \times W_2$ . Moreover, the group  $K_1 \cap T$  is contained in one of the factors of  $B \times B \times W_2$  isomorphic to  $B$ . Therefore, the centralizer

of  $K_1 \cap T$  contains a free abelian group of rank 2. Since  $K_1 \cap T$  is a normal subgroup of  $K_1$ , this contradicts the fact that the centralizer of  $K_1 \cap T$  is virtually a nonabelian free group by Property  $(P_{W_{n-2}})$  (1). Therefore, the splitting  $S'_0$  is a two-edge free splitting.

**Case 2** There exists a splitting  $S_0$  of  $A$  such that  $K$  fixes a vertex of  $S_0$  and such that one of the edge stabilizers of  $S_0$  is finite.

Let  $S'_0$  be the corresponding splitting of  $W_n$  constructed in the above discussion. If  $S_0$  has an edge  $e'$  with trivial stabilizer, then by collapsing every orbit of edges of  $S_0$  except the one containing  $e'$ , we obtain a splitting  $S_1$  of  $A$  such that  $K$  fixes a vertex of  $K$ . Then the corresponding splitting  $S'_1$  of  $W_n$  is a free splitting. Thus, we can apply Case 1.

Therefore, we may assume that every edge stabilizer of  $S_0$  is infinite or a nontrivial finite subgroup of  $W_n$ . By collapsing every edge of  $S_0$  with infinite stabilizer and by collapsing all but one orbit of edges with finite edge stabilizer, we may suppose that  $S_0$  is a one-edge splitting such that every edge stabilizer of  $S_0$  is a nontrivial finite subgroup of  $W_n$ . Every finite subgroup of  $W_n$  is isomorphic to  $F$  and is in fact a free factor of  $W_n$ . We claim that we can construct a splitting  $X_0$  of  $A$  which contains an edge with trivial stabilizer and such that  $K$  fixes a vertex of  $X_0$ . Indeed, let  $x_0$  be the vertex of  $S_0$  fixed by  $K$ , let  $f_0$  be an edge adjacent to  $x_0$  and let  $x_1$  be the vertex of  $f_0$  distinct from  $v_0$ . Let  $G_{x_0}$  be the stabilizer of  $x_0$ , let  $G_{x_1}$  be the stabilizer of  $x_1$  and let  $G_{f_0}$  be the stabilizer of  $f_0$ . Note that, since there does not exist HNN extensions in  $W_n$ , the groups  $G_{x_0}$  and  $G_{x_1}$  are not conjugate in  $W_n$ . The group  $G_{f_0}$  is a free factor of both  $G_{x_0}$  and  $G_{x_1}$ . Thus there exists a free factor  $A'$  of  $G_{x_1}$  such that  $G_{x_1} = G_{f_0} * A'$ . Let  $U$  be the splitting of  $A$  such that the underlying tree of  $W_n \setminus U$  is the same one as the underlying tree of  $W_n \setminus S_0$ , such that the stabilizer of every vertex which is not in the orbit of  $x_1$  is the same one as the stabilizer of the corresponding vertex in  $S_0$  and the stabilizer of  $x_1$  is  $A'$ . Then the edge  $f_0$  has trivial stabilizer in  $U$  and  $K$  fixes a vertex of  $U$ . This proves the claim. Therefore Case 2 is a consequence of Case 1.

**Case 3** The group  $A$  is one-ended relative to  $K$ .

We prove that this assumption leads to a contradiction. By Theorem 4.7.5, there exists a canonical splitting  $S_0$  of  $A$  whose edge stabilizers are virtually infinite cyclic, such that  $K$  fixes a point of  $S_0$  and such that every automorphism of  $A$  preserving  $K$  fixes the equivalence class of  $S_0$ . Let  $S'_0$  be the corresponding splitting of  $W_n$ , and let  $\mathcal{S}'_0$  be its equivalence class. Recall that the group  $K_1 \cap T$  is a normal subgroup of  $H$  contained in  $\text{Inn}(A)$ . Let  $k \in K$  and let  $f \in H$ . Let  $F$  be a representative of  $F$  which fixes  $x_n$  and which preserves  $A$ . As  $K_1 \cap T$  is a normal subgroup of  $H$ , there exists  $k' \in K$  such that

$$F \circ \text{ad}_k \circ F^{-1} = \text{ad}_{F(k)} = \text{ad}_{k'}.$$

Since the center of  $A$  is trivial, we have  $F(k) = k'$ . Hence the group  $H$  viewed as a subset of  $\text{Aut}(A)$  preserves  $K$ . Thus  $H$  preserves  $\mathcal{S}'_0$ .

Let  $v_0$  be the vertex of  $S'_0$  fixed by  $K$  and let  $e_0$  be the edge of  $S'_0$  between  $v_0$  and the point fixed by  $\langle x_n \rangle$ . By construction, the stabilizer of every edge of  $S'_0$  which is not in the orbit of  $e_0$  is virtually cyclic, that is it is isomorphic either to  $\mathbb{Z}$  or to  $W_2$ . By

Lemma 4.2.7, a twist about an edge whose stabilizer is isomorphic to  $\mathbb{Z}$  is central in a finite index subgroup of  $\text{Stab}_{\text{Out}(W_n)}(S'_0)$ . Since any finite index subgroup of  $H$  has finite center by Remark 4.6.1 (2), we see that the stabilizer of every edge of  $S'_0$  which is not in the orbit of  $e_0$  is isomorphic to  $W_2$ . Therefore, Remark 4.2.6 implies that the group of bitwists about every edge of  $S'_0$  which is not in the orbit of  $e_0$  is trivial. Thus, the group of bitwists  $T_0$  of  $S'_0$  is reduced to the group of twists about  $e_0$ .

Let  $\overline{W_n \setminus S'_0}$  be the graph associated with  $W_n \setminus S'_0$ . For every vertex  $v \in V(\overline{W_n \setminus S'_0})$ , let  $\text{Inc}_v$  be the set containing the conjugacy class of the edge group of every edge adjacent to  $v$  (seen as a subgroup of  $G_v$ ). Let  $\bar{v}_0$  be the image of  $v_0$  in  $\overline{W_n \setminus S'_0}$  and let  $\bar{e}_0$  be the image of  $e_0$  in  $\overline{W_n \setminus S'_0}$ . By Proposition 4.2.5 and Remark 4.2.6, up to taking a finite index subgroup of  $H$ , we have a natural homomorphism

$$\Psi: H \rightarrow \text{Out}(G_{v_0}, \{K, \text{Inc}_{\bar{v}_0}\}) \times \prod_{v \in V(\overline{W_n \setminus S'_0}), v \neq \bar{v}_0} \text{Out}(G_v, \text{Inc}_v),$$

whose kernel is  $T_0 \cap H$ . Note that every edge stabilizer is isomorphic to  $W_2$ , hence the outer automorphism group of every edge stabilizer is finite. Thus, up to taking a finite index subgroup of  $H$ , we may suppose that the image of  $\Psi$  is contained in

$$\text{Out}(G_{v_0}, \{K, \text{Inc}_{\bar{v}_0}^{(t)}\}) \times \prod_{v \in V(\overline{W_n \setminus S'_0}), v \neq \bar{v}_0} \text{Out}(G_v, \text{Inc}_v^{(t)}).$$

Recall that  $K_2 \cap T_0 = \{1\}$ , hence  $\Psi|_{K_2}$  is injective. Moreover, as  $K_2$  commutes with  $K_1$ , the group  $K_2$  is contained in  $\text{Out}(W_n, K^{(t)})$ . Recall that Theorem 4.7.5 (4) implies that the group  $\text{Out}(G_{v_0}, \{K, \text{Inc}_{\bar{v}_0}\}^{(t)})$  is finite. Since  $\Psi|_{K_2}$  is injective, this implies that

$$\prod_{v \in V(\overline{W_n \setminus S'_0}), v \neq \bar{v}_0} \text{Out}(G_v, \text{Inc}_v^{(t)})$$

is infinite. Since the graph  $\overline{W_n \setminus S'_0}$  is finite, there exists  $v \in V(\overline{W_n \setminus S'_0})$  such that  $v \neq \bar{v}_0$  and  $\text{Out}(G_v, \text{Inc}_v^{(t)})$  is infinite.

Suppose first that there exist two distinct vertices  $v$  and  $w$  of  $\overline{W_n \setminus S'_0}$  such that  $v, w \neq \bar{v}_0$  and both  $\text{Out}(G_v, \text{Inc}_v^{(t)})$  and  $\text{Out}(G_w, \text{Inc}_w^{(t)})$  are infinite. Since  $G_v$  and  $G_w$  are subgroups of  $W_n$  whose outer automorphism groups are infinite, they are virtually nonabelian free groups. Thus we can apply Theorem 4.6.4 to both  $(G_v, \text{Inc}_v)$  and  $(G_w, \text{Inc}_w)$  to show that there exist a  $\mathcal{Z}_{RC}$ -splitting  $U_v$  of  $G_v$  and  $U_w$  of  $G_w$  such that every group in  $\text{Inc}_v$  fixes a point in  $U_v$  and every group in  $\text{Inc}_w$  fixes a point in  $U_w$ . One can then blow-up the splittings  $U_v$  and  $U_w$  at the vertices  $v$  and  $w$  of  $\overline{W_n \setminus S'_0}$  and attach the edges adjacent to  $v$  and  $w$  in  $\overline{W_n \setminus S'_0}$  to the points fixed by their corresponding edge groups in  $U_v$  and  $U_w$ . This gives a refinement  $S_1$  of  $S'_0$ . Let  $\mathcal{S}_1$  be the equivalence class of  $S_1$ . Note that, since the group of twists about the edge  $e_0$  of  $S'_0$  is contained in the group of twists of  $\mathcal{S}_1$ , the group  $K_1 \cap T$  fixes  $\mathcal{S}_1$ . Note that the stabilizer of an edge in  $U_v$  or  $U_w$  is either finite or isomorphic to  $\mathbb{Z}$ . If there exists an edge in  $U_v$  or  $U_w$  with a finite edge stabilizer, as  $v$  and  $w$  come from vertices in  $S_0$ , we can apply Case 2 to conclude.

Suppose that every edge stabilizer of  $U_v$  and  $U_w$  is isomorphic to  $\mathbb{Z}$ . By Lemma 4.2.7, a twist about an edge whose stabilizer is isomorphic to  $\mathbb{Z}$  is central in a finite index subgroup of  $\text{Stab}_{\text{Out}(W_n)}(\mathcal{S}_1)$ . Hence  $K_1 \cap T$  has a finite index subgroup which is centralized by a free abelian group of rank 2. This contradicts Property  $(P_{W_{n-2}})$  (1).

Suppose now that there exists a unique vertex  $v \in V(\overline{W_n \setminus S'_0})$  such that  $v \neq \bar{v}_0$  and  $\text{Out}(G_v, \text{Inc}_v^{(t)})$  is infinite. Recall that the image of the homomorphism  $\Psi|_{K_2}$  is contained in

$$\text{Out}(G_{v_0}, \{K, \text{Inc}_{\bar{v}_0}\}^{(t)}) \times \prod_{w \in V(W_n \setminus S'_0), w \neq \bar{v}_0} \text{Out}(G_w, \text{Inc}_w^{(t)}).$$

In particular, as  $\text{Out}(G_{v_0}, \{K, \text{Inc}_{\bar{v}_0}\}^{(t)})$  is finite, up to taking a finite index subgroup of  $K_2$ , we may suppose that the image of  $\Psi|_{K_2}$  is contained in  $\text{Out}(G_v, \text{Inc}_v^{(t)})$ .

**Claim.** Let  $f \in K_2$  and let  $X$  be a connected subgraph of  $\overline{W_n \setminus S'_0}$  such that every vertex of  $X$  is distinct from  $v$  and such that the group associated with every edge of  $X$  is isomorphic to  $W_2$ . Then  $f$  has a representative which acts as the identity on  $\langle G_w \rangle_{w \in VX}$ .

**Proof.** We prove the result by induction on the number  $m$  of edges of  $X$ . If  $X$  is reduced to a vertex, then the conclusion is immediate. Suppose that  $|EX| = m \geq 1$ . Let  $w_1$  and  $w_2$  be two adjacent vertices in  $VX$  such that  $w_1$  is a leaf of  $X$ . Let  $e'$  be the edge in  $X$  between  $w_1$  and  $w_2$ . Let  $X'$  be the graph obtained from  $X$  by removing  $w_1$  and  $e'$ . The graph  $X'$  is a connected subgraph of  $\overline{W_n \setminus S'_0}$  which satisfies the hypothesis of the lemma and such that  $|EX'| = m - 1$ . By the induction hypothesis, the element  $f$  has a representative which acts as the identity on  $\langle G_w \rangle_{w \in VX'}$ . Let  $W_n \setminus S'_2$  be the graph of groups obtained from  $W_n \setminus S'_0$  by collapsing  $X'$  and let  $p: W_n \setminus S'_0 \rightarrow W_n \setminus S'_2$  be the natural projection. Since  $f$  has a representative which acts as the identity on  $\langle G_w \rangle_{w \in VX'}$ , the element  $f$  fixes the equivalence class of  $W_n \setminus S'_2$ . Note that the group associated with  $p(w_2)$  is  $\langle G_w \rangle_{w \in VX'}$  and that the group associated with  $p(w_1)$  is  $G_{w_1}$ . Moreover, the group associated with  $p(e')$  is  $G_{e'}$ , in particular, it is isomorphic to  $W_2$ . Thus for every  $i \in \{1, 2\}$ , the outer automorphism  $f$  has a representative  $F_i$  such that  $F_i(G_{p(w_i)}) = G_{p(w_i)}$  and  $F_i|_{G_{p(w_i)}} = \text{id}_{G_{p(w_i)}}$ . Thus, by Lemma 4.7.8 applied to  $W_n \setminus S'_2$ , the outer automorphism  $f$  has a representative which acts as the identity on

$$\langle G_{p(w_1)}, G_{p(w_2)} \rangle = \langle G_w \rangle_{w \in VX}.$$

The claim follows. □

Let  $e'$  be the edge adjacent to  $v$  in  $\overline{W_n \setminus S'_0}$  which is contained in the path between  $v$  and  $\bar{v}_0$  and let  $\tilde{e}'$  be a lift of  $e'$  in  $S'_0$ . Note that  $v$  is contained in the same connected component of  $\overline{W_n \setminus S'_0} - \bar{e}_0$  as  $\bar{v}_0$ . Thus the edges  $e'$  and  $e_0$  are not in the same orbit. Moreover the stabilizer of  $e'$  is isomorphic to  $W_2$ . Let  $S'_1$  be the splitting of  $W_n$  obtained from  $S'_0$  by collapsing every edge of  $S'_0$  which is not in the orbit of  $\tilde{e}'$  and  $e_0$ . Let  $\mathcal{S}'_1$  be the equivalence class of  $S'_1$ . Then  $S'_1$  has two orbits of edges. Let  $v_2$  be the vertex of  $S'_1$  fixed by  $K$  and let  $v_1$  be the vertex of  $S'_1$  adjacent to  $v_2$  which is fixed by a conjugate of  $G_v$ . Let  $e$  be the edge adjacent to  $v_1$  and  $v_2$ . Note that, up to taking a finite index subgroup of  $K_2$ , the group  $K_2$  fixes  $S'_1$ . Thus, by Proposition 4.2.5 and Remark 4.2.6,

we have a natural homomorphism  $\Phi: K_2 \rightarrow \text{Out}(G_{v_1}, G_e) \times \text{Out}(G_{v_2}, G_e)$  whose kernel is contained in  $K_2 \cap T_0 = \{1\}$ . Moreover, the claim applied to the connected component of  $\overline{W_n \setminus S'_0} - \dot{e} \cup \dot{\bar{e}}_0$  containing  $\bar{v}_0$  shows that every element  $f \in K_2$  has a representative which acts as the identity on  $G_{v_2}$ . Hence  $\Phi(K_2)$  is isomorphic to  $K_2$  and is contained in  $\text{Out}(G_{v_1}, G_e)$ . We also see that, as  $K$  is virtually a nonabelian free group, its centralizer in  $W_n$  is trivial. Hence every element  $f \in K_2$  has a unique representative which acts as the identity on  $K$ . Let  $f \in K_2$ . Recall that  $W_n = A * \langle x_n \rangle$ . Then the representative of  $f$  which preserves  $A$  and fixes  $x_n$  must fix  $K$  by Lemma 4.4.11 (since  $f \in K_2$  centralizes  $K_1$ ). As  $f$  has a representative which acts as the identity on  $G_{v_2}$  and as  $K \subseteq G_{v_2}$ , we see that  $f$  has a representative which acts as the identity on  $G_{v_2} * \langle x_n \rangle$ .

Note that the group  $G_{v_1} *_{G_e} G_{v_2}$  is a splitting of  $A$  such that  $G_e$  is isomorphic to  $W_2$ . Moreover, as  $K$  fixes  $v_2$ , the group  $G_{v_2}$  is not virtually cyclic. Since the group  $\text{Out}(G_v)$  is infinite, the group  $G_v$  is not virtually cyclic. Hence the group  $G_{v_1}$  is not virtually cyclic. Therefore we may apply Corollary 4.7.7 to  $G_{v_1} *_{G_e} G_{v_2}$ : there exist  $k_1, k_2 \geq 3$  such that for every  $i \in \{1, 2\}$ , the group  $G_{v_i}$  is isomorphic to  $W_{k_i}$ . Moreover, there exist  $i \in \{1, 2\}$  and  $j \in \{1, 2\} - \{i\}$  such that  $W_{k_i}$  is a free factor of  $W_{k_i}$  and  $G_e$  is a free factor of  $W_{k_j}$ .

Suppose first that  $W_{k_1}$  is a free factor of  $A$  and that  $G_e$  is a free factor of  $W_{k_2}$ . Let  $H$  be such that  $W_{k_2} = G_e * H$ . Then  $H$  is a free factor of  $A$  since

$$A = G_{v_1} *_{G_e} G_{v_2} = G_{v_1} *_{G_e} (G_e * H) = G_{v_1} * H.$$

Since  $k_2 \geq 3$ , the group  $H$  is not trivial. Let  $z$  be an infinite order element of  $G_e$ . Let  $F_1$  be the automorphism of  $W_n$  which acts as a global conjugation by  $z$  on  $H$  and which fixes  $x_n$  and  $G_{v_1}$  (recall that as  $W_n = H * G_{v_1} * \langle x_n \rangle$ , the automorphism  $F_1$  is uniquely determined). Let  $F_2$  be the automorphism of  $W_n$  which acts as a global conjugation by  $z$  on  $A$  and which fixes  $x_n$ . Then  $\langle [F_1], [F_2] \rangle$  is a subgroup of  $\text{Out}(W_n)$  isomorphic to a free abelian group of rank 2. Recall that every element of  $K_2$  has a representative which acts as the identity on  $G_{v_2} * \langle x_n \rangle$ . Since  $[F_1]$  and  $[F_2]$  have representatives whose support is contained in  $G_{v_2} * \langle x_n \rangle$ , the group  $\langle [F_1], [F_2] \rangle$  is contained in  $C_{\text{Out}(W_n)}(K_2)$ . This contradicts Property  $(P_{W_{n-2}})$  (1) which says that the centralizer of  $K_2$  is virtually a nonabelian free group.

Suppose now that  $W_{k_2}$  is a free factor of  $A$  and that  $G_e$  is a free factor of  $W_{k_1}$ . Let  $H$  be such that  $W_{k_1} = G_e * H$ . As before the group  $H$  is a free factor of  $A$  and  $A = H * G_{v_2}$ . But  $K$  is contained in  $G_{v_2}$ . This contradicts the fact that  $A$  is ond-ended relative to  $K$ . The conclusion in Case 3 follows.

Therefore, we have constructed a free splitting  $S'_0$  of  $W_n$  which is a two-edge free splitting fixed by  $K_1 \cap T$ . Moreover, the construction of the splitting is such that the vertex of the underlying graph of  $W_n \setminus S'_0$  whose associated group contains  $K$  is not a leaf. We now prove that  $S'_0$  is a  $W_{n-2}$ -star. Let  $C$  be the vertex stabilizer of  $S'_0$  containing  $K$ , and let  $C'$  be a vertex stabilizer of  $S'_0$  which is not a conjugate of  $C$  nor  $\langle x_n \rangle$ . Then  $C'$  is the vertex group of a leaf of the underlying graph of  $W_n \setminus S'_0$ . By Proposition 4.2.5 (3), the group of twists of  $S'_0$  is isomorphic to  $C \times C \times C'/Z(C')$ . Since the centralizer of

$K \cap T_1$  is virtually a nonabelian free group by Property  $(P_{W_{n-2}})$  (1), we conclude that  $C'/Z(C')$  is finite. Hence  $C'$  is isomorphic to  $F$  and  $S'_0$  is a  $W_{n-2}$ -star.

We now prove that  $H$  virtually fixes  $S'_0$ . By Proposition 4.2.5 (3), the group of twists of  $S'_0$  is isomorphic to  $W_{n-2} \times W_{n-2}$ . By Lemma 4.4.10, the group  $K_1 \cap T$  is contained in one of the factors isomorphic to  $W_{n-2}$  of the group of twists of  $S'_0$ . Therefore,  $K_1 \cap T$  is centralized by the other factor of the group of twists of  $S'_0$ . Since the centralizer of  $K_1 \cap T$  contains  $K_2$  as a finite index subgroup, the group  $K_2$  contains a twist  $f$  of infinite order about the edge  $e$  of  $S'_0$  which does not collapse onto  $S$ . This twist is a twist about a  $W_{n-1}$ -star obtained from  $S'_0$  by collapsing the orbit of edges which does not contain  $e$ . By Lemma 4.7.1, the group  $H$  virtually fixes  $S'_0$ . Moreover,  $K_1$  is commensurable with  $T \cap \text{Stab}(S'_0)$ , that is  $K_1$  is commensurable with the group of twists about one edge of  $S'_0$ . Lemma 4.6.5 then implies that  $K_1$  virtually fixes a unique equivalence class of  $W_{n-2}$ -stars. Therefore, since  $K_1$  is a normal subgroup of  $H$ , we see that  $H$  virtually fixes a unique equivalence class of  $W_{n-2}$ -stars. This concludes the proof.  $\square$

**Proposition 4.7.10.** *Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . Let  $\Psi \in \text{Comm}(\Gamma)$ . Then for every equivalence class  $\mathcal{S}$  of  $W_{n-2}$ -stars, there exists a unique equivalence class  $\mathcal{S}'$  of  $W_{n-2}$ -stars such that  $\Psi([\text{Stab}_\Gamma(\mathcal{S})]) = [\text{Stab}_\Gamma(\mathcal{S}')]$ .*

**Proof.** The uniqueness statement follows from Lemma 4.6.5 which shows that the stabilizer in finite index subgroups of  $\text{Out}(W_n)$  of two distinct equivalence classes of  $W_{n-2}$ -stars are not commensurable.

We now prove the existence statement. Let  $f: \Gamma_1 \rightarrow \Gamma_2$  be an isomorphism between finite index subgroups of  $\Gamma$  that represents  $\Psi$ . By Proposition 4.6.6, the group  $\text{Stab}_{\Gamma_1}(\mathcal{S})$  satisfies  $(P_{W_{n-2}})$ . As  $f$  is an isomorphism, we deduce that  $f(\text{Stab}_{\Gamma_1}(\mathcal{S}))$  also satisfies  $(P_{W_{n-2}})$ . Proposition 4.7.9 implies that there exists a unique equivalence class of  $W_{n-2}$ -stars  $\mathcal{S}'$  such that  $f(\text{Stab}_{\Gamma_1}(\mathcal{S})) \subseteq \text{Stab}_{\Gamma_2}(\mathcal{S}')$ , where the inclusion holds up to a finite index subgroup. Applying the same argument with  $f^{-1}$ , we see that there exists an equivalence class  $\mathcal{S}''$  of a  $W_{n-2}$ -star such that

$$\text{Stab}_{\Gamma_1}(\mathcal{S}) \subseteq f^{-1}(\text{Stab}_{\Gamma_2}(\mathcal{S}')) \subseteq \text{Stab}_{\Gamma_1}(\mathcal{S}''),$$

where the inclusion holds up to a finite index subgroup. Lemma 4.6.5 then implies that  $\mathcal{S}$  is the unique equivalence class of  $W_{n-2}$ -stars virtually fixed by  $\text{Stab}_{\Gamma_1}(\mathcal{S})$ . Therefore, we see that  $\mathcal{S} = \mathcal{S}''$  and we have equality everywhere. This completes the proof.  $\square$

## 4.8 Algebraic characterization of compatibility of $W_{n-2}$ -stars and conclusion

### 4.8.1 Algebraic characterization of compatibility of $W_{n-2}$ -stars

In this section, we give an algebraic characterization of the fact that two equivalence classes of  $W_{n-2}$ -stars have both a common collapse and a common refinement. This will imply that  $\text{Comm}(\text{Out}(W_n))$  preserves the set of pairs of commensurability classes of stabilizers of adjacent pairs in the graph  $X_n$  introduced in Definition 4.3.2 (2).



Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . We consider the following properties of a pair  $(H_1, H_2)$  of subgroups of  $\Gamma$ .

$(P_{\text{comp}})$  The pair  $(H_1, H_2)$  satisfies the following properties.

- (1) For every  $i \in \{1, 2\}$ , the group  $H_i$  satisfies  $(P_{W_{n-2}})$ .
- (2) For every normal subgroups  $K_1^{(1)} \times K_2^{(1)}$  of  $H_1$  and  $K_1^{(2)} \times K_2^{(2)}$  of  $H_2$  given by  $(P_{W_{n-2}})$  (1), there exist  $i, j \in \{1, 2\}$  such that  $K_i^{(1)} \cap K_j^{(2)}$  is infinite.
- (3) The group  $H_1 \cap H_2$  contains a subgroup isomorphic to  $\mathbb{Z}^{n-2}$ .

**Proposition 4.8.1.** *Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be two distinct equivalence classes of  $W_{n-2}$ -stars  $S_1$  and  $S_2$  and, for every  $i \in \{1, 2\}$ , let  $H_i = \text{Stab}_\Gamma(\mathcal{S}_i)$ . Then  $S_1$  and  $S_2$  have a refinement  $S$  which is a  $W_{n-3}$ -star if and only if  $(H_1, H_2)$  satisfies Property  $(P_{\text{comp}})$ .*

**Proof.** We first assume that  $S_1$  and  $S_2$  have a common refinement  $S$  which is a  $W_{n-3}$ -star. Let  $\mathcal{S}$  be the equivalence class of  $S$ . Let us prove that  $(H_1, H_2)$  satisfies  $(P_{\text{comp}})$ . By Proposition 4.6.6, for every  $i \in \{1, 2\}$ , the group  $H_i$  satisfies  $(P_{W_{n-2}})$ . This proves that the pair  $(H_1, H_2)$  satisfies  $(P_{\text{comp}})$  (1).

Let us check Property  $(P_{\text{comp}})$  (2). For every  $i \in \{1, 2\}$ , let  $T_1^{(i)} \times T_2^{(i)}$  be the group of twists of  $\mathcal{S}_i$  and let  $K_1^{(i)} = T_1^{(i)} \cap \Gamma$  and  $K_2^{(i)} = T_2^{(i)} \cap \Gamma$ . By Proposition 4.6.6, for every  $i \in \{1, 2\}$ , the group  $K_1^{(i)} \times K_2^{(i)}$  satisfies  $(P_{W_{n-2}})$  (1) and Lemma 4.7.1 implies that every normal subgroup of  $H_i$  given by  $(P_{W_{n-2}})$  (1) is commensurable with  $K_1^{(i)} \times K_2^{(i)}$ . Thus it suffices to check  $(P_{\text{comp}})$  (2) for  $K_1^{(1)} \times K_2^{(1)}$  and  $K_1^{(2)} \times K_2^{(2)}$ . The group of twists of  $\mathcal{S}$  is isomorphic to a direct product  $A_1 \times A_2 \times A_3$  of three copies of  $W_{n-3}$ . Since  $n \geq 5$ , we have  $n - 3 \geq 2$  and  $W_{n-3}$  is infinite. Since  $S$  is a common refinement of  $S_1$  and  $S_2$  and since  $S$  has 3 orbits of edges there exists a  $W_{n-1}$ -star  $S_0$  which is a common collapse of  $S_1$  and  $S_2$ . Moreover, there exists  $k \in \{1, 2, 3\}$  such that  $A_k$  is contained in the group of twists of  $S_0$ . Therefore, for every  $i \in \{1, 2\}$ , there exists  $j \in \{1, 2\}$  such that the group  $A_k$  is contained in  $T_j^{(i)}$ . Thus, there exist  $i, j \in \{1, 2\}$  such that  $A_k \cap \Gamma \subseteq K_i^{(1)} \cap K_j^{(2)}$ . In particular,  $K_i^{(1)} \cap K_j^{(2)}$  is infinite. This shows  $(P_{\text{comp}})$  (2).

Finally, since  $n \geq 5$ , the  $W_{n-2}$ -stars  $S_1$  and  $S_2$  have a common refinement which is a  $W_2$ -star (take any  $W_2$ -star which refines  $S$ ). Since the group of twists of a  $W_2$ -star contains a subgroup isomorphic to  $\mathbb{Z}^{n-2}$  by Proposition 4.2.5 (3), this shows  $(P_{\text{comp}})$  (3).

Conversely, suppose that  $(H_1, H_2)$  satisfies  $(P_{\text{comp}})$ . For  $i \in \{1, 2\}$ , let  $K_1^{(i)} \times K_2^{(i)}$  be the direct product of the groups of twists in  $\Gamma$  about the two edges of  $\mathcal{S}_i$ . Then for every  $i \in \{1, 2\}$ , the group  $(H_i \cap K_1^{(i)}) \times (H_i \cap K_2^{(i)})$  satisfies  $(P_{W_{n-2}})$  (1) by Proposition 4.6.6. Hence Property  $(P_{\text{comp}})$  (2) implies that there exists  $i, j \in \{1, 2\}$  such that

$$(H_1 \cap K_i^{(1)}) \cap (H_2 \cap K_j^{(2)})$$

is infinite. For  $i \in \{1, 2\}$ , let  $S_1^{(i)}$  and  $S_2^{(i)}$  be the two distinct  $W_{n-1}$ -stars on which  $S_i$  collapses. By Proposition 4.6.12, since  $H_1 \cap H_2$  fixes pointwise the set  $\{\mathcal{S}_1^{(1)}, \mathcal{S}_2^{(1)}, \mathcal{S}_1^{(2)}, \mathcal{S}_2^{(2)}\}$ , and since  $H_1 \cap H_2$  contains a subgroup isomorphic to  $\mathbb{Z}^{n-2}$  by  $(P_{\text{comp}})$  (3), the  $W_{n-1}$ -stars  $S_1^{(1)}, S_2^{(1)}, S_1^{(2)}$  and  $S_2^{(2)}$  are pairwise compatible. Hence  $S_1$  and  $S_2$  have a common refinement  $S$  which is either a  $W_{n-3}$ -star or a  $W_{n-4}$ -star. Since the groups of twists of  $\mathcal{S}_1$  and  $\mathcal{S}_2$  have infinite intersection, the refinement  $S$  cannot be a  $W_{n-4}$ -star since otherwise the  $W_{n-1}$ -stars  $S_1^{(1)}, S_2^{(1)}, S_1^{(2)}$  and  $S_2^{(2)}$  would be pairwise nonequivalent and hence their groups of twists would have trivial intersection. Thus  $S$  is a  $W_{n-3}$ -star. This concludes the proof.  $\square$

## 4.8.2 Conclusion

In this last section, we complete the proof of our main theorem.

**Theorem 4.8.2.** *Let  $n \geq 5$  and let  $\Gamma$  be a finite index subgroup of  $\text{Out}^0(W_n)$ . Then any isomorphism  $f: H_1 \rightarrow H_2$  between two finite index subgroups of  $\Gamma$  is given by conjugation by an element of  $\text{Out}(W_n)$  and the natural map:*

$$\text{Out}(W_n) \rightarrow \text{Comm}(\text{Out}(W_n))$$

*is an isomorphism.*

**Proof.** Suppose that  $\mathcal{S}$  and  $\mathcal{S}'$  are two distinct equivalence classes of  $W_{n-2}$ -stars. Then  $\text{Stab}_\Gamma(\mathcal{S})$  and  $\text{Stab}_\Gamma(\mathcal{S}')$  are not commensurable by Lemma 4.6.5. Proposition 4.7.10 shows that the collection  $\mathcal{I}$  of all commensurability classes of  $\Gamma$ -stabilizers of equivalence classes of  $W_{n-2}$ -stars is  $\text{Comm}(\Gamma)$ -invariant. Proposition 4.8.1 shows that the collection  $\mathcal{J}$  of all pairs  $([\text{Stab}_\Gamma(\mathcal{S})], [\text{Stab}_\Gamma(\mathcal{S}')])$  is also  $\text{Comm}(\Gamma)$ -invariant. Since the natural homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(X_n)$  is an isomorphism by Theorem 4.3.3, the conclusion follows from Proposition 4.2.1 and the fact that  $\text{Comm}(\Gamma)$  is isomorphic to  $\text{Comm}(\text{Out}(W_n))$  since  $\Gamma$  has finite index in  $\text{Out}(W_n)$ .  $\square$

## 4.9 Rigidity of the graph of $W_{n-1}$ -stars

The *graph of  $W_{n-1}$ -stars*, denoted by  $Y_n$ , is the graph whose vertices are the  $W_n$ -equivariant homeomorphism classes of  $W_{n-1}$ -stars, where two equivalence classes  $\mathcal{S}$  and  $\mathcal{S}'$  are joined by an edge if there exist  $S \in \mathcal{S}$  and  $S' \in \mathcal{S}'$  such that  $S$  and  $S'$  are compatible. This graph arises naturally in the study of  $\text{Out}(W_n)$  as it is isomorphic to the full subgraph of the free splitting graph  $\overline{K}_n$  of  $W_n$  whose vertices are equivalence classes of  $W_k$ -stars, with  $k$  varying in  $\{0, \dots, n-1\}$ . As  $\text{Aut}(W_n)$  acts on  $\overline{K}_n$  by precomposition of the marking, we have an induced action of  $\text{Aut}(W_n)$  on  $Y_n$ . As  $\text{Inn}(W_n)$  acts trivially on  $Y_n$ , the action of  $\text{Aut}(W_n)$  induces an action of  $\text{Out}(W_n)$ . We denote by  $\text{Aut}(Y_n)$  the group of graph automorphisms of  $Y_n$ . In this section we prove the following theorem.

**Theorem 4.9.1.** *Let  $n \geq 4$ . The natural homomorphism*

$$\text{Out}(W_n) \rightarrow \text{Aut}(Y_n)$$

is an isomorphism.

In order to prove this theorem, we take advantage of the action of  $\text{Out}(W_n)$  on the graph of  $\{0\}$ -stars and  $F$ -stars  $L_n$ . The strategy in order to prove Theorem 4.9.1 is to construct an injective homomorphism  $\Phi: \text{Aut}(Y_n) \rightarrow \text{Aut}(L_n)$  such that every automorphism in the image preserves the set of  $\{0\}$ -stars and the set of  $F$ -stars.

The homomorphism  $\Phi: \text{Aut}(Y_n) \rightarrow \text{Aut}(L_n)$  is defined as follows. Let  $f \in \text{Aut}(Y_n)$ . Let  $\mathcal{S}$  be the equivalence class of a  $\{0\}$ -star and let  $S$  be a representative of  $\mathcal{S}$ . By Theorem 4.3.7, there exist exactly  $n$   $W_{n-1}$ -stars  $S_1, \dots, S_n$  refined by  $S$ . Moreover, these  $W_{n-1}$ -stars are pairwise compatible. For  $i \in \{1, \dots, n\}$ , let  $\mathcal{S}_i$  be the equivalence class of  $S_i$ . Since  $f$  is an automorphism of  $Y_n$ ,  $f(\mathcal{S}_1), \dots, f(\mathcal{S}_n)$  are pairwise adjacent in  $Y_n$ . Let  $S'_1, \dots, S'_n$  be representatives of respectively  $f(\mathcal{S}_1), \dots, f(\mathcal{S}_n)$  that are pairwise compatible. Then Theorem 4.3.7 implies that there exists a unique common refinement  $S'$  of  $S'_1, \dots, S'_n$  with exactly  $n$  edges. Since, for every  $i \in \{1, \dots, n\}$ , the splitting  $S'_i$  is a  $W_{n-1}$ -star, the splitting  $S'$  is necessarily a  $\{0\}$ -star. Let  $\mathcal{S}'$  be the equivalence class of  $S'$ . We then define  $\Phi(f)(\mathcal{S}) = \mathcal{S}'$ . If  $\mathcal{T}$  is an  $F$ -star, we define  $\Phi(f)(\mathcal{T})$  similarly.

**Lemma 4.9.2.** *Let  $n \geq 4$ . Let  $f \in \text{Aut}(Y_n)$ . Let  $\Phi(f)$  be as above.*

- (1) *The map  $\Phi(f): VL_n \rightarrow VL_n$  induces a graph automorphism  $\tilde{\Phi}(f): L_n \rightarrow L_n$ .*
- (2) *If  $\tilde{\Phi}(f) = \text{id}_{L_n}$ , then  $f = \text{id}_{Y_n}$ .*

**Proof.** We prove the first statement. As  $\Phi(f) \circ \Phi(f^{-1}) = \Phi(f \circ f^{-1}) = \text{id}$ , we see that  $\Phi(f)$  is a bijection. Let  $\mathcal{S}$  be the equivalence class of a  $\{0\}$ -star and let  $\mathcal{T}$  be the equivalence class of an  $F$ -star. Suppose that  $\mathcal{S}$  and  $\mathcal{T}$  are adjacent in  $L_n$ . We prove that  $\Phi(f)(\mathcal{S})$  and  $\Phi(f)(\mathcal{T})$  are adjacent in  $L_n$ . Applying the same result to  $f^{-1}$ , this will prove that  $\mathcal{S}$  and  $\mathcal{T}$  are adjacent in  $L_n$  if and only if  $\Phi(f)(\mathcal{S})$  and  $\Phi(f)(\mathcal{T})$  are adjacent in  $L_n$ , and this will conclude the proof. Let  $S$  and  $T$  be representatives of respectively  $\mathcal{S}$  and  $\mathcal{T}$ . Let  $S_1, \dots, S_n$  be the  $n$   $W_{n-1}$ -stars refined by  $S$ , and let  $T_1, \dots, T_{n-1}$  be the  $n-1$   $W_{n-1}$ -stars refined by  $T$ . As  $S$  refines  $T$ , and as  $S$  refines exactly  $n$   $W_{n-1}$ -stars by Theorem 4.3.7, up to reordering, we can suppose that, for every  $i \in \{1, \dots, n-1\}$ , we have  $S_i = T_i$ . For  $i \in \{1, \dots, n\}$ , let  $\mathcal{S}_i$  be the equivalence class of  $S_i$ , and let  $S'_i$  be a representative of  $\Phi(f)(\mathcal{S}_i)$  such that for distinct  $i, j \in \{1, \dots, n\}$ ,  $S_i$  and  $S_j$  are compatible. Then, by Theorem 4.3.7, a representative  $T'$  of  $\Phi(f)(\mathcal{T})$  is the unique (up to  $W_n$ -equivariant homomorphism)  $F$ -star such that, for every  $j \in \{1, \dots, n-1\}$ ,  $T'$  is compatible with  $S'_j$ . Moreover, a representative  $S'$  of  $\Phi(f)(\mathcal{S})$  is the unique  $\{0\}$ -star such that, for every  $i \in \{1, \dots, n\}$ ,  $S'$  is compatible with  $S'_i$ . For  $i \in \{1, \dots, n\}$ , let  $x_i$  be the preimage by the marking of  $W_n \setminus S'_i$  (well defined up to global conjugation) of the generator of the vertex group isomorphic to  $F$  (which exists since  $S'_i$  is a  $W_{n-1}$ -star). Then the preimages by the marking of  $W_n \setminus T'$  of the generators of the groups associated with the  $n-1$  leaves of the underlying graph of  $W_n \setminus T'$  are  $x_1, \dots, x_{n-1}$  and the preimage by the marking of  $W_n \setminus T'$  of the generator of the group associated with the center of the underlying graph of  $W_n \setminus T'$  is  $x_n$ . Moreover, the preimages by the marking of  $W_n \setminus S'$  of the generators of the groups associated with the  $n$  leaves of the underlying graph of  $W_n \setminus S'$  are  $x_1, \dots, x_n$ . Let  $v_n$  be the leaf of the underlying graph of  $W_n \setminus S'$  such that the

preimage by the marking of  $W_n \setminus S'$  of the generator of the group associated with  $v_n$  is  $x_n$ . Then  $T'$  is obtained from  $S'$  by contracting the edge adjacent to  $v_n$ . Thus  $\Phi(f)(\mathcal{S})$  and  $\Phi(f)(\mathcal{T})$  are adjacent in  $L_n$ .

The proof of the second statement is identical to the proof of [Gue2, Lemma 5.4]. We add the proof for completeness as the statement of [Gue2, Lemma 5.4] is about automorphisms of  $\overline{K}_n$ . Let  $\mathcal{S} \in VY_n$  and let  $S$  be a representative of  $\mathcal{S}$ . We prove that  $f(\mathcal{S}) = \mathcal{S}$ . Let

$$W_n = \langle x_1, \dots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of  $W_n$  induced by  $S$ . Let  $S'$  be a representative of  $f(\mathcal{S})$ . Let  $\mathcal{X}$  be the equivalence class of the  $F$ -star  $X$  represented in Figure 3.13 on the left.

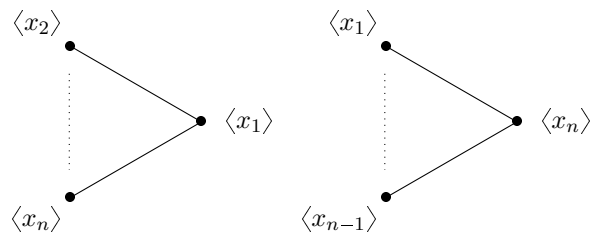


Figure 4.4: The  $F$ -stars  $X$  and  $X'$  of the proof of Lemma 4.9.2.

Since  $\Phi(f)(\mathcal{X}) = \mathcal{X}$ , the free splitting  $S'$  is a  $W_{n-1}$ -star obtained from  $X$  by collapsing  $n - 1$  edges. But if  $T$  is a  $W_{n-1}$ -star obtained from  $X$  by collapsing  $n - 1$  edges, then there exists  $i \in \{1, \dots, n\}$  such that the free factor decomposition of  $W_n$  induced by  $T$  is

$$W_n = \langle x_1, \dots, \hat{x}_i, \dots, x_n \rangle * \langle x_i \rangle.$$

For  $i \in \{1, \dots, n\}$ , we will denote by  $T_i$  the  $W_{n-1}$ -star with associated free factor decomposition  $\langle x_1, \dots, \hat{x}_i, \dots, x_n \rangle * \langle x_i \rangle$ , and by  $\mathcal{T}_i$  its equivalence class. For  $i \neq n$ , the free splitting  $T_i$  is a collapse of the  $F$ -star  $X'$  depicted in Figure 4.4 on the right, whereas  $S$  is not a collapse of  $X'$ .

Let  $\mathcal{X}'$  be the equivalence class of  $X'$ . Since  $\Phi(f)(\mathcal{X}') = \mathcal{X}'$ , there does not exist a representative of  $f(\mathcal{S})$  that is obtained from a representative of  $\mathcal{X}'$  by collapsing a forest. Thus, for all  $i \neq n$ , we have  $f(\mathcal{S}) \neq \mathcal{T}_i$ . Therefore, as  $\mathcal{S} = \mathcal{T}_n$ , we conclude that  $f(\mathcal{S}) = \mathcal{S}$ .  $\square$

**Proof of Theorem 4.9.1.** Let  $n \geq 4$ . We first prove injectivity. The homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$  is injective by Theorem 4.3.5. Moreover, the homomorphism  $\text{Out}(W_n) \rightarrow \text{Aut}(L_n)$  factors through  $\text{Out}(W_n) \rightarrow \text{Aut}(Y_n) \rightarrow \text{Aut}(L_n)$ . Therefore we deduce the injectivity of  $\text{Out}(W_n) \rightarrow \text{Aut}(Y_n)$ . We now prove surjectivity. Let  $f \in \text{Aut}(Y_n)$ . By Lemma 4.9.2 (1), we have a homomorphism  $\Phi: \text{Aut}(Y_n) \rightarrow \text{Aut}(L_n)$  whose image consists in automorphisms preserving the set of  $\{0\}$ -stars and the set of  $F$ -stars. By Theorem 4.3.5, the automorphism  $\Phi(f)$  is induced by an element  $\gamma \in \text{Out}(W_n)$ . Since the homomorphism  $\text{Aut}(Y_n) \rightarrow \text{Aut}(L_n)$  is injective by Lemma 4.9.2 (2),  $f$  is induced by  $\gamma$ . This concludes the proof.  $\square$

## Chapitre 5

# Currents relative to a malnormal subgroup system

### 5.1 Introduction

Let  $n \geq 2$ . This paper is the first of a sequence of papers where we study the exponential growth of elements of  $\text{Out}(F_n)$ , the outer automorphism group of a nonabelian free group  $F_n$  of rank  $n$ . Let  $[g]$  be the conjugacy class of a nontrivial element  $g$  of  $F_n$ , let  $\phi \in \text{Out}(F_n)$  and let  $\Phi \in \text{Aut}(F_n)$  be a representative of  $\phi$ . We say that  $[g]$  has *exponential growth under iterates of  $\phi$*  if there exists a basis  $\mathcal{B}$  of  $F_n$  such that the length of  $[\Phi^n(g)]$  with respect to the word metric relative to  $\mathcal{B}$  grows exponentially fast with  $n$ . It is known, using for instance the technology of relative train tracks (see [BH]) that, otherwise,  $[g]$  has polynomial growth under iterates of  $\phi$ . Let  $\text{Poly}(\phi)$  be the set of conjugacy classes of elements of  $F_n$  whose growth under iteration of  $\phi$  is polynomial. For a subgroup  $H$  of  $F_n$ , let  $\text{Poly}(H) = \bigcap_{\phi \in H} \text{Poly}(\phi)$ . The aim of these three papers is to prove the following result:

**Theorem 5.1.1** ([Gue6]). *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{Out}(F_n)$ . There exists  $\phi \in H$  such that  $\text{Poly}(\phi) = \text{Poly}(H)$ .*

Theorem 5.1.1 is proved using dynamical methods developed mainly in [Gue5]. In the present article, we introduce the topological space associated with the dynamics. Informally, Theorem 5.1.1 shows that the exponential growth of a subgroup  $H$  of  $\text{Out}(F_n)$  is captured by the exponential growth of a single element of  $H$ . In this paper, we construct a space which is well-adapted for our considerations, the *space of currents relative to a malnormal subgroup system*. These relative currents are positive  $F_n$ -invariant Radon measures on an appropriate subspace of the double boundary at infinity of  $F_n$ . Let  $\phi \in \text{Out}(F_n)$ . When the malnormal subgroup system is appropriately chosen, this space has the property that its points corresponding to conjugacy classes of elements in  $F_n - \text{Poly}(\phi)$  are dense in it (see Theorem 5.1.2).

The space of currents that we construct in this paper builds on objects introduced for similar purposes. For instance, the study of the mapping class group  $\text{Mod}(S)$  of a

connected, compact, oriented surface  $S$  has benefited from the study of the action of  $\text{Mod}(S)$  on the *space of geodesic currents*  $\text{Curr}(S)$ , introduced by Ruelle and Sullivan in [RS] (see also the work of Bonahon [Bon1]). It is defined as the space of  $\pi_1(S)$ -invariant and flip invariant nonnegative Radon measures on the double boundary  $\partial^2 \tilde{S}$  of a universal cover  $\tilde{S}$  of  $S$ , equipped with the weak-star topology. Considering the space of projective geodesic currents  $\mathbb{P}\text{Curr}(S)$ , one can show that  $\mathbb{P}\text{Curr}(S)$  can be viewed as a completion of the currents associated with weighted nontrivial homotopy classes of closed curves on  $S$ . The space  $\mathbb{P}\text{Curr}(S)$  is well-adapted to the study of  $\text{Mod}(S)$ . For instance, it can be used for counting closed geodesics whose length is bounded by a given constant when the surface  $S$  is equipped with a hyperbolic metric (see [EU] for a survey). Concerning dynamical properties, a result of Thurston ([Thu], see also [Uya1]) implies that pseudo-Anosov diffeomorphisms act with North-South dynamics on the space  $\mathbb{P}\text{Curr}(S)$ : every pseudo-Anosov element  $f \in \text{Mod}(S)$  has exactly two fixed points in  $\mathbb{P}\text{Curr}(S)$  and any other nonfixed point in  $\mathbb{P}\text{Curr}(S)$  converges to one of the fixed points under positive or negative iterates of  $f$ . Moreover, this convergence can be made uniform on compact subsets of  $\mathbb{P}\text{Curr}(S)$  which do not contain the fixed points.

In the specific context of free groups, building on [Bon3] for general hyperbolic groups, the space of currents  $\text{Curr}(F_n)$  was first studied by Martin [Mar]. It is defined as the space of  $F_n$ -invariant, flip invariant nonnegative Radon measure on the double boundary  $\partial^2 F_n$  of  $F_n$  equipped with the weak-star topology. Martin showed that the set of currents associated with conjugacy classes of nontrivial elements of  $F_n$  is dense in the space  $\mathbb{P}\text{Curr}(F_n)$  of projective currents. Currents for free groups have also been studied in [Kap, KapL, CHL]. Similarly to pseudo-Anosov elements of  $\text{Mod}(S)$  on  $\mathbb{P}\text{Curr}(S)$ , fully irreducible automorphisms of  $F_n$  and atoroidal automorphisms of  $F_n$  act with North-South type dynamics on  $\mathbb{P}\text{Curr}(F_n)$  (see [Uya1, Uya2]).

Currents on free groups have also been studied in a relative context, more precisely, in the context of *free factor systems*. A free factor system  $\mathcal{F}$  is a finite set of conjugacy classes  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$  of nontrivial subgroups  $A_1, \dots, A_k$  of  $F_n$  such that there exists a subgroup  $B$  of  $F_n$  with  $F_n = A_1 * \dots * A_k * B$ . Gupta [Gup1] (see also Guirardel-Horbez [GuH1]) introduced the space  $\text{Curr}(F_n, \mathcal{F})$  of currents relative to the free factor system  $\mathcal{F}$ . Relative currents are then  $F_n$ -invariant, flip invariant nonnegative Radon measures on a subspace of the double boundary of  $F_n$  which does not intersect the double boundary of any conjugate of  $A_i$ , equipped with the weak-star topology. Gupta [Gup1] then showed that the set of currents associated with conjugacy classes of *nonperipheral elements* of  $F_n$ , that is, elements of  $F_n$  that do not belong to any conjugate of some  $A_i$ , is dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F})$ . She then showed that fully irreducible outer automorphisms relative to  $\mathcal{F}$  act with a North-South type dynamics on  $\mathbb{P}\text{Curr}(F_n, \mathcal{F})$ .

In order to study the purely exponential growth part of an outer automorphism of  $F_n$ , we need to consider currents relative to a class of subgroup systems which is larger than the class of free factor systems. Indeed, if  $\phi \in \text{Out}(F_n)$ , the set of all maximal conjugacy classes of subgroups of  $F_n$  consisting of elements with polynomial growth under iterates of  $\phi$  is not necessarily a free factor system. However, Levitt [Lev2, Proposition 1.4] proved that this set is a *malnormal subgroup system*. A malnormal subgroup system  $\mathcal{A}$

is a finite set of conjugacy classes  $\mathcal{A} = \{[A_1], \dots, [A_k]\}$  of nontrivial subgroups of  $F_n$  such that, for every  $i \in \{1, \dots, k\}$ , the group  $A_i$  is malnormal and, for every subgroups  $B_1, B_2$  of  $F_n$  such that  $[B_1], [B_2] \in \mathcal{A}$ , if the intersection  $B_1 \cap B_2$  is nontrivial, then  $B_1 = B_2$ . A free factor system is, in particular, a malnormal subgroup system but the converse does not hold (see Section 5.2).

Let  $\mathcal{A} = \{[A_1], \dots, [A_k]\}$  be a malnormal subgroup system. We define the space  $\text{Curr}(F_n, \mathcal{A})$  of *currents relative to  $\mathcal{A}$*  as the space of  $F_n$ -invariant, flip invariant nonnegative Radon measures on a natural space  $\partial^2(F_n, \mathcal{A})$ , the double boundary of  $F_n$  relative to  $\mathcal{A}$  equipped with the weak-star topology. The space  $\partial^2(F_n, \mathcal{A})$  is a subspace of  $\partial^2 F_n$  which does not intersect the double boundary of any conjugate of  $A_i$  (see Section 5.2.4 for precise definitions). In this article, we prove the following result. An element of  $F_n$  is *non- $\mathcal{A}$ -peripheral* if it is not contained in any conjugate of any  $A_i$  with  $i \in \{1, \dots, k\}$ .

**Theorem 5.1.2.** *Let  $n \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system. The set of currents associated with conjugacy classes of non- $\mathcal{A}$ -peripheral elements of  $F_n$  is dense in the space  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$  of projective currents relative to  $\mathcal{A}$ .*

Let  $\phi \in \text{Out}(F_n)$ . If  $\mathcal{A}$  is the set of conjugacy classes of maximal polynomial subgroups of  $\phi$ , then Theorem 5.1.2 shows that the set of projective currents associated with exponentially growing elements of  $F_n$  under iterates of  $\phi$  is dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$ . Therefore, the space  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$  is a natural topological space for the study of the action of  $\phi$  on elements of  $F_n$  with exponential growth under iterates of  $\phi$ . A subsequent paper [Gue5] will then show that  $\phi$  acts with North-South type dynamics on  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$ . This North-South dynamics will be a central argument in the proof of Theorem 5.1.1.

We now give an outline of the proof of Theorem 5.1.2. The proof follows the one of a similar result in the context of currents relative to free factor systems due to Gupta [Gup1]. However, in the case of free factor systems, the proof relies on the existence of an adapted free basis of  $F_n$  associated with the free factor system, which does not necessarily exist in the case of malnormal subgroup systems. Our new argument in order to overcome this difficulty is the description of a finite set of elements of  $F_n$  associated with a malnormal subgroup system and a free basis of  $F_n$  which completely determines whether an element of  $F_n$  is contained in a conjugate of a subgroup of the malnormal subgroup system or not (see Lemma 5.2.3).

Let  $\mathcal{A}$  be a malnormal subgroup system and let  $\mu \in \mathbb{P}\text{Curr}(F_n, \mathcal{A})$ . We first show that  $\mu$  can be extended into a *signed measured current  $\tilde{\mu}$  on  $F_n$* , that is an  $F_n$ -invariant and flip invariant Radon measure on  $\partial^2 F_n$ . Even though  $\tilde{\mu}$  might have negative values, we show that  $\tilde{\mu}$  can be chosen so that  $\tilde{\mu}$  gives positive value to sufficiently many Borel subsets of  $\partial^2 F_n$ . One can then use the density of currents associated with conjugacy classes of nontrivial elements of  $F_n$  in the space  $\text{Curr}(F_n)$  in order to conclude the proof.

To our knowledge, the objects we construct in this paper have not been studied or constructed for larger classes of groups, such as relatively hyperbolic groups and quasi-convex almost malnormal subgroups of hyperbolic groups. Nevertheless, the extension of our definitions to this context seems natural since a result of Bowditch [Bow, Theorem 7.11] shows that the group  $F_n$  is always hyperbolic relative to a malnormal subgroup

system  $\mathcal{A}$ . But as we explained in Remark 5.2.8, the natural double boundary associated with a relative hyperbolic group will have less information than the boundary  $\partial^2(F_n, \mathcal{A})$ . Therefore, it would require new techniques to develop the notion of currents for relative hyperbolic groups or quasi-convex almost malnormal subgroups of hyperbolic groups.

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## 5.2 Malnormal subgroup systems

### 5.2.1 Malnormal subgroup systems

Let  $n$  be an integer greater than 1 and let  $F_n$  be a free group of rank  $n$ . In this section, we define, following Handel and Mosher [HaM4, Section I.1.1.2], malnormal subgroups systems and study some of their properties.

A *subgroup system* of  $F_n$  is a finite (possibly empty) set  $\mathcal{A}$  whose elements are conjugacy classes of nontrivial (that is distinct from  $\{1\}$  and  $F_n$ ) finite rank subgroups of  $F_n$ . Note that a subgroup system  $\mathcal{A}$  is completely determined by the set of subgroups  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$ . There exists a preorder on the set of subgroup systems of  $F_n$ , where  $\mathcal{A}_1 \leq \mathcal{A}_2$  if for every subgroup  $A_1$  of  $F_n$  such that  $[A_1] \in \mathcal{A}_1$ , there exists a subgroup  $A_2$  of  $F_n$  such that  $[A_2] \in \mathcal{A}_2$  and  $A_1$  is a subgroup of  $A_2$ . The *stabilizer in  $\text{Out}(F_n)$  of a subgroup system  $\mathcal{A}$* , denoted by  $\text{Out}(F_n, \mathcal{A})$ , is the set of all elements  $\phi \in \text{Out}(F_n)$  such that  $\phi(\mathcal{A}) = \mathcal{A}$ .

Recall that a subgroup  $A$  of  $F_n$  is *malnormal* if for every element  $x \in F_n - A$ , we have  $xAx^{-1} \cap A = \{e\}$ . A subgroup system  $\mathcal{A}$  is said to be *malnormal* if every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  is malnormal and, for any subgroups  $A_1, A_2$  of  $F_n$  such that  $[A_1], [A_2] \in \mathcal{A}$ , if  $A_1 \cap A_2$  is nontrivial then  $A_1 = A_2$ . There are equivalent formulations of malnormality which we present now (see [HaM4, Section I.1.1.2]). Let  $T$  be the Cayley graph of  $F_n$  with respect to some given free basis of  $F_n$ . For every subgroup  $A$  of  $F_n$ , let  $T_A$  be the minimal  $A$ -invariant subtree of  $T$ . Then a subgroup system  $\mathcal{A}$  made of conjugacy classes of malnormal subgroups is malnormal if and only if there exists a finite constant  $L > 0$  such that for any distinct subgroups  $A_1, A_2$  of  $F_n$  such that  $[A_1], [A_2] \in \mathcal{A}$ , the diameter of the intersection  $T_{A_1} \cap T_{A_2}$  is at most equal to  $L$ . Malnormality of a subgroup system  $\mathcal{A}$  made of conjugacy classes of malnormal subgroups is also equivalent to the fact that, for any distinct subgroups  $A_1$  and  $A_2$  of  $F_n$  such that  $[A_1], [A_2] \in \mathcal{A}$ , we have  $\partial_\infty T_{A_1} \cap \partial_\infty T_{A_2} = \emptyset$ .

### 5.2.2 Properness at infinity

Let  $\partial_\infty F_n$  be the Gromov boundary of  $F_n$ . Let  $\mathcal{B}$  be a free basis of  $F_n$  and let  $T$  be the Cayley graph of  $F_n$  with respect to  $\mathcal{B}$ . For convenience, we suppose that  $\mathcal{B}^{-1} = \mathcal{B}$ . The boundary of  $T$  is naturally homeomorphic to  $\partial_\infty F_n$ . For an element  $w \in F_n$ , we denote by  $\gamma_w$  the path in  $T$  starting from  $e$  corresponding to the word  $w$ . We denote by



$w^{+\infty}$  the element in  $\partial_\infty F_n$  corresponding to the quasi-geodesic starting at  $e$  obtained by concatenating paths in  $T$  labeled by  $w$ .

Let  $A$  be a subgroup of  $F_n$  of finite rank. The inclusion  $A \subseteq F_n$  induces an  $A$ -equivariant inclusion  $\partial_\infty A \hookrightarrow \partial_\infty F_n$ . Note that the  $F_n$ -orbit of the image of this map only depends on the conjugacy class of  $A$  in  $F_n$ .

Let  $\mathcal{A}$  be a subgroup system of  $F_n$ . The subgroup system  $\mathcal{A}$  is said to be *proper at infinity* if, for every element  $g$  of  $F_n$ , the following assertions are equivalent:

- there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  and  $g^{+\infty} \in \partial_\infty A$ ;
- there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  and  $g \in A$ .

For the proof of Lemma 5.2.2 below, we need the following result (see for instance [HaM4, Fact 1.2]). This is a particular case of the same result valid for all quasi-convex subgroups  $A_1, A_2$  of any word hyperbolic group, see [Swe], that has been for instance generalized in [Tra, Theorem 1.4].

**Lemma 5.2.1.** *For every finitely generated subgroups  $A_1$  and  $A_2$  of  $F_n$ , we have*

$$\partial_\infty(A_1 \cap A_2) = \partial_\infty A_1 \cap \partial_\infty A_2.$$

A subgroup  $A$  of  $F_n$  is *root-closed* if for every  $g \in F_n$  and every  $k \in \mathbb{N}^*$  such that  $g^k \in A$ , we have  $g \in A$ .

**Lemma 5.2.2.** *Let  $\mathcal{A}$  be a subgroup system. The following are equivalent:*

- (1) *the subgroup system  $\mathcal{A}$  is proper at infinity;*
- (2) *every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  is root-closed.*

*In particular, a malnormal subgroup system is proper at infinity.*

**Proof.** Suppose that  $\mathcal{A}$  is proper at infinity and let  $A$  be a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}$ . Let  $g \in F_n$  and  $k \in \mathbb{N}^*$  be such that  $g^k \in A$ . Let us prove that  $g \in A$ . Since  $g^k \in A$ , we see that  $g^{+\infty} \in \partial_\infty A$ . Since  $\mathcal{A}$  is proper at infinity, we have  $g \in A$ . Hence  $A$  is root-closed. Suppose now that every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  is root-closed. Let  $g \in F_n$  and let  $A$  be a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}$  and  $g^{+\infty} \in \partial_\infty A$ . By Lemma 5.2.1 applied to  $\langle g \rangle$  and  $A$ , there exists  $k \in \mathbb{N}^*$  such that  $g^k \in A$ . Since  $A$  is root-closed, we see that  $g \in A$ . Hence  $\mathcal{A}$  is proper at infinity. This shows the equivalence.

Let  $\mathcal{A}$  be a malnormal subgroup system and let  $A$  be a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}$ . We prove that  $A$  is root-closed. Let  $g \in F_n$  and let  $k \in \mathbb{N}^*$  be such that  $g^k \in A$ . We claim that  $g \in A$ . Indeed, suppose towards a contradiction that  $g \notin A$ . Then  $g^k = gg^k g^{-1}$  belongs to  $A \cap gAg^{-1}$  which is equal to  $\{e\}$ , a contradiction.  $\square$

Let  $\mathcal{A}$  be a malnormal subgroup system. An element  $g \in F_n$  is  *$\mathcal{A}$ -peripheral* (or simply *peripheral* if there is no ambiguity) if it is trivial or conjugate into one of the subgroups of  $\mathcal{A}$ , and *non- $\mathcal{A}$ -peripheral* otherwise. Note that, since  $\mathcal{A} \neq \{[F_n]\}$ , there always exists a non- $\mathcal{A}$ -peripheral element. Since  $\mathcal{A}$  is proper at infinity by Lemma 5.2.2,

we see that an element  $g$  of  $F_n$  is  $\mathcal{A}$ -peripheral if and only if there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  and  $g^{+\infty} \in \partial_\infty A$ .

Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system of  $F_n$ . For every element  $i \in \{1, \dots, r\}$ , let  $T_{A_i}$  be the minimal  $A_i$ -invariant subtree of  $T$ . Suppose that for every  $i \in \{1, \dots, r\}$ , the representative  $A_i$  of  $[A_i]$  is chosen so that the tree  $T_{A_i}$  contains the base point  $e$  of  $T$ .

By malnormality of  $\mathcal{A}$ , there exists  $L \in \mathbb{N}^*$  such that for any distinct subgroups  $A, B$  of  $F_n$  such that  $[A], [B] \in \mathcal{A}$ , the diameter of the intersection  $T_A \cap T_B$  is at most  $L$ . Let  $i \in \{1, \dots, r\}$ . Let  $\Gamma_i$  be the set of subgroups  $B$  of  $F_n$  such that there exists  $g_B \in F_n$  such that  $B = g_B A_i g_B^{-1}$  and the tree  $T_B$  contains the base point  $e$  of  $T$ . Note that, by malnormality of  $\mathcal{A}$ , for every  $i \in \{1, \dots, r\}$ , the set  $\Gamma_i$  is finite. Let  $C_i$  be the set of elements  $w$  of  $F_n$  such that the length of  $\gamma_w$  is equal to  $L + 2$  and, for every  $B \in \Gamma_i$ , the path  $\gamma_w$  is not contained in  $T_B$ . Let  $\mathcal{C} = \bigcap_{i=1}^r C_i$ . Since we are looking at geodesic paths of length equal to  $L + 2$ , the set  $\mathcal{C}$  is finite. If  $\gamma$  is a path in  $T$ , the element of  $F_n$  corresponding to  $\gamma$  is the element  $h \in F_n$  such that the path  $\gamma$  is labeled by  $h$ .

**Lemma 5.2.3.** *Let  $\mathcal{B}, T, \mathcal{A} = \{[A_1], \dots, [A_r]\}$ ,  $L \in \mathbb{N}^*$ ,  $\Gamma_1, \dots, \Gamma_r, \mathcal{C}$  be as above. The finite set  $\mathcal{C} = \mathcal{C}(A_1, \dots, A_r)$  is nonempty. Moreover, it satisfies the following:*

(1) *every element  $g \in F_n$  such that the length of  $\gamma_g$  is at least equal to  $L + 2$  and such that  $\gamma_g$  is not contained in a tree  $T_B$  with  $B \in \bigcup_{i=1}^r \Gamma_i$  contains an element of  $\mathcal{C}$  as a subword. In particular, every non- $\mathcal{A}$ -peripheral cyclically reduced element  $g \in F_n$  has a power which contains an element of  $\mathcal{C}$  as a subword;*

(2) *for every non- $\mathcal{A}$ -peripheral cyclically reduced element  $g \in F_n$ , if  $c_g$  is the geodesic ray in  $T$  starting from  $e$  obtained by concatenating edge paths labeled by  $g$ , there exists an edge path in  $c_g$  labeled by a word in  $\mathcal{C}$  at distance at most  $L + 2$  from  $\bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B$ ;*

(3) *if an element  $w \in F_n$  contains an element of  $\mathcal{C}$  as a subword, then for every  $i \in \{1, \dots, r\}$ , the element  $w$  is not contained in  $A_i$ .*

**Proof.** We first prove that (1) and (2) hold and that  $\mathcal{C}$  is nonempty. Let  $g$  be as in the first claim of Assertion (1). First note that, by the choice of  $L$ , for every  $i, j \in \{1, \dots, r\}$  and every distinct  $A \in \Gamma_i$  and  $B \in \Gamma_j$ , the intersection  $T_A \cap T_B$  is contained in the closed ball of radius  $L$  centered at  $e$ . We consider the geodesic path  $c_g: [0, 1] \rightarrow T$  such that  $c(0) = e$  and such that  $c_g(1)$  is the terminal endpoint of  $\gamma_g$ . Let  $i \in \{1, \dots, r\}$  and let

$$t_0 = \max \left\{ t \in [0, 1] \mid c_g(t) \in \bigcup_{i=1}^r \bigcup_{A \in \Gamma_i} T_A \right\}.$$

The point  $c_g(t_0)$  is a vertex and is distinct from  $c_g(1)$  by assumption. We denote by  $c_{\mathcal{A}}$  the geodesic segment  $c_g \cap \bigcup_{i=1}^r \bigcup_{A \in \Gamma_i} T_A$ .

Suppose first that the length of  $c_{\mathcal{A}}$  is at most equal to  $L + 1$ . Let  $c_0$  be the geodesic segment contained in  $c_g$  which originates at  $c_g(t_0)$  and such that the length of  $c_{\mathcal{A}}c_0$  is equal to  $L + 2$ . Such a path  $c_{\mathcal{A}}c_0$  exists since the length of  $\gamma_g$  is at least equal to  $L + 2$ .

Then the element  $h$  of  $F_n$  corresponding to  $c_{\mathcal{A}}c_0$  is in  $\mathcal{C}$  and is a subword of  $g$ . This concludes the proof in this case.

Suppose now that the length of  $c_{\mathcal{A}}$  is greater than  $L + 1$ . Let  $c_{\mathcal{A}}(t_0 - L - 1)$  be the vertex in  $c_{\mathcal{A}}$  at distance  $L + 1$  from  $c_g(t_0)$ , and let  $g_0$  be the corresponding element of  $F_n$ . Let  $s_0$  be the geodesic path between  $c_{\mathcal{A}}(t_0 - L - 1)$  and  $c_g(t_0)$ . Since the geodesic path  $s_0$  has length equal to  $L + 1$ , there exists a unique  $i \in \{1, \dots, r\}$  and a unique  $A \in \Gamma_i$  such that  $s_0$  is contained in  $T_A$ . Let  $e_0$  be the edge in  $c_g$  which originates at  $c_g(t_0)$ . Let  $h \in F_n$  be the element corresponding to the edge path  $s_1$  between  $c_g(t_0 - L - 1)$  and the terminal point of  $e_0$ . We claim that  $h \in \mathcal{C}$ . Indeed, suppose towards a contradiction that  $h \notin \mathcal{C}$ . Then there exists  $j \in \{1, \dots, r\}$  and  $B \in \Gamma_j$  such that the edge path  $\gamma_h$  is contained in  $T_B$ . Since  $\gamma_h$  has length equal to  $L + 2$ , the integer  $j$  and the subgroup  $B$  are unique. Remark that  $g_0^{-1}$  sends the geodesic path  $s_0$  to the initial segment of length  $L + 1$  of  $\gamma_h$ . Since  $g_0^{-1}s_0$  has length equal to  $L + 1$ , the subgroup  $B$  is the unique element of  $\bigcup_{\ell=1}^r \Gamma_{\ell}$  such that the tree  $T_B$  contains  $g_0^{-1}s_0$ . But  $s_0$  is contained in  $T_A$  and the tree  $T_A$  is sent by  $g_0^{-1}$  to the tree  $T_{g_0^{-1}Ag_0}$ . Therefore, we see that  $B = g_0^{-1}Ag_0$ . But  $g_0^{-1}$  induces an isometry between  $T_A$  and  $T_{g_0^{-1}Ag_0}$ . Therefore, since  $s_1$  is not contained in  $T_A$ , we see that  $\gamma_h = g_0^{-1}s_1$  is not contained in  $T_{g_0^{-1}Ag_0}$ . This leads to a contradiction. Hence  $h \in \mathcal{C}$  and  $h$  is a subword of  $g$ . This proves the first claim of Assertion (1). We now prove the second claim of Assertion (1). Let  $g$  be a non- $\mathcal{A}$ -peripheral cyclically reduced element of  $F_n$ . Let  $c'_g: \mathbb{R}_+ \rightarrow T$  be the geodesic ray in  $T$  starting from  $e$  obtained by concatenating edge paths labeled by  $g$ . Recall that, for every  $i \in \{1, \dots, r\}$ , the set  $\Gamma_i$  is finite. Therefore, since  $g$  is nonperipheral and since  $\mathcal{A}$  is proper at infinity by Lemma 5.2.2, the intersection of  $c'_g$  with  $\bigcup_{i=1}^r \bigcup_{A \in \Gamma_i} T_A$  is compact. Hence there exists a power of  $g$  which satisfies the first claim of Assertion (1). This proves (1). Moreover, the terminal endpoint of the path in  $c_g$  labeled by  $h$  which we have constructed is either at distance  $L + 2$  from  $e$  or is at distance at most 1 from  $\bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B$ . This proves (2). This also proves that  $\mathcal{C}$  is nonempty as there exists a non- $\mathcal{A}$ -peripheral element.

We now prove (3). Suppose towards a contradiction that there exist  $i \in \{1, \dots, r\}$  and  $a \in A_i$  such that  $a$  contains a word of  $\mathcal{C}$  as a subword. Thus there exist  $x \in \mathcal{C}$ ,  $b, c \in F_n$  such that  $a = bxc$  and the word  $bxc$  is reduced. Then since  $e$  is contained in  $T_{A_i}$ , the path  $\gamma_a$  is contained in  $T_{A_i}$ . But the element  $b^{-1}$  sends the tree  $T_{A_i}$  to the tree  $T_{b^{-1}A_i b}$ . Moreover, since  $T_{A_i}$  contains the vertex labeled by  $b$ , the tree  $T_{b^{-1}A_i b}$  contains the base point  $e$  of  $T$ . But then  $T_{b^{-1}A_i b}$  contains the geodesic segment  $\gamma_x$ . This contradicts the fact that  $x \in \mathcal{C} \subseteq C_i$ . This concludes the proof.  $\square$

### 5.2.3 Examples of malnormal subgroup systems

Let  $n$  be an integer greater than 1 and let  $F_n$  be a free group of rank  $n$ . In this section, we give some examples of malnormal subgroup systems. The first one that we describe, following Handel and Mosher [HaM4], is an  $\mathbb{R}$ -vertex group system. Let  $T$  be an  $\mathbb{R}$ -tree equipped with a minimal, isometric action of  $F_n$  for which no point or end of  $T$  is fixed by the whole group and with trivial arc stabilizers. A proper, nontrivial subgroup  $A$  of  $F_n$  is an  $\mathbb{R}$ -vertex group of  $T$  if there exists a point  $x \in T$  such that  $A = \text{Stab}(x)$ . Note

that every free factor of  $F_n$  is an  $\mathbb{R}$ -vertex group of some simplicial tree. Every  $\mathbb{R}$ -vertex group has rank at most equal to  $n$  (see [GaL]).

The  $\mathbb{R}$ -vertex group system of  $T$ , denoted by  $\mathcal{A}_T$ , is the set consisting of all conjugacy classes of nontrivial point stabilizers in  $T$ . The set  $\mathcal{A}_T$  is finite and its cardinality is bounded from above by a finite constant depending only on  $n$  (see [GaL]). Therefore the set  $\mathcal{A}_T$  is a subgroup system. Note that every free factor system of  $F_n$  is an  $\mathbb{R}$ -vertex group system of some simplicial tree. However, there exist  $\mathbb{R}$ -vertex group systems which are not free factor systems. For example, let  $S$  be a compact connected oriented hyperbolic surface with one totally geodesic boundary component such that  $\pi_1(S)$  is isomorphic to  $F_n$ . Let  $T$  be the  $\mathbb{R}$ -tree dual to the lift  $\tilde{\Lambda}$  to  $\mathbb{H}_2$  of a measured geodesic lamination  $\Lambda$  without compact leaves on  $S$ . An identification of  $\pi_1(S)$  with  $F_n$  induces an action of  $F_n$  on  $T$  which has trivial arc stabilizers. Moreover, the fundamental group of the connected component containing the boundary curve of  $S$  is the stabilizer of a point in  $T$ . Since the fundamental group of this connected component is not a free factor of  $F_n$ , this shows that  $\mathcal{A}_T$  is not a free factor system. More generally, Handel and Mosher [HaM4, Proposition 3.3] give general constructions of  $\mathbb{R}$ -vertex group systems which are not free factor systems.

**Lemma 5.2.4.** [HaM4, Lemma 3.1] *The subgroup system  $\mathcal{A}_T$  is a malnormal subgroup system.*

Another example of malnormal subgroup systems is the following. An outer automorphism  $\phi \in \text{Out}(F_n)$  is *exponentially growing* if there exists  $g \in F_n$  such that the length of the conjugacy class  $[g]$  of  $g$  in  $F_n$  with respect to some basis of  $F_n$  grows exponentially fast under iteration of  $\phi$ . If  $\phi \in \text{Out}(F_n)$  is not exponentially growing, then the length of the conjugacy class of every element of  $F_n$  is polynomially growing under iteration of  $\phi$  and  $\phi$  is said to be *polynomially growing*. One similarly says that an automorphism  $\alpha \in \text{Aut}(F_n)$  is exponentially growing or polynomially growing. Let  $\phi \in \text{Out}(F_n)$  be exponentially growing. A subgroup  $P$  of  $F_n$  is a *polynomial subgroup* of  $\phi$  if there exist  $k \in \mathbb{N}^*$  and a representative  $\alpha$  of  $\phi^k$  such that  $\alpha(P) = P$  and  $\alpha|_P$  is polynomially growing. By [Lev2, Proposition 1.4], there exist finitely many conjugacy classes  $[H_1], \dots, [H_k]$  of maximal polynomial subgroups of  $\phi$  and the set  $\mathcal{H} = \{[H_1], \dots, [H_k]\}$  is a malnormal subgroup system.

#### 5.2.4 Double boundary of $F_n$ relative to a malnormal subgroup system

In this section, we construct a boundary of  $F_n$  relative to a malnormal subgroup system. We follow a similar construction made by Gupta in [Gup1, Section 3.1] in the case of the boundary relative to a free factor system.

The *double boundary* of  $F_n$  is the quotient topological space

$$\partial^2 F_n = (\partial_\infty F_n \times \partial_\infty F_n \setminus \Delta) / \sim,$$

where  $\sim$  is the equivalence relation generated by the flip relation  $(x, y) \sim (y, x)$  and  $\Delta$  is the diagonal, endowed with the diagonal action of  $F_n$ . We denote by  $\{x, y\}$  the equivalence class of  $(x, y)$ .

Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system of  $F_n$ . Let  $\mathcal{B}, T, L \in \mathbb{N}^*$ ,  $\Gamma_1, \dots, \Gamma_r, \mathcal{C}$  be as above Lemma 5.2.3. The boundary of  $T$  is naturally homeomorphic to  $\partial_\infty F_n$  and the set  $\partial^2 F_n$  is then identified with the set of unoriented bi-infinite geodesics in  $T$ . Let  $\gamma$  be a finite geodesic path in  $T$ . The path  $\gamma$  determines a subset in  $\partial^2 F_n$  called the *cylinder set of  $\gamma$* , denoted by  $C(\gamma)$ , which consists in all unoriented bi-infinite geodesics in  $T$  that contain  $\gamma$ . Such cylinder sets form a basis for a topology on  $\partial^2 F_n$ , and in this topology, the cylinder sets are both open and compact, hence closed since  $\partial^2 F_n$  is Hausdorff. The action of  $F_n$  on  $\partial^2 F_n$  has a dense orbit.

Let  $A$  be a nontrivial subgroup of  $F_n$  of finite rank. The induced  $A$ -equivariant inclusion  $\partial_\infty A \hookrightarrow \partial_\infty F_n$  induces an inclusion  $\partial^2 A \hookrightarrow \partial^2 F_n$ . Let

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^r \bigcup_{g \in F_n} \partial^2 g A_i g^{-1}.$$

Let  $\partial^2(F_n, \mathcal{A}) = \partial^2 F_n - \partial^2 \mathcal{A}$  be the *double boundary of  $F_n$  relative to  $\mathcal{A}$* . This subset is invariant under the action of  $F_n$  on  $\partial^2 F_n$  and inherits the subspace topology of  $\partial^2 F_n$ , denoted by  $\tau$ .

**Lemma 5.2.5.** *Let  $\text{Cyl}(\mathcal{C})$  be the set of cylinder sets of the form  $C(\gamma)$ , where the element of  $F_n$  determined by the geodesic edge path  $\gamma$  contains an element of  $\mathcal{C}$  as a subword. We have*

$$\partial^2(F_n, \mathcal{A}) = \bigcup_{C(\gamma) \in \text{Cyl}(\mathcal{C})} C(\gamma).$$

*In particular, the space  $\partial^2(F_n, \mathcal{A})$  is an open subset of  $\partial^2 F_n$ .*

**Proof.** Let  $y \in \partial^2(F_n, \mathcal{A})$ . Let  $c$  be an oriented geodesic line  $c$  in  $T$  which belongs to the equivalence class  $y$ . Let  $v$  be a vertex of  $T$  contained in  $c$  and let  $g_0$  be the corresponding element of  $F_n$ .

Suppose first that the intersection  $c \cap g_0 \left( \bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B \right)$  is either compact or a half-line. In particular, the intersection  $c \cap g_0 \left( \bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B \right)$  has a terminal point  $v'$ . Let  $x$  be the vertex in  $c$  at distance  $L + 2$  from  $v'$ . Let  $g \in F_n$  be the element corresponding to the geodesic edge path between  $v$  and  $x$ . Note that the edge path  $\gamma_g$  is not contained in  $\bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B$  since, for every nontrivial subgroup  $A$  of  $F_n$  of finite rank, the element  $g_0$  sends  $T_A$  to  $T_{g_0 A g_0^{-1}}$ . By Lemma 5.2.3 (2), the word  $g$  contains a word of  $\mathcal{C}$  as a subword. Then  $y \in g_0 C(\gamma_g)$ , and  $g_0 C(\gamma_g) \in \text{Cyl}(\mathcal{C})$ .

Suppose now that the intersection  $c \cap g_0 \left( \bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B \right)$  is not compact. Since  $y \in \partial^2(F_n, \mathcal{A})$ , the path  $c$  cannot be contained in a single tree  $g_0 T_B$  with  $B \in \bigcup_{i=1}^r \Gamma_i$ . By the definition of  $L$ , there exist exactly two subgroups  $A, B \in \bigcup_{i=1}^r \Gamma_i$  such that  $c$  is contained in  $g_0 T_A \cup g_0 T_B$ . By the definition of the constant  $L$ , the intersection  $g_0 T_A \cap g_0 T_B$  has diameter at most equal to  $L$ . Let  $c_0$  be the subpath of  $c$  of length  $2L + 2$  whose middle point is  $v$  and whose starting point is in  $g_0 T_A$  and let  $g$  be the element of  $F_n$  corresponding to  $c_0$ . Let  $v'$  be the initial vertex of  $c_0$  and let  $g'$  be the element of  $F_n$  associated with  $v'$ . Note that the intersection of  $c_0$  with  $g_0 T_A$  and  $g_0 T_B$  has length at least equal to  $L + 1$ . Up to considering a larger path  $c_0$ , we may suppose

that  $g$  is cyclically reduced. We claim that  $g$  contains an element of  $\mathcal{C}$  as a subword. Indeed, suppose towards a contradiction that  $g$  does not contain an element of  $\mathcal{C}$  as a subword. By Lemma 5.2.3 (1), there exist  $i \in \{1, \dots, r\}$  and  $H \in \Gamma_i$  such that  $\gamma_g \subseteq T_H$ . But then  $g'\gamma_g = c_0$  and is contained in  $g'T_H$ . Thus the diameter of the intersection  $g'T_H$  with  $g_0T_A$  and  $g_0T_B$  is at least equal to  $L + 1$ . By definition of  $L$ , this means that  $g'T_H = g_0T_A = g_0T_B$ . This means that  $A = B$ , a contradiction. Hence  $g$  contains an element of  $\mathcal{C}$  as a subword. Thus we have  $y \in g_0C(\gamma_g)$ , with  $g_0C(\gamma_g) \in \text{Cyl}(\mathcal{C})$ . Therefore, we see that

$$\partial^2(F_n, \mathcal{A}) \subseteq \bigcup_{C(\gamma) \in \text{Cyl}(\mathcal{C})} C(\gamma).$$

Conversely, let  $\gamma$  be a geodesic path in  $T$  such that  $C(\gamma) \in \text{Cyl}(\mathcal{C})$ . Suppose towards a contradiction that there exists  $y \in \partial^2\mathcal{A}$  such that  $y \in C(\gamma)$ . Thus, there exist elements  $i \in \{1, \dots, r\}$ ,  $g \in F_n$  and  $a \in gA_i g^{-1}$  such that  $\{a^{+\infty}, a^{-\infty}\} \in C(\gamma)$ . Therefore, we see that  $\gamma$  is a subpath of  $T_{gA_i g^{-1}}$ . Decompose  $\gamma$  as  $\gamma = \tau_1 \delta \tau_2$  where  $\delta$  is labeled by a word  $w$  in  $\mathcal{C}$ . Let  $v$  be the origin of  $\delta$  and let  $h$  be the element of  $F_n$  corresponding to  $v$ . Then  $h^{-1}T_{gA_i g^{-1}} = T_{h^{-1}gA_i g^{-1}h} \in \Gamma_i$  and contains  $\gamma_w$  with  $w \in \mathcal{C}$ , a contradiction.  $\square$

Note that Lemma 5.2.5 implies that we can define a topology on  $\partial^2(F_n, \mathcal{A})$ , denoted by  $\tau'$ , where cylinder sets in  $\text{Cyl}(\mathcal{C})$  generate the topology. Lemma 5.2.5 also implies that the two topologies  $\tau$  and  $\tau'$  are equal.

Since  $\partial^2 F_n$  is locally compact and since  $\partial^2(F_n, \mathcal{A})$  is an open subset of  $\partial^2 F_n$  by Lemma 5.2.5, we have the following result.

**Lemma 5.2.6.** *The space  $\partial^2(F_n, \mathcal{A})$  is locally compact.*  $\square$

**Lemma 5.2.7.** *The action of  $F_n$  on  $\partial^2(F_n, \mathcal{A})$  has a dense orbit.*

**Proof.** Recall that there exists  $g \in F_n$  such that the action of  $g$  on  $\partial^2 F_n$  has a dense orbit. Since  $\partial^2(F_n, \mathcal{A})$  is an open subset of  $\partial^2 F_n$ , the element  $g$  also acts on  $\partial^2(F_n, \mathcal{A})$  with a dense orbit.  $\square$

**Remark 5.2.8.** *We now compare our definition with other natural constructions of double boundaries. The first one is to see the double boundary of  $F_n$  relative to a malnormal subgroup system as the double boundary of a Gromov hyperbolic space. Indeed, if  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  is a malnormal subgroup system, by a result of Bowditch (see [Bow, Theorem 7.11]), the group  $F_n$  is hyperbolic relative to  $\mathcal{A}$ . In particular, there is a natural (that is well-defined up to quasi-isometry) proper geodesic Gromov hyperbolic space  $X$  on which  $F_n$  acts by isometries and such that the subgroups of  $F_n$  whose conjugacy classes are in  $\mathcal{A}$  are precisely the maximal parabolic subgroups of the action of  $F_n$  on the Gromov-boundary of  $X$  (see [Bow] for a precise description of  $X$ ). Thus a natural construction for another type of double boundary of  $F_n$  relative to  $\mathcal{A}$  is to define it as the double boundary of  $X$ . This definition seems to extend to the more general case of relatively hyperbolic groups. However, the relative double boundary  $\partial^2(F_n, \mathcal{A})$  has the advantage of being an open subset of  $\partial^2 F_n$ , so that one can use the cylinder sets of  $\partial^2 F_n$  as a basis for the topology of  $\partial^2(F_n, \mathcal{A})$ . Moreover, the natural application from  $\partial F_n$  to  $\partial X$  sends*

the boundary of a parabolic subgroup to a point. Therefore, the relative double boundary  $\partial^2(F_n, \mathcal{A})$  seems to contain more information about the geodesic lines whose endpoints are in the Gromov boundary of distinct parabolic subgroups.

Another candidate for the double boundary of the pair  $(F_n, \mathcal{A})$  is the following. Let  $\hat{T}$  be the graph obtained from  $T$  by adding one vertex  $v(gA)$  for every coset  $gA$  with  $A$  a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}$  and by adding an edge from  $v(gA)$  to every vertex of  $T$  labeled by an element in  $gA$ . The graph  $\hat{T}$  is Gromov hyperbolic (see for instance [KR, Proposition 2.6] or [Bow]) and the Gromov boundary of  $\hat{T}$  is homeomorphic to the space  $\partial_\infty F_n - \bigcup_{i=1}^r \bigcup_{g \in F_n} \partial_\infty gA_i$  (see for instance [AM, Theorem 1.6] or [DT, Bow]). However, the double boundary  $\partial^2 \hat{T}$  does not contain any geodesic line whose endpoints are in distinct parabolic subgroups, which makes it a proper subspace of  $\partial^2(F_n, \mathcal{A})$  which does not seem to be a union of cylinder sets.

### 5.3 Currents relative to a malnormal subgroup system

In this section, we define currents of  $F_n$  relative to a malnormal subgroup system. We follow the construction of Gupta [Gup1, Section 3.2] of currents relative to a free factor system.

Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system of  $F_n$ . Let  $\mathcal{B}, T, L \in \mathbb{N}^*$ ,  $\Gamma_1, \dots, \Gamma_r, \mathcal{C}$  be as above Lemma 5.2.3.

A *relative current* on  $(F_n, \mathcal{A})$  is an  $F_n$ -invariant nonnegative Radon measure  $\mu$  on the locally compact space (by Lemma 5.2.6)  $\partial^2(F_n, \mathcal{A})$  (that is  $\mu$  gives finite measure to compact subsets of  $\partial^2(F_n, \mathcal{A})$ , is inner and outer regular). The set  $\text{Curr}(F_n, \mathcal{A})$  of all relative currents on  $\partial^2(F_n, \mathcal{A})$  is equipped with the weak-star topology: a sequence  $(\mu_n)_{n \in \mathbb{N}}$  in  $\text{Curr}(F_n, \mathcal{A})^{\mathbb{N}}$  converges to a current  $\mu \in \text{Curr}(F_n, \mathcal{A})$  if and only if for every disjoint clopen subsets  $S, S' \subseteq \partial^2(F_n, \mathcal{A})$ , the sequence  $(\mu_n(S \times S'))_{n \in \mathbb{N}}$  converges to  $\mu(S \times S')$ . The space  $\text{Curr}(F_n, \mathcal{A})$  is naturally identified with the space of non-negative,  $F_n$ -invariant, continuous linear functionals on the space  $C_c(\partial^2(F_n, \mathcal{A}))$  (equipped with the uniform norm) of continuous compactly supported functions of  $\partial^2(F_n, \mathcal{A})$  (see [Coh, Theorem 7.5.5]). Therefore, the space  $\text{Curr}(F_n, \mathcal{A})$  is homeomorphic to a subspace of  $C_c(\partial^2(F_n, \mathcal{A}))^*$  equipped with the weak-star topology. Equipped with the uniform structure induced by the weak-star topology on  $C_c(\partial^2(F_n, \mathcal{A}))^*$ , we see that the space  $\text{Curr}(F_n, \mathcal{A})$  is metrisable and complete (see [Bou, Chap. 3, Section 1, Proposition 14]).

The group  $\text{Out}(F_n, \mathcal{A})$  acts on  $\text{Curr}(F_n, \mathcal{A})$  as follows. Let  $\phi \in \text{Out}(F_n, \mathcal{A})$ , let  $\Phi$  be a representative of  $\phi$ , let  $\mu \in \text{Curr}(F_n, \mathcal{A})$  and let  $C$  be a Borel subset of  $\partial^2(F_n, \mathcal{A})$ . Then, since  $\phi$  preserves  $\mathcal{A}$ , we see that  $\Phi^{-1}(C)$  is a Borel subset of  $\partial^2(F_n, \mathcal{A})$ . Then we set

$$\phi(\mu)(C) = \mu(\Phi^{-1}(C)),$$

which is independent of the choice of the representative  $\Phi$  since  $\mu$  is  $F_n$ -invariant and the extension to the boundary of the action by conjugation and by left translation of  $F_n$  on itself coincide.

We now describe some coordinates for  $\text{Curr}(F_n, \mathcal{A})$ . Recall that  $\text{Cyl}(\mathcal{C})$  is the set of cylinder sets of the form  $C(\gamma)$ , where the element of  $F_n$  determined by the geodesic path

$\gamma$  contains an element of  $\mathcal{C}$  as a subword. Recall that

$$\partial^2(F_n, \mathcal{A}) = \bigcup_{C(\gamma) \in \text{Cyl}(\mathcal{C})} C(\gamma).$$

Let  $\eta \in \text{Curr}(F_n, \mathcal{A})$ . Let  $w \in F_n$  be such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$  and let  $w = w_1 \dots w_k$  be the reduced word associated with  $w$  written in the basis  $\mathcal{B}$ . Then  $C(\gamma_w) = \bigsqcup C(\gamma_{wb})$ , where the union is taken over all elements  $b$  of  $\mathcal{B} = \mathcal{B}^{-1}$  except  $b = w_k^{-1}$ . The  $\sigma$ -additivity of a relative current  $\eta$  implies that:

$$\eta(C(\gamma_w)) = \sum_{b \neq w_k^{-1}} \eta(C(\gamma_{wb})).$$

Finally, we note that, for every element  $w \in F_n$  such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ , we have  $\eta(C(\gamma_w)) = \eta(C(\gamma_{w^{-1}}))$ . Indeed, this follows from the fact that  $C(\gamma_w) = wC(\gamma_{w^{-1}})$  and from the  $F_n$ -invariance of  $\eta$ .

**Lemma 5.3.1.** *Let  $n \geq 3$  and let  $C$  be a compact open subset of  $\partial^2(F_n, \mathcal{A})$ . There exist finite geodesic edge paths  $\gamma_1, \dots, \gamma_k$  such that:*

- (1) *For every  $i \in \{1, \dots, k\}$ , we have  $C(\gamma_i) \in \text{Cyl}(\mathcal{C})$ ;*
- (2) *for every distinct  $i, j \in \{1, \dots, k\}$  we have  $C(\gamma_i) \cap C(\gamma_j) = \emptyset$ ;*
- (3) *we have  $C = \bigcup_{i=1}^k C(\gamma_i)$ .*

**Proof.** Since  $C$  is a compact open subset of  $\partial^2 F_n$ , using the topology  $\tau'$ , the set  $C$  can be written as a union of cylinder sets  $C(\gamma_1), \dots, C(\gamma_\ell)$ , where, for every  $i \in \{1, \dots, \ell\}$ , we have  $C(\gamma_i) \in \text{Cyl}(\mathcal{C})$ . We may suppose that for every distinct  $i, j \in \{1, \dots, \ell\}$ , we have  $C(\gamma_i) \not\subseteq C(\gamma_j)$ . In particular, there does not exist  $i, j \in \{1, \dots, \ell\}$  such that  $\gamma_i \subseteq \gamma_j$ . Let  $m$  be the number of pairs of distinct elements  $i, j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\gamma_j) \neq \emptyset$ . We prove Lemma 5.3.1 by induction on  $m$ . If for every distinct  $i, j \in \{1, \dots, \ell\}$ , we have  $C(\gamma_i) \cap C(\gamma_j) = \emptyset$ , then the set  $\{\gamma_1, \dots, \gamma_\ell\}$  satisfies the conclusion of the lemma. Suppose that there exists  $m$  pairs of distinct elements  $i, j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\gamma_j) \neq \emptyset$ , with  $m \geq 1$ .

**Claim.** Let  $i, j$  be as above. There exists finite geodesic paths  $\gamma_1^{(i)}, \dots, \gamma_{k_i}^{(i)}, \gamma_1^{(j)}, \dots, \gamma_{k_j}^{(j)}$  in  $T$  which satisfy the following:

- (a) for every  $s \in \{1, \dots, k_i\}$  and every  $t \in \{1, \dots, k_j\}$ , we have  $\gamma_i \subseteq \gamma_s^{(i)}$  and  $\gamma_j \subseteq \gamma_t^{(j)}$ ;
- (b) for every  $p \in \{i, j\}$ , for every distinct  $s, t \in \{1, \dots, k_p\}$ , we have  $C(\gamma_s^{(p)}) \cap C(\gamma_t^{(p)}) = \emptyset$ ;
- (c) for every  $s \in \{1, \dots, k_i\}$  and every  $t \in \{1, \dots, k_j\}$ , either  $C(\gamma_s^{(i)}) = C(\gamma_t^{(j)})$  or  $C(\gamma_s^{(i)}) \cap C(\gamma_t^{(j)}) = \emptyset$ ;
- (d) for every  $p \in \{i, j\}$ , we have

$$C(\gamma_p) = \bigcup_{s=1}^{k_p} C(\gamma_s^{(p)}).$$



**Proof.** See Figure 5.1 to follow the construction. Notice that we either have  $\gamma_i \cap \gamma_j = \emptyset$  or  $\gamma_i \cap \gamma_j \neq \emptyset$ . In both cases, we construct a path  $\tau$  and vertices  $v_i, v'_i, v_j, v'_j$  that we will use in the rest of the proof. First suppose that  $\gamma_i \cap \gamma_j = \emptyset$ . Let  $\tau$  be the unoriented geodesic path in  $T$  which realizes the distance between  $\gamma_i$  and  $\gamma_j$ . Since, by assumption,  $C(\gamma_i) \cap C(\gamma_j) \neq \emptyset$ , the endpoints of  $\tau$  are endpoints of  $\gamma_i$  and  $\gamma_j$ . For every  $p \in \{i, j\}$ , let  $v_p$  be the common endpoint of  $\gamma_p$  and  $\tau$  and let  $v'_p$  be the other endpoint of  $\gamma_p$ . Suppose now that  $\gamma_i \cap \gamma_j \neq \emptyset$ . Then, since  $C(\gamma_i) \cap C(\gamma_j) \neq \emptyset$  there exist three paths  $\tau, a_i$  and  $a_j$  such that, up to changing the orientation of  $\gamma_i$  and  $\gamma_j$ , we have:  $\gamma_i = a_i \tau$  and  $\gamma_j = \tau a_j$ . For every  $p \in \{i, j\}$ , let  $v_p$  be the common endpoint of  $a_p$  and  $\tau$  and let  $v'_p$  be the other endpoint of  $a_p$ .

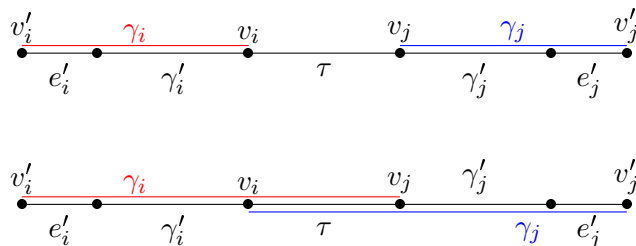


Figure 5.1: The paths constructed in the proof of Lemma 5.3.1.

For every  $p \in \{i, j\}$ , let  $e'_p$  be the edge of  $\gamma_p$  adjacent to  $v'_p$ , which exists since  $\gamma_p$  is not reduced to a vertex. For every  $p \in \{i, j\}$ , let  $\gamma'_p$  be the edge path such that either  $\gamma_p = \gamma'_p e'_p$  or  $\gamma_p = e'_p \gamma'_p$ . For every  $p \in \{i, j\}$  and  $\ell \in \{i, j\} - \{p\}$ , let  $\gamma_1^{(p)}, \dots, \gamma_{k_p}^{(p)}$  be the edge paths of  $T$  which start at  $v'_p$ , which properly contain  $\gamma_p$  and such that for every  $s \in \{1, \dots, k_p\}$ , the endpoint of  $\gamma_s^{(p)}$  distinct from  $v'_p$  is at distance exactly 1 from the minimal edge path of  $T$  which contains  $\tau$  and  $\gamma'_\ell$ . Note that for every  $p \in \{i, j\}$  and  $\ell \in \{i, j\} - \{p\}$ , there exists a unique  $s_p \in \{1, \dots, k_p\}$  such that  $\gamma_{s_p}^{(p)}$  contains  $e'_\ell$ . Note that for every  $p \in \{i, j\}$ , the integer  $s_p$  is the unique integer  $s \in \{1, \dots, k_p\}$  such that  $\gamma_s^{(p)}$  contains both  $\gamma_i$  and  $\gamma_j$ . Note also that  $\gamma_{s_i}^{(i)} = (\gamma_{s_j}^{(j)})^{-1}$ .

We claim that the paths  $\gamma_1^{(i)}, \dots, \gamma_{k_i}^{(i)}, \gamma_1^{(j)}, \dots, \gamma_{k_j}^{(j)}$  satisfy the conclusion of the claim. Indeed, (a) is satisfied by construction. We prove (b). Let  $p \in \{i, j\}$ . Let  $s, t \in \{1, \dots, k_p\}$  be distinct. Then  $\gamma_s^{(p)}$  and  $\gamma_t^{(p)}$  share the path  $\gamma_p$  as an initial segment. But, by construction of the paths  $\gamma_s^{(p)}$  and  $\gamma_t^{(p)}$ , the endpoints of  $\gamma_s^{(p)}$  and  $\gamma_t^{(p)}$  distinct from  $v'_p$  are at distance exactly 1 from the minimal edge path of  $T$  which contains  $\tau$  and  $\gamma'_\ell$ . Therefore, the endpoint of  $\gamma_s^{(p)}$  distinct from  $v'_p$  is not contained in  $\gamma_t^{(p)}$ . Hence the subtree of  $T$  generated by  $\gamma_s^{(p)}$  and  $\gamma_t^{(p)}$  is a tripod. This shows that  $C(\gamma_s^{(p)}) \cap C(\gamma_t^{(p)}) = \emptyset$  and this proves (b).

We now prove (c). Let  $s \in \{1, \dots, k_i\}$  and let  $t \in \{1, \dots, k_j\}$ . Suppose that we have  $C(\gamma_s^{(i)}) \cap C(\gamma_t^{(j)}) \neq \emptyset$ . Then there exists a path  $\gamma'$  of  $T$  such that  $\gamma'$  contains both  $\gamma_s^{(i)}$  and  $\gamma_t^{(j)}$ . Thus  $\gamma'$  contains both  $\gamma_i$  and  $\gamma_j$ . This implies that  $\gamma_s^{(i)} = \gamma_{s_i}^{(i)} = (\gamma_{s_j}^{(j)})^{-1} = (\gamma_t^{(j)})^{-1}$

and that  $C(\gamma_s^{(i)}) = C(\gamma_t^{(j)})$ . This proves (c). Finally, the fact that (d) holds follows from the fact that  $C(\gamma) = \bigcup_{b \in ET, \gamma b \notin \gamma} C(\gamma b)$ . This proves the claim.  $\square$

For every  $p \in \{i, j\}$ , replace  $\gamma_p$  by the paths  $\gamma_1^{(p)}, \dots, \gamma_{k_i}^{(p)}$ . Then we obtain a new set  $\{\gamma'_1, \dots, \gamma'_{\ell_1}\}$  such that, by the point (d) of the claim,  $C = \cup_{i=1}^{\ell_1} C(\gamma'_i)$ . Recall that for every  $p \in \{i, j\}$ , we have  $C(\gamma_p) \in \text{Cyl}(\mathcal{C})$ . By the point (a) of the claim, for every  $p \in \{i, j\}$  and every  $s \in \{1, \dots, k_i\}$ , we have  $\gamma_p \subseteq \gamma_s^{(p)}$ . Therefore, we see that for every  $p \in \{i, j\}$  and every  $s \in \{1, \dots, k_i\}$ , we have  $C(\gamma_s^{(p)}) \in \text{Cyl}(\mathcal{C})$ . Hence the set  $\{\gamma'_1, \dots, \gamma'_{\ell_1}\}$  satisfies (1). Point (a) of the claim also implies that, for every  $m' \in \{1, \dots, \ell\}$ , and every  $p \in \{i, j\}$ , if  $C(\gamma_{m'}) \cap C(\gamma_p) = \emptyset$  then for every  $s \in \{1, \dots, k_p\}$ , we have  $C(\gamma_{m'}) \cap C(\gamma_s^{(p)}) = \emptyset$ . Combined with points (b) and (c) of the claim, we see that the number of distinct elements  $m_1, m_2 \in \{1, \dots, \ell_1\}$  such that  $C(\gamma_{m_1}) \cap C(\gamma_{m_2}) \neq \emptyset$  is strictly less than  $m$ . An inductive argument then concludes the proof.  $\square$

We denote by  $F_n - \mathcal{A}$  the subset of  $F_n$  consisting in every element  $w \in F_n$  such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ . Note that  $F_n - \mathcal{A}$  is closed under inversion since  $\mathcal{C}$  is closed under inversion by Lemma 5.2.3. The next lemma gives a criterion to extend some functions defined on  $F_n - \mathcal{A}$  to a relative current in  $\text{Curr}(F_n, \mathcal{A})$  (see [Gup1, Lemma 3.9] for the free factor system case). First we need some definitions.

Let  $w \in F_n$ , and let  $k \in \mathbb{N}^*$ . A *length  $k$  extension of  $w$*  is a word  $w' = wx_1 \dots x_k$  where for every  $i \in \{1, \dots, k-1\}$ , we have  $x_i \neq x_{i+1}^{-1}$  and  $x_1$  is not the inverse of the last letter of  $w$ . An *extension of  $w$*  is a word  $w'$  such that there exists  $k \in \mathbb{N}^*$  such that  $w'$  is a length  $k$  extension of  $w$ .

**Lemma 5.3.2.** *Let  $\eta: F_n - \mathcal{A} \rightarrow \mathbb{R}_+$  be a function invariant under inversion and which satisfies, for every  $w \in F_n - \mathcal{A}$ :*

$$\eta(w) = \sum_{v \text{ is a length one extension of } w} \eta(v). \quad (5.1)$$

*There exists a unique element  $\tilde{\eta} \in \text{Curr}(F_n, \mathcal{A})$  such that for every element  $w \in F_n - \mathcal{A}$ , we have*

$$\eta(w) = \tilde{\eta}(C(\gamma_w)).$$

**Proof.** Since  $\partial^2(F_n, \mathcal{A})$  is totally disconnected and locally compact by Lemma 5.2.6, and since a relative current is a Radon measure, a relative current is uniquely determined by its values on compact open subsets of  $\partial^2(F_n, \mathcal{A})$ . Let  $C$  be a compact open subset of  $\partial^2(F_n, \mathcal{A})$ . By Lemma 5.3.1, the subset  $C$  is a disjoint union of cylinders of finitely many geodesic edge paths  $\gamma_1, \dots, \gamma_k$  such that for every  $i \in \{1, \dots, k\}$ , we have  $C(\gamma_i) \in \text{Cyl}(\mathcal{C})$ . For every  $i \in \{1, \dots, k\}$ , let  $g_i$  be the element of  $F_n$  which is the label of  $\gamma_i$ . For every  $i \in \{1, \dots, r\}$ , since  $g_i$  contains an element of  $\mathcal{C}$  as a subword, we have  $g_i \in F_n - \mathcal{A}$ . Hence we can set  $\tilde{\eta}(C) = \sum_{i=1}^k \eta(g_i)$ . We claim that the value  $\tilde{\eta}(C)$  does not depend on the choice of the paths  $\gamma_i$ . Indeed, let  $\alpha_1, \dots, \alpha_\ell$  be another set of geodesic edge paths given by Lemma 5.3.1 and let  $h_1, \dots, h_\ell$  be the corresponding elements in  $F_n$ . Note that

for every  $i \in \{1, \dots, k\}$  and every  $j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\alpha_j) \neq \emptyset$ , we have  $C(\gamma_i) \cap C(\alpha_j) = C(\beta_{i,j})$ , where  $\beta_{i,j}$  is a minimal edge path in  $T$  that contains both  $\gamma_i$  and  $\alpha_j$ .

We claim that for every  $i \in \{1, \dots, k\}$ , there do not exist distinct  $j_1, j_2 \in \{1, \dots, \ell\}$  and paths  $a_1$  and  $a_2$  such that  $\beta_{i,j_1} = a_1\gamma_i$  and  $\beta_{i,j_2} = \gamma_i a_2$ . Indeed, otherwise the path  $a_1\gamma_i a_2$  is a finite path that contains both  $\alpha_{j_1}$  and  $\alpha_{j_2}$ . Hence  $C(\alpha_{j_1}) \cap C(\alpha_{j_2}) \neq \emptyset$ , a contradiction. The claim follows.

For every  $i \in \{1, \dots, k\}$  and every  $j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\alpha_j) \neq \emptyset$ , let  $g_{i,j}$  be an element in  $F_n$  corresponding to  $\beta_{i,j}$ . By the above claim, for every  $i \in \{1, \dots, k\}$ , one of the following holds:

(a) for every  $j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\alpha_j) \neq \emptyset$ , the element  $g_{i,j}$  is an extension of  $g_i$ ;

(b) for every  $j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\alpha_j) \neq \emptyset$ , the element  $g_{i,j}^{-1}$  is an extension of  $g_i^{-1}$ .

Since  $\eta$  is invariant under inversion, we may suppose that for every  $i \in \{1, \dots, k\}$ , and for every  $j \in \{1, \dots, \ell\}$  such that  $C(\gamma_i) \cap C(\alpha_j) \neq \emptyset$ , the element  $g_{i,j}$  is an extension of  $g_i$ . Thus for every  $j \in \{1, \dots, \ell\}$ , and for every  $i \in \{1, \dots, k\}$  such that  $C(\gamma_i) \cap C(\alpha_j) \neq \emptyset$ , the element  $g_{i,j}^{-1}$  is an extension of  $h_j^{-1}$ .

Note that, since  $C = \cup_{i=1}^k C(\gamma_i) = \cup_{j=1}^{\ell} C(\alpha_j)$ , for every  $i \in \{1, \dots, k\}$ , the subset  $C(\gamma_i)$  is covered by a disjoint union of finitely many  $C(\alpha_j)$ . Hence, for every  $i \in \{1, \dots, k\}$ , Equation (5.1) implies that:

$$\eta(g_i) = \sum_{j \mid C(\gamma_i) \cap C(\alpha_j) \neq \emptyset} \eta(g_{i,j}).$$

Similarly, for every  $j \in \{1, \dots, \ell\}$ , we have:

$$\eta(h_j^{-1}) = \sum_{i \mid C(\gamma_i) \cap C(\alpha_j) \neq \emptyset} \eta(g_{i,j}^{-1}).$$

Thus, since  $\eta$  is invariant under inversion, we have:

$$\sum_{j=1}^{\ell} \eta(h_j) = \sum_{j=1}^{\ell} \eta(h_j^{-1}) = \sum_{j=1}^{\ell} \sum_{i \mid C(\gamma_i) \cap C(\alpha_j) \neq \emptyset} \eta(g_{i,j}^{-1}) = \sum_{i=1}^k \sum_{j \mid C(\gamma_i) \cap C(\alpha_j) \neq \emptyset} \eta(g_{i,j}) = \sum_{i=1}^k \eta(g_i).$$

Hence the value of  $\tilde{\eta}(C)$  does not depend on the choice of the paths  $\gamma_i$ .

Therefore  $\tilde{\eta}$  is an additive,  $F_n$ -invariant and nonnegative function on the set of compact open subsets of  $\partial^2(F_n, \mathcal{A})$ . We claim that  $\tilde{\eta}$  is in fact  $\sigma$ -additive. Indeed, by [Coh, Proposition 1.2.6], it suffices to prove that for every decreasing sequence  $(C_n)_{n \in \mathbb{N}}$  of compact open subsets of  $\partial^2(F_n, \mathcal{A})$  such that  $\bigcap_{n \in \mathbb{N}} C_n = \emptyset$ , we have  $\lim_{n \rightarrow \infty} \tilde{\eta}(C_n) = 0$ . But since a decreasing sequence of nonempty compact subsets is a nonempty compact subset, there exists  $n \in \mathbb{N}$  such that  $C_n = \emptyset$ . This proves the claim. By Carathéodory extension theorem (see [Coh, Proposition 1.2.6, Theorem 1.3.6]), the function  $\tilde{\eta}$  has a unique extension as a Radon measure on the  $\sigma$ -algebra of Borel sets of  $\partial^2(F_n, \mathcal{A})$ .  $\square$

Let

$$\mathbb{P}\text{Curr}(F_n, \mathcal{A}) = (\text{Curr}(F_n, \mathcal{A}) - \{0\}) / \mathbb{R}_+^*$$

be the set of projectivized relative currents (where  $\mathbb{R}_+^*$  acts on  $\text{Curr}(F_n, \mathcal{A})$  by homothety), equipped with the quotient topology which is metrizable. The next result is a generalization of [Gup1, Lemma 3.11].

**Lemma 5.3.3.** *The metrisable space  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$  is compact.*

**Proof.** Let  $([\eta_n])_{n \in \mathbb{N}}$  be a sequence of projective currents relative to  $\mathcal{A}$ . We prove that it has a convergent subsequence. Let  $\mathcal{C}$  be the finite set given by Lemma 5.2.3. For every  $n \in \mathbb{N}$ , let  $\eta_n$  be a representative of  $[\eta_n]$  such that, for every  $w \in \mathcal{C}$ , we have  $\eta(C(\gamma_w)) \leq 1$ , with equality for some  $w \in \mathcal{C}$ , independent of  $n$  up to extraction. The set  $\mathcal{C}$  being finite, there exists a subsequence  $(n_k)_{k \in \mathbb{N}}$  such that for every  $u \in \mathcal{C}$ , the sequence  $(\eta_{n_k}(C(\gamma_u)))_{k \in \mathbb{N}}$  converges. Moreover, there exists  $u_0 \in \mathcal{C}$  such that the limit  $\lim_{k \rightarrow \infty} (\eta_{n_k}(C(\gamma_{u_0})))_{k \in \mathbb{N}}$  is not equal to zero. Let  $w \in F_n$  be such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ . There exists  $u_w \in \mathcal{C}$  such that  $u_w$  is a subword of  $w$ . Therefore, for every  $k \in \mathbb{N}$ , we have

$$\eta_{n_k}(C(\gamma_w)) \leq \eta_{n_k}(C(\gamma_{u_w})) \leq 1.$$

Therefore, for every element  $w \in F_n - \mathcal{A}$ , the sequence  $(\eta_{n_k}(C(\gamma_w)))_{k \in \mathbb{N}}$  has a convergent subsequence. By a diagonal argument, up to extraction, for every  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ , the sequence  $(\eta_{n_k}(C(\gamma_w)))_{k \in \mathbb{N}}$  converges. Moreover, there exists  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$  such that  $(\eta_{n_k}(C(\gamma_w)))_{k \in \mathbb{N}}$  converges to a nonzero element.

Let  $\eta: F_n - \mathcal{A} \rightarrow \mathbb{R}_+$  be the function defined by, for every  $w \in F_n - \mathcal{A}$ :

$$\eta(w) = \lim_{k \rightarrow \infty} \eta_{n_k}(C(\gamma_w)).$$

Since for every  $k \in \mathbb{N}$ , the function  $\eta_{n_k}$  is a relative current, the function  $\eta$  satisfies the assumptions of Lemma 5.3.2. Therefore, by Lemma 5.3.2, there exists a unique relative current  $\tilde{\eta} \in \text{Curr}(F_n, \mathcal{A})$  such that for every element  $w \in F_n - \mathcal{A}$ , we have

$$\eta(w) = \tilde{\eta}(C(\gamma_w)).$$

Hence  $([\eta_{n_k}])_{k \in \mathbb{N}}$  converges to  $[\tilde{\eta}]$ . □

## 5.4 Density of rational currents

In this section, let  $n \geq 3$ . Let  $r \in \mathbb{N}$  and let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system of  $F_n$ . Let  $\mathcal{B}, T, L \in \mathbb{N}^*, \Gamma_1, \dots, \Gamma_r, \mathcal{C}$  be as above Lemma 5.2.3. Let  $\ell: F_n \rightarrow \mathbb{N}$  be the length function corresponding to  $\mathcal{B}$ .

Every conjugacy class of nonperipheral element  $g \in F_n$  determines a relative current  $\eta_g$  as follows. Suppose first that  $g$  is *root-free*, that is,  $g$  is not a proper power of any element in  $F_n$ . Let  $\gamma$  be a finite geodesic path in the Cayley graph  $T$  such that  $C(\gamma) \in \text{Cyl}(\mathcal{C})$ . Then  $\eta_g(C(\gamma))$  is the number of unoriented translation axes in  $T$  of conjugates of  $g$  that contain the path  $\gamma$ . If  $g = h^k$  with  $k \geq 2$  and  $h$  root-free, we set

$\eta_g = k \eta_n$ . Such currents are called *rational currents*. Note that for every nonperipheral element  $g \in F_n$ , the current  $\eta_g$  only depends on the conjugacy class of  $g$ . Therefore, we can talk about rational currents induced by conjugacy classes of nonperipheral elements of  $F_n$  and write  $\eta_{[g]}$  for the rational current associated with the conjugacy class of a nonperipheral element  $g \in F_n$ . We prove the following proposition.

**Proposition 5.4.1.** *Let  $n \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_n$ . The set of projectivized rational currents induced by conjugacy classes of nonperipheral elements of  $F_n$  is dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$ .*

We follow Gupta's proof ([Gup1, Proposition 3.12]) in the special case of free factor systems. The proof consists in approximating currents in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$  with *signed measured currents* on  $\partial^2 F_n$ , which are  $F_n$ -invariant and  $\sigma$ -additive real-valued functions on the set of Borel subsets of  $\partial^2 F_n$ . We will then conclude using the following lemma, due to Martin (see also [Gup1, Lemma 3.15]).

**Lemma 5.4.2.** *[Mar, Lemma 15] Let  $n \geq 3$ . Suppose that  $\mathcal{A} = \emptyset$ . Let  $k' \geq 1$ , let  $k \geq 2$  with  $k' \leq k$  and let  $\eta$  be a signed measured current such that, for every  $w \in F_n$  with  $k' \leq \ell(w) \leq k$ , we have  $\eta(C(\gamma_w)) \geq 0$ . Let  $P = 2n(2n-1)^{2n(2n-1)^{k-2}}$ . If there exists  $w_0 \in F_n$  such that  $\ell(w_0) = k$  and  $\eta(C(\gamma_{w_0})) \geq P$ , then there exists  $\alpha \in F_n - \{e\}$  such that, for every  $w \in F_n$  with  $k' \leq \ell(w) \leq k$ , we have  $\eta(C(\gamma_w)) \geq \eta_{[\alpha]}(C(\gamma_w))$ .*

**Remark 5.4.3.** (1) *The hypotheses in [Mar, Lemma 15] requires that  $k' = 1$ . However, the proof of Martin works by studying words of length exactly  $k$  and then extend the result to words of length at most  $k$  by additivity of the measures. Thus the proof with  $k' > 1$  is identical.*

(2) *For the rational current  $\eta_{[\alpha]}$  constructed in Lemma 5.4.2, there exists  $w \in F_n$  with  $k' \leq \ell(w) \leq k$  such that  $\eta_{[\alpha]}(C(\gamma_w)) > 0$ .*

Recall that  $\text{Cyl}(\mathcal{C})$  is the set of cylinder sets of the form  $C(\gamma_w)$ , where  $w$  is a word of  $F_n$  containing a word of  $\mathcal{C}$  as a subword. Let  $\eta_0 \in \text{Curr}(F_n, \mathcal{A})$  and let  $k \geq L + 2$ . Let  $\eta$  be a signed measured current such that, for every element  $w \in F_n$  with  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ , we have  $\eta(C(\gamma_w)) = \eta_0(C(\gamma_w))$  and for every element  $w \in F_n$  of length between  $L + 2$  and  $k$ , we have  $\eta(C(\gamma_w)) \geq 0$ . Then  $\eta$  is called a *k-extension* of  $\eta_0$ . The key lemma in order to prove Proposition 5.4.1 is the following result (see [Gup1, Lemma 3.15] for the same statement in the particular case free factor systems):

**Lemma 5.4.4.** *Let  $\eta_0$  be a relative current and let  $k \geq L + 2$ . There exists a signed measured current  $\eta: \partial^2 F_n \rightarrow \mathbb{R}$  which is a k-extension of  $\eta_0$ .*

Let  $\eta_0$  be a relative current. In order to prove Lemma 5.4.4, we need some preliminary results. We follow [Gup1, Section 8.1]. For  $k \in \mathbb{N}^*$ , let  $S_k$  be the set of elements of  $F_n$  of length  $k$  which do not contain an element of  $\mathcal{C}$  as a subword. Note that, since  $\mathcal{C}$  is closed under inversion by Lemma 5.2.3, we see that, for every  $k \in \mathbb{N}^*$ , the set  $S_k$  is closed under inversion. For  $k = 0$ , we set  $S_0 = \{e\}$ . Note also that, if  $k < L + 2$ , then  $S_k$  contains all words of length  $k$  since every element of  $\mathcal{C}$  has length equal to  $L + 2$ .

**Lemma 5.4.5.** (1) If  $\mathcal{A} \neq \emptyset$ , for every  $k \in \mathbb{N}^*$ , the set  $S_k$  is not empty.

(2) For every  $k \geq L + 2$  and every  $w \in S_k$ , there exist  $w' \in S_{k+1}$ ,  $i \in \{1, \dots, r\}$ ,  $g \in F_n$  and  $a \in gA_i g^{-1}$  such that  $w'$  is a length 1 extension of  $w$  and  $a$  is an extension of  $w'$ .

**Proof.** (1) Since the group  $A_1$  is infinite, the corresponding minimal subtree  $T_{A_1}$  is infinite. Recall that the tree  $T_{A_1}$  is supposed to contain the origin  $e$  of  $T$ . Let  $\gamma$  be a geodesic path contained in  $T_{A_1}$ , starting from  $e$  and of length equal to  $k$ , and let  $h \in F_n$  be the corresponding element of  $F_n$ . Then there exists  $a \in A_1$  such that  $a$  is an extension of  $h$ . We have  $h \in S_k$  as otherwise  $a$  would contradict Lemma 5.2.3 (3). This proves (1).

(2) Let  $k \geq L + 2$  and let  $w \in S_k$ . By Lemma 5.2.3 (1), there exist  $i \in \{1, \dots, r\}$  and  $g \in F_n$  such that  $\gamma_w$  is contained in  $T_{gA_i g^{-1}}$ . As  $T_{gA_i g^{-1}}$  does not contain any univalent vertex, there exists a geodesic ray  $c$  in  $T_{gA_i g^{-1}}$  starting from  $e$  which contains the path  $\gamma_w$ . Let  $\gamma'$  be the geodesic path in  $c$  of length  $k + 1$  containing  $\gamma_w$ , and let  $w'$  be the corresponding element in  $F_n$ . Then  $w' \in S_{k+1}$  and  $w'$  is a length 1 extension of  $w$ . This proves (2) and this concludes the proof.  $\square$

Let  $k \geq L + 2$ . Let  $S_k^0$  be a subset of  $S_k$  (chosen once and for all) such that for every  $w \in S_k$  exactly one of  $w$  or  $w^{-1}$  appears in  $S_k^0$ . In what follows, we adopt the convention that whenever an extension of a word  $w$  by a letter  $b \in \mathcal{B}$  is written as  $wb$  (resp.  $bw$ ), we assume that  $b$  is not the inverse of the last letter (resp. first letter) of the word  $w$ .

In order to construct the signed measured current which satisfies the conclusion of Lemma 5.4.4, we will define a signed measured current on cylinders of words in  $S_{k-1}$  and use those values together with the additivity laws in order to define  $\eta$  on cylinders of words of length  $k$ . First we set  $\eta(C(\gamma_b)) = 1$  for every letter  $b$  of  $\mathcal{B}$  not contained in  $\mathcal{C}$ . By induction, assume that for every element  $v \in S_{k-1}$ , the value  $\eta(C(\gamma_v))$  is defined. By additivity of a signed measured current, for every  $v \in S_{k-1}^0$ , we want to have:

$$\begin{aligned} \eta(C(\gamma_v)) &= \sum_{b \in \mathcal{B}, vb \in S_k} \eta(C(\gamma_{vb})) + \sum_{b \in \mathcal{B}, vb \notin S_k} \eta_0(C(\gamma_{vb})) \\ \eta(C(\gamma_{v^{-1}})) &= \sum_{b \in \mathcal{B}, v^{-1}b \in S_k} \eta(C(\gamma_{v^{-1}b})) + \sum_{b \in \mathcal{B}, v^{-1}b \notin S_k} \eta_0(C(\gamma_{v^{-1}b})) \end{aligned}$$

Since  $\eta$  is invariant under taking inverses, the equation obtained by using forward extensions of  $v^{-1}$  is the same one as the equation obtained by using backward extensions of  $v$ . After rearranging the equations in order to have the unknown terms on the left hand side, we obtain:

$$\begin{aligned} \sum_{b \in \mathcal{B}, vb \in S_k} \eta(C(\gamma_{vb})) &= \sum_{b \in \mathcal{B}, vb \notin S_k} \eta_0(C(\gamma_{vb})) - \eta(C(\gamma_v)) = c_v \\ \sum_{b \in \mathcal{B}, v^{-1}b \in S_k} \eta(C(\gamma_{v^{-1}b})) &= \sum_{b \in \mathcal{B}, v^{-1}b \notin S_k} \eta_0(C(\gamma_{v^{-1}b})) - \eta(C(\gamma_{v^{-1}})) = c_{v^{-1}}. \end{aligned} \tag{5.2}$$

Since  $\eta$  is invariant under taking inverse, this shows that there are  $|S_{k-1}|$  equations in  $|S_k|/2 = |S_k^0|$  variables.

Denote the system of equations (5.2) by  $E_{k-1}^1$ . These are equations obtained from length 1 extensions of words in  $S_{k-1}$ . Similarly, for every  $i \in \{1, \dots, k-1\}$ , we define  $E_{k-i}^i$  as the system of equations obtained from length  $i$  extensions of words in  $S_{k-i}$ .

Let  $[M|c]$  be the augmented matrix for the system of equations  $E_{k-1}^1$  with rows labeled by words in  $S_{k-1}$ , columns by words in  $S_k^0$  and such that for every  $w \in S_k^0$  and every  $v \in S_{k-1}$ , we have  $M_{v,w} = 1$  if there exists  $b \in \mathcal{B}$  such that  $w = vb$  or  $w^{-1} = vb$ ; and  $M_{v,w} = 0$  otherwise. Let  $c$  be the column vector indexed by words in  $S_{k-1}$  such that for every  $v \in S_{k-1}$ , the coordinate of  $c$  at  $v$  is equal to  $c_v$ . If  $v \in S_{k-1}$ , we will denote by  $r_v$  the corresponding row vector of  $M$ . Observe that each column has exactly two entries which are equal to 1. Indeed,  $M_{v,w}$  is equal to 1 exactly when  $w$  or  $w^{-1}$  is a length 1 extension of  $v$ . Observe also that any two distinct row vectors  $r_{v_1}$  and  $r_{v_2}$  can have at most one common coordinate which is equal to 1. Indeed, let  $w \in S_k^0$  be such that  $M_{v_1,w} = M_{v_2,w} = 1$ . Then there exist  $b_1, b_2 \in \mathcal{B}$  such that  $w = v_1 b_1$  or  $w = b_1^{-1} v_1^{-1}$  and  $w = v_2 b_2$  or  $w = b_2^{-1} v_2^{-1}$ . Therefore, the word  $v_1$  starts with  $b_2^{-1}$  and  $v_2$  starts with  $b_1^{-1}$ . This shows that  $w$  is uniquely determined.

The next lemma is the same one as [Gup1, Lemma 8.2] in the special case of free factor systems.

**Lemma 5.4.6.** (1) *For every  $i \geq 1$ , an equation in the system  $E_{k-i-1}^{i+1}$  is a linear combination of equations in the system  $E_{k-i}^i$ . Thus it is sufficient to look at the system  $E_{k-1}^1$  in order to obtain every constraint satisfied by  $\eta(C(\gamma_w))$  for every  $w \in S_k^0$ .*

(2) *Let  $u \in S_{k-2}$ . Then the following two linear combinations of rows of  $M$  are equal:*

$$\sum_{b \in \mathcal{B}, bu \in S_{k-1}} r_{bu} = \sum_{b \in \mathcal{B}, bu^{-1} \in S_{k-1}} r_{bu^{-1}}. \quad (5.3)$$

(3) *Every relation among the rows of  $M$  is a linear combination of relations in the set of relations (5.3) where  $u$  varies in  $S_{k-2}$ .*

(4) *We have*

$$\sum_{b \in \mathcal{B}, bu \in S_{k-1}} c_{bu} = \sum_{b \in \mathcal{B}, bu^{-1} \in S_{k-1}} c_{bu^{-1}},$$

where for every  $v \in S_{k-1}$ ,  $c_v$  is given by Equation (5.2).

(5) *The system of equations  $E_{k-1}^1$  is consistent and hence has a solution. Thus we can define  $\eta$  on words of length  $k$ .*

**Proof.** (1) Let  $i \geq 1$  and  $u \in S_{k-i-1}$ . Then by the system  $E_{k-i-1}^1$

$$\eta(C(\gamma_u)) = \sum_{b \in \mathcal{B}} \eta(C(\gamma_{ub})).$$

By the equations in  $E_{k-i}^i$ , we have, for every  $b \in \mathcal{B}$ :

$$\eta(C(\gamma_{ub})) = \sum_{y \in F_n, \ell(y)=i} \eta(C(\gamma_{uby})).$$

Adding all these equations over  $b \in \mathcal{B}$ , we have:

$$\eta(C(\gamma_u)) = \sum_{b,y \in F_n, \ell(b)=1, \ell(y)=i} \eta(C(\gamma_{uby})) = \sum_{z \in F_n, \ell(z)=i+1} \eta(C(\gamma_{uz})).$$

Thus we have recovered an equation in  $E_{k-i-1}^{i+1}$  as a linear combination of equations in  $E_{k-i}^i$ . This proves (1).

(2) Let  $u \in S_{k-2}$  and let  $w \in S_k^0$ . For every  $b \in \mathcal{B}$  such that  $bu \in S_{k-1}$ , we have  $M_{bu,w} \neq 0$  exactly when there exists  $y \in \mathcal{B}$  such that  $w = buy^{-1}$  or  $w = yu^{-1}b^{-1}$  (recall that the basis  $\mathcal{B}$  is supposed to be symmetric). Therefore, if  $M_{bu,w} \neq 0$ , there exists a unique  $y \in \mathcal{B}$  such that  $M_{yu^{-1},w} \neq 0$ . This proves (2).

(3) Let  $R$  be a relation given by  $\sum_{v \in S_{k-1}} d_v r_v = 0$ , where  $d_v \in \mathbb{R}$ . Suppose that the number of terms in the sum associated with  $R$  is minimal. Such an assumption is possible as every relation is a linear combination of relations whose number of terms is minimal. We can rescale the equation so that there exist  $b \in \mathcal{B}$  and  $u \in S_{k-2}$  such that  $d_{bu} = 1$ . For every  $y \in \mathcal{B}$  such that  $buy^{-1} \in S_k^0$ , we have

$$M_{bu, buy^{-1}} = M_{yu^{-1}, buy^{-1}} = 1.$$

This implies, as explained above the lemma, that the rows  $r_{bu}$  and  $r_{yu^{-1}}$  share exactly one common nonzero coordinate, which is  $buy^{-1}$ . Moreover, the rows  $r_{bu}$  and  $r_{yu^{-1}}$  are the only rows which have a nonzero coordinate in  $buy^{-1}$ . This shows that  $d_{yu^{-1}} = -1$ .

Let  $y \in \mathcal{B}$  be such that  $yu^{-1} \in S_{k-1}$ . For every  $z \in \mathcal{B}$  such that  $yu^{-1}z \in S_k^0$ , we have  $M_{yu^{-1}, yu^{-1}z} = M_{z^{-1}u, yu^{-1}z} = 1$ . Thus we have  $d_{z^{-1}u} = 1$ . Therefore we see that

$$\sum_{b \in \mathcal{B}, bu \in S_{k-1}} d_{bu} r_{bu} - \sum_{y \in \mathcal{B}, yu^{-1} \in S_{k-1}} d_{yu^{-1}} r_{yu^{-1}} = \sum_{b \in \mathcal{B}, bu \in S_{k-1}} r_{bu} - \sum_{y \in \mathcal{B}, yu^{-1} \in S_{k-1}} r_{yu^{-1}} = 0.$$

Hence the minimal relation  $R$  is just

$$\sum_{b \in \mathcal{B}, bu \in S_{k-1}} r_{bu} - \sum_{y \in \mathcal{B}, yu^{-1} \in S_{k-1}} r_{yu^{-1}} = 0.$$

(4) Let  $u \in S_{k-2}$ . We have, by the definition of  $c_v$ :

$$\begin{aligned} - \sum_{b \in \mathcal{B}, bu \in S_{k-1}} c_{bu} &= \sum_{b \in \mathcal{B}, bu \in S_{k-1}} \eta(C(\gamma_{bu})) - \sum_{b,y \in \mathcal{B}, bu \in S_{k-1}, buy \notin S_k} \eta(C(\gamma_{buy})) \\ &= \eta(C(\gamma_u)) - \sum_{b \in \mathcal{B}, bu \notin S_{k-1}} \eta(C(\gamma_{bu})) - \sum_{b,y \in \mathcal{B}, bu \in S_{k-1}, buy \notin S_k} \eta(C(\gamma_{buy})) \\ &= \eta(C(\gamma_u)) - \sum_{b,y \in \mathcal{B}, bu \notin S_{k-1}} \eta(C(\gamma_{buy})) - \sum_{b,y \in \mathcal{B}, bu \in S_{k-1}, buy \notin S_k} \eta(C(\gamma_{buy})). \end{aligned}$$

Note that we have:



$$\sum_{b,y \in \mathcal{B}, bu \notin S_{k-1}} \eta(C(\gamma_{buy})) = \sum_{b,y \in \mathcal{B}, bu \notin S_{k-1}, uy \in S_{k-1}} \eta(C(\gamma_{buy})) + \sum_{b,y \in \mathcal{B}, bu \notin S_{k-1}, uy \notin S_{k-1}} \eta(C(\gamma_{buy})) \quad (5.4)$$

Similarly, we have:

$$- \sum_{b \in \mathcal{B}, bu^{-1} \in S_{k-1}} c_{bu^{-1}} = \eta(C(\gamma_u)) - \sum_{b,y \in \mathcal{B}, bu^{-1} \notin S_{k-1}} \eta(C(\gamma_{bu^{-1}y})) - \sum_{b,y \in \mathcal{B}, bu^{-1} \in S_{k-1}, bu^{-1}y \notin S_k} \eta(C(\gamma_{bu^{-1}y})).$$

The right hand side is also equal to:

$$\eta(C(\gamma_u)) - \sum_{b,y \in \mathcal{B}, ub^{-1} \notin S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}})) - \sum_{b,y \in \mathcal{B}, ub^{-1} \in S_{k-1}, y^{-1}ub^{-1} \notin S_k} \eta(C(\gamma_{y^{-1}ub^{-1}})).$$

Observe that the sum  $\sum_{b,y \in \mathcal{B}, ub^{-1} \notin S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}}))$  equals:

$$\sum_{b,y \in \mathcal{B}, ub^{-1} \notin S_{k-1}, y^{-1}u \in S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}})) + \sum_{b,y \in \mathcal{B}, ub^{-1} \notin S_{k-1}, y^{-1}u \notin S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}})). \quad (5.5)$$

Suppose first that  $k \leq L + 2$ . Then  $S_{k-1}$  contains all words of length  $k - 1$ . Hence we have

$$- \sum_{b \in \mathcal{B}, bu \in S_{k-1}} c_{bu} = \eta(C(\gamma_u)) - \sum_{b,y \in \mathcal{B}, buy \notin S_k} \eta(C(\gamma_{buy}))$$

and

$$- \sum_{b \in \mathcal{B}, bu^{-1} \in S_{k-1}} c_{bu^{-1}} = \eta(C(\gamma_u)) - \sum_{b,y \in \mathcal{B}, y^{-1}ub^{-1} \notin S_k} \eta(C(\gamma_{y^{-1}ub^{-1}})),$$

so that Assertion (4) holds in this case with  $y = b^{-1}$ .

Suppose now that  $k > L + 2$ . Then since every element of  $\mathcal{C}$  has length equal to  $L + 2$ , an element of  $\mathcal{C}$  contained in a word  $x$  of length  $k$  is properly contained in  $x$ . Hence if  $b, y \in \mathcal{B}$  are such that  $bu \in S_{k-1}$  and  $buy \notin S_k$ , then  $uy \notin S_{k-1}$ . Thus, we see that:

$$\sum_{b,y \in \mathcal{B}, bu \in S_{k-1}, buy \notin S_k} \eta(C(\gamma_{buy})) = \sum_{b,y \in \mathcal{B}, bu \in S_{k-1}, uy \notin S_{k-1}} \eta(C(\gamma_{buy})). \quad (5.6)$$

Similarly, we have:

$$\sum_{b,y \in \mathcal{B}, ub^{-1} \in S_{k-1}, y^{-1}ub^{-1} \notin S_k} \eta(C(\gamma_{y^{-1}ub^{-1}})) = \sum_{b,y \in \mathcal{B}, ub^{-1} \in S_{k-1}, y^{-1}u \notin S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}})). \quad (5.7)$$

Using Equations (5.4), (5.5), (5.6) and (5.7) with  $y = b^{-1}$ , we see that

$$\begin{aligned}
& \sum_{b,y \in \mathcal{B}, bu \notin S_{k-1}} \eta(C(\gamma_{buy})) + \sum_{b,y \in \mathcal{B}, bu \in S_{k-1}, buy \notin S_k} \eta(C(\gamma_{buy})) \\
= & \sum_{b,y \in \mathcal{B}, ub^{-1} \notin S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}})) + \sum_{b,y \in \mathcal{B}, ub^{-1} \in S_{k-1}, y^{-1}ub^{-1} \notin S_k} \eta(C(\gamma_{y^{-1}ub^{-1}})).
\end{aligned}$$

This shows that

$$\sum_{b \in \mathcal{B}, bu \in S_{k-1}} c_{bu} = \sum_{b \in \mathcal{B}, bu^{-1} \in S_{k-1}} c_{bu^{-1}},$$

and this proves (4).

(5) By Assertions (3) and (4), if  $R$  is a linear combination of relations among the rows of  $M$  equal to zero, then the corresponding linear combination among coordinates of the vector  $c$  is also equal to zero. Therefore, the system  $[M|c]$  has a solution.  $\square$

*Proof of Lemma 5.4.4* Let  $\eta_0$  be a relative current. By Lemma 5.4.6, there exists a signed measured current  $\eta$  such that, for every element  $w$  of  $F_n$  which satisfies  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ , we have  $\eta_0(C(\gamma_w)) = \eta(C(\gamma_w))$ . This extension is not necessarily nonnegative on every element of length between  $L+2$  and  $k$ . Let

$$-M = \min_{w \in F_n, L+2 \leq \ell(w) \leq k} \eta(C(\gamma_w)).$$

Let  $S$  be a finite set of elements of  $\bigcup_{i=1}^r A_i$  such that for every element  $w \in S_k$ , there exists  $g_w \in S$  such that  $g_w$  is an extension of  $w$ . The set exists by Lemma 5.4.5 (2). Let

$$\eta_{\mathcal{A}} = \sum_{g \in S} \eta_{[g]}.$$

By Lemma 5.2.3 (3), for every  $w \in F_n$  such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ , we have  $\eta_{\mathcal{A}}(C(\gamma_w)) = 0$ . Moreover for every  $w \in \bigcup_{i=L+2}^k S_i$ , Lemma 5.4.5 (2) implies that there exists  $w' \in S_k$  such that  $w'$  is an extension of  $w$ . In particular, for every  $w \in \bigcup_{i=L+2}^k S_i$ , we have  $\eta_{\mathcal{A}}(C(\gamma_w)) > 0$ . By finiteness of  $\bigcup_{i=L+2}^k S_i$ , there exists a constant  $R > 0$  such that for every element  $w$  in  $\bigcup_{i=L+2}^k S_i$ , we have  $R\eta_{\mathcal{A}}(C(\gamma_w)) \geq M$ .

Then  $\eta + R\eta_{\mathcal{A}}$  is nonnegative on words of length between  $L+2$  and  $k$  and coincides with  $\eta_0$  on elements  $w \in F_n$  such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ . This concludes the proof.  $\square$

*Proof of Proposition 5.4.1* The proof follows [Gup1, Lemma 3.15] (see also [Mar]). Let  $\mathcal{C}$  be the set defined above Lemma 5.2.3. Let  $\eta_0$  be a relative current and let  $k \geq L+2$ . Note that every word in  $\mathcal{C}$  has length at most equal to  $k$ . Let  $P$  be the constant given by Lemma 5.4.2. Note that there exists an element  $w'$  in  $\mathcal{C}$  such that  $\eta_0(C(\gamma_{w'})) > 0$ . By additivity of  $\eta_0$ , there exists an element  $w_0 \in F_n$  with  $\ell(w_0) = k$  and  $C(\gamma_{w_0}) \in \text{Cyl}(\mathcal{C})$  and such that  $\eta_0(C(\gamma_{w_0})) > 0$ . Let  $R > 0$  be such that  $R\eta_0(C(\gamma_{w_0})) > P$ . By Lemma 5.4.4, there exists a signed measured current  $\eta$  which is a  $k$ -extension of  $\eta_0$ . By Lemma 5.4.2

applied to  $R\eta$  and  $k' = L + 2$ , there exists  $\alpha_1 \in F_n - \{e\}$  such that for every  $w \in F_n - \{e\}$  of length between  $L + 2$  and  $k$ , we have

$$R\eta(C(\gamma_w)) \geq \eta_{[\alpha_1]}(C(\gamma_w)).$$

Suppose first that for every  $w \in F_n$  of length between  $L + 2$  and  $k$ , we have

$$R\eta(C(\gamma_w)) \leq \eta_{[\alpha_1]}(C(\gamma_w)) + P.$$

Then we stop the process and choose  $\alpha_1$ . Otherwise, we apply Lemma 5.4.2 to  $R\eta - \eta_{[\alpha_1]}$  and  $k' = L + 2$ . This shows that there exists  $\alpha_2 \in F_n - \{e\}$  such that for every  $w \in F_n - \{e\}$  of length between  $L + 2$  and  $k$ , we have

$$R\eta(C(\gamma_w)) - \eta_{[\alpha_1]}(C(\gamma_w)) \geq \eta_{[\alpha_2]}(C(\gamma_w)).$$

Applying these arguments iteratively (the process stops by Remark 5.4.3 (2)), we see that there exist  $\alpha_1, \dots, \alpha_p \in F_n - \{e\}$  such that for every element  $w \in F_n - \{e\}$  of length between  $L + 2$  and  $k$ , we have:

$$\sum_{i=1}^p \eta_{[\alpha_i]}(C(\gamma_w)) \leq R\eta(C(\gamma_w)) \leq \sum_{i=1}^p \eta_{[\alpha_i]}(C(\gamma_w)) + P.$$

We claim that there exists  $i \in \{1, \dots, p\}$  such that  $\alpha_i$  is nonperipheral. Indeed, suppose towards a contradiction that for every  $i \in \{1, \dots, p\}$ , the element  $\alpha_i$  is peripheral. By Lemma 5.2.3 (3), we have

$$\sum_{i=1}^p \eta_{[\alpha_i]}(C(\gamma_{w_0})) = 0.$$

This implies that  $R\eta(C(\gamma_{w_0})) \leq P$ . This contradicts the construction of  $\eta$ . Therefore there exists  $i \in \{1, \dots, p\}$  such that  $\alpha_i$  is nonperipheral. Let  $S$  be the subset of  $\{\alpha_1, \dots, \alpha_p\}$  containing every nonperipheral element. Then, for every element  $w \in F_n$  of length  $k$  such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$  we have:

$$\left| \eta(C(\gamma_w)) - \frac{\sum_{\alpha \in S} \eta_{[\alpha]}(C(\gamma_w))}{R} \right| \leq \frac{P}{R}.$$

For  $\alpha \in S$ , let  $\bar{\eta}_{[\alpha]}$  be the restriction of  $\eta_{[\alpha]}$  to the Borel subsets of  $\partial^2(F_n, \mathcal{A})$ . By construction of  $\eta$ , for every element  $w \in F_n$  of length at most  $k$  such that  $C(\gamma_w) \in \text{Cyl}(\mathcal{C})$ , we have:

$$\left| \eta_0(C(\gamma_w)) - \frac{\sum_{\alpha \in S} \bar{\eta}_{[\alpha]}(C(\gamma_w))}{R} \right| \leq \frac{P}{R}.$$

Since  $R$  can be chosen arbitrarily large, we can approximate relative currents by sum of rational relative currents. For  $m \in \mathbb{N}^*$ , let  $\beta^m = \prod_{\alpha \in S} \alpha^m$  (for any total order on  $S$ ). Then there exists  $m \in \mathbb{N}^*$  such that  $\sum_{\alpha \in S} \eta_{[\alpha]}$  can be approximated by  $\frac{1}{m} \eta_{[\beta^m]}$ . This concludes the proof.  $\square$

## Chapitre 6

# North-South type dynamics of relative atoroidal automorphisms of free groups on a relative space of currents

### 6.1 Introduction

Let  $n \geq 2$ . This paper is the second of a sequence of three papers where we study the growth of the conjugacy classes of elements of  $F_n$  under iterations of elements of  $\text{Out}(F_n)$ , the outer automorphism group of a nonabelian free group of rank  $n$ . An outer automorphism  $\phi \in \text{Out}(F_n)$  is *exponentially growing* if there exist  $g \in F_n$ , a representative  $\Phi$  of  $\phi$ , a free basis  $\mathfrak{B}$  of  $F_n$  and a constant  $K > 0$  such that, for every  $m \in \mathbb{N}^*$ , we have

$$\ell_{\mathfrak{B}}(\Phi^m(g)) \geq e^{Km},$$

where  $\ell_{\mathfrak{B}}(\Phi^m(g))$  denotes the length of  $\Phi^m(g)$  in the basis  $\mathfrak{B}$ . Such an element  $g$  is said to be *exponentially growing under iteration of  $\phi$*  and the set of elements of  $F_n$  which have exponential growth under iteration of  $\phi$  is the *pure exponential part of  $\phi$* . It is known, using for instance the train track technology of Bestvina and Handel (see [BH]), that every element  $g$  of  $F_n$  which is not exponentially growing under iteration of  $\phi$  is *polynomially growing under iteration of  $\phi$* , that is, there exist  $\Phi \in \phi$  and an integer  $K \in \mathbb{N}$  such that, for every  $m \in \mathbb{N}^*$ , we have  $\ell_{\mathfrak{B}}(\Phi^m(g)) \leq (m+1)^K$ .

Initiated by Švarc, Milnor and Wolf, and particularly developed by Guivarc'h, Gromov and Grigorchuk, growth problems in groups is a major field of study in geometric and dynamical group theory, see for instance [LS, Man1, Hel]. Many works study the subfield of the element growths under iteration of group automorphisms (see for instance [BFH1, Lev2, CU2]), for instance in the context of hyperbolic groups. See in particular [Cou] for examples of intermediate growth rates. As another example, Dahmani and Krishna [DS] found a sufficient condition for the suspension of an automorphism

of a hyperbolic group to be relatively hyperbolic, and this condition is linked with the structure of the set of all elements of the hyperbolic group which have polynomial growth under iterations of the considered automorphism. Such exponentially growing outer automorphisms of  $F_n$  were already studied in distinct contexts. For instance, Bestvina, Feighn and Handel [BFH1] used them to prove the Tits alternative for  $\text{Out}(F_n)$ .

If  $\phi \in \text{Out}(F_n)$ , we denote by  $\text{Poly}(\phi)$  the set of elements  $g$  of  $F_n$  such that  $g$  is polynomially growing under iteration of  $\phi$ . Let  $\text{Poly}(H) = \bigcap_{\phi \in H} \text{Poly}(\phi)$ . The aim of this series of papers is to prove the following theorem.

**Theorem 6.1.1.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{Out}(F_n)$ . There exists  $\phi \in H$  such that  $\text{Poly}(\phi) = \text{Poly}(H)$ .*

Informally, Theorem 6.1.1 shows that the exponential growth of a subgroup  $H$  of  $\text{Out}(F_n)$  is captured by the exponential growth of a single element of  $H$ . Indeed, if  $g \in F_n$  has exponential growth for some element  $\psi \in H$ , then  $g$  has exponential growth for an element  $\phi \in H$  given by Theorem 6.1.1. The proof relies on dynamical properties of the action of outer automorphisms on some preferred topological space. In this article, we study the dynamical properties of the elements of the subgroup  $H$  of  $F_n$  that will be used in [Gue6] in order to construct an element  $\phi \in H$  given by Theorem 6.1.1.

Let  $\phi \in \text{Out}(F_n)$  be an exponentially growing outer automorphism. In this article, we construct natural (compact, metrizable) topological spaces  $X$  on which a subgroup of  $\text{Out}(F_n)$  containing  $\phi$  acts by homeomorphisms with the additional property that  $\phi$  acts with *North-South dynamics*: there exist two proper disjoint closed subsets of  $X$  such that every point of  $X$  which is not contained in these subsets converges to one of the two subsets under positive or negative iteration of  $\phi$ . North-South dynamics are preferred tools to apply ping-pong arguments similar to the ones of Tits [Tit1] and are used to obtain structural properties of some groups.

The topological space  $X$  that we use in the proof of Theorem 6.1.1 is constructed in such a way that it allows us to create a dictionary between dynamical properties of the action of  $\phi$  on  $X$  and growth properties of elements of  $F_n$  under iterations of  $\phi$ . In order to construct  $X$ , we first need to detect all the elements  $g$  of  $F_n$  such that the length of  $[g]$  with respect to any basis of  $F_n$  grows at most polynomially fast under iteration of  $\phi$ . Levitt [Lev2] proved that there exist finitely many finitely generated subgroups  $H_1, \dots, H_k$  of  $F_n$  such that the conjugacy class of an element  $g$  of  $F_n$  is not exponentially growing under iteration of  $\phi$  if and only if  $g$  is contained in a conjugate of some  $H_i$  for  $i \in \{1, \dots, k\}$ . Moreover, the set  $\mathcal{A}(\phi) = \{[H_1], \dots, [H_k]\}$  is a *malnormal subgroup system*: for every  $i \in \{1, \dots, k\}$ , the group  $H_i$  is a malnormal subgroup of  $F_n$  and for every distinct subgroups  $A$  and  $B$  such that  $[A], [B] \in \mathcal{A}(\phi)$ , we have  $A \cap B = \{e\}$ . Every element of  $F_n$  which is contained in a conjugate of some  $H_i$  with  $i \in \{1, \dots, k\}$  has polynomial growth under iteration of  $\phi$ . Moreover, we have  $\text{Poly}(\phi) = \bigcup_{i=1}^r \bigcup_{g \in F_n} gH_i g^{-1}$ .

In [Gue4], we construct a compact, metrizable space, called the space of projectivised currents relative to  $\mathcal{A}(\phi)$ , denoted by  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$ , which is the space of projectivised Radon measures on the double boundary of  $F_n$  relative to  $\mathcal{A}(\phi)$ , equipped with the weak-star topology (see Section 6.2.4 for precise definitions). In [Gue4], we proved that

the set of currents associated with  $\mathcal{A}(\phi)$ -nonperipheral conjugacy classes of elements of  $g$  of  $F_n$ , that is, such that  $g$  is not contained in the conjugacy class of some  $H_i$  with  $i \in \{1, \dots, k\}$ , is dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$ . Thus, the set of conjugacy classes of elements of  $F_n$  whose length grows exponentially fast under iteration of  $\phi$  is dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$ . If we denote by  $\text{Out}(F_n, \mathcal{A}(\phi))$  the subgroup of  $\text{Out}(F_n)$  consisting in every element  $\psi \in \text{Out}(F_n)$  such that  $\psi(\mathcal{A}(\phi)) = \mathcal{A}(\phi)$ , the group  $\text{Out}(F_n, \mathcal{A}(\phi))$  acts by homeomorphisms on  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$  by pushing forward the measures. In this article, we prove the following theorem.

**Theorem 6.1.2** (see Theorem 6.5.1). *Let  $n \geq 3$  and let  $\phi$  be an exponentially growing outer automorphism. The outer automorphism  $\phi$  acts with North-South dynamics on the space  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$ .*

In fact, we prove a slightly stronger result since we prove a *uniform North-South dynamics result*, that is, the convergence in the North-South dynamics statement can be made uniform on compact subsets of  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$ . As explained above, North-South dynamics results given by Theorem 6.1.2 will be a key point in the proof of Theorem 6.1.1.

Such dynamical results already appear in similar contexts. For instance, Tits proved in [Tit1] its alternative for linear groups using North-South dynamics and ping-pong arguments. In the context of the mapping class group  $\text{Mod}(S)$  of a compact connected orientable surface  $S$  of genus at least 2, pseudo-Anosov elements acts with North-South dynamics on the space of projectivised measured foliations ([Thu], see also the work of Ivanov [Iva1]) or the curve complex [MasM]. Using this North-South dynamics, Ivanov [Iva1] (see also the work of McCarthy [McC]) later proved a Tits alternative for subgroups of  $\text{Mod}(S)$ . Similarly, North-South dynamics results were obtained for certain classes of outer automorphisms of  $F_n$ . For instance, *fully irreducible outer automorphisms* act on the compactified Outer space [LL] or the space of projectivised currents ([Mar], see also the work of Uyanik [Uya1]) with a North-South dynamics and *atoroidal outer automorphisms* act on the space of projectivised currents with a North-South dynamics [LU2, Uya2]. Clay and Uyanik [CU2] applied this result in the proof of the fact that, for every subgroup  $H$  of  $\text{Out}(F_n)$ , either  $H$  contains an atoroidal outer automorphism or there exists a nontrivial element  $g$  of  $F_n$  such that, for every element  $\phi \in H$ , there exists  $k \in \mathbb{N}^*$  such that we have  $\phi^k([g]) = [g]$ . Such dynamical results were later extended to relative contexts by Gupta [Gup1, Gup2].

In order to prove Theorem 6.1.1, we will need a slightly stronger result than Theorem 6.1.2. Indeed, let  $\phi \in \text{Out}(F_n)$  and let  $\mathcal{A}(\phi) = \{[H_1], \dots, [H_k]\}$ . Suppose that  $\phi$  preserves the conjugacy class of a corank one free factor  $A$  of  $F_n$ . Let  $\mathcal{A}(\phi) \wedge A$  be the malnormal subgroup system consisting in the conjugacy classes of the intersection of the conjugates of the subgroups  $H_i$  with  $i \in \{1, \dots, k\}$  with  $A$ . Note that, by Theorem 6.1.2, there exist closed disjoint subsets  $\Delta_{\pm}(\phi|_A)$  such that the outer automorphism  $\phi|_A \in \text{Out}(A, \mathcal{A}(\phi) \wedge A)$  acts with North-South dynamics on  $\mathbb{P}\text{Curr}(A, \mathcal{A}(\phi) \wedge A)$  with respect to  $\Delta_{\pm}(\phi|_A)$ . There is a canonical embedding  $\mathbb{P}\text{Curr}(A, \mathcal{A}(\phi) \wedge A) \hookrightarrow \mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$ , and we denote by  $\Delta_{\pm}(\phi)$  the image of  $\Delta_{\pm}(\phi|_A)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$ . We will need to understand the dynamics of  $\phi$  on the space  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$ . As there might exist

elements in  $F_n$  which have polynomial growth under iterations of  $\phi$  and which are not contained in a conjugate of  $A$ , one cannot apply Theorem 6.1.2 to obtain a North-South dynamics result. However, we obtain the following result.

**Theorem 6.1.3** (see Theorem 7.2.7). *Let  $n \geq 3$  and let  $\phi \in \text{Out}(F_n)$  be an exponentially growing outer automorphism which preserves a corank one free factor  $A$ . There exist two convex compact subsets  $\widehat{\Delta}_{\pm}(\phi)$  of  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$  such that the following holds. Let  $U_{\pm}$  be open neighborhoods of  $\Delta_{\pm}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$  and  $\widehat{V}_{\pm}$  be open neighborhoods of  $\widehat{\Delta}_{\pm}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$ . There exists  $M \in \mathbb{N}^*$  such that for every  $n \geq M$ , we have*

$$\phi^{\pm n}(\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A) - \widehat{V}_{\mp}) \subseteq U_{\pm}.$$

In [CU2, Theorem 4.15], Clay and Uyanik proved an analogue of Theorem 6.1.3 in the context of atoroidal outer automorphisms of  $F_n$ . In Theorem 6.1.3, the two convex subsets  $\widehat{\Delta}_{\pm}(\phi)$  have nonempty intersection, so that Theorem 6.1.3 is not a North-South dynamics result as defined above. However, Theorem 6.1.3 gives a sufficiently precise description of the dynamics of  $\phi$  for our considerations. The intersection  $\widehat{\Delta}_{+}(\phi) \cap \widehat{\Delta}_{-}(\phi)$  corresponds informally to the polynomial growth part of  $\phi$ . This intersection, denoted by  $K_{PG}$  in the rest of the article, is the closure in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi) \wedge A)$  of the  $(\mathcal{A}(\phi) \wedge A)$ -nonperipheral elements of  $F_n$  which have polynomial growth under iteration of  $\phi$ . In Section 6.3.3, we give a complete study of the subspace  $K_{PG}$  in a more general context.

In fact, Section 6.3 is devoted to the study of the polynomial growth of an exponentially growing outer automorphism. Following the works of Bestvina, Feighn and Handel [BFH1, BFH2], of Feighn and Handel [FH] and of Handel and Mosher [HaM4], we use appropriate relative train track representatives of a power of an exponentially growing outer automorphism  $\phi$  in order to describe  $\mathcal{A}(\phi)$  geometrically. It gives rise to a (not necessarily connected) topological graph  $G^*$  such that the fundamental group of every connected component  $G_c^*$  of  $G^*$  injects into  $F_n$  and such that the set  $\{[\pi_1(G_c^*)]\}_{G_c^* \in \pi_0(G^*)}$  where  $\pi_1(G_c^*)$  is viewed as a subgroup of  $F_n$  is equal to  $\mathcal{A}(\phi)$  (see Proposition 6.3.13). We then use this characterization of  $\mathcal{A}(\phi)$  in Section 6.3.3 in order to describe the subset  $K_{PG}$ .

We now sketch a proof of Theorem 6.1.2. The proofs of Theorem 6.1.2 and Theorem 6.1.3 given in this paper are long and quite technical, this is why we postpone the proof of Theorem 6.1.1 in [Gue6]. Let  $\phi \in \text{Out}(F_n)$  be exponentially growing. The first step is to construct the closed subsets  $\Delta_{\pm}(\phi)$  associated with  $\phi$  as defined in Theorem 6.1.2. This is done in Section 6.4. In order to construct them, we use as inspiration the construction given by Lustig and Uyanik in [LU2] (see also [Uya2, Gup1]). We choose an appropriate relative train track representative  $f: G \rightarrow G$  of a power of  $\phi$ , where  $G$  is a graph whose fundamental group is isomorphic to  $F_n$ . A current of  $\Delta_{+}(\phi)$  is then constructed by considering occurrences of paths in  $\lim_{m \rightarrow \infty} f^m(e)$ , where  $e$  is an edge in  $G$  whose length grows exponentially fast under iteration of  $f$  (see Proposition 6.4.4). Currents of  $\Delta_{-}(\phi)$  are then defined similarly using a representative of a power of  $\phi^{-1}$ . We then prove Theorem 6.1.2 in Section 6.5. Let  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi)) - \Delta_{\pm}(\phi)$  be the current associated with a  $\mathcal{A}(\phi)$ -nonperipheral conjugacy class  $[w] \in F_n$ . Then

$[w]$  is represented by a circuit  $\gamma_w$  in the graph  $G$ . In order to show that we have  $\lim_{m \rightarrow \infty} \phi^m([\mu]) \in \Delta_+(\phi)$ , we prove that the proportion of the path  $f^m(\gamma_g)$  which grows exponentially fast under iteration of  $f$  tends to 1 as  $m$  goes to infinity. This fact is sufficient to prove that

$$\lim_{m \rightarrow \infty} \phi^m([\mu]) \in \Delta_+(\phi)$$

(see Lemma 6.5.20). We then conclude the proof using the density of currents associated with nonperipheral elements in  $F_n$  proved in [Gue4]. Theorem 6.1.3 is then proved in Section 6.6 using a combination of Theorem 6.1.2 and the description of the space  $K_{PG}$ .

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## 6.2 Preliminaries

### 6.2.1 Malnormal subgroup systems of $F_n$

Let  $n$  be an integer greater than 1 and let  $F_n$  be a free group of rank  $n$ . A *subgroup system* of  $F_n$  is a finite (possibly empty) set  $\mathcal{A}$  whose elements are conjugacy classes of nontrivial (that is distinct from  $\{1\}$ ) finite rank subgroups of  $F_n$ . There exists a partial order on the set of subgroup systems of  $F_n$ , where  $\mathcal{A}_1 \leq \mathcal{A}_2$  if for every subgroup  $A_1$  of  $F_n$  such that  $[A_1] \in \mathcal{A}_1$ , there exists a subgroup  $A_2$  of  $F_n$  such that  $[A_2] \in \mathcal{A}_2$  and  $A_1$  is a subgroup of  $A_2$ . The *stabilizer in  $\text{Out}(F_n)$  of a subgroup system  $\mathcal{A}$* , denoted by  $\text{Out}(F_n, \mathcal{A})$ , is the set of all elements  $\phi \in \text{Out}(F_n)$  such that  $\phi(\mathcal{A}) = \mathcal{A}$ .

Recall that a subgroup  $A$  of  $F_n$  is *malnormal* if for every element  $x \in F_n - A$ , we have  $xAx^{-1} \cap A = \{e\}$ . A subgroup system  $\mathcal{A}$  is said to be *malnormal* if every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  is malnormal and, for all subgroups  $A_1, A_2$  of  $F_n$  such that  $[A_1], [A_2] \in \mathcal{A}$ , if  $A_1 \cap A_2$  is nontrivial then  $A_1 = A_2$ . An element  $g \in F_n$  is  *$\mathcal{A}$ -peripheral* (or simply *peripheral* if there is no ambiguity) if it is trivial or conjugate into one of the subgroups of  $\mathcal{A}$ , and  *$\mathcal{A}$ -nonperipheral* otherwise.

An important class of examples of malnormal subgroup systems is given by the *free factor systems*. A *free factor system* of  $F_n$  is a (possibly empty) set  $\mathcal{F}$  of conjugacy classes  $\{[A_1], \dots, [A_r]\}$  of nontrivial subgroups  $A_1, \dots, A_r$  of  $F_n$  such that there exists an integer  $k \in \mathbb{N}$  with  $F_n = A_1 * \dots * A_r * F_k$ . The free factor system  $\mathcal{F}$  is *sporadic* if  $(k+r, k) \leq (2, 1)$  for the lexicographic order, and is *nonsporadic* otherwise. Therefore, the sporadic free factor systems are those of the form  $\{[C]\}$  where  $C$  has rank at least equal to  $n-1$  and those of the form  $\{[A], [B]\}$  with  $F_n = A * B$ . An ascending sequence of free factor systems  $\mathcal{F}_1 \leq \dots \leq \mathcal{F}_i = \{[F_n]\}$  of  $F_n$  is called a *filtration of  $F_n$* .

Given a free factor system  $\mathcal{F}$  of  $F_n$ , a *free factor of  $(F_n, \mathcal{F})$*  is a subgroup  $A$  of  $F_n$  such that there exists a free factor system  $\mathcal{F}'$  of  $F_n$  with  $[A] \in \mathcal{F}'$  and  $\mathcal{F} \leq \mathcal{F}'$ . When  $\mathcal{F} = \emptyset$ , we say that  $A$  is a *free factor of  $F_n$* . A free factor of  $(F_n, \mathcal{F})$  is *proper* if it is nontrivial, not equal to  $F_n$  and if its conjugacy class does not belong to  $\mathcal{F}$ .

Another class of examples of malnormal subgroup systems is the following one. An outer automorphism  $\phi \in \text{Out}(F_n)$  is *exponentially growing* if there exists  $g \in F_n$  such that



the length of the conjugacy class  $[g]$  of  $g$  in  $F_n$  with respect to some basis of  $F_n$  grows exponentially fast under iteration of  $\phi$ . If  $\phi \in \text{Out}(F_n)$  is not exponentially growing, then  $\phi$  is *polynomially growing*. For an automorphism  $\alpha \in \text{Aut}(F_n)$ , we say that  $\alpha$  is exponentially growing if there exists  $g \in F_n$  such that the length of  $g$  grows exponentially fast under iteration of  $\phi$ . Otherwise,  $\alpha$  is polynomially growing. Let  $\phi \in \text{Out}(F_n)$  be exponentially growing. A subgroup  $P$  of  $F_n$  is a *polynomial subgroup* of  $\phi$  if there exist  $k \in \mathbb{N}^*$  and a representative  $\alpha$  of  $\phi^k$  such that  $\alpha(P) = P$  and  $\alpha|_P$  is polynomially growing. By [Lev2, Proposition 1.4], there exist finitely many conjugacy classes  $[H_1], \dots, [H_k]$  of maximal polynomial subgroups of  $\phi$ . Moreover, the proof of [Lev2, Proposition 1.4] implies that the set  $\mathcal{H} = \{[H_1], \dots, [H_k]\}$  is a malnormal subgroup system. Indeed, Levitt shows that there exists a nontrivial  $\mathbb{R}$ -tree  $T$  in the boundary of Culler and Vogtmann Outer space [CV] on which  $F_n$  acts with trivial arc stabilizers, such that  $\phi$  preserves the homothety class of  $T$  and such that the groups  $H_1 \dots, H_k$  are elliptic in  $T$ . If two distinct subgroups  $A, B$  of  $F_n$  such that  $[A], [B] \in \mathcal{H}$  fix distinct points in  $T$ , then their intersection is trivial. If  $A$  and  $B$  fix the same point  $x$  in  $T$ , then (up to taking a power of  $\phi$ )  $\phi$  preserves  $[\text{Stab}(x)]$  and an inductive argument on the rank using  $\phi|_{\text{Stab}(x)}$  (the rank of  $\text{Stab}(x)$  is less than  $n$  by a result of Gaboriau-Levitt [GaL]) shows that the intersection of  $A$  and  $B$  is trivial. We denote this malnormal subgroup system by  $\mathcal{A}(\phi)$ . Note that, if  $H$  is a subgroup of  $F_n$  such that  $[H] \in \mathcal{A}(\phi)$ , there exists  $\Phi^{-1} \in \phi^{-1}$  such that  $\Phi^{-1}(H) = H$  and  $\Phi^{-1}|_H$  is polynomially growing. Hence we have  $\mathcal{A}(\phi) \leq \mathcal{A}(\phi^{-1})$ . By symmetry, we have

$$\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1}). \quad (6.1)$$

Let  $\mathcal{A}$  be a malnormal subgroup system and let  $\phi \in \text{Out}(F_n, \mathcal{A})$  be a relative outer automorphism. We say that  $\phi$  is *atoroidal relative to  $\mathcal{A}$*  if, for every  $k \in \mathbb{N}^*$ , the element  $\phi^k$  does not preserve the conjugacy class of any  $\mathcal{A}$ -nonperipheral element. We say that  $\phi$  is *expanding relative to  $\mathcal{A}$*  if  $\mathcal{A}(\phi) \leq \mathcal{A}$ . Note that an expanding outer automorphism relative to  $\mathcal{A}$  is in particular atoroidal relative to  $\mathcal{A}$ . When  $\mathcal{A} = \emptyset$ , then the outer automorphism  $\phi$  is expanding relative to  $\mathcal{A}$  if and only if for every nontrivial element  $g \in F_n$ , the length of the conjugacy class  $[g]$  of  $g$  in  $F_n$  with respect to some basis of  $F_n$  grows exponentially fast under iteration of  $\phi$ . Therefore, by a result of Levitt [Lev2, Corollary 1.6], the outer automorphism  $\phi$  is expanding relative to  $\mathcal{A} = \emptyset$  if and only if  $\phi$  is atoroidal relative to  $\mathcal{A} = \emptyset$ .

Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system and let  $\mathcal{F}$  be a free factor system. Let  $i \in \{1, \dots, r\}$ . By [SW, Theorem 3.14] for the action of  $A_i$  on one of its Cayley graphs, there exist finitely many subgroups  $A_i^{(1)}, \dots, A_i^{(k_i)}$  of  $A_i$  such that:

- (1) for every  $j \in \{1, \dots, k_i\}$ , there exists a subgroup  $B$  of  $F_n$  such that  $[B] \in \mathcal{F}$  and  $A_i^{(j)} = B \cap A_i$ ;
- (2) for every subgroup  $B$  of  $F_n$  such that  $[B] \in \mathcal{F}$  and  $B \cap A_i \neq \{e\}$ , there exists  $j \in \{1, \dots, k_i\}$  such that  $A_i^{(j)} = B \cap A_i$ ;
- (3) the subgroup  $A_i^{(1)} * \dots * A_i^{(k_i)}$  is a free factor of  $A_i$ .

Thus, one can define a new subgroup system as

$$\mathcal{F} \wedge \mathcal{A} = \bigcup_{i=1}^r \{[A_i^{(1)}], \dots, [A_i^{(k_i)}]\}.$$

Since  $\mathcal{A}$  is malnormal, and since, for every  $i \in \{1, \dots, r\}$ , the group  $A_i^{(1)} * \dots * A_i^{(k_i)}$  is a free factor of  $A_i$ , it follows that the subgroup system  $\mathcal{F} \wedge \mathcal{A}$  is a malnormal subgroup system of  $F_{\mathbf{n}}$ . We call it the *meet of  $\mathcal{F}$  and  $\mathcal{A}$* .

## 6.2.2 Graphs, markings and filtrations

Let  $\mathbf{n} \geq 2$ . A *marked graph* is a pointed (at a vertex  $*$ ), connected, finite graph  $G$  (in the sense of [Ser1]) whose fundamental group is isomorphic to  $F_{\mathbf{n}}$  which is equipped with a *marking*, that is an isomorphism  $\rho: F_{\mathbf{n}} \rightarrow \pi_1(G, *)$ .

We denote by  $VG$  (resp.  $\vec{E}G$ ) the set of vertices (resp. edges) of  $G$ . Given an edge  $e$  of  $G$ , we denote by  $o(e)$  the *origin* of  $e$ , by  $t(e)$  the *terminal point* of  $e$  and by  $e^{-1}$  the edge of  $G$  such that  $o(e^{-1}) = t(e)$  and  $t(e^{-1}) = o(e)$ . An *edge path*  $\gamma$  of length  $m$  is a concatenation of  $m$  edges  $\gamma = e_1 e_2 \dots e_m$  such that for every  $i \in \{1, \dots, m-1\}$ , we have  $t(e_i) = o(e_{i+1})$ . The length of  $\gamma$  is denoted by  $\ell(\gamma)$ . The edge path  $\gamma$  is *reduced* if for every  $i \in \{1, \dots, m-1\}$ , we have  $e_i \neq e_{i+1}^{-1}$ . A reduced edge path is *cyclically reduced* if  $t(e_m) = o(e_1)$  and  $e_m \neq e_1^{-1}$ . A cyclically reduced edge path is also called a *circuit*. For any edge path  $\gamma$ , there exists a unique reduced edge path homotopic to  $\gamma$  relatively to endpoints, we denote it by  $[\gamma]$ .

Let  $G$  and  $G'$  be two marked graphs. A *graph map* is a pointed homotopy equivalence  $f: G \rightarrow G'$  such that  $f(VG) \subseteq VG'$  and such that the restriction of  $f$  to the interior of an edge is an immersion. Thus, for every edge  $e \in \vec{E}G$ , the image  $f(e)$  determines a reduced edge path  $[f(e)]$ . Given  $\phi \in \text{Out}(F_{\mathbf{n}})$  and  $(G, \rho)$  a marked graph, a *topological representative* of  $\phi$  is a graph map  $f: G \rightarrow G$  such that the outer automorphism class of  $\rho^{-1} \circ f_* \circ \rho \in \text{Aut}(F_{\mathbf{n}})$  is  $\phi$ .

Let  $f: G \rightarrow G$  be a topological representative. Let  $w \in F_{\mathbf{n}}$ . We denote by  $\gamma_w$  the unique circuit in  $G$  which represents the conjugacy class of  $w$ .

Let  $f: G \rightarrow G$  be a topological representative. A *filtration* for  $G$  is an increasing sequence of  $f$ -invariant (not necessarily connected) subgraphs  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$ . Let  $r \in \{1, \dots, k\}$ . The  $r$ -th *stratum* in this filtration, denoted by  $H_r$  is the (not necessarily connected) closure of  $G_r - G_{r-1}$ . For every  $r \in \{1, \dots, k\}$ , there exists a square matrix  $M_r$  associated with the stratum  $H_r$  called the *transition matrix* of  $H_r$ . The rows and columns of  $M_r$  are indexed by the nonoriented edges in  $H_r$  and the entry associated with the pair of nonoriented edges defined by  $(e, e') \in (EH_r)^2$  is the number of occurrences of  $e'$  and  $e'^{-1}$  in  $[f(e)]$ .

Recall that a nonnegative square matrix  $M = (M_{i,j})_{i,j}$  is *irreducible* if for every  $(i, j)$ , there exists  $p = p(i, j)$  such that  $M_{i,j}^p > 0$  and that  $M$  is *primitive* if there exists  $p \in \mathbb{N}^*$  such that every entry of  $M^p$  is positive. For  $r \in \{1, \dots, k\}$ , we say that the stratum  $H_r$  is *irreducible* if its associated matrix is irreducible and we say that  $H_r$  is *primitive* if its associated matrix is primitive. Let  $r \in \{1, \dots, k\}$  and suppose that  $M_r$  is irreducible.

Then it has a unique real eigenvalue  $\lambda_r \geq 1$  called the *Perron-Frobenius* eigenvalue. Let  $H_r$  be an irreducible stratum. Then  $H_r$  is *exponentially growing (EG)* if  $\lambda_r > 1$  and is *nonexponentially growing (NEG)* otherwise. Finally, if the matrix associated with the stratum  $H_r$  is the zero matrix, then  $H_r$  is called a *zero stratum*.

Let  $G$  be a marked graph of  $F_n$  and let  $K$  be a (possibly disconnected) subgraph of  $G$ . The subgraph  $K$  determines a free factor system  $\mathcal{F}(K)$  of  $F_n$  as follows. Let  $C_1, \dots, C_k$  be the noncontractible connected components of  $K$ . Then, for every  $i \in \{1, \dots, k\}$ , the connected component  $C_i$  determines the conjugacy class  $[A_i]$  of a subgroup  $A_i$  of  $\pi_1(G)$ . Then the set  $\{[A_1], \dots, [A_k]\}$  is a free factor system  $\mathcal{F}(K)$  of  $F_n$ .

Let  $\mathcal{F}_1 \leq \dots \leq \mathcal{F}_i = \{[F_n]\}$  be a filtration of  $F_n$ . A *geometric realization of the filtration* is a marked graph  $G$  equipped with an increasing sequence

$$\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$$

of subgraphs of  $G$  such that for every  $k \in \{1, \dots, i\}$  there exists  $\ell \in \{1, \dots, j\}$  such that  $\mathcal{F}_k = \mathcal{F}(G_\ell)$ .

### 6.2.3 Train tracks and CTs

In this section we introduce the technology of *train tracks*. Train tracks are a type of graph maps introduced by Bestvina and Handel ([BH]). Even though there exist outer automorphisms of  $F_n$  which do not have a topological representative which is a train track, every outer automorphism has a power which has a topological representative called a *completely split train track map (CT)*. CT maps were introduced by Feighn and Handel ([FH]). The definition of a CT map being quite technical, we will only state the relevant properties needed for the rest of the article. First we need some preliminary definitions.

Let  $G$  be a marked graph of  $F_n$  and let  $f: G \rightarrow G$  be a graph map. The map  $f$  induces a *derivative map*  $Df: \vec{E}G \rightarrow \vec{E}G$  on the set of edges as follows. For every  $e \in \vec{E}G$ , the map  $Df(e)$  is equal to the first edge of the edge path  $f(e)$ . A *turn* in  $G$  is an unordered pair  $\{e_1, e_2\}$  of edges in  $G$  with  $o(e_1) = o(e_2)$ . A turn  $\{e_1, e_2\}$  is *degenerate* if  $e_1 = e_2$ , and is *nondegenerate* otherwise. A turn  $\{e_1, e_2\}$  is *illegal* if there exists  $k \in \mathbb{N}^*$  such that  $\{(Df)^k(e_1), (Df)^k(e_2)\}$  is degenerate, and is *legal* otherwise. An edge path  $\gamma = e_1 e_2 \dots e_i$  is *legal* if for every  $j \in \{1, \dots, i\}$ , the turn  $\{e_j^{-1}, e_{j+1}\}$  is legal.

In order to deal with relative outer automorphisms, we also need a notion of relative legal paths. Let  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$  be the geometric realization of some filtration of  $F_n$  which is  $f$ -invariant and let  $r \in \{1, \dots, j\}$ . We say that a turn  $\{e_1, e_2\}$  is contained in the stratum  $H_r$  if  $\{e_1, e_2\} \subseteq \vec{E}H_r$ . An edge path  $\gamma$  of  $G$  is  *$r$ -legal* if every turn in  $\gamma$  that is contained in  $H_r$  is legal. A *connecting path* for  $H_r$  is a nontrivial reduced path  $\gamma$  in  $G_{r-1}$  whose endpoints are in  $G_{r-1} \cap H_r$ . A path  $\gamma$  in  $G$  is  *$r$ -taken* (or *taken* if  $\gamma$  is  $r$ -taken for some  $r$ ) if it is contained in the reduced image of an iterate of an edge  $e \in \vec{E}H_r$ , where  $H_r$  is an irreducible stratum. The *height of a path*  $\gamma$  is the maximal  $r$  such that  $\gamma$  contains an edge of  $H_r$ . We can now define the notion of a *relative train track map* due to Bestvina and Handel ([BH]).

**Definition 6.2.1.** Let  $n \geq 3$ . Let  $G$  be a marked graph and let  $f: G \rightarrow G$  be a graph map equipped with a  $f$ -invariant filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$ . The map  $f$  is a *relative train track map* if, for each exponentially growing stratum  $H_r$ , the following holds:

- (1) for every edge  $e \in \vec{E}H_r$  and every  $k \in \mathbb{N}^*$ , we have  $(Df)^k(e) \in \vec{E}H_r$ ;
- (2) for every connecting path  $\gamma$  for  $H_r$ , the reduced path  $[f(\gamma)]$  is also a connecting path for  $H_r$ ;
- (3) if  $\gamma$  is a height  $r$  reduced edge path which is  $r$ -legal, then so is  $[f(\gamma)]$ .

In order to explain the properties of CT maps that we will use in this paper, we will need some further definitions regarding edge paths in a graph.

**Definition 6.2.2.** Let  $n \geq 3$  and let  $G$  be a marked graph of  $F_n$  equipped with an  $f$ -invariant filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$ . Let  $\gamma$  be an edge path of  $G$ .

- (1) The path  $\gamma$  is a *periodic Nielsen path* if there exists  $k \in \mathbb{N}^*$  such that  $[f^k(\gamma)] = \gamma$ . The minimal such  $k$  is the *period*, and if  $k = 1$ , then  $\gamma$  is a *Nielsen path*.
- (2) A *(periodic) indivisible Nielsen path ((p)INP)* is a (periodic) Nielsen path that cannot be written as a nontrivial concatenation of (periodic) Nielsen paths.
- (3) The path  $\gamma$  is an *exceptional path* if there exist a cyclically reduced Nielsen path  $w$ , edges  $e_1, e_2 \in \vec{E}G$  and integers  $d_1, d_2, p \in \mathbb{Z}^*$  such that for every  $i \in \{1, 2\}$ , we have  $f(e_i) = e_i w^{d_i}$  and  $\gamma = e_1 w^p e_2^{-1}$ . The value  $|p|$  is called the *width* of  $\gamma$ .

**Definition 6.2.3.** Let  $n \geq 3$ , let  $G$  be a marked graph of  $F_n$  and let  $f: G \rightarrow G$  be a relative train track map equipped with a filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$ . Let  $\gamma$  be a reduced edge path or a circuit of  $G$ .

- (1) A *splitting* of  $\gamma$  is a decomposition of  $\gamma$  into edge subpaths  $\gamma = \gamma_1 \gamma_2 \dots \gamma_i$  such that for every  $k \in \mathbb{N}^*$ , we have

$$[f^k(\gamma)] = [f^k(\gamma_1)] \dots [f^k(\gamma_i)],$$

that is one can tighten the image of  $f^k(\gamma)$  by tightening the image of every  $f^k(\gamma_j)$  (where  $o(\gamma)$  is the base point in the case where  $\gamma$  is a circuit).

- (2) Let  $\gamma$  be a circuit. A *circuit splitting* is a splitting  $\gamma = \gamma_1 \dots \gamma_i$  of  $\gamma$  such that for every  $k \in \mathbb{N}^*$ , the concatenation  $[f^k(\gamma_1)] \dots [f^k(\gamma_i)]$  defines a path whose initial and terminal directions are distinct.

- (3) Let  $\gamma = \gamma_1 \gamma_2 \dots \gamma_i$  be a splitting of  $\gamma$ . The splitting is *complete* if for every  $j \in \{1, \dots, i\}$ , the subpath  $\gamma_j$  is one of the following:

- an edge in an irreducible stratum;
- an INP;
- an exceptional path;
- a connecting path in a zero stratum that is both maximal (for the inclusion in  $\gamma$ ) and taken.

Let  $n \geq 2$ , let  $G$  be a marked graph of  $F_n$  and let  $f: G \rightarrow G$  be a relative train track map with respect to a filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_j = G$ . Let  $\gamma$  be an edge path of  $G$ . Such paths in the above list are called *splitting units*. When  $\gamma$  has a complete splitting, we say that  $\gamma$  is *completely split*.

**Definition 6.2.4.** [HaM4, Fact 2.16] Let  $p \in \{0, \dots, j\}$ . Let  $\gamma = \gamma_1 \gamma_2 \dots \gamma_i$  be a splitting of  $\gamma$ . This splitting is *complete relatively to  $G_p$* , or *relatively complete* if there is no ambiguity, if for every  $j \in \{1, \dots, i\}$ , the subpath  $\gamma_j$  is one of the following:

- a splitting unit of height at least equal to  $p + 1$ ;
- a subpath in  $G_p$ .

We now describe some properties of CT maps whose complete definition can be found in [FH, Definition 4.7].

**Proposition 6.2.5.** *Let  $n \geq 3$  and let  $G$  be a marked graph of  $F_n$ . Let  $f: G \rightarrow G$  be a completely split train track (CT) map. Then  $f$  satisfies the following properties.*

- (1) *The map  $f$  is a relative train track map and every stratum in  $G$  is either irreducible or a zero stratum ([FH, Definition 4.7]).*
- (2) *If  $H_r$  is an NEG stratum, then  $H_r$  consists of a single edge  $e_r$ . Moreover, either  $e_r$  is fixed by  $f$  or  $f(e_r) = e_r u_r$  where  $u_r$  is a nontrivial completely split circuit in  $G_{r-1}$ . The terminal endpoint of each NEG stratum is fixed ([FH, Lemma 4.21]).*
- (3) *For every filtration element  $G_r$ , the stratum  $H_r$  is a zero stratum if and only if  $H_r$  is a contractible component of  $G_r$  ([FH, Lemma 4.15]).*
- (4) *For every zero stratum  $H_r$ , there exists a unique  $\ell > r$  such that  $H_\ell$  is an EG stratum and, for every vertex  $v \in V H_r$ , we have  $v \in V H_r \cap V H_\ell$  and the link of  $v$  is contained in  $V H_r \cup V H_\ell$  ([FH, Definition 4.7]).*
- (5) *Every periodic Nielsen path has period one ([FH, Lemma 4.13]).*
- (6) *For every edge  $e$  in an irreducible stratum, the reduced path  $f(e)$  is completely split. For every taken connecting path  $\gamma$  in a zero stratum,  $[f(\gamma)]$  is completely split.*
- (7) *Every completely split path or circuit has a unique complete splitting.*
- (8) *If  $\gamma$  is an edge path, there exists  $k_0 \in \mathbb{N}^*$  such that for every  $k \geq k_0$ , the reduced path  $[f^k(\gamma)]$  is completely split ([FH, Lemma 4.25]).*
- (9) *If  $H_r$  is an EG stratum, there is at most one INP  $\rho_r$  of height  $r$ . The initial edges of  $\rho_r$  and  $\rho_r^{-1}$  are distinct oriented edges in  $H_r$  ([FH, Corollary 4.19]).*
- (10) *If  $H_r$  is a zero stratum, no Nielsen path intersects  $H_r$  in at least one edge ([HaM4, Fact I.1.43]).*
- (11) *Let  $H_r$  be an NEG stratum such that  $H_r = \{e_r\}$ , such that  $f(e_r) = e_r u_r$  and such that  $u_r$  is not trivial. There exists an INP  $\sigma$  which intersects  $H_r$  nontrivially if and only if  $u_r$  is a Nielsen path and there exists  $s \in \mathbb{Z}$  such that  $\sigma = e_r u_r^s e_r^{-1}$  ([FH, Definition 4.7]).*

**Definition 6.2.6.** Let  $n \geq 2$  and let  $G$  be a marked graph of  $F_n$ . Let  $f: G \rightarrow G$  be a completely split train track (CT) map. Let  $H_r$  be an NEG stratum and let  $e_r$  be the edge of  $H_r$ . Let  $u_r$  be such that  $f(e_r) = e_r u_r$ . The edge  $e_r$  is called a *fixed edge* if  $u_r$  is trivial, a *linear edge* if  $u_r$  is a Nielsen path and a *superlinear edge* otherwise.

**Lemma 6.2.7.** [HaM4, Fact 1.39] Let  $n \geq 2$  and let  $G$  be a marked graph of  $F_n$ . Let  $f: G \rightarrow G$  be a CT map. Let  $\gamma$  be a Nielsen path. Then  $\gamma$  is completely split, and all terms in the complete splitting of  $\gamma$  are fixed edges and INPs.

**Lemma 6.2.8.** [HaM4, Fact 1.41] Let  $n \geq 2$  and let  $G$  be a marked graph of  $F_n$ . Let  $f: G \rightarrow G$  be a CT map.

(1) Let  $H_r$  be a zero stratum and let  $H_\ell$  be the EG stratum given by Proposition 6.2.5 (4). There does not exist an INP of height  $\ell$ .

(2) Let  $H_r$  be an EG stratum and let  $\rho_r$  be an INP of height  $r$ . Then  $\rho_r$  has a decomposition  $\rho_r = a_0 b_1 a_1 \dots b_k a_k$  where, for every  $i \in \{0, \dots, k\}$ , the subpath  $a_i$  is a nontrivial path contained in  $H_r$  and for every  $i \in \{1, \dots, k\}$ , the subpath  $b_i$  is a Nielsen path contained in  $G_{r-1}$ .

An INP is an *EG INP* if the maximal stratum it intersects is an EG stratum and is an *NEG INP* otherwise. Note that, by Proposition 6.2.5 (9), there exists only finitely many EG INPs.

**Lemma 6.2.9.** Let  $n \geq 2$ . Let  $\phi \in \text{Out}(F_n)$ . Suppose that there exists a CT map  $f: G \rightarrow G$  representing a power of  $\phi$ . Let  $\gamma'$  be a nontrivial path in a zero stratum. There does not exist a reduced edge path  $\gamma = \alpha \gamma'$  where  $\alpha$  is either an INP or a fixed edge.

**Proof.** Suppose towards a contradiction that such a path  $\gamma = \alpha \gamma'$  exists. Let  $H_r$  be the zero stratum containing  $\gamma'$ . Note that, by Proposition 6.2.5 (10), the path  $\alpha$  does not contain edges in  $H_r$ . By Proposition 6.2.5 (4), there exists  $\ell > r$  such that  $H_\ell$  is an EG stratum and such that any edge adjacent to a vertex in  $H_r$  and not contained in  $H_r$  is in  $H_\ell$ . Hence  $\alpha$  has height at least  $\ell$ . Since  $H_\ell$  is an EG stratum, the path  $\alpha$  is not a fixed edge. Hence  $\alpha$  is an INP. By Lemma 6.2.8 (1), the height of  $\alpha$  is not equal to  $\ell$ . Let  $j > \ell$  be the height of  $\alpha$ . We distinguish between three cases according to the nature of the stratum  $H_j$ . By Proposition 6.2.5 (10), the stratum  $H_j$  is not a zero stratum. Hence, by Proposition 6.2.5 (1), the stratum  $H_j$  is irreducible. By Proposition 6.2.5 (11), if  $H_j$  is an NEG stratum, then  $\alpha$  is of the form  $\alpha = e_j w^k e_j^{-1}$ , where  $e_j \in H_j$ ,  $k$  is an integer and  $w$  is a closed Nielsen path in  $G_{j-1}$ . But then  $e_j^{-1}$  is adjacent to a vertex in  $H_r$ . This contradicts Proposition 6.2.5 (4) since  $j > \ell$ . If  $H_j$  is an EG stratum, then by Lemma 6.2.8 (2), the path  $\alpha$  is the concatenation of subpaths in  $H_j$  and Nielsen paths of height at most  $j - 1$ , and  $\alpha$  ends with an edge in  $H_j$ . By Proposition 6.2.5 (4), we see that  $j = \ell$ . This contradicts Lemma 6.2.8 (1).  $\square$

The next theorem due to Feighn and Handel is the main existence theorem of the CT maps.

**Theorem 6.2.10.** [FH, Theorem 4.28, Lemma 4.42] Let  $n \geq 3$ . There exists a uniform constant  $M = M(n) \geq 1$  such that for every  $\phi \in \text{Out}(F_n)$  and every  $\phi^M$ -invariant filtration  $\mathcal{C}$  of  $F_n$ , there exists a CT map  $f: G \rightarrow G$  that represents  $\phi^M$  and realizes  $\mathcal{C}$ .

#### 6.2.4 Relative currents

In this section, we define the notion of *currents of  $F_n$  relative to a malnormal subgroup system*. The section follows [Gue4] (see the work of Gupta [Gup1] for the particular case of free factor systems and Guirardel and Horbez [GuH1] in the context of free products of groups). It is closely related to the notion of conjugacy classes of  $\mathcal{A}$ -nonperipheral elements of  $F_n$ .

Let  $\partial_\infty F_n$  be the Gromov boundary of  $F_n$ . The *double boundary of  $F_n$*  is the quotient topological space

$$\partial^2 F_n = (\partial_\infty F_n \times \partial_\infty F_n \setminus \Delta) / \sim,$$

where  $\sim$  is the equivalence relation generated by the flip relation  $(x, y) \sim (y, x)$  and  $\Delta$  is the diagonal, endowed with the diagonal action of  $F_n$ . We denote by  $\{x, y\}$  the equivalence class of  $(x, y)$ .

Let  $T$  be the Cayley graph of  $F_n$  with respect to a free basis  $\mathfrak{B}$ . The boundary of  $T$  is naturally homeomorphic to  $\partial_\infty F_n$  and the set  $\partial^2 F_n$  is then identified with the set of unoriented bi-infinite geodesics in  $T$ . Let  $\gamma$  be a finite geodesic path in  $T$ . The path  $\gamma$  determines a subset in  $\partial^2 F_n$  called the *cylinder set of  $\gamma$* , denoted by  $C(\gamma)$ , which consists in all unoriented bi-infinite geodesics in  $T$  that contain  $\gamma$ . Such cylinder sets form a basis for a topology on  $\partial^2 F_n$ , and in this topology, the cylinder sets are both open and closed, hence compact. The action of  $F_n$  on  $\partial^2 F_n$  has a dense orbit.

For every nontrivial subgroup  $A$  of  $F_n$ , let  $T_A$  be the minimal  $A$ -invariant subtree of  $T$ . Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system of  $F_n$ . By malnormality of  $\mathcal{A}$ , there exists  $L \in \mathbb{N}^*$  such that for all distinct subgroups  $A, B$  of  $F_n$  such that  $[A], [B] \in \mathcal{A}$ , the diameter of the intersection  $T_A \cap T_B$  is at most  $L$  (see for instance [HaM4, Section I.1.1.2]). Let  $i \in \{1, \dots, r\}$ . Let  $\Gamma_i$  be the set of subgroups  $B$  of  $F_n$  such that there exists  $g_B \in F_n$  such that  $B = g_B A_i g_B^{-1}$  and the tree  $T_B$  contains the base point  $e$  of  $T$ . Note that, by malnormality of  $\mathcal{A}$ , for every  $i \in \{1, \dots, r\}$ , the set  $\Gamma_i$  is finite. For an element  $w \in F_n$ , let  $\widehat{\gamma}_w$  be the geodesic path in  $T$  starting at  $e$  and labeled by  $w$ . Let  $C_i$  be the set of elements  $w$  of  $F_n$  such that the length of  $\widehat{\gamma}_w$  is equal to  $L + 2$  and, for every  $B \in \Gamma_i$ , the path  $\widehat{\gamma}_w$  is not contained in  $T_B$ . Let  $\mathcal{C} = \bigcap_{i=1}^r C_i$ . Since we are looking at geodesic paths of length equal to  $L + 2$ , the set  $\mathcal{C}$  is finite. Moreover, it only depends on the choice of  $\mathcal{A}$ ,  $\mathfrak{B}$  and  $L$ .

**Lemma 6.2.11.** [Gue4, Lemma 2.3] Let  $\mathfrak{B}$ ,  $T$ ,  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$ ,  $L \in \mathbb{N}^*$ ,  $\Gamma_1, \dots, \Gamma_r$ ,  $\mathcal{C}$  be as above. The finite set  $\mathcal{C} = \mathcal{C}(A_1, \dots, A_k)$  is nonempty. Moreover, it satisfies the following properties:

- (1) every  $\mathcal{A}$ -nonperipheral cyclically reduced element  $g \in F_n$  has a power which contains an element of  $\mathcal{C}$  as a subword;

(2) for every  $\mathcal{A}$ -nonperipheral cyclically reduced element  $g \in F_{\mathbf{n}}$ , if  $c_g$  is the geodesic ray in  $T$  starting from  $e$  obtained by concatenating infinitely many edge paths labeled by  $g$ , there exists an edge path in  $c_g$  labeled by a word in  $\mathcal{C}$  at distance at most  $L + 2$  from  $\bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B$ ;

(3) if  $\gamma$  is a path in  $T$  which contains a subpath labeled by an element of  $\mathcal{C}$ , then for every  $i \in \{1, \dots, r\}$  and every  $g \in F_{\mathbf{n}}$ , the path  $\gamma$  is not contained in  $T_{gA_i g^{-1}}$ .

Let  $A$  be a nontrivial subgroup of  $F_{\mathbf{n}}$  of finite rank. The induced  $A$ -equivariant inclusion  $\partial_{\infty} A \hookrightarrow \partial_{\infty} F_{\mathbf{n}}$  induces an inclusion  $\partial^2 A \hookrightarrow \partial^2 F_{\mathbf{n}}$ . Let

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^r \bigcup_{g \in F_{\mathbf{n}}} \partial^2 (gA_i g^{-1}).$$

Let  $\partial^2(F_{\mathbf{n}}, \mathcal{A}) = \partial^2 F_{\mathbf{n}} - \partial^2 \mathcal{A}$  be the *double boundary* of  $F_{\mathbf{n}}$  relative to  $\mathcal{A}$ . This subset is invariant under the action of  $F_{\mathbf{n}}$  on  $\partial^2 F_{\mathbf{n}}$  and inherits the subspace topology of  $\partial^2 F_{\mathbf{n}}$ .

**Lemma 6.2.12.** [Gue4, Lemma 2.5] Let  $\text{Cyl}(\mathcal{C})$  be the set of cylinder sets of the form  $C(\gamma)$ , where the element of  $F_{\mathbf{n}}$  determined by the geodesic edge path  $\gamma$  contains an element of  $\mathcal{C}$  as a subword. We have

$$\partial^2(F_{\mathbf{n}}, \mathcal{A}) = \bigcup_{C(\gamma) \in \text{Cyl}(\mathcal{C})} C(\gamma).$$

In particular, the space  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$  is an open subset of  $\partial^2 F_{\mathbf{n}}$ .

**Lemma 6.2.13.** [Gue4, Lemma 2.6, Lemma 2.7] Let  $\mathbf{n} \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_{\mathbf{n}}$ . The space  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$  is locally compact and the action of  $F_{\mathbf{n}}$  on  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$  has a dense orbit.

We can now define a *relative current*. Let  $\mathbf{n} \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_{\mathbf{n}}$ . A *relative current* on  $(F_{\mathbf{n}}, \mathcal{A})$  is a (possibly zero)  $F_{\mathbf{n}}$ -invariant Radon measure  $\mu$  on  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$ . The set  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  of all relative currents on  $(F_{\mathbf{n}}, \mathcal{A})$  is equipped with the weak-\* topology: a sequence  $(\mu_n)_{n \in \mathbb{N}}$  in  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})^{\mathbb{N}}$  converges to a current  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  if and only if for all disjoint clopen subsets  $S, S' \subseteq \partial^2(F_{\mathbf{n}}, \mathcal{A})$ , the sequence  $(\mu_n(S \times S'))_{n \in \mathbb{N}}$  converges to  $\mu(S \times S')$ .

The group  $\text{Out}(F_{\mathbf{n}}, \mathcal{A})$  acts on  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  as follows. Let  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{A})$ , let  $\Phi$  be a representative of  $\phi$ , let  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  and let  $C$  be a Borel subset of  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$ . Then, since  $\phi$  preserves  $\mathcal{A}$ , we see that  $\Phi^{-1}(C) \in \partial^2(F_{\mathbf{n}}, \mathcal{A})$ . Then we set

$$\phi(\mu)(C) = \mu(\Phi^{-1}(C)),$$

which is well-defined since  $\mu$  is  $F_{\mathbf{n}}$ -invariant.

Every conjugacy class of nonperipheral element  $g \in F_{\mathbf{n}}$  determines a relative current  $\eta_{[g]}$  as follows. Suppose first that  $g$  is *root-free*, that is  $g$  is not a proper power of any element in  $F_{\mathbf{n}}$ . Let  $\gamma$  be a finite geodesic path in the Cayley graph  $T$ . Then  $\eta_{[g]}(C(\gamma))$



is the number of axes in  $T$  of conjugates of  $g$  that contain the path  $\gamma$ . If  $g = h^k$  with  $k \geq 2$  and  $h$  root-free, we set  $\eta_{[g]} = k \eta_{[h]}$ . Such currents are called *rational currents*.

Let  $G$  be a pointed connected graph whose fundamental group is isomorphic to  $F_{\mathbf{n}}$ . Let  $\tilde{G}$  be the universal cover of  $G$ . There exists a (nonunique, but fixed)  $F_{\mathbf{n}}$ -equivariant quasi-isometry  $\tilde{m}: \tilde{G} \rightarrow T$  which extends uniquely to a homeomorphism  $\hat{m}: \partial_{\infty} G \rightarrow \partial_{\infty} F_{\mathbf{n}}$ . Therefore, if  $\tilde{\gamma}$  is a reduced edge path in  $\tilde{G}$ , we can define the cylinder set in  $\partial^2 F_{\mathbf{n}}$  defined by  $\tilde{\gamma}$  as

$$C_{\tilde{m}}(\tilde{\gamma}) = C([\tilde{m}(\tilde{\gamma})]).$$

Let  $\gamma$  be a reduced edge path in  $G$  and let  $\tilde{\gamma}$  be a lift of  $\gamma$  in  $\tilde{G}$ . Let  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$ . We define the *number of occurrences of  $\gamma$  in  $\mu$*  as

$$\langle \gamma, \mu \rangle_{\tilde{m}} = \mu(C_{\tilde{m}}(\tilde{\gamma})). \quad (6.2)$$

For every such graph  $G$ , we fix once and for all the quasi-isometry  $\tilde{m}: \tilde{G} \rightarrow T$ . Therefore, when the graph  $G$  is fixed, we will generally omit the mention of  $\tilde{m}$ . We also define the *simplicial length of  $\mu$*  as:

$$\|\mu\| = \sum_{e \in \tilde{E}G} \langle e, \mu \rangle.$$

For any given reduced edge path  $\gamma$ , the functions  $\langle \gamma, \cdot \rangle$  and  $\|\cdot\|$  are continuous, linear functions of  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$ .

Let  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$ . The *support of  $\mu$* , denoted by  $\text{Supp}(\mu)$ , is the support of the Borel measure  $\mu$  on  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$ . We recall that  $\text{Supp}(\mu)$  is a closed subset of  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$ .

In the rest of the article, rather than considering the space of relative currents itself, we will consider the set of *projectivized relative currents*, denoted by

$$\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{A}) = (\text{Curr}(F_{\mathbf{n}}, \mathcal{A}) - \{0\}) / \sim,$$

where  $\mu \sim \nu$  if there exists  $\lambda \in \mathbb{R}_+^*$  such that  $\mu = \lambda\nu$ . The projective class of a current  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  will be denoted by  $[\mu]$ . We have the following properties.

**Lemma 6.2.14.** *[Gue4, Lemma 3.3] Let  $\mathbf{n} \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_{\mathbf{n}}$ . The space  $\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  is compact.*

**Proposition 6.2.15.** *[Gue4, Theorem 1.1] Let  $\mathbf{n} \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_{\mathbf{n}}$ . The set of projectivized rational currents about nonperipheral elements of  $F_{\mathbf{n}}$  is dense in  $\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$ .*

### 6.3 The polynomially growing subgraph of a CT map

In this section, let  $\mathbf{n} \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_{\mathbf{n}}$ . Let  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{F})$ . Let  $f: G \rightarrow G$  be a CT map with filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$  representing a power of  $\phi$  and such that there exists  $p \in \{1, \dots, k-1\}$  such that  $\mathcal{F}(G_p) = \mathcal{F}$ .

We construct a subgraph of  $G$ , called the *polynomially growing subgraph* of  $G$  and denoted by  $G_{PG}$ , which encaptures the information regarding polynomial growth in the graph  $G$ . We then define a notion of length relative to  $G_{PG}$ , called the *exponential length*, which measures the time spent by an edge path outside of  $G_{PG}$ . Finally, we construct a subspace of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F})$  which consists in the currents whose support maps to  $G_{PG}$ .

### 6.3.1 Definitions and first properties

We define in this section the *polynomially growing subgraph*  $G_{PG}$  of  $G$  and proves some of its properties.

**Definition 6.3.1.** (1) Let  $G_{PG}$  be the (not necessarily connected) subgraph of  $G$  whose edges are the edges  $e$  of  $G$  in an NEG stratum such that for every  $k \in \mathbb{N}^*$ , the path  $[f^k(e)]$  does not contain a splitting unit which is an edge in an EG stratum.

(2) Let  $\mathcal{N}'_{PG}$  be the set of all Nielsen paths in  $G$ .

(3) Let  $\mathcal{N}_{PG}$  be the subset of  $\mathcal{N}'_{PG}$  consisting in all Nielsen paths which are either EG INPs or concatenations of (at least 2) nonclosed EG INPs.

(4) Let  $\mathcal{Z}$  be the subgraph of  $G$  whose edges are the edges contained in a zero stratum.

Note that, by Lemma 6.2.7, every path in  $\mathcal{N}'_{PG}$  (and hence every path in  $\mathcal{N}_{PG}$ ) has a complete splitting consisting in fixed edges and INPs. Since a complete splitting is unique by Proposition 6.2.5 (7), if  $\gamma$  is a reduced path in  $\mathcal{N}_{PG}$ , then the splitting of  $\gamma$  given in Definition 6.3.1 (3) is the complete splitting of  $\gamma$ . Moreover,  $\gamma$  is either an EG INP or the complete splitting of  $\gamma$  has at least two splitting units and all of them are nonclosed EG INPs. In particular, the set  $\mathcal{N}_{PG}$  does not contain Nielsen paths such that one of their splitting units is either a fixed edge or an NEG INP. Moreover, a Nielsen path which is a concatenation of at least 2 splitting units and such that one of them is a closed EG INP is not in  $\mathcal{N}_{PG}$ . Excluding such paths from  $\mathcal{N}_{PG}$  ensures a finiteness result for  $\mathcal{N}_{PG}$  (see Lemma 6.3.4 (1)). Informally, paths in  $\mathcal{N}_{PG}$  play the role of low-dynamics bridges between connected components of  $G_{PG}$  (see Figure 6.1). We will see in Proposition 6.3.13 that a cycle in  $G$  has polynomial growth under iteration of  $f$  if and only if is a concatenation of paths in  $G_{PG}$  and paths in  $\mathcal{N}_{PG}$ .

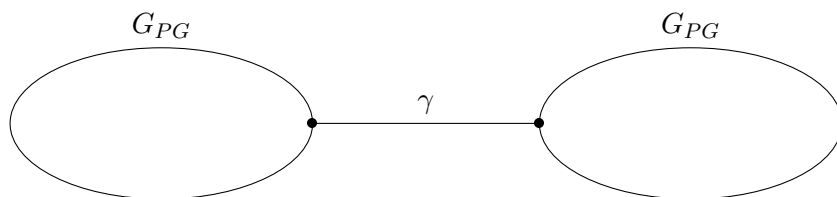


Figure 6.1: A path  $\gamma$  in  $\mathcal{N}_{PG}$  between two connected components of  $G_{PG}$ .

Note that, with  $p$  defined at the beginning of Section 6.3, one can similarly define the *polynomially growing subgraph of  $G_p$* , denoted by  $G_{PG, \mathcal{F}}$ , which is the subgraph

$G_{PG} \cap G_p$ . We can also define similarly  $\mathcal{N}'_{PG, \mathcal{F}}$ ,  $\mathcal{N}_{PG, \mathcal{F}}$  and  $\mathcal{Z}_{\mathcal{F}}$  by considering the paths of  $\mathcal{N}'_{PG}$ ,  $\mathcal{N}_{PG}$  and  $\mathcal{Z}$  contained in  $G_p$ .

We now recall a lemma due to Bestvina and Handel regarding  $r$ -legal paths.

**Lemma 6.3.2.** [BH, Lemma 5.8] *Let  $f: G \rightarrow G$  be a relative train track map. Let  $H_r$  be an EG stratum. Suppose that  $\sigma = a_1 b_1 a_2 \dots a_\ell b_\ell$  is the decomposition of an  $r$ -legal path into subpaths  $a_j \subseteq H_r$  and  $b_j \subseteq G_{r-1}$  (where  $a_1$  and  $b_\ell$  might be trivial). Then for every  $i \in \{1, \dots, \ell\}$ , the path  $f(a_\ell)$  is a reduced edge path and*

$$[f(\sigma)] = f(a_1)[f(b_1)]f(a_2) \dots f(a_\ell)[f(b_\ell)].$$

Note that, if  $H_r$  is an EG stratum and if  $\sigma = a_1 b_1 a_2 \dots a_\ell b_\ell$  is an  $r$ -legal path as in Lemma 6.3.2, then for every  $i \in \{1, \dots, \ell\}$ , as  $a_i \subseteq H_r$ , the path  $a_i$  grows exponentially fast under iteration of  $f$ . Hence, by Lemma 6.3.2 the path  $\sigma$  grows exponentially fast under iteration of  $f$ . We now prove some results regarding paths in  $\mathcal{N}_{PG}$ .

**Lemma 6.3.3.** *Let  $\sigma$  be an EG INP.*

- (1) *There do not exist nontrivial subpaths  $c, d$  of  $\sigma$  such that  $\sigma = cdc$ .*
- (2) *Let  $\gamma \in \{\sigma^{\pm 1}\}$ . There do not exist paths  $\gamma_1, \gamma_2, \gamma_3$  such that  $\gamma_2$  is nontrivial,  $\gamma_1$  or  $\gamma_3$  is nontrivial and  $\sigma = \gamma_1 \gamma_2$  and  $\gamma = \gamma_2 \gamma_3$ .*

**Proof.** (1) Let  $r$  be the height of  $\sigma$ . Suppose towards a contradiction that such a decomposition  $\sigma = cdc$  exists. By [BH, Lemma 5.11], there exist two distinct  $r$ -legal paths  $\alpha$  and  $\beta$  such that  $\sigma = \alpha\beta$  and such that the turn  $\{Df(\alpha^{-1}), Df(\beta)\}$  is the only height  $r$  illegal turn. Moreover, there exists a path  $\tau$  such that  $[f(\alpha)] = \alpha\tau$  and  $[f(\beta)] = \tau^{-1}\beta$ . Hence  $c$  is contained in  $\alpha$  and in  $\beta$  and is  $r$ -legal. Thus, there exist two paths  $d_1$  and  $d_2$  such that  $\alpha = cd_1$  and  $\beta = d_2c$ .

First we claim that for every  $k \in \mathbb{N}^*$ , there exists a path  $\tau_k$  such that  $[f^k(\alpha)] = \alpha\tau_k$  and  $[f^k(\beta)] = \tau_k^{-1}\beta$ . The proof is by induction on  $k$ . The base case follows from the existence of  $\tau$ . Suppose now that  $\tau_{k-1}$  exists. We have:

$$[f^k(\alpha)] = [f(\alpha\tau_{k-1})] = [f(\alpha)][f(\tau_{k-1})] = \alpha\tau[f(\tau_{k-1})] = \alpha\tau_k,$$

where the second equality comes from the fact that  $\alpha$  is  $r$ -legal, that  $\alpha$  ends with an edge in  $H_r$  and from Lemma 6.3.2. Similarly, we have  $[f^k(\beta)] = \tau_k^{-1}\beta$ . This proves the claim.

We now claim that, up to taking a power of  $f$ , there exists a cycle  $e$  such that  $[f(c)] = \alpha e \beta$ . Indeed, by Proposition 6.2.5 (9), the path  $\sigma$  starts and ends with an edge in  $H_r$ . Hence the path  $c$  starts and ends with an edge in  $H_r$ . Since  $c$  is  $r$ -legal, we see that the length of  $[f^k(c)]$  goes to infinity as  $k$  goes to infinity by Lemma 6.3.2. But, for every  $k \in \mathbb{N}^*$ , there exists a path  $\tau_k$  such that  $[f^k(\alpha)] = \alpha\tau_k$  and  $[f^k(\beta)] = \tau_k^{-1}\beta$ . By Lemma 6.3.2, since  $c$  is the initial segment of  $\alpha$  and since  $\alpha$  is  $r$ -legal, there is no identification between  $[f(c)]$  and  $[f(d_1)]$ . Thus, there exists  $k_1 \in \mathbb{N}^*$  such that  $[f^{k_1}(c)]$  starts with  $\alpha$ . Similarly, there exists  $k_2 \in \mathbb{N}^*$  such that  $[f^{k_2}(c)]$  ends with  $\beta$ . Thus, up

to taking a power of  $f$ , and since the paths  $\alpha$  and  $\beta$  are  $r$ -legal, we may suppose that there exists a (reduced) cycle  $e$  such that  $[f(c)] = \alpha e \beta$ .

Finally, we claim that the cycle  $e$  is trivial. Indeed, since the paths  $\alpha$  and  $\beta$  are  $r$ -legal, and since  $c$  starts and ends with an edge in  $H_r$ , we see that

$$[f(\alpha)] = [f(c)][f(d_1)] = \alpha e \beta [f(d_1)]$$

and

$$[f(\beta)] = [f(d_2)][f(c)] = [f(d_2)]\alpha e \beta.$$

Recall that there exists  $k \in \mathbb{N}^*$  such that  $[f(\alpha)] = \alpha \tau_k$  and  $[f(\beta)] = \tau_k^{-1} \beta$ . This implies that  $\tau_k = e \beta [f(d_1)]$  and that  $\tau_k^{-1} = [f(d_2)] \alpha e$ , that is  $\tau_k = e^{-1} \alpha^{-1} [f(d_2)]^{-1}$ . This shows that  $e = e^{-1}$ , that is,  $e$  is trivial. This proves the claim.

Therefore, we see that  $[f(c)] = \alpha \beta = \sigma$ . But  $\sigma$  contains a height  $r$  illegal turn, whereas  $c$  is an  $r$ -legal path. This contradicts Proposition 6.2.5 (1) and Definition 6.2.1 (3). This concludes the proof of (1).

(2) Let  $\sigma, \gamma$  be as in the assertion of the lemma. Suppose towards a contradiction that there exist three paths  $\gamma_1, \gamma_2, \gamma_3$  such that  $\gamma_2$  is nontrivial and  $\sigma = \gamma_1 \gamma_2$  and  $\gamma = \gamma_2 \gamma_3$ . Suppose first that  $\gamma = \sigma$ . Then either a nontrivial initial segment of  $\gamma_2$  is its terminal segment or there exists a path  $\gamma_4$  such that  $\sigma = \gamma_2 \gamma_4 \gamma_2$ . The first case is not possible as otherwise  $\sigma$  would contain two illegal turns. This contradicts the fact that  $\sigma$  contains a unique illegal turn (see [BH, Lemma 5.11]). The second case is not possible by Lemma 6.3.3 (1). Suppose now that  $\gamma = \sigma^{-1}$ . But  $\sigma^{-1} = \gamma_2^{-1} \gamma_1^{-1}$ . Therefore we see that  $\gamma_2^{-1} = \gamma_2$ , that is,  $\gamma_2$  is trivial. This leads to a contradiction. This concludes the proof.  $\square$

**Lemma 6.3.4.** (1) *There are only finitely many paths in  $\mathcal{N}_{PG}$ .*

(2) *Let  $\gamma, \gamma'$  be paths in  $\mathcal{N}_{PG}$ . Suppose that  $\gamma$  has a decomposition  $\gamma = \gamma_1 \gamma_2$  such that  $\gamma_2$  is an initial segment of  $\gamma'$ . Then  $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$  and  $\gamma_1 \gamma' \in \mathcal{N}_{PG}$ .*

(3) *Let  $\gamma, \gamma'$  be paths in  $\mathcal{N}_{PG}$ . Suppose that  $\gamma' \subseteq \gamma$ . Then one of the following holds:*

(a) *there exist (possibly trivial) paths  $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$  such that  $\gamma = \gamma_1 \gamma' \gamma_2$ ;*

(b) *there exists an INP  $\sigma$  in the complete splitting of  $\gamma$  such that  $\gamma' \not\subseteq \sigma$  and  $\gamma'$  is not an initial or a terminal segment of  $\sigma$ .*

(4) *Let  $\gamma, \gamma'$  be two paths in  $\mathcal{N}_{PG}$ . Suppose that there exist three paths  $\gamma_1, \gamma_2$  and  $\gamma_3$  such that  $\gamma = \gamma_1 \gamma_2$ ,  $\gamma' = \gamma_2^{-1} \gamma_3$  and the path  $\gamma_1 \gamma_3$  is reduced. Then  $\gamma_2 \in \mathcal{N}_{PG}$  and  $\gamma_1 \gamma_3 \in \mathcal{N}_{PG}$ .*

**Proof.** (1) First note that, since there are only finitely many EG strata in  $G$ , there are only finitely many EG INPs by Proposition 6.2.5 (9). Let  $\gamma$  be a path in  $\mathcal{N}_{PG}$  which is a concatenation of at least 2 nonclosed EG INPs. Let  $\gamma = \sigma_1 \dots \sigma_k$  be the complete splitting of  $\gamma$  given by Lemma 6.2.7. As  $\gamma$  is a concatenation of nonclosed EG INPs, every splitting unit of  $\gamma$  is a nonclosed EG INP. By Proposition 6.2.5 (9), an INP contained in the complete splitting of  $\gamma$  is entirely determined by the highest stratum

$H_r$  such that  $\gamma$  contains an edge of  $H_r$ . For every  $i \in \{1, \dots, k\}$ , let  $r_i$  be the height of  $\sigma_i$ . Let  $i \in \{2, \dots, k\}$ . Since  $\sigma_i$  is not closed, by [HaM4, Fact 1.42(1)(a)], one of the endpoints of  $\sigma_i$  is not contained in  $G_{r_{i-1}}$ . Since there exists a unique INP of height  $r_i$  by Proposition 6.2.5 (9), either  $r_{i-1} < r_i$  or  $r_i < r_{i-1}$ . We treat the case  $r_1 < r_2$ , the case  $r_2 < r_1$  being similar. We claim that, for every  $i \in \{1, \dots, k-1\}$ , we have  $r_{i+1} > r_i$ . The proof is by induction on  $i$ . The base case is true by hypothesis. Let  $i \in \{2, \dots, k-1\}$ . Since  $r_{i-1} < r_i$ , the origin of  $\sigma_i$  is contained in  $G_{r_{i-1}}$  and the terminal point of  $\sigma_i$  is not contained in  $G_{r_{i-1}}$ . Thus, the first edge of  $\sigma_{i+1}$  is contained in  $\overline{G - G_{r_{i-1}}}$ . Since there exists a unique INP of height  $r_i$  we necessarily have  $r_i < r_{i+1}$ . Thus, the sequence of maximal heights of INPs in  $\gamma$  is (strictly) monotonic. Since there are only finitely many EG strata, there are only finitely many paths in  $\mathcal{N}_{PG}$ . This concludes the proof of (1).

(2) Let  $\gamma' \in \mathcal{N}_{PG}$  and let  $\gamma = \gamma_1\gamma_2$  be as in the assertion of the lemma. We claim that  $\gamma_2 \in \mathcal{N}_{PG}$  and that the splitting units of  $\gamma_2$  are splitting units of both  $\gamma$  and  $\gamma'$ . This will conclude the proof of Assertion (2) because  $\gamma_1$  will be a concatenation of splitting units of  $\gamma$ , that is, it will be either an EG INP or a concatenation of nonclosed EG INPs (cf Definition 6.3.1 (3)). Hence we will have  $\gamma_1 \in \mathcal{N}_{PG}$  and  $\gamma_1\gamma' \in \mathcal{N}_{PG}$ . We show that  $\gamma_2$  is a concatenation of INPs which are splitting units of  $\gamma'$ . A similar proof will show that the splitting units of  $\gamma_2$  will also be splitting units of  $\gamma$ . Indeed, the path  $\gamma'$  has a splitting  $\gamma' = \sigma'_1\sigma'_2 \dots \sigma'_k$  which consists in EG INPs. Let  $r'$  be the height of  $\sigma'_1$ . By Proposition 6.2.5 (9), there exists a unique unoriented INP of height  $r'$  and this INP starts and ends with an edge in  $H_{r'}$ . Let  $\sigma$  be the INP of  $\gamma$  which has a decomposition  $\sigma = \sigma_1\sigma_2$ , where  $\sigma_2$  is a nontrivial initial segment of  $\gamma'$ . As every splitting unit of  $\gamma$  is an EG INP, so is  $\sigma$ . Let  $r$  be the height of  $\sigma$ . Since the first edge of  $\sigma'_1$  is of height  $r'$ , we cannot have  $r' > r$ . If  $r = r'$ , then by the uniqueness statement in Proposition 6.2.5 (9), we have  $\sigma'_1 \in \{\sigma, \sigma^{-1}\}$ . Note that, if  $\sigma_1$  is nontrivial, there exist reduced paths  $\tau_1, \tau_2$  such that  $\sigma = \sigma_1\tau_1$  and  $\sigma'_1 = \tau_1\tau_2$ . This contradicts Lemma 6.3.3 (2) applied to  $\sigma$  and  $\sigma'_1$ . Thus, we see that  $\sigma = \sigma'_1$  and  $\sigma'_1 \subseteq \gamma_2$ . If  $r' < r$ , then by Lemma 6.2.8 (2), the path  $\sigma$  has a decomposition  $\sigma = a_1b_1 \dots b_{k-1}a_k$  such that, for every  $i \in \{1, \dots, k\}$ , the path  $a_i$  is a path in  $H_r$  and for every  $i \in \{1, \dots, k-1\}$ , the path  $b_i$  is a Nielsen path in  $G_{r-1}$ . Hence there exists  $i \in \{1, \dots, k-1\}$  such that  $\sigma'_1$  is contained in  $b_i$ . Therefore, we see that  $\sigma'_1 \subseteq \sigma \subseteq \gamma$ . As  $\sigma'_1 \subseteq \gamma'$ , we see that  $\sigma'_1 \subseteq \gamma \cap \gamma' = \gamma_2$ . If  $\gamma_2 = \sigma'_1$ , then we are done. Otherwise, the path  $\gamma_2$  contains an edge of  $\sigma'_2$ . As  $\sigma'_2$  is an EG INP, the same argument as for  $\sigma'_1$  shows that  $\sigma'_2 \subseteq \gamma_2$ , and an inductive argument shows that  $\gamma_2$  is a concatenation of INPs in the splitting of  $\gamma'$ . Hence  $\gamma_2$  is a Nielsen path. Therefore, we see that  $\gamma_2 \in \mathcal{N}_{PG}$  and that  $\gamma_2$  is composed of splitting units of  $\gamma'$ . Similarly, we see that  $\gamma_2$  is composed of splitting units which are splitting units of both  $\gamma$  and  $\gamma'$ . Hence  $\gamma_1$  is composed of splitting units of  $\gamma$ . This concludes the proof of (2).

(3) Let  $\gamma, \gamma'$  be as in the assertion of the lemma. Let  $\gamma = \sigma_1 \dots \sigma_k$  be the complete splitting of  $\gamma$  and let  $\gamma' = \sigma'_1 \dots \sigma'_m$  be the complete splitting of  $\gamma'$ , which exist by Lemma 6.2.7. Recall that every splitting unit of both  $\gamma$  and  $\gamma'$  is an EG INP. There exists  $i \in \{1, \dots, k\}$  such that  $\sigma_i$  contains an initial segment of  $\sigma'_1$ . We claim that  $\sigma'_1$  is either equal to  $\sigma_i$  or  $\gamma'$  is strictly contained in  $\sigma_i$ . Indeed, let  $r$  be the height of  $\sigma_i$  and

let  $r'$  be the height of  $\sigma'_1$ . Since the first edge of  $\sigma'_1$  is of height  $r'$ , we cannot have  $r' > r$ .

Suppose first that  $r' < r$ . By Lemma 6.2.8 (2), the path  $\sigma_i$  has a decomposition  $\sigma_i = a_1 b_1 \dots b_{p-1} a_p$  such that, for every  $i \in \{1, \dots, p\}$ , the path  $a_i$  is a path in  $H_r$  and for every  $j \in \{1, \dots, p-1\}$ , the path  $b_j$  is a Nielsen path in  $G_{r-1}$ . Hence there exists  $j \in \{1, \dots, p-1\}$  such that  $\sigma'_1$  is contained in  $b_j$ . We claim that, for every  $\ell \in \{1, \dots, m\}$ , the splitting unit  $\sigma'_\ell$  is contained in  $b_j$ . The proof is by induction on  $\ell$ . For the base case, we already know that  $\sigma'_1 \subseteq b_j$ . Suppose that for some  $\ell \in \{2, \dots, m\}$ , the path  $\sigma'_{\ell-1}$  is contained in  $b_j$ . By Proposition 6.2.5 (9), the path  $\sigma_i$  ends with an edge in  $H_r$ . Hence the path  $a_p$  is nontrivial. Since  $\sigma'_{\ell-1}$  is contained in  $b_j$ , the path  $\sigma'_\ell$  intersects  $\sigma_i$  nontrivially. Let  $r_\ell$  be the height of  $\sigma'_\ell$ . Recall that  $\sigma'_\ell$  is an EG INP. By Proposition 6.2.5 (9), the path  $\sigma'_\ell$  starts with an edge in  $H_{r_\ell}$ . Hence  $r_\ell \leq r$ . Suppose towards a contradiction that  $r_\ell = r$ . Then, by the uniqueness statement of Proposition 6.2.5 (9), we see that  $\sigma'_\ell \in \{\sigma_i^{\pm 1}\}$ . As  $\sigma_i$  contains an initial segment of  $\sigma'_\ell$ , there exist three paths  $\gamma_1, \gamma_2$  and  $\gamma_3$  of  $G$  such that  $\gamma_2$  is nontrivial and  $\sigma_i = \gamma_1 \gamma_2$  and  $\sigma'_\ell = \gamma_2 \gamma_3$ . Since  $\sigma'_{\ell-1}$  is contained in  $\sigma_i$ , the path  $\gamma_1$  is nontrivial. This contradicts Lemma 6.3.3 (2). Therefore we have  $r_\ell < r$ . But then  $\sigma'_\ell$  cannot intersect  $a_{j+1}$ . This implies that  $\sigma'_\ell$  is contained in  $b_j$ . This proves the claim and the fact that  $\gamma' \not\subseteq \sigma_i$  and  $\gamma'$  is not an initial or a terminal segment of  $\sigma_i$ .

Suppose now that  $r = r'$ . By the uniqueness statement of Proposition 6.2.5 (9), we see that  $\sigma'_1 \in \{\sigma_i^{\pm 1}\}$ . As  $\sigma_i$  contains an initial segment of  $\sigma'_1$ , there exist three paths  $\gamma_1, \gamma_2$  and  $\gamma_3$  of  $G$  such that  $\gamma_2$  is nontrivial and  $\sigma_i = \gamma_1 \gamma_2$  and  $\sigma'_1 = \gamma_2 \gamma_3$ . By Lemma 6.3.3 (2), we necessarily have that  $\gamma_1$  and  $\gamma_3$  are trivial. Thus, we see that  $\sigma_i = \sigma'_1$ . Therefore,  $\gamma'$  is an initial segment of  $\sigma_i \dots \sigma_k$  and is a Nielsen path. By [FH, Corollary 4.12], for every  $j \in \{1, \dots, m\}$ , we have  $\sigma_{i+j-1} = \sigma'_j$ . Thus, there exist (possibly trivial) paths  $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$  such that  $\gamma = \gamma_1 \gamma' \gamma_2$ . This concludes the proof of (3).

(4) Let  $\gamma, \gamma', \gamma_1, \gamma_2$  and  $\gamma_3$  be as in the assertion of the lemma. Let  $\gamma = \alpha_1 \dots \alpha_k$  and  $\gamma' = \beta_1 \dots \beta_\ell$  be the complete splittings of  $\gamma$  and  $\gamma'$  given by Lemma 6.2.7. By definition of  $\mathcal{N}_{PG}$ , every splitting unit of  $\gamma$  and  $\gamma'$  is an EG INP. Let  $i \in \{1, \dots, k\}$  be such that  $\alpha_i$  contains the first edge of  $\gamma_2$ . Let  $j \in \{1, \dots, \ell\}$  be such that  $\beta_j$  contains the last edge of  $\gamma_2^{-1}$ . We claim that  $\alpha_i \subseteq \gamma_2$  and that  $\beta_j \subseteq \gamma_2^{-1}$ . By [FH, Corollary 4.12] applied to  $\gamma_2^{-1}$  and  $\gamma^{-1}$ , there exists a path  $\delta_i$  contained in  $\alpha_i$  such that the decomposition  $\gamma_2 = \delta_i \alpha_{i+1} \dots \alpha_k$  is a splitting of  $\gamma_2$ . Similarly, there exists a path  $\delta'_j$  in  $\beta_j$  such that  $\gamma_2^{-1} = \beta_1 \dots \beta_{j-1} \delta'_j$  is a splitting of  $\gamma_2^{-1}$ . By Proposition 6.2.5 (9), an EG INP starts with an edge of highest height and an EG INP is entirely determined by its height. Hence  $\alpha_k = \beta_1^{-1}$ . Note that the paths  $\delta_i \alpha_{i+1} \dots \alpha_{k-1}$  and  $\beta_2 \dots \beta_{j-1} \delta'_j$  satisfy the same hypotheses as  $\delta_i \alpha_{i+1} \dots \alpha_k$  and  $\beta_1 \dots \beta_{j-1} \delta'_j$ . Applying the same arguments, we see that  $i = j$  and for every  $s \in \{1, \dots, j-1\}$ , we have  $\beta_s = \alpha_{k-s+1}^{-1}$ . Hence we see that  $\delta_i = \delta'_j$ . Let  $r$  be the height of  $\alpha_i$  and let  $r'$  be the height of  $\beta_j$ . Note that by Proposition 6.2.5 (9) applied to  $\alpha_i$  and  $\beta_j$ , the path  $\delta_i$  ends with an edge in  $H_r$  and  $\delta'_j$  ends with an edge in  $H_{r'}$ . Therefore, we see that  $r = r'$ . By uniqueness of EG INPs of height  $r_i$  given by Proposition 6.2.5 (9), and since  $\gamma_1 \gamma_3$  is reduced, we see that  $\alpha_i = \beta_j^{-1}$ , that  $\alpha_i \subseteq \gamma_2$  and that  $\beta_j \subseteq \gamma_2^{-1}$ . This shows that  $\gamma_2$  is a path in  $\mathcal{N}_{PG}$ . By Assertion (2) applied to  $\gamma$  and

$\gamma_2$ , the path  $\gamma_1$  is contained in  $\mathcal{N}_{PG}$ . Similarly, we see that the path  $\gamma_3$  is contained in  $\mathcal{N}_{PG}$ . Since the path  $\gamma_1\gamma_3$  is reduced, we see that  $\gamma_1\gamma_3 \in \mathcal{N}_{PG}$ . This concludes the proof.  $\square$

**Lemma 6.3.5.** *Let  $\gamma$  and  $\gamma'$  be two reduced edge paths in  $G$  which are concatenations of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Suppose that there exist three paths  $\gamma_1, \gamma_2$  and  $\gamma_3$  such that  $\gamma = \gamma_1\gamma_2$ ,  $\gamma' = \gamma_2^{-1}\gamma_3$  and  $\gamma_1\gamma_3$  is reduced. Then  $\gamma_2$  and  $\gamma_1\gamma_3$  are concatenations of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ .*

**Proof.** Let  $\gamma = b_0a_1b_1 \dots a_kb_k$  be the decomposition of the path  $\gamma$  such that for every  $i \in \{0, \dots, k\}$ , the path  $b_i$  is in  $G_{PG}$  and for every  $i \in \{1, \dots, k\}$ , the path  $a_i$  is a maximal subpath of  $\gamma$  contained in  $\mathcal{N}_{PG}$ . The existence of the paths  $a_i$  follows from Lemma 6.3.4 (2). Let  $\gamma' = d_0c_1d_1 \dots c_\ell d_\ell$  be the similar decomposition of  $\gamma'$ . Let  $e$  be the initial edge of  $\gamma_2$ .

**Claim.** There exists  $i \in \{0, \dots, k\}$  such that  $b_i$  contains  $e$  if and only if there exists  $j \in \{0, \dots, \ell\}$  such that the edge  $e^{-1}$  is contained in  $d_j$ .

**Proof.** The proof of the two directions being similar, we only prove one direction. Suppose that there exists  $i \in \{0, \dots, k\}$  such that  $b_i$  contains  $e$ . Suppose towards a contradiction that there exists  $j \in \{1, \dots, \ell\}$  such that  $e^{-1}$  is contained in  $c_j$ . It follows that there exists an EG INP  $\sigma$  of  $c_j$  such that  $e^{-1}$  is contained in  $\sigma$ . Let  $r$  be the height of  $\sigma$ . Let  $\delta^{-1}$  be the subpath of  $\sigma$  contained in  $\gamma_2^{-1}$ . Note that, as  $\gamma_2^{-1}$  is an initial segment of  $\gamma'$ , the path  $\delta^{-1}$  is an initial segment of  $\sigma$ . By Proposition 6.2.5 (9), the path  $\delta^{-1}$  starts with an edge in  $H_r$ . As  $\delta$  is contained in  $\gamma$ , the terminal edge of  $\delta$  is an edge in an EG stratum. Since every edge in  $G_{PG}$  is contained in an NEG stratum, there exists  $s \in \{1, \dots, k\}$  such that  $a_s$  contains a terminal segment of  $\delta$ . Since the initial edge  $e$  of  $\gamma_2$  is not contained in  $a_s$  by hypothesis, the path  $\delta$  contains the initial segment  $\delta'$  of  $a_s$ . Hence the terminal segment  $\delta'^{-1}$  of  $a_s^{-1}$  is the initial segment  $\delta'^{-1}$  of  $\sigma$ . By Lemma 6.3.4 (2) applied to  $a_s^{-1}$  and  $\sigma$  and [FH, Corollary 4.12], the path  $\delta'^{-1}$  is contained in  $\mathcal{N}_{PG}$  and is a concatenation of splitting units of  $\sigma$ . As  $\sigma$  contains a unique splitting unit, this implies that  $\delta' = \sigma$ . As  $\delta' \subseteq \delta^{-1} \subseteq \sigma$ , we see that  $\delta^{-1} = \sigma$ . Note that the edge  $\delta^{-1}$  ends with  $e^{-1}$ . But  $\sigma$  ends with an edge in an EG stratum by Proposition 6.2.5 (9), that is,  $e^{-1}$  is an edge in an EG stratum. But every edge in  $b_i$  is contained in an NEG stratum by definition of  $G_{PG}$ . This contradicts the fact that  $e \subseteq b_i$ . This concludes the proof of the claim.  $\square$

Suppose first that there exists  $i \in \{1, \dots, k\}$ , such that  $e$  is contained in  $b_i$ . By the above claim, there exists  $j \in \{0, \dots, \ell\}$  such that  $e^{-1}$  is contained in  $d_j$ . Let  $\tau$  and  $\tau'$  be such that  $\gamma = b_0a_1b_1 \dots a_i\tau\gamma_2$  and  $\gamma' = \gamma_2^{-1}\tau'c_{j+1} \dots d_\ell$ . Note that  $\tau \subseteq b_i$  and  $\tau' \subseteq d_j$ . Then we have  $\gamma_1 = b_0a_1b_1 \dots a_i\tau$  and  $\gamma_3 = \tau'c_{j+1} \dots d_\ell$ . Since the path  $\gamma_1\gamma_3$  is reduced, so is  $\tau\tau'$ . Moreover the reduced edge path  $\tau\tau'$  is contained in  $G_{PG}$  and  $\gamma_1\gamma_3 = b_0a_1b_1 \dots a_i\tau\tau'c_{j+1} \dots d_\ell$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Moreover, let  $\delta''$  be the maximal subpath of  $b_i$  contained in  $\gamma_2$ . Then  $\gamma_2 = \delta''a_{i+1} \dots b_k$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .

Suppose now that there exists  $i \in \{1, \dots, k\}$  such that the initial edge  $e$  of  $\gamma_2$  is contained in  $a_i$ . By the above claim, there exists  $j \in \{1, \dots, \ell\}$  such that  $e^{-1}$  is contained in  $c_j$ . Let  $\delta'$  be the terminal segment of  $a_i$  contained in  $\gamma_2$ . By Proposition 6.2.5 (9), the terminal edge  $e'$  of  $\delta'$  is an edge in an EG stratum. Since  $G_{PG}$  does not contain any edge in an EG stratum, there exists  $s \leq j$  such that  $c_s$  contains  $e'^{-1}$ . We claim that  $s = j$ . Indeed, suppose towards a contradiction that  $s < j$ . Let  $\delta^{-1}$  be the terminal segment of  $c_s$  whose first edge is  $e'^{-1}$ . Then  $\delta$  is a terminal segment of  $a_i$  and  $\delta$  is an initial segment of  $c_s^{-1}$ . By Lemma 6.3.4 (2) applied to  $a_i$  and  $c_s^{-1}$ , the path  $\delta$  is a concatenation of splitting units of  $a_i$  and  $c_s^{-1}$ . If  $\delta$  is properly contained in  $\delta'$ , there exists an EG INP  $\sigma$  which is a splitting unit of  $a_i$  and such that the last edge of  $\sigma$  is the last edge of  $\delta'$  not contained in  $\delta$ . But, by Proposition 6.2.5 (9), the terminal edge  $e_\sigma$  of  $\sigma$  is in an EG stratum. However, the first edge of  $d_s$  (which is the edge  $e_\sigma^{-1}$ ) is in  $G_{PG}$ . This leads to a contradiction. Hence  $\delta = \delta'$ . But  $\delta$  intersects  $c_j$  nontrivially. Hence we have  $s = j$ .

Therefore,  $\delta'^{-1}$  is contained in  $c_j$ . We claim that  $\delta'^{-1}$  is an initial segment of  $c_j$ . Indeed, otherwise let  $\epsilon'$  be the initial segment of  $c_j$  whose endpoint is the origin of  $\delta'^{-1}$ . By Proposition 6.2.5 (9), the first edge of  $\epsilon'$  is an edge in an EG stratum. Hence there exists  $p > i$  such that  $a_p$  contains the terminal edge of  $\epsilon'^{-1}$ . Let  $\epsilon^{-1}$  be the subpath of  $\epsilon'^{-1}$  contained in  $a_p$ . Then  $\epsilon^{-1}$  is an initial segment of  $a_p$  and  $\epsilon$  is an initial segment of  $c_j$ . By Lemma 6.3.4 (2) applied to  $a_p^{-1}$  and  $c_j$ , the path  $\epsilon$  is a concatenation of splitting units of  $a_p^{-1}$  and  $c_j$ . But since  $\epsilon$  is properly contained in  $c_j$  as it does not intersect  $\delta'^{-1}$ , the path  $\epsilon$  is adjacent to a splitting unit of  $c_j$ . Since an EG INP starts with an edge in an EG stratum by Proposition 6.2.5 (9), the path  $b_{p-1}$  ends with an edge in an EG stratum. This contradicts the fact that  $b_{p-1}$  is contained in  $G_{PG}$ .

Hence  $\delta'^{-1}$  is an initial segment of  $c_j$  and  $\delta'$  is a terminal segment of  $a_i$ . Let  $\tau$  and  $\tau'$  be two paths such that  $a_i = \tau\delta'$  and  $c_j = \delta'^{-1}\tau'$ . By Lemma 6.3.4 (4) applied to  $a_i$  and  $c_j$ , the path  $\delta'$  is in  $\mathcal{N}_{PG}$  and the path  $\tau\tau'$  is in  $\mathcal{N}_{PG}$ . Hence  $\gamma_2 = \tau b_i a_{i+1} \dots b_k$  and  $\gamma_1 \gamma_3 = b_0 a_1 b_1 \dots a_i \tau \tau' c_{j+1} \dots d_\ell$  are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . This concludes the proof.  $\square$

**Lemma 6.3.6.** *Let  $\gamma$  be a closed Nielsen path of  $G$ . Then  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

**Proof.** Let  $\gamma$  be a closed Nielsen path of  $G$ . We prove the result by induction on the height  $r$  of  $\gamma$ . If  $r = 0$ , there is nothing to prove. Assume that  $r \geq 1$ . By Lemma 6.2.7, the path  $\gamma$  is completely split, and every splitting unit in its complete splitting is either an INP or a fixed edge. Let  $\gamma = \sigma_1 \dots \sigma_k$  be the complete splitting of  $\gamma$ . For every  $i \in \{1, \dots, k\}$ , let  $r_i$  be the height of  $\sigma_i$ . We prove that for every  $i \in \{1, \dots, k\}$ , the path  $\sigma_i$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $i \in \{1, \dots, k\}$ . If  $\sigma_i$  is a fixed edge, it is contained in  $G_{PG}$ . Suppose that  $\sigma_i$  is an NEG INP. By Proposition 6.2.5 (11), there exists an edge  $e_{r_i} \in EH_{r_i}$ , a Nielsen path  $w$  in  $G_{r_i-1}$  and an integer  $s \in \mathbb{Z}^*$  such that  $\sigma_i = e_{r_i} w^s e_{r_i}^{-1}$ . Moreover, we have  $f(e_{r_i}) = e_{r_i} w$ . Hence for every  $j \in \mathbb{N}^*$ , we have  $[f^j(e_{r_i})] = e_{r_i} w^j$ . Since  $w$  is a Nielsen path, by Lemma 6.2.7, the path  $w$  is completely split and its complete splitting is made of fixed edges and INPs. Thus, for every  $j \in \mathbb{N}^*$ , the complete splitting of  $[f^j(e_{r_i})]$  does not contain splitting units which are edges in



EG strata. By definition of  $G_{PG}$ , we have  $e_{r_i} \in \vec{E}G_{PG}$ . Moreover, by the induction hypothesis, the path  $w^s$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Hence  $\sigma_i$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Finally, if  $\sigma_i$  is an EG INP, then it is contained in  $\mathcal{N}_{PG}$ . Hence  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .  $\square$

**Lemma 6.3.7.** *Let  $\gamma$  be either an NEG INP or an exceptional path. Then  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

**Proof.** We claim that there exist edges  $e_1, e_2$  and a closed Nielsen path  $w$  such that  $\gamma = e_1 w e_2^{-1}$  and, for every  $i \in \{1, 2\}$ , we have  $f(e_i) = e_i w^{d_i}$  for some  $d_i \in \mathbb{Z}^*$ . If  $\gamma$  is an exceptional path, it follows from the definition. If  $\gamma$  is an NEG INP, let  $r$  be the height of  $\gamma$ . Then  $H_r$  is an NEG stratum. As  $\gamma$  is a Nielsen path, we can apply Proposition 6.2.5 (11) to conclude the proof of the claim. Since  $e_1$  and  $e_2$  are linear edges, for every  $k \in \mathbb{N}^*$ , the paths  $[f^k(e_1)]$  and  $[f^k(e_2)]$  do not contain splitting units which are edges in EG strata. Thus  $e_1$  and  $e_2$  are contained in  $G_{PG}$ . By Lemma 6.3.6, the path  $w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Hence  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . This concludes the proof.  $\square$

**Lemma 6.3.8.** *Let  $\gamma$  be a Nielsen path in  $G$ . Then  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

**Proof.** By Lemma 6.2.7, the path  $\gamma$  is completely split, and every splitting unit in its complete splitting is either an INP or a fixed edge. Let  $\gamma = \sigma_1 \dots \sigma_k$  be the complete splitting of  $\gamma$ . Let  $i \in \{1, \dots, k\}$ . If  $\sigma_i$  is a fixed edge, then  $\sigma_i$  is contained in  $G_{PG}$ . If  $\sigma_i$  is an NEG INP then, by Lemma 6.3.7, the path  $\sigma_i$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . If  $\sigma_i$  is an EG INP then, by definition, we have  $\sigma_i \in \mathcal{N}_{PG}$ . Hence  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .  $\square$

**Lemma 6.3.9.** (1) *Let  $\gamma$  be an edge in  $G_{PG}$  (resp. an edge in  $G_{PG, \mathcal{F}}$ ). The path  $[f(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  (resp. a concatenation of paths in  $G_{PG, \mathcal{F}}$  and in  $\mathcal{N}_{PG, \mathcal{F}}$ ).*

(2) *Let  $\gamma$  be an edge path contained in  $G_{PG}$  (resp. an edge path in  $G_{PG, \mathcal{F}}$ ). The path  $[f(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  (resp. a concatenation of paths in  $G_{PG, \mathcal{F}}$  and in  $\mathcal{N}_{PG, \mathcal{F}}$ ).*

(3) *Let  $\gamma$  be an edge path which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  (resp. a concatenation of paths in  $G_{PG, \mathcal{F}}$  and in  $\mathcal{N}_{PG, \mathcal{F}}$ ). The path  $[f(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  (resp. a concatenation of paths in  $G_{PG, \mathcal{F}}$  and in  $\mathcal{N}_{PG, \mathcal{F}}$ ).*

**Proof.** We prove Assertions (1), (2), (3) for paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , the proofs for paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$  being similar, using the fact that  $f(G_p) = G_p$ .

(1) Let  $\gamma$  be an edge of  $G_{PG}$ . By definition of  $G_{PG}$ , the edge  $\gamma$  is an edge in an NEG stratum. By Proposition 6.2.5 (6), the path  $[f(\gamma)]$  is completely split. Let  $[f(\gamma)] = \gamma_1 \dots \gamma_m$  be the complete splitting of  $[f(\gamma)]$ . Since  $\gamma$  is an edge in an NEG stratum, by Proposition 6.2.5 (2), we have  $\gamma_1 = \gamma$ . Suppose towards a contradiction that  $[f(\gamma)]$

is not a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . It follows that there exists  $i \in \{1, \dots, m\}$  and an edge  $e$  of  $\gamma_i$  which is not contained in  $G_{PG}$  and is not contained in a subpath of  $[f(\gamma)]$  contained in  $\mathcal{N}_{PG}$ . Hence  $\gamma_i$  is not an EG INP nor a fixed edge. By Lemma 6.3.7, the path  $\gamma_i$  cannot be an NEG INP or an exceptional path. Hence  $\gamma_i$  is either an edge in an irreducible stratum or a maximal taken connecting path in a zero stratum. Suppose first that  $\gamma_i$  is a maximal taken connecting path in a zero stratum. By Proposition 6.2.5 (4), the path  $\gamma_i$  cannot be adjacent to an edge in an NEG stratum nor an edge in a zero stratum. As  $\gamma_1 = \gamma$ , we see that  $i \geq 3$  and that  $\gamma_{i-1}$  ends with an edge in an EG stratum. By Lemma 6.2.9 (applied to  $\gamma = \gamma_{i-1}\gamma_i$ ), the path  $\gamma_{i-1}$  is not an EG INP. Therefore we see that  $\gamma_{i-1}$  is an edge in an EG stratum. This contradicts the definition of the edges in  $G_{PG}$ . Hence we are reduced to the case where  $\gamma_i$  is an edge in an irreducible stratum. Therefore, we have  $\gamma_i = e$ . By definition of  $G_{PG}$  and as  $e \notin \vec{E}G_{PG}$ , there exists  $k \in \mathbb{N}^*$  such that  $[f^k(\gamma_i)]$  contains a splitting unit which is an edge in an EG stratum. This contradicts the fact that  $\gamma$  is contained in  $G_{PG}$ . This concludes the proof of (1).

(2) Let  $\gamma$  be a path in  $G_{PG}$ . We prove by induction on the length of  $\gamma$  that  $[f(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . The case where  $\gamma$  is an edge follows from (1). Suppose now that the length of  $\gamma$  is at least equal to 2. Let  $e$  be the last edge of  $\gamma$  and let  $\gamma'$  be an edge path such that  $\gamma = \gamma'e$ . Hence  $\gamma'$  and  $e$  are paths in  $G_{PG}$ . By the induction hypothesis, the paths  $[f(\gamma')]$  and  $[f(e)]$  are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . It remains to show that identifications between  $[f(\gamma')]$  and  $[f(e)]$  do not create paths which are not concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $\alpha$ ,  $\beta$  and  $\sigma$  be paths such that  $[f(\gamma')] = \alpha\sigma$ ,  $[f(e')] = \sigma^{-1}\beta$  and  $\alpha\beta$  is reduced. By Lemma 6.3.5 applied to  $[f(\gamma')]$  and  $[f(e')]$ , the path  $[f(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . This concludes the proof of (2).

(3) Let  $\gamma$  be a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $\gamma = \gamma'_0\gamma'_1\gamma'_1\dots\gamma'_k\gamma'_k$  be a decomposition of  $\gamma$  such that for every  $i \in \{1, \dots, k\}$ , the path  $\gamma_i$  is a maximal subpath of  $\gamma$  in  $\mathcal{N}_{PG}$  and for every  $i \in \{0, \dots, k\}$ , the path  $\gamma'_i$  is a path in  $G_{PG}$ . Such a decomposition is possible by Lemma 6.3.4 (2). We prove the result by induction on  $k$ . If  $k = 0$ , the proof follows from Assertion (2). Suppose that the result is true for  $k' < k$ . Then the paths  $\gamma' = \gamma'_0\gamma'_1\gamma'_1\dots\gamma'_{k-1}\gamma'_{k-1}$  and  $\gamma'' = \gamma_k\gamma'_k$  satisfy the induction hypothesis. Hence the paths  $[f(\gamma')]$  and  $[f(\gamma'')]$  are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $\alpha$ ,  $\beta$  and  $\sigma$  be three paths such that  $[f(\gamma')] = \alpha\beta$ ,  $[f(\gamma'')] = \beta^{-1}\sigma$  and  $\alpha\beta$  is reduced. By Lemma 6.3.5, the path  $[f(\gamma)] = \alpha\sigma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . This concludes the proof.  $\square$

For the next lemma, we recall a definition due to Bestvina, Feighn and Handel ([BFH1, Section 6], see also [HaM4, Definition III.1.2]). Let  $H_{r_+}$  be the EG stratum of  $G$  of maximal height  $r_+$ . By Proposition 6.2.5 (9), there exists at most one unoriented INP  $\rho_{r_+}$  of height  $r_+$  (we suppose that  $\rho_{r_+}$  is a point if such a nontrivial INP does not exist). Following [HaM4, Definition III.1.2], let  $Z_{r_+}$  be the subgraph of  $G$  consisting in

all edges  $e'$  such that for every  $m \in \mathbb{N}^*$  and every splitting unit  $\sigma$  of  $[f^m(e')]$ , the path  $\sigma$  is not an edge in  $H_{r_+}$ . Let  $\langle Z_{r_+}, \rho_{r_+} \rangle$  be the set consisting in the following paths:

- (i) paths in  $Z_{r_+}$ ;
- (ii) paths in  $\{\rho_{r_+}, \rho_{r_+}^{-1}\}$ ;
- (iii) concatenations of paths in  $Z_{r_+}$  and in  $\{\rho_{r_+}, \rho_{r_+}^{-1}\}$ .

Note that  $\langle Z_{r_+}, \rho_{r_+} \rangle$  contains every path in  $G_{r_+-1}$ .

**Lemma 6.3.10.** *The set  $\langle Z_{r_+}, \rho_{r_+} \rangle$  contains every path which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

**Proof.** It suffices to prove that  $\langle Z_{r_+}, \rho_{r_+} \rangle$  contains every edge of  $G_{PG}$  and every EG INP. Let  $e$  be an edge in  $G_{PG}$ . By definition of  $G_{PG}$ , for every  $k \in \mathbb{N}^*$ , the complete splitting of  $[f^k(e)]$  does not contain a splitting unit which is an edge in an EG stratum. In particular, for every  $k \in \mathbb{N}^*$ , the complete splitting of  $[f^k(e)]$  does not contain a splitting unit which is an edge in  $H_{r_+}$ . Hence  $e \subseteq Z_{r_+}$  and  $G_{PG}$  is a subgraph of  $Z_{r_+}$ . Let  $\rho$  be an EG INP and let  $r$  be the height of  $\rho$ . By definition of  $r_+$ , we have  $r \leq r_+$ . If  $r = r_+$ , by Proposition 6.2.5 (9), we have  $\rho \in \{\rho_{r_+}, \rho_{r_+}^{-1}\}$ , hence we have  $\rho \in \langle Z_{r_+}, \rho_{r_+} \rangle$ . If  $r < r_+$ , then  $\rho$  is contained in  $G_{r_+-1}$ . Hence  $\rho$  is contained in  $\langle Z_{r_+}, \rho_{r_+} \rangle$  by the above remark.  $\square$

We now define a graph which will be used in the proof of Lemma 6.3.12. Let  $G^*$  be the finite, not necessarily connected, graph defined as follows:

- (a) vertices of  $G^*$  are the vertices in  $G_{PG}$  and the endpoints of EG INPs in  $G$  which are not in  $G_{PG}$ ;
- (b) we add one edge between two vertices corresponding to vertices in  $G_{PG}$  if there exists an edge in  $G_{PG}$  between the corresponding vertices of  $G_{PG}$ ;
- (c) we add one edge between two vertices corresponding to the endpoints of an EG INP.

Note that we have a natural continuous application  $p_{G^*} : G^* \rightarrow G$  which sends an edge as defined in (b) to the corresponding edge in  $G_{PG}$  and which sends an edge as defined in (c) to the corresponding EG INP in  $G$ . Let  $x \in VG^*$ .

**Lemma 6.3.11.** (1) *If  $\gamma$  is a nontrivial reduced path in  $G^*$ , so is  $p_{G^*}(\gamma)$ .*

(2) *The homomorphism*

$$p'_{G^*} : \pi_1(G^*, x) \rightarrow \pi_1(G, p_{G^*}(x))$$

*induced by  $p_{G^*}$  is injective.*

**Proof.** (1) Let  $\gamma$  be a reduced path in  $G^*$ . Suppose towards a contradiction that  $p_{G^*}(\gamma)$  is not a reduced path in  $G$ . Thus, there exist an edge  $e \in \vec{E}G$  and two paths  $a$  and  $b$  such that  $p_{G^*}(\gamma) = aee^{-1}b$ . Let  $e^*$  be an arc in  $\gamma$  such that  $p_{G^*}(e^*) = ee^{-1}$ . Note that, by definition of  $p_{G^*}$ , the application  $p_{G^*}$  sends edges of  $G^*$  to reduced edge paths in  $G$ . In particular, the path  $e^*$  is not contained in a single edge of  $G^*$ . As the image of an edge in  $G^*$  by  $p_{G^*}$  is either an edge in  $G$  or an edge path, we see that the path  $e^*$  is contained in at most two edges of  $G^*$ . Let  $e_1, e_2 \in G^*$  be such that  $e^* \subseteq e_1e_2$ . Suppose

first that  $p_{G^*}(e_1)$  and  $p_{G^*}(e_2)$  are edges in  $G_{PG}$ . Then  $p_{G^*}(e_1) = e$  and  $p_{G^*}(e_2) = e^{-1}$ . But, as  $\gamma$  is reduced, we have  $e_1 \neq e_2^{-1}$ . Thus we have  $p_{G^*}(e_1) \neq p_{G^*}(e_2)^{-1}$ . Suppose now that  $p_{G^*}(e_1)$  is an edge in  $G_{PG}$  and  $p_{G^*}(e_2)$  is an EG INP. By Proposition 6.2.5 (9), the first edge of  $p_{G^*}(e_2)$  is an edge in an EG stratum. By definition, every edge in  $G_{PG}$  is an edge in an NEG stratum. Hence the turn  $\{p_{G^*}(e_1)^{-1}, p_{G^*}(e_2)\}$  is nondegenerate. Therefore, we see that  $p_{G^*}(e^*) \neq ee^{-1}$ . Finally, suppose that  $p_{G^*}(e_1)$  and  $p_{G^*}(e_2)$  are EG INPs. for every  $i \in \{1, 2\}$ , let  $r_i$  be the height of  $p_{G^*}(e_i)$ . By Proposition 6.2.5 (9), the last edge of  $p_{G^*}(e_1)$  is in  $H_{r_1}$  whereas the first edge of  $p_{G^*}(e_2)$  is in  $H_{r_2}$ . Hence if  $r_1 \neq r_2$ , there is no identification between  $p_{G^*}(e_1)$  and  $p_{G^*}(e_2)$ . Hence  $p_{G^*}(e^*) \neq ee^{-1}$ . If  $r_1 = r_2$ , then by the uniqueness statement in Proposition 6.2.5 (9), we have  $p_{G^*}(e_2) \in \{p_{G^*}(e_1), p_{G^*}(e_1)^{-1}\}$ . Hence  $e_2 \in \{e_1, e_1^{-1}\}$ . As  $\gamma$  is a reduced path, we see that  $e_2 = e_1$ . Hence  $e_1$  is a loop and  $p_{G^*}(e_1)$  is a closed EG INP. By Proposition 6.2.5 (9), the initial and terminal edges of  $p_{G^*}(e_1)$  are distinct unoriented edges. Hence the path  $p_{G^*}(e_1)p_{G^*}(e_2)$  is a reduced path and  $p_{G^*}(e^*) \neq ee^{-1}$ . As we have ruled out every case, we see that such a path  $e^*$  does not exist. This concludes the proof of Assertion (1).

(2) Let  $\gamma$  be a nontrivial reduced closed path in  $G^*$  based at  $x$ . By Assertion (1), the path  $p_{G^*}(\gamma)$  is a nontrivial reduced closed path in  $G$ . Hence the kernel of  $p'_{G^*}$  is trivial.  $\square$

**Lemma 6.3.12.** *The application  $[f]$  which sends a circuit  $\alpha$  in  $G$  to  $[f(\alpha)]$  preserves the set of circuits which are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Moreover,  $[f]$  restricts to a bijection on the set of circuits which are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

**Proof.** The first part follows from Lemma 6.3.9 (3). By [HaM4, Lemma III.1.6 (2), (5)], the application  $[f]$  preserves  $\langle Z_{r_+}, \rho_{r_+} \rangle$  and restricts to a bijection on the set of circuits of  $\langle Z_{r_+}, \rho_{r_+} \rangle$ . By Lemma 6.3.10 concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  are contained in  $\langle Z_{r_+}, \rho_{r_+} \rangle$ . By Lemma 6.3.9, the application  $[f]$  preserves concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . In particular, this shows that  $[f]$  is injective when restricted to the set of paths which are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .

For surjectivity, let  $\alpha$  be a circuit in  $G$  which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  and let  $x$  be a vertex in  $\alpha$  which is either an endpoint of an edge in  $G_{PG}$  or an endpoint of an EG INP contained in  $\alpha$ . Note that by Proposition 6.2.5 (2), the endpoint of every edge in  $G_{PG}$  is fixed by  $f$ . Moreover, the endpoint of every EG INP is fixed by  $f$ . Therefore,  $f$  fixes  $x$ . The circuit  $\alpha$  naturally corresponds to a circuit  $\alpha'$  in  $G^*$ . Let  $x'$  be the vertex of  $\alpha'$  corresponding to  $x$  (which exists by the choices made on  $x$ ). Since  $[f]$  preserves concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  by Lemma 6.3.9, the application  $[f]$  induces an application

$$[f]_{G^*}: \pi_1(G^*, x') \rightarrow \pi_1(G^*, x').$$

Note that, by Lemma 6.3.11, the group  $\pi_1(G^*, x')$  is naturally identified with a subgroup of  $\pi_1(G, x)$ . By [BFH1, Lemma 6.0.6], the application  $[f]_{G^*}$  is a bijection. Hence there exists a closed path  $\beta'$  in  $G^*$  such that  $[f]_{G^*}([\beta']) = \alpha'$ . Let  $\beta$  be the circuit

corresponding to  $\beta'$  in  $G$ . Then  $\beta$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  and  $[f(\beta)] = \alpha$ . This concludes the proof.  $\square$

**Proposition 6.3.13.** *Let  $n \geq 3$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an exponentially growing outer automorphism, let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$ . Let  $w \in F_n$ . There exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and  $w \in A$  if and only if the circuit  $\gamma_w$  of  $G$  associated with  $w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

**Proof.** Suppose first that  $\gamma_w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . We claim that  $\gamma_w$  has polynomial growth under iteration of  $f$ . By Proposition 6.2.5 (8), there exists  $m \in \mathbb{N}^*$  such that  $[f^m(\gamma_w)]$  is completely split. By Lemma 6.3.9 (3), the path  $[f^m(\gamma_w)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Hence every splitting unit of  $[f^m(\gamma_w)]$  is either an edge of  $G_{PG}$  or an INP. Let  $[f^m(\gamma_w)] = \gamma_1 \dots \gamma_k$  be the complete splitting of  $[f^m(\gamma_w)]$ . For every  $i \geq m$ , we have

$$\ell([f^i(\gamma_w)]) = \sum_{j=1}^k \ell([f^i(\gamma_j)]).$$

Therefore, it suffices to prove that, for every  $j \in \{1, \dots, k\}$ , there exists a polynomial  $P_j \in \mathbb{Z}[X]$  such that for every  $i \in \mathbb{N}^*$ , we have

$$\ell([f^i(\gamma_j)]) = O(P(i)).$$

**Claim.** There exists a polynomial  $P \in \mathbb{Z}[X]$  such that for every edge  $e \in \vec{E}G_{PG}$  and every  $i \in \mathbb{N}^*$ , we have

$$\ell([f^i(e)]) = O(P(i)).$$

**Proof.** Since there are finitely many edges in  $G_{PG}$ , it suffices to prove the claim for a single edge  $e \in \vec{E}G_{PG}$ . Let  $e \in \vec{E}G_{PG}$ . By Proposition 6.2.5 (2), there exists a cyclically reduced, completely split circuit  $w$  of height less than the one of  $e$  and such that  $f(e) = ew$ . By Lemma 6.3.9 (1), the path  $w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . We prove the claim by induction on the height of  $e$ . Suppose first that  $e$  has minimal height in  $G_{PG}$ . By minimality of  $e$ , the path  $w$  does not contain a splitting unit which is an edge in  $G_{PG}$ . Hence  $w$  is either trivial or a path in  $\mathcal{N}_{PG}$ , that is, a closed Nielsen path. If  $w$  is trivial then  $e$  is a fixed edge and  $P = 1$  satisfies the claim. Suppose that  $w$  is a closed Nielsen path. For every  $i \in \mathbb{N}^*$ , we have  $[f^i(e)] = ew^i$ . Hence  $\ell([f^i(e)]) \leq i\ell(w) + 1$ . Then the polynomial  $P(i) = i\ell(w) + 1$  satisfies the assertion of the claim. This proves the base case. Suppose now that  $e$  has height  $r$ . Let  $w = w_1 \dots w_k$  be the complete splitting of  $w$ . Recall that, for every reduced path  $x$  in  $G$ , we have  $[f([f(x)])] = [f^2(x)]$ . Thus, for every  $i \in \mathbb{N}^*$ , we have

$$[f^i(e)] = ew_1 \dots w_k [f(w_1)] \dots [f(w_k)] \dots [f^{i-1}(w_1)] \dots [f^{i-1}(w_k)].$$

Hence, for every  $i \in \mathbb{N}^*$ , we have

$$\ell([f^i(e)]) = 1 + \sum_{\ell=1}^k \sum_{j=0}^{i-1} \ell([f^j(w_\ell)]).$$

Hence it suffices, for every  $\ell \in \{1, \dots, k\}$ , to find a polynomial  $P_\ell \in \mathbb{Z}[X]$  such that, for every  $i \in \mathbb{N}^*$ , we have

$$\ell([f^i(w_\ell)]) = O(P_\ell(i)).$$

Let  $\ell \in \{1, \dots, k\}$ . As  $w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , every splitting unit of  $w$  is either an edge in  $G_{PG}$  or an INP. If  $w_\ell$  is an edge in  $G_{PG}$ , the polynomial  $P_\ell$  exists using the induction hypothesis. If  $w_\ell$  is an INP, then the polynomial  $P_\ell(i) = \ell(w_\ell)$  satisfies the conclusion of the claim. This proves the existence of the polynomial  $P$ .  $\square$

Let  $j \in \{1, \dots, k\}$ . If  $\gamma_k$  is an edge in  $G_{PG}$  which is a splitting unit of  $[f^m(\gamma_w)]$ , by the above claim, the polynomial  $P_j$  exists. If  $\gamma_j$  is an INP, then the polynomial  $P_\ell(x) = \ell(\gamma_j)$  satisfies the conclusion. Thus, the path  $\gamma_w$  has polynomial growth under iteration of  $[f]$ . Therefore,  $[w]$  has polynomial growth under iterates of  $\phi$ . By the definition of  $\mathcal{A}(\phi)$ , there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and  $w \in A$ .

Conversely, suppose that there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and  $w \in A$ . Let  $m \in \mathbb{N}^*$  be such that  $[f^m(\gamma_w)]$  is completely split, which exists by Proposition 6.2.5 (7). Since  $[w]$  has polynomial growth under iteration of  $\phi$ , there does not exist a splitting unit of  $[f^m(\gamma_w)]$  which is an edge in an EG stratum or a superlinear edge with exponential growth. Suppose towards a contradiction that a splitting unit  $\sigma$  of  $[f^m(\gamma_w)]$  is contained in a zero stratum. By Proposition 6.2.5 (3), every zero stratum of  $G$  is contractible. As  $[f^m(\gamma_w)]$  is a cycle, it is not contained in a zero stratum. By Proposition 6.2.5 (4), every edge adjacent to  $\sigma$  and not contained in the same stratum as  $\sigma$  is in an EG stratum. Hence there exists a splitting unit  $\sigma'$  of  $[f^m(\gamma_w)]$  such that  $\sigma\sigma' \subseteq [f^m(\gamma_w)]$  and  $\sigma'$  the first edge of  $\sigma$  is in an EG stratum. Hence  $\sigma'$  is either an edge in an EG stratum or an INP. But, by Lemma 6.2.9, the path  $\sigma'$  is not an INP. Hence  $\sigma'$  is an edge in an EG stratum. This contradicts the fact that  $[w]$  has polynomial growth under iteration of  $\phi$ . Hence every splitting unit of  $[f^m(\gamma_w)]$  is either an INP, an exceptional path or an edge in an NEG stratum whose iterates by  $f$  do not contain splitting units which are edges in EG strata. Edges in the last category are precisely the edges in  $G_{PG}$ . By Lemma 6.3.7 and Lemma 6.3.8 every INP and every exceptional path is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Thus, the path  $[f^m(\gamma_w)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.12, the circuit  $\gamma_w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .  $\square$

Let  $\mathcal{F}$  be a nonsporadic free factor system of  $F_n$  and let  $\phi \in \text{Out}(F_n, \mathcal{F})$ . We say that  $\phi$  is *fully irreducible relative to  $\mathcal{F}$*  if no power of  $\phi$  preserves a proper free factor system  $\mathcal{F}'$  of  $F_n$  such that  $\mathcal{F} < \mathcal{F}'$ . The following corollary will be used in [Gue6]. It is a well-known result but we did not find a precise statement in the literature.

**Corollary 6.3.14.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a nonsporadic free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be a fully irreducible outer automorphism relative to  $\mathcal{F}$ . There exists at most one (up to taking inverse) conjugacy class  $[g]$  of root-free  $\mathcal{F}$ -nonperipheral element of  $F_n$  which has polynomial growth under iteration of  $\phi$ . Moreover, the conjugacy class  $[g]$  is  $\phi$ -periodic.*

**Proof.** Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  and let  $G'$  be a subgraph of  $G$  such that  $\mathcal{F}(G') = \mathcal{F}$ . Since  $\phi$  is irreducible relative to  $\mathcal{F}$  and since  $\mathcal{F}$  is nonsporadic, we see that  $\overline{G - G'}$  is an EG stratum  $H_r$ . Let  $[g]$  be the conjugacy class of a root-free  $\mathcal{F}$ -nonperipheral element  $g$  of  $F_n$ . Then  $\gamma_g$  has height  $r$ . Suppose that  $[g]$  has polynomial growth with respect to  $\phi$ . By Proposition 6.3.13, the circuit  $\gamma_g$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Since  $\gamma_g$  has height  $r$  and since  $H_r$  is an EG stratum, every subpath  $\alpha$  of  $\gamma_g$  contained in  $H_r$  is contained in a concatenation of INPs of height  $r$ . By Proposition 6.2.5 (9), there exists at most one INP  $\sigma$  of height  $r$ . Moreover, one of its endpoints is not contained in  $G' = G_{r-1}$  (see [HaM4, I.Fact 1.42]). Hence  $\sigma$  is necessarily a closed EG INP. Since the endpoint of  $\sigma$  is not in  $G_{r-1}$  and since  $\gamma_g$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ , we see that  $\gamma_g$  is an iteration of the closed path  $\sigma$ . Since  $g$  is root-free, we have  $\gamma_g = \sigma^{\pm 1}$ . This concludes the proof.  $\square$

### 6.3.2 The exponential length of a CT map

In this section, we define the *exponential length function*  $\ell_{exp}$ , and its relative version  $\ell_{\mathcal{F}}$ , of paths in CT maps. We compute its value for some paths in  $G$ . Let  $G'_{PG} = G_{PG} \cup \mathcal{Z}$  (see Definition 6.3.1) and let  $G'_{PG,\mathcal{F}} = G_{PG,\mathcal{F}} \cup \mathcal{Z}_{\mathcal{F}}$ .

Let  $\gamma$  be a reduced edge path in  $G$ . By Lemma 6.3.4 (2), every path of  $\mathcal{N}_{PG}$  which is contained in  $\gamma$  is contained in a unique maximal subpath of  $\gamma$  contained in  $\mathcal{N}_{PG}$ . Thus, the path  $\gamma$  has a unique decomposition into edge paths  $\gamma = \gamma_0 \gamma'_1 \gamma_1 \dots \gamma_k \gamma'_k$  where:

- (1) for every  $i \in \{0, \dots, k\}$ , the path  $\gamma_i$  is a maximal path in  $\mathcal{N}_{PG}$  contained in  $\gamma$  (where  $\gamma_0$  and  $\gamma'_k$  might be trivial);
- (2) for every  $\gamma' \in \mathcal{N}_{PG}$  contained in  $\gamma$ , there exists  $i \in \{1, \dots, k\}$  such that  $\gamma' \subseteq \gamma_i$ .

Such a decomposition of  $\gamma$  is called the *exponential decomposition* of  $\gamma$ . Note that the exponential decomposition of  $\gamma$  is not necessarily a splitting of  $\gamma$ . We denote by  $\mathcal{N}_{PG}^{\max}(\gamma)$  the set consisting in all paths  $\gamma_i$ , with  $i \in \{0, \dots, k\}$ . Similarly,  $\gamma$  has a decomposition  $\gamma = \alpha_0 \alpha'_1 \alpha_1 \dots \alpha_m \alpha'_m$ , where for every  $i \in \{0, \dots, m\}$ , the path  $\alpha_i$  is a maximal path in  $\mathcal{N}_{PG,\mathcal{F}}$  and for every  $\gamma' \in \mathcal{N}_{PG,\mathcal{F}}$  contained in  $\gamma$ , there exists  $i \in \{1, \dots, m\}$  such that  $\gamma' \subseteq \alpha_i$ . Such a decomposition is called the  *$\mathcal{F}$ -exponential decomposition* of  $\gamma$ . We denote by  $\mathcal{N}_{PG,\mathcal{F}}^{\max}(\gamma)$  the set consisting in all paths  $\alpha_i$ , with  $i \in \{0, \dots, m\}$ .

**Definition 6.3.15.** (1) Let  $\gamma$  be a reduced edge path in  $G$ . The *exponential length* of  $\gamma$ , denoted by  $\ell_{exp}(\gamma)$  is:

$$\ell_{exp}(\gamma) = \ell\left(\gamma \cap \overline{G - G'_{PG}}\right) - \sum_{\alpha \in \mathcal{N}_{PG}^{\max}(\gamma)} \ell\left(\alpha \cap \overline{G - G'_{PG}}\right).$$

(2) Let  $\gamma$  be a reduced edge path in  $G$ . The  *$\mathcal{F}$ -exponential length* of  $\gamma$ , denoted by  $\ell_{\mathcal{F}}(\gamma)$  is:

$$\ell_{\mathcal{F}}(\gamma) = \ell\left(\gamma \cap \overline{G - G'_{PG,\mathcal{F}}}\right) - \sum_{\alpha \in \mathcal{N}_{PG,\mathcal{F}}^{\max}(\gamma)} \ell\left(\alpha \cap \overline{G - G'_{PG,\mathcal{F}}}\right).$$

(3) Let  $\gamma$  be a reduced edge path in  $G$  and let  $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_k\gamma_k$  be the exponential decomposition of  $\gamma$ . A *PG-relative complete splitting* of the path  $\gamma$  is a splitting  $\gamma = \delta_1 \dots \delta_m$  such that for every  $i \in \{1, \dots, m\}$ , the path  $\delta_i$  is one of the following paths:

- a splitting unit of positive exponential length not contained in some  $\gamma_i$  for  $i \in \{0, \dots, k\}$ ;
- a maximal taken connecting path in a zero stratum;
- a subpath of  $\gamma$  which is a concatenation of subpaths contained in  $G_{PG}$  and Nielsen paths in  $\mathcal{N}_{PG}$ .

We call the above paths *PG-relative splitting units*. If  $\gamma$  is a circuit, a *PG-relative circuitual complete splitting* of  $\gamma$  is a circuitual splitting of  $\gamma$  which is a *PG-relative complete splitting* of  $\gamma$ .

(4) A *factor* of a *PG-relative completely split edge path*  $\gamma$  is a concatenation of *PG-relative splitting units* of some given *PG-relative complete splitting* of  $\gamma$ .

Note that if  $\gamma$  is an edge path of  $G$ , then  $\ell_{exp}(\gamma) \geq 0$ . Indeed, two paths  $\gamma_1$  and  $\gamma_2$  contained in  $\mathcal{N}_{PG}^{\max}(\gamma)$  are either equal or disjoint. Let  $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma'_k\gamma_k$  be the exponential decomposition of  $\gamma$ . For every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\gamma'_i) = \ell(\gamma'_i \cap \overline{G - G'_{PG}})$  and

$$\ell_{exp}(\gamma) = \sum_{i=1}^k \ell_{exp}(\gamma'_i).$$

We prove the existence of *PG-relative complete splittings* in Lemma 6.3.19. Note that a *PG-relative complete splitting* of a reduced edge path  $\gamma$  is not necessarily unique. Indeed, it might be possible that one can split a *PG-relative splitting unit* of  $\gamma$  which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  into two *PG-relative splitting units* which are concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .

In the rest of the section, we describe some properties of the exponential length.

**Lemma 6.3.16.** *Let  $\gamma$  be a reduced edge path in  $G$  and let  $\gamma = \gamma_1\gamma_2$  be a decomposition of  $\gamma$  into two edge paths. We have:*

$$\ell_{exp}(\gamma) \leq \ell_{exp}(\gamma_1) + \ell_{exp}(\gamma_2).$$

**Proof.** It is immediate that

$$\ell(\gamma \cap \overline{G - G'_{PG}}) = \ell(\gamma_1 \cap \overline{G - G'_{PG}}) + \ell(\gamma_2 \cap \overline{G - G'_{PG}}).$$

Let  $i \in \{1, 2\}$ . Let  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_i)$ . Then there exists  $\gamma'' \in \mathcal{N}_{PG}^{\max}(\gamma)$  such that  $\gamma' \subseteq \gamma''$ . In particular, we have

$$\sum_{\gamma'' \in \mathcal{N}_{PG}^{\max}(\gamma)} \ell(\gamma'' \cap \overline{G - G'_{PG}}) \geq \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_1)} \ell(\gamma' \cap \overline{G - G'_{PG}}) + \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_2)} \ell(\gamma' \cap \overline{G - G'_{PG}}).$$

By definition of the exponential length, this concludes the proof.  $\square$



Note that we do not necessarily have equality in Lemma 6.3.16. Indeed, let  $\gamma = \gamma_1\gamma_2$  be as in Lemma 6.3.16. Suppose that the endpoint of  $\gamma_1$  is contained in a path  $\gamma'$  of  $\mathcal{N}_{PG}^{\max}(\gamma)$ . Then  $\gamma'$  is not necessarily a concatenation of paths in  $\mathcal{N}_{PG}^{\max}(\gamma_1)$  and  $\mathcal{N}_{PG}^{\max}(\gamma_2)$ . Therefore, we might have:

$$\sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)} \ell(\gamma' \cap \overline{G - G'_{PG}}) > \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_1)} \ell(\gamma' \cap \overline{G - G'_{PG}}) + \sum_{\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_2)} \ell(\gamma' \cap \overline{G - G'_{PG}}),$$

and a strict inequality in Lemma 6.3.16. In particular, a proper subpath of  $\gamma$  might have greater exponential length than  $\gamma$  itself. For instance, if  $\gamma$  is a reduced path in  $G$  such that  $\ell_{exp}(\gamma) = 0$ , it is possible that there exists a proper subpath  $\gamma'$  of  $\gamma$  such that  $\ell_{exp}(\gamma') > 0$ . However, there exists a bound, depending only on  $G$ , on the difference of the exponential length of a subpath of  $\gamma$  and the exponential length of  $\gamma$  (see Lemma 6.5.6).

If  $\gamma$  is a path in  $G$  such that  $\ell_{exp}(\gamma) = 0$ , we do not necessarily have  $\ell_{exp}([f(\gamma)]) = 0$ . Indeed, if  $\gamma$  is an edge in a zero stratum such that  $[f(\gamma)]$  contains a splitting unit which is an edge in an EG stratum, we have  $\ell_{exp}([f(\gamma)]) > 0$ . However, the following lemma describes an important situation where the map  $f$  preserves the property of having zero exponential length.

**Lemma 6.3.17.** *Let  $\gamma$  be a reduced edge path which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . For every  $n \in \mathbb{N}$ , we have  $\ell_{exp}([f^n(\gamma)]) = 0$ .*

**Proof.** Since the  $[f]$ -image of a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  by Lemma 6.3.9, it suffices to prove the result for  $n = 0$ . Let  $\gamma$  be a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma_k\gamma'_k$  be the exponential decomposition of  $\gamma$ : for every  $i \in \{1, \dots, k\}$ , the path  $\gamma_i$  is a maximal subpath of  $\gamma$  in  $\mathcal{N}_{PG}$  and for every  $i \in \{0, \dots, k\}$ , the path  $\gamma'_i$  is a path in  $G_{PG}$ . Note that for every  $i \in \{1, \dots, k\}$ , we have  $\gamma_i \in \mathcal{N}_{PG}^{\max}(\gamma)$ . By definition of the exponential length, we have  $\ell_{exp}(\gamma) = \sum_{i=0}^k \ell_{exp}(\gamma'_i) = 0$ .  $\square$

**Corollary 6.3.18.** *Let  $\gamma$  be a path of  $\mathcal{N}'_{PG}$ . Then  $\ell_{exp}(\gamma) = 0$ . In particular, if  $\gamma$  is either a closed Nielsen path, an NEG INP or an exceptional path, we have  $\ell_{exp}(\gamma) = 0$ .*

**Proof.** By Lemma 6.3.8, the path  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.17, we have  $\ell_{exp}(\gamma) = 0$ . The second assertion follows from Lemmas 6.3.6 and 6.3.7.  $\square$

**Lemma 6.3.19.** *Let  $\gamma$  be a completely split edge path and let  $\gamma = \gamma_1 \dots \gamma_m$  be its complete splitting. Let  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$ . Then either  $\gamma'$  is a concatenation of splitting units of  $\gamma$  or there exists  $i \in \{1, \dots, m\}$  such that  $\gamma' \not\subseteq \gamma_i$ . Moreover, the complete splitting of  $\gamma$  is a PG-relative complete splitting of  $\gamma$ .*

**Proof.** Let  $e$  be the first edge of  $\gamma'$  and let  $i \in \{1, \dots, m\}$  be such that  $e$  is contained in  $\gamma_i$ . Let  $\sigma$  be the splitting unit of  $\gamma'$  containing  $e$ . By Proposition 6.2.5 (9), the edge  $e$  is in an EG stratum. Hence  $\gamma_i$  is either an edge in an EG stratum, an exceptional path or

an INP. Since  $\gamma'$  is a Nielsen path, and since  $\gamma_i$  is a splitting unit of  $\gamma$ , we see that  $\gamma_i$  is not an edge in an EG stratum. If  $\gamma_i$  is either an NEG INP or an exceptional path, then Proposition 6.2.5 (11) implies that  $\gamma_i$  starts and ends with edges in NEG strata whose height are strictly higher than the one of  $e$ . Since the height of  $e$  is equal to the height of  $\sigma$ , we see that  $\gamma_i$  contains  $\sigma$ . An inductive argument shows that  $\gamma'$  is contained in  $\gamma_i$ .

Suppose now that  $\gamma_i$  is an EG INP. By Lemma 6.3.4 (2) applied to  $\gamma_i$  and  $\gamma'$ , either  $\gamma'$  is contained in  $\gamma_i$  or  $\gamma_i$  is the initial segment of  $\gamma'$ . If  $\gamma'$  is contained in  $\gamma_i$ , by maximality of  $\gamma'$ , we see that  $\gamma' = \gamma_i$ . Suppose that  $\gamma'$  is the initial segment of the completely split edge path  $\gamma_i \dots \gamma_k$ . Then [FH, Corollary 4.12] implies that  $\gamma'$  is a factor of  $\gamma$ .

The last assertion of the lemma follows from the following observations. Every splitting unit of  $\gamma$  which is either an INP or an exceptional path is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  by Lemma 6.3.7. Moreover, by the first assertion of the lemma, every splitting unit of  $\gamma$  which is an edge in an irreducible stratum not contained in  $G_{PG}$  does not intersect a path in  $\mathcal{N}_{PG}^{\max}(\gamma)$ . Hence the complete splitting of  $\gamma$  is a  $PG$ -relative complete splitting.  $\square$

$PG$ -relative completely split edge paths are well-adapted to the computation of the exponential length as explained by the following lemma.

**Lemma 6.3.20.** *Let  $\gamma$  be a  $PG$ -relative completely split edge path and let  $\gamma = \alpha_1 \dots \alpha_\ell$  be a  $PG$ -relative complete splitting.*

(1) *For every path  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$ , there exists a minimal concatenation of  $PG$ -relative splitting units  $\delta$  of  $\gamma$  such that  $\gamma' \subseteq \delta$ ; every  $PG$ -relative splitting unit of  $\delta$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ ; for every  $PG$ -relative splitting unit  $\delta'$  of  $\delta$ , the intersection  $\delta' \cap \gamma'$  is an element of  $\mathcal{N}_{PG}^{\max}(\delta')$ .*

(2) *We have  $\ell_{exp}(\gamma) = \sum_{i=1}^{\ell} \ell_{exp}(\alpha_i)$  and  $\ell_{\mathcal{F}}(\gamma) = \sum_{i=1}^{\ell} \ell_{\mathcal{F}}(\alpha_i)$ .*

**Proof.** (1) Let  $\gamma = \gamma_0 \gamma'_1 \gamma_1 \dots \gamma'_k \gamma_k$  be the exponential decomposition of  $\gamma$  where, for every  $i \in \{0, \dots, k\}$ , we have  $\gamma_i \in \mathcal{N}_{PG}^{\max}(\gamma)$ . Let  $i \in \{0, \dots, k\}$ . Let  $j \in \{1, \dots, \ell\}$  be such that  $\alpha_j$  contains an initial segment of  $\gamma_i$ . By Proposition 6.2.5 (10), the splitting unit  $\alpha_j$  is not contained in a zero stratum. Moreover, by definition of the  $PG$ -relative splitting units, if  $\alpha_j$  is an edge in an irreducible stratum of positive exponential length, it is not contained in  $\gamma_i$ . Hence, by the description of  $PG$ -relative splitting units, the path  $\alpha_j$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Proposition 6.2.5 (9), the path  $\gamma_i$  starts with an edge in an EG stratum. Hence there exists a path  $\beta_j$  in  $\mathcal{N}_{PG}^{\max}(\alpha_j)$  which contains an initial segment of  $\gamma_i$ . By maximality of  $\gamma_i$ , we see that  $\beta_j \subseteq \gamma_i$ . Suppose first that  $\beta_j = \gamma_i$ . Then setting  $\delta = \alpha_j$  proves the first assertion. Suppose now that  $\beta_j \subsetneq \gamma_i$ . By Lemma 6.3.4 (2) applied to  $\gamma = \gamma_i^{-1}$  and  $\gamma' = \beta_j^{-1}$ , the path  $[\beta_j^{-1} \gamma_i]$  is a path in  $\mathcal{N}_{PG}$ . Therefore, by Proposition 6.2.5 (9), the path  $[\beta_j^{-1} \gamma_i]$  starts with an edge in an EG stratum. Note that, as  $\alpha_j$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , if  $\alpha_j$  contains the first edge  $e$  of  $[\beta_j^{-1} \gamma_i]$ , then  $e$  would be contained in an EG INP contained in  $\alpha_j$ . Since  $\beta_j$  is a maximal subpath of  $\alpha_j$  in  $\mathcal{N}_{PG}$ , we see that  $[\beta_j^{-1} \gamma_i]$  is contained in  $\gamma'' = \alpha_{j+1} \dots \alpha_\ell$  and is in  $\mathcal{N}_{PG}^{\max}(\gamma'')$ . We can thus apply the same arguments to the paths  $[\beta_j^{-1} \gamma_i]$  and  $\gamma''$ . This concludes the proof of (1).

The proof of (2) follows as the exponential length and the  $\mathcal{F}$ -length are computed by removing paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . As all subpaths in  $G_{PG}$  are contained in a splitting unit of  $\gamma$  and as subpaths in  $\mathcal{N}_{PG}$  are obtained by concatenating paths in  $\coprod_{j=1}^{\ell} \mathcal{N}_{PG}^{\max}(\alpha_j)$ , we see that  $\ell_{exp}(\gamma) = \sum_{i=1}^{\ell} \ell_{exp}(\alpha_i)$  and  $\ell_{\mathcal{F}}(\gamma) = \sum_{i=1}^{\ell} \ell_{\mathcal{F}}(\alpha_i)$ .  $\square$

The following property of the exponential length allows us to pass, if needed, to a further iterate of the CT map  $f$ .

**Lemma 6.3.21.** *For every edge  $e$  of  $\overline{G - G'_{PG}}$ , we have*

$$\lim_{n \rightarrow \infty} \ell_{exp}([f^n(e)]) = \infty \text{ and } \lim_{n \rightarrow \infty} \ell_{\mathcal{F}}([f^n(e)]) = \infty.$$

Moreover, the sequences  $(\ell_{exp}([f^n(e)]))_{n \in \mathbb{N}}$  and  $(\ell_{\mathcal{F}}([f^n(e)]))_{n \in \mathbb{N}}$  grows exponentially fast.

**Proof.** We prove the result concerning  $\ell_{exp}$ , the proof of the result concerning  $\ell_{\mathcal{F}}$  follows from the fact that for every reduced edge path  $\gamma$  in  $G$ , we have  $\ell_{exp}(\gamma) \leq \ell_{\mathcal{F}}(\gamma)$ . Let  $e$  be an edge of  $\overline{G - G'_{PG}}$ . Since every iterate of  $e$  is completely split by Proposition 6.2.5 (6) and since there exists an iterate of  $e$  which contains a splitting unit which is an edge in an EG stratum, we may suppose that  $e$  is an edge in an EG stratum  $H_r$ . Since  $H_r$  is an EG stratum, the number of edges in  $[f^n(e)] \cap H_r$  grows exponentially fast as  $n$  goes to infinity. Therefore the number of splitting units of  $[f^n(e)]$  which are edges of  $H_r$  grows exponentially fast and  $\lim_{n \rightarrow \infty} \ell_{exp}([f^n(e)]) = \infty$ .  $\square$

**Lemma 6.3.22.** *Let  $\gamma$  be a  $PG$ -relative completely split edge path. There exists  $n_0 \in \mathbb{N}^*$  such that for every  $k \geq n_0$ , we have  $\ell_{exp}([f^k(\gamma)]) \geq \ell_{exp}(\gamma)$ .*

**Proof.** Let  $\gamma = \gamma_1 \dots \gamma_k$  be a  $PG$ -relative complete splitting of  $\gamma$ . By Lemma 6.3.20, it suffices to prove the assertion for every subpath  $\gamma_i$ , with  $i \in \{1, \dots, k\}$ . Let  $i \in \{1, \dots, k\}$ . If  $\gamma_i$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , then  $\ell_{exp}([f(\gamma_i)]) = \ell_{exp}(\gamma_i) = 0$  by Lemma 6.3.17. If  $\gamma_i$  is a maximal taken connecting path in a zero stratum, we have  $\ell_{exp}(\gamma_i) = 0$ . Hence  $\ell_{exp}([f(\gamma_i)]) \geq \ell_{exp}(\gamma_i)$ . In the other cases,  $\gamma_i$  is an edge in an irreducible stratum which is not contained in  $G_{PG}$ . By Lemma 6.3.21, we have  $\lim_{n \rightarrow \infty} \ell_{exp}([f^n(\gamma_i)]) = \infty$ . Hence there exists  $n_0 \in \mathbb{N}^*$  such that, for every  $k \geq n_0$ , we have  $\ell_{exp}([f^k(\gamma_i)]) \geq \ell_{exp}(\gamma_i)$ , and  $n_0$  may be chosen to be independent of  $\gamma_i$  with  $i \in \{1, \dots, k\}$ .  $\square$

The last lemma in this section shows that the exponential length of a  $PG$ -relative completely split edge path captures the splitting units which are edges with exponential growth under iterates of  $f$ .

**Lemma 6.3.23.** *Let  $\gamma$  be a  $PG$ -relative completely split edge path, let  $\gamma = \gamma_1 \dots \gamma_k$  be a  $PG$ -relative complete splitting and let  $i \in \{1, \dots, k\}$ . Then  $\ell_{exp}(\gamma_i) > 0$  if and only if  $\gamma_i$  is an edge in an irreducible stratum not contained in  $G_{PG}$ . In particular, the value  $\ell_{exp}(\gamma)$  is the number of splitting units which are edges in  $\overline{G - G'_{PG}}$ .*

**Proof.** Suppose first that  $\gamma_i$  is either a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or a maximal taken connecting path in a zero stratum. By Lemma 6.3.17, we have  $\ell_{exp}(\gamma_i) = 0$ . Suppose that  $\gamma_i$  is an edge in an irreducible stratum which is not contained in  $G_{PG}$ . Since there does not exist an EG INP of length 1, by definition of the exponential length, we have  $\ell_{exp}(\gamma_i) = 1 > 0$ . This concludes the proof of the first part of the lemma. The computation of  $\ell_{exp}(\gamma)$  follows from Lemma 6.3.20 (2).  $\square$

### 6.3.3 The space of polynomially growing currents

In this section, let  $\mathcal{F}$  be a free factor system and let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an exponentially growing outer automorphism. Recall the definition of  $\mathcal{A}(\phi)$  and  $\mathcal{F} \wedge \mathcal{A}(\phi)$  from Section 6.2.1. We define a subspace of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ , called the *space of polynomially growing currents*. It consists in the currents whose support is contained in  $\partial^2 \mathcal{A}(\phi)$  (see Lemma 6.3.27). In order to define it, we first need to show that the exponential length extends to a continuous function  $\Psi: \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}$ . The space of polynomially growing currents will then be defined as a level set of  $\Psi$ .

We first need some preliminary results concerning paths in  $\mathcal{N}_{PG}$ . For a path  $\gamma \in \mathcal{N}_{PG}$ , let  $\mathcal{N}_{PG}^{++}(\gamma)$  be the subset of  $\mathcal{N}_{PG}$  which consists in all paths  $\gamma' \in \mathcal{N}_{PG}$  such that  $\gamma \subsetneq \gamma'$  and  $\gamma'$  is minimal for this property. Let  $\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)$ . By Lemma 6.3.4 (3), either  $\gamma$  is properly contained in an INP  $\sigma$  of the complete splitting of  $\gamma'$ , or there exist (possibly trivial) paths  $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}$  such that  $\gamma' = \gamma_1 \gamma \gamma_2$ . By minimality, either  $\gamma_1$  or  $\gamma_2$  is trivial. Moreover, a result of Feighn and Handel ([FH, Corollary 4.12]) shows that, in this case, splitting units of the complete splittings of  $\gamma_1$ ,  $\gamma_2$  and  $\gamma$  are splitting units of  $\gamma'$ . Thus the set  $\mathcal{N}_{PG}^{++}(\gamma)$  can be partitioned into three disjoint subsets:

$$\mathcal{N}_{PG}^{++}(\gamma) = \mathcal{N}_{PG,INP}^{++}(\gamma) \sqcup \mathcal{N}_{PG,left}^{++}(\gamma) \sqcup \mathcal{N}_{PG,right}^{++}(\gamma),$$

where  $\mathcal{N}_{PG,INP}^{++}(\gamma)$  is the set of paths in  $\mathcal{N}_{PG}^{++}(\gamma)$  such that one of their splitting units properly contains  $\gamma$ ,  $\mathcal{N}_{PG,left}^{++}(\gamma)$  is the set of paths  $\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)$  such that  $\gamma' = \gamma_1 \gamma$  and  $\mathcal{N}_{PG,right}^{++}(\gamma)$  is the set of paths  $\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)$  such that  $\gamma' = \gamma \gamma_2$ . One can also define similarly the three sets  $\mathcal{N}_{PG,INP,\mathcal{F}}^{++}(\gamma)$ ,  $\mathcal{N}_{PG,left,\mathcal{F}}^{++}(\gamma)$  and  $\mathcal{N}_{PG,right,\mathcal{F}}^{++}(\gamma)$  as the restriction to the paths in  $\mathcal{N}_{PG,INP}^{++}(\gamma)$ ,  $\mathcal{N}_{PG,left}^{++}(\gamma)$  and  $\mathcal{N}_{PG,right}^{++}(\gamma)$  contained in  $G_p$ . We emphasize on the fact that a path in  $\mathcal{N}_{PG,INP}^{++}(\gamma)$  might contain several occurrences of the path  $\gamma$ . However, a path in  $\mathcal{N}_{PG,left}^{++}(\gamma)$  or in  $\mathcal{N}_{PG,right}^{++}(\gamma)$  contains a unique occurrence of  $\gamma$ . Indeed, let  $\gamma' \in \mathcal{N}_{PG,left}^{++}(\gamma)$  (the proof for  $\mathcal{N}_{PG,right}^{++}(\gamma)$  being similar). Then  $\gamma' = \gamma_1 \gamma_2$  with  $\gamma_1 \in \mathcal{N}_{PG}$  and  $\gamma_2 = \gamma$ . Let  $\gamma_3$  be an occurrence of  $\gamma$  which contains an edge of  $\gamma_1$ . By Lemma 6.3.3 (2), the path  $\gamma_3$  cannot intersect  $\gamma_2$  nontrivially. Hence  $\gamma_3 \subseteq \gamma_1$ . Hence  $\gamma_1 \in \mathcal{N}_{PG}$  and  $\gamma_1$  contains an occurrence of  $\gamma$ . This contradicts the minimality of  $\gamma'$ .

**Lemma 6.3.24.** *Let  $\gamma$  be a path in  $\mathcal{N}_{PG}$ . Let  $\gamma_1, \gamma_2$  be two distinct paths in  $\mathcal{N}_{PG}^{++}(\gamma)$ . Suppose that there exist three paths  $\mu_1, \mu_2, \mu_3$  such that  $\gamma_1 = \mu_1 \mu_2$ ,  $\gamma_2 = \mu_2 \mu_3$  and  $\gamma$  is contained in  $\mu_2$ . Then  $\gamma_1 \in \mathcal{N}_{PG,left}^{++}(\gamma)$ ,  $\gamma_2 \in \mathcal{N}_{PG,right}^{++}(\gamma)$  and  $\mu_2 = \gamma$ .*

**Proof.** By Lemma 6.3.4 (2), the path  $\mu_2$  belongs to  $\mathcal{N}_{PG}$  and contains  $\gamma$ . Since  $\gamma_1$  and  $\gamma_2$  are minimal paths of  $\mathcal{N}_{PG}$  for the property of properly containing  $\gamma$ , we have  $\mu_2 = \gamma$ . Therefore, we see that  $\gamma_1 = \mu_1\gamma$  and  $\gamma_2 = \gamma\mu_3$ . This shows that  $\gamma_1 \in \mathcal{N}_{PG, \text{left}}^{++}(\gamma)$  and that  $\gamma_2 \in \mathcal{N}_{PG, \text{right}}^{++}(\gamma)$ .  $\square$

Lemma 6.3.24 implies that an occurrence of  $\gamma$  in the intersection of paths in  $\mathcal{N}_{PG}^{++}(\gamma)$  is well-controlled. Following Lemma 6.3.24, we then define  $\mathcal{N}_{PG, lr}^{++}(\gamma)$  to be the set of paths of the form  $\gamma_1\gamma\gamma_2$ , where  $\gamma_1\gamma \in \mathcal{N}_{PG, \text{left}}^{++}(\gamma)$  and  $\gamma\gamma_2 \in \mathcal{N}_{PG, \text{right}}^{++}(\gamma)$ . We define similarly the set  $\mathcal{N}_{PG, lr, \mathcal{F}}^{++}(\gamma)$  to be the set of all paths in  $\mathcal{N}_{PG, lr}^{++}(\gamma)$  contained in  $G_p$ . As for  $\mathcal{N}_{PG, \text{left}}^{++}(\gamma)$  and  $\mathcal{N}_{PG, \text{right}}^{++}(\gamma)$ , a path in  $\mathcal{N}_{PG, lr}^{++}(\gamma)$  contains a unique occurrence of  $\gamma$ .

Given two paths  $\gamma$  and  $\gamma'$  of  $G$  let  $N(\gamma', \gamma)$  be the number of occurrences of  $\gamma$  and  $\gamma^{-1}$  in  $\gamma'$ . Using the finiteness of  $\mathcal{N}_{PG}$  (see Lemma 6.3.4 (1)), we denote by

$$\Psi'_0: \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}$$

the continuous function

$$\Psi'_0(\nu) = \sum_{\gamma \in \mathcal{N}_{PG}} \left( \langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG, lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \right) \ell \left( \gamma \cap \overline{G - G'_{PG}} \right),$$

and by  $\Psi_0: \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}$  the continuous linear function

$$\begin{aligned} \Psi_0(\nu) &= \frac{1}{2} \left( \sum_{e \in \vec{E}(\overline{G - G'_{PG}})} \langle e, \nu \rangle - \Psi'_0(\nu) \right) \\ &= \frac{1}{2} \left( \sum_{e \in \vec{E}(\overline{G - G'_{PG}})} \left( \langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left( \langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG, lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \right) N(\gamma, e) \right) \right). \end{aligned}$$

**Definition 6.3.25.** The space of polynomially growing currents, denoted by  $K_{PG}(f)$ , is the compact subset of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  consisting in all projective classes of currents  $[\nu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  such that:

$$\Psi_0(\nu) = 0.$$

Finally, we define the  $\mathcal{F}$ -simplicial length function  $\|\cdot\|_{\mathcal{F}}: \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}$  as

$$\|\nu\|_{\mathcal{F}} = \frac{1}{2} \left( \sum_{e \in \vec{E}(\overline{G - G'_{PG, \mathcal{F}}})} \langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG, \mathcal{F}}} \left( \langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG, \mathcal{F}}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG, lr, \mathcal{F}}^{++}(\gamma)} \langle \gamma', \nu \rangle \right) \ell \left( \gamma \cap \overline{G - G'_{PG}} \right) \right).$$

**Lemma 6.3.26.** Let  $w \in F_n$  be a nonperipheral element with conjugacy class  $[w]$ , associated rational current  $\eta_{[w]}$  and associated reduced edge path  $\gamma_w$  in  $G$ . Then

$$\begin{aligned} \Psi_0(\eta_{[w]}) &= \ell_{\text{exp}}(\gamma_w); \\ \|\eta_{[w]}\|_{\mathcal{F}} &= \ell_{\mathcal{F}}(\gamma_w). \end{aligned}$$

Therefore  $\eta_{[w]} \in K_{PG}(f)$  if and only if

$$\ell_{exp}(\gamma_w) = 0.$$

In particular, there exist a basis  $\mathfrak{B}$  of  $F_{\mathbf{n}}$  and a constant  $C > 0$  such that, for every  $\mathcal{F} \wedge \mathcal{A}(\phi)$ -nonperipheral element  $g \in F_{\mathbf{n}}$ , we have  $\|\eta_{[g]}\|_{\mathcal{F}} \in \mathbb{N}^*$  and

$$\ell_{\mathfrak{B}}([g]) \geq C \|\eta_{[g]}\|_{\mathcal{F}}.$$

**Proof.** We prove the result for  $\Psi_0$ , the proof for  $\|\eta_{[w]}\|_{\mathcal{F}}$  being similar. First note that

$$\sum_{e \in \vec{E}(\overline{G - G'_{PG}})} \langle e, \eta_{[w]} \rangle = 2\ell(\gamma_w \cap \overline{G - G'_{PG}}),$$

where the factor 2 follows from the fact that the sum on the left hand side is over oriented edges. Therefore, it remains to prove that

$$\Psi'_0(\eta_{[w]}) = \sum_{\gamma \in \mathcal{N}_{PG}^{\max}(\gamma_w)} \ell\left(\gamma \cap \overline{G - G'_{PG}}\right). \quad (6.3)$$

Let  $\gamma \in \mathcal{N}_{PG}$ . Then the value

$$\langle \gamma, \eta_{[w]} \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle$$

measures the number of occurrences of  $\gamma$  or  $\gamma^{-1}$  in  $\gamma_w$  which are not induced by an occurrence of a path  $\gamma' \in \mathcal{N}_{PG}$  containing properly  $\gamma$  or  $\gamma^{-1}$  and contained in  $\gamma_w$ . Indeed, an occurrence of  $\gamma$  in a path  $\gamma' \in \mathcal{N}_{PG}$  containing properly  $\gamma$  will be counted in  $\sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma)$ . Moreover, if an occurrence of  $\gamma$  is contained in two distinct paths  $\gamma_1, \gamma_2 \in \mathcal{N}_{PG}^{++}(\gamma)$ , Lemma 6.3.24 ensures that this occurrence is contained in a path  $\gamma_3 \in \mathcal{N}_{PG,lr}^{\max}(\gamma)$ . Therefore, the value

$$- \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle$$

measures an occurrence of  $\gamma$  or  $\gamma^{-1}$  in a larger path, and each such occurrence will be counted exactly once. Therefore, the equation below Equation (6.3) measures the number of occurrences of  $\gamma$  and  $\gamma^{-1}$  in  $\mathcal{N}_{PG}^{\max}(\gamma_w)$ . Thus, the equality (6.3) holds. The last assertions of Lemma 6.3.26 then follows by definition of  $K_{PG}(f)$  and of  $\ell_{\mathcal{F}}$ .  $\square$

Note that in the proof of Lemma 6.3.26, we show that, for every edge  $e \in \vec{E}(\overline{G - G'_{PG}})$  and every nonperipheral element  $w \in F_{\mathbf{n}}$ , the value:

$$\langle e, \eta_{[w]} \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left( \langle \gamma, \eta_{[w]} \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle \right) N(\gamma, e)$$

measures the number of occurrences of  $e$  in  $\gamma_w$  which are not contained in a path of  $\mathcal{N}_{PG}^{\max}(\gamma_w)$ . Thus, for every nonperipheral element and every edge  $e \in \vec{E}(\overline{G - G'_{PG}})$ , we have:

$$\langle e, \eta_{[w]} \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left( \langle \gamma, \eta_{[w]} \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \eta_{[w]} \rangle \right) N(\gamma, e) \geq 0.$$

The density of rational currents given by Proposition 6.2.15 and the continuity of  $\langle \cdot, \cdot \rangle$  then shows that for every current  $\nu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and every edge  $e \in \vec{E}(\overline{G - G'_{PG}})$ , we have :

$$\langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left( \langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \right) N(\gamma, e) \geq 0.$$

**Lemma 6.3.27.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system. Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an exponentially growing outer automorphism. Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$ .*

(1) *If  $[\nu] \in K_{PG}(f)$ , then  $\text{Supp}(\nu) \subseteq \partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)$ . In particular, if  $\phi$  is expanding relative to  $\mathcal{F}$ , then  $K_{PG}(f) = \emptyset$ .*

(2) *Conversely, if  $\nu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  is such that the support  $\text{Supp}(\nu)$  of  $\nu$  is contained in  $\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)$ , then  $[\nu] \in K_{PG}(f)$ . Thus we have*

$$K_{PG}(f) = \{[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}) \mid \text{Supp}(\mu) \subseteq \partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)\}.$$

(3) *If  $\nu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ , we have  $\|\nu\|_{\mathcal{F}} = 0$  if and only if  $\nu = 0$ .*

**Proof.** The proof of (3) being identical to the proof of (1) and (2) replacing  $G'_{PG}$  and  $\mathcal{N}_{PG}$  by  $G'_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$ , we only prove (1) and (2). For the proof of both (1) and (2), let  $\mathcal{B}$  be a free basis of  $F_n$  and let  $T$  be the Cayley graph of  $F_n$  associated with  $\mathcal{B}$ . Let  $\mathcal{C}(\mathcal{A}(\phi))$  be the set of elements of  $F_n$  associated with  $\mathcal{A}(\phi)$  given by Lemma 6.2.11. Recall that  $\text{Cyl}(\mathcal{C}(\mathcal{A}(\phi)))$  is the set of cylinder subsets of the form  $C(\gamma)$ , where  $\gamma$  is a geodesic edge path in  $T$  starting at the base point whose associated element  $w \in F_n$  contains a word of  $\mathcal{C}(\mathcal{A}(\phi))$  as a subword.

(1) Let  $\nu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  nonzero be such that  $\text{Supp}(\nu)$  is not contained in  $\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)$ . Then  $\text{Supp}(\nu) \cap \partial^2(F_n, \mathcal{A}(\phi)) \neq \emptyset$ . Hence the restriction of  $\nu$  to  $\partial^2(F_n, \mathcal{A}(\phi))$  induces a nonzero current  $\nu' \in \text{Curr}(F_n, \mathcal{A}(\phi))$ . By Lemma 6.2.12 applied to  $\mathcal{A} = \mathcal{A}(\phi)$  and  $\nu'$ , there exists  $C(\gamma) \in \mathcal{C}(\mathcal{A}(\phi))$  such that  $\nu(C(\gamma)) > 0$ . Let  $w$  be the element of  $F_n$  associated with  $\gamma$ , and let  $\gamma'_w$  be the reduced circuit in  $G$  associated with the conjugacy class of  $w$ . Up to taking a larger geodesic edge path  $\gamma'' \supseteq \gamma$  in  $T$  such that  $\nu(C(\gamma'')) > 0$  (which exists by additivity of  $\nu$ ), we may suppose that  $w$  is cyclically reduced. By Lemma 6.2.11 (3), the path  $\gamma$  is not contained in any tree  $T_A$  such that  $[A] \in \mathcal{A}(\phi)$ . As  $w$  is cyclically reduced, the translation axis in  $T$  of  $w$  contains  $\gamma$ . Hence

$\{w^{+\infty}, w^{-\infty}\} \notin \partial^2 \mathcal{A}(\phi)$  and  $w$  is not contained in any subgroup  $A$  such that  $[A] \in \mathcal{A}(\phi)$ . By Proposition 6.3.13, the circuit  $\gamma'_w$  is not a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Therefore, there exists an edge  $e$  of  $G$  such that

$$\langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left( \langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG, lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \right) N(\gamma, e) > 0.$$

Thus, we see that  $\Psi_0(\nu) > 0$  and that  $[\nu] \notin K_{PG}(f)$ . The second part of (1) follows from the fact that, if  $\phi$  is expanding relative to  $\mathcal{F}$ , then  $\partial^2 \mathcal{A}(\phi) \subseteq \partial^2 \mathcal{F}$ . This proves (1).

(2) Let  $\nu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  be such that  $\text{Supp}(\nu) \subseteq \partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)$ . Let  $e$  be an edge such that  $\langle e, \nu \rangle > 0$ . By Lemma 6.3.4 (1), there exists a constant  $C_1 > 0$  such that, for every path  $\gamma' \in \mathcal{N}_{PG}$ , we have  $\ell(\gamma') \leq C_1$ . Recall the definition of the graph  $G^*$  and the application  $p_{G^*}: G^* \rightarrow G$ . from Lemma 6.3.11. Let  $C_2$  be the length of a maximal path in a maximal forest of  $p_{G^*}(G^*)$ . Let  $C = \max\{2C_1, C_2\}$ .

**Claim.** Let  $\gamma, \delta_1$  and  $\delta_2$  be reduced paths such that  $\gamma = \delta_1 e \delta_2$ ,  $\ell(\delta_1), \ell(\delta_2) \geq 2C$  and  $\langle \gamma, \nu \rangle > 0$ . Let  $\gamma = \gamma_0 \gamma'_1 \gamma_1 \dots \gamma_k \gamma'_k$  be the exponential decomposition of  $\gamma$  (where, for every  $i \in \{0, \dots, k\}$ , the path  $\gamma_i$  is contained in  $\mathcal{N}_{PG}$ ). Either  $e \in \vec{E}G'_{PG}$  or  $e$  is contained in an EG stratum and there exists  $i \in \{0, \dots, k\}$  such that  $e \subseteq \gamma_i$ .

**Proof.** Since  $\text{Supp}(\nu) \subseteq \partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \cap \partial^2 \mathcal{A}(\phi)$ , there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$ , and two elements  $a$  and  $b$  of  $A$  such that the geodesic path in  $\tilde{G}$  representing  $\{a^{+\infty}, b^{+\infty}\} \in \partial^2 A$  contains a lift of  $\gamma$ . If  $b = a^{-1}$ , then  $\gamma$  is contained in an iterate of  $a$  and, by Proposition 6.3.13,  $\gamma$  is contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . The claim follows in this case. So we may assume that  $b \neq a^{-1}$ . Suppose first that the axes  $\text{Ax}(a)$  and  $\text{Ax}(b)$  of  $a$  and  $b$  are disjoint. Then  $\gamma$  is contained in the axis of  $a^{-1}b$ . Thus, by Proposition 6.3.13,  $\gamma$  is contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$  and the claim follows in this case.

Suppose now that  $\text{Ax}(a) \cap \text{Ax}(b) \neq \emptyset$ . Let  $\gamma'_a$  and  $\gamma'_b$  be the reduced circuit in  $G$  associated with  $a$  and  $b$ . Then  $\gamma$  is contained in the union of  $\gamma'_a \cup \gamma'_b$ . Recall that, by Proposition 6.3.13, the paths  $\gamma'_a$  and  $\gamma'_b$  are concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Hence there exist reduced circuits  $\alpha$  and  $\beta$  in  $G^*$  and reduced arcs  $\tau, \tau_e$  in  $G^*$  such that  $p_{G^*}(\alpha) = \gamma'_a$  and  $p_{G^*}(\beta) = \gamma'_b$  and such that  $p_{G^*}(\tau) = \gamma$  and  $p_{G^*}(\tau_e) = e$ . By the choice of  $C$ , and as  $\ell(\delta_1), \ell(\delta_2) \geq 2C$ , one can remove an initial and a terminal segment of  $\tau$  so that the resulting path  $\tau'$  is nontrivial, is contained in a subgraph of  $G^*$  with no leaf and is such that  $\ell(p_{G^*}(\tau')) \geq 2C + 1$ . Thus, there exist subpaths  $\tau'_1, \tau''_1, \tau'_2, \tau''_2$  of  $\tau$  and a reduced circuit  $\delta$  of  $G^*$  such that:

- (i)  $\ell(p_{G^*}(\tau'_1)), \ell(p_{G^*}(\tau'_2)) \geq C$ ,
- (ii)  $\tau = \tau''_1 \tau'_1 e \tau'_2 \tau''_2$ ,
- (iii)  $\tau' = \tau'_1 e \tau'_2 \subseteq \delta$ .

By Lemma 6.3.11 (1), the path  $p_{G^*}(\delta)$  is a reduced circuit which contains  $e$ . Since  $\ell(p_{G^*}(\tau'_1)), \ell(p_{G^*}(\tau'_2)) \geq C \geq 2C_1$ , if  $\gamma' \in \mathcal{N}_{PG}^{\max}(p_{G^*}(\delta))$  is such that  $e \subseteq \gamma'$ , then  $\gamma' \subseteq \tau'_1 e \tau'_2$ . Hence it suffices to prove the claim for  $\gamma = p_{G^*}(\delta)$ . As  $\delta$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , the claim follows.  $\square$



Suppose towards a contradiction that there exists an edge  $e \in \overline{G - G'_{PG}}$  such that:

$$\langle e, \nu \rangle - \sum_{\gamma \in \mathcal{N}_{PG}, e \subseteq \gamma} \left( \langle \gamma, \nu \rangle - \sum_{\gamma' \in \mathcal{N}_{PG}^{++}(\gamma)} \langle \gamma', \nu \rangle N(\gamma', \gamma) + \sum_{\gamma' \in \mathcal{N}_{PG,lr}^{++}(\gamma)} \langle \gamma', \nu \rangle \right) N(\gamma, e) > 0. \quad (6.4)$$

By additivity of  $\nu$ , there exists a reduced path  $\gamma$  of length  $4C + 1$  such that the path  $\gamma$  has a decomposition  $\gamma = \gamma_1 e \gamma_2$ , where for every  $i \in \{1, 2\}$ , the path  $\gamma_i$  has length equal to  $2C$  and we have  $\nu(C(\gamma)) > 0$ . By Equation 6.4, we can choose  $\gamma$  such that if  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$ , then  $\gamma'$  does not contain  $e$ . Hence  $e \notin G'_{PG}$  and  $e$  is not contained in a subpath of  $\mathcal{N}_{PG}^{\max}(\gamma)$ . This contradicts the above claim and this concludes the proof.  $\square$

Let  $\mathcal{F}$  be a free factor system and let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an exponentially growing outer automorphism. Note that, by Lemma 6.3.27 and since for every  $k \in \mathbb{N}^*$ , we have  $\mathcal{A}(\phi) = \mathcal{A}(\phi^k)$ , the space  $K_{PG}(f)$  does not depend on the CT map  $f$  and does not depend on the chosen power of  $\phi$ . Therefore, we will simply write  $K_{PG}(\phi)$  instead. Moreover, since  $\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1})$ , we see that  $K_{PG}(\phi) = K_{PG}(\phi^{-1})$ .

For the next proposition, let  $C_1 > 0$  be a constant such that for every path  $\gamma \in \mathcal{N}_{PG}$ , we have  $\ell(\gamma) \leq C_1$ . It exists since  $\mathcal{N}_{PG}$  is finite by Lemma 6.3.4 (1). Let  $L$  be the malnormality constant associated with  $\mathcal{A}(\phi)$  as defined above Lemma 6.2.11 and let  $C_0 = \max\{C_1, L\}$ . Let  $\mathcal{C}$  be the set of elements of  $F_n$  associated with  $\mathcal{F} \wedge \mathcal{A}(\phi)$  given above Lemma 6.2.11. Let  $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  be the set of reduced paths  $\gamma$  in  $G$  such that  $C(\gamma) \in \text{Cyl}(\mathcal{C})$ ,  $\ell(\gamma) > C_0$  and  $\gamma$  is not contained in a concatenation of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$ .

**Lemma 6.3.28.** *Let  $n \geq 3$ , let  $\mathcal{F}$  be a free factor system of  $F_n$  and let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an exponentially growing outer automorphism. We have*

$$\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))} C(\gamma).$$

**Proof.** Let  $A_1, \dots, A_r$  be subgroups of  $F_n$  such that  $\mathcal{F} \wedge \mathcal{A}(\phi) = \{[A_1], \dots, [A_r]\}$  and  $\mathcal{C} = \mathcal{C}(A_1, \dots, A_r)$ . By Lemma 6.2.12, we have

$$\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{C(\gamma) \in \text{Cyl}(\mathcal{C})} C(\gamma).$$

Note that, for every path  $\gamma \subseteq G$ , we have

$$C(\gamma) = \bigcup_{e \in \vec{E}G, \ell(\gamma e) > \ell(\gamma)} C(\gamma e).$$

Hence we have

$$\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{C(\gamma) \in \text{Cyl}(\mathcal{C}), \ell(\gamma) > C_0} C(\gamma).$$

So it suffices to prove that we can restrict our considerations to paths  $\gamma$  which are not contained in a concatenation of paths in  $G_{PG,\mathcal{F}}$  and  $\mathcal{N}_{PG,\mathcal{F}}$ . Let  $\gamma$  be a path such that  $C(\gamma) \in \text{Cyl}(\mathcal{C})$  and  $\ell(\gamma) > C_0$ . By Lemma 6.2.11 (3), the path  $\gamma$  is not contained in any tree  $T_{gA_i g^{-1}}$  with  $g \in F_{\mathbf{n}}$  and  $i \in \{1, \dots, r\}$ . Moreover, it is not contained in any path of  $\mathcal{N}_{PG}$  since  $\ell(\gamma) > C_1$ . Suppose that  $\gamma$  is contained in a concatenation of paths in  $G_{PG,\mathcal{F}}$  and  $\mathcal{N}_{PG,\mathcal{F}}$ . Suppose first that there does not exist a circuit which contains  $\gamma$  and which is a concatenation of paths in  $G_{PG,\mathcal{F}}$  and  $\mathcal{N}_{PG,\mathcal{F}}$ . Recall the definition of  $G^*$  and  $p_{G^*}$  from Lemma 6.3.11 and let  $G_{\mathcal{F}}^* = p_{G^*}^{-1}(G_p)$ . By assumption, either there does not exist an immersed path (not necessarily an edge path)  $\gamma^*$  in  $G_{\mathcal{F}}^*$  such that  $p_{G^*}(\gamma^*) = \gamma$  or there exists an immersed path  $\gamma^*$  in  $G_{\mathcal{F}}^*$  such that  $p_{G^*}(\gamma^*) = \gamma$  and  $\gamma^*$  is not contained in a circuit of  $G_{\mathcal{F}}^*$  (recall that  $G^*$  might contain univalent vertices). In the first case, we have  $\ell_{\mathcal{F}}(\gamma) > 0$ . In the second case, since  $G^*$  is finite, by Lemma 6.3.11, up to considering  $\gamma^{-1}$ , there exists  $d \in \mathbb{N}^*$  such that for every path  $\gamma'$  such that  $\gamma\gamma'$  is a reduced path in  $G$  and  $\ell(\gamma\gamma') = \ell(\gamma) + d$ , the path  $\gamma\gamma'$  is not the image by  $p_{G^*}$  of an immersed path in  $G_{\mathcal{F}}^*$ . Thus we have  $\ell_{\mathcal{F}}(\gamma\gamma') > 0$ . Using the fact that

$$C(\gamma) = \bigcup_{e \in \vec{E}G, \ell(\gamma e) > \ell(\gamma)} C(\gamma e),$$

we can replace  $\gamma$  by paths  $\gamma''$  such that  $\gamma \subseteq \gamma''$  and  $\gamma''$  is not contained in a concatenation of paths in  $G_{PG,\mathcal{F}}$  and  $\mathcal{N}_{PG,\mathcal{F}}$ . This concludes the proof.  $\square$

Let  $\nu$  be a nonzero current in  $\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ . By Lemma 6.3.27 (3), we have  $\|\nu\|_{\mathcal{F}} \neq 0$ . The following result characterizes limits in  $\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ . The result is due to Kapovich [Kap, Lemma 3.5] for a nonrelative context.

**Lemma 6.3.29.** *Let  $\mathbf{n} \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_{\mathbf{n}}$ . Let  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{F})$  be an exponentially growing outer automorphism. Let  $([\mu_n])_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$  and let  $[\mu] \in \mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ . Let  $G$  be a graph whose fundamental group is isomorphic to  $F_{\mathbf{n}}$  and such that there exists a subgraph  $G_p$  of  $G$  such that  $\mathcal{F}(G_p) = \mathcal{F}$ . Then  $\lim_{n \rightarrow \infty} [\mu_n] = [\mu]$  if and only if, for every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , we have*

$$\lim_{n \rightarrow \infty} \frac{\langle \gamma, \mu_n \rangle}{\|\mu_n\|_{\mathcal{F}}} = \frac{\langle \gamma, \mu \rangle}{\|\mu\|_{\mathcal{F}}}. \quad (6.5)$$

**Proof.** Suppose first that  $\lim_{n \rightarrow \infty} [\mu_n] = [\mu]$ . Thus there exists a sequence  $(\lambda_n)_{n \in \mathbb{N}^*}$  of positive real numbers such that  $\lim_{n \rightarrow \infty} \lambda_n \mu_n = \mu$ . By continuity of  $\|\cdot\|_{\mathcal{F}}$ , we have  $\lim_{n \rightarrow \infty} \|\lambda_n \mu_n\|_{\mathcal{F}} = \|\mu\|_{\mathcal{F}}$ . By linearity of  $\|\cdot\|_{\mathcal{F}}$  and  $\langle \cdot, \cdot \rangle$  in the second variable, for every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , we have

$$\lim_{n \rightarrow \infty} \frac{\langle \gamma, \lambda_n \mu_n \rangle}{\|\lambda_n \mu_n\|_{\mathcal{F}}} = \lim_{n \rightarrow \infty} \frac{\langle \gamma, \mu_n \rangle}{\|\mu_n\|_{\mathcal{F}}} = \frac{\langle \gamma, \mu \rangle}{\|\mu\|_{\mathcal{F}}}.$$

Suppose now that for every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , Equation (6.5) holds. By Lemma 6.3.28, for every Borel subset  $B$  of  $\partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$  such that  $\mu(\partial B) = 0$ ,

we have

$$\lim_{n \rightarrow \infty} \frac{\mu_n(B)}{\|\mu_n\|_{\mathcal{F}}} = \frac{\mu(B)}{\|\mu\|_{\mathcal{F}}}.$$

Hence we have  $\lim_{n \rightarrow \infty} [\mu_n] = [\mu]$ . □

## 6.4 Stable and unstable currents for relative atoroidal outer automorphisms

Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an atoroidal outer automorphism relative to  $\mathcal{F}$ . In this section, under additional hypotheses on  $\phi$ , we construct two  $\phi$ -invariant convex subsets of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F})$ . We will then show in the following section that, with respect to these convex subsets, the outer automorphism  $\phi$  acts with *generalized north-south dynamics*.

In order to define the extremal points of these simplices, we need some results regarding substitution dynamics.

### 6.4.1 Substitution dynamics

Let  $A$  be a finite set with cardinality at least equal to 2. Let  $\zeta$  be a *substitution on  $A$* , that is, a map from  $A$  to the set of nonempty finite words on  $A$ . The substitution  $\zeta$  induces a map on the set of all finite words on  $A$  by concatenation, which we still denote by  $\zeta$ . We can therefore iterate the substitution  $\zeta$ . For a word  $w$  on  $A$ , we will denote by  $|w|$  the length of  $w$  on the alphabet  $A$ .

To the substitution  $\zeta$  one can associate its *transition matrix*  $M$ , which is a square matrix whose rows and columns are indexed by letters in  $A$  and, for all  $a, b \in A$ ,  $M(a, b)$  is the number of occurrences of  $a$  in  $\zeta(b)$ . Likewise, for  $n \geq 1$ , the matrix  $M^n$  is the transition matrix for  $\zeta^n$ . We say that a substitution  $\zeta$  is *irreducible* if its transition matrix is irreducible, and that the substitution is *primitive* if its transition matrix is.

Let  $\ell \in \mathbb{N}^*$  and let  $A_\ell$  be the set of words on  $A$  of length  $\ell$ . As defined in [Que, Section 5.4.1], the substitution  $\zeta$  induces a substitution  $\zeta_\ell$  on  $A_\ell$  as follows. Let  $w = x_1 \dots x_\ell \in A_\ell$ . Then  $\zeta_\ell(w) = w_1 w_2 \dots w_{|\zeta(x_1)|}$ , where, for every  $i \in \{1, \dots, |\zeta(x_1)|\}$ , the word  $w_i$  is the subword of  $\zeta(w)$  of length  $\ell$  starting at the  $i^{\text{th}}$  position of  $\zeta(x_1)$ . Therefore,  $\zeta_\ell$  is the concatenation of the  $|\zeta(x_1)|$  first subwords of  $\zeta(w)$  of length  $\ell$ . Note that the number of  $i \in \{1, \dots, |\zeta(x_1)|\}$  such that  $w_i$  that is not contained in  $\zeta(x_1)$  is bounded by  $\ell - 1$ . Let  $|\cdot|_\ell$  be the length of words on  $A_\ell$ . Then  $|\zeta_\ell(w)|_\ell = |\zeta(x_1)|$ . Denote by  $M_\ell$  the transition matrix of  $\zeta_\ell$ . Note that, for every  $n, \ell \geq 1$ , we have  $(\zeta^n)_\ell = (\zeta_\ell)^n$  as applications on the set of words on  $A_\ell$  and thus  $(M^n)_\ell = (M_\ell)^n$ .

Consider now a partition of the alphabet  $A = \coprod_{i=0}^k B_i$ . Suppose that the transition matrix associated with the substitution  $\zeta$  is lower block triangular with respect to this partition. Therefore, for every  $i \in \{0, \dots, k\}$ , for every  $x \in B_i$  and for every  $j < i$ , the word  $\zeta(x)$  does not contain letters in  $B_j$ . In the remainder of the article, for every  $i \in \{0, \dots, k\}$  the diagonal block in  $M$  corresponding to the block  $B_i$  will be denoted by  $M_{B_i}$ .

The partition of  $A$  induces a partition of  $A_\ell$  as follows. For every  $i \in \{0, \dots, k\}$ , let  $\tilde{B}_i \subseteq A_\ell$  be the set of all words on  $A$  of length  $\ell$  which start with a letter in  $B_i$  and which, for every  $j < i$  do not contain a letter in  $B_j$ . Let  $\bar{B}_i$  be the set of all words  $w$  on  $A$  of length  $\ell$  which start with a letter in  $B_i$  and such that there exists  $j < i$  such that  $w$  contains a letter in  $B_j$  (note that  $\bar{B}_0$  is empty). Then  $\tilde{B}_i \cup \bar{B}_i$  is the set of all words on  $A$  of length  $\ell$  which starts with a letter in  $B_i$ . The hypothesis on the substitution  $\zeta$  implies that the transition matrix  $M_\ell$  is lower block triangular with respect to the partition

$$\tilde{B}_0 \amalg \bar{B}_1 \amalg \tilde{B}_1 \amalg \dots \amalg \bar{B}_k \amalg \tilde{B}_k$$

of  $A_\ell$ . As before, for every  $i \in \{0, \dots, k\}$ , we will denote by  $M_{\ell, \bar{B}_i}$  the diagonal block in  $M_\ell$  corresponding to  $\bar{B}_i$  and by  $M_{\ell, \tilde{B}_i}$  the diagonal block in  $M_\ell$  corresponding to  $\tilde{B}_i$ .

**Lemma 6.4.1.** [*Gup1, Lemma 8.8*] *Let  $A$  be a finite alphabet equipped with a partition  $A = \amalg_{i=0}^k B_i$ . Let  $\zeta$  be a substitution and let  $M$  be its transition matrix. Let  $\ell \in \mathbb{N}^*$ .*

(1) *The eigenvalues of  $M_{\ell, \tilde{B}_i}$  are those of  $M_{B_i}$  with possibly additional eigenvalues of absolute value at most equal to 1.*

(2) *The eigenvalues of  $M_{\ell, \bar{B}_i}$  have absolute value at most equal to 1.*

Fix an integer  $p \in \{0, \dots, k\}$ . For every  $i \geq p$ , let  $\bar{B}_i^{(p)}$  be the subset of  $\bar{B}_i$  consisting in all words  $w$  of length  $\ell$  which start with a letter in  $B_i$  and such that there exists  $j < p$  such that  $w$  contains a letter in  $B_j$ . Then, for every  $i \geq p$ , the block  $M_{\ell, \bar{B}_i}$  decomposes into a lower triangular block matrix where the columns and rows corresponding to  $\bar{B}_i^{(p)}$  are on the top left. Let  $M_{\ell, \bar{B}_i^{(p)}}$  be the corresponding block matrix. By Lemma 6.4.1 (2), the eigenvalues of  $M_{\ell, \bar{B}_i^{(p)}}$  have absolute value at most 1. Moreover, for every  $i, j \geq p$ , for every word  $w$  contained in  $\tilde{B}_j \cup \bar{B}_j - \bar{B}_j^{(p)}$ , the word  $\zeta_\ell(w)$  considered as a word on  $A_\ell$  does not contain any word of  $\bar{B}_i^{(p)}$ . Let  $M_\ell(p)$  be the matrix obtained from  $M_\ell$  by deleting, for every  $i \geq p$ , every row and column corresponding to elements in  $\tilde{B}_i$ , and every row and columns corresponding to elements of  $\bar{B}_i$  which do not belong to  $\bar{B}_i^{(p)}$ . Note that, by Lemma 6.4.1 (1), the eigenvalues of  $M_\ell(p)$  are those of every block  $M_{B_j}$  with  $j < p$  with possibly additional eigenvalues of absolute value at most 1.

We can now prove a result concerning the number of occurrences of words in iterates of a letter. For words  $w, v$  on  $A$ , we denote by  $(w, v)$  the number of occurrences of  $w$  in  $v$ , so that  $M = ((a, \zeta(b))_{a, b \in A})$ . For a word  $w$  on  $A$ , we denote by  $\|w\|_{(p)}$  the number of letters in  $w$  which are contained in some  $B_j$  for  $j < p$ .

**Proposition 6.4.2.** *Let  $A$  be an alphabet equipped with a partition  $A = \amalg_{i=0}^k B_i$ . Let  $\zeta$  be a substitution on  $A$  and let  $M$  be its transition matrix. Suppose that  $M$  is lower triangular by block with respect to the partition of  $A$ . Let  $p \in \mathbb{N}^*$ . Let  $a \in \bigcup_{j < p} B_j$  be such that  $\zeta(a)$  starts with  $a$ . Suppose that there exists  $j < p$  such that  $M_{B_j}$  is a primitive block whose Perron-Frobenius eigenvalue is greater than 1 and such that there exists  $n_j \geq 1$  such that  $\zeta^{n_j}(a)$  contains a letter of  $B_j$ . Let  $w$  be a word such that  $w$  contains a letter in  $B_j$ . Then*

$$\lim_{n \rightarrow \infty} \frac{(w, \zeta^n(a))}{\|\zeta^n(a)\|_{(p)}}$$

exists and is finite. Furthermore there exists a word  $w$  containing a letter in some  $B_j$  with  $j < p$  such that this limit is positive.

**Proof.** The proof follows [Gup1, Lemma 8.9] (see also [LU1] for similar statements). First, up to replacing  $A$  by the smallest  $\zeta$ -invariant subalphabet of  $A$  containing  $a$  (which still satisfies the hypotheses of Proposition 6.4.2), we may suppose that, for every letter  $x \in A$ , there exists  $n_x \geq 1$  such that  $\zeta^{n_x}(a)$  contains the letter  $x$ . Let  $\alpha$  be a word on  $A$  with length  $\ell \geq 1$  that starts with  $a$ . Note that, since  $a \in \cup_{j < p} B_j$ , the word  $\alpha$  defines a column and a row in  $M_\ell(p)$ . Recall that for every  $n$  the number of occurrences of a word  $w$  in  $\zeta^n(a)$  differs from the number of occurrences of the letter  $w \in A_\ell$  in  $\zeta_\ell^n(\alpha)$  by at most  $\ell - 1$ . Moreover, we have  $(w, \zeta_\ell^n(\alpha)) = M_\ell^n(p)(w, \alpha)$ .

Let  $S$  be the set of all  $s < p$  such that  $M_{B_s}$  is a primitive block with associated Perron-Frobenius eigenvalue greater than 1. By assumption, the set  $S$  is a nonempty finite set. Let  $S'$  be the subset of  $S$  consisting in all such  $B_s$  such that the associated Perron-Frobenius eigenvalue is maximal. Call this eigenvalue  $\lambda$ . By Lemma 6.4.1, the eigenvalue  $\lambda$  is also the maximal eigenvalue of the matrix  $M_\ell(p)$ . Let  $d_\lambda$  be the size of the maximal Jordan block of  $M_\ell(p)$  associated with  $\lambda$ . Then the growth under iterates of the maximal Jordan block of  $\frac{M_\ell(p)}{\lambda}$  is polynomial of degree  $d_\lambda$ . Therefore, we have

$$\lim_{n \rightarrow \infty} \frac{(w, \zeta^n(a))}{\lambda^n n^{d_\lambda}} = \lim_{n \rightarrow \infty} \frac{(w, \zeta_\ell^n(\alpha))}{\lambda^n n^{d_\lambda}} = \lim_{n \rightarrow \infty} \frac{M_\ell^n(p)(w, \alpha)}{\lambda^n n^{d_\lambda}} = d_{w,a},$$

where  $d_{w,a}$  is a real number. Moreover, the limit does not depend on the choice of  $\alpha$  since, for any  $n$ , and for any two columns of  $M_\ell^n(p)$  corresponding to words starting with the same letter, the sum of the values of each column differ by at most  $\ell - 1$  (see [Gup1, Lemma 8.6]). Moreover, there exists a word  $w$  such that the limit is positive since we quotiented by the growth of the iterates of the Jordan block with maximal eigenvalue.

Let  $\|\cdot\|$  be the  $L_1$ -norm on  $\mathbb{R}^{|A_\ell|}$ . By [LU1, Remark 4.1], since  $\lim_{n \rightarrow \infty} \frac{M_\ell^n(p)(w, \alpha)}{\lambda^n n^{d_\lambda}}$  exists, so does

$$\lim_{n \rightarrow \infty} \frac{M_\ell^n(p)(w, \alpha)}{\|M_\ell^n(p)(\alpha)\|},$$

where  $\|M_\ell^n(p)(\alpha)\|$  is the norm of the column of  $M_\ell^n(p)$  corresponding to  $\alpha$ .

**Claim.** Suppose that there exists  $C \geq 1$  such that for every  $n \in \mathbb{N}$ , we have

$$\|\zeta^n(a)\|_{(p)} \leq \|M_\ell^n(p)(\alpha)\| \leq C \|\zeta^n(a)\|_{(p)}.$$

Then

$$\lim_{n \rightarrow \infty} \frac{(w, \zeta^n(a))}{\|\zeta^n(a)\|_{(p)}}$$

exists for all words  $w$  on  $A$  and is positive for some word  $w$ .

**Proof.** Recall that two sequences  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  with values in  $\mathbb{R}$  are equivalent if there exists a sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  tending to zero such that  $u_n = (1 + \epsilon_n)v_n$ . Recall that there exists  $C' > 0$  such that the sequence  $(\|M_\ell^n(p)(\alpha)\|)_{n \in \mathbb{N}}$  is equivalent to  $(C'\lambda^n n^{d_\lambda})_{n \in \mathbb{N}}$ . Recall also that for every  $n$ , the value of  $\|\zeta^n(a)\|_{(p)}$  is the norm of  $M^n(p)(v_a)$ , where  $v_a$  is the vector whose coordinates is 1 on the coordinate associated with  $a$  and 0 otherwise. Hence, since the matrix  $M^n(p)$  is nonnegative and not the zero matrix, there exist  $C_a, \lambda_a \in \mathbb{R}_+^*$  and  $d_a \in \mathbb{N}$  such that the sequence  $(\|\zeta^n(a)\|_{(p)})_{n \in \mathbb{N}}$  is equivalent to  $(C_a \lambda_a^n n^{d_a})_{n \in \mathbb{N}}$ . Thus, by the assumption of the claim, since the limit

$$\lim_{n \rightarrow \infty} \frac{M_\ell^n(p)(w, \alpha)}{\|M_\ell^n(p)(\alpha)\|}$$

exists, and is not equal to zero for some  $w$ , the same is true for

$$\lim_{n \rightarrow \infty} \frac{(w, \zeta^n(a))}{\|\zeta^n(a)\|_{(p)}}.$$

This proves the claim.  $\square$

Therefore, in order to conclude the proof of the proposition, it remains to prove that the hypothesis of the claim is true in our context. Let  $\zeta^n(a) = x_1 \dots x_{|\zeta^n(a)|}$  and let

$$\zeta_\ell^n(\alpha) = w_1 \dots w_{|\zeta^n(a)|}.$$

Let  $X^n(a)$  be the list  $x_1, \dots, x_{|\zeta^n(a)|}$  and let  $X_{<p}^n(a)$  be the sublist of  $X^n(a)$  consisting in all letters in  $\cup_{i=1}^{p-1} B_i$ . Let  $X^{(\ell, n)}(\alpha)$  be the list  $w_1, \dots, w_{|\zeta^n(a)|}$  and let  $X_{<p}^{(\ell, n)}(\alpha)$  be the sublist of  $X^{(\ell, n)}(\alpha)$  which consists in all elements of  $X^{(\ell, n)}(\alpha)$  that do not belong to  $\cup_{i \leq p} \tilde{B}_i \cup \bar{B}_i - \bar{B}_i^{(p)}$ . Note that  $|X_{<p}^{(\ell, n)}(\alpha)| = \|M_\ell^n(p)(\alpha)\|$  and that  $|X_{<p}^n(a)| = \|\zeta^n(a)\|_{(p)}$ . The fact that  $\|\zeta^n(a)\|_{(p)} \leq \|M_\ell^n(p)(\alpha)\|$  follows from the fact that we have an injection from  $X_{<p}^n(a)$  to  $X_{<p}^{(\ell, n)}(\alpha)$  by sending the letter  $x_i \in X_{<p}^n(a)$  to  $w_i \in X_{<p}^{(\ell, n)}(\alpha)$ . Since every word of length  $\ell$  contained in  $X_{<p}^{(\ell, n)}(\alpha)$  contains a letter in  $X_{<p}^n(a)$ , we have an application from  $X_{<p}^{(\ell, n)}(\alpha)$  to  $X_{<p}^n(a)$  defined as follows. Let  $w \in X_{<p}^{(\ell, n)}(\alpha)$  and let  $j_w \in \{1, \dots, |\zeta^n(a)|\}$  be the minimal integer such that  $x_{j_w} \in X_{<p}^n(a)$  and  $x_{j_w}$  is a letter in  $w$ . Then the application sends  $w$  to  $x_{j_w}$ . By construction, the cardinal of the preimage of any  $x \in X_{<p}^n(a)$  is at most equal to  $\ell$ . Therefore, we have

$$\|\zeta^n(a)\|_{(p)} \leq \|M_\ell^n(p)(\alpha)\| \leq \ell \|\zeta^n(a)\|_{(p)}.$$

This concludes the proof.  $\square$

## 6.4.2 Construction of the attractive and repulsive currents for relative almost atoroidal automorphisms

Let  $n \geq 3$  and let  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$  be a free factor system of  $F_n$ . We first define a class of outer automorphisms of  $F_n$  which we will study in the rest of the article. If

$\phi \in \text{Out}(F_n, \mathcal{F})$  and  $\phi$  preserves the conjugacy class of every  $A_i$  with  $i \in \{1, \dots, k\}$ , we denote by  $\phi|_{\mathcal{F}}$  the element  $([\phi_1|_{A_1}], \dots, [\phi_k|_{A_k}])$ , where, for every  $i \in \{1, \dots, k\}$ , the element  $\phi_i$  is a representative of  $\phi$  such that  $\phi_i(A_i) = A_i$  and  $[\phi_i|_{A_i}]$  is an element of  $\text{Out}(A_i)$ . Note that the outer class of  $\phi_i|_{A_i}$  in  $\text{Out}(A_i)$  does not depend on the choice of  $\phi_i$ .

**Definition 6.4.3.** Let  $n \geq 3$  and let  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be exponentially growing. The outer automorphism  $\phi$  is *almost atoroidal relative to  $\mathcal{F}$*  if  $\phi$  preserves the conjugacy class of every  $A_i$  with  $i \in \{1, \dots, k\}$  and  $\phi$  is one of the following:

- (1) an atoroidal outer automorphism relative to  $\mathcal{F}$ .
- (2) an outer automorphism which preserves a sequence of free factor systems  $\mathcal{F} \leq \mathcal{F}_1 \leq \{F_n\}$  with  $\mathcal{F}_1 = \{[B_1], \dots, [B_\ell]\}$  and such that:
  - (a)  $\mathcal{F}_1 \leq \{F_n\}$  is sporadic,
  - (b)  $\phi$  preserves the conjugacy class of every  $B_i$  with  $i \in \{1, \dots, \ell\}$ , the element  $\phi|_{\mathcal{F}_1}$  is an expanding atoroidal outer automorphism relative to  $\mathcal{F}$  and  $\phi$  is not expanding relative to  $\mathcal{F}$  ( $\mathcal{F}$  might be equal to  $\mathcal{F}_1$ ).

The main example of an almost atoroidal automorphism is the following. Suppose that  $\mathcal{F}_1 = [A]$  and let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be such that  $\phi([A]) = [A]$ . Then  $\phi$  is almost atoroidal if  $\phi|_{[A]}$  is expanding relative to  $\mathcal{F}$ . Indeed, either  $\phi$  is expanding relative to  $\mathcal{F}$  and in this case  $\phi$  satisfies (1) or  $\phi$  is not expanding relative to  $\mathcal{F}$  and  $\phi$  satisfies (2). Almost atoroidality allows us to deal with sporadic extensions.

Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$ . In this section, we construct a nontrivial convex compact subset in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  associated with  $\phi$ . We follow the construction of [Uya2] in the context of atoroidal automorphisms. By Theorem 6.2.10, there exists  $M \geq 1$  such that  $\phi^M$  is represented by a CT map  $f: G \rightarrow G$  with filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$  and such that there exists  $p \in \{1, \dots, k\}$  such that  $\mathcal{F}(G_p) = \mathcal{F}$ . For a splitting unit  $\sigma$  in  $G$ , we say that  $\sigma$  is *expanding* if  $\lim_{m \rightarrow \infty} \ell_{exp}([f^m(\sigma)]) = +\infty$ . Note that, by Lemma 6.3.23, this is equivalent to saying that there exists  $N \in \mathbb{N}^*$  such that  $[f^N(\sigma)]$  contains a splitting unit which is an edge in an EG stratum. Moreover, a splitting unit  $\sigma$  which is an expanding splitting unit is either an edge in  $\overline{G - G'_{PG}}$  or a maximal taken connecting path in zero stratum such that a reduced iterate of  $\sigma$  contains an edge in  $\overline{G - G'_{PG}}$  as a splitting unit. In particular, there are finitely many expanding splitting units by Proposition 6.2.5 (3).

Let  $\gamma$  and  $\gamma'$  be two finite reduced subpaths of  $G$ . We denote by  $\#(\gamma, \gamma')$  the number of occurrences of  $\gamma$  in  $\gamma'$  and by  $\langle \gamma, \gamma' \rangle$  the sum

$$\langle \gamma, \gamma' \rangle = \#(\gamma, \gamma') + \#(\gamma^{-1}, \gamma'). \quad (6.6)$$

The next proposition shows the existence of relative currents associated with relative atoroidal outer automorphisms. Once we have constructed these currents for relative atoroidal outer automorphisms, we will also be able to construct attractive and repulsive simplices for every almost atoroidal outer automorphisms relative to  $\mathcal{F}$ . The proposition

and its proof are inspired by the same result in the absolute context due to Uyanik ([Uya2, Proposition 3.3]) and by the proof due to Gupta in the relative fully irreducible context ([Gup1, Proposition 8.13]). Recall the definition of  $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  before Lemma 6.3.28 and  $\mathcal{C}$  before Lemma 6.2.11.

**Proposition 6.4.4.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an atoroidal outer automorphism relative to  $\mathcal{F}$ . Let  $f: G \rightarrow G$  be a CT map that represents a power of  $\phi$  with filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$  and such that there exists  $p \in \{1, \dots, k\}$  such that  $\mathcal{F}(G_p) = \mathcal{F}$ . Let  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  and let  $\sigma$  be an expanding splitting unit with fixed initial direction.*

(1) *The limit*

$$\sigma_\gamma = \lim_{m \rightarrow \infty} \frac{\langle \gamma, [f^m(\sigma)] \rangle}{\ell_{\mathcal{F}}([f^m(\sigma)])}$$

*exists and is finite.*

(2) *There exists a unique current  $\eta_\sigma \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  such that, for every finite reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , we have:*

$$\eta_\sigma(C(\gamma)) = \sigma_\gamma.$$

**Proof.** (1) We may suppose that  $\gamma$  occurs in a reduced iterate of  $\sigma$  as otherwise  $\sigma_\gamma = 0$ . We first treat the case where  $\sigma$  is an expanding splitting unit which is an edge in an irreducible stratum. Let  $r$  be the height of  $\sigma$ . In order to prove the proposition in this case, we want to apply Proposition 6.4.2 to the CT map  $f$  seen as a substitution on the set of splitting units contained in iterates of  $\sigma$ . However, the set of splitting units might be infinite since exceptional paths may have arbitrarily large widths and INPs arbitrarily large lengths. Instead, we construct a finite alphabet  $A_\gamma$  depending on  $\gamma$ . The alphabet is constructed as follows by associating a letter to every splitting unit occurring in a reduced iterate of  $\sigma$ . However some letters will correspond to infinitely many splitting units.

(a) We add one letter for each of the finitely many edges in irreducible strata that are contained in a reduced iterate of  $\sigma$ .

(b) We add one letter for each reduced maximal taken connecting path in a zero stratum contained in a reduced iterate of  $\sigma$ .

(c) We add one letter for each INP contained in a reduced iterate of  $\sigma$  and such that the stratum of maximal height it intersects is an EG stratum.

(d) Let  $\delta$  be an INP such that the stratum of maximal height it intersects is an NEG stratum and such that it appears in a reduced iterate of  $\sigma$ . By Proposition 6.2.5 (11), there exist an edge  $e$ , an integer  $s \in \mathbb{Z}$  and a closed Nielsen path  $w$  such that  $\delta = ew^se^{-1}$ . Note that  $\gamma$  is not contained in  $w^s$  since  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  and  $w^s$  is a concatenation of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$  by Lemma 6.3.7 and the fact that  $\phi$  is atoroidal relative to  $\mathcal{F}$ . Hence if  $\gamma$  is contained in  $\delta$ , it is either an initial or a terminal segment of  $\delta$ . Let  $M_1$  be the maximal integer  $|d|$  such that  $\gamma$  contains an INP of the form  $ew^de^{-1}$ . Let



$M_2$  be the minimal integer  $|d|$  such that  $\gamma \cap (ew^de^{-1})$  is either an initial or a terminal segment of  $ew^de^{-1}$ . Let  $M_3$  be the maximal integer  $|d|$  such that  $ew^de^{-1}$  is contained in  $[f(\sigma')]$  with  $\sigma'$  a splitting unit which is either an edge in an irreducible stratum or a maximal taken connecting path in a zero stratum. Let  $M = \max\{M_1, M_2, M_3\}$ . We add one letter for each  $ew^de^{-1}$  with  $|d| \leq M + 1$ . We add exactly one letter representing every  $ew^de^{-1}$  with  $|d| > M + 1$ .

(e) Let  $\delta$  be an exceptional path appearing in a reduced iterate of  $\sigma$ . There exist edges  $e_1, e_2$ , a nonzero integer  $s$  and a closed Nielsen path  $w$  such that  $\delta = e_1w^se_2^{-1}$ . Note that  $\gamma$  is not contained in  $w^s$  since  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  and  $w^s$  is a concatenation of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$  by Lemma 6.3.7 and the fact that  $\phi$  is atoroidal relative to  $\mathcal{F}$ . Let  $M_4$  be the maximal integer  $|d|$  such that  $\gamma$  contains an exceptional path of the form  $e_1w^de_2^{-1}$ . Let  $M_5$  be the minimal integer  $|d|$  such that  $\gamma \cap e_1w^de_2^{-1}$  is either a proper initial or terminal segment of  $e_1w^de_2^{-1}$ . Let  $M_6$  be the maximal integer  $|d|$  such that  $e_1w^de_2^{-1}$  is contained in  $[f(\sigma')]$  with  $\sigma'$  a splitting unit which is either an edge in an irreducible stratum or a maximal taken connecting path in a zero stratum. Let  $M' = \max\{M_4, M_5, M_6\}$ . We add one letter for each  $e_1w^de_2^{-1}$  with  $|d| \leq M' + 1$ . We add one letter representing every  $e_1w^de_2^{-1}$  with  $|d| > M' + 1$ .

We claim that the alphabet  $A_\gamma$  is finite. Indeed, since the graph  $G$  is finite, so is the number of letters in the first category. By Proposition 6.2.5 (3), the zero strata of  $G_{r-1}$  are exactly the contractible components of  $G_{r-1}$ . Hence the number of letters in the second category is finite. The number of letters in the third category is finite by Proposition 6.2.5 (9). The remaining letters of  $A_\gamma$  are finite by definition. Let  $\zeta$  be the following substitution on  $A_\gamma$ . If  $a \in A_\gamma$  represents a unique path in  $G$ , we set  $\zeta(a) = [f(a)]$ . If  $a \in A_\gamma$  represents several paths in  $G$ , we set  $\zeta(a) = a$ . We claim that  $\zeta$  is a well-defined substitution. Indeed, by Proposition 6.2.5 (6), if  $a$  is a letter in  $A_\gamma$  which represents a unique path in  $G$ , then  $[f(a)]$  is completely split and every splitting unit in  $[f(a)]$  is represented by a unique letter by the construction of letters in the fourth and fifth category. Moreover, if  $a \in A_\gamma$  represents several paths, then the definition of  $\zeta$  does not depend on the choice of a representative of  $a$ . Hence  $\zeta$  is a well-defined substitution.

We claim that if  $a \in A_\gamma$  represents several paths in  $G$ , then, for every representative  $\alpha$  of  $a$ , the path  $[f(\alpha)]$  is represented by  $a$ . Indeed, the claim is immediate when  $a$  represents several INPs, so we focus on the case where  $a$  represents several exceptional paths. Let  $e_1, e_2$  be edges in  $G$ , let  $w$  be a closed Nielsen path in  $G$  and let  $d \in \mathbb{Z}$  be such that  $e_1w^de_2^{-1}$  is represented by the letter  $a$ . There exist a splitting unit  $\sigma'$  of a reduced iterate of  $\sigma$  by  $[f]$ , an integer  $N \in \mathbb{N}^*$  and an integer  $d_1 \in \mathbb{Z}$  such that  $e_1w^{d_1}e_2^{-1}$  is a subpath of  $[f^N(\sigma')]$ . Thus, using the constants given in (e), we have  $|d_1| \leq M_6 \leq M$ . By the construction of the alphabet  $A_\gamma$ , there exists a letter  $a'$  in  $A_\gamma$  corresponding to the path  $e_1w^{d_1}e_2^{-1}$  and  $a'$  represents a unique path. For every  $n \in \mathbb{N}$ , let  $d_n \in \mathbb{Z}$  be such that  $[f^n(e_1w^{d_1}e_2^{-1})] = e_1w^{d_n}e_2^{-1}$ . Then the sequence  $(d_n)_{n \in \mathbb{N}}$  is monotonic. Let  $m_0$  be the minimal integer such that the path  $e_1w^{d_{m_0}}e_2^{-1}$  is represented by  $a$ . Note that  $m_0 > 1$  as  $a'$  represents a unique path. By monotonicity,  $d_{m_0} \neq d_1$ . Thus, if  $d_{m_0} > d_1$ , then for every  $m \geq m_0$ , we have  $d_m \geq d_{m_0}$  and if  $d_{m_0} < d_1$ , then for every  $m \geq m_0$ , we have  $d_m \leq d_{m_0}$ . Hence for every  $m \geq m_0$ , the path  $e_1w^{d_{m+1}}e_2^{-1}$  is represented by  $a$ . This

shows that if  $\alpha \in a$  then  $[f(\alpha)] \in a$ . This concludes the proof of the claim. Hence  $\zeta$  only depends on the function  $[f(\cdot)]$ .

By reordering columns and rows, we may suppose that, if  $M$  is the matrix associated with  $\zeta$ , then columns and rows of  $M$  with index greater than  $p$  are precisely the letters in  $A_\gamma$  representing splitting units which are concatenations of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$ . By Lemma 6.3.9, iterates by  $\zeta$  of letters of  $A_\gamma$  representing concatenations of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$  are words on  $A_\gamma$  whose letters represent concatenations of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$ . Thus, the matrix  $M$  is a lower block triangular matrix, where every block of index at most  $p$  corresponds to either edges in a common stratum, or the 0 matrix when the associated letter is a maximal taken connecting path in a zero stratum.

Since  $\sigma$  is expanding, it has a reduced iterate which contains splitting units which are edges in EG strata. Hence if  $a_\sigma$  is the letter in  $A_\gamma$  corresponding to  $\sigma$ , the iterates  $\zeta^n(a_\sigma)$  contain letters of  $A_\gamma$  in a Perron-Frobenius block with eigenvalue greater than 1. Since the initial direction of  $\sigma$  is fixed by Proposition 6.4.2, for every word  $w$  in the alphabet  $A_\gamma$ , the limit

$$\lim_{m \rightarrow \infty} \frac{(w, [\zeta^m(\sigma)])}{\|\zeta^m(\sigma)\|_{(p)}}$$

exists and is finite. Hence the limit

$$\lim_{m \rightarrow \infty} \frac{\langle w, [\zeta^m(\sigma)] \rangle}{\|\zeta^m(\sigma)\|_{(p)}}$$

exists and is finite.

**Claim.** There exists a matrix  $M'$  obtained from  $M$  by multiplying rows and columns by positive scalars and such that, for every  $m \in \mathbb{N}^*$ , we have  $\ell_{\mathcal{F}}([f^m(\sigma)]) = \|M'^m(\sigma)\|_{(p)}$ .

**Proof.** Remark that if  $e_1 w^s e_2^{-1}$  is an exceptional path, and if  $e_1 w^d e_2^{-1}$  is an exceptional path with distinct width, then their  $\mathcal{F}$ -lengths are equal and at most equal to 2. Indeed, since  $\phi$  is an atoroidal outer automorphism relative to  $\mathcal{F}$ , every closed Nielsen path of  $G$  is contained in  $G_p$ . Since  $w$  is a closed Nielsen path, we see that  $w$  is a concatenation of paths in  $G_{PG, \mathcal{F}}$  and  $\mathcal{N}_{PG, \mathcal{F}}$  by Lemma 6.3.6. Hence we have

$$\ell_{\mathcal{F}}(e_1 w^s e_2^{-1}) = \ell_{\mathcal{F}}(e_1) + \ell_{\mathcal{F}}(e_2) \leq 2.$$

Similarly, if  $ew^s e^{-1}$  and  $ew^d e^{-1}$  are INP intersecting the same maximal NEG stratum, then their  $\mathcal{F}$ -length are equal and at most equal to 2. Let  $M'$  be the matrix obtained from  $M$  by multiplying every row corresponding to either an exceptional path not contained in  $G_p$ , an INP not contained in  $G_p$ , a collection of exceptional paths not contained in  $G_p$ , a collection of INPs not contained in  $G_p$  or a maximal taken connecting path not contained in  $G_p$ , by the corresponding  $\mathcal{F}$ -length. Note that, by the above remarks, this does not depend on the choice of a representative when the letter corresponds to a collection of paths. Then for every  $m \in \mathbb{N}^*$ , the value  $\|M'^m(\sigma)\|_{(p)}$  corresponds to the sum of the  $\mathcal{F}$ -length of every splitting unit in  $[f^m(\sigma)]$  not contained in  $G_p$ . By Lemma 6.3.19, complete splittings are  $PG$ -relative complete splittings. By Lemma 6.3.20 (2), we have  $\ell_{\mathcal{F}}([f^m(\sigma)]) = \|M'^m(\sigma)\|_{(p)}$ . This proves the claim.  $\square$

By the claim, we see that for every  $m \in \mathbb{N}^*$ , there exists a constant  $K$  such that we have

$$\frac{1}{K} \|\zeta^m(\sigma)\|_{(p)} \leq \ell_{\mathcal{F}}([f^m(\sigma)]) \leq K \|\zeta^m(\sigma)\|_{(p)}.$$

Using the claim in the proof of Proposition 6.4.2 (replacing  $\|M_{\ell}^n(p)(\alpha)\|$  by  $\ell_{\mathcal{F}}([f^n(\sigma)])$  which is possible since  $\ell_{\mathcal{F}}([f^n(\sigma)])$  is the norm of a matrix by the claim), the limit

$$\lim_{m \rightarrow \infty} \frac{\langle w, [f^m(\sigma)] \rangle}{\ell_{\mathcal{F}}([f^m(\sigma)])}$$

exists and is finite. We now construct a finite set of words  $W(\gamma)$  in the alphabet  $A_{\gamma}$  such that for every  $m \in \mathbb{N}^*$ , there exists a bijection between occurrences of  $\gamma$  in  $[f^m(\sigma)]$  and occurrences of a word  $w \in W(\gamma)$  in  $[\zeta^m(\sigma)]$ . This will conclude the proof of Case 2. Let  $W(\gamma)$  be the set of words in  $A_{\gamma}$  consisting in every path contained in a reduced iterate of  $\sigma$  which contains  $\gamma$ , which is completely split and which is minimal for these properties. By construction, every occurrence of  $\gamma$  in a reduced iterate of  $\sigma$  is contained in a word in  $W(\gamma)$ . We claim that the set  $W(\gamma)$  is finite. Indeed, let  $w$  be a word in  $W(\gamma)$ . Then  $w$  corresponds to a path in a reduced iterate of  $\sigma$  which is a concatenation of splitting units  $w = \sigma_1 \dots \sigma_k$ . By minimality of  $w$ , if  $w' \in W(\gamma)$  is distinct from  $w$ , then the number of splitting units in  $w'$  is at most equal to  $k$  and  $w'$  might differ from  $w$  by changing  $\sigma_1$  and  $\sigma_k$ . Thus,  $W(\gamma)$  is finite. For every  $w \in W(\gamma)$ , let  $m_w$  be the number of occurrences of  $\gamma$  in  $w$ . Since  $\gamma$  is not contained in  $G_p$ , the value  $m_w$  does not depend on the choice of a representative of  $w$  if  $w$  represents a collection of paths. Therefore, for every  $m \in \mathbb{N}^*$ , we have

$$\langle \gamma, f^m(\sigma) \rangle = \sum_{w \in W(\gamma)} m_w \langle w, f^m(\sigma) \rangle.$$

This shows that the limit

$$\sigma_{\gamma} = \lim_{m \rightarrow \infty} \frac{\langle \gamma, f^m(\sigma) \rangle}{\ell_{\mathcal{F}}(f^m(\sigma))}$$

exists and is finite. This proves Assertion (1) of the proposition when  $\sigma$  is an edge in an irreducible stratum.

Suppose now that  $\sigma$  is a maximal taken connecting path in a zero stratum. We prove the proposition by induction on the height  $r$  of the splitting unit  $\sigma$ . Suppose first that  $\sigma$  is an expanding splitting unit which is a maximal taken connecting path in a zero stratum of minimal height  $r$ . Then  $[f(\sigma)]$  has height  $r - 1$ , hence it does not contain splitting units which are maximal taken connecting path in zero strata. In this case, the proof follows from the above case. Suppose now that  $\sigma$  is a maximal taken connecting path in a zero stratum. Then its reduced image is completely split and has height at most  $r - 1$ . In this case the claim follows by induction applied to  $[f(\sigma)]$ . This concludes the proof of Assertion (1).

(2) Let us prove that for every element  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , we have:

- (i)  $0 \leq \sigma_{\gamma} < \infty$ ;
- (ii)  $\sigma_{\gamma} = \sigma_{\gamma-1}$ ;

(iii)  $\sigma_\gamma = \sum_{e \in E} \sigma_{\gamma e}$ , where  $E$  is the subset of  $\vec{E}G$  consisting in all edges that are incident to the endpoints of  $\gamma$  and distinct from the inverse of the last edge of  $\gamma$ .

The point (i) follows from Assertion (1). The second point follows from the definition of  $\langle \gamma, f^m(\sigma) \rangle$ . In order to prove the third point, remark that  $\langle \gamma, f^m(\sigma) \rangle$  and  $\sum_{e \in E} \langle \gamma e, f^m(\sigma) \rangle$  differ only when  $[f^m(\sigma)]$  ends with  $\gamma$  or  $\gamma^{-1}$ . Therefore the difference between  $\langle \gamma, f^m(\sigma) \rangle$  and  $\sum_{e \in E} \langle \gamma e, f^m(\sigma) \rangle$  is at most 2. This implies that

$$\left| \frac{\langle \gamma, f^m(\sigma) \rangle}{\ell_{\mathcal{F}}(f^m(\sigma))} - \sum_{e \in E} \frac{\langle \gamma e, f^m(\sigma) \rangle}{\ell_{\mathcal{F}}(f^m(\sigma))} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves the third point. By [Gue4, Lemma 3.2], since the map  $\gamma \mapsto \sigma_\gamma$  satisfies the conditions (i) – (iii), it determines a projective relative current  $n_\sigma \in \mathbb{P}\text{Curr}(F_n, \mathcal{F})$ . This current is unique since a relative current is entirely determined by its set of values on cylinders of finite paths  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  by Lemma 6.3.28. This concludes the proof.  $\square$

**Definition 6.4.5.** Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$  and let  $\mathcal{F}_1$  be a free factor system such that  $\mathcal{F} \leq \mathcal{F}_1$  and such that the extension  $\mathcal{F}_1 \leq \{F_n\}$  is sporadic and such that  $\phi|_{\mathcal{F}_1}$  is atoroidal relative to  $\mathcal{F}$ . In the case that  $\phi$  is atoroidal relative to  $\mathcal{F}$ , we assume that  $\mathcal{F}_1 = \{[F_n]\}$ . Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  with filtration

$$\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G,$$

such that there exists  $i \in \{1, \dots, k-1\}$  with  $\mathcal{F}(G_i) = \mathcal{F}_1$ . We define the *simplex of attraction* of  $\phi$ , denoted by  $\Delta_+(\phi)$ , as the set of projective classes of nonnegative linear combinations of currents  $\mu_\sigma$  obtained from Proposition 6.4.4 applied to  $\phi|_{\mathcal{F}_1}$  and  $f$  and which correspond to splitting units  $\sigma$  whose exponential length grows exponentially fast under iteration of  $f$ . The *simplex of repulsion* of  $\phi$ , denoted by  $\Delta_-(\phi)$ , is  $\Delta_+(\phi^{-1})$ .

**Remark 6.4.6.** *The definitions of attractive and repulsive currents given in Definition 6.4.5 rely on the choice of CT maps representing powers of the almost atoroidal outer automorphisms  $\phi$  and  $\phi^{-1}$ . However, it will be a consequence of Proposition 6.4.12 and Proposition 6.5.24 that the attractive and repulsive currents depend only on  $\phi$ .*

We now prove properties of the subsets  $\Delta_\pm(\phi)$ . As explained above Proposition 6.4.4, there are only finitely many expanding splitting units. Hence the subsets  $\Delta_\pm(\phi)$  are closed. Since  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  is a Hausdorff, compact space by Lemma 6.2.14 and since  $\Delta_\pm(\phi)$  are closed subsets, we have the following.

**Lemma 6.4.7.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an atoroidal outer automorphism relative to  $\mathcal{F}$ . The subsets  $\Delta_\pm(\phi)$  are compact and contain finitely many extremal points.  $\square$*

Note that one compute  $\|\mu(\sigma)\|_{\mathcal{F}}$  by counting the number of occurrences of every  $PG$ -relative splitting unit of positive  $\mathcal{F}$ -length in a reduced iterate of  $\sigma$  and taking the limit. This is precisely the limit of the  $\mathcal{F}$ -length of reduced iterates of  $\sigma$  by Lemma 6.3.20. Hence we have the following result.

**Lemma 6.4.8.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an atoroidal outer automorphism relative to  $\mathcal{F}$ . We have  $\|\mu(\sigma)\|_{\mathcal{F}} = 1$ .  $\square$*

We now prove that the subsets  $\Delta_{\pm}(\phi)$  are  $\phi$ -invariant. We first recall some lemmas.

**Lemma 6.4.9.** *[Coop, Bounded Cancellation] Let  $n \geq 2$  and let  $G$  be a marked graph of  $F_n$ . Let  $f: G \rightarrow G$  be a graph map. There exists a constant  $C_f$  such that for any reduced path  $\rho = \rho_1\rho_2$  in  $G$  we have*

$$\ell([f(\rho)]) \geq \ell([f(\rho_1)]) + \ell([f(\rho_2)]) - 2C_f.$$

**Lemma 6.4.10.** *[LU2, Lemma 5.7] For any graph  $G$  without valence 1 vertices there exists a constant  $K \geq 0$  such that for any finite reduced edge path  $\gamma$  in  $G$  there exists an edge path  $\gamma'$  of length at most  $K$  such that the concatenation  $\gamma\gamma'$  exists and is a reduced circuit.*

**Lemma 6.4.11.** *Let  $f: G \rightarrow G$  be as in Proposition 6.4.4. Let  $K_1 \geq 0$  be any constant, let  $\sigma$  be an expanding splitting unit and let  $\eta_{\sigma}$  be the current associated with  $\sigma$  given by Proposition 6.4.4 (2). Let  $m \in \mathbb{N}$  and let  $\gamma'_m$  be a reduced edge path of length at most  $K_1$ . Let  $\gamma_m = [f^m(\sigma)]^*\gamma'_m$ , where  $[f^m(\sigma)]^*$  is obtained from  $[f^m(\sigma)]$  by erasing an initial and a terminal subpath of length  $K_1$ . For every element  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , we have*

$$\lim_{m \rightarrow \infty} \frac{\langle \gamma, \gamma_m \rangle}{\ell_{\mathcal{F}}(\gamma_m)} = \langle \gamma, \eta_{\sigma} \rangle.$$

**Proof.** The proof follows [LU2, Lemma 5.8]. Since  $\ell(\gamma'_m) \leq K_1$ , we have

$$\ell_{\mathcal{F}}([f^m(\sigma)]^*) \geq \ell_{\mathcal{F}}([f^m(\sigma)]) - 2K_1.$$

Since  $\sigma$  is expanding, we have  $\lim_{m \rightarrow \infty} \ell_{\mathcal{F}}([f^m(\sigma)]) = +\infty$ . Hence we have

$$\lim_{m \rightarrow \infty} \frac{\langle \gamma, \gamma_m \rangle}{\langle \gamma, [f^m(\sigma)] \rangle} = 1$$

and

$$\lim_{m \rightarrow \infty} \frac{\ell_{\mathcal{F}}(\gamma_m)}{\ell_{\mathcal{F}}([f^m(\sigma)])} = 1.$$

Hence the result follows from Proposition 6.4.4 (1).  $\square$

**Proposition 6.4.12.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$ . Let  $f: G \rightarrow G$  be as in Proposition 6.4.4. Let  $\sigma$  be an expanding splitting unit and let  $\eta_{\sigma}$  be the current associated with  $\sigma$  given by Proposition 6.4.4 (2). There exists  $\lambda_{\sigma} > 1$  such that*

$$\phi(\eta_{\sigma}) = \lambda_{\sigma}\eta_{\sigma}.$$

**Proof.** The proof follows [LU2, Proposition 5.9]. Let  $K \geq 0$  be the constant associated with  $G$  given by Lemma 6.4.10. Let  $m \in \mathbb{N}$ , and let  $\gamma'_m$  be the path of length at most  $K$  given by Lemma 6.4.10 such that  $\gamma_m = [f^m(\sigma)]\gamma'_m$  is a reduced circuit. Since  $\lim_{t \rightarrow \infty} \ell_{exp}([f^t(\sigma)]) = +\infty$ , for large values of  $m$ , we have  $\ell_{exp}(\gamma_m) > 0$ . Let  $w_m$  be an element of  $F_n$  whose conjugacy class is represented by  $\gamma_m$ . Note that, by Lemma 6.3.26, we have  $\ell_{\mathcal{F}}(\gamma_m) = \|\eta_{w_m}\|_{\mathcal{F}}$ . By Proposition 6.3.13, since  $\ell_{exp}(\gamma_m) > 0$ , we see that  $w_m$  is  $\mathcal{F} \wedge \mathcal{A}(\phi)$ -nonperipheral, hence  $w_m$  defines a current  $\eta_{[w_m]} \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ .

Let  $\alpha_m = [f^{m+1}(\sigma)][f(\gamma'_m)]$ . Note that since  $\ell(\gamma'_m) \leq K$ , the value  $\ell([f(\gamma'_m)])$  is bounded by a constant  $K_0$  which only depends on  $K$ . Let  $C'$  be the constant given by Lemma 6.4.9 and let  $K_1 = \max\{K_0, C'\}$ . Then, with the notations of Lemma 6.4.11, the reduced circuit  $\gamma''_m = [\alpha_m]$  can be written as a product  $\gamma''_m = [f^m(\sigma)]^* \beta_m$  where  $\ell(\beta_m) \leq K_1$  and  $\ell_{\mathcal{F}}([f^m(\sigma)]^*) \geq \ell_{\mathcal{F}}([f^m(\sigma)]) - 2K_1$ . Applying Lemma 6.4.11 twice, we see that, for every element  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , we have

$$\lim_{m \rightarrow \infty} \frac{\langle \gamma, \gamma_m \rangle}{\ell_{\mathcal{F}}(\gamma_m)} = \langle \gamma, \eta_{\sigma} \rangle$$

and

$$\lim_{m \rightarrow \infty} \frac{\langle \gamma, \gamma''_m \rangle}{\ell_{\mathcal{F}}(\gamma''_m)} = \langle \gamma, \eta_{\sigma} \rangle.$$

By Lemma 6.3.29, we have

$$\lim_{m \rightarrow \infty} \frac{\eta_{[w_m]}}{\|\eta_{[w_m]}\|_{\mathcal{F}}} = \eta_{\sigma}.$$

From the continuity of the  $\text{Out}(F_n)$ -action on  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and from  $\phi(\eta_{[w_m]}) = \eta_{\phi([w_m])}$ , we see that

$$\lim_{m \rightarrow \infty} \frac{\eta_{\phi([w_m])}}{\|\eta_{\phi([w_m])}\|_{\mathcal{F}}} = \phi(\eta_{\sigma}).$$

Since the reduced circuit  $\gamma''_m$  represents the conjugacy class  $\phi([w_m])$ , the second of the above equalities implies that

$$\lim_{m \rightarrow \infty} \frac{\eta_{\phi([w_m])}}{\|\eta_{\phi([w_m])}\|_{\mathcal{F}}} = \eta_{\sigma}.$$

Recall that  $\lim_{m \rightarrow \infty} \frac{\ell_{\mathcal{F}}(\gamma_m)}{\ell_{\mathcal{F}}([f^m(\sigma)])} = 1$ , that  $\lim_{m \rightarrow \infty} \frac{\ell_{\mathcal{F}}(\gamma''_m)}{\ell_{\mathcal{F}}([f^{m+1}(\sigma)])} = 1$ , that  $\ell_{\mathcal{F}}(\gamma_m) = \|\eta_{[w_m]}\|_{\mathcal{F}}$  and that  $\ell_{\mathcal{F}}(\gamma''_m) = \|\eta_{\phi([w_m])}\|_{\mathcal{F}}$ . Recall from the claim in the proof of Proposition 6.4.4 that  $\ell_{\mathcal{F}}([f(\sigma)])$  is the norm of a matrix. By [LU1, Remark 3.3], there exists  $\lambda_{\sigma} \geq 1$  such that

$$\lim_{m \rightarrow \infty} \frac{\ell_{\mathcal{F}}([f^{m+1}(\sigma)])}{\ell_{\mathcal{F}}([f^m(\sigma)])} = \lambda_{\sigma}.$$

Since  $\sigma$  is expanding, we have in fact  $\lambda_{\sigma} > 1$  and this concludes the proof.  $\square$

We now prove a lemma which will be used in [Gue6].

**Lemma 6.4.13.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an expanding outer automorphism relative to  $\mathcal{F}$ . Let  $f: G \rightarrow G$  be as in Proposition 6.4.4. Let  $\sigma$  be an expanding splitting unit and let  $\eta_\sigma$  be the current associated with  $\sigma$  given by Proposition 6.4.4 (2).*

(1) *There exists a projective current  $[\eta] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  whose support is contained in the support of  $\eta_\sigma$  and such that  $\text{Supp}(\eta)$  is uniquely ergodic. In particular, the support of every extremal current of  $\Delta_\pm(\phi)$  contains a closed subset which is uniquely ergodic.*

(2) *There exists only finitely many projective currents  $[\eta] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  whose support is contained in the support of  $\eta_\sigma$  and such that  $\text{Supp}(\eta)$  is uniquely ergodic.*

**Proof.** (1) Note that, since  $\phi$  is expanding relative to  $\mathcal{F}$ , we have  $\mathcal{F} \wedge \mathcal{A}(\phi) = \mathcal{A}(\phi)$ . Let  $r \in \mathbb{N}$  be the minimal integer such that  $H_r$  is an EG stratum and a reduced iterate of  $\sigma$  contains an edge of  $H_r$ . Such a stratum  $H_r$  exists since  $\sigma$  is expanding. Let  $e$  be an edge of  $H_r$  with fixed initial direction and let  $\eta_e$  be the current in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A}(\phi))$  associated with  $e$  given by Proposition 6.4.4 (2).

**Claim.** The support of  $\eta_e$  is uniquely ergodic.

**Proof.** By minimality of  $r$ , every edge contained in a reduced iterate of  $e$  is either in  $H_r$  or in  $G'_{PG}$ . Let  $G'$  be the minimal subgraph of  $G$  which contains every reduced iterate of  $e$  and let  $A$  be a subgroup of  $F_n$  such that  $\pi_1(G')$  is a conjugate of  $A$  when  $\pi_1(G)$  is identified with  $F_n$ . Then  $G'$  is  $f$ -invariant and hence  $[A]$  is  $\phi$ -invariant. Let  $G'_1, \dots, G'_k$  be the connected component of  $\overline{G' - H_r}$  and let  $\mathcal{F}'$  be the free factor system of  $F_n$  determined by  $G'_1, \dots, G'_k$ . Let  $\Phi \in \phi$  be such that  $\Phi(A) = A$ . Note that  $[\Phi|_A] \in \text{Out}(A)$  is fully irreducible relative to  $\mathcal{F}'$ . For every  $i \in \{1, \dots, k\}$ , we have  $G'_i \subseteq G'_{PG}$ . By Proposition 6.2.5 (3), for every  $i \in \{1, \dots, k\}$ , either  $G'_i$  is contractible or  $G'_i \subseteq G_{PG}$ . By Proposition 6.3.13 for every subgroup  $H$  of  $F_n$  such that  $[H] \in \mathcal{F}'$ , there exists a subgroup  $H'$  of  $F_n$  such that  $[H'] \in \mathcal{A}(\phi)$  and  $H \subseteq H'$ . Hence we have  $\mathcal{F}' \leq \mathcal{A}(\phi)$ . Moreover by Proposition 6.3.13 and Proposition 6.2.5 (9), if  $\gamma$  is a cyclically reduced circuit of  $G'$  of height  $r$  whose growth under iteration of  $f$  is polynomial, then  $\gamma$  contains (up to taking inverse) the only height  $r$  EG INP  $\sigma_r$ . As one of the endpoints of  $\sigma_r$  is not contained in  $G_{r-1}$  by [HaM4, Fact I.1.42], we see that either  $\sigma_r$  is not closed and  $\gamma$  does not exist or  $\sigma_r$  is closed and  $\gamma$  is an iterate of  $\sigma_r$  or  $\sigma_r^{-1}$ . Let  $b \in F_n$  be the (possibly trivial) element associated with  $\sigma_r$ . Then, we have

$$\partial^2 A \cap \partial^2(F_n, \mathcal{A}(\phi)) = \partial^2(A, \mathcal{F}' \cup \{[b]\}).$$

Let  $\mathbb{P}\text{Curr}(\text{Supp}(\eta_e))$  be the set of projective currents in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  whose support is contained in  $\text{Supp}(\eta_e)$ . We now construct an injective application

$$\Theta: \mathbb{P}\text{Curr}(\text{Supp}(\eta_e)) \rightarrow \mathbb{P}\text{Curr}(A, \mathcal{F}')$$

such that for every projective current  $\mu \in \mathbb{P}\text{Curr}(\text{Supp}(\eta_e))$  we have

$$\text{Supp}(\Theta([\mu])) = \text{Supp}([\mu]) \cap \partial^2 A.$$

Let  $\mathcal{P}(\mathcal{F}')$  be the set of paths in a Cayley tree of  $F_{\mathbf{n}}$  defined above Lemma 6.3.28 (replacing  $\mathcal{F} \wedge \mathcal{A}(\phi)$  by  $\mathcal{F}'$ ). Let  $\mathcal{P}_A(\mathcal{F}')$  be the set of paths in  $\mathcal{P}(\mathcal{F}')$  contained in  $G'$ . By Lemma 6.3.28, the set consisting in elements  $C(\gamma)$  with  $\gamma \in \mathcal{P}(\mathcal{F}')$  covers  $\partial^2(A, \mathcal{F}')$ . Thus, by [Gue4, Lemma 3.2], it suffices to prove that for every projective current  $\eta \in \mathbb{P}\text{Curr}(\text{Supp}(\eta_e))$ , we can associate a function  $\tilde{\eta}: \mathcal{P}_A(\mathcal{F}') \rightarrow \mathbb{R}$  such that for every  $\gamma \in \mathcal{P}_A(\mathcal{F}')$ , we have

$$(i) \quad 0 \leq \tilde{\eta}(\gamma) < \infty;$$

$$(ii) \quad \tilde{\eta}(\gamma) = \sigma_{\gamma^{-1}};$$

(iii)  $\tilde{\eta}(\gamma) = \sum_{e \in E} \sigma_{\gamma e}$ , where  $E$  is the subset of  $\vec{E}G'$  consisting in all edges that are incident to the endpoints of  $\gamma$  and distinct from the inverse of the last edge of  $\gamma$ .

Let  $\eta \in \mathbb{P}\text{Curr}(\text{Supp}(\eta_e))$ . If  $\gamma \in \mathcal{P}_A(\mathcal{F}')$  is not contained in the axis of a conjugate of  $b$ , we may set  $\tilde{\eta}(\gamma) = \eta(C(\gamma))$ . Since  $\sigma_e$  is  $r$ -legal, a reduced iterate of  $\sigma_e$  cannot contain the only height  $r$  EG INP. Thus, we may set, for every path  $\gamma \in \mathcal{P}_A(\mathcal{F}')$  contained in the axis of a conjugate of  $b$ :  $\tilde{\eta}(\gamma) = 0$ . The function  $\tilde{\eta}$  satisfies conditions (i) – (iii) as  $\eta$  is a relative currents, hence it defines a unique current in  $\mathbb{P}\text{Curr}(A, \mathcal{F}')$ , which we still denote by  $\tilde{\eta}$ . Note that for every element  $\gamma \in \mathcal{P}_A(\mathcal{F}')$ , we have

$$\tilde{\eta}(C(\gamma) \cap \partial^2 A \cap \partial^2(F_{\mathbf{n}}, \mathcal{A}(\phi))) = \eta(C(\gamma) \cap \partial^2 A \cap \partial^2(F_{\mathbf{n}}, \mathcal{A}(\phi))),$$

so that the application  $\mathbb{P}\text{Curr}(\text{Supp}(\eta_e)) \rightarrow \mathbb{P}\text{Curr}(A, \mathcal{F}')$  is injective. Moreover, we have  $\text{Supp}(\tilde{\eta}) = \text{Supp}(\eta) \cap \partial^2 A$ .

Hence  $\eta_e$  defines a current  $\tilde{\eta}_e \in \mathbb{P}\text{Curr}(A, \mathcal{F}')$ . This current coincides with the attractive projective current associated with  $[\Phi|_A]$  defined by Gupta in [Gup1, Proposition 8.12]. By [Gup2, Lemma 4.17], the support of  $\tilde{\eta}_e$  is uniquely ergodic. Thus the support of  $\eta_e$  is uniquely ergodic.  $\square$

By the claim, it remains to prove that  $\text{Supp}(\eta_e) \subseteq \text{Supp}(\eta_\sigma)$ . But an element  $\eta \in \partial^2(F_{\mathbf{n}}, \mathcal{A}(\phi))$  is contained in the support of  $\eta_\sigma$  if for every element  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  such that  $\beta \in C(\gamma)$ , we have  $\eta_\sigma(C(\gamma)) > 0$ . Thus, the support of  $\eta_\sigma$  contains all the cylinder sets of the form  $C(\gamma)$  where  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  and  $\gamma$  is contained in a reduced iterate of  $\sigma$ . In particular, since  $e$  is contained in a reduced iterate of  $\sigma$ , we have  $\text{Supp}(\eta_e) \subseteq \text{Supp}(\eta_\sigma)$ . This proves Assertion (1).

(2) Suppose towards a contradiction that there exist infinitely many pairwise distinct projective currents  $([\eta_m])_{m \in \mathbb{N}} \in \mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$  whose support is contained in the support of  $\eta_\sigma$  and such that for every  $m \in \mathbb{N}$ , the support  $\text{Supp}(\eta_m)$  is uniquely ergodic. By compactness of  $\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$  (see Lemma 6.2.14) up to passing to a subsequence, there exists a projective current  $[\eta] \in \mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$  such that  $\lim_{m \rightarrow \infty} [\eta_m] = [\eta]$ . Let  $K \in \mathbb{N}^*$  be such that  $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  contains reduced edge paths of length equal to  $K$ . By additivity of  $\eta$ , there exists  $\gamma_1, \dots, \gamma_t \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  of length equal to  $K$  such that the support  $\text{Supp}(\eta)$  is contained in  $\bigcup_{j=1}^t C(\gamma_j)$  and for every  $j \in \{1, \dots, m\}$ , we have  $\eta(C(\gamma_j)) > 0$ . Then, there exists  $N \in \mathbb{N}^*$  such that, for every



$m \geq N$  and every  $j \in \{1, \dots, t\}$ , we have  $\eta_m(C(\gamma_j)) > 0$ . Hence for every  $m \geq N$ , we have

$$\text{Supp}(\eta) \subseteq \bigcup_{j=1}^t C(\gamma_j) \subseteq \text{Supp}(\eta_m).$$

By unique ergodicity, for every  $m \geq N$ , we have  $[\eta] = [\eta_m]$ , a contradiction.  $\square$

## 6.5 North-South dynamics for almost atoroidal outer automorphisms

Let  $X$  be a compact metric space and let  $G$  be a group acting on  $X$  by homeomorphisms. We say that an element  $g \in G$  acts on  $X$  with *generalized north-south dynamics* if the action of  $g$  on  $X$  has two invariant disjoint closed subsets  $\Delta_-$  and  $\Delta_+$  such that, for every open neighborhood  $U_{\pm}$  of  $\Delta_{\pm}$  and every compact set  $K_{\pm} \subseteq X - \Delta_{\mp}$ , there exists  $M > 0$  such that, for every  $n \geq M$ , we have

$$g^{\pm n} K_{\pm} \subseteq U_{\pm}.$$

In this section we prove the following theorem. Recall that a relative expanding outer automorphism is relative atoroidal, hence relative almost atoroidal.

**Theorem 6.5.1.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be a relative expanding outer automorphism. Let  $\Delta_+(\phi)$  and  $\Delta_-(\phi)$  be the simplexes of attraction and repulsion of  $\phi$ . Then  $\phi$  acts on  $\mathbb{P}\text{Curr}(F_n, \mathcal{F})$  with generalized north-south dynamics with respect to  $\Delta_+(\phi)$  and  $\Delta_-(\phi)$ .*

Theorem 6.1.2 in the introduction follows from Theorem 6.5.1 since every exponentially growing element of  $\text{Out}(F_n)$  is expanding relative to its polynomial part.

### 6.5.1 Relative exponential length and goodness

Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$ . In this section we define and prove the properties of the objects needed in order to prove Theorem 6.5.1. Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  with filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$  and let  $p \in \{1, \dots, k\}$  be such that  $\mathcal{F}(G_p) = \mathcal{F}$ . The proof of Theorem 6.5.1 relies on the study of  $PG$ -relative completely split edge paths. More precisely, given a reduced circuit  $\gamma$  of  $G$ , we study the proportion of subpaths of  $\gamma$  which have  $PG$ -relative complete splittings. This proportion will be measured using the exponential length. However, the lack of equality in Lemma 6.3.16 shows that the exponential length is not well-adapted to study the exponential length of a path by comparing it with the exponential length of its subpaths. Instead, we define a notion of *exponential length of a subpath relative to  $\gamma$* . We first need some preliminary results regarding splittings of edge paths.

**Definition 6.5.2.** Let  $\gamma$  be a reduced edge path in  $G$  and let  $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma_k\gamma'_k$  be the exponential decomposition of  $\gamma$  (see the beginning of Section 6.3.2). Let  $\alpha$  be a subpath of  $\gamma$ . The *exponential length of  $\alpha$  relative to  $\gamma$* , denoted by  $\ell_{exp}^\gamma(\alpha)$  is:

$$\ell_{exp}^\gamma(\alpha) = \sum_{i=1}^k \ell_{exp}(\alpha \cap \gamma'_i).$$

We define the  $\mathcal{F}$ -length of  $\alpha$  relative to  $\gamma$  similarly replacing  $\ell_{exp}$  by  $\ell_{\mathcal{F}}$  and the exponential decomposition by the  $\mathcal{F}$ -exponential decomposition.

Note that, for every reduced edge path  $\gamma$  of  $G$ , we have  $\ell_{exp}^\gamma(\gamma) = \ell_{exp}(\gamma)$ . The exponential length relative to a path  $\gamma$  is well-adapted to compute the exponential length of  $\gamma$  using its subpaths, as shown by the following lemma.

**Lemma 6.5.3.** *Let  $\gamma$  be a reduced edge path and let  $\gamma' = \alpha\beta \subseteq \gamma$  be a subpath of  $\gamma$ . Then*

$$\ell_{exp}^\gamma(\gamma') = \ell_{exp}^\gamma(\alpha) + \ell_{exp}^\gamma(\beta).$$

*In particular, when  $\gamma' = \gamma$ , we have*

$$\ell_{exp}(\gamma) = \ell_{exp}^\gamma(\alpha) + \ell_{exp}^\gamma(\beta).$$

*The same statement is true replacing  $\ell_{exp}^\gamma$  by  $\ell_{\mathcal{F}}^\gamma$ .*

**Proof.** The proof is similar for both  $\ell_{exp}^\gamma$  and  $\ell_{\mathcal{F}}^\gamma$ , so we only do the proof for  $\ell_{exp}^\gamma$ . Let  $\gamma = \gamma_0\gamma'_1\gamma_1 \dots \gamma_k\gamma'_k$  be the exponential decomposition of  $\gamma$ . Then, for every  $i \in \{1, \dots, k\}$ , the paths  $\alpha \cap \gamma'_i$  and  $\beta \cap \gamma'_i$  do not contain a subpath of a path in  $\mathcal{N}_{PG}^{\max}(\gamma)$ . In particular, for every  $i \in \{1, \dots, k\}$ , one computes  $\ell_{exp}(\alpha \cap \gamma'_i)$  and  $\ell_{exp}(\beta \cap \gamma'_i)$  by removing edges from  $G'_{PG}$ . Since  $\ell_{exp}^\gamma(\gamma')$  is computed by removing edges in  $G'_{PG}$  from every  $\gamma'_i$  with  $i \in \{1, \dots, k\}$ , the proof follows.  $\square$

In Lemma 6.5.6, we will show that if  $\gamma$  is a reduced edge path in  $G$  and that  $\alpha$  is a subpath of  $\gamma$ , then  $\ell_{exp}(\alpha)$  and  $\ell_{exp}^\gamma(\alpha)$  differ by a uniform additive constant. This will allow us to compute directly  $\ell_{exp}(\alpha)$  rather than  $\ell_{exp}^\gamma(\alpha)$ .

Let  $\gamma$  be a reduced edge path in  $G$  and let  $\gamma = \gamma_1 \dots \gamma_m$  be a splitting of  $\gamma$ . Let  $J_{CS,PG} \subseteq \{\gamma_1, \dots, \gamma_m\}$  be the subset consisting in all subpaths which have a  $PG$ -relative complete splitting. If  $\ell_{exp}(\gamma) > 0$ , let

$$\mathfrak{g}_{CT,PG}(\gamma; \gamma_1, \dots, \gamma_m) = \frac{\sum_{\gamma_i \in J_{CS,PG}} \ell_{exp}^\gamma(\gamma_i)}{\ell_{exp}(\gamma)}.$$

The *goodness of  $\gamma$* , denoted by  $\mathfrak{g}(\gamma)$ , is the least upperbound of  $\mathfrak{g}_{CT,PG}(\gamma)$  over all splittings of  $\gamma$  if  $\ell_{exp}(\gamma) > 0$ , and is equal to 0 otherwise. When  $\gamma$  is a circuit, the value  $\mathfrak{g}_{CT,PG}(\gamma)$  is defined using only circuitual splittings.

Since there are only finitely many decompositions of a finite edge path into subpaths, the value  $\mathfrak{g}(\gamma)$  is realized for some splitting of  $\gamma$ . A splitting for which  $\mathfrak{g}(\gamma)$  is realized is called an *optimal splitting* of  $\gamma$ , and an *optimal circuitual splitting* when  $\gamma$  is a circuit.

A subpath of  $\gamma$  which is the concatenation of consecutive splitting units of an optimal splitting of  $\gamma$  is called a *factor* of  $\gamma$ . When  $\ell_{exp}(\gamma) = 0$ , we use the convention that the only factor of  $\gamma$  is  $\gamma$  itself. The factors of  $\gamma$  that admit a  $PG$ -relative complete splitting are called *complete factors*. The factors in an optimal splitting which do not admit  $PG$ -relative complete splittings are said to be *incomplete*. Remark that, by Proposition 6.2.5 (6), (8) and by Lemma 6.3.9, the  $[f]$ -image of a  $PG$ -relative complete path is  $PG$ -relative complete, and the reduced iterates of an incomplete factor are eventually  $PG$ -relative complete.

Using Lemma 6.5.3, we have the following result.

**Lemma 6.5.4.** *Let  $\gamma$  be a reduced edge path and let  $\gamma = \gamma'_0\gamma_1\gamma'_1\dots\gamma_m\gamma'_m$  be an optimal splitting of  $\gamma$ , where, for every  $i \in \{0, \dots, m\}$ , the path  $\gamma'_i$  is an incomplete factor of  $\gamma$  and, for every  $i \in \{1, \dots, m\}$ , the path  $\gamma_i$  is complete. Then*

$$\mathfrak{g}(\gamma) = \frac{\sum_{i=1}^m \ell_{exp}^\gamma(\gamma_i)}{\sum_{i=1}^m \ell_{exp}^\gamma(\gamma_i) + \sum_{j=0}^m \ell_{exp}^\gamma(\gamma'_j)}. \quad \square$$

**Definition 6.5.5.** Let  $\mathfrak{n}, \mathcal{F}, \phi, f, p$  be as in the beginning of Section 6.5.1. Let  $K \geq 1$ . The CT map  $f$  is  $3K$ -expanding if for every edge  $e$  of  $G - G'_{PG}$ , we have

$$\ell_{exp}([f(e)]) \geq 3K.$$

Note that, by Lemma 6.3.21, for every  $K \geq 1$ , the CT map  $f$  has a power which is  $3K$ -expanding. Note that, since  $\phi$  is exponentially growing, we have  $G \neq G'_{PG}$ , so that the definition of  $3K$ -expanding is not empty.

In the rest of the section, let  $K \geq 1$  be a constant such that, for every reduced edge path  $\sigma$  which is either in  $\mathcal{N}_{PG}$  or a path in a zero stratum, we have  $\ell(\sigma) \leq \frac{K}{2}$ . Such a  $K$  exists since  $\mathcal{N}_{PG}$  is finite by Lemma 6.3.4 (1) and since every zero stratum is contractible by Proposition 6.2.5 (3). We fix a constant  $C_f$  given by Lemma 6.4.9. Let

$$C = \max\{K, C_f\}. \quad (6.7)$$

Recall that, if  $\sigma$  is a  $PG$ -relative splitting unit,  $\sigma$  is either an edge in an irreducible stratum, a path in a zero stratum or a path in  $\mathcal{N}_{PG}$ . Thus, the choice of  $K$  implies that for every  $PG$ -relative splitting unit  $\sigma$ , we have  $\ell_{exp}(\sigma) \leq \frac{K}{2}$ .

**Lemma 6.5.6.** *Let  $\gamma$  be a reduced edge path in  $G$  and let  $\gamma'$  be a subpath of  $\gamma$ . Let  $\gamma = \gamma_0\gamma'_1\gamma_1\dots\gamma_k\gamma'_k$  be the exponential decomposition of  $\gamma$ . There exist three (possibly empty) subpaths  $\delta_1, \delta_2$  and  $\tau$  of  $\gamma$  such that for every  $i \in \{1, 2\}$ , the path  $\delta_i$  is a proper subpath of a splitting unit of some  $\gamma_j$ , we have  $\ell_{exp}(\tau) = \ell_{exp}^\gamma(\tau) = \ell_{exp}^\gamma(\gamma')$  and  $\gamma' = \delta_1\tau\delta_2$ . In particular, we have*

$$\ell_{exp}^\gamma(\gamma') \leq \ell_{exp}(\gamma') \leq \ell_{exp}^\gamma(\gamma') + 2C \leq \ell_{exp}(\gamma) + 2C.$$

The same statement is true replacing  $\ell_{exp}$  by  $\ell_{\mathcal{F}}$  and  $\ell_{exp}^\gamma$  by  $\ell_{\mathcal{F}}^\gamma$ .

**Proof.** The proof is similar for both  $\ell_{exp}$  and  $\ell_{\mathcal{F}}$ , so we only do the proof for  $\ell_{exp}$ . Since  $\gamma'$  is a subpath of  $\gamma$ , there exist three (possibly trivial) paths  $\delta'_1$ ,  $\tau'$  and  $\delta'_2$  such that:

- (a) for every  $i \in \{1, 2\}$ , there exists  $k_i \in \{0, \dots, k\}$  such that the path  $\delta'_i$  is a subpath of some  $\gamma_{k_i}$ ;
- (b) for every  $j \in \{0, \dots, k\}$ , either  $\gamma_j$  is contained in  $\tau'$  or  $\gamma_j$  does not contain edges of  $\tau'$ ;
- (c) we have  $\gamma' = \delta'_1 \tau' \delta'_2$ .

The path  $\delta'_1$  has a decomposition  $\delta'_1 = \delta_1 f_1$ , where  $f_1$  is a (possibly trivial) factor of  $\gamma_{k_1}$  and  $\delta_1$  is properly contained in a splitting unit of  $\gamma_{k_1}$  for some fixed choice of optimal splitting of  $\gamma_{k_1}$ . Similarly, the path  $\delta'_2$  has a decomposition  $\delta'_2 = f_2 \delta_2$ , where  $f_2$  is a (possibly trivial) factor of  $\gamma_{k_2}$  and  $\delta_2$  is properly contained in a splitting unit of  $\gamma_{k_2}$  for some fixed choice of optimal splitting of  $\gamma_{k_2}$ . Let  $\tau = f_1 \tau' f_2$ . Then  $\gamma' = \delta_1 \tau \delta_2$ . It remains to show that  $\ell_{exp}(\tau) = \ell'_{exp}(\tau) = \ell'_{exp}(\gamma')$ . Since for every  $i \in \{1, 2\}$ , the path  $f_i$  is a path in  $\mathcal{N}_{PG}$ , we have  $\ell_{exp}(\tau) = \ell_{exp}(\tau')$ . By (b), one obtains  $\ell_{exp}(\gamma')$  by deleting edges in  $G'_{PG}$  and every path of  $\mathcal{N}_{PG}^{\max}(\gamma)$  contained in  $\tau'$ . Hence we have

$$\ell'_{exp}(\tau') = \sum_{i=1}^k \ell_{exp}(\tau' \cap \gamma'_i) = \sum_{i=1}^k \ell_{exp}(\tau \cap \gamma'_i) = \ell'_{exp}(\tau).$$

Since  $\delta_1$  and  $\delta_2$  are contained in paths of  $\mathcal{N}_{PG}^{\max}(\gamma)$ , we have  $\ell'_{exp}(\gamma') = \ell'_{exp}(\tau)$ , that is, the second equality holds.

We now prove the final inequalities in the lemma. The first inequality follows from the fact that every path in  $\mathcal{N}_{PG}^{\max}(\gamma')$  is a subpath of some  $\gamma_i$  for  $i \in \{0, \dots, k\}$ . Thus, we have  $\ell'_{exp}(\gamma') \leq \ell_{exp}(\gamma')$ . By Lemma 6.3.16, we have

$$\ell_{exp}(\gamma') \leq \ell_{exp}(\delta_1) + \ell_{exp}(\tau) + \ell_{exp}(\delta_2) \leq \ell'_{exp}(\gamma') + \ell(\delta_1) + \ell(\delta_2).$$

By definition of the constant  $K$  and the fact that  $K \leq C$ , we have:

$$\ell'_{exp}(\gamma') + \ell(\delta_1) + \ell(\delta_2) \leq \ell'_{exp}(\gamma') + 2C \leq \ell_{exp}(\gamma) + 2C,$$

where the last inequality follows from Lemma 6.5.3. □

**Lemma 6.5.7.** *Let  $f: G \rightarrow G$  be a  $3K$ -expanding CT map. Let  $\gamma$  be a PG-relative completely split edge path of positive exponential length. Then*

$$\ell_{exp}([f(\gamma)]) \geq 3 \ell_{exp}(\gamma).$$

**Proof.** Consider a PG-relative complete splitting  $\gamma = \gamma'_0 \gamma_1 \gamma'_1 \dots \gamma_m \gamma'_m$  of  $\gamma$ , where, for every  $i \in \{0, \dots, m\}$ , the path  $\gamma'_i$  is either a (possibly trivial) concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or a (possibly trivial) reduced maximal taken connecting path in a zero stratum and, for every  $i \in \{1, \dots, m\}$ , the path  $\gamma_i$  is an edge in an irreducible stratum of positive exponential length. By Lemma 6.3.23, we have

$$\ell_{exp}(\gamma) = \sum_{i=1}^m \ell(\gamma_i).$$

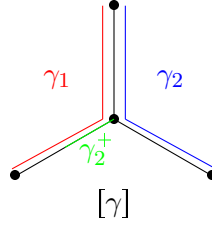


Figure 6.2: Illustration of Lemma 6.5.8. If a complete factor of  $\gamma_1$  contained in  $[\gamma]$  is not contained in  $\gamma_2^+$ , then it is a complete factor of  $[\gamma]$ .

Since  $f$  is  $3K$ -expanding, for every  $i \in \{1, \dots, m\}$ , we have

$$\ell_{exp}([f(\gamma_i)]) \geq 3K \ell_{exp}(\gamma_i).$$

Since the reduced image of a  $PG$ -relative complete splitting is a  $PG$ -relative complete splitting by Lemma 6.3.9, by Lemma 6.3.20 (2), we see that

$$\ell_{exp}([f(\gamma)]) \geq \sum_{i=1}^m \ell_{exp}([f(\gamma_i)]) \geq \sum_{i=1}^m 3K \ell_{exp}(\gamma_i) \geq 3 \ell_{exp}(\gamma).$$

This concludes the proof.  $\square$

**Lemma 6.5.8.** *Let  $f: G \rightarrow G$  be a  $3K$ -expanding CT map. Let  $\gamma = \gamma_1 \gamma_2$  be a (not necessarily reduced) edge path of positive exponential length, where  $\gamma_1$  and  $\gamma_2$  are reduced edge paths. Let  $\gamma_1 = a_1 b_1 \dots a_k b_k$  be an optimal splitting of  $\gamma_1$  where for every  $i \in \{1, \dots, k\}$ , the path  $a_i$  is an incomplete factor and for every  $i \in \{1, \dots, k\}$  the path  $b_i$  is complete. For every  $i \in \{1, 2\}$ , let  $\gamma'_i$  be the subpath of  $\gamma_i$  contained in  $[\gamma]$ . Let  $\gamma'_1 = \gamma_1^- \gamma_1^+$  be a decomposition of  $\gamma'_1$  into two subpaths where  $\gamma_1^+$  is the maximal terminal segment of  $\gamma'_1$  such that  $\sum_{i=1}^k \ell_{exp}(\gamma_1^+ \cap b_i) = 2C$ . Then every  $PG$ -relative complete factor  $b'$  of  $\gamma_1$  contained in  $\gamma_1^-$  (for the given optimal splitting) is also a  $PG$ -relative complete factor of  $[\gamma]$ .*

**Remark 6.5.9.** (1) *We emphasize that, in the statement of Lemma 6.5.8, if the path  $\gamma_1$  is  $PG$ -relative completely split, the path  $\gamma'_1$  is not necessarily  $PG$ -relative completely split. Indeed, there might be some identification with the path  $\gamma_2$  that might create incomplete factors in  $\gamma'_1$ .*

(2) *Lemma 6.5.8 also implies that if  $\gamma_1$  is  $PG$ -relative completely split, the intersection of an incomplete factor of  $[\gamma]$  with  $\gamma'_1$  is contained in a terminal segment of  $\gamma'_1$  of exponential length at most equal to  $2C$  (see Figure 6.2). Indeed, the claim in the proof of Lemma 6.5.8 shows that the path  $\gamma_1^-$  is a complete factor of  $\gamma_1$ , hence a complete factor of  $[\gamma]$  by Lemma 6.5.8. Moreover, we have  $k = 1$ ,  $a_1$  is trivial and  $\ell_{exp}(\gamma_1^+) = \ell_{exp}(\gamma_1^+ \cap b_1)$ .*

**Proof.** Let  $t \in \{1, \dots, k\}$  be the minimal integer such that  $\gamma_1^-$  is contained in  $\delta' = a_1 b_1 \dots a_t b_t$ . Let  $b_t = \delta_1 \dots \delta_{s'}$  be a  $PG$ -relative complete splitting of  $b_t$ . Let  $s \in$

$\{1, \dots, s'\}$  be the minimal integer such that  $\gamma_1^-$  is contained in  $\delta = a_1 b_1 \dots a_t \delta_1 \dots \delta_s$ . The integer  $s$  exists since, by maximality of  $\gamma_1^+$ , for every  $i \in \{1, \dots, k\}$ , either  $\gamma_1^+ \cap a_i = a_i$  or  $\gamma_1^+ \cap a_i = \emptyset$ .

**Claim.** We have  $\delta = \gamma_1^-$ .

**Proof.** By minimality of  $t$  and  $s$ , the path  $\gamma_1^-$  contains an edge of  $\delta_s$ . We claim that  $\delta_s$  is contained in  $\gamma_1'$ . Indeed, it is clear if  $\delta_s$  is an edge. Suppose towards a contradiction that  $\delta_s$  is not contained in  $\gamma_1'$ . Then the concatenation point of  $\gamma_1'$  and  $\gamma_2'$  is contained in  $\delta_s$ . If  $\delta_s$  is a maximal taken connecting path in a zero stratum, then, by the choice of  $K$ , we have  $\ell(\delta_s) \leq \frac{K}{2} \leq \frac{C}{2}$ . Since  $\ell(\gamma_1^+) \geq 2C$ , the path  $\delta_s \cap \gamma_1'$  would be contained in  $\gamma_1^+$ , contradicting the fact that  $\gamma_1^-$  contains the first edge of  $\delta_s$ . Suppose that  $\delta_s$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Then  $\delta_s$  has a decomposition  $\delta_s = \beta_1^{(s)} \alpha_1^{(s)} \beta_1^{(s)} \dots \alpha_{k_s-1}^{(s)} \beta_{k_s}^{(s)} \alpha_{k_s}^{(s)}$ , where for every  $i \in \{1, \dots, k_s\}$ , the path  $\beta_i^{(s)}$  is contained in  $G_{PG}$ , for every  $i \in \{1, \dots, k_s - 1\}$ , the path  $\alpha_i^{(s)}$  is contained in  $\mathcal{N}_{PG}^{\max}(\sigma)$  and  $\alpha_{k_s}^{(s)}$  is a subpath of a path in  $\mathcal{N}_{PG}^{\max}(\delta_s)$ . By the choice of  $K$ , we have  $\ell_{exp}(\delta_s) \leq \ell(\alpha_{k_s}) \leq \frac{K}{2} \leq \frac{C}{2}$ . Since  $\ell_{exp}(\gamma_1^+) \geq 2C$ , the path  $\delta_s \cap \gamma_1'$  would be contained in  $\gamma_1^+$ , contradicting the fact that  $\gamma_1^-$  contains the first edge of  $\delta_s$ . Hence, in every case, the path  $\delta_s$  is contained in  $\gamma_1'$ . Note that, since  $\gamma_1^+$  is the maximal subpath of  $\gamma_1'$  for the property that  $\sum_{i=1}^k \ell_{exp}(\gamma_1^+ \cap b_i) = 2C$ , the  $PG$ -relative splitting unit  $\delta_s$  is not a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or a maximal taken connecting path in a zero stratum. Indeed, otherwise it is properly contained in  $\gamma_1^+$ , contradicting the fact that  $\gamma_1^-$  intersects  $\delta_s$ . Hence  $\delta_s$  is an edge and  $\delta = \gamma_1^-$ .  $\square$

By the claim, we see that  $\gamma_1^- = a_1 b_1 \dots a_t \delta_1 \dots \delta_s$  is an optimal splitting of  $\gamma_1^-$ . Let  $r \in \{1, \dots, k\}$  be the minimal integer such that  $\gamma_1'$  is contained in  $a_1 b_1 \dots a_r b_r$ . The last edge of  $\gamma_1'$  is either contained in  $a_r$  or in  $b_r$ . In the first case, for every  $i \in \{1, \dots, k\}$ , either  $b_i$  is contained in  $\gamma_1'$  or  $b_i \cap \gamma_1'$  is at most a point. In the second case, it is possible that  $b_r \cap \gamma_1' \neq b_r$  and that  $b_r \cap \gamma_1'$  contains an edge. Let  $\alpha'$  be the (possibly trivial) terminal segment of  $\gamma_1^+$  which is properly contained in a splitting unit  $\sigma$  of  $b_r$ . If  $\sigma$  is a maximal taken connecting path in a zero stratum, then, by the choice of  $K$ , we have  $\ell_{exp}(\alpha') \leq \ell(\alpha') \leq \ell(\sigma) \leq \frac{K}{2} \leq \frac{C}{2}$ . Suppose that  $\sigma$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Then  $\alpha'$  has a decomposition  $\alpha' = \beta_1 \alpha_1 \beta_1 \dots \alpha_{\ell-1} \beta_{\ell} \alpha_{\ell}$ , where for every  $i \in \{1, \dots, \ell\}$ , the path  $\beta_i$  is contained in  $G_{PG}$ , for every  $i \in \{1, \dots, \ell - 1\}$ , the path  $\alpha_i$  is contained in  $\mathcal{N}_{PG}^{\max}(\sigma)$  and  $\alpha_{\ell}$  is a subpath of a path in  $\mathcal{N}_{PG}^{\max}(\sigma)$ . By the choice of  $K$ , we have  $\ell_{exp}(\alpha') \leq \ell(\alpha_{\ell}) \leq \frac{K}{2} \leq \frac{C}{2}$ . Since  $\ell_{exp}(\gamma_1^+) \geq 2C$ , there exists a  $PG$ -relative complete factor  $\alpha_0$  of  $b_r$  such that  $\gamma_1^+ = \delta_{s+1} \dots \delta_{s'} a_{t+1} b_{t+1} \dots a_r \alpha_0 \alpha' = \alpha \alpha'$  and

$$\sum_{i=1}^k \ell_{exp}(\alpha \cap b_i) \geq C.$$

We now prove that every  $PG$ -relative complete factor of  $\gamma_1$  contained in  $\gamma_1^-$  is a  $PG$ -relative complete factor of  $\gamma$ . Note that the decomposition  $\gamma_1^- \alpha$  is a splitting. Thus, it suffices to prove that, for every  $k \in \mathbb{N}^*$ , the path  $[f^k(\gamma_1^-)]$  is contained in  $[f^k(\gamma)]$  as any

identification in order to obtain  $[f^k(\gamma)]$  which involves a path in  $f^k(\gamma_1^-)$  will be induced by an identification in order to obtain  $[f^k(\gamma_1^-)]$  from  $f^k(\gamma_1^-)$ . By Lemma 6.5.7 applied to  $\delta_{s+1}, \dots, \delta_{s'}$ , to the paths  $b_i$  with  $i \in \{1, \dots, k\}$  such that  $b_i \subseteq \alpha$  and to  $\alpha_0$ , we have

$$\begin{aligned} \sum_{i=1}^k \ell_{exp}([f(\alpha)] \cap [f(b_i)]) &\geq \sum_{i=s+1}^{s'} \ell_{exp}([f(\delta_i)]) + \sum_{i=t+1}^{r-1} \ell_{exp}([f(b_i)]) + \ell_{exp}([f(\alpha_0)]) \\ &\geq 3 \sum_{i=1}^k \ell_{exp}(\alpha \cap b_i) \geq 3C, \end{aligned}$$

where the first inequality follows from the fact that the decomposition

$$\alpha = \delta_{s+1} \dots \delta_{s'} a_{t+1} b_{t+1} \dots a_r \alpha_0$$

is an optimal splitting of  $\alpha$ . Note that, since the decomposition  $\gamma_1^- \alpha$  is a splitting, for every  $k \in \mathbb{N}^*$ , the path  $[f^k(\alpha)]$  is contained in  $[f^k(\gamma_1^- \alpha)]$ . Remark that Lemma 6.4.9 implies that the segment of  $[f(\gamma_1^- \alpha)]$  which is  $C$  away from the concatenation point between  $[f(\gamma_1^- \alpha)]$  and  $[f(\alpha' \gamma_2')]$  remains in  $[f([\gamma])]$ . In particular, the edges of  $[f(\gamma_1^- \alpha)]$  which are cancelled with edges of  $[f(\alpha' \gamma_2')]$  are contained in  $[f(\alpha)]$ . Recall that  $\sum_{i=1}^k \ell_{exp}([f(\alpha)] \cap [f(b_i)]) \geq 3C$  and that the subpath of  $[f(\alpha)]$  which is contained in  $[f([\gamma])]$  is obtained by the concatenation of at most  $C$  edges of  $[f(\alpha)]$ . Thus, we see that the sum over  $i$  of the exponential length of the subpaths of  $[f(\alpha)] \cap [f(b_i)]$  which are contained in  $[f([\gamma])]$  is at least equal to  $2C$ . Hence the path  $[f(\gamma_1^-)]$  is a subpath of  $[f([\gamma])]$  and  $\sum_{i=1}^k \ell_{exp}([f(\gamma_1^-)] \cap [f(b_i)] \cap [f([\gamma])]) \geq 2C$ . Thus, we can apply the same arguments to show that for every  $k \geq 1$ , the path  $[f^k(\gamma_1^-)]$  is contained in  $[f^k([\gamma])]$  and the exponential length of the subpath of  $[f^k(\alpha)]$  contained in  $[f^k([\gamma])]$  is at least equal to  $2C$ . Hence every  $PG$ -relative complete factor of the path  $\gamma_1$  contained in  $\gamma_1^-$  is a complete factor of an optimal splitting of  $[\gamma]$ .  $\square$

**Lemma 6.5.10.** (1) *Let  $\gamma = \alpha\beta$  be a reduced path. Let  $N \in \mathbb{N}^*$  be such that  $[f^N(\alpha)]$  has a  $PG$ -relative complete splitting and that  $[f^N(\beta)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . For every  $m \geq N$ , let  $\alpha_m, \beta_m$  and  $\sigma_m$  be paths such that  $[f^m(\alpha)] = \alpha_m \sigma_m$  and  $[f^m(\beta)] = \sigma_m^{-1} \beta_m$ . For every  $m \geq N$ , we have  $\ell_{exp}(\sigma_m) \leq C$ ,  $\ell_{exp}(\alpha_m) \geq \ell_{exp}([f^m(\alpha)]) - C$  and  $\ell_{exp}(\beta_m) \leq C$ .*

(2) *Let  $\gamma = \beta^{(1)} \alpha \beta^{(2)}$  be a reduced path. Let  $N \in \mathbb{N}^*$  be such that  $[f^N(\alpha)]$  has a  $PG$ -relative complete splitting and, for every  $i \in \{1, 2\}$ , the path  $[f^N(\beta^{(i)})]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . For every  $m \geq N$ , let  $\alpha_m, \beta_m^{(1)}, \beta_m^{(2)}$ , and  $\sigma_m^{(1)}, \sigma_m^{(2)}$  be paths such that  $[f^m(\alpha)] = \sigma_m^{(1)} \alpha_m \sigma_m^{(2)}$ ,  $[f^m(\beta^{(1)})] = \beta_m^{(1)} \sigma_m^{(1)-1}$  and  $[f^m(\beta^{(2)})] = \sigma_m^{(2)-1} \beta_m^{(2)}$ . For every  $m \geq N$ , either  $\ell_{exp}(\alpha_m) \leq 2C$  or we have  $\ell_{exp}(\sigma_m^{(1)}, \ell_{exp}(\sigma_m^{(2)}) \leq C$ ,  $\ell_{exp}(\alpha_m) \geq \ell_{exp}([f^m(\alpha)]) - 2C$  and  $\ell_{exp}(\beta_m^{(1)}, \ell_{exp}(\beta_m^{(2)}) \leq C$ .*

**Proof.** The proof of Assertion (2) follows from Assertion (1) by applying Assertion (1) twice: one with  $\gamma = \alpha\beta^{(2)}$  and one with  $\gamma = \alpha^{-1}\beta^{(1)}$ . If for some  $m \in \mathbb{N}^*$ ,  $\ell_{exp}(\alpha_m) \geq 2C$ , there is no identification between  $[f^m(\beta^{(1)})]$  and  $[f^m(\beta^{(2)})]$ , so Assertion (2) follows from Assertion (1). Therefore, we focus on the proof of Assertion (1). Let  $m \geq N$ . When  $\sigma_m$  is reduced to a point, we have  $\ell_{exp}(\alpha_m) = \ell_{exp}([f^m(\alpha)])$  and  $\ell_{exp}(\beta_m) = \ell_{exp}([f^m(\beta)]) = 0$

by Lemma 6.3.17. This concludes the proof in this case. So we may suppose that  $\sigma_m$  is nontrivial. Let  $[f^m(\alpha)] = a_1 \dots a_k$  be a  $PG$ -relative complete splitting of  $[f^m(\alpha)]$ . Suppose that, for every  $i \in \{1, \dots, k\}$  such that  $a_i$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ , the path  $a_i$  is a maximal subpath of  $[f^m(\alpha)]$  for the property of being a factor which is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . For every  $j \in \{1, \dots, k\}$ , let  $r_j$  be the height of  $a_j$ . Let  $i \in \{1, \dots, k\}$  be such that  $a_i$  contains the first edge of  $\sigma_m$ . Let  $\sigma' \in \mathcal{N}_{PG}^{\max}(\sigma_m)$ . Note that there exists  $\sigma'' \in \mathcal{N}_{PG}^{\max}([f^m(\alpha)])$  such that  $\sigma' \subseteq \sigma''$ . By Lemma 6.3.20 (1) applied to  $\sigma''$  and  $[f^m(\alpha)]$ , the path  $\sigma''$  is contained in a factor which is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . By the maximality assumption, there exists  $j \in \{1, \dots, k\}$  such that  $\sigma' \subseteq \sigma'' \subseteq a_j$ . Hence we can compute  $\ell_{exp}(\sigma_m)$  by removing, for every  $j \in \{1, \dots, k\}$ , paths in the intersection  $\sigma_m \cap a_j$ . Thus, we have,

$$\ell_{exp}(\sigma_m) = \sum_{j>i} \ell_{exp}(a_j) + \ell_{exp}(a_i \cap \sigma_m).$$

Note that, by Lemma 6.3.9, the path  $[f^m(\beta)] = \sigma_m^{-1} \beta_m$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $j \in \{i, \dots, k\}$ .

**Claim.** If  $j > i$ , then either  $a_j$  is not an edge in an EG stratum and  $\ell_{exp}(a_j \cap \sigma_m) = 0$ , or  $\ell_{exp}((a_i \dots a_j) \cap \sigma_m) \leq C$ . If  $j = i$ , then  $\ell_{exp}(a_j \cap \sigma_m) \leq C$ .

**Proof.** We distinguish several cases, according to the nature of  $a_j$ .

(i) Suppose that  $a_j$  is maximal taken connecting path in a zero stratum. By definition we have  $\ell_{exp}(a_j \cap \sigma_m) = 0$ .

(ii) Suppose that  $a_j$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . If  $j > i$ , we have  $a_j \cap \sigma_m = a_j$ . By Lemma 6.3.17 applied to  $a_j$ , we have  $\ell_{exp}(a_j \cap \sigma_m) = 0$ . Suppose that  $i = j$ . Suppose that the first edge of  $\sigma_m$  is not contained in a path in  $\mathcal{N}_{PG}^{\max}(a_i)$ . Then  $a_i$  has a decomposition  $a_i = a_i^0 a_i^1 a_i^2$  where  $a_i^1$  is a path contained in  $G_{PG}$  such that the first edge of  $\sigma_m$  is contained in  $a_i^1$  and such that, for every path  $\delta \in \mathcal{N}_{PG}^{\max}(a_i)$ , either  $\delta \subseteq a_i^0$  or  $\delta \subseteq a_i^2$ . Note that a terminal segment of  $a_i$  whose first edge is contained in  $a_i^1$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . In particular, the path  $a_i \cap \sigma_m$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.17 applied to  $a_i \cap \sigma_m$ , we have  $\ell_{exp}(a_i \cap \sigma_m) = 0$ . Suppose now that the first edge of  $\sigma_m$  is contained in a path  $\delta \in \mathcal{N}_{PG}^{\max}(a_i)$ . Then  $a_i$  has a decomposition  $a_i^1 \delta a_i^2$ , where the first edge of  $\sigma_m$  is contained in  $\delta$ . Note that  $a_i^2$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  which is contained in  $\sigma_m$ . By Lemma 6.3.16 applied to  $a_i \cap \sigma_m = (\delta \cap \sigma_m) a_i^2$ , by Lemma 6.3.17 applied to  $a_i^2$  and by definition of the constant  $K$ , we have

$$\ell_{exp}(\sigma_m \cap a_i) \leq \ell_{exp}(\sigma_m \cap \delta) + \ell_{exp}(a_i^2) = \ell_{exp}(\delta \cap \sigma_m) \leq \ell(\delta) \leq K \leq C.$$

(iii) Suppose that  $a_j$  is an edge in an irreducible stratum with positive exponential length. Since  $[f^m(\beta)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , there exists a path  $\gamma' \in \mathcal{N}_{PG}^{\max}([f^m(\beta)])$  such that  $a_j$  is contained in  $\gamma'$ . By Lemma 6.3.20 (1), every path in  $\mathcal{N}_{PG}^{\max}([f^m(\alpha)])$  is contained in a minimal factor of  $[f^m(\alpha)]$  consisting in  $PG$ -relative splitting units which are concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Since  $a_j$  is a  $PG$ -relative splitting unit of  $[f^m(\alpha)]$  which is not a concatenation of paths in  $G_{PG}$  and



in  $\mathcal{N}_{PG}$ , the path  $a_j$  is not contained in a path of  $\mathcal{N}_{PG}^{\max}([f^m(\alpha)])$ . Hence the path  $\gamma'$  is not contained in  $\sigma_m$  as otherwise it would be contained in a path of  $\mathcal{N}_{PG}^{\max}([f^m(\alpha)])$ . Therefore, we see that  $(a_i \dots a_j) \cap \sigma_m \subseteq \gamma'$ . Hence, by the choice of  $K$ , we have

$$\ell_{exp}((a_i \dots a_j) \cap \sigma_m) \leq \ell((a_i \dots a_j) \cap \sigma_m) \leq \ell(\gamma') \leq C.$$

This proves the claim as we considered all possible  $PG$ -relative splitting units.  $\square$

Let  $m \in \mathbb{N}^*$ . By the claim, either  $\ell_{exp}((a_i \dots a_j) \cap \sigma_m) \leq C$  or, for every  $j > i$ , we have  $\ell_{exp}(a_j \cap \sigma_m) = 0$ . In the second case, we have

$$\ell_{exp}(\sigma_m) = \sum_{j>i} \ell_{exp}(a_j) + \ell_{exp}(a_i \cap \sigma_m) = \ell_{exp}(a_i \cap \sigma_m) \leq C,$$

where the last inequality follows from the case  $j = i$  of the claim. Hence, for every  $m \in \mathbb{N}^*$ , we have  $\ell_{exp}(\sigma_m) \leq C$ . Note that, by Lemma 6.3.16 applied to  $[f^m(\alpha)] = \alpha_m \sigma_m$ , we have

$$\ell_{exp}(\alpha_m) \geq \ell_{exp}([f^m(\alpha)]) - \ell_{exp}(\sigma_m) \geq \ell_{exp}([f^m(\alpha)]) - C.$$

It remains to prove that  $\ell_{exp}(\beta_m) \leq C$ . But  $\beta_m$  can be written as  $\beta_m = \delta_1 \delta_2$  where  $\delta_2$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  and  $\delta_1$  is a (possibly trivial) path contained in a path of  $\mathcal{N}_{PG}^{\max}([f^m(\beta)])$ . By Lemma 6.3.17 applied to  $\delta_2$  and by the choice of  $K$  (since  $\delta_1$  is a subpath of a path in  $\mathcal{N}_{PG}$ ), we have

$$\ell_{exp}(\beta_m) \leq \ell_{exp}(\delta_1) + \ell_{exp}(\delta_2) = \ell_{exp}(\delta_1) \leq \ell(\delta_1) \leq C.$$

This concludes the proof.  $\square$

**Lemma 6.5.11.** *Let  $L \geq 1$ . There exists  $n_0 = n_0(L) \in \mathbb{N}^*$  which satisfies the following properties. Let  $\gamma$  be a reduced edge path of  $G$  such that  $\ell_{exp}(\gamma) \leq L$ . For every  $n \geq n_0$  and every optimal splitting of  $[f^n(\gamma)]$ , either  $[f^n(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or the following two assertions hold:*

(a) *the path  $[f^n(\gamma)]$  contains a complete factor of exponential length at least equal to  $10C$ ;*

(b) *the exponential length of an incomplete factor of  $[f^n(\gamma)]$  is at most equal to  $8C$ .*

**Proof.** By Lemma 6.3.21, there exists an integer  $m' \in \mathbb{N}^*$  depending only on  $f$  such that for every edge  $e$  of  $\overline{G - G'_{PG}}$  and every  $n \geq m'$ , we have  $\ell_{exp}[f^n(e)] \geq 16C + 1$ . Let  $\gamma = \gamma_0 \gamma'_1 \gamma_1 \dots \gamma_\ell \gamma'_\ell$  be the exponential decomposition of  $\gamma$ . Let

$$\gamma = \beta_0 \alpha_1 \beta_1 \dots \alpha_k \beta_k$$

be a nontrivial decomposition of  $\gamma$  such that, for every  $i \in \{0, \dots, k\}$ , the path  $\beta_i$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  and for every  $i \in \{1, \dots, k\}$ , the path  $\alpha_i$  is a concatenation of edges in irreducible strata not contained in some  $\gamma_j$  with  $j \in \{0, \dots, \ell\}$  and paths in zero strata. The main point of the proof is to show that, up to applying

an iterate of  $[f]$ , there is no cancellation between the subpaths  $\alpha_i$ . By definition of the exponential length, for every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\gamma) = \sum_{i=1}^k \ell_{exp}(\alpha_i)$ . Therefore, since  $\ell_{exp}(\gamma) \leq L$ , for every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\alpha_i) \leq L$ . Note that, for every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\alpha_i) = \ell(\alpha_i) - \ell(\alpha_i \cap \mathcal{Z})$  where  $\mathcal{Z}$  is the subgraph of  $G$  consisting in all zero strata. By the choice of  $C$  the length of every path contained in a zero stratum is at most equal to  $C$ . Hence for every  $i \in \{1, \dots, k\}$ , we have  $\ell(\alpha_i) \leq CL$ . By Proposition 6.2.5 (8) there exists  $m'' \in \mathbb{N}^*$  depending only on  $L$  such that, for all  $i \in \{1, \dots, k\}$  and  $m \geq m''$ , the path  $[f^m(\alpha_i)]$  is completely split. Let  $m = m' + m''$ . By Lemma 6.3.20 (2), for every  $n \geq m$  and every  $i \in \{1, \dots, k\}$ , since  $[f^{n-m'}(\alpha_i)]$  is completely split, one compute its exponential length by adding the exponential length of all its splitting units. Thus, if  $[f^{n-m'}(\alpha_i)]$  contains a splitting unit which is an edge  $e$  in  $\overline{G - G'_{PG}}$ , we have

$$\ell_{exp}([f^n(\alpha_i)]) \geq \ell_{exp}([f^{m'}(e)]) \geq 16C + 1. \quad (6.8)$$

Let  $C_m$  be a bounded cancellation constant for  $f^m$  given by Lemma 6.4.9. Note that, if there exists  $i \in \{1, \dots, k-1\}$  such that  $\ell(\beta_i) < C_m$ , then there might exist some identifications between  $[f^m(\alpha_{i-1})]$  and  $[f^m(\alpha_i)]$  when reducing the paths in order to obtain  $[f^m(\gamma)]$ . This is why we replace the decomposition  $\gamma = \beta_0 \alpha_1 \beta_1 \dots \alpha_k \beta_k$  of  $\gamma$  by a new one. This new decomposition is defined as follows. Since every lift of  $f^m$  to the universal cover of  $G$  is a quasi-isometry, there exists  $M_m > 0$  depending only on  $m$  such that, for every reduced edge path of length  $\ell(\beta) > M_m$ , we have  $\ell([f^m(\beta)]) \geq 2C_m + 1$ . Let  $\Gamma_m = \{\beta_i \mid \ell(\beta_i) \leq M_m\}$ . Note that  $|\Gamma_m| \leq k + 1$ . Note that, by Lemma 6.2.9 and Proposition 6.2.5 (4) and Lemma 6.2.9, for every  $i \in \{1, \dots, k\}$ , if  $\beta_{i-1}$  or  $\beta_i$  is not trivial, then  $\alpha_i$  is not contained in a zero stratum. In particular, we may suppose that, for every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\alpha_i) > 0$ . Thus, since  $\ell_{exp}(\gamma) = \sum_{i=1}^k \ell_{exp}(\alpha_i) \leq L$ , and, for every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\alpha_i) > 0$ , we see that  $k \leq L$ . Hence we have  $|\Gamma_m| \leq k + 1 \leq L + 1$ .

**Claim.** There exist  $m_1 \geq m$  depending only on  $|\Gamma_m|$  (and hence on  $L$ ) and a decomposition  $\gamma = \beta_0^{(1)} \alpha_1^{(1)} \beta_1^{(1)} \dots \alpha_{k_1}^{(1)} \beta_{k_1}^{(1)}$  such that:

- (a') for every  $i \in \{1, \dots, k_1\}$ , the path  $[f^{m_1}(\alpha_i^{(1)})]$  is completely split;
- (b') for every  $i \in \{0, \dots, k_1\}$ , the path  $\beta_i^{(1)}$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ ;
- (c') for every  $i \in \{0, \dots, k_1\}$ , the subpath of  $[f^{m_1}(\beta_i^{(1)})]$  contained in  $[f^{m_1}(\gamma)]$  is not reduced to a point;
- (d') for every  $i \in \{1, \dots, k_1\}$ , for every  $n \geq m'$ , if  $[f^{n-m'}(\alpha_i^{(1)})]$  contains a splitting unit which is an edge in  $\overline{G - G'_{PG}}$  then  $\ell_{exp}([f^n(\alpha_i^{(1)})]) \geq 16C + 1$ .

**Proof.** The proof is by induction on  $|\Gamma_m|$ . Suppose first that  $\Gamma_m = \emptyset$ . By the definition of  $|\Gamma_m|$  and  $M_m$ , for every  $i \in \{0, \dots, k\}$ , the path  $[f^m(\beta_i)]$  has length at least equal to  $2C_m + 1$ . By Lemma 6.4.9, for every  $i \in \{0, \dots, k\}$ , the subpath of  $[f^m(\beta_i)]$  contained in  $[f^m(\gamma)]$  is not reduced to a point. So the integer  $m_1 = m$  and the decomposition  $\gamma = \beta_0 \alpha_1 \beta_1 \dots \alpha_k \beta_k$  satisfy the assertions of the claim (Assertion (d') follows from Equation (6.8)).

Suppose now that  $\Gamma_m \neq \emptyset$ . Then

$$\sum_{i=1}^k \ell(\alpha_i) + \sum_{\beta_i \in \Gamma_m} \ell(\beta_i) \leq kCL + M_m L \leq CL^2 + M_m L.$$

Let  $m'_2 \geq m$  be such that for every path  $\beta$  of length at most equal to  $CL^2 + M_m L$  and every  $n \geq m'_2$ , the path  $[f^n(\beta)]$  is completely split. Then  $\gamma$  has a decomposition  $\gamma = \beta_0^{(2)} \alpha_1^{(2)} \beta_2^{(2)} \dots \alpha_{k_2}^{(2)} \beta_{k_2}^{(2)}$  such that, for every  $i \in \{1, \dots, k_2\}$ , the path  $[f^{m'_2}(\alpha_i^{(2)})]$  is completely split and for every  $i \in \{0, \dots, k_2\}$ , the path  $\beta_i^{(2)}$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  of length greater than  $M_m$ . Let  $m_2 = m'_2 + m'$ . Then for every  $i \in \{1, \dots, k_2\}$ , the paths  $[f^{m_2}(\alpha_i^{(2)})]$  and  $[f^{m_2-m'}(\alpha_i^{(2)})]$  are completely split. Moreover, if  $[f^{m_2-m'}(\alpha_i^{(2)})]$  contains a splitting unit which is an edge in  $\overline{G - G'_{PG}}$ , then  $\ell_{exp}([f^{m_2}(\alpha_i^{(2)})]) \geq 16C + 1$  as in Equation (6.8). Let  $C_{m_2}$  be a bounded cancellation constant associated with  $f^{m_2}$  and let  $M_{m_2} \geq M_m$  be such that, for every reduced edge path of length  $\ell(\beta) > M_{m_2}$ , we have  $\ell([f^{m_1}(\beta)]) \geq 2C_{m_2} + 1$ . Let  $\Gamma_{m_2} = \{\beta_i^{(2)} \mid \ell(\beta_i) \leq M_{m_2}\}$ . Note that  $|\Gamma_{m_2}| < |\Gamma_m|$ . Hence we can apply the induction hypothesis to the decomposition  $\gamma = \beta_0^{(2)} \alpha_1^{(2)} \beta_2^{(2)} \dots \alpha_{k_2}^{(2)} \beta_{k_2}^{(2)}$  to obtain the desired decomposition of  $\gamma$ . This concludes the proof of the claim.  $\square$

Let  $m_1$  and  $\gamma = \beta_0^{(1)} \alpha_1^{(1)} \beta_1^{(1)} \dots \alpha_{k_1}^{(1)} \beta_{k_1}^{(1)}$  be as in the assertion of the claim. By Assertion (c') of the claim, for every  $i \in \{1, \dots, k_1\}$ , there is no identification between edges of  $[f^{m_1}(\alpha_i^{(1)})]$ ,  $[f^{m_1}(\alpha_{i-1}^{(1)})]$  and  $[f^{m_1}(\alpha_{i+1}^{(1)})]$  when reducing in order to obtain  $[f^{m_1}(\gamma)]$ .

For every  $i \in \{1, \dots, k_1\}$ , since  $[f^{m_1}(\alpha_i^{(1)})]$  is  $PG$ -relative completely split, we can distinguish three possible cases for  $[f^{m_1}(\alpha_i^{(1)})]$ :

- (i) the path  $[f^{m_1}(\alpha_i^{(1)})]$  contains a  $PG$ -relative splitting unit which is an edge in  $\overline{G - G'_{PG}}$  (by Lemma 6.3.23 this case happens exactly when  $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) > 0$ );
- (ii)  $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) = 0$  and the path  $[f^{m_1}(\alpha_i^{(1)})]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ ;
- (iii)  $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) = 0$  and  $[f^{m_1}(\alpha_i^{(1)})]$  contains a maximal taken connecting path in a zero stratum.

We claim that if there exists  $i \in \{1, \dots, k_1\}$  such that  $[f^{m_1}(\alpha_i^{(1)})]$  satisfies (iii), then  $[f^{m_1}(\gamma)]$  is contained in a zero stratum. Indeed, suppose that  $[f^{m_1}(\alpha_i^{(1)})]$  satisfies (iii). By Lemma 6.3.23 applied to the  $PG$ -relative completely split edge path  $[f^{m_1}(\alpha_i^{(1)})]$ , since  $\ell_{exp}([f^{m_1}(\alpha_i^{(1)})]) = 0$  the path  $[f^{m_1}(\alpha_i^{(1)})]$  does not contain an edge in  $\overline{G - G'_{PG}}$ . Therefore, the path  $[f^{m_1}(\alpha_i^{(1)})]$  is a concatenation of paths in  $G'_{PG}$  and in  $\mathcal{N}_{PG}$ . By Proposition 6.2.5 (4) and Lemma 6.2.9, there is no path in a zero stratum which is adjacent to a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Hence  $[f^{m_1}(\alpha_i^{(1)})] = \sigma$ , where  $\sigma$  is a maximal taken connecting path in a zero stratum not contained in  $G_{PG}$ . But the

endpoints of  $\sigma$  are the endpoints of  $[f^{m_1}(\beta_{i-1}^{(1)})]$  and  $[f^{m_1}(\beta_i^{(1)})]$ , which are concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . As above, this implies that  $[f^{m_1}(\gamma)] = \sigma$ . Since zero strata are contractible, there exists  $m_3 \in \mathbb{N}^*$  such that  $[f^{m_3}(\gamma)]$  is  $PG$ -relative completely split. Hence Assertion (b) of Lemma 6.5.11 follows. Applying a further power of  $[f]$  (which can be chosen uniformly as there are finitely many reduced edge paths contained in a zero stratum), there exists  $m_4 \in \mathbb{N}^*$  such that  $[f^{m_4}(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or it satisfies Assertion (a) of Lemma 6.5.11. This concludes the proof of Lemma 6.5.11 in case (iii).

Hence we may suppose that for every  $i \in \{1, \dots, k_1\}$ , the path  $[f^{m_1}(\alpha_i^{(1)})]$  satisfies either (i) or (ii). Note that, if  $i \in \{1, \dots, k_1\}$  is such that the path  $[f^{m_1}(\alpha_i^{(1)})]$  satisfies (i), then  $[f^{m_1}(\alpha_i^{(1)})]$  also satisfies the hypothesis of Assertion (d') of the claim. Thus

$$\ell_{exp}([f^{m_1+m'}(\alpha_i^{(1)})]) \geq 16C + 1.$$

Let  $m'_1 = m_1 + m'$  and let  $n' \geq m'_1$ . Let  $\Lambda_{exp} = \{\alpha_i^{(1)} \mid \ell_{exp}([f^{n'}(\alpha_i^{(1)})]) \geq 16C + 1\}$ . For every  $j \in \{1, \dots, k_1\}$  and every  $n \in \mathbb{N}^*$ , let  $\alpha_j^{(n)}$  be the subpath of  $[f^n(\alpha_j^{(1)})]$  contained in  $[f^n(\gamma)]$ . For every  $j \in \{0, \dots, k_1\}$  and every  $n \in \mathbb{N}^*$ , let  $\beta_j^{(n)}$  be the subpath of  $[f^n(\beta_j^{(1)})]$  contained in  $[f^n(\gamma)]$ . Suppose first that  $\Lambda_{exp}$  is not empty and let  $\alpha_i^{(1)} \in \Lambda_{exp}$ . By Lemma 6.5.10 (2) applied to  $\beta^{(1)} = [f^{n'}(\beta_{i-1}^{(1)})]$ ,  $\alpha = [f^{n'}(\alpha_i^{(1)})]$  and  $\beta^{(2)} = [f^{n'}(\beta_i^{(1)})]$ , we have  $\ell_{exp}(\alpha_i^{(n')}) \geq 14C + 1$ . By Remark 6.5.9 (2) applied twice (once with  $\gamma_1 = [f^{n'}(\alpha_i^{(1)})]$  and  $\gamma_2 = [f^{n'}(\beta_i^{(1)} \dots \alpha_{k_1}^{(1)} \beta_{k_1}^{(1)})]$ , and once with  $\gamma_1 = [f^{n'}(\alpha_i^{(1)})]^{-1}$  and  $\gamma_2 = [f^{n'}(\beta_0^{(1)} \dots \alpha_{i-1}^{(1)} \beta_{i-1}^{(1)})]^{-1}$ ), the path  $\alpha_i^{(n')}$  contains a complete factor of  $[f^{n'}(\gamma)]$  of exponential length at least equal to  $14C + 1 - 4C = 10C + 1$ . This proves Assertion (a) of Lemma 6.5.11. Moreover, Remark 6.5.9 (2) implies that the intersection of an incomplete factor of  $[f^{n'}(\gamma)]$  with  $\alpha_i^{(n')}$  is contained in the union of an initial and a terminal segment of  $\alpha_i^{(n')}$  of exponential lengths at most  $2C$ . For every  $i \in \{1, \dots, k_1\}$  such that  $\alpha_i^{(1)} \in \Lambda_{exp}$ , let  $\tau_i^1$  be the maximal initial segment of  $\alpha_i^{(n')}$  of exponential length equal to  $2C$  and let  $\tau_i^2$  be the maximal terminal segment of  $\alpha_i^{(n')}$  of exponential length equal to  $2C$ .

We now prove Assertion (b) of Lemma 6.5.11. Suppose that there exists  $i \in \{1, \dots, k_1\}$  such that  $\alpha_i^{(1)} \notin \Lambda_{exp}$ , so that in particular  $[f^{m_1}(\alpha_i^{(1)})]$  does not satisfy (i). Then  $[f^{m_1}(\alpha_i^{(1)})]$  satisfies (ii) and is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.9 (3), the path  $[f^{n'}(\alpha_i^{(1)})]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.5, the path  $[[f^{n'}(\beta_{i-1}^{(1)})][f^{n'}(\alpha_i^{(1)})][f^{n'}(\beta_i^{(1)})]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Thus, the path  $\beta_{i-1}^{(n')} \alpha_i^{(n')} \beta_i^{(n')}$  is a subpath of a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Hence  $[f^{n'}(\gamma)]$  has a decomposition

$$[f^{n'}(\gamma)] = \epsilon_1 \alpha_1^{(n',+)} \epsilon_2 \dots \alpha_{k_2}^{(n',+)} \epsilon_{k_2}$$

where for every  $j \in \{1, \dots, k_2\}$ , the path  $\alpha_j^{(n',+)}$  is in  $\Lambda_{exp}$  and for every  $j \in \{0, \dots, k_2\}$ , the path  $\epsilon_j$  is contained in a path  $\iota_j$  which is a concatenation of paths in  $G_{PG}$  and in

$\mathcal{N}_{PG}$ . Hence, for every  $j \in \{0, \dots, k_2\}$ , we have  $\ell_{exp}(\iota_j) = 0$  by Lemma 6.3.17 and, by Lemma 6.5.6, we have  $\ell_{exp}(\epsilon_j) \leq 2C$ .

If  $\gamma'$  is an incomplete factor of  $[f^{n'}(\gamma)]$ , as explained above, there exists  $i \in \{1, \dots, k_2\}$  such that  $\gamma'$  is contained in  $\tau_{i-1}^2 \epsilon_{i-1} \tau_i^1$ . By Lemma 6.5.6, we have

$$\ell_{exp}(\gamma') \leq \ell_{exp}(\tau_{i-1}^2 \epsilon_{i-1} \tau_i^1) + 2C.$$

By Lemma 6.3.16, the exponential length of  $\gamma'$  is at most equal to

$$\ell_{exp}(\tau_{i-1}^2) + \ell_{exp}(\epsilon_{i-1}) + \ell_{exp}(\tau_i^1) + 2C \leq 6C + \ell_{exp}(\epsilon_{i-1}) \leq 8C.$$

This proves (b).

Finally, suppose that  $\Lambda_{exp}$  is empty. For every  $j \in \{1, \dots, k_1\}$ , the path  $[f^{m_1}(\alpha_j^{(1)})]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.5, the path  $[f^{m_1}(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.9, for every  $n' \geq m_1$ , the path  $[f^{n'}(\gamma)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . This concludes the proof.  $\square$

**Lemma 6.5.12.** *Let  $f: G \rightarrow G$  be a  $3K$ -expanding CT map. There exists  $N \in \mathbb{N}^*$  such that for every reduced edge path  $\gamma$  and every  $m \geq N$ , the total exponential length of incomplete factors in any optimal splitting of  $[f^m(\gamma)]$  is uniformly bounded by  $8C\ell_{exp}(\gamma)$ .*

**Proof.** By Proposition 6.2.5 (8), there exists  $N \in \mathbb{N}^*$  such that, for every reduced edge path  $\alpha$  of length at most equal to  $C + 1$ , the path  $[f^N(\alpha)]$  is completely split. Suppose first that  $\ell_{exp}(\gamma) = 0$ . Then, by definition of the exponential length, the path  $\gamma$  is a concatenation of paths in  $G'_{PG}$  and in  $\mathcal{N}_{PG}$ . By Proposition 6.2.5 (4), every edge in a zero stratum is adjacent to either an edge in a zero stratum or an edge in an EG stratum. Moreover, by Lemma 6.2.9, there does not exist a subpath of  $\gamma$  contained in a zero stratum which is adjacent to a Nielsen path. Hence  $\gamma$  is either a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or a path in a zero stratum. In the first case, the path  $\gamma$  is  $PG$ -relative completely split. In the second case, by the definition of the constant  $K$  and Equation (6.7), we have  $\ell(\gamma) \leq K \leq C$ . By the choice of  $N$ , for every  $m \geq N$ , the path  $[f^m(\gamma)]$  is completely split. By Lemma 6.3.19, for every  $m \geq N$ , the path  $[f^m(\gamma)]$  is  $PG$ -relative completely split. By Lemma 6.3.17, for every  $m \geq N$ , we have  $\ell_{exp}([f^m(\gamma)]) = 0$ .

So we may suppose that  $\ell_{exp}(\gamma) > 0$ . Let  $\gamma = \gamma_0 \gamma'_1 \gamma_1 \dots \gamma_\ell \gamma'_\ell$  be the exponential decomposition of  $\gamma$  (see the beginning of Section 6.3.2). By Lemma 6.2.9, there does not exist a subpath of  $\gamma$  contained in a zero stratum which is adjacent to a Nielsen path. Therefore, the path  $\gamma$  has a decomposition  $\alpha_0 \beta_1 \alpha_1 \dots \beta_k \alpha_k$  where, for every  $i \in \{0, \dots, k\}$ , the path  $\alpha_i$  is a (possibly trivial) concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  and, for every  $i \in \{1, \dots, k\}$ , the path  $\beta_i$  is a concatenation of a (possibly trivial) maximal reduced path in a zero stratum and an edge in an irreducible stratum not contained in  $G_{PG}$  or in some  $\gamma_i$ . By construction of  $K$ , for every  $i \in \{1, \dots, k\}$ , we have  $\ell(\beta_i) \leq C + 1$ . By the choice of  $N$ , for every  $m \geq N$ , the path  $[f^m(\beta_i)]$  is completely split. Note that, for

every  $i \in \{1, \dots, k\}$ , we have  $\ell_{exp}(\beta_i) = 1$  and that

$$\ell_{exp}(\gamma) = \sum_{i=1}^k \ell_{exp}(\beta_i) = k.$$

By Lemma 6.3.9, for every  $i \in \{0, \dots, k\}$  and every  $m \geq M$ , the path  $[f^m(\alpha_i)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Lemma 6.3.17, for every  $m \geq M$ , we have  $\ell_{exp}([f^m(\alpha_i)]) = 0$ . By Lemma 6.5.6, the exponential length of the subpath of  $[f^m(\alpha_i)]$  contained in  $[f^m(\gamma)]$  is at most equal to  $2C$ . For every  $i \in \{0, \dots, k\}$  (resp.  $i \in \{1, \dots, k\}$ ) and every  $m \geq N$ , let  $\alpha_{i,m}$  (resp.  $\beta_{i,m}$ ) be the subpath of  $[f^m(\alpha_i)]$  (resp.  $[f^m(\beta_i)]$ ) contained in  $[f^m(\gamma)]$ . By Remark 6.5.9 (2), for every  $i \in \{1, \dots, k\}$  and every  $m \geq N$ , the exponential length of any incomplete factor in  $\beta_{i,m}$  is at most equal to  $4C$ . By Lemma 6.3.16, for every  $m \geq N$ , the sum of the exponential lengths of the incomplete factors in  $[f^m(\gamma)]$  is at most equal to

$$\sum_{i=0}^k \ell_{exp}(\alpha_{i,m}) + 4Ck \leq 2C(k+1) + 4kC \leq 4Ck + 4Ck = 8Ck = 8C\ell_{exp}(\gamma).$$

The conclusion of the lemma follows.  $\square$

**Lemma 6.5.13.** *Let  $f: G \rightarrow G$  be a  $3K$ -expanding CT map. Let  $\gamma$  be a reduced edge path in  $G$ . Suppose that  $\gamma$  has a splitting  $\gamma = b_1 a b_2$  where, for every  $i \in \{1, 2\}$ , the path  $b_i$  is a possibly trivial PG-relative completely split. If  $\ell_{exp}^{\gamma}(a) = 0$  then  $\ell_{exp}(a) = 0$ .*

**Proof.** Let  $\gamma = \gamma_0 \gamma_1' \gamma_1 \dots \gamma_k \gamma_k'$  be the exponential decomposition of  $\gamma$ . By Lemma 6.5.6, there exist three (possibly trivial) paths  $\delta_1$ ,  $\delta_2$  and  $\tau$  such that for every  $i \in \{1, 2\}$ , the path  $\delta_i$  is a proper initial or terminal subpath of a splitting unit of some  $\gamma_j$ , we have  $\ell_{exp}(\tau) = \ell_{exp}^{\gamma}(\tau) = \ell_{exp}^{\gamma}(a)$  and  $a = \delta_1 \tau \delta_2$ . Since  $\ell_{exp}^{\gamma}(a) = 0$ , we have  $\ell_{exp}(\tau) = 0$ . Hence  $\tau$  is a concatenation of paths in  $G'_{PG}$  and in  $\mathcal{N}_{PG}$ . By Proposition 6.2.5 (4), every edge in a zero stratum is adjacent to either an edge in a zero stratum or an edge in an EG stratum. Moreover, by Lemma 6.2.9, there does not exist a subpath of  $\gamma$  contained in a zero stratum which is adjacent to a Nielsen path. Hence  $\tau$  is either a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or a path in a zero stratum. If  $\tau$  is contained in a zero stratum, by Lemma 6.2.9, we see that  $\delta_1$  and  $\delta_2$  are trivial, that is,  $a = \tau$ . Thus, we have  $\ell_{exp}(a) = \ell_{exp}(\tau) = 0$ .

So we may suppose that  $\tau$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Suppose towards a contradiction that there exists  $i \in \{1, 2\}$  such that  $\delta_i$  is not trivial. For every  $i \in \{1, 2\}$  such that  $\delta_i \neq \emptyset$ , let  $\sigma_i$  be the splitting unit of some  $\gamma_j$  containing  $\delta_i$  and let  $r_i$  be the height of  $\sigma_i$ . By [BH, Lemma 5.11], for every  $i \in \{1, 2\}$  such that  $\delta_i$  is not trivial, there exist two distinct  $r_i$ -legal paths  $\alpha_i$  and  $\beta_i$  such that  $\sigma_i = \alpha_i \beta_i$  and such that the turn  $\{Df(\alpha_i^{-1}), Df(\beta_i)\}$  is the only height  $r_i$  illegal turn. Moreover, there exists a path  $\tau_i'$  such that  $[f(\alpha_i)] = \alpha_i \tau_i'$  and  $[f(\beta_i)] = \tau_i'^{-1} \beta_i$ . Let  $\epsilon_1^{(1)}, \epsilon_1^{(2)}$  be two paths such that  $\sigma_1 = \epsilon_1^{(1)} \epsilon_1^{(2)}$ , the path  $\epsilon_1^{(1)}$  is contained in  $b_1$  and the path  $\epsilon_1^{(2)}$  is contained in  $a$ . Similarly, let  $\epsilon_2^{(1)}, \epsilon_2^{(2)}$  be two paths such that  $\sigma_2 = \epsilon_2^{(1)} \epsilon_2^{(2)}$ , the path  $\epsilon_2^{(2)}$  is contained in  $b_2$  and the path  $\epsilon_2^{(1)}$  is contained in  $a$ .

**Claim.** (1) For every path  $b \in \mathcal{N}_{PG}^{\max}(b_1)$  (resp.  $b \in \mathcal{N}_{PG}^{\max}(b_2)$ ), the path  $b$  does not contain edges of  $\epsilon_1^{(1)}$  (resp.  $\epsilon_2^{(2)}$ ).

(2) The path  $\epsilon_1^{(1)}$  is  $r_1$ -legal and the path  $\epsilon_2^{(2)}$  is  $r_2$ -legal.

**Proof.** We prove the claim for  $b_1$ , the proof for  $b_2$  being similar.

(1) Let  $b \in \mathcal{N}_{PG}^{\max}(b_1)$ . There exists  $c \in \mathcal{N}_{PG}^{\max}(\gamma)$  such that  $b \subseteq c$ . Moreover, by Lemma 6.3.4 (3) applied to  $\gamma' = b$  and  $\gamma = c$ , either  $b$  is a concatenation of splitting units of  $c$ , or  $b$  is properly contained in a splitting unit of  $c$  and is not an initial or a terminal segment of  $c$ . Since  $b_1$  is an initial segment of  $\gamma$ , the second case cannot occur. Hence  $b$  is a concatenation of splitting units of  $c$ . Since  $\sigma_1$  is not contained in  $b_1$ , the path  $b$  cannot contain edges of  $\sigma_1$ . Since  $\epsilon_1^{(1)} \subseteq \sigma_1$ , the path  $b$  cannot contain edges of  $\epsilon_1^{(1)}$ .

(2) Suppose towards a contradiction that  $\epsilon_1^{(1)}$  is not  $r_1$ -legal. Then it contains the illegal turn  $\{Df(\alpha_1^{-1}), Df(\beta_2)\}$ . Recall that the path  $b_1$  is  $PG$ -relative completely split. By the description of  $PG$ -relative splitting units, the illegal turn must be contained in a  $PG$ -relative splitting unit of  $b_1$  which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Since the last edge of  $\alpha_1$  is an edge in an EG stratum, the last edge of  $\alpha_1$  must be contained in a path contained in  $\mathcal{N}_{PG}$ . Hence  $\epsilon_1$  intersects a path in  $\mathcal{N}_{PG}^{\max}(b_1)$ . This contradicts Assertion (1).  $\square$

By Assertion (2) of the claim, for every  $i \in \{1, 2\}$  such that  $\sigma_i$  is not trivial, the path  $\epsilon_i^{(i)}$  is  $r_i$ -legal. Moreover, by Assertion (1) of the claim an INP contained in  $b_i$  cannot intersect the path  $\epsilon_i^{(i)}$ . Since the paths  $b_1$  and  $b_2$  are  $PG$ -relative completely split, the paths  $b_1$  and  $b_2$  split respectively at the origin of  $\epsilon_1^{(1)}$  and at the end of  $\epsilon_2^{(2)}$ . So we may suppose that  $b_1 = \epsilon_1^{(1)}$  and  $b_2 = \epsilon_2^{(2)}$ . Therefore, there exists a (possibly trivial) path  $\tau_1$  such that, up to taking a power of  $f$  so that the length of  $[f(b_1)]$  is greater than  $\alpha_1$ , we have  $[f(b_1)] = \alpha_1 \tau_1$  and  $[f(\epsilon_1^{(2)})] = \tau_1^{-1} \beta_1$ . Similarly, there exists a path  $\tau_2$  such that  $[f(\epsilon_2^{(1)})] = \alpha_2 \tau_2$  and  $[f(b_2)] = \tau_2^{-1} \beta_2$ .

Since  $\gamma$  splits at the concatenation points of  $b_1$ ,  $a$  and  $b_2$ , the paths  $\tau_1^{-1}$  and  $\tau_2$  contained in  $[f(\epsilon_1^{(2)})][f(\tau)][f(\epsilon_2^{(1)})]$  must be identified when passing to  $[f(a)]$ . Suppose first that  $[f(\tau)]$  is a point. Then since the EG INPs  $\sigma_1$  and  $\sigma_2$  are uniquely determined by their initial and terminal edges by Proposition 6.2.5 (9), we see that  $\sigma_1 = \sigma_2^{-1}$ . But then there are some identifications between  $b_1$  and  $b_2$ , which contradicts the fact that  $b_1 a b_2$  is a splitting.

Thus, we may suppose that  $[f(\tau)]$  is nontrivial. By Lemma 6.3.9, since  $\tau$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  so is  $[f(\tau)]$ . Note that, since an EG INP is completely determined by its initial and terminal edges by Proposition 6.2.5 (9), if  $[f(\tau)]$  contains the initial or the terminal edge of an EG INP  $\sigma$ , then  $\sigma$  is contained in  $[f(\tau)]$ . Note that there are identifications between edges of  $[f(\epsilon_1^{(2)})]$  and  $[f(\tau)]$  or between edges of  $[f(\tau)]$  and  $[f(\epsilon_2^{(1)})]$ . Therefore,  $[f(\tau)]$  starts with  $\sigma_1^{-1}$  or  $[f(\tau)]$  ends with  $\sigma_2^{-1}$ . Thus, one of the following holds:

- (a)  $[f(\tau)] = \sigma_1^{-1}\tau'$  with  $\tau'$  a (possibly trivial) path which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  which does not end by  $\sigma_2^{-1}$ ;
- (b)  $[f(\tau)] = \tau'\sigma_2^{-1}$  with  $\tau'$  a (possibly trivial) path which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  which does not start by  $\sigma_1^{-1}$ ;
- (c)  $[f(\tau)] = \sigma_1^{-1}\tau'\sigma_2^{-1}$  with  $\tau'$  a (possibly trivial) path.

Note that  $\sigma_1^{-1}\tau'\sigma_2^{-1}$  is reduced, so that there is no identification between  $\alpha_1^{-1}$  and  $\tau'$  and between  $\tau'$  and  $\beta_2^{-1}$ . Let  $e_{\sigma_1}$  be the terminal edge of  $\sigma_1$  and let  $e_{\sigma_2}$  be the initial edge of  $\sigma_2$ . By Proposition 6.2.5 (9), both  $e_{\sigma_1}$  and  $e_{\sigma_2}$  are edges in EG strata. Since  $f$  is  $3K$ -expanding, for every  $i \in \{1, 2\}$ , the path  $[f(e_{\sigma_i})]$  has length at least equal to  $3K$ . Recall that, for every  $i \in \{1, 2\}$ , by definition of  $K$ , we have  $\ell(\sigma_i) \leq K$ , so that  $\ell(\alpha_i), \ell(\beta_i) \leq K$ . Since  $[f(\epsilon_1^{(2)})] = \alpha_1\tau_1$  and  $[f(\epsilon_2^{(1)})] = \alpha_2\tau_2$ , the path  $[f(e_{\sigma_1})]$  contains a nondegenerate terminal segment of  $\tau_1^{-1}$  and the path  $[f(e_{\sigma_2})]$  contains a nondegenerate initial segment of length  $2K$  of  $\tau_2$ . As  $e_{\sigma_1}$  is  $r_1$ -legal and as  $f$  is a relative train track by Proposition 6.2.5 (1), we see that the last edge of  $\tau_1^{-1}$  is not the last edge of  $\alpha_1$ . Similarly, the first edge of  $\tau_2$  is not the first edge of  $\beta_2$ . Therefore, we have  $[\tau_1^{-1}\beta_1\sigma_1^{-1}] = \tau_1^{-1}\alpha_1^{-1}$  and  $[\sigma_2^{-1}\alpha_2\tau_2] = \beta_2^{-1}\tau_2$ . Thus we have

$$[[f(\epsilon_1^{(2)})][f(\tau)][f(\epsilon_2^{(1)})]] = [\tau_1^{-1}\beta_1\sigma_1^{-1}\tau'\sigma_2^{-1}\alpha_2\tau_2] = [\tau_1^{-1}\alpha_1^{-1}\tau'\beta_2^{-1}\tau_2],$$

and there is no identification between  $\tau_1^{-1}$  and  $\alpha_1^{-1}$ ,  $\alpha_1^{-1}$  and  $\tau'$ ,  $\tau'$  and  $\beta_2^{-1}$  and  $\beta_2^{-1}$  and  $\tau_2$ . Therefore, if  $\tau'$  is not trivial, then we have a contradiction as  $\tau_1^{-1}$  and  $\tau_2$  are not identified in  $[f(a)]$ . Suppose that  $\tau'$  is trivial. Then the paths  $\tau_1^{-1}$  and  $\tau_2$  are identified in  $[f(a)]$  only if a terminal segment of  $\alpha_1^{-1}$  is identified with an initial segment of  $\beta_2^{-1}$ . Since EG INP are uniquely determined by their initial and terminal edges by Proposition 6.2.5 (9), we see that  $\sigma_1 = \sigma_2^{-1}$ . Hence  $\alpha_1^{-1} = \beta_2$  and either  $\tau_1^{-1}$  is an initial segment of  $\tau_2^{-1}$  or  $\tau_2$  is an initial segment of  $\tau_1$ . Up to changing the orientation of  $\gamma$ , we may suppose that  $\tau_1^{-1}$  is an initial segment of  $\tau_2^{-1}$ . If  $\tau_1^{-1} = \tau_2^{-1}$ , then  $[f(a)]$  is a vertex. Moreover, as  $\sigma_1 = \sigma_2^{-1}$ , the segment  $b_1 = \epsilon_1^{(1)}$  is equal to  $b_2^{-1}$ . Therefore, a terminal segment of  $b_1$  is identified with an initial segment of  $b_2$ , a contradiction. If  $\tau_1^{-1}$  is a proper initial segment of  $\tau_2^{-1}$ , then  $\tau_2$  is identified with edges in  $b_1$ , a contradiction. As we have considered every case, we see that  $\delta_1$  and  $\delta_2$  are trivial and  $\ell_{exp}(a) = \ell_{exp}(\tau) = 0$ .  $\square$

**Lemma 6.5.14.** *Let  $f: G \rightarrow G$  be a  $3K$ -expanding CT map. There exists  $n_0 \in \mathbb{N}^*$  such that for every  $n \geq n_0$ , and every closed reduced edge path  $\gamma$  of  $G$ , we have the following relation between the goodness of  $\gamma$  and the one of  $[f^n(\gamma)]$ :*

$$\mathfrak{g}([f^n(\gamma)]) \geq \mathfrak{g}(\gamma).$$

**Proof.** By Lemma 6.3.22, there exists  $N_0 \in \mathbb{N}^*$  such that, for every  $n \geq N_0$  and every  $PG$ -relative splitting unit  $\sigma$ , the exponential length of the path  $[f^n(\sigma)]$  is at least equal to the one of  $\sigma$ . By Lemma 6.5.12, there exists  $N_1$  such that for every  $n \geq N_1$  and every closed reduced edge path  $\gamma$  of  $G$ , the total exponential length of incomplete segments in any



optimal splitting of  $[f^n(\gamma)]$  is bounded by  $8C\ell_{exp}(\gamma)$ . Let  $N_2 = \lceil \log_3(10C + 16C^2) \rceil \in \mathbb{N}^*$  be such that for every  $x, y \geq 0$  such that  $(x, y) \neq (0, 0)$ , we have

$$\frac{(3^{N_2} - 2C)x}{(3^{N_2} - 2C)x + 8C(1 + 2C)y} \geq \frac{x}{x + y}.$$

Let  $n_0 = \max\{N_0, N_1, N_2\}$ .

Let  $\gamma$  be a closed reduced edge path in  $G$ . All splittings of  $\gamma$  are circuital splittings in what follows. Let  $\gamma = \alpha_0\beta_1\alpha_1 \dots \beta_k\alpha_k$  be an optimal splitting of  $\gamma$ , where for every  $i \in \{0, \dots, k\}$ , the path  $\alpha_i$  is an incomplete factor of  $\gamma$  and for every  $i \in \{1, \dots, k\}$ , the path  $\beta_i$  is a  $PG$ -relative complete factor of  $\gamma$ . First note that, for every  $i \in \{1, \dots, k\}$ , and for every  $n \geq 1$ , the path  $[f^n(\beta_i)]$  is  $PG$ -relative completely split by Proposition 6.2.5 (6) and Lemma 6.3.9. Therefore, if  $n \geq n_0 \geq N_0$ , the total exponential length of such  $PG$ -relative complete segments is nondecreasing under  $[f^n]$ . We now distinguish two cases, according to the growth of the paths  $\beta_i$ .

Suppose first that for every  $i \in \{1, \dots, k\}$ , the exponential length of  $\beta_i$  relative to  $\gamma$  is equal to zero. Since the splitting  $\gamma = \alpha_0\beta_1\alpha_1 \dots \beta_k\alpha_k$  is optimal and since for every  $i \in \{1, \dots, k\}$ , we have  $\ell'_{exp}(\beta_i) = 0$ , we have  $\mathfrak{g}(\gamma) = 0$ . Therefore, for every  $n \in \mathbb{N}^*$ , we have  $\mathfrak{g}([f^n(\gamma)]) \geq \mathfrak{g}(\gamma)$ .

Suppose now that there exists  $i \in \{1, \dots, k\}$  such that the exponential length of  $\beta_i$  relative to  $\gamma$  is positive. By Lemma 6.3.21, the sequence  $(\ell_{exp}([f^n(\beta_i)]))_{n \in \mathbb{N}^*}$  grows exponentially with  $n$ . We can now modify the splitting of  $\gamma$  into the following splitting:  $\gamma = \alpha'_0\beta'_1\alpha'_1 \dots \beta'_m\alpha'_m$  where:

- (a) for every  $j \in \{0, \dots, m\}$ , the path  $\alpha'_j$  is a concatenation of incomplete factors and complete factors of zero exponential length relative to  $\gamma$  of the previous splitting;
- (b) for every  $j \in \{1, \dots, m\}$ , the path  $\beta'_j$  is a complete factor of positive exponential length relative to  $\gamma$  of the previous splitting.

Note that, by definition of the exponential length relative to  $\gamma$ , for every  $i \in \{1, \dots, m\}$  and every path  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$ , the path  $\beta'_i$  is not contained in  $\gamma'$ . Therefore, if there exists  $j \in \{0, \dots, m\}$  and  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma)$  such that  $\alpha'_j$  intersects  $\gamma'$  nontrivially, then  $\gamma'$  is contained in  $\beta'_{j-1}\alpha'_j\beta'_j$ . In particular, Lemma 6.5.13 applies and for every  $j \in \{0, \dots, m\}$ , if  $\ell'_{exp}(\alpha'_j) = 0$ , then  $\ell_{exp}(\alpha'_j) = 0$ . Let  $\Lambda$  be the subset of  $\{0, \dots, m\}$  such that for every  $j \in \Lambda$ , we have  $\ell'_{exp}(\alpha'_j) > 0$ .

By Lemma 6.5.6 and Lemma 6.5.7, for every  $j \in \{1, \dots, m\}$  and every  $M \in \mathbb{N}^*$ , we have

$$\ell^{[f^M(\gamma)]}_{exp}([f^M(\beta'_i)]) \geq \ell_{exp}([f^M(\beta'_i)]) - 2C \geq 3^M \ell_{exp}(\beta'_i) - 2C' \geq (3^M - 2C)\ell'_{exp}(\beta'_i).$$

By Lemma 6.5.6, for every  $j \in \{0, \dots, m\}$ , we have  $\ell'_{exp}(\alpha'_j) \leq \ell_{exp}(\alpha'_j)$ . Note that, for every  $i \in \{1, \dots, m\}$ , and every  $n \in \mathbb{N}^*$ , the path  $[f^n(\beta'_i)]$  is  $PG$ -relative completely split. In particular, for every  $n \in \mathbb{N}^*$ , any incomplete factor of  $[f^n(\gamma)]$  is contained in a reduced iterate of some  $\alpha'_i$ . Thus, by Lemma 6.5.12, for every  $n \geq n_0 \geq N_1$ , the total exponential length of incomplete segments in  $[f^n(\gamma)]$  is bounded by  $8C \sum_{j=1}^k \ell_{exp}(\alpha'_j) = 8C \sum_{j \in \Lambda} \ell_{exp}(\alpha'_j)$ . Note that the function

$$x \mapsto \frac{x}{x + 8C \sum_{j \in \Lambda} \ell_{exp}(\alpha'_j)}$$

is nondecreasing. Recall that, for every  $n \in \mathbb{N}^*$ , the goodness function is a supremum over splittings of  $[f^n(\gamma)]$ . Thus, by Lemma 6.5.4, for every  $n \geq n_0$ , we have:

$$\mathfrak{g}([f^n(\gamma)]) \geq \frac{(3^n - 2C) \sum_{i=1}^m \ell_{exp}^\gamma(\beta'_i)}{(3^n - 2C) \sum_{i=1}^m \ell_{exp}^\gamma(\beta'_i) + 8C \sum_{j \in \Lambda} \ell_{exp}(\alpha'_j)}.$$

By Lemma 6.5.6, we have

$$8C \sum_{j \in \Lambda} \ell_{exp}(\alpha'_j) \leq 8C \sum_{j \in \Lambda} (\ell_{exp}^\gamma(\alpha'_j) + 2C) \leq 8C(1 + 2C) \sum_{j \in \Lambda} \ell_{exp}^\gamma(\alpha'_j),$$

where the last inequality follows from the fact that, for every  $j \in \Lambda$ , we have  $\ell_{exp}^\gamma(\alpha'_j) \geq 1$ . Therefore, since  $n_0 \geq N_2$ , for every  $n \geq n_0$ , we have:

$$\frac{(3^n - 2C) \sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j)}{(3^n - 2C) \sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j) + 8C(1 + 2C) \sum_{j \in \Lambda} \ell_{exp}^\gamma(\alpha'_j)} \geq \frac{\sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j)}{\sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j) + \sum_{j \in \Lambda} \ell_{exp}^\gamma(\alpha'_j)}.$$

By Lemma 6.5.3, we have

$$\ell_{exp}(\gamma) = \sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j) + \sum_{j=0}^m \ell_{exp}^\gamma(\alpha'_j) = \sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j) + \sum_{j \in \Lambda} \ell_{exp}^\gamma(\alpha'_j).$$

Thus, we see that

$$\frac{\sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j)}{\sum_{j=1}^m \ell_{exp}^\gamma(\beta'_j) + \sum_{j \in \Lambda} \ell_{exp}^\gamma(\alpha'_j)} = \mathfrak{g}(\gamma),$$

which gives the result.  $\square$

**Remark 6.5.15.** *In the next lemmas, we will adopt the following conventions.*

Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$ . Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  with filtration

$$\emptyset = G_0 \subsetneq \dots \subsetneq G_k = G.$$

Let  $p \in \{1, \dots, k-1\}$  be such that  $\mathcal{F}(G_p) = \mathcal{F}$ . By Lemma 6.3.21, up to taking a power of  $f$ , we may suppose that  $f$  is  $3K$ -expanding. By Lemma 6.5.14, up to passing to a power of  $f$ , we may suppose that for every closed reduced edge path  $\gamma$  of  $G$ , we have  $\mathfrak{g}([f(\gamma)]) \geq \mathfrak{g}(\gamma)$ .

**Lemma 6.5.16.** *Let  $f: G \rightarrow G$  be as in Remark 6.5.15.*

(1) *For every  $\delta > 0$ , there exists  $m \in \mathbb{N}^*$  such that for every reduced edge path  $\gamma$  such that  $\mathfrak{g}(\gamma) \geq \delta$  and every  $n \geq m$ , the total exponential length relative to  $[f^n(\gamma)]$  of complete factors in  $[f^n(\gamma)]$  denoted by  $TEL(n, \gamma)$  is at least*

$$TEL(n, \gamma) \geq \mathfrak{g}(\gamma) \ell_{exp}(\gamma) (3^n - 2C).$$

(2) *For every  $\delta > 0$  and every  $\epsilon > 0$ , there exists  $m \in \mathbb{N}^*$  such that for every cyclically reduced circuit  $\gamma$  such that  $\ell_{exp}(\gamma) > 0$ ,  $\mathfrak{g}(\gamma) \geq \delta$  and every  $n \geq m$ , we have  $\mathfrak{g}([f^n(\gamma)]) \geq 1 - \epsilon$ .*

**Proof.** Let  $\gamma = \alpha_0\beta_1\alpha_1 \dots \alpha_k\beta_k$  be an optimal splitting, where for every  $i \in \{0, \dots, k\}$ , the path  $\alpha_i$  is an incomplete factor of  $\gamma$  and for every  $i \in \{1, \dots, k\}$ , the path  $\beta_i$  is a  $PG$ -relative complete factor of  $\gamma$ . We may assume that  $\ell_{exp}(\gamma) > 0$ , otherwise  $\mathfrak{g}(\gamma) = 0$  and the result is immediate. Note that, since  $\mathfrak{g}(\gamma) \geq \delta > 0$ , there exists  $i \in \{1, \dots, k\}$  such that  $\ell_{exp}(\beta_i) > 0$ . Let  $\Lambda_\gamma$  be the set consisting in all complete factors  $\beta_i$  of  $\gamma$  whose exponential length relative to  $\gamma$  is positive. Let  $\ell_{exp}^\gamma(\Lambda_\gamma)$  be the sum of the exponential lengths relative to  $\gamma$  of all factors that belongs to  $\Lambda_\gamma$ . Note that

$$\ell_{exp}^\gamma(\Lambda_\gamma) = \sum_{\beta_i \in \Lambda_\gamma} \ell_{exp}^\gamma(\beta_i) = \mathfrak{g}(\gamma)\ell_{exp}(\gamma).$$

Note that, for every  $n \in \mathbb{N}^*$ , the value  $TEL(n, \gamma)$  is a supremum over all splittings of  $[f^n(\gamma)]$ . Thus, by Lemma 6.5.6 and Lemma 6.5.7, for every  $n \in \mathbb{N}^*$ , we have:

$$TEL(n, \gamma) \geq \sum_{\beta_i \in \Lambda_\gamma} \ell_{exp}^{[f^n(\gamma)]}([f^n(\beta_i)]) \geq (3^n - 2C)\ell_{exp}^\gamma(\Lambda_\gamma) \geq (3^n - 2C)\mathfrak{g}(\gamma)\ell_{exp}(\gamma).$$

This proves (1). We now prove (2). By Lemma 6.5.12, there exists  $n_0 \in \mathbb{N}^*$  such that for every  $n \geq n_0$ , the total exponential length of incomplete segments in  $[f^n(\gamma)]$  is bounded by  $8C\ell_{exp}(\gamma)$ . By Lemma 6.5.6, the total exponential length relative to  $\gamma$  of incomplete segments in  $[f^n(\gamma)]$  is hence bounded by  $10C\ell_{exp}(\gamma)$ . Note that, for every  $n \in \mathbb{N}^*$ , the value  $\mathfrak{g}([f^n(\gamma)])$  is a supremum over all splittings of  $[f^n(\gamma)]$ . Thus, by Lemma 6.5.4, for every  $n \geq n_0$ , we have:

$$\begin{aligned} \mathfrak{g}([f^n(\gamma)]) &\geq \frac{\mathfrak{g}(\gamma)\ell_{exp}(\gamma)(3^n - 2C)}{10C\ell_{exp}(\gamma) + \mathfrak{g}(\gamma)\ell_{exp}(\gamma)(3^n - 2C)} \\ &= \frac{\mathfrak{g}(\gamma)(3^n - 2C)}{10C + \mathfrak{g}(\gamma)(3^n - 2C)} \geq \frac{\delta(3^n - 2C)}{10C + \delta(3^n - 2C)}. \end{aligned}$$

The last term is independent of  $\gamma$  and converges to 1 as  $n$  goes to infinity. Therefore the conclusion of Lemma 6.5.16 holds for some  $n$  large enough which does not depend on  $\gamma$ . This proves (2) and this concludes the proof.  $\square$

## 6.5.2 North-South dynamics for a relative atoroidal outer automorphisms

Let  $\mathfrak{n} \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_{\mathfrak{n}}$ . Let  $\phi \in \text{Out}(F_{\mathfrak{n}}, \mathcal{F})$  be an almost atoroidal automorphism relative to  $\mathcal{F}$ . In this subsection we prove Theorem 6.5.1. The proof of Theorem 6.5.1 is inspired by the proof of the same result due to Uyanik ([Uya2]) in the context of an atoroidal outer automorphism for  $\text{Out}(F_{\mathfrak{n}})$ , that is, in the special case when  $\mathcal{F} = \emptyset$ . The proof relies on the study of splittings of reduced edge paths in the graph associated with a CT map representing a power of  $\phi$ . Indeed, we show that, when a cyclically reduced edge path representing  $w \in F_{\mathfrak{n}}$  has a splitting which is close to a complete splitting, then some iterate of  $\phi$  sends  $[w]$  into an open neighborhood of  $\Delta_+(\phi)$  (see Definition 6.4.5), and this iterate can be chosen uniformly (see Lemma 6.5.20).

Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism which satisfies Definition 6.4.3 (2). Let  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2 = \{[F_n]\}$  be a sequence of free factor system given in this definition. Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  with filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$  and such that there exist  $p$  and  $i$  in  $\{1, \dots, k\}$  such that  $\mathcal{F}(G_p) = \mathcal{F}$  and  $\mathcal{F}(G_i) = \mathcal{F}_1$ . We denote by  $\text{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$  the set of currents of  $\text{Curr}(F_n, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$  whose support is contained in  $\partial^2 \mathcal{F}_1$ . Note that, since the extension  $\mathcal{F}_1 \leq \{[F_n]\}$  is sporadic, either  $\mathcal{F}_1 = \{[H_1], [H_2]\}$  or  $\mathcal{F}_1 = \{[H]\}$  for some subgroups  $H_1, H_2$  of  $F_n$ . Up to assuming that  $H_2$  is the trivial group, we may assume that  $\mathcal{F}_1 = \{[H_1], [H_2]\}$ . Moreover, we have  $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \{[A_1], \dots, [A_s], [B_1], \dots, [B_t]\}$  where, for every  $j \in \{1, \dots, s\}$ , the group  $A_j$  is contained in  $H_1$  and for every  $j \in \{1, \dots, t\}$ , the group  $B_j$  is contained in  $H_2$ . Since  $\mathcal{F}_1 \wedge \mathcal{A}(\phi)$  is a malnormal subgroup system, the set  $\{[A_1], \dots, [A_s]\}$  is a malnormal subgroup system of  $H_1$  and the set  $\{[B_1], \dots, [B_t]\}$  is a malnormal subgroup system of  $H_2$ .

Let

$$X(\mathcal{F}_1) = \text{Curr}(H_1, \{[A_1], \dots, [A_s]\}) \times \text{Curr}(H_2, \{[B_1], \dots, [B_t]\}).$$

Let  $\mu \in \text{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . We set  $\psi_1(\mu) = (\mu|_{\partial^2 H_1}, \mu|_{\partial^2 H_2}) \in X(\mathcal{F}_1)$ . Since  $\mu$  is  $F_n$ -invariant,  $\psi_1(\mu)$  does not depend on the choice of the representatives of the conjugacy classes of  $H_1$  and  $H_2$ . Let  $(\mu_1, \mu_2) \in X(\mathcal{F}_1)$ . Since the subgroup system  $\mathcal{F}_1 \wedge \mathcal{A}(\phi)$  is malnormal, for every  $j \in \{1, 2\}$ , the current  $\mu_j$  can be extended in a canonical way to a current  $\mu_j^* \in \text{Curr}(F_n, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . The current  $\mu_j^*$  is such that, for every Borel subset  $B$  of  $\partial^2(F_n, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ , we have

$$\mu_j^*(B) = \mu_j^*(B \cap \partial^2 H_j) = \mu_j(B \cap \partial^2 H_j).$$

We set  $\psi_2((\mu_1, \mu_2)) = \mu_1^* + \mu_2^*$ . By the property of  $\mu_j^*$  described above, we see that  $\psi_2((\mu_1, \mu_2)) \in \text{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . The maps  $\psi_1$  and  $\psi_2$  are clearly continuous.

**Lemma 6.5.17.** *The space  $\text{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$  is homeomorphic to  $X(\mathcal{F}_1)$ .*

**Proof.** We prove that  $\psi_1$  and  $\psi_2$  are inverse from each other. Let  $\mu \in \text{Curr}(F_n, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . Then  $\psi_2 \circ \psi_1(\mu) = (\mu|_{\partial^2 H_1})^* + (\mu|_{\partial^2 H_2})^*$ . Note that  $\mu$  and  $\psi_2 \circ \psi_1(\mu)$  coincide on Borel subsets contained in  $\partial^2 \mathcal{F}_1$ . Since both have supports contained in  $\partial^2 \mathcal{F}_1$ , they are equal. Conversely, let  $(\mu_1, \mu_2) \in X(\mathcal{F}_1)$ . Then

$$\psi_1 \circ \psi_2((\mu_1, \mu_2)) = ((\mu_1^* + \mu_2^*)|_{\partial^2 H_1}, (\mu_1^* + \mu_2^*)|_{\partial^2 H_2}).$$

But  $\mu_2^*|_{\partial^2 H_1} = 0$  and  $\mu_1^*|_{\partial^2 H_2} = 0$ . Hence we have

$$((\mu_1^* + \mu_2^*)|_{\partial^2 H_1}, (\mu_1^* + \mu_2^*)|_{\partial^2 H_2}) = (\mu_1^*|_{\partial^2 H_1}, \mu_2^*|_{\partial^2 H_2}) = (\mu_1, \mu_2).$$

This concludes the proof. □

For every  $\phi \in \text{Out}(F_n, \mathcal{F})$ , we refer to the definition of  $\mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  given above Lemma 6.3.28.

**Lemma 6.5.18.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism which satisfies Definition 6.4.3 (2). Let  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2 = \{F_n\}$  be a sequence of free factor systems given in this definition. Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  with filtration  $\emptyset = G_0 \subsetneq G_1 \subsetneq \dots \subsetneq G_k = G$  and such that there exist  $p$  and  $i$  in  $\{0, \dots, k-1\}$  such that  $\mathcal{F}(G_p) = \mathcal{F}$  and  $\mathcal{F}(G_i) = \mathcal{F}_1$ .*

(1) *The graph  $\overline{G - G_i}$  either is a topological arc whose endpoints are in  $G_i$  or it retracts onto a circuit  $C$  and there exists exactly one topological arc that connects  $C$  and  $G_i$ .*

(2) *There do not exist an EG stratum or a zero stratum of height greater than  $i$ . If  $\overline{G - G_i}$  is a topological arc, every edge in  $\overline{G - G_i}$  is contained in  $G_{PG}$ . Otherwise every edge of the circuit  $C$  in  $\overline{G - G_i}$  is contained in  $G_{PG}$ .*

(3) *Let  $\gamma$  be a path of  $G_i$  which is not contained in a concatenation of paths of  $G_{PG, \mathcal{F}_1}$  and  $\mathcal{N}_{PG, \mathcal{F}_1}$ . Then  $\gamma$  is not contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ .*

(4) *We have*

$$\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) = \bigcup_{\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))} C(\gamma).$$

*In particular, we have*

$$\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) = \mathbb{P}\text{Curr}(F_n, \mathcal{F}_1 \wedge \mathcal{A}(\phi)).$$

(5) *For every edge path  $\gamma$  in  $G$ , the value  $\ell_{\mathcal{F}_1}(\gamma) - \ell_{\text{exp}}(\gamma)$  is the number of edges of  $\overline{G - G_i}$  contained in  $\gamma$ . In particular, for every path  $\gamma$  contained in  $G_i$ , we have*

$$\ell_{\mathcal{F}_1}(\gamma) = \ell_{\text{exp}}(\gamma)$$

*and for every current  $\mu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  whose support is contained in  $\partial^2 \mathcal{F}_1$ , we have*

$$\Psi_0(\mu) = \|\mu\|_{\mathcal{F}_1}.$$

(6) *Let  $\gamma$  be a circuit in  $G$ . For every  $m \in \mathbb{N}^*$ , we have*

$$\ell_{\mathcal{F}_1}([f^m(\gamma)]) - \ell_{\text{exp}}([f^m(\gamma)]) = \ell_{\mathcal{F}_1}(\gamma) - \ell_{\text{exp}}(\gamma).$$

(7) *Suppose that  $\mathcal{F} \wedge \mathcal{A}(\phi) = \{[A_1], \dots, [A_r]\}$ . One of the following holds.*

- *There exist distinct  $i, j \in \{1, \dots, r\}$  such that*

$$\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi)) - \{[A_i], [A_j]\} \cup \{[A_i * A_j]\}.$$

- *There exists  $i \in \{1, \dots, r\}$  and an element  $g \in F_n$  such that*

$$\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi)) - \{[A_i]\} \cup \{[A_i * \langle g \rangle]\}.$$

*In that case, there exists a subgroup  $A$  of  $F_n$  such that  $\mathcal{F} = \{[A]\}$  and  $F_n = A * \langle g \rangle$ .*

- There exists  $g \in F_n$  such that  $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{\langle g \rangle\}$ . In that case, there exists a subgroup  $A$  of  $F_n$  such that  $\mathcal{F} = \{[A]\}$  and  $F_n = A * \langle g \rangle$ .

**Proof.** (1) It is a consequence of [HaM4, Lemma II.2.5]. Note that, in the terminology of [HaM4, Lemma 2.2.5], the first case is called a *one-edge extension* and the second case is called a *lollipop extension*.

(2) By Proposition 6.2.5 (4), it suffices to show that there does not exist an EG stratum of height greater than  $i$ . This follows from [BFH1, Corollary 3.2.2] (where the stratum described in it is the whole graph  $\overline{G - G_i}$ ). We now prove the second part of Assertion (2). Let  $w$  be an element of  $F_n$  represented by  $\gamma$ . Then there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and  $w \in A$ . Since  $\phi|_{\mathcal{F}_1}$  is expanding relative to  $\mathcal{F}$  but  $\phi$  is not expanding relative to  $\mathcal{F}$  by Definition 6.4.3 (2), there exists a reduced circuit  $\gamma$  in  $G$  which is not contained in  $G_i$  which has polynomial growth under iterates of  $f$ . By Proposition 6.3.13, the circuit  $\gamma$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By the first part of Assertion (2), the intersection  $\gamma \cap \overline{G - G_i}$  does not contain EG INPs, hence consists in edges in  $G_{PG}$ . Hence if  $\overline{G - G_i}$  is a lollipop, then the circuit  $C$  in  $\overline{G - G_i}$  is contained in  $\gamma$ , hence is contained in  $G_{PG}$ . If  $\overline{G - G_i}$  is a topological arc, the graph  $\overline{G - G_i}$  is contained in  $\gamma$ , hence consists in edges in  $G_{PG}$ . This proves (2).

(3) Let  $\gamma$  be as in Assertion (3). By Assertion (2), every edge of  $\overline{G - G_i}$  is contained in an NEG stratum. In particular, there does not exist an EG INP of height greater than  $i$ . Hence  $\mathcal{N}_{PG} = \mathcal{N}_{PG, \mathcal{F}_1}$ . Since  $\gamma$  is contained in  $G_i$  and since  $G_{PG} \cap G_i = G_{PG, \mathcal{F}_1}$ , the path  $\gamma$  is not contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ .

(4) Since  $\phi|_{\mathcal{F}_1}$  is expanding relative to  $\mathcal{F}$ , we see that  $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$ . Thus, we have  $\partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) = \partial^2(F_n, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . Assertion (4) then follows from Lemma 6.3.28 applied to  $\mathcal{F}_1 \wedge \mathcal{A}(\phi)$ .

(5) By Assertion (2), there does not exist an EG INP of height at least  $i + 1$ . Hence  $\ell_{\mathcal{F}_1}(\gamma)$  differs from  $\ell_{exp}(\gamma)$  by the number of edges in  $G_{PG}$  of height at least  $i + 1$ . Since every edge in  $\overline{G - G_i}$  is in  $G_{PG}$  by Assertion (2), the conclusion of the first claim of Assertion (5) follows. The claim about paths contained in  $G_i$  is then a direct consequence. Let  $\mu$  be a current in  $\text{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . By Lemma 6.5.17, there exists  $(\mu_1, \mu_2) \in X(\mathcal{F}_1)$  such that  $\mu = \mu_1^* + \mu_2^*$ . Since rational currents are dense in  $\text{Curr}(H_1, \{[A_1], \dots, [A_s]\})$  and  $\text{Curr}(H_2, \{[B_1], \dots, [B_t]\})$  by Proposition 6.2.15, linear combination of rational currents are dense in  $\text{Curr}(\mathcal{F}_1, \mathcal{F}_1 \wedge \mathcal{A}(\phi))$ . The last claim of Assertion (5) then follows from the linearity and continuity of  $\Psi_0$  and  $\|\cdot\|_{\mathcal{F}_1}$ .

(6) Let  $m \in \mathbb{N}^*$ . By Assertion (5), it suffices to prove that the number of edges in  $\overline{G - G_i}$  contained in  $[f^m(\gamma)]$  is equal to the number of edges in  $\overline{G - G_i}$  contained in  $\gamma$ . In the case that  $\overline{G - G_i}$  is a lollipop extension and that  $\gamma$  is the circuit  $C$  in  $\overline{G - G_i}$ , then  $\gamma$  is fixed by  $f$  by [HaM4, Definition I.1.29 (3)] (that is the filtration associated with  $f$  is *reduced*). Hence  $[f^m(\gamma)] = \gamma$  and the claim follows. Otherwise, if  $\overline{G - G_i}$  is either a one-edge extension or a lollipop extension, the circuit  $\gamma$  is not contained in  $\overline{G - G_i}$ . Moreover, if  $\gamma$  or  $[f^m(\gamma)]$  contains an edge in  $\overline{G - G_i}$ , then it contains  $\overline{G - G_i}$ . Hence it suffices to count the number of occurrences of  $\overline{G - G_i}$  in  $\gamma$  and  $[f^m(\gamma)]$ . Since

$f$  preserves  $G_i$ , the result follows from Assertion (1) and [BFH1, Corollary 3.2.2] (where the stratum in it is the graph  $\overline{G - G_i}$ ).

(7) Note that since  $\phi|_{\mathcal{F}_1}$  is expanding relative to  $\mathcal{F}$ , we have  $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$ . Recall the definition of the graph  $G^*$  and the map  $p_{G^*}: G^* \rightarrow G$  from above Lemma 6.3.11. By Proposition 6.3.13 and Lemma 6.3.11 (2), the malnormal subgroup system  $\mathcal{A}(\phi)$  is precisely the subgroup system associated with the fundamental groups of the connected components of  $G^*$ . Moreover, the malnormal subgroup system associated with  $\mathcal{F}_1 \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$  is the subgroup system associated with the connected components of  $p_{G^*}^{-1}(G_i)$ . By Assertion (1), the graph  $\overline{G - G_i}$  is either a topological arc or a lollipop. Suppose first that  $\overline{G - G_i}$  is a topological arc. By Assertion (2), the graph  $\overline{G - G_i}$  consists in edges in  $G_{PG}$ . Thus, the graph  $G^*$  is obtained from  $p_{G^*}^{-1}(G_i)$  by adding a topological arc  $\tau$ . If the endpoints of  $\tau$  are in two distinct connected components of  $G^*$ , then the first case of Assertion (7) occurs and otherwise the second case of Assertion (7) occurs. Moreover, if the second case occurs, the extension  $\mathcal{F} \leq \{[F_n]\}$  is an HNN extension. Thus there exists a subgroup  $A$  of  $F_n$  such that  $\mathcal{F} = \{[A]\}$ . By [BFH1, Corollary 3.2.2], one can obtain an element  $g$  of  $F_n$  such that  $F_n = A * g$  by taking a circuit in the image of  $p_{G^*}$  which contains  $\overline{G - G_i}$  exactly once. Suppose now that  $\overline{G - G_i}$  is a lollipop extension. By Assertion (2), the circuit  $C$  in  $\overline{G - G_i}$  consists in edges in  $G_{PG}$ . Thus, either  $G^*$  is obtained from  $p_{G^*}^{-1}(G_i)$  by adding a lollipop extension, or  $G^*$  is obtained from  $p_{G^*}^{-1}(G_i)$  by adding a connected component which is homotopy equivalent to a circle. If  $G^*$  is obtained from  $p_{G^*}^{-1}(G_i)$  by adding a lollipop extension, the second case of Assertion (7) occurs. If  $G^*$  is obtained from  $p_{G^*}^{-1}(G_i)$  by adding a connected component which is homotopy equivalent to a circle, the third case of Assertion (7) occurs. The proof of the fact about HNN extension is similar to the proof for the one-edge extension case. This concludes the proof.  $\square$

**Remark 6.5.19.** *By Lemma 6.5.18 (1),  $\overline{G - G_i}$  is either a topological arc or it retracts onto a circuit  $C$  and there exists exactly one topological arc that connects  $C$  and  $G_i$ . In the second case, we will adopt the convention that  $\overline{G - G_i} = C$ , so that, by Lemma 6.5.18 (2), in both cases of Lemma 6.5.18 (1), every edge in  $\overline{G - G_i}$  is in  $G_{PG}$ .*

**Lemma 6.5.20.** *Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  and let  $f: G \rightarrow G$  be as in Remark 6.5.15.*

(1) *Let  $U$  be an open neighborhood of  $\Delta_+(\phi)$ , let  $V$  be a neighborhood of  $K_{PG}(\phi)$  (see Definition 6.3.25). There exist  $N \in \mathbb{N}^*$  and  $\delta \in (0, 1)$  such that for every  $m \geq 1$  and every  $w \in F_n$  with  $\mathfrak{g}(\gamma_w) > \delta$  and  $\eta_{[w]} \notin V$ , we have*

$$(\phi^N)^m(\eta_{[w]}) \in U.$$

(2) *Suppose that  $\phi$  is an almost atoroidal outer automorphism relative to  $\mathcal{F}$  as in Definition 6.4.3 (2). Let  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2$  be an associated sequence of free factor systems.*

*For every  $\epsilon > 0$  and  $L > 0$ , there exists  $\delta \in (0, 1)$  and  $M > 0$  such that, for every  $n \geq M$ , for every nonperipheral element  $w \in F_n$  with  $\mathfrak{g}(\gamma_w) > \delta$ , there exists*

$[\mu_w] \in \Delta_+(\phi)$  such that for every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  of length at most  $L$  contained in  $G_i$ :

$$\left| \frac{\langle \gamma, [f^n(\gamma_w)] \rangle}{\ell_{exp}([f^n(\gamma_w)])} - \frac{\langle \gamma, [\mu_w] \rangle}{\|[\mu_w]\|_{\mathcal{F}_1}} \right| < \epsilon.$$

**Proof.** The proof is similar to the one of [LU2, Lemma 6.1]. By Lemma 6.5.3 and Lemma 6.5.16 (1), up to passing to a power of  $f$ , we may assume that for every  $w \in F_n$  such that  $\mathfrak{g}(\gamma_w) \geq \frac{1}{2}$ , and every  $n \in \mathbb{N}^*$ , we have  $\mathfrak{g}([f^n(\gamma_w)]) \geq \mathfrak{g}(\gamma_w)$  and

$$\ell_{exp}([f^n(\gamma_w)]) \geq TEL(n, \gamma) \geq (3^n - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w). \quad (6.9)$$

Let  $N \in \mathbb{N}^*$  be such that  $3^N > 2C$ . Let  $\lambda > 0$  be such that, for every edge  $e \in \vec{E}G$  and every  $n \in \mathbb{N}^*$ , we have

$$\ell([f^n(e)]) \leq \lambda^n. \quad (6.10)$$

By Lemma 6.3.29, a sequence  $([\nu_m])_{m \in \mathbb{N}}$  of projective relative currents tends to a projective current  $[\nu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  if for every  $\epsilon > 0$  and  $R > 0$  there exists  $M \in \mathbb{N}^*$  such that, for every  $m \geq M$  and every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  with  $\ell(\gamma) \leq R$ , we have

$$\left| \frac{\langle \gamma, \nu \rangle}{\|\nu\|_{\mathcal{F}}} - \frac{\langle \gamma, \nu_m \rangle}{\|\nu_m\|_{\mathcal{F}}} \right| < \epsilon. \quad (6.11)$$

For every  $\mathcal{F}$ -expanding splitting unit  $\sigma$ , we denote by  $\mu(\sigma)$  the corresponding current given by Proposition 6.4.4. By Lemma 6.4.8, we have  $\|\mu(\sigma)\|_{\mathcal{F}} = 1$ . Since  $\Delta_+(\phi)$  is compact by Lemma 6.4.7, there exist  $\epsilon, R > 0$  such that for every  $m \geq M$ , if there exists  $\nu \in \Delta_+(\phi)$  such that  $\nu_m, \nu, R, \epsilon$  satisfy Equation (6.11), then  $\nu_m \in U$ . Since there are only finitely many expanding splitting units of positive exponential length and finitely many edge paths  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  such that  $\ell(\gamma) \leq R$ , there exists  $M_0 \in \mathbb{N}^*$  such that for every  $m \geq M_0$ , for every expanding splitting unit  $\sigma$  and for every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  with  $\ell(\gamma) \leq R$ , we have:

$$\left| \frac{\langle \gamma, [f^m(\sigma)] \rangle}{\ell_{\mathcal{F}}([f^m(\sigma)])} - \langle \gamma, \mu(\sigma) \rangle \right| < \frac{\epsilon}{6}.$$

Recall that  $\langle \gamma, \mu(\sigma) \rangle$  is equal to  $\mu(\sigma)(C(\gamma))$  by definition of the number of occurrences of  $\gamma$  in  $\mu(\sigma)$ . Let  $\gamma'$  be a reduced edge path in  $G$ . By Lemma 6.5.6, for every reduced edge path  $\sigma$  of  $G$  contained in  $\gamma'$ , we have  $\ell_{\mathcal{F}}(\sigma) \geq \ell'_{\mathcal{F}}(\sigma) \geq \ell_{\mathcal{F}}(\sigma) - 2C$ . Hence there exists  $M_1 \in \mathbb{N}^*$  such that for every  $m \geq M_1$ , for every expanding splitting unit  $\sigma$ , for every edge path  $\gamma'$  containing  $\sigma$  as a splitting unit and for every reduced edge path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  with  $\ell(\gamma) \leq R$ , we have:

$$\left| \frac{\langle \gamma, [f^m(\sigma)] \rangle}{\ell'_{\mathcal{F}}([f^m(\sigma)])} - \langle \gamma, \mu(\sigma) \rangle \right| < \frac{\epsilon}{6}. \quad (6.12)$$



Recall the definition of the continuous function  $\Psi_0: \text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}$  given above Definition 6.3.25. Recall that, by Lemma 6.3.27 (3), for every current  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ , we have  $\|\mu\|_{\mathcal{F}} > 0$ . Let

$$\begin{aligned} \Psi: \text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) &\rightarrow \mathbb{R} \\ [\nu] &\mapsto \frac{\Psi_0(\nu)}{\|\nu\|_{\mathcal{F}}}. \end{aligned}$$

Since  $\Psi$  is continuous and since  $\mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - V$  is compact, there exists  $s > 0$  such that for every  $\nu \in \mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) - V$ , we have:

$$\Psi([\nu]) \geq s.$$

In particular, by Lemma 6.3.26, for every nonperipheral element  $w \in F_{\mathbf{n}}$  such that  $\eta_{[w]} \notin V$ , we have

$$\frac{\ell_{\text{exp}}(\gamma_w)}{\ell_{\mathcal{F}}(\gamma_w)} = \frac{\Psi_0(\eta_{[w]})}{\|\eta_{[w]}\|_{\mathcal{F}}} = \Psi([\eta_{[w]}]) \geq s. \quad (6.13)$$

Now let  $w \in F_{\mathbf{n}}$  be a nonperipheral element such that  $\mathbf{g}(\gamma_w) \geq \frac{1}{2}$  and  $\eta_{[w]} \notin V$ . Let  $\gamma_w = \alpha_0\beta_1\alpha_1 \dots \alpha_k\beta_k$  be an optimal splitting of  $\gamma_w$ , where for every  $i \in \{0, \dots, k\}$ , the path  $\alpha_i$  is an incomplete factor of  $\gamma_w$  and for every  $i \in \{1, \dots, k\}$ , the path  $\beta_i$  is a complete factor of  $\gamma$ . Using this optimal splitting, we construct another decomposition of  $\gamma_w$  (which is not necessarily a splitting of  $\gamma_w$ ). Since concatenations of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  have zero exponential length by Lemma 6.3.18, we change the decomposition in such a way that every subpath of  $\gamma_w$  which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  is in some  $\alpha_i$  for  $i \in \{1, \dots, k\}$ . In particular, for every  $i \in \{1, \dots, k\}$ , the exponential lengths of  $\beta_i$  and  $\alpha_i$  are equal to their exponential lengths relative to  $\gamma_w$ . Let  $i \in \{0, \dots, k\}$ . The path  $\alpha_i$  has a decomposition  $\alpha_i = \alpha_i^{(1)} \alpha_i^{(1')} \dots \alpha_i^{(k_i)} \alpha_i^{(k'_i)}$  where, for every  $j \in \{1, \dots, k_i\}$ , the path  $\alpha_i^{(j)}$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$  and, for every  $j \in \{1, \dots, k_i\}$ , the path  $\alpha_i^{(j')}$  is a path in  $\overline{G - G_{PG}}$  such that every edge of  $\alpha_i^{(j')}$  either has positive exponential length relative to  $\gamma_w$  or is in a zero stratum. Note that, by Proposition 6.2.5 (4), for every  $j \in \{1, \dots, k_i\}$  and every maximal subpath  $\tau$  of  $\alpha_i^{(j')}$  contained in some zero stratum, the path  $\tau$  is adjacent to a path in  $\gamma_w$  of positive exponential length. Suppose that  $\tau$  is nontrivial. Since no zero path is adjacent to a path which is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$  by Lemma 6.2.9 and Proposition 6.2.5 (4), either  $\alpha_i = \tau$  or  $\ell_{\text{exp}}(\alpha_i^{(j')}) > 0$ . In the first case, we have  $\ell(\tau) \leq C$  by definition of  $C$ . Thus, there exists  $n \in \mathbb{N}^*$  such that  $[f^n(\tau)]$  is completely split. Thus, if the first case occurs, we may suppose, up to taking a power of  $f$ , that  $\alpha_i$  is a completely split and is a splitting unit of some  $\beta_j$ . Let  $i \in \{1, \dots, k\}$ . Since  $\beta_i$  does not contain splitting units which are concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ , every splitting unit of  $\beta_i$  is an edge in  $\overline{G - G'_{PG}}$  or a maximal taken connecting path in a zero stratum. By Lemma 6.3.21, every splitting unit of  $\beta_i$  which is an edge in  $\overline{G - G'_{PG}}$  is expanding. Let  $\sigma'$  be a splitting unit of  $\beta_i$  which is a maximal taken connecting path in a zero stratum and which is not expanding. Let  $n \in \mathbb{N}^*$  be such that  $[f^n(\sigma')]$  is completely split. By

Lemma 6.3.21 and Lemma 6.3.20, the path  $[f^n(\sigma')]$  does not contain splitting units which are edges in  $\overline{G - G_{PG}}$ . If  $[f^n(\sigma')]$  contains a splitting unit which is contained in a zero stratum, then an inductive argument shows that, up to taking a larger  $n$ , the path  $[f^n(\sigma')]$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Thus, the  $\mathcal{F}$ -length of  $\sigma'$  grows at most polynomially fast under iterates of  $f$ . Thus, we see that  $\gamma_w$  has a decomposition

$$\gamma_w = a_0 b_0 a_1 c_1^{(1)} c_2^{(1)} \dots c_{k_1}^{(1)} a_2 b_2 \dots a_t c_1^{(t)} c_2^{(t)} \dots c_{k_t}^{(t)} a_{t+1} b_{t+1} a_{t+2},$$

where:

- (a) for every  $i \in \{0, \dots, t+2\}$ , the path  $a_i$  is either possibly trivial, a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or a maximal taken connecting path whose  $\mathcal{F}$ -length grows at most polynomially fast;
- (b) for every  $i \in \{0, \dots, t+1\}$ , the path  $b_i$  is a subpath of positive exponential length relative to  $\gamma_w$  of an incomplete path of  $\gamma_w$  such that every edge of  $b_i$  either has positive exponential length relative to  $\gamma_w$  or is in a zero stratum;
- (c) for every  $i \in \{1, \dots, t\}$  and every  $j \in \{1, \dots, k_i\}$ , the path  $c_j^{(i)}$  is a (possibly trivial) expanding splitting unit of a complete factor of  $\gamma_w$ .

Recall that the length of every path in a zero stratum is bounded by  $C$ . Thus, for every  $i \in \{0, \dots, t+1\}$ , we have

$$\ell(b_i) \leq C \ell_{exp}(b_i).$$

We claim that the exponential length relative to  $\gamma_w$  of one of the edges at the concatenation point of two consecutive nontrivial paths of the form  $a_i b_i$ ,  $b_i a_{i+1}$ ,  $a_i c_1^{(i)}$ ,  $c_j^{(i)} c_{j+1}^{(i)}$  or  $c_{k_i}^{(i)} a_{i+1}$  is positive. Indeed, for every  $i \in \{1, \dots, t\}$  (resp.  $i \in \{0, \dots, t+1\}$ ) and every  $j \in \{1, \dots, k_i\}$ , the path  $c_j^{(i)}$  (resp.  $b_i$ ) either has positive exponential length relative to  $\gamma_w$  or is contained in a zero stratum. Note that by hypothesis, for every  $i \in \{0, \dots, t+1\}$ , the path  $b_i$  is not contained in a zero stratum. Moreover, if  $b_i$  is adjacent to a path  $a_i$ , then the first edge of  $b_i$  is not in a zero stratum by Proposition 6.2.5 (4), Lemma 6.2.9 and the fact that the paths in zero strata that we consider in our subdivision are maximal. Hence one of the edges at the concatenation point of every path of the form  $a_i b_i$ ,  $b_i a_{i+1}$  has positive exponential length relative to  $\gamma_w$ . By maximality of the splitting units contained in zero strata, one of the splitting unit in a path  $c_j^{(i)} c_{j+1}^{(i)}$  is an edge in  $\overline{G - G'_{PG}}$ , hence has positive exponential length relative to  $\gamma_w$ . Since paths in zero strata and concatenations of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$  cannot be adjacent by Proposition 6.2.5 (4) and Lemma 6.2.9, paths of the form  $a_i c_1^{(i)}$  and  $c_{k_i}^{(i)} a_{i+1}$  have positive exponential length since in this case  $c_1^{(i)}$  or  $c_{k_i}^{(i)}$  is an edge in  $\overline{G - G'_{PG}}$ . This proves the claim.

Remark that, by construction and the definition of goodness of a reduced path, we have

$$\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{exp}(c_j^{(i)}) = \ell_{exp}(\gamma_w) \mathfrak{g}(\gamma_w).$$

Note that the length of reduced iterates of edges in  $G_{PG}$  grows at most polynomially fast, hence the  $\mathcal{F}$ -length of reduced iterates of edges in  $G_{PG}$  grows at most polynomially fast. Let  $C' > 0$  and  $k \in \mathbb{N}^*$  be such that, for every splitting unit  $\sigma'$  which is either an edge in  $G_{PG}$  or a maximal taken connecting path in a zero stratum whose  $\mathcal{F}$ -length grows at most polynomially fast, and every  $m \in \mathbb{N}^*$ , we have:

$$\ell_{\mathcal{F}}([f^m(\sigma')]) \leq C' m^k \ell_{\mathcal{F}}(\sigma').$$

The constants  $C'$  and  $k$  exist by the claim in Proposition 6.3.13. Let  $i \in \{0, \dots, t+2\}$  and let  $a_i = \alpha_0 \dots \alpha_{\ell_i}$  be a decomposition of  $a_i$  such that, for every  $j \in \{0, \dots, \ell_i\}$ ,  $\alpha_{\ell_i}$  is either an edge in  $G_{PG}$ , a path in  $\mathcal{N}_{PG}^{\max}(a_i)$  or a maximal taken connecting path in a zero stratum whose  $\mathcal{F}$ -length grows at most polynomially fast. By Lemma 6.3.16, for every  $m \in \mathbb{N}^*$ , we have

$$\ell_{\mathcal{F}}([f^m(a_i)]) \leq \sum_{j=0}^{\ell_i} \ell_{\mathcal{F}}([f^m(\alpha_j)]) \leq C' m^k \sum_{j=1}^{\ell_i} \ell_{\mathcal{F}}(\alpha_j) = C' m^k \ell_{\mathcal{F}}(a_i),$$

where the last equality follows from the fact that a path in  $\mathcal{N}_{PG}$  is contained in some subpath  $\alpha_j$  by hypothesis. In particular,

$$\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^n(a_i)]) \leq C' m^k \sum_{i=0}^{t+2} \ell_{\mathcal{F}}(a_i) \leq C' \ell_{\mathcal{F}}(\gamma_w) n^k, \quad (6.14)$$

where the last inequality follows from the fact that, by hypothesis, every path in  $\mathcal{N}_{PG}^{\max}(\gamma)$  is contained in some  $a_i$ . Thus, if  $\mathfrak{g}(\gamma_w) \geq \frac{1}{2}$ , there exists  $C'' > 0$  such that, for every  $n \geq N$ , by Equations (6.9), (6.14) and (6.13), we have:

$$\frac{\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^n(a_i)])}{\ell_{exp}([f^n(\gamma_w)])} \leq \frac{C' \ell_{\mathcal{F}}(\gamma_w) n^k}{(3^n - 2C) \mathfrak{g}(\gamma_w) \ell_{exp}(\gamma_w)} \leq \frac{C' \frac{1}{s} \ell_{exp}(\gamma_w) n^k}{(3^n - 2C) \mathfrak{g}(\gamma_w) \ell_{exp}(\gamma_w)} \leq C'' \frac{n^k}{(3^n - 2C) \mathfrak{g}(\gamma_w)}.$$

Recall that, for every reduced edge path  $\gamma$  of  $G$ , we have

$$\ell_{exp}(\gamma) \leq \ell_{\mathcal{F}}(\gamma).$$

Up to taking a larger  $N \in \mathbb{N}^*$ , we may suppose that, for every  $n \geq N$ , we have

$$C'' \frac{n^k}{(3^n - 2C) \mathfrak{g}(\gamma_w)} \leq \frac{\epsilon}{48 \mathfrak{g}(\gamma_w) R}. \quad (6.15)$$

For every  $n \geq N$  and every nonperipheral element  $w \in F_n$  such that  $\mathfrak{g}(\gamma_w) \geq \frac{1}{2}$ , by Equation (6.9), we have

$$\frac{2R \ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^n(\gamma_w)])} \leq \frac{2R \ell_{exp}(\gamma_w)}{(3^n - 2C) \mathfrak{g}(\gamma_w) \ell_{exp}(\gamma_w)} = \frac{2R}{(3^n - 2C) \mathfrak{g}(\gamma_w)}.$$

Up to taking a larger  $N$ , we may assume that for every  $n \geq N$  and every  $w \in F_n$  such that  $\mathfrak{g}(\gamma_w) \geq \frac{1}{2}$ , we have:

$$\frac{2R \ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^n(\gamma_w)])} \leq \frac{2R}{(3^n - 2C)\mathfrak{g}(\gamma_w)} \leq \frac{\epsilon}{12\mathfrak{g}(\gamma_w)}. \quad (6.16)$$

Let

$$\delta = \max \left\{ \frac{1}{1 + \frac{\epsilon}{6}}, \frac{1}{1 + \frac{2RC\epsilon\lambda^N}{(3^N - 2C)6}}, \frac{1}{2} \right\}.$$

Thus, in order to prove the first assertion of Lemma 6.5.20, it suffices to show that for every  $m \geq N$  and every  $w \in F_{\mathfrak{n}}$  such that  $\mathfrak{g}(\gamma_w) > \delta$  and  $\eta_{[w]} \notin V$ , the projective current  $[\nu_m] = \phi^m([\eta_w])$  is close to an element  $[\nu]$  in  $\Delta_+(\phi)$  in the sense of Equation (6.11). Since the goodness function is monotone by Remark 6.5.15, it suffices to prove it for  $m = N$ . Let  $w \in F_{\mathfrak{n}}$  such that  $\mathfrak{g}(\gamma_w) > \delta$  and  $\eta_{[w]} \notin V$ .

By Equation (6.15) and the fact that  $\mathfrak{g}(\gamma_w) \geq \delta \geq \frac{1}{2}$ , we have

$$\begin{aligned} \frac{\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} &\leq \frac{\sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\ell_{exp}([f^n(\gamma_w)])} \\ &\leq C'' \frac{n^k}{(3^N - 2C)\mathfrak{g}(\gamma_w)} \leq C'' \frac{n^k}{(3^N - 2C)\delta} \leq \frac{\epsilon}{24R}. \end{aligned} \quad (6.17)$$

Moreover, by Equation (6.16) and the fact that  $\mathfrak{g}(\gamma_w) \geq \delta \geq \frac{1}{2}$ , we have

$$\frac{2R \ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leq \frac{\epsilon}{6}. \quad (6.18)$$

Note that, for every  $w \in F_{\mathfrak{n}}$  such that  $\mathfrak{g}(\gamma_w) > \delta$  and  $\eta_{[w]} \notin V$ , we have:

$$\begin{aligned} \frac{2RC\lambda^N(1 - \mathfrak{g}(\gamma_w))\ell_{exp}(\gamma_w)}{(3^N - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} &= 2RC \frac{\lambda^N}{3^N - 2C} \left( \frac{1}{\mathfrak{g}(\gamma_w)} - 1 \right) \\ &\leq 2RC \frac{\lambda^N}{3^N - 2C} \left( \frac{1}{\delta} - 1 \right) \leq \frac{\epsilon}{6}, \end{aligned} \quad (6.19)$$

where the last inequality follows from the definition of  $\delta$ .

Let  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$  be of length at most  $R$ . By the triangle inequality, we have

$$\begin{aligned}
& \left| \frac{\langle \gamma, [f^N(\gamma_w)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \frac{\langle \gamma, \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})]) \mu(c_j^{(i)}) \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} \right| \\
& \leq \left| \frac{\langle \gamma, [f^N(\gamma_w)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \sum_{i=1}^t \sum_{j=1}^{k_i} \frac{\langle \gamma, [f^N(c_j^{(i)})] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \right| \\
& + \left| \sum_{i=1}^t \sum_{j=1}^{k_i} \frac{\langle \gamma, [f^N(c_j^{(i)})] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} \right| \\
& + \left| \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} - \frac{\langle \gamma, \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})]) \mu(c_j^{(i)}) \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma_w)]}([f^N(c_j^{(i)})])} \right|. \tag{6.20}
\end{aligned}$$

Note that an occurrence of  $\gamma$  or  $\gamma^{-1}$  in  $[f^N(\gamma_w)]$  might happen either in some  $[f^N(c_j^{(i)})]$  or in some  $[f^N(a_i)]$  or in some  $[f^N(b_i)]$  or it might cross over the concatenation points. Recall that one of the edges at the concatenation point of paths of the form  $a_i b_i$ ,  $b_i a_{i+1}$ ,  $a_i c_1^{(i)}$ ,  $c_j^{(i)} c_{j+1}^{(i)}$  or  $c_{k_i}^{(i)} a_{i+1}$  has positive exponential length relative to  $\gamma_w$ . Recall also that the length of  $\gamma$  is at most equal to  $R$ . Thus the number of such crossings is at most  $2R\ell_{exp}(\gamma_w)$ . Thus:

$$\left| \frac{\langle \gamma, [f^N(\gamma_w)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} - \sum_{i=1}^t \sum_{j=1}^{k_i} \frac{\langle \gamma, [f^N(c_j^{(i)})] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \right| \leq \frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \sum_{i=0}^{t+2} \frac{\langle \gamma, [f^N(a_i)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \sum_{i=0}^{t+1} \frac{\langle \gamma, [f^N(b_i)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])}.$$

Since  $\gamma$  is not contained in a concatenation of paths in  $G_{PG,\mathcal{F}}$  and  $\mathcal{N}_{PG,\mathcal{F}}$ , if  $\gamma$  is contained in  $[f^N(a_i)]$  for  $i \in \{1, \dots, t+1\}$ , then  $\gamma$  contains an edge of  $[f^N(a_i)]$  of positive  $\mathcal{F}$ -length relative to  $[f^N(a_i)]$ . Hence we have  $\langle \gamma, [f^N(a_i)] \rangle \leq \ell_{\mathcal{F}}([f^N(a_i)])$ . By Equations (6.18) and (6.17) with  $n = N$ , we have

$$\frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \sum_{i=0}^{t+2} \frac{\langle \gamma, [f^N(a_i)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leq \frac{2R\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} + \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(a_i)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leq \frac{\epsilon}{4}.$$

Moreover, since for every  $i \in \{0, \dots, t+1\}$ , we have  $\ell(b_i) \leq C\ell_{exp}(b_i)$  and by Equations (6.9), (6.13) and (6.19), we see that:

$$\begin{aligned}
\sum_{i=0}^{t+1} \frac{\langle \gamma, [f^N(b_i)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} & \leq \sum_{i=0}^{t+1} \frac{\ell([f^N(b_i)])}{\ell_{\mathcal{F}}([f^N(\gamma_w)])} \leq \sum_{i=0}^{t+1} \frac{C\lambda^N \ell_{exp}(b_i)}{(3^N - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} \\
& \leq \frac{C\lambda^N (1 - \mathfrak{g}(\gamma_w))\ell_{exp}(\gamma_w)}{(3^N - 2C)\mathfrak{g}(\gamma_w)\ell_{exp}(\gamma_w)} \leq \frac{\epsilon}{6}.
\end{aligned}$$

For the third term of Inequality (6.20), note that, since  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , it is not contained in a concatenation of paths in  $G_{PG, \mathcal{F}}$  and in  $\mathcal{N}_{PG, \mathcal{F}}$ . Hence an occurrence of  $\gamma$  always appear with an edge  $e$  of  $c$  such that  $\ell_{\mathcal{F}}^c(e) = 1$ . Since  $\ell(\gamma) \leq R$ , such an edge  $e$  can be crossed by at most  $R$  occurrences of  $\gamma$  in  $c$ . Thus, for every reduced edge path  $c$  in  $G$ , we have  $\langle \gamma, c \rangle \leq 2R\ell_{\mathcal{F}}(c)$ . Hence we have

$$\left| \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \leq 2R.$$

Since

$$\ell_{\mathcal{F}}([f^N(\gamma w)]) = \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) + \sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(a_i b_i a_{i+1})]),$$

using Lemma 6.5.3 and Lemma 6.5.6 for the last inequality we have:

$$\begin{aligned} & \left| \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma w)])} - \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \\ &= \left| \frac{\left( \sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle \right) \left( \sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(a_i b_i a_{i+1})]) \right)}{\left( \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) \right) \left( \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) + \sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(a_i b_i a_{i+1})]) \right)} \right| \\ &\leq \left| \frac{\left( \sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle \right) \left( \sum_{i=0}^{t+1} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(a_i b_i a_{i+1})]) \right)}{\left( \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) \right) \left( \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) \right)} \right| \\ &\leq \left| \frac{\left( \sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle \right) \left( \sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) + 2 \sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)]) \right)}{\left( \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) \right) \left( \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) \right)} \right| \\ &\leq 2R \left| \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) + 2 \sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right|. \end{aligned}$$

Recall that we have

$$\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{exp}(c_j^{(i)}) = \ell_{exp}(\gamma w) \mathfrak{g}(\gamma w)$$

and, for every  $i \in \{1, \dots, t\}$  and every  $j \in \{1, \dots, k_i\}$ , we have either  $\ell_{exp}(c_j^{(i)}) = 1$  or  $\ell_{exp}(c_j^{(i)}) = 0$ . Hence, we have:

$$\begin{aligned} \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) &\geq \sum_{i=1}^t \sum_{j=1}^{k_i} (\ell_{\mathcal{F}}([f^N(c_j^{(i)})]) - 2C) \geq \sum_{i=1}^t \sum_{j=1}^{k_i} (3^N - 2C) \\ &\geq (3^N - 2C) \mathfrak{g}(\gamma w) \ell_{exp}(\gamma w), \end{aligned}$$

where the first inequality follows from Lemma 6.5.6 and the second inequality follows from the fact that  $f$  is  $3K$ -expanding and  $K \geq 1$ . Thus, we have

$$\begin{aligned}
& 2R \left| \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) + 2 \sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \\
& \leq 2R \left| \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) + 2 \sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \\
& \leq 2R \left| \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)])}{(3^N - 2C)\mathfrak{g}(\gamma w)\ell_{exp}(\gamma w)} \right| + 2R \left| \frac{2 \sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{(3^N - 2C)\delta\ell_{exp}(\gamma w)} \right|.
\end{aligned}$$

By Equation (6.10), we have

$$\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)]) \leq \sum_{i=0}^{t+1} \ell([f^N(b_i)]) \leq \lambda^N \sum_{i=0}^{t+1} \ell(b_i) \leq C\lambda^N \sum_{i=0}^{t+1} \ell_{exp}(b_i) \leq C\lambda^N \ell_{exp}(\gamma w)(1 - \mathfrak{g}(\gamma w)).$$

Hence we have:

$$\begin{aligned}
& 2R \left| \frac{\sum_{i=0}^{t+1} \ell_{\mathcal{F}}([f^N(b_i)])}{(3^N - 2C)\mathfrak{g}(\gamma w)\ell_{exp}(\gamma w)} \right| + 2R \left| \frac{2 \sum_{i=0}^{t+2} \ell_{\mathcal{F}}([f^N(a_i)])}{(3^N - 2C)\delta\ell_{exp}(\gamma w)} \right| \\
& \leq 2R \left| \frac{C\lambda^N(1 - \mathfrak{g}(\gamma w))\ell_{exp}(\gamma w)}{(3^N - 2C)\mathfrak{g}(\gamma w)\ell_{exp}(\gamma w)} \right| + 2R \left| \frac{2C'\ell_{\mathcal{F}}(\gamma w)n^k}{(3^N - 2C)\delta\ell_{exp}(\gamma w)} \right| \text{ by Equation (6.14)} \\
& \leq 2R \left| \frac{C\lambda^N(1 - \mathfrak{g}(\gamma w))\ell_{exp}(\gamma w)}{(3^N - 2C)\mathfrak{g}(\gamma w)\ell_{exp}(\gamma w)} \right| + 2R \left| \frac{2C''n^k}{(3^N - 2C)\delta} \right| \\
& \leq \frac{2\epsilon}{6} \text{ by Equation (6.17) and (6.19).}
\end{aligned}$$

Finally, using Equation (6.12) and the fact that for every  $i \in \{1, \dots, t\}$  and every  $j \in \{1, \dots, k_i\}$ , the splitting unit  $c_j^{(i)}$  is expanding, we have:

$$\begin{aligned}
& \left| \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \langle \gamma, [f^N(c_j^{(i)})] \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} - \frac{\langle \gamma, \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])\mu(c_j^{(j)}) \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \\
& = \left| \frac{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})]) \left( \frac{\langle \gamma, [f^N(c_j^{(i)})] \rangle}{\ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} - \langle \gamma, \mu(c_j^{(j)}) \rangle \right)}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \\
& \leq \frac{\frac{\epsilon}{6} \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} = \frac{\epsilon}{6}.
\end{aligned}$$

Combining all inequalities, we have

$$\left| \frac{\langle \gamma, [f^N(\gamma w)] \rangle}{\ell_{\mathcal{F}}([f^N(\gamma w)])} - \frac{\langle \gamma, \sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])\mu(c_j^{(j)}) \rangle}{\sum_{i=1}^t \sum_{j=1}^{k_i} \ell_{\mathcal{F}}^{[f^N(\gamma w)]}([f^N(c_j^{(i)})])} \right| \leq \frac{\epsilon}{4} + \frac{\epsilon}{6} + \frac{2\epsilon}{6} + \frac{\epsilon}{6} \leq \epsilon.$$

This concludes the proof of Assertion (1) of Lemma 6.5.20 since for every  $i \in \{1, \dots, t\}$  and every  $j \in \{1, \dots, k_i\}$ , we have  $\mu(c_j^{(i)}) \in \Delta_+(\phi)$ .

The proof of Assertion (2) is the same one as the proof of Assertion (1), replacing  $\ell_{\mathcal{F}}$  and  $\ell_{\mathcal{F}}^{\gamma}$  by  $\ell_{exp}$  and  $\ell_{exp}^{\gamma}$ , adding the following arguments. Let  $\gamma$  and  $w \in F_n$  be as in

Assertion (2). Then  $\gamma$  is not contained in a contonation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  by Lemma 6.5.18 (3). If

$$\gamma_w = a_0 b_0 a_1 c_1^{(1)} c_2^{(1)} \dots c_{k_1}^{(1)} a_2 b_2 \dots a_t c_1^{(t)} c_2^{(t)} \dots c_{k_t}^{(t)} a_{t+1} b_{t+1} a_{t+2},$$

is the same decomposition of  $\gamma_w$  as in the proof of Assertion (1), then for every  $m \in \mathbb{N}$  and every  $i \in \{1, \dots, t+2\}$ , the path  $\gamma$  is not contained in  $[f^m(a_i)]$  by Lemma 6.3.9. Similarly, for every  $m \in \mathbb{N}^*$  and every  $i \in \{1, \dots, t+2\}$ , we have  $\ell_{exp}([f^m(a_i)]) = 0$ . Hence we do not need Equation (6.17). By Lemma 6.5.18 (5), we have

$$\ell_{exp}(\gamma) = \ell_{\mathcal{F}_1}(\gamma).$$

Moreover, by Lemma 6.5.18 (5), for every current  $[\mu] \in \Delta_+(\phi)$ , we have  $\Psi_0(\mu) = \|\mu\|_{\mathcal{F}_1}$ . Replacing  $\ell_{\mathcal{F}}$  and  $\ell_{\mathcal{F}}^{\gamma}$  by  $\ell_{exp}$  and  $\ell_{exp}^{\gamma}$  in the equations in the proof of Assertion (1) concludes the proof.  $\square$

For the next lemma, we need to compute the exponential length of incomplete segments in a circuit  $\gamma$  in  $G$ . Let  $\ell_{exp}(\text{Inc}(\gamma))$  be the sum of the exponential lengths of the incomplete segments of an optimal splitting of  $\gamma$ . Let  $\ell_{exp}^{\gamma}(\text{Inc}(\gamma))$  be the sum of the exponential lengths relative to  $\gamma$  of the incomplete segments of an optimal splitting of  $\gamma$ . Note that  $\ell_{exp}^{\gamma}(\text{Inc}(\gamma))$  do not depend on the choice of an optimal splitting. Note that

$$\ell_{exp}^{\gamma}(\text{Inc}(\gamma)) = (1 - \mathfrak{g}(\gamma))\ell_{exp}(\gamma) \leq \ell_{exp}(\gamma).$$

**Lemma 6.5.21.** *Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  and let  $f: G \rightarrow G$  be as in Remark 6.5.15. Let  $\delta \in (0, 1)$ , and let  $R > 1$ . There exists  $n_0 \in \mathbb{N}^*$  such that for every  $n \geq n_0$  and every nonperipheral element  $w \in F_n$  such that  $\eta_{[w]} \notin K_{PG}(\phi)$ , we either have*

$$\mathfrak{g}([f^n(\gamma_w)]) \geq \delta$$

or

$$\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq \frac{10C}{R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \text{ and } \ell_{exp}([f^n(\gamma_w)]) \leq \frac{10C}{(1-\delta)R} \ell_{exp}(\gamma_w).$$

**Proof.** Let  $w \in F_n$  be a nonperipheral element such that  $\eta_{[w]} \notin K_{PG}(\phi)$ . Suppose that  $n \in \mathbb{N}^*$  is such that  $\mathfrak{g}([f^n(\gamma_w)]) < \delta$ . Assuming for now that we have proved that

$$\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq \frac{10C}{R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)),$$

we deduce that  $\ell_{exp}([f^n(\gamma_w)]) \leq \frac{10C}{(1-\delta)R} \ell_{exp}(\gamma_w)$ . Indeed, we have

$$\ell_{exp}^{[f^n(\gamma)]}(\text{Inc}([f^n(\gamma)])) = (1 - \mathfrak{g}([f^n(\gamma)]))\ell_{exp}([f^n(\gamma)]) \geq (1 - \delta)\ell_{exp}([f^n(\gamma)]).$$

Thus we have

$$\begin{aligned} \ell_{exp}([f^n(\gamma_w)]) &\leq \frac{1}{1-\delta} \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq \frac{10C}{(1-\delta)R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \\ &\leq \frac{10C}{(1-\delta)R} \ell_{exp}(\gamma_w). \end{aligned}$$



Therefore, it suffices to prove that there exists  $n_0 \in \mathbb{N}^*$  such that for every  $n \geq n_0$ , if  $\mathfrak{g}([f^n(\gamma_w)]) < \delta$ , then

$$\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq \frac{10C}{R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)).$$

Consider an optimal splitting  $\gamma_w = \alpha'_0 \beta'_1 \alpha'_1 \dots \alpha'_m \beta'_m$ , where for every  $i \in \{0, \dots, m\}$ , the path  $\alpha'_i$  is an incomplete factor of  $\gamma_w$  and for every  $i \in \{0, \dots, m\}$ , the path  $\beta'_i$  is a  $PG$ -relative complete factor of  $\gamma_w$ . We can modify the splitting of  $\gamma_w$  in a new splitting  $\gamma_w = \alpha_0 \beta_1 \alpha_1 \dots \beta_k \alpha_k$  where:

- (i) for every  $i \in \{0, \dots, k\}$ , the path  $\alpha_i$  is a concatenation of incomplete factors and complete factors of zero exponential length relative to  $\gamma_w$  of the old splitting;
- (ii) for every  $i \in \{1, \dots, k\}$ , the path  $\beta_i$  is a complete factor of positive exponential length relative to  $\gamma_w$  of the old splitting.

In the remainder of the proof, we still refer to the paths  $\alpha_i$  as incomplete factors. By the last claim of Remark 6.5.15, we may suppose that  $\mathfrak{g}(\gamma_w) < \delta$ , that is:

$$\ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) = \sum_{i=0}^k \ell_{exp}^{\gamma_w}(\alpha_i) \geq (1 - \delta) \ell_{exp}(\gamma_w). \quad (6.21)$$

**Claim.** For every  $i \in \{0, \dots, k\}$  and every  $m \in \mathbb{N}^*$ , we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\text{Inc}([f^m(\alpha_i)])) \leq 24C^2 \ell_{exp}^{\gamma_w}(\alpha_i).$$

Similarly, for every  $m \in \mathbb{N}^*$ , we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\text{Inc}([f^m(\gamma_w)])) \leq 24C^2 \ell_{exp}(\gamma_w).$$

**Proof.** Since a reduced iterate of a complete factor is complete, every incomplete factor of  $[f^m(\gamma_w)]$  is contained in a reduced iterate of some  $\alpha_i$ . Thus, we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\text{Inc}([f^m(\gamma_w)])) \leq \sum_{i=0}^k \ell_{exp}^{[f^m(\gamma_w)]}(\text{Inc}([f^m(\alpha_i)])).$$

Hence it suffices to prove the result for the paths  $\alpha_i$  with  $i \in \{0, \dots, k\}$ . By Property (ii) for every  $i \in \{1, \dots, k\}$ , the path  $\beta_i$  has positive exponential length relative to  $\gamma_w$ . Therefore, if there exists  $\gamma' \in \mathcal{N}_{PG}^{\max}(\gamma_w)$  such that  $\alpha_i$  intersects  $\gamma'$  nontrivially, then  $\gamma'$  is contained in  $\beta_i \alpha_i \beta_{i+1}$ . In particular, Lemma 6.5.13 applies and for every  $i \in \{0, \dots, k\}$ , if  $\ell_{exp}^{\gamma_w}(\alpha_i) = 0$ , then  $\ell_{exp}(\alpha_i) = 0$ .

Let  $i \in \{0, \dots, k\}$ . Suppose first that  $\ell_{exp}^{\gamma_w}(\alpha_i) = 0$ . By the above, we have  $\ell_{exp}(\alpha_i) = 0$ . By Lemma 6.5.12, there exists  $N \in \mathbb{N}^*$  such that for every  $m \geq N$ , such that the total exponential length of incomplete factors in any optimal splitting of  $[f^m(\alpha_i)]$  is equal to 0. Hence for every  $m \geq N$ , the path  $[f^m(\alpha_i)]$  is  $PG$ -relative completely split. Up to

taking a power of  $f$ , we may assume that  $N = 1$ . So this concludes the proof of the claim in the case when  $\ell_{exp}^{\gamma_w}(\alpha_i) = 0$ .

So we may assume that  $\ell_{exp}^{\gamma_w}(\alpha_i) > 0$ . By Lemma 6.5.12, for every  $m \in \mathbb{N}^*$ , the total exponential length of incomplete factors in  $[f^m(\alpha_i)]$  is at most equal to  $8C\ell_{exp}(\alpha_i)$ . By Lemma 6.5.6, for every  $i \in \{1, \dots, k\}$ , we have

$$\ell_{exp}(\alpha_i) \leq \ell_{exp}^{\gamma_w}(\alpha_i) + 2C \leq 3C\ell_{exp}^{\gamma_w}(\alpha_i).$$

Hence again by Lemma 6.5.6, we have

$$\ell_{exp}^{[f^m(\gamma_w)]}(\text{Inc}([f^m(\alpha_i)])) \leq \ell_{exp}(\text{Inc}([f^m(\alpha_i)])) \leq 24C^2\ell_{exp}^{\gamma_w}(\alpha_i).$$

This proves the claim.  $\square$

Let  $\Lambda_{\gamma_w}$  be the set consisting in all incomplete factors  $\alpha_i$  of  $\gamma_w$  whose exponential length relative to  $\gamma_w$  is at least equal to  $(3.10^8)R^6C^{12} + 1$ . Let  $\Lambda'_{\gamma_w}$  be the set consisting in all incomplete factors  $\alpha_i$  of  $\gamma_w$  which are not in  $\Lambda_{\gamma_w}$ . Let  $\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})$  (resp.  $\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})$ ) be the sum of the exponential lengths relative to  $\gamma_w$  of all incomplete factors of  $\gamma$  that belongs to  $\Lambda_{\gamma_w}$  (resp.  $\Lambda'_{\gamma_w}$ ). We distinguish between two cases, according to the proportion of  $\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})$  in the exponential length relative to  $\gamma_w$  of incomplete factors in  $\gamma_w$ .

**Case 1** Suppose that

$$\frac{\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))} < \frac{1}{(24C^2R)^2}.$$

This implies that

$$\frac{\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))} \geq \frac{(24C^2R)^2 - 1}{(24C^2R)^2}. \quad (6.22)$$

Note that, by Lemma 6.5.6, every path in  $\Lambda'_{\gamma_w}$  has exponential length at most equal to  $(3.10^8)C^{12}R^6 + 1 + 2C$ . By Lemma 6.5.11, there exists  $n_0 \in \mathbb{N}^*$  such that, for every edge path  $\beta$  of exponential length at most equal to  $(3.10^8)R^6C^{12} + 1 + 2C$  and every  $n \geq n_0$  either  $[f^n(\beta)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or  $[f^{n_0}(\beta)]$  contains a complete factor of exponential length at least equal to  $10C$ . By Lemma 6.5.6, in the second case, the path  $[f^{n_0}(\beta)]$  has a complete factor of positive exponential length relative to  $[f^{n_0}(\beta)]$ . Let  $\Gamma_{\gamma_w}$  be the set consisting in all incomplete paths  $\alpha_i$  of  $\gamma_w$  such that  $\alpha_i \in \Lambda'_{\gamma_w}$  and  $[f^{n_0}(\alpha_i)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Let  $\Gamma'_{\gamma_w}$  be the set consisting in all incomplete paths  $\alpha_i$  of  $\gamma_w$  such that  $\alpha_i \in \Lambda'_{\gamma_w}$  and  $[f^{n_0}(\alpha_i)]$  has at least one complete factor of positive exponential length relative to  $[f^{n_0}(\alpha_i)]$ . Note that  $\Lambda'_{\gamma_w} = \Gamma_{\gamma_w} \cup \Gamma'_{\gamma_w}$ . Let  $\ell_{exp}^{\gamma_w}(\Gamma_{\gamma_w})$  (resp.  $\ell_{exp}^{\gamma_w}(\Gamma'_{\gamma_w})$ ) be the sum of the exponential lengths relative to  $\gamma_w$  of paths in  $\Gamma_{\gamma_w}$  (resp.  $\Gamma'_{\gamma_w}$ ).

**Subcase 1** Suppose that

$$\frac{\ell_{exp}^{\gamma_w}(\Gamma_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w})} \geq \frac{24C^2R}{24C^2R + 1}.$$

Then

$$\ell_{exp}^{\gamma_w}(\Gamma_{\gamma_w}) \geq \frac{24C^2R}{24C^2R+1} \ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w}) \geq \frac{24C^2R-1}{24C^2R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)).$$

Note that, for every  $n \geq n_0$  and every path  $\alpha_i \in \Gamma_{\gamma_w}$ , we have  $\ell_{exp}([f^n(\alpha_i)]) = 0$  by Lemma 6.3.17. By the claim, for every path  $\alpha_i$  such that  $\alpha_i \in \Lambda'_{\gamma_w}$  and  $\alpha_i \notin \Gamma_{\gamma_w}$ , and for every  $n \in \mathbb{N}^*$ , the total exponential length of incomplete factors in  $[f^n(\alpha_i)]$  relative to  $[f^n(\alpha_i)]$  is at most equal to  $24C^2 \ell_{exp}^{\gamma_w}(\alpha_i)$ . Thus, for every  $n \geq n_0$ , we have:

$$\begin{aligned} \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) &\leq \sum_{\alpha_i \in \Lambda_{\gamma_w} \cup \Lambda'_{\gamma_w}} \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\alpha_i)])) \\ &\leq \sum_{\alpha_i \in \Lambda_{\gamma_w} \cup (\Lambda'_{\gamma_w} - \Gamma_{\gamma_w})} 24C^2 \ell_{exp}^{\gamma_w}(\alpha_i) \\ &\leq 24C^2 \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) - 24C^2 \frac{24C^2R-1}{24C^2R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \\ &\leq \frac{1}{R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)). \end{aligned}$$

This concludes the proof of Lemma 6.5.21 when Subcase 1 occurs.

**Subcase 2** Suppose that

$$\frac{\ell_{exp}(\Gamma_{\gamma_w})}{\ell_{exp}(\Lambda'_{\gamma_w})} < \frac{24C^2R}{24C^2R+1}.$$

Note that the assumption of Subcase 2 and Equation (6.22) imply that

$$\ell_{exp}^{\gamma_w}(\Gamma'_{\gamma_w}) \geq \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\Lambda'_{\gamma_w}) \geq \frac{(24C^2R)^2 - 1}{(24C^2R)^2} \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)).$$

Since every path in  $\Gamma'_{\gamma_w}$  has exponential length at most equal to  $(3 \cdot 10^8)R^6C^{12} + 1 + 2C$ , by Lemma 6.5.7, up to taking a larger  $n_0$ , for every path  $\alpha_i \in \Gamma'_{\gamma_w}$  such that  $\ell_{exp}(\alpha_i) > 0$  and every  $n \geq n_0$ , the exponential length of a complete factor in  $[f^n(\alpha_i)]$  is at least equal to  $3^{n-n_0} \ell_{exp}(\alpha_i)$ . Moreover, for every path  $\alpha_i \in \Gamma'_{\gamma_w}$  such that  $\ell_{exp}(\alpha_i) = 0$  and every  $n \geq n_0$ , the exponential length of a complete factor in  $[f^n(\alpha_i)]$  is at least equal to  $3^{n-n_0}$ . By Lemma 6.5.6, for every  $n \geq n_0$  and every path  $\alpha_i \in \Gamma'_{\gamma_w}$  such that  $\ell_{exp}(\alpha_i) > 0$ , the exponential length relative to  $[f^n(\alpha_i)]$  of a complete factor in  $[f^n(\alpha_i)]$  is at least equal to

$$3^{n-n_0} \ell_{exp}(\alpha_i) - 2C \geq (3^{n-n_0} - 2C) \ell_{exp}(\alpha_i).$$

Thus, for every  $n \geq n_0$  and every path  $\alpha_i \in \Gamma'_{\gamma_w}$ , the exponential length relative to  $[f^n(\alpha_i)]$  of a complete factor in  $[f^n(\alpha_i)]$  is at least equal to

$$(3^{n-n_0} - 2C) \ell_{exp}(\alpha_i).$$

Therefore, for every  $n \geq n_0$ , the sum of the exponential lengths of complete factors in  $[f^n(\gamma_w)]$  is at least equal to

$$(3^{n-n_0} - 2C) \ell_{exp}^{\gamma_w}(\Gamma'_{\gamma_w}) \geq (3^{n-n_0} - 2C) \frac{(24C^2R)^2 - 1}{(24C^2R)^2} \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)). \quad (6.23)$$

By the claim, for every  $n \in \mathbb{N}^*$ , we have  $\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq 24C^2 \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))$ . Recall that the goodness function is a supremum over splittings of the considered path. Thus, by Equation (6.23) for every  $n \geq n_0$ , since the maps  $t \mapsto \frac{t}{t+a}$  are nonincreasing for every  $a > 0$ , we have

$$\begin{aligned} \mathfrak{g}([f^n(\gamma_w)]) &\geq \frac{(3^{n-n_0}-2C) \frac{(24C^2R)^2-1}{(24C^2R)^2} \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))}{(3^{n-n_0}-2C) \frac{(24C^2R)^2-1}{(24C^2R)^2} \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) + \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)]))} \\ &\geq \frac{(3^{n-n_0}-2C) \frac{(24C^2R)^2-1}{(24C^2R)^2} \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))}{(3^{n-n_0}-2C) \frac{(24C^2R)^2-1}{(24C^2R)^2} \frac{1}{24C^2R+1} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) + 24C^2 \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))} \\ &\geq \frac{(3^{n-n_0}-2C) \frac{(24C^2R)^2-1}{(24C^2R)^2} \frac{1}{24C^2R+1}}{(3^{n-n_0}-2C) \frac{(24C^2R)^2-1}{(24C^2R)^2} \frac{1}{24C^2R+1} + 24C^2}, \end{aligned}$$

which goes to 1 as  $n$  goes to infinity. Hence there exists  $n_1 \in \mathbb{N}$  which is independent of  $\gamma_w$ , such that, for every path  $\gamma_w$  as in Subcase 2 and every  $n \geq n_1$ , we have:  $\mathfrak{g}([f^n(\gamma_w)]) \geq \delta$ . This concludes the proof of Lemma 6.5.21 when Case 1 occurs.

**Case 2** Suppose that, contrarily to Case 1, we have

$$\frac{\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w})}{\ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w))} \geq \frac{1}{(24C^2R)^2}.$$

Let  $\alpha \in \Lambda_{\gamma_w}$  and consider the decomposition of the reduced path  $\alpha$  into maximal subsegments  $\alpha^{(1)} \dots \alpha^{(k_\alpha)}$  of exponential length relative to  $\gamma_w$  equal to  $2000R^3C^6$ , except possibly the last one of exponential length relative to  $\gamma_w$  less than or equal to  $2000R^3C^6$ . Let

$$\Lambda_{\gamma_w}^{(1)} = \left\{ \alpha^{(j)} \mid \alpha \in \Lambda_{\gamma_w}, j \in \{1, \dots, k_\alpha\}, \ell_{exp}^{\gamma_w}(\alpha^{(j)}) = 2000R^3C^6 \right\},$$

$$\Lambda_{\gamma_w}^{(2)} = \left\{ \alpha^{(j)} \mid \alpha \in \Lambda_{\gamma_w}, j \in \{1, \dots, k_\alpha\}, \ell_{exp}^{\gamma_w}(\alpha) < 2000R^3C^6 \right\}.$$

Note that, since for every  $\alpha \in \Lambda_{\gamma_w}$ , we have  $\ell_{exp}^{\gamma_w}(\alpha) \geq (3.10^8)R^6C^{12} + 1$ , we see that

$$|\Lambda_{\gamma_w}^{(1)}| \geq 120000R^3C^6 |\Lambda_{\gamma_w}^{(2)}|. \quad (6.24)$$

Note that every element in  $\Lambda_{\gamma_w}^{(1)} \cup \Lambda_{\gamma_w}^{(2)}$  has exponential length at most equal to  $2000R^3C^6 + 1 + 2C$  by Lemma 6.5.6. By Lemma 6.5.11, there exists  $M \in \mathbb{N}^*$  depending only on  $f$  such that for every  $n \geq M$  and every reduced edge path  $\alpha$  of exponential length at most equal to  $(3.10^8)R^6C^{12} + 1 + 2C$ , either  $[f^n(\alpha)]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  or the following holds:

- (a) there exists a complete factor of  $[f^n(\alpha)]$  whose exponential length is at least equal to  $10C$ ;
- (b) the exponential length of an incomplete factor of  $[f^n(\alpha)]$  is at most equal to  $8C$ .

This applies in particular to every element  $\alpha \in \Lambda_{\gamma_w}^{(1)} \cup \Lambda_{\gamma_w}^{(2)}$  and to every element  $\alpha \in \Lambda'_{\gamma_w}$ . For every  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(1)}$  and every  $n \geq M$ , let  $\alpha^{(j,n)}$  be the (possibly degenerate) subpath of  $[f^n(\alpha^{(j)})]$  contained in  $[f^n(\alpha)]$ . Let  $\Lambda_{\gamma_w}^{(3)}$  be the subset of  $\Lambda_{\gamma_w}^{(1)}$  consisting in all  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(1)}$  such that  $\ell_{exp}(\alpha^{(j,M)}) \leq 80C^2$ , and let  $\Lambda_{\gamma_w}^{(4)} = \Lambda_{\gamma_w}^{(1)} - \Lambda_{\gamma_w}^{(3)}$ .

Suppose first that

$$|\Lambda_{\gamma_w}^{(4)}| > \frac{1}{30000R^3C^6} |\Lambda_{\gamma_w}^{(3)}|. \quad (6.25)$$

Therefore, as  $|\Lambda_{\gamma_w}^{(1)}| = |\Lambda_{\gamma_w}^{(3)}| + |\Lambda_{\gamma_w}^{(4)}|$ , by Equation (6.24), we have

$$|\Lambda_{\gamma_w}^{(2)}| \leq \frac{30001R^3C^6}{120000R^3C^6} |\Lambda_{\gamma_w}^{(4)}| = K_0 |\Lambda_{\gamma_w}^{(4)}|,$$

where  $K_0$  is a constant depending only on  $C$  and  $R$ . Note that  $\Lambda_{\gamma_w} = \Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(3)} \cup \Lambda_{\gamma_w}^{(4)}$  and for every  $j \in \{2, 3, 4\}$ , every path in  $\Lambda_{\gamma_w}^{(j)}$  has exponential length at most equal to  $2000R^3C^6$ . Thus, we see that

$$\ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w}) \leq 2000R^3C^6 (|\Lambda_{\gamma_w}^{(2)}| + |\Lambda_{\gamma_w}^{(3)}| + |\Lambda_{\gamma_w}^{(4)}|) \leq K'_0 |\Lambda_{\gamma_w}^{(4)}|$$

for some constant  $K'_0$  depending only on  $C$  and  $R$ .

Recall that if  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}$ , then  $\ell_{exp}(\alpha^{(j,M)}) > 80C^2$ . Suppose towards a contradiction that  $[f^M(\alpha^{(j)})]$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Since  $\alpha^{(j,M)}$  is a subpath of  $[f^M(\alpha^{(j)})]$ , we have  $\ell_{exp}^{[f^M(\alpha^{(j)})]}(\alpha^{(j,M)}) = 0$ . By Lemma 6.5.6, we see that  $\ell_{exp}(\alpha^{(j,M)}) \leq \ell_{exp}^{[f^M(\alpha^{(j)})]}(\alpha^{(j,M)}) + 2C = 2C$ , which leads to a contradiction. Hence  $[f^M(\alpha^{(j)})]$  satisfies (a) and (b). Note that  $\alpha^{(j,M)}$  is a subpath of  $[f^M(\alpha^{(j)})]$ . Since  $\ell_{exp}(\alpha^{(j,M)}) > 80C^2$ , since every incomplete factor of  $[f^M(\alpha^{(j)})]$  has exponential length at most equal to  $8C$  by (b) and since an incomplete factor of  $[f^M(\alpha^{(j)})]$  is followed by a complete factor of  $[f^M(\alpha^{(j)})]$ , we see that  $\alpha^{(j,M)}$  contains a subpath of a complete factor of  $[f^M(\alpha^{(j)})]$ . Since  $\ell_{exp}(\alpha^{(j,M)}) > 80C^2$  and since every incomplete subpath of  $[f^M(\alpha^{(j)})]$  has exponential length at most equal to  $8C$ , the path  $\alpha^{(j,M)}$  must contain a subpath  $\alpha^{(j,M)'}$  such that the total exponential length of complete factors of  $\alpha^{(j,M)'}$  is at least equal to  $10C$ . Let  $\alpha_0^{(j,M)}$  be the minimal concatenation of splittings of a fixed optimal splittings of  $[f^m(\alpha^{(j)})]$  which contains  $\alpha^{(j,M)'}$ . Let  $\tau_1^{(j,M)}$  and  $\tau_2^{(j,M)}$  be paths such that  $[f^M(\alpha^{(j)})] = \tau_1^{(j,M)} \alpha_0^{(j,M)} \tau_2^{(j,M)}$ . By Lemma 6.5.8 applied twice (once with  $\gamma = \alpha_0^{(j,M)} \tau_2^{(j,M)} [f^M(\alpha^{(j+1)} \dots \alpha_k^{(k\alpha_k)})]$  and  $\gamma_1 = \alpha_0^{(j,M)}$  and once with  $\gamma^{-1} = [f^M(\alpha_1^{(1)} \dots \alpha^{(j-1)})] \tau_1^{(j,M)} \alpha_0^{(j,M)}$  and  $\gamma_1^{-1} = \alpha_0^{(j,M)}$ ), we see that  $\alpha^{(j,M)}$  contains a complete factor of  $[f^M(\gamma_w)]$  of exponential length at least equal to  $10C - 4C = 6C$ . By Lemma 6.5.6, the path  $\alpha^{(j,M)}$  contains a complete factor of  $[f^M(\gamma_w)]$  of exponential length relative to  $[f^M(\gamma_w)]$  at least equal to  $C$ . By Lemma 6.5.7 (with  $\gamma$  a complete factor contained in  $\alpha^{(j,M)}$ ), for every  $n \geq M$  and every  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}$ , the path  $\alpha^{(j,n)}$  contains a complete subpath of  $[f^n(\gamma_w)]$  of exponential length at least equal to  $3^{n-M}C$ . By Lemma 6.5.6, for every  $n \geq M$  and every  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}$ , the path  $\alpha^{(j,n)}$  contains a complete subpath of  $[f^n(\gamma_w)]$  of exponential length relative to  $[f^n(\gamma_w)]$  at least equal

to  $3^{n-M}C - 2C$ . Hence for every  $n \geq M$ , the sum of the exponential length relative to  $[f^n(\gamma_w)]$  of complete factors contained in  $[f^n(\gamma_w)]$  is at least equal to  $(3^{n-M}C - 2C)|\Lambda_{\gamma_w}^{(4)}|$ .

By the claim, for every  $n \geq M$ , we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq 24C^2 \ell_{exp}^{\gamma_w}(\gamma_w) \leq 24C^2 \frac{1}{1-\delta} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)),$$

where the last inequality holds by Equation (6.21). Using the above equations and the assumptions of Case 2, we see that

$$\begin{aligned} \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) &\leq 24C^2 \frac{1}{1-\delta} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \\ &\leq 24C^2 \frac{1}{1-\delta} (24C^2 R)^2 \ell_{exp}^{\gamma_w}(\Lambda_{\gamma_w}) \\ &\leq 24C^2 \frac{1}{1-\delta} (24C^2 R)^2 K'_0 |\Lambda_{\gamma_w}^{(4)}| = K_1 |\Lambda_{\gamma_w}^{(4)}|, \end{aligned}$$

where  $K_1$  is a constant depending only on  $C$ ,  $R$  and  $\delta$ . Thus, since the goodness function is a supremum over all splittings of the considered path, for every  $n \geq M$ , we have:

$$\begin{aligned} \mathfrak{g}([f^n(\gamma_w)]) &\geq \frac{(3^{n-M}C - 2C)|\Lambda_{\gamma_w}^{(4)}|}{(3^{n-M}C - 2C)|\Lambda_{\gamma_w}^{(4)}| + \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\alpha)]))} \\ &\geq \frac{(3^{n-M}C - 2C)|\Lambda_{\gamma_w}^{(4)}|}{(3^{n-M}C - 2C)|\Lambda_{\gamma_w}^{(4)}| + K_1 |\Lambda_{\gamma_w}^{(4)}|} \\ &= \frac{3^{n-M}C - 2C}{3^{n-M}C - 2C + K_1}, \end{aligned}$$

which converges to 1 as  $n$  goes to infinity. Hence there exists  $M' \in \mathbb{N}^*$  depending only on  $f$  such that for every  $n \geq M$ , we have  $\mathfrak{g}([f^n(\gamma_w)]) \geq \delta$ . This proves Lemma 6.5.21 in this case.

Suppose now that contrarily to Equation (6.25), we have

$$|\Lambda_{\gamma_w}^{(4)}| \leq \frac{1}{30000R^3C^6} |\Lambda_{\gamma_w}^{(3)}|. \quad (6.26)$$

Then

$$|\Lambda_{\gamma_w}^{(1)}| = |\Lambda_{\gamma_w}^{(3)}| + |\Lambda_{\gamma_w}^{(4)}| \leq \left(1 + \frac{1}{30000R^3C^6}\right) |\Lambda_{\gamma_w}^{(3)}|.$$

**Claim 2** Let  $n \geq M$ , let  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(4)}$ . The total exponential length of incomplete factors of  $[f^n(\gamma_w)]$  contained in  $\alpha^{(j,n)}$  is at most equal to  $12C \ell_{exp}(\alpha^{(j)})$ .

**Proof.** Let  $\sigma$  be an incomplete factor of  $[f^n(\gamma_w)]$  which is contained in  $\alpha^{(j,M)}$ . Then one of the following holds:

- (i) the path  $\sigma$  is an incomplete factor of  $[f^n(\alpha^{(j)})]$ ;
- (ii) the path  $\sigma$  contains a subpath which is complete in  $[f^n(\alpha^{(j)})]$ .

Note that the total exponential length of incomplete factors of  $[f^n(\gamma_w)]$  which satisfy (i) is bounded by the total exponential length of incomplete factors of  $[f^n(\alpha^{(j)})]$ . Thus, by Lemma 6.5.12, the total exponential length of incomplete factors of  $[f^n(\gamma_w)]$  which satisfy (i) is bounded by  $8C \ell_{exp}(\alpha^{(j)})$ . Suppose that  $\sigma$  satisfies (ii). Let  $\alpha^{(j,n)} = a_1 c a_2$

be a decomposition of  $\alpha^{(j,n)}$  where for every  $i \in \{1, 2\}$ , the total exponential length of complete factors of  $[f^n(\alpha^{(j)})]$  contained in  $a_i$  is equal to  $2C$ . By Lemma 6.5.8 applied to  $\gamma = [f^n(\alpha^{(j)})][f^n(\alpha^{(j+1)}) \dots \alpha_k^{(k_{\alpha_k})}]$  and  $\gamma_1 = [f^n(\alpha^{(j)})]$  and to  $\gamma^{-1} = [f^n(\alpha_1^{(1)}) \dots \alpha^{(j-1)}][f^n(\alpha^{(j)})]$  and  $\gamma_1^{-1} = [f^n(\alpha^{(j)})]$ , the path  $\sigma$  is contained in either  $a_1$  or  $a_2$ . For every  $t \in \{1, 2\}$ , let  $a_t = b_1^{(t)} b_1^{(t)'} \dots b_s^{(t)} b_s^{(t)'}$  be a decomposition of  $a_t$  where, for every  $i \in \{1, \dots, s_t\}$ , the path  $b_i^{(t)}$  is an incomplete factor of  $[f^n(\alpha^{(j)})]$  and for every  $i \in \{1, \dots, s_t\}$ , the path  $b_i^{(t)'}$  is a complete factor of  $[f^n(\alpha^{(j)})]$  contained in  $a_t$ .

Suppose that there exists  $i \in \{1, \dots, s_1\}$  such that  $b_i^{(1)'}$  is a complete factor of  $[f^n(\gamma_w)]$ . We claim that for every  $j \geq i + 1$ , the path  $b_j^{(1)'}$  is a complete factor of  $[f^n(\gamma_w)]$ . Indeed, let  $n' \geq n$  and let  $j \geq i + 1$ . Then there is no identification between an initial segment of  $[f^{n'}(b_i^{(1)'})]$  and an initial segment of  $[f^{n'}(\gamma_w)]$  not intersecting  $\alpha^{(j,n')}$  as otherwise there would exist identifications with  $[f^{n'}(b_i^{(1)'})]$ , contradicting the fact that  $b_i^{(1)'}$  is complete. Similarly, there is no identification between a terminal segment of  $[f^{n'}(b_i^{(1)'})]$  and a terminal segment of  $[f^{n'}(\gamma_w)]$  not intersecting  $\alpha^{(j,n')}$  as otherwise there would exist identifications with  $[f^{n'}(c)]$ . The claim follows. Similarly, if there exists  $i \in \{1, \dots, s_2\}$  such that  $b_i^{(2)'}$  is a complete factor of  $[f^n(\gamma_w)]$ , then for every  $j < i$ , the path  $b_j^{(2)'}$  is a complete factor of  $[f^n(\gamma_w)]$ . Hence we may assume that for every  $t \in \{1, 2\}$  and every  $s \in \{1, \dots, s_t\}$ , the path  $b_s^{(t)'}$  is incomplete in  $[f^n(\gamma_w)]$ . Therefore, for every  $t \in \{1, 2\}$ , the whole path  $a_t$  is incomplete in  $[f^n(\gamma_w)]$ . Therefore, in order to prove the claim, it suffices to bound the exponential lengths of  $a_1$  and  $a_2$ . Let  $t \in \{1, 2\}$ . By Lemma 6.3.16, we have

$$\ell_{exp}(a_t) \leq \sum_{i=1}^{s_t} \ell_{exp}(b_i^{(t)}) + \ell_{exp}(b_i^{(t)'}).$$

For every  $i \in \{1, \dots, s_t\}$ , the path  $b_i^{(t)}$  satisfies (i) and we already have a bound on the total exponential length of such paths. Moreover, since the total exponential length of complete factors of  $\alpha^{(j,n)}$  contained in  $a_t$  is at most equal to  $2C$ , we have

$$\sum_{i=1}^{s_t} \ell_{exp}(b_i^{(t)'}) \leq 2C.$$

Thus, the total exponential length of incomplete factors of  $[f^n(\gamma_w)]$  contained in  $\alpha^{(j,M)}$  is at most equal to

$$8C \ell_{exp}(\alpha^{(j)}) + \sum_{t=1}^2 \sum_{i=1}^{s_t} \ell_{exp}(b_i^{(t)'}) \leq 8C \ell_{exp}(\alpha^{(j)}) + 4C \leq 12C \ell_{exp}(\alpha^{(j)}),$$

where the last inequality follows from the fact that every element of  $\Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(4)}$  has positive exponential length.  $\square$

By Claim 2 and Lemma 6.5.6, for every  $n \geq M$  and every  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(4)}$ , the total exponential length relative to  $[f^n(\gamma_w)]$  of incomplete factors in the subpath of  $[f^n(\gamma_w)]$  contained in  $[f^n(\alpha^{(j)})]$  is at most equal to  $12C\ell_{exp}^{\gamma_w}(\alpha^{(j)}) + 2C \leq 14C\ell_{exp}^{\gamma_w}(\alpha^{(j)})$ . Hence by definition, for every  $n \geq M$  and every path  $\alpha^{(j)} \in \Lambda_{\gamma_w}^{(2)} \cup \Lambda_{\gamma_w}^{(4)}$ , we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)]) \cap \alpha^{(j,n)}) \leq 14C\ell_{exp}(\alpha^{(j)}).$$

We claim that, for every  $n \geq M$ , every element in  $\Lambda_{[f^n(\gamma_w)]}$  is contained in an iterate of an element in  $\Lambda_{\gamma_w}$ . Indeed, note that, by the choice of  $M$  (in the above application of Lemma 6.5.11), for every element  $\alpha \in \Lambda_{\gamma_w}'$ , the exponential length of an incomplete factor in  $[f^n(\alpha)]$  is at most equal to  $8C$ . Hence an incomplete factor of  $[f^n(\alpha)]$  whose exponential length is at least equal to  $(3.10^8)R^6C^{12} + 1$  cannot be contained in an iterate of an element of  $\Lambda_{\gamma_w}$ . The claim follows. Therefore, using Equation (6.26) for the third inequality, the value of  $\ell_{exp}(\Lambda_{[f^M(\gamma_w)]})$  is at most equal to

$$\begin{aligned} & \sum_{\alpha^{(j)} \in \Lambda_{\gamma_w}^{(3)}} \ell_{exp}(\alpha^{(j,M)}) + \sum_{\alpha^{(j)} \in \Lambda_{\gamma_w}^{(4)}} \ell_{exp}^{[f^M(\gamma_w)]}(\text{Inc}([f^M(\gamma_w)]) \cap \alpha^{(j,M)}) \\ & + \sum_{\alpha^{(j)} \in \Lambda_{\gamma_w}^{(2)}} \ell_{exp}^{[f^M(\gamma_w)]}(\text{Inc}([f^M(\gamma_w)]) \cap \alpha^{(j,M)}) \\ & \leq 80C^2|\Lambda_{\gamma_w}^{(3)}| + 14C \sum_{\beta \in \Lambda_{\gamma_w}^{(4)}} \ell_{exp}(\beta) + 14C \sum_{\alpha \in \Lambda_{\gamma_w}^{(2)}} \ell_{exp}(\alpha) \\ & \leq 80C^2|\Lambda_{\gamma_w}^{(3)}| + 14C(2000R^3C^6)|\Lambda_{\gamma_w}^{(4)}| + 14C \sum_{\alpha \in \Lambda_{\gamma_w}^{(2)}} \ell_{exp}(\alpha) \\ & \leq 80C^2|\Lambda_{\gamma_w}^{(3)}| + C|\Lambda_{\gamma_w}^{(3)}| + 14C \sum_{\alpha \in \Lambda_{\gamma_w}^{(2)}} \ell_{exp}(\alpha) \\ & \leq 81C^2|\Lambda_{\gamma_w}^{(3)}| + 14C \sum_{\alpha \in \Lambda_{\gamma_w}^{(2)}} \ell_{exp}(\alpha). \end{aligned}$$

Since by Equation (6.24)

$$\left(1 + \frac{1}{30000R^3C^6}\right) |\Lambda_{\gamma_w}^{(3)}| \geq |\Lambda_{\gamma_w}^{(1)}| \geq 120000R^3C^6|\Lambda_{\gamma_w}^{(2)}|,$$

we have  $|\Lambda_{\gamma_w}^{(3)}| \geq 60000R^3C^6|\Lambda_{\gamma_w}^{(2)}|$ . Hence we have

$$\begin{aligned} \ell_{exp}^{[f^M(\gamma_w)]}(\Lambda_{[f^M(\gamma_w)]}) & \leq 81C^2|\Lambda_{\gamma_w}^{(3)}| + 14C \sum_{\alpha \in \Lambda_{\gamma_w}^{(2)}} \ell_{exp}(\alpha) \\ & \leq 81C^2|\Lambda_{\gamma_w}^{(3)}| + (14C)(2000R^3C^6)|\Lambda_{\gamma_w}^{(2)}| \\ & \leq 81C^2|\Lambda_{\gamma_w}^{(3)}| + 2C|\Lambda_{\gamma_w}^{(3)}| = 83C^2|\Lambda_{\gamma_w}^{(3)}|. \end{aligned}$$

Let  $n \geq M$ . Suppose first that

$$\frac{\ell_{exp}^{[f^n(\gamma_w)]}(\Lambda_{[f^n(\gamma_w)]})}{\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)]))} < \frac{1}{(24C^2R)^2}.$$



Then we can apply Case 1 to conclude the proof of Lemma 6.5.21. Otherwise, we have

$$(24C^2R)^2 \ell_{exp}^{[f^n(\gamma_w)]}(\Lambda_{[f^n(\gamma_w)]}) \geq \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])).$$

By Lemma 6.5.12 and Lemma 6.5.6, we have

$$\begin{aligned} \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) &\leq \ell_{exp}(\text{Inc}([f^n(\gamma_w)])) \leq 8C \ell_{exp}(\text{Inc}([f^M(\gamma_w)])) \\ &\leq 10C \ell_{exp}^{[f^M(\gamma_w)]}(\text{Inc}([f^M(\gamma_w)])). \end{aligned}$$

Hence we have

$$\begin{aligned} \frac{\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)]))}{\ell_{exp}^w(\text{Inc}(\gamma_w))} &\leq \frac{(24C^2R)^2 \ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \ell_{exp}^{[f^M(\gamma_w)]}(\text{Inc}([f^M(\gamma_w)]))}{\ell_{exp}^{[f^M(\gamma_w)]}(\text{Inc}([f^M(\gamma_w)])) \ell_{exp}^w(\text{Inc}(\gamma_w))} \\ &\leq \frac{10C(24C^2R)^2 \ell_{exp}^{[f^n(\gamma_w)]}(\Lambda_{[f^n(\gamma_w)]})}{\ell_{exp}^{[f^n(\gamma_w)]}(\Lambda_{\gamma_w})} \\ &\leq \frac{10C(24C^2R)^2(83C^2|\Lambda_{\gamma_w}^{(3)}|)}{2000R^3C^6|\Lambda_{\gamma_w}^{(3)}|} \\ &\leq \frac{10C}{R}. \end{aligned}$$

This concludes the proof of Lemma 6.5.21.  $\square$

In the next proposition, we need to work with CT maps that represent both an almost atoroidal outer automorphism and its inverse. We therefore introduce the following conventions:

Let  $f': G' \rightarrow G'$  be a CT map representing  $\phi^{-M}$ , which exists by Theorem 6.2.10. We denote by  $K'$  the constant similar to the constant  $K$  given above Lemma 6.5.6 and by  $C_{f'}$  the bounded cancellation constant given by Lemma 6.4.9. We set  $C' = \max\{K', C_{f'}\}$  as in Equation (6.7). We denote by  $G_{p'}$  the invariant subgraph of  $G'$  such that  $\mathcal{F}(G_{p'}) = \mathcal{F}$ , by  $\ell_{\mathcal{F}'}$  the corresponding  $\mathcal{F}$ -length and by  $\ell_{exp'}$  the corresponding exponential length. Let  $\mathbf{g}'$  be the corresponding goodness function. If  $w \in F_n$ , we denote by  $\gamma'_w$  the corresponding circuit in  $G'$ .

We also need a result which shows that the exponential length is invariant by  $F_n$ -equivariant quasi-isometry. In order to prove this, we need some additional definitions. Let  $G$  be a connected (pointed) graph whose fundamental group is isomorphic to  $F_n$  and let  $\tilde{G}$  be the universal cover of  $G$ . Let  $\phi \in \text{Out}(F_n)$  be an exponentially growing outer automorphism. Let  $\hat{G}$  be the graph obtained from  $\tilde{G}$  as follows. We add one vertex  $v_{gA}$  for every left class  $gA$ , with  $g \in F_n$  and  $A$  is a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and we add one edge between  $v_{gA}$  and a vertex  $v$  of  $\tilde{G}$  if and only if the vertex  $v$  is contained in the tree  $T_{gAg^{-1}}$ . The graph  $\hat{G}$  is known as the *electrification of  $\tilde{G}$*  (see for instance [Bow]). For a path  $\gamma$  in  $G$ , we denote by  $\tilde{\gamma}$  a lift of  $\gamma$  in  $\tilde{G}$ . Let  $\hat{\gamma}$  be the path in  $\hat{G}$  constructed as follows. Let  $\tilde{\gamma} = a_1b_1 \dots a_kb_k$  be the decomposition of  $\tilde{\gamma}$  such that, for every  $i \in \{1, \dots, k\}$ , the path  $b_i$  is contained in some tree  $T_{g_iA_i g_i^{-1}}$  with  $g_i \in F_n$ ,  $A_i$  a subgroup of  $F_n$  such that  $[A_i] \in \mathcal{A}(\phi)$  and  $b_i$  is maximal for the property of being contained in such a tree  $T_{g_iA_i g_i^{-1}}$ . Then  $\hat{\gamma}$  is a path  $\hat{\gamma} = a_1c_1 \dots a_kc_k$  where, for every

$i \in \{1, \dots, k\}$ , the path  $c_i$  is the two-edge path whose endpoints are the endpoints of  $b_i$  and the middle vertex of  $c_i$  is  $v_{g_i A_i}$ . Note that the path  $\widehat{\gamma}$  is not uniquely determined. Indeed, it is possible that there exists  $i \in \{1, \dots, k\}$  such that  $b_i$  is contained in two distinct trees  $T_A$  and  $T_B$  with  $[A], [B] \in \mathcal{A}(\phi)$ . However, if  $\widehat{\gamma}$  and  $\widehat{\gamma}'$  are two such paths associated with  $\widetilde{\gamma}$ , then  $\ell(\widehat{\gamma}) = \ell(\widehat{\gamma}')$ .

**Proposition 6.5.22.** *Let  $n \geq 3$ , let  $\phi \in \text{Out}(F_n)$  and let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$ .*

(1) *There exists a constant  $B_0 \geq 1$  such that, for every element  $w \in F_n$  with  $\ell_{\text{exp}}(\gamma_w) > 0$ , we have:*

$$\frac{1}{B_0} \ell_{\text{exp}}(\gamma_w) \leq \ell(\widehat{\gamma}_w) \leq B_0 \ell_{\text{exp}}(\gamma_w).$$

(2) *Let  $f': G' \rightarrow G'$  be a CT map representing a power of  $\phi^{-1}$ . There exists a constant  $B > 0$  such that, for every element  $w \in F_n$ , we have:*

$$\frac{1}{B} \ell_{\text{exp}'}(\gamma'_w) \leq \ell_{\text{exp}}(\gamma_w) \leq B \ell_{\text{exp}'}(\gamma'_w).$$

**Proof.** (1) Recall the definition of the graph  $G^*$  from just above Lemma 6.3.11. We can turn the graph  $G^*$  into a metric graph by assigning, to every edge  $e \in \vec{E}G^*$ , the length equal to the length of the path  $p_{G^*}(e)$  in  $G$ . Since the graph  $G^*$  is finite, there exists a constant  $B'$  such that the diameter of every maximal subtree of  $G^*$  is at most  $B'$ . Let  $B_0 = 2B' + 2$ .

Let  $w \in F_n$ . Let  $\gamma_w = a_1 b_1 \dots a_k b_k$  be the decomposition of  $\gamma_w$  with  $a_1$  and  $b_k$  possibly empty such that, for every  $i \in \{1, \dots, k\}$ , the path  $b_i$  is a maximal concatenation of paths in  $G'_{PG}$  and in  $\mathcal{N}_{PG}$  and, for every  $i \in \{1, \dots, k\}$  and every edge  $e$  of  $a_i$ , we have  $\ell_{\text{exp}}^w(e) = 1$ . Note that by the definition of the exponential length we have

$$\ell_{\text{exp}}(\gamma_w) = \sum_{i=1}^k \ell(a_i).$$

Let  $A$  be a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$ . Let  $i \in \{1, \dots, k\}$  and let  $\alpha$  be a subpath of  $a_i$  whose lift is contained in  $T_A$ . By Proposition 6.3.13, the subpath  $\alpha$  is contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Since  $a_i$  does not contain any concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ , the path  $\alpha$  is a proper subpath of an EG INP. By the definition of  $C$  (see Equation (6.7)), we see that  $\ell(\alpha) \leq C$ . Thus, we have:  $\ell(a_i) \leq C \ell(\widehat{a}_i)$  and

$$\ell_{\text{exp}}(\gamma_w) \leq C \sum_{i=1}^k \ell(\widehat{a}_i).$$

**Claim.** Let  $A$  be a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$ . Let  $\beta$  be a subpath of  $\gamma_w$  such that a lift of  $\beta$  is contained in  $T_A$ . There does not exist  $i \in \{1, \dots, k\}$  such that both  $\beta \cap b_i$  and  $\beta \cap b_{i+1}$  are not reduced to a point.

**Proof.** Suppose towards a contradiction that such an element  $i \in \{1, \dots, k\}$  exists. Then  $a_{i+1}$  is contained in  $\beta$ . By the above, the path  $a_{i+1}$  is contained in an EG INP  $\sigma$ . Since

both  $b_i$  and  $b_{i+1}$  are concatenations of paths in  $G'_{PG}$  and  $\mathcal{N}_{PG}$ , the path  $a_{i+1}$  must contain the initial or the terminal segment of  $\sigma$ . Since  $\beta$  is contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  by Proposition 6.3.13, the EG INP  $\sigma$  must be contained in  $\beta$  and  $\beta \cap a_{i+1} \subseteq \sigma$ . This contradicts the maximality of the paths  $b_i$  and  $b_{i+1}$ .  $\square$

Hence  $\beta$  is either contained in  $b_i a_{i+1}$  or in  $a_{i+1} b_{i+1}$ . Let  $i \in \{1, \dots, k\}$  and let  $\beta$  be a maximal subpath of  $\gamma_w$  containing edges of  $a_i$  and such that a lift of  $\beta$  is contained in some  $T_A$  with  $A$  a subgroup of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$ . By the claim, the path  $a_i$  has a decomposition  $a_i = c_i^+ d_i c_i^-$  such that  $c_i^+$  and  $c_i^-$  are possibly trivial, lifts of  $c_i^+$  and  $c_i^-$  are contained in trees  $T_{A_+}$  and  $T_{A_-}$  with  $A_+$  and  $A_-$  subgroups of  $F_n$  such that  $[A_+], [A_-] \in \mathcal{A}(\phi)$  and one of the following holds:

- (a)  $\beta \subseteq d_i$ ;
- (b)  $\beta \cap a_i \neq \beta$  and  $\beta \cap a_i \in \{c_i^+, c_i^-\}$ .

Note that for every  $i \in \{1, \dots, k\}$ , we have  $\ell(\hat{a}_i) \leq \ell(\hat{d}_i) + 4$ . Then

$$\ell(\widehat{\gamma}_w) \geq \sum_{i=1}^k \ell(\hat{d}_i) \geq \sum_{i=1}^k (\ell(\hat{a}_i) - 4) = \sum_{i=1}^k \ell(\hat{a}_i) - 4k.$$

Moreover, if  $\beta$  is a path which satisfies the hypothesis of the claim, then there exists at most one  $i \in \{1, \dots, k\}$  such that  $\beta \cap b_i$  is not reduced to a point. Therefore, we see that  $\ell(\widehat{\gamma}_w) \geq k$ . Thus, we have

$$\ell_{exp}(\gamma_w) \leq C \sum_{i=1}^k \ell(\hat{a}_i) \leq C(\ell(\widehat{\gamma}_w) + 4k) \leq 5C\ell(\widehat{\gamma}_w).$$

This proves the first inequality of Assertion (1). We now prove the second inequality. For every  $i \in \{1, \dots, k\}$ , there exists a unique edge path  $b_i^* \subseteq G^*$  such that  $p^*(b_i^*) = b_i$ . Let  $i \in \{1, \dots, k\}$ . Since  $G^*$  is a finite graph, there exist (possibly trivial) reduced paths  $\beta_i^*, \delta_i^*$  and  $\delta_i^{*'}$  such that:

- (i) the path  $\beta_i^*$  is a circuit;
- (ii) the paths  $\delta_i^*$  and  $\delta_i^{*'}$  are contained in maximal trees of  $G^*$ ;
- (iii) we have  $b_i^* = \delta_i^* \beta_i^* \delta_i^{*'}$ .

By Lemma 6.3.11 (1), the paths  $p^*(\delta_i^*), p^*(\beta_i^*)$  and  $p^*(\delta_i^{*'})$  are reduced edge paths of  $G$ . By definition of  $B'$ , we have  $\ell(\delta_i^*), \ell(\delta_i^{*'}) \leq B'$ . Since  $p^*(\beta_i^*)$  is a circuit which is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , by Proposition 6.3.13, there exists a subgroup  $H_i$  of  $F_n$  such that  $[H_i] \in \mathcal{A}(\phi)$  and the conjugacy classes of elements of  $F_n$  represented by  $p^*(\beta_i^*)$  are contained in  $[H_i]$ . Hence the length of  $\widehat{p^*(\beta_i^*)}$  is bounded by 2. Hence the length of the path  $\widehat{b}_i$  is bounded by  $2 + 2B' = B_0$ . Therefore, since  $\ell_{exp}(\gamma_w) > 0$ , we have

$$\ell(\widehat{\gamma}_w) = \sum_{i=1}^k \ell(a_i) + \ell(\widehat{b}_i) \leq \sum_{i=1}^k (\ell(a_i) + B_0) \leq (B_0 + 1) \sum_{i=1}^k \ell(a_i) = (B_0 + 1) \ell_{exp}(\gamma_w).$$

This proves Assertion (1).

(2) Let  $f'$  be as in Assertion (2) and let  $w \in F_n$ . Suppose first that  $\ell_{exp}(\gamma_w) = 0$ . Then  $\gamma_w$  is a concatenation of paths in  $G'_{PG}$  and in  $\mathcal{N}_{PG}$ . By Proposition 6.2.5 (4) and Lemma 6.2.9, there does not exist an edge in a zero stratum which is adjacent to a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Since zero strata are contractible by Proposition 6.2.5 (3), it follows that  $\gamma_w$  is a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . By Proposition 6.3.13, there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and  $w \in A$ . Since  $\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1})$  by Equation (6.1), by Proposition 6.3.13, we have  $\ell_{exp'}(\gamma'_w) = 0$ . So we may suppose that  $\ell_{exp}(\gamma_w) > 0$  and that  $\ell_{exp'}(\gamma'_w) > 0$ . By Assertion (1), in order to prove Assertion (2), it suffices to prove that  $\widehat{G}$  and  $\widehat{G}'$  are  $F_n$ -equivariantly quasi-isometric. Since  $\mathcal{A}(\phi)$  is a malnormal subgroup system, this follows from [Bow, Theorem 7.11] and [Hru, proof of Theorem 5.1].  $\square$

**Proposition 6.5.23.** *Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  and let  $f: G \rightarrow G$  be as in Remark 6.5.15. Let  $f': G' \rightarrow G'$  be as in the above convention. Let  $\delta \in (0, 1)$  and let  $W$  be a neighborhood of  $K_{PG}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ . There exists  $n_0 \in \mathbb{N}^*$  such that for every  $n \geq n_0$  and every nonperipheral element  $w \in F_n$  such that  $\eta_{[w]} \notin W$ , one of the following holds:*

$$\mathfrak{g}([f^n(\gamma_w)]) \geq \delta$$

or

$$\mathfrak{g}'([f'^n(\gamma'_w)]) \geq \delta.$$

**Proof.** Let  $w \in F_n$  be a nonperipheral element such that  $\eta_{[w]} \notin W$ . Let  $R = \frac{10C}{(1-\delta)^2} 8C' B^2$ . We use the alternative given by Lemma 6.5.21 with the constants  $\delta$  and  $R$ . If the first alternative of Lemma 6.5.21 occurs, then we are done. Suppose that  $\mathfrak{g}([f^n(\gamma_w)]) < \delta$ . There exists  $n_0 \in \mathbb{N}^*$  depending only on  $f$  such that for every  $n \geq n_0$ , we have

$$\ell_{exp}^{[f^n(\gamma_w)]}(\text{Inc}([f^n(\gamma_w)])) \leq \frac{10C}{R} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)).$$

By Lemma 6.5.14, since  $\mathfrak{g}([f^n(\gamma_w)]) < \delta$ , we have  $\mathfrak{g}(\gamma_w) < \delta$ . Thus, we see that

$$\ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \geq (1 - \delta) \ell_{exp}(\gamma_w).$$

Let  $\gamma''$  be the reduced circuit in  $G$  such that  $[f^{n_0}(\gamma'')] = \gamma_w$ . Since  $\mathfrak{g}(\gamma_w) < \delta$  and  $[\eta_{[w]}] \notin K_{PG}(\phi)$ , by Lemma 6.5.21, we see that

$$\ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \leq \frac{10C}{R} \ell_{exp}^{\gamma''}(\text{Inc}(\gamma'')).$$

We have

$$\begin{aligned} \ell_{exp'}([f'^{n_0}(\gamma'_w)]) &\geq \frac{1}{B} \ell_{exp}(\gamma'') \geq \frac{1}{B} \ell_{exp}^{\gamma''}(\text{Inc}(\gamma'')) \\ &\geq \frac{1}{B} \frac{R}{10C} \ell_{exp}^{\gamma_w}(\text{Inc}(\gamma_w)) \geq \frac{1}{B} \frac{(1-\delta)R}{10C} \ell_{exp}(\gamma_w) \\ &\geq \frac{1}{B^2} \frac{(1-\delta)R}{10C} \ell_{exp'}(\gamma'_w) = 8C' \frac{1}{1-\delta} \ell_{exp'}(\gamma'_w). \end{aligned}$$

But by Lemma 6.5.12, we have:

$$\ell_{exp'}^{[f'^{n_0}(\gamma'_w)]}(\text{Inc}(f'^{n_0}(\gamma'_w))) \leq \ell_{exp'}(\text{Inc}(f'^{n_0}(\gamma'_w))) \leq 8C' \ell_{exp'}(\gamma'_w).$$

Therefore, we see that

$$\mathfrak{g}'([f^{m_0}(\gamma'_w)]) = 1 - \frac{\ell^{[f^{m_0}(\gamma'_w)]}(\text{Inc}([f^{m_0}(\gamma'_w)]))}{\ell_{\text{exp}'([f^{m_0}(\gamma'_w)])}} \geq 1 - (1 - \delta) = \delta > 0.$$

By Lemma 6.5.16, we see that there exists  $n_1 \geq n_0$  depending only on  $f'$  such that for every  $n \geq n_1$ ,

$$\mathfrak{g}'([f'^n(\gamma'_w)]) \geq \delta.$$

This concludes the proof.  $\square$

**Proposition 6.5.24.** *Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  and let  $f: G \rightarrow G$  be as in Remark 6.5.15. Let  $U_+$  be a neighborhood of  $\Delta_+(\phi)$ , let  $U_-$  be a neighborhood of  $\Delta_-(\phi)$ , let  $V$  be a neighborhood of  $K_{PG}(\phi)$ . There exists  $N \in \mathbb{N}^*$  such that for every  $n \geq 1$  and every  $\mathcal{F} \wedge \mathcal{A}(\phi)$ -nonperipheral  $w \in F_n$  such that  $\eta_{[w]} \notin V$ , one of the following holds*

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \quad \text{or} \quad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

**Proof.** Let  $\delta \in (0, 1)$  and let  $w \in F_n$  be a nonperipheral element with  $\eta_{[w]} \notin V$ . By Proposition 6.5.23, there exists  $n_0 \in \mathbb{N}^*$  such that for every  $n \geq n_0$ , we have  $\mathfrak{g}([f^n(\gamma_w)]) \geq \delta$  or  $\mathfrak{g}'([f'^n(\gamma'_w)]) \geq \delta$ . By Lemma 6.5.20 (1), there exists  $n_1 \geq n_0$  such that for every  $n \geq n_1$ , we have

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \quad \text{or} \quad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

This concludes the proof.  $\square$

The above proposition gives a result of North-South dynamics outside of a neighborhood of  $K_{PG}(\phi)$ . As  $K_{PG}(\phi)$  is empty for a relative expanding outer automorphism by Lemma 6.3.27 (1), we can now prove Theorem 6.5.1.

*Proof of Theorem 6.5.1.* Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an expanding outer automorphism relative to  $\mathcal{F}$ . By Lemma 6.3.27, we have  $K_{PG}(\phi) = \emptyset$ . Let  $U_+$  be a neighborhood of  $\Delta_+(\phi)$  and let  $U_-$  be a neighborhood of  $\Delta_-(\phi)$ . By Proposition 6.5.24, there exists  $N \in \mathbb{N}^*$  such that for every  $n \geq 1$  and every nonperipheral element  $w \in F_n$ , we have

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \quad \text{or} \quad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

Recall that, by Proposition 6.2.15, the rational currents are dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ . Hence we can apply [LU2, Proposition 3.3] to see that  $\phi^{2N}$  has generalized North-South dynamics. Then, using [LU2, Proposition 3.4], we conclude that  $\phi$  has generalized North-South dynamics.  $\square$

## 6.6 North-South dynamics for almost atoroidal relative outer automorphism

Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism which satisfies Definition 6.4.3 (2). Let  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2 = \{[F_n]\}$  be a sequence of free factor system given in this definition. We use the convention of Remark 6.5.19. We will show a result of North-South type dynamics for  $\phi$  (see Theorem 7.2.7). Note that, if  $\mathcal{A}(\phi) \neq \{[F_n]\}$  the simplices  $\Delta_{\pm}(\phi)$  are still defined. Note that, by Lemma 6.3.27 (3) and Lemma 6.5.18 (4), for every current  $\mu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ , we have  $\|\mu\|_{\mathcal{F}_1} > 0$ . Let  $K_{PG}(\phi)$  be the set of polynomially growing currents of  $\phi$ . Note that, combining Lemma 6.4.8 and Lemma 6.5.18 (5), we have  $K_{PG}(\phi) \cap \Delta_{\pm}(\phi) = \emptyset$ . Let

$$\widehat{\Delta}_{\pm}(\phi) = \{[t\mu + (1-t)\nu] \mid t \in [0, 1], [\mu] \in \Delta_{\pm}(\phi), [\nu] \in K_{PG}(\phi), \|\mu\|_{\mathcal{F}_1} = \|\nu\|_{\mathcal{F}_1} = 1\}$$

be the *convexes of attraction and repulsion* of  $\phi$ .

In order to promote a global North-South type dynamics, we need to construct contracting neighborhoods of  $\widehat{\Delta}_{+}(\phi)$ . To this end, following [CU2], we introduce a notion of *goodness for currents* of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ .

Let  $f: G \rightarrow G$  be as in Remark 6.5.15. By Lemma 6.3.21, let  $N \in \mathbb{N}^*$  be such that, for every edge  $e$  of  $\overline{G} - G'_{PG}$ , we have  $\ell_{exp}([f^N(e)]) \geq 4C + 1$ . Let  $C_N = C_{f^N}$  be a constant associated with  $f^N$  given by Lemma 6.4.9. Let  $L > 0$  be such that for every path  $\gamma$  of  $G$  of length at least  $L$ , we have  $\ell([f^N(\gamma)]) \geq C_N + 1$ . The constant  $L$  exists since  $f^N$  lifts to a quasi-isometry on the universal cover of  $G$ . Let  $\mathcal{P}_{cs}$  be the finite set of paths of the form  $\gamma = \gamma_1 e \gamma_2$ , where, for every  $i \in \{1, 2\}$ , the path  $\gamma_i$  has length equal to  $L$ , the path  $e$  is an edge in  $\overline{G} - G'_{PG}$  and  $\gamma_1 e \gamma_2$  is a splitting of  $\gamma$ . In Lemma 6.6.1 (2), we prove in particular that  $\mathcal{P}_{cs}$  is not empty. We will denote by  $\widehat{\gamma}$  the edge  $e$ .

Let  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ . Recall the definition of  $\Psi_0$  just above Definition 6.3.25. By Lemma 6.3.27 (1), (2), we have  $\phi(K_{PG}(\phi)) = K_{PG}(\phi)$ . Hence, for every current  $[\mu] \notin K_{PG}(\phi)$ , we have  $\Psi_0(\phi(\mu)) > 0$ . Thus, for every current  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$ , we can define the *completely split goodness*  $\overline{\mathfrak{g}}(\mu)$  of  $\mu$  by

$$\overline{\mathfrak{g}}(\mu) = \frac{1}{\Psi_0(\phi^N(\mu))} \sum_{\gamma \in \mathcal{P}_{cs}} \langle \gamma, \mu \rangle.$$

Observe that the function  $\overline{\mathfrak{g}}$  is continuous and that it defines a well-defined continuous function  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi) \rightarrow \mathbb{R}$ .

**Lemma 6.6.1.** *Let  $f: G \rightarrow G$  be as in Remark 6.5.15.*

(1) *Let  $w \in F_n$  be such that  $\ell_{exp}(\gamma_w) > 0$ . We have  $\mathfrak{g}([f^N(\gamma_w)]) \geq \overline{\mathfrak{g}}(\eta_{[w]})$ .*

(2) *For every  $[\mu] \in \Delta_{+}(\phi)$ , we have  $\overline{\mathfrak{g}}([\mu]) > 0$*

**Proof.** (1) The proof of this assertion is similar to the one of [CU2, Lemma 4.9 (2)]. Let  $\gamma \in \mathcal{P}_{cs}$  be such that  $\langle \gamma, \eta_{[w]} \rangle > 0$ . Then  $\gamma \subseteq \gamma_w$ . For every occurrence of  $\gamma$  in  $\gamma_w$ , by

the choice of  $L$ ,  $C_N$  and by Lemma 6.4.9, the path  $[f^N(\gamma_w)]$  contains  $[f^N(\hat{\gamma})]$ , which has exponential length at least equal to  $4C_N + 1$ . Therefore, Lemma 6.5.8 implies that the path  $[f^N(\gamma_w)]$  contains a subpath of  $[f^N(\hat{\gamma})]$  of exponential length at least 1 which is a complete factor of  $[f^N(\gamma_w)]$  relative to  $G_{PG}$ . Hence we have:

$$\ell_{exp}([f^N(\gamma_w)])\mathfrak{g}([f^N(\gamma_w)]) \geq \sum_{\gamma \in \mathcal{P}_{cs}} \langle \gamma, \eta_{[w]} \rangle.$$

By Lemma 6.3.26, we have

$$\Psi_0(\phi^N(\eta_{[w]})) = \ell_{exp}([f^N(\gamma_w)]) = \Psi_0(\eta_{[\phi^N(w)]}) = \ell_{exp}(\gamma_{\phi^N([w])}).$$

Therefore, we have

$$\mathfrak{g}([f^N(\gamma_w)]) \geq \bar{\mathfrak{g}}(\eta_{[w]}).$$

(2) Let  $[\mu] \in \Delta_+(\phi)$ . Since  $[\mu]$  is a convex combination of extremal points of  $\Delta_+(\phi)$  and since for every element  $\gamma \in \mathcal{P}_{cs}$ , the application  $\langle \gamma, \cdot \rangle$  is linear, it suffices to prove the result for every extremal point of  $\Delta_+(\phi)$ . So we may suppose that  $[\mu]$  is an extremal point of  $\Delta_+(\phi)$ . Let  $G_i$  be the minimal subgraph of  $G$  such that  $\mathcal{F}(G_i) = \mathcal{F}_1$ . Since  $[\mu]$  is extremal and since  $\phi|_{\mathcal{F}_1}$  is expanding relative to  $\mathcal{F}$ , by Proposition 6.4.4, there exists an expanding splitting unit  $\sigma$  in  $G_i$  whose initial direction is fixed by  $f$  and such that, for every path  $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$ , we have

$$\langle \gamma, \mu \rangle = \mu(C(\gamma)) = \lim_{n \rightarrow \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{\mathcal{F}_1}([f^n(\sigma)])}.$$

By Lemma 6.5.18 (5), since the path  $[f^n(\sigma)]$  is contained in  $G_i$  and, for every path  $\gamma \in \mathcal{P}(\mathcal{F} \wedge \mathcal{A}(\phi))$ , the above limit is finite, we have

$$\lim_{n \rightarrow \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{\mathcal{F}_1}([f^n(\sigma)])} = \lim_{n \rightarrow \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{exp}([f^n(\sigma)])}.$$

Hence it suffices to prove that there exists  $\gamma \in \mathcal{P}_{cs}$  such that

$$\lim_{n \rightarrow \infty} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{exp}([f^n(\sigma)])} > 0.$$

Let  $e$  be an edge of  $\overline{G - G'_{PG}}$ . Note that, since  $\sigma$  is a splitting unit, for every  $m \in \mathbb{N}^*$ , the path  $[f^m(\sigma)]$  is completely split. Hence an occurrence of  $e$  in  $\lim_{m \rightarrow \infty} [f^m(\sigma)]$  is contained in a splitting unit of  $\lim_{m \rightarrow \infty} [f^m(\sigma)]$  which is either an INP or is equal to  $e$ . By Lemma 6.3.7 if an INP  $\gamma'$  contains  $e$ , there exists  $\gamma'_0 \in \mathcal{N}_{PG}$  such that  $e \subseteq \gamma'_0 \subseteq \gamma'$ . For every  $m \in \mathbb{N}^*$ , we denote by  $N(m, e)$  the number of occurrences of  $e$  or  $e^{-1}$  in  $[f^m(\sigma)]$  which are splitting units of  $[f^m(\sigma)]$  and by  $EGINP(e)$  the set of all EG INPs containing  $e$ . Note that, since the set  $\mathcal{N}_{PG}$  is finite by Lemma 6.3.4, so is the limit

$$\lim_{n \rightarrow \infty} \sum_{\gamma \in EGINP(e)} \frac{\langle \gamma, [f^n(\sigma)] \rangle}{\ell_{exp}([f^n(\sigma)])}.$$

Since for every  $m \in \mathbb{N}^*$ , we have

$$\langle e, [f^m(\sigma)] \rangle = N(m, e) + \sum_{\gamma \in EGINP(e)} \langle \gamma, [f^m(\sigma)] \rangle,$$

we see that the limit

$$\lim_{m \rightarrow \infty} \frac{N(m, e)}{\ell_{exp}([f^m(\sigma)])}$$

exists. We claim that there exists an edge  $e$  of  $\overline{G - G'_{PG}}$  such that

$$\lim_{m \rightarrow \infty} \frac{N(m, e)}{\ell_{exp}([f^m(\sigma)])} > 0.$$

Indeed, note that, by Lemma 6.3.23 since  $[f^m(\sigma)]$  is  $PG$ -relative completely split, we have

$$\ell_{exp}([f^m(\sigma)]) = \sum_{e \in \tilde{E}(G - G'_{PG})} N(m, e).$$

Hence

$$\sum_{e \in \tilde{E}(G - G'_{PG})} \lim_{m \rightarrow \infty} \frac{N(m, e)}{\ell_{exp}([f^m(\sigma)])} = 1,$$

which implies the claim. Let  $e_0$  be an edge of  $\overline{G - G'_{PG}}$  which satisfies the claim. Since, for every  $m \in \mathbb{N}^*$ , the path  $[f^m(\sigma)]$  is completely split, if an occurrence of  $e_0$  or  $e_0^{-1}$  in  $[f^m(\sigma)]$  is a splitting unit and if  $\gamma$  is a path in  $[f^m(\sigma)]$  of the form  $\gamma = \gamma_1 e_0 \gamma_2$  or  $\gamma = \gamma_1 e_0^{-1} \gamma_2$ , then such a decomposition of  $\gamma$  is a splitting of  $\gamma$ . Thus, if  $\ell(\gamma_1) = \ell(\gamma_2) = L$ , then the path  $\gamma$  is in  $\mathcal{P}_{cs}$  and it contains the occurrence of  $e_0$ . Hence since  $\mu = \mu(\sigma)$ , we have

$$\lim_{m \rightarrow \infty} \frac{N(m, e)}{\ell_{exp}([f^m(\sigma)])} = \sum_{\gamma \in \mathcal{P}_{cs}, e_0 \subseteq \gamma} \langle \gamma, \mu \rangle > 0.$$

Therefore, there exists  $\gamma \in \mathcal{P}_{cs}$  such that  $\langle \gamma, \mu \rangle > 0$  and  $\bar{\mathfrak{g}}([\mu]) > 0$ .  $\square$

**Lemma 6.6.2.** *Let  $f: G \rightarrow G$  be as in Remark 6.5.15. Let  $U_{\pm}$  be open neighborhoods of  $\Delta_{\pm}(\phi)$ . There exist open neighborhoods  $U'_{\pm} \subseteq U_{\pm}$  of  $\Delta_{\pm}(\phi)$  such that  $\phi^{\pm 1}(U'_{\pm}) \subseteq U'_{\pm}$ .*

**Proof.** The proof is similar to the one of [CU2, Lemma 4.13]. We prove the result for  $\Delta_+(\phi)$ , the proof for  $\Delta_-(\phi)$  being symmetric. By Lemma 6.6.1 (2), for every  $[\mu] \in \Delta_+(\phi)$ , we have  $\bar{\mathfrak{g}}([\mu]) > 0$ . By compactness of  $\Delta_+(\phi)$  and continuity of  $\bar{\mathfrak{g}}$ , there exists  $\delta_0 > 0$  such that, for every  $\mu \in \Delta_+(\phi)$ , we have  $\bar{\mathfrak{g}}(\mu) \geq \delta_0$ . Let  $\delta \in (0, \delta_0)$ . Let  $U_+$  be a neighborhood of  $\Delta_+(\phi)$ . Since the function  $\bar{\mathfrak{g}}$  is continuous, there exists an open neighborhood  $U_+^0 \subseteq U_+$  of  $\Delta_+(\phi)$  such that, for every  $[\mu] \in U_+^0$ , we have  $\bar{\mathfrak{g}}([\mu]) > \delta$ . Up to taking a smaller  $U_+^0$ , we may suppose that  $K_{PG}(\phi) \cap U_+^0 = \emptyset$  (this is possible since  $K_{PG}(\phi)$  is compact and  $\Delta_+(\phi) \cap K_{PG}(\phi) = \emptyset$ ). In particular, by Lemma 6.3.26, for every nonperipheral element  $w \in F_n$  such that  $\eta_{[w]} \in U_+^0$ , we have  $\ell_{exp}(\gamma_w) > 0$ .



Let  $w \in F_n$  be a nonperipheral element such that  $\eta_{[w]} \in U_0^+$ . By Lemma 6.6.1 (1), we have

$$\mathfrak{g}([f^N(\gamma_w)]) \geq \bar{\mathfrak{g}}(\eta_{[w]}) > \delta.$$

By Lemma 6.5.20 (1), there exists  $M \geq N$  such that, for every  $w \in F_n$  such that  $\eta_{[w]} \in U_+^0$ , we have  $\phi^M([\eta_{[w]}]) \in U_+^0$ . Let

$$U'_+ = \bigcap_{i=0}^{M-1} \phi^i(U_+^0).$$

Since  $\phi(\Delta_+(\phi)) = \Delta_+(\phi)$  by Proposition 6.4.12, the set  $U'_+$  is an open neighborhood of  $\Delta_+(\phi)$  which is stable by  $\phi$  by density of rational currents (see Proposition 6.2.15) and continuity of  $\phi$ . This concludes the proof.  $\square$

**Lemma 6.6.3.** *Let  $f: G \rightarrow G$  be as in Remark 6.5.15. Suppose that the outer automorphism  $\phi$  is of type (2) in Definition 6.4.3. Let  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2 = \{F_n\}$  be as in the beginning of Section 6.6. Let  $i \in \{1, \dots, k-1\}$  be such that  $\mathcal{F}(G_i) = \mathcal{F}_1$ . Let  $\widehat{V}_\pm$  be open neighborhoods of  $\widehat{\Delta}_\pm(\phi)$ . There exist open neighborhoods  $\widehat{V}'_\pm$  of  $\widehat{\Delta}_\pm(\phi)$  contained in  $\widehat{V}_\pm$  such that  $\phi^\pm(\widehat{V}'_\pm) \subseteq \widehat{V}'_\pm$ .*

**Proof.** The proof follows [CU2, Lemma 4.14]. We prove the result for  $\widehat{\Delta}_+(\phi)$ , the proof for  $\widehat{\Delta}_-(\phi)$  being symmetric. Given  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$ , a finite set of reduced edge paths  $\mathcal{P}$  in  $G$  and some  $\epsilon > 0$  determine an open neighborhood  $N([\mu], \mathcal{P}, \epsilon)$  of  $[\mu]$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$  as follows:

$$N([\mu], \mathcal{P}, \epsilon) = \left\{ [\nu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi) \mid \forall \gamma \in \mathcal{P}, \left| \frac{\langle \gamma, \nu \rangle}{\Psi_0(\nu)} - \frac{\langle \gamma, \mu \rangle}{\Psi_0(\mu)} \right| < \epsilon \right\}.$$

Since  $K_{PG}(\phi)$  is compact, if  $\epsilon$  is small enough, this defines an open neighborhood of  $[\mu]$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$ . For a subset  $X \subseteq \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - K_{PG}(\phi)$ , let

$$N(X, \mathcal{P}, \epsilon) = \bigcup_{[\mu] \in X} N([\mu], \mathcal{P}, \epsilon).$$

For  $L > 0$ , let  $\mathcal{P}_+(L)$  be the set of reduced edge paths in  $G_i$  of length at most equal to  $L$  which are not contained in any concatenation of paths in  $G_{PG, \mathcal{F}_1}$  and  $\mathcal{N}_{PG, \mathcal{F}_1}$ . By Lemma 6.5.18 (3), the set  $\mathcal{P}_+(L)$  is also the set of reduced edge paths in  $G_i$  of length at most equal to  $L$  which are not contained in any concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Let  $[\mu] \in \Delta_+(\phi)$  and let  $t \in [0, 1]$ . Let

$$K_{PG}([\mu], t) = \{[(1-t)\nu + t\mu] \mid [\nu] \in K_{PG}(\phi), \|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1\}.$$

Remark that

$$\widehat{\Delta}_+(\phi) = \bigcup_{[\mu] \in \Delta_+(\phi), t \in [0, 1]} K_{PG}([\mu], t).$$

Let  $\epsilon > 0$ . Let  $V_{poly}(\epsilon) = [\Psi_0^{-1}((-\epsilon, \epsilon))]$ . It is clear, by the continuity of  $\Psi_0$  and the definition 6.3.25 of  $K_{PG}(\phi)$ , that  $\bigcap_{\epsilon > 0} V_{poly}(\epsilon) = K_{PG}(\phi)$ . Let  $t \in (0, 1]$  and  $[\mu] \in \Delta_+(\phi)$  and let  $\mu$  be such that  $\|\mu\|_{\mathcal{F}_1} = 1$ . By Lemma 6.5.18 (5), we have  $\Psi_0(\mu) = 1$ . Let

$$V_{poly}([\mu], t, \epsilon) = \left\{ [\nu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \mid \begin{array}{l} \|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1, \\ t(1 + \epsilon) > \Psi_0(\nu) > t(1 - \epsilon) \end{array} \right\}.$$

Note that, since  $\Psi_0(\mu) = 1$ , we have  $[\nu] \in V_{poly}([\mu], t, \epsilon)$  if for  $[\nu]$  such that  $\|\nu\|_{\mathcal{F}_1} = 1$ , we have

$$t\Psi_0(\mu)(1 + \epsilon) > \Psi_0(\nu) > t\Psi_0(\mu)(1 - \epsilon).$$

Let

$$V_\infty([\mu], t) = \bigcap_{L \rightarrow \infty, \epsilon \rightarrow 0} N(K_{PG}([\mu], t), \mathcal{P}_+(L), \epsilon) \cap V_{poly}([\mu], t, \epsilon).$$

**Claim 1.** For every  $[\mu] \in \Delta_+(\phi)$  and every  $t \in (0, 1]$ , we have  $V_\infty([\mu], t) = K_{PG}([\mu], t)$ .

**Proof.** The inclusion  $K_{PG}([\mu], t) \subseteq V_\infty([\mu], t)$  being immediate since  $\Psi_0$  is linear and vanishes on  $K_{PG}(\phi)$ , we prove the converse inclusion. Let  $\nu \in V_\infty([\mu], t)$ . By definition 6.4.5 of  $\Delta_+(\phi)$ , for every  $[\mu'] \in \Delta_+(\phi)$  and for every reduced edge path  $\gamma$  not contained in  $G_i$ , we have  $\langle \gamma, \mu' \rangle = 0$ . Hence, by Lemma 6.5.18 (4), the current  $[\mu]$  is entirely determined by the cylinder sets determined by reduced edge paths contained in  $G_i$  which are not contained in concatenation of paths in  $G_{PG, \mathcal{F}_1}$  and  $\mathcal{N}_{PG, \mathcal{F}_1}$ . By Lemma 6.5.18 (3), the current  $[\mu]$  is entirely determined by the cylinder sets determined by reduced edge paths contained in  $G_i$  which are not contained in concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . Let  $\gamma$  be a reduced edge path which is contained in  $G_i$  and which is not contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . By Lemma 6.3.27, for every projective current  $[\nu'] \in K_{PG}(\phi)$ , the support of  $\nu'$  is contained in  $\partial^2 \mathcal{A}(\phi)$ . By Proposition 6.3.13, if  $g \in F_n$  is such that there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}(\phi)$  and  $g \in A$ , then  $\gamma_g$  is a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . In particular, if  $\gamma'$  is a path of  $G$  such that  $\{g^{+\infty}, g^{-\infty}\} \in C(\gamma')$ , then  $\gamma'$  is contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . In particular, since  $\gamma$  is not contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ , for every projective current  $[\nu'] \in K_{PG}(\phi)$ , we have  $\langle \gamma, \nu' \rangle = 0$ .

Suppose that  $\|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1$ . By Lemma 6.5.18 (5), we also have  $\Psi_0(\mu) = 1$ . There exists  $\lambda > 0$  such that for every path  $\gamma$  which is contained in  $G_i$  and which is not contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ , we have  $\langle \gamma, \nu \rangle = \langle \gamma, \lambda t \mu \rangle$ . We claim that  $\nu - \lambda t \mu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and that  $[\nu - \lambda t \mu] \in K_{PG}(\phi)$ . Indeed, for the first part, it suffices to show that for every path  $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$ , we have  $(\nu - \lambda t \mu)(C(\gamma)) \geq 0$ . This follows from the fact that, for every path  $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$  such that  $\gamma \subseteq G_i$ , the path  $\gamma$  is not contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$ . Hence we have  $\langle \gamma, \nu \rangle = \langle \gamma, \lambda t \mu \rangle$ . Moreover, if  $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$ , then we have  $\mu(C(\gamma)) = 0$ . This shows that  $\nu - \lambda t \mu \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ .

We now prove that  $[\nu - \lambda t \mu] \in K_{PG}(\phi)$ . Otherwise, by Lemma 6.3.27, the support of  $\nu - \lambda t \mu$  is not contained in  $\partial^2 \mathcal{A}(\phi)$ . By Proposition 6.3.13, there exists a path  $\gamma$  which is not contained in a concatenation of paths in  $G_{PG}$  and in  $\mathcal{N}_{PG}$  such that

$$\langle \gamma, \nu - \lambda t \mu \rangle > 0.$$

Consider a decomposition of  $\gamma = a_1 b_1 \dots a_k b_k$  where, for every  $j \in \{1, \dots, k\}$ , the path  $a_j$  is contained in  $\overline{G - G_i}$  and, for every  $j \in \{1, \dots, k\}$ , the path  $b_j$  is contained in  $G_i$  with  $a_1$  and  $b_k$  possibly empty. By Lemma 6.5.18 (1), (2) and Remark 6.5.19, up to taking a larger path  $\gamma$ , we may suppose that  $b_1$  is nontrivial. By Lemma 6.5.18 (2) and Remark 6.5.19, for every  $j \in \{1, \dots, k\}$ , the path  $a_j$  is contained in  $G_{PG}$ . Since  $\gamma$  is not contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ , there exists  $j \in \{1, \dots, k\}$  such that  $b_j$  is not contained in a concatenation of paths in  $G_{PG}$  and  $\mathcal{N}_{PG}$ . But then  $\langle b_j, \nu \rangle = \langle b_j, \lambda t \mu \rangle$ , that is  $\langle b_j, \nu - \lambda t \mu \rangle = 0$ . By additivity of  $\nu - \lambda t \mu$ , we have

$$\langle \gamma, \nu - \lambda t \mu \rangle \leq \langle b_j, \nu - \lambda t \mu \rangle = 0.$$

This contradicts the choice of  $\gamma$ . Hence  $[\nu - \lambda t \mu] \in K_{PG}(\phi)$ . Therefore, we have  $\Psi_0(\nu - \lambda t \mu) = 0$ . Since  $[\nu] \in V_\infty([\mu], t)$  and since  $\|\nu\|_{\mathcal{F}_1} = \|\mu\|_{\mathcal{F}_1} = 1$ , we see that

$$\Psi_0(\nu) = t\Psi_0(\mu).$$

By linearity of  $\Psi_0$  and the fact that  $\Psi_0(\mu) = 1$ , we have

$$t = t\Psi_0(\mu) = \Psi_0(\nu) = \lambda t\Psi_0(\mu) = \lambda t.$$

Since  $t > 0$  and  $\Psi_0(\mu) = 1$ , we have  $\lambda = 1$ . Suppose first that  $t \neq 1$ . Let  $\nu' = \frac{1}{1-t}(\nu - t\mu)$ , so that  $[\nu'] \in K_{PG}(\phi)$  and  $\|\nu'\|_{\mathcal{F}} = 1$ . Then  $[\nu] = [(1-t)\nu' + t\mu] \in K_{PG}([\mu], t)$ . Thus, we have  $V_\infty([\mu], t) = K_{PG}([\mu], t)$ .

Suppose now that  $t = 1$ . Then  $\Psi_0(\nu) = 1 = \|\nu\|_{\mathcal{F}}$ . We claim that if  $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$  is such that  $\nu(C(\gamma)) > 0$ , then  $\gamma \subseteq G_i$ . Indeed, otherwise there would exist an edge  $e$  contained in  $\overline{G - G_i}$  such that  $\nu(C(e)) > 0$ . By the description of  $\overline{G - G_i}$  given in Lemma 6.5.18 (1), (2) and additivity of the current  $\nu$ , we can choose the edge  $e \in \overline{G - G_i}$  in such a way that  $e \in G_{PG}$ . This would imply that  $\|\nu\|_{\mathcal{F}_1} > \Psi_0(\nu)$ , a contradiction. The claim follows. But, since for every path  $\gamma \in \mathcal{P}(\mathcal{F}_1 \wedge \mathcal{A}(\phi))$  such that  $\gamma \subseteq G_i$ , we have  $\nu(C(\gamma)) = \mu(C(\gamma))$ , we see that  $\nu = \mu$  and that  $\nu \in K_{PG}([\mu], 1)$ . This concludes the proof of the claim.  $\square$

Since  $\widehat{\Delta}_+(\phi)$  is compact, there exist  $L > 0$  and  $\epsilon > 0$  such that, for every  $[\mu] \in \Delta_+(\phi)$  and every  $t \in (0, 1]$ , we have

$$V([\mu], t, L, \epsilon) = N(K_{PG}([\mu], t), \mathcal{P}_+(L), \epsilon) \cap V_{poly}(t, \epsilon) \subseteq \widehat{V}_+.$$

When  $t = 0$ , there exists  $\epsilon > 0$  such that  $V_{poly}(\epsilon) \subseteq \widehat{V}_+$ . Let  $s \in (0, 1)$ , and let  $V$  be an open neighborhood of  $K_{PG}(\phi)$  such that, for every  $[\nu] \in V$  with  $\|\nu\|_{\mathcal{F}_1} = 1$ , we have:

$$\Psi_0(\nu) < s. \tag{6.27}$$

For every  $[\mu] \in \left(N(\widehat{\Delta}_+(\phi), \widehat{\mathcal{P}}_+(L), \epsilon) - V\right) \cap \widehat{\Delta}_+(\phi)$ , there exist  $[\mu_{poly}] \in K_{PG}(\phi)$ ,  $[\mu_{exp}] \in \Delta_+(\phi)$  and  $t \in (0, 1]$  such that

$$[\mu] = [t\mu_{exp} + (1-t)\mu_{poly}].$$

By Lemma 6.6.1 (2), for every  $[\mu] \in \Delta_+(\phi)$ , we have  $\bar{\mathfrak{g}}([\mu]) > 0$ . By compactness of  $\Delta_+(\phi)$  and continuity of  $\bar{\mathfrak{g}}$ , there exists  $\delta_1 > 0$  such that, for every  $\mu \in \Delta_+(\phi)$ , we have  $\bar{\mathfrak{g}}(\mu) \geq \delta_1$ . Since  $N(\widehat{\Delta}_+(\phi), \widehat{\mathcal{P}}_+(L), \epsilon) - V \cap \widehat{\Delta}_+(\phi)$  is compact, and since the function  $\bar{\mathfrak{g}}$  is continuous, there exists  $\delta'_0 > 0$  such that the set  $U = \bar{\mathfrak{g}}^{-1}((\delta'_0, +\infty))$  is an open neighborhood of  $(N(\widehat{\Delta}_+(\phi), \widehat{\mathcal{P}}_+(L), \epsilon) - V) \cap \widehat{\Delta}_+(\phi)$  intersecting  $V$ . Note that  $U \cap K_{PG}(\phi) = \emptyset$ . We set

$$\widehat{V}'_+ = \left( \bigcup_{[\mu] \in \Delta_+(\phi), t \in (0,1]} V([\mu], t, L, \epsilon) \cup V_{poly}(\epsilon) \right) \cap (U \cup V).$$

Let  $\delta_0$  and  $M_0$  be the constants given by Lemma 6.5.20 (2) for the above choice of  $\epsilon > 0$  and  $L > 0$ . By replacing  $\delta_0$  with a smaller constant and  $M_0$  with a larger one, we may suppose that  $\delta_0$  and  $M_0$  also satisfy the conclusion of Lemma 6.5.20 (1) for  $U$  as well (where the open neighborhood  $W$  of  $K_{PG}(\phi)$  needed in Lemma 6.5.20 (1) is such that  $W \subseteq V - U$ ).

**Claim 2** There exists  $N \in \mathbb{N}^*$  such that  $\phi^N(\widehat{V}'_+) \subseteq \widehat{V}'_+$ .

**Proof.** Let  $w \in F_{\mathbf{n}}$  be a nonperipheral element such that  $\eta_{[w]} \in \widehat{V}'_+$ . Suppose first that  $\eta_{[w]} \in U \cap \widehat{V}'_+$ . Since  $\eta_{[w]} \notin K_{PG}(\phi)$ , by Lemma 6.3.26, we have  $\ell_{exp}(\gamma_w) > 0$ . By Lemma 6.6.1 (1), we have:

$$\mathfrak{g}([f^N(\gamma_w)]) \geq \bar{\mathfrak{g}}(\eta_{[w]}) > \delta'_0.$$

By Lemma 6.5.20 (1), there exists  $M \geq M_0 + N$  such that, for every  $w \in F_{\mathbf{n}}$  such that  $\eta_{[w]} \in U \cap \widehat{V}'_+$  and every  $n \geq 1$ , we have  $\phi^{Mn}([\eta_{[w]}]) \in U \cap \widehat{V}'_+ \subseteq \widehat{V}'_+$ .

Suppose now that  $\eta_{[w]} \in V \cap \widehat{V}'_+$ . By Lemma 6.3.27 (3) and Lemma 6.5.18 (4) for every projective current  $[\mu] \in \mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ , we have  $\|\mu\|_{\mathcal{F}_1} > 0$ . For a projective current  $[\mu] \in \mathbb{P}\text{Curr}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi))$ , let

$$\Psi_{\mathcal{F}_1}([\mu]) = \frac{\Psi_0(\mu)}{\|\mu\|_{\mathcal{F}_1}}.$$

Then, by definition of  $V$  and by Lemma 6.3.26, we have

$$\Psi_{\mathcal{F}_1}([\eta_{[w]}]) = \frac{\ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}_1}(\gamma_w)} < s.$$

If  $[\eta_{[w]}] \in K_{PG}(\phi)$ , then since  $\phi(K_{PG}(\phi)) = K_{PG}(\phi)$ , we are done. Therefore, we may suppose that  $[\eta_{[w]}] \notin K_{PG}(\phi)$  and, by Lemma 6.3.26, for every  $n \in \mathbb{N}^*$ , we have  $\ell_{exp}([f^n(\gamma_w)]) \geq 1$ . Let  $R > 1$  be such that  $\frac{1}{1 + \frac{R(1-\delta_0)}{10C}(1-s)} \leq \epsilon$ . By Lemma 6.5.21, one of the following assertion holds:

- (1)  $\mathfrak{g}([f^M(\gamma_w)]) > \delta_0$ ,
- (2)  $\ell_{exp}([f^M(\gamma_w)]) < \frac{10C}{(1-\delta_0)R} \ell_{exp}(\gamma_w)$ .

First assume that Assertion (1) holds. Let  $[\mu_{[\phi^M([w])]}] \in \Delta_+(\phi)$  be the projective current associated with  $[\phi^M([w])]$  given by Lemma 6.5.20 (2). Let

$$t = \Psi_{\mathcal{F}_1}([\eta_{[\phi^M([w])]}]).$$

We claim that  $[\eta_{[\phi^M([w])]}] \in V([\mu_{[\phi^M([w])]}], t, L, \epsilon)$ . Indeed, we clearly have

$$[\eta_{[\phi^M([w])]}] \in V_{poly}([\mu_{[\phi^M([w])]}], t, \epsilon).$$

By Lemma 6.5.20 (2), for every reduced edge path  $\gamma \in \mathcal{P}_+(L)$ , we have

$$\left| \frac{\langle \gamma, \eta_{[\phi^M([w])]} \rangle}{\Psi_0(\eta_{[\phi^M([w])]})} - \frac{\langle \gamma, \mu_{[\phi^M([w])]} \rangle}{\Psi_0(\mu_{[\phi^M([w])]})} \right| < \epsilon.$$

Therefore we have  $[\eta_{[\phi^M([w])]}] \in N(K_{PG}([\mu_{[\phi^M([w])]}], t), \mathcal{P}_+(L), \epsilon)$ . The claim follows by Equation (6.27). By definition of  $\widehat{V}'_+$ , we see that  $\phi^M([\eta_{[w]}]) = [\eta_{[\phi^M([w])]}] \in \widehat{V}'_+$ .

Suppose now that Assertion (2) holds. We claim that  $[\eta_{[\phi^M([w])]}] \in V_{poly}(\epsilon)$ . By Lemma 6.5.18 (1),(2) and Remark 6.5.19, the graph  $\overline{G} - \overline{G}_i$  consists in edges in  $G_{PG}$ . By Lemma 6.5.18 (6), we have

$$\ell_{\mathcal{F}_1}([f^M(\gamma_w)]) - \ell_{exp}([f^M(\gamma_w)]) = \ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w).$$

Hence we have

$$\begin{aligned} \Psi_{\mathcal{F}_1}([\eta_{[\phi^M(\gamma_w)]}]) &= \frac{\ell_{exp}([f^M(\gamma_w)])}{\ell_{\mathcal{F}_1}([f^M(\gamma_w)])} = \frac{\ell_{exp}([f^M(\gamma_w)])}{\ell_{exp}([f^M(\gamma_w)]) + \ell_{\mathcal{F}_1}([f^M(\gamma_w)]) - \ell_{exp}([f^M(\gamma_w)])} \\ &= \frac{\ell_{exp}([f^M(\gamma_w)])}{\ell_{exp}([f^M(\gamma_w)]) + \ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w)} \\ &= \frac{1}{1 + \frac{\ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w)}{\ell_{exp}([f^M(\gamma_w)])}} \leq \frac{1}{1 + \frac{R(1-\delta_0)}{10C} \frac{\ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w)}{\ell_{exp}(\gamma_w)}} \\ &\leq \frac{1}{1 + \frac{R(1-\delta_0)}{10C} \frac{\ell_{\mathcal{F}_1}(\gamma_w) - \ell_{exp}(\gamma_w)}{\ell_{\mathcal{F}_1}(\gamma_w)}} \leq \frac{1}{1 + \frac{R(1-\delta_0)}{10C}(1-s)} \leq \epsilon. \end{aligned}$$

Note that  $\psi_{\mathcal{F}_1}^{-1}((0, \epsilon)) \subseteq V_{poly}(\epsilon)$ . Thus, we have  $\Phi^M([\eta_{[w]}]) = [\eta_{[\phi^M([w])]}] \in V_{poly}(\epsilon) \subseteq \widehat{V}'_+$ . Therefore, by density of the rational currents (see Proposition 6.2.15) and continuity of  $\phi$ , we have  $\phi^M(\widehat{V}'_+) \subseteq \widehat{V}'_+$ . This proves Claim 2.  $\square$

Let

$$\widehat{V}''_+ = \bigcap_{i=0}^{M-1} \phi^i(\widehat{V}'_+).$$

Since  $\phi(\widehat{\Delta}_+(\phi)) = \widehat{\Delta}_+(\phi)$ , the set  $\widehat{V}''_+$  is an open neighborhood of  $\widehat{\Delta}_+(\phi)$  which is stable by  $\phi$  by construction. This concludes the proof.  $\square$

**Theorem 6.6.4.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2$  be a sequence of free factor systems such that the extension  $\mathcal{F}_1 \leq \mathcal{F}_2$  is sporadic. Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be such that  $\phi$  preserves  $\mathcal{F} \leq \mathcal{F}_1 \leq \mathcal{F}_2$  and  $\phi|_{\mathcal{F}_1}$  is an expanding automorphism relative to  $\mathcal{F}$ . Let  $\hat{\Delta}_{\pm}(\phi)$  be the convexes of attraction and repulsion of  $\phi$  and  $\Delta_{\pm}(\phi)$  be the simplices of attraction and repulsion of  $\phi$ . Let  $U_{\pm}$  be open neighborhoods of  $\Delta_{\pm}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and  $\hat{V}_{\pm}$  be open neighborhoods of  $\hat{\Delta}_{\pm}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ . There exists  $M \in \mathbb{N}^*$  such that for every  $n \geq M$ , we have*

$$\phi^{\pm n}(\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \hat{V}_{\mp}) \subseteq U_{\pm}.$$

**Proof.** The proof is similar to [CU2, Theorem 4.15]. We replace  $\phi$  by a power so that  $\phi$  satisfies Remark 6.5.15. By Lemmas 6.6.2 and 6.6.3, we may suppose that  $\phi(U_+) \subseteq U_+$  and that  $\phi(\hat{V}_+) \subseteq \hat{V}_+$ . Let  $M$  be the exponent given by Proposition 6.5.24 by using  $U_+ = U_+$  and  $U_- = V = \hat{V}_-$ . For every current  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \phi^M(\hat{V}_+)$ , we have  $\phi^M([\mu]) \in U_+$  since  $\phi^{-M}([\mu]) \notin \hat{V}_-$ . Therefore, for every  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \hat{V}_-$ , we have  $\phi^{2M}([\mu]) \in U_+$  and for every  $n \geq M$ , we have  $\phi^{2n}([\mu]) \in U_+$  since  $\phi(U_+) \subseteq U_+$ . Therefore for every  $n \geq M$ , we see that

$$\phi^{2n}(\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \hat{V}_-) \subseteq U_+.$$

A symmetric argument for  $\phi^{-1}$  shows that  $\phi^2$  acts with generalized North-South dynamics. By [LU2, Proposition 3.4], we see that  $\phi$  acts with generalized North-South dynamics. This concludes the proof.  $\square$

**Corollary 6.6.5.** *For every open neighborhood  $\hat{V}_- \subseteq \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  of  $\hat{\Delta}_-$ , there exist  $M \in \mathbb{N}^*$  and a constant  $L_0$  such that, for every current  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \hat{V}_-$ , and every  $m \geq M$ , we have*

$$\|\phi^m(\mu)\|_{\mathcal{F}} \geq 3^{m-M} L_0 \|\mu\|_{\mathcal{F}}.$$

**Proof.** Let  $f: G \rightarrow G$  be as in Remark 6.5.15. By Lemma 6.6.1 (2), there exist a constant  $\delta > 0$  and an open neighborhood  $U$  of  $\Delta_+(\phi)$  such that, for every projective current  $[\mu] \in U$ , we have  $\mathfrak{g}([\mu]) \geq \delta$ . We first prove Corollary 6.6.5 for currents  $[\mu] \in U$ . By Proposition 6.2.15, it suffices to prove the result for rational currents. By Lemma 6.6.1 (1), since  $U \cap K_{PG}(\phi) = \emptyset$ , for every element  $w \in F_n$  such that  $[\eta_{[w]}] \in U$ , we have  $\mathfrak{g}([f^N(\gamma_w)]) \geq \delta$ . By Lemma 6.5.16 (1) and Lemma 6.5.3, there exists a constant  $K_1 > 0$  depending on  $\delta$  such that for every  $m \geq N$  and for every element  $w \in F_n$  such that  $[\eta_w] \in U$ , we have  $\ell_{exp}([f^m(\gamma_w)]) \geq TEL(m - N, [f^N(\gamma_w)]) \geq 3^{m-N} K_1 \ell_{exp}([f^N(\gamma_w)])$ . Since  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \hat{V}_-$  is compact and since  $K_{PG}(\phi) \subseteq \hat{V}_-$ , by Lemma 6.3.26 and Lemma 6.3.27 (3), there exists a constant  $K_2 > 0$  such that for every  $m \geq N$  and for every element  $w \in F_n$  such that  $[\eta_{[w]}] \in U$ , we have  $\frac{\ell_{exp}([f^m(\gamma_w)])}{\ell_{\mathcal{F}}([f^m(\gamma_w)])} \geq K_2$ . Thus, we have

$$\ell_{\mathcal{F}}([f^m(\gamma_w)]) \geq \ell_{exp}([f^m(\gamma_w)]) \geq 3^{m-N} K_1 \ell_{exp}([f^N(\gamma_w)]) \geq 3^{m-M} K_1 K_2 \ell_{\mathcal{F}}([f^N(\gamma_w)]).$$

We set  $K_3 = K_1 K_2$ . By compactness of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and Lemma 6.3.27 (3), there exists  $L > 0$  such that for every current  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ , we have  $\frac{\|\phi^N(\mu)\|_{\mathcal{F}}}{\|\mu\|_{\mathcal{F}}} \geq L$ . Hence for every  $m \geq N$  and for every element  $w \in F_n$  such that  $[\eta_{[w]}] \in U$ , we have

$$\ell_{\mathcal{F}}([f^m(\gamma_w)]) \geq 3^{m-N} K_3 L \ell_{\mathcal{F}}(\gamma_w).$$

Hence the proof follows when  $[\mu] \in U$ .

By Theorem 6.6.4, there exists  $M_1 \in \mathbb{N}^*$  such that, for every  $m \geq M_1$  and every  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_-$ , we have  $\phi^m([\mu]) \in U$ . Let  $M = M_1 + N$ . By the above, Lemma 6.3.26, the density of rational currents (see Proposition 6.2.15) and continuity of  $\phi$ , for every current  $[\mu] \notin \widehat{V}_-$ , for every  $n \geq M$ , we have

$$\|\phi^n(\mu)\|_{\mathcal{F}} \geq 3^{n-M} K_3 L \|\phi^{M_1}(\mu)\|_{\mathcal{F}}.$$

By compactness of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and Lemma 6.3.27 (3), there exists  $L' > 0$  such that for every current  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ , we have  $\frac{\|\phi^{M_1}(\mu)\|_{\mathcal{F}}}{\|\mu\|_{\mathcal{F}}} \geq L'$ . Hence for every  $n \geq M$ , we have

$$\|\phi^n(\mu)\|_{\mathcal{F}} \geq 3^{n-M} K_3 L L' \|\mu\|_{\mathcal{F}}.$$

This concludes the proof. □

# Chapitre 7

## Polynomial growth and subgroups of $\text{Out}(F_n)$

### 7.1 Introduction

Let  $n \geq 3$ . This paper, which is the last of a series of three papers, studies the exponential growth of elements in  $\text{Out}(F_n)$ . An outer automorphism  $\phi \in \text{Out}(F_n)$  is *exponentially growing* if there exist  $g \in F_n$ , a representative  $\Phi$  of  $\phi$ , a free basis  $\mathfrak{B}$  of  $F_n$  and a constant  $K > 0$  such that, for every  $m \in \mathbb{N}^*$ , we have

$$\ell_{\mathfrak{B}}(\Phi^m(g)) \geq e^{Km}, \quad (7.1)$$

where  $\ell_{\mathfrak{B}}(\Phi^m(g))$  denotes the length of  $\Phi^m(g)$  in the basis  $\mathfrak{B}$ . If  $g \in F_n$  satisfies Equation (7.1) for every representative  $\Phi$  of  $\phi$ , then  $g$  is said to be *exponentially growing under iteration of  $\phi$* . Otherwise, one can show, using for instance the technology of relative train tracks introduced by Bestvina and Handel [BH], that  $g$  has *polynomial growth under iteration of  $\phi$* , replacing  $\geq e^{Km}$  by  $\leq (m+1)^K$  in Equation (7.1) (see also [Lev2] for a complete description of all growth types that can occur under iteration of an outer automorphism  $\phi$ ). We denote by  $\text{Poly}(\phi)$  the set of elements of  $F_n$  which have polynomial growth under iteration of  $\phi$ . If  $H$  is a subgroup of  $F_n$ , we set  $\text{Poly}(H) = \bigcap_{\phi \in H} \text{Poly}(\phi)$ . In this article, we prove the following theorem.

**Theorem 7.1.1.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{Out}(F_n)$ . There exists  $\phi \in H$  such that  $\text{Poly}(\phi) = \text{Poly}(H)$ .*

In other words, there exists an element of  $H$  which encaptures all the exponential growth of  $H$ : there exists  $\phi \in H$  such that if  $g \in F_n$  has exponential growth for some element of  $H$ , then  $g$  has exponential growth for  $\phi$ . Theorem 6.1.2 has analogues in other contexts. For instance, one has a similar result in the context of the mapping class group of a closed, connected, orientable surface  $S$  equipped with a hyperbolic structure. Indeed, a consequence of the Nielsen-Thurston classification (see for instance [FarM, Theorem 13.2]) and the work of Thurston [FLP, Proposition 9.21] is that the growth



of the length of the geodesic representative of a homotopy class of an essential closed curve under iteration of an element of  $\text{Mod}(S)$  is either exponential or linear. Moreover, linear growth comes from twists about essential curves while exponential growth comes from pseudo-Anosov homeomorphisms of subsurfaces of  $S$ . In [Iva1] (see also the work of McCarthy [McC]), Ivanov proved that, for every subgroup  $H$  of  $\text{Mod}(S)$ , up to taking a finite index subgroup of  $H$ , there exists finitely many homotopy classes of pairwise disjoint essential closed curves  $C_1, \dots, C_k$  elementwise fixed by  $H$  and such that, for every connected component  $S'$  of  $S - \bigcup_{i=1}^k C_i$ , the restriction  $H|_{S'} \subseteq \text{Mod}(S')$  is either the trivial group or contains a pseudo-Anosov element. One can then construct an element  $f \in H$  such that, for every connected component  $S - \bigcup_{i=1}^k C_i$  such that the restriction  $H|_{S'} \subseteq \text{Mod}(S')$  contains a pseudo-Anosov element, the element  $f|_{S'} \in \text{Mod}(S')$  is a pseudo-Anosov.

In the context of  $\text{Out}(F_n)$ , Clay and Uyanik [CU2] proved Theorem 6.1.2 when  $H$  is a subgroup of  $\text{Out}(F_n)$  such that  $\text{Poly}(H) = \{1\}$ . Indeed, by a result of Levitt [Lev2, Proposition 1.4, Lemma 1.5], if  $\phi \in \text{Out}(F_n)$  and if  $\text{Poly}(\phi) \neq \{1\}$ , then there exists a nontrivial element  $g \in F_n$  and  $k \in \mathbb{N}^*$  such that  $\phi^k([g]) = [g]$ . In this context, Clay and Uyanik proved that, if  $H$  does not virtually preserve the conjugacy class of a nontrivial element of  $F_n$ , there exists an element  $\phi \in H$  which is *atoroidal*: no power of  $\phi$  fixes the conjugacy class of a nontrivial element of  $F_n$ . From Clay and Uyanik's theorem, one can then ask the following question. If  $H$  is a subgroup of  $\text{Out}(F_n)$  such that  $H$  virtually fixes the conjugacy class of a nontrivial subgroup  $A$  of  $F_n$ , is it true that either  $H$  virtually fixes the conjugacy class of a nontrivial element  $g \in F_n$  such that  $g$  is not contained in a conjugate of  $A$ , or there exists  $\phi \in H$  such that the only conjugacy classes of elements of  $F_n$  virtually fixed by  $\phi$  are contained in a conjugate of  $A$ ?

Unfortunately, such a result is not true. Indeed, let  $F_3 = \langle a, b, c \rangle$  be a nonabelian free group of rank 3. Let  $\phi_a$  (resp.  $\phi_b$ ) be the automorphism of  $F_3$  which fixes  $a$  and  $b$  and which sends  $c$  to  $ca$  (resp.  $c$  to  $cb$ ), and let  $H = \langle [\phi_a], [\phi_b] \rangle \subseteq \text{Out}(F_3)$ . Then every element  $\phi \in H$  has a representative which fixes  $\langle a, b \rangle$  and sends  $c$  to  $cg_\phi$  with  $g_\phi \in \langle a, b \rangle$ . Thus,  $\phi$  fixes the conjugacy class of  $g_\phi c g_\phi^{-1}$ . However, there always exist  $\phi' \in H$ , such that  $\phi'$  does not preserve the conjugacy class of  $g_\phi c g_\phi^{-1}$ . This example illustrates the main difficulty which appears when generalizing Clay and Uyanik's theorem: the fact that  $\text{Poly}(H) \neq \{1\}$  implies that every element of  $H$  has periodic conjugacy classes which might not be fixed by the whole group. However, for the above example, we have  $\text{Poly}(H) = F_3$  and every element of  $H$  satisfies Theorem 6.1.2. Therefore, Theorem 6.1.2 is, from this viewpoint, the right generalization of Clay and Uyanik's theorem.

We now sketch the proof of Theorem 6.1.2. It is inspired by the proof of [CU2, Theorem A]. However, technical difficulties emerge due to the presence of elements of  $F_n$  with polynomial growth under iteration of elements of the considered subgroup of  $\text{Out}(F_n)$ . The main difficulties are dealt with in the second article of the series [Gue5]. Let  $H$  be a subgroup of  $\text{Out}(F_n)$ . We first consider  $H$ -invariant *free factor systems*  $\mathcal{F}$  of  $F_n$ , that is,  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$ , where, for every  $i \in \{1, \dots, k\}$ ,  $[A_i]$  is the conjugacy class of a subgroup  $A_i$  of  $F_n$  and there exists a subgroup  $B$  of  $F_n$  such that  $F_n = A_1 * \dots * A_k * B$ . There exists a partial order on the set of free factor systems of  $F_n$ ,

where  $\mathcal{F}_1 \leq \mathcal{F}_2$  if for every subgroup  $A_1$  of  $F_n$  such that  $[A_1] \in \mathcal{F}_1$ , there exists a subgroup  $A_2$  of  $F_n$  such that  $[A_2] \in \mathcal{F}_2$  and  $A_1$  is a subgroup of  $A_2$ . Hence we may consider a maximal  $H$ -invariant sequence of free factor systems

$$\emptyset = \mathcal{F}_0 \leq \mathcal{F}_1 \leq \dots \leq \mathcal{F}_k = \{[F_n]\}.$$

The proof is now by induction on  $i \in \{1, \dots, k\}$ : for every  $i \in \{0, \dots, k\}$ , we construct an element  $\phi_i \in H$  such that  $\text{Poly}(\phi_i|_{\mathcal{F}_i}) = \text{Poly}(H|_{\mathcal{F}_i})$  (we define the sense of the restrictions in Section 7.2.3). Let  $i \in \{1, \dots, k\}$  and suppose that we have constructed  $\phi_{i-1}$ . There are two cases to consider. If the extension  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$  is *nonsporadic* (see the definition in Section 7.2.1) then the construction of  $\phi_i$  from  $\phi_{i-1}$  follows from the works of Handel-Mosher [HaM4], Guirardel-Horbez [GuH2] and Clay-Uyanik [CU1]. If the extension  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$  is *sporadic*, the construction of  $\phi_i$  relies on the action of  $H$  on some natural (compact, metrizable) space that we introduced in [Gue4]. This space is called the *space of currents relative to*  $\text{Poly}(H|_{\mathcal{F}_{i-1}})$ , denoted by  $\mathbb{P}\text{Curr}(F_n, \text{Poly}(H|_{\mathcal{F}_{i-1}}))$ . It is defined as a subspace of the space of Radon measures on a natural space  $\partial^2(F_n, \text{Poly}(H|_{\mathcal{F}_{i-1}}))$ , the double boundary of  $F_n$  relative to  $\text{Poly}(H|_{\mathcal{F}_{i-1}})$  (see Section 7.2.2 for precise definitions). In [Gue5], we proved that the element  $\phi_{i-1}$  that we have constructed acts with a *North-South dynamics* on  $\mathbb{P}\text{Curr}(F_n, \text{Poly}(H|_{\mathcal{F}_{i-1}}))$ : there exist two proper disjoint closed subsets of  $\mathbb{P}\text{Curr}(F_n, \text{Poly}(H|_{\mathcal{F}_{i-1}}))$  such that every point of  $\mathbb{P}\text{Curr}(F_n, \text{Poly}(H|_{\mathcal{F}_{i-1}}))$  which is not contained in these subsets converges to one of the two subsets under positive or negative iteration of  $\phi_{i-1}$ . This North-South dynamics result allows us, applying classical ping-pong arguments similar to the one of Tits [Tit1], to construct the element  $\phi_i \in H$  such that  $\text{Poly}(\phi_i|_{\mathcal{F}_i}) = \text{Poly}(H|_{\mathcal{F}_i})$ , which concludes the proof.

The element constructed in Theorem 6.1.2 is in general not unique. Indeed, when the subgroup  $H$  of  $\text{Out}(F_n)$  is such that  $\text{Poly}(H) = \{1\}$ , Clay and Uyanik [CU2, Theorem B] give necessary and sufficient conditions for  $H$  to contain a nonabelian free subgroup consisting in atoroidal elements.

We now outline some consequences of Theorem 6.1.2. The first one is a result concerning the periodic subset of a subgroup of  $\text{Out}(F_n)$ . Let  $H$  be a subgroup of  $\text{Out}(F_n)$ . We denote by  $\text{Per}(H)$  the set of conjugacy classes of  $F_n$  fixed by a power of every element of  $H$ . In the above example, we constructed a subgroup  $H$  of  $\text{Out}(F_n)$  such that  $\text{Per}(H)$  contains the conjugacy class of a nonabelian subgroup of rank 2. This is in fact the lowest possible rank where a generalization of the theorem of Clay and Uyanik using  $\text{Per}(H)$  instead of  $\text{Poly}(H)$  cannot work, as shown by the following result.

**Theorem 7.1.2** (Corollary 7.5.3). *Let  $n \geq 3$  and let  $g_1, \dots, g_k$  be nontrivial root-free elements of  $F_n$ . Let  $H$  be subgroup of  $\text{Out}(F_n)$  such that, for every  $i \in \{1, \dots, k\}$ , every element of  $H$  has a power which fixes the conjugacy class of  $g_i$ . Then one of the following (mutually exclusive) statements holds.*

- (1) *There exists  $g_{k+1} \in F_n$  such that  $[g_{k+1}] \notin \{[g_1], \dots, [g_k]\}$  and whose conjugacy class is fixed by a power of every element of  $H$ .*
- (2) *There exists  $\phi \in H$  such that  $\text{Per}(\phi) = \{[\langle g_1 \rangle], \dots, [\langle g_k \rangle]\}$ .*

As we show with Corollary 7.5.4, Case (2) of Theorem 7.1.2 naturally occurs when we are working with a subgroup of a mapping class group of a compact, connected surface  $S$  whose fundamental group is identified with  $F_n$ . Finally, we give in Proposition 7.5.6 a method, using JSJ decompositions of  $F_n$ , allowing to compute  $\text{Poly}(H)$  for subgroups  $H$  of  $\text{Out}(F_n)$  which act by global conjugations on some subgroups of  $F_n$ .

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## 7.2 Preliminaries

### 7.2.1 Malnormal subgroup systems of $F_n$

Let  $n$  be an integer greater than 1 and let  $F_n$  be a free group of rank  $n$ . A *subgroup system* of  $F_n$  is a finite (possibly empty) set  $\mathcal{A}$  whose elements are conjugacy classes of nontrivial (that is distinct from  $\{1\}$ ) finite rank subgroups of  $F_n$ . Note that a subgroup system  $\mathcal{A}$  is completely determined by the set of subgroups  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$ . There exists a partial order on the set of subgroup systems of  $F_n$ , where  $\mathcal{A}_1 \leq \mathcal{A}_2$  if for every subgroup  $A_1$  of  $F_n$  such that  $[A_1] \in \mathcal{A}_1$ , there exists a subgroup  $A_2$  of  $F_n$  such that  $[A_2] \in \mathcal{A}_2$  and  $A_1$  is a subgroup of  $A_2$ . In this case we say that  $\mathcal{A}_2$  is an *extension* of  $\mathcal{A}_1$ . The *stabilizer in  $\text{Out}(F_n)$  of a subgroup system  $\mathcal{A}$* , denoted by  $\text{Out}(F_n, \mathcal{A})$ , is the set of all elements  $\phi \in \text{Out}(F_n)$  such that  $\phi(\mathcal{A}) = \mathcal{A}$ . If  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are two subgroup systems, we set  $\text{Out}(F_n, \mathcal{A}_1, \mathcal{A}_2) = \text{Out}(F_n, \mathcal{A}_1) \cap \text{Out}(F_n, \mathcal{A}_2)$ .

Recall that a subgroup  $A$  of  $F_n$  is *malnormal* if for every element  $x \in F_n - A$ , we have  $xAx^{-1} \cap A = \{e\}$ . A subgroup system  $\mathcal{A}$  is said to be *malnormal* if every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{A}$  is malnormal and, for all subgroups  $A_1, A_2$  of  $F_n$  such that  $[A_1], [A_2] \in \mathcal{A}$ , if  $A_1 \cap A_2$  is nontrivial then  $A_1 = A_2$ . An element  $g \in F_n$  is  *$\mathcal{A}$ -peripheral* (or simply *peripheral* if there is no ambiguity) if it is trivial or conjugate into one of the subgroups of  $\mathcal{A}$ , and  *$\mathcal{A}$ -nonperipheral* otherwise.

An important class of examples of malnormal subgroup systems is given by the *free factor systems*. A *free factor system* of  $F_n$  is a (possibly empty) set  $\mathcal{F}$  of conjugacy classes  $\{[A_1], \dots, [A_r]\}$  of nontrivial subgroups  $A_1, \dots, A_r$  of  $F_n$  such that there exists a subgroup  $B$  of  $F_n$  with  $F_n = A_1 * \dots * A_r * B$ . An extension  $\mathcal{F}_1 \leq \mathcal{F}_2 = \{[A_1], \dots, [A_k]\}$  of free factor systems is *sporadic* if there exists  $\ell \in \{1, \dots, k\}$  such that, for every  $j \in \{1, \dots, k\} - \{\ell\}$ , we have  $[A_j] \in \mathcal{F}_1$  and if one of the following holds:

- (a) there exist subgroups  $B_1, B_2$  of  $F_n$  such that  $[B_1], [B_2] \in \mathcal{F}_1$  and  $A_\ell = B_1 * B_2$ ;
- (b) there exists a subgroup  $B$  of  $F_n$  such that  $[B] \in \mathcal{F}_1$  and  $A_\ell$  is an HNN extension of  $B$  over the trivial group;
- (c) there exists  $g \in F_n$  such that  $\mathcal{F}_2 = \mathcal{F}_1 \cup \{[g]\}$  and  $A_\ell = \langle g \rangle$ .

Otherwise, the extension  $\mathcal{F}_1 \leq \mathcal{F}_2$  is *nonsporadic*. A free factor system  $\mathcal{F}$  of  $F_n$  is *sporadic* (resp. *nonsporadic*) if the extension  $\mathcal{F} \leq \{[F_n]\}$  is sporadic (resp. nonsporadic). An ascending sequence of free factor systems  $\mathcal{F}_1 \leq \dots \leq \mathcal{F}_i = \{[F_n]\}$  of  $F_n$  is called a *filtration* of  $F_n$ .

Given a free factor system  $\mathcal{F}$  of  $F_n$ , a *free factor of  $(F_n, \mathcal{F})$*  is a subgroup  $A$  of  $F_n$  such that there exists a free factor system  $\mathcal{F}'$  of  $F_n$  with  $[A] \in \mathcal{F}'$  and  $\mathcal{F} \leq \mathcal{F}'$ . When  $\mathcal{F} = \emptyset$ , we say that  $A$  is a *free factor of  $F_n$* . A free factor of  $(F_n, \mathcal{F})$  is *proper* if it is nontrivial, not equal to  $F_n$  and if its conjugacy class does not belong to  $\mathcal{F}$ .

In general, we will work in a finite index subgroup of  $\text{Out}(F_n)$  defined as follows. Let

$$\text{IA}_n(\mathbb{Z}/3\mathbb{Z}) = \ker(\text{Out}(F_n) \rightarrow \text{Aut}(H_1(F_n, \mathbb{Z}/3\mathbb{Z}))).$$

For every  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ , we have the following properties:

- (1) any  $\phi$ -periodic conjugacy class of free factor of  $F_n$  is fixed by  $\phi$  [HaM4, Theorem II.3.1];
- (2) any  $\phi$ -periodic conjugacy class of elements of  $F_n$  is fixed by  $\phi$  [HaM4, Theorem II.4.1].

Another class of examples of malnormal subgroup systems is the following one. Let  $g \in F_n$  and let  $\mathfrak{B}$  be a free basis of  $F_n$ . The length of the conjugacy class of  $g$  with respect to  $\mathfrak{B}$  is

$$\ell_{\mathfrak{B}}([g]) = \min_{h \in [g]} \ell_{\mathfrak{B}}(h),$$

where  $\ell_{\mathfrak{B}}(h)$  is the word length of  $h$  with respect to the basis  $\mathfrak{B}$ . An outer automorphism  $\phi \in \text{Out}(F_n)$  is *exponentially growing* if there exists  $g \in F_n$  such that the length of the conjugacy class  $[g]$  of  $g$  in  $F_n$  with respect to some basis of  $F_n$  grows exponentially fast under positive iteration of  $\phi$ . One can show that if  $g$  is exponentially growing with respect to some free basis of  $F_n$ , then it is exponentially growing for every free basis of  $F_n$ . If  $\phi \in \text{Out}(F_n)$  is not exponentially growing, one can show, using for instance the technology of train tracks due to Bestvina and Handel [BH], that for every  $g \in F_n$ , the element  $g$  has polynomial growth under positive iteration of  $\phi$ . In this case, we say that  $\phi$  is *polynomially growing*. A result of Levitt [Lev2, Proposition 1.4 (1)] shows that this definition is equivalent to the definition given in the introduction. For an automorphism  $\alpha \in \text{Aut}(F_n)$ , we say that  $\alpha$  is exponentially growing if there exists  $g \in F_n$  such that the word length of  $g$  grows exponentially fast under iteration of  $\alpha$ . Otherwise,  $\alpha$  is polynomially growing. Let  $\phi \in \text{Out}(F_n)$  be exponentially growing. A subgroup  $P$  of  $F_n$  is a *polynomial subgroup* of  $\phi$  if there exist  $k \in \mathbb{N}^*$  and a representative  $\alpha$  of  $\phi^k$  such that  $\alpha(P) = P$  and  $\alpha|_P$  is polynomially growing. By [Lev2, Proposition 1.4], there exist finitely many conjugacy classes  $[H_1], \dots, [H_k]$  of maximal polynomial subgroups of  $\phi$ . Moreover, the proof of [Lev2, Proposition 1.4] implies that the set  $\mathcal{H} = \{[H_1], \dots, [H_k]\}$  is a malnormal subgroup system (see [Gue5, Section 2.1]). We denote this malnormal subgroup system by  $\mathcal{A}(\phi)$ . Note that, if  $H$  is a subgroup of  $F_n$  such that  $[H] \in \mathcal{A}(\phi)$ , there exists  $\Phi^{-1} \in \phi^{-1}$  such that  $\Phi^{-1}(H) = H$  and  $\Phi^{-1}|_H$  is polynomially growing. Hence we have  $\mathcal{A}(\phi) \leq \mathcal{A}(\phi^{-1})$ . By symmetry, we have

$$\mathcal{A}(\phi) = \mathcal{A}(\phi^{-1}). \tag{7.2}$$

Moreover, for every element  $\psi \in \text{Out}(F_n)$ , we have

$$\mathcal{A}(\psi\phi\psi^{-1}) = \psi(\mathcal{A}(\phi)).$$

In order to distinguish between the set of elements of  $F_{\mathbf{n}}$  which have polynomial growth under positive iteration of  $\phi$  and the associated malnormal subgroup system, we will denote by  $\text{Poly}(\phi)$  the former. We have  $\text{Poly}(\phi) = \text{Poly}(\phi^{-1})$  by Equation (7.2). If  $H$  is a subgroup of  $\text{Out}(F_{\mathbf{n}})$ , we set  $\text{Poly}(H) = \bigcap_{\phi \in H} \text{Poly}(\phi)$ .

Let  $\mathcal{A}$  be a malnormal subgroup system and let  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{A})$  be a relative outer automorphism. We say that  $\phi$  is *atoroidal relative to  $\mathcal{A}$*  if, for every  $k \in \mathbb{N}^*$ , the element  $\phi^k$  does not preserve the conjugacy class of any  $\mathcal{A}$ -nonperipheral element. We say that  $\phi$  is *expanding relative to  $\mathcal{A}$*  if  $\mathcal{A}(\phi) \leq \mathcal{A}$ . Note that an expanding outer automorphism relative to  $\mathcal{A}$  is in particular atoroidal relative to  $\mathcal{A}$ . When  $\mathcal{A} = \emptyset$ , then the outer automorphism  $\phi$  is expanding relative to  $\mathcal{A}$  if and only if for every nontrivial element  $g \in F_{\mathbf{n}}$ , the length of the conjugacy class  $[g]$  of  $g$  in  $F_{\mathbf{n}}$  with respect to some basis of  $F_{\mathbf{n}}$  grows exponentially fast under iteration of  $\phi$ . Therefore, by a result of Levitt [Lev2, Corollary 1.6], the outer automorphism  $\phi$  is expanding relative to  $\mathcal{A} = \emptyset$  if and only if  $\phi$  is atoroidal relative to  $\mathcal{A} = \emptyset$ .

Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system and let  $\mathcal{F}$  be a free factor system. Let  $i \in \{1, \dots, r\}$ . By [SW, Theorem 3.14] for the action of  $A_i$  on one of its Cayley graphs, there exist finitely many subgroups  $A_i^{(1)}, \dots, A_i^{(k_i)}$  of  $A_i$  such that:

- (1) for every  $j \in \{1, \dots, k_i\}$ , there exists a subgroup  $B$  of  $F_{\mathbf{n}}$  such that  $[B] \in \mathcal{F}$  and  $A_i^{(j)} = B \cap A_i$ ;
- (2) for every subgroup  $B$  of  $F_{\mathbf{n}}$  such that  $[B] \in \mathcal{F}$  and  $B \cap A_i \neq \{e\}$ , there exists  $j \in \{1, \dots, k_i\}$  such that  $A_i^{(j)} = B \cap A_i$ ;
- (3) the subgroup  $A_i^{(1)} * \dots * A_i^{(k_i)}$  is a free factor of  $A_i$ .

Thus, one can define a new subgroup system as

$$\mathcal{F} \wedge \mathcal{A} = \bigcup_{i=1}^r \{[A_i^{(1)}], \dots, [A_i^{(k_i)}]\}.$$

Since  $\mathcal{A}$  is malnormal, and since, for every  $i \in \{1, \dots, r\}$ , the group  $A_i^{(1)} * \dots * A_i^{(k_i)}$  is a free factor of  $A_i$ , it follows that the subgroup system  $\mathcal{F} \wedge \mathcal{A}$  is a malnormal subgroup system of  $F_{\mathbf{n}}$ . We call it the *meet of  $\mathcal{F}$  and  $\mathcal{A}$* . If  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{F}, \mathcal{A})$  then  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A})$ .

## 7.2.2 Relative currents

In this section, we define the notion of *currents of  $F_{\mathbf{n}}$  relative to a malnormal subgroup system  $\mathcal{A}$* . The section follows [Gue4, Gue5] (see the work of Gupta [Gup1] for the particular case of free factor systems and Guirardel and Horbez [GuH1] in the context of free products of groups). It can be thought of as a functional space in which densely live the  $\mathcal{A}$ -nonperipheral elements of  $F_{\mathbf{n}}$ .

Let  $\partial_{\infty} F_{\mathbf{n}}$  be the Gromov boundary of  $F_{\mathbf{n}}$ . The *double boundary of  $F_{\mathbf{n}}$*  is the Hausdorff locally compact, totally disconnected quotient topological space

$$\partial^2 F_{\mathbf{n}} = (\partial_{\infty} F_{\mathbf{n}} \times \partial_{\infty} F_{\mathbf{n}} \setminus \Delta) / \sim,$$

where  $\sim$  is the equivalence relation generated by the flip relation  $(x, y) \sim (y, x)$  and  $\Delta$  is the diagonal, endowed with the diagonal action of  $F_{\mathbf{n}}$ . We denote by  $\{x, y\}$  the equivalence class of  $(x, y)$ .

Let  $T$  be the Cayley graph of  $F_{\mathbf{n}}$  with respect to a free basis  $\mathfrak{B}$ . The boundary of  $T$  is naturally homeomorphic to  $\partial_{\infty} F_{\mathbf{n}}$  and the set  $\partial^2 F_{\mathbf{n}}$  is then identified with the set of unoriented bi-infinite geodesics in  $T$ . Let  $\gamma$  be a finite geodesic path in  $T$ . The path  $\gamma$  determines a subset in  $\partial^2 F_{\mathbf{n}}$  called the *cylinder set of  $\gamma$* , denoted by  $C(\gamma)$ , which consists in all unoriented bi-infinite geodesics in  $T$  that contains  $\gamma$ . Such cylinder sets form a basis for a topology on  $\partial^2 F_{\mathbf{n}}$ , and in this topology, the cylinder sets are both open and closed, hence compact. The action of  $F_{\mathbf{n}}$  on  $\partial^2 F_{\mathbf{n}}$  has a dense orbit.

Let  $A$  be a nontrivial subgroup of  $F_{\mathbf{n}}$  of finite rank. The induced  $A$ -equivariant inclusion  $\partial_{\infty} A \hookrightarrow \partial_{\infty} F_{\mathbf{n}}$  induces an inclusion  $\partial^2 A \hookrightarrow \partial^2 F_{\mathbf{n}}$ . Let  $\mathcal{A} = \{[A_1], \dots, [A_r]\}$  be a malnormal subgroup system. Let

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^r \bigcup_{g \in F_{\mathbf{n}}} \partial^2 (gA_i g^{-1}).$$

Let  $\partial^2(F_{\mathbf{n}}, \mathcal{A}) = \partial^2 F_{\mathbf{n}} - \partial^2 \mathcal{A}$  be the *double boundary of  $F_{\mathbf{n}}$  relative to  $\mathcal{A}$* . This subset is invariant under the action of  $F_{\mathbf{n}}$  on  $\partial^2 F_{\mathbf{n}}$  and inherits the subspace topology of  $\partial^2 F_{\mathbf{n}}$ .

**Lemma 7.2.1.** [Gue4, Lemmas 2.5, 2.6, 2.7] *Let  $\mathbf{n} \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_{\mathbf{n}}$ . The space  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$  is an open subspace of  $\partial^2 F_{\mathbf{n}}$ , hence is locally compact, and the action of  $F_{\mathbf{n}}$  on  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$  has a dense orbit.*

We can now define a *relative current*. Let  $\mathbf{n} \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_{\mathbf{n}}$ . A *relative current on  $(F_{\mathbf{n}}, \mathcal{A})$*  is a (possibly zero)  $F_{\mathbf{n}}$ -invariant nonnegative Radon measure  $\mu$  on  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$ . The set  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  of all relative currents on  $(F_{\mathbf{n}}, \mathcal{A})$  is equipped with the weak- $*$  topology: a sequence  $(\mu_n)_{n \in \mathbb{N}}$  in  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})^{\mathbb{N}}$  converges to a current  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  if and only if for any pair of disjoint clopen subsets  $S, S' \subseteq \partial^2(F_{\mathbf{n}}, \mathcal{A})$ , the sequence  $(\mu_n(S \times S'))_{n \in \mathbb{N}}$  converges to  $\mu(S \times S')$ .

The group  $\text{Out}(F_{\mathbf{n}}, \mathcal{A})$  acts on  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  as follows. Let  $\phi \in \text{Out}(F_{\mathbf{n}}, \mathcal{A})$  and let  $\Phi$  be a representative of  $\phi$ . The automorphism  $\Phi$  acts diagonally by homeomorphisms on  $\partial^2 F_{\mathbf{n}}$ . If  $\Phi' \in \phi$ , then the action of  $\Phi'$  on  $\partial^2 F_{\mathbf{n}}$  differs from the action of  $\Phi$  by a translation by an element of  $F_{\mathbf{n}}$ . Let  $\mu \in \text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  and let  $C$  be a Borel subset of  $\partial^2(F_{\mathbf{n}}, \mathcal{A})$ . Then, since  $\phi$  preserves  $\mathcal{A}$ , we see that  $\Phi^{-1}(C) \in \partial^2(F_{\mathbf{n}}, \mathcal{A})$ . Then we set

$$\phi(\mu)(C) = \mu(\Phi^{-1}(C)),$$

which is well-defined since  $\mu$  is  $F_{\mathbf{n}}$ -invariant.

Every conjugacy class of nonperipheral element  $g \in F_{\mathbf{n}}$  determines a relative current  $\eta_{[g]}$  as follows. Suppose first that  $g$  is *root-free*, that is  $g$  is not a proper power of any element in  $F_{\mathbf{n}}$ . Let  $\gamma$  be a finite geodesic path in the Cayley graph  $T$ . Then  $\eta_{[g]}(C(\gamma))$  is the number of axes in  $T$  of conjugates of  $g$  that contain the path  $\gamma$ . By [Gue4, Lemma 3.2],  $\eta_{[g]}$  extends uniquely to a current in  $\text{Curr}(F_{\mathbf{n}}, \mathcal{A})$  which we still denote by

$\eta_{[g]}$ . If  $g = h^k$  with  $k \geq 2$  and  $h$  root-free, we set  $\eta_{[g]} = k \eta_{[h]}$ . Such currents are called *rational currents*.

Let  $\mu \in \text{Curr}(F_n, \mathcal{A})$ . The *support* of  $\mu$ , denoted by  $\text{Supp}(\mu)$ , is the support of the Borel measure  $\mu$  on  $\partial^2(F_n, \mathcal{A})$ . We recall that  $\text{Supp}(\mu)$  is a *lamination* of  $\partial^2(F_n, \mathcal{A})$ , that is, a closed  $F_n$ -invariant subset of  $\partial^2(F_n, \mathcal{A})$ .

In the rest of the article, rather than considering the space of relative currents itself, we will consider the set of *projectivized relative currents*, denoted by

$$\mathbb{P}\text{Curr}(F_n, \mathcal{A}) = (\text{Curr}(F_n, \mathcal{A}) - \{0\}) / \sim,$$

where  $\mu \sim \nu$  if there exists  $\lambda \in \mathbb{R}_+^*$  such that  $\mu = \lambda\nu$ . The projective class of a current  $\mu \in \text{Curr}(F_n, \mathcal{A})$  will be denoted by  $[\mu]$ . For every  $\phi \in \text{Out}(F_n, \mathcal{A})$ , the action  $\phi: \mu \mapsto \phi(\mu)$  is positively linear. Therefore, the action of  $\text{Out}(F_n, \mathcal{A})$  on  $\text{Curr}(F_n, \mathcal{A})$  induces an action on  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$ . We have the following properties.

**Lemma 7.2.2.** [Gue4, Lemma 3.3] *Let  $n \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_n$ . The space  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$  is compact.*

**Proposition 7.2.3.** [Gue4, Theorem 1.2] *Let  $n \geq 3$  and let  $\mathcal{A}$  be a malnormal subgroup system of  $F_n$ . The set of projectivised rational currents about nonperipheral elements of  $F_n$  is dense in  $\mathbb{P}\text{Curr}(F_n, \mathcal{A})$ .*

### 7.2.3 Currents associated with an almost atoroidal outer automorphism of $F_n$

Let  $n \geq 3$  and let  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$  be a free factor system of  $F_n$ . If  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  preserves  $\mathcal{F}$ , we denote by

$$\phi|_{\mathcal{F}} = ([\Phi_1|_{A_1}], \dots, [\Phi_k|_{A_k}]) \in \prod_{i=1}^k \text{Out}(A_i)$$

where, for every  $i \in \{1, \dots, k\}$ , the element  $\Phi_i$  is a representative of  $\phi$  such that  $\Phi_i(A_i) = A_i$ . Note that the outer class of  $\Phi_i|_{A_i}$  in  $\text{Out}(A_i)$  does not depend on the choice of  $\Phi_i$  since  $A_i$  is a malnormal subgroup of  $F_n$ . Note that, for every  $i \in \{1, \dots, k\}$ , the element  $[\Phi_i|_{A_i}]$  is expanding relative to  $\mathcal{F} \wedge \{[A_i]\} = \{[A_i]\}$ . Hence we will say that  $\phi|_{\mathcal{F}}$  is *expanding relative to  $\mathcal{F}$* . Let

$$\text{Poly}(\phi|_{\mathcal{F}}) = \bigcup_{i=1}^k \bigcup_{g \in F_n} g \text{Poly}([\Phi_i|_{A_i}]) g^{-1} \subseteq F_n.$$

If  $H$  is a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  which preserves  $\mathcal{F}$ , we set  $\text{Poly}(H|_{\mathcal{F}}) = \bigcap_{\phi \in H} \text{Poly}(\phi|_{\mathcal{F}})$ .

We now define a class of outer automorphisms of  $F_n$  which we will study in the rest of the article.

**Definition 7.2.4.** Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ . The outer automorphism  $\phi$  is *almost atoroidal relative to  $\mathcal{F}$*  if  $\text{Poly}(\phi) \neq \{[F_n]\}$  and if one of the following holds:

- (1)  $\phi$  is an atoroidal outer automorphism relative to  $\mathcal{F}$ ;
- (2) the extension  $\mathcal{F} \leq \{[F_n]\}$  is sporadic.

Let  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$ . In this section, we recall from [Gue5] the definition and some properties of some subsets of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  associated with  $\phi$ . Let  $K_{PG}(\phi)$  be the subspace of all currents in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  whose support is contained in  $\partial^2 \mathcal{A}(\phi) \cap \partial^2(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ . The subspace  $K_{PG}(\phi)$  is called the *space of polynomially growing currents associated with  $\phi$* .

**Proposition 7.2.5.** [Gue5, Proposition 4.4, Proposition 4.12, Proposition 5.23] Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$ . There exist two unique proper compact  $\phi$ -invariant subsets  $\Delta_{\pm}(\phi)$  of  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  such that the following holds. For every  $[\mu] \in \Delta_+(\phi) \cup \Delta_-(\phi)$ , the support of  $\mu$  is contained in  $\partial^2 \mathcal{F}$ . Let  $U_+$  be a neighborhood of  $\Delta_+(\phi)$ , let  $U_-$  be a neighborhood of  $\Delta_-(\phi)$ , let  $V$  be a neighborhood of  $K_{PG}(\phi)$ . There exists  $N \in \mathbb{N}^*$  such that for every  $n \geq 1$  and every  $(\mathcal{F} \wedge \mathcal{A}(\phi))$ -nonperipheral  $w \in F_n$  such that  $\eta_{[w]} \notin V$ , one of the following holds

$$\phi^{Nn}(\eta_{[w]}) \in U_+ \quad \text{or} \quad \phi^{-Nn}(\eta_{[w]}) \in U_-.$$

The subsets  $\Delta_+(\phi)$  and  $\Delta_-(\phi)$  are called the *simplices of attraction and repulsion of  $\phi$* . Let  $\mathcal{F} \leq \mathcal{F}_1 = \{[A_1], \dots, [A_k]\}$  be two free factor systems of  $F_n$ . Let  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z}) \cap \text{Out}(F_n, \mathcal{F}, \mathcal{F}_1)$ . We say that  $\phi|_{\mathcal{F}_1}$  is *almost atoroidal relative to  $\mathcal{F}$*  if, for every  $i \in \{1, \dots, k\}$ , the outer automorphism  $[\Phi_i|_{A_i}]$  is almost atoroidal relative to  $\mathcal{F} \wedge \{[A_i]\}$ . Let  $i \in \{1, \dots, k\}$ . If  $\phi|_{\mathcal{F}_1}$  is almost atoroidal relative to  $\mathcal{F}$ , we denote by  $\Delta_{\pm}([A_i], \phi) \subseteq \mathbb{P}\text{Curr}(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}([\Phi_i|_{A_i}]))$  the convexes of attraction and repulsion of  $[\Phi_i|_{A_i}]$ . If  $\psi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  preserves the conjugacy class of  $A_i$  and  $\mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}([\Phi_i|_{A_i}])$ , then  $\Delta_{\pm}([A_i], \psi\phi\psi^{-1}) = \psi(\Delta_{\pm}([A_i], \phi))$ .

We will also need the following result which gives the existence and properties of an approximation of the length function of the conjugacy class of an element of  $F_n$  in the context of the space of currents.

**Proposition 7.2.6.** [Gue5, Lemma 3.26, Lemma 3.27 (3)] Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$  of type (2). There exists a continuous, positively linear function  $\|\cdot\|_{\mathcal{F}}: \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}_+$  such that the following holds.

- (1) There exist a basis  $\mathfrak{B}$  of  $F_n$  and a constant  $C > 1$  such that, for every  $\mathcal{F} \wedge \mathcal{A}(\phi)$ -nonperipheral element  $g \in F_n$ , we have  $\|\eta_{[g]}\|_{\mathcal{F}} \in \mathbb{N}^*$  and

$$\ell_{\mathfrak{B}}([g]) \geq C \|\eta_{[g]}\|_{\mathcal{F}}.$$



(2) For every  $\eta \in \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ , if  $\|\eta\|_{\mathcal{F}} = 0$ , then  $\eta = 0$ .

Let

$$\widehat{\Delta}_{\pm}(\phi) = \{[t\mu + (1-t)\nu] \mid t \in [0, 1], [\mu] \in \Delta_{\pm}(\phi), [\nu] \in K_{PG}(\phi), \|\mu\|_{\mathcal{F}} = \|\nu\|_{\mathcal{F}} = 1\}$$

be the *convexes of attraction and repulsion* of  $\phi$ . We have the following result.

**Theorem 7.2.7.** [Gue5, Theorem 6.4] *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z}) \cap \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$  of type (2). Let  $\widehat{\Delta}_{\pm}(\phi)$  be the convexes of attraction and repulsion of  $\phi$  and  $\Delta_{\pm}(\phi)$  be the simplices of attraction and repulsion of  $\phi$ . Let  $U_{\pm}$  be open neighborhoods of  $\Delta_{\pm}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  and  $\widehat{V}_{\pm}$  be open neighborhoods of  $\widehat{\Delta}_{\pm}(\phi)$  in  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ . There exists  $M \in \mathbb{N}^*$  such that for every  $n \geq M$ , we have*

$$\phi^{\pm n}(\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_{\mp}) \subseteq U_{\pm}.$$

**Proposition 7.2.8.** [Gue5, Corollary 6.5] *Let  $n \geq 3$  and let  $\mathcal{F}$  be a free factor system of  $F_n$ . Let  $\phi \in \text{Out}(F_n, \mathcal{F})$  be an almost atoroidal outer automorphism relative to  $\mathcal{F}$  of type (2). There exists a continuous, positively linear function  $\|\cdot\|_{\mathcal{F}}: \text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) \rightarrow \mathbb{R}_+$  such that the following holds.*

*For every open neighborhood  $\widehat{V}_{-} \subseteq \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  of  $\widehat{\Delta}_{-}(\phi)$ , there exists  $M \in \mathbb{N}^*$  and a constant  $L_0 > 0$  such that, for every current  $[\mu] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi)) - \widehat{V}_{-}$ , and every  $m \geq M$ , we have*

$$\|\phi^m(\mu)\|_{\mathcal{F}} \geq 3^{m-M} L_0 \|\mu\|_{\mathcal{F}}.$$

### 7.3 Nonsporadic extensions and fully irreducible outer automorphisms

Let  $n \geq 3$  and let  $\mathcal{F}$  and  $\mathcal{F}_1 = \{[A_1], \dots, [A_k]\}$  be two free factor systems of  $F_n$  with  $\mathcal{F} \leq \mathcal{F}_1$  such that the extension  $\mathcal{F} \leq \mathcal{F}_1$  is nonsporadic. Let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  which preserves  $\mathcal{F}$  and  $\mathcal{F}_1$ . We suppose that  $H$  is *irreducible with respect to  $\mathcal{F} \leq \mathcal{F}_1$* , that is, there does not exist a proper, nontrivial free factor system  $\mathcal{F}'$  of  $F_n$  preserved by  $H$  with  $\mathcal{F} < \mathcal{F}' < \mathcal{F}_1$ . Suppose that there exists  $\phi \in H$  such that  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . In this section, we show that there exists  $\widehat{\phi} \in H$  such that  $\text{Poly}(\widehat{\phi}|_{\mathcal{F}_1}) = \text{Poly}(H|_{\mathcal{F}_1})$ . The key point is to construct *fully irreducible outer automorphisms relative to  $\mathcal{F}$*  in  $H$  in the following sense. Let  $\phi \in \text{Out}(F_n, \mathcal{F})$ . We say that  $\phi$  is *fully irreducible relative to  $\mathcal{F}$*  if no power of  $\phi$  preserves a proper free factor system  $\mathcal{F}'$  of  $F_n$  such that  $\mathcal{F} < \mathcal{F}'$ . If  $\phi \in \text{Out}(F_n, \mathcal{F}, \mathcal{F}_1)$ , we say that  $\phi|_{\mathcal{F}_1}$  is *fully irreducible relative to  $\mathcal{F}$*  (resp. *atoroidal relative to  $\mathcal{F}$* ) if, for every  $i \in \{1, \dots, k\}$ , the outer automorphism  $[\Phi_i|_{A_i}]$  is fully irreducible relative to  $\mathcal{F} \wedge \{[A_i]\}$  (resp. atoroidal relative to  $\mathcal{F} \wedge \{[A_i]\}$ ). If  $H$  is a subgroup of  $\text{Out}(F_n, \mathcal{F}, \mathcal{F}_1)$ , we say that  $H|_{\mathcal{F}_1}$  is *atoroidal relative to  $\mathcal{F}$*  if there does not exist a conjugacy class of  $F_n$  which is  $H$ -invariant,  $\mathcal{F}$ -nonperipheral and  $\mathcal{F}_1$ -peripheral. First, we recall some properties of fully irreducible outer automorphisms.

**Proposition 7.3.1.** *Let  $n \geq 3$  and let  $\mathcal{F}$  be a nonsporadic free factor system of  $F_n$ . Let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  which preserves  $\mathcal{F}$  and such that  $H$  is irreducible with respect to the extension  $\mathcal{F} \leq \{[F_n]\}$ . Let  $\phi \in H$  be a fully irreducible outer automorphism relative to  $\mathcal{F}$ .*

(1) *[Gue5, Corollary 3.14] There exists at most one (up to taking inverse) conjugacy class  $[g]$  of root-free  $\mathcal{F}$ -nonperipheral element of  $F_n$  which has polynomial growth under iteration of  $\phi$ . Moreover, the conjugacy class  $[g]$  is fixed by  $\phi$ .*

(2) *[GuH2, Theorem 7.4] One of the following holds:*

(a) *there exists  $\psi \in H$  such that  $\psi$  is a fully irreducible, atoroidal outer automorphism relative to  $\mathcal{F}$ ;*

(b) *if  $\phi$  fixes the conjugacy class of a root-free  $\mathcal{F}$ -nonperipheral element  $g$  of  $F_n$ , then  $[g]$  is fixed by  $H$ .*

Hence Proposition 7.3.1 suggests that, if  $H$  is a subgroup of  $F_n$  which satisfies the hypotheses in Proposition 7.3.1, one step in order to prove Theorem 6.1.2 is to construct relative fully irreducible (atoroidal) outer automorphisms in  $H$ . This is contained in the following theorem.

**Theorem 7.3.2.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ . Let*

$$\emptyset = \mathcal{F}_0 < \mathcal{F}_1 < \dots < \mathcal{F}_k = \{[F_n]\}$$

*be a maximal  $H$ -invariant sequence of free factor systems. There exists  $\phi \in H$  such that for every  $i \in \{1, \dots, k\}$  such that the extension  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$  is nonsporadic, the element  $\phi|_{\mathcal{F}_i}$  is fully irreducible relative to  $\mathcal{F}_{i-1}$ . Moreover, if  $H|_{\mathcal{F}_i}$  is atoroidal relative to  $\mathcal{F}_{i-1}$ , one can choose  $\phi$  so that  $\phi|_{\mathcal{F}_i}$  is atoroidal relative to  $\mathcal{F}_{i-1}$ .*

**Proof.** The proof follows [CU1, Theorem 6.6] (see also [CU2, Corollary 3.4]). Let

$$S = \{j \mid \text{the extension } \mathcal{F}_{j-1} \leq \mathcal{F}_j \text{ is nonsporadic}\}$$

and let  $j \in S$ .

**Claim.** There exists a unique conjugacy class  $[B_j]$  of a subgroup  $B_j$  in  $F_n$  such that  $[B_j] \in \mathcal{F}_j$  and  $[B_j] \notin \mathcal{F}_{j-1}$ .

**Proof.** There exists at least one such conjugacy class since  $\mathcal{F}_{j-1} < \mathcal{F}_j$ . Suppose towards a contradiction that there exist two distinct subgroups  $B_+$  and  $B_-$  of  $F_n$  such that  $[B_+] \neq [B_-]$ ,  $[B_+], [B_-] \in \mathcal{F}_j$  and  $[B_+], [B_-] \notin \mathcal{F}_{j-1}$ . Then

$$\mathcal{F}'([B_-]) = (\mathcal{F}_j - \{[B_+]\}) \cup (\mathcal{F}_{j-1} \wedge \{[B_+]\})$$

is  $H$ -invariant and  $\mathcal{F}_{j-1} < \mathcal{F}'([B_-]) < \mathcal{F}_j$ , which contradicts the maximality hypothesis of the sequence of free factor systems. The claim follows.  $\square$

Let  $B_j$  be a subgroup of  $F_n$  given by the claim. Let  $A_{j,1}, \dots, A_{j,s}$  be the subgroups of  $B_j$  with pairwise disjoint conjugacy classes such that  $\mathcal{A}_{j-1} = \{[A_{j,1}], \dots, [A_{j,s}]\} \subseteq \mathcal{F}_{j-1}$

and  $s$  is maximal for this property. By [HaM4, Theorem D], for every  $j \in S$ , there exists  $\phi \in H$  such that  $[\Phi_j|_{\mathcal{A}_j}] \in \text{Out}(B_j, \mathcal{A}_{j-1})$  is fully irreducible relative to  $\mathcal{A}_{j-1}$ . By Proposition 7.3.1 (2), for every  $j \in S$  such that  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , there exists  $\phi \in H$  such that  $[\Phi_j|_{\mathcal{A}_j}] \in \text{Out}(B_j, \mathcal{A}_{j-1})$  is fully irreducible and atoroidal relative to  $\mathcal{A}_{j-1}$ . Note that, for every  $j \in S$ , the free factor system  $\mathcal{A}_{j-1}$  is a nonsporadic free factor system of  $B_j$  by the claim and since the extension  $\mathcal{F}_{j-1} \leq \mathcal{F}_j$  is nonsporadic. Let  $S_1$  be the subset of  $S$  consisting in every  $j \in S$  such that  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , and let  $S_2 = S - S_1$ . By [GuH2, Theorem 4.1,4.2] (see also [Man2, Man3, Hor2, Gup2]), for every  $j \in S_1$  (resp.  $j \in S_2$ ) there exists a Gromov-hyperbolic space  $X_j$  (the  $\mathcal{Z}$ -factor complex of  $B_j$  relative to  $\mathcal{A}_{j-1}$  when  $j \in S_1$  and the free factor complex of  $B_j$  relative to  $\mathcal{A}_{j-1}$  otherwise) on which  $\text{Out}(B_j, \mathcal{A}_{j-1})$  acts by isometries and such that  $\phi_0 \in \text{Out}(B_j, \mathcal{A}_{j-1})$  is a loxodromic element if and only if  $\phi_0$  is fully irreducible atoroidal relative to  $\mathcal{A}_{j-1}$  (resp. fully irreducible relative to  $\mathcal{A}_{j-1}$ ). The conclusion then follows from [CU1, Theorem 5.1].  $\square$

## 7.4 Sporadic extensions and polynomial growth

Let  $n \geq 3$  and let  $\mathcal{F}$  and  $\mathcal{F}_1 = \{[A_1], \dots, [A_k]\}$  be two free factor systems of  $F_n$  with  $\mathcal{F} \leq \mathcal{F}_1$ . Suppose that the extension  $\mathcal{F} \leq \mathcal{F}_1$  is sporadic. Let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z}) \cap \text{Out}(F_n, \mathcal{F}, \mathcal{F}_1)$ .

In order to prove Theorem 6.1.2, we need to show that if  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ , there exists  $\psi \in H$  such that  $\text{Poly}(\psi|_{\mathcal{F}_1}) = \text{Poly}(H|_{\mathcal{F}_1})$ . Let  $\phi \in H$  be such that  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . Note that, for every element  $g$  of  $\text{Poly}(\phi|_{\mathcal{F}})$ , there exists a subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{F} \wedge \mathcal{A}(\phi)$  and  $g \in A$ . Conversely, for every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{F} \wedge \mathcal{A}(\phi)$  and every element  $g \in A$ , we have  $g \in \text{Poly}(\phi|_{\mathcal{F}})$ . Thus  $\mathcal{F} \wedge \mathcal{A}(\phi)$  is the natural malnormal subgroup system associated with  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . Thus, we see that  $H$  preserves  $\mathcal{F} \wedge \mathcal{A}(\phi)$  and hence  $H$  acts by homeomorphisms on  $\mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$ .

**Lemma 7.4.1.** *Let  $n \geq 3$ , let  $\mathcal{F}$  be a sporadic free factor system of  $F_n$  and let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z}) \cap \text{Out}(F_n, \mathcal{F})$  which is irreducible with respect to  $\mathcal{F} \leq \{[F_n]\}$ . Suppose that there exists  $\phi \in H$  such that  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . If  $\text{Poly}(\phi) \neq \text{Poly}(H)$ , there exists an infinite subset  $X \subseteq H$  such that for all distinct  $\psi_1, \psi_2 \in X$ , we have  $\psi_1(K_{PG}(\phi)) \cap \psi_2(K_{PG}(\phi)) = \emptyset$ .*

**Proof.** Let  $\mathcal{F} \wedge \mathcal{A}(\phi) = \{[A_1], \dots, [A_r]\}$ . Suppose towards a contradiction that  $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi)$ . Then

$$\text{Poly}(\phi) = \text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}}) = \text{Poly}(H).$$

This contradicts the fact that  $\text{Poly}(\phi) \neq \text{Poly}(H)$ . Thus, we have  $\mathcal{A}(\phi) \neq \mathcal{F} \wedge \mathcal{A}(\phi)$ . By [Gue5, Lemma 5.18 (7)], one of the following holds.

(i) There exist distinct  $i, j \in \{1, \dots, r\}$  such that, up to replacing  $A_i$  by a conjugate, we have  $\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i], [A_j]\}) \cup \{[A_i * A_j]\}$ .

(ii) There exists  $i \in \{1, \dots, r\}$  and an element  $g \in F_n$  such that  $\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i]\}) \cup \{[A_i * \langle g \rangle]\}$ .

(iii) There exists  $g \in F_n$  such that  $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g \rangle]\}$ .

**Case 1** Suppose that there exist distinct  $i, j \in \{1, \dots, r\}$  such that

$$\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i], [A_j]\}) \cup \{[A_i * A_j]\}.$$

Since  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$  and  $\text{Poly}(\phi) \neq \text{Poly}(H)$ , there exists  $\psi \in H$  such that, for every  $n \in \mathbb{N}^*$ , the element  $\psi^n$  does not preserve  $[A_i * A_j]$  while preserving  $[A_i]$  and  $[A_j]$ . Hence there exist a representative  $\Psi$  of  $\psi$  such that, for every  $n \in \mathbb{N}^*$ , there exists  $g_n \in F_n - A_i * A_j$  such that  $\Psi^n(A_i) = A_i$  and  $\Psi^n(A_j) = g_n A_j g_n^{-1}$ .

**Claim 1.** For every  $n \in \mathbb{N}^*$ , every  $g \in F_n$  and every  $h \in F_n$ , if

$$h \in (g\Psi^n(A_i * A_j)g^{-1}) \cap (A_i * A_j),$$

then  $h$  is either contained in a conjugate of  $A_i$  or a conjugate of  $A_j$ .

**Proof.** Let  $n \in \mathbb{N}^*$  and let  $h \in (g\Psi^n(A_i * A_j)g^{-1}) \cap (A_i * A_j)$ . Suppose towards a contradiction that  $h$  is not contained in a conjugate of  $A_i$  or a conjugate of  $A_j$ . By [Lev2, Lemma 1.2], there exists a nontrivial  $\mathbb{R}$ -tree  $T$  equipped with a minimal, isometric action of  $F_n$  with trivial edge stabilizers and such that every polynomial subgroup of  $\phi$  fixes a point in  $T$ .

The groups  $A_i * A_j$ ,  $gg_n A_i * A_j g_n^{-1} g^{-1}$  and  $g A_i * A_j g^{-1}$  fix points in  $T$ . Note that, if we have  $gg_n \in A_i * A_j$ , then, since  $g_n \notin A_i * A_j$ , we have  $g \notin A_i * A_j$ . By malnormality of  $A_i * A_j$ , we have  $(A_i * A_j) \cap (g A_i * A_j g^{-1}) = \{1\}$ . Thus, for every  $g \in F_n$ , one of the following holds:  $(A_i * A_j) \cap (g A_i * A_j g^{-1}) = \{1\}$  or  $(A_i * A_j) \cap (gg_n A_i * A_j g_n^{-1} g^{-1}) = \{1\}$ . If  $A_i * A_j$ ,  $g A_i * A_j g^{-1}$  and  $gg_n A_i * A_j g_n^{-1} g^{-1}$  fix the same point  $x$ , then, by induction on the rank of  $\text{Stab}(x)$  (which is less than  $n$  by [GaL]), one can construct a nontrivial  $\mathbb{R}$ -tree  $T'$  equipped with a minimal, isometric action of a subgroup  $B'$  of  $F_n$  containing  $A_i * A_j$ ,  $g A_i * A_j g^{-1}$  and  $gg_n A_i * A_j g_n^{-1} g^{-1}$  with trivial arc stabilizers, such that  $A_i * A_j$  fixes a point  $x_1$ ,  $g A_i * A_j g^{-1}$  fixes a point  $x_2$ ,  $gg_n A_i * A_j g_n^{-1} g^{-1}$  fixes a point  $x_3$  and one of the following holds:  $x_1 \neq x_2$  or  $x_1 \neq x_3$ .

Suppose first that  $x_2 = x_3$ . Then  $x_1 \neq x_2$ . Since  $g\Psi^n(A_i * A_j)g^{-1} = g A_i * (g_n A_j g_n^{-1}) g^{-1}$ ,  $h$  fixes both  $x_1$  and  $x_2$ . This contradicts the fact that  $T'$  has trivial arc stabilizers.

Suppose now that  $x_2 \neq x_3$ . Since  $g\Psi^n(A_i * A_j)g^{-1} = g A_i * (g_n A_j g_n^{-1}) g^{-1}$ , and since  $h$  is not contained in a conjugate of  $A_i$  or a conjugate of  $A_j$ , the element  $h$  can be written as a product of elements  $a_1 b_1 \dots a_k b_k$  where, for every  $i \in \{1, \dots, k\}$ ,  $a_i$  fixes  $x_2$  and  $b_i$  fixes  $x_3$ .

We claim that  $h$  is loxodromic in  $T'$ . Let  $G = \langle \text{Stab}(x_2), \text{Stab}(x_3) \rangle$ . The minimal tree  $T'_G$  in  $T'$  of  $G$  is simplicial with trivial edge stabilizers and the quotient  $T'_G/G$  has exactly one edge. Hence if  $g' \in G$  stabilizes a point in  $T'$  it is either contained in a conjugate of  $\text{Stab}(x_2)$  or a conjugate of  $\text{Stab}(x_3)$ . We may suppose that  $h$  is a cyclically

reduced element when written in the generating set  $\{\text{Stab}(x_2), \text{Stab}(x_3)\}$ . In particular,  $h$  either fixes  $x_2$  or  $x_3$ . Since  $h$  is not contained in a conjugate of  $A_i$  or a conjugate of  $A_j$ , we have  $k \geq 2$ . Hence  $h$  cannot fix  $x_2$  or  $x_3$  and  $h$  is a loxodromic element. Therefore  $h$  does not fix  $x_1$  and  $h \notin A_i * A_j$ , a contradiction.  $\square$

Claim 1 implies that, for every distinct  $m, n \in \mathbb{N}^*$  and every  $x \in F_{\mathbf{n}}$ , the intersection  $\Psi^n(A_i * A_j) \cap (x\Psi^m(A_i * A_j)x^{-1})$  is either contained in a conjugate of  $A_i$  or a conjugate of  $A_j$ . By for instance [HaM4, Fact I.1.2], for every distinct  $m, n \in \mathbb{N}^*$  and every  $x \in F_{\mathbf{n}}$ , we have

$$\begin{aligned} \partial^2(\Psi^n(A_i * A_j)) \cap \partial^2(x\Psi^m(A_i * A_j)x^{-1}) &= \frac{\partial^2(\Psi^n(A_i * A_j) \cap x\Psi^m(A_i * A_j)x^{-1})}{\bigcup_{y \in F_{\mathbf{n}}} (\partial^2(yA_i y^{-1}) \cup \partial^2(yA_j y^{-1}))}. \end{aligned}$$

By definition of  $K_{PG}(\phi)$ , we have  $[\mu] \in K_{PG}(\phi)$  if and only if

$$\text{Supp}(\mu) \subseteq \partial^2 \mathcal{A}(\phi) \cap \partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \partial^2\{[A_i * A_j]\} \cap \partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)).$$

Moreover, if  $n \in \mathbb{N}^*$  and if  $[\mu] \in \psi^n(K_{PG}(\phi))$ , then

$$\text{Supp}(\mu) \subseteq \partial^2 \psi^n(\mathcal{A}(\phi)) \cap \partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) = \partial^2\{[A_i * g_n A_j g_n^{-1}]\} \cap \partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)).$$

Let  $n, m \in \mathbb{N}^*$  be distinct. Suppose towards a contradiction that

$$\psi^n(K_{PG}(\phi)) \cap \psi^m(K_{PG}(\phi)) \neq \emptyset$$

and let  $[\mu] \in \psi^n(K_{PG}(\phi)) \cap \psi^m(K_{PG}(\phi))$ . By  $F_{\mathbf{n}}$ -invariance of  $\mu$ , there exists  $x \in F_{\mathbf{n}}$  such that  $\mu$  gives positive measure to

$$\begin{aligned} &\partial^2(A_i * g_n A_j g_n^{-1}) \cap \partial^2(x(A_i * g_m A_j g_m^{-1})x^{-1}) \cap \partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \\ &\subseteq \left( \bigcup_{y \in F_{\mathbf{n}}} \partial^2(yA_i y^{-1}) \cup \partial^2(yA_j y^{-1}) \right) \cap \partial^2(F_{\mathbf{n}}, \mathcal{F} \wedge \mathcal{A}(\phi)) \end{aligned}$$

and the last intersection is empty by the definition of the relative boundary, a contradiction.

**Case 2** Suppose that either there exists  $i \in \{1, \dots, r\}$  and an element  $g \in F_{\mathbf{n}}$  such that  $\mathcal{A}(\phi) = (\mathcal{F} \wedge \mathcal{A}(\phi) - \{[A_i]\}) \cup \{[A_i * \langle g \rangle]\}$  or there exists  $g \in F_{\mathbf{n}}$  such that  $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g \rangle]\}$ .

In order to treat both cases simultaneously, in the case that there exists  $g \in F_{\mathbf{n}}$  such that  $\mathcal{A}(\phi) = \mathcal{F} \wedge \mathcal{A}(\phi) \cup \{[\langle g \rangle]\}$ , we fix  $A_i = \{e\}$ . Case (2) only occurs when the extension  $\mathcal{F} \leq \{[F_{\mathbf{n}}]\}$  is an HNN extension over the trivial group. In particular, we have  $\mathcal{F} = \{[A]\}$  for some subgroup  $A$  of  $F_{\mathbf{n}}$  and, up to changing the representative of  $[A]$ , we have  $F_{\mathbf{n}} = A * \langle g \rangle$  and  $A_i \subseteq A$ . In particular, since  $H$  preserves the extension  $\mathcal{F} \leq \{[F_{\mathbf{n}}]\}$ , for every  $\psi \in H$ , there exist a unique representative  $\Psi_0$  of  $\psi$  and  $g_\psi \in A$  such that  $\Psi_0(A) = A$  and  $\Psi_0(g) = gg_\psi$ . Since  $H$  is irreducible with respect to  $\mathcal{F} \leq \{[F_{\mathbf{n}}]\}$ , the subgroup  $H$  does not preserve the free factor system  $\mathcal{F} \cup \{[g]\}$ . Thus, there exists

$\psi' \in H$  such that  $g_{\psi'} \neq 1$ . We claim that there exist  $\psi \in H$  with  $g_\psi \notin A_i$ , a representative  $\Psi$  of  $\psi$  and  $h_1 \in A$  such that  $\Psi(A_i) = A_i$  and  $\Psi(g) = h_1 g g_\psi h_1^{-1}$ . Indeed, if  $g_{\psi'} \notin A_i$ , we are done. Otherwise, since  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$  and  $\text{Poly}(\phi) \neq \text{Poly}(H)$ , there exist  $\psi \in H$  and  $h \in A - A_i$  such that either  $g_\psi \notin A_i$ , or  $\Psi_0(A) = A$  and  $\Psi_0(A_i) = h A_i h^{-1}$ . In the first case we are done. Otherwise, we have  $\Psi_0 \circ \Psi'_0(g) = g g_\psi h g_\psi h^{-1}$ . Since  $A_i$  is malnormal, we have  $h g_\psi h^{-1} \notin A_i$  and  $g_\psi h g_\psi h^{-1} \notin A_i$ . The claim follows. Thus, for every  $n \in \mathbb{N}^*$  and  $\psi \in H$  as in the claim, we have  $g_{\psi^n} \notin A_i$  and there exists  $h_n \in A$  such that  $\Psi^n(A_i) = A_i$  and  $\Psi^n(g) = h_n g g_{\psi^n} h_n^{-1}$ .

**Claim 2.** For every  $n \in \mathbb{N}^*$  and every  $a, h \in F_n$ , if

$$h \in (a\Psi^n(A_i * \langle g \rangle)a^{-1}) \cap (A_i * \langle g \rangle),$$

then  $h$  is contained in a conjugate of  $A_i$ .

**Proof.** Let  $n \in \mathbb{N}^*$ , let  $a \in F_n$  and let  $h \in (a\Psi^n(A_i * \langle g \rangle)a^{-1}) \cap (A_i * \langle g \rangle)$ . Suppose towards a contradiction that  $h$  is not contained in a conjugate of  $A_i$ . First note that  $ah_n g g_{\psi^n} h_n^{-1} a^{-1} \notin aA_i * \langle g \rangle a^{-1}$ . Indeed, since  $F_n = A * \langle g \rangle$ , the elements  $h_n g g_{\psi^n} h_n^{-1}$  can be written uniquely as a reduced product of elements in  $A$  and elements in  $\langle g \rangle$ . Since  $h_n, g_{\psi^n} \in A$ , if we have  $ah_n g g_{\psi^n} h_n^{-1} a^{-1} \in aA_i * \langle g \rangle a^{-1}$ , then  $h_n \in A_i$  and  $g_{\psi^n} h_n^{-1} \in A_i$ . Therefore,  $g_{\psi^n} \in A_i$ , a contradiction. Thus, we have  $ah_n g g_{\psi^n} h_n^{-1} a^{-1} \notin aA_i * \langle g \rangle a^{-1}$ .

We claim that there exist a subgroup  $B'$  of  $F_n$  containing  $A_i * \langle g \rangle$ ,  $aA_i * \langle g \rangle a^{-1}$  and  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$ , and an  $\mathbb{R}$ -tree  $T'$  equipped with a minimal, isometric action of  $B'$  with trivial arc stabilizers and such that  $A_i * \langle g \rangle$  fixes a point  $x'_1$  in  $T'$ ,  $aA_i * \langle g \rangle a^{-1}$  fixes a point  $x'_2$  in  $T'$  and  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  either fixes a point in  $T'$  distinct from  $x'_1$  or  $x'_2$  or is loxodromic. Indeed, by [Lev2, Lemma 1.2], there exists a nontrivial  $\mathbb{R}$ -tree  $T$  equipped with a minimal, isometric action of  $F_n$  with trivial arc stabilizers and such that every polynomial subgroup of  $\phi$  fixes a point in  $T$ . In particular,  $A_i * \langle g \rangle$  fixes a point  $x_1$  in  $T$  and  $aA_i * \langle g \rangle a^{-1}$  fixes a point  $x_2$  in  $T$ . If  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  either fixes a point in  $T$  distinct from  $x_1$  or  $x_2$  or is loxodromic, we may take  $T = T'$ . Otherwise  $x_1 = x_2$ ,  $ah_n g g_{\psi^n} h_n^{-1} a^{-1} \in \text{Stab}(x_1)$  and an induction on the rank of  $\text{Stab}(x_1)$  (which is less than  $n$  by [GaL] and invariant by a power of  $\phi$ ) allows us to conclude since  $ah_n g g_{\psi^n} h_n^{-1} a^{-1} \notin aA_i * \langle g \rangle a^{-1}$ .

Suppose first that  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  fixes  $x'_2$ . Then  $x'_1 \neq x'_2$ . Since  $a\Psi^n(A_i * \langle g \rangle)a^{-1}$  fixes  $x'_2$ ,  $h$  fixes both  $x'_1$  and  $x'_2$ . This contradicts the fact that  $T'$  has trivial arc stabilizers.

Suppose now that  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  does not fix  $x'_2$ . We claim that  $h$  is loxodromic in  $T'$ . Indeed, note that, since  $h \in a\Psi^n(A_i * \langle g \rangle)a^{-1}$ ,  $h$  can be written as a product of elements of  $aA_i a^{-1}$  and powers of  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$ . Since  $h$  is not contained in a conjugate of  $A_i$ , we may suppose that:

- (i) the word  $h$  contains at least one occurrence of a nontrivial element in  $aA_i a^{-1}$  and one occurrence of a nontrivial power of  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$ ;
- (ii) the word  $h$  is cyclically reduced when written in the generating set

$$\{aA_i a^{-1}, ah_n g g_{\psi^n} h_n^{-1} a^{-1}\}.$$

Suppose first that  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  fixes a point  $x'$  in  $T'$  (which is distinct from  $x'_2$ ). Then the minimal tree  $T'_0$  in  $T'$  of the subgroup  $B$  of  $F_n$  generated by  $\text{Stab}(x')$  and  $\text{Stab}(x'_2)$  is simplicial and its vertex stabilizers are conjugates of  $\text{Stab}(x')$  and  $\text{Stab}(x'_2)$ . We conclude as in the proof of Claim 1 that  $h$  is loxodromic. Suppose now that  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  is loxodromic and that its axis does not contain  $x'_2$ . Assertion (ii) implies that, if  $h$  is not loxodromic, then it fixes  $x'_2$ . Then a ping pong argument shows, since  $h$  satisfies Assertion (i), that  $h$  is loxodromic. Finally, suppose that  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  is loxodromic and that its axis contains  $x'_2$ . Assertion (ii) implies that, if  $h$  is not loxodromic, then it fixes  $x'_2$ . Then the minimal tree  $T'_0$  in  $T'$  of the subgroup  $B$  of  $F_n$  generated by  $\text{Stab}(x'_2)$  and  $ah_n g g_{\psi^n} h_n^{-1} a^{-1}$  is simplicial and its vertex stabilizers are conjugate of  $\text{Stab}(x'_2)$  (it is an HNN extension). In particular,  $h$  is loxodromic as it satisfies Assertion (i). Thus  $h$  is loxodromic in  $T'$ . Hence  $h$  cannot fix  $x'_1$  and  $h \notin A_i * \langle g \rangle$ , a contradiction.  $\square$

Claim 2 implies that, for every distinct  $n, m \in \mathbb{N}^*$  and every  $x \in F_n$ , we have

$$\Psi^n(A_i * \langle g \rangle) \cap x \Psi^m(A_i * \langle g \rangle) x^{-1} \subseteq \bigcup_{y \in F_n} y A_i y^{-1}$$

and by [HaM4, Fact I.1.2], we have

$$\partial^2 \Psi^n(A_i * \langle g \rangle) \cap \partial^2 (x \Psi^m(A_i * \langle g \rangle) x^{-1}) \subseteq \overline{\bigcup_{y \in F_n} \partial^2 (y A_i y^{-1})}.$$

The rest of the proof is then similar to the one of Case 1.  $\square$

**Lemma 7.4.2.** *Let  $n \geq 3$ , let  $\mathcal{F}$  and  $\mathcal{F}_1 = \{[A_1], \dots, [A_k]\}$  be two free factor systems of  $F_n$  with  $\mathcal{F} \leq \mathcal{F}_1$  such that the extension  $\mathcal{F} \leq \mathcal{F}_1$  is sporadic. Let  $H$  be a subgroup of  $\text{Out}(F_n, \mathcal{F}, \mathcal{F}_1) \cap \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  such that  $H$  is irreducible with respect to  $\mathcal{F} \leq \mathcal{F}_1$ . Suppose that there exists  $\phi \in H$  such that  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . Suppose that  $\text{Poly}(\phi|_{\mathcal{F}_1}) \neq \text{Poly}(H|_{\mathcal{F}_1})$ . There exists  $\psi \in H$  such that for every  $i \in \{1, \dots, k\}$ , we have  $\psi(K_{PG}([\Phi_i|_{A_i}])) \cap K_{PG}([\Phi_i|_{A_i}]) = \emptyset$  and*

$$\Delta_+([A_i], \phi) \cap \psi(\Delta_-([A_i], \phi)) = \Delta_-([A_i], \phi) \cap \psi(\Delta_+([A_i], \phi)) = \emptyset.$$

**Proof.** The proof follows [CU2, Lemma 5.1]. Recall that, since the extension  $\mathcal{F} \leq \mathcal{F}_1$  is sporadic, there exists  $\ell \in \{1, \dots, k\}$  such that, for every  $j \in \{1, \dots, k\} - \{\ell\}$ , we have  $[A_j] \in \mathcal{F}$ . By Lemma 7.4.1 applied to the image of  $H$  in  $\text{Out}(A_\ell)$  (which is contained in  $\text{IA}(A_\ell, \mathbb{Z}/3\mathbb{Z})$ ), there exists an infinite subset  $X \subseteq H$  such that, for any distinct  $h_1, h_2 \in X$ , we have

$$h_1(K_{PG}([\Phi_\ell|_{A_\ell}])) \cap h_2(K_{PG}([\Phi_\ell|_{A_\ell}])) = \emptyset.$$

We now prove that there exist  $h_1, h_2 \in X$  such that  $h_2^{-1} h_1$  satisfies the assertion of Lemma 7.4.2. Note that, for any distinct  $h_1, h_2 \in X$ , we have  $h_2^{-1} h_1(K_{PG}([\Phi_\ell|_{A_\ell}])) \cap K_{PG}([\Phi_\ell|_{A_\ell}]) = \emptyset$ . Hence it suffices to find two distinct  $h_1, h_2$  such that  $\psi = h_2^{-1} h_1$  satisfies the second assertion of Lemma 7.4.2. Let  $j \in \{1, \dots, k\}$  and let  $[\mu]$  be an extremal

point of  $\Delta_+([A_j], \phi)$  or  $\Delta_-([A_j], \phi)$ . By [Gue5, Lemma 4.13], the support  $\text{Supp}(\mu)$  contains the support of finitely many projective currents  $[\mu_1], \dots, [\mu_s] \in \mathbb{P}\text{Curr}(F_n, \mathcal{F} \wedge \mathcal{A}(\phi))$  such that, for every  $t \in \{1, \dots, s\}$ , the support of  $\mu_t$  is uniquely ergodic. Let  $E_\mu = \{[\mu_1], \dots, [\mu_s]\}$ . Let  $E_\phi = \bigcup E_\mu$ , where the union is taken over all  $j \in \{1, \dots, k\}$  and extremal points of  $\Delta_+(A_j, \phi)$  and  $\Delta_-(A_j, \phi)$ . The set  $E_\phi$  is finite by [Gue5, Lemma 4.7].

Since the set  $E_\phi$  is finite, up to taking an infinite subset of  $X$ , we may suppose that, for every  $s \in E_\phi$ , either  $h_1 s = h_2 s$  for every  $h_1, h_2 \in X$  or for every distinct  $h_1, h_2 \in X$ , we have  $h_1 s \neq h_2 s$ . Let  $E_1 \subseteq E_\phi$  be the subset for which the first alternative occurs and let  $E_\infty = E_\phi - E_1$ .

Let  $h_1 \in X$  and, for every  $s \in E_\infty$ , let

$$X_s = \{h \in X \mid h_1 s = h s' \text{ for some } s' \in E_\infty\}.$$

Note that  $X_s$  is a finite set. Let  $h_2 \in X - \bigcup_{s \in E_\infty} X_s$ . For every  $s, s' \in E_\infty$ , we have  $h_1 s \neq h_2 s'$ . If there exists  $s' \in E_1$  such that  $h_1 s = h_2 s'$ , then  $s = h_1^{-1} h_2 s' = s'$ , contradicting the fact that  $s \in E_\infty$ . Thus, for every  $s \in E_\infty$ , we have  $h_2^{-1} h_1 s \notin E_\phi$  and for every  $s \in E_1$ , we have  $h_2^{-1} h_1 s = s$ . Let  $\psi = h_2^{-1} h_1$ . Then, for every  $s \in E_\phi$ , either  $\psi(s) = s$  or  $\psi(s) \notin E_\phi$ .

Let  $j \in \{1, \dots, k\}$ , let  $[\mu] \in \Delta_-([A_j], \phi)$  and suppose for a contradiction that we have  $\psi([\mu]) \in \Delta_+([A_j], \phi)$ . There exist extremal measures  $\mu_1^-, \dots, \mu_m^-$  of  $\Delta_-([A_j], \phi)$  and  $\lambda_1, \dots, \lambda_m \in \mathbb{R}_+$  such that  $\mu = \sum_{i=1}^m \lambda_i \mu_i^-$ . Similarly, there exist extremal measures  $\mu_1^+, \dots, \mu_n^+$  of  $\Delta_+([A_j], \phi)$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$  such that  $\psi(\mu) = \sum_{i=1}^n \alpha_i \mu_i^+$ .

Thus, we have

$$\sum_{i=1}^m \lambda_i \psi(\mu_i^-) = \psi(\mu) = \sum_{i=1}^n \alpha_i \mu_i^+.$$

In particular, we have

$$\bigcup_{i=1}^m \text{Supp}(\psi(\mu_i^-)) = \bigcup_{i=1}^n \text{Supp}(\mu_i^+).$$

Let  $\Lambda \subseteq \text{Supp}(\mu_1^-)$  be the uniquely ergodic support of a current in  $E_\phi$ . Let  $\Psi$  be a representative of  $\psi$  and let  $\partial^2 \Psi$  be the homeomorphism of  $\partial^2 F_n$  induced by  $\Psi$ . Since uniquely ergodic laminations are minimal, there exists  $i \in \{1, \dots, n\}$  such that we have  $\partial^2 \Psi(\Lambda) \subseteq \text{Supp}(\mu_i^+)$ . Thus, we have  $\psi([\mu_1^- |_\Lambda]) = [\mu_i^+ |_\Lambda]$ . This contradicts the fact that  $[\mu_1^- |_\Lambda]$  and  $[\mu_i^+ |_\Lambda]$  are distinct elements of  $E_\phi$  since  $\Delta_+([A_j], \phi) \cap \Delta_-([A_j], \phi) = \emptyset$ .  $\square$

**Proposition 7.4.3.** *Let  $n \geq 3$ , let  $\mathcal{F}$  and  $\mathcal{F}_1 = \{[A_1], \dots, [A_k]\}$  be two free factor systems of  $F_n$  with  $\mathcal{F} \leq \mathcal{F}_1$  such that the extension  $\mathcal{F} \leq \mathcal{F}_1$  is sporadic. Let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z}) \cap \text{Out}(F_n, \mathcal{F}, \mathcal{F}_1)$  such that  $H$  is irreducible with respect to  $\mathcal{F} \leq \mathcal{F}_1$ . Suppose that there exists  $\phi \in H$  such that  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . Suppose that  $\text{Poly}(\phi|_{\mathcal{F}_1}) \neq \text{Poly}(H|_{\mathcal{F}_1})$ . There exist  $\psi \in H$  and a constant  $M > 0$  such that, for all  $m, n \geq M$ , if  $\theta = \psi \phi \psi^{-1}$ , we have  $\text{Poly}(\theta^m \phi^n |_{\mathcal{F}_1}) = \text{Poly}(H|_{\mathcal{F}_1})$ .*

**Proof.** The proof follows [CU2, Proposition 5.2]. Let  $\psi \in H$  be an element given by Lemma 7.4.2 and let  $\theta = \psi \phi \psi^{-1}$ . For every  $i \in \{1, \dots, k\}$ , let  $\Theta_i$  be a representative of



$\theta$  such that  $\Theta_i(A_i) = A_i$  and  $\Phi_i$  be a representative of  $\phi$  such that  $\Phi_i(A_i) = A_i$ . Note that, since for every  $i \in \{1, \dots, k\}$ ,  $[\Phi_i|_{A_i}]$  is almost atoroidal relative to  $\mathcal{F}$ , so is  $[\Theta_i|_{A_i}]$ . Moreover, for every  $i \in \{1, \dots, k\}$ , we have  $K_{PG}([\Theta_i|_{A_i}]) = [\Psi_i|_{A_i}](K_{PG}([\Phi_i|_{A_i}]))$ . Let  $i \in \{1, \dots, k\}$ . Let  $A_i \wedge \mathcal{F}$  be the free factor system of  $A_i$  induced by  $\mathcal{F}$ : it is the free factor system of  $A_i$  consisting in the intersection of  $A_i$  with every subgroup  $A$  of  $F_n$  such that  $[A] \in \mathcal{F}$ . It is well-defined by for instance [SW, Theorem 3.14].

**Claim.** We have  $\widehat{\Delta}_+([A_i], \phi) \cap \psi(\widehat{\Delta}_-([A_i], \phi)) = \emptyset$  and  $\widehat{\Delta}_-([A_i], \phi) \cap \psi(\widehat{\Delta}_+([A_i], \phi)) = \emptyset$ .

**Proof.** We prove the first equality, the other one being similar. By Lemma 7.4.2, we have  $\Delta_+([A_i], \phi) \cap \psi(\Delta_-([A_i], \phi)) = \emptyset$  and  $\psi(K_{PG}([\Phi_i|_{A_i}])) \cap K_{PG}([\Phi_i|_{A_i}]) = \emptyset$ . Let  $[\mu] \in \widehat{\Delta}_+([A_i], \phi) \cap \psi(\widehat{\Delta}_-([A_i], \phi))$ . By definition, there exist  $[\mu_1] \in \Delta_+([A_i], \phi)$ ,  $[\nu_1] \in K_{PG}([\Phi_i|_{A_i}])$ ,  $t \in [0, 1]$ , and  $[\mu_2] \in \psi(\Delta_-([A_i], \phi))$ ,  $[\nu_2] \in \psi(K_{PG}([\Phi_i|_{A_i}]))$ ,  $s \in [0, 1]$  such that

$$[\mu] = [t\mu_1 + (1-t)\nu_1] = [s\mu_2 + (1-s)\nu_2].$$

Note that

$$\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi)) = \emptyset.$$

Moreover, since  $\text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ , we have  $\text{Poly}(\theta|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . Therefore, we see that  $\mathcal{F} \wedge \mathcal{A}(\phi) = \mathcal{F} \wedge \psi(\mathcal{A}(\phi))$ . Thus, we have

$$\partial^2(\mathcal{F} \wedge \{[A_i]\}) \cap \psi(\partial^2\mathcal{A}(\phi)) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi)) = \emptyset.$$

Recall that, by Proposition 7.2.5, the supports of the currents in

$$\Delta_+([A_i], \phi) \cup \psi(\Delta_-([A_i], \phi))$$

are contained in  $\partial^2(\mathcal{F} \wedge \{[A_i]\})$ . Moreover, by definition, the supports of currents in  $K_{PG}([\Phi_i|_{A_i}])$  are contained in  $\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))$  and the supports of currents in  $\psi(K_{PG}([\Phi_i|_{A_i}]))$  are contained in  $\psi(\partial^2\mathcal{A}(\phi)) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))$ . Thus, we have

$$\mu_1(\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))) = \mu_2(\partial^2\mathcal{A}(\phi) \cap \partial^2(A_i, \mathcal{F} \wedge \{[A_i]\} \wedge \mathcal{A}(\phi))) = 0.$$

Hence the support of  $\nu_1$  is contained in the support of  $\nu_2$ . By definition of  $\psi(K_{PG}([\Phi_i|_{A_i}]))$ , this implies that

$$\nu_1 \in K_{PG}([\Phi_i|_{A_i}]) \cap \psi(K_{PG}([\Phi_i|_{A_i}])) = \emptyset.$$

Thus, we necessarily have  $t = 1$ . Similarly, we have  $s = 1$ . This implies that  $[\mu_1] = [\mu_2]$  and that  $\Delta_+([A_i], \phi) \cap \psi(\Delta_-([A_i], \phi)) \neq \emptyset$ , a contradiction.  $\square$

By the claim, there exist subsets  $U, V, \widehat{U}, \widehat{V}$  of  $\mathbb{P}\text{Curr}(A_i, (A_i \wedge \mathcal{F}) \wedge \mathcal{A}(\phi))$  such that:

- (1)  $\Delta_+([A_i], \phi) \subseteq U$ ,  $\widehat{\Delta}_+([A_i], \phi) \subseteq \widehat{U}$ ,  $\Delta_-([A_i], \phi) \subseteq V$ ,  $\widehat{\Delta}_-([A_i], \phi) \subseteq \widehat{V}$ ;
- (2)  $U \subseteq \widehat{U}$ ,  $V \subseteq \widehat{V}$ ;
- (3)  $\widehat{U} \cap \psi(\widehat{V}) = \emptyset$  and  $\widehat{V} \cap \psi(\widehat{U}) = \emptyset$ .

Let  $\mathfrak{B}$  and  $C > 0$  be respectively the basis of  $F_n$  and the constant given by Proposition 7.2.6 (1). Let  $M_0(\phi)$  (resp.  $M_0(\theta^{-1})$ ) be the constant associated with  $\phi$ ,  $U$  and  $\widehat{V}$  (resp.  $\theta^{-1}$ ,  $\psi(V)$  and  $\psi(\widehat{U})$ ) given by Theorem 7.2.7. Let  $M_1(\phi)$  and  $L_1(\phi)$ , (resp.  $M_1(\theta)$  and  $L_1(\theta)$ ) be the constants associated with  $[\Phi_i|_{A_i}]$  and  $\widehat{V}$  (resp.  $[\Theta_i|_{A_i}]$  and  $\psi(\widehat{V})$ ) given by Proposition 7.2.8. Similarly, let  $M_1(\phi^{-1})$  and  $L_1(\phi^{-1})$ , (resp.  $M_1(\theta^{-1})$  and  $L_1(\theta^{-1})$ ) be the constants associated with  $[\Phi_i|_{A_i}^{-1}]$  and  $\widehat{U}$  (resp.  $[\Theta_i|_{A_i}^{-1}]$  and  $\psi(\widehat{U})$ ) given by Proposition 7.2.8. Let

$$M = \max\{M_0(\phi), M_0(\theta^{-1}), M_1(\phi), M_1(\theta), M_1(\phi^{-1}), M_1(\theta^{-1})\}$$

and let

$$L = \min\{L_1(\phi), L_1(\theta), L_1(\phi^{-1}), L_1(\theta^{-1})\} > 0.$$

Let  $M'$  be such that  $3^{M'}L^2 > 1$ . Let  $m, n \geq M + M'$  and let  $\mu \in \text{Curr}(A_i, A_i \wedge \mathcal{F} \wedge \mathcal{A}(\phi))$  be a nonzero current. Suppose first that  $[\mu] \notin \widehat{V}$ . Then by Theorem 7.2.7, we have  $\phi^n(\mu) \in U$ . By Proposition 7.2.8, we have  $\|\phi^n(\mu)\|_{\mathcal{F}} \geq 3^{n-M}L\|\mu\|_{\mathcal{F}}$ . Since  $U \cap \psi(\widehat{V}) = \emptyset$ , by Proposition 7.2.8, we have

$$\|\theta^m \phi^n(\mu)\|_{\mathcal{F}} \geq 3^{m-M}L\|\phi^n(\mu)\|_{\mathcal{F}} \geq 3^{m+n-2M}L^2\|\mu\|_{\mathcal{F}}.$$

Note that, since  $\widehat{V} \cap \psi(\widehat{U}) = \emptyset$ , we have  $\theta^m \phi^n(\mu) \notin \widehat{V}$ . Therefore, we can apply the same arguments replacing  $\mu$  by  $\theta^m \phi^n(\mu)$  and an inductive argument shows that, for every  $n' \in \mathbb{N}^*$ , we have

$$\|(\theta^m \phi^n)^{n'}(\mu)\|_{\mathcal{F}} \geq 3^{n'(m+n-2M-M')} (3^{M'}L^2)^{n'}\|\mu\|_{\mathcal{F}}.$$

Therefore, if  $\mu$  is the current associated with a nonperipheral element  $g \in A_i$  with  $[\mu] \notin \widehat{V}$ , for every  $n' \geq 1$ , by Proposition 7.2.6 (1) we have

$$\ell_{\mathfrak{B}}((\theta^m \phi^n)^{n'}([g])) \geq 3^{n'(m+n-2M-M')} (3^{M'}L^2)^{n'}C\|\mu\|_{\mathcal{F}} \geq 3^{n'(m+n-2M-M')}C.$$

Hence we have  $g \notin \text{Poly}([\Theta_i^m \Phi_i^n|_{A_i}])$ . Suppose now that  $[\mu] \in \widehat{V}$ . Therefore, we have  $[\mu] \notin \psi(\widehat{U})$ . By Theorem 7.2.7, we have  $\theta^{-m}([\mu]) \in \psi(V)$ . By Proposition 7.2.8, we have  $\|\theta^{-m}(\mu)\|_{\mathcal{F}} \geq 3^{m-M}L\|\mu\|_{\mathcal{F}}$ . Moreover, since  $\psi(V) \cap \widehat{U} = \emptyset$ , we have  $\theta^{-m}([\mu]) \notin \widehat{U}$  and

$$\|\phi^{-n} \theta^{-m}(\mu)\|_{\mathcal{F}} \geq 3^{n-M}L\|\theta^{-m}(\mu)\|_{\mathcal{F}} \geq 3^{n+m-2M-M'} (3^{M'}L^2)\|\mu\|_{\mathcal{F}}.$$

Note that, since  $\widehat{V} \cap \psi(\widehat{U}) = \emptyset$ , we have  $\phi^{-n} \theta^{-m}(\mu) \notin \psi(\widehat{U})$ . Therefore, we can apply the same arguments replacing  $\mu$  by  $\phi^{-n} \theta^{-m}(\mu)$  and an inductive argument shows that, for every  $n' \in \mathbb{N}^*$ , we have

$$\|(\phi^{-n} \theta^{-m})^{n'}(\mu)\|_{\mathcal{F}} \geq 3^{n'(m+n-2M-M')} (3^{M'}L^2)^{2n'}\|\mu\|_{\mathcal{F}}.$$

Therefore, if  $\mu$  is the current associated with a nonperipheral element  $g \in A_i$  with  $[\mu] \in \widehat{V}$ , for every  $n' \geq 1$ , we have

$$\ell_{\mathfrak{B}}((\phi^{-n} \theta^{-m})^{n'}([g])) \geq 3^{n'(m+n-2M-M')} (3^{M'}L^2)^{2n'}C\|\mu\|_{\mathcal{F}} \geq 3^{n'(m+n-2M)}C.$$

Hence we have  $g \notin \text{Poly}([\Phi_i^{-n}\Theta_i^{-m}|_{A_i}]) = \text{Poly}([\Theta_i^m\Phi_i^n|_{A_i}])$ . Therefore,  $\theta^m\phi^n|_{\mathcal{F}_1}$  is expanding relative to  $\mathcal{F} \wedge \mathcal{A}(\phi)$ . Hence if  $g \in \text{Poly}(\theta^m\phi^n|_{\mathcal{F}_1})$ , there exists a subgroup  $A$  of  $F_n$  such that  $g \in A$  and  $[A] \in \mathcal{F} \wedge \mathcal{A}(\phi)$ . Note that, if  $g \in F_n$  is such that there exists a subgroup  $A$  of  $F_n$  such that  $g \in A$  and  $[A] \in \mathcal{F} \wedge \mathcal{A}(\phi)$ , then  $g \in \text{Poly}(\phi|_{\mathcal{F}}) = \text{Poly}(H|_{\mathcal{F}})$ . Thus, we have  $\text{Poly}(\theta^m\phi^n|_{\mathcal{F}_1}) = \text{Poly}(H|_{\mathcal{F}_1})$ . This concludes the proof.  $\square$

**Proposition 7.4.4.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ . Let*

$$\emptyset = \mathcal{F}_0 < \mathcal{F}_1 < \dots < \mathcal{F}_k = \{[F_n]\}$$

*be a maximal  $H$ -invariant sequence of free factor systems. Let  $2 \leq i \leq k$ . Suppose that  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$  is sporadic. Suppose that there exists  $\phi \in H$  such that*

- (a)  $\text{Poly}(H|_{\mathcal{F}_{i-1}}) = \text{Poly}(\phi|_{\mathcal{F}_{i-1}})$ ;
- (b) *for every  $j \in \{1, \dots, k\}$ , if the extension  $\mathcal{F}_{j-1} \leq \mathcal{F}_j$  is nonsporadic, then  $\phi|_{\mathcal{F}_j}$  is fully irreducible relative to  $\mathcal{F}_{j-1}$  and if  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , so is  $\phi|_{\mathcal{F}_j}$ .*

*Then there exists  $\hat{\phi} \in H$  such that:*

- (1)  $\text{Poly}(H|_{\mathcal{F}_i}) = \text{Poly}(\hat{\phi}|_{\mathcal{F}_i})$ ;
- (2) *for every  $j \in \{1, \dots, k\}$ , if the extension  $\mathcal{F}_{j-1} \leq \mathcal{F}_j$  is nonsporadic, then  $\hat{\phi}|_{\mathcal{F}_j}$  is fully irreducible relative to  $\mathcal{F}_{j-1}$  and if  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , so is  $\hat{\phi}|_{\mathcal{F}_j}$ .*

**Proof.** The proof follows [CU2, Proposition 5.3]. If  $\text{Poly}(H|_{\mathcal{F}_i}) = \text{Poly}(\phi|_{\mathcal{F}_i})$ , we may take  $\hat{\phi} = \phi$ . Otherwise, by Proposition 7.4.3, there exists  $\psi \in H$  and a constant  $M > 0$  such that, for every  $m, n \geq M$ , if  $\theta = \psi\phi\psi^{-1}$ , we have  $\text{Poly}(\theta^m\phi^n|_{\mathcal{F}_i}) = \text{Poly}(H|_{\mathcal{F}_i})$ . Therefore, for every  $m, n \geq M$ , the element  $\hat{\phi} = \theta^m\phi^n$  satisfies (1). It remains to show that there exist  $m, n \geq M$  such that  $\theta^m\phi^n$  satisfies (2). Let

$$S = \{j \mid \text{the extension } \mathcal{F}_{j-1} \leq \mathcal{F}_j \text{ is nonsporadic}\}$$

and let  $j \in S$ .

Let  $B_j$  be a subgroup of  $F_n$  given by the claim in the proof of Theorem 7.3.2. Let  $A_{j,1}, \dots, A_{j,s}$  be the subgroups of  $B_j$  with pairwise disjoint conjugacy classes such that  $\mathcal{A}_{j-1} = \{[A_{j,1}], \dots, [A_{j,s}]\} \subseteq \mathcal{F}_{j-1}$  and  $s$  is maximal for this property. By Hypothesis (b), the outer automorphism  $[\Phi_j|_{B_j}] \in \text{Out}(B_j, \mathcal{A}_{j-1})$  is fully irreducible relative to  $\mathcal{A}_{j-1}$ . Note that  $\mathcal{A}_{j-1}$  is a nonsporadic free factor system of  $B_j$  by the claim and since the extension  $\mathcal{F}_{j-1} \leq \mathcal{F}_j$  is nonsporadic. Let  $S_1$  be the subset of  $S$  consisting in every  $j \in S$  such that  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , and let  $S_2 = S - S_1$ . By [GuH2, Theorem 4.1,4.2] (see also [Man2, Man3, Hor2, Gup2]), for every  $j \in S_1$  (resp.  $j \in S_2$ ) there exists a Gromov-hyperbolic space  $X_j$  (the  $\mathcal{Z}$ -factor complex of  $B_j$  relative to  $\mathcal{A}_{j-1}$  when  $j \in S_1$  and the free factor complex of  $B_j$  relative to  $\mathcal{A}_{j-1}$  otherwise) on which  $\text{Out}(B_j, \mathcal{A}_{j-1})$  acts by isometries and such that  $\phi_0 \in \text{Out}(B_j, \mathcal{A}_{j-1})$  is a loxodromic element if and only if  $\phi_0$  is fully irreducible atoroidal relative to  $\mathcal{A}_{j-1}$  (resp. fully irreducible relative to  $\mathcal{A}_{j-1}$ ). In particular, since  $H$  preserves  $\mathcal{F}_{j-1} < \mathcal{F}_j$ , and hence  $\psi$  preserves  $[B_j]$ , the elements  $[\Phi_j|_{B_j}]$  and  $[\Theta_j|_{B_j}]$  are loxodromic elements of  $X_j$ .

Recall that two loxodromic isometries of a Gromov-hyperbolic space  $X$  are *independent* if their fixed point sets in  $\partial_\infty X$  are disjoint and are *dependent* otherwise. Let  $I \subseteq S$

be the subset of indices where for every  $j \in I$ , the elements  $[\Phi_j|_{B_j}]$  and  $[\Theta_j|_{B_j}]$  are independent and let  $D = S - I$ . By standard ping pong arguments (see for instance [CU1, Proposition 4.2, Theorem 3.1]), there exist constants  $m, n_0 \geq M$  such that for every  $n \geq n_0$ , the element  $[\Theta_j^m \Phi_j^n|_{B_j}]$  acts loxodromically on  $X_j$ . By [CU1, Proposition 3.4], there exists  $n \geq n_0$  such that, for every  $j \in D$ , the element  $[\Theta_j^m \Phi_j^n|_{B_j}]$  acts loxodromically on  $X_j$ . Hence for every  $j \in S_1$ , the element  $\theta^m \phi^n|_{\mathcal{F}_j}$  is fully irreducible atoroidal relative to  $\mathcal{F}_{j-1}$  and for every  $j \in S_2$ , the element  $\theta^m \phi^n|_{\mathcal{F}_j}$  is fully irreducible relative to  $\mathcal{F}_{j-1}$ . This concludes the proof.  $\square$

## 7.5 Proof of the main result and applications

We are now ready to complete the proof of our main theorem.

**Theorem 7.5.1.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{Out}(F_n)$ . There exists  $\phi \in H$  such that  $\text{Poly}(\phi) = \text{Poly}(H)$ .*

**Proof.** Since  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  is a finite index subgroup of  $\text{Out}(F_n)$  and since for every  $\psi \in H$  and every  $n \in \mathbb{N}^*$ , we have  $\text{Poly}(\psi^k) = \text{Poly}(\psi)$ , we see that  $\text{Poly}(H) = \text{Poly}(H \cap \text{IA}_n(\mathbb{Z}/3\mathbb{Z}))$ . Hence we may suppose that  $H$  is a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ . Let

$$\emptyset = \mathcal{F}_0 < \mathcal{F}_1 < \dots < \mathcal{F}_k = \{[F_n]\}$$

be a maximal  $H$ -invariant sequence of free factor systems. By Theorem 7.3.2, there exists  $\phi \in H$  such that for every  $j \in \{1, \dots, k\}$  such that the extension  $\mathcal{F}_{j-1} \leq \mathcal{F}_j$  is nonsporadic, the element  $\phi|_{\mathcal{F}_j}$  is fully irreducible relative to  $\mathcal{F}_{j-1}$  and if  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , so is  $\phi|_{\mathcal{F}_{j-1}}$ .

We now prove by induction on  $i \in \{0, \dots, k\}$  that for every  $i \in \{0, \dots, k\}$ , there exists  $\phi_i \in H$  such that

- (a)  $\text{Poly}(\phi_i|_{\mathcal{F}_i}) = \text{Poly}(H|_{\mathcal{F}_i})$ ;
- (b) for every  $j \in \{1, \dots, k\}$  such that the extension  $\mathcal{F}_{j-1} \leq \mathcal{F}_j$  is nonsporadic, the element  $\phi_i|_{\mathcal{F}_j}$  is fully irreducible relative to  $\mathcal{F}_{j-1}$  and if  $H|_{\mathcal{F}_j}$  is atoroidal relative to  $\mathcal{F}_{j-1}$ , so is  $\phi_i|_{\mathcal{F}_{j-1}}$ .

For the base case  $i = 0$ , we set  $\phi_0 = \phi$ . Let  $i \in \{1, \dots, k\}$  and suppose that  $\phi_{i-1} \in H$  has been constructed. We distinguish between two cases, according to the nature of the extension  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$ . Suppose first that the extension  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$  is nonsporadic. We set  $\phi_i = \phi_{i-1}$ . We claim that  $\phi_i$  satisfies the hypotheses. Indeed, it clearly satisfies (b). For (a), since  $\text{Poly}(\phi_{i-1}|_{\mathcal{F}_{i-1}}) = \text{Poly}(H|_{\mathcal{F}_{i-1}})$ , it suffices to show that for every element  $g \in F_n$  which is  $\mathcal{F}_i$ -peripheral but  $\mathcal{F}_{i-1}$ -nonperipheral, if  $g \in \text{Poly}(\phi_i|_{\mathcal{F}_i})$ , then  $g \in \text{Poly}(H|_{\mathcal{F}_i})$ . Note that, if  $\phi_i|_{\mathcal{F}_i}$  is atoroidal relative to  $\mathcal{F}_{i-1}$ , by Proposition 7.3.1 (1), we have  $\text{Poly}(\phi_i|_{\mathcal{F}_i}) = \text{Poly}(\phi_i|_{\mathcal{F}_{i-1}})$ . Hence we have  $\text{Poly}(H|_{\mathcal{F}_i}) = \text{Poly}(\phi_i|_{\mathcal{F}_i})$ . So we may suppose that  $\phi_i|_{\mathcal{F}_i}$  is not atoroidal relative to  $\mathcal{F}_{i-1}$ .

Let  $g \in \text{Poly}(\phi_i|_{\mathcal{F}_i})$  be an element which is  $\mathcal{F}_i$ -peripheral but  $\mathcal{F}_{i-1}$ -nonperipheral. By Proposition 7.3.1 (1), there exists at most one (up to taking inverse)  $h \in F_n$  such that

$g \in \langle h \rangle$  and  $[h]$  is fixed by  $\phi_i$ . By Proposition 7.3.1 (2)(b), the conjugacy class of  $[h]$  is fixed by  $H$ . Hence the conjugacy class of  $[g]$  is fixed by  $H$  and  $g \in \text{Poly}(H|_{\mathcal{F}_i})$ .

Suppose now that the extension  $\mathcal{F}_{i-1} \leq \mathcal{F}_i$  is sporadic. If  $\text{Poly}(\phi_{i-1}|_{\mathcal{F}_i}) = \text{Poly}(H|_{\mathcal{F}_i})$ , we set  $\phi_i = \phi_{i-1}$ . Then  $\phi_i$  satisfies (a) and (b). Suppose that  $\text{Poly}(\phi_{i-1}|_{\mathcal{F}_i}) \neq \text{Poly}(H|_{\mathcal{F}_i})$ . By Proposition 7.4.4, there exists  $\hat{\phi}_{i-1} \in H$  such that  $\hat{\phi}_{i-1}$  satisfies (a) and (b). Then we set  $\phi_i = \hat{\phi}_{i-1}$ . This completes the induction argument. In particular, we have  $\text{Poly}(\phi_m) = \text{Poly}(H)$ . This concludes the proof.  $\square$

We now give some applications of Theorem 7.5.1. The first one is a straightforward consequence using the fact that for every  $\phi \in \text{Out}(F_n)$ , there exists a natural malnormal subgroup system associated with  $\text{Poly}(\phi)$ .

**Corollary 7.5.2.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{Out}(F_n)$  such that  $\text{Poly}(H) \neq \{1\}$ . There exist nontrivial maximal subgroups  $A_1, \dots, A_k$  of  $F_n$  such that*

$$\text{Poly}(H) = \bigcup_{i=1}^k \bigcup_{g \in F_n} gA_i g^{-1}$$

and  $\mathcal{A} = \{[A_1], \dots, [A_k]\}$  is a malnormal subgroup system.  $\square$

If  $H$  is a subgroup of  $\text{Out}(F_n)$  is such that  $\text{Poly}(H) \neq \{1\}$ , we denote by  $\mathcal{A}(H)$  the malnormal subgroup system given by Corollary 7.5.2. If  $\text{Poly}(H) = \{1\}$ , we set  $\mathcal{A}(H) = \emptyset$ .

The following result is a generalization of [CU2, Theorem A] regarding fixed conjugacy classes. If  $\phi \in \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ , we denote by  $\text{Fix}(\phi)$  the set of conjugacy classes of  $F_n$  fixed by  $\phi$ . Note that, if  $g \in F_n$  is such that  $[g] \in \text{Fix}(\phi)$ , then  $g \in \text{Poly}(\phi)$ . Moreover, by [Lev2, Lemma 1.5], if  $\text{Poly}(\phi) \neq \{1\}$ , the set  $\text{Fix}(\phi)$  is nonempty. If  $P$  is a subgroup of  $F_n$ , we denote by  $\text{Out}(F_n, P^{(t)})$  the subgroup of  $\text{Out}(F_n)$  consisting in every element  $\phi \in \text{Out}(F_n)$  such that there exists  $\Phi \in \phi$  such that  $\Phi(P) = P$  and  $\Phi|_P = \text{id}_P$ .

**Corollary 7.5.3.** *Let  $n \geq 3$  and let  $g_1, \dots, g_k$  be nontrivial root-free elements of  $F_n$ . Let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$  such that, for every  $i \in \{1, \dots, k\}$ , every element of  $H$  fixes the conjugacy class of  $g_i$ . Then one of the following (mutually exclusive) statements holds.*

(1) *There exists  $g_{k+1} \in F_n$  such that  $[g_{k+1}] \notin \{[g_1], \dots, [g_k]\}$  and whose conjugacy class is fixed by every element of  $H$ .*

(2) *There exists  $\phi \in H$  such that  $\text{Fix}(\phi) = \{[\langle g_1 \rangle], \dots, [\langle g_k \rangle]\}$ .*

*Moreover, if (1) holds, either there exist  $\ell \geq k + 1$  and  $g_{k+1}, \dots, g_\ell \in F_n$  such that*

$$\text{Fix}(H) = \mathcal{A}(H) = \{[\langle g_1 \rangle], \dots, [\langle g_\ell \rangle]\}$$

*or  $H$  virtually fixes the conjugacy class of a nonabelian free subgroup of  $F_n$  of rank 2.*

**Proof.** First assume that  $H$  is finitely generated. Suppose that (2) does not hold. In particular, by Theorem 7.5.1, we see that  $\mathcal{A}(H) \neq \{[\langle g_1 \rangle], \dots, [\langle g_k \rangle]\}$ . Let  $\mathcal{A}(H) = \{[P_1], \dots, [P_\ell]\}$ , where for every  $i \in \{1, \dots, \ell\}$ ,  $P_i$  is a malnormal subgroup of  $F_n$ . Note that, for every  $i \in \{1, \dots, \ell\}$ , since  $P_i$  is malnormal, we have a natural homomorphism  $H \rightarrow \text{Out}(P_i)$  whose image, denoted by  $H|_{P_i}$ , is contained in the set of polynomially growing outer automorphisms of  $P_i$ . Since  $H$  is finitely generated, up to taking a finite index subgroup of  $H$ , we can apply the Kolchin theorem for  $\text{Out}(F_n)$  (see [BFH3, Theorem 1.1]): there exists a  $H|_{P_i}$ -invariant sequence of free factor systems of  $P_i$

$$\emptyset = \mathcal{F}_0^{(i)} < \mathcal{F}_1^{(i)} < \dots < \mathcal{F}_{k_i}^{(i)} = \{[P_i]\}$$

such that, for every  $j \in \{1, \dots, k_i\}$ , the extension  $\mathcal{F}_{j-1}^{(i)} \leq \mathcal{F}_j^{(i)}$  is sporadic.

Note that, since  $\mathcal{A}(H) \neq \{[\langle g_1 \rangle], \dots, [\langle g_k \rangle]\}$ , either  $\ell > k$  or there exists  $i \in \{1, \dots, \ell\}$  such that the rank of  $P_i$  is at least equal to 2. Suppose that  $\ell > k$ . Let  $i \in \{1, \dots, \ell\}$ . Since for every  $j \in \{1, \dots, k_i\}$ , the extension  $\mathcal{F}_{j-1}^{(i)} \leq \mathcal{F}_j^{(i)}$  is sporadic, for every  $i \in \{1, \dots, \ell\}$ , the free factor system  $\mathcal{F}_1^{(i)}$  contains a unique element and the rank of the associated subgroup is 1. Thus, the group  $H$  fixes at least  $\ell$  distinct conjugacy classes of elements of  $F_n$  and (1) holds.

Otherwise, let  $i \in \{1, \dots, \ell\}$  be such that the rank of  $P_i$  is at least equal to 2. Since, for every  $j \in \{1, \dots, k_i\}$ , the extension  $\mathcal{F}_{j-1}^{(i)} \leq \mathcal{F}_j^{(i)}$  is sporadic we have  $k_i \geq 2$ . Moreover, there exists  $j_0 \in \{1, \dots, k_i\}$  and a subgroup  $U_{j_0}$  of  $P_i$  such that  $[U_{j_0}] \in \mathcal{F}_{j_0}^{(i)}$  and one of the following holds:

- (a) there exist two subgroups  $B_1$  and  $B_2$  of  $P_i$  such that  $\text{rank}(B_1) = \text{rank}(B_2) = 1$ ,  $[B_1], [B_2] \in \mathcal{F}_{j_0-1}$  and  $U_{j_0} = B_1 * B_2$ ;
- (b) there exists a subgroup  $B$  of  $P_i$  such that  $\text{rank}(B) = 1$ ,  $[B] \in \mathcal{F}_{j_0-1}$  and  $U_{j_0}$  is an HNN extension of  $B$  over the trivial group.

If Case (a) occurs, then  $H$  acts as the identity on  $U_{j_0}$  since  $\text{rank}(U_{j_0}) = 2$  and since every element of  $H$  fixes elementwise a set of conjugacy classes of generators of  $U_{j_0}$  (recall that the abelianization homomorphism  $F_2 \rightarrow \mathbb{Z}^2$  induces an isomorphism  $\text{Out}(F_2) \simeq \text{GL}(2, \mathbb{Z})$ ). Hence Assertion (1) holds.

If Case (b) occurs, let  $b$  be a generator of  $B$  and let  $t \in U_{j_0}$  be such that  $U_{j_0} = \langle b \rangle * \langle t \rangle$ . Then, since  $H \subseteq \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ , for every element  $\psi$  of  $H$ , there exist  $\Psi \in \psi$  and  $k \in \mathbb{Z}$  such that  $\psi(b) = b$  and  $\psi(t) = tb^k$ . In particular, for every  $\psi \in H$ , the automorphism  $\Psi$  fixes the group generated by  $b$  and  $tbt^{-1}$  and (1) holds.

The moreover part follows since either for every  $i \in \{1, \dots, \ell\}$ , the group  $P_i$  has rank 1 or there exists  $i \in \{1, \dots, k\}$  such that the rank of  $P_i$  is at least equal to 2. In the first case, since  $H \subseteq \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ , for every  $i \in \{1, \dots, \ell\}$ , the conjugacy class  $[P_i]$  is fixed by  $H$ . In the later case, the subgroup  $H$  fixes the conjugacy class of a nonabelian subgroup of rank 2 as explained above. This concludes the proof when  $H$  is finitely generated.

Suppose now that  $H$  is not finitely generated and let  $(H_m)_{m \in \mathbb{N}}$  be an increasing sequence of finitely generated subgroups of  $H$  such that  $H = \bigcup_{m \in \mathbb{N}} H_m$ . For every  $m \in \mathbb{N}$ , we have  $H_m \subseteq \text{Out}(F_n, \text{Fix}(H_m)^{(t)})$  and for every  $m_1, m_2 \in \mathbb{N}$  such that  $m_1 \leq m_2$ , we have  $\text{Fix}(H_{m_2}) \subseteq \text{Fix}(H_{m_1})$ . By [GuL3, Theorem 1.5], there exists  $N \in \mathbb{N}$  such that,

for every  $m \geq N$ , we have  $\text{Out}(F_n, \text{Fix}(H_m)^{(t)}) = \text{Out}(F_n, \text{Fix}(H_N)^{(t)})$ . In particular, we have  $\text{Fix}(H_N) = \text{Fix}(H)$ . The result now follows from the finitely generated case.  $\square$

The following result might be folklore, but we did not find a precise statement in the literature. If  $S$  is a compact, connected surface, we denote by  $\text{Mod}(S)$  the group of homotopy classes of homeomorphisms that preserve the boundary of  $S$ .

**Corollary 7.5.4.** *Let  $n \geq 3$  and let  $H$  be a subgroup of  $\text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ . The following assertions are equivalent:*

- (1)  $\mathcal{A}(H) = \{[\langle g \rangle]\}$ , where  $g$  is an element of  $F_n$  not contained in a proper free factor of  $F_n$ ;
- (2) *there exists a connected, compact surface  $S$  with exactly one boundary component and an identification of  $\pi_1(S)$  with  $F_n$  such that  $H$  is identified with a subgroup of  $\text{Mod}(S)$  and  $H$  contains a pseudo-Anosov element.*

**Proof.** Suppose that (2) holds. Let  $\phi \in H$  be identified with a pseudo-Anosov element of  $S$ . In particular,  $\phi$  is a fully irreducible element of  $\text{Out}(F_n)$ . By Proposition 7.3.1 (1) with  $\mathcal{F} = \emptyset$ , the element  $\phi$  fixes exactly one (up to taking inverse) conjugacy class  $[g]$  of a root-free element  $g$  of  $F_n$ . Since  $\phi$  fixes the conjugacy class of the element of  $F_n$  identified with the boundary component of  $S$ , the conjugacy class  $[g]$  is identified with the conjugacy class in  $\pi_1(S)$  of the element associated with the homotopy class of the boundary component of  $S$ . Hence  $g$  is not contained in any proper free factor of  $F_n$ . Moreover, since  $H$  is identified with a subgroup of  $\text{Mod}(S)$ , every element of  $H$  fixes  $[g]$ . Hence we have  $\mathcal{A}(H) = \{[\langle g \rangle]\}$ .

Suppose now that (1) holds. Let  $\phi \in H$  be an element given by Theorem 7.5.1. Then  $\mathcal{A}(\phi) = \mathcal{A}(H) = \{[\langle g \rangle]\}$ . In particular, since  $H \subseteq \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ , the conjugacy class of  $g$  is fixed by every element of  $H$ . Let  $f: G \rightarrow G$  be a CT map representing a power of  $\phi$  (see the definition in [FH, Definition 4.7]).

**Claim.** The graph  $G$  consists in a single stratum and this stratum is an EG stratum.

**Proof.** Let  $H_r$  be the highest stratum in  $G$ . We first prove that  $H_r$  is an EG stratum. Indeed,  $H_r$  is either a zero stratum, an EG stratum or a NEG stratum. The stratum  $H_r$  cannot be a zero stratum by [FH, Definition 4.7 (6)]. Moreover,  $H_r$  cannot be a NEG stratum as otherwise by [CU2, Proposition 4.1], the element  $g$  would be a basis element of  $F_n$ , contradicting the fact that  $g$  is not contained in any proper free factor of  $F_n$ . Hence  $H_r$  is an EG stratum. Since  $g$  is not contained in any proper free factor of  $F_n$ , the reduced circuit  $\gamma_g$  in  $G$  representing the conjugacy class of  $g$  has height  $r$  and is fixed by  $f$ . By [HaM4, Fact I.2.3], the stratum  $H_r$  is a geometric stratum. By [HaM4, Proposition I.2.18], the element  $\phi$  fixes elementwise a finite set  $\mathcal{C} = \{[g], [c_1], \dots, [c_k]\}$  of conjugacy classes of elements of  $F_n$ . Moreover, by [HaM4, Proposition I.2.18 (5)], by the definition of a geometric stratum in [HaM4] and the fact that  $G$  is connected, we have  $\mathcal{C} = \{[g]\}$  if and only if  $G_{r-1}$  is reduced to a point, that is, if and only if  $G$  consists in the single stratum  $H_r$ .  $\square$

By the claim and [HaM4, Fact I.2.3], the outer automorphism  $\phi$  is *geometric*: there exist a connected, compact surface  $S$  with exactly one boundary component and an identification of  $\pi_1(S)$  with  $F_n$  such that  $\phi$  is identified with a pseudo-Anosov element of  $\text{Mod}(S)$ . Moreover, the conjugacy class  $[g]$  is identified with the conjugacy class in  $\pi_1(S)$  of the element associated with the homotopy class of the boundary component of  $S$ . Since  $[g]$  is fixed by every element of  $H$ , by the Dehn-Nielsen-Baer theorem (see for instance [FarM, Theorem 8.8] for the orientable case and [Fuj, Section 3] for the nonorientable case), the group  $H$  is identified with a subgroup of  $\text{Mod}(S)$ .  $\square$

We now give a method to compute  $\mathcal{A}(H)$  for some subgroups  $H$  of  $\text{Out}(F_n)$ . Let  $P$  be a subgroup of  $F_n$ . Let  $F$  be the minimal free factor of  $F_n$  which contains  $P$ . Then  $F$  is one-ended relative to  $P$ . Let  $T$  be the JSJ tree of  $F$  relative to  $P$  over cyclic subgroups given by [GuL5, Theorem 9.14]. Let  $v$  be a vertex of  $T$ . Let  $G_v$  be the stabilizer of  $v$  in  $F$ . Let  $\text{Inc}_v$  be the finite set of all conjugacy classes of groups associated with edges in  $T$  which are incident to  $v$ . Following the terminology of [GuL5], either  $v$  is a *rigid* vertex or  $v$  is *flexible*. When  $G_v$  is cyclic, we use the convention that  $v$  is rigid. If  $v$  is flexible, by [GuL5, Theorem 9.14 (2)], there exists a compact connected hyperbolic surface  $S_v$  such that  $\pi_1(S_v)$  is isomorphic to  $G_v$  and, for every subgroup  $G_e$  of  $F$  such that  $[G_e] \in \text{Inc}_v$ , the group  $G_e$  is conjugate to a subgroup of  $\pi_1(S_v)$  associated with a boundary connected component of  $S_v$ . Since the JSJ tree constructed by Guirardel and Levitt is a *tree of cylinders*, if  $v$  is a flexible vertex of  $T$ , the fundamental group of every boundary component  $c$  of  $S_v$  fixes at most one edge  $e_c$  adjacent to  $v$  and the stabilizer of the endpoint of  $e_c$  distinct from  $v$  is cyclic and included in the group generated by the homotopy class of  $c$ . For every flexible vertex  $v$  of  $T$ , let  $\text{BC}_v$  be the finite set of conjugacy classes of subgroups of  $F_n$  generated by the homotopy classes of the boundary components of  $S_v$  which do not fix an edge in  $T$ . Let  $V_f$  be the set of flexible vertices of  $T$ . Let  $T'$  be the tree obtained from  $T$  by collapsing every edge of  $T$  which is not adjacent to a flexible vertex. For every vertex  $C$  of  $F \setminus (T' - V_f)$ , let  $G_C$  be the associated vertex stabilizer. Let

$$\mathcal{A}_P = \{[G_C]\}_{C \in V(F \setminus (T' - V_f))} \cup \bigcup_{v \in V_f} (\text{Inc}_v \cup \text{BC}_v),$$

which is a finite set of conjugacy classes of finitely generated subgroups of  $F_n$ . Note that, by [Lev2, Proposition 2.1], if  $v \in VT$  and  $\phi \in \text{Out}(G_v)$ , then  $\phi$  extends to an outer automorphism  $\hat{\phi}$  of  $\text{Out}(F_n)$  which preserves  $T$  and  $T'$ .

**Lemma 7.5.5.** *Let  $v \in V_f$  and let  $\phi'$  be the outer automorphism class of  $\text{Out}(G_v)$  associated with a pseudo-Anosov element of  $S_v$ . Let  $T'_v$  be the tree obtained from  $T$  by collapsing every edge of  $T$  which is not contained in the orbit of an edge adjacent to  $v$ . Then  $\hat{\phi}'$  preserves  $T'_v$ . Moreover, if  $g \in F$  is loxodromic in  $T'_v$ , then  $g$  has exponential growth under iteration of  $\hat{\phi}'$ .*

**Proof.** The fact that  $\hat{\phi}'$  preserves  $T'_v$  follows from the fact that  $\hat{\phi}'$  preserves  $T$  and the fact that  $\hat{\phi}'$  acts as the identity on the graph associated with  $F_n \setminus T$ . In order to prove



the second part of Lemma 7.5.5, we first construct an  $\mathbb{R}$ -tree  $T_v$  with an equivariant map  $T_v \rightarrow T'_v$ . Let  $T_0$  be the dilating arational  $G_v$ -tree associated with  $\phi'$  and let  $\lambda > 1$  be the stretching factor of  $\phi'$ . There exists a homothety  $H_0: T_0 \rightarrow T_0$  whose stretching factor is equal to  $\lambda$  and such that, for every  $h \in G_v$  and every  $x \in T_0$ , we have  $H_0(hx) = \phi'(h)H_0(x)$ . The arational tree  $T_0$  is such that every arc stabilizer is trivial and the only point stabilizers are cyclic and conjugate to the groups generated by the homotopy classes of boundary components of  $S_v$ . Since the edge stabilizers of  $T'_v$  are precisely groups which are conjugates of groups generated by the homotopy classes of boundary components of  $S_v$ , one can replace the vertex  $v$  in  $T'_v$  by the tree  $T_0$  and attach the edges to their corresponding point stabilizers. Extending this construction equivariantly, we obtain a tree  $T_v$  with an equivariant map  $T_v \rightarrow T'_v$ . Moreover, since  $\hat{\phi}'$  preserves  $T'_v$ , the map  $H_0$  extends to an equivariant map  $H': T_v \rightarrow T_v$  such that, for all  $x, y \in T_v$ , we have  $d(H'(x), H'(y)) \leq \lambda d(x, y)$ . We now follow the construction given by Gaboriau, Jaeger, Levitt and Lustig in [GJLL]. For every  $k \in \mathbb{N}^*$ , let  $d_k$  be the pseudo-distance in  $T_v$  given by, for all  $x, y \in T_v$ :

$$d_k(x, y) = \frac{d(H'^k(x), H'^k(y))}{\lambda^k},$$

and let  $d_\infty$  be the limit of these pseudo-distances. Then  $d_\infty$  induces a distance on the set

$$T_v^\infty = T_v / \sim,$$

where  $\sim$  is the equivalence relation generated by  $x \sim y$  if and only if  $d_\infty(x, y) = 0$ . Moreover, the metric space  $T_v^\infty$  is a nontrivial  $\mathbb{R}$ -tree equipped with a minimal, nontrivial action of  $F_n$  by isometries. Finally,  $H'$  induces a homothety  $H: T_v^\infty \rightarrow T_v^\infty$  with stretching factor equal to  $\lambda$  and such that, for every  $h \in F_n$  and every  $x \in T_v^\infty$ , we have  $H(hx) = \hat{\phi}'(h)H(x)$ . Note that, for every  $g \in F_n$  and every  $n \in \mathbb{N}^*$ , the translation length of  $\phi'^n(g)$  in  $T_v^\infty$  is equal to  $\lambda^n$  times the translation length of  $g$  in  $T_v^\infty$ . Therefore, if  $g$  has polynomial growth under iteration of  $\phi'$ , then  $g$  must fix a point in  $T_v^\infty$ .

Let  $g \in F$  be loxodromic in  $T'_v$ . By equivariance of the map  $T_v \rightarrow T'_v$ , the element  $g$  is loxodromic in  $T_v$ . By the construction of  $T'_v$ , the axis of  $g$  in  $T_v$  contains a vertex in the orbit of  $v$ . Since the group generated by the homotopy class of every boundary component of  $S_v$  fixes at most one edge in  $T'_v$ , if the axis of  $g$  in  $T'_v$  contains a vertex in the orbit of  $v$ , then the axis of  $g$  in  $T_v$  contains a nondegenerate arc  $[x, y]$  in a copy of  $T_0$  in  $T_v$ . Since  $H_0$  is a homothety of  $T_0$  of stretching factor equal to  $\lambda$ , the homothety  $H$  restricts to a homothety of stretching factor  $\lambda$  in the copy of  $T_0$  in  $T_v$ . Thus we have  $d_\infty(x, y) = d(x, y) > 0$ . Hence the characteristic set of  $g$  in  $T_v^\infty$ , which is the projection of the characteristic set of  $g$  in  $T_v$ , contains a nondegenerate arc, that is,  $g$  is loxodromic in  $T_v^\infty$ . Hence  $g$  has exponential growth under iteration of  $\phi'$ .  $\square$

**Proposition 7.5.6.** *Let  $n \geq 3$  and let  $P$  be a finitely generated subgroup of  $F_n$ . Suppose that  $\text{Out}(F_n, P^{(t)})$  is infinite. Then either  $\mathcal{A}(\text{Out}(F_n, P^{(t)})) = \{[F_n]\}$  or we have  $\mathcal{A}(\text{Out}(F_n, P^{(t)})) = \mathcal{A}_P$ . Moreover, in the second case, there exists  $\phi \in \text{Out}(F_n, P^{(t)})$  such that  $\mathcal{A}(\phi) = \mathcal{A}_P$ .*

**Proof.** The moreover part follows from the first part of Proposition 7.5.6 using Theorem 7.5.1, so we focus on the first part. Let  $H = \text{Out}(F_n, P^{(t)}) \cap \text{IA}_n(\mathbb{Z}/3\mathbb{Z})$ . Then  $\text{Poly}(H) = \text{Poly}(\text{Out}(F_n, P^{(t)}))$ .

Note that  $H$  preserves the conjugacy class of  $F$  and we have an induced homomorphism  $\Lambda: H \rightarrow \text{Out}(F, P^{(t)})$ . By [GuL5, Theorem 9.14], the group  $\Lambda(H)$  fixes the  $F$ -equivariant homeomorphism class  $\mathcal{T}$  of the above JSJ tree  $T$ . Moreover, up to taking a finite index subgroup of  $H$ , we may suppose that the group  $H$  fixes the conjugacy class of every vertex group of  $T$  and that  $\Lambda(H)$  acts as the identity on the graph associated with  $F \backslash T$ . Since edge stabilizers of  $T$  are cyclic,  $H$  fixes the conjugacy class of the generator of every edge group. In particular, we have  $\bigcup_{v \in V_f} \text{Inc}_v \leq \mathcal{A}(H)$ . Moreover, up to taking a finite index subgroup of  $H$ , for every flexible vertex  $v$  of  $T$ , the group  $H$  fixes the conjugacy classes of subgroups of  $F_n$  generated by the homotopy classes of the boundary components of  $S_v$ . Thus, we have

$$\bigcup_{v \in V_f} \text{Inc}_v \cup \text{BC}_v \leq \mathcal{A}(H). \quad (7.3)$$

**Claim 1.** Let  $C'$  be a connected component of  $F \backslash (T - V_f)$  and let  $C$  be a connected subgraph of  $C'$  which contains at least one vertex of the graph associated with  $F \backslash T$ . Let  $F \backslash (T_C - V_f)$  be the graph of groups obtained from  $F \backslash (T - V_f)$  by collapsing the edges of  $F \backslash T$  contained in  $C$  to a vertex  $c$  and let  $G_c$  be the corresponding vertex group. We have  $G_c \subseteq \text{Poly}(H)$ .

**Proof.** An *interior edge* of  $C$  is an edge of the graph associated with  $F \backslash (T - V_f)$  entirely contained in  $C$ . We remark that the statement of the claim is made in such a way that we are able to apply an induction argument on the number  $m$  of interior edges of  $C$ . If  $C$  does not contain an interior edge, then  $C$  contains at most one vertex  $v$  of  $F \backslash T$  (recall that  $C$  is connected). Moreover,  $v$  is a rigid vertex. Since  $v$  is rigid, by [GuL4, Theorem 3.9], the group  $H$  has trivial image in  $\text{Out}(G_v)$ . Hence the statement is true when  $C$  has no interior edge. Suppose that the number of interior edges  $m$  of  $C$  is at least equal to 1. Let  $e$  be an interior edge of  $C$ . Suppose first that  $\overline{C - e}$  has two connected components  $A_1$  and  $A_2$ , where the closure is taken in  $C$ . For every  $i \in \{1, 2\}$ , let  $G_{a_i}$  be the subgroup of  $F_n$  corresponding to  $A_i$  as in the statement of the claim. By induction, for every  $i \in \{1, 2\}$ , we have  $G_{a_i} \subseteq \text{Poly}(H)$ . Since  $F$  is one-ended relative to  $P$ , edge stabilizers are nontrivial. Thus, we have  $G_{a_1} \cap G_{a_2} \neq \{1\}$ . Since  $\mathcal{A}(H)$  is a malnormal subgroup system, there exists a subgroup  $B$  of  $F_n$  such that  $[B] \in \mathcal{A}(H)$  and  $G_c = \langle G_{a_1}, G_{a_2} \rangle \subseteq B$ . Suppose now that  $\overline{C - e}$  has one connected component  $A$ . Let  $G_a$  be the subgroup of  $F_n$  corresponding to  $A$  as in the statement of the claim. By induction, we have  $G_a \subseteq \text{Poly}(H)$ . Moreover, there exists  $t \in F$  such that  $G_c = \langle G_a, t \rangle$ . Note that  $H$  preserves the conjugacy classes of  $G_a$  and  $G_c$  as every element of  $\Lambda(H)$  acts as the identity on the graph associated with  $F \backslash T$ . Thus, every element  $\psi$  of  $H$  has a representative  $\Psi$  such that  $\Psi(G_a) = G_a$ , every element of  $G_a$  has polynomial growth under iteration of  $\Psi$  and  $\Psi$  sends  $t$  to  $ta_\Psi$  with  $a_\Psi \in G_a$ . Since  $a_\Psi$  has polynomial growth under iteration of  $\Psi$ , there exist  $s > 0$ ,  $n \in \mathbb{N}$  and a free basis  $\mathfrak{B}$  of  $F_n$  such that, for

every  $k \in \mathbb{N}$ , we have

$$\ell_{\mathfrak{B}}(\Psi^k(a_{\Psi})) \leq s(k+1)^n.$$

Hence, for every  $k \in \mathbb{N}$ , we have

$$\ell_{\mathfrak{B}}(\Psi^k(t)) \leq 1 + \sum_{i=0}^{k-1} \ell_{\mathfrak{B}}(\Psi^i(a_{\Psi})) \leq 1 + s(k+1)^{n+1}. \quad (7.4)$$

Therefore,  $t$  has polynomial growth under iteration of  $\Psi$  and every element of  $G_c = \langle G_a, t \rangle$  has polynomial growth under iteration of  $\Psi$ . Thus, for every  $\psi \in H$ , we have  $G_c \subseteq \text{Poly}(\psi)$  and  $G_c \subseteq \text{Poly}(H)$ . This proves the claim.  $\square$

By Claim 1 and Equation (7.3), we have  $\mathcal{A}_P \leq \mathcal{A}(H)$ . We now prove that either  $\mathcal{A}(H) = \{[F_n]\}$  or  $\mathcal{A}_P = \mathcal{A}(H)$ . Let  $K$  be a subgroup of  $F_n$  such that  $F_n = F * K$ . Suppose first that the rank of  $K$  is at most equal to 1 and that the set of vertices  $V_f$  is empty. Let  $k$  be a (possibly trivial) generator of  $K$ . Recall the definition of  $T'$  above Lemma 7.5.5. Then  $F \setminus T'$  is reduced to a vertex  $v$ . Therefore, we have  $\mathcal{A}_P = \{[F]\}$ . Moreover, since  $H$  preserves the sporadic free factor system  $\{[F]\}$ , every element of  $H$  has a representative which sends  $F$  to  $F$  and  $k$  to  $kg$  with  $g \in F$ . In particular, as in Equation (7.4), we have  $k \in \text{Poly}(H)$ ,  $F * K \subseteq \text{Poly}(H)$  and  $\mathcal{A}(H) = \{[F_n]\}$ .

**Claim 2.** If either the rank of  $K$  is at least equal to 2 or  $V_f$  is nonempty, then  $\mathcal{A}(H) \leq \{[F]\}$ .

**Proof.** We distinguish between two cases, according to the rank of  $K$ . When the rank of  $K$  is equal to 0, the proof is trivial.

**Case 1.** Suppose that the rank of  $K$  is equal to 1 and that  $V_f$  is not empty.

Let  $k$  be a generator of  $K$ . Let  $v \in V_f$ , let  $\phi'$  be a pseudo-Anosov element of the surface  $S_v$  associated with  $\pi_1(G_v)$ . As explained above Lemma 7.5.5, the outer automorphism  $\phi'$  induces an outer automorphism  $\hat{\phi}'$  of  $F$ . Let  $g' \in G_v$  be such that  $g'$  is not contained in the conjugacy class of the group generated by the homotopy class of any boundary component of  $S_v$ . Let  $\hat{\Phi}'$  be a representative of  $\hat{\phi}'$ . Let  $\hat{\Phi}$  be an automorphism of  $F_n$  which acts as  $\hat{\Phi}'$  on  $F$  and sends  $k$  to  $kg'$ , and let  $\hat{\phi}$  be the outer automorphism class of  $\hat{\Phi}$ . Suppose that  $\mathcal{A}(\hat{\phi}) \not\leq \{[F]\}$ . By [Gue5, Lemma 5.18 (7)] applied to  $\mathcal{F} = \{[F]\}$  with the element  $\hat{\phi}' \in \text{Out}(F_n, \mathcal{F})$  (recall that  $F$  is a sporadic free factor of  $F_n$ ), there exists  $g \in F_n$  such that  $F_n = F * \langle g \rangle$  and either  $\mathcal{A}(\hat{\phi}) = \mathcal{A}(\hat{\phi}') \cup \{[\langle g \rangle]\}$  or there exists a subgroup  $A$  of  $F$  such that  $[A] \in \mathcal{A}(\hat{\phi}')$  and  $\mathcal{A}(\hat{\phi}) = (\mathcal{A}(\hat{\phi}') - \{[A]\}) \cup \{[A * \langle g \rangle]\}$ . In the first case, let  $h \in F$  be nontrivial. Let  $\Psi$  be the automorphism of  $F_n$  such that  $\Psi(F) = F$ ,  $\Psi|_F = \hat{\Phi}'$  and  $\Psi$  sends  $g$  to  $gh$  with  $h \in F$  nontrivial and let  $\psi$  be the outer automorphism class of  $\Psi$ . Note that  $\psi \in \text{Out}(F_n, P^{(t)})$ . Then  $\psi$  does not preserve the conjugacy class of  $g$ . Thus, we have  $\mathcal{A}(H) \leq \mathcal{A}(\psi) \leq \{[F]\}$ .

Suppose that there exists a subgroup  $A$  of  $F$  such that  $[A] \in \mathcal{A}(\hat{\phi}')$  and  $\mathcal{A}(\hat{\phi}) = (\mathcal{A}(\hat{\phi}') - \{[A]\}) \cup \{[A * \langle g \rangle]\}$ . Note that  $\hat{\phi}$  has a representative  $\hat{\Phi}_0$  such that  $\hat{\Phi}_0(F) = F$ ,  $\hat{\Phi}_0(A) = A$  and  $\hat{\Phi}_0(A * \langle g \rangle) = A * \langle g \rangle$ . Then, up to composing  $\hat{\Phi}_0$  by an inner automorphism  $\text{ad}_{a_0}$  with  $a_0 \in A$ , we may suppose that  $\hat{\Phi}_0$  sends  $g$  to  $ga$  with  $a \in A$ .

Moreover, since we have  $\text{Poly}(H) \subseteq \text{Poly}(\widehat{\phi})$ , if  $h' \in F_n$  is an  $\{[F]\}$ -nonperipheral element such that  $h' \in \text{Poly}(H)$ , then  $h'$  is contained in a conjugate of  $A * \langle g \rangle$ . Let  $h \in F$  be an  $\mathcal{A}(\widehat{\phi})$ -nonperipheral element and let  $\psi \in \text{Out}(F_n, P^{(t)})$  be such that there exists  $\Psi \in \psi$  with  $\Psi(F) = F$ ,  $\Psi|_F = \widehat{\Phi}_0|_F$  and  $\Psi$  sends  $g$  to  $gh$ . The element  $h$  exists since  $\phi'$  is a pseudo-Anosov of  $S_v$ , in particular,  $\widehat{\phi}'$  is an exponentially growing outer automorphism of  $F$ . By Claim 2 in the proof of Lemma 7.4.1, for every  $a' \in F_n$ , the intersection  $(a'\Psi(A * \langle g \rangle)a'^{-1}) \cap (A * \langle g \rangle)$  is contained in conjugates of  $A$ . Note that  $\Psi(A * \langle g \rangle)$  is the only (up to conjugacy) polynomial subgroup of  $\psi\widehat{\phi}\psi^{-1}$  which contains  $\{[F]\}$ -nonperipheral element. Thus, every element of  $A * \langle g \rangle$  which is not contained in a conjugate of  $A$  has exponential growth under iteration of  $\psi\widehat{\phi}\psi^{-1}$ . In particular, we have  $\mathcal{A}(H) \leq \{[F]\}$ .

**Case 2.** Suppose that the rank of  $K$  is at least equal to 2.

Note that we have  $\text{Out}(F_n, F^{(t)}) \subseteq \text{Out}(F_n, P^{(t)})$  and that  $F$  is a nonsporadic free factor of  $F_n$ . By [GuH2, Theorem 7.4], since  $\text{Out}(F_n, F^{(t)})$  does not preserve the conjugacy class of any  $\{[F]\}$ -peripheral element of  $F_n$ , the group  $\text{Out}(F_n, F^{(t)})$  contains a fully irreducible atoroidal element  $\phi''$  of  $F_n$  relative to  $\{[F]\}$ . By Proposition 7.3.1 (1), there does not exist an  $\{[F]\}$ -nonperipheral element of  $F_n$  which has polynomial growth under iteration of  $\phi''$ . Thus we have  $\mathcal{A}(\phi'') \leq \{[F]\}$ . Thus, we have  $\mathcal{A}(H) \leq \{[F]\}$ . This proves Claim 2.  $\square$

By Claim 2 and the paragraph above Claim 2, either  $\mathcal{A}(H) = \{[F_n]\}$  or  $\mathcal{A}(H) \leq \{[F]\}$ . We are thus left with the case  $\mathcal{A}(H) \leq \{[F]\}$ . In this case, we prove that  $\mathcal{A}_P = \mathcal{A}(H)$ . Since  $\mathcal{A}_P \leq \mathcal{A}(H)$ , it remains to prove that every  $\mathcal{A}_P$ -nonperipheral element of  $F$  is  $\mathcal{A}(H)$ -nonperipheral. Let  $g \in F$  be  $\mathcal{A}_P$ -nonperipheral. Recall that  $T'$  is the tree obtained from  $T$  by collapsing every edge of  $T$  which is not adjacent to a flexible vertex. Note that, if a vertex  $v$  of  $T'$  is not the image of a flexible vertex of  $T$ , then its stabilizer is a conjugate of some  $G_c$  with  $C$  a connected component of  $F \setminus (T - V_f)$ . In particular, we have  $[G_c] \in \mathcal{A}_P$ . Suppose first that  $g$  fixes a point in  $T'$ . Since  $g$  is  $\mathcal{A}_P$ -nonperipheral, the element  $g$  fixes a flexible vertex  $v$  of  $F_n$  and is not conjugate to an element of  $F_n$  contained in the group generated by the homotopy class of a boundary component of  $S_v$ . Let  $\phi' \in \text{Out}(G_v)$  be the outer automorphism associated with a pseudo-Anosov element of  $S_v$ . Then  $g$  has exponential growth under iteration of  $\widehat{\phi}' \in \text{Out}(F_n, P^{(t)})$ . Thus, we have  $g \notin \text{Poly}(H)$ . Suppose now that  $g$  is loxodromic in  $T'$ . Then its axis contains the image of a flexible vertex  $v \in V_f$ . By Lemma 7.5.5,  $g$  has exponential growth under iteration of  $\widehat{\phi}' \in \text{Out}(F_n, P^{(t)})$ . Therefore, every  $\mathcal{A}_P$ -nonperipheral element of  $F$  is  $\mathcal{A}(H)$ -nonperipheral. Thus, we have  $\mathcal{A}(H) = \mathcal{A}_P$ .  $\square$

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**Titre :** Géométrie, dynamique et rigidité de groupes d'automorphismes de produits libres

**Mots clés:** groupes, théorie géométrique des groupes, rigidité, dynamique

**Résumé :** Dans cette thèse, nous étudions des résultats de rigidité de groupes d'automorphismes de certains groupes hyperboliques au sens de Gromov. La première partie de la thèse est consacrée à l'étude du groupe  $\text{Out}(W_n)$  des automorphismes extérieurs d'un groupe de Coxeter universel de rang  $n$ , produit libre de  $n$  copies d'un groupe cyclique d'ordre 2. Nous montrons que tout isomorphisme entre sous-groupes d'indices fini de  $\text{Out}(W_n)$  est induit par une conjugaison globale par un élément de  $\text{Out}(W_n)$ . Dans une deuxième partie, nous étudions le groupe

$\text{Out}(F_n)$  des automorphismes extérieurs d'un groupe libre non abélien de rang  $n$ . Soit  $H$  un sous-groupe de  $\text{Out}(F_n)$ . Nous étudions l'existence d'éléments génériques dans  $H$  au sens suivant. Un élément  $f$  de  $H$  est dit générique si, pour toute classe de conjugaison  $c$  d'éléments de  $F_n$ , nous avons la propriété suivante : il existe un polynôme  $P$  à coefficients dans  $\mathbb{R}$  tel que la longueur de  $f^n(c)$  est équivalente à  $P(n)$  si, et seulement si, pour tout élément  $h$  de  $H$ , il existe un polynôme  $Q_h$  à coefficients dans  $\mathbb{R}$  tel que la longueur de  $h^n(c)$  est équivalente à  $Q_h(n)$ .

**Title:** Geometry, dynamics and rigidity of automorphism groups of free products

**Keywords:** groups, geometric group theory, rigidity, dynamics

**Abstract:** In this thesis we study rigidity properties of automorphism groups of some Gromov hyperbolic groups. In the first part of the thesis, the main group of interest is the group  $\text{Out}(W_n)$  of outer automorphisms of a universal Coxeter group of rank  $n$ , which is the free product of  $n$  copies of a cyclic group of order 2. We prove that every isomorphism between finite index subgroups of  $\text{Out}(W_n)$  is induced by a global conjugation by an element of  $\text{Out}(W_n)$ . In the second part of the thesis, we study the group  $\text{Out}(F_n)$

of outer automorphisms of a nonabelian free group of rank  $n$ . Let  $H$  be a subgroup of  $\text{Out}(F_n)$ . We study the existence of generic elements of  $H$  in the following sense. We say that an element  $f$  of  $H$  is generic if, for every conjugacy class  $c$  of elements of  $F_n$ , we have the following property: there exists a polynomial  $P$  with coefficients in  $\mathbb{R}$  such that the length of  $f^n(c)$  is equivalent to  $P(n)$  if and only if for every element  $h$  of  $H$ , there exists a polynomial  $Q_h$  with coefficients in  $\mathbb{R}$  such that the length of  $h^n(c)$  is equivalent to  $Q_h(n)$ .