Locally analytic completed cohomology of Shimura varieties and overconvergent BGG maps

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Numéro National de Thèse : 2022LYSEN027

THÈSE DE DOCTORAT DE L'UNIVERSITÉ DE LYON

opérée par

l'Ecole Normale Supérieure de Lyon

Ecole Doctorale N° 512

École Doctorale en Informatique et Mathématiques de Lyon

Discipline: **Mathématiques**

Soutenue publiquement le 29/06/2022, par:

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Cohomologie completée localement analytique des variétés de Shimura et applications de BGG surconvergentes

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Abstract

In this thesis, we study the Hodge-Tate structure of the proétale cohomology of Shimura varieties. This document is divided in four main issues.

First, we construct an integral model of the perfectoid modular curve. Using this formal scheme, we prove some vanishing results for the coherent cohomology of the perfectoid modular curve, we also provide a description of the dual completed cohomology as an inverse limit of integral modular forms of weight 2 by normalized traces.

Secondly, we construct theoverconvergent Eichler-Shimura map for the first coherent cohomology group, complementing the work of Andreatta-Iovita-Stevens. More precisely, we construct a map from the overconvergent cohomology with compact support of Boxer-Pilloni to the locally analytic modular symbols of Ash-Stevens. We reinterpret the construction of these maps in terms of the Hodge-Tate period map and the perfectoid modular curve. We also reprove the classical Faltings's Eichler-Shimura decomposition using the Hodge-Tate period map, and the dual BGG resolution of irreducible representations of GL_2. We show that the overconvergent Eichler-Shimura maps are compatible with the U_p-operator, and that their small slope vectors interpolate the classical Eichler-Shimura maps.

Thirdly, in a joint work with Joaquín Rodrigues Jacinto, we develop the classical theory of locally analytic representations of p-adic Lie groups in the context of condensed mathematics. Inspired from foundational works of Lazard, Schneider-Teitelbaum and Emerton, we define a notion of solid locally analytic representation for a compact p-adic Lie group. We prove that the category of solid locally analytic representations can be described as modules over algebras of analytic distributions. As an application, we prove a cohomological comparison theorem between solid group cohomology, solid group cohomology of the (derived) locally analytic vectors, and Lie algebra cohomology.

Finally, we generalize the work of Lue Pan to arbitrary Shimura varieties. We construct a geometric Sen operator for a class of proetale $\widehat{\mathscr{O}}$-modules $\mathscr{F}$ which we call relative locally analytic. We prove that this Sen operator is related with the p-adic Simpson correspondence, and that it computes the proétale cohomology of $\mathscr{F}$. We apply this theory to Shimura varieties, obtaining that the computation of proétale cohomology can be translated in terms of Lie algebra cohomology over the flag variety via the Hodge-Tate period map. In particular, we prove that the $\mathbb{C}_p$-extension of scalars of the locally analytic completed cohomology can be described as the analytic cohomology of the infinite-at-p level Shimura variety, of the locally analytic sections of the structural sheaf. This implies a rational version of the Calegari-Emerton conjectures for any Shimura variety without the hypothesis of the infinite-at-p level Shimura variety to be perfectoid. Then, we study the λ-isotypic part of the locally analytic completed cohomology for the action of a Borel subalgebra. Using the interpretation as Lie algebra cohomology over the flag variety, we construct overconvergent BGG maps generalizing the previous work for the modular curve. In addition, we give a local proof of the classical Hodge-Tate decompositions for Shimura varieties, using the dual BGG resolution and the Hodge-Tate period map.

Résumé

Dans ce manuscrit, nous étudions la structure de Hodge-Tate de la cohomologie proétale des variétés de Shimura. Cette thèse est divisée dans quatre parties.

D'abord, nous construisons un modèle entière de la courbe modulaire perfectoïde. Avec ce schema formel, on montre quelques résultats d'annulation de la cohomologie cohérente en niveau infini, et

nous donnons une description du dual de la cohomologie completée en termes de formes modulaires intégrales de poids 2 et de traces normalisées.

Dans un second temps, on construit l'application surconvergente d'Eichler-Shimura pour le premier groupe de cohomologie cohérente, il s'agit d'un morphisme de la cohomologie surconvergente à support compact de Boxer-Pilloni vers les symboles modulaires localement analytiques d'Ash-Stevens, qui interpole l'application d'Eichler-Shimura classique. Nous réinterprétons les constructions précédentes en termes du morphisme des périodes de Hodge-Tate et de la courbe perfectoïde. Avec cette technique, on donne une nouvelle demonstration d'un théorème de Faltings sur la décomposition d'Eichler-Shimura classique en utilisant le morphisme des périodes et la resolution BGG-dual de GL_2. On prouve que les applications d'Eichler-Shimura surconvergentes sont compatibles avec l'action de l'opérateur U_p, et que les petites pentes interpolent la décomposition d'Eichler-Shimura classique.

Ensuite, dans un travail un commun avec Joaquín Rodrigues Jacinto, nous introduisons le concept de représentation localement analytique solide pour un groupe de Lie p-adique compact G. Nous nous inspirons des travaux de Lazard, Schneider-Teitelbaum et Emerton pour réinterpréter la propriété localement analytique dans la catégorie des représentations solides de G, et nous voyons que les objets obtenus peuvent être décrit en termes de modules sur des algèbres de distributions analytiques. En guise d'une application, nous démontrons quelques théorèmes de comparaison entre la cohomologie solide des groupes et la cohomologie de l'algèbre de Lie des vecteurs localement analytiques derivés.

Pour finir, nous généralisons à des variétés de Shimura quelconque les travaux de Lue Pan sur la cohomologie complétée localement analytique des courbes modulaires. Le premier point technique est l'existence d'un opérateur de Sen géométrique qui est lié à la correspondence de Simpson p-adique. On montre que cet opérateur calcule la cohomologie proétale des $\widehat{\mathscr{O}}$-modules dans un sens précis. En appliquant cette théorie dans le cas des variétés de Shimura, nous arrivons à réduire le calcule de la cohomologie proétale de certains faisceaux de $\widehat{\mathscr{O}}$-modules à celui de la cohomologie de Lie des $\mathcal{D}$-modules sur la variété de drapeaux. En particulier, nous prouvons que l'extension des scalaires à $\mathbb{C}_p$ de la cohomologie completée localement analytique se calcule comme la cohomologie des sections localement analytiques du faisceau structural de la variété de Shimura de niveau infini en p sur le site analytique. Comme corollaire, on en déduit une version rationnelle des conjectures de Calegari-Emerton sur l'annulation de la cohomologie completée. Ensuite, nous étudions la partie λ-isotypique de la cohomologie completée localement analytique pour l'action d'un Borel. En utilisant le dictionnaire entre cohomologie proétale et cohomologie de Lie des faisceaux sur la variété de drapeaux, on arrive à construire des applications de BGG surconvergentes. De plus, nous donnons une preuve locale de la décomposition de Hodge-Tate avec coefficients, en utilisant la résolution BGG-dual et le morphisme des périodes de Hodge-Tate.

General introduction

This thesis concerns the study of the Hodge-Tate structure of the proétale cohomology of Shimura varieties. More precisely, we use perfectoid techniques to describe the Hodge-Tate structure of Emerton's completed cohomology. Before addressing the general case, let us explain what can be done for modular curves.

0.1. The case of modular curves

Let us fix p a prime number, and let $C = \mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. We let $\mathbb{A}_{\mathbb{Q}}^{\infty}$ denote the ring of finite adèles (resp. $\mathbb{A}_{\mathbb{Q}}^{\infty,p}$ denote the ring of finite prime-to-p adèles). Let $K \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a neat compact open subgroup. The modular curve of level K is the algebraic curve Y_K over $\mathbb{Q}$ parametrizing elliptic curves and torsion structure of level K. For instance, let $N \geq 3$ be an integer and denote $\widehat{\Gamma}(N) := \ker(\mathrm{GL}_2(\widehat{\mathbb{Z}})) \to \mathrm{GL}_2(\mathbb{Z}/N)$, then $Y(N) := Y_{\widehat{\Gamma}(N)}$ is the modular curve parametrizing elliptic curves E and a trivialization of the N-torsion $\psi : (\mathbb{Z}/N)^2 \cong E[N]$.

Let $\mathcal{H}^{\pm} = \mathbb{C}\backslash\mathbb{R}$ be the upper and lower half planes, and consider the left action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{H}^{\pm}$ by Möbius transformations. The complex analytic variety associated to Y_K is the locally symmetric space

$$Y_K(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K,$$

where $\mathrm{GL}_2(\mathbb{Q})$ acts diagonally, and K only acts on the second factor. This implies that $Y_K(\mathbb{C})$ is a disjoint union of finitely many quotients of $\mathcal{H}^+$ by congruence subgroups. For example, let $\mu_N \subset \overline{\mathbb{Q}}^{\times}$ be the subgroup of N-th roots of unity, one has that

$$Y(N)(\mathbb{C}) = \bigsqcup_{\mu_N} \Gamma(N)\backslash\mathcal{H}^+$$

where $\Gamma(N) = \ker(\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N))$ is the principal congruence subgroup of level N.

We fix a tame level $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$, e.g. $K^p := \widehat{\Gamma}(N) \cap \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$ for $N \geq 3$ prime to p, and given an open compact subgroup $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ we will denote by $Y_{K^pK_p}$ the modular curve of level K^pK_p. From now on we will see the modular curves as schemes over $\mathbb{Q}_p$, and we will denote by $\mathcal{Y}_K$ the analytification of Y_K to an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, cf. [Hub96]. We let X_K be the compactification of Y_K by adding cusps [DR73], and let E^{sm}/X_K be the semiabelian scheme extending the elliptic curve E. We let $\mathcal{X}_K$ denote the analytification of the compact modular curve . Let $e : X_K \to E^{sm}$ be the unit section and $\omega_E := e^*\Omega^1_{E^{sm}/X_K}$ be the modular sheaf. For $k \in \mathbb{Z}$ we denote $\omega_E^k = \omega_E^{\otimes k}$.

0.1.1. Completed cohomology and an integral model of the perfectoid modular curve

We fix $N \geq 3$ prime to p, and consider the modular curves $Y(Np^n)$ and $X(Np^n)$ over $\mathrm{Spec}\,\mathbb{Q}_p$ for $n \geq 0$. We shall follow the theory of [KM85]. The schemes $Y(Np^n)$ and $X(Np^n)$ admit integral models $Y(Np^n)_{\mathbb{Z}_p}$ and $X(Np^n)_{\mathbb{Z}_p}$ which are constructed as moduli spaces of Drinfeld basis of elliptic curves.

Definition 0.1.1. Let R be a $\mathbb{Z}_p$-algebra and E an elliptic curve over R. Let $M \geq 1$ be an integer, a Drinfeld basis of $E[M]$ is a map of finite flat group schemes over R

$$\psi : (\mathbb{Z}/M)^2 \to E[M]$$

such that we have an equality of divisors over E

$$E[M] = \sum_{v \in (\mathbb{Z}/M)^2} \psi(v).$$

Remark 0.1.2. If M is invertible in R, a Drinfeld basis is the same as a trivialization of $E[M]$.

The moduli problem sending R to the isomorphism classes of elliptic curves E and Drinfeld basis $\psi : (\mathbb{Z}/M)^2 \to E[M]$ is representable for $M \geq 3$ by a regular scheme $Y(M)_{\mathbb{Z}_p}$ whose generic fiber is the modular curve of level M. The j-invariant of the universal elliptic curve over $Y(M)_{\mathbb{Z}_p}$ defines a map $j : Y(M)_{\mathbb{Z}_p} \to \mathbb{A}^1_{\mathbb{Z}_p} \subset \mathbb{P}^1_{\mathbb{Z}_p}$ whose normalization is the compactified curve $X(M)_{\mathbb{Z}_p}$. We let $\mathfrak{X}(M)/\operatorname{Spf}\mathbb{Z}_p$ be the p-adic completion of $X(M)_{\mathbb{Z}_p}$.

The p-multiplication of Drinfeld basis induces finite flat maps $\mathfrak{X}(Np^{n+1}) \to \mathfrak{X}(Np^n)$, we are interested in the tower $\{\mathfrak{X}(Np^n)\}_{n\geq 1}$. Let $\mathfrak{X}(Np^\infty) := \varprojlim_n \mathfrak{X}(Np^n)$ be the infinite level modular curve seen as a p-adic formal scheme over $\operatorname{Spf}\mathbb{Z}_p$. The Weil pairing of the Drinfeld basis $\langle \psi(1,0), \psi(0,1)\rangle$ turns out to be a primitive Np^n-th root of unity in the sense that it is killed by the Np^n-th cyclotomic polynomial. Taking the inverse limit one has a map $\mathfrak{X}(Np^\infty) \to \operatorname{Spf}\mathbb{Z}_p^{cyc}[\mu_N]$. We have the following theorem

Theorem 0.1.3 (Part I Theo. 3.2.1)**.** *The formal scheme $\mathfrak{X}(Np^\infty)$ is a perfectoid formal scheme over $\operatorname{Spf}\mathbb{Z}_p^{\mathrm{cyc}}[\mu_N]$. Moreover, the rigid analytic fiber of $\mathfrak{X}(Np^\infty)$ is naturally isomorphic to the perfectoid modular curve $\mathcal{X}(Np^\infty) = \varprojlim_n \mathcal{X}(Np^n)$.*

Remark 0.1.4. The construction of $\mathfrak{X}(Np^\infty)$ was also given by Lurie [Lur20] using the language of stacks. Lurie reduces the proof of perfectoidness to the ordinary locus by a mixed characteristic analogue of Kunz theorem. The study of the ordinary locus in both approaches is similar an better explained in Part I §3.3.

The proof of Theorem 0.1.3 is rather elementary and reduces to the study of the deformation rings of the finite level modular curves by faitfully flat descent. In the ordinary locus we use the Serre-Tate coordinates [Kat81], for the cusps we use the Tate curve [DR73], and for the supersingular points we use some explicit generators of the maximal ideal of the completed local ring and the formal group law.

With the integral perfectoid modular curve we can prove some vanishing results for the coherent cohomology at infinite level. Let $\omega_{E,n}$ be the modular sheaf over $\mathfrak{X}(Np^n)$, and let $D_n \subset \mathfrak{X}(Np^n)$ be the cusp divisor. We let $\omega^k_{E,\infty} = (\varinjlim_n \omega^k_{E,n})^{\wedge -p}$ and $\omega^k_{E,\infty}(-D_\infty) = (\varinjlim_n \omega^k_{E,n}(-D_n))^{\wedge -p}$ be the p-adic completions of the colimits of the modular sheaves as $n \to \infty$. These are quasi-coherent sheaves over $\mathfrak{X}(Np^\infty)$, and $\omega^k_{E,\infty}$ is equal to the pullback of $\omega^k_{E,n}$ for any n. The sheaf $\mathscr{O}_{\mathfrak{X}(Np^\infty)}(-D_\infty) := \omega^0_{E,\infty}(-D_\infty)$ is the ideal defining the boundary divisor of $\mathfrak{X}(Np^\infty)$ and one has $\omega^k_{E,\infty}(-D_\infty) = \omega^k_{E,\infty} \otimes_{\mathscr{O}_{\mathfrak{X}(Np^\infty)}} \mathscr{O}_{\mathfrak{X}(Np^\infty)}(-D_\infty)$.

Let $p^{1/p} \in \mathbb{Z}_p^{\mathrm{cyc}}$ be such that $|p^{1/p}| = |p|^{1/p}$. The perfectoid nature of $\mathfrak{X}(Np^\infty)$ shows that the Frobenius map is an isomorphism

$$\varphi : \mathfrak{X}(Np^\infty)/p \xrightarrow{\sim} \mathfrak{X}(Np^\infty)/p^{1/p},$$

such that $\varphi^*(\omega^k_{E,\infty}/p^{1/p}) = \omega^{kp}_{E,\infty}/p$ and $\varphi^*(\omega^k_{E,\infty}(-D_\infty)/p^{1/p}) = \omega^{kp}_{E,\infty}(-D_\infty)/p$. This fact, together with the vanishing of $H^1(\mathfrak{X}(Np^n), \omega^k_{E,n})$ and $H^0(\mathfrak{X}(Np^n), \omega^{-k}_{E,n})$ for $k >> 0$ (resp. for the sheaves $\omega^k_{E,n}(-D_n)$) imply the following theorem

Theorem 0.1.5 (Part I Theo. 5.2.4)**.** *The cohomology complexes $R\Gamma(\mathfrak{X}(Np^\infty), \omega^k_{E,\infty}(-D_\infty))$ and $R\Gamma(\mathfrak{X}(Np^\infty), \omega^k_{E,\infty})$ are concentrated in degrees $[0,1]$. Moreover,*

1. $H^1(\mathfrak{X}(Np^\infty), \omega^k_{E,\infty}) = H^1(\mathfrak{X}(Np^\infty), \omega^k_{E,\infty}(-D_\infty)) = 0$ *for* $k \geq 1$.

2. $H^0(\mathfrak{X}(Np^\infty), \omega^k_{E,\infty}) = H^0(\mathfrak{X}(Np^\infty), \omega^k_{E,\infty}(-D_\infty)) = 0$ *for* $k \leq -1$.

Remark 0.1.6. What could be surprising is the vanishing for the H^1 cohomology of the sheaves $\omega^1_{E,\infty}$ and $\omega^1_{E,\infty}(-D_\infty)$. Indeed, the H^1 cohomology at finite levels in non zero for all n; this follows by Serre duality that there are modular forms of weight 1. This phenomena is a consequence of the fact that the pullbacks of the $H^1(\mathfrak{X}(Np^n), \omega^1_{E,n}/p)$ are duals to the traces of $H^0(\mathfrak{X}(Np^n), \omega^1_{E,n}(-D_n)/p)$ whose composition is eventually zero in the tower.

It is left to study the coherent cohomology of the structural sheaf and the ideal of the cusps. It is clear that $H^0(\mathfrak{X}(Np^\infty), \mathscr{O}_{\mathfrak{X}(Np^\infty)}) = \mathbb{Z}_p^{cyc}[\mu_n]$ and $H^0(\mathfrak{X}(Np^\infty), \mathscr{O}_{\mathfrak{X}(Np^\infty)}(-D_\infty)) = 0$, so we only care about the H^1 cohomology; this one is related with completed cohomology.

Definition 0.1.7. We define the following completed cohomology groups of tame level $K^p = \widehat{\Gamma}(N) \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$

$$\widehat{H}^1_c(K^p, \mathbb{Z}_p) = \varprojlim_s \varinjlim_n H^1_{\text{ét},c}(Y(Np^n)_C, \mathbb{Z}/p^s)$$
$$\widehat{H}^1(K^p, \mathbb{Z}_p) = \varprojlim_s \varinjlim_n H^1_{\text{ét}}(X(Np^n)_C, \mathbb{Z}/p^s).$$

Theorem IV.2.1 of [Sch15] gives a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant almost quasi-isomorphism

$$\widehat{H}^1_c(K^p, \mathbb{Z}_p)\widehat{\otimes}\mathcal{O}_C =^{ae} H^1_{\text{an}}(\mathcal{X}(Np^\infty)_C, \mathscr{O}_{\mathcal{X}(Np^\infty)}(-D_\infty)),$$

where $\mathcal{X}(Np^\infty)_C$ is the C-base change of the modular curve, and $\mathscr{O}_{\mathcal{X}(Np^\infty)}(-D_\infty)$ is the ideal defining the cusps in $\mathcal{X}(Np^\infty)_C$. It turns out that the same argument gives the following proposition

Proposition 0.1.8. *We have a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant almost isomorphism*

$$\widehat{H}^1(K^p, \mathbb{Z}_p)\widehat{\otimes}\mathcal{O}_C =^{ae} H^1_{\text{an}}(\mathcal{X}(Np^\infty)_C, \mathscr{O}_{\mathcal{X}(Np^\infty)}).$$

Taking an affine cover of $\mathfrak{X}(Np^\infty)$ and Čech cohomology, one also has $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant almost isomorphisms

$$H^1_{\text{an}}(\mathcal{X}(Np^\infty)_C, \mathscr{O}_{\mathcal{X}(Np^\infty)}(-D_\infty)) =^{ae} H^1(\mathfrak{X}(Np^\infty)_{\mathcal{O}_C}, \mathscr{O}_{\mathfrak{X}(Np^\infty)}(-D_\infty))$$
$$H^1_{\text{an}}(\mathcal{X}(Np^\infty)_C, \mathscr{O}_{\mathcal{X}(Np^\infty)}) =^{ae} H^1(\mathfrak{X}(Np^\infty)_{\mathcal{O}_C}, \mathscr{O}_{\mathfrak{X}(Np^\infty)}).$$

The modular curves $\mathfrak{X}(Np^n)$ are local complete intersections over $\mathbb{Z}_p$, in particular they admit a dualizing sheaf which turns out to be isomorphic to $\omega^2_{E,n}(-D_n)$. Serre duality at finite level provides the following description for the dual of completed cohomology

Theorem 0.1.9 (Part I Theo. 5.3.1)**.** *There are* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant almost isomorphisms*

$$\mathrm{Hom}_{\mathbb{Z}_p}(\widehat{H}^1_c(K^p, \mathbb{Z}_p), \mathbb{Z}_p)\widehat{\otimes}\mathcal{O}_C =^{ae} \varprojlim_{\widetilde{\mathrm{Tr}}} H^0(\mathfrak{X}(Np^n)_{\mathcal{O}_C}, \omega^2_{E,n})$$
$$\mathrm{Hom}_{\mathbb{Z}_p}(\widehat{H}^1(K^p, \mathbb{Z}_p), \mathbb{Z}_p)\widehat{\otimes}\mathcal{O}_C =^{ae} \varprojlim_{\widetilde{\mathrm{Tr}}} H^0(\mathfrak{X}(Np^n)_{\mathcal{O}_C}, \omega^2_{E,n}(-D_n))$$

where $\widetilde{\mathrm{Tr}}$ *are normalized traces.*

The previous theorem relates completed cohomology with coherent cohomology at finite levels. In the rest of the document we will encounter with analogous descriptions of the locally analytic vectors of the completed cohomology in terms of overconvergent modular forms as in [Pan20].

0.1.2. Classical Eichler-Shimura maps via π_{HT}

In Parts II and IV we will work with the pro-Kummer-étale site as in [DLLZ19]. It is a generalization of the proétale site of [Sch13a] to (fs) log adic spaces, allowing some controlled ramification at the boundary. One of the main achievements of [DLLZ19] is that the pro-Kummer-étale site and the proétale site behave formally in the same way; both sites have natural fiber products, they have enough perfectoid spaces, and after adding enough ramification one ends up working with the proétale site of a perfectoid space. In particular, there is a purity on torsion local systems for complements of normal crossings divisors [DLLZ19, Theo. 4.6.1], and a primitive comparison theorem for proper fs log smooth adic spaces [DLLZ19, Theo. 6.2.1]. Moreover, by [DLLZ18] the pro-Kummer-étale site has period sheaves $\mathbb{B}^+_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{dR}}$, $\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}}$, $\mathscr{O}\mathbb{B}_{\mathrm{dR,log}}$ and $\mathscr{O}\mathbb{C}_{\log} := \mathrm{gr}^0\mathscr{O}\mathbb{B}_{\mathrm{dR,log}}$.

We have fixed a neat compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$, and for $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ a compact open subgroup we let $\mathcal{Y}_{K^pK_p}$ and $\mathcal{X}_{K^pK_p}$ denote the modular curves of level K^pK_p, seen as adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We let $\mathcal{Y}_{K^p} = \varprojlim_{K_p} \mathcal{Y}_{K^pK_p}$ and $\mathcal{X}_{K^p} = \varprojlim_{K^pK_p} \mathcal{X}_{K^pK_p}$ be the perfectoid modular curves and $\pi_{\mathrm{HT}} : \mathcal{X}_{K^p} \to \mathbb{P}^1$ the Hodge-Tate period map, see [Sch15]. The map $\pi_{K_p} : \mathcal{X}_{K^p} \to \mathcal{X}_{K^pK_p}$ is a pro-Kummer-étale (right) K_p-torsor. From now on we write $\mathscr{O}^{(+)}$ and $\widehat{\mathscr{O}}^{(+)}$ for the uncompleted and the completed structural sheaves of $\mathcal{X}_{K^pK_p,\mathrm{prokét}}$.

Let $\mathbf{B} \subset \mathrm{GL}_2$ be the upper triangular Borel, $\mathbf{N}$ its unipotent radical consisting on upper triangular unipotent matrices and $\mathbf{T}$ the diagonal torus. Let $\lambda : \mathbf{T} \to \mathbb{G}_m$ be a character that we identify as a pair $\lambda = (k_1, k_2)$ such that $\lambda\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = t_1^{k_1}t_2^{k_2}$. We say that λ is dominant if $k_1 \geq k_2$, and we let $X^*(\mathbf{T})^+$ denote the cone of dominant characters. The irreducible representations of GL_2 are parametrized by $X^*(\mathbf{T})^+$. More precisely, let St be the standard representation of GL_2 and det the determinant, the irreducible representation of highest weight λ is then

$$V_\lambda = \mathrm{Sym}^{k_1-k_2}\,\mathrm{St} \otimes (\det)^{k_2}.$$

By an abuse of notation we will see V_λ as a representation over $\mathbb{Q}_p$.

Let $V_{\lambda,\text{ét}}$ be the proétale local system over $\mathcal{Y}_{K^pK_p}$ attached to V_λ via the K_p-torsor $\pi_{K_p} : \mathcal{Y}_{K^p} \to \mathcal{Y}_{K^pK_p}$. Let $j : \mathcal{Y}_{K^pK_p} \subset \mathcal{X}_{K^pK_p}$ be the inclusion map and $j_{\mathrm{prokét}} : \mathcal{Y}_{K^pK_p,\mathrm{proét}} \to \mathcal{X}_{K^pK_p,\mathrm{prokét}}$ the induced map of sites. By purity of torsion local systems, the complex $Rj_{\mathrm{prokét},*}V_{\lambda,\text{ét}}$ is a pro-Kummer-étale local system concentrated in degree 0, which we denote as $V_{\lambda,\text{két}}$. The sheaf $V_{\lambda,\text{két}}$ is the pro-Kummer-étale local system attached to V_λ by the pro-Kummer-étale K_p-torsor $\pi_{K_p} : \mathcal{X}_{K^p} \to \mathcal{X}_{K^pK_p}$. The previous shows that

$$R\Gamma_{\mathrm{proét}}(\mathcal{Y}_{K^pK_p,C}, V_{\lambda,\text{ét}}) = R\Gamma_{\mathrm{prokét}}(\mathcal{X}_{K^pK_p,C}, V_{\lambda,\text{két}}),$$

namely, we can compute the proétale cohomology of $V_{\lambda,\text{ét}}$ in the pro-Kummer-étale site of $\mathcal{X}_{K^pK_p}$.

We can recover $V_{\lambda,\text{két}}$ with the Hodge-Tate period map. First of all, let us recall its construction. Let $T_pE = \varprojlim_n E[p^n]$ be the Tate module of the universal elliptic curve over $\mathcal{Y}_{K^pK_p}$, by an abuse of notation we write in the same way its extension to a pro-Kummer-étale local system over $\mathcal{X}_{K^pK_p}$. We have the Hodge-Tate exact sequence of $\widehat{\mathscr{O}}$-sheaves over $\mathcal{X}_{K^pK_p,\mathrm{prokét}}$

$$0 \to \omega_E^{-1} \otimes \widehat{\mathscr{O}}(1) \to T_pE \otimes \widehat{\mathscr{O}} \to \omega_E \otimes \widehat{\mathscr{O}} \to 0. \tag{0.1.1}$$

Taking pullbacks via the universal trivialization $\psi : \mathbb{Z}_p^2 \xrightarrow{\sim} T_pE$ over $\mathcal{X}_{K^p}$, one has a line subbundle $\psi^*(\omega_E^{-1} \otimes \widehat{\mathscr{O}}(1)|_{\mathcal{X}_{K^p,\mathrm{an}}}) \subset (\widehat{\mathscr{O}}|_{\mathcal{X}_{K^p,\mathrm{an}}})^2$ which defines the map $\pi_{\mathrm{HT}} : \mathcal{X}_{K^p} \to \mathbb{P}^1$. By construction, the map π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. Moreover, it pullbacks the constant $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant local system V_λ over $\mathscr{F}\ell$ to the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant local system $V_{\lambda,\text{két}}|_{\mathcal{X}_{K^p}}$. The Weyl pairing induces an isomorphism $\det T_pE = \mathbb{Z}_p(1)$, hence one finds that

$$V_{\lambda,\text{két}} = \mathrm{Sym}^{k_1-k_2} T_pE \otimes \mathbb{Q}_p(k_2).$$

The Hodge-Tate decompositions of the proétale cohomology of the sheaves $V_{\lambda,\text{ét}}$ can be described in terms of modular forms. Let us recall the following theorem of Faltings, see [Fal87].

Theorem 0.1.10 (Faltings). *The maps $\omega_E^{k_2-k_1}\widehat{\otimes}\widehat{\mathscr{O}}(k_1) \to V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}}$ and $V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}} \to \omega_E^{k_1-k_2} \otimes \widehat{\mathscr{O}}(k_2)$ arising from symmetric powers and a Tate twist of (0.1.1) induce surjective (resp. injective) Eichler-Shimura maps*

$$ES : H^1_{\text{proét}}(\mathcal{Y}_{K^pK_p,C}, V_{\lambda,\text{ét}}) \otimes C \to H^0_{\text{an}}(\mathcal{X}_{K^pK_p,C}, \omega_E^{k_1-k_2+2}) \otimes C(k_2-1)$$
$$ES^\vee : H^1_{\text{an}}(\mathcal{X}_{K^pK_p,C}, \omega_E^{k_2-k_1}) \otimes C(k_1) \to H^1_{\text{proét}}(\mathcal{Y}_{K^pK_p,C}, V_{\lambda,\text{ét}}) \otimes C.$$

Furthermore, the maps ES and $ES^\vee$ are naturally split and provide the Eichler-Shimura (ES) decomposition

$$H^1_{\text{proét}}(\mathcal{Y}_{K^pK_p,C}, V_{\lambda,\text{ét}}) = H^0_{\text{an}}(\mathcal{X}_{K^pK_p,C}, \omega_E^{k_1-k_2+2}) \otimes C(k_2-1) \oplus H^1_{\text{an}}(\mathcal{X}_{K^pK_p,C}, \omega_E^{k_2-k_1}) \otimes C(k_1).$$

Our first goal is to give a purely local proof of Faltings's Eichler-Shimura decomposition using the geometry of π_{HT}.

Convention. Given a K_p-equivariant sheaf $\mathscr{F}$ over $\mathbb{P}^1$ (or a K_p-equivariant $\mathscr{O}_{\mathbb{P}^1}$-sheaf), we will identify $\pi_{\text{HT}}^*(\mathscr{F})$ with the pro-Kummer-étale sheaf over $\mathcal{X}_{K^pK_p}$ it defines by descending along the torsor $\pi_{K_p} : \mathcal{X}_{K^p} \to \mathcal{X}_{K^pK_p}$.

By the primitive comparison theorem, we have that

$$R\Gamma_{\text{prokét}}(\mathcal{X}_{K^pK_p,C}, V_{\lambda,\text{két}}) = R\Gamma_{\text{prokét}}(\mathcal{X}_{K^pK_p,C}, V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}}).$$

The sheaf $V_{\lambda,\text{ét}} \otimes \widehat{\mathscr{O}}$ can also be described as the pullback by π_{HT} of the GL_2-equivariant vector bundle $V_\lambda \otimes \mathscr{O}_{\mathbb{P}^1}$ over $\mathbb{P}^1$. Our strategy to study the Hodge-Tate structure of proétale cohomology, and in particular to prove Faltings's ES decomposition, is to understand the pullbacks of GL_2-equivariant quasi-coherent sheaves of $\mathbb{P}^1$ by π_{HT}. We need some notation.

Let $\mathbb{P}^1$ be written as the quotient $\mathbb{P}^1 = \mathbf{B}\backslash \text{GL}_2$, by taking the base point $e = [0:1]$. This provides an equivalence of categories between finite dimensional $\mathbf{B}$-representations and GL_2-equivariant vector bundle over $\mathbb{P}^1$; explicitly one maps a $\mathbf{B}$-representation V to the vector bundle $\mathcal{V} := \text{GL}_2 \times^{\mathbf{B}} V := \mathbf{B}\backslash(\text{GL}_2 \times V)$ over $\mathbb{P}^1$. Note that this construction can be extended to Ind-algebraic representations of $\mathbf{B}$.

Definition 0.1.11. Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be a character, and let $w_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the non trivial element of the Weyl group of GL_2. We define the GL_2-equivariant line bundle over $\mathbb{P}^1$ of weight κ as

$$\mathcal{L}(\kappa) = \text{GL}_2 \times^{\mathbf{B}} w_0(\kappa).$$

The line bundles $\mathcal{L}(\kappa)$ have the property that, if λ is a dominant character, then $\Gamma(\mathbb{P}^1, \mathcal{L}(\lambda)) = V_\lambda$. They also admit a different description in terms of a $\mathbf{T}$-torsor.

Definition 0.1.12. Let $\mathbf{T}_{\mathbb{P}^1} = \mathbf{N}\backslash \text{GL}_2$ be the natural (left) $\mathbf{T}$-torsor over $\mathbb{P}^1$, where $\mathbf{N} \subset \mathbf{B}$ is the unipotent radical. We let $\star_1$ denote the left regular action of $\mathbf{T}$ on $\mathbf{T}_{\mathbb{P}^1}$ and let $k : \mathbf{T}_{\mathbb{P}^1} \to \mathbb{P}^1$ be the natural projection, we have that

$$\mathcal{L}(\kappa) = k_* \mathscr{O}_{\mathbf{T}_{\mathbb{P}^1}}[-w_0(\kappa)_{\star_1}].$$

The $\mathbf{B}$-filtration of the standard representation is given by

$$0 \to (1,0) \to \text{St} \to (0,1) \to 0.$$

Taking the associated vector bundle over $\mathbb{P}^1$ and pullbacks via π_{HT} one recovers the Hodge-Tate exact sequence of T_pE. This shows that

$$\pi_{\mathrm{HT}}^*(\mathcal{L}(0,1)) = \omega_E^{-1} \otimes \widehat{\mathscr{O}}(1) \text{ and } \pi_{\mathrm{HT}}^*(\mathcal{L}(1,0)) = \omega_E \otimes \widehat{\mathscr{O}}.$$

Therefore, if $\kappa = (k_1, k_2)$, one has that $\pi_{\mathrm{HT}}^*(\mathcal{L}(\kappa)) = \omega_E^{k_1-k_2} \otimes \widehat{\mathscr{O}}(k_2)$. This gives a complete description of the pullbacks of GL_2-equivariant vector bundle over $\mathbb{P}^1$ arising from semisimple representations of $\mathbf{B}$. It is left to understand the case of non-semisimple representations.

Let $\mathscr{O}(\mathbf{N})$ be the ring of algebraic regular functions of the unipotent group $\mathbf{N}$, it has a natural action of $\mathbf{T}$ given by $t \cdot f(n) = f(t^{-1}nt)$ for $f \in \mathscr{O}(\mathbf{N})$ and $t \in \mathbf{T}$. Moreover, the right regular action of $\mathbf{N}$ on $\mathscr{O}(\mathbf{N})$ is faithful. This endows $\mathscr{O}(\mathbf{N})$ with a structure of $\mathbf{B}$-module. The ring $\mathscr{O}(\mathbf{B})$ of algebraic regular functions of $\mathbf{B}$ endowed with the right regular action is isomorphic to

$$\mathscr{O}(\mathbf{B}) = \bigoplus_{\kappa \in X^*(\mathbf{T})} \kappa \otimes \mathscr{O}(\mathbf{N}).$$

Thus, we only need to consider the GL_2-equivariant quasi-coherent sheaf over $\mathbb{P}^1$ arising from the $\mathbf{B}$-module $\mathscr{O}(\mathbf{N})$. We have the following theorem.

Theorem 0.1.13 (Part II Prop. 10.1.3)**.** *Let $\underline{\mathscr{O}(\mathbf{N})}$ be the GL_2-equivariant quasi-coeherent sheaf over $\mathbb{P}^1$ associated to $\mathscr{O}(\mathbf{N})$. Let $\underline{\mathscr{O}(\mathbf{N})^{\leq 1}}$ be the GL_2-equivariant sheaf associated to the subrepresentation $\mathscr{O}(\mathbf{N})^{\leq 1} \subset \mathscr{O}(\mathbf{N})$ of polynomials of degree ≤ 1. Then*

$$\pi_{\mathrm{HT}}^*(\underline{\mathscr{O}(\mathbf{N})^{\leq 1}}) = \mathrm{gr}^1 \mathscr{O}\mathbb{B}_{\mathrm{dR},\log}^+ \otimes \widehat{\mathscr{O}}(-1). \tag{0.1.2}$$

In particular, taking n-th symmetric powers and colimits as $n \to \infty$ we have that

$$\pi_{\mathrm{HT}}^*(\underline{\mathscr{O}(\mathbf{N})}) = \mathscr{O}\mathbb{C}_{\log}.$$

Remark 0.1.14. The equality (0.1.2) was already known by Faltings [Fal87, Theo. 5] and Lue Pan [Pan20, Theo. 4.2.2]. Indeed, let $\Omega^1(\log)$ be the sheaf of log differentials of $\mathcal{X}_{K^pK_p}$, and consider the short exact sequence of $\mathbf{B}$-modules

$$0 \to \mathbb{Q}_p \to \mathscr{O}(\mathbf{N}) \to (\mathrm{Lie}\,\mathbf{N})^\vee \to 0, \tag{0.1.3}$$

where in the last arrow we identify the elements of $(\mathrm{Lie}\,\mathbf{N})^\vee$ with homogeneous linear forms of $\mathbf{N}$. Faltings and Pan have shown, in our terminology, that the pullback by π_{HT} of the GL_2-equivariant complex associated to (0.1.3) is isomorphic to (minus) the Faltings extension

$$0 \to \widehat{\mathscr{O}} \to \mathrm{gr}^1 \mathscr{O}\mathbb{B}_{\mathrm{dR},\log}^+ \otimes \widehat{\mathscr{O}}(-1) \to \Omega^1(\log) \otimes \widehat{\mathscr{O}}(-1) \to 0.$$

Once we have understood the pullbacks of GL_2-equivariant quasi-coherent sheaves by π_{HT}, we can reprove Faltings's ES decomposition using the dual BGG resolution. Let $\mathfrak{b} = \mathrm{Lie}\,\mathbf{B}$ be the Borel subalgebra of $\mathfrak{gl}_2$ and $\bar{\mathfrak{b}}$ the opposite Borel subalgebra of lower triangular matrices. We let $\mathfrak{h} = \mathrm{Lie}\,\mathbf{T}$ be the Cartan subalgebra.

Definition 0.1.15. Let $\kappa : \mathfrak{h} \to \mathbb{Q}_p$ be a weight of $\mathfrak{h}$, we define the Verma module of $(\mathfrak{g}, \bar{\mathfrak{b}})$ of weight κ to be the coinduction

$$\mathrm{Ver}_{\bar{\mathfrak{b}}}^{\mathfrak{g}}(\kappa) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} \kappa.$$

We let $V(-\kappa) \subset \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{Ver}_{\bar{\mathfrak{b}}}^{\mathfrak{g}}(\kappa), \mathbb{Q}_p)$ be the admissible dual consisting of $\mathfrak{h}$-finite vectors, seen as a $\mathfrak{g}$-module.

Remark 0.1.16. The Verma module $\mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{b}}}(\kappa)$ is an object in the category $\overline{\mathcal{O}}$, i.e. the category $\mathcal{O}$ with respect to $\overline{\mathfrak{b}}$. Then, its dual $V(-\kappa)$ is an object in the category $\mathcal{O}$ with respect to $\mathfrak{b}$. If κ arises as the derivation of a character of $\mathbf{T}$, the $\mathfrak{b}$-action on $V(-\kappa)$ can be integrated to an Ind-algebraic $\mathbf{B}$-action. As $\mathbf{B}$-module we have an isomorphism

$$V(-\kappa) = (-\kappa) \otimes \mathscr{O}(\mathbf{N}).$$

Definition 0.1.17. Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight, and consider the BGG resolution of $V_\lambda^\vee$ in the category $\overline{\mathcal{O}}$

$$0 \to \mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{b}}}(-w_0(\lambda)+\alpha) \to \mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{b}}}(-\lambda) \to V_\lambda^\vee \to 0, \tag{0.1.4}$$

where $\alpha = (1,-1) \in X^*(\mathbf{T})$. The dual BGG resolution of V_λ is the admissible dual of (0.1.4)

$$0 \to V_\lambda \to V(\lambda) \to V(w_0(\lambda)-\alpha) \to 0. \tag{0.1.5}$$

We have the following theorem

Theorem 0.1.18 (Part II Theo. 10.1.5)**.** *The pullback by π_{HT} of the GL_2-equivariant complex defined by the $\mathbf{B}$-representation (0.1.5) is quasi-isomorphic to*

$$0 \to V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}} \to \omega_E^{k_2-k_1} \otimes \mathscr{O}\mathbb{C}_{\log}(k_1) \to \omega_E^{k_1-k_2+2} \otimes \mathscr{O}\mathbb{C}_{\log}(k_2-1) \to 0.$$

We call this resolution of $V_{\lambda,\text{ét}} \otimes \widehat{\mathscr{O}}$ the Hodge-Tate dual BGG *complex, c.f. [FC90, Prop. VI.5.4].*

Let $\nu : \mathcal{X}_{K^pK_p,C,\text{prokét}} \to \mathcal{X}_{K^pK_p,C,\text{két}}$ be the projection of sites, then

$$R\nu_*(V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}}) = \omega_E^{k_2-k_1} \otimes C(k_1) \oplus \omega_E^{k_1-k_2+2} \otimes C(k_2-1)[-1].$$

Taking projections to the analytic site and H^1-cohomology one recovers Faltings's Eichler-Shimura decomposition

$$H^1_{\text{proét}}(\mathcal{Y}_{K^pK_p,C}, V_{\lambda,\text{ét}}) \otimes C = H^1(\mathcal{X}_{K^pK_p}, \omega_E^{k_2-k_1}) \otimes C(k_1) \oplus H^0(\mathcal{X}_{K^pK_p}, \omega_E^{k_1-k_2+2}) \otimes C(k_2-1). \tag{0.1.6}$$

As one can appreciate from the previous theorem, the key step to understanding the Hodge-Tate decompositions is the computation of the derived projection $R\nu_*(V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}})$. Following this idea, we can construct p-adic interpolations of (0.1.6) by first constructing p-adic interpolations of the maps $\omega_E^{k_2-k_1} \otimes \widehat{\mathscr{O}}(k_1) \to V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}}$ and $V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}} \to \omega_E^{k_1-k_2} \otimes \widehat{\mathscr{O}}(k_w)$ appearing in Theorem 0.1.10.

0.1.3. Interpolation of the Eichler-Shimura maps

Our next goal is to p-adically interpolate the maps (0.1.6). This requires a p-adic variation of the modular sheaves ω_E^k and the local systems $V_{\lambda,\text{két}}$. The former will be given by sheaves of overconvergent modular forms as in [Pil13]. Let us explain how to interpolate the later. Let $\mathscr{O}(\mathrm{GL}_2)$ be the ring of regular functions of GL_2, it is a $\mathrm{GL}_2 \times \mathrm{GL}_2$-representation given by the left $\star_1$ and right $\star_2$ regular actions respectively. In other words, $[g_2 \star_2 (g_1 \star_1 f)](h) = f(g_1^{-1}hg_2)$ for $f \in \mathscr{O}(\mathrm{GL}_2)$ and $g_1, g_2, h \in \mathrm{GL}_2$. As $\mathrm{GL}_2 \times \mathrm{GL}_2$-representation $\mathscr{O}(\mathrm{GL}_2)$ is isomorphic to ([Mil17])

$$\mathscr{O}(\mathrm{GL}_2) = \bigoplus_{\lambda \in X^*(\mathbf{T})^+} V_\lambda^\vee \otimes V_\lambda,$$

where we map a tensor $f \otimes v \in V_\lambda^\vee \otimes V_\lambda$ to the regular function $g \mapsto f(gv)$. Recall that $V_\lambda^\vee = V_{-w_0(\lambda)}$, hence

$$V_\lambda = \mathscr{O}(\mathrm{GL}_2)^{\mathbf{B}_{\star_2} = -w_0(\lambda)}. \tag{0.1.7}$$

Let us take $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ an open compact subgroup admitting an Iwahori decomposition $K_p = \overline{N}_{K_p} T_{K_p} N_{K_p}$ as product of lower triangular unipotent matrices, diagonal matrices, and upper triangular unipotent matrices. We denote $B_{K_p} = T_{K_p} N_{K_p}$. Let (R, R^+) be an uniform affinoid $\mathbb{Q}_p$-algebra and $\chi : T_{K_p} \to R^\times$ a character. We need the following easy lemma

Lemma 0.1.19. *There is $\delta > 0$ such that χ enhances to a character of rigid spaces*

$$\chi : T_{K_p}(1 + p^{\delta}\mathbb{G}_a) \to \mathbb{G}_{m,R}.$$

We say that χ is a locally analytic character of T_{K_p}.

Definition 0.1.20. Let $C^{la}(K_p, R)$ be the space of locally analytic functions of K_p with values in R. The locally analytic principal series of K_p of weight χ is the space

$$A_{\chi}^{la} = C^{la}(K_p, R)^{B_{K_p,\star_2}=-w_0(\chi)}.$$

In other words, A_{χ}^{la} is the space of locally analytic functions $f : K_p \to R$ such that $f(xb) = w_0(\chi)(b^{-1})f(x)$ for all $x \in K_p$ and $b \in T_{K_p}N_{K_p}$.

If $r : R \to \mathbb{Q}_p$ is a specialization of R such that $r \circ \chi = \lambda$ is a dominant weight of GL_2, we have a diagram

$$\begin{array}{ccc} A_{\chi}^{la} & \xrightarrow{\widehat{\otimes}_{R,r}\mathbb{Q}_p} & A_{\lambda}^{la} \\ & & \uparrow \\ & & \mathscr{O}(\mathrm{GL}_2)^{\mathbf{B}_{\star_2}=-w_0(\lambda)} = V_{\lambda}. \end{array}$$

Therefore, the spaces A_{χ}^{la} are good candidates to interpolate the algebraic representations V_{λ}. We can define the following p-adic variations of the proétale cohomology of the local systems $V_{\lambda,\text{ét}}$.

Definition 0.1.21. Let $C^{la}(K_p, R)_{\text{két}}$ and $A_{\chi,\text{két}}^{la}$ be the pro-Kummer-étale sheaves over $\mathcal{X}_{K^pK_p}$ associated to $C^{la}(K_p, R)$ and A_{χ}^{la} respectively, where these spaces are endowed with the $\star_1$-action of K_p.

1. The locally analytic completed cohomology with coefficients in R is the complex
$$R\Gamma_{\text{proét}}(\mathcal{Y}_{K^p,C}, R)^{la} := R\Gamma_{\text{proét}}(\mathcal{X}_{K^pK_p,C}, C^{la}(K_p, R)_{\text{két}})$$
2. The derived $B_{K_p} = -w_0(\chi)$ isotipic part of the locally analytic completed cohomology is the complex
$$R\Gamma_{\text{proét}}(\mathcal{Y}_{K^p,C}, R)^{la,RB_{K_p,\star_2}=-w_0(\chi)} := R\Gamma_{\text{proét}}(\mathcal{X}_{K^pK_p,C}, A_{\chi,\text{két}}^{la}).$$

Remark 0.1.22. As the previous notation suggests, the locally analytic completed cohomology is independent of K_p. This follows from Shapiro's lemma, and the fact that $C^{la}(K_p, R)$ can be written as an induction of $C^{la}(K_p', R)$ for $K_p' \subset K_p$.

Remark 0.1.23. We saw in §0.1.1 that Emerton's completed cohomology (with coefficients in R) can be computed as the cohomology groups of the complex $R\Gamma_{\text{proét}}(\mathcal{Y}_{K^p,C}, R)$. Moreover, Emerton proved that the completed cohomology groups are admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. By Part III Proposition 14.5.3, this implies that the cohomology groups of $R\Gamma_{\text{proét}}(\mathcal{Y}_{K^p,C}, R)^{la}$ are nothing but the locally analytic vectors of Emerton's completed cohomology.

Similarly as for $V_{\lambda,\text{két}}$, the pro-Kummer-étale sheaf $A_{\chi,\text{két}}^{la}$ can be constructed using the Hodge-Tate period map, namely, $A_{\chi,\text{két}}^{la}$ is the pullback by π_{HT} of the K_p-equivariant constant sheaf A_{χ}^{la} over $\mathbb{P}^1$. In order to interpolate the ES maps, we need to study the Hodge-Tate structure of the locally analytic completed cohomology. More precisely, we want to study the complex

$$R\Gamma_{\text{proét}}(\mathcal{Y}_{K^p,C}, R)^{la,RB_{K_p,\star_2}=-w_0(\chi)}\widehat{\otimes}C = R\Gamma_{\text{prokét}}(\mathcal{X}_{K^pK_p}, A_{\chi,\text{két}}^{la})\widehat{\otimes}C.$$

By the primitive comparison theorem, and a devisage in terms of finite local systems, we have that

$$R\Gamma_{\text{prokét}}(\mathcal{X}_{K^pK_p}, A^{la}_{\chi,\text{két}})\widehat{\otimes}C = R\Gamma_{\text{prokét}}(\mathcal{X}_{K^pK_p}, A^{la}_{\chi}\widehat{\otimes}\widehat{\mathscr{O}}),$$

and the sheaf $A^{la}_{\chi}\widehat{\otimes}\widehat{\mathscr{O}}$ is the pullback by π_{HT} of the K_p-equivariant quasi-coherent sheaf $A^{la}_{\chi}\widehat{\otimes}\mathscr{O}_{\mathbb{P}^1}$ over $\mathbb{P}^1$.

In the following we will construct interpolations of the maps of pro-Kummer-étale sheaves of Theorem 0.1.10 by studying the restriction of the sheaves $A^{la}_{\chi}\widehat{\otimes}\mathscr{O}_{\mathbb{P}^1}$ to overconvergent neighbourhoods of 1 and w_0 in $\mathbb{P}^1$. In the process, we also construct sheaves of overconvergent modular forms à la higher Coleman theory [BP21]. We need some notations.

For an affine scheme Z over $\mathbb{Q}_p$ let Z^{an} denote the analytification of Z to an adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $w \in \{1, w_0\}$ be an element in the Weyl group of GL_2, and let us consider a basis of K_p-equivariant neighbourhoods $U_w(\epsilon)$ of $[w] \in \mathbb{P}^1$ converging to the K_p-orbit of w. For instance, let $\mathcal{N}(\epsilon) \subset \mathbf{N}^{\text{an}}$ be the open subgroup of elements $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $x \equiv 0 \mod p^{\epsilon}$, and let $\overline{\mathcal{N}}(\epsilon) = w_0\mathcal{N}(\epsilon)w_0$. Then, we can take $U_1(\epsilon) = [1]\overline{\mathcal{N}}(\epsilon)K_p$ and $U_{w_0} = [w_0]\mathcal{N}(\epsilon)K_p$. By [Sch15, Theo. III.3.18] the inverse images $\pi^{-1}_{\text{HT}}(U_w(\epsilon))$ are affinoid perfectoid subspaces of $\mathcal{X}_{K^p}$, and there are open affinoid spaces $\mathcal{X}_{w,K_p}(\epsilon) \subset \mathcal{X}_{K^pK_p}$ such that $\pi^{-1}_{\text{Iw}_n}(\mathcal{X}_{w,K_p}(\epsilon)) = \pi^{-1}_{\text{HT}}(U_w(\epsilon))$. These affinoid spaces are overconvergent neighbourhoods of the w-ordinary locus of $\mathcal{X}_{K^pK_p}$, i.e. the locus $\overline{\mathcal{X}^{\text{ord},w}_{K^pK_p}} = \pi_{K^p}(\pi^{-1}_{HT}([w]K_p))$.

Over the neighbourhoods $U_w(\epsilon)$ we a a trivialization of the $\mathbf{T}$-torsor $\mathbf{T}_{\mathbb{P}^1} = \mathbf{N}\backslash\text{GL}_2$, namely, by the Iwahori decomposition we have isomorphisms of adic spaces

$$U_1(\epsilon) \cong \overline{\mathcal{N}}(\epsilon)\overline{N}_{K_p} \quad \text{and} \quad U_{w_0}(\epsilon) \cong w_0\mathcal{N}(\epsilon)N_{K_p},$$

we can then take a section to GL_2 whose image to $\mathbf{T}_{\mathbb{P}^1}$ gives the desired trivialization. It will be useful to work with overconvergent neighbourhoods of the trivialization. More precisely, let $\delta > 0$ and let $\mathcal{T}(\delta) \subset \mathbf{T}^{\text{an}}$ be the open subgroup of diagonal matrices which are congruent to the identity modulo p^{δ}.

Definition 0.1.24. We let $\mathcal{T}_{w,\mathbb{P}^1}(\delta)$ be the K_p-equivariant (left) $T_{K_p}\mathcal{T}(\delta)$-torsor over $U_w(\epsilon)$ defined as the quotient

$$\mathcal{T}_{w,\mathbb{P}^1}(\delta) := \mathbf{N}^{\text{an}}\backslash\mathbf{N}^{\text{an}}(T_{K_p}\mathcal{T}(\delta))\overline{\mathcal{N}}(\epsilon)wK_p = \begin{cases} \mathbf{N}^{\text{an}}\backslash\mathbf{N}^{\text{an}}(T_{K_p}\mathcal{T}(\delta))(\overline{\mathcal{N}}(\epsilon)\overline{N}_{K_p}) & \text{if } w = 1, \\ \mathbf{N}^{\text{an}}\backslash\mathbf{N}^{\text{an}}T_{K_p}\mathcal{T}(\delta)w_0(\mathcal{N}(\epsilon)N_{K_p}) & \text{if } w = w_0. \end{cases}$$

The torsors $\mathcal{T}_{w,\mathbb{P}^1}(\delta)$ admit a left $T_{K_p}\mathcal{T}(\delta)$-action $\star_1$ and a right K_p action $\star_2$, we let $C(\mathcal{T}_{w,\mathbb{P}^1}(\delta))$ denote the algebra of functions of $\mathcal{T}_{w,\mathbb{P}^1}(\delta)$ over $U_w(\epsilon)$, this is an affinoid algebra over $\mathscr{O}_{U_w(\epsilon)}$ endowed with locally analytic actions of T_{K_p} and K_p.

Definition 0.1.25. Let $\chi : T_{K_p} \to R^{\times}$ be a δ-analytic character, i.e. a character which extends to a rigid function of $T_{K_p}\mathcal{T}(\delta)$, we define the line bundle of weight χ over $U_w(\epsilon)$ to be the $\mathscr{O}_{U_w(\epsilon)}\widehat{\otimes}R$-line bundle $\mathcal{L}_w(\chi) := C(\mathcal{T}_{w,\mathbb{P}^1}(\delta))[-w_0(\chi)_{\star_1}]$. The sheaves $\mathcal{L}_w(\chi)$ are independent of δ and ϵ.

With these line bundles attached to locally analytic characters we can finally construct overconvergent modular sheaves

Definition 0.1.26. Let $\eta : \mathcal{X}_{w,K_p}(\epsilon)_{C,\text{proét}} \to \mathcal{X}_{w,K_p}(\epsilon)_{C,\text{an}}$ be the projection of sites. The pullback $\pi^*_{\text{HT}}(\mathcal{L}_w(\chi))$ defines an $\widehat{\mathscr{O}}\widehat{\otimes}R$-line bundle over $\mathcal{X}_{w,K_p}(\epsilon)_{\text{prokét}}$, we define the sheaf of overconvergent modular forms of weight χ over $\mathcal{X}_{w,K_p}(\epsilon)_C$ to be the projection

$$\omega^{\chi}_E := \eta_*\pi^*_{\text{HT}}(\mathcal{L}_w(\chi)).$$

Remark 0.1.27. The reader might ask why we did take the pushforward for the extension of scalars to C if the overconvergent modular sheaves are objects that actually live over $\mathcal{X}_{w,K_p}(\epsilon)$. The reason is that, in order to descend to a Banach sheaf over $\mathcal{X}_{w,K_p}(\epsilon)$, we need to take a Tate twist. More precisely, let us write $\chi = (\chi_1, \chi_2)$, and let $\omega_E^{\chi_1-\chi_2}$ be the overconvergent modular sheaf of weight $\chi_1 - \chi_2$ of [Pil13] (equivalently be the sheaf $\omega_E^{\chi_1-\chi_2} = \eta_*\pi_{\mathrm{HT}}^*(\mathcal{L}_w(\chi_1 - \chi_2, 0))$). Let $R(\chi_2)$ be the Galois representation given as the composition of the cyclotomic character and χ_2. We have that

$$\omega_E^{\chi} = \omega_E^{\chi_1-\chi_2} \otimes_R R(\chi_2).$$

We will keep working with the overconvergent modular sheaves $\omega_E^{\chi_1-\chi_2}$.

Let $\lambda = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight, the maps $\omega_E^{k_2-k_1} \otimes \widehat{\mathscr{O}}(k_1) \to V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}}$ and $V_{\lambda,\text{két}} \otimes \widehat{\mathscr{O}} \to \omega_E^{k_1-k_2} \otimes \widehat{\mathscr{O}}(k_2)$ are the pullbacks by π_{HT} of the map of GL_2-equivariant maps over $\mathbb{P}^1$

$$\mathcal{L}(w_0(\lambda)) \to V_\lambda \otimes \mathscr{O}_{\mathbb{P}^1} \text{ and } V_\lambda \otimes \mathscr{O}_{\mathbb{P}^1} \to \mathcal{L}(\lambda).$$

They are attached to the highest weight vector injection $\lambda \to V_\lambda$, and the lowest weight vector quotient $V_\lambda \to w_0(\lambda)$ respectively. Using this idea, we can construct interpolations by restricting to $U_w(\epsilon)$. The construction is divided in two cases:

i. Over $U_1(\epsilon)$, the sheaves $A_\chi^{la}\widehat{\otimes}\mathscr{O}_{U_1(\epsilon)}$ and $\mathcal{L}_1(\chi)$ can be constructed as the K_p-equivariant sheaf attached to the locally analytic representations $w_0(\chi)$ and A_χ^{la} of B_{K_p} by a procedure analogous to that of GL_2-equivariant vector bundles. In particular, the highest weight vector map $\chi \to A_\chi^\delta$ induces a K_p-equivariant map of sheaves $\mathcal{L}_1(w_0(\chi)) \to A_\chi^{la}\widehat{\otimes}\mathscr{O}_{U_1(\epsilon)}$. The map $\chi \to A_\chi^{la}$ factors through the N_{K_p}-invariants of A_χ^{la}.

ii. Let $\overline{B}_{K_p} = \overline{N}_{K_p}T_{K_p}$. Over $U_{w_0}(\epsilon)$, the sheaves $A_\chi^{la}\widehat{\otimes}\mathscr{O}_{U_{w_0}(\epsilon)}$ and $\mathcal{L}_{w_0}(\chi)$ are constructed from the analytic $\overline{B}_{K_p}$-representations χ and A_χ^{la} by taking as base point $[w_0] \in U_{w_0}(\epsilon)$. In this case, we have a $\overline{B}_{K_p}$-equivariant quotient map $A_\chi^{la} \to \chi$ which factors through the $\overline{N}_{K_p}$-coinvariants of A_χ^{la}. This induces a map $A_\chi^{la}\widehat{\otimes}\mathscr{O}_{U_{w_0}(\epsilon)} \to \mathcal{L}_{w_0}(\chi)$.

Remark 0.1.28. The phenomena of taking invariants and coinvariants for different unipotent subgroups, depending on w, is just an incarnation of taking $\mathfrak{n}^0$-cohomology, where $\mathfrak{n}^0(x) = x^{-1}\,\mathrm{Lie}\,\mathbf{N}x$ for $x \in \mathbb{P}^1$, is the horizontal unipotent Lie algebra. See §0.2.3 and §0.2.4 down below.

Taking pullbacks by π_{HT} and pushforwards to the analytic site we get the following theorem

Theorem 0.1.29 (Part II Theo. 10.3.2)**.** *There are maps of pro-Kummer-étale sheaves*

$$\omega_E^{\chi_2-\chi_1}\widehat{\otimes}\widehat{\mathscr{O}}(\chi_1) \to A^{la}_{\chi,\text{két}}\widehat{\otimes}\widehat{\mathscr{O}} \text{ over } \mathcal{X}_{1,K_p}(\epsilon)$$
$$A^{la}_{\chi,\text{két}}\widehat{\otimes}\widehat{\mathscr{O}} \to \omega_E^{\chi_1-\chi_2} \otimes \widehat{\mathscr{O}}(\chi_2) \text{ over } \mathcal{X}_{w_0,K_p}(\epsilon)$$

interpolating the two extremes of the Hodge-Tate filtration of $V_{\lambda,\text{két}}$. Let $\eta : X_{K_p,C,\text{prokét}} \to X_{K_p,C,\text{an}}$ be the projection of sites. We have Galois equivariant maps

$$\omega_E^{\chi_2-\chi_1}\widehat{\otimes}R(\chi_1) \to R\eta_*(A^{la}_{\chi,\text{két}}\widehat{\otimes}\widehat{\mathscr{O}})|_{\mathcal{X}_{1,K_p}(\epsilon)} \text{ over } \mathcal{X}_{1,K_p}(\epsilon)$$
$$R\eta_*(A^{la}_{\chi,\text{két}}\widehat{\otimes}\widehat{\mathscr{O}})|_{\mathcal{X}_{w_0,K_p}(\epsilon)} \to \omega_E^{\chi_1-\chi_2+2}\widehat{\otimes}R(\chi_2 - 1)[-1] \text{ over } \mathcal{X}_{w_0,K_p}(\epsilon).$$

Taking cohomology and cohomology with closed supports, we have interpolations of the ES maps (0.1.6)

$$H^1_{w,\epsilon}(\mathcal{X}_{K^pK_p,C}, \omega_E^{\chi_2-\chi_1})\widehat{\otimes}R(\chi_1) \xrightarrow{ES_A^\vee} H^1_{\text{proét}}(\mathcal{Y}_{K^pK_p,C}, A_\chi^{la})\widehat{\otimes}C \xrightarrow{ES_A} H^0(\mathcal{X}_{w_0,K_p}(\epsilon), \omega^{\chi_1-\chi_2+2})\widehat{\otimes}R(\chi_2 - 1),$$

where the first is a cohomology group with closed supports as in [BP20], and the third is a space of overconvergent modular forms.

More can be said about the overconvergent ES maps. For instance, one can define an action of the U_p-operator on all the cohomologies involved, and show that $ES_A^\vee$ and ES_A are U_p-equivariant. Analogous staments hold for the pro-Kummer-étale cohomology with compact supports of $A^\delta_{\chi,\text{két}}$, see Part II §10.3.

0.2. Generalization to Shimura varieties

In the generalization to Shimura varieties, we will study the Hodge-Tate structure of the locally analytic completed cohomology. This requires a series of technical preparations which will reduce the problem to the study of some $\mathcal{D}$-modules over flag varieties.

The objects we work with are naturally topological $\mathbb{Q}_p$-vector spaces, in order to deal with commutative algebra and topology we shall work in the category of solid $\mathbb{Q}_p$-vector spaces, defined by Clausen and Scholze via condensed mathematics, c.f. [Sch19]. Moreover, the locally analytic completed cohomology is a complex which should be considered as a locally analytic representation of the p-adic Lie group K_p. In the joint work with Joaquín Rodrigues Jacinto [RJRC21], we develop a solid foundation of the theory of locally analytic representations. We use this theory in the current paper to formalize the notion of a locally analytic representation in the derived category of solid K_p-representations over $\mathbb{Q}_p$, and to deal with the technical passage from group cohomology to Lie algebra cohomology of the locally analytic vectors, see §0.2.1 for a brief introduction and Part III for the complete document.

We also need to extend Lue Pan's construction of the geometric Sen operator [Pan20, §3.3 and 3.4] to arbitrary (log) smooth rigid analytic spaces, and to a larger class of sheaves. We use a generalized version of the Berger-Colmez's Sen formalism [BC08], which can be applied to semilinear representations arising from locally analytic representations. Then, by a glueing process, we define the geometric Sen operator as a Higgs bundle of $\widehat{\mathscr{O}}$-modules which is related with the p-adic Simpson correspondence. We show in particular that the Sen bundle computes proétale cohomology in a precise sense; this is one of the most important tools we use to understand the Hodge-Tate structure of proétale cohomology. We refer to §0.2.2 down below for an introduction to the subject, and Part IV Chapters 19 and 20 for a complete treatment.

Finally, we specialize to Shimura varieties. We compute the Sen operator, and prove that it arises from a $\mathbf{G}$-equivariant Lie algebra over the flag variety $\mathscr{F}\ell = \mathbf{P}_\mu\backslash\mathbf{G}$ via the Hodge-Tate period map π_{HT}. This provides a dictionary between Lie algebra cohomology of locally analytic sheaves over $\mathscr{F}\ell$, and proétale cohomology of their pullbacks by π_{HT}. We consider the $\mathfrak{b} = \lambda$ isotypic part of the locally analytic completed cohomology for the action of a Borel subalgebra, and invoking the previous dictionary, we construct the overconvergent BGG maps by studying the Lie algebra cohomology of the corresponding locally analytic sheaf over $\mathscr{F}\ell$. See §0.2.3 and §0.2.4 for an introduction, and Part IV Chapters 21-23 for the original work.

0.2.1. Solid locally analytic representations

Let G be a compact p-adic Lie group. The theory of locally analytic representations dates back to the work of Lazard [Laz65], Schneider-Teitelbaum [ST02, ST03], Emerton [Eme17], *et. al.* Roughly speaking, a locally analytic representation of G on a complete locally convex $\mathbb{Q}_p$-vector space V is a continuous representation such that, for all $v \in V$ the orbit map $O_v : G \to V$ can be written as a power series, locally for the choice of some coordinates of G. Using Emerton's approach to define locally analytic vectors, one can extend this notion to arbitrary solid $\mathbb{Q}_p$-vector spaces.

Let $*_{\text{proét}}$ be the proétale site of a geometric point, namely, the category of profinite sets with covers given by finitely many jointly surjective maps.

Definition 0.2.1 ([Sch20, Def. 11.7]). Let $\mathscr{C}$ be a category that admits all (small) colimits and finite limits. The category of condensed $\mathscr{C}$-objects Cond($\mathscr{C}$) is the category of sheaves $T : *_{\text{proét}} \to \mathscr{C}$.

Thus, one has the category of condensed sets Cond(Set), the category of condensed abelian groups Cond(Ab), the category of condensed rings Cond(Ring), etc.

On the other hand, by [Sch20, Prop. 1.2] there is a fully faithful embedding from the category of compactly generated weak Hausdorff spaces to Cond(Set) given by mapping X to the condensed set

$$\underline{X} : *_{\text{proét}} \to \text{Set}, \quad S \mapsto \text{Cont}(S, X).$$

In order to do algebraic geometry and p-adic functional analysis, Clausen and Scholze have constructed the category Solid($\mathbb{Z}$) of solid abelian groups. Let $S = \varprojlim_i S_i$ be a profinite set written as an inverse limit of finite sets, we define

$$\mathbb{Z}_\blacksquare[S] := \varprojlim_i \mathbb{Z}[S_i].$$

Definition 0.2.2. A solid abelian group is an object $A \in \text{Cond(Ab)}$ such that, for all $S \in *_{\text{proét}}$ one has

$$A[S] = \text{Hom}(\mathbb{Z}_\blacksquare[S], A).$$

By Theorem [Sch19, Theo. 5.8], the category of Solid abelian groups is an abelian full subcategory of Cond(Ab) stable by colimits, limits and extensions. It satisfies some Grothendieck axioms, and the objects $\mathbb{Z}_\blacksquare[S]$ for $S \in *_{\text{proét}}$ form a family of compact projective generators. The category Solid($\mathbb{Z}$) also has an internal $\underline{\text{Hom}}$ which is the same internal $\underline{\text{Hom}}$ of Cond(Ab), and a solid tensor product $\otimes_{\mathbb{Z},\blacksquare}$. Furthermore, its derived category if the full subcategory of $D(\text{Cond(Ab)})$ of objects C such that $H^i(C) \in \text{Solid}(\mathbb{Z})$ for all $i \in \mathbb{Z}$. In addition, there is a symmetric monoidal solidification functor

$$(-)_\blacksquare : D(\text{Cond(Ab)}) \to D(\text{Solid}(\mathbb{Z}))$$

which is uniquely determined by $(\mathbb{Z}[S])_\blacksquare = \mathbb{Z}_\blacksquare[S]$ for all $S \in *_{\text{proét}}$.

We let $\mathbb{Q}_p$ be considered as a condensed field via the functor $X \mapsto \underline{X}$. The field $\mathbb{Q}_p$ can be written as $\mathbb{Q}_p = \varinjlim_n p^{-n}\mathbb{Z}_p$, and $\mathbb{Z}_p = \varprojlim_s \mathbb{Z}/p^s$. In particular, $\mathbb{Q}_p \in \text{Solid}(\mathbb{Z})$. We let Solid($\mathbb{Q}_p$) denote the category of $\mathbb{Q}_p$-modules in Solid($\mathbb{Z}$), we call Solid($\mathbb{Q}_p$) the category of solid $\mathbb{Q}_p$-vector spaces. More generally, for any ring $A \in \text{Solid}(\mathbb{Z})$ let us denote by Solid(A) the category of A-modules in Solid($\mathbb{Z}$), and by $D(A)$ the category of solid A-modules.

Definition 0.2.3. A solid G-representation over $\mathbb{Q}_p$ is a G-module in Solid($\mathbb{Q}_p$). Equivalently, let $\mathbb{Q}_{p,\blacksquare}[G] = \mathbb{Z}_\blacksquare[G] \otimes_{\mathbb{Z},\blacksquare} \mathbb{Q}_p$ be the Iwasawa algebra of G. A solid G-representation is an object in Solid($\mathbb{Q}_{p,\blacksquare}[G]$).

In short, we have replaced the category of continuous G-representations on complete locally convex $\mathbb{Q}_p$-vector spaces by the category Solid($\mathbb{Q}_{p,\blacksquare}[G]$), it has the advantage to be an abelian category stable under all limits, colimits and extensions in the category of G-modules on condensed abelian groups.

Let us now discuss the notion of locally analytic representation. By the work of Lazard, a compact p-adic Lie group G has local coordinates, this means that we can find a normal open subgroup $G_0 \subset G$, and elements $g_1, \dots, g_k \in G_0$ such that:

1. The map $\psi : \mathbb{Z}_p^k \to G_0$ sending $(x_1, \dots, x_k) \mapsto g_1^{x_1} \dots g_k^{x_k}$ is a homeomorphism.
2. The multiplication law $(g, h) \mapsto gh^{-1}$ is written as a power series with bounded coefficients in the coordinates ψ.

The choice of ψ allows us to embed G in a rigid analytic group $\mathbb{G}$ as a Zariski dense subspace. The group $\mathbb{G}$ is constructed by adding polydiscs of radius 1 around each coset of G/G_0. Moreover, we can define a decreasing family of rigid analytic groups $\mathbb{G}^{(h)} \subset \mathbb{G}$ with $\bigcap_h \mathbb{G}^{(h)} = G$, where each $\mathbb{G}^{(h)}$ is constructed by adding finitely many polydiscs of radius p^{-h} in cosets $G/G_0^{p^h}$ (modulo a rescaling of the valuation of G_0). We have the following definition

Definition 0.2.4. We let $C(\mathbb{G}^{(h)}, \mathbb{Q}_p)$ be the ring of global functions of $\mathbb{G}$, and let $\mathcal{D}^{(h)}(G, \mathbb{Q}_p) = \underline{\mathrm{Hom}}_{\mathbb{Q}_p}(C(\mathbb{G}^{(h)}, \mathbb{Q}_p), \mathbb{Q}_p)$ be the algebra of h-analytic distributions.

The ring $C(\mathbb{G}^{(h)}, \mathbb{Q}_p)$ has a natural structure of analytic ring in the sense of [Sch19, Def. 7.4] provided by its subring of bounded functions. We let $C(\mathbb{G}^{(h)}, \mathbb{Q}_p)_\blacksquare$ denote this analytic ring structure.

Definition 0.2.5. Let $V \in \mathrm{Solid}(\mathbb{Q}_{p,\blacksquare}[G])$. The derived $\mathbb{G}^{(h)}$-analytic vectors of V is the G-solid representation

$$V^{R\mathbb{G}^{(h)}-an} = R\underline{\mathrm{Hom}}_{\mathbb{Q}_{p,\blacksquare}[G]}(\mathbb{Q}_p, (V \otimes_{\mathbb{Q}_p,\blacksquare} C(\mathbb{G}^{(h)}, \mathbb{Q}_p)_\blacksquare)_{\star_{1,3}}),$$

where $\otimes_{\mathbb{Q}_p,\blacksquare} C(\mathbb{G}^{(h)}, \mathbb{Q}_p)_\blacksquare$ is the base change of analytic rings (see [Sch19, Prop. 7.7]), and the $\star_{1,3}$ action is given by $g \star_{1,3} f(x) = gf(g^{-1}x)$ (see [RJRC21] for a precise definition). The G-module structure on $V^{R\mathbb{G}^{(h)}-an}$ is induced by the right regular action $\star_2$ on $C(\mathbb{G}^{(h)}, \mathbb{Q}_p)$. We let $V^{\mathbb{G}^{(h)}-an} = H^0(V^{R\mathbb{G}^{(h)}-an})$ denote the $\mathbb{G}^{(h)}$-analytic vectors of V. We also define $V^{RG-la} = \varinjlim_{h\to\infty} V^{R\mathbb{G}^{(h)}-an}$ to be the derived locally analytic vectors of V, and $V^{G-la} = H^0(V^{RG-la})$ the locally analytic vectors.

When V is a Banach representation (or an LF representation in more generality), the previous definition coincides with the one of Emerton [Eme17].

The following theorem relates the locally analytic vectors with the distribution algebras

Theorem 0.2.6 (Part III Theo. 14.3.9)**.** *Let $V \in \mathrm{Solid}(\mathbb{Q}_{p,\blacksquare}[G])$, then*

$$V^{R\mathbb{G}^{(h)}-an} = R\underline{\mathrm{Hom}}_{\mathbb{Q}_{p,\blacksquare}[G]}(\mathcal{D}^{(h)}(G, \mathbb{Q}_p), V),$$

where we see $\mathcal{D}^{(h)}(G, \mathbb{Q}_p)$ as a $\mathbb{Q}_{p,\blacksquare}[G]$-module via the left multiplication.

The Theorem 0.1.3 is an application of the adjunction of the shriek functors for the map $\mathbb{G}^{(h)} \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. An informal way to read the previous theorem is that the $\mathbb{G}^{(h)}$-analytic vectors of V are those for which the action of G enhances to an action of $\mathcal{D}^{(h)}(G, \mathbb{Q}_p)$. In order to relate locally analytic vectors with modules over distribution algebras we need to consider a variant of $\mathbb{G}^{(h)}$. Let $\mathbb{G}^{(h^+)} = \bigcup_{h'>h} \mathbb{G}^{(h')}$ be the Stein neighbourhood of G by adding open polydiscs of radius p^{-h} on finitely many cosets. Let us denote by $C(\mathbb{G}^{(h^+)}, \mathbb{Q}_p) = \varprojlim_{h'>h} C(\mathbb{G}^h, \mathbb{Q}_p)$ its global functions and by $\mathcal{D}^{(h^+)}(G, \mathbb{Q}_p) = \underline{\mathrm{Hom}}_{\mathbb{Q}_p}(C(\mathbb{G}^{(h^+)}, \mathbb{Q}_p), \mathbb{Q}_p)$ the algebra of $\mathbb{G}^{(h^+)}$-analytic distributions. Notice that

$$\mathcal{D}^{(h^+)}(G, \mathbb{Q}_p) = \varinjlim_{h'>h} \mathcal{D}^{(h')}(G, \mathbb{Q}_p).$$

The algebra $\mathcal{D}^{(h^+)}(G, \mathbb{Q}_p)$ has the advantage to be a localization of $\mathbb{Q}_{p,\blacksquare}[G]$, namely, that

$$\mathcal{D}^{(h^+)}(G, \mathbb{Q}_p) \otimes^L_{\mathbb{Q}_{p,\blacksquare}[G]} \mathcal{D}^{(h^+)}(G, \mathbb{Q}_p) = \mathcal{D}^{(h^+)}(G, \mathbb{Q}_p),$$

see [RJRC21, Prop. 5.9]. In particular, $D(\mathcal{D}^{(h^+)}(G, \mathbb{Q}_p))$ is a full subcategory of $D(\mathbb{Q}_{p,\blacksquare}[G])$. Let us define the derived $\mathbb{G}^{(h^+)}$-analytic vectors of V to be

$$V^{R\mathbb{G}^{(h^+)}-an} = R\varprojlim_{h'>h} V^{R\mathbb{G}^{(h')}-an},$$

we set $V^{\mathbb{G}^{(h^+)}-an} = H^0(V^{R\mathbb{G}^{(h^+)}-an})$.

Hence, Theorem 0.2.6 implies that

$$V^{R\mathbb{G}^{(h^+)}-an} = R\underline{\mathrm{Hom}}_{\mathbb{Q}_{p,\blacksquare}[G]}(\mathcal{D}^{(h^+)}(G,\mathbb{Q}_p),V).$$

We obtain the following corollary

Corollary 0.2.7. *A solid G-representation V over $\mathbb{Q}_p$ is $\mathbb{G}^{(h^+)}$-analytic (i.e. $V^{\mathbb{G}^{(h^+)}} = V$) if and only if it is derived $\mathbb{G}^{(h^+)}$-analytic (i.e. $V^{R\mathbb{G}^{(h^+)}} = V$) if and only if $V \in \mathrm{Solid}(\mathcal{D}^{(h^+)}(G,\mathbb{Q}_p))$.*

In other words, the notion of being an analytic representation (for an open radius of analyticity) is equivalent to being a module over a distribution algebra! This fact suggests the following definition

Definition 0.2.8. The category $\mathrm{Solid}_{\mathbb{Q}_p}(G^{la})$ of solid locally analytic representations is the full subcategory of $\mathrm{Solid}(\mathbb{Q}_{p,\blacksquare}[G])$ generated by all the $\mathcal{D}^{(h^+)}(G,\mathbb{Q}_p)$-modules for all $h > 0$. We define derived category $D_{\mathbb{Q}_p}(G^{la})$ of locally analytic representations of G to be the derived category of $\mathrm{Solid}_{\mathbb{Q}_p}(G^{la})$.

Remark 0.2.9. Being precise, one first has to define the categories of κ-small locally analytic representations for κ an uncountable strong limit cardinal, and then take colimits along all the κ, this requires the use of ∞-categories. The previous definition does not appear in [RJRC21], and the study of the derived category $D_{\mathbb{Q}_p}(G^{la})$ is still work in progress with J. Rodrigues Jacinto.

One has the following corollary

Corollary 0.2.10. *A representation $V \in \mathrm{Solid}(\mathbb{Q}_{p,\blacksquare}[G])$ is locally analytic (i.e. $V^{G-la} = V$) if and only if it is derived locally analytic (i.e. $V^{RG-la} = V$) if and only if $V \in \mathrm{Solid}_{\mathbb{Q}_p}(G^{la})$.*

We can finally state the cohomological comparison theorems which are going to be applied in Sen theory

Theorem 0.2.11 (Part III Theo. 15.2.1 and 15.2.3)**.** *Let $V \in \mathrm{Solid}(\mathbb{Q}_{p,\blacksquare}[G])$, we have quasi-isomorphisms of solid $\mathbb{Q}_p$-vector spaces*

$$R\underline{\mathrm{Hom}}_{\mathbb{Q}_{p,\blacksquare}[G]}(\mathbb{Q}_p,V) = R\underline{\mathrm{Hom}}_{\mathbb{Q}_{p,\blacksquare}[G]}(\mathbb{Q}_p,V^{RG-la}) = R\underline{\mathrm{Hom}}_{\mathrm{Lie}\,G}(\mathbb{Q}_p,V^{RG-la})^G,$$

where in the last equality we use the fact that $R\underline{\mathrm{Hom}}_{\mathrm{Lie}\,G}(\mathbb{Q}_p,V^{RG-la})$ is a smooth representation of G, and that taking G-invariants is exact for smooth G representations in characteristic 0 since G is compact.

0.2.2. Geometric Sen theory over rigid spaces: the p-adic Simpson correspondence

We go back to rigid analytic geometry. For simplicity in the exposition we are only going to discuss the case of rigid spaces, all the relevant constructions and the main theorems extend to log adic spaces as is shown in Part IV §20.

Let (K,K^+) be a complete nonarchimedean extension of $\mathbb{Q}_p$ with K a discretely valued field, let $C = \widehat{\overline{K}}$ be the p-adic completion of an algebraic closure of K. Let X be a smooth locally noetherian adic space over $\mathrm{Spa}(K,K^+)$, we let $\widehat{\mathscr{O}}^{(+)}$ and $\mathscr{O}^{(+)}$ denote the completed and uncompleted sheaves of $X_{\mathrm{proét}}$ respectively. Let Ω^1_X be the sheaf of differential forms of X. Let us begin with the statement of the main theorem of Part IV §20, we need a definition.

Definition 0.2.12. A proétale $\widehat{\mathscr{O}}$-module $\mathscr{F}$ over X is said relative locally analytic if there is an étale cover $\{U_i\}_{i\in I}$ of X such that, for all i, the restriction $\mathscr{F}|_{U_i}$ admits a p-adically complete $\widehat{\mathscr{O}}^+$-lattice $\mathscr{F}_i^0$, and there is $\epsilon > 0$ (depending on i) such that $\mathscr{F}_i^0/p^\epsilon =^{ae} \bigoplus_I \mathscr{O}^+/p^\epsilon$.

Theorem 0.2.13 (Part IV Theo. 20.1.15)**.** *Let $\mathscr{F}$ be a relative locally analytic $\widehat{\mathscr{O}}$-module over X. There is an $\widehat{\mathscr{O}}$-linear map*

$$\theta_X : \mathscr{F} \to \mathscr{F} \otimes \Omega^1_X \otimes \widehat{\mathscr{O}}(-1)$$

called the Sen operator of $\mathscr{F}$. The map θ_X satisfies the following properties

1. *The formation of θ_X is functorial on $\mathscr{F}$ and compatible with pullbacks.*
2. *θ_X is a Higgs bundle, namely, $\theta_X \wedge \theta_X = 0$.*
3. *Let $\nu : X_{C,\text{proét}} \to X_{C,\text{ét}}$ be the projection of sites, then*

$$R\nu_*(\mathscr{F}) = \nu_* R\Gamma(\theta_X, \mathscr{F}).$$

   In other words, $R^i\nu_*\mathscr{F} = \nu_* H^i(R\Gamma(\theta_X, \mathscr{F}))$.

4. *Let $\eta : X_{C,\text{proét}} \to X_{C,\text{an}}$ be the projection of sites. If in addition we can take the cover $\{U_i\}_{i\in I}$ to be a cover in the analytic topology, then*

$$R\eta_*(\mathscr{F}) = \eta_*(R\Gamma(\theta_X, \mathscr{F})).$$

The property of being a relative locally analytic $\widehat{\mathscr{O}}$-module might look a bit mysterious. Nevertheless, these sheaves arise naturally when studying locally analytic vectors of proétale cohomology. Let us explain in which context they appear. Let G be a compact p-adic Lie group and $\widetilde{X} \to X$ a G-torsor (e.g. take X a finite level modular curve and $\widetilde{X}$ the perfectoid modular curve). Let V be a $\mathbb{Q}_p$-Banach locally analytic representation of G, for example, we can take $V = C(\mathbb{G}, \mathbb{Q}_p)$ for some group affinoid neighbourhood $\mathbb{G}$ of G. Then V defines a proétale sheaf $V_{\text{ét}}$ over X by descending the G-representation V along the torsor $\widetilde{X} \to X$. By Part IV Corollary 17.2.6, there is a lattive $V^0 \subset V$, $\epsilon > 0$, and an open subgroup $G_0 \subset G$, such that G_0 stabilizes V^0, and that the action of G_0 on V^0/p^ϵ is trivial. Therefore, the proétale $\widehat{\mathscr{O}}$-module $V_{\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}$ is a relative locally analytic sheaf. Indeed, the restriction of $V_{\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}$ to $X_{G_0} := \widetilde{X}/G_0$ satisfies the conditions of Definition 0.2.12. Furthermore, in this situation we have a more refined result

Theorem 0.2.14 (Part IV Theo. 20.1.15)**.** *] We keep the previous notation. Let $\mathfrak{g} = \operatorname{Lie} G$ endowed with the adjoint action, and let V be a $\mathbb{Q}_p$-Banach locally analytic representation of G. We have an action of $\mathfrak{g}$ on V by derivations. Then, the Sen operator θ_X of $V_{\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}$ factors through a morphism of sheaves*

$$\mathscr{F} \xrightarrow{\text{id}\otimes\theta} \mathfrak{g}_{\text{ét}} \otimes \mathscr{F} \otimes \Omega^1_X \otimes \widehat{\mathscr{O}}(-1) \to \mathscr{F} \otimes \Omega^1_X \otimes \widehat{\mathscr{O}}(-1)$$

where the second map is induced by the derivation $\mathfrak{g} \otimes V \to V$, and θ is the Sen operator of the G-torsor $\widetilde{X} \to X$:

$$\theta : \widehat{\mathscr{O}} \to \mathfrak{g}_{\text{ét}} \otimes \Omega^1_X \otimes \widehat{\mathscr{O}}(-1). \tag{0.2.1}$$

Moreover, let $\eta : X_{C,\text{proét}} \to X_{C,\text{an}}$ be the projection of sites. Then $R\eta_*(V\widehat{\otimes}\widehat{\mathscr{O}}) = \eta_*(R\Gamma(\theta, V\widehat{\otimes}\widehat{\mathscr{O}}))$.

As an example of the previous theorems, let's perform the construction of the Sen operator for the simplest case of a torus $\mathbb{T} = \operatorname{Spa}(K\langle T^{\pm 1}\rangle, K^+\langle T^{\pm 1}\rangle)$.

One of the main ideas of [Pan20] is to use the axiomatic framework of Sen theory à la Berger-Colmez [BC08], to compute proétale cohomology of $\widehat{\mathscr{O}}$-sheaves over $\mathbb{T}$. Let $\mathscr{F}$ be a relative locally analytic sheaf over $\mathbb{T}$, suppose in addition that $U = \mathbb{T}$ satifies the condition of Definition 0.2.12. In other words, there is a p-adically complete lattice $\mathscr{F}^0 \subset \mathscr{F}$, and $\epsilon > 0$, such that $\mathscr{F}^0/p^\epsilon =^{ae} \bigoplus_I \mathscr{O}^+/p^\epsilon$.

We want to compute the (geometric) proétale cohomology $R\Gamma_{\text{proét}}(\mathbb{T}_C, \mathscr{F})$. Let

$$\mathbb{T}_{n,K} = \operatorname{Spa}(K\langle T^{\pm 1/p^n}\rangle, K^+\langle T^{\pm 1/p^n}\rangle),$$

and let $\mathbb{T}_{\infty,K} = \varprojlim_n \mathbb{T}_{n,K}$ be the preperfectoid torus. The perfectoid torus $\mathbb{T}_{\infty,C}$ is a Galois cover of $\mathbb{T}_C$ with group $\Gamma = \mathbb{Z}_p(1)$. By Scholze's almost acyclicity of $\mathscr{O}^+/p$ in affinoid perfectoid spaces [Sch12, Prop. 7.13], one deduces that

$$R\Gamma_{\text{proét}}(\mathbb{T}_C, \mathscr{F}) = R\mathrm{Hom}_\Gamma(\mathbb{Q}_p, \mathscr{F}(\mathbb{T}_{\infty,C})).$$

Using Theorem 0.2.11 one has

$$\begin{aligned} R\Gamma_{\text{proét}}(\mathbb{T}_C, \mathscr{F}) &= R\mathrm{Hom}_\Gamma(\mathbb{Q}_p, \mathscr{F}(\mathbb{T}_{\infty,C})^{R\Gamma-la}) \\ &= R\mathrm{Hom}_{\mathrm{Lie}\,\Gamma}(\mathbb{Q}_p, \mathscr{F}(\mathbb{T}_{\infty,C})^{R\Gamma-la})^\Gamma. \end{aligned}$$

In other words, we have separated the problem of computing proétale cohomology in three steps: first, we need to compute the derived locally analytic vectors of $\mathscr{F}(\mathbb{T}_{\infty,C})$. Second, we take the Lie algebra cohomology of $\mathscr{F}(\mathbb{T}_{\infty,C})$, and finally, we take the Γ-invariants of a smooth representation.

Let us focus in the first step which seems to be the more subtle. For $n \geq m$ there are normalized traces

$$R^n_m : C\langle T^{\pm 1/p^n}\rangle \to C\langle T^{\pm 1/p^m}\rangle$$

where $R^n_m = \frac{1}{p^{n-m}} \sum_{\sigma \in p^m\Gamma/p^n\Gamma} \sigma$. These extend to Tate traces

$$R_m : C\langle T^{\pm 1/p^\infty}\rangle \to C\langle T^{\pm 1/p^m}\rangle$$

such that, for any $f \in C\langle T^{\pm 1/p^\infty}\rangle$, the sequence $(R_m(f))_m$ converges to f. Furthermore, the tuple $(C\langle T^{\pm 1/p^\infty}\rangle, \Gamma)$ satisfies the Colmez-Sen-Tate axioms of [BC08], see Part IV §19.1 for a generalization.

Let $(\zeta_{p^n})_n$ be a compatible system of primitive p-th power roots of unity, and let $\psi : \mathbb{Z}_p \xrightarrow{\sim} \Gamma$ be the induced isomorphism. Using ψ we define the affinoid group $\mathbb{G}_n$ which is a copy of the additive group of radius p^{-n} (i.e. $\mathbb{G}_n(\mathbb{Q}_p) = p^n\mathbb{Z}_p$). We will keep using the expression "$p^n\Gamma$-analytic" instead of $\mathbb{G}_n$-analytic. The following theorem is a generalization of [BC08, Prop. 3.3.1] to relative ϵ-analytic representations, it can be seen as a decompletion theorem à la Kedlaya-Liu [KL19].

Theorem 0.2.15 (Part IV Theo. 19.3.3)**.** *There exists $m >> 0$ depending on ϵ such that*

$$\mathscr{F}(\mathbb{T}_{\infty,C}) = C\langle T^{\pm 1/p^\infty}\rangle \widehat{\otimes}_{C\langle T^{\pm 1/p^m}\rangle} \mathscr{F}(\mathbb{T}_{\infty,C})^{p^m\Gamma-an}.$$

Moreover, we have that

$$\mathscr{F}(\mathbb{T}_{\infty,C})^{R\Gamma-la} = \mathscr{F}(\mathbb{T}_{\infty,C})^{\Gamma-la} = \varinjlim_n C\langle T^{\pm 1/p^n}\rangle \widehat{\otimes}_{C\langle T^{\pm 1/p^m}\rangle} \mathscr{F}(\mathbb{T}_{\infty,C})^{p^m\Gamma-an}.$$

The previous theorem shows that, under certain conditions on $\mathscr{F}$, the derived locally analytic vectors of $\mathscr{F}(\mathbb{T}_{\infty,C})$ are concentrated in degree 0, and that all the relevant information is already encoded in the $p^m\Gamma$-analytic vectors for some $m >> 0$. In particular, we have that

$$R\Gamma_{\text{proét}}(\mathbb{T}_{\infty,C}, \mathscr{F}) = R\Gamma_{\mathrm{Lie}\,\Gamma}(\mathbb{Q}_p, \mathscr{F}(\mathbb{T}_{\infty,C})^{p^m\Gamma-an})^\Gamma. \tag{0.2.2}$$

Thus, the problem of computing proétale cohomology has been reduced to a problem of computing Lie algebra cohomology. The module $\mathscr{F}(\mathbb{T}_{\infty,C})^{p^m\Gamma-an}$ is not mysterious at all, it is basically the p-adic Simpson correspondence of $\mathscr{F}$, see Part IV §20.2.3 for a discussion in this direction.

Now, the action of $\mathrm{Lie}\,\Gamma$ is $C\langle T^{\pm 1/p^m}\rangle$-linear and Γ-equivariant. It induces a Γ-equivariant $C\langle T^{\pm 1/p^\infty}\rangle$-linear action on $\mathscr{F}(\mathbb{T}_{\infty,C})$ by extending scalars. Moreover, this action is Gal_K-equivariant, where Gal_K acts on $\mathrm{Lie}\,\Gamma$ via the cyclotomic character. On the other hand, we can identify $\mathrm{Lie}\,\Gamma \otimes C\langle T^{\pm 1/p^\infty}\rangle = \Omega^{1,\vee}_{\mathbb{T}}(\mathbb{T}) \otimes C\langle T^{\pm 1/p^\infty}\rangle \otimes \mathbb{Z}_p(1)$. This shows that the action of $\mathrm{Lie}\,\Gamma$ defines an $\widehat{\mathscr{O}}$-linear map of proétale sheaves over $\mathbb{T}$

$$\theta_{\mathbb{T}} : \Omega^{1,\vee}_{\mathbb{T}} \otimes \mathscr{F} \otimes \widehat{\mathscr{O}}(1) \to \mathscr{F},$$

equivalently, it defines the Sen operator

$$\theta_{\mathbb{T}} : \mathscr{F} \to \mathscr{F} \otimes \Omega^1_{\mathbb{T}} \otimes \widehat{\mathscr{O}}(-1).$$

Thus, the equation (0.2.2) implies that $R\eta_*(\mathscr{F}) = \eta_* R\Gamma(\theta_X, \mathscr{F})$, obtaining Theorem 0.2.13 for the torus.

Remark 0.2.16. The notion of relative locally analytic $\widehat{\mathscr{O}}$-module given in this document is ad hoc, but it suffices for the main objective of the paper. We believe that there should be a (derived) category of solid relative locally analytic $\widehat{\mathscr{O}}$-modules over X for which Theorem 0.2.13 holds.

0.2.3. The classical BGG decompositions of Shimura varieties

We specialize to Shimura varieties. Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ and $(\mathbf{G}, X)$ a Shimura datum. Let E be the reflex field of $(\mathbf{G}, X)$ and $\mu : \mathbb{G}_m \to \mathbf{G}_E$ a fixed Hodge cocharacter. Let $K \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a neat compact open subgroup and $\mathrm{Sh}_K / \operatorname{Spec} E$ the canonical model of the Shimura variety of level K. The complex points of Sh_K is the locally symmetric space

$$\mathrm{Sh}_K(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty}) / K.$$

For simplicity in the exposition, let us suppose that the Shimura varieties are compact, and that the maximal $\mathbb{Q}$-rational non split subtorus in the center of $\mathbf{G}$ which is $\mathbb{R}$-split is trivial. The second condition guarantees that, if $K' \subset K$ is an open normal subgroup, then $\mathrm{Sh}_{K'} \to \mathrm{Sh}_K$ is a Galois cover of group K/K'.

From now on we fix $K^p \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$ a neat compact open subgroup, and given $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ compact open subgroup we let $\mathrm{Sh}_{K^p K_p}$ denote the Shimura variety of level $K^p K_p$. Let $L/\mathbb{Q}_p$ be a finite extension for which $\mathbf{G}$ is split, and such that there is an immersion $E \hookrightarrow L$. We will let $\mathcal{S}h_{K,L}$ denote the analytification of the L-extension of scalars of the Shimura variety Sh_K.

Consider the infinite level-at-p Shimura variety $\mathcal{S}h_{K^p,L} = \varprojlim_{K_p} \mathcal{S}h_{K^p K_p,L}$, it is a proétale right K_p-torsor over $\mathcal{S}h_{K^p K_p,L}$. Let $\mathrm{Rep}_L(\mathbf{G})$ be the category of algebraic representations of $\mathbf{G}$ over L. Given $V \in \mathrm{Rep}_L(\mathbf{G})$ we denote by $V_{\text{ét}}$ the proétale local system over $\mathcal{S}h_{K^p K_p,L}$ defined as

$$V_{\text{ét}} = \mathcal{S}h_{K^p,L} \times^{K_p} V.$$

By [DLLZ18], the sheaves $V_{\text{ét}}$ are de Rham, and the associated filtered vector bundle with integrable connection V_{dR}, which is provided by the p-adic Riemann-Hilbert correspondence, is compatible with the formation of the Betti local system over $\mathrm{Sh}_{K^p K_p}(\mathbb{C})$, and the classical Riemann-Hilbert correspondence.

Let $\mathbf{P}_\mu$ and $\mathbf{P}_\mu^{\mathrm{std}}$ denote the parabolic subgroups

$$\mathbf{P}_\mu = \{g \in \mathbf{G}_L \mid \lim_{t\to 0} \mathrm{Ad}(\mu(t))(g) \text{ converges}\}$$
$$\mathbf{P}_\mu^{\mathrm{std}} = \{g \in \mathbf{G}_L \mid \lim_{t\to \infty} \mathrm{Ad}(\mu(t))(g) \text{ converges}\}.$$

Equivalently, let $\mathrm{Fil}_\bullet(\mu)$ and $\mathrm{Fil}^\bullet(\mu)$ be the decreasing and ascending filtrations of $\mathrm{Rep}_L(\mathbf{G})$ defined as

$$\mathrm{Fil}_i V = \bigoplus_{k \geq -i} V_k$$
$$\mathrm{Fil}^i V = \bigoplus_{k \leq -i} V_k,$$

where $V = \bigoplus_{k\in\mathbb{Z}} V_k$ is the decomposition in isotypic components with respect to the action of μ, with $\mu(t)v = t^k v$ for $t \in \mathbb{G}_m$ and $v \in V_k$. Then $\mathbf{P}_\mu$ (resp. $\mathbf{P}_\mu^{\mathrm{std}}$) is the parabolic subgroup stabilizing $\mathrm{Fil}_\bullet(\mu)$ (resp. $\mathrm{Fil}^\bullet(\mu)$). We let $\mathbf{M}_\mu$ be the centralizer of μ in $\mathbf{G}_L$.

The Hodge filtration $\mathrm{Fil}^\bullet$ of the vector bundle V_{dR} defines a right $\mathbf{P}_\mu^{\mathrm{std}}$-torsor $\mathbf{P}_{\mu,\mathrm{dR}}^{\mathrm{std}}$, we let $\mathbf{M}_{\mu,\mathrm{dR}}$ denote its pushforward by the map $\mathbf{P}_\mu^{\mathrm{std}} \to \mathbf{M}_\mu$. It will be convenient to consider $\mathbf{M}_{\mu,\mathrm{dR}}$ as a left $\mathbf{M}_\mu$-torsor by redefining the action $mx := xm^{-1}$ for $(m,x) \in \mathbf{M}_\mu \times \mathbf{M}_{\mu,\mathrm{dR}}$. Thanks to the p-adic Riemann-Hilbert correspondence, we can define an ascending Hodge-Tate filtration for $V_{\text{ét}} \otimes \widehat{\mathscr{O}}$, see Part IV §22.1.1. The graded pieces of the Hodge filtration and the Hodge-Tate filtration are related by the formula

$$\mathrm{gr}_i(V_{\text{ét}} \otimes \widehat{\mathscr{O}}) = \mathrm{gr}^i V_{\mathrm{dR}} \otimes \widehat{\mathscr{O}}(-i). \tag{0.2.3}$$

Let $\mathrm{FL} = \mathbf{P}_\mu \backslash \mathbf{G}_L$ and $\mathrm{FL}^{\mathrm{std}} = \mathbf{P}_\mu^{\mathrm{std}} \backslash \mathbf{G}_L$ be the flag varieties, and denote by $\mathscr{F}\ell$ and $\mathscr{F}\ell^{\mathrm{std}}$ their analytification to adic spaces over $\mathrm{Spa}(L, \mathcal{O}_L)$. The Hodge-Tate filtration defines a $\mathbf{G}(\mathbb{Q}_p)$-equivariant map of diamonds over $\mathrm{Spd}\, L$ (c.f. [CS17])

$$\pi_{\mathrm{HT}}^\diamond : \mathcal{S}h_{K^p,L}^\diamond \to \mathscr{F}\ell^\diamond,$$

inducing a morphism of ringed topoi $\pi_{\mathrm{HT}} : (\mathcal{S}h_{K^p,L,\text{proét}}, \widehat{\mathscr{O}}) \to (\mathscr{F}\ell_{\mathrm{an}}, \mathscr{O}_{\mathscr{F}\ell})$. Therefore, similarly as in the case of the modular curve, the map π_{HT} pullbacks $\mathbf{G}$-equivariant quasi-coherent sheaves over $\mathscr{F}\ell$ to $\mathbf{G}(\mathbb{Q}_p)$-equivariant $\widehat{\mathscr{O}}$-modules over $\mathcal{S}h_{K^p,L}$. We will identify the pullback $\pi_{\mathrm{HT}}^*(\mathcal{V})$ of a $\mathbf{G}$-equivariant quasi-coherent sheaf $\mathcal{V}$ with the proétale $\widehat{\mathscr{O}}$-module over $\mathcal{S}h_{K^pK_p,L}$ obtained by descending $\pi_{\mathrm{HT}}^*(\mathcal{V})$ along the K_p-torsor $\pi_{K_p} : \mathcal{S}h_{K^p,L} \to \mathcal{S}h_{K^pK_p,L}$.

Let $\mathbf{N}_\mu \subset \mathbf{P}_\mu$ be the unipotent radical and $\mathbf{M}_{\mu,\mathscr{F}\ell} = \mathbf{N}_\mu \backslash \mathbf{G}_L$ the natural $\mathbf{M}_\mu$-torsor over $\mathscr{F}\ell$. The equation (0.2.3) provides an isomorphism of $\mathbf{G}(\mathbb{Q}_p)$-equivariant tosors over $\mathcal{S}h_{K^p,L}$ (see Part IV Corollary 22.1.5)

$$\pi_{\mathrm{HT}}^*(\mathbf{M}_{\mu,\mathscr{F}\ell}) = \pi_{K_p}^*(\mathbf{M}_{\mu,\mathrm{dR}}) \times^\mu \mathbb{Z}_p(-1)^\times. \tag{0.2.4}$$

Let us explain what this isomorphism means in terms of automorphic sheaves. Let $\mathbf{B} \subset \mathbf{P}_\mu$ be a Borel subgroup and $\mathbf{T} \subset \mathbf{B}$ a maximal torus contained in $\mathbf{M}_\mu$, let $\mathbf{B}_{\mathbf{M}_\mu} = \mathbf{M}_\mu \cap \mathbf{B}$ be the Borel subgroup of $\mathbf{M}_\mu$. We let $X^*(\mathbf{T})^+$ and $X^*(\mathbf{T})^+_{\mathbf{M}_\mu}$ denote the cone of dominant characters for $\mathbf{G}_L$ and $\mathbf{M}_\mu$ respectively. Given $\lambda \in X^*(\mathbf{T})^+$ (resp. $\kappa \in X^*(\mathbf{M}_\mu)^+$) we let V_λ (resp. W_κ) denote the irreducible representation of $\mathbf{G}_L$ of highest weight λ (resp. the irreducible representation of $\mathbf{M}_\mu$ of highest weight κ). We let W and $W_{\mathbf{M}_\mu}$ denote the Weyl groups of $\mathbf{G}_L$ and $\mathbf{M}_\mu$ respectively, and let $w_0 \in W$ and $w_{0,\mathbf{M}_\mu} \in W_{\mathbf{M}_\mu}$ denote the longest elements.

Definition 0.2.17. Let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}_\mu}$.

1. We define the $\mathbf{G}$-equivariant vector bundle of weight κ over $\mathscr{F}\ell$ to be $\mathcal{W}(\kappa) = \mathbf{G}_L \times^{\mathbf{P}_\mu} W^\vee_{-w_0(\kappa)}$. Equivalently, denote $k : \mathbf{M}_{\mu,\mathscr{F}\ell} \to \mathscr{F}\ell$ and let $\star_1$ be the left regular action of $\mathbf{M}_\mu$ on $k_*(\mathscr{O}_{\mathbf{M}_{\mu,\mathscr{F}\ell}})$, then
$$\mathcal{W}(\kappa) = k_*(\mathscr{O}_{\mathbf{M}_{\mu,\mathscr{F}\ell}})[-w_0(\kappa)_\star]$$
where we take isotypic parts with respect to $\mathbf{B}_{\mathbf{M}_\mu}$.

2. We define the automorphic vector bundle of weight κ over $\mathcal{S}h_{K^pK_p,L}$ to be $\mathcal{M}(\kappa) = \mathbf{M}_{\mu,\mathrm{dR}} \times^{\mathbf{M}_\mu} W^\vee_{-w_0(\kappa)}$. Equivalently, let $f : \mathbf{M}_{\mu,\mathrm{dR}} \to \mathcal{S}h_{K^pK_p,L}$, then
$$\mathcal{M}(\kappa) = f_*(\mathscr{O}_{\mathbf{M}_{\mu,\mathrm{dR}}})[-w_0(\kappa)_{\star_1}].$$

Thus, the equation (0.2.4) says that

$$\pi_{\mathrm{HT}}^*(\mathcal{W}(\kappa)) = \mathcal{M}(\kappa) \otimes \widehat{\mathscr{O}}(w_0(\kappa)(\mu)). \tag{0.2.5}$$

Remark 0.2.18. The convention on $\mathcal{W}(\kappa)$ is made in such a way that, if $\lambda \in X^*(\mathbf{T})^+$, the global sections $\Gamma(\mathscr{F}\ell, \mathcal{W}(\lambda))$ are isomorphic to V_λ. Thus, π_{HT} preserves the weights between $\mathbf{G}$-equivariant vector bundles of $\mathscr{F}\ell$ and automorphic vector bundles. In the notation of [BP20] we have $\mathcal{M}(\kappa) = \mathcal{V}^\vee_{-w_0(\kappa)}$.

Our strategy to give a local proof of the classical Hodge-Tate decompositions of the proétale cohomology of the local systems $V_{\lambda,\text{ét}}$ (see. [FC90]), is first to compute the pullback by π_{HT} of all the $\mathbf{G}$-equivariant vector bundles over $\mathscr{F}\ell$, and then apply this construction to the dual BGG resolution.

Recall that the category of $\mathbf{G}$-equivariant vector bundles over $\mathscr{F}\ell$ is equivalent to the category $\mathrm{Rep}_L(\mathbf{P}_\mu)$ of algebraic representations of $\mathbf{P}_\mu$. By (0.2.5), we already know how the semisimple representations of $\mathbf{P}_\mu$ are transformed via π_{HT}, it is left to understand the pullback of non-semisimple representations. Let $\mathscr{O}(\mathbf{N}_\mu)$ be the ring of algebraic functions of $\mathbf{N}_\mu$, it is endowed with an action of $\mathbf{M}_\mu$ by conjugation, and with the right regular action of $\mathbf{N}_\mu$. This provides an action of $\mathbf{P}_\mu$ on $\mathscr{O}(\mathbf{N}_\mu)$. Let $\mathscr{O}(\mathbf{P}_\mu)$ be the ring of algebraic functions of $\mathbf{P}_\mu$ endowed with the right regular action, we have an isomorphism as $\mathbf{P}_\mu$-modules

$$\mathscr{O}(\mathbf{P}_\mu) = \bigoplus_{\lambda\in X^*(\mathbf{T})^+_{\mathbf{M}_\mu}} (W_\lambda^\vee)_0 \otimes W_\lambda \otimes \mathscr{O}(\mathbf{N}_\mu),$$

where $\mathbf{P}_\mu$ acts trivially on $(W_\lambda^\vee)_0$. Therefore, it suffices to compute the pullback by π_{HT} of the $\mathbf{G}$-equivariant sheaf associated to $\mathscr{O}(\mathbf{N}_\mu)$. We have the following theorem

Theorem 0.2.19 (Part IV Theo. 22.2.2)**.** *Let $\underline{\mathscr{O}(\mathbf{N}_\mu)}$ be the $\mathbf{G}$-equivariant quasi-coherent sheaf over $\mathscr{F}\ell$ defined by $\mathscr{O}(\mathbf{N}_\mu)$, and let $\mathscr{O}(\mathbf{N}_\mu)^{\leq 1} \subset \mathscr{O}(\mathbf{N}_\mu)$ denote the subrepresentation of polynomials of degree ≤ 1. Let $\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR}}$ be the Faltings extension of $\mathcal{S}h_{K^pK_p,L}$, and $\mathscr{O}\mathbb{C} = \mathrm{gr}^0\mathscr{O}\mathbb{B}_{\mathrm{dR}}$ the Hodge-Tate period sheaf. We have a short exact sequence of $\mathbf{P}_\mu$-representations*

$$0 \to L \to \mathscr{O}(\mathbf{N}_\mu)^{\leq 1} \to (\mathrm{Lie}\,\mathbf{N}_\mu)^\vee \to 0, \tag{0.2.6}$$

where we see $(\mathrm{Lie}\,\mathbf{N}_\mu)^\vee$ as homogeneous linear forms of $\mathbf{N}_\mu$. Then, the pullback by π_{HT} of the $\mathbf{G}$-equivariant complex associated to (0.2.6) is isomorphic to (minus) the Faltings extension

$$0 \to \widehat{\mathscr{O}} \to \mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR}} \otimes \widehat{\mathscr{O}}(-1) \to \Omega^1 \otimes \widehat{\mathscr{O}}(-1) \to 0,$$

where Ω^1 is the sheaf of differentials of $\mathcal{S}h_{K^pK_p}$. Taking n-th symmetric powers and colimits as $n \to \infty$ we have a natural isomorphism

$$\pi_{\mathrm{HT}}^*(\underline{\mathscr{O}(\mathbf{N}_\mu)}) = \mathscr{O}\mathbb{C}.$$

Having understanding the pullbacks of $\mathbf{G}$-equivariant sheaves over $\mathscr{F}\ell$ by π_{HT}, we can prove the BGG decomposition for the local systems $V_{\lambda,\text{ét}}$. Let us briefly introduce the dual BGG resolution, we let $\mathfrak{g}$, $\mathfrak{p}$, $\overline{\mathfrak{p}}$ and $\mathfrak{m}$ denote the Lie algebras of $\mathbf{G}_L$, $\mathbf{P}_\mu$, $\mathbf{P}_\mu^{\mathrm{std}}$ and $\mathbf{M}_\mu$ respectively.

Definition 0.2.20. Let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}_\mu}$ be a dominant weight for the Levi subgroup. The Verma module of $(\mathfrak{g}, \overline{\mathfrak{p}})$ of weight κ is the coinduction

$$\mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(\kappa) = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{p}})} W_\kappa.$$

We let $(\mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(\kappa))^\vee$ denote the admissible dual of $\mathfrak{m}$-finite vectors.

Remark 0.2.21. As in the case of GL_2, the Verma module $\mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(\kappa)$ is an object in the relative category $\mathcal{O}^{\overline{\mathfrak{p}}}$, cf. Part IV Definition 21.3.9. Its admissible dual is an object in the category $\mathcal{O}^{\mathfrak{p}}$, in particular, the action of $\mathfrak{p}$ integrates to an Ind-algebraic action of $\mathbf{P}_\mu$. As $\mathbf{P}_\mu$-modules we have

$$(\mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(\kappa))^\vee = W_\kappa^\vee \otimes \mathscr{O}(\mathbf{N}_\mu) = W_{-w_{0,\mathbf{M}}(\kappa)} \otimes \mathscr{O}(\mathbf{N}_\mu).$$

Definition 0.2.22. Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight for $\mathbf{G}_L$. Let $\mathrm{BGG}(-w_0(\lambda))$ be the BGG resolution of $V_\lambda^\vee$ in the category $\mathcal{O}^{\overline{\mathfrak{p}}}$

$$0 \to \mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(-w_{0,\mathbf{M}_\mu}(w_0^{\mathbf{M}_\mu}\cdot\lambda)) \to \cdots \bigoplus_{\substack{w\in{}^{\mathbf{M}_\mu}W\\ \ell(w)=k}} \mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(-w_{0,\mathbf{M}_\mu}(w\cdot\lambda)) \to \cdots \to \mathrm{Ver}^{\mathfrak{g}}_{\overline{\mathfrak{p}}}(-w_{0,\mathbf{M}_\mu}(\lambda)) \to V_\lambda^\vee \to 0,$$

where

1. ${}^{\mathbf{M}_\mu}W$ is the set of minimal length representatives of $W_{\mathbf{M}_{mu}}\backslash W$, and ℓ is the length function of W.
2. Let ρ be the sum of the positive roots with respect to the Borel $\mathbf{B}$, then $w\cdot\kappa = w(\kappa+\rho)-\rho$ is the dot action.
3. $w_0^{\mathbf{M}} \in {}^{\mathbf{M}_\mu}W$ is the longest element.

The dual BGG resolution of V_λ is the admissible dual $\mathrm{BGG}^\vee(-w_0(\lambda))$ of the BGG resolution of $V_\lambda^\vee$.

Remark 0.2.23. In Part IV Corollary 21.3.22 we show that the dual BGG resolution of V_λ is isomorphic as $\mathbf{P}_\mu$-module to the following complex concentrated in degrees $[0,d]$, with $d=\dim\mathscr{F}\ell=\dim\mathcal{S}h$:

$$\mathrm{BGG}^\vee(-w_0(\lambda)) = [W_\lambda\otimes\mathscr{O}(\mathbf{N}_\mu)\to\cdots\to\bigoplus_{\substack{w\in{}^{\mathbf{M}_\mu}W\\ \ell(w)=k}} W_{w\cdot\lambda}\otimes\mathscr{O}(\mathbf{N}_\mu)\to\cdots\to W_{w_0^{\mathbf{M}_\mu}\cdot\lambda}\otimes\mathscr{O}(\mathbf{N}_\mu).$$

We can finally state the main theorem of this section

Theorem 0.2.24 (Part. IV Theo. 22.2.6)**.** *Let $\lambda\in X^*(\mathbf{T})^+$ be a dominant weight. Let* $\mathrm{BGG}^\vee(-w_0(\lambda))$ *be the dual* BGG *resolution of V_λ and* $\underline{\mathrm{BGG}^\vee(-w_0(\lambda))}$ *its associated* $\mathbf{G}$*-equivariant complex over $\mathscr{F}\ell$. The following hold*

1. *The Hodge-Tate dual* BGG *complex $\pi_{\mathrm{HT}}^*(\underline{\mathrm{BGG}^\vee(-w_0(\lambda))})$ is a resolution of $V_{\lambda,\text{ét}}\otimes\widehat{\mathscr{O}}$, whose k-th term is isomorphic to*
$$\bigoplus_{\substack{w\in{}^{\mathbf{M}_\mu}W\\ \ell(w)=k}} \mathcal{M}^\vee(-w_0(w\cdot\kappa))\otimes\mathscr{O}\mathbb{C}(w\cdot\lambda(\mu)).$$
2. *Let $\eta_{K_p}:\mathcal{S}h_{K^pK_p,C,\text{proét}}\to\mathcal{S}h_{K^pK_p,C,\text{an}}$ be the projection of sites. Then*
$$R\eta_{K_p,*}(V_{\lambda,\text{ét}}\otimes\widehat{\mathscr{O}}) = \bigoplus_{w\in{}^{\mathbf{M}_\mu}W}\mathcal{M}^\vee(-w_0(w\cdot\lambda))\otimes C(w\cdot\lambda(\mu))[-\ell(w)].$$
Taking analytic cohomology and applying the primitive comparison theorem we obtain the Hodge-Tate BGG *decomposition*
$$H^\bullet_{\text{proét}}(\mathcal{S}h_{K^pK_p,C},V_{\lambda,\text{ét}})\otimes C = \bigoplus_{w\in{}^{\mathbf{M}_\mu}W} H^{\bullet-\ell(w)}_{\text{an}}(\mathcal{S}h_{K^pK_p,C},\mathcal{M}^\vee(-w_0(w\cdot\lambda)))\otimes C(w\cdot\lambda(\mu)). \quad (0.2.7)$$

0.2.4. The geometric Sen operator of Shimura varieties and the overconvergent BGG maps

Using Theorem 0.2.19 one can compute the Sen operator of the Shimura varieties $\mathcal{S}h_{K^pK_p,L}$. Let us denote by $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{n}$, $\mathfrak{m}$, $\mathfrak{b}$, $\mathfrak{b}_{\mathbf{M}_\mu}$ and $\mathfrak{h}$ the Lie algebras of $\mathbf{G}_L$, $\mathbf{P}_\mu$, $\mathbf{N}_\mu$, $\mathbf{M}_\mu$, $\mathbf{B}$, $\mathbf{B}_{\mathbf{M}_\mu}$ and $\mathbf{T}$ respectively. We let $\mathfrak{n}^0$ and $\mathfrak{p}^0$ be the subbundles of $\mathscr{O}_{\mathscr{F}\ell}\otimes\mathfrak{g}$ given by $\mathfrak{n}^0=\mathbf{G}_L\times^{\mathbf{P}_\mu}\mathfrak{n}$ and $\mathfrak{p}^0=\mathbf{G}_L\times^{\mathbf{P}_\mu}\mathfrak{p}$. Equivalently, $\mathfrak{n}^0$ and $\mathfrak{p}^0$ are the subbundles whose fiber at $x\in\mathscr{F}\ell$ is equal to $\mathrm{Lie}\,\tilde{x}^{-1}\mathfrak{n}\tilde{x}$ and $\tilde{x}^{-1}\mathfrak{p}\tilde{x}$ respectively, with $\tilde{x}\in\mathbf{G}_L$ any lift of x. The lie algebra $\mathfrak{g}$ acts on $\mathscr{O}_{\mathscr{F}\ell}$ by derivations, this induces an action of the Lie algebroid $\mathscr{O}_{\mathscr{F}\ell}\otimes\mathfrak{g}$. Since $\mathrm{FL}=\mathbf{P}_\mu\backslash\mathbf{G}_L$ one easily shows that $\mathfrak{p}^0\subset\mathscr{O}_{\mathscr{F}\ell}\otimes\mathfrak{g}$ acts trivially on $\mathscr{O}_{\mathscr{F}\ell}$.

Let $\mathcal{S}h_{K^p,L}=\varprojlim_{K_p}\mathcal{S}h_{K^pK_p,L}$ be the infinite-at-p level Shimura variety, it is a proétale K_p-torsor of $\mathcal{S}h_{K^pK_p,L}$. Theorem 0.2.14 says that the Sen operator of the torsor $\mathcal{S}h_{K^p,L}$ is given by a map of proétale sheaves
$$\theta:\widehat{\mathscr{O}}\to\mathfrak{g}_{\text{ét}}\otimes\Omega^1\otimes\widehat{\mathscr{O}}(-1),$$
or equivalently,
$$\theta:\Omega^{1,\vee}\otimes\widehat{\mathscr{O}}(1)\to\mathfrak{g}_{\text{ét}}\otimes\widehat{\mathscr{O}}.$$
The following is a consequence of Theorem 0.2.19.

Theorem 0.2.25 (Part IV Theo. 22.2.1)**.** *The Sen operator θ of the K_p-torsor $\pi_{K_p} : \mathcal{S}h_{K^pK_p,L} \to \mathcal{S}h_{K^pK_p,L}$ is isomorphic to the pullback by π_{HT} of the morphism of $\mathbf{G}$-equivariant vector bundles $\mathfrak{n}^0 \subset \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}$. In other words, the Sen operator $\theta : \Omega^{1,\vee} \otimes \widehat{\mathscr{O}}(1) \to \mathfrak{g}_{\text{ét}} \otimes \widehat{\mathscr{O}}$ factors through*

$$\begin{array}{ccc} \Omega^{1,\vee} \otimes \widehat{\mathscr{O}}(1) & \xrightarrow{\theta} & \mathfrak{g}_{\text{ét}} \otimes \widehat{\mathscr{O}} \\ & \searrow & \uparrow \\ & & \pi_{\mathrm{HT}}^*(\mathfrak{n}^0), \end{array}$$

and the map $\Omega^{1,\vee} \otimes \widehat{\mathscr{O}}(1) \to \pi_{\mathrm{HT}}^*(\mathfrak{n}^0)$ is an isomorphism of proétale sheaves.

The previous theorem provides a way to compute proétale cohomology of $\mathcal{S}h_{K^pK_p,C}$ in terms of Lie algebra cohomology. More precisely, let V be a locally analytic representation of K_p and $V_{\text{ét}}$ the proétale sheaf over $\mathcal{S}h_{K^pK_p,L}$ it defines. Let $\eta_{K_p} : \mathcal{S}h_{K^pK_p,C,\text{proét}} \to \mathcal{S}h_{K^pK_p,C,\text{an}}$ be the projection of sites, Theorem 0.2.13 implies that

$$R\eta_{K_p,*}(V_{\text{ét}} \otimes \widehat{\mathscr{O}}) = \eta_{K_p,*}\pi_{\mathrm{HT}}^*(R\Gamma(\mathfrak{n}^0, V \otimes \mathscr{O}_{\mathscr{F}\ell})).$$

Applying this result to the sheaf $V = C^{la}(K_p, L)$ endowed with the left regular action $\star_1$, one obtains the following generalization of [Pan20, Theo. 4.4.6]

Theorem 0.2.26 (Part IV Theo. 22.3.16)**.** *Let $\mathscr{O}_{\mathcal{S}h}^{la}$ be the subsheaf of locally analytic sections of $\widehat{\mathscr{O}}|_{\mathcal{S}h_{K^p,C,\text{an}}}$ for the action of $\mathbf{G}(\mathbb{Q}_p)$. Let $R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, L)^{la} := R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, C^{la}(K_p, L)_{\text{ét}})$ be the locally analytic completed cohomology with coefficients over L. We have a $Gal_L \times \mathbf{G}(\mathbb{Q}_p)$-equivariant quasi-isomorphism*

$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, L)^{la}\widehat{\otimes}_L C = R\Gamma_{\text{an}}(\mathcal{S}h_{K^p,C}, \mathscr{O}_{\mathcal{S}h}^{la}). \tag{0.2.8}$$

Remark 0.2.27. Let us briefly justify the equality (0.2.8). We denote $C^{la}(K_p, \widehat{\mathscr{O}}) = C^{la}(K_p, L)_{\text{ét}}\widehat{\otimes}\widehat{\mathscr{O}}$. By Theorem 0.2.25 we know that $R\eta_{K_p,*}(C^{la}(K_p, \widehat{\mathscr{O}})) = \eta_{K_p,*}(R\Gamma(\mathfrak{n}^0, C^{la}(K_p, \mathscr{O}_{\mathscr{F}\ell})))$ where $\mathfrak{n}^0$ acts via the $\star_{1,3}$-action $g\star_{1,3}f(h) = gf(g^{-1}h)$. But $R\Gamma(\mathfrak{n}^0, C^{la}(K_p, \mathscr{O}_{\mathscr{F}\ell}))$ is equal to the invariants $C^{la}(K_p, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0}$ by the Poincaré lemma. This implies that $R\eta_{K_p,*}(C^{la}(K_p, \widehat{\mathscr{O}})) = \eta_{K_p,*}\pi_{\mathrm{HT}}^*(C^{la}(K_p, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0})$ is concentrated in degree 0. Taking colimits as $K_p \to 1$, one can show that the sheaf $\varinjlim_{K_p \to 1} \eta_{K_p,*}\pi_{\mathrm{HT}}^*(C^{la}(K_p, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0})$ is equal to $\mathscr{O}_{\mathcal{S}h}^{la}$. Finally, taking analytic cohomology over $\mathcal{S}h_{K^p,C}$ one recovers (0.2.8).

As a corollary one deduces a rational version of the Calegari-Emerton conjectures (cf. [Sch15] and [HJ20])

Corollary 0.2.28 (Part IV Coro. 22.3.17)**.** *The completed cohomology complex $R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \mathbb{Q}_p)$ is concentrated in degrees $[0, d]$.*

A sketch of the proof of the previous corollary is the following: first, since the completed cohomology groups are admissible representations of $\mathbf{G}(\mathbb{Q}_p)$, in order to prove vanishing for degrees $> d$, it is enough to prove the vanishing for the locally analytic completed cohomology. Then, Theorem 0.2.26 implies that the locally anlaytic completed cohomology can be computed as the analytic cohomology of some sheaf over $\mathcal{S}h_{K^p,C}$. But the topological space $|\mathcal{S}h_{K^p,C}|$ can be written as an inverse limit of noetherian spaces of dimension d (e.g. by writing $|\mathcal{S}h_{K^p,C}|$ as the inverse limit of the formal models of finite level Shimura varieties). One obtains the vanishing by Grothendieck's bound on the cohomology of noetherian spaces, c.f. [Sch15, Coro. IV. 2.2].

We finish this introduction with the definition of the overconvergent BGG maps, generalizing Theorem 0.1.29. To motivate this construction, let us write down the classical BGG decomposition independently of K_p, we need some more notation:

Definition 0.2.29. The space of algebraic functions of $\mathfrak{g}$ with coefficients in L is the colimit of the spaces of locally algebraic functions $C^{alg}(\mathfrak{g}, L) = \varinjlim_{K_p \to 1} C^{lalg}(K_p, L)$. Analogously, the space of locally analytic functions of $\mathfrak{g}$ with coefficients in L is the colimit $C^{la}(\mathfrak{g}, L) = \varinjlim_{K_p \to 1} C^{la}(K_p, L)$.

Convention. Let $(\mathscr{F}_{K_p})_{K_p}$ be a family of proétale sheaves over $\mathcal{S}h_{K^pK_p,L}$ with compatible transition maps $\mathscr{F}_{K_p}|_{\mathcal{S}h_{K^pK'_p,L}} \to \mathscr{F}_{K'_p}$ for $K'_p \subset K_p$. Let $\eta_{K_p} : \mathcal{S}h_{K^pK_p,C,\text{proét}} \to \mathcal{S}h_{K^pK_p,C,\text{an}}$ be the projection of sites. Let us write $\mathscr{F} = \varinjlim_{K_p} \mathscr{F}_{K_p}$ for a formal inductive limit of the $\mathscr{F}_{K_p}$, we define

$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \mathscr{F}) := \varinjlim_{K_p \to 1} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \mathscr{F}_{K_p}) \text{ and } R\eta_{\infty,*}(\mathscr{F}) := \varinjlim_{K_p} R\eta_{K_p,*}(\mathscr{F}_{K_p}),$$

where the last is a sheaf in the analytic site of $\mathcal{S}h_{K^p,C}$.

Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight, it is not hard to check that $V_\lambda = C^{alg}(\mathfrak{g}, L)^{\mathfrak{b}_{\star_2}=-w_0(\lambda)}$ as $\mathfrak{g}$-module via the left regular action $\star_1$, compare with equation (0.1.7). Hence, we have that

$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, C^{alg}(\mathfrak{g}, L)^{\mathfrak{b}_{\star_2}=-w_0(\lambda)}_{\text{ét}}) = \varinjlim_{K_p} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, V_{\lambda,\text{ét}}).$$

Thus, in order to interpolate the BGG decomposition (0.2.7), we have to study the isotypic parts of the locally analytic completed cohomology for the action of the Borel subalgebra $\mathfrak{b}$. Let $\lambda : \mathfrak{h} \to C$ be a (not necessarily algebraic) character, by applying the primitive comparison theorem and projecting to the analytic site, one finds that

$$\begin{aligned} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, L)^{la,R\mathfrak{b}=-w_0(\lambda)}\widehat{\otimes}C &:= R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, C^{la}(\mathfrak{g}, L)^{\mathfrak{b}_{\star_2}=-w_0(\lambda)}_{\text{ét}})\widehat{\otimes}C \\ &= R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, C^{la}(\mathfrak{g}, \widehat{\mathscr{O}})^{\mathfrak{b}_{\star_2}=-w_0(\lambda)}) \\ &= R\Gamma_{\text{an}}(\mathcal{S}h_{K^p,C}, R\eta_{\infty,*}C^{la}(\mathfrak{g}, \widehat{\mathscr{O}})^{\mathfrak{b}_{\star_2}=-w_0(\lambda)}). \end{aligned}$$

Theorems 0.2.13 and 0.2.25 allow us to compute the projection

$$R\eta_{\infty,*}C^{la}(\mathfrak{g}, \widehat{\mathscr{O}})^{\mathfrak{b}_{\star_2}=-w_0(\lambda)} = \eta_{\infty,*}\pi^*_{\text{HT}}(R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{b}_{\star_2}=-w_0(\lambda)})).$$

This suggests that the overconvergent BGG maps should appear naturally by studying the locally anlaytic sheaf $R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{b}_{\star_2}=-w_0(\lambda)})$ over $\mathscr{F}\ell$. This is indeed the case, and the maps are parametrized by the Bruhat stratification of $\mathscr{F}\ell$.

Let $w \in {}^{\mathbf{M}_\mu}W$ and let $C_w = \mathbf{P}_\mu \backslash \mathbf{P}_\mu w \mathbf{B}$ be the w-Bruhat cell of $\mathscr{F}\ell$. For $i = 0, \dots, d = \dim \mathcal{S}h = \dim \mathscr{F}\ell$ denote

$$Y_i = \bigsqcup_{\substack{w \in {}^{\mathbf{M}_\mu}W \\ \ell(w) \geq i}} C_w.$$

We have an open filtration of $\mathscr{F}\ell$: $\emptyset \subset Y_d \subset Y_{d-1} \subset \cdots Y_0 = \mathscr{F}\ell$ with graded pieces $Y_i \backslash Y_{i+1} = \bigsqcup_{\substack{w \in {}^{\mathbf{M}_\mu}W \\ \ell(w)=i}} C_w$.

Definition 0.2.30 (Part IV §18)**.** For $w \in {}^{\mathbf{M}_\mu}W$ we let $j_w : \pi^{-1}_{\text{HT}}(C_w) \subset \mathcal{S}h_{K^p,C}$ be the inclusion. Let $\mathscr{F}$ be a sheaf over $\mathcal{S}h_{K^p,C}$, we define the following cohomology with compact supports

$$R\Gamma_{c,w}(\mathcal{S}h_{K^p,C}, \mathscr{F}) := R\Gamma_{\text{an}}(\mathcal{S}h_{K^p,C}, j_{w,!}j_w^{-1}\mathscr{F}).$$

We have the following theorem

Theorem 0.2.31 (Part IV Theo. 23.2.1)**.** *Consider the open filtration* $\emptyset \subset \pi^{-1}_{\text{HT}}(Y_d) \subset \cdots \subset \pi^{-1}_{\text{HT}}(Y_0) = \mathcal{S}h_{K^p,C}$.

1. *We have*
$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, L)^{la,R\mathfrak{b}=-w_0(\lambda)}\widehat{\otimes}C = R\Gamma_{\text{an}}(\mathcal{S}h_{K^p,C}, \mathscr{O}_{\mathcal{S}h}^{la,R\mathfrak{b}=-w_0(\lambda)}).$$

2. *For $w \in {}^{\mathbf{M}_\mu}W$ the complex $j_w^{-1}(\mathscr{O}_{\mathcal{S}h}^{la,R\mathfrak{b}=-w_0(\lambda)})$ is concentrated in degrees $[0, d-\ell(w)]$.*

3. *Let $\iota(w) = w_{0,\mathbf{M}}ww_0$ be the involution of ${}^{\mathbf{M}_\mu}W$ so that $\ell(\iota(w)) = d - \ell(w)$. There are surjective highest weight vector maps*
$$\Upsilon_{\iota(w)} : j_{\iota(w)}^{-1}(\mathscr{O}_{\mathcal{S}h}^{la,R\mathfrak{b}=-w_0(\lambda)}) \to C(\mathcal{M}^\dagger_{\iota(w),\mathrm{dR}})^{\mathfrak{b}_{\mathbf{M}_\mu},\star_1=-w_{0,\mathbf{M}}(w\cdot\lambda)}[-\ell(w)],$$
where $C(\mathcal{M}^\dagger_{\iota(w),\mathrm{dR}})$ is an overconvergent automorphic sheaf endowed with an horizontal left regular action $\star_1$ of $\mathfrak{m}$.

4. *There is an spectral sequence*
$$E_1^{p,q} = \bigoplus_{\substack{w\in{}^{\mathbf{M}_\mu}W\\ \ell(w)=p}} H^{p+q}_{c,w}(\mathcal{S}h_{K^p,C}, \mathscr{O}_{\mathcal{S}h}^{la,R\mathfrak{b}=-w_0(\lambda)}) \Rightarrow H^{p+q}(R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, L)^{la,R\mathfrak{b}=-w_0(\lambda)}\widehat{\otimes}C),$$
and overconvergent BGG *maps induced by $\Upsilon_{\iota(w)}$*
$$\mathrm{BGG}_{\iota(w)} : R\Gamma_{c,\iota(w)}(\mathcal{S}h_{K^p,C}, \mathscr{O}_{\mathcal{S}h}^{la,R\mathfrak{b}=-w_0(\lambda)}) \to R\Gamma_{c,\iota(w)}(\mathcal{S}h_{K^p,C}, C(\mathcal{M}^\dagger_{\iota(w),\mathrm{dR}})^{\mathfrak{b}_{\mathbf{M}_\mu},\star_1=-w_{0,\mathbf{M}}(w\cdot\lambda)})[-\ell(w)].$$

Many other interesting properties can be said about the locally analytic completed cohomology. For example, it admits an arithmetic Sen operator which is given by the horizontal action of $\theta_\mu = \mathrm{Lie}\,\mu$ (Part IV Theo. 22.3.18). We expect that the BGG maps are the only terms contributing to the $\mathfrak{b} = -w_0(\lambda)$ isotypic part of the locally analytic completed cohomology for generic λ. Furthermore, we expect that if λ is algebraic, the cohomologies $R\Gamma_{c,\iota(w)}(\mathcal{S}h_{K^p,C}, \mathscr{O}_{\mathcal{S}h}^{la,R\mathfrak{b}=-w_0(\lambda)})$ can be completely described in terms of overconvergent automorphic forms. The idea behind is that the complex $R\Gamma(\mathfrak{n}^0, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{b}\star_2=-w_0(\lambda)})$ has an additional horizontal action by the centralizer $\mathcal{Z}(\mathfrak{m})$ of the enveloping algebra of $\mathfrak{m}$. The action of $\mathcal{Z}(\mathfrak{m})$ is determined, via a polynomial equation, by the action of the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$, which is given by the infinitesimal character χ_λ. Then, imposing some regularity conditions on λ, we can take isotypic parts for the action of $\mathcal{Z}(\mathfrak{m})$ obtaining a more refined complex; this is still work in progress.

0.3. An overview of the thesis

This thesis is divided in four Parts, each one corresponds to a different paper of the author. Because of this, the notations might differ all across the document. Each part has its own introduction, we hope this helps the reader who is interested in a particular work addressed in this thesis.

Parts I, II and III are divided in sections, and their sections are divided in subsections. Part IV is divided in chapters, the chapters are divided in sections, and the sections are divided in subsections.

Part I is the content of the paper [RC21b], it studies the integral model of the perfecoid modular curve and its relation with completed cohomology. Part II is the submitted version of [RC21a], it concerns the overconvergent Eichler-Shimura maps of the modular curve. Part III is a joint work with Joaquín Rodrigues Jacinto [RJRC21], it develops the theory of solid locally analytic representations of a compact p-adic Lie group. Part IV is a paper in preparation, it involves geometric Sen theory of rigid spaces, and the construction of the overconvergent BGG maps for Shimura varieties.

Acknowledgements

First and foremost, I wish to extend my special thanks to my advisor Vincent Pilloni for having guided me throughout the M2 and the Ph. D. program. It is thanks to him that I was introduced to p-adic Hodge theory and the p-adic geometry of Shimura varieties, I feel grateful for the many hours and several important exchanges he shared with me during these last four years. I would like to thank George Boxer, Joaquín Rodrigues Jacinto and Damien Junger for a lot of useful conversations, and their support as colleagues and friends during the last years.

I want to thank my professors of the M2 at the ENS de Lyon for the excellent courses I had the chance to follow. I am grateful with Laurent Berger and Gabriel Dospinescu for many exchanges with respect to the last work addressed in this document concerning Sen theory and locally analytic completed cohomology. I am grateful with Valentin Hernández, Andrew Graham and Arthur-César Le Bras for many fruitful conversations and their interest in my work.

I am also indebted with my professors in the Javeriana university in Colombia for my mathematical formation during my bachelors. I want to share my special thanks to Johan Bogoya, Leonardo Chacón, Germán Combariza, Jesús Ochoa, Jorge Plazas, Andrés Vargas and Mario Velásquez. I also want to thank Guillermo Mantilla for his guidance during my passage in Los Andes University in Colombia. I am grateful with my high school professors Carlos Torres and Faber Arredondo who encouraged me to study math when I was a teenager.

Last but not least, I would like to thank my family for their love and their support during my whole life. I want to dedicate this thesis to my parents and my sister who have been the pillar in my formation as a person and mathematician. I want to thank my grandfather Alvaro and my grandmother Rubiela for their wisdom and their company since I was a child. Finally, I want to thank my partner Jessica for her love and her patience in the distance.

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Part I.

An integral model of the perfectoid modular curve

1. Introduction

Throughout this document we fix a prime number p, $\mathbb{C}_p$ the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and $\{\zeta_m\}_{m\in\mathbb{N}} \subset \mathbb{C}_p$ a compatible system of primitive roots of unity. Given a non-archimedean field K we let $\mathcal{O}_K$ denote its valuation ring. We let $\overline{\mathbb{F}}_p$ be the residue field of $\mathcal{O}_{\mathbb{C}_p}$ and $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p) \subset \mathbb{C}_p$ the ring of Witt vectors. Let $\mathbb{Z}_p^{cyc}$ and $\breve{\mathbb{Z}}_p^{cyc}$ denote the p-adic completions of the p-adic cyclotomic extensions of $\mathbb{Z}_p$ and $\breve{\mathbb{Z}}_p$ in $\mathbb{C}_p$ respectively.

Let $M \geq 1$ be an integer and $\Gamma(M) \subset \mathrm{GL}_2(\mathbb{Z})$ the principal congruence subgroup of level M. We fix $N \geq 3$ an integer prime to p. For $n \geq 0$ we denote by $Y(Np^n)/\operatorname{Spec}\mathbb{Z}_p$ the integral modular curve of level $\Gamma(Np^n)$ and $X(Np^n)$ its compactification, cf. [KM85]. We denote by $\mathfrak{X}(Np^n)$ the completion of $X(Np^n)$ along its special fiber, and by $\mathcal{X}(Np^n)$ its analytic generic fiber seen as an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, cf. [Hub96].

In [Sch15], Scholze constructed the perfectoid modular curve of tame level $\Gamma(N)$. He proved that there exists a perfectoid space $\mathcal{X}(Np^\infty)$, unique up to a unique isomorphism, satisfying the tilde limit property

$$\mathcal{X}(Np^\infty) \sim \varprojlim_n \mathcal{X}(Np^n),$$

see definition 2.4.1 of [SW13] and definition 2.4.2 of [Hub96].

The first result of this paper is the existence of a Katz-Mazur integral model of the perfectoid modular curve. More precisely, we prove the following theorem, see Section 3 for the notion of a perfectoid formal scheme

Theorem 1.0.1. *The inverse limit $\mathfrak{X}(Np^\infty) = \varprojlim_n \mathfrak{X}(Np^n)$ is a perfectoid formal scheme over $\operatorname{Spf}\mathbb{Z}_p^{cyc}$ whose analytic generic fiber is naturally isomorphic to the perfectoid modular curve $\mathcal{X}(Np^\infty)$.*

The integral perfectoid modular curve $\mathfrak{X}(Np^\infty)$ was previously constructed by Lurie in [Lur20], his method reduces the proof of perfectoidness to the ordinary locus via a mixed characteristic version of Kunz Theorem. The strategy in this paper is more elementary: we use faithfully flat descent to deduce perfectoidness of $\mathfrak{X}(Np^\infty)$ from the description of the stalks at the $\overline{\mathbb{F}}_p$-points. Then, we deal with three different kind of points:

- The ordinary points where we use the Serre-Tate parameter to explicitly compute the deformation rings, cf. [Kat81, §2].
- The cusps where we have explicit descriptions provided by the Tate curve, cf. [KM85, §8-10].
- The supersingular points where even though we do not compute explicitly the stalk, one can proves that the Frobenius map is surjective modulo p.

It worth to mention that the study of the ordinary locus in Lurie's approach and the one presented in this document are very related, see Proposition 2.2 of [Lur20] and Proposition 2.2.2 down below.

As an application of the integral model we can prove vanishing results for the coherent cohomology of the perfectoid modular curve. Let $E^{sm}/X(N)$ be the semi-abelian scheme extending the universal elliptic curve over $Y(N)$, cf. [DR73]. Let $e : X(N) \to E^{sm}$ be the unit section and $\omega_E = e^*\Omega^1_{E^{sm}/X(N)}$ the sheaf of invariant differentials. For $n \geq 0$ we denote by $\omega_{E,n}$ the pullback of ω_E to $X(Np^n)$, and $D_n \subset X(Np^n)$ the reduced cusp divisor. Let $k \in \mathbb{Z}$, we denote $\omega_{E,n}^k = \omega_{E,n}^{\otimes k}$ and $\omega_{E,n,cusp}^k = \omega_{E,n}^k(-D_n)$.

Let $\omega^k_{E,\infty}$ be the pullback of ω^k_E to $\mathfrak{X}(Np^\infty)$, and $\omega^k_{E,\infty,cusp}$ the p-adic completion of the direct limit of the cuspidal modular sheaves $\omega^k_{E,n,cusp}$. In the following we consider almost mathematics with respect to the maximal ideal of $\mathbb{Z}_p^{cyc}$.

Theorem 1.0.2. *Let $\mathcal{F} = \omega^k_{E,\infty}$ or $\omega^k_{E,\infty,cusp}$ and $\mathcal{F}^+_\mu = \mathcal{F}\widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}(Np^\infty)}}\mathscr{O}^+_{\mathcal{X}(Np^\infty)}$. There is an almost quasi-isomorphism of complexes*

$$R\Gamma_{an}(\mathcal{X}(Np^\infty), \mathcal{F}^+_\mu) \simeq^{ae} R\Gamma(\mathfrak{X}(Np^\infty), \mathcal{F}).$$

Moreover, the following holds

1. *The cohomology complex $R\Gamma(\mathfrak{X}(Np^\infty), \mathcal{F})$ is concentrated in degree 0 if $k > 0$, degree $[0, 1]$ for $k = 0$, and degree 1 if $k < 0$.*
2. *For $k \in \mathbb{Z}$ and $i, s \geq 0$, we have $\mathrm{H}^i(\mathfrak{X}(Np^\infty), \mathcal{F})/p^s = \mathrm{H}^i(\mathfrak{X}(Np^\infty), \mathcal{F}/p^s)$ and $\mathrm{H}^i(\mathfrak{X}(Np^\infty), \mathcal{F}) = \varprojlim_s \mathrm{H}^i(\mathfrak{X}(Np^\infty), \mathcal{F}/p^s)$.*
3. *The cohomology groups $\mathrm{H}^i(\mathfrak{X}(Np^\infty), \mathcal{F})$ are torsion free.*

Next, we use Serre duality and Pontryagin duality to construct a local duality theorem for the modular curves at finite level. In the limit one obtains the following theorem

Theorem 1.0.3. *Let $\mathfrak{X}_n$ be the connected component of $\mathfrak{X}(Np^n)_{\check{\mathbb{Z}}_p}$ defined as the locus where the Weil pairing of the universal basis of $E[N]$ is equal to ζ_N. We denote $\mathfrak{X}_\infty = \varprojlim_n \mathfrak{X}_n$. Let $\mathcal{F} = \omega^k_{E,\infty}$ or $\omega^k_{E,\infty,cusp}$ and $\mathcal{F}_n = \omega^k_{E,n}$ or $\omega^k_{E,n,cusp}$ respectively. There is a natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism*

$$\mathrm{Hom}_{\check{\mathbb{Z}}_p^{cyc}}(\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}), \check{\mathbb{Z}}_p^{cyc}) = \varprojlim_{n, \widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X_n, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}),$$

where the transition maps in the RHS are given by normalized traces, and $\mathcal{F}_n^\vee$ is the dual sheaf of $\mathcal{F}_n$.

Finally, we specialize to the case $\mathcal{F} = \mathscr{O}_{\mathfrak{X}_\infty}$ where the completed cohomology appears. Let X_n be a connected component of $X(Np^n)_{\check{\mathbb{Z}}_p}$ as in the previous theorem. Let $i \geq 0$ and let $\widetilde{\mathrm{H}}^i = \varprojlim_s \varinjlim_n \mathrm{H}^i_{et}(X_{n,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z})$ be the completed i-th cohomology group, where $X_{n,\mathbb{C}_p} = X_n \times_{\mathrm{Spec}\,\check{\mathbb{Z}}_p[\zeta_{p^n}]} \mathrm{Spec}\,\mathbb{C}_p$. Note that this is a slightly different version of Emerton's completed cohomology [Eme06], where one considers the étale cohomology with compact supports of $Y_{n,\mathbb{C}_p} \subset X_{n,\mathbb{C}_p}$. Nevertheless, both cohomologies are related via the open and closed immersions $Y_n \subset X_n \supset D_n$. Following the same ideas of [Sch15, §4.2] one can show that $\widetilde{\mathrm{H}}^i\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_{\mathbb{C}_p}$ is almost equal to $\mathrm{H}^i_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}^+_{\mathcal{X}_\infty})$, in particular it vanishes for $i \geq 2$ and $\widetilde{\mathrm{H}}^0 = \mathbb{Z}_p$. Using the theorem above we obtain the following result

Theorem 1.0.4. *There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant almost isomorphism of almost $\mathcal{O}_{\mathbb{C}_p}$-modules*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\widetilde{\mathrm{H}}^1\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}_{\mathbb{C}_p}, \mathcal{O}_{\mathbb{C}_p}) =^{ae} \varprojlim_{n, \widetilde{\mathrm{Tr}}_n} \mathrm{H}^0(X_{n,\mathbb{C}_p}, \omega^2_{E,\infty,cusp}).$$

The outline of the paper is the following. In Section 2 we recall the construction of the integral modular curves at finite level; they are defined as the moduli space parametrizing elliptic curves endowed with a Drinfeld basis of the torsion subgroups, we will follow [KM85]. Then, we study the deformation rings of the modular curves at $\overline{\mathbb{F}}_p$-points. For ordinary points we use the Serre-Tate parameter to describe the deformation ring at level $\Gamma(Np^n)$. We show that it represents the moduli problem parametrizing deformations of the p-divisible group $E[p^\infty]$, and a split of the connected-étale short exact sequence

$$0 \to \widehat{E} \to E[p^\infty] \to E[p^\infty]^{et} \to 0.$$

For cusps we refer to the explicit computations of [KM85, §8 and 10]. Finally, in the case of a supersingular point we prove that any element of the local deformation ring at level $\Gamma(Np^n)$ admits a p-th root modulo p at level $\Gamma(Np^{n+1})$.

In Section 3 we introduce the notion of a perfectoid formal scheme. We prove Theorem 1.0.1 reducing to the formal deformation rings at $\overline{\mathbb{F}}_p$-points via faithfully flat descent. We will say some words regarding Lurie's construction of $\mathfrak{X}(Np^\infty)$. It is worth to mention that the tame level $\Gamma(N)$ is taken only for a more clean exposition, by a result of Kedlaya-Liu about quotients of perfectoid spaces by finite group actions (Theorem 3.3.26 of [KL19]), there are integral models of any tame level.

In Section 4, we use Serre and Pontryagin duality to define a local duality pairing for the coherent cohomology of vector bundles over an lci projective curve over a finite extension of $\mathbb{Z}_p$.

In Section 5, we compute the dualizing complexes of the modular curves at finite level. We prove the cohomological vanishing of Theorem 1.0.2 and its comparison with the cohomology of the perfectoid modular curve. We prove the duality theorem at infinite level, Theorem 1.0.3, and specialize to $\mathcal{F} = \mathcal{O}_{\mathfrak{X}_\infty}$ to obtain Theorem 1.0.4.

2. A brief introduction to the Katz-Mazur integral modular curves

Let $N \geq 3$ be an integer prime to p and $n \in \mathbb{N}$. Let $\Gamma(Np^n) \subset \mathrm{GL}_2(\mathbb{Z})$ be the principal congruence subgroup of level Np^n.

2.1. Drinfeld bases

We recall the definition of a Drinfeld basis for the M-torsion of an elliptic curve

Definition 2.1.1. Let M be a positive integer, S a scheme and E an elliptic curve over S. A *Drinfeld basis* of $E[M]$ is a morphism of group schemes $\psi : (\mathbb{Z}/M\mathbb{Z})^2 \to E[M]$ such that the following equality of effective divisors holds

$$E[M] = \sum_{(a,b)\in(\mathbb{Z}/M\mathbb{Z})^2} \psi(a,b). \tag{2.1.1}$$

We also write $(P, Q) = (\psi(1,0), \psi(0,1))$ for the Drinfeld basis ψ.

Remark 2.1.2. The left-hand-side of (2.1.1) is an effective divisor of E/S being a finite flat group scheme over S. The right-hand-side is a sum of effective divisors given by the sections $\psi(a,b)$ of S to E. Furthermore, if M is invertible over S, a homomorphism $\psi : \mathbb{Z}/M\mathbb{Z} \to E[M]$ is a Drinfeld basis if and only if it is an isomorphism of group schemes, cf. [KM85, Lem. 1.5.3].

Proposition 2.1.3. *Let E/S be an elliptic curve. Let (P, Q) be a Drinfeld basis of $E[M]$ and $e_M : E[M] \times E[M] \to \mu_M$ the Weil pairing. Then $e_M(P, Q) \in \mu_M^\times(S)$ is a primitive root of unity , i.e. a root of the M-th cyclotomic polynomial.*

Proof. [KM85, Theo. 5.6.3]. □

Let $M \geq 3$. From Theorem 5.1.1 and Scholie 4.7.0 of [KM85], the moduli problem parametrizing elliptic curves E/S and Drinfeld bases (P, Q) of $E[M]$ is representable by an affine and regular curve over $\mathbb{Z}$. We denote this curve by $Y(M)$ and call it the *(affine) integral modular curve of level* $\Gamma(M)$. By an abuse of notation, we will write $Y(M)$ for its scalar extension to $\mathbb{Z}_p$.

The j-invariant is a finite flat morphism of $\mathbb{Z}_p$-schemes $j : Y(M) \longrightarrow \mathbb{A}^1_{\mathbb{Z}_p}$. The *compactified integral modular curve of level* $\Gamma(M)$, denoted by $X(M)$, is the normalization of $\mathbb{P}^1_{\mathbb{Z}_p}$ in $Y(M)$ via the j-invariant. The *cusps* or the *boundary divisor* D is the closed reduced subscheme of $X(M)$ defined by $\frac{1}{j} = 0$. The curve $X(M)$ is projective over $\mathbb{Z}_p$ and a regular scheme. We refer to $X(M)$ and $Y(M)$ simply as the modular curves of level $\Gamma(M)$.

Let $E_{univ}/Y(M)$ be the universal elliptic curve and $(P_{univ,M}, Q_{univ,M})$ the universal Drinfeld basis of $E_{univ}[M]$. Let $\Phi_M(X)$ be the M-th cyclotomic polynomial, and let $\mathbb{Z}_p[\mu_M^\times]$ denote the ring $\mathbb{Z}_p[X]/(\Phi_M(X))$. The Weil pairing of $(P_{univ,M}, Q_{univ,M})$ induces a morphism of $\mathbb{Z}_p$-schemes

$$e_M : Y(M) \to \operatorname{Spec} \mathbb{Z}_p[\mu_M^\times].$$

The map e_M extends uniquely to a map $e_M : X(M) \to \operatorname{Spec} \mathbb{Z}_p[\mu_M^\times]$ by normalization. In addition, e_M is geometrically reduced, and has geometrically connected fibers.

Taking N as in the beginning of the section, and $n \in \mathbb{N}$ varying, we construct the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X(Np^{n+1}) & \longrightarrow & X(Np^{n}) & \longrightarrow & X(Np^{n-1}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_p[\mu^{\times}_{Np^{n+1}}]) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_p[\mu^{\times}_{Np^{n}}]) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}_p[\mu^{\times}_{Np^{n-1}}]) & \longrightarrow & \cdots, \end{array}$$

the upper horizontal arrows being induced by the map

$$(P_{univ,Np^{n+1}}, Q_{univ,Np^{n+1}}) \mapsto (pP_{univ,Np^{n+1}}, pQ_{univ,Np^{n+1}}) = (P_{univ,Np^{n}}, Q_{univ,Np^{n}}),$$

and the lower horizontal arrows by the natural inclusions. In fact, the commutativity of the diagram is a consequence of the compatibility of the Weil pairing with multiplication by p

$$e_{Np^{n+1}}(pP_{univ,Np^{n+1}}, pQ_{univ,Np^{n+1}}) = e_{Np^{n}}(P_{univ,Np^{n}}, Q_{univ,Np^{n}})^p$$

cf. Theorems 5.5.7 and 5.6.3 of [KM85].

2.2. Deformation rings at $\overline{\mathbb{F}}_p$-points

Let $k = \overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Let $\{\zeta_{Np^n}\}_{n\in\mathbb{N}}$ be a fixed sequence of compatible primitive Np^n-th roots of unity, set $\zeta_{p^n} = \zeta^N_{Np^n}$. Let $\breve{\mathbb{Z}}_p = W(k)$ denote the ring of integers of the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. In the next paragraphs we will study the deformation rings of the modular curve at the closed points $X(Np^n)(k)$. We let $X(Np^n)_{\breve{\mathbb{Z}}_p}$ denote the compactified modular curve over $\breve{\mathbb{Z}}_p$ of level $\Gamma(Np^n)$. Proposition 8.6.7 of [KM85] implies that $X(Np^n)_{\breve{\mathbb{Z}}_p} = X(Np^n) \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spec}\,\breve{\mathbb{Z}}_p$.

There is an isomorphism $\breve{\mathbb{Z}}_p[\mu^{\times}_{Np^n}] \cong \prod_{k\in(\mathbb{Z}/N\mathbb{Z})^\times} \breve{\mathbb{Z}}_p[\zeta_{p^n}]$ given by fixing a primitive N-th root of unity in $\breve{\mathbb{Z}}_p$. Let $X(Np^n)^{\circ}_{\breve{\mathbb{Z}}_p}$ be the connected component of the modular curve which corresponds to the root ζ_N. In other words, $X(Np^n)^{\circ}_{\breve{\mathbb{Z}}_p}$ is the locus of $X(Np^n)_{\breve{\mathbb{Z}}_p}$ where $e_{Np^n}(P_{univ,Np^n}, Q_{univ,Np^n}) = \zeta_{Np^n}$. We denote $P^{(n)}_{univ} := NP_{univ,Np^n}$ and $Q^{(n)}_{univ} := NQ_{univ,Np^n}$.

Finally, given an elliptic curve E/S, we denote by $\widehat{E}$ the completion of E along the identity section.

The ordinary points

Let Art_k be the category of local artinian rings with residue field k, whose morphisms are the local ring homomorphisms compatible with the reduction to k. Any object in Art_k admits an unique algebra structure over $\breve{\mathbb{Z}}_p$. Let $\breve{\mathbb{Z}}_p[\zeta_{p^n}]$- Art_k denote the subcategory of Art_k of objects endowed with an algebra structure of $\breve{\mathbb{Z}}_p[\zeta_{p^n}]$ compatible with the reduction to k. Following [Kat81], we use the Serre-Tate parameter to describe the deformation rings at ordinary k-points of $X(Np^n)_{\breve{\mathbb{Z}}_p}$.

Let E_0 be an ordinary elliptic curve over k and R an object in Art_k. A *deformation of E_0 to R* is a pair (E, ι) consisting of an elliptic curve E/R and an isomorphism $\iota : E \otimes_R k \to E_0$. We define the deformation functor $\mathrm{Ell}_{E_0} : \mathrm{Art}_k \to \mathrm{Sets}$ by the rule

$$R \mapsto \{(E, \iota) : \text{ deformation of } E_0 \text{ to } R\}/\sim .$$

Then Ell_{E_0} sends an artinian ring R to the set of deformations of E_0 to R modulo isomorphism.

Let Q be a generator of the physical Tate module $T_pE_0(k) = T_p(E_0[p^\infty]^{et})$. Let $\mathbb{G}_m$ be the multiplicative group over $\breve{\mathbb{Z}}_p$ and $\widehat{\mathbb{G}}_m$ its formal completion along the identity. We have the following pro-representability theorem

Theorem 2.2.1. *[Kat81, Theo. 2.1]*

1. *The Functor* Ell_{E_0} *is pro-representable by the formal scheme*

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_pE_0(k) \otimes T_pE_0(k), \widehat{\mathbb{G}}_m).$$

The isomorphism is given by the Serre-Tate parameter q, which sends a deformation E/R *of* E_0 *to a bilinear form*

$$q(E/R; \cdot, \cdot) : T_pE_0(k) \times T_pE_0(k) \to \widehat{\mathbb{G}}_m(R).$$

By evaluating at the fixed generator Q *of* $T_pE_0(k)$*, we obtain the more explicit description*

$$\mathrm{Ell}_{E_0} = \mathrm{Spf}(\breve{\mathbb{Z}}_p[[X]])$$

where $X = q(E_{univ}/\mathrm{Ell}_{E_0}; Q, Q) - 1$.

2. *Let* E_0 *and* E_0' *be ordinary elliptic curves over k, let* $\pi_0 : E_0 \longrightarrow E_0'$ *be a homomorphism and* $\pi_0^t : E_0' \longrightarrow E_0$ *its dual. Let E and* E' *be liftings of* E_0 *and* E_0' *to R respectively. A necessary and sufficient condition for* π_0 *to lift to a homomorphism* $\pi : E \longrightarrow E'$ *is that*

$$q(E/R; \alpha, \pi^t(\beta)) = q(E'/R; \pi(\alpha), \beta)$$

for every $\alpha \in T_pE(k)$ *and* $\beta \in T_pE'(k)$.

We deduce the following proposition describing the ordinary deformation rings of finite level:

Proposition 2.2.2. *Let* $x \in X(Np^n)^{\circ}_{\breve{\mathbb{Z}}_p}(k)$ *be an ordinary point, say given by a triple* (E_0, P_0, Q_0)*, and write* $(P_0^{(n)}, Q_0^{(n)}) = (NP_0, NQ_0)$*. Let* A_x *denote the deformation ring of* $X(Np^n)^{\circ}_{\breve{\mathbb{Z}}_p}$ *at x. Then there is an isomorphism*

$$A_x \cong \breve{\mathbb{Z}}_p[\zeta_{p^n}][[X]][T]/((1+T)^{p^n} - (1+X)) = \breve{\mathbb{Z}}_p[\zeta_{p^n}][[T]] \tag{2.2.1}$$

such that:

i. *the map (2.2.1) is* $\breve{\mathbb{Z}}_p[\zeta_{p^n}]$*-linear.*

ii. *the variable* $1 + X$ *is equal to the Serre-Tate parameter* $q(E_{univ}/A_{\bar{x}}; Q, Q)$*;*

iii. *the variable* $1 + T$ *is equal to the Serre-Tate parameter* $q(E'_{univ}/A_{\bar{x}}, (\pi^t)^{-1}(Q), (\pi^t)^{-1}(Q))$ *of the universal deformation* $\pi : E_{univ} \to E'_{univ}$ *of the étale isogeny* $\pi_0 : E_0 \to E_0/C_0$*, with* $C_0 = E_0[p^n]^{et}$.

Proof. The group scheme $E_0[N]$ is finite étale over k, which implies that a deformation of (E_0, P_0, Q_0) is equivalent to a deformation of $(E_0, P_0^{(n)}, Q_0^{(n)})$. The group $\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})$ acts transitively on the set of Drinfeld bases of $E_0[p^n]$ with Weil pairing ζ_{p^n}. Without loss of generality, we can assume that $P_0^{(n)} = 0$ and that $Q_0^{(n)}$ generates $E_0[p^n](k)$, see Theorem 5.5.2 of [KM85]. Let E_{univ} denote the universal elliptic curve over A_x and $C \subset E_{univ}[p^n]$ the subgroup generated by $Q_{univ}^{(n)}$, it is an étale group lifting the étale group $C_0 = E_0[p^n]^{et}$. The base $(P_{univ}^{(n)}, Q_{univ}^{(n)})$ provides a splitting of the exact sequence

$$0 \longrightarrow \widehat{E}_{univ} \longrightarrow E_{univ}[p^n] \underset{Q_{univ}^{(n)}}{\longleftarrow} C_0 \longrightarrow 0.$$

Conversely, let R be an object in $\breve{\mathbb{Z}}_p[\zeta_{p^n}]$- Art_k and E/R a deformation of E_0. Let C be an étale subgroup of $E[p^n]$ of rank p^n. Then there exists a unique $Q^{(n)} \in C$ reducing to $Q_0^{(n)}$ modulo the maximal ideal. By Cartier duality, there is a unique $P^{(n)} \in \widehat{E}[p^n]$ such that $e(P^{(n)}, Q^{(n)}) = \zeta_{p^n}$. The pair $(P^{(n)}, Q^{(n)})$ is

then a Drinfeld basis of $E[p^n]$ lifting $(P_0^{(n)}, Q_0^{(n)})$ (cf. Proposition 1.11.2 of [KM85]). We have proved the equivalence of functors of $\breve{\mathbb{Z}}_p[\zeta_{p^n}]$- Art_k

$$\left\{\begin{array}{c}\text{Deformations } E \text{ of } E_0 \text{ and}\\ \text{Drinfeld bases of } E[p^n]\\ \text{with Weil pairing } \zeta_{p^n}\end{array}\right\}/\sim \longleftrightarrow \left\{\begin{array}{c}\text{Deformations } E \text{ of } E_0 \text{ and}\\ \text{étale subgroup } C\subset E[p^n] \text{ of rank } p^n\end{array}\right\}/\sim .$$

We also have a natural equivalence

$$\left\{\begin{array}{c}\text{Deformations } E \text{ of } E_0 \text{ and}\\ \text{étale subgroup } C\subset E[p^n] \text{ of rank } p^n\end{array}\right\}/\sim \longleftrightarrow \left\{\begin{array}{c}\text{Deformations of the étale isogeny}\\ \pi_0 : E_0 \to E_0/C_0\end{array}\right\}/\sim .$$

Let $E'_{univ}/\breve{\mathbb{Z}}_p[\zeta_{p^n}][[T]]$ denote the universal deformation of E_0/C_0. The universal étale point $Q^{(n)}_{univ}$ induces an étale isogeny of degree p^n over A_x

$$\pi : E_{univ} \to E'_{univ}$$

lifting the quotient $\pi_0 : E_0 \to E_0/C_0$. Furthermore, the dual morphism $\pi^t : E'_{univ} \to E_{univ}$ induces an isomorphism of the physical Tate modules $\pi^t : T_pE'_{univ}(k) \xrightarrow{\sim} T_pE_{univ}(k)$. Let $Q \in T_pE_{univ}(k)$ be the fixed generator, and $Q' \in T_pE'_{univ}(k)$ its inverse under π^t. Theorem 2.2.1 implies

$$q(E_{univ}; Q, Q) = q(E_{univ}; Q, \pi^t(Q')) = q(E'_{univ}; \pi(Q), Q') = q(E'_{univ}; Q', Q')^{p^n}.$$

We obtain the isomorphism

$$A_x \cong \breve{\mathbb{Z}}_p[\zeta_{p^n}][[X,T]]/((1+T)^{p^n} - (1+X)) = \breve{\mathbb{Z}}_p[\zeta_{p^n}][[T]]$$

where $X = q(E_{univ}; Q, Q) - 1$ and $T = q(E'_{univ}; Q', Q') - 1$. □

Remark 2.2.3. Let $x \in X(Np^n)(k)$ be a closed ordinary point. The special fiber of the map $e_{Np^n} : X(Np^n) \to \mathrm{Spec}\,\mathbb{Z}_p[\mu_{Np^n}]$ is a union of Igusa curves with intersections at the supersingular points [KM85, Theo. 13.10.3]. The Igusa curves are smooth over $\mathbb{F}_p$ [KM85, Theo. 12.6.1], which implies that the deformation ring of $X(Np^n)$ at x is isomorphic to a power series ring $\breve{\mathbb{Z}}_p[[T_n]]$ (cf. discussion after Remark 3.4.4 of [Wei13]). The content of the previous proposition is the explicit relation between the variables T_n in the modular tower, see also Proposition 2.2 of [Lur20].

The cusps

Let $\mathrm{Tate}(q)/\mathbb{Z}_p((q))$ be the Tate curve, we recall from [KM85, Ch. 8.8] that it has j-invariant equal to

$$1/q + 744 + \cdots .$$

We consider the ring $\mathbb{Z}_p[[q]]$ as the completed stalk of $\mathbb{P}^1_{\mathbb{Z}_p}$ at infinity. The Tate curve provides a description of the modular curve locally around the cusps, for that reason one can actually compute the formal deformation rings by means of this object, see [KM85] and [DR73]. In fact, let $\widehat{\mathrm{Cusps}}[\Gamma(Np^n)]$ be the completion of the modular curve $X(Np^n)_{\breve{\mathbb{Z}}_p}$ along the cusps. From the theory developed in [KM85, Ch. 8 and 10], more precisely Theorems 8.11.10 and 10.9.1, we deduce the following proposition:

Proposition 2.2.4. *We have an isomorphism of formal $\breve{\mathbb{Z}}_p[[q]]$-schemes*

$$\widehat{\mathrm{Cusps}}([\Gamma(Np^n)]) \xrightarrow{\sim} \bigsqcup_{\Lambda\in \mathrm{HomSurj}((\mathbb{Z}/Np^n\mathbb{Z})^2, \mathbb{Z}/Np^n\mathbb{Z})/\pm 1} \mathrm{Spf}(\breve{\mathbb{Z}}_p[\zeta_{p^n}][[q^{1/Np^n}]]).$$

The morphism $\widehat{\mathrm{Cusps}}[\Gamma(Np^{n+1})] \to \widehat{\mathrm{Cusps}}[\Gamma(Np^n)]$ is induced by the natural inclusion

$$\breve{\mathbb{Z}}_p[\zeta_{p^n}][[q^{1/Np^n}]] \to \breve{\mathbb{Z}}_p[\zeta_{p^{n+1}}][[q^{1/Np^{n+1}}]]$$

on each respective connected component.

The supersingular points

Let $(x_n \in X(Np^n)_{\breve{\mathbb{Z}}_p})_{n\in\mathbb{N}}$ be a sequence of compatible supersingular points and E_0 the elliptic curve defined over x_n. We denote by A_{x_n} the deformation ring of $X(Np^n)_{\breve{\mathbb{Z}}_p}$ at x_n. Let E_{univ}/A_{x_n} be the universal elliptic curve and $(P^{(n)}_{univ}, Q^{(n)}_{univ})$ the universal Drinfeld basis of $E_{univ}[p^n]$. We fix a formal parameter T of $\widehat{E}_{univ}$. Since x_n is supersingular, any p-power torsion point belongs to $\widehat{E}_{univ}$. We will use the following lemma as departure point:

Lemma 2.2.5. *[KM85, Theo. 5.3.2]. The maximal ideal of the local ring A_{x_n} is generated by $T(P^{(n)}_{univ})$ and $T(Q^{(n)}_{univ})$.*

By the Serre-Tate Theorem [Kat81, Theo. 1.2.1], and the general moduli theory of 1-dimensional formal groups over k [LT66], the deformation ring of $X(N)_{\breve{\mathbb{Z}}_p}$ at a supersingular point is isomorphic to $\breve{\mathbb{Z}}_p[[X]]$. Moreover, the p-multiplication modulo p can be written as $[p](T) \equiv V(T^p) \mod p$, with $V \in k[[X]][[T]]$ the Verschiebung map $V : E_0^{(p)} \to E_0$. Without loss of generality we assume that V has the form

$$V(T) = XT + \cdots u(X)T^p + \cdots,$$

with $V(T) \equiv T^p \mod X$. Using the Weierstrass Preparation Theorem we factorize $V(T)$ as

$$V(T) = T(X + \cdots \tilde{u}(X)T^{p-1})(1 + XTR(X,T)), \tag{2.2.2}$$

where $\tilde{u}(0) = 1$ and $R \in k[[X,T]]$.

Proposition 2.2.6. *The parameter X is a p-power in A_{x_1}/p. Moreover, the generators $T(P^{(n)}_{univ})$ and $T(Q^{(n)}_{univ})$ of the maximal ideal of A_{x_n} are p-powers in $A_{x_{n+1}}/p$.*

Proof. The second claim follows from the first and the equality $[p](T) \equiv V(T^p) \mod p$. Consider $n = 1$ and write $P = P^{(1)}_{univ}$ and $Q = Q^{(1)}_{univ}$. Let $F : E_0 \to E_0^{(p)}$ and $V : E_0^{(p)} \to E_0$ denote the Frobenius and Verschiebung homomorphisms respectively. Using the action of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, we can assume that P and $F(Q)$ are generators of $\ker F$ and $\ker V$ respectively (cf. Theorem 5.5.2 of [KM85]). We have the equality of divisors on $E^{(p)}_{univ}/(A_{x_n}/p)$

$$\ker V = \sum_{i=0}^{p-1} [i \cdot F(Q)]. \tag{2.2.3}$$

The choice of the formal parameter T gives a formal parameter of $E^{(p)}_{univ}$ such that $T(F(Q)) = T(Q)^p$. Therefore, from (2.2.3) we see that the roots of $V(T)/T$ are $\{[i]^{(p)}(T(Q)^p)\}_{1\le i\le p-1}$ where $[i]^{(p)}(T)$ is the i-multiplication of the formal group of $E^{(p)}_{univ}$. We obtain from (2.2.2)

$$\frac{X}{\tilde{u}(X)} = (-1)^{p-1} \prod_{i=1}^{p-1} ([i]^{(p)}(T(Q)^p)) = \left(\prod_{i=1}^{p-1} [i](T(Q)) \right)^p,$$

proving that $\frac{X}{\tilde{u}(X)}$ is a p-power in A_{x_1}/p. As $k[[X]] = k[[X/\tilde{u}(X)]]$ we are done. □

Corollary 2.2.7. *The Frobenius $\varphi : \varinjlim_n A_{x_n}/p \to \varinjlim_n A_{x_n}/p$ is surjective.*

Proof. By induction on the graded pieces of the filtration defined by the ideal $(T(P^n), T(Q^n))$, one shows that A_{x_n}/p is in the image of the Frobenius restricted to $A_{x_{n+1}}/p$. □

Remark 2.2.8. The completed local ring at a geometric supersingular point $\overline{x}$ of $X(Np^n)$ is difficult to describe. For example, its reduction modulo p is the quotient of the power series ring $k[[X, Y]]$ by some explicit principal ideal which is written in terms of the formal group law of E at $\overline{x}$ [KM85, Theo. 13.8.4]. Weinstein gives in [Wei16] an explicit description of the deformation ring at a supersingular point of the modular curve at level $\Gamma(Np^\infty)$. In fact, Weinstein finds an explicit description of the deformation ring at infinite level of the Lubin-Tate space parametrizing 1-dimensional formal O_K-modules of arbitrary height. In particular, he proves that the $\mathfrak{m}_{x_0}$-adic completion of the direct limit $\varinjlim_n A_{x_n}$ is a perfectoid ring. The Corollary 2.2.7 says that the p-adic completion of $\varinjlim_n A_{x_n}$ is perfectoid, which is a slightly stronger result.

3. Construction of the perfectoid integral model

3.1. Perfectoid Formal spaces

In this section we introduce a notion of perfectoid formal scheme which is already considered in [BMS18, Lemma 3.10], though not explicitly defined. We start with the affine pieces

Definition 3.1.1. An *integral perfectoid ring* is a topological ring R containing a non zero divisor π such that $p \in \pi^p R$, satisfying the following conditions:

i. the ring R is endowed with the π-adic topology. Moreover, it is separated and complete.

ii. the Frobenius morphism $\varphi : R/\pi R \to R/\pi^p R$ is an isomorphism.

We call π satisfying the previous conditions a *pseudo-uniformizer* of R.

Remark 3.1.2. The previous definition of integral perfectoid rings is well suited for p-adic completions of formal schemes. We do not consider the case where the underlying topology is not generated by a non-zero divisor, for example, the ring $W(\overline{\mathbb{F}}_p)[[X^{1/p^\infty}, Y^{1/p^\infty}]]$ which is the (p, X, Y)-adic completion of the ring $W(\overline{\mathbb{F}}_p)[X^{1/p^\infty}, Y^{1/p^\infty}]$. As is pointed out in Remark 3.8 of [BMS18], the notion of being integral perfectoid does not depend on the underlying topology, however to construct a formal scheme it is necessary to fix one.

Let R be an integral perfectoid ring with pseudo-uniformizer π, we attach to R the formal scheme $\operatorname{Spf} R$ defined as the π-adic completion of $\operatorname{Spec} R$. We say that $\operatorname{Spf} R$ is a perfectoid formal affine scheme. The following lemma says that the standard open subschemes of $\operatorname{Spf} R$ are perfectoid

Lemma 3.1.3. *Let $f \in R$. Then $R\langle f^{-1}\rangle = \varprojlim_n R/\pi^n[f^{-1}]$ is an integral perfectoid ring.*

Proof. Let $n, k \geq 0$, as π is not a zero divisor we have a short exact sequence

$$0 \to R/\pi^n \xrightarrow{\pi^k} R/\pi^{n+k} \to R/\pi^k \to 0.$$

Localizing at f and taking inverse limits on n we obtain

$$0 \to R\langle f^{-1}\rangle \xrightarrow{\pi^k} R\langle f^{-1}\rangle \to R/\pi^k[f^{-1}] \to 0.$$

Then $R\langle f^{-1}\rangle$ is π-adically complete and π is not a zero-divisor. On the other hand, localizing at f the Frobenius map $\varphi : R/\pi \xrightarrow{\sim} R/\pi^p$ one gets

$$\varphi : R/\pi[f^{-1}] \xrightarrow{\sim} R/\pi^p[f^{-p}] = R/\pi^p[f^{-1}]$$

which proves that $R\langle f^{-1}\rangle$ is an integral perfectoid ring. □

Definition 3.1.4. A *perfectoid formal scheme* $\mathfrak{X}$ is a formal scheme which admits an affine cover $\mathfrak{X} = \bigcup_i U_i$ by perfectoid formal affine schemes.

Let F be equal to $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, $\mathcal{O}_F$ denote the ring of integers of F and ϖ be a uniformizer of $\mathcal{O}_F$. Let $\mathfrak{Int}\text{-}\mathfrak{Perf}_{\mathcal{O}_F}$ be the category of perfectoid formal schemes over $\mathcal{O}_F$ whose structural morphism is adic, i.e. the category of perfectoid formal schemes $\mathfrak{X}/\operatorname{Spf}\mathcal{O}_F$ such that $\varpi\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathcal{O}_{\mathfrak{X}}$. Let $\mathscr{P}\mathrm{erf}_F$ be the category of perfectoid spaces over $\operatorname{Spa}(F,\mathcal{O}_F)$.

Proposition 3.1.5. *Let R be an integral perfectoid ring and π a pseudo-uniformizer. The ring $R[\frac{1}{\pi}]$ is a perfectoid ring in the sense of Fontaine [Fon13]. Furthermore, there is a unique "generic fiber" functor*

$$(-)_\eta : \mathfrak{Int}\text{-}\mathfrak{Perf}_{\mathcal{O}_F} \to Perf_F$$

extending $\operatorname{Spf} R \rightsquigarrow \operatorname{Spa}(R[\frac{1}{\varpi}], R^+)$, *where R^+ is the integral closure of R in $R[\frac{1}{\varpi}]$. Moreover, given $\mathfrak{X}$ a perfectoid formal scheme over $\mathcal{O}_F$, its generic fiber is universal for morphisms from perfectoid spaces to $\mathfrak{X}$. Namely, if $\mathcal{Y}$ is a perfectoid space and $(\mathcal{Y},\mathcal{O}_{\mathcal{Y}}^+) \to (\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ is a morphism of locally and topologically ringed spaces, then there is a unique map $\mathcal{Y} \to \mathfrak{X}_\mu$ making the following diagram commutative*

$$\begin{array}{ccc} (\mathcal{Y},\mathcal{O}_{\mathcal{Y}}^+) & \longrightarrow & (\mathfrak{X}_\mu,\mathcal{O}_{\mathfrak{X}_\mu}^+) \\ & \searrow \quad \swarrow & \\ & (\mathfrak{X},\mathcal{O}_{\mathfrak{X}}) & \end{array}$$

Remark 3.1.6. The universal property of the functor $(-)_\mu$ is Huber's characterization of the generic fiber of formal schemes in the case of perfectoid spaces, see [Hub94, Prop. 4.1].

Proof. The first statement is Lemma 3.21 of [BMS18]. For the construction of the functor, let $\mathfrak{X}$ be a perfectoid formal scheme over $\mathcal{O}_F$. One can define $\mathfrak{X}_\eta$ to be the glueing of the affinoid spaces $\operatorname{Spa}(R[\frac{1}{\varpi}],,R^+)$ for $\operatorname{Spf} R \subset \mathfrak{X}$ an open perfectoid formal afine subscheme, this is well defined after Lemma 3.1.3.

We prove the universal property of the generic fiber functor. Let $\mathcal{Y} \in \mathscr{P}\mathrm{erf}_K$ and let $f : (\mathcal{Y},\mathcal{O}_{\mathcal{Y}}^+) \to (\mathfrak{X},\mathcal{O}_{\mathfrak{X}})$ be a morphism of locally and topologically ringed spaces. First, if $\mathcal{Y} = \operatorname{Spa}(S,S^+)$ is affinoid perfectoid and $\mathfrak{X} = \operatorname{Spf} R$ is perfectoid formal affine, f is determined by the global sections map $f^* : R \to S^+$. Then, there exists a unique map of affinoid perfectoid rings $f_\eta^* : (R[\frac{1}{\varpi}],R^+) \to (S,S^+)$ extending f^*. By glueing morphisms from affinoid open subsets for a general $\mathcal{Y}$, one gets that $\mathfrak{X}_\eta := \operatorname{Spa}(R[\frac{1}{\varpi}],R^+)$ satisfies the universal property. For an arbitrary $\mathfrak{X}$, one can glue the generic fibers of the open perfectoid formal affine subschemes of $\mathfrak{X}$. □

We end this subsection with a theorem which reduces the proof of the perfectoidness of the integral modular curve at any tame level to the level $\Gamma(Np^\infty)$.

Theorem 3.1.7 (Kedlaya-Liu)**.** *Let A be a perfectoid ring on which a finite group G acts by continuous ring homomorphisms. Then the invariant subring A^G is a perfectoid ring. Moreover, if $R \subset A$ is an open integral perfectoid subring of A then R^G is an open integral perfectoid subgring of A^G.*

Proof. The first statement is Theorem 3.3.26 of [KL19]. The second statement follows from the description of open perfectoid subrings of A as p-power closed subrings of A°, i.e open subrings of A° such that $x^p \in R$ implies $x \in R$, see Corollary 2.2 of [Mor17]. □

3.2. The main construction

Let $\mathcal{X}(Np^\infty)$ denote Scholze's perfectoid modular curve [Sch15]. Let $\mathbb{Z}_p^{cyc}$ be the p-adic completion of the p-adic cyclotomic integers $\varinjlim_n \mathbb{Z}_p[\mu_{p^n}]$. Let $\mathfrak{X}(Np^n)$ be the completion of $X(Np^n)$ along its special fiber. We have the following theorem

Theorem 3.2.1. *The inverse limit* $\mathfrak{X}(Np^\infty) := \varprojlim_n \mathfrak{X}(Np^n)$ *is a p-adic perfectoid formal scheme, it admits a structural map to* $\operatorname{Spf} \mathbb{Z}_p^{cyc}[\mu_N]$*, and its generic fiber is naturally isomorphic to the perfectoid modular curve* $\mathcal{X}(Np^\infty)$*. Furthermore, let* $n \geq 0$*, let* $\operatorname{Spec} R \subset X(Np^n)$ *be an affine open subscheme,* $\operatorname{Spf} \widehat{R}$ *its p-adic completion and* $\operatorname{Spf} \widehat{R}_\infty$ *the inverse image in* $\mathfrak{X}(Np^\infty)$*. Then* $\widehat{R}_\infty = \left(\widehat{R}_\infty[\frac{1}{p}]\right)^\circ$ *and*

$$(\operatorname{Spf} \widehat{R}_\infty)_\eta = \operatorname{Spa}(\widehat{R}_\infty[\frac{1}{p}], \widehat{R}_\infty).$$

Remark 3.2.2. The previous result gives a different proof of Scholze's theorem that the generic fiber $\mathcal{X}(Np^\infty)$ is a perfectoid space by more elementary means.

Proof. The maps between the (formal) modular curves are finite and flat. Then $\mathfrak{X}(Np^\infty) := \varprojlim_n \mathfrak{X}(Np^n)$ is a flat p-adic formal scheme over $\mathbb{Z}_p$. Fix $n_0 \geq 0$, let $\operatorname{Spec} R \subset X(Np^{n_0})$ and $\operatorname{Spf} \widehat{R} \subset \mathfrak{X}(Np^{n_0})$ be as in the theorem. For $n \geq n_0$, let $\operatorname{Spec} R_n$ (resp. $\operatorname{Spf} \widehat{R}_n$) denote the inverse image of $\operatorname{Spec} R$ (resp. $\operatorname{Spf} \widehat{R}$) in $\operatorname{Spec} X(Np^n)$ (resp. $\mathfrak{X}(Np^n)$). Let $R_\infty := \varinjlim_n R_n$ and let $\widehat{R}_\infty$ be its p-adic completion.

Claim. $\widehat{R}_\infty$ is an integral perfectoid ring, equal to $(\widehat{R}_\infty[\frac{1}{p}])^\circ$.

Suppose that the claim holds, it is left to show that $\mathfrak{X}(Np^\infty)_\eta$ is the perfectoid modular curve $\mathcal{X}(Np^\infty)$. There are natural maps of locally and topologically ringed spaces

$$(\mathcal{X}(Np^n), \mathscr{O}^+_{\mathcal{X}(Np^n)}) \to (\mathfrak{X}(Np^n), \mathscr{O}_{\mathfrak{X}(Np^n)}).$$

We have $\mathcal{X}(Np^\infty) \sim \varprojlim_n \mathcal{X}(Np^n)$, where we use the notion of tilde limit [SW13, Def. 2.4.1]. Then, by p-adically completing the inverse limit of the tower, we obtain a map of locally and topologically ringed spaces

$$(\mathcal{X}(Np^\infty), \mathscr{O}^+_{\mathcal{X}(Np^\infty)}) \to (\mathfrak{X}(Np^\infty), \mathscr{O}_{\mathfrak{X}}(Np^\infty)).$$

This provides a map $f : \mathcal{X}(Np^\infty) \to (\mathfrak{X}(Np^\infty))_\eta$. Since

$$(\operatorname{Spf} \widehat{R}_\infty)_\eta = \operatorname{Spa}(\widehat{R}_\infty[1/p], \widehat{R}_\infty) \sim \varprojlim_n \operatorname{Spa}(\widehat{R}_n[1/p], \widehat{R}_n),$$

and the tilde limit is unique in the category of perfectoid spaces [SW13, Prop. 2.4.5], the map f is actually an isomorphism. □

Proof of the Claim. First, by Lemma A.2.2.3 of [Heu19] the ring $(\widehat{R}_n[\frac{1}{p}])^\circ$ is the integral closure of $\widehat{R}_n$ in its generic fiber. By Lemma 5.1.2 of [Bha17] and the fact that R_n is a regular ring one gets that $\widehat{R}_n = (\widehat{R}_n[\frac{1}{p}])^\circ$ for all $n \geq n_0$. As R_∞ is faithfully flat over R_n for all n, one easily checks that $\widehat{R}_\infty \cap \widehat{R}_n[\frac{1}{p}] = \widehat{R}_n$. Moreover, R_∞ is integrally closed in its generic fiber, and by Lemma 5.1.2 of *loc. cit.* again one obtains that $\widehat{R}_\infty$ is integrally closed in $\widehat{R}_\infty[\frac{1}{p}]$. Let $x \in \widehat{R}_n[\frac{1}{p}]$ be power bounded in $\widehat{R}_\infty[\frac{1}{p}]$, then $px^s \in \widehat{R}_\infty$ for all $s \in \mathbb{N}$, in particular $\{px^s\}_{s\in\mathbb{N}} \subset \widehat{R}_n$ which implies that $x \in \widehat{R}_n$. This shows that $\varinjlim_n \widehat{R}_n$ is dense in $(\widehat{R}_\infty[\frac{1}{p}])^\circ$, taking p-adic completions one gets $\widehat{R}_\infty = (\widehat{R}_\infty[\frac{1}{p}])^\circ$.

The Weil pairings evaluated at the universal Drinfeld basis $(P_{univ,Np^n}, Q_{univ,Np^n})$ of $E[Np^n]$ induce compatible morphisms $\mathfrak{X}(Np^n) \to \operatorname{Spf} \mathbb{Z}_p[\mu_{Np^n}]$. Taking inverse limits one gets the structural map $\mathfrak{X}(Np^\infty) \to \operatorname{Spf} \mathbb{Z}_p^{cyc}[\mu_N]$. In particular, there exists $\pi \in \widehat{R}_\infty$ such that $\pi^p = pa$ with $a \in \mathbb{Z}_p^{cyc,\times}$. To prove that $\widehat{R}_\infty$ is integral perfectoid we need to show that the absolute Frobenius map

$$\varphi : R_\infty/\pi \longrightarrow R_\infty/p$$

is an isomorphism. The strategy is to prove this fact for the completed local rings of the stalks of $\operatorname{Spec} R_\infty/p$ and use faithfully flat descent.

Injectivity is easy, it follows from the fact that R_∞ is integrally closed in $R_\infty[1/p]$. To show that φ is surjective, it is enough to prove that the absolute Frobenius is surjective after a profinite étale base change. Indeed, the relative Frobenius is an isomorphism for profinite étale base changes. Let $S = R \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$ and let $\widehat{S} = \widehat{R\otimes}_{\mathbb{Z}_p}\breve{\mathbb{Z}}_p$ be the p-adic completion of S. We use similar notation for $S_n = R_n \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p$, $\widehat{S}_n$, S_∞ and $\widehat{S}_\infty$. We have to show that the absolute Frobenius

$$\varphi : S_\infty/\pi \to S_\infty/p$$

is surjective.

Let $x = (x_{n_0}, x_{n_0+1}, \cdots, x_n, \cdots)$ be a $\overline{\mathbb{F}}_p$-point of $\operatorname{Spf}\widehat{S}_\infty$ which is an inverse limit of $\overline{\mathbb{F}}_p$-points of $\operatorname{Spf}\widehat{S}_n$. Write x_{n_0} simply by x_0. Then, it is enough to show that φ is surjective after taking the stalk at x. Let S_{n,x_n} be the localization of S_n at the prime x_n and $S_{\infty,x} = \varinjlim_n S_{n,x_n}$. Let $\widehat{S_{n,x_n}}$ be the completion of S_{n,x_n} along its maximal ideal. Recall that the ring S_n is finite flat over S, this implies that $\widehat{S_{n,x_n}} = S_{n,x_n} \otimes_{S_{x_0}} \widehat{S_{x_0}}$.

The scheme $X(Np^n)$ is of finite type over $\mathbb{Z}_p$, in particular every point has a closed point as specialization. Thus, by faithfully flat descent, we are reduced to prove that for every $\overline{\mathbb{F}}_p$-point $x \in \operatorname{Spf} S_\infty/p = \varprojlim \operatorname{Spf} S_n/p$, the $\widehat{S_{x_0}}$-base change of

$$\varphi : S_{\infty,x}/\pi \to S_{\infty,x}/p$$

is surjective (even an isomorphism). We have the following commutative diagram

$$\begin{array}{ccc} S_{\infty,x}/\pi \otimes_{S_{x_0}} \widehat{S_{x_0}} & \xrightarrow{\varphi\otimes\mathrm{id}} & S_{\infty,x}/p \otimes_{\varphi,S_{x_0}} \widehat{S_{x_0}} \\ \| & & \| \\ \varinjlim \widehat{S_{n,x_n}}/\pi & \longrightarrow & \varinjlim(\widehat{S_{n,x_n}}/p \otimes_{\varphi,\widehat{S_{x_0}}} \widehat{S_{x_0}}). \end{array}$$

The ring R_n is of finite type over $\mathbb{Z}_p$, so that the absolute Frobenius $\varphi : R_n/\pi \to R_n/p$ is finite. This implies that $\widehat{S_{n,x_n}}/p$ is a finite $\widehat{S_{x_0}}$-module via the module structure induced by the Frobenius. Then, the following composition is an isomorphism

$$\widehat{S_{n,x_n}}/p \otimes_{\varphi,\widehat{S_{x_0}}} \widehat{S_{x_0}} \to \varprojlim_m \widehat{S_{n,x_n}}/(p, \mathfrak{m}_S^{mp}) \cong \widehat{S_{n,x_n}}/p,$$

where $\mathfrak{m}_S$ is the maximal ideal of S_{x_0}. Thus, we are reduced to prove that the absolute Frobenius $\varphi : \varinjlim_n \widehat{S_{n,x_n}}/\pi \to \varinjlim_n \widehat{S_{n,x_n}}/p$ is surjective. Finally, we deal with the cusps, the supersingular and the ordinary points separately; we use the descriptions of Section 2:

- **In the ordinary case**, the local ring $\widehat{S_{n,x_n}}$ is isomorphic to $\breve{\mathbb{Z}}_p[\zeta_{p^n}][[X_n]]$. From the proof of Proposition 2.2.2, one checks that the inclusion $\widehat{S_{n,x_n}} \to \widehat{S_{n+1,x_{n+1}}}$ is given by $X_n = (1+X_{n+1})^p - 1$. Then, one obtains the surjectivity of Frobenius when reducing modulo p.
- **The supersingular case** is Corollary 2.2.7.
- Finally, if we are dealing with a **cusp** x, the ring $\widehat{S_{n,x_n}}$ is isomorphic to $\breve{\mathbb{Z}}_p[\zeta_{p^n}][[q^{1/Np^n}]]$ and $\widehat{S_{n,x_n}} \to \widehat{S_{n+1,x_{n+1}}}$ is the natural inclusion by Proposition 2.2.4. The surjectivity of φ is clear.

□

3.3. Relation with Lurie's stack

In this subsection we make more explicitly the relation between Lurie's construction of $\mathfrak{X}(Np^\infty)$ and the one presented in this document. The key result is the following theorem

Theorem 3.3.1 ([Lur20, Theo. 1.9]). *Let $\pi \in \mathbb{Z}_p[\mu_{p^2}]$ be a pseudo-uniformizer such that $\pi^p = ap$ where a is a unit. For $n \geq 3$ there exists a unique morphism $\theta : \mathfrak{X}(Np^n)/\pi \to \mathfrak{X}(Np^{n-1})/p$ making the following diagram commutative*[1]

$$\begin{array}{ccc} \mathfrak{X}(Np^n)/p & \xrightarrow{\varphi} & \mathfrak{X}(Np^n)/\pi \\ \downarrow & \swarrow_{\theta} & \downarrow \\ \mathfrak{X}(Np^{n-1})/p & \xrightarrow{\varphi} & \mathfrak{X}(Np^{n-1})/\pi \end{array}$$

where φ is the absolute Frobenius.

This theorem can be deduced from the local computations made in Section 2. Indeed, let $x_n \in \mathfrak{X}(Np^n)(\overline{\mathbb{F}}_p)$ be a $\overline{\mathbb{F}}_p$-point and $x_{n-1} \in \mathfrak{X}(Np^{n-1})(\overline{\mathbb{F}}_p)$ its image. We have proven that there exists a unique map of the deformation rings at the points x_{n-1} and x_n

$$\theta^* : \widehat{\mathscr{O}}_{\mathfrak{X}(Np^{n-1}),x_{n-1}}/p \to \widehat{\mathscr{O}}_{\mathfrak{X}(Np^n),x_n}/\pi$$

making the following diagram commutative

$$\begin{array}{ccc} \widehat{\mathscr{O}}_{\mathfrak{X}(Np^n),x_n}/p & \xleftarrow{\varphi^*} & \widehat{\mathscr{O}}_{\mathfrak{X}(Np^n),x_n}/\pi \\ \uparrow & \nearrow_{\theta^*} & \uparrow \\ \widehat{\mathscr{O}}_{\mathfrak{X}(Np^{n-1}),x_{n-1}}/p & \xleftarrow{\varphi^*} & \widehat{\mathscr{O}}_{\mathfrak{X}(Np^{n-1}),x_{n-1}}/\pi \end{array}$$

This corresponds to Propositions 2.2.2, 2.2.4 and 2.2.6 for x_n ordinary, a cusp and a supersingular point respectively. Then, one constructs θ using faithfully flat descent from the completed local rings to the localized local rings at x_n, and glueing using the uniqueness of θ^*.

[1] The assumption $n \geq 3$ is only to guarantee that $\mathscr{O}(\mathfrak{X}(Np^{n-1}))$ contains π.

4. Cohomology and local duality for curves over $\mathcal{O}_K$

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ its valuation ring. In this section we recall the Grothendieck-Serre duality theorem for local complete intersection (lci) projective curves over $\mathcal{O}_K$, we will follow [Har66]. Then, we use Pontryagin duality to define a local duality paring of coherent cohomologies.

Let X be a locally noetherian scheme and $D(X)$ the derived category of $\mathscr{O}_X$-modules. We use subscripts c, qc on $D(X)$ for the derived category of $\mathscr{O}_X$-modules with coherent and quasi-coherent cohomology, the subscript fTd refers to the subcategory of complexes with finite Tor dimension. We use superscripts $+, -, b$ for the derived category of bounded below, bounded above and bounded complexes respectively. For instance, $D^b_c(X)_{fTd}$ is the derived category of bounded complexes of $\mathscr{O}_X$-modules of finite Tor dimension and coherent cohomology. If $X = \operatorname{Spec} A$ is affine, we set $D(A) := D_{qc}(X)$, the derived category of A-modules.

Definition 4.0.1. Let $f : X \to Y$ be a morphism of schemes.

1. The map f is *embeddable* if it factors as $X \xrightarrow{\iota} S \to Y$ where ι is a finite morphism and S is smooth over Y.

2. The map f is *projectively embeddable* if it factors as composition $X \xrightarrow{\iota} \mathbb{P}^n_Y \to Y$ for some $n \geq 0$, where ι a finite morphism.

3. The map f is a *local complete intersection* if locally on Y and X it factors as $X \xrightarrow{\iota} S \to Y$, where S is a smooth Y-scheme, and ι is a closed immersion defined by a regular sequence of S. The length of the regular sequence is called the codimension of X in S.

Theorem 4.0.2 (Hartshorne)**.** *Let $f : X \to Y$ be a projectively embeddable morphism of noetherian schemes of finite Krull dimension. Then there exist an exceptional inverse image functor $f^! : \mathrm{D}(Y) \to \mathrm{D}(X)$, a trace map $\mathrm{Tr} : \mathrm{R}f_* f^! \to 1$ in $\mathrm{D}^+_{qc}(Y)$, and an adjunction*

$$\theta : Rf_* R\mathscr{H}om_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} R\mathscr{H}om_Y(Rf_*\mathcal{F}, \mathcal{G})$$

for $\mathscr{F} \in \mathrm{D}^-_{qc}(X)$ and $\mathscr{G} \in \mathrm{D}^+_{qc}(Y)$.

Moreover, the formation of the exceptional inverse image is functorial. More precisely, given a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ with f, g and gf projectively embeddable, there is a natural isomorphism $(gf)^! \cong f^! g^!$. This functor commutes with flat base change. Namely, let $u : Y' \to Y$ be a flat morphism, $f' : X' \to Y'$ the base change of X to Y' and $v : X' \to X$ the projection. Then there is a natural isomorphism of functors $v^* f^! = f'^! u^*$.

Proof. We refer to [Har66, Theo. III. 8.7] for the existence of $f^!$, its functoriality and compatibility with flat base change. See Theorems III. 10.5 and III 11.1 of *loc. cit.* for the existence of Tr and the adjunction θ respectively. □

Example 4.0.3. Let $f : X \to Y$ be a morphism of finite type of noetherian schemes of finite Krull dimension.

1. We can define the functor $f^!$ for finite morphisms as

$$f^!\mathcal{F} = f^{-1}R\mathcal{H}om_{O_Y}(f_*\mathcal{O}_X, \mathcal{F}) \text{ for } \mathcal{F} \in D(Y).$$

   The duality theorem in this case is equivalent to the (derived) $\otimes$-Hom adjunction, see [Har66, §III. 6].

2. Let f be smooth of relative dimension n, then one has $f^!\mathcal{F} = \mathcal{F} \otimes \omega^\circ_{X/Y}[n]$ where $\omega^\circ_{X/Y} = \bigwedge^n \Omega^1_{X/Y}$, see [Har66, §III.2].

Lemma 4.0.4. *Let $f : X \to Y$ be an lci morphism of relative dimension n between locally noetherian schemes of finite Krull dimension. Then $f^!\mathcal{O}_Y = \omega^\circ_{X/Y}[n]$ with $\omega^\circ_{X/Y}$ an invertible $\mathcal{O}_X$-module.*

Proof. Working locally on Y and X, we may assume that f factors as $X \xrightarrow{\iota} S \xrightarrow{g} Y$, where g is a smooth morphism of relative dimension m, and ι is a regular closed imersion of codimension $m-n$ defined by an ideal $\mathcal{I} = (f_1, \dots, f_{m-n})$. Let $\omega^\circ_{S/Y} = \bigwedge^m \Omega^1_{S/Y}$ be the sheaf of m-differentials of S over Y, then

$$\begin{aligned} f^!\mathcal{O}_Y &= \iota^! g^! \mathcal{O}_Y \\ &= \iota^{-1} R\mathcal{H}om_{\mathcal{O}_S}(\iota_*\mathcal{O}_X, \omega^\circ_{S/Y}[m]) \\ &= \iota^{-1} R\mathcal{H}om_{\mathcal{O}_S}(\mathcal{O}_S/\mathcal{I}, \omega^\circ_{S/Y})[m] \end{aligned}$$

Let $K(\underline{f})$ be the Koszul complex of the regular sequence $\underline{f} = (f_1, \dots, f_{m-n})$. Then $K(\underline{f})$ is a flat resolution of $\mathcal{O}_S/\mathcal{I}$, its dual $K(\underline{f})^\vee = \mathcal{H}om_{\mathcal{O}_S}(K(\underline{f}), \mathcal{O}_S)$ is a flat resolution of $(\mathcal{O}_S/\mathcal{I})[-(m-n)]$. Therefore

$$\begin{aligned} f^!\mathcal{O}_Y &\simeq \iota^{-1}\mathcal{H}om_{\mathcal{O}_S}(K(\underline{f}), \omega^\circ_{S/Y})[m] \\ &= \iota^{-1} K(\underline{f})^\vee \otimes \omega^\circ_{S/Y}[m] \\ &\simeq \iota^{-1}(\mathcal{O}_S/\mathcal{I} \otimes^L \omega^\circ_{S/Y}[n]) \\ &= \iota^{-1}((\omega^\circ_{S/Y}/\mathcal{I})[n]) = \iota^*\omega^\circ_{S/Y}[n] \end{aligned}$$

which is an invertible sheaf of $\mathcal{O}_X$-modules as required. □

Remark 4.0.5. Let $f : X \to Y$ be a regular closed immersion of codimension n defined by the ideal $\mathcal{I}$. From the proof of Lemma 4.0.4 one can deduce that $f^!\mathcal{O}_Y = \bigwedge^n f^*(\mathcal{I}/\mathcal{I}^2)^\vee[-n]$ is the normal sheaf concentrated in degree n.

The compatibility of $f^!$ with tensor products allows us to compute $f^!\mathcal{F}$ in terms of $f^*\mathcal{F}$ and $f^!\mathcal{O}_Y$:

Proposition 4.0.6 ([Har66, Prop. III.8.8])**.** *Let $f : X \to Y$ be an embeddable morphism of locally noetherian schemes of finite Krull dimension. Then there are functorial isomorphisms*

1. $f^!\mathcal{F} \otimes^L f^*\mathcal{G} \to f^!(\mathcal{F} \otimes^{\mathrm{L}} \mathcal{G})$ *for* $\mathcal{F} \in \mathrm{D}^+_{qc}(Y)$ *and* $\mathcal{G} \in \mathrm{D}^b_{qc}(Y)_{fTd}$.
2. $\mathrm{R}\mathcal{H}om_X(\mathrm{L}f^*\mathcal{F}, f^!\mathcal{G}) \to f^!(R\mathcal{H}om_Y(\mathcal{F}, \mathcal{G}))$ *for* $\mathcal{F} \in D^-_c(Y)$ *and* $\mathcal{G} \in D^+_{qc}(Y)$.

Moreover, if f is an lci morphism, then $f^!\mathcal{O}_Y$ is invertible and we have $f^!\mathcal{G} \cong f^!\mathcal{O}_Y \otimes \mathrm{L}f^*\mathcal{G}$ for $\mathcal{G} \in D^b_{qc}(Y)_{fTd}$. We call $f^!\mathcal{O}_Y$ the dualizing sheaf of f.

We now prove the local duality theorem for vector bundles over lci projective curves:

Proposition 4.0.7. *Let $f : X \to \operatorname{Spec} O_K$ be an lci projective curve, and let ω°_{X/O_K} be the dualizing sheaf of f, i.e. the invertible sheaf such that $\omega^\circ_{X/O_K}[1] = f^!O_K$. Let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module of finite rank, then:*

1. *$Rf_*\mathcal{F}$ is representable by a perfect complex of lenght* $[0,1]$*;*
2. *we have a perfect pairing*

$$\mathrm{H}^0(X, \mathcal{F} \otimes K/\mathcal{O}_K) \times \mathrm{H}^1(X, \mathcal{F}^\vee \otimes \omega^\circ_{X/\mathcal{O}_K}) \to K/\mathcal{O}_K$$

given by the composition of the cup product and the trace $\mathrm{Tr} : Rf_*\omega^\circ_{X/\mathcal{O}_K} \to \mathcal{O}_K$.

Proof. As $\mathcal{F}$ is a vector bundle and f is projective of relative dimension 1, the cohomology groups $R^i f_*\mathcal{F}$ are finitely generated over $\mathcal{O}_K$ and concentrated in degrees 0 and 1. Then, $Rf_*\mathcal{F}$ is quasi-isomorphic to a complex $0 \longrightarrow M_0 \xrightarrow{d} M_1 \longrightarrow 0$ with M_1 and M_2 finite free $\mathcal{O}_K$-modules. Moreover, the complex $0 \longrightarrow M_0 \otimes K/\mathcal{O}_K \xrightarrow{d\otimes 1} M_1 \otimes K/\mathcal{O}_K \longrightarrow 0$ is quasi-isomorphic to $Rf_*(\mathcal{F} \otimes K/\mathcal{O}_K)$ in $D(\mathcal{O}_K)$, see [Mum08, Theo. 5.2].

Duality theorem 4.0.2 gives a quasi-isomorphism

$$Rf_*(\mathcal{F}^\vee \otimes \omega^\circ_{X/\mathcal{O}_K})[1] = Rf_*R\mathscr{H}om_X(\mathcal{F}, f^!\mathcal{O}_K) \simeq R\mathrm{Hom}_{\mathcal{O}_K}(Rf_*\mathcal{F}, \mathcal{O}_K).$$

This implies that $Rf_*(\mathcal{F}^\vee \otimes \omega^\circ_{X/\mathcal{O}_K})$ is quasi-isomorphic to $0 \longrightarrow M_1^\vee \xrightarrow{d^\vee} M_0^\vee \longrightarrow 0$. Finally, Pontryagin duality for $\mathcal{O}_K$ implies $\mathrm{Hom}_{\mathcal{O}_K}(\ker(d \otimes 1), K/\mathcal{O}_K) = \mathrm{coker}\, d^\vee$, which translates in the desired statement. □

Remark 4.0.8. The previous proposition relates two notions of duality. Namely, Serre and Pontryagin duality. We can deduce the following facts:

1. The $\mathcal{O}_K$-module $\mathrm{H}^0(X, \mathcal{F} \otimes K/\mathcal{O}_K)$ is co-free of rank r, that is isomorphic to $(K/\mathcal{O}_K)^r$, if and only if $\mathrm{H}^1(X, \mathcal{F}^\vee \otimes \omega^\circ_{X/\mathcal{O}_K})$ is free of rank r. In that case, the module $\mathrm{H}^0(X, \mathcal{F})$ is free and $\mathrm{H}^0(X, \mathcal{F})/p^n \to \mathrm{H}^0(X, \mathcal{F}/p^n)$ is an isomorphism for all $n \in \mathbb{N}$. Furthermore, Serre duality provides a perfect pairing

$$\mathrm{H}^0(X, \mathcal{F}) \times \mathrm{H}^1(X, \mathcal{F}^\vee \otimes \omega^\circ_{X/\mathcal{O}_K}) \to \mathcal{O}_K.$$

2. The $\mathcal{O}_K$-module $\mathrm{H}^0(X, \mathcal{F})$ (resp. $\mathrm{H}^1(X, \mathcal{F} \otimes K/\mathcal{O}_K)$) is free (resp. co-free) for any finite locally free $\mathscr{O}_X$-module.

3. In the notation of the previous proof, Pontryagin duality implies

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathrm{coker}(d \otimes 1), K/\mathcal{O}_K) = \ker d^\vee,$$

which is equivalent to a perfect pairing

$$\mathrm{H}^1(X, \mathcal{F} \otimes K/\mathcal{O}_K) \times \mathrm{H}^0(X, \mathcal{F}^\vee \otimes \omega^\circ_{X/\mathcal{O}_K}) \to K/\mathcal{O}_K.$$

5. Cohomology of modular sheaves

Let $N \geq 3$ be an integer prime to p. Let $X(Np^n)$ be the modular curve over $\mathbb{Z}_p$ of level $\Gamma(Np^n)$. Let $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$ and let $X(Np^n)_{\breve{\mathbb{Z}}_p}$ be the extension of scalars of $X(Np^n)$ to $\breve{\mathbb{Z}}_p$. We denote by $X_n := X(Np^n)^\circ_{\breve{\mathbb{Z}}_p}$ the connected component of $X(Np^n)_{\breve{\mathbb{Z}}_p}$ given by fixing the Weil pairing $e_N(P_N, Q_N) = \zeta_N$, where (P_N, Q_N) is the universal basis of $E[N]$ and $\zeta_N \in \breve{\mathbb{Z}}_p$ a primitive N-th root of unity. We also write $X = X_0$. Let $\mathcal{O}_n = \breve{\mathbb{Z}}_p[\mu_{p^n}]$ be the n-th cyclotomic extension of $\breve{\mathbb{Z}}_p$, $\mathcal{O}^{cyc}$ the p-adic completion of $\varinjlim_n \mathcal{O}_n$, K_n and K^{cyc} the field of fractions of $\mathcal{O}_n$ and $\mathcal{O}^{cyc}$ respectively. We set $\mathcal{O} = \breve{\mathbb{Z}}_p$ and $K = \mathcal{O}[\frac{1}{p}]$. Let $\pi_n : X_n \to \operatorname{Spec} \mathcal{O}_n$ denote the structural map defined by the Weyl pairing of the universal basis of $E[p^n]$. We also denote $p_n : X_n \to X_{n-1}$ the natural morphism induced by p-multiplication of Drinfeld bases.

Let E^{sm}/X be the semi-abelian scheme over X extending the universal elliptic curve to the cusps, cf [DR73]. Let $e : X \to E^{sm}$ be the unit section and $\omega_E := e^*\Omega^1_{E^{sm}/X}$ the modular sheaf, i.e., the sheaf of invariant differentials of E^{sm} over X. For $k \in \mathbb{Z}$ we define $\omega_E^k = \omega_E^{\otimes k}$ the sheaf of modular forms of weight k, we denote by $\omega^k_{E,n}$ the pullback of ω^k_E to X_n. Let $D_n \subset X_n$ be the (reduced) cusp divisor and $\omega^k_{E,n,cusp} := \omega^k_{E,n}(-D_n)$ the sheaf of cusp forms of weight k over X_n. By an abuse of notation we will also write D_n for the pullback $p^*_{n+1}D_n$ to X_{n+1}, by Proposition 2.2.4 we have that $D_n = pD_{n+1}$.

Finally, we let $\mathfrak{X}_n$ be the completion of X_n along its special fiber and $\mathfrak{X}_\infty = \varprojlim_n \mathfrak{X}_n$ the integral perfectoid modular curve, see Theorem 3.2.1. Let $\mathcal{X}_n$ be the analytic generic fiber of X_n and $\mathcal{X}_\infty \sim \varinjlim_n \mathcal{X}_n$ the Scholze's perfectoid modular curve.

5.1. Dualizing sheaves of modular curves

Consider the tower of modular curves

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{n+1} & \xrightarrow{p_{n+1}} & X_n & \xrightarrow{p_n} & X_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \pi_{n+1}} & & \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} & & \\
\cdots & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{n+1}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_n) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{n-1}) & \longrightarrow & \cdots .
\end{array}$$

Since X_n is regular of finite type over $\mathcal{O}_n$, it is a local complete intersection. This implies that the sheaf $\omega^\circ_n := \pi^!_n\mathcal{O}_n$ is invertible. The modular curve $X/\mathcal{O}$ is smooth of relative dimension 1, then we have that $\omega^\circ_0 = \Omega^1_{X_0/\mathcal{O}}$, cf. Example 4.0.3 (2). On the other hand, the Kodaira-Spencer map provides an isomorphism $KS : \omega^2_{E,cusp} \cong \Omega^1_{X/\mathcal{O}}$.

Let $X'_{n-1} = X_{n-1} \times_{\operatorname{Spec}\mathcal{O}_{n-1}} \operatorname{Spec}\mathcal{O}_n$, and by an abuse of notation $p_n : X_n \to X'_{n-1}$ the induced map. Let $\pi'_{n-1} : X'_{n-1} \to \mathcal{O}_n$ be the structural map and $pr_1 : X'_{n-1} \to X_{n-1}$ the first projection. We also write $\omega^k_{E,n-1}$ for the pullback of $\omega^k_{E,n-1}$ to X'_{n-1}. Note that the compatibility of the exceptional inverse image functor with flat base change (Theorem 4.0.2) implies that $\pi'^!_{n-1}\mathcal{O}_n \cong pr_1^*\omega^\circ_{n-1} = \omega^\circ_{n-1} \otimes_{\mathcal{O}_{n-1}} \mathcal{O}_n$.

Proposition 5.1.1. *There exists a natural isomorphism $\xi_n : p_n^*(\omega^\circ_{n-1})(D_{n-1} - D_n) \xrightarrow{\sim} \omega^\circ_n$ induced by the normalized trace $\frac{1}{p}\operatorname{Tr}_n : \mathcal{O}_{X_n} \to p^!_n\mathcal{O}_{X'_{n-1}}$. Moreover, the composition of $\xi_n \circ \cdots \circ \xi_1$ with the Kodaira-Spencer map gives an isomorphism $\omega^2_{E,n,cusp} \cong \omega^\circ_n$.*

Proof. By Proposition 4.0.6 we have an isomorphism

$$\xi'_n : p^!_n\mathcal{O}_{X'_{n-1}} \otimes p_n^*\omega^\circ_{n-1} \xrightarrow{\sim} p^!_n\omega^\circ_{n-1} = \omega^\circ_n. \tag{5.1.1}$$

The map p_n is finite flat, then $p_n^!\mathscr{O}_{X'_{n-1}} = p_n^{-1}\mathscr{H}om_{\mathscr{O}_{X'_{n-1}}}(p_{n,*}\mathscr{O}_{X_n}, \mathscr{O}_{X'_{n-1}})$ by Example 4.0.3 (1). By Lemma 4.0.4, the sheaf $p_n^!\mathscr{O}_{X'_{n-1}}$ is invertible as X'_{n-1} is an lci projective curve. We claim that the trace $\mathrm{Tr}_n : \mathscr{O}_{X_n} \to p_n^!\mathscr{O}_{X'_{n-1}}$ induces an isomorphism $\frac{1}{p}\mathrm{Tr}_n : \mathscr{O}_{X_n}(D_{n-1} - D_n) \cong p_n^!\mathscr{O}_{X'_{n-1}}$. It suffices to consider the ordinary points and the cusps, indeed, the supersingular points are of codimension 2 in X_n.

Let $x \in X'_{n-1}(\overline{\mathbb{F}}_p)$ be an ordinary point. We have a cartesian square

$$\begin{array}{ccc} \bigsqcup_{x_n \mapsto x} \mathrm{Spf}\, \widehat{\mathscr{O}}_{X_n,x_n} & \longrightarrow & X_n \\ \downarrow & & \downarrow{\scriptstyle p_n} \\ \mathrm{Spf}\, \widehat{\mathscr{O}}_{X'_{n-1},x} & \longrightarrow & X'_{n-1}. \end{array} \tag{5.1.2}$$

By Proposition 2.2.2 we have isomorphisms

$$\widehat{\mathscr{O}}_{X'_{n-1},x} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[T_{n-1}]], \quad \widehat{\mathscr{O}}_{X_n,x_n} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[T_n]]$$

with relations $(1+T_n)^p = 1 + T_{n-1}$. Taking the different ideal of the finite flat extension $\widehat{\mathscr{O}}_{X_n,x_n}/\widehat{\mathscr{O}}_{X'_{n-1},x}$, one finds

$$\mathscr{H}om_{\widehat{\mathscr{O}}_{X'_{n-1},x}}(\widehat{\mathscr{O}}_{X_n,x_n}, \widehat{\mathscr{O}}_{X'_{n-1},x}) \cong \frac{1}{p}\widehat{\mathscr{O}}_{X_n,x_n} \cdot \mathrm{Tr}_n\,.$$

On the other hand, let $x \in X'_{n-1}(\overline{\mathbb{F}}_p)$ be a cusp. We have a cartesian square (5.1.2) and by Proposition 2.2.4 isomorphisms

$$\widehat{\mathscr{O}}_{X'_{n-1},x} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[q^{1/p^{n-1}}]], \quad \widehat{\mathscr{O}}_{X_n,x_n} \cong W(\overline{\mathbb{F}}_p)[\zeta_{p^n}][[q^{1/p^n}]].$$

Taking the different ideal we obtain the equality

$$\mathscr{H}om_{\widehat{\mathscr{O}}_{X'_{n-1},x}}(\widehat{\mathscr{O}}_{X_n,x_n}, \widehat{\mathscr{O}}_{X'_{n-1},x}) \quad \cong \quad \frac{1}{p}q^{1/p^n - 1/p^{n-1}}\widehat{\mathscr{O}}_{X_n,x_n} \cdot \mathrm{Tr}_n\,.$$

The previous computations show that the trace of $\mathscr{O}_{X_n}/\mathscr{O}_{X'_{n-1}}$ induces an isomorphism of invertible sheaves

$$\frac{1}{p}\mathrm{Tr}_n : \mathscr{O}_{X_n}(D_{n-1} - D_n) \xrightarrow{\sim} p_n^!\mathscr{O}_{X'_{n-1}}.$$

Then, from (5.1.1) we have an isomorphism

$$\xi_n : \mathscr{O}_{X_n}(D_{n-1} - D_n) \otimes p_n^*\omega^\circ_{n-1} \to \omega^\circ_n$$

with $\xi_n = \xi'_n \circ (\frac{1}{p}\mathrm{Tr}_n \otimes 1)$.

The isomorphism $\omega^2_{E,n,cusps} \cong \omega^\circ_n$ follows by a straightforward induction on the composition $\xi_n \circ \cdots \circ \xi_1$, and the Kodaira-Spencer map $KS : \omega^2_{E,cusp} \cong \Omega^1_{X/O}$. □

Lemma 5.1.2. *Let $x \in X'_{n-1}(\overline{\mathbb{F}}_p)$ be an an ordinary point or a cusp. Let $\widetilde{\mathrm{Tr}}_n : p_{n,*}\mathscr{O}_{X_n}(D_{n-1} - D_n) \to \mathscr{O}_{X'_{n-1}}$ be the normalized trace map $\frac{1}{p}\mathrm{Tr}_n$. Then the completed localization of $\widetilde{\mathrm{Tr}}_n$ at x is surjective. Moreover, if $\mathcal{F}$ is a quasi-coherent sheaf over X'_{n-1}, the composition $\mathcal{F} \to p_{n,*}p_n^*\mathcal{F} \xrightarrow{\widetilde{\mathrm{Tr}}_n} \mathcal{F}$ is multiplication by p.*

Proof. Localizing at x we find

$$\widetilde{\mathrm{Tr}}_n = \oplus(\frac{1}{p}\mathrm{Tr}_n) : \bigoplus_{x_n \mapsto x} \widehat{\mathscr{O}}_{X_n,x_n} \otimes (q^{1/p^n - 1/p^{n-1}}) \to \widehat{\mathscr{O}}_{X'_{n-1},x}$$

where $q^{1/p^{n-1}}$ is invertible if x is ordinary, or a generator of D_{n-1} if it is a cusp. The explicit descriptions found in the previous proposition show that $\widetilde{\mathrm{Tr}}_n$ is surjective on each direct summand. Finally, looking at an ordinary point x, it is clear that there are p different points x_n in the fiber of x, this implies $\widetilde{\mathrm{Tr}}_n(1) = p$. □

5.2. Vanishing of coherent cohomology

In order to prove vanishing theorems for the coherent cohomology over the perfectoid modular curve, we first need some vanishing results at finite integral level. We have the following proposition

Proposition 5.2.1. *For all $n \in \mathbb{N}$ the following holds*

1. $\mathrm{H}^0(X_n, \omega_{E,n}^k \otimes_{O_n} K_n/O_n) = \mathrm{H}^0(X_n, \omega_{E,n,cusp}^k \otimes_{O_n} K_n/O_n) = 0$ *for* $k < 0$.
2. $\mathrm{H}^1(X_n, \omega_{E,n}^k) = \mathrm{H}^1(X_n, \omega_{E,n,cusp}^k) = 0$ *for* $k > 2$.
3. $\mathrm{H}^0(X_n, \mathscr{O}_{X_n}(-D_n) \otimes_{O_n} K_n/O_n) = \mathrm{H}^1(X_n, \omega_{E,n}^2) = 0$ *and* $\mathrm{H}^0(X_n, \mathscr{O}_{X_n} \otimes_{O_n} K_n/O_n) = \mathrm{H}^1(X_n, \omega_{E,n,cusp}^2) \otimes_{O_n} (K_n/O_n) = K_n/O_n$.

Proof. By Propositions 4.0.7 and 5.1.1, (1) and (2) are equivalent. Similarly, by (1) of Remark 4.0.8, and Proposition 5.1.1, it is enough to show (3) for $\mathscr{O}_{X_n}$ and $\mathscr{O}_{X_n}(-D_n)$.

Let ν_n be the closed point of $\operatorname{Spec} O_n$ and $\varpi \in O_n$ a uniformizer, we write $\nu = \nu_0$ for the closed point of $\operatorname{Spec} O$. It suffices to prove $\mathrm{H}^0(X_n, \omega_{E,n}^k/\varpi) = \mathrm{H}^0(X_{n,\nu_n}, \omega_{E,n}^k) = 0$ for $k < 0$. Indeed, for $s \geq 1$, the short exact sequence

$$0 \to \omega_{E,n}^k/\varpi^s \xrightarrow{\varpi} \omega_{E,n}^k/\varpi^{s+1} \to \omega_{E,n}^k/\varpi \to 0$$

induces a left exact sequence in global sections

$$0 \to \mathrm{H}^0(X_n, \omega_{E,n}^k/\varpi^s) \to \mathrm{H}^0(X_n, \omega_{E,n}^k/\varpi^{s+1}) \to \mathrm{H}^0(X_n, \omega_{E,n}^k/\varpi).$$

An inductive argument on s shows $\mathrm{H}^0(X_n, \omega_{E,n}^k/\varpi^s) = 0$ for all $s \geq 1$.

Let $\lambda \in \mathrm{H}^0(X_{n,\nu_n}, \omega_{E,n}^k)$ be non-zero. Applying the action of $\mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})$, we can assume that λ is non-zero in an open dense subscheme of X_{n,ν_n}. In fact, this holds for some linear combination $\sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z}/p^n\mathbb{Z})} a_n \gamma^* \lambda$ with $a_n \in \overline{\mathbb{F}}_p$. The norm $\mathbb{N}_{X_{n,\nu_n}/X_\nu}(\omega_{E,n}^k)$ of $\omega_{E,n}^k$ to X_ν is ω_E^{kd}, where $d = \deg(X_{n,\nu_n}/X_\nu)$. Hence if $k < 0$, the sheaf $\mathbb{N}_{X_{n,\nu_n}/X_\nu}(\omega_{E,n}^k)$ has negative degree in the smooth curve X_ν. This implies that $\mathrm{H}^0(X_\nu, \mathbb{N}_{X_{n,\nu_n}/X_\nu}(\omega_{E,n}^k)) = 0$ and $\mathbb{N}_{X_{n,\nu_n}/X_\nu}(\lambda) = 0$, a contradiction. Therefore $\mathrm{H}^0(X_{n,\nu_n}, \omega_{E,n}^k) = 0$ for $k < 0$. Since $\omega_{E,n,cusp}^k = \omega_{E,n}^k(-D_n)$, we trivially deduce $\mathrm{H}^0(X_{n,\nu_n}, \omega_{E,n,cusp}^k) = 0$.

The results for $\mathscr{O}_{X_n}$ and $\mathscr{O}_{X_n}(-D_n)$ are clear as X_n/O_n is proper, flat, geometrically connected and has geometrically reduced fibers.

□

Remark 5.2.2. Strictly speaking, we can apply Proposition 4.0.7 only for projective curves over a finite extension of $\mathbb{Z}_p$. However, as the formation of coherent cohomology is compatible with affine flat base change of the base, the conclusion of *loc. cit.* holds in the situation of the previous proposition.

Corollary 5.2.3. *Let $\mathcal{F} = \omega_{E,n}^k$ or $\omega_{E,n,cusp}^k$ for $k \neq 1$, the following holds*

1. *The cohomology groups $\mathrm{H}^0(X_n, \mathcal{F} \otimes K/O)$ and $\mathrm{H}^1(X_n, \mathcal{F})$ are cofree and free O_n-modules respectively.*
2. *We have a perfect duality pairing*

$$\mathrm{H}^0(X_n, \mathcal{F} \otimes K/O) \times \mathrm{H}^1(X_n, \mathcal{F}^\vee \otimes \omega_{n,cusp}^2) \to K_n/O_n.$$

Proof. Part (2) is Proposition 4.0.7. Part (1) follows from Remark 4.0.8 i) and the previous proposition. Indeed, if $k < 0$, the vanishing of $\mathrm{H}^0(X_n, \mathcal{F} \otimes K_n/O_n)$ implies that $\mathrm{H}^1(X_n, \mathcal{F})$ is torsion free. As the cohomology group is of finite type over O_n, it is a finite free O_n-module. The other cases are proved in a similar way.

□

Next, we will prove some cohomological vanishing results for the modular sheaves ω_E^k and $\omega_{E,cusp}^k$ at infinite level. Particularly, we will show that the cohomology of ω_E^k over $\mathfrak{X}_\infty$ is concentrated in degree 0 if $k > 0$. The case $k > 2$ will follow from Proposition 5.2.1, one can also argue directly for $k = 2$. What is remarkable is the vanishing for $k = 1$, in which case we use the perfectoid nature of $\mathfrak{X}_\infty$.

Let $\omega_{E,\infty}^k$ be the pullback of ω_E^k to $\mathfrak{X}_\infty$. Let $m \geq n$, note that we have an inequality of divisors $D_m \leq D_n$. Then, $\mathscr{O}_{X_m}(-D_n) \subset \mathscr{O}_{X_m}(-D_m)$, and the pullback of $\omega_{E,n,cusp}^k$ injects into $\omega_{E,m,cusp}^k$. We define $\omega_{E,\infty,cusp}^k$ as the p-adic completion of the direct limit $\varinjlim_n \omega_{E,n,cusp}^k$, if $k = 0$ we simply write $\mathscr{O}_{\mathfrak{X}_\infty}(-D_\infty)$ for $\omega_{E,\infty,cusp}^0$. The sheaf $\omega_{E,\infty,cusp}^k$ is no longer a coherent sheaf over $\mathfrak{X}_\infty$; its reduction modulo p is a direct limit of line bundles which is not stationary at the cusps. One way to think about an element in $\omega_{E,\infty,cusp}^k$ is via q-expansions: the completed localization of $\omega_{E,\infty}^k$ at a cusp $x = (x_0, x_1, \cdots) \in \mathfrak{X}_\infty$ is isomorphic to

$$\mathcal{O}^{cyc}[[q^{1/p^\infty}]] := \varprojlim_s (\varinjlim_n \mathcal{O}^{cyc}[[q^{1/p^n}]])/(p,q)^s.$$

Then, an element $f \in \omega_{E,\infty,x}^k$ can be written as a power series

$$f = \sum_{m \in \mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_m q^m$$

satisfying certain convergence conditions. The element f belongs to the localization at x of $\omega_{E,\infty,cusp}^k$ if and only if $a_0 = 0$. For a detailed treatment of the cusps at perfectoid level we refer to [Heu20], particularly Theorem 3.17.

Theorem 5.2.4. *The following holds*

1. *The cohomology complexes $R\Gamma(\mathfrak{X}_\infty, \omega_{E,\infty}^k)$ and $R\Gamma(\mathfrak{X}_\infty, \omega_{E,\infty,cusp}^k)$ are concentrated in degree $[0,1]$ for all $k \in \mathbb{Z}$.*
2. *For all $m, i \geq 0$ and $k \in \mathbb{Z}$, we have $\mathrm{H}^i(\mathfrak{X}_\infty, \omega_{E,\infty}^k/p^m) = \varinjlim_n \mathrm{H}^i(X_n, \omega_{E,n}^k/p^m)$ and $\mathrm{H}^i(\mathfrak{X}_\infty, \omega_{E,\infty,cusp}^k/p^m) = \varinjlim_n \mathrm{H}^i(X_n, \omega_{E,n,cusp}^k/p^m)$.*
3. *The sheaves $\omega_{E,\infty}^k$ and $\omega_{E,\infty,cusp}^k$ have cohomology concentrated in degree 0 for $k > 0$. Similarly, the sheaves $\omega_{E,\infty}^k$ and $\omega_{E,\infty,cusp}^k$ have cohomology concentrated in degree 1 for $k < 0$.*
4. *$\mathrm{H}^0(\mathfrak{X}_\infty, \mathscr{O}_{\mathfrak{X}_\infty}(-D_\infty)) = 0$ and $\mathrm{H}^0(\mathfrak{X}_\infty, \mathscr{O}_{\mathfrak{X}_\infty}) = \mathcal{O}^{cyc}$.*

Proof. Let $\mathcal{F} = \omega_{E,\infty}^k$ or $\omega_{E,\infty,cusp}^k$ and $\mathcal{F}_n = \omega_{E,n}^k$ or $\omega_{E,n,cusp}^k$ respectively. We show (1) assuming part (2). By evaluating $\mathcal{F}$ at formal affine perfectoids of $\mathfrak{X}_\infty$ arising from finite level, one can use Lemma 3.18 of [Sch13a] to deduce that $\mathcal{F} = R\varprojlim_s \mathcal{F}/p^s$: the case $\mathcal{F} = \omega_{E,\infty}^k$ is clear as it is a line bundle. Otherwise, we know that $\mathcal{F}/p^s = \varinjlim_n \mathcal{F}_n/p^s = \varinjlim_n (\mathcal{F}_n/p^s \otimes_{X_n} \mathscr{O}_{\mathfrak{X}_\infty})$ is a direct limit of $\mathscr{O}_{\mathfrak{X}_\infty}/p^s$-line bundles, so that it is a quasi-coherent sheaf over $\mathfrak{X}_\infty$, and the system $\{\mathcal{F}/p^s\}_{s\in\mathbb{N}}$ satisfies the Mittag-Leffler condition on formal affine perfectoids. One obtains the quasi-isomorphism

$$R\Gamma(\mathfrak{X}_\infty, \mathcal{F}) = R\varprojlim_s R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s)$$

whose cohomology translates into short exact sequences

$$0 \to R^1 \varprojlim_s \mathrm{H}^{i-1}(\mathfrak{X}_\infty, \mathcal{F}/p^s) \to \mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}) \to \varprojlim_s \mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}/p^s) \to 0. \tag{5.2.1}$$

But part (2) implies that $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}/p^s) = \varinjlim_n \mathrm{H}^i(X_n, \mathcal{F}_n/p^s)$ for all $s \in \mathbb{N}$. As X_n is a curve over $\mathcal{O}_n$ and $\mathcal{F}_n/p^s$ is supported in its special fiber, we know that $\mathrm{H}^i(X_n, \mathcal{F}_n/p^s) = 0$ for $i \geq 2$ and that the inverse

system $\{\mathrm{H}^1(X_n, \mathcal{F}_n/p^s)\}_{s\in\mathbb{N}}$ satisfies the ML condition. This implies that $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}/p^s) = 0$ for $i \geq 2$ and that the ML condition holds for $\{\mathrm{H}^1(\mathfrak{X}_\infty, \mathcal{F}/p^s)\}_{s\in\mathbb{N}}$. From (5.2.1) one obtains that $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}) = 0$ for $i \geq 2$.

We prove part (2). Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a finite affine cover of X, let $\mathfrak{U}_n$ (resp. $\mathfrak{U}_\infty$) be its pullback to X_n (resp. $\mathfrak{X}_\infty$). As $\mathcal{F}/p^s = \varinjlim_n \mathcal{F}_n/p^s$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X}_\infty}/p^s$-module, and the (formal) schemes $\mathfrak{X}_\infty$ and X_n are separated, we can use the Čech complex of $\mathfrak{U}_n$ (resp. $\mathfrak{U}_\infty$) to compute the cohomology groups. By definition we have

$$\mathscr{C}^\bullet(\mathfrak{U}_\infty, \mathcal{F}/p^s) \;=\; \varinjlim_n \mathscr{C}^\bullet(\mathfrak{U}_n, \mathcal{F}_n/p^s),$$

then (2) follows as filtered direct limits are exact.

The vanishing results of Proposition 5.2.1 imply (3) for $k < 0$ and $k > 2$. Let $k = 1, 2$ and $p^{1/p} \in \mathcal{O}^{cyc}$ be such that $|p^{1/p}| = |p|^{1/p}$. As $\mathfrak{X}_\infty$ is integral perfectoid, the Frobenius $F : \mathfrak{X}_\infty/p \to \mathfrak{X}_\infty/p^{1/p}$ is an isomorphism. Moreover, $F^*(\omega^k_{E,\infty}/p^{1/p}) = \omega^{pk}_{E,\infty}/p$ and $F^*(\omega^k_{E,\infty,cusp}/p^{1/p}) = \omega^{pk}_{E,\infty,cusp}/p$ (notice that $F^*(D_n) = pD_n = D_{n-1}$). Then, Proposition 5.2.1 (2) implies

$$\mathrm{H}^1(\mathfrak{X}_\infty, \omega^k_{E,\infty}/p^{1/p}) \cong \mathrm{H}^1(\mathfrak{X}_\infty, \omega^{pk}_{E,\infty}/p) = 0, \tag{5.2.2}$$

similarly for $\omega^k_{E,\infty,cusp}$. By induction on s, one shows that $\mathrm{H}^1(\mathfrak{X}_\infty, \omega^k_{E,\infty}/p^s) = 0$ and that $\mathrm{H}^0(\mathfrak{X}_\infty, \omega^k_{E,\infty}/p^{s+1}) \to \mathrm{H}^0(\mathfrak{X}_\infty, \omega^k_{E,\infty}/p^s)$ is surjective for all $s \in \mathbb{N}$ (resp. for $\omega^k_{E,\infty,cusp}$). Taking derived inverse limits one gets $\mathrm{H}^1(\mathfrak{X}_\infty, \omega^k_{E,\infty}) = \mathrm{H}^1(\mathfrak{X}_\infty, \omega^k_{E,\infty,cusps}) = 0$ and $\mathrm{H}^0(\mathfrak{X}_\infty, \omega^k_{E,\infty}) = \varprojlim_s \mathrm{H}^0(\mathfrak{X}_\infty, \omega^k_{E,\infty}/p^s)$ (resp. for $\omega^k_{E,\infty,cusp}$). This proves (3) for $k = 1, 2$.

Finally, part (4) follows from part (2), Proposition 5.2.1 (3), and the fact that

$$\mathrm{H}^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}) = \varprojlim_s \mathrm{H}^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}/p^s)$$

by (5.2.1) (resp. for $\mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)$).

□

Corollary 5.2.5. *Let $\mathcal{F} = \omega^k_{E,\infty}$ or $\omega^k_{E,\infty,cusp}$ for $k \in \mathbb{Z}$. Then $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F})/p^s = \mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}/p^s)$ and $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}) = \varprojlim_s \mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}/p^s)$ for all $i, s \geq 0$. In particular, the cohomology groups $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F})$ are p-adically complete and separated. Moreover, they are all torsion free.*

Proof. The case $k \neq 0$ follows since the cohomology complexes $R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s)$ are concentrated in only one degree, and $R\Gamma(\mathfrak{X}_\infty, \mathcal{F}) = R\varprojlim_s R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s)$. The case $k = 0$ follows by part (4) of the previous theorem. Namely, $\mathrm{H}^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}(-D_\infty)/p^s) = 0$ and $\mathrm{H}^0(\mathfrak{X}_\infty, \mathcal{O}_{\mathfrak{X}_\infty}/p^s) = \mathcal{O}^{cyc}/p^s$ for all $s \geq 0$. Hence, the inverse system of H^0-cohomology groups satisfy the Mittag-Leffler condition, and the $R^1\varprojlim$ appearing in the derived inverse limit disappears for the H^1-cohomology. □

As an application of the previous vanishing theorem, we obtain vanishing results for the coherent cohomology of the perfectoid modular curve. Let $(\mathcal{X}_\infty, \mathcal{O}^+_{\mathcal{X}_\infty}) \to (\mathfrak{X}_\infty, \mathcal{O}_{\mathcal{O}_{\mathfrak{X}_\infty}})$ be the natural map of locally and topologically ringed spaces provided by the generic fiber functor, see Proposition 3.1.5 and Theorem 3.2.1. We define $\omega^{k,+}_{E,\eta} := \omega^k_{E,\infty} \otimes_{\mathcal{O}_{\mathfrak{X}_\infty}} \mathcal{O}^+_{\mathcal{X}_\infty}$ and $\omega^{k,+}_{E,cusp,\eta} := \omega^k_{E,\infty,cusp} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{X}_\infty}} \mathcal{O}^+_{\mathcal{X}_\infty}$, where the completed tensor product is with respect to the p-adic topology. As usual, we denote $\mathcal{O}^+_{\mathcal{X}_\infty}(-D_\infty) = \omega^{0,+}_{E,cusp}$. In the following we consider almost mathematics with respect to the maximal ideal of $\mathcal{O}^{cyc}$.

Corollary 5.2.6. *The following holds*

1. *The cohomology complexes $R\Gamma_{an}(\mathcal{X}_\infty, \omega^{k,+}_{E,\eta})$ and $R\Gamma_{an}(\mathcal{X}_\infty, \omega^{k,+}_{E,cusp,\eta})$ of almost $\mathcal{O}^{cyc}$-modules are concentrated in degrees $[0, 1]$ for all $k \in \mathbb{Z}$.*

2. *The sheaves* $\omega_{E,\eta}^{k,+}$ *and* $\omega_{E,cusp,\eta}^{k,+}$ *have cohomology almost concentrated in degree* 0 *for* $k > 0$. *Similarly, the sheaves* $\omega_{E,\eta}^{k,+}$ *and* $\omega_{E,cusp,\eta}^{k,+}$ *have cohomology almost concentrated in degree* 1 *for* $k < 0$.

3. $\mathrm{H}^0_{an}(\mathcal{X}_\infty, \mathscr{O}^+_{\mathcal{X}_\infty}(-D_\infty)) = 0$ *and* $\mathrm{H}^0_{an}(\mathcal{X}_\infty, \mathscr{O}^+_{\mathcal{X}_\infty}) = \mathcal{O}^{cyc}$.

Proof. We first prove the corollary for $\mathcal{F} = \omega^k_{E,\infty}$. Let $\mathcal{F}^+_\eta$ denote the pullback of $\mathcal{F}$ to $(\mathcal{X}_\infty, \mathscr{O}^+_{\mathcal{X}_\infty})$. Let $\mathfrak{U} = \{\mathcal{U}_i\}_{i\in I}$ be an open cover of $\mathfrak{X}_\infty$ given by formal affine perfectoids arising from finite level such that $\omega_{E,\infty}|_{\mathcal{U}_i}$ is trivial. By Theorem 3.2.1, the generic fiber $\mathcal{U}_{i,\eta}$ of $\mathcal{U}_i$ is an open affinoid perfectoid subspace of $\mathcal{X}_\infty$. Let $\mathfrak{U}_\eta := \{\mathcal{U}_{i,\eta}\}_{i\in I}$, note that $\mathfrak{U}_\eta$ is a covering of $\mathcal{X}_\infty$ and that the restriction of $\mathcal{F}^+_\eta$ to $\mathcal{U}_{i,\eta}$ is trivial. By Scholze's Almost Acyclicity Theorem for affinoid perfectoids, $\mathcal{F}^+_\eta|_{\mathcal{U}_{i,\eta}}$ is almost acyclic for all $i \in I$. The Čech-to-derived functor spectral sequence gives us an almost quasi-isomorphism

$$\mathscr{C}^\bullet(\mathfrak{U}_\eta, \mathcal{F}^+_\eta) \simeq R\Gamma_{an}(\mathcal{X}_\infty, \mathcal{F}^+_\eta).$$

On the other hand, by the proof of Theorem 5.2.4 there is a quasi-isomorphism

$$\mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F}) \simeq R\Gamma(\mathfrak{X}_\infty, \mathcal{F}).$$

But by definition of $\mathcal{F}^+_\eta$, and the fact that $\mathscr{O}^+_{\mathcal{X}_\infty}(\mathcal{U}_{i,\eta}) = \mathscr{O}_{\mathfrak{X}_\infty}(\mathcal{U}_i)$ by Theorem 3.2.1, we actually have an almost equality $\mathscr{C}^\bullet(\mathfrak{U}_\eta, \mathcal{F}^+_\eta) =^{ae} \mathscr{C}^\bullet(\mathfrak{U}, \mathcal{F})$. In other words, there is an almost quasi-isomorphism $R\Gamma_{an}(\mathcal{X}_\infty, \mathcal{F}^+_\eta) \simeq^{ae} R\Gamma(\mathfrak{X}_\infty, \mathcal{F})$.

Let $\mathcal{F}_{cusp} = \omega^k_{E,\infty,cusp}$ and $\mathcal{F}^+_{cusp,\eta}$ its pullback to $(\mathcal{X}_\infty, \mathscr{O}^+_{\mathcal{X}_\infty})$. To prove that $R\Gamma_{an}(\mathcal{X}_\infty, \mathcal{F}^+_{cusp,\eta}) \simeq^{ae} R\Gamma(\mathfrak{X}_\infty, \mathcal{F}_{cusp})$ we argue as follows: note that we can write $\mathcal{F}_{cusp} = \mathcal{F} \otimes_{\mathscr{O}_{\mathfrak{X}_\infty}} \mathscr{O}_{\mathfrak{X}_\infty}(-D_\infty)$. To apply the same argument as before we only need to show that $\mathscr{O}^+_{\mathcal{X}_\infty}(-D_\infty)$ is almost acyclic over affinoid perfectoids of $\mathcal{X}_\infty$. Let $V(D_\infty) \subset \mathcal{X}_\infty$ be the perfectoid closed subspace defined by the cusps. Note that $\mathscr{O}_{\mathcal{X}_\infty}(-D_\infty)$ is the ideal sheaf of $V(D_\infty)$, see the proof of [Sch15, Theo. IV.2.1] or the explicit description of the completed stalks at the cusps of the integral perfectoid modular curve. Then, we have an almost short exact sequence for all $s \in \mathbb{N}$

$$0 \to \mathscr{O}^+_{\mathcal{X}_\infty}(-D_\infty)/p^s \to \mathscr{O}^+_{\mathcal{X}_\infty}/p^s \to \mathscr{O}^+_{V(D_\infty)}/p^s \to 0. \tag{5.2.3}$$

As the intersection of an affinoid perfectoid of $\mathcal{X}_\infty$ with $V(D_\infty)$ is affinoid perfectoid, and the second map of (5.2.3) is surjective when evaluating at affinoid perfectoids of $\mathcal{X}_\infty$, Scholze's almost acyclicity implies that $\mathscr{O}^+_{\mathcal{X}_\infty}(-D_\infty)/p^s$ is almost acyclic in affinoid perfectoids. Taking inverse limits and noticing that $\{\mathscr{O}^+_{\mathcal{X}_\infty}(-D_\infty)/p^s\}_{s\in\mathbb{N}}$ satisfies the ML condition in affinoid perfectoids, we get that $\mathscr{O}^+_{\mathcal{X}_\infty}(-D_\infty)$ is almost acyclic in affinoid perfectoids of $\mathcal{X}_\infty$. The corollary follows from the vanishing results at the level of formal schemes. □

Remark 5.2.7. As it was mentioned to me by Vincent Pilloni, the cohomological vanishing of the modular sheaves at infinite level provides many different exact sequences involving modular forms and the completed cohomology of the modular tower (to be defined in the next subsection). Namely, the primitive comparison theorem permits to compute the $\mathbb{C}_p$-scalar extension of the completed cohomology as $\mathrm{H}^1_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}_{\mathcal{X}_\infty})$. On the other hand, the Hodge-Tate exact sequence

$$0 \to \omega_E^{-1} \otimes_{\mathscr{O}_\mathcal{X}} \widehat{\mathscr{O}}_\mathcal{X} \to T_pE \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_\mathcal{X} \to \omega_E \otimes_{\mathscr{O}_\mathcal{X}} \widehat{\mathscr{O}}_\mathcal{X} \to 0$$

gives a short exact sequence over $\mathcal{X}_\infty$

$$0 \to \omega^{-1}_{E,\eta} \to \mathscr{O}^{\oplus 2}_{\mathcal{X}_\infty,\mathbb{C}_p} \to \omega_{E,\eta} \to 0 \tag{5.2.4}$$

via the universal trivialization of T_pE. Then, taking the cohomology of (5.2.4) one obtains an exact sequence

$$0 \to \mathbb{C}_p^{\oplus 2} \to \mathrm{H}^0_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \omega_{E,\eta}) \to \mathrm{H}^1_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \omega^{-1}_{E,\eta}) \to \mathrm{H}^1_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}_{\mathcal{X}_\infty})^{\oplus 2} \to 0.$$

Another is example is given by tensoring (5.2.4) with ω_E and taking cohomology. One finds

$$0 \to \mathbb{C}_p \to \mathrm{H}^0_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \omega_{E,\eta})^{\oplus 2} \to \mathrm{H}^0_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \omega^2_{E,\eta}) \to \mathrm{H}^1_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}_{\mathcal{X}_\infty}) \to 0.$$

It may be interesting a more careful study of these exact sequences.

5.3. Duality at infinite level

Let $\mathcal{F} = \omega^k_{E,\infty}$ or $\omega^k_{E,\infty,cusp}$ for $k \in \mathbb{Z}$, let $\mathcal{F}_n = \omega^k_{E,n}$ or $\omega^k_{E,ncusp}$ respectively. Let C be a non archimedean field extension of K^{cyc} and $\mathcal{O}_C$ its valuation ring. Let $\mathfrak{X}_{\infty,C}$ be the extension of scalars of the integral modular curve to $\mathcal{O}_C$. Corollary 5.2.5 says that the cohomology groups $\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F})$ are torsion free, p-adically complete and separated. In particular, we can endow $\mathrm{H}^i(\mathfrak{X}_{\infty,C}, \mathcal{F})[\frac{1}{p}]$ with an structure of C-Banach space with unit ball $\mathrm{H}^i(\mathfrak{X}_{\infty,C}, \mathcal{F})$. The local duality theorem extends to infinite level in the following way

Theorem 5.3.1. *Let $\mathcal{F}$ and $\mathcal{F}_n$ be as above, and let $\mathcal{F}_n^\vee = \mathscr{H}om_{\mathscr{O}_{X_n}}(\mathcal{F}_n, \mathscr{O}_{X_n})$ be the dual sheaf of $\mathcal{F}_n$. There is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism of topological $\mathcal{O}_C$-modules*

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathrm{H}^i(\mathfrak{X}_{\infty,C}, \mathcal{F}), \mathcal{O}_C) \cong \varprojlim_{n,\widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X_{n,\mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}). \tag{5.3.1}$$

The LHS is endowed with the weak topology, the RHS is endowed with the inverse limit topology, $\widetilde{\mathrm{Tr}}_n$ are the normalized traces of Proposition 5.1.1, and the extension of scalars is given by $X_{n,\mathcal{O}_C} = X_n \times_{\mathrm{Spec}\, \mathcal{O}_n} \mathrm{Spec}\, \mathcal{O}_C$.

Remark 5.3.2.

1. We could restate the previous theorem using $\omega_n^\circ = \pi_n^! \mathcal{O}_n$ instead of $\omega^2_{E,n,cusp}$, the trace $\widetilde{\mathrm{Tr}}_n$ would be replaced by the Serre duality trace relative to the morphism $X_{n+1,\mathcal{O}_C} \to X_{n,\mathcal{O}_C}$. Note that even though the ring $\mathcal{O}_C$ is not noetherian, all the objects involved are defined as pullbacks of objects which live over a finite extension of $\mathbb{Z}_p$, see Remark 5.2.2.

2. Let $\mathcal{F}_\eta^+ = \mathcal{F} \widehat{\otimes}_{\mathscr{O}_{\mathfrak{X}_\infty}} \mathscr{O}^+_{\mathcal{X}_\infty}$ be the pullback of $\mathcal{F}$ to $\mathcal{X}_\infty$, denote $\mathcal{F}_\eta = \mathcal{F}_\eta^+[\frac{1}{p}]$. By Corollary 5.2.6 we know that
$$\mathrm{H}^i(\mathcal{X}_\infty, \mathcal{F}_\eta) = \mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F})[\frac{1}{p}].$$
Thus, $\mathrm{H}^i(\mathcal{X}_{\infty,C}, \mathcal{F}_\eta)$ can be endowed with an structure of C-Banach space. Its dual is given by
$$\mathrm{H}^i(\mathcal{X}_{\infty,C}, \mathcal{F}_\eta)^* = (\varprojlim_{m,\widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X_{n,\mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}))[\frac{1}{p}].$$

3. Let $R_n : \mathbb{Z}_p[\zeta_N]^{cyc} \to \mathbb{Z}_p[\zeta_{Np^n}]$ denote the n-th normalized Tate trace, and let X'_n be the connected component of $X(N, p^n)_{\mathbb{Z}_p[\zeta_N]}$ corresponding to ζ_N. There is a natural injective map
$$\varprojlim_{m,\widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X'_{n,\mathbb{Z}_p[\zeta_N]^{cyc}}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}) \to \varprojlim_{n,R_n \circ \widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X'_n, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}).$$
However, this map is not surjective in general; the RHS is profinite while the LHS is not compact.

Before proving Theorem 5.3.1 let us say some words about the inverse limit of (5.3.1), it can be described as the kernel of the map

$$\prod_n \mathrm{H}^{1-i}(X_{n,\mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}) \xrightarrow{1-\widetilde{\mathrm{Tr}}_n} \prod_n \mathrm{H}^{1-i}(X_{n,\mathcal{O}_C}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}).$$

Moreover, the Corollary 5.2.5 says that the factors in the products are p-adically complete, separated and torsion free. The following lemma implies that the inverse limit is always p-adically complete and separated

Lemma 5.3.3. *Let N, M be torsion free, p-adically complete and separated $\mathbb{Z}_p$-modules, and $f : N \to M$ a $\mathbb{Z}_p$-linear map. Then* $\ker f$ *is torsion free, p-adically complete and separated.*

Proof. It is clear that $\ker f$ is torsion free. The map f is continuous for the p-adic topology, in particular $\ker f \subset N$ is a closed sub-module. Since M is torsion free, one has that $\ker f \cap p^s N = p^s \ker f$ for all $s \geq 1$. Then,

$$\ker f = \varprojlim_s (\ker f / (\ker f \cap p^s N)) = \varprojlim_s \ker f / p^s \ker f$$

proving the lemma. □

Next, we recall the $\mathrm{GL}_2(\mathbb{Q}_p)$-action in both sides of (5.3.1). Without loss of generality we take $C = K^{cyc}$. Let $\chi : \mathrm{Gal}(O^{cyc}/O) \xrightarrow{\sim} \mathbb{Z}_p^\times$ be the cyclotomic character. We define $\psi : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{Gal}(O^{cyc}/O)$ to be $\psi(g) = \chi^{-1}(p^{-v_p(\det g)} \det g)$, where $v_p : \mathbb{Q}_p^* \to \mathbb{Z}$ denotes the p-adic valuation. Fix $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $n \geq 0$. Let $m \geq 1$ be such that $\Gamma(p^m) \subset \Gamma(p^n) \cap g\Gamma(p^n)g^{-1}$, write $c_g : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{Q}_p)$ for the conjugation $x \mapsto gxg^{-1}$. We denote by $X(Np^n)_{c(g)}$ be the modular curve of level $\Gamma(N) \cap \Gamma(p^n) \cap g\Gamma(p^n)g^{-1}$, let $X_{n,c(g)}$ be the locus where the Weil pairing of the universal basis of $E[N]$ is equal to $\zeta_N \in \breve{\mathbb{Z}}_p$. We let ω° be the dualizing sheaf of $X_{n,c(g)}$, i.e. the exceptional inverse image of $O_{n,c(g)} := \mathrm{H}^0(X_{n,c(g)}, \mathscr{O}_{X_{n,c(g)}})$ over X_{n,c_g}.

The maps

$$\Gamma(p^m) \hookrightarrow \Gamma(p^n) \cap g\Gamma(p^n)g^{-1} \xrightarrow{c_{g^{-1}}} g^{-1}\Gamma(p^n)g \cap \Gamma(p^n) \hookrightarrow \Gamma(p^n)$$

induce maps of modular curves

$$X_m \xrightarrow{q_1} X_{n,c(g)} \xrightarrow{g} X_{n,c(g^{-1})} \xrightarrow{q_2} X_n,$$

with g an isomorphism. Notice that the modular sheaves ω_E^k are preserved by the pullbacks of q_1, q_2 and g. Let $\mathcal{F}$ and $\mathcal{F}_n$ be as in Theorem 5.3.1, we have induced maps of cohomology

$$R\Gamma(X_n, \mathcal{F}_n/p^s) \xrightarrow{q_1^* \circ g^* \circ q_2^*} R\Gamma(X_m, \mathcal{F}_m/p^s).$$

Taking direct limits we obtain a map

$$R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s) \xrightarrow{g^*} R\Gamma(\mathfrak{X}_\infty, \mathcal{F}/p^s).$$

Finally, taking derived inverse limits one gets the action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ on the cohomology $R\Gamma(\mathfrak{X}_\infty, \mathcal{F})$.

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on cohomology is not O^{cyc}-linear. In fact, it is ψ-semi-linear; this can be shown by considering the Cartan decomposition

$$\mathrm{GL}_2(\mathbb{Q}_p) = \bigsqcup_{n_1 \geq n_2} \mathrm{GL}_2(\mathbb{Z}_p) \begin{pmatrix} p^{n_1} & 0 \\ 0 & p^{n_2} \end{pmatrix} \mathrm{GL}_2(\mathbb{Z}_p)$$

and using the compatibility of the Weil pairing with the determinant.

The action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\varprojlim_{n, \widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X_{n,O^{cyc}}, \mathcal{F}_n^\vee \otimes \omega_{E,n,cusp}^2)$ is defined in such a way that the isomorphism (5.3.1) is equivariant. Namely, there is a commutative diagram of local duality pairings

provided by the functoriality of Serre duality

$$\begin{array}{ccc}
\mathrm{H}^{1-i}(X_{m,\mathcal{O}^{cyc}}, \mathcal{F}_m^\vee \otimes \omega^2_{E,m,cusp}) \times \mathrm{H}^i(X_{m,\mathcal{O}^{cyc}}, \mathcal{F}_m \otimes K/\mathcal{O}) & \longrightarrow & K^{cyc}/\mathcal{O}^{cyc} \\
\downarrow \widetilde{\mathrm{Tr}}_{q_1} \quad \uparrow q_1^* & & \| \\
\mathrm{H}^{1-i}(X_{n,c(g),\mathcal{O}^{cyc}}, (g\circ q_2)^*\mathcal{F}_n^\vee \otimes \omega^\circ) \times \mathrm{H}^i(X_{n,c(g),\mathcal{O}^{cyc}}, (g\circ q_2)^*\mathcal{F}_n \otimes K/\mathcal{O}) & \to & K^{cyc}/\mathcal{O}^{cyc} \\
\downarrow g^{-1^*} \quad \uparrow g^* & & \psi(g) \uparrow \\
\mathrm{H}^{1-i}(X_{n,c(g^{-1}),\mathcal{O}^{cyc}}, q_2^*\mathcal{F}_n^\vee \otimes \omega^\circ) \times \mathrm{H}^i(X_{n,c(g^{-1}),\mathcal{O}^{cyc}}, q_2^*\mathcal{F} \otimes K/\mathcal{O}) & \longrightarrow & K^{cyc}/\mathcal{O}^{cyc} \\
\downarrow \widetilde{\mathrm{Tr}}_{q_2} \quad \uparrow q_2^* & & \| \\
\mathrm{H}^{1-i}(X_{n,\mathcal{O}^{cyc}}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}) \times \mathrm{H}^i(X_{n,\mathcal{O}^{cyc}}, \mathcal{F}_n \otimes K/\mathcal{O}) & \longrightarrow & K^{cyc}/\mathcal{O}^{cyc}.
\end{array} \tag{5.3.2}$$

The maps $\widetilde{\mathrm{Tr}}_{q_1}$ and $\widetilde{\mathrm{Tr}}_{q_2}$ are induced by the Serre duality traces of q_1 and q_2 respectively, cf. Remark 5.3.2 (1). Thus, the right action of $g \in \mathrm{GL}_2(\mathbb{Q}_p)$ on a tuple $\underline{f} = (f_n) \in \varprojlim_{n,\widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X_{n,\mathcal{O}^{cyc}}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp})$ is given by $\underline{f}|_g = ((\underline{f}|_g)_n)_{n\in\mathbb{N}}$, where

$$(\underline{f}|_g)_n \quad = \quad \widetilde{\mathrm{Tr}}_{q_2} \circ g^{-1^*} \circ \widetilde{\mathrm{Tr}}_{q_1}(f_m)$$

for m big enough, and q_1, q_2 as in (5.3.2).

Proof of Theorem 5.3.1. Without loss of generality we take $C = K^{cyc}$. Let $\mathcal{F} = \omega^k_{E,\infty}$ or $\omega^k_{E,\infty,cusp}$. By Corollary 5.2.5 we have

$$\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}) \otimes (K/\mathcal{O}) = \mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F} \otimes K/\mathcal{O}).$$

Therefore

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}^{cyc}}(\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}), \mathcal{O}^{cyc}) &= \mathrm{Hom}_{\mathcal{O}^{cyc}}(\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}) \otimes K/\mathcal{O}, K^{cyc}/\mathcal{O}^{cyc}) \\
&= \mathrm{Hom}_{\mathcal{O}^{cyc}}(\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F} \otimes K/\mathcal{O}), K^{cyc}/\mathcal{O}^{cyc}).
\end{aligned}$$

On the other hand, we have

$$\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F} \otimes K/\mathcal{O}) = \varinjlim_{n,p_n^*} \mathrm{H}^i(X_{n,\mathcal{O}^{cyc}}, \mathcal{F}_n \otimes K/\mathcal{O})$$

where the transition maps are given by pullbacks. By local duality, Proposition 4.0.7, we have a natural isomorphism

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}^{cyc}}(\mathrm{H}^i(\mathfrak{X}_\infty, \mathcal{F}), \mathcal{O}^{cyc}) &= \varprojlim_{n,p_n^*} \mathrm{Hom}_{\mathcal{O}^{cyc}}(\mathrm{H}^i(X_{n,\mathcal{O}^{cyc}}, \mathcal{F}_n \otimes K/\mathcal{O}), K^{cyc}/\mathcal{O}^{cyc}) \\
&= \varprojlim_{n,\widetilde{\mathrm{Tr}}_n} \mathrm{H}^{1-i}(X_{n,\mathcal{O}^{cyc}}, \mathcal{F}_n^\vee \otimes \omega^2_{E,n,cusp}).
\end{aligned}$$

The isomorphism is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant by the diagram (5.3.2). □

We end this section with an application of the local duality theorem at infinite level to the completed cohomology. We let $\mathcal{X}_{n,proet}$ be the pro-étale site of the finite level modular curve as in §3 of [Sch13a], and $\mathcal{X}_{\infty,proet}$ the pro-étale site of the perfectoid modular curve as in Lecture 8 of [SW20].

Definition 5.3.4. Let $i \geq 0$. The *i-th completed cohomology group* of the modular tower $\{\mathcal{X}_n\}_{n\geq 0}$ is defined as

$$\widetilde{\mathrm{H}}^i := \varprojlim_s \varinjlim_n \mathrm{H}^i_{et}(\mathcal{X}_{n,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}).$$

Remark 5.3.5. The previous definition of completed cohomology is slightly different from the one of [Eme06]. Indeed, Emerton consider the étale cohomology with compact support of the affine modular curve Y_n. Let $j : Y_n \to X_n$ be the inclusion and $\iota : D_n \to X_n$ be the cusp divisor, both constructions are related by taking the cohomology of the short exact sequence

$$0 \to j_!(\mathbb{Z}/p^s\mathbb{Z}) \to \mathbb{Z}/p^s\mathbb{Z} \to \iota_*\iota^*\mathbb{Z}/p^s\mathbb{Z} \to 0.$$

Moreover, the cohomology at the cusps can be explicitely computed, and many interesting cohomology classes already appear in $\widetilde{\mathrm{H}}^1$.

We recall some important completed sheaves in the pro-étale site. Let $\mathcal{W}$ denote $\mathcal{X}_n$ or $\mathcal{X}_\infty$

- We denote $\widehat{\mathbb{Z}}_p = \varprojlim_s \mathbb{Z}/p^s\mathbb{Z}$, the p-adic completion over $\mathcal{W}_{proet}$ of the locally constant sheaf $\mathbb{Z}$.
- Let $\widehat{\mathscr{O}}^+_{\mathcal{W}} = \varprojlim_s \mathscr{O}^+_{\mathcal{W}}/p^s$ be the p-adic completion of the structural sheaf of bounded functions over $\mathcal{W}_{proet}$.

By Lemma 3.18 of [Sch13a] the sheaf $\widehat{\mathscr{O}}^+_{\mathcal{W}}$ is the derived inverse limit of the projective system $\{\mathscr{O}^+_{\mathcal{W}}/p^s\}_s$. On the other hand, the repleteness of the proétale site and Proposition 3.1.10 of [BS14] implies that $\widehat{\mathbb{Z}}_p$ is also the derived inverse limit of $\{\mathbb{Z}/p^s\mathbb{Z}\}_s$. We have the following proposition

Proposition 5.3.6. *Let $i \geq 0$, there is a short exact sequence*

$$0 \to R^1 \varprojlim_s \mathrm{H}^{i-1}_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}) \to \mathrm{H}^i_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \widehat{\mathbb{Z}}_p) \to \widetilde{\mathrm{H}}^i \to 0.$$

Proof. As $\widehat{\mathbb{Z}}_p = R\varprojlim_s \mathbb{Z}/p^s\mathbb{Z}$, the Grothendieck spectral sequence for derived limits gives short exact sequences for $i \geq 0$

$$0 \to R^1 \varprojlim_s \mathrm{H}^{i-1}_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}) \to \mathrm{H}^i_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \widehat{\mathbb{Z}}_p) \to \varprojlim_s \mathrm{H}^i_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}) \to 0.$$

Lemma 3.16 of [Sch13a] implies that $\mathrm{H}^i_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}) = \mathrm{H}^i_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z})$. On the other hand, Corollary 7.18 of [Sch12] gives an isomorphism

$$\mathrm{H}^i_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}) = \varinjlim_n \mathrm{H}^i_{et}(\mathcal{X}_{n,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}),$$

the proposition follows. □

Next, we relate the completed cohomologies $\widetilde{\mathrm{H}}^i$ with the coherent cohomology of $\mathcal{X}_\infty$ via the Primitive Comparison Theorem. This strategy is the same as the one presented by Scholze in Chapter IV of [Sch15] for Emerton's completed cohomology. In the following we work with the almost-setting with respect to the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}$

Proposition 5.3.7 ([Sch15, Theo. IV.2.1])**.** *There are natural almost isomorphisms*

$$\widetilde{\mathrm{H}}^i \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} =^{ae} \mathrm{H}^i_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \widehat{\mathscr{O}}^+_{\mathcal{X}_\infty}) =^{ae} \mathrm{H}^i(\mathfrak{X}_\infty, \mathscr{O}_{\mathfrak{X}_\infty}) \widehat{\otimes}_{\mathcal{O}^{cyc}} \mathcal{O}_{\mathbb{C}_p}. \tag{5.3.3}$$

In particular, $\widetilde{\mathrm{H}}^i = 0$ for $i \geq 2$, the $R^1 \varprojlim_s$ of Proposition 5.3.6 vanishes, and the $\widetilde{\mathrm{H}}^i$ are torsion free, p-adically complete and separated.

Proof. By the Primitive Comparison Theorem [Sch13a, Theo. 5.1], there are almost quasi-isomorphisms for all $n, s, i \in \mathbb{N}$

$$\mathrm{H}^i_{et}(\mathcal{X}_{n,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} =^{ae} \mathrm{H}^i_{et}(\mathcal{X}_{n,\mathbb{C}_p}, \mathscr{O}^+_{\mathcal{X}_n}/p^s).$$

Taking direct limits on n, and using Corollary 7.18 of [Sch12] one gets

$$\mathrm{H}^i_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p} =^{ae} \mathrm{H}^i_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}^+_{\mathcal{X}_\infty}/p^s). \tag{5.3.4}$$

Namely, we have $\mathscr{O}^+_{\mathcal{X}_\infty}/p^s = \varinjlim_n \mathscr{O}^+_{\mathcal{X}_n}/p^s$ as sheaves in the étale site of $\mathcal{X}_\infty$. In fact, let U_∞ be an affinoid perfectoid in the étale site of $\mathcal{X}_\infty$ which factors as a composition of rational localizations and finite étale maps. By Lemma 7.5 of [Sch12] there exists $n_0 \geq 0$ and an affinoid space $U_{n_0} \in \mathcal{X}_{n,et}$ such that $U_\infty = \mathcal{X}_\infty \times_{\mathcal{X}_{n_0}} U_{n_0}$. For $n \geq n_0$ denote the pullback of U_{n_0} to $\mathcal{X}_{n,et}$ by U_n, then $U_\infty \sim \varprojlim_{n \geq n_0} U_n$ and $\mathscr{O}^+(U_\infty)/p^s = \varinjlim_{n \geq n_0} \mathscr{O}^+(U_n)/p^s$.

The sheaf $\mathscr{O}^+_{\mathcal{X}_\infty}/p^s$ is almost acyclic on affinoid perfectoids, this implies that the RHS of (5.3.4) is equal to $\mathrm{H}^i_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}^+_{\mathcal{X}_\infty}/p^s)$. Then, the proof of Corollary 5.2.6 allows us to compute the above complex using the formal model $\mathfrak{X}_\infty$

$$\mathrm{H}^i_{an}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathscr{O}^+_{\mathcal{X}_\infty}/p^s) =^{ae} \mathrm{H}^i(\mathfrak{X}_\infty, \mathscr{O}_{\mathfrak{X}_\infty}/p^s) \otimes_{\mathcal{O}^{cyc}} \mathcal{O}_{\mathbb{C}_p}. \tag{5.3.5}$$

The Corollary 5.2.5 shows that the inverse system $\{\mathrm{H}^i(\mathfrak{X}_\infty, \mathscr{O}_{\mathfrak{X}_\infty}/p^s)\}_s$ satisfy the Mittag-Leffler condition. As $\mathcal{O}_{\mathbb{C}_p}/p^s$ is a faithfully flat $\mathbb{Z}/p^s\mathbb{Z}$-algebra, the inverse system $\{\mathrm{H}^i_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z})\}_s$ also satisfies the Mittag-Leffler condition. One deduces from Proposition 5.3.6 that

$$\mathrm{H}^i_{proet}(\mathcal{X}_{\infty,\mathbb{C}_p}, \widehat{\mathbb{Z}}_p) = \widetilde{\mathrm{H}}^i. \tag{5.3.6}$$

We also obtain that $\widetilde{\mathrm{H}}^i/p^s = \mathrm{H}^i_{et}(\mathcal{X}_{\infty,\mathbb{C}_p}, \mathbb{Z}/p^s\mathbb{Z})$ for all $i \in \mathbb{N}$. Taking inverse limits in (5.3.4), and using (5.3.5) and (5.3.6) one obtains the corollary. □

We obtain a description of the dual of the completed cohomology in terms of cuspidal modular forms of weight 2:

Theorem 5.3.8. *There is a* $\mathrm{GL}_2(\mathbb{Q}_p)$*-equivariant isomorphism of almost* $\mathcal{O}_{\mathbb{C}_p}$*-modules*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\widetilde{\mathrm{H}}^1 \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{C}_p}, \mathcal{O}_{\mathbb{C}_p}) =^{ae} \varprojlim_{n, \mathrm{Tr}_n} \mathrm{H}^0(X_{n,\mathcal{O}_{\mathbb{C}_p}}, \omega^2_{E,n,cusp}).$$

Proof. This is a consequence of Proposition 5.3.7 and the particular case of Theorem 5.3.1 when $\mathcal{F} = \mathscr{O}_{\mathfrak{X}_\infty}$ and $C = \mathbb{C}_p$. □

Part II.

p-adic Eichler-Shimura maps for the modular cuve

6. Introduction

Let p be a prime number, $\mathbb{A}_{\mathbb{Q}}^{\infty}$ the finite adeles of $\mathbb{Q}$, $\mathbb{A}_{\mathbb{Q}}^{\infty,p}$ the finite prime-to-p adeles, and $\mathbb{Z}_p$ the ring of p-adic integers. Let $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and $G_{\mathbb{Q}_p} = \mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$ the absolute Galois group. From now on we fix a neat compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$. Let $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be an open compact subgroup, we denote by $Y_{K_p}^{\mathrm{alg}}$ the modular curve over $\mathrm{Spec}\,\mathbb{Q}_p$ of level $K^pK_p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \times \mathrm{GL}_2(\mathbb{Q}_p)$, and by $X_{K_p}^{\mathrm{alg}}$ its compactification by adding cusps. Let Y_{K_p} and X_{K_p} be the rigid analytic varieties attached to the modular curves, seen as adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, cf. [Hub96]. Let $D = X_{K_p} \backslash Y_{K_p}$ be the cusp divisor. We endow X_{K_p} with the log-structure defined by D.

Given an fs log adic space Z and $? \in \{\mathrm{an}, \text{ét}, \text{két}, \text{proét}, \text{prokét}\}$, we denote by $Z_?$ its analytic, étale, Kummer-étale, proétale and pro-Kummer-étale sites respectively, see [Sch13a] and [DLLZ19].

In [Fal87], Faltings computed the Hodge-Tate decomposition of the étale cohomology (with coefficients) of the modular curve Y_{K_p}. More precisely, let E be the universal elliptic curve over Y_{K_p}, it admits an extension to a semi-abelian adic space E^{sm} over X_{K_p} (cf. [DR73]). Let $e : X_{K_p} \to E^{sm}$ be the unit section, $\omega_E = e^*\Omega^1_{E^{sm}/X}$ the modular sheaf and $T_pE = \varprojlim_n E[p^n]$ the Tate module over Y_{K_p}. Let $Y_{K_p,\mathbb{C}_p}$ and $X_{K_p,\mathbb{C}_p}$ denote the extension of scalars of the modular curves, we have the following theorem

Theorem 6.0.1 (Faltings)**.** *Let $k \geq 0$, there exists a Galois and Hecke equivariant isomorphism*

$$H^1_{\text{ét}}(Y_{K_p,\mathbb{C}_p}, \mathrm{Sym}^k T_pE) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(1) = H^0_{\mathrm{an}}(X_{K_p,\mathbb{C}_p}, \omega_E^{k+2}) \oplus H^1_{\mathrm{an}}(X_{K_p,\mathbb{C}_p}, \omega_E^{-k})(k+1) \tag{6.0.1}$$

called the Eichler-Shimura (ES) decomposition.

The first result of this paper is a new proof of Faltings' ES decomposition using BGG methods and the Hodge-Tate period map. Our proof is the proétale analogue of the BGG decomposition for the de Rham cohomology of Faltings-Chai [FC90, Ch. 5 Theo. 5.5]. Let us develop the ideas behind:

Let $X_\infty := \varprojlim_{K_p} X_{K_p}$ be Scholze's perfectoid modular curve and $\pi_{\mathrm{HT}} : X_\infty \to \mathbb{P}^1_{\mathbb{Q}_p}$ the Hodge-Tate period map [Sch15]. The morphism π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant where we see $\mathbb{P}^1_{\mathbb{Q}_p}$ as the left quotient of GL_2 by the upper triangular Borel $\mathbf{B}$. Let $\pi_{K_p} : X_\infty \to X_{K_p}$ be the natural map, we can see X_∞ as a pro-Kummer-étale K_p-torsor over X_{K_p}. We let $\widehat{\mathbb{Z}}_p$ and $\widehat{\mathscr{O}}_{X_{K_p}}$ denote the p-adic completions of the constant sheaf $\mathbb{Z}_p$ and the structural sheaf respectively, seen as sheaves over $X_{K_p,\text{prokét}}$. Let $\widehat{\mathbb{Z}}_p(1) = \varprojlim_n \mu_{p^n}$ be the Tate twist and $\widehat{\mathscr{O}}_{X_{K_p}}(i)$ the i-th twist of $\widehat{\mathscr{O}}_{X_{K_p}}$. We see the Tate module as a $\widehat{\mathbb{Z}}_p$-local system over $Y_{K_p,\text{proét}}$. By [DLLZ19, Theo. 4.6.1], T_pE admits a natural extension to the pro-Kummer-étale site of X_{K_p} which we denote in the same way. From now on we fix the level K_p and write $Y = Y_{K_p}$ and $X = X_{K_p}$.

The Hodge-Tate map is defined from the Hodge-Tate exact sequence

$$0 \to \omega_E^{-1} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) \xrightarrow{\mathrm{HT}^\vee} T_pE \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_X \xrightarrow{\mathrm{HT}} \omega_E \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X \to 0, \tag{6.0.2}$$

which is the variation in families of the Hodge-Tate decomposition for elliptic curves (cf. [Tat67]), via the universal trivialization of the Tate module $\Psi : \widehat{\mathbb{Z}}_p^2 \xrightarrow{\sim} T_pE$ over X_∞.

The $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariance of π_{HT} recovers (6.0.2) from a short exact sequence of GL_2-equivariant sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$. Indeed, let $\mathrm{Alg}\text{-}\mathbf{B}\text{-}\mathrm{Rep}$ be the category of finite dimensional algebraic representations of $\mathbf{B}$, $\mathrm{GL}_2\text{-}\mathrm{VB}_{\mathbb{P}^1_{\mathbb{Q}_p}}$ the category of GL_2-equivariant vector bundles (VB) over $\mathbb{P}^1_{\mathbb{Q}_p}$, and $\mathrm{GL}_2(\mathbb{Q}_p)\text{-}\widehat{\mathscr{O}}_{X_\infty}\text{-}\mathrm{Mod}$ the category of $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant $\widehat{\mathscr{O}}_{X_\infty}$-sheaves over X_∞. We have the following functors

$$\begin{array}{ccccccc} \mathrm{Alg}\text{-}\mathbf{B}\text{-}\mathrm{Rep} & \longrightarrow & \mathrm{GL}_2\text{-}\mathrm{VB}_{\mathbb{P}^1_{\mathbb{Q}_p}} & \xrightarrow{\pi_{\mathrm{HT}}^*} & \mathrm{GL}_2(\mathbb{Q}_p)\text{-}\widehat{\mathscr{O}}_{X_\infty}\text{-}\mathrm{Mod} & \longrightarrow & \widehat{\mathscr{O}}_X\text{-}\mathrm{Mod} \\ V & \longmapsto & \mathrm{GL}_2 \times^{\mathbf{B}} V & & \mathcal{F} & \longmapsto & (\pi_{K_p,*}\mathcal{F})^{K_p} \\ \mathcal{V}|_{[\mathbf{B}]} & \longmapsfrom & \mathcal{V} & & & & \end{array} \tag{6.0.3}$$

with $\mathrm{GL}_2 \times^{\mathbf{B}} V$ being the quotient of $\mathrm{GL}_2 \times V$ by the left diagonal action of $\mathbf{B}$. Let $\mathcal{V}$ be a GL_2-equivariant VB over $\mathbb{P}^1_{\mathbb{Q}_p}$, in the following we shall denote $\Upsilon(\mathcal{V})$ for the sheaf $(\pi_{K_p,*}(\pi_{\mathrm{HT}}^*(\mathcal{V})))^{K_p}$ over $X_{\text{prokét}}$.

Let $\mathbf{T} \subset \mathbf{B}$ be the diagonal torus and $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be a character, we see κ as a $\mathbf{B}$-representation by letting the unipotent radical of $\mathbf{B}$ act trivially. Let $W = \{1, w_0\}$ be the Weyl group of GL_2. We denote by $\mathscr{L}(\kappa)$ the GL_2-equivariant sheaf over $\mathbb{P}^1_{\mathbb{Q}_p}$ obtained from $w_0(\kappa)$ via (6.0.3). The standard representation St has a $\mathbf{B}$-filtration

$$0 \to (1,0)\mathbb{Q}_p \to \mathrm{St} \to (0,1)\mathbb{Q}_p \to 0. \tag{6.0.4}$$

By construction of π_{HT}, the pullback of (6.0.4) via Υ is equal to the Hodge-Tate-exact sequence (6.0.2). In particular, one obtains that $\Upsilon(\mathrm{Sym}^k \mathrm{St} \otimes \mathscr{O}_{\mathbb{P}^1}) = \mathrm{Sym}^k T_pE \otimes \widehat{\mathscr{O}}_X$ and that $\Upsilon(\mathscr{L}(\kappa)) = \omega_E^{k_1-k_2} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(k_2)$.

Let $\mathrm{LAlg}\text{-}\mathbf{B}\text{-}\mathrm{Rep}$ be the cateogory of locally algebraic representations of $\mathbf{B}$, in other words, the category of those representations which are written as colimits of finite dimensional representations. It is straightforward to check that the functor (6.0.3) naturally extends to the category $\mathrm{LAlg}\text{-}\mathbf{B}\text{-}\mathrm{Rep}$. Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be a dominant weight (i.e. such that $k_1 \geq k_2$) and V_κ the irreducible representation of GL_2 of highest weight κ (isomorphic to $\mathrm{Sym}^{k_1-k_2} \mathrm{St} \otimes (\det)^{k_2}$). We have the following theorem, cf. Theorem 10.1.5

Theorem 6.0.2. *Let $\alpha = (1,-1) \in X^*(\mathbf{T})$, and $\mathrm{BGG}(\kappa)$ be the dual BGG complex of κ (see §8.4)*

$$0 \to V_\kappa \to V(\kappa) \to V(w_0(\kappa) - \alpha) \to 0.$$

The pullback of $\mathrm{BGG}(\kappa)$ via the functor (6.0.3) is a short exact sequence

$$0 \to \mathrm{Sym}^{k_1-k_2} T_pE \otimes \otimes \widehat{\mathscr{O}}_X(k_2) \to \omega_E^{k_2-k_1} \otimes \mathscr{OC}_{\log}(k_1) \to \omega_E^{k_1-k_2+2} \otimes \mathscr{OC}_{\log}(k_2 - 1) \to 0,$$

where $\mathscr{OC}_{\log} = \mathrm{gr}^0 \mathscr{OB}_{\mathrm{dR},\log}$ and $\mathscr{OB}_{\mathrm{dR},\log}$ is Scholze's relative (log) de Rham period sheaf (see [Sch13a, §5] or [DLLZ18, §2.2]). Furthermore, let $\nu_{\mathrm{an}} : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\mathrm{an}}$ be the projection of sites. Then

$$R\nu_{\mathrm{an},*}(\mathrm{Sym}^k T_pE \otimes \widehat{\mathscr{O}}_X(1)) = \omega_E^{-k}(k+1)[0] \oplus \omega_E^{k+2}[-1].$$

Taking H^1-cohomology in the analytic site of $X_{\mathbb{C}_p}$ we recover the Eichler-Shimura decomposition of Theorem 6.0.1.

The proof of the previous theorem is a consequence of the isomorphism between $T_pE \otimes \widehat{\mathscr{O}}_X \otimes \omega_E$ and the Faltings extension $gr^1 \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$, cf. Proposition 10.1.3. This isomorphism was already known by Faltings, and used in his proof of the ES decomposition. This new proof provides a more explicit definition of the ES maps in terms of cocycles and can be generalized to Siegel varieties. Moreover, it shows immediately the degeneration of the spectral sequence appearing in [Fal87], as well as its natural splitting without using the Galois equivariance. However, notice that the existence of the period sheaf $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ is subject to the hypothesis that our rigid space is defined over a discretely valued field. It is worth to mention that the isomorphism between the twist of the Tate module and the Faltings extension was used by Lue Pan in [Pan20] to compute the relative Sen operator of the modular curve.

The second goal of this paper is the interpolation of the Eichler-Shimura decomposition (6.0.1). The H^0 of the overconvergent ES maps was previously constructed by Andreatta-Iovita-Stevens in [AIS15]. The strategy followed in this document is close to the construction of the ES map for Shimura curves in [CHJ17]. Roughly speaking, we apply a variant of the functor (6.0.3) to certain locally analytic sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$. In this way, we interpolate all the terms appearing in the Hodge-Tate exact sequence (6.0.2): we get overconvergent modular sheaves whose cohomology are the object of study in higher Coleman theory developed by Boxer-Pilloni [BP20, BP21]. The interpolation of the symmetric powers will be given by locally analytic principal series or locally analytic distributions as in [AS08]. Finally, the Hodge-Tate maps HT and $\mathrm{HT}^\vee$ can be put in families, obtaining the dlog map of [AIS15] as a particular case.

Let us sketch the main steps of the construction. Let $n \geq 1$ be an integer and

$$\mathrm{Iw}_n := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

be the Iwahori group modulo p^n. We will take $X = X_{\mathrm{Iw}_n}$, in such a way that $\pi_{\mathrm{Iw}_n} : X_\infty \to X$ is Galois of Galois group Iw_n. Let $\epsilon \geq \delta \geq n$ be rational numbers and (R, R^+) a uniform Tate algebra over $\mathbb{Q}_p$ which we may assume to be sheafy (i.e. such that the pre-sheaf of rational functions in $\mathrm{Spa}(R, R^+)$ is an actual sheaf). Let $T = \mathbf{T}(\mathbb{Z}_p)$ denote the $\mathbb{Z}_p$-points of the diagonal torus and $\chi = (\chi_1, \chi_2) : T \to R^{+,\times}$ a δ-analytic character (cf. Proposition 8.2.1). We denote by $R^+\widehat{\otimes}_{\mathbb{Z}_p}\mathscr{O}^+_{X,\text{két}}$ the p-adically complete tensor product in the Kummer-étale site of X, and by $R^+\widehat{\otimes}\mathscr{O}^+_X$ its inverse image to $X_{\text{prokét}}$. We let $R^+\widehat{\otimes}\widehat{\mathscr{O}}^+_X$ be the p-adically complete tensor product of R^+ and the completed sheaf $\widehat{\mathscr{O}}^+_X$. We warn that $R^+\widehat{\otimes}\mathscr{O}^+_X \neq R^+\widehat{\otimes}\widehat{\mathscr{O}}^+_X$. Given a character $\lambda : \mathbb{Z}_p^\times \to R^{+,\times}$ we denote by $R^+(\lambda)$ the $G_{\mathbb{Q}_p}$-module induced by the composition $G_{\mathbb{Q}_p} \xrightarrow{\chi_{\mathrm{cyc}}} \mathbb{Z}_p^\times \xrightarrow{\lambda} R^{+,\times}$. Finally, we write $\widehat{\mathscr{O}}^+_X(\lambda) := R^+(\lambda)\widehat{\otimes}\widehat{\mathscr{O}}^+_X$. We use similar notations and definitions for the structural sheaves $\mathscr{O}_X$ and $\widehat{\mathscr{O}}_X$.

We begin with the construction of all the sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$: for $w \in W = \{1, w_0\}$ we define a family of overconvergent neighbourhoods $\{U_w(\epsilon)\,\mathrm{Iw}_n\}_{\epsilon > n}$ of $w\,\mathrm{Iw}_n$ in $\mathbb{P}^1_{\mathbb{Q}_p}$. The affinoid spaces $U_w(\epsilon)\,\mathrm{Iw}_n$ admit sections of the quotient map $\mathrm{GL}_2 \to \mathbb{P}^1_{\mathbb{Q}_p}$. In particular, the $\mathbf{T}$-torsor $\mathbf{N}\backslash\mathrm{GL}_2 \to \mathbb{P}^1_{\mathbb{Q}_p}$, where $\mathbf{N}$ is the unipotent radical of $\mathbf{B}$, has a trivialization over $U_w(\epsilon)\,\mathrm{Iw}_n$. We define a $R\widehat{\otimes}\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}$ line bundle $\mathscr{L}(\chi)$ in the analytic site of $U_w(\epsilon)\,\mathrm{Iw}_n$ in the same way we have defined the line bundles $\mathscr{L}(\kappa)$ for $\kappa \in X^*(\mathbf{T})$. Then, we define the space of δ-analytic principal series of weight χ to be the R-Banach space

$$A^\delta_\chi = \Gamma(U_{w_0}(\delta)\,\mathrm{Iw}_n, \mathscr{L}(\chi)),$$

we let D^δ_χ be its dual. The space A^δ_χ has a natural action of Iw_n, so that it defines a constant Iw_n-equivariant sheaf on $\mathbb{P}^1_{\mathbb{Q}_p}$. Concretely, we see A^δ_χ and D^δ_χ as sheaves in the proétale site of $\mathbb{P}^1_{\mathbb{Q}_p}$, denoted by $\mathscr{A}^\delta_\chi$ and $\mathscr{D}^\delta_\chi$ respectively, as in Definition 7.2.3.

It is easy to construct maps

$$\begin{aligned} \chi R &\xrightarrow{\iota} A^{\delta}_{\chi} \text{ equivariant for the action of } \mathbf{B}(\mathbb{Z}_p) \cap \mathrm{Iw}_n \\ A^{\delta}_{\chi} &\xrightarrow{\mathrm{ev}_{w_0}} \chi R \text{ equivariant for the action of } w_0^{-1}\mathbf{B}(\mathbb{Z}_p)w_0 \cap \mathrm{Iw}_n, \end{aligned}$$

with ι being the highest weight vector, and ev_{w_0} the evaluation at w_0. We prove that these maps give rise morphisms of proétale Iw_n-equivariant sheaves

$$\begin{aligned} \mathscr{L}(w_0(\chi))\widehat{\otimes}_{\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}} \widehat{\mathscr{O}}_{\mathbb{P}^1_{\mathbb{Q}_p}} &\to \mathcal{A}^{\delta}_{\chi}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_{\mathbb{P}^1_{\mathbb{Q}_p}} \text{ over } U_1(\epsilon)\,\mathrm{Iw}_n \\ \mathcal{A}^{\delta}_{\chi}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_{\mathbb{P}^1_{\mathbb{Q}_p}} &\to \mathscr{L}(\chi)\widehat{\otimes}_{\mathscr{O}_{\mathbb{P}^1_{\mathbb{Q}_p}}} \widehat{\mathscr{O}}_{\mathbb{P}^1_{\mathbb{Q}_p}} \text{ over } U_{w_0}(\epsilon)\,\mathrm{Iw}_n\,. \end{aligned} \tag{6.0.5}$$

The next step is to translate all the previous constructions to the modular curve X. We start by defining the strict neighbourhoods of the w-ordinary locus $\{X_{w,\mathrm{Iw}_n}(\epsilon)\}_{\epsilon>n}$; they are equal to $\pi_{\mathrm{Iw}_n}(\pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n))$. The second object we descend to X are the overconvergent modular sheaves ω_E^{χ}; they are $R\widehat{\otimes}\mathscr{O}_X$-line bundles in the étale site of $X_{w,\mathrm{Iw}_n}(\epsilon)$. We refer to [BP21] for the general construction of these sheaves. We will follow essentially the same idea of *loc. cit.*, except that in some occasions we use more classical arguments involving the canonical group. The dictionary (6.0.3) gives then

$$\Upsilon(\mathscr{L}(\chi)) = \omega_E^{\chi} \otimes \widehat{\otimes}_{R\widehat{\otimes}\mathscr{O}_X} \widehat{\mathscr{O}}_X(\chi_2). \tag{6.0.6}$$

We continue with the pullback of the δ-analytic principal series and distributions, seen as Iw_n-equivariant proétale sheaves over $\mathbb{P}^1_{\mathbb{Q}_p}$. They define sheaves over $X_{\text{prokét}}$ which we keep denoting as $\mathcal{A}^{\delta}_{\chi}$ and $\mathcal{D}^{\delta}_{\chi}$ respectively. Finally, we pullback the maps (6.0.5) obtaining $G_{\mathbb{Q}_p}$-equivariant overconvergent Hodge-Tate maps of pro-Kummer-étale sheaves

$$\begin{aligned} \omega_E^{w_0(\chi)} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(\chi_1) &\xrightarrow{\mathrm{HT}^{\vee}} \mathcal{A}^{\delta}_{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X \text{ over } X_{1,\mathrm{Iw}_n}(\epsilon) \\ \mathcal{A}^{\delta}_{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X &\xrightarrow{\mathrm{HT}} \omega_E^{\chi} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(\chi_2) \text{ over } X_{w_0,\mathrm{Iw}_n}(\epsilon), \end{aligned}$$

similarly for $\mathcal{D}^{\delta}_{\chi}$ after applying $R\widehat{\otimes}\widehat{\mathscr{O}}_X$-duals. Taking pro-Kummer-étale cohomology one obtains the following theorem, which we state only for δ-analytic principal series and the transpose of the U_p-operator, see Theorem 10.3.2.

Theorem 6.0.3. *There are overconvergent Eichler-Shimura maps*

$$0 \to H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_{\epsilon}(\chi_1) \xrightarrow{ES^{\vee}_{\mathcal{A}}} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}^{\delta}_{\chi}\widehat{\otimes}\widehat{\mathscr{O}}_X) \xrightarrow{ES_{\mathcal{A}}} H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_{\epsilon}(\chi_2 - 1) \to 0 \tag{6.0.7}$$

satisfying the following properties:

1. *The composition $ES_{\mathcal{A}} \circ ES^{\vee}_{\mathcal{A}}$ is zero.*
2. *Assume that $\mathcal{V} = \mathrm{Spa}(R, R^+)$ is an affinoid subspace of the weight space $\mathcal{W}_T$ of T, and let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a dominant weight of $\mathbf{T}$. Let $\alpha = (1, -1) \in X^*(\mathbf{T})$ and let $\chi = \chi^{un}_{\mathcal{V}}$ be the universal character of $\mathcal{V}$. Then there is a commutative diagram*

$$\begin{array}{ccccc}
H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) & \xrightarrow{ES^\vee_{\mathcal{A}}} & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X) & \xrightarrow{ES_{\mathcal{A}}} & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})_\epsilon(k_1) & \longrightarrow & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}^\delta_\kappa \widehat{\otimes} \widehat{\mathscr{O}}_X) & \longrightarrow & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})_\epsilon(k_2 - 1) \\
\downarrow \text{Cor} & & \uparrow & & \uparrow \text{Res} \\
H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})(k_1) & \xrightarrow{ES^\vee} & H^1_{\text{ét}}(Y_{\mathbb{C}_p}, V_\kappa) \otimes \mathbb{C}_p & \xrightarrow{ES} & H^0_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})(k_2 - 1)
\end{array}$$

3. *The maps of (2) are Galois and U_p^t equivariant with respect to the good nomalizations of the U_p^t-operators. In particular, the diagram above restricts to the finite slope part with respect to the U_p^t-action.*

4. *Let $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-slope part of the restriction of (6.0.7) to $\mathcal{V}'$ is a short exact sequence of finite free $\mathbb{C}_p \widehat{\otimes}_{\mathbb{Q}_p} \mathscr{O}(\mathcal{V}')$-modules.*

5. *Keep the hypothesis of (4), and let χ be the universal character of $\mathcal{V}'$. Let $\widetilde{\chi} = \chi_1 - \chi_2 + 1 : \mathbb{Z}_p^\times \to R^{+,\times}$, and $b = \frac{d}{dt}|_{t=1}\widetilde{\chi}(t)$. Then we have a Galois-equivariant split after inverting b*

$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X)^{\leq h}_b = [H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})^{\leq h}_\epsilon(\chi_1)]_b \oplus [H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})^{\leq h}_\epsilon(\chi_2 - 1)]_b.$$

Remark 6.0.4.

1. The group $H^0_w(X_{\mathbb{C}_p}, -)_\epsilon$ is the overconvergent cohomology and $H^1_{w,c}(X_{\mathbb{C}_p}, -)_\epsilon$ the overconvergent cohomology with closed supports around the w-ordinary locus of X, see [BP20] and Definition 9.2.9 down below.

2. A similar statement holds for the distribution sheaves $\mathcal{D}^\delta_\chi$, in this case the overconvergent Eichler-Shimura map of [AIS15] is $ES_{\mathcal{D}}$.

3. Note that if $\kappa = (k_1, k_2)$ with $k_1 + 1 \neq k_2$, i.e. when the Hodge-Tate weights are not equal, one can choose $\mathcal{V}'$ small enough such that $b \neq 0$.

We finish the paper with the compatibility of the oveconvergent ES maps (6.0.7) with the Poincaré and Serre pairings. One can define a Poincaré pairing between the overconvergent proétale cohomologies

$$\langle -, - \rangle_P \quad : \quad H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathcal{D}^\delta_\chi(1)) \times H^1_{\text{proét}}(Y_{\mathbb{C}_p}, \mathcal{A}^\delta_\chi) \to \mathscr{O}(\mathcal{V}') \tag{6.0.8}$$

where the left hand side is the proétale cohomology with compact supports. On the other hand, one also has Serre pairings between overconvergent coherent cohomologies

$$\begin{aligned}
\langle -, - \rangle_S \quad &: \quad H^1_{w,c}(X, \omega_E^{-\chi}(-D))_\epsilon \times H^0_w(X, \omega_E^{\chi+\alpha})_\epsilon \to \mathscr{O}(\mathcal{V}') \\
\langle -, - \rangle_S \quad &: \quad H^1_{w,c}(X, \omega_E^{w_0(\chi)})_\epsilon \times H^0_w(X, \omega_E^{-w_0(\chi)+\alpha}(-D))_\epsilon \to \mathscr{O}(\mathcal{V}').
\end{aligned} \tag{6.0.9}$$

We have the following theorem

Theorem 6.0.5.

1. *The Theorem 6.0.3 holds for the cohomology with compact supports $H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathcal{D}^\delta_\chi)$ and the overconvergent cohomologies $H^0_w(X_{\mathbb{C}_p}, \omega_E^{-w_0(\chi)+\alpha}(-D))_\epsilon$ and $H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_\epsilon$.*

2. *The Poincaré and Serre pairings (6.0.8) and (6.0.9) are compatible with the U_p-operators and the overconvergent Eichler-Shimura maps.*

3. *Let $\mathcal{V} \subset \mathcal{W}_T$ be an open affinoid, and let χ be the universal character of $\mathcal{V}$. Let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a dominant weight and fix $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-part of the pairings (6.0.8) and (6.0.9) are perfect pairings of finite free $\mathbb{C}_p\widehat{\otimes}\mathscr{O}(\mathcal{V}')$-modules compatible with the Eichler-Shimura decomposition.*

The outline of the document is the following. In Section 7 we briefly recall the period sheaves over the pro-Kummer-étale site of an fs log adic space, cf. [Sch13a, DLLZ18, DLLZ19]. Then, we set the conventions of the representation theory of GL_2 and construct the dictionary (6.0.3).

In Section 8 we develop the overconvergent theory over the flag variety. We define the affinoid subspaces $U_w(\epsilon)\,\mathrm{Iw}_n$ and the sheaves $\mathscr{L}(\chi)$. We construct the δ-analytic principal series A^δ_χ and the maps (6.0.5). We recall some facts of the BGG theory for irreducible representations of GL_2, in particular we define the dual BGG complex $\mathrm{BGG}(\kappa)$.

Then in Section 9, we translate all the previous constructions from $\mathbb{P}^1_{\mathbb{Q}_p}$ to the modular curves via π_{HT}. We define the strict neighbourhoods of the w-ordinary locus, the overconvergent modular sheaves and the overconvergent Hodge-Tate maps. We give the good normalizations of the Hecke operators and show that the HT-maps are compatible with the normalized U_p-correspondance.

Finally, in Section 10, we show how to obtain the classical ES decomposition from the dual BGG complex, proving Theorems 6.0.2 and 6.0.1. We prove the analogous theorem for the cohomology with compact supports, and prove the compatibility of the pairings for the classical ES decomposition. Next, we construct the overconvergent ES maps and obtain Theorem 6.0.3. We prove the analogous theorem for the cohomology with compact supports and the compatibility of Poincaré and Serre duality for the overconvergent ES maps obtaining Theorem 6.0.5.

Notation

Throughout this document we fix a prime number p, we fix an algebraic closure of $\mathbb{Q}_p$ and denote by $\mathbb{C}_p$ its p-adic completion. We will work with adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which are either locally topologically of finite type over a non-archimedean extension K of $\mathbb{Q}_p$, or with perfectoid spaces. An adic space will be denote by Roman font letters X, Y, Z, W, V, etc. except in Definition 8.2.3 where we will use $\mathcal{W}_\Pi$ for the weight space attached to a finite $\mathbb{Z}_p$-module Π. In general, we will denote by $\mathcal{V}$ an affinoid admitting a character of Π, i.e. an affinoid endowed with a map $\mathcal{V} \to \mathcal{W}_\Pi$. We will denote the algebraic schemes over K as X^{alg}, Y^{alg}, Z^{alg}, etc., in such a way that X is the p-adic analytification of X^{alg} to an adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. We will use Calligraphy font $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$, etc. for sheaves over different sites of our spaces.

For affine algebraic groups over $\mathbb{Q}_p$ we will use bold font $\mathbf{H}$, $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$, etc., we denote its analytification to an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ by $\mathcal{H}, \mathcal{G}, \mathcal{B}, \mathcal{T}$, etc. Moreover, if $\mathbf{H}$ admits an integral model $\mathbf{H}_{\mathbb{Z}_p}$ over $\mathrm{Spec}\,\mathbb{Z}_p$, we denote by $\mathcal{H}^0$ the rigid generic fibre of the p-adic completion of $\mathbf{H}_{\mathbb{Z}_p}$. We have an open immersion $\mathcal{H}^0 \subset \mathcal{H}$. In addition, we will write $H = \mathbf{H}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ for its $\mathbb{Z}_p$-points. We define in a similar way the profinite groups G, B, T, etc. if any of these algebraic groups admit an integral model.

7. Preliminaries

7.1. Period sheaves in the pro-Kummer-étale site

Let K be a complete discretely valued field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and perfect residue field $\mathbb{F}$. Let C/K denote the p-adic completion of the algebraic closure of K, and let $G_K = \mathrm{Gal}(C/K)$ denote its absolute Galois group. Throughout this document we will work with adic spaces which are either locally topologically of finite type over a non archimedean field, or perfectoid spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. All of them are étale sheafy in the sense that their étale site admits a basis by affinoid adic spaces stable under rational localizations and finite étale maps (cf. [Hub96, §1.7] and [SW20, §1.7]). We consider the proétale site as in [Sch13a] and its extension to the pro-Kummer-étale site in [DLLZ19].

Let X be an adic space as before, we denote by X_{an}, $X_{\text{ét}}$ and $X_{\text{proét}}$ the analytic, étale and proétale sites of X respectively. If X is endowed with a log structure we let $X_{\text{két}}$ and $X_{\text{prokét}}$ be the Kummer-étale and the pro-Kummer-étale sites. For $? \in \{\mathrm{an}, \text{ét}, \text{két}\}$ we denote by $\mathscr{O}^+_{X,?}$ and $\mathscr{O}_{X,?}$ the structural sheaves. Let $\nu : X_{\text{proét}} \to X_{\text{ét}}$ be the natural projection of sites, we briefly recall the definition of the period sheaves over the proétale site of X, see §6 of [Sch13a]:

0) The structural sheaves $\mathscr{O}^+_X = \nu^{-1}(\mathscr{O}^+_{X,\text{ét}})$ and $\mathscr{O}_X = \nu^{-1}(\mathscr{O}_{X,\text{ét}})$.

1) The completed structural sheaves $\widehat{\mathscr{O}}^+_X = \varprojlim_s \mathscr{O}^+_X/p^s$ and $\widehat{\mathscr{O}}_X = \mathscr{O}^+_X[\frac{1}{p}]$.

2) The tilted sheaves $\widehat{\mathscr{O}}^{\flat,+}_X = \varprojlim_\Phi \widehat{\mathscr{O}}^+_X = \varprojlim_\Phi \mathscr{O}^+_X/p$ and $\widehat{\mathscr{O}}^\flat_{\mathcal{W}} = \varprojlim_\Phi \widehat{\mathscr{O}}_{\mathcal{W}}$ where $\Phi : x \mapsto x^p$.

3) The Fontaine's period sheaf $\mathbb{A}_{\mathrm{inf},X}$ defined as the derived p-adic completion of the Witt vectors $W(\widehat{\mathscr{O}}^{\flat,+}_X)$, the sheaf $\mathbb{B}_{\mathrm{inf},X} = \mathbb{A}_{\mathrm{inf},X}[\frac{1}{p}]$.

There is a natural surjective map $\theta : \mathbb{B}_{\mathrm{inf},X} \to \widehat{\mathscr{O}}_X$, which locally on perfectoid objects $\mathrm{Spa}(R, R^+)$ is the Fontaine's map

$$\theta : W(R^{\flat,+})[\frac{1}{p}] \to R \quad \theta([(x_n)]) = \lim_n \tilde{x}_n^{p^n},$$

where $(x_n)_{n\in\mathbb{N}} \in R^{\flat,+} = \varprojlim_\Phi R^+/p$ and $\tilde{x}_n$ is any lift of x_n to R^+. The kernel of θ is, locally proétale, generated by a non zero divisor ξ. For example, if $\mathrm{Spa}(R, R^+) \in X_{\text{proét}}$ admits a map to $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cyc}}, \mathbb{Z}_p^{\mathrm{cyc}})$, and $\epsilon = (\zeta_{p^n})_{n\in\mathbf{N}}$ is a compatible sequence of p-th power roots of unit, then $\omega = \frac{[\epsilon]-1}{[\epsilon]^{1/p}-1}$ is a generator of $\ker(\theta)$.

4) We have the de Rham period sheaf $\mathbb{B}^+_{\mathrm{dR},X} = \varprojlim_s \mathbb{B}_{\mathrm{inf},X}/(\ker\theta)^s$. We endow $\mathbb{B}^+_{\mathrm{dR},X}$ with the decreasing adic filtration defined by $\ker\theta$. We set $\mathbb{B}_{\mathrm{dR},X} = \mathbb{B}^+_{\mathrm{dR}}[\frac{1}{\xi}]$ and define $\mathrm{Fil}^r\,\mathbb{B}_{\mathrm{dR},X} = \xi^r\mathbb{B}^+_{\mathrm{dR},X}$ locally proétale.

We denote by $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}(C)$ the Tate twist, where G_K acts via the cyclotomic character $\chi_{\mathrm{cyc}} : G_K \to \mathbb{Z}_p^\times$. We will denote by $\mathbb{Z}_p$ the constant sheaf over $X_{\text{ét}}$, we will write $\widehat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ for the p-adic completion of $\mathbb{Z}_p$ on the proétale site. Let $\mathscr{F}$ be a sheaf over $X_{\text{proét}}$, we denote by $\mathscr{F}(i) = \mathscr{F} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ its i-th Tate twist, if $\mathscr{F}$ is a $\widehat{\mathbb{Z}}_p$-module then $\mathscr{F}(i) = \mathscr{F} \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathbb{Z}}_p(i)$. Let $B^+_{\mathrm{dR}} = \mathbb{B}^+_{\mathrm{dR}}(C, \mathcal{O}_C)$

be the Fontaine's de Rham period ring, the element $\log([\epsilon])$ converges to an element $t \in B^+_{\mathrm{dR}}$ for which G_K acts as multiplication by the cyclotomic character. Moreover, t generates $\ker\theta$ so that $\mathrm{gr}^i\mathbb{B}_{\mathrm{dR},X} = \widehat{\mathscr{O}}_X(i)$.

5) Let $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},X}$ be the sheafification of the pre-sheaf which sends an affinoid perfectoid $\mathrm{Spa}(R,R^+) = \varprojlim_i \mathrm{Spa}(R_i,R_i^+)$ to the direct limit of the $(\ker\tilde{\theta})$-adic completion of

$$(R_i^+\widehat{\otimes}_{W(\mathbb{F})}\mathbb{A}_{\mathrm{inf},X}(R,R^+))[\frac{1}{p}],$$

where $\tilde{\theta}$ is the map sending $r_i\otimes\alpha \mapsto r_i\theta(\alpha)$, and the completed tensor product is with respect to the p-adic topology. We endow $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},X}$ with the $(\ker\tilde{\theta})$-adic filtration. We define $\mathscr{O}\mathbb{B}_{\mathrm{dR},X} = \mathscr{O}\mathbb{B}^+_{\mathrm{dR},X}[\frac{1}{\xi}]$ and $\mathrm{Fil}^r\,\mathscr{O}\mathbb{B}_{\mathrm{dR},X} = \sum_{i+j=r}\xi^i\,\mathrm{Fil}^j\,\mathscr{O}\mathbb{B}^+_{\mathrm{dR},X}$.

In the following we shall omit the subscript X. Let X be an fs log-smooth adic space over $\mathrm{Spa}(K,O_K)$, whose log structure is given by a morphism of sheaves of monoids $\alpha : \mathcal{M}_{\text{ét}} \to \mathscr{O}_{X,\text{ét}}$ over the étale site of X, see Definition 2.2.2 of [DLLZ19]. Let $\mathcal{M}_{\text{két}} : X_{\text{két}} \to Sets$ be the sheaf of monoids of the Kummer-étale site mapping $U \rightsquigarrow \mathcal{M}_{\text{ét},U}(U)$.

We define the following sheaves over the pro-Kummer-étale site of X, cf. [DLLZ18, DLLZ19].

0) Let $\mathcal{M}$, $\mathscr{O}_X^+$, and $\mathscr{O}_X$ be the inverse image of $\mathcal{M}_{\text{két}}$, $\mathscr{O}^+_{X,\text{két}}$ and $\mathscr{O}_{X,\text{két}}$ from the Kummer-étale to the pro-Kummer-étale site respectively.

1) Let $\widehat{\mathscr{O}}_X^+ = \varprojlim_s \mathscr{O}_X^+/p^s$ and $\widehat{\mathscr{O}}_X = \widehat{\mathscr{O}}_X^+[\frac{1}{p}]$ be the completed structural sheaves of the pro-Kummer-étale site. Let $\widehat{\mathcal{M}}$ be the completed log structure of X, i.e. the pushout $\widehat{\mathscr{O}}_X^\times \xleftarrow{\iota} \mathscr{O}_X^\times \xrightarrow{\alpha^{-1}} \mathcal{M}$ of saturated monoids.

We can recover the sheaf $\mathcal{M}_{\text{két}}$ from the sheaf $\widehat{\mathcal{M}}$:

Lemma 7.1.1. *Let $\nu : X_{\text{prokét}} \to X_{\text{két}}$ be the projection of sites. We have $\nu_*\widehat{\mathcal{M}} = \mathcal{M}_{\text{két}}$.*

Proof. As $\widehat{\mathscr{O}}_X^\times$ and $\mathscr{O}_X^\times$ are sheaves on groups, and $\mathcal{M}$ is a saturated monoid, $\widehat{\mathcal{M}}$ is equal to the quotient $(\widehat{\mathscr{O}}_X^\times\oplus\mathcal{M})/\mathscr{O}_X^\times$ with respect to the inclusion $(-\iota,\alpha^{-1})$. Then, since $R\nu_*\mathscr{O}_X^\times = \mathscr{O}^\times_{X,\text{két}}$, one gets

$$\nu_*((\widehat{\mathscr{O}}_X^\times\oplus\mathcal{M})/\mathscr{O}_X^\times) = (\nu_*(\widehat{\mathscr{O}}_X^\times)\oplus\mathcal{M}_{\text{két}})/\mathscr{O}^\times_{\mathcal{W},\text{két}} = \mathcal{M}_{\text{két}}.$$

□

2) We define the tilted sheaves $\widehat{\mathscr{O}}_X^{\flat,+} = \varprojlim_\Phi \widehat{\mathscr{O}}_X^+ = \varprojlim_\Phi \mathscr{O}_X^+/p$, $\widehat{\mathscr{O}}_X^\flat = \varprojlim_\Phi \widehat{\mathscr{O}}_X$ and $\widehat{\mathcal{M}}^\flat = \varprojlim_\Phi \widehat{\mathcal{M}}$. We have a natural map of multiplicative monoids $\alpha^\flat : \widehat{\mathcal{M}} \to \widehat{\mathscr{O}}_X^\flat$.

3) The Fontaine's period sheaves $\mathbb{A}_{\mathrm{inf}}$, $\mathbb{B}_{\mathrm{inf}}$ $\mathbb{B}^+_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{dR}}$ are defined in the same way as for the proétale site. Moreover, there is a θ-map onto $\widehat{\mathscr{O}}_X$ whose kernel is locally proétale generated by a non-zero divisor ξ, and the sheaves $\mathbb{B}^+_{\mathrm{dR}}$ and $\mathbb{B}_{\mathrm{dR}}$ are filtered with graded pieces $\mathrm{gr}^i\mathbb{B}_{\mathrm{dR}} = \widehat{\mathscr{O}}_X(i)$.

Next, we recall the construction of the relative de Rham sheaves $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}$ and $\mathscr{O}\mathbb{B}_{\mathrm{dR},\log,X}$ over $X_{\text{prokét}}$ [DLLZ18, §2.2]. We write $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ and $\mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$ if X is clear from the context.

Construction 7.1.2. We see $\mathcal{M}$ as a multiplicative monoid. Let $\mathrm{Spa}(R,R^+)$ be a log perfectoid affinoid space in $X_{\text{prokét}}$, and $\mathrm{Spa}(R,R^+) = \varprojlim_i \mathrm{Spa}(R_i,R_i^+)$ a pro-Kummer-étale presentation over X modeled in a p-divisible monoid $M = \varprojlim_i M_i$. Consider the monoid algebra

$$(R_i\widehat{\otimes}_{W(\mathbb{F})}\mathbb{A}_{\mathrm{inf}}(R,R^+))[M_i\times_{\widehat{\mathcal{M}}(R,R^+)}\widehat{\mathcal{M}}^\flat(R,R^+)][\frac{1}{p}], \tag{7.1.1}$$

the completed tensor product being with respect to the p-adic topology. Given an element $a \in M_i \times_{\widehat{\mathcal{M}}(R,R^+)} \widehat{\mathcal{M}}^\flat(R,R^+)$ we denote by $e^{\log a}$ its image in (7.1.1). Let $\mathscr{I}$ be the ideal of (7.1.1) generated by $\{\alpha(a)\otimes 1-(1\otimes[\alpha^\flat(a)])e^{\log a}\}_a$ for $a \in M_i \times_{\widehat{\mathcal{M}}} \widehat{\mathcal{M}}^\flat$. Let $\tilde{S}_i$ denote the quotient of 7.1.1 by $\mathscr{I}$. We consider the map $\tilde{\theta}_{\log} : \tilde{S}_i \to \widehat{\mathscr{O}}_X(R,R^+) = R$ sending $e^{\log a} \mapsto 1$, and whose restriction to $R_i\widehat{\otimes}_{W(\mathbb{F})}\mathbb{A}_{\inf}(R,R^+)$ is the map $\tilde{\theta}$. Let S_i denote the $(\ker\tilde{\theta}_{\log})$-adic completion of $\tilde{S}_i$.

We define the relative log de Rham period sheaf $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ to be the sheafification of the presheaf

$$\mathrm{Spa}(R,R^+) \mapsto \varinjlim_i S_i.$$

We endow $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ with the $(\ker\tilde{\theta}_{\log})$-filtration. We define $\mathscr{O}\mathbb{B}_{\mathrm{dR},\log} = \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}[\frac{1}{\xi}]$ endowed with the convolution filtration as for the de Rham sheaf in the proétale case. Finally, we denote $\mathscr{O}\mathbb{C}_{\log}(i) = \mathrm{gr}^i\mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$ and $\mathscr{O}\mathbb{C}_{\log} = \mathscr{O}\mathbb{C}_{\log}(0)$.

Remark 7.1.3. We do not work with the completed de Rham sheaf $\mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$ of [DLLZ18]. Indeed, the main results of *loc. cit.* hold for the sheaf $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}[\xi^{-1}]$, proper log-smooth maps and the higher derived images of the constant sheaf $\widehat{\mathbb{Z}}_p$. In fact, the same arguments of [Sch13a] can be applied to this situation. Moreover, the sheaves we are interested in are constructed in this way, see §2.2 and §5.1 down below.

Let $\Omega^1_X(\log)$ denote the sheaf of log differentials of X [DLLZ19, §3]. The log connection $(d,\delta) : \mathscr{O}_X[\mathcal{M}] \to \Omega^1_X(\log)$ has a natural extension to $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ in the following way: for $x = (s\otimes\gamma)e^{\log a}$ in the monoid algebra (7.1.1) we define $\nabla(x) = \gamma e^{\log a}ds + (s\otimes\gamma)\delta(a)e^{\log a}$, where $\delta(a) = \delta(\mathrm{pr}_1(a))$ is applied to the projection in the first component. As $\ker\tilde{\theta}_{\log}$ is generated by $\ker\tilde{\theta}$ and the elements $e^{\log a}-1$ with $a \in M_i \times_{\widehat{\mathcal{M}}} \widehat{\mathcal{M}}^\flat$, one easily checks that $\nabla(\ker\tilde{\theta}_{\log})^k \subset \ker\tilde{\theta}_{\log}^{k-1}\otimes_{\mathscr{O}_X}\Omega^1_X(\log) \subset S_i\otimes_{\mathscr{O}_X}\Omega^1_X(\log)$. By completing with respect to $\ker\tilde{\theta}_{\log}$, and taking direct limits on i, one obtains an integrable log-connection

$$\nabla_{\log} : \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log} \to \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}\otimes_{\mathscr{O}_X}\Omega^1_X(\log)$$

satisfying Griffiths transversality. We get a log de Rham complex

$$0 \to \mathbb{B}^+_{\mathrm{dR}} \to \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log} \xrightarrow{\nabla_{\log}} \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}\otimes_{\mathscr{O}_X}\Omega^1_X(\log) \to \cdots \xrightarrow{\nabla_{\log}} \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}\otimes_{\mathscr{O}_X}\Omega^d_X(\log) \to 0. \tag{7.1.2}$$

By inverting ξ one has a log connection for $\mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$. Then, taking graded pieces one gets a Higgs bundle structure for $\mathscr{O}\mathbb{C}_{\log}$ [DLLZ18].

One of the most important features of (7.1.2) is that the Poincaré lemma holds [DLLZ18, Cor. 2.4.2]. It is a formal consequence of the fact that locally pro-Kummer-étale, the ring $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ is a power series ring over $\mathbb{B}^+_{\mathrm{dR}}$, [DLLZ18, Prop. 2.3.15] and [Sch13a, Prop. 6.10].

7.2. The geometric setup

Let $\mathbb{A}^\infty_{\mathbb{Q}}$ be the finite adeles of $\mathbb{Q}$ and $\mathbb{A}^{\infty,p}_{\mathbb{Q}}$ the prime-to-p finite adeles. From now on we fix a neat compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}^{\infty,p}_{\mathbb{Q}})$. Let $n \geq 0$, we denote by $\Gamma(p^n)$, $\Gamma_1(p^n)$ and $\Gamma_0(p^n)$ the principal congruence subgroups

$$\Gamma(p^n) = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \equiv 1 \mod p^n\}$$
$$\Gamma_1(p^n) = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \mod p^n\}$$
$$\Gamma_0(p^n) = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \equiv \begin{pmatrix}* & *\\ 0 & *\end{pmatrix} \mod p^n\}.$$

Let $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be a compact open subgroup, we denote by $Y^{\mathrm{alg}}_{K_p}$ and $X^{\mathrm{alg}}_{K_p}$ the modular and compactified modular curves of level K^pK_p over $\mathrm{Spec}\,\mathbb{Q}_p$ [DR73]. We let $Y^{\mathrm{alg}}(p^n)$, $Y^{\mathrm{alg}}_1(p^n)$ and $Y^{\mathrm{alg}}_0(p^n)$ denote the modular curves of level $K^p\Gamma(p^n)$, $K^p\Gamma_1(p^n)$ and $K^p\Gamma_0(p^n)$ respectively (similarly for the compactified modular curves). We let Y_{K_p} and X_{K_p} denote their p-adic analytification to adic spaces over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ [Hub96]. We endow X_{K_p} with the log structure defined by the cusp divisor $D = X_{K_p} \backslash Y_{K_p}$.

Let $E^{\mathrm{alg}}/Y^{\mathrm{alg}}_{K_p}$ be the universal elliptic curve and $E^{\mathrm{alg,sm}}/X^{\mathrm{alg}}_{K_p}$ its extension to a semi-abelian scheme. Let $e : X^{\mathrm{alg}}_{K_p} \to E^{\mathrm{alg,sm}}$ be the unit section and $\omega_E = e^*\Omega^1_{E^{\mathrm{alg,sm}}/X^{\mathrm{alg}}}$ the modular sheaf. The semi-abelian scheme $E^{\mathrm{alg,sm}}$ has a relative compactification to a log smooth morphism $f : \overline{E}^{\mathrm{alg}} \to X^{\mathrm{alg}}_{K_p}$ [DR73]. The analytification $\overline{E}$ of $\overline{E}^{\mathrm{alg}}$ is a log adic space, and the map $f : \overline{E} \to X_{K_p}$ is a proper log smooth morphism of adic spaces. Given an integer $k \in \mathbb{Z}$ we denote by $\omega^k_E = \omega^{\otimes k}_E$ the modular sheaf of weight k.

Let $E[p^n]/Y_{K_p}$ be the local system of p^n-torsion points of the universal elliptic curve. The étale sheaf $E[p^n]$ has a natural extension to a Kummer-étale local system over X_{K_p}, which by an abuse of notation we also write as $E[p^n]$. Indeed, let $j_{\mathrm{két}} : Y_{K_p,\mathrm{ét}} \to X_{K_p,\mathrm{két}}$ be the natural map of sites, then, by purity on p-torsion local systems ([DLLZ19, Theo. 4.6.1]), $Rj_{\mathrm{két},*}E[p^n]$ is concentrated in degree 0 and $j_{\mathrm{két},*}E[p^n]$ is a Kummer-étale local system. Moreover, let $\mu_{p^n,\overline{E}}$ be the local system of p^n-th roots of unity over $\overline{E}$. Using the principal polarization of E, and purity on local systems, one obtains

$$E[p^n] = \mathrm{Pic}^0(\overline{E}/X)[p^n] = R^1 f_{\mathrm{két},*}(\mu_{p^n,\overline{E}}).$$

The Tate module is the sheaf over $X_{\mathrm{prokét}}$ defined as the inverse limit $T_pE = \varprojlim_n E[p^n]$. By Lemma 3.18 of [Sch13a] one has $T_pE = R\varprojlim_n E[p^n]$. Moreover, we have

$$T_pE = R^1 f_{\mathrm{prokét},*} T_p\mathbb{G}_{m,\overline{E}},$$

where $T_p\mathbb{G}_{m,\overline{E}} = \varprojlim \mu_{p^n,\overline{E}} = \widehat{\mathbb{Z}}_p(1)$.

From now on an space without underlying log structure will be endowed with the trivial one. Fiber products are always fiber products of fs log adic spaces unless otherwise specified, cf. [DLLZ19, Prop. 2.3.27].

7.2.1. The Hodge-Tate period map

Let $\mathbb{Q}^{\mathrm{cyc}}_p$ be the p-adic completion of the p-adic cyclotomic field $\mathbb{Q}_p(\mu_{p^\infty})$. Scholze proved in [Sch15] that the inverse limit $X(p^\infty) = \text{“}\varprojlim_n X(p^n)\text{”}$ has a natural structure of a perfectoid space. Furthermore, he constructed a Hodge-Tate period map $\pi_{\mathrm{HT}} : X(p^\infty) \to \mathbb{P}^1_{\mathbb{Q}_p}$ parametrizing the Hodge-Tate filtration of elliptic curves at geometric points. More precisely, we have the following theorem

Theorem 7.2.1 (Theorem III.3.18 [Sch15])**.** *There exists a prefectoid space $X(p^\infty)$ over $\mathbb{Q}^{\mathrm{cyc}}_p$ satisfying the tilde limit property of [SW13, Def. 2.4.1]*

$$X(p^\infty) \sim \varprojlim_n X(p^n).$$

Moreover, there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT}} : X(p^\infty) \to \mathbb{P}^1_{\mathbb{Q}_p}$$

such that for any open rational subset U of $U_0 = \{[x : y] \| x/y| \leq 1\}$ or $U_1 = \{[x : y] \| y/x| \leq 1\}$ of $\mathbb{P}^1_{\mathbb{Q}_p}$, the inverse image $\pi^{-1}_{\mathrm{HT}}(U) \subset X(p^\infty)$ is an affinoid perfectoid subspace, and there is $n >> 0$ and an open affinoid $V_n \subset X(p^n)$ whose inverse image to $X(p^\infty)$ is equal to $\pi^{-1}_{\mathrm{HT}}(U)$.

For simplicity let us fix the level K_p and write $X = X_{K_p}$. We briefly recall how π_{HT} is defined. Let $f : \overline{E} \to X$ be the relative compactification of the universal elliptic curve. The primitive comparison Theorem ([Sch13a, Cor. 5.11] or [DLLZ19, Theo. 6.2.1]) gives an isomorphism

$$R^1 f_{\text{prokét},*}(T_p\mathbb{G}_m \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_{\overline{E}}) = T_pE \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_X.$$

On the other hand, we have a natural boundary map

$$R^1 f_{\text{an},*}\mathscr{O}_{\overline{E}} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) \to R^1 f_{\text{prokét},*}(T_p\mathbb{G}_m \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_{\overline{E}}).$$

Deformation theory provides an isomorphism

$$R^1 f_*\mathscr{O}_{\overline{E}} = \operatorname{Lie} E^{sm} = \omega_E^{-1},$$

taking $\widehat{\mathscr{O}}_X(1)$-duals, and using the principal polarization of E one obtains a map

$$T_pE \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_X \to \omega_E \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X.$$

Joining both arrows one obtains the Hodge-Tate exact sequence (see [Tat67] and [Sch15, Prop. III.3.1])

$$0 \to \omega_E^{-1} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) \xrightarrow{\mathrm{HT}^\vee} T_pE \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}}_X \xrightarrow{\mathrm{HT}} \omega_E \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X \to 0. \tag{7.2.1}$$

Now, the perfectoid space $X(p^\infty)$ trivializes the Tate module T_pE. Let $\psi^{\mathrm{un}} : \widehat{\mathbb{Z}}_p^2 \xrightarrow{\sim} T_pE$ be the universal trivialization, the pullback of (7.2.1) by $\psi^{un} \otimes \mathscr{O}_{X(p^\infty)}$ gives a line subbundle $\mathscr{L}_{\mathrm{HT}} \subset \mathscr{O}_{X(p^\infty)}^{\oplus 2}$ which defines the morphism π_{HT}.

7.2.2. Representation theory of GL_2

Let $\mathbf{B} \subset \mathrm{GL}_2$ be the Borel subgroup of upper triangular matrices, $\mathbf{N} \subset \mathbf{B}$ its unipotent radical and $\mathbf{T} \subset \mathbf{B}$ the diagonal torus. Let $\overline{\mathbf{B}}$ and $\overline{\mathbf{N}}$ be the opposite Borel of lower triangular matrices and its unipotent radical respectively. We let $W = \{1, w_0\}$ denote the Weyl group of GL_2, it acts on $\mathbf{T}$ by permutations of the diagonal components, we take $w_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ as a representative of w_0.

Let $X^*(T)$ be the character group of T, it is identified with $\mathbb{Z} \times \mathbb{Z}$ via the isomorphism $\mathbb{G}_m^2 \to \mathbf{T} : (t_1, t_2) \mapsto \mathrm{diag}(t_1, t_2)$. We see a character $\kappa \in X^*(\mathbf{T})$ as a character of $\mathbf{B}$ or $\overline{\mathbf{B}}$ by extending trivially to the unipotent radicals. We write $\mathfrak{g}$, $\mathfrak{b}$, $\overline{\mathfrak{b}}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$ and $\mathfrak{h}$ for the Lie algebras of GL_2, $\mathbf{B}$, $\overline{\mathbf{B}}$, $\mathbf{N}$, $\overline{\mathbf{N}}$ and $\mathbf{T}$ respectively. We see $\mathfrak{g}$ as a GL_2-representation via the adjoint action $(g, X) \mapsto gXg^{-1}$ for $g \in \mathrm{GL}_2$ and $X \in \mathfrak{g}$. Its restriction to $\mathbf{T}$ admits a weight decomposition $\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$, where $\mathbf{T}$ acts on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ by the character $\alpha = (1, -1)$ and $-\alpha$ respectively. Let $\alpha^\vee : \mathbb{G}_m \to \mathbf{T}$ be the cocharacter $\alpha(t) = \mathrm{diag}(t, t^{-1})$, we say that a character (or a weight) $\kappa = (k_1, k_2) \in X^*(T)$ is dominant if $\langle \kappa, \alpha^\vee \rangle \geq 0$, i.e. if $k_1 \geq k_2$. We denote by $X^*(\mathbf{T})^+$ the cone of dominant weights. A dominant weight is of the form $k\alpha \otimes \det^s$ for unique $k \in \mathbb{N}$ and $s \in \mathbb{Z}$.

Let $\mathcal{GL}_2$ / $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the analytification of GL_2 / $\mathrm{Spec}(\mathbb{Q}_p)$, and $\mathcal{GL}_2^0 \subset \mathcal{GL}_2$ the open subgroup whose (R, R^+)-points are given by

$$\mathcal{GL}_2^0(R, R^+) = \mathrm{GL}_2(R^+).$$

We use similar notation for the upper and lower triangular Borel subgroups of GL_2, their unipotent radicals and the diagonal torus. Let $\mathbb{G}_a \cong \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}$ be the analytic additive group and $\mathbb{G}_a^+ \subset \mathbb{G}_a$ the unit disc subgroup, i.e. $\mathbb{G}_a^+ = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$.

Let $FL := \mathbf{B}\backslash \mathrm{GL}_2$ be the flag variety of GL_2, we denote by $\mathscr{FL}$ its analytification. There is a GL_2-equivariant isomorphism $FL = \mathbb{P}^1_{\mathbb{Q}_p}$ induced by the right action

$$[x : y]\begin{pmatrix} a & b \\ c & d \end{pmatrix} = [ax + cy : bx + dy],$$

and taking $[0:1] \in \mathbb{P}^1_{\mathbb{Q}_p}$ as a base point. Since FL is proper, there are isomorphisms of adic spaces [Hub96, Prop. 1.9.6]

$$\mathscr{F}\ell = \mathcal{B}^0 \backslash \mathcal{GL}_2^0 = \mathcal{B} \backslash \mathcal{GL}_2 .$$

Let St be the left standard representation of GL_2. There is a complete description of the isomorphism classes of irreducible algebraic representation of GL_2 in terms of dominant weights [Mil17]

$$\begin{array}{ccc} X^*(\mathbf{T})^+ & \longleftrightarrow & \text{Irr-Rep } \mathrm{GL}_2 / \sim \\ \kappa = (k_1, k_2) & \mapsto & V_\kappa := \mathrm{Sym}^{k_1-k_2} \mathrm{St} \otimes \det{}^{k_2}. \end{array}$$

The representation V_κ is the irreducible representation of highest weight $\kappa = (k_1, k_2)$. Let V be a representation of GL_2, we denote by $V^\vee$ its contragradient representation: given $f \in V^\vee$ and $g \in \mathrm{GL}_2$, then $(gf)(v) = f(g^{-1}v)$ for $v \in V$. For irreducible representations we have isomorphisms

$$V_\kappa^\vee \cong V_{-w_0(\kappa)} = \mathrm{Sym}^{k_1-k_2} \mathrm{St} \otimes \det{}^{-k_1}. \tag{7.2.2}$$

A representative of V_κ can be constructed as a Borel induction:

$$V_\kappa = \{f : \mathrm{GL}_2 \to \mathbb{A}^1 \mid f(bg) = w_0(\kappa)(b) f(g) \text{ for } b \in \mathbf{B}\}. \tag{7.2.3}$$

Indeed, let $v_0^\vee \in V_\kappa^\vee$ be a highest weight vector, and $\langle -, - \rangle$ denote the natural pairing of $V_\kappa^\vee$ and V_κ. Then

$$v \mapsto f_v : g \mapsto \langle v_0^\vee, gv \rangle$$

defines an isomorphism as in (7.2.3). We can describe V_κ using the opposite Borel instead:

$$V_\kappa = \{f : \mathrm{GL}_2 \to \mathbb{A}^1 | f(\bar{b}g) = \kappa(\bar{b}) f(g) \text{ for } \bar{b} \in \overline{\mathbf{B}}\},$$

the isomorphism being $f \mapsto L_{w_0} f$, where $(L_{w_0} f)(g) = f(w_0 g)$.

Let VB_{FL} denote the category of vector bundles (VB) over FL with linear morphisms. We let GL_2-VB_{FL} be the subcategory of VB_{FL} whose objects are GL_2-equivariant vector bundles, and whose morphisms are GL_2-equivariant linear maps. Let Alg-$\mathbf{B}$-Rep be the category of finite dimensional algebraic representations of $\mathbf{B}$. There is an equivalence of categories

$$\begin{array}{ccc} \text{Alg-}\mathbf{B}\text{-Rep} & \leftrightarrows & \mathrm{GL}_2\text{-VB}_{FL} \\ V & \mapsto & \mathcal{V} := \mathrm{GL}_2 \times^{\mathbf{B}} V \\ V := \mathcal{V}|_{[1]} & \hookleftarrow & \mathcal{V} \end{array} \tag{7.2.4}$$

Where $\mathrm{GL}_2 \times^{\mathbf{B}} V$ is the quotient of $\mathrm{GL}_2 \times V$ by the left $\mathbf{B}$-action: $(g, v) \sim (bg, bv)$, and $[1] \in FL$ is the class of $1 \in \mathrm{GL}_2$. Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$, we denote by $\mathscr{L}(\kappa)$ the GL_2-equivariant line bundle over FL given by $\mathrm{GL}_2 \times^{\mathbf{B}} w_0(\kappa)$. Note that if κ is dominant then V_κ is equal to the global sections of $\mathscr{L}(\kappa)$. If V is the restriction of a GL_2-representation to $\mathbf{B}$, then $\mathcal{V}$ is isomorphic to $\mathscr{O}_{FL}^{\dim V}$ as a VB over FL, but not as a GL_2-equivariant VB in general. In particular, the line bundle $\mathscr{L}(\det)$ attached to $\det = (1, 1)$ is trivial. Moreover, the isomorphism class of $\mathscr{L}(\kappa)$ only depends on $k_1 - k_2$; it is isomorphic to the twisted sheaf $\mathscr{O}(k_1 - k_2)$ over $\mathbb{P}^1_{\mathbb{Q}_p}$.

Let $\widetilde{FL} = \mathbf{N} \backslash \mathrm{GL}_2$ be the natural left $\mathbf{T}$ torsor over FL, and let $\pi : \widetilde{FL} \to FL$ be the projection map. We have

$$\mathscr{L}(\kappa) = \pi_* \mathscr{O}_{\widetilde{FL}}[-w_0(\kappa)].$$

Indeed, let $U \subset FL$ be an open subscheme, then $\mathscr{O}_{\widetilde{FL}}(U)$ is the ring of regular functions $f \in \mathbf{B}U \subset \mathrm{GL}_2$ such that $f(nx) = f(x)$ for all $x \in \mathbf{B}U$ and $n \in \mathbf{N}$. The action of $\mathbf{T}$ over $\mathscr{O}_{\widetilde{FL}}(U)$ is given by the left regular action:

$$(t \cdot f)(x) = f(t^{-1}x).$$

Therefore, $f \in \mathscr{L}(\kappa)$ if and only if $(t \cdot f) = -w_0(\kappa)(t) f$. The previous shows that

$$\widetilde{FL} \cong \underline{\mathrm{Isom}}(\mathscr{O}_{FL}, \mathscr{L}(0,1)) \times \underline{\mathrm{Isom}}(\mathscr{O}_{FL}, \mathscr{L}(1,0))$$

as GL_2-equivariant left $\mathbf{T}$-torsors over FL.

7.2.3. Pullbacks via π_{HT}

Let $X = X_{K_p}$ be the modular curve of level K^pK_p, so that $\pi_{K_p} : X(p^\infty) \to X$ is a pro-Kummer-étale torsor of Galois group K_p.

Construction 7.2.2. Let K_p-Sh$(X(p^\infty)_{\text{prokét}})$ be the category of K_p-equivariant sheaves over the pro-Kummer-étale site of the prefectoid modular curve. There is a natural equivalence of categories

$$\begin{array}{rcl} \mathrm{Sh}(X_{\text{prokét}}) & \rightleftarrows & K_p\text{-Sh}(X(p^\infty)_{\text{prokét}}) \\ \mathscr{F} & \mapsto & \mathscr{F}|_{X(p^\infty)} \\ (\pi_{K_p,*}\mathscr{G})^{K_p} & \hookleftarrow & \mathscr{G}. \end{array}$$

On the other hand, the π_{HT}-map gives rise a functor

$$\begin{gathered} \Upsilon : K_p\text{-Sh}(\mathscr{F}\ell_{\text{proét}}) \to K_p\text{-Sh}(X(p^\infty)_{\text{prokét}}) = \mathrm{Sh}(X_{proket}) \\ \mathscr{F} \mapsto \pi_{\mathrm{HT}}^{-1}(\mathscr{F}) \mapsto (\pi_{K_p,*}(\pi_{\mathrm{HT}}^{-1}(\mathscr{F})))^{K_p}. \end{gathered}$$

If we restrict to GL_2-equivariant $\widehat{\mathscr{O}}_{\mathscr{F}\ell}$-vector bundles and compose with the functor (7.2.4) one gets

$$\Upsilon : \text{Alg-}\mathbf{B}\text{-Rep} \to \mathrm{GL}_2\text{-VB}_{\mathscr{F}\ell} \xrightarrow{\pi_{\mathrm{HT}}^*} \mathrm{GL}_2(\mathbb{Q}_p)\text{-}\widehat{\mathscr{O}}\text{-VB}_{X(p^\infty)} \xrightarrow{\pi_{K_p,*}(\ )^{K_p}} \widehat{\mathscr{O}}\text{-VB}_X.$$

with $\pi_{\mathrm{HT}}^*(\mathcal{W}) = \widehat{\mathscr{O}}_{X(p^\infty)} \otimes_{\pi_{\mathrm{HT}}^{-1}\widehat{\mathscr{O}}_{\mathscr{F}\ell}} \pi_{\mathrm{HT}}^{-1}(\mathcal{W})$.

Definition 7.2.3. Let $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ be a compact open subgroup and S be a topological space endowed with a continuous action of K_p. We define the sheaf $\underline{S}$ on $X_{\text{proét}}$ to be the K_p-equivariant sheaf over $X(p^\infty)$ whose points at an object $U \in X(p^\infty)_{\text{proét}}$ are

$$\underline{S}(U) = C^{\text{cont}}(|U|, \underline{S}).$$

Let V be a finite dimensional GL_2-representation, we write $\underline{V}$ for the pro-Kummer-étale local system defined by V via $X(p^\infty) \to X$, in this case we have $\Upsilon(V) = \underline{V} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_X$.

Example 7.2.4. Let St be the standard representation of GL_2, and consider its $\mathbf{B}$-filtration

$$0 \to (1,0)\mathbb{Q} \to \mathrm{St} \to (0,1)\mathbb{Q} \to 0.$$

It induces a short exact sequence of GL_2-equivariant VB

$$0 \to \mathscr{L}(0,1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}(1,0) \to 0 \tag{7.2.5}$$

which is the universal filtration of $\mathscr{O}_{\mathscr{F}\ell}^2$ over $\mathscr{F}\ell$. Then, by construction of π_{HT}, the pullback of (7.2.5) is the Hodge-Tate filtration (7.2.1). In particular,

$$\Upsilon(\mathscr{L}(1,0)) = \omega_E \otimes \widehat{\mathscr{O}}_X, \quad \Upsilon(\mathscr{L}(0,1)) = \omega_E^{-1} \otimes \widehat{\mathscr{O}}_X(1) \text{ and } \Upsilon(\mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell}) = T_pE \otimes \widehat{\mathscr{O}}_X.$$

For a character $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ we denote

$$\omega_E^\kappa := \omega_E^{k_1 - k_2},$$

with this convention we have

$$\Upsilon(\mathscr{L}(\kappa)) = \omega_E^\kappa \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(k_2). \tag{7.2.6}$$

8. Overconvergent theory over the flag variety

Let W denote the Weyl group of GL_2. The goal of this section is to introduce some overconvergent affinoid neighbourhoods of $w \in W$ in $\mathcal{GL}_2$ and in the flag variety. Then, over these affinoids, we define locally analytic line bundles interpolating the $\mathscr{L}(\kappa)$ of the previous section. We define the locally analytic principal series as the global sections of these line bundles, and show that there are highest weight vector and lowest weight vector maps. Finally, we put these weight vector maps in families over the overconvergent neighbourhoods of the flag variety, and show that they are compatible with respect to the Hecke action.

Throughout this section we follow the conventions of §7.2. Let $\mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ its valuation ring. Given $\delta \in \mathbb{Q}$ we denote $p^\delta \in \mathbb{C}_p$ an element with p-adic valuation $|p^\delta| = |p|^\delta$; we will only consider valuations of this form.

Definition 8.0.1. Let $\mathbf{H}$ be an affine group scheme over $\operatorname{Spec}\mathbb{Z}_p$. We let $\mathcal{H}$ the analytification of $\mathbf{H}_{\mathbb{Q}_p}$ and $\mathcal{H}^0$ the rigid generic fiber of the p-adic completion of $\mathbf{H}$. We denote the profinite group $H = \mathbf{H}(\mathbb{Z}_p)$. Define the following groups:

1. Let $m \geq 1$ be a positive integer, we set $H_m := \ker(\mathbf{H}(\mathbb{Z}_p) \to \mathbf{H}(\mathbb{Z}/p^m\mathbb{Z}))$.
2. Let $\delta \in \mathbb{Q}_{>0}$, we define $\mathcal{H}(\delta) \subset \mathcal{H}^0$ as the open subgroup whose (R, R^+)-points are given by

$$\mathcal{H}(\delta)(R, R^+) = \ker(\mathbf{H}(R^+) \to \mathbf{H}(R^+/p^\delta R^+)).$$

   We call $\mathcal{H}(\delta)$ the δ-neighbourhood of the identity in $\mathcal{H}$.

Let Z be an fs log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, we the following group-sheaves over the pro-Kummer-étale site of Z:

$$\mathcal{H}_Z := \mathbf{H}(\mathscr{O}_Z), \quad \widehat{\mathcal{H}}_Z := \mathbf{H}(\widehat{\mathscr{O}}_Z), \quad \mathcal{H}^0_Z := \mathbf{H}(\mathscr{O}^+_Z) \text{ and } \widehat{\mathcal{H}}^0_Z := \mathbf{H}(\widehat{\mathscr{O}}^+_Z)$$

(3) We define the δ-neighbourhoods of the identity in $\mathcal{H}_Z$ and $\widehat{\mathcal{H}}_Z$ to be the subgroups

$$\mathcal{H}(\delta)_Z := \ker(\mathbf{H}(\mathscr{O}^+_Z) \to \mathbf{H}(\mathscr{O}^+_Z/p^\delta)) \text{ and } \widehat{\mathcal{H}}(\delta)_Z := \ker(\mathbf{H}(\widehat{\mathscr{O}}^+_Z) \to \mathbf{H}(\mathscr{O}^+_Z/p^\delta)).$$

   We say that $\widehat{\mathcal{H}}_Z$ (resp. $\widehat{\mathcal{H}}^0_Z$ and $\widehat{\mathcal{H}}(\delta)_Z$) is the completion of $\mathcal{H}_Z$ (resp. $\mathcal{H}^0_Z$ and $\mathcal{H}(\delta)_Z$) in the pro-Kummer-étale site of Z.

(4) Let W be an adic space topologically of finite type over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let Z be an fs log adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We denote by W_Z the inverse image to $Z_{\text{prokét}}$ of the sheaf represented by W in the Kummer-étale site of Z. We let $\widehat{W}_Z$ be the completion of $\mathcal{M}_W$ in $\mathcal{W}_{\text{prokét}}$, in other words, $\widehat{W}_Z$ is the sheafification of the pre-sheaf whose points at an affinoid perfectoid $\mathrm{Spa}(R, R^+) \in Z_{\text{prokét}}$ are equal to $W(R, R^+)$.

If Z is clear from the context we omit the subscript in the sheaves of (3) and (4) above.

Remark 8.0.2. Let W and Z as in (4), let $U \in Z_{\text{prokét}}$ be an affinoid perfectoid with $U = \mathrm{Spa}(R, R^+)$ and $U = \varprojlim_i \mathrm{Spa}(R_i, R_i^+)$ a pro-Kummer-étale presentation. Then $W_Z(U) = \varinjlim_i W(R_i, R_i^+)$ while $\widehat{W}_Z(U) = W(R, R^+)$, that's the reason we think of $\widehat{W}_Z$ as a completion of W_Z.

Example 8.0.3. 1. Let $\mathbf{H}$ be the diagonal torus $\mathbf{T}$ or the upper unipotent $\mathbf{N}$, we have

$$\mathcal{T}(\delta) = \begin{pmatrix} 1 + p^\delta \mathbb{G}_a^+ & 0 \\ 0 & 1 + p^\delta \mathbb{G}_a^+ \end{pmatrix} \text{ and } \mathcal{N}(\delta) = \begin{pmatrix} 1 & p^\delta \mathbb{G}_a^+ \\ 0 & 1 \end{pmatrix}.$$

Let $m \geq 1$ be an integer, we have

$$T_m = \begin{pmatrix} 1 + p^m \mathbb{Z}_p & 0 \\ 0 & 1 + p^m \mathbb{Z}_p \end{pmatrix} \text{ and } N_m = \begin{pmatrix} 1 & p^m \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}.$$

2. Let $\delta > 0$, then

$$\mathcal{GL}_2(\delta) = \begin{pmatrix} 1 + p^\delta \mathbb{G}_a^+ & p^\delta \mathbb{G}_a^+ \\ p^\delta \mathbb{G}_a^+ & 1 + p^\delta \mathbb{G}_a^+ \end{pmatrix}.$$

Definition 8.0.4. Let $\mathcal{H}$ be an analytic group over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and Z an adic space.

1. Let $\mathfrak{M}$ be a $\mathcal{H}_Z$-torsor over $Z_{\text{prokét}}$, we define its completion as the $\widehat{\mathcal{H}}$-torsor $\widehat{\mathfrak{M}} := \widehat{\mathcal{H}}_Z \times^{\mathcal{H}_Z} \mathfrak{M}$.
2. Let $f : Z' \to Z$ be a morphism of adic spaces and $\widehat{\mathfrak{M}}$ a $\widehat{\mathcal{H}}$-torsor on $Z_{\text{prokét}}$. We define its pullback to a $\widehat{\mathcal{H}}$-torsor on $Z'_{\text{prokét}}$ as $f^*(\widehat{\mathfrak{M}}) = \widehat{\mathcal{H}}_{Z'} \times^{f^{-1}(\widehat{\mathcal{H}}_Z)} f^{-1}(\widehat{\mathfrak{M}})$ (similar definition for $\mathcal{H}_Z$-torsors).

Let $n \geq 1$ and $\mathrm{Iw}_n = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{Z}_p)$ so that the perfectoid modular curve $X(p^\infty)$ is a pro-Kummer-étale Iw_n-torsor over $X_0(p^n)$.

Definition 8.0.5. Let $\epsilon \geq \delta$ be positive rational numbers.

1. We denote
$$\mathcal{GL}_2(\epsilon, \delta) := \mathcal{N}(\delta) \times \mathcal{T}(\delta) \times \overline{\mathcal{N}}(\epsilon) \subset \mathcal{GL}_2^0 .$$
2. Suppose that $\delta \geq n$. The δ-neighbourhood of Iw_n in $\mathcal{GL}_2^0$ is the open subgroup
$$\mathcal{I}\mathrm{w}_n(\delta) := \mathrm{Iw}_n \, \mathcal{GL}_2(\delta) = \mathcal{GL}_2(\delta) \, \mathrm{Iw}_n .$$
The analytic group $\mathcal{I}\mathrm{w}_n(\delta)$ will be called an affinoid Iwahori subgroup of $\mathcal{GL}_2^0$.

8.1. Open affinoid subspaces of the flag variety

Let $W = \{1, w_0\}$ be the Weyl group of GL_2 and $w \in W$. Let $\epsilon > 0$ be rational and consider the ϵ-analytic neighbourhood of w

$$w \, \mathcal{GL}_2(\epsilon) \subset \mathcal{GL}_2^0,$$

we denote by $U_w(\epsilon)$ its image into $\mathscr{Fl}$.

Lemma 8.1.1. *1. The collection $\{U_w(\epsilon)\}_\epsilon$ of open affinoid subspaces of $\mathscr{Fl}$ is a basis of open affinoid neighbourhoods of $w \in \mathscr{Fl}$. Moreover, we have a decomposition*

$$\mathcal{GL}_2(\epsilon) = \mathcal{N}(\epsilon) \times \mathcal{T}(\epsilon) \times \overline{\mathcal{N}}(\epsilon) = \mathcal{GL}_2(\epsilon, \epsilon).$$

and a natural isomorphism

$$\overline{\mathcal{N}}(\epsilon) w \xrightarrow{\sim} U_w(\epsilon).$$

2. *The Iwahori subgroups admit Iwahori decompositions (cf Definition 8.0.1 for the conventions)*

$$\mathcal{I}\mathrm{w}_n(\epsilon) = (\overline{N}_n\overline{\mathcal{N}}(\epsilon)) \times (T\mathcal{T}(\epsilon)) \times (N\mathcal{N}(\epsilon))$$

3. *Let $\epsilon \geq \delta \geq n \geq 1$. We have Iwahori decompositions*

$$\mathcal{GL}_2(\epsilon,\delta)\,\mathrm{Iw}_n = (N\mathcal{N}(\delta)) \times (T\mathcal{T}(\delta)) \times (\overline{N}_n\overline{\mathcal{N}}(\epsilon))$$
$$\mathcal{GL}_2(\epsilon,\delta)w_0\,\mathrm{Iw}_n = (N_n\mathcal{N}(\delta)) \times (T\mathcal{T}(\delta)) \times (\overline{N\mathcal{N}}(\epsilon))w_0.$$

Proof. We prove (2), the points (1) and (3) are done in a similar way. It suffices to show the equality at (R,R^+)-points, with (R,R^+) a uniform affinoid $(\mathbb{Q}_p,\mathbb{Z}_p)$-algebra. By definition we have

$$\mathcal{I}\mathrm{w}_n(\epsilon) = \begin{pmatrix} \mathbb{Z}_p^\times(1+p^\epsilon\mathbb{G}_a^+) & \mathbb{Z}_p + p^\epsilon\mathbb{G}_a^+ \\ p^n\mathbb{Z}_p + p^\epsilon\mathbb{G}_a^+ & \mathbb{Z}_p^\times(1+p^\epsilon\mathbb{G}_a^+) \end{pmatrix}.$$

Then

$$\mathcal{I}\mathrm{w}_n(\epsilon)(R,R^+) = \begin{pmatrix} \mathbb{Z}_p^\times(1+p^\epsilon R^+) & \mathbb{Z}_p + p^\epsilon R^+ \\ p^n\mathbb{Z}_p + p^\epsilon R^+ & \mathbb{Z}_p^\times(1+p^\epsilon R^+) \end{pmatrix}$$

Let $g \in \mathcal{I}\mathrm{w}_n(\epsilon)(R,R^+)$. Writing

$$g = \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & 0 \\ 0 & x_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_3 & 1 \end{pmatrix}$$

and solving the equations one finds $x_3 \in p^n\mathbb{Z}_p + p^\epsilon R^+$, x_1 and $x_4 \in \mathbb{Z}_p^\times(1+p^\epsilon R^+)$, and $x_2 \in \mathbb{Z}_p + p^\epsilon R^+$ which gives (1). □

In a first approximation to describe the dynamics of the U_p-operators (cf. §9.1), we need to understand the action of $\mathbf{T}(\mathbb{Q}_p)$ over $\mathscr{F}\ell = \mathbb{P}^1_{\mathbb{Q}_p}$. This action has only two fixed points, represented by the elements of the Weyl group $W = \{1, w_0\}$. We have the following lemma

Lemma 8.1.2. *Let $\varpi = \mathrm{diag}(1,p)$. The following holds*

1. *$U_1(\epsilon)\varpi = U_1(\epsilon-1)$ and $U_{w_0}(\epsilon)\varpi = U_{w_0}(\epsilon+1)$.*
2. *Let $\epsilon \geq n \geq 1$, then $U_1(\epsilon)\,\mathrm{Iw}_n\,\varpi = U_1(\epsilon-1)\,\mathrm{Iw}_{n-1}$ and $U_{w_0}(\epsilon)\,\mathrm{Iw}_n\,\varpi = U_{w_0}(\epsilon+1)N_1$.*

Proof. It follows from Lemma 8.1.1 and the computation

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix}.$$

□

Let $\widetilde{FL} = \mathbf{N}\backslash\,\mathrm{GL}_2$ be the natural $\mathbf{T}$-torsor over FL. We denote $\widetilde{\mathscr{F}\ell} = \mathcal{N}\backslash\,\mathcal{GL}_2$ and $\widetilde{\mathscr{F}\ell}^0 = \mathcal{N}^0\backslash\,\mathcal{GL}_2^0$. The $\mathcal{T}$-torsor $\widetilde{\mathscr{F}\ell}$ is the rigid space defined by the generic fiber of $\widetilde{FL}$, while the $\mathcal{T}^0$-torsor $\widetilde{\mathscr{F}\ell}^0$ is the rigid generic fiber of the p-adic completion of $\widetilde{FL}$.

Definition 8.1.3. Let $w \in W$ and $\epsilon \geq \delta \geq n$ be positive rational numbers. We define the following trivial $\mathcal{T}(\delta)$-torsor over $U_w(\epsilon)$

$$\widetilde{U}_w(\epsilon,\delta) := \mathcal{N}(\delta)\backslash\,\mathcal{GL}_2(\epsilon,\delta)w = \mathcal{N}^0\backslash\mathcal{N}^0\mathcal{T}(\delta)\overline{\mathcal{N}}(\epsilon)w \xleftarrow{\sim} \mathcal{T}(\delta)\times\overline{\mathcal{N}}(\epsilon)w.$$

The space $\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n$ is a trivial $T\mathcal{T}(\delta)$-torsor over $U_w(\epsilon)\,\mathrm{Iw}_n$.

Remark 8.1.4. Suppose that $\delta \geq n$, by Lemma 8.1.1 we have the decomposition

$$\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n \xleftarrow{\sim} \begin{cases} T\mathcal{T}^0(\delta) \times \overline{N}_n \overline{\mathcal{N}}(\epsilon) & \text{if } w = 1 \\ w_0(T\mathcal{T}(\delta) \times N\mathcal{N}(\epsilon)) & \text{if } w = w_0 \end{cases}.$$

Furthermore, there are commutative diagrams of torsors

$$\begin{array}{ccc} \widetilde{U}_w(\epsilon,\delta) & \longrightarrow & \widetilde{\mathscr{F}\ell}^0 \\ {\scriptstyle T\mathcal{T}(\delta)}\downarrow & & \downarrow{\scriptstyle \mathcal{T}^0} \\ U_w(\epsilon) & \longrightarrow & \mathscr{F}\ell^0 \end{array} \qquad \begin{array}{ccc} \widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n & \longrightarrow & \widetilde{\mathscr{F}\ell}^0 \\ {\scriptstyle T\mathcal{T}(\delta)}\downarrow & & \downarrow{\scriptstyle \mathcal{T}^0} \\ U_w(\epsilon)\,\mathrm{Iw}_n & \longrightarrow & \mathscr{F}\ell^0. \end{array} \tag{8.1.1}$$

Let $\Lambda \subset \mathbf{T}(\mathbb{Q}_p)$ be the subgroup generated by $c = \mathrm{diag}(p,p)$ and $\varpi = \mathrm{diag}(1,p)$. Let $\Lambda^+ \subset \Lambda$ be the multiplicative monoid generated by $c^{\pm 1}$ and ϖ. To interpolate the U_p operators we have to normalize the action of Λ over the torsors (8.1.1). We normalize the action of Λ over $\widetilde{\mathscr{F}\ell}$ by considering the quotient $\Lambda\backslash\widetilde{\mathscr{F}\ell}$, note that it is a $\Lambda\backslash\mathcal{T}$-torsor over $\mathscr{F}\ell$.

Lemma 8.1.5. *The maps*

$$\widetilde{U}_w(\epsilon,\delta) \to \mathcal{N}\Lambda\backslash\mathcal{N}\Lambda\,\mathcal{GL}_2(\epsilon,\delta)w \subset \Lambda\backslash\widetilde{\mathscr{F}\ell}$$
$$\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n \to \mathcal{N}\Lambda\backslash\mathcal{N}\Lambda\,\mathcal{GL}_2(\epsilon,\delta)w\,\mathrm{Iw}_n \subset \Lambda\backslash\widetilde{\mathscr{F}\ell}$$

are isomorphisms of adic spaces. Moreover,

$$\widetilde{U}_1(\epsilon,\delta)\,\mathrm{Iw}_n\,\varpi = \widetilde{U}_1(\epsilon-1,\delta)\,\mathrm{Iw}_{n-1} \text{ and } \widetilde{U}_{w_0}(\epsilon,\delta)\,\mathrm{Iw}_n\,\varpi = \widetilde{U}_{w_0}(\epsilon+1,\delta)TN_1$$

where in the first equation we assume $\delta \leq \epsilon - 1$.

Proof. The second statement follows from Lemma 8.1.2. To prove the first statement, it suffices to take $w = 1$. Note that $\mathcal{N}\Lambda\mathcal{T}(\delta)\overline{\mathcal{N}}(\epsilon) = \bigsqcup_{\lambda\in\Lambda}\mathcal{N}\lambda\mathcal{T}(\delta)\overline{\mathcal{N}}(\epsilon)$. Then, it is enough to show that $\mathcal{N}^0\backslash\mathcal{N}^0\mathcal{T}(\delta)\overline{\mathcal{N}}(\epsilon) = \mathcal{N}\backslash\mathcal{N}\mathcal{T}(\delta)\overline{\mathcal{N}}(\epsilon)$ which is obvious by the Iwahori decomposition. □

8.2. Overconvergent line bundles

Let $T = \mathbf{T}(\mathbb{Z}_p)$ be the $\mathbb{Z}_p$-points of the diagonal torus. Let $\mathcal{V} = \mathrm{Spa}(R,R^+)$ be an affinoid adic space with R an uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T \to R^{+,\times}$ a continuous character. We need the following elementary proposition

Proposition 8.2.1. *Let* $\mathcal{V}$ *and* χ *as above. There exists* $\delta > 0$ *such that* χ *extends to a character*

$$\chi : T\mathcal{T}(\delta) \times \mathcal{V} \to \mathbb{G}_m. \tag{8.2.1}$$

If the previous holds we say that χ *is a* δ*-analytic character of* T.

Proof. Writing $\mathbf{T}$ as product of multiplicative groups, it is enough to prove the proposition for a character $\chi : \mathbb{Z}_p^\times \to R^{+,\times}$. By continuity, there is $n \geq 2$ such that $|\chi(1+p^n)-1| < |p|^{1/(p-1)}$. On the other hand, we have an isomorphism of groups $1 + p^n\mathbb{Z}_p \xrightarrow{\log} p^n\mathbb{Z}_p$ mapping $1+p^n$ to a generator. Then, for $x \in \mathbb{Z}_p$, we have

$$\chi((1+p^n)^x) = (\chi(1+p^n))^x = \sum_{k\geq 0}\binom{x}{k}(\chi(1+p^n)-1)^k.$$

The bound on $|\chi(1+p^n)-1|$ implies that

$$\frac{|\chi(1+p^n)-1|^k}{|k!|} \to 0 \text{ as } k \to \infty.$$

This shows that the function $x \mapsto \chi((1+p^n)^x)$ is analytic, and defines a character

$$\chi : (1+p^n\mathbb{G}_a^+) \times \mathcal{V} \to \mathbb{G}_m.$$

□

Remark 8.2.2. The same proof shows that given a finite $\mathbb{Z}_p$-module Π of rank s and a character $\chi : \Pi \to R^{+,\times}$, there exists a family of free elements $\{e_1, \dots, e_s\} \subset \Pi$ and $\delta > 0$ such that χ extends to an analytic character

$$\chi : \Pi\left(\prod_{i=1}^{s}(1+p^\delta\mathbb{G}_a^+)e_i\right) \times \mathcal{V} \to \mathbb{G}_m.$$

Definition 8.2.3. Let Π be a finite $\mathbb{Z}_p$-module and let $\mathfrak{W}_\Pi = \operatorname{Spf}\mathbb{Z}_p[[\Pi]]$ denote the weight space of its Iwasawa algebra. We denote by $\mathcal{W}_\Pi$ its rigid generic fiber over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. The space $\mathcal{W}_\Pi$ is described in (S, S^+)-points by

$$\mathcal{W}_\Pi(S, S^+) = \bigcup_{S_0 \subset S^+} \operatorname{Hom}_{\text{cont}}(\Pi, S_0^\times) = \operatorname{Hom}_{\text{cont}}(\Pi, S^+)$$

where S_0 runs over all the subrings of definition of S contained in S^+ (the last equality is follows from the fact that Π is finitely generated over $\mathbb{Z}_p$).

In the following we will take $\Pi = T = \mathbf{T}(\mathbb{Z}_p)$ or $\Pi = \mathbb{Z}_p^\times$. Let $\mathcal{V} = \operatorname{Spa}(R, R^+) \subset \mathcal{W}_T$ be an open affinoid subspace. The universal character $\chi^{\text{un}} : T \to \mathbb{Z}_p[[T]]^\times$ restricts to a character $\chi^{\text{un}}_{\mathcal{V}} : T \to R^{+,\times}$ which, by the previous proposition, extends to a δ-neighbourhood of T in $\mathcal{T}^0$ for some $\delta > 0$. The same holds for open affinoid subspaces of the weight space $\mathcal{W}_{\mathbb{Z}_p^\times}$.

Definition 8.2.4. Let $\epsilon \geq \delta \geq n \geq 1$. Let $\mathcal{V} = \operatorname{Spa}(R, R^+)$ be an uniform affinoid space over $\operatorname{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and $\chi : T \to R^{+,\times}$ a δ-analytic character. Let $w \in W = \{1, w_0\}$ be an element in the Weyl group of GL_2, consider the open neighbourhood $U_w(\epsilon)\operatorname{Iw}_n$ of w of §8.1. We define the $R^+\widehat{\otimes}_{\mathbb{Z}_p}\mathscr{O}^+_{\mathscr{F}\ell,\text{an}}$-line bundle $\mathscr{L}^+(\chi)_\epsilon$ over $U_w(\epsilon)\operatorname{Iw}_n$ to be the quotient

$$\mathscr{L}^+(\chi)_\epsilon = T\mathcal{T}(\delta)\backslash(\widetilde{U}_w(\epsilon,\delta)\operatorname{Iw}_n \times_{w_0(\chi)} R^+)$$

(see Definition 8.1.3). Equivalently, let $f : \widetilde{U}_w(\epsilon,\delta)\operatorname{Iw}_n \to U_w(\delta)$ be the projection map, then

$$\mathscr{L}^+(\chi)_\epsilon = f_*(\mathscr{O}^+_{\widetilde{U}_w(\epsilon,\delta)\operatorname{Iw}_n}\widehat{\otimes}R^+)[-w_0(\chi)].$$

We denote $\mathscr{L}(\chi)_\epsilon := \mathscr{L}^+(\chi)_\epsilon[\frac{1}{p}]$.

Remark 8.2.5. The sheaf $\mathscr{L}^+(\chi)_\epsilon$ is completely determined by its global sections as $\widetilde{U}_w(\epsilon,\delta)$ is a trivial $T\mathcal{T}(\delta)$-torsor. Indeed $\mathscr{L}^+(\chi)_\epsilon$ is the trivial line bundle whose global sections are functions $f : w\operatorname{Iw}_n \to R^+$ satisfying the following conditions:

i. f extends to an analytic function of $\mathcal{GL}_2(\epsilon,\delta)w\operatorname{Iw}_n$.

ii. $f(bx) = w_0(\chi)(b)f(x)$ for $b \in \mathcal{B} \cap w\mathcal{I}\mathrm{w}_n(\delta)w^{-1}$.

From now on all the affinoid pairs (R, R^+) endowed with a character of T will be supposed to be sheafy, i.e. we suppose that $\operatorname{Spa}(R, R^+)$ is an adic space. In fact, we will be only interested in $\mathcal{V} = \operatorname{Spa}(R, R^+)$ an open affinoid of the weight space of T, or $(R, R^+) = (K, O_K)$ a non-archimedean extension of $\mathbb{Q}_p$.

Proposition 8.2.6. *Keep the notation of Definition 8.2.4. The following holds*

1. *Let ϵ be fixed. The formation of $\mathscr{L}^+(\chi)_\epsilon$ is independent of $\epsilon \geq \delta' \geq \delta$.*
2. *If $\epsilon' \geq \epsilon$, then $\mathscr{L}^+(\chi)_\epsilon|_{U_w(\epsilon')\mathrm{Iw}_n} = \mathscr{L}^+(\chi)_{\epsilon'}$.*
3. *Let $\phi : (R, R^+) \to (R', R'^+)$ be a map of uniform affinoid rings, then $\mathscr{L}^+(\chi)_\epsilon \widehat{\otimes}_{R^+,\phi} R'^+ = \mathscr{L}^+(\phi \circ \chi)_\epsilon$.*

Proof. The proposition follows from the compatibility with respect to δ and ϵ of the isomorphisms

$$\widetilde{U}(\epsilon,\delta)\,\mathrm{Iw}_n \cong \begin{cases} T\mathcal{T}(\delta) \times \overline{N}_n \overline{\mathcal{N}}(\epsilon) \text{ if } w = 1 \\ w_0(T\mathcal{T}(\delta) \times N\mathcal{N}(\epsilon)) \text{ if } w = w_0 \end{cases} \qquad U_w(\epsilon) \cong \begin{cases} \overline{N}_n \overline{\mathcal{N}}(\epsilon) \text{ if } w = 1 \\ N\mathcal{N}(\epsilon) \text{ if } w = w_0 \end{cases},$$

and the obvious functoriality with respect to the affinoid algebra (R, R^+). □

As the above proposition shows, the line bundle $\mathscr{L}(\chi)_\epsilon$ is independent of ϵ and δ provided we work over a small enough neighbourhood of $w \in W$. In the following we will simply write $\mathscr{L}^+(\chi)$ instead of $\mathscr{L}^+(\chi)_\epsilon$. Let $c = \mathrm{diag}(p, p)$ and $\varpi = \mathrm{diag}(1, p)$. We now define a normalized action of $\Lambda^+ = \langle c^{\pm 1}, \varpi \rangle$ over $\mathscr{L}^+(\chi)$. By Lemma 8.1.5 we can see $U_w(\epsilon,\delta)\,\mathrm{Iw}_n$ as a subspace of $\Lambda\backslash\widetilde{\mathscr{F}\ell}$

Definition 8.2.7. We let $c^{\pm 1}$ act trivially on $\mathscr{L}^+(\chi)$. We let ϖ act on $\mathscr{L}^+(\chi)$ as the pullback of right multiplication on $\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n \subset \Lambda\backslash\widetilde{\mathscr{F}\ell}$. More precisely, let $f : \widetilde{U}_w(\epsilon,\delta) \to U_w(\epsilon)$ be the projection and consider the right multiplication by ϖ

$$R_\varpi : \widetilde{U}_1(\epsilon,\delta)\,\mathrm{Iw}_n \to \widetilde{U}_1(\epsilon-1,\delta)\,\mathrm{Iw}_n \text{ and } R_\varpi : \widetilde{U}_{w_0}(\epsilon,\delta)\,\mathrm{Iw}_n \to \widetilde{U}_{w_0}(\epsilon+1,\delta)\,\mathrm{Iw}_n\,.$$

Since the map R_ϖ commutes with the left action of $T\mathcal{T}(\delta)$, it induces a pullback map between the sheaves $\mathscr{L}^+(\chi) = f_*(\mathscr{O}^+_{\widetilde{U}_w(\epsilon,\delta)\mathrm{Iw}_n} \widehat{\otimes} R^+)[-w_0(\chi)]$ which we define to be the action of ϖ. Notice that, depending on w, ϖ expands or contracts de domain of $\mathscr{L}^+(\chi)$.

8.3. Locally analytic principal series

Let $\delta \geq n \geq 1$. Let $\mathcal{V} = \mathrm{Spa}(R, R^+)$ be an uniform affinoid space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ and $\chi : T \to R^{+,\times}$ a δ-analytic character. We use the line bundles $\mathscr{L}(\chi)$ over $U_{w_0}(\delta)\,\mathrm{Iw}_n$ to define δ-analytic principal series and distributions of weight χ. We will follow the conventions of [AS08].

Definition 8.3.1. We define the left $R^+[[\mathrm{Iw}_n]]$-module of δ-analytic principal series of weight χ as the global sections of $\mathscr{L}^+(\chi)$

$$A^{\delta,+}_\chi := \Gamma(U_{w_0}(\delta)\,\mathrm{Iw}_n, \mathscr{L}^+(\chi)), \quad A^\delta_\chi = \Gamma(U_{w_0}(\delta)\,\mathrm{Iw}_n, \mathscr{L}(\chi))$$

endowed with the right regular action. Suppose that χ is δ'-analytic for $\delta \geq \delta'$, by remark 8.2.5 we can describe $A^{\delta,+}_\chi$ as the R^+-module of functions $f : \mathrm{Iw}_n \to R^+$ such that

i. f extends to an analytic function of $\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n$,

ii. $f(bx) = w_0(\chi)(b)f(x)$ for $b \in \mathcal{B} \cap w_0\,\mathcal{I}\mathrm{w}_n(\delta')w_0^{-1}$.

The left $R^+[[\mathrm{Iw}_n]]$-module of δ-analytic distributions of weight χ is by definition

$$D^{\delta,+}_\chi := \mathrm{Hom}_{R^+}(A^{\delta,+}_\chi, R^+), \quad D^\delta_\chi = D^{\delta,+}_\chi[\frac{1}{p}]$$

endowed with the contragradient action.

Remark 8.3.2. The space $N\mathcal{N}(\delta)$ is a disjoint union of closed discs (cf. Example 8.0.3), in particular $\mathscr{O}^+(N\mathcal{N}(\delta))$ is an ON $\mathbb{Z}_p$-algebra. Let $\{e_i\}_{i\in I}$ be an ON basis of $\mathscr{O}^+(N\mathcal{N}(\delta))$, using the Iwahori decomposition one has isomorphisms of R^+-modules

$$A_\chi^{\delta,+} \cong \widehat{\bigoplus}_{i\in I} R^+ e_i \text{ and } D_\chi^{\delta,+} \cong \prod_{i\in I} R^+ e_i^\vee.$$

Remark 8.3.3. It is easy to compare the δ-analytic principal series and distributions defined above with those used in [AIS15]. Let $\chi : T \to R^{+,\times}$ be a δ-analytic character written as $\chi = (\chi_1, \chi_2)$. Consider the set $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ endowed with the right multiplication by Iw_n and the left multiplication by $\mathbb{Z}_p^\times$. We let $A_{\chi_1-\chi_2}^{\delta,+}$ be the space of functions $f : \mathbb{Z}_p^\times \times \mathbb{Z}_p \to R^{+,\times}$ satisfying the following conditions

i. $f|_{1\times\mathbb{Z}_p}$ extends to an analytic function of $\mathbb{Z}_p + p^\delta \mathbb{G}_a^+$,

ii. $f(tx) = (\chi_1 - \chi_2)(t) f(x)$ for $t \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Z}_p^\times \times \mathbb{Z}_p$.

Note that $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ endowed with the action of Iw_n and $\mathbb{Z}_p^\times$ is isomorphic to the quotient

$$\mathbb{Z}_p^\times \times \mathbb{Z}_p = \begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \backslash \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \backslash \mathrm{Iw}_n .$$

Thus, we have an isomorphism

$$A_\chi^{\delta,+} = A_{\chi_1-\chi_2}^{\delta,+} \otimes (\det)^{\chi_2}.$$

Let $c = \mathrm{diag}(p,p)$, $\varpi = \mathrm{diag}(1,p)$, $\Lambda = \langle c^{\pm 1}, \varpi^{\pm 1}\rangle \subset \mathbf{T}(\mathbb{Q}_p)$, $\Lambda^+ = \langle c^{\pm 1}, \varpi\rangle$ and $\Lambda^- = \langle c^{\pm 1}, \varpi^{-1}\rangle$. We want to endow $A_\chi^{\delta,+}$ with an action of Λ^+ compatible with the one of Iw_n (dually, we want to endow $D_\chi^{\delta,+}$ with an action of Λ^-). Let $\epsilon \geq \delta$, by Lemma 8.1.5 we can see $\widetilde{U}_{w_0}(\epsilon,\delta)\,\mathrm{Iw}_n$ as a subspace of $\Lambda\backslash\widetilde{\mathscr{F}\ell}$. Moreover, we have

$$\widetilde{U}_{w_0}(\epsilon,\delta)\,\mathrm{Iw}_n\,\varpi\,\mathrm{Iw}_n = \widetilde{U}_{w_0}(\epsilon+1,\delta)\,\mathrm{Iw}_n \subset \widetilde{U}_{w_0}(\epsilon,\delta)\,\mathrm{Iw}_n,$$

this leads to the following lemma

Lemma 8.3.4. *Let $\delta \geq \delta'$ and $\chi : T \to R^{+,\times}$ a δ'-analytic character. There is an action of the multiplicative monoid $\Sigma^+ := \mathrm{Iw}_n\,\Lambda^+\,\mathrm{Iw}_n$ over $A_\chi^{\delta,+}$ which is induced by right multiplication on $U_{w_0}(\delta,\delta')\,\mathrm{Iw}_n$. Furthermore, the action of ϖ on $A_\chi^{\delta+1,+}$ factors through $A_\chi^{\delta,+}$. Dually, there is an action of $\Sigma^- := \mathrm{Iw}_n\,\Lambda^-\,\mathrm{Iw}_n$ over $D_\chi^{\delta,+}$, and the action of ϖ^{-1} on $D_\chi^{\delta,+}$ factors through $D_\chi^{\delta+1,+}$. In particular, ϖ and ϖ^{-1} are compact operators of $A_\chi^{\delta,+}$ and $D_\chi^{\delta,+}$ respectively.*

Proof. To define the action of Σ^+ over $A_\chi^{\delta,+}$ it is enough to extend the character χ to Λ by setting $\chi(c) = \chi(\varpi) = 1$. Note that the inclusion $\widetilde{U}_{w_0}(\delta,\delta') \subset \Lambda\backslash\widetilde{\mathscr{F}\ell}$ is stable by multiplication of Σ^+. Then, the action is defined by taking pulbacks of $-w_0(\chi)$-equivariant functions of the $T\mathcal{T}(\delta')$-torsor $\widetilde{U}_{w_0}(\delta,\delta') \to U_{w_0}(\delta)$ (i.e. of the sheaf $\mathscr{L}^+(\chi)$).

It remains to study the action of ϖ. We have a commutative diagram

$$\begin{array}{ccc} \widetilde{U}_{w_0}(\delta,\delta')\,\mathrm{Iw}_n & \xrightarrow{\varpi} & \widetilde{U}_{w_0}(\delta+1,\delta')\,\mathrm{Iw}_n \\ \downarrow & & \downarrow \\ U_{w_0}(\delta)\,\mathrm{Iw}_n & \xrightarrow{\varpi} & U_{w_0}(\delta+1)\,\mathrm{Iw}_n . \end{array}$$

Taking global sections of the line bundle $\mathscr{L}^+(\chi)$ we get an action of ϖ over $A_\chi^{\delta,+}$ which is explicitly given by

$$(\varpi f)(x) = f(\varpi^{-1} x \varpi).$$

Moreover, it factors through $\varpi : A_\chi^{\delta+1,+} \to A_\chi^{\delta,+} \subset A_\chi^{\delta+1,+}$, which shows that it is a compact operator as $U_{w_0}(\delta+1) \to U_{w_0}(\delta)$ is a strict immersion. The statements for $D_\chi^{\delta,+}$ are clear by taking duals. □

In the definition of overconvergent modular symbols we are going to consider $A_\chi^{\delta,+}$ and $D_\chi^{\delta,+}$ as sheaves over the pro-Kummer-étale site of $X_0(p^n)$, the strategy is to use Construction 7.2.2 and Definition 7.2.3. In order to study the pro-Kummer-étale cohomology of these sheaves we need a devissage in finite local systems of the modular curve. We know that $A_\chi^{\delta,+}$ and $D_\chi^{\delta,+}$ have natural underlying topologies (the p-adic topology and the weak topology respectively). Furthermore, they have presentations as projective limits of inductive limits of finite Iw_n-modules. Using such a presentation we can describe the sheaves $\underline{A_\chi^{\delta,+}}$ and $\underline{D_\chi^{\delta,+}}$ over $\mathscr{F}\ell_{\text{proét}}$ as projective limits of inductive limits of finite constant sheaves. Then, we take the inverse image via $\pi_{\mathrm{HT}} : X(p^\infty) \to \mathscr{F}\ell$ and the Iw_n-equivariance to define the local systems over $X_0(p^n)$. We need a couple of lemmas.

Lemma 8.3.5. *Let (F, O_F) be a non archimedean field. Let $\mathcal{H} = \mathrm{Spa}(A, A^+)$ be an affinoid adic analytic group over F, and $Z = \mathrm{Spa}(R, R^+)$ an affinoid adic space topologically of finite type over $\mathrm{Spa}(F, O_F)$. Let $\Theta : \mathcal{H} \times Z \to Z$ be an action of $\mathcal{H}$ over Z. Then for all $N > 0$ there exists a neighbourhood $1 \in U \subset \mathcal{H}$ such that for all $g \in U$, $z \in Z$ and $f \in \mathscr{O}^+(Z)$, we have $|f(z) - f(gz)| \leq |p|^N$.*

Proof. As $\mathscr{O}^+(Z) = R^+$ is topologically of finite type over O_F, it suffices to prove the proposition for a single $f \in R^+$. Let $\Theta^* : R^+ \to (A^+ \widehat{\otimes}_{O_K} R^+)^+$ be the pullback of the multiplication map. Let $V \subset \mathcal{H} \times Z$ be the open affinoid subspace defined by the equation

$$|1 \otimes f - \Theta^*(f)| \leq |p|^N.$$

As V contains $1 \times Z$ and this is a quasi-compact closed subset of $\mathcal{H} \times Z$, there exists $1 \in U_f \subset U$ such that $U_f \times Z \subset V$. Therefore, for all $g \in U_f$ and $z \in Z$ we have $|f(z) - f(gz)| \leq |p|^N$. □

Lemma 8.3.6. *Let $s \geq 1$ be an integer. There exists a positive integer r such that $\Gamma(p^{s+r}) = \ker(\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Z}/p^{s+r}\mathbb{Z}))$ acts trivially on $A_\chi^{\delta,+}/p^s A_\chi^{\delta,+}$. In particular, we can write $A_\chi^{\delta,+} = \varprojlim_s \varinjlim_i F_{s,i}$ where $F_{s,i}$ are finite Iw_n-modules.*

Proof. It is enough to show that there exists $r > 0$ such that $T_{s+r}N_{s+r}$ acts trivially on $A_\chi^{\delta,+}/p^s A_\chi^{\delta,+}$. Let $f \in A_\chi^{\delta,+}$ and $g = tn \in TN$, then $(gf)(x) = \chi(t) f(t^{-1}xtn)$ for all $x \in N\mathcal{N}(\delta) \cong U_{w_0}(\delta)\,\mathrm{Iw}_n$. Since χ is continuous, we can find $r_1 >> 0$ such that $\chi(t) \equiv 1 \mod p^s$ for all $t \in T_{r_1}$. Applying Lemma 8.3.5, one finds $r_2 >> 0$ such that $f(t^{-1}xtn) \equiv f(x) \mod p^s$ for all $x \in N\mathcal{N}(\delta)$ and $tn \in T_{r_2}N_{r_2}$. Taking $r = \max\{r_1, r_2\}$ we have the first statement of the lemma. The second statement follows from the fact that $\mathrm{Iw}_n/\Gamma(p^{s+r})$ is a finite group. □

In order to write $D_\chi^{\delta,+}$ as a projective limit of inductive limits of finite Iw_n-modules we first give a filtration defining the weak topology

Definition 8.3.7. We let $\mathrm{Fil}^s D_\chi^{\delta,+}$ be the kernel of $D_\chi^{\delta,+} \to D_\chi^{\delta-1,+}/p^s D_\chi^{\delta-1,+}$.

Lemma 8.3.8. *We have an isomorphism $D_\chi^{\delta,+} = \varprojlim_s D_\chi^{\delta,+}/\mathrm{Fil}^s D_\chi^{\delta,+}$. Moreover, $D_\chi^{\delta,+}/\mathrm{Fil}^s D_\chi^{\delta,+}$ is a finite R^+-module and the inverse limit topology is the weak topology. In particular, writing $D_\chi^{\delta,+}/\mathrm{Fil}^s D_\chi^{\delta,+} = \varinjlim_i F'_{s,i}$ with $F'_{s,i}$ a finite Iw_n-module, we get $D_\chi^{\delta,+} = \varprojlim_s \varinjlim_i F'_{s,i}$.*

Proof. This follows from the fact that the injective map $D_\chi^{\delta,+} \to D_\chi^{\delta-1,+}$ is the dual of a compact map of ON R^+-modules. □

Corollary 8.3.9. *Let F denote $A_\chi^{\delta,+}$ or $D_\chi^{\delta,+}$, and let $F = \varprojlim_s \varinjlim_i F_{s,i}$ be the presentation as a projective limit of a direct limit of finite Iw_n-modules of Lemmas 8.3.6 and 8.3.8. Then the sheaf $\underline{F}$ of Definition 7.2.3 can be written as*

$$\underline{F} = \varprojlim_s \varinjlim_i \underline{F_{s,i}}$$

in the proétale site of $\mathscr{F}\ell$.

Proof. Let $U \in \mathscr{F}\ell_{\text{proét}}$ be an affinoid perfectoid. As the presentation $F = \varprojlim_s \varinjlim_i F_{s,i}$ is compatible with the topology of each sheaf (the p-adic or the weak topology), it is enough to show that

$$C^{\text{cont}}(|U|, \varprojlim_s \varinjlim_i F_{s,i}) = \varprojlim_s \varinjlim_i C^{\text{cont}}(|U|, F_{s,i}),$$

but this is clear as $|U|$ is a spectral space and $\varinjlim_i F_{s,i}$ is a discrete set. □

We end this subsection relating the δ-analytic principal series and distributions with the algebraic representations of GL_2. Let $\delta \geq \delta'$, we have an Iw_n-equivariant map of torsors

$$\begin{array}{ccc} \widetilde{U}_{w_0}(\delta,\delta')\,\mathrm{Iw}_n & \longrightarrow & \widetilde{\mathscr{F}\ell}^0 \\ {\scriptstyle T\mathcal{T}(\delta')}\downarrow & & \downarrow{\scriptstyle \mathcal{T}^0} \\ U_{w_0}(\delta)\,\mathrm{Iw}_n & \longrightarrow & \mathscr{F}\ell\,. \end{array} \tag{8.3.1}$$

Let $\kappa \in X^*(\mathbf{T})$ be a dominant weight, $\mathscr{L}(\kappa)$ the line bundle over $\mathscr{F}\ell$ defined by the character $w_0(\kappa)$ and the functor (7.2.4). Moreover, let $\mathscr{L}^+(\kappa) \subset \mathscr{L}(\kappa)$ be the $\mathscr{O}^+_{\mathscr{F}\ell}$-subsheaf defined by the lattice $w_0(\kappa)\mathbb{Z}_p \subset w_0(\kappa)\mathbb{Q}_p$ via $\widetilde{\mathscr{F}\ell}^0 \to \mathscr{F}\ell$. We set $V^+_\kappa := \Gamma(\mathscr{F}\ell, \mathscr{L}^+(\kappa))$. Recall from §7.2 that $V_\kappa = \Gamma(\mathscr{F}\ell, \mathscr{L}(\kappa))$ is the irreducible representation of GL_2 of highest weight κ.

Proposition 8.3.10. *Let $\kappa \in X^*(\mathbf{T})$ be a dominant weight.*

1. *There is a natural Iw_n-equivariant inclusion $V^+_\kappa \to A^{\delta,+}_\kappa$ for all $\delta \geq n$.*
2. *There is a natural Iw_n-equivariant map $D^{\delta,+}_\kappa \to V^{+,\vee}_\kappa = V^+_{-w_0(\kappa)}$ which becomes surjective after inverting p.*

Proof. From diagram (8.3.1), taking global sections of the line bundle $\mathscr{L}(\kappa)$ we obtain a natural Iw_n-equivariant map $V^+_\kappa \to A^{\delta,+}_\kappa$ which is clearly an inclusion. The map $D^{\delta,+}_\kappa \to V^+_{-w_0(\kappa)}$ is defined by taking duals. □

Remark 8.3.11. Let $c = \mathrm{diag}(p,p)$ and $\varpi = \mathrm{diag}(1,p)$, let $\Lambda^+ = \langle c^{\pm 1}, \varpi\rangle$ and $\Lambda^- = \langle c^{\pm 1}, \varpi^{-1}\rangle$ be multiplicative monoids. Set $\Sigma^+ = \mathrm{Iw}_n\,\Lambda^+\,\mathrm{Iw}_n$ and $\Sigma^- = \mathrm{Iw}_m\,\Lambda^-\,\mathrm{Iw}_n$. In order to make the maps of Proposition 8.3.10 $\Sigma^\pm$-equivariant we have to normalize the action of Λ on V^+_κ. Namely, we let $t \in \Lambda$ act on V^+_κ by $(tf)(x) := \frac{1}{\kappa(t)} f(xt)$. Thus, under this normalization, the maps $V^+_\kappa \to A^{\delta,+}_\kappa$ and $D^{\delta,+}_\kappa \to V^+_{-w_0(\kappa)}$ are Σ^+ and Σ^--equivariant respectively.

Definition 8.3.12. Let (R, R^+) be an uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T \to R^{+,\times}$ a δ-analytic character. We define the following Iw_n-equivariant sheaves over $\mathscr{F}\ell_{\text{proét}}$

1. The sheaf of δ-analytic principal series

$$\mathcal{A}^{\delta,+}_\chi = \underline{A^{\delta,+}_\chi} = \varprojlim_s \underline{(A^{\delta,+}_\chi / p^s A^{\delta,+}_\chi)} \text{ and } \mathcal{A}^\delta_\chi = \mathcal{A}^{\delta,+}_\chi[\frac{1}{p}] = \underline{A^\delta_\chi}.$$

   We let $\mathcal{A}^{\delta,+}_\chi \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ be the completed tensor product with respect to the p-adic topology.

2. The sheaf of δ-analytic distributions

$$\mathcal{D}^{\delta,+}_\chi = \underline{D^{\delta,+}_\chi} = \varprojlim_s \underline{D^{\delta,+}_\chi / \mathrm{Fil}^s} \text{ and } \mathcal{D}^\delta_\chi = \mathcal{D}^{\delta,+}_\chi[\frac{1}{p}] = \underline{D^\delta_\chi}.$$

   We let $\mathcal{D}^{\delta,+}_\chi \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ be the completed tensor product with respect to the filtration of $\mathcal{D}^{\delta,+}_\chi$.

Proposition 8.3.13. *The sheaves $\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ and $\mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ admit an $\widehat{\mathcal{I}\mathrm{w}}_n(\delta)$-equivariant action compatible with the $\mathcal{I}\mathrm{w}_n(\delta)$-action over the flag variety.*

Proof. See Definition 8.0.1 for the description of the completed sheaf $\widehat{\mathcal{I}\mathrm{w}}_n(\delta)_{\mathscr{F}\ell}$. First, we show that the sheaf $\widehat{\mathcal{I}\mathrm{w}}_n(\delta)_{\mathscr{F}\ell}$ is compatible with the action of $\mathcal{I}\mathrm{w}_n(\delta)$ over $\mathscr{F}\ell$ and the right multiplication. It is enough to prove that $\widehat{\mathcal{GL}}_{2,\mathscr{F}\ell}$ is $\mathcal{GL}_2$-equivariant for the right multiplication and the action over $\mathscr{F}\ell$. Consider the following diagram

$$\begin{array}{ccccc} & & \mathscr{F}\ell\times\mathcal{GL}_2 & & \\ & \overset{\mathrm{pr}_1}{\swarrow} & & \overset{m}{\searrow} & \\ \mathscr{F}\ell & & & & \mathscr{F}\ell \end{array}$$

where pr_1 is the projection onto the first factor and m the multiplication map. A $\mathcal{GL}_2$-equivariant action over $\widehat{\mathcal{GL}}_{2,\mathscr{F}\ell}$ is the same as an isomorphism

$$\mathrm{pr}_1^*(\widehat{\mathcal{GL}}_{2,\mathscr{F}\ell}) \tilde{\to} m^*(\widehat{\mathcal{GL}}_{2,\mathscr{F}\ell})$$

of $\widehat{\mathcal{GL}}_2$-torsors over $(\mathscr{F}\ell\times\mathcal{GL}_2)_{\mathrm{pro\acute{e}t}}$ satisfying a cocycle condition. Let $\mathrm{Spa}(S,S^+)$ be a perfectoid affinoid space in $(\mathscr{F}\ell\times\mathcal{GL}_2)_{\mathrm{pro\acute{e}t}}$, we have induced maps $x_0:\mathrm{Spa}(S,S^+)\to\mathscr{F}\ell$ and $g_0:\mathrm{Spa}(S,S^+)\to\mathcal{GL}_2$. We define the isomorphism

$$\begin{array}{ccccc} \mathrm{GL}_2(S)\cong\mathrm{pr}_1^*(\widehat{\mathcal{GL}}_{2,\mathscr{F}\ell})(S,S^+) & \tilde{\to} & m^*(\widehat{\mathcal{GL}}_{2,\mathscr{F}\ell})\cong\widehat{\mathcal{GL}}_2(S,S^+)\cong\mathrm{GL}_2(S) \\ g & \mapsto & gg_0 \end{array}$$

It is straightforward to check that this isomorphism provides the desired action.

Assume that χ is δ'-analytic for $\delta\geq\delta'$. We want to describe the sheaf $\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ as $\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$-valued functions over the sheaf $\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}$ satisfying the conditions of Definition 8.3.1. By Lemma 8.1.1, we have an Iwahori decomposition

$$\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}=(\underline{N_n}\widehat{\mathcal{N}}(\delta'))_{\mathscr{F}\ell}\times(\underline{T}\widehat{\mathcal{T}}(\delta'))_{\mathscr{F}\ell}\times(\underline{\overline{N}}\widehat{\overline{\mathcal{N}}}(\delta))_{\mathscr{F}\ell}w_0.$$

We define $\mathscr{OA}_\chi^{\delta,+}$ over $\mathscr{F}\ell_{\mathrm{pro\acute{e}t}}$ to be the sheaf whose points at an affinoid perfectoid $U=\mathrm{Spa}(S,S^+)\to\mathscr{F}\ell$ is the space of functions $f\in\mathscr{OA}_\chi^{\delta,+}(U)\subset\underline{\mathrm{Hom}}(\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}|_U,\widehat{\mathscr{O}}^+_U)$ satisfying:

i. f is induced by an analytic function $\mathrm{Hom}(\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n,\mathbb{G}_a^+)\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_U=\mathscr{O}^+(\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n)\widehat{\otimes}\widehat{\mathscr{O}}^+_U$.

ii. $f(bx)=w_0(\chi)(b)f(x)$ for $b\in\widehat{\mathcal{B}}\cap w_0\widehat{\mathcal{I}\mathrm{w}}_n(\delta')w_0^{-1}|_U$.

The Iwahori decomposition of $\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n$ shows that we have an isomorphism of sheaves over $\mathscr{F}\ell_{\mathrm{pro\acute{e}t}}$

$$\mathscr{OA}_\chi^{\delta,+}\cong\mathscr{O}^+(w_0N\mathcal{N}(\delta))\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}.$$

By Remark 8.3.2, we see that the natural map $\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}\to\mathscr{OA}_\chi^{\delta,+}$ is in fact an isomorphism of Iw_n-equivariant sheaves. Since $\mathscr{OA}_\chi^{\delta,+}$ has a natural $\widehat{\mathcal{I}\mathrm{w}}_n(\delta)$-equivariant action induced by the right regular action, we have proven the proposition for $\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$. We obtain the result for $\mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ by taking $\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$-duals (recall that the completion in this case is with respect to the weak topology, cf. Lemma 8.3.8). □

Corollary 8.3.14. *Let $\kappa\in X^*(\mathbf{T})$ be a dominant character. The maps $V_\kappa^+\to A_\kappa^{\delta,+}$ and $D_\kappa^{\delta,+}\to V_{-w_0(\kappa)}^+$ induce morphisms of $\widehat{\mathcal{I}\mathrm{w}}_n(\delta)$-equivariant sheaves over $\mathscr{F}\ell_{\mathrm{pro\acute{e}t}}$*

$$\underline{V_\kappa^+}\otimes_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}\to\mathcal{A}_\kappa^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}\text{ and }\mathcal{D}_\kappa^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}\to\underline{V_{-w_0(\kappa)}^+}\otimes_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}.$$

Proof. In the notation of the previous proposition, it is enough to consider the inclusion of sheaves over $\mathscr{F}\ell_{\text{proét}}$

$$\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}}_n \subset \widehat{\mathcal{GL}}_2$$

and take analytic functions with values in $\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ satisfying $f(bx) = w_0(\kappa)(b)f(x)$ for $b \in \widehat{\mathcal{B}}$. □

8.4. The dual BGG complex and highest weight vector maps

We begin this subsection with a brief construction of the dual BGG complexes for GL_2, we refer to [Hum08] for a more general treatment of the subject. Let $W = \{1, w_0\}$ be the Weyl group of GL_2 and $\mathbf{B}w_0\mathbf{N} \subset \mathrm{GL}_2$ the big cell. We have a commutative diagram of torsors

$$\begin{array}{ccc} \mathbf{B}w_0\mathbf{N} & \longrightarrow & \mathrm{GL}_2 \\ {\scriptstyle\mathbf{B}}\downarrow & & \downarrow{\scriptstyle\mathbf{B}} \\ \mathbf{B}\backslash\mathbf{B}w_0\mathbf{N} & \longrightarrow & FL \end{array} \tag{8.4.1}$$

where the left vertical arrow is in fact a trivial $\mathbf{B}$-torsor. Let $\kappa \in X^*(\mathbf{T})$ be a character and $\mathscr{L}(\kappa)/FL$ the line bundle attached to $w_0(\kappa)$ by the functor (7.2.4), in other words, the line bundle given by the quotient $\mathbf{B}\backslash(\mathrm{GL}_2 \times w_0(\kappa))$. If κ is dominant, let $V_\kappa := \Gamma(\mathscr{F}\ell, \mathscr{L}(\kappa))$ be the irreducible representation of GL_2 of highest weight κ.

Definition 8.4.1. We define the $(\mathfrak{g}, \mathbf{B})$-representation $V(\kappa) := \Gamma(\mathbf{B}\backslash\mathbf{B}w_0\mathbf{N}, \mathscr{L}(\kappa))$, where the action of $(\mathfrak{g}, \mathbf{B})$ is induced by the right regular action on the big cell.

As $\mathbf{B}$-module, $V(\kappa)$ is a twist of the algebra of regular functions of $w_0\mathbf{N}$. Indeed, there is an isomorphism $\mathbf{B}\backslash\mathbf{B}w_0\mathbf{N} \cong w_0\mathbf{N}$, and one has

$$V(\kappa) = \kappa \otimes V(1) = \kappa \otimes \Gamma(w_0\mathbf{N}, \mathscr{O}_{\mathbf{N}}), \tag{8.4.2}$$

where the action of $\mathbf{B}$ over $\Gamma(w_0\mathbf{N}, \mathscr{O}_{\mathbf{N}})$ is induced by the adjoint action $(n, b) \mapsto b^{-1}nb$ for $(n, b) \in \mathbf{N} \times \mathbf{B}$.

Remark 8.4.2. The $(\mathfrak{g}, \mathbf{B})$-module $V(\kappa)$ is in fact the admissible dual of the Verma module of highest weight κ, see §3.10 of [AS08].

Let κ be a dominant weight. Taking the global sections of $\mathscr{L}(\kappa)$ in the diagram (8.4.1), one obtains a map

$$V_\kappa \to V(\kappa).$$

Writing $\kappa = (k_1, k_2) \in \mathbb{Z}^2$, and $\mathbb{G}_a \cong \mathbf{N}$ via $X \mapsto \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$, the map of V_κ in $V(\kappa)$ is identified in (8.4.2) with the inclusion $\mathbb{Q}_p[X]_{k_1-k_2} \subset \mathbb{Q}_p[X] \cong C(w_0\mathbf{N}, \mathbb{Q}_p)$ of polynomials of degree $\leq k_1 - k_2$. We have the following proposition

Proposition 8.4.3. *Let $\alpha = (1, -1) \in \mathbb{Z}^2 \cong X^*(\mathbf{T})$ and $\kappa = (k_1, k_2)$ a dominant weight. There is a short exact sequence of $(\mathfrak{g}, \mathbf{B})$-representations*

$$\mathrm{BGG}(\kappa): \quad 0 \to V_\kappa \to V(\kappa) \to V(w_0(\kappa) - \alpha) \to 0$$

called the dual BGG complex of weight κ. As $\mathbf{B}$-representations it is identified with the short exact sequence

$$0 \to \kappa \otimes \mathbb{Q}_p[X]_{\leq k_1-k_2} \to \kappa \otimes \mathbb{Q}_p[X] \xrightarrow{(\frac{d}{dX})^{k_1-k_2+1}} (w_0(\kappa) - \alpha) \otimes \mathbb{Q}_p[X] \to 0. \tag{8.4.3}$$

Proof. We have a weight decomposition of $V(\kappa)$ with respect to $\mathbf{T}$

$$V(\kappa) = \bigoplus_{n\geq 0}(\kappa - n\alpha)\mathbb{Q}_p,$$

where $(\kappa - n\alpha)\mathbb{Q}_p$ is identified with $\kappa \otimes \mathbb{Q}_p X^n$ under the isomorphism (8.4.2). As V_κ is the irreducible representation of highest weight κ, it has a weight decomposition $V_\kappa \cong \bigoplus_{0\leq n\leq k_1-k_2}(\kappa - n\alpha)\mathbb{Q}_p$. This shows that $V_\kappa \subset V(\kappa)$ is identified with the inclusion $\kappa \otimes \mathbb{Q}_p[X]_{\leq k_1-k_2} \subset \kappa \otimes \mathbb{Q}_p[X]$. As $\kappa \otimes X^{k_1-k_2+1}$ has weight $(w_0(\kappa) - \alpha)$, the isomorphism of $\mathrm{BGG}(\kappa)$ with (8.4.3) as $\mathbf{B}$-representations is clear. □

Let $\kappa \in X^*(\mathbf{T})$ be a character, note that there are natural maps

$$\kappa \xrightarrow{\iota} V(\kappa) \text{ as } \mathbf{B}\text{-modules, and } V(\kappa) \xrightarrow{\mathrm{ev}_{w_0}} \kappa \text{ as } \bar{\mathfrak{b}}\text{-modules.} \tag{8.4.4}$$

In fact, the action of $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \bar{\mathfrak{n}}$ maps $\kappa \otimes X^n$ to $(k_1 - k_2 - n)(\kappa \otimes X^{n+1})$. They are the highest weight vector and lowest weight quotient of $V(\kappa)$ with respect to the actions of $\mathbf{B}$ and $\bar{\mathfrak{b}}$ respectively. Futhermore, recall that the δ-analytic principal series A_κ^δ is by definition the global sections of $\mathscr{L}(\kappa)$ restricted to the open analytic subspace $U_{w_0}(\delta)\,\mathrm{Iw}_n \subset \mathscr{F}\ell$. Let $B = \mathbf{B}(\mathbb{Z}_p)$, since $U_{w_0}(\delta)\,\mathrm{Iw}_n \subset (\mathbf{B}\backslash\mathbf{B}w_0\mathbf{N})^{\mathrm{an}} \subset \mathscr{F}\ell$, there is a natural $(\mathfrak{g}, B)$-inclusion $V(\kappa) \subset A_\kappa^\delta$ which can be improved to an equivariant $(\mathfrak{g}, B\Lambda^+)$-inclusion by normalizing the action of $c = \mathrm{diag}(p, p)$ and $\varpi = \mathrm{diag}(1, p)$ on $V(\kappa)$ as in Remark 8.3.11. The maps (8.4.4) can be extended to maps between δ-analytic characters and δ-analytic principal series:

Proposition 8.4.4. *Let $\epsilon \geq \delta \geq n$. Let (R, R^+) be a uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{\times,+}$ a δ-analytic character as in §8.3.*

1. *There is a $\mathcal{B} \cap \mathcal{I}\mathrm{w}_n(\delta)$-equivariant map $\iota : \chi R^+ \to A_\chi^{\delta,+}$ (the highest weight vector map). Moreover, it induces a morphism of $\widehat{\mathcal{I}\mathrm{w}}_n(\epsilon)$-equivariant sheaves over $U_1(\epsilon)\,\mathrm{Iw}_n$.*

$$\mathscr{L}^+(w_0(\chi))\widehat{\otimes}_{\mathscr{O}^+_{\mathscr{F}\ell}}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \to \mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}.$$

   Dually, we have equivariant maps $D_\chi^{\delta,+} \to (-\chi)R^+$ and $\mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \to \mathscr{L}^+(-w_0(\chi))\widehat{\otimes}_{\mathscr{O}^+_{\mathscr{F}\ell}}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$.

2. *There is a $\overline{\mathcal{B}} \cap \mathcal{I}\mathrm{w}_n(\delta)$-equivariant map $\mathrm{ev}_{w_0} : A_\chi^{\delta,+} \to \chi R^+$ (the lowest weight vector quotient). Moreover, it induces a morphism of $\widehat{\mathcal{I}\mathrm{w}}_n(\epsilon)$-equivariant sheaves over $U_{w_0}(\epsilon)\,\mathrm{Iw}_n$*

$$\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \to \mathscr{L}^+(\chi)\widehat{\otimes}_{\mathscr{O}^+_{\mathscr{F}\ell}}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}.$$

   Dually, we have equivariant maps $(-\chi)R^+ \to D_\chi^{\delta,+}$ and $\mathscr{L}^+(-\chi)\widehat{\otimes}_{\mathscr{O}^+_{\mathscr{F}\ell}}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \to \mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$.

Proof. Let $w \in W = \{1, w_0\}$. By duality it is enough to construct the maps for the δ-analytic principal series. It is obvious that the maps ι and ev_{w_0} are morphisms of $B \cap \mathrm{Iw}_n$ and $\overline{B} \cap \mathrm{Iw}_n$-modules respectively. Moreover, it is clear that they extend to a morphism of $\mathcal{B} \cap \mathcal{I}\mathrm{w}_n(\delta)$ and $\overline{\mathcal{B}} \cap \mathcal{I}\mathrm{w}_n(\delta)$-representations in the sense that for any affinoid ring (S, S^+), the extension of scalars to S^+ of the maps ι and ev_{w_0} commutes with the action of $(\mathcal{B} \cap \mathcal{I}\mathrm{w}_n(\delta))(S, S^+)$ and $(\overline{\mathcal{B}} \cap \mathcal{I}\mathrm{w}_n(\delta))(S, S^+)$ respectively. Denote $Z_w := \mathcal{GL}_2(\epsilon, \delta)w\,\mathrm{Iw}_n$, the quotient maps

$$\mathcal{B} \cap \mathcal{I}\mathrm{w}_n(\delta)\backslash(Z_1 \times \chi R^+) \to \mathcal{B} \cap \mathcal{I}\mathrm{w}_n(\delta)\backslash(Z_1 \times A_\chi^{\delta,+})$$
$$(\mathcal{B} \cap w_0\,\mathcal{I}\mathrm{w}_n(\delta)w_0^{-1})\backslash(Z_{w_0} \times A_\chi^{\delta,+}) \to (\mathcal{B} \cap w_0\,\mathcal{I}\mathrm{w}_n(\delta)w_0^{-1})\backslash(Z_{w_0} \times w_0(\chi)R^+)$$

induce maps $\mathscr{L}(w_0(\chi)) \to \mathcal{A}_\chi^{\delta,+}|_{an}\widehat{\otimes}\mathscr{O}_{\mathscr{F}\ell}$ and $\mathcal{A}_\chi^{\delta,+}|_{an}\widehat{\otimes}\mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}^+(\chi)$ over the analytic site of $U_1(\epsilon)\,\mathrm{Iw}_n$ and $U_{w_0}(\epsilon)\,\mathrm{Iw}_n$ respectively. Their p-adic completions in the pro-étale site give rise the maps of the proposition. □

Second proof. Let us provide a slightly different proof whose ideas are used in the forthcoming sections. We assume that χ is δ'-analytic for $\delta \geq \delta'$. We saw in Proposition 8.3.13 that the sheaf $\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}_\chi^+$ occurs as the sheaf of $w_0(\chi)$-equivariant functions of a trivial sheaf $\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}$-sheaf for the left action of $\widehat{\mathcal{B}} \cap w_0\widehat{\mathcal{I}\mathrm{w}}_n(\delta')w_0^{-1}$. In the next we will construct the maps of the abelian sheaves by first constructing maps between analytic spaces over $U_w(\epsilon)\,\mathrm{Iw}_n$.

Let $Z_w = \mathcal{GL}_2(\epsilon)w\,\mathrm{Iw}_n$, consider the following map of left $\mathcal{B} \cap w\,\mathcal{I}\mathrm{w}_n(\epsilon)w^{-1}$-spaces

$$\begin{gathered} Z_w \times (\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n\,w^{-1}) \to Z_w \times (\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n)_0 \\ (x,g) \mapsto (x,gx) \end{gathered} \tag{8.4.5}$$

where the action of $b \in \mathcal{B} \cap w\,\mathcal{I}\mathrm{w}_n(\epsilon)w^{-1}$ is $b(x,g) = (bx, gb^{-1})$ in the left hand side, and $b(x,g) = (bx,g)$ in the right hand side[1]. By the Iwahori decomposition (Lemma 8.1.1) we know that

$$\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n \cong (T\mathcal{T}(\delta')) \times (\overline{N}\overline{\mathcal{N}}(\delta))w_0.$$

Therefore, the map (8.4.5) is well defined and it is in fact an isomorphism. On the other hand, we have maps of $\mathcal{B} \cap w\,\mathcal{I}\mathrm{w}_n(\epsilon)w^{-1}$-equivariant spaces

$$\begin{gathered} Z_{w_0} \times T\mathcal{T}(\delta') = Z_{w_0} \times N\mathcal{N}(\delta')\backslash B\mathcal{B}(\delta') \subset Z_{w_0} \times (\mathcal{N}\backslash\mathcal{N}\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n\,w_0^{-1}) \\ Z_1 \times (\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n) \to Z_1 \times (\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n\,\mathcal{N}^0/\mathcal{N}^0) \cong Z_1 \times w_0T\mathcal{T}(\delta') \end{gathered}$$

Composing the previous maps with the isomorphism (8.4.5) we obtain $\mathcal{B} \cap w\,\mathcal{I}\mathrm{w}_n(\epsilon)w^{-1}$-equivariant maps

$$\begin{gathered} Z_{w_0} \times T\mathcal{T}(\delta) \to Z_{w_0} \times (\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n)_0 \\ (x,t) \mapsto (w_0x, tx) \\ Z_1 \times (\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n)_0 \to Z_1 \times w_0T\mathcal{T}(\delta) \\ (x, w_0g) \mapsto (x, w_0gx^{-1}) = (x, w_0\overline{n}_{x,g}t_{x,g}n_{x,g}) \mapsto (x, w_0t_{x,g}), \end{gathered}$$

where $gx^{-1} = \overline{n}_{x,g}t_{x,g}n_{x,g}$ is the Iwahori decomposition of gx^{-1}, and the action of the upper triangular Borel on $T\mathcal{T}(\delta)$ is $b \cdot t = tt_b^{-1}$ with t_b the image of b in the torus. Dividing by the action of the Borel we obtain maps of $\mathcal{I}\mathrm{w}_n(\epsilon)$-equivariant spaces over $U_w(\epsilon)\,\mathrm{Iw}_n$

$$\begin{gathered} \widetilde{U}_{w_0}(\epsilon,\delta')\,\mathrm{Iw}_n \to U_{w_0}(\epsilon)\,\mathrm{Iw}_n \times_{\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n} \\ U_1(\epsilon)\,\mathrm{Iw}_n \times_{\mathcal{N}^0\backslash\mathcal{N}^0\,\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n} \to \widetilde{U}_1(\epsilon,\delta')\,\mathrm{Iw}_n\,. \end{gathered} \tag{8.4.6}$$

Indeed, we have an isomorphism $\widetilde{\mathscr{F}\ell} \cong \mathcal{B}\backslash(\mathcal{GL}_2 \times \mathcal{T})$ where the action of $b \in \mathcal{B}$ on $(g,t) \in \mathcal{GL}_2 \times \mathcal{T}$ is given by $b(g,t) = (bg, tt_b^{-1})$. Notice that the first map is equivariant for the left multiplication of $T\mathcal{T}(\delta)$ while the second map is w_0-equivariant. Taking completions in the proétale site of $U_w(\epsilon)\,\mathrm{Iw}_n$, and analytic functions in $\underline{\mathrm{Hom}}_{\underline{T\widehat{\mathcal{T}}(\delta)}}(-, w_0(\chi)\widehat{\mathscr{O}}^+_{\widetilde{\mathscr{F}\ell}})$, one obtains the desired maps

$$\begin{gathered} \mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\widetilde{\mathscr{F}\ell}} \to \mathscr{L}^+(\chi)\widehat{\otimes}_{\mathscr{O}^+_{\widetilde{\mathscr{F}\ell}}}\widehat{\mathscr{O}}^+_{\widetilde{\mathscr{F}\ell}} \text{ over } U_{w_0}(\epsilon)\,\mathrm{Iw}_n \\ \mathscr{L}^+(w_0(\chi))\widehat{\otimes}_{\mathscr{O}^+_{\widetilde{\mathscr{F}\ell}}}\widehat{\mathscr{O}}^+_{\widetilde{\mathscr{F}\ell}} \to \mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\widetilde{\mathscr{F}\ell}} \text{ over } U_1(\epsilon)\,\mathrm{Iw}_n\,. \end{gathered}$$

□

Let $\kappa \in X^*(\mathbf{T})$ be a dominant weight and V_κ the irreducible representation of GL_2 of highest weight κ. The highest and lowest weight vectors of V_κ induce maps of $\mathbf{B}$-representations

$$V_\kappa \to w_0(\kappa)\mathbb{Q}_p \text{ and } \kappa\mathbb{Q}_p \to V_\kappa.$$

[1] The zero subscript in the right hand side meaning that the action of the Borel is trivial in that component

Passing through the functor (7.2.4) one gets maps of $\mathcal{GL}_2$-equivariant sheaves over $\mathscr{F}\ell$

$$V_\kappa \otimes_{\mathbb{Q}_p} \mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}(\kappa) \text{ and } \mathscr{L}(w_0(\kappa)) \to V_\kappa \otimes_{\mathbb{Q}_p} \mathscr{O}_{\mathscr{F}\ell}.$$

Recall that we have integral structures $\mathscr{L}^+(\kappa)$ and $V_\kappa^+ = \Gamma(\mathscr{F}\ell, \mathscr{L}^+(\kappa))$ which are compatible with the previous maps . We have the following corollary

Corollary 8.4.5. *Let $\kappa \in X^*(\mathbf{T})$ be a dominant character. There are commutative diagrams of $\widehat{\mathcal{I}\mathrm{w}}_n(\delta)$-equivariant sheaves*

$$\begin{array}{ccc} \mathcal{A}_\kappa^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} & \longrightarrow & \mathscr{L}^+(\kappa)\otimes_{\mathscr{O}^+_{\mathscr{F}\ell}}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \\ \uparrow & \nearrow & \\ \underline{V_\kappa^+}\otimes\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} & & \end{array} \qquad \begin{array}{ccc} \mathscr{L}^+(w_0(\kappa))\otimes_{\mathscr{O}^+_{\mathscr{F}\ell}}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} & \longrightarrow & \mathcal{A}_\kappa^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \\ & \searrow & \uparrow \\ & & \underline{V_\kappa^+}\otimes\widehat{\mathscr{O}}^+_{\mathscr{F}\ell} \end{array}$$

A similar statement holds for $\mathcal{D}_\kappa^{\delta,+}$.

Proof. It is enough to prove the commutativity of the diagrams at the level of $\mathcal{B}\cap\mathcal{I}\mathrm{w}_n(\delta)$-representations for $w = 1$ (resp. as $\overline{\mathcal{B}} \cap \mathcal{I}\mathrm{w}_n(\delta)$-representations for $w = w_0$) which is easy to check. □

Corollary 8.4.6. *Let $\epsilon \geq \delta \geq n$. Let (R, R^+) be an uniform affinoid Tate algebra over $\mathbb{Q}_p$ and $\chi : T \to R^{+,\times}$ a δ'-analytic character. The maps of Proposition 8.4.4 are compatible with the action of Λ^+ for $\mathcal{A}_\chi^{\delta,+}$, and with the action of Λ^- for $\mathcal{D}_\chi^{\delta,+}$.*

Proof. Assume that the character is δ'-analytic for $\delta \geq \delta'$. It is enough to check that the maps of spaces in (8.4.6) are compatible with the action of Λ. But this follows from the normalization of the action of Λ in $\Lambda\backslash\widetilde{\mathscr{F}\ell}$ and Lemma 8.1.5. □

9. Overconvergent theory over the modular curves

The theory of the canonical group is fundamental in the construction of the overconvergent modular forms and the perfectoid modular curve. One of the main ideas of the recent work of Boxer-Pilloni in higher Coleman theory [BP20] is to use the Hodge-Tate period map and the perfectoid Shimura variety to define an overconvergent theory of automorphic forms. Following their method, one can describe the overconvergent modular forms only via the Hodge-Tate period morphism, not using explicitly the canonical group but in the construction of the perfectoid modular curve.

The goal of this section is to translate the overconvergent (or locally analytic) objects over $\mathscr{F}\ell$ defined in §8 into overconvergent objects over the modular curves via the Hodge-Tate period map. We will define the sheaves of overconvergent modular forms of [Pil13] and [AIS14], the overconvergent modular symbols of [AS08], and the overconvergent dlog map of [AIS15] interpolating the Hodge-Tate map $\mathrm{HT}^k : \mathrm{Sym}^k T_p E \otimes \widehat{\mathscr{O}}_X \to \omega_E^k \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X$. Moreover, we also construct a "dual" of the above map interpolating $\mathrm{HT}^{\vee,k} : \omega^{-k} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(k) \to \mathrm{Sym}^k T_p E \otimes \widehat{\mathscr{O}}_X$. Later in §10, we will use these maps to construct the interpolations of the Eichler-Shimura decomposition, recovering the "first half" of [AIS15].

We keep the conventions of §7. Throughout this section we shall denote $X = X_0(p^n)$ for the modular curve of level $K^p \mathrm{Iw}_n$, and $X_\infty = X(p^\infty)$ for the perfectoid modular curve.

9.1. Overconvergent neighbourhoods of the ordinary locus

We start with the definition of the overconvergent of neighbourhoods of the ordinary locus of X. Let $\pi_{\mathrm{HT}} : X_\infty \to \mathscr{F}\ell = \mathbb{P}^1_{\mathbb{Q}_p}$ be the Hodge-Tate period map and $\pi_{\mathrm{Iw}_n} : X_\infty \to X$ the natural projection map. Let $\overline{X^{\mathrm{ord}}_\infty} \subset X_\infty$ be the closure of the ordinary locus of the perfectoid modular curve, and $\overline{X^{\mathrm{ord}}}$ the closure of the ordinary locus of X. By [Sch15] we know that $\overline{X^{\mathrm{ord}}_\infty} = \pi_{\mathrm{HT}}^{-1}(\mathscr{F}\ell(\mathbb{Q}_p))$. In particular, we can see the topological space $\overline{X^{\mathrm{ord}}}$ as $\mathrm{pr}(\pi_{\mathrm{HT}}^{-1}(\mathscr{F}\ell(\mathbb{Q}_p)))$

Let $C_n^{\mathrm{can}} \subset E[p^n]$ be the canonical subgroup over $\overline{X^{\mathrm{ord}}}$ and $w \in W = \{1, w_0\}$. We let $\overline{X}^{\mathrm{ord}}_{w,\mathrm{Iw}_n} \subset \overline{X^{\mathrm{ord}}}$ denote the w-ordinary locus, i.e. the ordinary locus where C_n^{can} has relative position w with respect to the universal group $H_n \subset E[p^n]$. In other words, $\overline{X}^{\mathrm{ord}}_{1,\mathrm{Iw}_n}$ is the ordinary locus where $C_n^{\mathrm{can}} = H_n$ and $\overline{X}^{\mathrm{ord}}_{w_0,\mathrm{Iw}_n}$ the locus where $C_n^{\mathrm{can}} \cap H_n = 0$. We can also describe the closed subspace $\overline{X}^{\mathrm{ord}}_{w,\mathrm{Iw}_n}$ as $\pi_{\mathrm{Iw}_n}(\pi_{\mathrm{HT}}^{-1}(w\,\mathrm{Iw}_n))$.

Let $\epsilon \geq n \geq 1$ and $w \in W = \{1, w_0\}$. In §8.1 we have defined overconvergent affinoid neighbourhoods $\{U_w(\epsilon)\,\mathrm{Iw}_n\}_{\epsilon > n}$ of $w \in \mathscr{F}\ell$.

Proposition/Definition 9.1.1. There exists a unique open affinoid subspace $X_{w,\mathrm{Iw}_n}(\epsilon) \subset X$ such that $\pi_{\mathrm{Iw}_n}^{-1}(X_{w,\mathrm{Iw}_n}(\epsilon)) = \pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n)$. Moreover, it satisfies the following properties:

i. $X_{w,\mathrm{Iw}_n}(\epsilon') \subset X_{w,\mathrm{Iw}_n}(\epsilon)$ is a strict immersion for $\epsilon' > \epsilon$.

ii. $\{X_{w,\mathrm{Iw}_n}(\epsilon)\}_\epsilon$ is a basis of strict neighbourhoods of $\overline{X}^{\mathrm{ord}}_{w,\mathrm{Iw}_n}$.

Proof. By Theorem III 3.18 of [Sch15] we know that $\pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n)$ is an affinoid perfectoid subspace, and that there is some finite level $K_p \subset \mathrm{GL}_2(\mathbb{Z}_p)$ and some open affinoid subset $U \subset X_{K_p}$ whose preimage to X_∞ is $\pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n)$. As the underlying topological space of the affinoid perfectoid is Iw_n-invariant, we can actually take $K_p = \mathrm{Iw}_n$ so that $X_{w,\mathrm{Iw}_n}(\epsilon) = U$. Let $\epsilon' > \epsilon$. Since X is proper, to see that $X_{w,\mathrm{Iw}_n}(\epsilon') \subset X_{w,\mathrm{Iw}_n}(\epsilon)$ is a strict immersion it suffices to prove that $\overline{X_{w,\mathrm{Iw}_n}(\epsilon')} \subset X_{w,\mathrm{Iw}_n}(\epsilon)$. This is a consequence of the construction and the fact that $\overline{U_w(\epsilon')\,\mathrm{Iw}_n} \subset U_w(\epsilon)\,\mathrm{Iw}_n$. Finally, to prove that $\{X_{w,\mathrm{Iw}_n}(\epsilon)\}_{\epsilon>n}$ is a basis of strict neighbourhoods of $\overline{X}^{\mathrm{ord}}_{w,\mathrm{Iw}_n}$ it is enough to notice that they are quasi-compact open subsets of a qcqs adic space, and that $\bigcap_\epsilon X_w(\epsilon) = \overline{X}^{\mathrm{ord}}_{w,\mathrm{Iw}_n}$. The last equality is a consequence of the equality (in $\mathscr{F}\ell$) $w\,\mathrm{Iw}_n = \bigcap_\epsilon U_w(\epsilon)\,\mathrm{Iw}_n$, and the description of the ordinary locus of the perfectoid modular curve as $\pi_{\mathrm{HT}}^{-1}(\mathscr{F}\ell(\mathbb{Q}_p))$. □

We briefly recall some properties of the canonical group in the overconvergent neighbourhoods of the canonical locus, see [Far11] and [Sch15, §III.2].

Theorem 9.1.1. *The following holds*

1. *Let $m \geq n$ be an integer. There exists $\epsilon \geq n$ such that $X_{w,\mathrm{Iw}_n}(\epsilon)$ admits a canonical subgroup of order p^m.*
2. *Let $\mathfrak{X} = \mathfrak{X}_0(p^n)$ be the formal integral modular curve of level $K^p\,\mathrm{Iw}_n$ over $\mathrm{Spf}\,\mathbb{Z}_p$, cf. [KM85]. Let $\omega_{E,\mathfrak{X}}$ be the modular sheaf over $\mathfrak{X}$, and $\omega_{E,\mathfrak{X}} \otimes_{\mathscr{O}_\mathfrak{X}} \mathscr{O}_X^+$ the induced integral structure of ω_E. The map $\mathrm{HT} : T_pE \otimes \widehat{\mathscr{O}}_X \to \omega_E \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X$ restricts to a map*
$$T_pE \otimes \widehat{\mathscr{O}}_X^+ \to \omega_{E,\mathfrak{X}} \otimes_{\mathscr{O}_\mathfrak{X}} \widehat{\mathscr{O}}_X^+$$
with cokernel of p-torsion.
3. *Let $m \geq n$ and $\epsilon \geq n$ such that $X_{w,\mathrm{Iw}_n}(\epsilon)$ admits a canonical subgroup of order p^m. There is $0 < r < 1$ such that the restriction of the Hode-Tate map to $X_{w,\mathrm{Iw}_n}(\epsilon)$ factors through*
$$\begin{array}{ccc} T_pE \otimes \widehat{\mathscr{O}}_X^+ & \longrightarrow & \omega_{E,\mathfrak{X}} \otimes_{\mathscr{O}_\mathfrak{X}} \widehat{\mathscr{O}}_X^+ \\ \downarrow & & \downarrow \\ C_m^{\mathrm{can},\vee} \otimes \widehat{\mathscr{O}}_X^+/p^{m-r} & \longrightarrow & \omega_{E,\mathfrak{X}}/p^{m-r} \end{array}$$
where $C_m^{\mathrm{can},\vee}$ is the Cartier dual of the canonical group C_m^{can}.

Proof. Part (1) is of Théorème 6 (1) in [Far11] where there are explicit bounds for ϵ. Parts (2) and (3) follows by Théorème 6 (7) and Théorème 4 of *loc. cit.*, see [AIP15, §3] for more details. □

The affine modular curve $Y \subset X$ parametrises triples (E, H_n, ψ_N) where E is an elliptic curve E, ψ_N is some prime-to-p level structure, and $H_n \subset E[p^n]$ is a cyclic subgroup of order p^n. Let $\varpi = \mathrm{diag}(1, p)$. We finish this subsection with the dynamics of the U_p-correspondance of X_n.

Definition 9.1.2. The U_p-correspondance of X is the finite flat correspondance C
$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & X \end{array} \tag{9.1.1}$$
parametrising (E, ψ_N, H_n, H'), where $(E, H_n, \psi_N) \in X$ and $H' \subset E[p]$ is a cyclic subgroup of order p such that $H_n \cap H' = 0$. We define $p_1(E, \psi_N, H_n, H') = (E, \psi_N, H_n)$ and $p_2(E, \psi_N, H_n, H') = (E/H', \overline{\psi}_N, \overline{H}_n)$, where $\overline{\psi}_N$ and $\overline{H}_n$ are the images of ψ_N and H_n in the quotient E/H'. Let $\pi : p_1^*E \to p_2^*E$ be the universal isogeny over C and $\pi^\vee : p_2^*E \to p_1^*E$ its dual. For a subspace $Z \subset X$ we denote $U_p(Z) = p_1(p_2^{-1}(Z))$ and $U_p^t(Z) = p_2(p_1^{-1}(Z))$.

Lemma 9.1.3. *Let $\epsilon \geq n$, the following holds*

1. *$U_p^t(X_{1,\mathrm{Iw}_n}(\epsilon)) \subset X_{1,\mathrm{Iw}_n}(\epsilon+1)$ and $U_p(X_{1,\mathrm{Iw}_n}(\epsilon)) \supset X_{1,\mathrm{Iw}_n}(\epsilon-1)$ if $\epsilon \geq n+1$.*
2. *$U_p(X_{w_0,\mathrm{Iw}_n}(\epsilon)) \subset X_{w_0,\mathrm{Iw}_n}(\epsilon+1)$ and $U_p^t(X_{w_0,\mathrm{Iw}_n}(\epsilon)) \supset X_{w_0,\mathrm{Iw}_n}(\epsilon-1)$ if $\epsilon \geq n+1$.*

Proof. The perfectoid modular curve X_∞ parametrises $(E, \psi_N, (e_1, e_2))$ where $(E, \psi_N) \in X(1)$ and (e_1, e_2) is a basis of T_pE. Let $C_\infty = X_\infty \times_{X, p_1} C$. The perfectoid curve C_∞ parametrizes $(E, \psi_N, (e_1, e_2), H')$ where $(E, \psi_N, (e_1, e_2)) \in X_\infty$ and $H' \subset E[p^n]$ is a cyclic subgroup of oder p such that $\langle \overline{e}_1 \rangle \cap H' = 0$ mod p. Write $C_\infty = \bigsqcup_{a \in \mathbb{F}_p} C_{\infty,a}$ with $C_{\infty,a}$ the locus where $H' = \langle \overline{e}_2 + a\overline{e}_1 \rangle$. Note that the map $p_1 : C_{\infty,a} \to X_\infty$ is an isomorphism for all a. We have a diagram

$$\begin{array}{ccccc} & & \bigsqcup_a C_{\infty,a} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X_\infty & & & & X_\infty \end{array}$$

with $p_1(E, \psi_N, (e_1, e_2), H') = (E, \psi_N, (e_1, e_2))$ and $p_2(E, \psi_N, (e_1, e_2), H') = (E/H', \overline{\psi}_N, (\pi(e_1), \tilde{e}_2))$, such that the restriction of p_2 to $C_{\infty,a}$ is given by $\tilde{e}_2 = \frac{1}{p}(\pi(e_2) + a\pi(e_1))$ for $0 \leq a < p$ lifting a. Let $U_{p,a} := \begin{pmatrix} 1 & -a \\ 0 & p \end{pmatrix}$. Composing with the Hodge-Tate period map $\pi_{\mathrm{HT}} : X_\infty \to \mathscr{F}\ell$ we have a commutative diagram

$$\begin{array}{ccccc} & & C_{\infty,a} & & \\ & \swarrow^{\pi_{\mathrm{HT}} \circ p_1} & & \searrow^{\pi_{\mathrm{HT}} \circ p_2} & \\ \mathscr{F}\ell & & \xleftarrow{U_{p,a}} & & \mathscr{F}\ell\,. \end{array}$$

By Lemma 8.1.2 we obtain

$$\begin{aligned} &[(\pi_{\mathrm{HT}} \circ p_1)((\pi_{\mathrm{HT}} \circ p_2)^{-1}(U_{w_0}(\epsilon)\,\mathrm{Iw}_n)] \cdot \mathrm{Iw}_n \subset U_{w_0}(\epsilon+1)\,\mathrm{Iw}_n \\ &[(\pi_{\mathrm{HT}} \circ p_1)((\pi_{\mathrm{HT}} \circ p_2)^{-1}(U_1(\epsilon)\,\mathrm{Iw}_n)] \cdot \mathrm{Iw}_n \supset U_1(\epsilon-1)\,\mathrm{Iw}_n \\ &[(\pi_{\mathrm{HT}} \circ p_2)((\pi_{\mathrm{HT}} \circ p_1)^{-1}(U_{w_0}(\epsilon)\,\mathrm{Iw}_n)] \cdot \mathrm{Iw}_n \supset U_{w_0}(\epsilon-1)\,\mathrm{Iw}_n \\ &[(\pi_{\mathrm{HT}} \circ p_2)((\pi_{\mathrm{HT}} \circ p_1)^{-1}(U_1(\epsilon)\,\mathrm{Iw}_n)] \cdot \mathrm{Iw}_n \subset U_1(\epsilon+1)\,\mathrm{Iw}_n\,. \end{aligned} \tag{9.1.2}$$

On the other hand, there is a natural map of correspondances

$$\begin{array}{ccccc} X_\infty & \xleftarrow{p_1} & C_\infty & \xrightarrow{p_2} & X_\infty \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ X & \xleftarrow{p_1} & C & \xrightarrow{p_2} & X \end{array}$$

with both squares being cartesian. The lemma follows from the equations (9.1.2) and the equality $X_{w,\mathrm{Iw}_n}(\epsilon) = \pi_{\mathrm{Iw}_n}(\pi_{\mathrm{HT}}^{-1}(U_w(\epsilon)\,\mathrm{Iw}_n))$. □

Corollary 9.1.4. *Let $\overline{X}^{\mathrm{ord}}_{1,\mathrm{Iw}_n} \subset U$ be an open quasi-compact overconvergent neighbourhood of the 1-ordinary locus of X. Suppose that $\overline{X}^{\mathrm{ord}}_{w_0,\mathrm{Iw}_n} \subset X \backslash U$ is an overconvergent neighbourhood. Then there exists $M >> 0$ such that $(U_p^t)^M(U) \subset U$ is a strict open immersion. We have a similar behaviour with U_p and exchanging 1 and w_0.*

Proof. The affinoids $X_{w,\mathrm{Iw}_n}(\epsilon)$ form a basis of oveconvergent neighbourhoods of $\overline{X}^{\mathrm{ord}}_{w,\mathrm{Iw}_n}$. Hence, there is $\epsilon > n$ such that $X_{w_0,\mathrm{Iw}_n}(\epsilon) \subset X \backslash U$ and $X_{1,\mathrm{Iw}_n}(\epsilon) \subset U$. It is enough to show that there exists $M >> 0$ such that $X \backslash (U_p^{t,M}(X_{w_0,\mathrm{Iw}_n}(\epsilon))) \subset X_{1,\mathrm{Iw}_n}(\epsilon+1)$. Working over the perfectoid modular curve and the

perfectoid finite flat correspondance of the previous lemma, the question is reduced to show that there exists $M >> 0$ such that

$$\mathscr{F}\ell \setminus \left(U_{w_0}(\epsilon)\,\mathrm{Iw}_n \begin{pmatrix} p^M & 0 \\ 0 & 1 \end{pmatrix} \right) \subset U_1(\epsilon+1)\,\mathrm{Iw}_n\,.$$

Let $\mathscr{F}\ell \to \mathbb{P}^1_{\mathbb{Q}_p}$ be the isomorphism mapping a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $(c : d)$. Then $U_{w_0}(\epsilon)\,\mathrm{Iw}_n$ contains the locus $\{(x : y) \mid |y/x| \leq |p|^\epsilon\}$. We get

$$\{(x : y) \mid |y/x| \leq |p|^\epsilon\} \begin{pmatrix} p^M & 0 \\ 0 & 1 \end{pmatrix} = \{(x : y) \mid |y/x| \leq |p|^{\epsilon - M}\},$$

its complement is the locus $\{(x : y) \mid |x/y| < |p|^{M-\epsilon}\}$, taking $M > 2\epsilon + 1$ we have the desired result. □

9.2. The sheaf of overconvergent modular forms

The goal of this subsection is to define the sheaf of overconvergent modular forms at the overconvergent neighbourhoods of the w-ordinary locus of X. Given an uniform affinoid Tate algebra (R, R^+) and a δ-analytic character $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$, we construct (for $\epsilon >> n$) a $R\widehat{\otimes}\mathscr{O}_{X_{\mathrm{Iw}_n}(\epsilon),\mathrm{an}}$-line bundle ω_E^χ over $X_{w,\mathrm{Iw}_n}(\epsilon)$ interpolating the classical modular sheaves ω_E^κ for $\kappa \in X^*(\mathbf{T})$. To achieve this goal, we first construct an overconvergent torsor of modular forms $\mathcal{T}_{\mathrm{mod}}(\delta)$ whose $-w_0(\chi)$-equivariant sections give the line bundle ω_E^χ, cf. Definition 9.2.7. The torsor $\mathcal{T}_{\mathrm{mod}}(\delta)$ is essentially the image of the torsor $\Upsilon(\widetilde{U}_w(\epsilon,\delta)) := \pi_{\mathrm{Iw}_*}(\pi_{\mathrm{HT}}^*(\widetilde{U}_w(\epsilon,\delta)))^{\mathrm{Iw}_n}$, obtained by Construction 7.2.2, from the pro-Kummer-étale site of $X_{w,\mathrm{Iw}_n}(\epsilon)$ to the étale site, cf. Proposition 9.2.1. As a consequence we will prove that the sheaves over $X_{w,\mathrm{Iw}_n}(\epsilon)_{\mathrm{prokét}}$ arising from $\mathscr{L}(\chi)$ via the functor Υ of §7.2 are written in terms of ω_E^χ and Tate twists, cf. Proposition 9.2.8.

Let $\mathscr{F}\ell$ be the flag variety of $\mathcal{GL}_2$, $\widetilde{\mathscr{F}\ell}^0 = \mathcal{N}^0\backslash\mathcal{GL}_2^0$ and $\widetilde{\mathscr{F}\ell} = \mathcal{N}\backslash\mathcal{GL}_2$ the natural $\mathcal{T}^0$ and $\mathcal{T}$ torsors over $\mathscr{F}\ell$. We see both torsors as $\mathcal{GL}_2^0$ (resp. $\mathcal{GL}_2$)-equivariant sheaves in the étale site of $\mathscr{F}\ell$ as usual. We denote in the same way their extension to the proétale site and by $\widehat{\widetilde{\mathscr{F}\ell}^0}$ and $\widehat{\widetilde{\mathscr{F}\ell}}$ their completions, see Definition 8.0.1. Let $f : \widetilde{\mathscr{F}\ell}^0 \to \mathscr{F}\ell$ be the projection map and $\kappa \in X^*(\mathbf{T})$ an algebraic weight. By definition we have $\mathscr{L}^+(\kappa) = f_*(\mathscr{O}^+_{\widetilde{\mathscr{F}\ell}^0})[-w_0(\kappa)]$ with respect to the left regular action of $\mathcal{T}^0$. In other words, we have isomorphisms of torsors

$$\begin{aligned} \widetilde{\mathscr{F}\ell}^0 &= \underline{\mathrm{Isom}}(\mathscr{O}^+_{\mathscr{F}\ell}, \mathscr{L}^+(0,1)) \times \underline{\mathrm{Isom}}(\mathscr{O}^+_{\mathscr{F}\ell}, \mathscr{L}^+(1,0)). \\ \widetilde{\mathscr{F}\ell} &= \underline{\mathrm{Isom}}(\mathscr{O}_{\mathscr{F}\ell}, \mathscr{L}(0,1)) \times \underline{\mathrm{Isom}}(\mathscr{O}_{\mathscr{F}\ell}, \mathscr{L}(1,0)). \end{aligned} \tag{9.2.1}$$

Let $K_p \subset \mathrm{GL}_2(\mathbb{Z}_p)$ be an open subgroup and X_{K_p} the modular curve of level K^pK_p.

Definition 9.2.1. We define the modular $\mathcal{T}$-torsor over the analytic site of X_{K_p} as

$$\mathcal{T}_{\mathrm{mod,an}} := \underline{\mathrm{Isom}}(\mathscr{O}_{X_{K_p}}, \omega_E^{-1}) \times \underline{\mathrm{Isom}}(\mathscr{O}_{X_{K_p}}, \omega_E)$$

whose left $\mathcal{T}$-action is induced by the actions of $\mathbb{G}_m$ over ω_E^{-1} and ω_E respectively. The torsor has a natural enhancement to a torsor over the étale (resp. Kummer-étale) site of X_{K_p} denoted by $\mathcal{T}_{\mathrm{mod,ét}}$ (resp. $\mathcal{T}_{\mathrm{mod,két}}$). We see the torsor as a sheaf $\mathcal{T}_{\mathrm{mod}}$ in the pro-Kummer-étale site by taking the inverse image of $\mathcal{T}_{\mathrm{mod,két}}$ via the projection map $X_{K_p,\mathrm{prokét}} \to X_{K_p,\mathrm{két}}$.

We let $\widehat{\mathcal{T}}(1,0)$ denote the Tate twist

$$\widehat{\mathcal{T}}(1,0) = \widehat{\mathcal{T}} \otimes \begin{pmatrix} \widehat{\mathbb{Z}}_p(1)^\times & 0 \\ 0 & \widehat{\mathbb{Z}}_p^\times \end{pmatrix}.$$

where $\widehat{\mathbb{Z}}_p(1)^\times = \underline{\mathrm{Isom}}(\widehat{\mathbb{Z}}_p, \widehat{\mathbb{Z}}_p(1))$.

Proposition 9.2.2 ([CS17, Prop. 2.3.9]). *There is a Galois-equivariant isomorphism of $\widehat{\mathcal{T}}$-torsors*

$$\Upsilon(\widehat{\mathscr{F}\ell}) = \widehat{\mathcal{T}}(1,0) \times^{\mathcal{T}} \mathcal{T}_{\mathrm{mod}}.$$

Proof. Let St be the standard representation of GL_2 and consider the short exact sequence of GL_2-equivariant sheaves over $\mathscr{F}\ell$ (Example 7.2.4)

$$0 \to \mathscr{L}(0,1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}(1,0) \to 0.$$

We have $\Upsilon(\mathscr{L}(0,1)) = \omega_E^{-1} \otimes_{\mathscr{O}_{X_{K_p}}} \widehat{\mathscr{O}}_{X_{K_p}}(1)$ and $\Upsilon(\mathscr{L}(1,0)) = \omega_E \otimes_{\mathscr{O}_{X_{K_p}}} \widehat{\mathscr{O}}_{X_{K_p}}$. The proposition follows from the equation (9.2.1) and the definition of $\mathcal{T}_{\mathrm{mod}}$. □

In order to construct overconvergent modular sheaves we have to find refinements of the torsor $\mathcal{T}_{\mathrm{mod,an}}$. It turns out that the torsor $\mathcal{T}_{\mathrm{mod}}$ admits an integral reduction to an étale torsor as is shown in the following theorem

Theorem 9.2.3 ([BP20, §4.6]). *There exists an étale $\mathcal{T}^0_{\text{ét}}$-torsor $\mathcal{T}^0_{\mathrm{mod,ét}}$ over X_{K_p} such that*

$$\mathcal{T}_{\mathrm{mod,ét}} = \mathcal{T}_{\text{ét}} \times^{\mathcal{T}^0_{\text{ét}}} \mathcal{T}^0_{\mathrm{mod,ét}} \text{ and } \Upsilon(\widehat{\mathscr{F}\ell}^0) = \widehat{\mathcal{T}}(1,0) \times^{\mathcal{T}^0_{\text{ét}}} \mathcal{T}^0_{\mathrm{mod,ét}}.$$

We denote by $\mathcal{T}^0_{\mathrm{mod,két}}$ its pullback to a torsor over the Kummer-étale site and by $\mathcal{T}^0_{\mathrm{mod}}$ the inverse image of $\mathcal{T}^0_{\mathrm{mod,két}}$ to the pro-Kummer-étale site of X_{K_p}.

Remark 9.2.4. The existence of the integral torsor holds in greater generality for Shimura varieties when the reductive group G is quasi-split over $\mathbb{Q}_p$. We will sketch the original construction involving the canonical subgroup, see [AIP15] for more details.

Before proving Theorem 9.2.3 we need to define some sheaves over the pro-Kummer-étale site of the modular curve.

Definition 9.2.5. Let $\mathrm{HT} : T_pE \otimes \widehat{\mathscr{O}}_{X_{K_p}} \to \omega_E \otimes_{\mathscr{O}_{X_{K_p}}} \widehat{\mathscr{O}}_{X_{K_p}}$ be the Hodge-Tate period map. We define the integral modular sheaves over $X_{K_p,\text{prokét}}$ as

$$\widehat{\omega}^+_E := \mathrm{HT}(T_pE \otimes \widehat{\mathscr{O}}^+_{X_{K_p}}) \text{ and } \omega^+_E := \omega_E \cap \widehat{\omega}^+_E.$$

Proposition 9.2.6. *1. The sheaf ω^+_E is a locally free $\mathscr{O}^+_{X_{K_p},\text{ét}}$-module.*

2. We have $\omega^+_E[\frac{1}{p}] = \omega_E$ and $\omega^+_E \otimes_{\mathscr{O}^+_{X_{K_p}}} \widehat{\mathscr{O}}^+_{X_{K_p}} = \widehat{\omega}^+_E$ over $X_{\text{prokét}}$. In particular, $\omega^+_E/p^s = \widehat{\omega}^+_E/p^s$ for all $s > 0$.

Proof. Let $\mathfrak{X}_{K_p}$ be the completion along the special fibre of the integral modular curve of level K^pK_p. Let D denote the cusp divisor. By Theorem 9.1.1 the Hodge-Tate map restricts to a map

$$\mathrm{HT} : T_pE \otimes \mathscr{O}^+_{X_{K_p}} \to \omega_{E,\mathfrak{X}_{K_p}} \otimes_{\mathscr{O}_{\mathfrak{X}_{K_p}}} \widehat{\mathscr{O}}^+_{X_{K_p}}$$

with torsion cokernel. In particular, by Nakayama's lemma the restriction of ω^+_E to $X_{K_p} \times_{X(1)} (X(p)\backslash D)$ is locally free for the analytic topology. It remains to show that it is locally free for the étale topology around the cusps. Let $D^{ac} \subset D \subset X_1(p)$ be the locus of the cusps where the universal group is not canonical. Then $X_1(p)\backslash D^{ac}$ is étale surjective over $X(1)$ and by Nakayama's lemma the restriction of ω^+_E to $X_{K_p} \times_{X(1)} (X_1(p)\backslash D^{ac})$ is locally free around the cusps, this proves (1). Part (2) is clear because $\omega^+_E \subset \omega_{\mathfrak{X}_{K_p},E} \otimes_{\mathscr{O}_{\mathfrak{X}_{K_p}}} \mathscr{O}^+_{X_{K_p}}$ has torsion cokernel and $\omega^+_E \subset \widehat{\omega}^+_E$ is dense when restricted to the perfectoid modular curve. □

Proof of Theorem 9.2.3. Define the $\mathcal{T}^0$-torsor over $X_{K_p,\text{prokét}}$

$$\mathcal{T}^0_{\text{mod}} = \underline{\text{Isom}}(\mathscr{O}^+_{X_{K_p}}, \omega_E^{+,-1}) \times \underline{\text{Isom}}(\mathscr{O}^+_{X_{K_p}}, \omega_E^+).$$

By Proposition 9.2.6 the torsor $\mathcal{T}^0_{\text{mod}}$ is trivial locally for the étale topology of X_{K_p}. Moreover, since $\omega_E^+[\frac{1}{p}] = \omega_E$ we have $\mathcal{T}_{\text{mod}} = \mathcal{T} \times^{\mathcal{T}^0} \mathcal{T}^0_{\text{mod}}$. On the other hand, by Example 7.2.4 we have

$$\Upsilon(\mathscr{L}^+(1,0)) = \widehat{\omega}_E^+ \text{ and } \Upsilon(\mathscr{L}^+(0,1)) = \widehat{\omega}_E^+(1).$$

By the equation (9.2.1) one obtains the isomorphism of torsors

$$\Upsilon(\widehat{\widetilde{\mathscr{F}\ell}^0}) = \widehat{\mathcal{T}}^0(1,0) \times^{\mathcal{T}^0} \mathcal{T}^0_{\text{mod}}.$$

□

We do not expect that the torsor $\mathcal{T}^0_{\text{mod}}$ can be refined to a smaller torsor over the whole modular curve. This phenomenon already appears over the flag variety where to find reductions of the torsor $\widetilde{\mathscr{F}\ell}^0$ over $\mathscr{F}\ell$ we need to restrict to affinoid subspaces admitting a section to $\mathcal{GL}_2^0$.

Recall that $X = X_0(p^n)$ denotes the modular curve of level $K^p \operatorname{Iw}_n$. The strategy of constructing objects over the modular curves via pullbacks from the flag variety carries us to the following definition

Proposition/Definition 9.2.1. Let $\epsilon \geq \delta \geq n \geq 1$ and $w \in W = \{1, w_0\}$. Consider the trivial $T\mathcal{T}(\delta)$-torsor of §8.1

$$\widetilde{U}_w(\epsilon,\delta)\operatorname{Iw}_n \to U_w(\epsilon)$$

and its completion to an $\underline{T\mathcal{T}}\widehat{(\delta)}$-torsor

$$\widehat{\widetilde{U}}_w(\epsilon,\delta)\operatorname{Iw}_n := \underline{T\widehat{\mathcal{T}}}(\delta) \times^{T\mathcal{T}(\delta)} \widetilde{U}_w(\epsilon,\delta)\operatorname{Iw}_n .$$

The restriction of $\mathcal{T}^0_{\text{mod}}$ to $X_{w,\operatorname{Iw}_n}(\epsilon)$ admits a reduction to an étale $T\mathcal{T}(\delta)$-torsor $\mathcal{T}_{\text{mod}}(\delta)$ such that

$$\Upsilon(\widehat{\widetilde{U}}_w(\epsilon,\delta)\operatorname{Iw}_n) = \underline{T\widehat{\mathcal{T}}}(\delta)(1,0) \times^{T\mathcal{T}(\delta)} \mathcal{T}_{\text{mod}}(\delta), \tag{9.2.2}$$

where $\underline{T\widehat{\mathcal{T}}}(\delta)(1,0)$ is the Tate twist of $\underline{T\widehat{\mathcal{T}}}(\delta)$ by $\begin{pmatrix} \widehat{\mathbb{Z}}_p(1)^\times & 0 \\ 0 & \widehat{\mathbb{Z}}_p^\times \end{pmatrix}$.

Proof. Similarly as for the torsor $\mathcal{T}^0_{\text{mod}}$, one can construct $\mathcal{T}_{\text{mod}}(\delta)$ by a pro-finite-étale descent argument on topological spaces, cf. [BP20]. We provide the more classical construction involving the canonical subgroup, see [AIS14] and [AIP15] for more details.

Let $m \geq \delta + 2$ be an integer, by Theorem 9.1.1 there exists $\epsilon \geq \delta \geq n$ such that $X_{w,\operatorname{Iw}_n}(\epsilon)$ admits a canonical subgroup of order p^m. The same theorem says that we have a factorization

$$\begin{array}{ccc} T_pE \otimes \widehat{\mathscr{O}}^+_X & \xrightarrow{\text{HT}} & \omega_{E,\mathfrak{x}} \otimes_{\mathscr{O}_{\mathfrak{x}}} \widehat{\mathscr{O}}^+_X \\ \downarrow & & \downarrow \\ C_m^{\text{can},\vee} \otimes \mathscr{O}/p^{\delta+1} & \xrightarrow{\text{HT}} & \omega_{\mathfrak{x}}/p^{\delta+1}. \end{array}$$

Define the étale sheaf

$$\omega_E^\times(\delta) := \{v \in \omega_{E,\mathfrak{x}} \mid \overline{v} \in \text{HT}((C_m^{\text{can},\vee})^\times \mod p^\delta\},$$

where $(C_m^{\text{can},\vee})^\times$ denotes the points of order p^m. By Proposition 9.2.6, $\omega_E^\times(\delta)$ is an étale $\mathbb{Z}_p^\times(1 + p^\delta\mathbb{G}_a^+)$-torsor over $X_{w,\operatorname{Iw}_n}(\epsilon)$. Let

$$\omega_E^{\times,-1}(\delta) = \underline{\text{Isom}}(\omega_E^\times(\delta), \mathbb{Z}_p^\times(1 + p^\delta\mathbb{G}_a^+))$$

be the dual torsor. We define the $\mathcal{TT}(\delta)$-torsor

$$\mathcal{T}_{\mathrm{mod}}(\delta) := \omega_E^{\times,-1}(\delta) \times \omega_E^{\times}(\delta).$$

Then $\mathcal{T}_{\mathrm{mod}}(\delta)$ is an étale $T\mathcal{T}(\delta)$-torsor which by construction satisfies $\mathcal{T}^0_{\mathrm{mod}} = \mathcal{T}^0 \times^{T\mathcal{T}(\delta)} \mathcal{T}_{\mathrm{mod}}(\delta)$. Finally, since $\omega_E^+/p^\delta = \widehat{\omega}_E^+/p^\delta$ and the pullback of

$$0 \to \mathscr{L}^+(0,1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}^+(1,0) \to 0$$

via $\Upsilon = \pi_{\mathrm{Iw}_n}(\pi_{\mathrm{HT}}^*(-))^{\mathrm{Iw}_n}$ is equal to

$$0 \to \widehat{\omega}_E^{+,-1}(1) \to T_p E \otimes \widehat{\mathscr{O}}_X^+ \to \widehat{\omega}_E^+ \to 0,$$

one deduces

$$\Upsilon(\widehat{\widetilde{U}}_w(\epsilon,\delta)\,\mathrm{Iw}_n) = \underline{T\widehat{\mathcal{T}}}(\delta)(1,0) \times^{T\mathcal{T}(\delta)} \mathcal{T}_{\mathrm{mod}}(\delta).$$

□

Definition 9.2.7. Let (R, R^+) be a uniform affinoid Tate $\mathbb{Q}_p$-algebra, and $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$ a δ-analytic character. We define the overconvergent modular sheaf of weight χ over $X_{w,\mathrm{Iw}_n}(\epsilon)_{\text{ét}}$ as the space of equivariant functions

$$\omega^{+,\chi}_{E,\text{ét}} := \underline{\mathrm{Hom}}_{T\mathcal{T}(\delta)}(\mathcal{T}_{\mathrm{mod}}(\delta), w_0(\chi) R^+\widehat{\otimes}\mathscr{O}^+_{X,\text{ét}}), \quad \omega^{\chi}_{E,\text{ét}} = \omega^{+,\chi}_{E,\text{ét}}[\frac{1}{p}].$$

We denote by $\omega_E^{+,\chi}$ and ω_E^{χ} the extension of $\omega^{+,\chi}_{E,\text{ét}}$ and $\omega^{\chi}_{E,\text{ét}}$ to the pro-Kummer-étale site as $\mathscr{O}_X^+$ and $\mathscr{O}_X$-modules respectively. We also let $R^+\widehat{\otimes}\mathscr{O}_X^+$ be the inverse image of $R^+\widehat{\otimes}\mathscr{O}^+_{X,\text{két}}$ to the pro-Kummer-étale site. Note that $R^+\widehat{\otimes}\mathscr{O}_X^+ \neq R^+\widehat{\otimes}\widehat{\mathscr{O}}_X^+$. Indeed, if $U = \varprojlim U_i$ is an affinoid perfectoid with pro-Kummer-étale presentation then

$$(R^+\widehat{\otimes}\mathscr{O}_X^+)(U) = \varinjlim_i R^+\widehat{\otimes}\mathscr{O}_X^+(U_i) \text{ and } (R^+\widehat{\otimes}\widehat{\mathscr{O}}_X^+)(U) = R^+\widehat{\otimes}\widehat{\mathscr{O}}_X(U).$$

Proposition 9.2.8. *Let (R, R^+) and δ be as in Definition 9.2.7, write $\chi = (\chi_1, \chi_2)$. Let $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ be the cyclotomic character and $\chi_2 \circ \chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to R^{+,\times}$ its composition with χ_2. We denote the Tate twist $\widehat{\mathscr{O}}_X^+(\chi_2) := R^+(\chi_2 \circ \chi_{\mathrm{cyc}})\widehat{\otimes}\widehat{\mathscr{O}}_X^+$. There is a Galois equivariant isomorphism of sheaves over $X_{w,\mathrm{Iw}_n}(\epsilon)_{\text{prokét}}$*

$$\omega_E^{+,\chi} \otimes_{R^+\widehat{\otimes}\mathscr{O}_X^+} \widehat{\mathscr{O}}_X^+(\chi_2) = \Upsilon(\mathscr{L}^+(\chi)).$$

Proof. This follows from (9.2.2) of Proposition 9.2.1, see Definition 8.2.4. □

We can finally define the overconvergent modular forms and the overconvergent cohomology classes appearing in Coleman and higher Coleman theory. We refer to [Urb11] for the notion of perfect Banach complexes and compact operators of perfect Banach complexes. See [Sta20, Tag 0A39] for the definition of cohomology with supports in a closed subspace.

Definition 9.2.9. Let (R, R^+) be a uniform Tate $\mathbb{Q}_p$-algebra and $\chi : \mathbb{Z}_p \to R^{+,\times}$ a δ-analytic character. Let $w \in W = \{1, w_0\}$.

1. The overconvergent modular forms of weight χ over $X_{w,\mathrm{Iw}_n}(\epsilon)$ is the cohomology complex

$$R\Gamma_w(X, \omega_E^{\chi})_\epsilon := R\Gamma_{\mathrm{an}}(X_{w,\mathrm{Iw}_n}(\epsilon), \omega_E^{\chi}).$$

We denote $H^0_w(X, \omega_E^{\chi})_\epsilon := H^0_{\mathrm{an}}(X_{w,\mathrm{Iw}_n}(\epsilon), \omega_E^{\chi})$.

2. Let $X_{w,\mathrm{Iw}_n}(>\epsilon) = \bigcup_{\epsilon'>\epsilon} X_{w,\mathrm{Iw}_n}(\epsilon')$. We define the overconvergent cohomology with supports to be the complex
$$R\Gamma_{w,c}(X,\omega_E^\chi)_\epsilon := R\Gamma_{\mathrm{an},\overline{X_{w,\mathrm{Iw}_n}(>\epsilon+1)}}(X_{w,\mathrm{Iw}_n}(\epsilon),\omega_E^\chi).$$
We denote $H^1_{w,c}(X,\omega_E^\chi)_\epsilon = H^1_{\mathrm{an},\overline{X_{w,\mathrm{Iw}_n}(>\epsilon+1)}}(X_{w,\mathrm{Iw}_n}(\epsilon),\omega_E^\chi)$.

Remark 9.2.10. We can define the previous cohomologies for any complex in the analytic site of X.

Proposition 9.2.11. *1. The restriction of ω_E^χ to the analytic site of $X_{w,\mathrm{Iw}_n}(\epsilon)$ is an ON Banach sheaf attached to their global sections.*

2. The cohomology complexes $R\Gamma_w(X,\omega_E^\chi)_\epsilon$ and $R\Gamma_{w,c}(X,\omega_E^\chi)_\epsilon$ are represented by perfect Banach complexes of length $[0,1]$. Moreover, we have quasi-isomorphisms
$$H^0_w(X,\omega_E^\chi)_\epsilon[0] \simeq R\Gamma_w(X,\omega_E^\chi)_\epsilon \text{ and } H^1_{w,c}(X,\omega_E^\chi)_\epsilon[-1] \simeq R\Gamma_{w,c}(X,\omega_E^\chi)_\epsilon.$$

Proof. Part (2) is a consequence of part (1) as is shown in [BP20, Lem. 5.2]. For part (1) we can assume that the character χ is the universal character of an open affinoid $\mathcal{V}$ of the weight space of $T = \mathbf{T}(\mathbb{Z}_p)$. Then, one uses the fact that $\omega_E^{+,\chi}$ is trivial locally étale over $U_w(\epsilon)\,\mathrm{Iw}_n$ and the main result of [BG98] to prove that ω_E^χ can be seen as a line bundle over $(U_w(\epsilon)\,\mathrm{Iw}_n)\times\mathcal{V}$, the proposition follows. □

9.2.1. Classical Hecke operators

We end this section with the definition of the U_p-operators for the overconvergent modular forms. First, let us recall the definition for the classical modular sheaves. Let $X \xleftarrow{p_1} C \xrightarrow{p_2} X$ be the U_p-correspondance. We let $\pi : p_1^*E \to p_2^*E$ be the universal isogeny over C and $\pi^\vee : p_2^*E \to p_1^*E$ its dual. We denote by $\pi^* : p_2^*\omega_E \to p_1^*\omega_E$ and $\pi_* : p_1^*\omega_E^{-1} \to p_2^*\omega_E^{-1}$ the pullback and pushforward maps of π (resp. for $\pi^\vee$). For a quasicoherent sheaf $\mathscr{F}$ over X we let $\mathrm{Tr}_{p_i} : p_{i,*}p_i^*\mathscr{F} \to \mathscr{F}$ be the trace map of p_i. Let $\kappa = (k_1,k_2) \in X^*(\mathbf{T})$ be a weight of $\mathbf{T}$, recall that we have made the convention $\omega_E^\kappa = \omega_E^{k_1}\otimes\omega_E^{-k_2} = \omega_E^{k_1-k_2}$.

Definition 9.2.12. The Hecke operator $U_{p,\kappa}$ acting over $R\Gamma_{\mathrm{an}}(X,\omega_E^\kappa)$ is the composition
$$R\Gamma_{\mathrm{an}}(X,\omega_E^\kappa) \xrightarrow{p_2^*} R\Gamma_{\mathrm{an}}(C,p_2^*\omega_E^\kappa) \xrightarrow{(\pi^{\vee,*,-1})^{\otimes k_1}\otimes(\pi_*^{-1})^{\otimes k_2}} R\Gamma_{\mathrm{an}}(C,p_1^*\omega_E^\kappa) \xrightarrow{\mathrm{Tr}_{p_1}} R\Gamma_{\mathrm{an}}(X,\omega_E^\kappa).$$

We define the $U^t_{p,\kappa}$ operator shifting the roles of p_1 and p_2, and composing with the map $(\pi^{\vee,*})^{\otimes k_1}\otimes(\pi_*)^{\otimes k_2}$.

Remark 9.2.13. The $U_{p,\kappa}$ above is equal to the operator $p^{-k_1}U^{\mathrm{naive}}_{p,k_1-k_2}$ of [BP20]. Indeed, $(\pi^{\vee,*,-1})^{\otimes k_1} = p^{-k_1}(\pi^*)^{\otimes k_1}$ and $(\pi_*^{-1})^{\otimes k_2} = (\pi^*)^{\otimes -k_2}$. In other words, $U^{\mathrm{naive}}_{p,k} = U_{p,(0,-k)}$.

Let us justify this definition of the Hecke operators, they are normalized by the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariance of $\pi_{\mathrm{HT}} : X_\infty \to \mathscr{F}\ell$. Let E be the universal elliptic curve and $E^\vee$ its dual, in the next discussion we will not identify both elliptic curves. For $\kappa = (k_1,k_2) \in X^*(\mathbf{T})$ is an algebraic weight, we set $\omega_E^\kappa = \omega_{E^\vee}^{k_1}\otimes\omega_E^{-k_2}$. Then, the Hodge-Tate exact sequence of E has the form
$$0 \to \omega_E^{(0,1)}\otimes_{\mathscr{O}_X}\widehat{\mathscr{O}}_X(1) \xrightarrow{\mathrm{HT}^\vee} T_pE\otimes\widehat{\mathscr{O}}_X \xrightarrow{\mathrm{HT}} \omega_E^{(1,0)}\otimes_{\mathscr{O}_X}\widehat{\mathscr{O}}_X \to 0.$$

Let $\psi : \mathbb{Z}_p^2 \to T_pE$ be the universal trivialization over X_∞, and $V_pE = T_pE[\frac{1}{p}]$. Let $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, the action of g on X_∞ is given by right composition $\psi \mapsto \psi\circ g$. Let g_*E be the image of the universal

elliptic curve by the action of g. Then, there exists a unique quasi-isogeny $\pi : E \dashrightarrow g_*E$ making the following diagram commutative

$$\begin{array}{ccc} \mathbb{Q}_p^2 & \xrightarrow{g_*\psi} & V_p g_* E \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle V_p\pi} \\ \mathbb{Q}_p^2 & \xrightarrow{\psi} & V_p E. \end{array} \tag{9.2.3}$$

Let $\pi^\vee : g_*E^\vee \dashrightarrow E^\vee$ be the dual of π. We have a commutative square of Hodge-Tate exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_E^{(0,1)} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) & \xrightarrow{\mathrm{HT}^\vee} & V_p E \otimes \widehat{\mathscr{O}}_X & \xrightarrow{\mathrm{HT}} & \omega_E^{(1,0)} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle V_p\pi} & & \downarrow{\scriptstyle \pi^{\vee,*}} & & \\ 0 & \longrightarrow & \omega_{g_*E}^{(0,1)} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) & \xrightarrow{\mathrm{HT}^\vee} & V_p g_* E \otimes \widehat{\mathscr{O}}_X & \xrightarrow{\mathrm{HT}} & \omega_{g_*E}^{(1,0)} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X & \longrightarrow & 0, \end{array} \tag{9.2.4}$$

similarly for $\pi^\vee$. Let St be the left standard representation of GL_2, recall from §7.2 that we have a right GL_2-equivariant exact sequence of sheaves over $\mathscr{F}\ell$

$$0 \to \mathscr{L}(0,1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell} \to \mathscr{L}(1,0) \to 0,$$

whose pullback via $\Upsilon = \pi_{\mathrm{Iw}_n,*}(\pi_{\mathrm{HT}}^*(-))^{\mathrm{Iw}_n}$ is the upper row of (9.2.4). The action of $g : \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell} \to g_*(\mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell}) = \mathrm{St} \otimes g_* \mathscr{O}_{\mathscr{F}\ell}$ is given by left multipication on St and the usual action over $\mathscr{O}_{\mathscr{F}\ell}$. Thus, the action of g over the Hodge-Tate exact sequence translates in the diagram (9.2.4). In particular, the actions of g over $\omega_E^{(1,0)}$ and $\omega_E^{(0,1)}$ are given by $\pi^{\vee,*}$ and π_* respectively.

In the situation of Definition 9.2.12, we can work at perfectoid level with the correspondance C_∞ of Lemma 9.1.3. There are maps

$$\begin{array}{ccc} \mathbb{Q}_p^2 & \longrightarrow & V_p p_2^* E \\ {\scriptstyle U_{p,a}}\uparrow & & \uparrow{\scriptstyle V_p\pi} \\ \mathbb{Q}_p^2 & \longrightarrow & V_p p_1^* E \end{array}$$

with $U_{p,a} = \begin{pmatrix} 1 & -a \\ 0 & p \end{pmatrix}$ and $0 \leq a < p$. Thus, the action of U_p^t is given by the double coset $[\mathrm{Iw}_n \, \varpi \, \mathrm{Iw}_n]$ and is induced by $(\pi^{\vee,*})^{\otimes k_1} \otimes (\pi_*)^{\otimes k_2} : p_1^* \omega_E^\kappa \to p_2^* \omega_E^\kappa$. Dually, the action of U_p is defined using the quasi-isogeny π^{-1}, equivalently it is given by the double coset $[\mathrm{Iw}_n \, \varpi^{-1} \, \mathrm{Iw}_n]$, and induced by $(\pi^{\vee,*,-1})^{\otimes k_1} \otimes (\pi_*^{-1})^{\otimes k_2} : p_2^* \omega_E^\kappa \to p_1^* \omega_E^\kappa$.

Remark 9.2.14. We highlight that to define the Hecke action over the modular sheaves we have to fix a $\mathrm{GL}_2(\mathbb{Q}_p)$ action at infinite level, this action arises naturally from the pullback of a GL_2-equivariant sheaf over the flag variety. In particular, ω_E^k for $k \in \mathbb{Z}$ can be endowed with different Hecke actions, all of them differing by a power of p, the choice of an isomorphism $\omega_E^k \cong \omega_E^{(k_1,k_2)}$ with $k_1 - k_2 = k$ fixes such an action.

Before defining the U_p-operators for overconvergent modular forms let us show how the operators $U_{p,\kappa}$ are described using the torsor $\mathcal{T}_{\mathrm{mod}} = \underline{\mathrm{Isom}}(\mathscr{O}_X, \omega_E^{-1}) \times \underline{\mathrm{Isom}}(\mathscr{O}_X, \omega_E)$. By Proposition 9.2.2 we have

$$\widehat{\mathcal{T}}(1,0) \times^{\mathcal{T}} \mathcal{T}_{\mathrm{mod}} = \pi_{HT}^*(\widetilde{\mathscr{F}\ell})$$

where

$$\widetilde{\mathscr{F}\ell} = \underline{\mathrm{Isom}}(\mathscr{O}_{\mathscr{F}\ell}, \mathscr{L}(0,1)) \times \underline{\mathrm{Isom}}(\mathscr{O}_{\mathscr{F}\ell}, \mathscr{L}(1,0)). \tag{9.2.5}$$

Let $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, from diagram (9.2.3) one deduces that the action of g on $\mathcal{T}_{\mathrm{mod}}$ is given by

$$\pi_* \times \pi^{\vee,*} : \mathcal{T}_{\mathrm{mod}} \to g_* \mathcal{T}_{\mathrm{mod}}.$$

Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})$ be an algebraic weight, taking $w_0(\kappa)$-equivariant functions $\underline{\mathrm{Hom}}_{\mathcal{T}}(-, \mathscr{O}_{X,\mathrm{an}}(w_0(\kappa)))$ in (9.2.5) and then taking duals, one recovers the map $(\pi^{\vee,*})^{\otimes k_1} \otimes (\pi_*)^{\otimes k_2} : \omega_E^\kappa \to g_* \omega_E^\kappa$.

9.2.2. Overconvergent Hecke operators

Let $\epsilon \geq \delta \geq n$. By Lemma 9.1.3 the U_p-correspondance restricts to the following diagrams

$$\begin{array}{ccc} & p_1^{-1}(X_{1,\mathrm{Iw}_n}(\epsilon)) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X_{1,\mathrm{Iw}_n}(\epsilon) & & X_{1,\mathrm{Iw}_n}(\epsilon+1) \end{array} \qquad \begin{array}{ccc} & p_2^{-1}(X_{w_0,\mathrm{Iw}_n}(\epsilon)) & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X_{w_0,\mathrm{Iw}_n}(\epsilon+1) & & X_{w_0,\mathrm{Iw}_n}(\epsilon). \end{array}$$

Moreover, we have the inclusions $p_1(p_2^{-1}(X_{1,\mathrm{Iw}_n}(\epsilon))) \supset X_{1,\mathrm{Iw}_n}(\epsilon-1)$ and $p_2(p_1^{-1}(X_{w_0,\mathrm{Iw}_n}(\epsilon))) \supset X_{w_0,\mathrm{Iw}_n}(\epsilon-1)$. Let $\mathcal{T}_{\mathrm{mod}}(\delta)$ be the $T\mathcal{T}(\delta)$-torsor of overconvergent modular forms (see Definition 9.2.1), and $\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n$ the $T\mathcal{T}(\delta)$-torsor over $U_w(\epsilon) \subset \mathscr{F}\ell$ (see Definition 8.1.3).

Let $c = \mathrm{diag}(p,p)$, $\varpi = \mathrm{diag}(1,p)$, $\Lambda = \langle c^{\pm 1}, \varpi^{\pm 1}\rangle$, $\Lambda^+ = \langle c^{\pm 1}, \varpi\rangle$ and $\Lambda^- = \langle c^{\pm 1}, \varpi^{-1}\rangle$. We have proven in Lemma 8.1.5 that

$$\widetilde{U}_w(\epsilon,\delta) = \mathcal{N}\Lambda\backslash\mathcal{N}\Lambda\, \mathcal{GL}_2(\epsilon,\delta) w_0\, \mathrm{Iw}_n \subset \Lambda\backslash\widetilde{\mathscr{F}\ell}.$$

Proposition/Definition 9.2.2. There are well defined maps of torsors

$$\frac{1}{p}\pi_*^\vee \times \pi^* : p_2^*(\mathcal{T}_{\mathrm{mod}}(\delta)) \to p_1^*(\mathcal{T}_{\mathrm{mod}}(\delta)) \text{ over } p_1^{-1}(X_{1,\mathrm{Iw}_n}(\epsilon))$$
$$\pi_*^\vee \times \frac{1}{p}\pi^* : p_2^*(\mathcal{T}_{\mathrm{mod}}(\delta)) \to p_1^*(\mathcal{T}_{\mathrm{mod}}(\delta)) \text{ over } p_2^{-1}(X_{w_0,\mathrm{Iw}_n(\epsilon)}).$$

Let (R, R^+) be an uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T \to R^{+,\times}$ a δ-analytic character. Taking $w_0(\chi)$-equivariant functions $\underline{\mathrm{Hom}}_{T\mathcal{T}(\delta)}(-, \mathscr{O}^+_{X,\text{ét}}(w_0(\chi)))$ with respect to the above maps, and then taking duals, we obtain the overconvergent U_p correspondances

$$\pi_{-\chi}^\vee : p_1^*(\omega_E^{-\chi,+}) \to p_2^*(\omega_E^{-\chi,+}) \text{ and } \pi_\chi : p_2^*(\omega_E^{\chi,+}) \to p_1^*(\omega_E^{\chi,+}).$$

1. We define the U_p-operator over $R\Gamma_1(X, \omega_E^\chi)_\epsilon$ to be the composition

$$\begin{aligned} R\Gamma_1(X,\omega_E^\chi)_\epsilon &= R\Gamma_{\mathrm{an}}(X_{1,\mathrm{Iw}_n}(\epsilon), \omega_E^\chi) \\ &\xrightarrow{p_2^*} R\Gamma_{\mathrm{an}}(p_2^{-1}(X_{1,\mathrm{Iw}_n}), p_2^*\omega_E^\chi) \\ &\xrightarrow{\pi_\chi} R\Gamma_{\mathrm{an}}(p_2^{-1}(X_{1,\mathrm{Iw}_n}), p_1^*\omega_E^\chi) \\ &\xrightarrow{\mathrm{Tr}_{p_1}} R\Gamma_{\mathrm{an}}(p_1(p_2^{-1}(X_{1,\mathrm{Iw}_n}(\epsilon))), \omega_E^\chi) \\ &\xrightarrow{\mathrm{Res}} R\Gamma_{\mathrm{an}}(X_{1,\mathrm{Iw}_n}(\epsilon-1), \omega_E^\chi) \\ &= R\Gamma_1(X,\omega_E^\chi))_{\epsilon-1}. \end{aligned}$$

We define in a similar way the operator $U_p^t : R\Gamma_{w_0}(X, \omega_E^{\chi,+})_\epsilon \to R\Gamma_{w_0}(X, \omega_E^{\chi,+})_{\epsilon-1}$ switching the roles of p_1 and p_2, and using $\pi_\chi^\vee$ instead of π_χ.

2. We define the U_p-operator over $R\Gamma_{w_0,c}(X,\omega_E^\chi)_\epsilon$ to be the composition

$$\begin{aligned}
R\Gamma_{w_0,c}(X,\omega_E^{\chi,+})_\epsilon &= R\Gamma_{\mathrm{an},\overline{X_{w_0,\mathrm{Iw}_n}(>\epsilon+1)}}(X_{w_0,\mathrm{Iw}_n}(\epsilon),\omega_E^\chi)\\
&\xrightarrow{p_2^*} R\Gamma_{\mathrm{an},p_2^{-1}(\overline{X_{w_0,\mathrm{Iw}_n}(>\epsilon+1)})}(p_2^{-1}(X_{w_0,\mathrm{Iw}_n}(\epsilon)),p_2^*\omega_E^\chi)\\
&\xrightarrow{\pi_\chi} R\Gamma_{\mathrm{an},p_2^{-1}(\overline{X_{w_0,\mathrm{Iw}_n}(>\epsilon+1)})}(p_2^{-1}(X_{w_0,\mathrm{Iw}_n}(\epsilon)),p_1^*\omega_E^\chi)\\
&\xrightarrow{\mathrm{Tr}_{p_1}} R\Gamma_{\mathrm{an},p_1(p_2^{-1}(\overline{X_{w_0,\mathrm{Iw}_n}(>\epsilon+1)}))}(p_1(p_2^{-1}(X_{w_0,\mathrm{Iw}_n}(\epsilon))),\omega_E^\chi)\\
&\xrightarrow{\mathrm{Cor}} R\Gamma_{\mathrm{an},\overline{X_{w_0}(>\epsilon+2)}}(X_{w_0,\mathrm{Iw}_n}(\epsilon),\omega_E^\chi)\\
&= R\Gamma_{w_0,c}(X,\omega_E^\chi)_{\epsilon+1}.
\end{aligned}$$

We define in a similar way the operator $U_p^t : R\Gamma_{1,c}(X,\omega_E^\chi)_\epsilon \to R\Gamma_{1,c}(X,\omega_E^\chi)_{\epsilon+1}$ switching the roles of p_1 and p_2, and using $\pi_\chi^\vee$ instead of π_χ.

Remark 9.2.15. In the definition of the Hecke operators we always consider analytic cohomology, i.e. cohomology in the analytic site of X.

Proof. By the discussion before Remark 9.2.14 and Proposition 9.2.1, one reduces the problem to translating the action of $U_{p,a} = \begin{pmatrix} 1 & -a \\ 0 & p \end{pmatrix}$ over $\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n$ to $\mathcal{T}_{\mathrm{mod}}(\delta)$ via π_{HT}. Indeed, the Hecke operators for modular forms are constructed via the $w_0(\chi)$-equivariant functions $\underline{\mathrm{Hom}}_{T\mathcal{T}(\delta)}(-,\mathscr{O}^+_{X,\text{ét}}(w_0(\chi)))$ of a Hecke correspondence between the torsors. From Proposition 9.2.1 we have that $\underline{T}\widehat{\mathcal{T}}(\delta)(-1,0)\times^{T\mathcal{T}(\delta)}\mathcal{T}_{\mathrm{mod}}(\delta) = \Upsilon(\widetilde{\widetilde{U}}_w(\epsilon,\delta)\,\mathrm{Iw}_n)$ with

$$\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n \cong \begin{cases} \mathcal{N}\Lambda\backslash(\mathcal{N})\times(\Lambda T\mathcal{T}(\delta))\times(\overline{N}_n\overline{\mathcal{N}}(\epsilon)) \text{ if } w=1\\ \mathcal{N}\Lambda\backslash(\mathcal{N})\times(\Lambda T\mathcal{T}(\delta))\times(\overline{N\mathcal{N}}(\epsilon))w_0 \text{ if } w=w_0. \end{cases}$$

Then, the normalized action of $U_{p,a}$ on $\widetilde{U}_w(\epsilon,\delta)\,\mathrm{Iw}_n$ is given by

$$x\cdot U_{p,a} = w\varpi^{-1}w^{-1}xU_{p,a}$$

when seen as a subspace of $\widetilde{\mathscr{F}\ell} = \mathcal{N}\backslash\mathcal{GL}_2$.

As $\widetilde{\mathscr{F}\ell} = \underline{\mathrm{Isom}}(\mathscr{O}_{\mathscr{F}\ell},\mathscr{L}(0,1))\times\underline{\mathrm{Isom}}(\mathscr{O}_{\mathscr{F}\ell},\mathscr{L}(1,0))$, the equation (9.2.2) and the diagram (9.2.3) show that the normalized action of $U_{p,a}$ pullbacks via Υ to the $\widehat{\mathscr{O}}_X$-extension of scalars of the isomorphisms

$$\begin{aligned}
&\pi_* : p_1^*\omega_E^{-1}\to p_2^*\omega_E^{-1} \text{ and } \frac{1}{p}\pi^{\vee,*} : p_1^*\omega_E\to p_2^*\omega_E \text{ if } w=1\\
&\frac{1}{p}\pi_* : p_1^*\omega_E^{-1}\to p_2^*\omega_E^{-1} \text{ and } \pi^{\vee,*} : p_1^*\omega_E\to p_2^*\omega_E \text{ if } w=w_0.
\end{aligned}$$

Taking the inverses one obtains the desired isomorphisms of torsors

$$\begin{aligned}
&\frac{1}{p}\pi_*^\vee\times\pi^* : p_2^*(\mathcal{T}_{\mathrm{mod}}(\delta))\to p_1^*(\mathcal{T}_{\mathrm{mod}}(\delta)) \text{ if } w=1\\
&\pi_*^\vee\times\frac{1}{p}\pi^* : p_2^*(\mathcal{T}_{\mathrm{mod}}(\delta))\to p_1^*(\mathcal{T}_{\mathrm{mod}}(\delta)) \text{ if } w=w_0.
\end{aligned}$$

□

Lemma 9.2.16. *The U_p-operators acting on overconvergent modular forms and overconvergent cohomology classes are compact operators of Banach complexes.*

Proof. This follows from the fact that the restriction and correstriction maps

$$R\Gamma_w(X,\omega_E^\chi)_{\epsilon+1} \xrightarrow{\text{Res}} R\Gamma_w(X,\omega_E^\chi)_\epsilon \text{ and } R\Gamma_{w,c}(X,\omega_E^\chi)_\epsilon \xrightarrow{\text{Cor}} R\Gamma_{w,c}(X,\omega_E^\chi)_{\epsilon+1}$$

are compact maps of Banach complexes. □

Corollary 9.2.17. *The finite slope part of $H^0_w(X,\omega_E^\chi)_\epsilon$ (resp. of $H^1_{w,c}(X,\omega_E^\chi)_\epsilon$) with respect to the U_p-operators is independent of ϵ.*

We end this section with some bounds for the finite slope cohomology.

Lemma 9.2.18 ([BP20, Lemma 5.3])**.** *Let $\kappa=(k_1,k_2)\in X^*(\mathbf{T})$ be an algebraic weight.*

1. *The $U_{p,\kappa}$ operator has slopes $\geq 1-k_1$ on $R\Gamma_1(X,\omega_E^\kappa)_\epsilon$ and slopes $\geq -k_2$ on $R\Gamma_{w_0,c}(X,\omega_E^\kappa)_\epsilon$.*
2. *The $U^t_{p,\kappa}$ operator has slopes $\geq -k_2$ on $R\Gamma_{1,c}(X,\omega_E^\kappa)_\epsilon$ and slopes $\geq 1-k_1$ on $R\Gamma_{w_0}(X,\omega_E^\kappa)_\epsilon$.*

Proof. It follows from the bounds of *loc. cit.*, Remark 9.2.13, and the fact that U_p^t is the dual of U_p. □

Definition 9.2.19. Let (R,R^+) be a uniform Tate $\mathbb{Q}_p$-algebra and $\chi: T\to R^{+,\times}$ a δ-analytic character. Let $\kappa\in X^*(\mathbf{T})$. We define normalizations of U_p and U_p^t

$$U_p^{good}=\begin{cases}\frac{1}{p}U_{p,\chi} & \text{over } R\Gamma_1(X,\omega_E^\chi)_\epsilon,\\ U_{p,\chi} & \text{over } R\Gamma_{w_0,c}(X,\omega_E^\chi)_\epsilon\\ p^{-\min\{1-k_1,-k_2\}}U_{p,\kappa} & \text{over } R\Gamma_{\text{an}}(X,\omega^k)\end{cases},$$

$$U_p^{t,good}=\begin{cases}\frac{1}{p}U^t_{p,\chi} & \text{over } R\Gamma_{w_0}(X,\omega_E^\chi)_\epsilon,\\ U^t_{p,\chi} & \text{over } R\Gamma_{1,c}(X,\omega_E^\chi)_\epsilon\\ p^{-\min\{1-k_1,-k_2\}}U^t_{p,\kappa} & \text{over } R\Gamma_{\text{an}}(X,\omega_E^\kappa).\end{cases}$$

Theorem 9.2.20. *Let $\kappa=(k_1,k_2)\in X^*(\mathbf{T})$ be an algebraic weight.*

1. *The U_p^{good}-operator has slopes ≥ 0 on $H^0_1(X,\omega_E^\kappa)_\epsilon$ and $H^1_{w_0,c}(X,\omega_E^\kappa)_\epsilon$.*
2. *The $U_p^{t,good}$ operator has slopes ≥ 0 on $H^0_{w_0}(X,\omega_E^\kappa)_\epsilon$ and $H^1_{1,c}(X,\omega_E^\kappa)_\epsilon$.*

Futhermore, we have isomorphisms of small slope cohomologies

$$H^0_1(X,\omega_E^\kappa)_\epsilon^{U_p^{good}<k_1-k_2-1}=H^0_{\text{an}}(X,\omega_E^\kappa)^{U_p^{good}<k_1-k_2-1},$$
$$H^1_{w_0,c}(X,\omega_E^\kappa)_\epsilon^{U_p^{good}<1+k_2-k_1}=H^1_{\text{an}}(X,\omega_E^\kappa)^{U_p^{good}<1+k_2-k_1},$$
$$H^0_{w_0}(X,\omega_E^\kappa)_\epsilon^{U_p^{t,good}<k_1-k_2-1}=H^0_{\text{an}}(X,\omega_E^\kappa)^{U_p^{t,good}<k_1-k_2-1},$$
$$H^1_{1,c}(X,\omega_E^\kappa)_\epsilon^{U_p^{t,good}<1+k_2-k_1}=H^1_{\text{an}}(X,\omega_E^\kappa)^{U_p^{t,good}<1+k_2-k_1}.$$

Proof. The theorem is an immediate consequence of Lemma 9.2.18, Corollary 9.1.4, and the distinguished triangle

$$R\Gamma_{\text{an},X\setminus X_{w,\text{Iw}_n}(\epsilon)}(X,\omega_E^\kappa)\to R\Gamma_{\text{an}}(X,\omega_E^\kappa)\to R\Gamma_{\text{an}}(X_{w,\text{Iw}_n}(\epsilon),\omega_E^\kappa)\xrightarrow{+1}.$$

□

9.3. Overconvergent modular symbols

Let (R, R^+) be a uniform Tate $\mathbb{Q}_p$-algebra and $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$ a δ-analytic character. Let $A_\chi^{\delta,+}$ be the δ-analytic principal series of weight χ defined in §8.3, and $D_\chi^{\delta,+} = \mathrm{Hom}_{R^+}(A_\chi^{\delta,+}, R^+)$ its dual. We have associated to $A_\chi^{\delta,+}$ and $D_\chi^{\delta,+}$ Iw_n-equivariant sheaves $\mathcal{A}_\chi^{\delta,+}$ and $\mathcal{D}_\chi^{\delta,+}$ over $\mathscr{F}\ell_{\text{proét}}$, cf. Definition 8.3.12. Let $\pi_{\mathrm{HT}} : X_\infty \to \mathscr{F}\ell$ be the Hodge-Tate period map. By an abuse of notation we will write $\mathcal{A}_\chi^{\delta,+}$ and $\mathcal{D}_\chi^{\delta,+}$ for the pro-Kummer-étale sheaves over X defined by the Iw_n-equivariant pro-Kummer-étale sheaves $\pi_{\mathrm{HT}}^{-1}(\mathcal{A}_\chi^{\delta,+})$ and $\pi_{\mathrm{HT}}^{-1}(\mathcal{D}_\chi^{\delta,+})$ respectively, i.e. the sheaves $\Upsilon(\mathcal{A}_\chi^{\delta,+})$ and $\Upsilon(\mathcal{D}_\chi^{\delta,+})$ over $X_{\text{prokét}}$, see Construction 7.2.2. We denote $A_\chi^{\delta} = A_\chi^{\delta,+}[\frac{1}{p}]$ and $D_\chi^{\delta} = D_\chi^{\delta,+}[\frac{1}{p}]$. By Lemmas 8.3.6 and 8.3.8 we have a devisage of the R^+-modules $A_\chi^{\delta,+}$ and $D_\chi^{\delta,+}$ of the form $\varprojlim_s \varinjlim_i \mathscr{F}_{s,i}$, where the $\mathscr{F}_{s,i}$ are finite Iw_n-modules. Then, we have finite local systems $\underline{\mathscr{F}_{s,i}}$ over $X_{\text{prokét}}$ and one can write $\mathcal{A}_\chi^{\delta,+}$ or $\mathcal{D}_\chi^{\delta,+}$ as $\varprojlim_s \varinjlim_i \underline{\mathscr{F}_{s,i}}$, see Corollary 8.3.9.

Definition 9.3.1. We define the overconvergent modular symbols as the cohomology complexes

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{\delta}) \text{ and } R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{\delta}).$$

We want to show that the modular symbols are represented by good topological complexes. We need the following theorem of Borel-Serre.

Theorem 9.3.2 ([BS73]). *Let $\Gamma \subset \mathrm{GL}_2(\mathbb{Z})$ denote the level of the algebraic modular curve X^{alg}. There exists a finite free resolution of the trivial representation Γ by a complex $P^\bullet \to \mathbb{Z}$ of the form*

$$0 \to \mathbb{Z}[\Gamma]^{r_d} \to \cdots \to \mathbb{Z}[\Gamma]^{r_1} \to \mathbb{Z}[\Gamma]^{r_0} \to \mathbb{Z} \to 0.$$

Proposition 9.3.3 ([AS08]). *The overconvergent modular symbols are represented by the complexes*

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{+,\delta}) \simeq \mathrm{Hom}_\Gamma(P^\bullet, A_\chi^{\delta,+})$$
$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{+,\delta}) \simeq \mathrm{Hom}_\Gamma(P^\bullet, D_\chi^{\delta,+})$$

Proof. Let $\mathscr{F}$ denote $\mathcal{A}_\chi^{\delta,+}$ or $\mathcal{D}_\chi^{\delta,+}$, we let $\mathscr{F}_s$ denote $\mathcal{A}_\chi^{\delta,+}/p^s$ or $D_\chi^{\delta,+}/\mathrm{Fil}^s$. We can write $\mathscr{F}_s = \varinjlim_i \mathscr{F}_{s,i}$ where $\mathscr{F}_{s,i}$ are finite local systems over X. By Lemma 3.18 of [Sch13a], and the Mittag-Leffler property for the inverse system $(\mathscr{F}_s)_{s\in\mathbb{N}}$, we have $\mathscr{F} = R\lim_s \mathrm{hocolim}_i \mathscr{F}_{s,i}$ (see [Sta20, Tag 0A5K] for the definition of homotopic colimits).

Let $\mathbb{C}_p \cong \mathbb{C}$ be an isomorphism of fields, we have

$$\begin{aligned}
R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathscr{F}) &= R\varprojlim_s \mathrm{hocolim}_i R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathscr{F}_{s,i}) \\
&= R\varprojlim_s \mathrm{hocolim}_i R\Gamma_{\text{ét}}(Y_{\mathbb{C}_p}, \mathscr{F}_{s,i}) \\
&= R\varprojlim_s \mathrm{hocolim}_i R\Gamma_{\text{ét}}(Y^{\mathrm{alg}}_{\mathbb{C}}, \mathscr{F}_{s,i}) \\
&= R\varprojlim_s \mathrm{hocolim}_i R\Gamma_{\text{ét}}(Y^{\mathrm{alg}}(\mathbb{C}), \mathscr{F}_{s,i}) \\
&= R\varprojlim_s \mathrm{hocolim}_i \mathrm{Hom}_\Gamma(P^\bullet, \mathscr{F}_{s,i}) \\
&= \mathrm{Hom}_\Gamma(P^\bullet, \mathscr{F}).
\end{aligned}$$

The first equality is formal from the definition of derived limits and homotopic colimits, and the fact that X is qcqs in the pro-Kummer-étale site, see [Sta20, Tag 0739]. The second equality is the purity of torsion local systems [DLLZ19, Theo. 4.6.1]. The third equality is the comparison of étale cohomology between an algebraic scheme and its p-adic analytification [Hub96]. The fourth equality is Artin's comparison theorem [Art68]. Finally, the last two equalities are a consequence of Theorem 9.3.2, the fact that $\pi_1(Y^{\mathrm{alg}}(\mathbb{C})) = \Gamma$ and that the universal cover of $Y^{\mathrm{alg}}(\mathbb{C})$ is contractible. □

Corollary 9.3.4. *Let $\mathcal{A}^{\delta,+}_\chi\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_X$ and $\mathcal{D}^{\delta,+}_\chi\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_X$ denote the completed tensor products with respect to the p-adic topology of $\mathcal{A}^{\delta,+}_\chi$ and the filtration of $\mathcal{D}^{\delta,+}_\chi$. There are almost quasi-isomorphisms*

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p},\mathcal{A}^{\delta,+}_\chi\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_X) =^{ae} \mathrm{Hom}_\Gamma(P^\bullet, A^{\delta,+}_\chi\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})$$
$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p},\mathcal{D}^{\delta,+}_\chi\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathscr{O}}^+_X) =^{ae} \mathrm{Hom}_\Gamma(P^\bullet, D^{\delta,+}_\chi\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}).$$

Proof. The corollary follows from the devisage in the proof of the previous proposition and the primitive comparison theorem [Sch13a, Theo. 5.1] and [DLLZ19, Theo. 6.2.1]. □

Next, we define the U_p-operators for overconvergent modular symbols. To keep the construction in the same direction as for overconvergent modular forms, we will use the U_p-correspondance and certain sheaves over $X_{\text{proét}}$ defined by the open subspaces $\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n \subset \mathcal{GL}_2^0$, see Definition 8.0.5 for the affinoids $\mathcal{GL}_2(\delta,\delta')$.

Definition 9.3.5. Let $\delta \geq \delta' \geq n$. We see $\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n$ as a right $\mathcal{I}\mathrm{w}_n(\delta)$-equivariant sheaf over $\mathscr{F}\ell$, we denote by $\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}$ its completion to a proétale sheaf, see Definition 8.0.1. We define the sheaf over $X_{\text{prokét}}$

$$\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')} := \Upsilon(\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}) = \pi^*_{\mathrm{HT}}(\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n})/\,\mathrm{Iw}_n,$$

equivalently, as the quotient sheaf $(X_\infty\times\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n})/\,\mathrm{Iw}_n$ with respect to right multiplication in both components. We also write $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta)} = \widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta)}$.

Remark 9.3.6. The sheaf $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}$ has a left action by $\widehat{\mathcal{B}}^0_X\cap w_0\widehat{\mathcal{I}\mathrm{w}}_n(\delta')_X w_0^{-1}$ which at perfectoid level is given by left multiplication on $\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}$.

Let $H_n\subset E[p^n]$ be the universal subgroup over X. The sheaf $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta)}$ has the following modular description:

Proposition 9.3.7. *Let $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n}$ be the pro-Kummer-étale sheaf over X given by the quotient $(X_\infty\times w_0\underline{\mathrm{Iw}_n})/\,\mathrm{Iw}_n$. There is a natural isomorphism of left $w_0\,\mathrm{Iw}_n\,w_0^{-1}$-torsors*

$$\widehat{\mathcal{T}or}_{\mathrm{Iw}_n} = \underline{\mathrm{Isom}}((H_n,T_pE),(0\oplus\mathbb{Z}/p^n\mathbb{Z},\mathbb{Z}_p^2)). \tag{9.3.1}$$

Furthermore, $w_0\widehat{\mathcal{I}\mathrm{w}}_n(\delta)_X w_0^{-1}\times^{w_0\,\mathrm{Iw}_n\,w_0^{-1}}\widehat{\mathcal{T}or}_{\mathrm{Iw}_n} = \widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta)}$.

Proof. Let $\psi:\mathbb{Z}_p^2\to T_pE$ be the universal trivialization over X_∞. Write $\widehat{\mathcal{T}or}'_{\mathrm{Iw}_n}$ for the right-hand-side of (9.3.1). Let $f\in\widehat{\mathcal{T}or}'_{\mathrm{Iw}_n}$, note that the isomorphism $f\circ\psi:\mathbb{Z}_p^2\to\mathbb{Z}_p^2$ is induced by left multiplication by an element of $w_0\,\mathrm{Iw}_n$. This provides a map

$$\widehat{\mathcal{T}or}'_{\mathrm{Iw}_n}|_{X_\infty}\to X_\infty\times w_0\underline{\mathrm{Iw}_n}$$

which descends to (9.3.1). □

Remark 9.3.8. Let $\delta\geq\delta'\geq n$ and $\chi: T\to R^{+,\times}$ a δ'-analytic character. In §8.3 we have proven that the sheaf $\mathcal{A}^{\delta,+}_\chi\widehat{\otimes}\widehat{\mathscr{O}}^+_{\mathscr{F}\ell}$ over $\mathscr{F}\ell_{\text{proét}}$ is constructed by taking the analytic functions of the sheaf $(\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n})_{\mathscr{F}\ell}$ which are $w_0(\chi)$-equivariant with respect to the left multiplication of $\widehat{\mathcal{B}}\cap w_0\widehat{\mathcal{I}\mathrm{w}}_n(\delta')w_0^{-1}$, see Definitions 8.3.1 and Proposition 8.3.13. The same holds for the abelian sheaf $\mathcal{A}^{\delta,+}_\chi\widehat{\otimes}\widehat{\mathscr{O}}^+_X$ and the sheaf $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}$. More precisely, $\mathcal{A}^{\delta,+}_\chi\widehat{\otimes}\widehat{\mathscr{O}}^+_X$ is the sheaf of functions $f:\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}\to R^+\widehat{\otimes}\widehat{\mathscr{O}}^+_X$ satisfying the following conditions

i. $f|_{X_\infty}:\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}|_{X_\infty}\cong\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n}\to R^+\widehat{\otimes}\widehat{\mathscr{O}}^+_X$ is a function in $\mathscr{O}(\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n)\widehat{\otimes}(R^+\widehat{\otimes}\widehat{\mathscr{O}}^+_X)$.

ii. $f(bx)=w_0(\chi)(b)f(x)$ for $x\in\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}$ and $b\in\widehat{\mathcal{B}}\cap w_0\widehat{\mathcal{I}\mathrm{w}}_n(\delta)_X w_0^{-1}$.

9.3.1. Hecke operators

Let $X \xleftarrow{p_1} C \xrightarrow{p_2} X$ be the U_p-correspondance. To define the U_p-operator of modular symbols we need to construct a map of sheaves following the idea of Definition 9.2.2.

Proposition 9.3.9. *Let $\delta \geq \delta' \geq n$. There is a left $\underline{T}\widehat{\mathcal{T}}(\delta')$-equivariant map of sheaves*

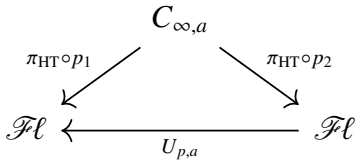
$$\pi_{\mathrm{Iw}_n} : p_2^*(\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}) \xrightarrow{\sim} p_1^*(\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta+1,\delta')}) \tag{9.3.2}$$

induced by the right multiplication $\widetilde{U}_{w_0}(\delta,\delta')\varpi \subset \widetilde{U}_{w_0}(\delta+1,\delta')$.

Proof. Let $C_\infty = C \times_{p_1,X} X_\infty$ be the correspondance over X_∞ constructed in the proof of Lemma 9.1.3. It is enough to define a $\underline{T}\widehat{\mathcal{T}}(\delta)$ equivariant map of sheaves

$$\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash(\widehat{\mathcal{GL}}_2(\delta+1,\delta')w_0\underline{\mathrm{Iw}_n})_C = p_2^*(\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta+1,\delta')}) \xrightarrow{\pi_{\mathrm{Iw}_n}}$$
$$\to p_1^*(\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}) = \underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash(\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n})_C.$$

Let $0 \leq a < p$ and let $C_{\infty,a} \subset C_\infty$ be the locus where we have the commutative diagram



with $U_{p,a} = \begin{pmatrix} 1 & -a \\ 0 & p \end{pmatrix}$. The action by $U_{p,a}$ over the constant sheaf $N\mathcal{N}(\delta')\backslash\mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n$ over $\mathscr{F}\ell$ translates to the right multiplication by $U_{p,a}$ over $C_{\infty,a}$:

$$\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash p_2^*(\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')})|_{C_\infty} = \underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash(\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n})_{C_\infty} \xrightarrow{U_{\infty,a}}$$
$$\to \underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash(\widehat{\mathcal{GL}}_2(\delta+1,\delta')w_0\underline{\mathrm{Iw}_n})_{C_\infty} = \underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash p_1^*(\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta+1,\delta')})|_{C_\infty}.$$

Taking Iw_n-invariants from C_∞ to C one gets the map (9.3.2). □

Corollary 9.3.10. *Let $\delta \geq \delta' \geq n$. Let (R,R^+) be a uniform Tate algebra and $\chi : T \to R^{+,\times}$ a δ'-analytic character. We have maps*

$$\pi_{\mathcal{A}}^\vee : p_1^*(\mathcal{A}_\chi^{\delta+1,+}\widehat{\otimes}\widehat{\mathscr{O}}_X^+) \to p_2^*(\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}_X^+)$$
$$\pi_{\mathcal{D}} : p_2^*(\mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}}_X^+) \to p_1^*(\mathcal{D}_\chi^{\delta+1,+}\widehat{\otimes}\widehat{\mathscr{O}}_X^+).$$

arising from the map π_{Iw_n} in (9.3.2).

Proof. Taking analytic $\underline{T}\widehat{\mathcal{T}}(\delta')$-equivariant functions with values in $\widehat{\mathscr{O}}_X^+(w_0(\chi))$ in the equation (9.3.2) we obtain the map $\pi_{\mathcal{A}}^\vee$. The map $\pi_{\mathcal{D}}$ is obtained by taking duals. □

Remark 9.3.11. The proof of Proposition 9.3.9 also works for the proétale sheaves $\mathcal{A}_\chi^{\delta,+}$ and $\mathcal{D}_\chi^{\delta,+}$ and the torsor $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n}$, (9.3.1) of Proposition 9.3.7. In particular, we also have maps $\pi_{\mathcal{A}}^\vee : p_1^*\mathcal{A}_\chi^{\delta+1,+} \to p_2^*\mathcal{A}_\chi^{\delta,+}$ and $\pi_{\mathcal{D}} : \mathcal{D}_\chi^{\delta+1,+} \to \mathcal{D}_\chi^{\delta,+}$. Furthermore their extensions of scalars by $\widehat{\mathscr{O}}_X^+$ are equal to the maps of the previous corollary.

Definition 9.3.12. 1. We define the U_p^t-operator acting over $R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{\delta,+})$ to be the composition

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{\delta,+}) \xrightarrow{p_1^*} R\Gamma_{\text{prokét}}(C_{\mathbb{C}_p}, p_1^*(\mathcal{A}_\chi^{\delta,+})) \xrightarrow{\pi_{\mathcal{A}}^\vee} R\Gamma_{\text{prokét}}(C_{\mathbb{C}_p}, p_2^*\mathcal{A}_{\text{Iw}_n}^{\delta-1,+}) \xrightarrow{\text{Tr}_{p_2}} \\ \to R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_{\text{Iw}_n}^{\delta-1,+}) \to R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_{\text{Iw}_n}^{\delta,+}).$$

Similarly for $\mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}_X^+}$.

2. We define the U_p-operator acting over $R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{\delta,+})$ to be the composition

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{\delta,+}) \xrightarrow{p_2^*} R\Gamma_{\text{prokét}}(C_{\mathbb{C}_p}, p_2^*(\mathcal{D}_\chi^{\delta,+})) \xrightarrow{\pi_{\mathcal{D}}} R\Gamma_{\text{prokét}}(C_{\mathbb{C}_p}, p_1^*\mathcal{D}_{\text{Iw}_n}^{\delta+1,+}) \xrightarrow{\text{Tr}_{p_1}} \\ \to R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_{\text{Iw}_n}^{\delta+1,+}) \to R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_{\text{Iw}_n}^{\delta,+}).$$

Similarly for $\mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}_X^+}$.

Lemma 9.3.13. *The U_p-operators on overconvergent modular symbols are compact.*

Proof. It follows from Proposition 9.3.3 and the fact that $A_\chi^{\delta-1} \to A_\chi^\delta$ and $D_\chi^{\delta+1} \to D_\chi^\delta$ are compact maps. □

Corollary 9.3.14. *We have isomorphisms of finite slope cohomology*

$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta)^{fs} = H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{\delta+1})^{fs}$$
$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{\delta+1})^{fs} = H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta)^{fs}.$$

Similarly for $\mathcal{A}_\chi^\delta\widehat{\otimes}\widehat{\mathscr{O}_X}$ and $\mathcal{D}_\chi^\delta\widehat{\otimes}\widehat{\mathscr{O}_X}$. Moreover, we have isomorphisms

$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta)^{fs} \otimes_{\mathbb{Z}_p} \mathbb{C}_p = H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta\widehat{\otimes}\widehat{\mathscr{O}_X})^{fs}$$
$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta)^{fs} \otimes_{\mathbb{Z}_p} \mathbb{C}_p = H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta\widehat{\otimes}\widehat{\mathscr{O}_X})^{fs}.$$

Proof. The first assertion follows from Lemma 9.3.13 and the finite slope theory of compact operators of Banach complexes, cf. [Buz07] or [Urb11]. The second statement follows from the almost quasi-isomorphisms

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{\delta,+})\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p} =^{ae} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}_X^+})$$
$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{\delta,+})\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p} =^{ae} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^{\delta,+}\widehat{\otimes}\widehat{\mathscr{O}_X^+})$$

□

The following lemma provides a bound for the action of the U_p-operators for overconvergent modular symbols.

Lemma 9.3.15 ([AS08, Theo. 3.11.1]). *Let $\varpi = \text{diag}(1, p)$, let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight and V_κ the irreducible representation of GL_2 of highest weight κ. Let $\psi_\kappa : V_\kappa \to A_\kappa^\delta$ be the natural inclusion and $\psi_\kappa^\vee : D_\kappa^\delta \to V_{-w_0(\kappa)}$ its dual. Then*

$$\|\varpi\|_{\text{coker}\,\psi_\kappa} \leq |p|^{k_1-k_2+1} \text{ and } \|\varpi^{-1}\|_{\ker\psi_\kappa^\vee} \leq |p|^{k_1-k_2+1}.$$

Proof. Let $V(\kappa) = \kappa \otimes \Gamma(\mathbf{B}w_0\mathbf{N})$, it is a dense subrepresentation of A_κ^δ whose action by ϖ is given by $(\varpi \cdot f)(x) = f(\varpi^{-1}x\varpi)$. We proved in Proposition 8.4.3 that $V_\kappa \subset V(\kappa)$. In particular, $V(\kappa)/V_\kappa$ is a dense subspace of $\text{coker}\,\psi_\kappa$. We have the weight decomposition

$$V(\kappa)/V_\kappa = \bigoplus_{n \geq k_1-k_2+1} \kappa \otimes \mathbb{Q}_p T^n$$

where $\mathscr{O}(\mathbf{N}) = \mathbb{Q}_p[T]$. Since $\varpi \cdot (\kappa \otimes T^n) = p^n(\kappa \otimes T^n)$, we obtain by density that $\|\varpi\|_{\text{coker}\,\psi_\kappa} \leq |p|^{k_1-k_2+1}$. One deduces the second inequality by taking duals. □

We deduce the following classicity result for modular symbols

Theorem 9.3.16. *Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight. The maps $D_\kappa^\delta \to V_{-w_0(\kappa)}$ and $V_\kappa \to \widehat{A}_\kappa^\delta$ induce isomorphisms of the $(< k_1 - k_2 + 1)$-slope part for the action of the (normalized) U_p-operators*

$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\kappa^\delta)^{U_p < k_1-k_2+1} \xrightarrow{\sim} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, V_{-w_0(\kappa)})^{U_p < k_1-k_2+1}$$
$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, V_\kappa)^{U_p^t < k_1-k_2+1} \xrightarrow{\sim} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\kappa^\delta)^{U_p^t < k_1-k_2+1}.$$

Proof. This follows from Lemma 9.3.15 and the definition of the Hecke operators via correspondances. Notice that we are considering the U_p and U_p^t on $R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, V_\kappa)$ by normalizing the action of ϖ as in Remark 8.3.11. □

9.4. The overconvergent Hodge-Tate maps

We end with this section with the definition of overconvergent HT-maps interpolating the morphisms $\mathrm{HT}^k : \mathrm{Sym}^k T_p E \otimes \widehat{\mathscr{O}}_X \to \omega_E^k \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X$ and $\mathrm{HT}^{k,\vee} : \omega_E^{-k} \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(k) \to \mathrm{Sym}^k T_p E \otimes \widehat{\mathscr{O}}_X$.

Definition 9.4.1. Let $\epsilon \geq \delta \geq n$, (R, R^+) a uniform affinoid Tate $\mathbb{Q}_p$-algebra and $\chi = (\chi_1, \chi_2) : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$ a δ-analytic character.

1. We define the map over $X_{1,\mathrm{Iw}_n}(\epsilon)$

$$\mathrm{HT}^{\chi,\vee}_{\mathcal{A}} : \omega_E^{w_0(\chi)} \otimes_{R\widehat{\otimes}\mathscr{O}_X} \widehat{\mathscr{O}}_X(\chi_1) \to \mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X$$

   to be the pullback of the highest weight vector map $\mathscr{L}(w_0(\chi)) \to \mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_{\mathscr{F}\ell}$ over $U_1(\epsilon)\,\mathrm{Iw}_n \subset \mathscr{F}\ell$ via $\Upsilon = \pi_{\mathrm{Iw}_n,*}(\pi_{\mathrm{HT}}^*(-))^{K_p}$, cf. Proposition 8.4.4. We define the map $\mathrm{HT}^\chi_{\mathcal{D}} : \mathcal{D}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X \to \omega_E^{-w_0(\chi)} \otimes_{R\widehat{\otimes}\mathscr{O}_X} \widehat{\mathscr{O}}_X(-\chi_1)$ as the dual of $\mathrm{HT}^\chi_{\mathcal{A}}$.

2. We define the map over $X_{w_0,\mathrm{Iw}_n}(\epsilon)$

$$\mathrm{HT}^\chi_{\mathcal{A}} : \mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X \to \omega_E^\chi \otimes_{R\widehat{\otimes}\mathscr{O}_X} \widehat{\mathscr{O}}_X(\chi_2)$$

   to be the pullback of the lowest weight vector map $\mathcal{A}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_{\mathscr{F}\ell} \to \mathscr{L}(\chi)$ over $U_{w_0}(\epsilon)\,\mathrm{Iw}_n \subset \mathscr{F}\ell$ via Υ. We define the map $\mathrm{HT}^{\chi,\vee}_{\mathcal{D}} : \omega_E^{-\chi} \otimes_{R\widehat{\otimes}\mathscr{O}_X} \widehat{\mathscr{O}}_X(-\chi_2) \to \mathcal{D}^\delta_\chi \widehat{\otimes} \widehat{\mathscr{O}}_X$ as the dual of $\mathrm{HT}^\chi_{\mathcal{A}}$.

Proposition 9.4.2. *Let $\epsilon \geq \delta \geq \delta' \geq n$ and denote $\widehat{\mathcal{T}}_{\mathrm{mod}}(\delta') := \underline{T}\widehat{\mathcal{T}}(\delta')(1,0) \times^{T\mathcal{T}(\delta')} \mathcal{T}_{\mathrm{mod}}(\delta')$. The maps HT^χ and $\mathrm{HT}^{\chi,\vee}$ are compatible with respect to the U_p-correspondance. More precisely, we have $\underline{T}\widehat{\mathcal{T}}(\delta)$-equivariant maps of sheaves*

$$\mathrm{HT} : \underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')} \to \widehat{\mathcal{T}}_{\mathrm{mod}}(\delta') \text{ over } X_{1,\mathrm{Iw}_n}(\epsilon)$$
$$\mathrm{HT}^\vee : w_0\widehat{\mathcal{T}}_{\mathrm{mod}}(\delta') \to \underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')} \text{ over } X_{w_0,\mathrm{Iw}_n}(\epsilon)$$

satisfying the following commutative diagrams

$$\begin{array}{ccc} p_1^*(\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}) & \xrightarrow{\mathrm{HT}} & p_1^*(\widehat{\mathcal{T}}_{\mathrm{mod}}(\delta')) \\ {\scriptstyle \pi_{\mathrm{Iw}_n}}\uparrow & & \uparrow{\scriptstyle \frac{1}{p}\pi_*^\vee \times \pi^*} \\ p_2^*(\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}) & \xrightarrow{\mathrm{HT}} & p_2^*(\widehat{\mathcal{T}}_{\mathrm{mod}}(\delta')) \end{array} \qquad \begin{array}{ccc} p_1^*(w_0\widehat{\mathcal{T}}_{\mathrm{mod}}(\delta')) & \xrightarrow{\mathrm{HT}^\vee} & p_1^*(\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}) \\ {\scriptstyle \frac{1}{p}\pi^* \times \pi_*^\vee}\uparrow & & \uparrow{\scriptstyle \pi_{\mathrm{Iw}_n}} \\ p_2^*(w_0\widehat{\mathcal{T}}_{\mathrm{mod}}(\delta')) & \xrightarrow{\mathrm{HT}^\vee} & p_2^*(\underline{N_n}\widehat{\mathcal{N}}(\delta')\backslash\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')}). \end{array}$$

Proof. The maps HT and $\mathrm{HT}^\vee$ are defined as the pullbacks via Υ of the (completion of the) maps of sheaves

$$\begin{aligned} \widetilde{U}_{w_0}(\epsilon,\delta')\,\mathrm{Iw}_n &\to U_{w_0}(\epsilon)\,\mathrm{Iw}_n \times_{\mathcal{N}^0\backslash\mathcal{N}^0} \mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n \\ U_1(\epsilon)\,\mathrm{Iw}_n \times_{\mathcal{N}^0\backslash\mathcal{N}^0} \mathcal{GL}_2(\delta,\delta')w_0\,\mathrm{Iw}_n &\to \widetilde{U}_1(\epsilon,\delta')\,\mathrm{Iw}_n\,. \end{aligned} \tag{9.4.1}$$

of the equation (8.4.6) in the proof of Proposition 8.4.4. Indeed, by Definition/Propostion 9.2.1 we have

$$\Upsilon(\widehat{\widetilde{U}}_w(\epsilon,\delta)\,\mathrm{Iw}_n) = \underline{T}\widehat{\mathcal{T}}(\delta)(1,0) \times^{T\mathcal{T}(\delta)} \mathcal{T}_{\mathrm{mod}}(\delta),$$

and by definition $\widehat{\mathcal{T}or}_{\mathrm{Iw}_n(\delta,\delta')} = \Upsilon((\widehat{\mathcal{GL}}_2(\delta,\delta')w_0\underline{\mathrm{Iw}_n})_{\mathscr{Fl}})$. The equivariance with respect to the U_p-correspondance follows from the compatibility of (9.4.1) with respect to right multiplication by $\Sigma^+ = \mathrm{Iw}_n\,\Lambda^+\,\mathrm{Iw}_n$. Finally, one obtains the compatibility of the Hecke correspondance between the overconvergent modular sheaves and the δ-analytic principal series or the δ-analytic distributions by taking analytic $w_0(\chi)$-equivariant functions in $\underline{\mathrm{Hom}}_{\underline{T}\widehat{\mathcal{T}}(\delta)}(-, \widehat{\mathscr{O}}_X(w_0(\chi)))$. □

Corollary 9.4.3. *The maps* HT^χ *and* $\mathrm{HT}^{\chi,\vee}$ *of Definition 9.4.1 are Galois equivariant.*

Proof. This follows from the fact that the maps HT and $\mathrm{HT}^\vee$ of the previous proposition are Galois equivariant, since they are constructed as the pullback of a Iw_n-equivariant map of Iw_n-sheaves over $\mathscr{Fl}_{\mathrm{proét}}$. □

10. p-adic Eichler-Shimura decompositions

Let $K^p \subset \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}^{\infty,p}})$ be a neat compact open subgroup. Given $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$ an open compact subgroup we let $Y = Y_{K_p}$ and $X = X_{K_p}$ be the affine and compactified modular curves of level K^pK_p over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We denote by $D = X \backslash Y$ the cusp divisor. Let $f : E^{sm} \to X$ be the semi-abelian scheme extending the universal elliptic curve over Y, and $\overline{E}$ its relative compactification to a log smooth adic space over X. We denote by $\mathrm{DR}_X(\overline{E})$ the relative log de Rham complex of $\overline{E}$ over X, and $\mathscr{H}^1_{\mathrm{dR}} := R^1 f_{\mathrm{an},*}(\mathrm{DR}_X(\overline{E}))$ the first relative de Rham cohomology group. The sheaf $\mathscr{H}^1_{\mathrm{dR}}$ is endowed with a log connection

$$\nabla : \mathscr{H}^1_{\mathrm{dR}} \to \mathscr{H}^1_{\mathrm{dR}} \otimes_{\mathscr{O}_X} \Omega^1_X(\log)$$

and the Hodge filtration $0 \to \omega_E \to \mathscr{H}^1_{dR} \to \omega_E^{-1} \to 0$ with $\mathrm{Fil}^0 \mathscr{H}^1_{\mathrm{dR}} = \mathscr{H}^1_{\mathrm{dR}}$, $\mathrm{Fil}^1 \mathscr{H}^1_{\mathrm{dR}} = \omega_E$ and $\mathrm{Fil}^2 \mathscr{H}^1_{\mathrm{dR}} = 0$, satisfying Griffiths transversality. This last section is dedicated to the construction of the Eichler Shimura decomposition for the étale cohomology of the modular curves. We start with a new proof of Faltings' Eichler-Shimura (ES) decomposition of the local systems V_κ (cf. [Fal87]) involving the Hodge-Tate period map and the dual BGG resolution of V_κ of §8.4. Next, we use the overconvergent HT maps of §9.4 to define overconvergent ES maps. We will recover the results of [AIS15] as well as a new map from the H^1-cohomology with compact supports of overconvergent modular forms to overconvergent modular symbols. Finally, we show that the overconvergent ES maps are compatible with the Poincaré and Serre duality pairings, and that, for small slope, we have a perfect pairing.

10.1. A proétale Eichler Shimura decomposition

Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight and V_κ the irreducible representation of highest weight κ, by an abuse of notation we denote by V_κ the pro-Kummer-étale local system over X defined by V_κ. Let $\alpha = (1, -1) \in X^*(\mathbf{T})$. We state the main theorem of the section

Theorem 10.1.1 (Faltings)**.** *There is a Hecke and Galois equivariant isomorphism*

$$H^1_{\text{ét}}(Y_{\mathbb{C}_p}, V_\kappa) \otimes_{\mathbb{Z}_p} \mathbb{C}_p = H^1_{\mathrm{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})(k_1) \oplus H^0_{\mathrm{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})(k_2 - 1)$$

The main ingredient of the proof is an explicit relation between the Faltings extension $\mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}}$ and the Tate module T_pE.

Remark 10.1.2. The next proposition is the key tool necessary to compute the relative Sen operator for the modular curve in Lue Pan's locally analytic vectors, cf. [Pan20].

Proposition 10.1.3 ([Fal87, Theo. 5])**.** *There is an isomorphism of extensions*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathscr{O}}_X(1) & \xrightarrow{\mathrm{HT}^\vee} & T_pE \otimes \widehat{\mathscr{O}}_E \otimes \omega_E & \xrightarrow{\mathrm{HT}} & \omega_E^2 \otimes \widehat{\mathscr{O}}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle -KS} & & \\
0 & \longrightarrow & \widehat{\mathscr{O}}_X(1) & \longrightarrow & \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}} & \longrightarrow & \Omega^1_X(\log) \otimes \widehat{\mathscr{O}}_X & \longrightarrow & 0
\end{array}$$

where KS is the Kodaira-Spencer isomorphism.

Proof. We provide a modern proof of this theorem using the pro-Kummer-étale site and the comparison between (Kummer-)étale local systems and filtered vector bundles with integral connection (satisfying Griffiths transversality). We refer to §7.1 for the definition of the period sheaves we use down below. Let $\theta : \mathbb{B}^+_{\mathrm{dR},X} \to \widehat{\mathscr{O}}_X$ be the Fontaine's map and ξ a generator of $\ker\theta$. First, let us recall how the de Rham comparison theorem for the Tate module is deduced using the period sheaves (and the main results of §8 in [Sch13a]).

Let $\mathscr{F}$ be a sheaf endowed with an integral log connection ∇, we denote by $\mathrm{DR}(\mathscr{F},\nabla)$ the log de Rham complex of $\mathscr{F}$. We have a quasi-isomorphism of complexes over $\overline{E}_{\text{prokét}}$

$$T_p\mathbb{G}_m \otimes_{\widehat{\mathbb{Z}}_p} \mathbb{B}_{\mathrm{dR},\overline{E}} \simeq T_p\mathbb{G}_m \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{DR}(\mathscr{O}\mathbb{B}_{\mathrm{dR},\log,\overline{E}}, d) = \mathrm{DR}(\mathscr{O}\mathbb{B}_{\mathrm{dR},\log,\overline{E}}, d)(1).$$

Taking $R^1 f_{\text{prokét},*}$ one obtains by [DLLZ18, Theo 3.2.7 (5)] or [Sch13a, Theo. 8.8]

$$T_pE \otimes \mathbb{B}_{\mathrm{dR},X} \simeq T_pE \otimes_{\widehat{\mathbb{Z}}_p} \mathrm{DR}(\mathscr{O}\mathbb{B}_{\mathrm{dR},\log,X}, d) \cong \mathrm{DR}(\mathscr{H}^1_{dR} \otimes \mathscr{O}\mathbb{B}_{\mathrm{dR},\log,X}, \nabla)(1). \tag{10.1.1}$$

Let $\mathbb{M} := T_pE(-1) \otimes \mathbb{B}^+_{\mathrm{dR},X} = (T_pE(-1) \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X})^{\nabla=0}$ and $\mathbb{M}_0 = (\mathscr{H}^1_{\mathrm{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X})^{\nabla=0}$. Both $\mathbb{M}_0$ and $\mathbb{M}$ are $\mathbb{B}^+_{\mathrm{dR},X}$-lattices of $T_pE(-1) \otimes \mathbb{B}_{\mathrm{dR},X}$. The Hodge Filtration of $\mathscr{H}^1_{\mathrm{dR}}$ is concentrated in degrees 0 and 1, and equal to

$$0 \to \omega_E \to \mathscr{H}^1_{\mathrm{dR}} \to \omega_E^{-1} \to 0.$$

This implies that $\xi\mathbb{M} \subset \mathbb{M}_0 \subset \mathbb{M}$, and that $(\mathrm{Fil}^1(\mathscr{H}^1_{\mathrm{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}))^{\nabla=0} = \xi\mathbb{M}$. Then, Proposition 7.9 of [Sch13a] implies

$$\mathbb{M}_0/\xi\mathbb{M} = \mathrm{gr}^0\mathscr{H}^1_{\mathrm{dR}} \otimes \widehat{\mathscr{O}}_X = \omega_E^{-1} \otimes \widehat{\mathscr{O}}_X$$
$$\mathbb{M}/\mathbb{M}_0 = \mathrm{gr}^1\mathscr{H}^1_{\mathrm{dR}} \otimes \widehat{\mathscr{O}}_X(-1) = \omega_E \otimes \widehat{\mathscr{O}}_X(-1).$$

In particular,

$$0 \to \xi\mathbb{M}_0/\xi^2\mathbb{M} \to \xi\mathbb{M}/\xi^2\mathbb{M} \to \xi\mathbb{M}/\xi\mathbb{M}_0 \to 0$$

is just the Hodge-Tate exact sequence of $T_pE \otimes \widehat{\mathscr{O}}_X$ (note the multiplication by ξ induced by the Tate twist in (10.1.1)), and

$$0 \to \xi\mathbb{M}/\xi\mathbb{M}_0 \to \mathbb{M}_0/\xi\mathbb{M}_0 \to \mathbb{M}_0/\xi\mathbb{M} \to 0$$

is the Hodge exact sequence of $\mathscr{H}^1_{dR} \otimes \widehat{\mathscr{O}}_X$.

Consider the map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{M} & \longrightarrow & \mathbb{M} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} & \xrightarrow{d} & \mathbb{M} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} \otimes \Omega^1_X(\log) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{M}_0 & \longrightarrow & \mathbb{M}_0 \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} & \xrightarrow{d} & \mathbb{M}_0 \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} \otimes \Omega^1_X(\log) & \longrightarrow & 0
\end{array} \tag{10.1.2}$$

and let $\tilde{\theta} : \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} \to \widehat{\mathscr{O}}_X$ be the Fontaine's map.

Taking the first graded piece in the upper short exact sequence one finds

$$0 \to \xi\mathbb{M}/\xi^2\mathbb{M} \to \frac{\mathbb{M} \otimes (\ker\tilde{\theta})}{\mathbb{M} \otimes (\ker\tilde{\theta})^2} \xrightarrow{\overline{\nabla}} \frac{\mathbb{M} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}}{\mathbb{M} \otimes (\ker\tilde{\theta})} \otimes \Omega^1_X(\log) \to 0.$$

Since $\xi\mathbb{M} \subset \mathbb{M}_0$, taking the intersection with the image of the lower short exact sequence in (10.1.2) one obtains a short exact sequence

$$0 \to \frac{\xi\mathbb{M}}{\xi^2\mathbb{M}} \to \frac{\mathbb{M}_0 \otimes (\ker\tilde{\theta}) + \xi\mathbb{M} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}}{\mathbb{M}_0 \otimes (\ker\tilde{\theta})^2 + \xi\mathbb{M} \otimes (\ker\tilde{\theta})} \xrightarrow{\nabla} \frac{\mathbb{M}_0 \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}}{\mathbb{M}_0 \otimes (\ker\tilde{\theta}) + \xi\mathbb{M} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}} \otimes \Omega^1_X(\log) \to 0 \tag{10.1.3}$$

The right term of (10.1.3) is equal to $\mathbb{M}_0/\xi\mathbb{M} \otimes \Omega^1_X(\log) = \omega_E^{-1} \otimes \Omega^1_X(\log) \otimes \widehat{\mathscr{O}}_X$. The middle term is equal to

$$gr^1(\mathscr{H}^1_{\mathrm{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}) = \omega_E \otimes \widehat{\mathscr{O}}_X \oplus \omega_E^{-1} \otimes gr^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X}.$$

Note that the restriction of $\overline{\nabla}$ to $\omega_E \otimes \widehat{\mathscr{O}}_X$ is the Kodaira-Spencer map by definition. Indeed, if $\nabla : \mathscr{H}^1_{dR} \to \mathscr{H}^1_{dR} \otimes \Omega^1_X(\log)$ is the connection, taking the first graded piece we get the map

$$KS : \omega_E \to \omega_E^{-1} \otimes \Omega^1_X(\log).$$

Therefore, we have constructed a short exact sequence

$$0 \to T_pE \otimes \widehat{\mathscr{O}}_X \xrightarrow{\mathrm{HT}\oplus\alpha} \omega_E \otimes \widehat{\mathscr{O}}_X \oplus \omega_E^{-1} \otimes gr^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} \xrightarrow{KS\oplus\overline{\nabla}} \omega_E^{-1} \otimes \Omega^1_X(\log) \otimes \widehat{\mathscr{O}}_X \to 0.$$

Thus, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_E^{-1} \otimes \widehat{\mathscr{O}}_X(1) & \xrightarrow{\mathrm{HT}^\vee} & T_pE \otimes \widehat{\mathscr{O}}_X & \xrightarrow{\mathrm{HT}} & \omega_E \otimes \widehat{\mathscr{O}}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle -KS} & & \\
0 & \longrightarrow & \omega_E^{-1} \otimes \widehat{\mathscr{O}}_X(1) & \longrightarrow & \omega_E^{-1} \otimes gr^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log,X} & \xrightarrow{\nabla} & \omega_E^{-1} \otimes \Omega^1_X(\log) \otimes \widehat{\mathscr{O}}_X & \longrightarrow & 0
\end{array}$$

which gives the proposition. □

In §7.1 we defined the period sheaf $\mathscr{O}\mathbb{C}_{\log}$ as the 0-th graded piece of $\mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$. It can also be described as

$$\mathscr{O}\mathbb{C}_{\log} = \varinjlim_n \xi^{-n} gr^n \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}.$$

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{h}$ denote the Lie algebra of GL_2, $\mathbf{B}$, $\mathbf{N}$ and $\mathbf{T}$ respectively. We refer to §8.4 for the definition of the dual BGG short exact sequence and §7.2 for the conventions regarding pullbacks of GL_2-equivariant sheaves over the flag variety via π_{HT}. For a character $\kappa \in X(\mathbf{T})$, let $V(\kappa) = \{f : \mathbf{B}w_0\mathbf{B} \to \mathbb{A}^1_{\mathbb{Q}_p} \mid f(bg) = \kappa(b)f(g)\}$ be the (admissible) dual of the Verma module of highest weight κ, and denote by $\mathscr{V}(\kappa)$ the GL_2-equivariant quasi-coherent sheaf over $\mathscr{F}\ell$ defined by $V(w_0(\kappa))$ via the functor (7.2.4). Let $\Upsilon = \pi_{K_p,*}(\pi_{\mathrm{HT}}^*(-))^{K_p}$ be the functor of Construction 7.2.2. We have the following lemma

Lemma 10.1.4. *There is a natural isomorphism of sheaves over $X_{\mathrm{prokét}}$*

$$\Upsilon(\mathscr{V}(0)) \cong \mathscr{O}\mathbb{C}_{\log}.$$

Proof. Let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight and V_κ the irreducible representation of highest weight κ. By Proposition 8.4.3 we have an inclusion $V_\kappa \subset V(\kappa)$, where under the identification $V(\kappa) \cong \kappa \otimes \mathscr{O}(w_0\mathbf{N})$, the representation V_κ maps bijectively onto the polynomials of degree $\leq k_1 - k_2$. Let $\kappa_0 = (1, 0)$ and $V_{\kappa_0} = \mathrm{St}$ be the standard representation. The previous implies that we have an isomorphism of $\mathbf{B}$-modules

$$V(0) = \varinjlim_n \mathrm{Sym}^n(St \otimes -\kappa_0)$$

where, if v_0 denotes the function in $V(\kappa_0)$ whose restriction to $w_0\mathbf{N}$ is constant equal to 1, the transition maps are given by $v \mapsto v \otimes v_0(-\kappa_0)$. We obtain a GL_2-equivariant isomorphism of quasi-coherent sheaves over $\mathscr{F}\ell$

$$\mathscr{V}(0) = \varinjlim_n \mathrm{Sym}^n(\mathrm{St} \otimes \mathscr{V}(0, -1)).$$

Taking pullbacks by π_{HT} and using Proposition 10.1.3, we obtain an isomorphism of sheaves over $X_{\text{prokét}}$

$$\begin{aligned}\Upsilon(\mathscr{V}(0)) &= \varinjlim_n \mathrm{Sym}^n(T_pE \otimes \widehat{\mathscr{O}}_X \otimes \omega_E(-1)) \\ &= \varinjlim_n \mathrm{Sym}^n(\xi^{-1}\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}}) \\ &= \varinjlim_n \xi^{-n}\mathrm{gr}^n\mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}} \\ &= \mathscr{O}\mathbb{C}_{\log}(0).\end{aligned}$$

□

Theorem 10.1.5. *Let $\alpha = (1,-1)$. Let $\kappa = (k_1,k_2) \in X^*(\mathbf{T})^+$ be a dominant weight and* BGG(κ) *the BGG complex of Section 8.4*

$$\mathrm{BGG}(\kappa): \quad [0 \to V_\kappa \to V(\kappa) \to V(w_0(\kappa)-\alpha) \to 0].$$

Let BGG(κ)$_{\mathscr{F}\ell}$ *be the* GL_2*-equivariant complex of sheaves defined by* BGG(κ)*. We have an isomorphism of complexes over $X_{\text{prokét}}$*

$$\Upsilon(\mathrm{BGG}(\kappa)_{\mathscr{F}\ell}) = [0 \to V_\kappa \otimes \widehat{\mathscr{O}}_X \to \omega_E^{w_0(\kappa)} \otimes \mathscr{O}\mathbb{C}_{\log}(k_1) \to \omega_E^{\kappa+\alpha} \otimes \mathscr{O}\mathbb{C}_{\log}(k_2-1) \to 0].$$

Proof. Note that $V(\kappa) = \kappa \otimes V(0)$ as $\mathbf{B}$-module. The theorem follows from Lemma 10.1.4 and the fact that $\Upsilon(\mathscr{L}(\kappa)) = \omega_E^\kappa \otimes \widehat{\mathscr{O}}_X(k_2)$. □

Corollary 10.1.6. *Let $\nu : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{két}}$ be the projection of sites. In the notation of the previous theorem we have*

$$R\nu_*(V_\kappa \otimes \widehat{\mathscr{O}}_X) = \omega_E^{w_0(\kappa)}(k_1)[0] \oplus \omega_E^{\kappa+\alpha}(k_2-1)[-1].$$

Proof. By [DLLZ18, Lem. 3.3.15] or [Sch13a, Prop. 6.16] we know that $R\nu_*\mathscr{O}\mathbb{C}_{\log} = \mathscr{O}_{X_{\mathbb{C}_p},\text{két}}$. Therefore,

$$R\nu_*(V_\kappa \otimes \widehat{\mathscr{O}}_X) = [\omega_E^{w_0(\kappa)} \to \omega_E^{\kappa+\alpha}],$$

but the above map is zero, as the sheaf $\omega_E^{w_0(\kappa)}$ already factors through $V_\kappa \otimes \widehat{\mathscr{O}}_X$ via $\mathrm{HT}^{\kappa,\vee} : \omega_E^{w_0(\kappa)} \otimes \widehat{\mathscr{O}}_X(k_1) \to V_\kappa \otimes \widehat{\mathscr{O}}_X$. The corollary follows. □

Proof of Theorem 10.1.1. By the previous corollary we have

$$R\nu_*(V_\kappa \otimes \widehat{\mathscr{O}}_X) = \omega_E^{w_0(\kappa)}(k_1)[0] \oplus \omega_E^{\kappa+\alpha}(k_2-1)[-1] \tag{10.1.4}$$

over the Kummer-étale site of $X_{\mathbb{C}_p}$. Projecting to the analytic site, and using that the modular sheaves are coherent, the equation (10.1.4) holds true over $X_{\mathbb{C}_p,\mathrm{an}}$. By taking analytic cohomology for $X_{\mathbb{C}_p}$, using purity on torsion local systems [DLLZ19, Theo 4.6.1] and the primitive comparison theorem, one obtains

$$\begin{aligned}R\Gamma_{\text{ét}}(Y_{\mathbb{C}_p}, V_\kappa) \otimes_{\mathbb{Q}_p} \mathbb{C}_p &= R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, V_\kappa \otimes \widehat{\mathscr{O}}_X) \\ &= R\Gamma_{\mathrm{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0}(\kappa))(k_1) \oplus R\Gamma_{\mathrm{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})(k_2-1)[-1].\end{aligned}$$

The theorem follows by taking H^1-cohomology. □

10.1.1. Étale cohomology with compact supports

In the next paragraph we prove the Eichler-Shimura decomposition for the cohomology with compact supports. Let $j : Y \to X$ and $\iota : D \to X$ be the immersions of the affine modular curve and the cusps divisors in X. Let $\mathscr{O}_D^+$ be the uncompleted bounded structural sheaf of $D_{\text{prokét}}$ and $\mathscr{I}_D^+ := \ker(\mathscr{O}_X^+ \to \iota_*\mathscr{O}_D^+)$ the bounded elements of the ideal of definition of $D_{\text{két}}$, we let $\mathscr{I}_D = \mathscr{I}_D^+[\frac{1}{p}]$. We warn that $\mathscr{I}_D \neq \mathscr{O}_X(-D)$ is not defined by the boundary divisor as the ramification at the cusps is not stable in the Kummer-étale site of X. For instance, if $D_m \subset X(p^m)$ denotes the (reduced) cusps divisor for all $m \geq 0$ and $X_\infty \sim \varprojlim_m X_n$ is the perfectoid modular curve, then $\mathscr{I}_D|_{X_{\infty,an}} = \varinjlim_m \mathscr{O}_{X_m,an}(-D_m)$. We let $\widehat{\mathscr{I}_D^+}$ denote the p-adic completion of $\mathscr{I}_D^+$ and $\widehat{\mathscr{I}_D} = \widehat{\mathscr{I}_D^+}[\frac{1}{p}]$. Since $\mathscr{O}_D^+$ is torsion free, the sheaf $\widehat{\mathscr{I}_D^+}$ is equal to the kernel of $\widehat{\mathscr{O}_X^+} \to \iota_*\widehat{\mathscr{O}_D^+}$ (similarly for $\widehat{\mathscr{I}_D}$).

Let $\mathbb{L}$ be a finite étale local system over Y, and let $R\Gamma_{\text{ét},c}(Y_{\mathbb{C}_p}, \mathbb{L}) = R\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, j_{\text{ét},!}\mathbb{L})$ be the étale cohomology with compact supports. Lemma 4.4.27 of [DLLZ19] implies that this cohomology can be computed in the Kummer-étale site, i.e. that we have a quasi-isomorphism

$$R\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, j_{\text{ét},!}\mathbb{L}) = R\Gamma_{\text{két}}(X_{\mathbb{C}_p}, j_{\text{két},!}\mathbb{L}).$$

Let $\mathscr{F}$ be a sheaf over $Y_{\text{proét}}$ which is written as $\mathscr{F} = \varprojlim_{s\in\mathbb{N}} \varinjlim_i \mathscr{F}_{s,i}$ where $\mathscr{F}_{s,i}$ are étale local systems of Y, and such that the projective limit satisfies the Mittag-Leffler property. We can define $j_!\mathscr{F} := \varprojlim_s \varinjlim_i j_{\text{két},!}\mathscr{F}_{s,i}$ and $R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathscr{F}) := R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, j_!\mathscr{F})$. As $j_{\text{két},!}$ is exact, one gets

$$R\Gamma_{\text{prokét},c}(Y_{\mathbb{C}_!}, \mathscr{F}) = R\varprojlim_s \operatorname{hocolim}_i R\Gamma_{\text{két}}(X_{\mathbb{C}_p}, j_{\text{két},!}\mathscr{F}_{s,i}).$$

Following the ideas of Scholze in [Sch15, Theo. IV 2.1] for describing the completed cohomology in terms of the perfectoid modular curve, one obtains the following proposition

Proposition 10.1.7. *Let $\mathscr{F}$ be a proétale sheaf over Y as before. The inclusion map $j_!\mathscr{F} \to \mathscr{F}\widehat{\otimes}_{\widehat{\mathbb{Z}}_p}\widehat{\mathscr{I}_D^+}$ induces an almost quasi-isomorphism*

$$R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathscr{F})\widehat{\otimes}^L_{\mathbb{Z}_p}\mathcal{O}_{\mathbb{C}_p} =^{ae} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathscr{F}\widehat{\otimes}_{\widehat{\mathbb{Z}}_p}\widehat{\mathscr{I}_D^+}),$$

where the completed tensor products are taken with respect to the filtration of $\mathscr{F}$.

Proof. We have a commutative diagram with short exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_!\mathscr{F} & \longrightarrow & \mathscr{F} & \longrightarrow & \iota_*\mathscr{F}|_D & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}\widehat{\otimes}\widehat{\mathscr{I}_D^+} & \longrightarrow & \mathscr{F}\widehat{\otimes}\widehat{\mathscr{O}_X^+} & \longrightarrow & \mathscr{F}\widehat{\otimes}\iota_*\widehat{\mathscr{O}_D^+} & \longrightarrow & 0. \end{array}$$

The primitive comparison theorem for log-smooth adic spaces gives us almost quasi-isomorphisms

$$\begin{aligned} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathscr{F}\widehat{\otimes}\widehat{\mathscr{O}_X^+}) &=^{ae} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathscr{F})\widehat{\otimes}^L\mathcal{O}_{\mathbb{C}_p} \\ R\Gamma_{\text{prokét}}(D_{\mathbb{C}_p}, \mathscr{F}|_D\widehat{\otimes}\widehat{\mathscr{O}_D^+}) &=^{ae} R\Gamma_{\text{prokét}}(D_{\mathbb{C}_p}, \mathscr{F}|_D)\widehat{\otimes}^L\mathcal{O}_{\mathbb{C}_p}. \end{aligned}$$

The proposition follows by taking the corresponding distinguished triangles in cohomology. □

Lemma 10.1.8. *Let $\alpha = (1,-1)$, let $\kappa = (k_1,k_2) \in X^*(\mathbf{T})^+$ be a dominant character and V_κ the irreducible representation of highest weight κ. Let $\nu_{\text{an}} : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{an}}$ denote the projection of sites, then*

$$R\nu_{\text{an},*}(V_\kappa \otimes \iota_*\widehat{\mathscr{O}_D}) = \omega_E^{w_0(\kappa)}|_D(k_1)[0] \oplus \omega_E^{\kappa+\alpha}|_D(k_2-1)[-1] \tag{10.1.5}$$

$$R\nu_{\text{an},*}(V_\kappa \otimes \iota_*\widehat{\mathscr{I}_D}) = \omega_E^{w_0(\kappa)}(-D)(k_1)[0] \oplus \omega_E^{\kappa+\alpha}(-D)(k_2-1)[-1]. \tag{10.1.6}$$

Proof. The projection (10.1.6) follows from (10.1.5) and Corollary 10.1.6 by taking $R\nu_{\mathrm{an},*}$ of the distinguished triangle

$$0 \to V_\kappa \otimes \widehat{\mathscr{I}_D} \to V_\kappa \otimes \widehat{\mathscr{O}_X} \to V_\kappa \otimes \iota_* \widehat{\mathscr{O}_D} \to 0.$$

To prove (10.1.5) it is enough to work over a geometric cusp $\xi \in D_{\mathbb{C}_p}$. Let $N_\xi \subset K_p$ be its unipotent monodromy. Let X_∞ denote the perfectoid modular curve, then ξ admits a lift to $X_{\infty,\mathbb{C}_p}/N_\xi$ by fixing a compatible system of roots of unity. Let $\widetilde{\xi} = X_{\infty,\mathbb{C}_p} \times_{X_{\infty,\mathbb{C}_p}/N_\xi} \xi$ be the log perfectoid point lying over ξ. Then $\mathrm{Gal}(\widetilde{\xi}/\xi) = N_\xi$ and

$$R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, V_\kappa \otimes \iota_* \widehat{\mathscr{O}_\xi}) = R\Gamma_{\text{prokét}}(\xi, V_\kappa \otimes \widehat{\mathscr{O}_\chi}) = R\Gamma(N_\xi, V_\kappa \otimes_{\mathbb{Z}_p} \mathbb{C}_p).$$

Consider the short exact sequence provided by the dual BGG complex (Theorem 10.1.5)

$$0 \to V_\kappa \widehat{\otimes} \widehat{\mathscr{O}_X} \to \omega_E^{w_0(\kappa)} \otimes_{\mathscr{O}_X} \mathscr{O}\mathbb{C}_{\log}(k_1) \to \omega_E^{\kappa+\alpha} \otimes_{\mathscr{O}_X} \mathscr{O}\mathbb{C}_{\log}(k_2 - 1) \to 0.$$

The sheaf $\mathscr{O}\mathbb{C}_{\log}$ is a free $\widehat{\mathscr{O}_X}$-sheaf locally pro-Kummer-étale on X, hence it remains exact after tensoring with $\widehat{\mathscr{O}_\xi}$. By Lemma 10.1.4 one has that $\mathscr{O}\mathbb{C}_{\log}|_{\widetilde{\xi}} \cong C^{pol}(N_\xi, \mathbb{C}_p)$ is isomorphic to the ring of of polynomial functions of N_ξ with coefficients in $\mathbb{C}_p$. Thus, one computes

$$\begin{aligned} R\Gamma(N_\xi, V_\kappa \otimes_{\mathbb{Z}_p} \mathbb{C}_p) &\simeq R\Gamma(N_\xi, \left[\omega_E^{w_0(\kappa)}|_\xi(k_1) \otimes_{\mathbb{C}_p} C^{pol}(N_\xi, \mathbb{C}_p) \to \omega_E^{\kappa+\alpha}|_\xi(k_2 - 1) \otimes_{\mathbb{C}_p} C^{pol}(N_\xi, \mathbb{C}_p)\right]) \\ &= \omega_E^{w_0(\kappa)}|_\xi(k_1)[0] \oplus \omega_E^{\kappa+\alpha}|_\xi(k_2 - 1)[-1], \end{aligned}$$

the lemma follows. □

Theorem 10.1.9 (ES case of compact supports)**.** *Let $\alpha = (1, -1)$, let $\kappa = (k_1, k_2) \in X^*(\mathbf{T})^+$ be a dominant weight and V_κ the irreducible representation of highest weight κ. There is a natural Hecke and Galois equivariant isomorphism*

$$H^1_{\text{ét},c}(Y_{\mathbb{C}_p}, V_\kappa) = H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)}(-D))(k_1) \oplus H^0(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha}(-D))(k_2 - 1).$$

Proof. The theorem follows by Proposition 10.1.7 and (10.1.6) of Lemma 10.1.8 by taking H^1-cohomology in the analytic site of $X_{\mathbb{C}_p}$. □

10.2. Classical p-adic Eichler-Shimura decomposition and duality

To prove the compatibility of Poincaré and Serre duality in the ES decomposition it suffices to show the compatibility of Poincaré and Serre duality traces for algebraic curves in the pro-étale framework. We recall the Lemma 3.24 of [Sch13b]

Lemma 10.2.1. *Let Z be a proper smooth adic space over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Consider the exact sequence in Z_{proet}*

$$0 \to \widehat{\mathbb{Z}}_p(1) \to \varprojlim_p \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

The boundary map $\mathbb{G}_{m,et} \to R^1\nu_*\mathbb{Z}_p(1)$ induces a commutative diagram of sheaves over $\mathcal{W}_{et}$

$$\begin{array}{ccc} \mathbb{G}_{m,et} & \longrightarrow & R^1\nu_*\widehat{\mathbb{Z}}_p(1) \\ \downarrow \mathrm{dlog} & & \downarrow \\ \Omega^1_{Z,et} & \xrightarrow{FE} & R^1\nu_*\widehat{\mathscr{O}_X}(1). \end{array}$$

Furthermore, the lower horizontal map is induced by the Faltings's extension if Z arises from a base change of a rigid space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

Proposition 10.2.2. *Let Z^{alg} be a proper smooth curve over $\mathrm{Spec}(\mathbb{C}_p)$, and let Z be its analytification over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Let $\nu : Z_{\mathrm{pro\acute{e}t}} \to Z_{\acute{e}t}$ be the projection of sites. The following holds*

1. $H^2_{\mathrm{pro\acute{e}t}}(Z, \widehat{\mathcal{O}}_X(1)) = H^1_{\acute{e}t}(Z, R^1\nu_*\widehat{\mathcal{O}}_X(1))$.
2. *Let $\mathrm{Tr}_P : H^2_{\acute{e}t}(Z, \mathbb{Z}_p(1)) \to \mathbb{Z}_p$ denote the Poincaré duality trace. Then the composition*

$$H^1(Z, \Omega^1_Z) \xrightarrow{FE} H^2_{proet}(Z, \widehat{\mathcal{O}}_Z(1)) = H^2_{et}(Z, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{\mathrm{Tr}_P} \mathbb{C}_p$$

   is the Serre duality Trace.

3. *Let $D^{\mathrm{alg}} \subset Z^{\mathrm{alg}}$ be a reduced divisor given by finitely many points. Let $U^{\mathrm{alg}} = Z^{\mathrm{alg}} \backslash D^{\mathrm{alg}}$ be its complement and U its analytification as adic space over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$. Then the natural map*

$$H^2_{\acute{e}t,c}(U, \mathbb{Z}_p(1)) \to H^2_{\acute{e}t}(Z, \mathbb{Z}_p(1)) \tag{10.2.1}$$

   is an isomorphism.

Proof. Consider the spectral sequence

$$H^p_{\acute{e}t}(Z, R^q\nu_*\widehat{\mathcal{O}}_Z(1)) \Rightarrow H^{p+q}_{\mathrm{pro\acute{e}t}}(Z, \widehat{\mathcal{O}}_Z(1)).$$

We know that $R^p\nu_*\widehat{\mathcal{O}}_{\mathcal{W}}(1)$ is a coherent sheaf for all p, then $H^p_{et}(Z, R^p\nu_*\widehat{\mathcal{O}}_Z(1)) = 0$ for $q > 1$. The degeneracy of the spectral sequence implies part (1).

For (2) recall that the isomorphism $H^2_{\acute{e}t}(Z, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_p = H^2_{\mathrm{pro\acute{e}t}}(Z, \widehat{\mathcal{O}}_Z(1))$ is induced by the inclusion $\widehat{\mathbb{Z}}_p(1) \hookrightarrow \widehat{\mathcal{O}}_Z(1)$. By Lemma 10.2.1 we have a commutative diagram

$$\begin{array}{ccccc}
H^1(Z^{\mathrm{alg}}, \mathbb{G}_m) & \longrightarrow & H^2_{\acute{e}t}(Z^{\mathrm{alg}}, \mathbb{Z}_p(1)) & & \\
\downarrow{\scriptstyle \mathrm{dlog}} & & \downarrow & \searrow^{\mathrm{Tr}_P} & \\
H^1_{\acute{e}t}(Z, \Omega^1_{Z,\acute{e}t}) & \xrightarrow{FE} & H^2_{\acute{e}t}(Z, \widehat{\mathcal{O}}_Z(1)) & \xrightarrow{\mathrm{Tr}_P} & \mathbb{C}_p
\end{array}$$

Since the image of dlog is dense (as Z arises from an algebraic curve), it suffices to show that for $\alpha \in H^1(Z^{\mathrm{alg}}, \mathbb{G}_m)$, the composition $\mathrm{Tr}_P \circ FE(\mathrm{dlog}(\alpha))$ is the Serre duality trace. Let $\mathscr{L}/Z^{\mathrm{alg}}$ be a line bundle, we can see $\mathscr{L}$ as a subsheaf of the rational functions $\mathscr{K}_{Z^{\mathrm{alg}}}$ of Z^{alg}. Let $\{U_i\}_i$ be a covering of Z^{alg} by affine schemes trivializing $\mathscr{L}$, and $\mathscr{L}|_{U_i} = \mathcal{O}_{U_i} e_i \subset \mathscr{K}_{Z^{\mathrm{alg}}}$ a trivialization. We write as usual $U_{ij} = U_i \cap U_j$. Let $c_{ij} = e_i e_j^{-1} \in \mathcal{O}^\times_{U_{ij}}$, then $\mathrm{dlog}(\mathscr{L}) = (\frac{dc_{ij}}{c_{ij}})_{ij} \in H^1_{\acute{e}t}(Z, \Omega^1_{Z,et})$. We have that

$$\mathrm{Tr}_P \mathscr{L} = \sum_{x \in Z^{\mathrm{alg}}} v_x(\mathscr{L})$$

where for $x \in U_i$, $v_x(\mathscr{L}) = v_x(e_i)$ is the valuation at x. On the other hand, the Serre trace of $\mathrm{dlog}(\mathscr{L})$ is

$$\sum_{x \in X} \mathrm{res}_x(\mathrm{dlog}(\mathscr{L}))$$

where $\mathrm{res}_x(\mathrm{dlog}(\mathscr{L}))$ is the residue at x of $\mathrm{dlog}\, e_i$ for $x \in U_i$. But we know that

$$\mathrm{res}_x(\mathrm{dlog}\, e_i) = v_x(e_i),$$

which finishes the proof of (2).

Finally, let $j : U \to Z$ and $\iota : D \to Z$ denote the immersion maps. As the étale cohomology with and without supports of $\mathbb{F}_p$ of U and Z is concentrated in degrees $[0, 2]$, it is enough to show that the natural map

$$H^2_{\text{ét},c}(U, \mathbb{F}_p(1)) \to H^2_{\text{ét}}(Z, \mathbb{F}_p(1)) \tag{10.2.2}$$

is an isomorphism. But the fundamental exact sequence

$$0 \to j_!\mathbb{F}_p \to \mathbb{F}_p \to \iota_*\mathbb{F}_p \to 0$$

and the fact that $H^i_{\text{ét}}(Z, \iota_*\mathbb{F}_p) = H^i_{\text{ét}}(D, \mathbb{F}_p) = 0$ for $i \geq 1$ as D is a disjoint union of geometric points, imply that (10.2.2) is an isomorphism. □

Corollary 10.2.3. *The Poincaré pairing*

$$H^1_{\text{ét}}(Y_{\mathbb{C}_p}, V_\kappa)(1) \times H^1_{\text{ét},c}(Y_{\mathbb{C}_p}, V_{-w_0(\kappa)}) \xrightarrow{\cup} H^2_{\text{ét},c}(Y_{\mathbb{C}_p}, \mathbb{Q}_p(1)) \xrightarrow{\text{Tr}_P} \mathbb{Q}_p$$

and the Serre pairing

$$H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{-\kappa}) \times H^0_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha}(-D)) \xrightarrow{KS\circ\cup} H^1_{\text{an}}(X_{\mathbb{C}_p}, \Omega^1_X) \xrightarrow{\text{Tr}_S} \mathbb{C}_p$$

(resp. $\omega_E^{w_0(\kappa)}(-D)$ and $\omega_E^{-w_0(\kappa)}$) are compatible with the Eichler-Shimura decomposition.

Proof. Let $\alpha = (1, -1) \in X^*(\mathbf{T})$. By functoriality of the Yoneda pairing, one is reduce to show that the Poincaré and Serre duality traces are compatible over X, which is exactly Proposition 10.2.2. Indeed, let $\nu_{\text{an}} : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{an}}$ be the projection of sites and $f : X_{\mathbb{C}_p} \to \text{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$ the structural morphism. By the compatibility of the cup product with compositions [Sta20, Tag 0FP6], the cup product

$$R(f \circ \nu_{\text{an}})_*(V_\kappa(1) \otimes \widehat{\mathscr{O}}_X) \otimes^L R(f \circ \nu_{\text{an}})_*(V_{-w_0(\kappa)} \otimes \widehat{\mathscr{I}}_X) \xrightarrow{\cup_{f\circ\nu_{\text{an}}}} R\Gamma(f \circ \nu_{\text{an}})_*(\widehat{\mathscr{I}}(1))$$

is equal to the composition of cup products

$$Rf_*R\nu_{\text{an},*}(V_\kappa(1) \otimes \widehat{\mathscr{O}}_X) \otimes^L Rf_*R\nu_{\text{an},*}(V_{-w_0(\kappa)} \otimes \widehat{\mathscr{I}}_X) \xrightarrow{\cup_f} Rf_*(R\nu_{\text{an},*}(V_\kappa(1) \otimes \widehat{\mathscr{O}}_X) \otimes^L R\nu_{\text{an},*}(V_{-w_0(\kappa)} \otimes \widehat{\mathscr{I}}_X))$$

and

$$R\nu_{\text{an},*}(V_\kappa(1) \otimes \widehat{\mathscr{O}}_X) \otimes^L R\nu_{\text{an},*}(V_{-w_0(\kappa)} \otimes \widehat{\mathscr{I}}_D) \xrightarrow{\cup_{\nu_{\text{an}}}} R\nu_{\text{an},*}(\widehat{\mathscr{I}}_D(1)).$$

But using the formulas

$$\begin{aligned} R\nu_{\text{an},*}(V_\kappa(1) \otimes \widehat{\mathscr{O}}_X) &= \omega_E^{w_0(\kappa)}(k_1 + 1)[0] \oplus \omega_E^{\kappa+\alpha}(k_2)[-1] \\ R\nu_{\text{an},*}(V_{-w_0(\kappa)} \otimes \widehat{\mathscr{I}}_D) &= \omega_E^{-\kappa}(-D)(-k_2)[0] \oplus \omega_E^{-w_0(\kappa)}(-D)(-k_1 - 1)[-1] \\ R\nu_{\text{an},*}(\widehat{\mathscr{I}}_D(1)) &= \mathscr{O}_X(-D)(1)[0] \oplus \omega_E^{\alpha}(-D)[-1] \end{aligned}$$

one deduces that the projection of $\cup_{\nu_{\text{an}}}$ onto the factor $\omega_E^\alpha(-D)[-1]$ is given by the Yoneda pairings

$$\begin{gathered} \omega_E^{-w_0(\kappa)}(k_1 + 1) \otimes^L \omega_E^{w_0(\kappa)+\alpha}(-D)(-k_1 - 1) \to \omega_E^\alpha(-D) \cong^{KS} \Omega^1_X \\ \omega_E^{-\kappa}(-D)(-k_2) \otimes^L \omega_E^{\kappa+\alpha}(k_2) \to \omega_E^\alpha(-D) \cong^{KS} \Omega^1_X, \end{gathered}$$

proving that the pairings of the corollary with image in

$$H^2_{\text{ét},c}(Y_{\mathbb{C}_p}, \mathbb{Q}_p(1)) \otimes \mathbb{C}_p \cong H^1_{\text{an}}(X_{\mathbb{C}_p}, \Omega^1_X) \cong^{KS} H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega^\alpha(-D))$$

are compatible. The corollary follows from (2) and (3) of Proposition 10.2.2. □

10.3. The overconvergent Eichler-Shimura maps

Let $n \geq 1$ be a fixed integer. In the next two sections we will take $Y = Y_0(p^n)$ and $X = X_0(p^n)$ the modular curves of level $K^p \operatorname{Iw}_n$. Let $\epsilon \geq \delta \geq n$ be rational numbers, (R, R^+) a uniform affinoid Tate $\mathbb{Q}_p$-algebra and $\chi = (\chi_1, \chi_2) : T \to R^{+,\times}$ a δ-analytic character. Let $w \in W = \{1, w_0\}$ be an element in the Weyl group of GL_2 and $X_{w,\mathrm{Iw}_n}(\epsilon)$ the ϵ-neighbourhood of the w-ordinary locus defined in §9.1. Let ω_E^κ be the sheaf of overconvergent modular forms over $X_{w,\mathrm{Iw}_n}(\epsilon)$ (§9.2), $\mathcal{A}_\chi^\delta$ and $\mathcal{D}_\chi^\delta$ the sheaves of δ-analytic principal series and distributions (§9.3). In Section 9.4 we have defined HT morphisms between the previous sheaves, interpolating the HT maps $V_\kappa \otimes \widehat{\mathscr{O}}_X \to \omega_E^\kappa \otimes \widehat{\mathscr{O}}_X(k_2 - 1)$ and $\omega_E^{w_0(\kappa)}(k_1) \otimes \widehat{\mathscr{O}}_X \to V_\kappa \otimes \widehat{\mathscr{O}}_X$. We have the following lemma

Lemma 10.3.1. *Let $\alpha = (1, -1) \in X^*(\mathbf{T})$. The overconvergent Hodge-Tate maps (Definition 9.4.1) give rise Galois and U_p^t-equivariant maps of cohomology groups (with the good normalizations for the overconvergent modular forms, see Definition 9.2.19)*

$$\begin{aligned} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) &\xrightarrow{ES_{\mathcal{A}}} H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \\ H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) &\xrightarrow{ES^\vee_{\mathcal{A}}} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X). \end{aligned} \tag{10.3.1}$$

Dually, we have Galois and U_p-equivariant maps of cohomology groups

$$\begin{aligned} H^1_{w_0,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi})_\epsilon(-\chi_2) &\xrightarrow{ES^\vee_{\mathcal{D}}} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X)) \\ H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) &\xrightarrow{ES_{\mathcal{D}}} H^0_1(X_{\mathbb{C}_p}, \omega_E^{-w_0(\chi)+\alpha})_\epsilon(-\kappa_1 - 1). \end{aligned}$$

Proof. Let $\nu_{\text{an}} : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{an}}$ be the natural projection. First, let us show that

$$R\nu_{\text{an},*}(\omega_E^\chi \widehat{\otimes} \widehat{\mathscr{O}}_X) = \omega_E^\chi[0] \oplus \omega_E^{\chi+\alpha}(-1)[-1]. \tag{10.3.2}$$

By Theorem 9.2.3 the sheaf $\omega_E^{\chi,+}$ is an ON-$\mathscr{O}^+_{X,\text{ét}}$ sheaf locally for the étale topology of X. Let $\nu_{\text{két}} : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{két}}$ be the natural projection of sites. Then, locally étale, we can write $\omega_E^{\chi,+} \widehat{\otimes} \widehat{\mathscr{O}}_X = \widehat{\bigoplus}_i \widehat{\mathscr{O}}_X^+ e_i$. Thus, we get

$$\begin{aligned} R\nu_{\text{két},*}(\omega_E^\chi \widehat{\otimes} \widehat{\mathscr{O}}_X) &= R\nu_{\text{két},*}(\omega_E^{\chi,+} \widehat{\otimes} \widehat{\mathscr{O}}_X^+)[\frac{1}{p}] \\ &= R\varprojlim_s R\nu_{\text{két},*}(\bigoplus (\mathscr{O}_X^+/p^s)e_i)[\frac{1}{p}] \\ &= R\varprojlim_s \bigoplus_s (R\nu_{\text{két},*}\widehat{\mathscr{O}}_X^+/p^s)e_i[\frac{1}{p}] \\ &= \omega_E^\chi \widehat{\otimes}^L R\nu_{\text{két},*}\widehat{\mathscr{O}}_X. \end{aligned}$$

Then, by Therorem 10.1.5 we know that $R_{\nu,\text{két}}\widehat{\mathscr{O}}_X = \mathscr{O}_{X,\text{két}}[0] \oplus \omega_E^\alpha(-1)[-1]$. Lemmas [Sch13a, 5.5] or [DLLZ19, 6.17] imply that the integral structure obtained by $R\nu_{\text{két},*}(\widehat{\mathscr{O}}_X^+)$ defines the same topology of the one given by $\mathscr{O}^+_{X,\text{két}}[0] \oplus \omega_E^{\alpha,+}(-1)[-1]$. Therefore

$$R\nu_{\text{két},*}(\omega_E^\chi \widehat{\otimes} \widehat{\mathscr{O}}_X) = \omega_E^\chi[0] \oplus \omega_E^{\chi+\alpha}(-1)[0]$$

over the Kummer-étale site of $X_{w,\mathrm{Iw}_m}(\epsilon)$. Finally, let $\mu_{\text{két}} : X_{\mathbb{C}_p,\text{két}} \to X_{\mathbb{C}_p,\text{an}}$ be the projection map. In order to descend to the analytic site we recall that ω_E^χ is a projective Banach sheaf over $X_{w,\mathrm{Iw}_n}(\epsilon)$ (cf. [BP20, §5.5.2]). Thus, it is a direct summand of an ON Banach sheaf $\widehat{\bigoplus}_i \mathscr{O}_X$ over $X_{w,\mathrm{Iw}_n}(\epsilon)$. But

we know that the Kummer-étale cohomology of $\mathscr{O}_X^+$ in affinoids admitting a Kummer-étale map to a torus $\mathbb{T} = \mathrm{Spa}(\mathbb{Q}_p\langle T^{\pm 1}\rangle, \mathbb{Z}_p\langle T^{\pm 1}\rangle)$ or a disc $\mathbb{D} = \mathrm{Spa}(\mathbb{Q}_p\langle U\rangle, \mathbb{Z}_p\langle U\rangle)$ has bounded torsion (using the Lemma 5.5 of [Sch13a] for example). A similar argument as before using derived limits shows that $R\mu_{\text{két},*}(\widehat{\bigoplus}_i \mathscr{O}_{X,\text{két}}) = \widehat{\bigoplus}_i \mathscr{O}_{X,\text{an}}$, whence $R\mu_{\text{két},*}\omega_E^\chi = \omega_E^\chi$.

Consider the overconvergent HT maps of §9.4

$$\omega_E^{w_0(\chi)} \widehat{\otimes} \widehat{\mathscr{O}}_X(\chi_1) \to \mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X \text{ over } X_{1,\mathrm{Iw}_n}(\epsilon)$$
$$\mathcal{A}_{\mathrm{Iw}_n}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X \to \omega_E^\chi \widehat{\otimes} \widehat{\mathscr{O}}_X(\chi_2) \text{ over } X_{w_0,\mathrm{Iw}_n}(\epsilon).$$

Taking the projection from the pro-Kummer-étale site to the analytic site, one gets maps

$$\omega_E^{w_0(\chi)} \widehat{\otimes} R\nu_{\text{an},*}\widehat{\mathscr{O}}_X(\chi_1) \to R\nu_{\text{an},*}(\mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) \text{ over } X_{1,\mathrm{Iw}_n}(\epsilon)$$
$$R\nu_{\text{an},*}(\mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) \to \omega_E^\chi \widehat{\otimes} R\nu_{\text{an},*}\widehat{\mathscr{O}}_X(\chi_2) \text{ over } X_{w_0,\mathrm{Iw}_n}(\epsilon).$$

Taking the cohomologies of Definition 9.2.9 and using (10.3.2) we obtain maps

$$R\Gamma_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) \to R\Gamma_{1,c}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\mathcal{A}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon$$
$$R\Gamma_{w_0}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\mathcal{A}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon \to R\Gamma_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha}))_\epsilon(\chi_2 - 1)[-1].$$

On the other hand, we have restriction and correstriction maps

$$R\Gamma_{1,c}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\mathcal{A}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon \xrightarrow{\text{Cor}} R\Gamma_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X^+) \xrightarrow{\text{Res}} R\Gamma_{w_0}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\mathcal{A}^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon. \quad (10.3.3)$$

Taking H^1-cohomology and the composition of the previous morphisms, on obtains the maps (10.3.1). A similar construction yields for the distributions $\mathcal{D}_\chi^\delta$. The Galois equivariance is clear as the HT-maps are Galois equivariant. The compatibility with respect to the good normalization of the U_p-operators follows from Proposition 9.4.2 and the equality $U_{p,\alpha}^{good} = U_{p,\alpha}$, see Definition 9.2.19. □

We can finally state the main theorem of this paper. We will focus in the case of the δ-analytic principal series, the statements and the proof for the δ-analytic distributions being totally analogous, cf. [AIS15, Theo. 6.1].

Theorem 10.3.2. *Let $\epsilon \geq \delta \geq n$, (R, R^+) an uniform affinoid Tate $\mathbb{Q}_p$-algebra and $\chi : T = \mathbf{T}(\mathbb{Z}_p) \to R^{+,\times}$ a δ-analytic character. The following holds*

1. *The composition of the Eichler-Shimura maps $ES_{\mathcal{A}} \circ ES_{\mathcal{A}}^\vee$ is zero:*

$$0 \to H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) \xrightarrow{ES_{\mathcal{A}}^\vee} H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) \xrightarrow{ES_{\mathcal{A}}} H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \to 0. \quad (10.3.4)$$

2. *Assume that $\mathcal{V} = \mathrm{Spa}(R, R^+)$ is an affinoid subspace of the weight space $\mathcal{W}_T$ of T, an let $\kappa = (k_1, k_2) \in \mathcal{V}$ be a dominant weight of $\mathbf{T}$. Let $\alpha = (1, -1) \in X^*(\mathbf{T})$ and let $\chi = \chi_{\mathcal{V}}^{un}$ be the universal character of $\mathcal{V}$. The following diagram commutes*

$$\begin{array}{ccccc}
H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})_\epsilon(\chi_1) & \xrightarrow{ES_{\mathcal{A}}^\vee} & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) & \xrightarrow{ES_{\mathcal{A}}} & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon(\chi_2 - 1) \\
\downarrow \nearrow & & \downarrow \nearrow_{\text{Cor}} \quad \nearrow_{\text{Res}} & & \downarrow \nearrow \\
H^1_{1,c}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon & & H^1_{w_0}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\mathcal{A}_\chi^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X))_\epsilon & & \downarrow \\
H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})_\epsilon(k_1) & \longrightarrow & H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\kappa^\delta \widehat{\otimes} \widehat{\mathscr{O}}_X) & \longrightarrow & H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})_\epsilon(k_2 - 1) \\
\downarrow \text{Cor} & & \uparrow & & \uparrow \text{Res} \\
H^1_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{w_0(\kappa)})(k_1) & \xrightarrow{ES^\vee} & H^1_{\text{ét}}(Y_{\mathbb{C}_p}, V_\kappa) \otimes \mathbb{C}_p & \xrightarrow{ES} & H^0_{\text{an}}(X_{\mathbb{C}_p}, \omega_E^{\kappa+\alpha})(k_2 - 1)
\end{array}$$

3. *The maps of (2) are Galois and U_p^t equivariant with respect to the good nomalizations (Definition 9.2.19). In particular, the diagram above restricts to the finite slope part with respect to the U_p^t action.*

4. *Let $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-slope part of the restriction of (10.3.4) to $\mathcal{V}'$ is a short exact sequence of finite free $\mathbb{C}_p\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{O}(\mathcal{V}')$-modules.*

5. *Keep the hypothesis of (4), and let χ be the universal character of $\mathcal{V}'$. Let $\widetilde{\chi} = \chi_1 - \chi_2 + 1 : \mathbb{Z}_p^\times \to R^{+,\times}$, and $b = \frac{d}{dt}|_{t=1}\widetilde{\chi}(t)$. Then we have a Galois-equivariant split after inverting b*

$$H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}^\delta_\chi\widehat{\otimes}\widehat{\mathscr{O}}_X)^{\leq h}_b = [H^1_{1,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})^{\leq h}_\epsilon(\chi_1)]_b \oplus [H^0_{w_0}(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})^{\leq h}_\epsilon(\chi_2 - 1)]_b.$$

Proof. Part (1) follows from the fact that the composition of the restriction and correstriction maps (10.3.3) is zero.

Parts (2) and (3) follows from Lemma 10.3.1, and the compatibility of the formation of $\mathcal{A}^\delta_\chi$ and ω_E^χ with the character χ. The commutation of the lower diagram is a direct consequence of the constructions and Corollary 8.4.5.

For part (4) we follow the same arguments of [AIS15]: the finite slope theory (cf. [Urb11, Buz07]) implies that there is an affinoid open subspace $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-part of the sequence (10.3.4) restricted to $\mathcal{V}'$ is a sequence of finite free $\mathbb{C}_p\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{O}(\mathcal{V}')$-modules. Moreover, by the classicity theorems 9.2.20 and 9.3.16, and the classical Eichler-Shimura decomposition (Theorem 10.1.1), we can take $\mathcal{V}'$ such that the sequence (10.3.4) is short and exact.

Finally, we briefly sketch the argument for part (5). Let $\mathcal{V}'$ be as in (4), let $R = \mathscr{O}(\mathcal{V}')$, and consider the short exact sequence of the $(\leq h)$-part of (10.3.4). Taking basis and tensoring with the Tate twist $R(1-\chi_2)$ we are left to prove that the localization by b of $H^1(G_{\mathbb{Q}_p}, \mathbb{C}_p\widehat{\otimes}R(\chi_1 - \chi_2 + 1))$ vanishes. By almost étale descent one has

$$H^1(G_{\mathbb{Q}_p}, \mathbb{C}_p\widehat{\otimes}R(\chi_1 - \chi_2 + 1)) = H^1(\text{Gal}(\mathbb{Q}_p^{\text{cyc}}/\mathbb{Q}_p), \mathbb{Q}_p^{\text{cyc}}\widehat{\otimes}_{\mathbb{Q}_p}R(\chi_1 - \chi_2 + 1)). \tag{10.3.5}$$

We identify $\text{Gal}(\mathbb{Q}_p^{\text{cyc}}/\mathbb{Q}_p)$ with $\mathbb{Z}_p^\times$ via χ_{cyc}. By Sen theory, to show that (10.3.5) is of b-torsion it is enough to prove that $H^1(\text{Lie}\,\mathbb{Z}_p^\times, R(\chi_1 - \chi_2 + 1))_b = 0$, but this is clear as $H^1(\text{Lie}\,\mathbb{Z}_p^\times, R(\chi_1 - \chi_2 + 1)) \cong R/bR$. □

10.4. Compatibility with Poincaré and Serre duality

We finish this paper with a construction of the Poincaré paring for overconvergent modular symbols, then we prove its compatibility with the Eichler-Shimura maps and the Serre pairing of overconvergent modular forms. We need a couple of lemmas

Lemma 10.4.1. *Let $\mathscr{F}$ denote $\mathcal{A}^\delta_\chi$ or $\mathcal{D}^\delta_\chi$. Then the complex $R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathscr{F})$ admits, locally on the weight space, finite slope decompositions.*

Proof. This follows from the distinguished triangle

$$R\Gamma_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathscr{F}) \longrightarrow R\Gamma_{\text{proét}}(Y_{\mathbb{C}_p}, \mathscr{F}) \longrightarrow R\Gamma_{\text{prokét}}(D_{\mathbb{C}_p}, \mathscr{F}) \xrightarrow{+}$$

and the fact that $R\Gamma_{\text{prokét}}(D_{\mathbb{C}_p}, \mathscr{F})$ is represented by direct sums of complexes of the form

$$\mathscr{F}(\tilde{\xi}) \xrightarrow{n_\xi - 1} \mathscr{F}(\tilde{\xi})$$

where $\xi \in D$, $\tilde{\xi} \in X_\infty$ is a log perfectoid point over ξ, and $n_\xi \in N_\xi \subset \text{Iw}_n$ is the monodromy of $\widetilde{\xi}$ over ξ. □

We have short exact sequences over $X_{w,\mathrm{Iw}_n}(\epsilon)$

$$0 \to \omega_E^{\chi}\widehat{\otimes}_{\mathscr{O}_X}\widehat{\mathscr{I}_D} \to \omega_E^{\chi}\widehat{\otimes}_{\mathscr{O}_X}\widehat{\mathscr{O}}_X \to \omega_E^{\chi}\widehat{\otimes}_{\mathscr{O}_X}\iota_*\widehat{\mathscr{O}}_D \to 0$$

where the completed tensor products are with respect to the integral modular sheaves $\omega_E^{\chi,+}$ and the p-adic topology.

Lemma 10.4.2. *Let $\nu_{\mathrm{an}} : X_{w,\mathrm{Iw}_n}(\epsilon)_{\mathrm{prokét}} \to X_{w,\mathrm{Iw}_n}(\epsilon)_{\mathrm{an}}$ be the projection of sites, then*

$$R\nu_{\mathrm{an},*}(\omega_E^{\chi}\widehat{\otimes}_{\mathscr{O}_X}\widehat{\mathscr{I}_D}) = \omega_E^{\chi}(-D)[0] \oplus \omega_E^{\chi+\alpha}(-D)(-1)[-1].$$

Proof. This follows from the proof of Lemma 10.3.1 which gives $R\nu_{\mathrm{an},*}(\omega_E^{\chi}\widehat{\otimes}_{\mathscr{O}_X}\widehat{\mathscr{O}}_X) = \omega_E^{\chi}[0] \oplus \omega_E^{\chi}(-1)[-1]$, and (10.1.5) of Lemma 10.1.8 which implies that $R\nu_{\mathrm{an},*}(\omega_E^{\chi}\widehat{\otimes}_{\mathscr{O}_X}\iota_*\widehat{\mathscr{O}}_D) = \omega_E^{\chi}|_D[0] \oplus \omega_E^{\chi+\alpha}(-1)|_D[-1]$. □

On the other hand, Proposition 10.1.7 provides an isomorphism of distinguished triangles

$$\begin{array}{ccccccc}
R\Gamma_{\mathrm{proét},c}(Y_{\mathbb{C}_p},\mathscr{F})\widehat{\otimes}\mathbb{C}_p & \longrightarrow & R\Gamma_{\mathrm{proét}}(Y_{\mathbb{C}_p},\mathscr{F})\widehat{\otimes}\mathbb{C}_p & \longrightarrow & R\Gamma_{\mathrm{prokét}}(D_{\mathbb{C}_p},\mathscr{F})\widehat{\otimes}\mathbb{C}_p & \xrightarrow{+} & \\
\| & & \| & & \| & & \\
R\Gamma_{\mathrm{prokét},c}(X_{\mathbb{C}_p},\mathscr{F}\widehat{\otimes}\widehat{\mathscr{I}_D}) & \longrightarrow & R\Gamma_{\mathrm{prokét}}(X_{\mathbb{C}_p},\mathscr{F}\widehat{\otimes}\widehat{\mathscr{O}}_X) & \longrightarrow & R\Gamma_{\mathrm{prokét}}(D_{\mathbb{C}_p},\mathscr{F}\widehat{\otimes}\widehat{\mathscr{O}}_D) & \xrightarrow{+} &
\end{array}$$

where the completed tensor products are with respect to the filtrations of $\mathscr{A}_{\chi}^{\delta,+}$ and $\mathscr{D}_{\chi}^{\delta,+}$. We deduce the following analogous of Theorem **??** for the cohomology with compact supports

Corollary 10.4.3. *With the hypothesis of Theorem **??**, we have Galois and Hecke equivariant Eichler-Shimura maps*

$$0 \to H^1_{1,c}(X_{\mathbb{C}_p},\omega_E^{w_0(\chi)}(-D))_{\epsilon}(\chi_1) \xrightarrow{ES^{\vee}_{\mathscr{A}}} H^1_{\mathrm{proét},c}(Y_{\mathbb{C}_p},\mathscr{A}_{\chi}^{\delta})\widehat{\otimes}\mathbb{C}_p \xrightarrow{ES_{\mathscr{A}}} H^0_{w_0}(X_{\mathbb{C}_p},\omega_E^{\chi+\alpha}(-D))_{\epsilon}(\chi_2-1) \longrightarrow 0$$

$$0 \to H^1_{w_0,c}(X_{\mathbb{C}_p},\omega_E^{-\chi}(-D))_{\epsilon}(-\chi_2) \xrightarrow{ES^{\vee}_{\mathscr{D}}} H^1_{\mathrm{proét},c}(Y_{\mathbb{C}_p},\mathscr{D}_{\chi}^{\delta})\widehat{\otimes}\mathbb{C}_p \xrightarrow{ES_{\mathscr{D}}} H^0_1(X_{\mathbb{C}_p},\omega_E^{-w_0(\chi)+\alpha}(-D))(-\chi_1-1) \to 0. \tag{10.4.1}$$

where D is the cusp divisor. Furthermore, the following holds

1. *The maps (10.4.1) are a sequence, i.e. the composition of two consecutive arrows is zero.*
2. *Let $\mathcal{V} \subset \mathcal{W}_T$ be an open affinoid of the weight space of $T = \mathbf{T}(\mathbb{Z}_p)$, let $R = \mathscr{O}(\mathcal{V})$ and χ be the universal character over $\mathcal{V}$. Let $\kappa = (k_1,k_2) \in \mathcal{V}$ be a dominant weight and $h < k_1 - k_2 + 1$. Then there exists an affinoid subspace $\mathcal{V}' \subset \mathcal{V}$ containing κ such that the $(\leq h)$-slope part of (10.4.1) is a short exact sequence of finite free $\mathbb{C}_p\widehat{\otimes}\mathscr{O}(\mathcal{V}')$-modules.*
3. *With the hypothesis as in (2), suppose that $\mathcal{V} = \mathcal{V}'$. Let $\widetilde{\chi} = \chi_1 - \chi_2 + 1$ and $b = \frac{d}{dt}|_{t=1}\widetilde{\chi}(t)$. Then, after inverting b, the $(\leq h)$-slope part of (10.4.1) splits as a Galois representation.*

Proof. This is a consequence of Lemmas 10.4.1 and 10.4.2 and the proof of Theorem **??**. □

10.4.1. The pairings

Let $\epsilon \geq \delta \geq n$, (R, R^+) and χ be as in previous sections. By construction there is a natural pairing between the δ-principal series and distributions

$$A_\chi^\delta \times D_\chi^\delta \to R.$$

It is easy to see that it induces a Poincaré pairing

$$\langle -,-\rangle_P : H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta(1)) \times H^1_{\text{proét}}(Y_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta) \to H^2_{\text{proét},c}(Y_{\mathbb{C}_p}, R(1)) \xrightarrow{\text{Tr}_P} R$$

where the first arrow is a Yoneda pairing, and the last arrow is induced by the Poincaré trace $H^1_{\text{ét},c}(Y_{\mathbb{C}_p}, \mathbb{Z}_p(1)) \xrightarrow{\text{Tr}_P} \mathbb{Z}_p$.

On the other hand, in [BP20] the authors define overconvergent Serre pairings in families

$$\langle -,-\rangle_S : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_\epsilon \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon \to R$$

compatible with the classical Serre pairings. They are constructed by taking the Yoneda's product

$$\cup : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_\epsilon \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon \xrightarrow{\cup} H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{\alpha}(-D))_\epsilon \cong H^1_{w,c}(X_{\mathbb{C}_p}\Omega^1_X \widehat{\otimes} R)_\epsilon$$

and composing with the Serre trace map of X

$$\text{Tr}_S : H^1_{w,c}(X_{\mathbb{C}_p}, \Omega^1_X \widehat{\otimes} R)_\epsilon \xrightarrow{\text{Cor}} H^1_{\text{an}}(X_{\mathbb{C}_p}, \Omega^1_X \widehat{\otimes} R) \to R.$$

Theorem 10.4.4. *Keep the notation of Theorem* **??**. *The following holds*

1. *The Poincaré and Serre pairings of overconvergent cohomologies are compatible with the good normalizations of the U_p-operators (Definition 9.2.19). Moreover, they are compatible with the Eichler-Shimura maps of Corollary 10.4.3.*
2. *Let $\mathcal{V} \subset \mathcal{W}_T$ be an open affinoid, and let $\chi = \chi^{\text{un}}_{\mathcal{V}}$ be the universal character of $\mathcal{V}$. Let $\kappa = (k_1, k_2) \in \mathcal{V}$ and fix $h < k_1 - k_2 + 1$. There exists an open affinoid $\mathcal{V}' \subset \mathcal{V}$ containing κ such that we have perfect pairings of finite free $\mathbb{C}_p\widehat{\otimes}\mathscr{O}(\mathcal{V}')$-modules*

$$\langle -,-\rangle_P : H^1_{\text{proét},c}(Y_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta(1))^{\leq h} \times H^1_{\text{proét}}(Y_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta)^{\leq h} \to \mathscr{O}(\mathcal{V}')$$

and

$$\langle -,-\rangle_S : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))^{\leq h}_\epsilon \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})^{\leq h}_\epsilon \to \mathscr{O}(\mathcal{V}')$$
$$\langle -,-\rangle_S : H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{w_0(\chi)})^{\leq h}_\epsilon \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{-w_0(\chi)+\alpha}(-D))^{\leq h}_\epsilon \to \mathscr{O}(\mathcal{V}'),$$

compatible with the overconvergent Eichler-Shimura maps.

Proof. The Hecke operators are compatible with the pairings by their definition via finite flat correspondances, see Definitions 9.2.2 and 9.3.12.

In the following we forget the Galois action. Let $\nu_{\text{an}} : X_{\mathbb{C}_p,\text{prokét}} \to X_{\mathbb{C}_p,\text{an}}$ be the projection of sites. We have a commutative diagram of Yoneda's products

$$\begin{array}{ccc}
H^1_{1,c}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\omega_E^{w_0(\chi)}\widehat{\otimes}\widehat{\mathscr{O}_X}))_\epsilon \times H^1_1(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\omega_E^{-w_0(\kappa)}\widehat{\otimes}\widehat{\mathscr{I}_D}))_\epsilon & \to & H^2_{1,c}(X_{\mathbb{C}_p}, R\nu_{an,*}(R\widehat{\otimes}\widehat{\mathscr{I}_D}))_\epsilon \\
\downarrow \quad \uparrow & & \downarrow \text{Cor} \\
H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{A}_\chi^\delta\widehat{\otimes}\widehat{\mathscr{O}_X}) \times H^1_{\text{prokét}}(X_{\mathbb{C}_p}, \mathcal{D}_\chi^\delta\widehat{\otimes}\widehat{\mathscr{I}_D}) & \longrightarrow & H^2_{\text{prokét}}(X_{\mathbb{C}_p}, R\widehat{\otimes}\widehat{\mathscr{I}_D}) \\
\downarrow \quad \uparrow & & \uparrow \text{Cor} \\
H^1_{w_0}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{O}_X}))_\epsilon \times H^1_{w_0,c}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\omega_E^{-\chi}\widehat{\otimes}\widehat{\mathscr{I}_D}))_\epsilon & \to & H^2_{w_0,c}(X_{\mathbb{C}_p}, R\nu_{an,*}(R\widehat{\otimes}\widehat{\mathscr{I}_D}))_\epsilon
\end{array}$$

On the other hand, we also have compatible pairings provided by the Faltings extension

$$\begin{array}{ccc} H^1_{w,c}(X_{\mathbb{C}_p}, \omega_E^{-\chi}(-D))_\epsilon \times H^0_w(X_{\mathbb{C}_p}, \omega_E^{\chi+\alpha})_\epsilon & \xrightarrow{\text{Cor}\circ\cup} & H^1_{\text{an}}(X_{\mathbb{C}_p}, R\widehat{\otimes}\Omega^1_X) \\ \downarrow & & \downarrow FE \\ H^1_{w_0,c}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\omega_E^{-\chi}\widehat{\otimes}\widehat{\mathscr{I}_D}))_\epsilon \times H^1_{w_0}(X_{\mathbb{C}_p}, R\nu_{\text{an},*}(\omega_E^{\chi}\widehat{\otimes}\widehat{\mathscr{O}_X}))_\epsilon & \xrightarrow{\text{Cor}\circ\cup} & H^2_{\text{prokét}}(X_{\mathbb{C}_p}, R\widehat{\otimes}\widehat{\mathscr{I}_D}). \end{array}$$

The compatibility of Poincaré and Serre traces (Proposition 10.2.2) implies part (1). Part (2) follows the same lines of the proof of Theorem **??** and Corollary 10.4.3 using the fact that the pairings are perfect for the classical Eichler-Shimura decomposition. □

Part III.

Solid locally analytic representations of p-adic Lie groups

Joint work with Joaquín Rodrigues Jacinto

11. Introduction

The theory of p-adic representations of p-adic groups has a long history and it has played a key role in the field of Number Theory during the last decades, as witnessed, e.g., in the study of the p-adic Langlands correspondence [Col10], [CDP14].

In this article, we intend to reformulate the theory of locally analytic representations of p-adic Lie groups as developed in [ST02], [ST01], [ST03], [ST05], [Eme17], using the theory of condensed mathematics developed by Clausen and Scholze. We define and study the notions of analytic and locally analytic representations of p-adic Lie groups on solid modules. One of our main new results, which was the departing point of our investigations, is a generalisation of Lazard's comparison theorem [Laz65] between continuous and locally analytic cohomology of a finite dimensional representation of a compact p-adic Lie group over $\mathbb{Q}_p$ to arbitrary solid locally analytic representations[1]. Generalisations of Lazard's comparison between locally analytic and Lie algebra cohomology have already been considered in [HKN11], [Lec12], [Tam15]. We also give, following the lines of [Tam15], a proof of this result in the solid context. Our second main new result is a comparison between continuous cohomology of solid representation and the continuous cohomology of its locally analytic vectors. This results can be seen as a p-adic analogue of a theorem of G. D. Mostow [Mos61] and P. Blanc [Bla79].

11.1. Background

Let p be a prime number and G a compact p-adic Lie group. The theory of p-adic representations of G comes in different flavours: one has the notion of continuous, analytic and locally analytic representations V of G according to whether the orbit map $o_v : G \to V$, $g \mapsto g \cdot v$ ($v \in V$ fixed) is a continuous, resp. analytic, resp. locally analytic function of the p-adic variety G.

One usual way of studying such representations is through their cohomology. In a good situation, the cohomology groups are Ext-groups in certain abelian categories. For instance, one defines the cohomology of an abstract group H by working on the abelian category of $\mathbb{Z}[H]$-modules: if M is an H-module, the cohomology groups $H^i(H, M)$ are by definition $\mathrm{Ext}^i_{\mathbb{Z}[H]}(\mathbb{Z}, M)$, where $\mathbb{Z}$ denotes the trivial representation.

In the case where V is a continuous representation of G, it is natural to consider continuous group cohomology. Unfortunately, the category of topological abelian groups is not abelian. The absence of a good formalism leads to ad-hoc definitions and conditions to get some basic results. For instance, the restriction to strict short exact sequences in order to obtain a long exact sequence in cohomology, or the necessity of proving by hand certain results such as Hochschild-Serre's spectral sequence (cf. [Ser94]), which would otherwise be formal consequences of a well behaved theory.

A possible way to remedy this issue is to cut out some subcategory of objects which is better behaved. Let us focus on a case of special interest to us. Let K be a finite extension of $\mathbb{Q}_p$. The category

[1] In particular, to any complete compactly generated locally convex vector space, e.g. metrizable.

of locally analytic representations of G is not abelian. But one can consider the subcategory of admissible locally analytic representations [ST03]. These objects carry a natural topology and form an abelian category.

The theory of condensed mathematics developed by Clausen and Scholze [Sch19], [Sch20], [CS] also provides a natural approach to deal with these kind of difficulties. Very vaguely, the condensed objects in a category $\mathscr{C}$ can be defined as sheaves on the proétale site of a point with values in $\mathscr{C}$. It is shown in [Sch19] that the category of condensed abelian groups is an abelian category satisfying Grothendieck's axioms. Hence, one is on a good footing for doing homological algebra. Notice that, as opposed to the approach described in the previous paragraph, this one consists not in imposing restrictions but rather enlarging the universe where several topological problems disappear, thus giving a very general theory.

11.2. Statement of the main results

Let us now describe with some more detail what is carried out in this article.

11.2.1. Solid non-archimedean functional analysis

Let K be a finite extension of $\mathbb{Q}_p$. The field K naturally defines a condensed ring which, moreover, has an analytic ring structure $K_\blacksquare$ in the sense of [Sch19], usually called the solid ring structure on K. We let $\mathrm{Solid}(K)$ be the category of solid K-vector spaces. This category is stable under limits, colimits and extensions, it has a tensor product $\otimes_{K_\blacksquare}$ and an internal Hom denoted by $\underline{\mathrm{Hom}}_K(-,-)$ [CS]. Let us point out that all the important spaces in the classical theory of non-archimedean functional analysis [Sch02] live naturally in $\mathrm{Solid}(K)$. Indeed, there is a natural functor

$$\mathcal{LC}_K \to \mathrm{Solid}(K) \tag{11.2.1}$$

from the category of complete locally convex K-vector spaces to solid K-vector spaces, as any complete locally convex K-vector space can be written as a cofiltered limit of Banach spaces. Moreover, it is fully faithful on a very large class of complete locally convex K-vector spaces, e.g. all compactly generated ones, e.g. all metrizable ones. The main notions of the theory of condensed non-archimedean functional analysis we use are due to Clausen and Scholze [Sch19], [CS], [Bos21]. All the vector spaces considered in this text are solid K-vector spaces, unless otherwise specified.

Our first result is an anti-equivalence between two special families of solid K-vector spaces. Let us first give some definitions. A Smith space is a K-vector space of the form $\underline{\mathrm{Hom}}_K(V,K)$, where V is a Banach space. In classical terms, a Smith space is the dual of a Banach space equipped with the compact-open topology. An LS space is a countable filtered inductive limit of Smith spaces with injective transition maps. We then have the following result.

Theorem 11.2.1 (Theorem 13.3.13)**.** *The functor $V \mapsto V^\vee := \underline{\mathrm{Hom}}_K(V,K)$ induces an anti-equivalence between Fréchet and LS spaces such that $\underline{\mathrm{Hom}}_K(V,V') = \underline{\mathrm{Hom}}_K(V'^\vee, V^\vee)$.*

Remark 11.2.2. The previous theorem restricts in particular to an anti-equivalence between classical nuclear Fréchet spaces and LB spaces of compact type (see, e.g., [ST02, Theorem 1.3]).

11.2.2. Representation theory

Let G be a compact p-adic Lie group. A representation of G on a solid K-vector space V is a map of condensed sets $G \times V \to V$ satisfying the usual axioms. Define the Iwasawa algebra of G with

coefficients in K as $K_\blacksquare[G]$; explicitly,

$$K_\blacksquare[G] = (\varprojlim_N O_K[G/N])[\frac{1}{p}],$$

where N runs over all the open normal subgroups of G. This is the solid algebra defined by the classical Iwasawa algebra endowed with the weak topology. The category of G-representations on solid K-vector spaces is equivalent to the category $\mathrm{Solid}(K_\blacksquare[G])$ of solid $K_\blacksquare[G]$-modules. Observe that the category of continuous representations of G on complete locally convex K-vector spaces lives naturally in $\mathrm{Solid}(K_\blacksquare[G])$ via the functor (11.2.1).

Inspired by Emerton's treatment [Eme17], we define analytic and locally analytic vectors of solid representations of G. Roughly speaking, they are defined as those vectors whose induced orbit map is analytic or locally analytic. One advantage of our approach is that definitions make sense at the level of derived categories, so one can speak about derived (locally) analytic vectors of complexes $C \in D(K_\blacksquare[G])$ in the derived category of $K_\blacksquare[G]$-modules. The derived functors of the locally analytic vectors (for Lie groups over finite extensions of $\mathbb{Q}_p$) for admissible representations have been considered in [Sch09].

More precisely, let $\mathbb{G}$ be an analytic affinoid group over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ such that $G = \mathbb{G}(\mathbb{Q}_p, \mathbb{Z}_p)$ and suppose in addition that $\mathbb{G}$ is isomorphic to a finite disjoint union of polydiscs. In practice, the group $\mathbb{G}$ will be constructed using some local charts of G, see Remark 11.2.4 below for a more detailed description. Let $C(\mathbb{G}, K) := \mathscr{O}(\mathbb{G}) \otimes_{\mathbb{Q}_p} K$ be the algebra of functions of $\mathbb{G}$. The affinoid algebra $C(\mathbb{G}, K)$ has a natural analytic ring structure denoted by $C(\mathbb{G}, K)_\blacksquare$, see [And21]. We denote by $\mathcal{D}(\mathbb{G}, K) = \mathscr{O}(\mathbb{G})^\vee \otimes_{\mathbb{Q}_p} K$ the distribution algebra of the affinoid group $\mathbb{G}$. We define the derived $\mathbb{G}$-analytic vectors of an object $C \in D(K_\blacksquare[G])$ to be the complex[2]

$$C^{R\mathbb{G}-an} := R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C \otimes^L_{K_\blacksquare} C(\mathbb{G}, K)_\blacksquare) \tag{11.2.2}$$

where K is the trivial representation, and the G-action on $C \otimes^L_{K_\blacksquare} C(\mathbb{G}, K)_\blacksquare$ is the diagonal one induced by the action on C and the left regular action on $C(\mathbb{G}, K)$. We endow $C^{R\mathbb{G}-an}$ with the right regular action of G. It turns out that there is a natural map $C^{\mathbb{G}-an} \to C$, and we say that C is derived $\mathbb{G}$-analytic if this map is a quasi-isomorphism. If V is a Banach G-representation, then $V \otimes^L_{K_\blacksquare} C(\mathbb{G}, K)_\blacksquare = V \otimes_{K_\blacksquare} C(\mathbb{G}, K)$ coincides with the projective tensor product of Banach spaces. Thus, our definition of derived $\mathbb{G}$-analytic vectors is the derived extension of [Eme17, Definition 3.3.13].

Now let $\mathring{\mathbb{G}}$ be a Stein analytic group over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ such that $G = \mathring{\mathbb{G}}(\mathbb{Q}_p, \mathbb{Z}_p)$. Suppose in addition that $\mathring{\mathbb{G}} = \bigcup_h \mathbb{G}^{(h)}$ is written as an increasing union of affinoid groups $\mathbb{G}^{(h)}$, where each $\mathbb{G}^{(h)}$ is as in the previous paragraph. In practice, $\mathring{\mathbb{G}}$ will be as in Remark 11.2.4 below. We also denote by $\mathcal{D}(\mathring{\mathbb{G}}, K) := \mathscr{O}(\mathring{\mathbb{G}})^\vee \otimes_{\mathbb{Q}_p} K = \varinjlim_h \mathcal{D}(\mathbb{G}^{(h^+)}, K)$ the distribution algebra of $\mathring{\mathbb{G}}$. Then the derived $\mathring{\mathbb{G}}$-analytic vectors of $C \in D(K_\blacksquare[G])$ are defined as the complex

$$C^{R\mathring{\mathbb{G}}-an} := R\varprojlim_h C^{R\mathbb{G}^{(h)}-an}, \tag{11.2.3}$$

and we say that C is derived $\mathring{\mathbb{G}}$-analytic if the natural map $C^{R\mathring{\mathbb{G}}-an} \to C$ is a quasi-isomorphism. Again, if V is a Banach G-representation, this definition is compatible with the $\mathring{\mathbb{G}}$-analytic vectors of [Eme17, Definition 3.4.1]. The main theorem is the following:

[2] The tensor product $\otimes_{K_\blacksquare} C(\mathbb{G}, K)_\blacksquare$ is the derived base change of modules over analytic rings, see [Sch19, Proposition 7.7].

Theorem 11.2.3 (Theorem 10.3.2)**.** *A complex $C \in D(K_\blacksquare[G])$ is derived $\mathring{\mathbb{G}}$-analytic if and only if it is a module over $\mathcal{D}(\mathring{\mathbb{G}}, K)$.*

This theorem is a generalisation of the integration map constructed by Schneider and Teitelbaum to solid G-modules, cf. [ST02, Theorem 2.2]. It will serve us as a bridge between solid analytic representations and solid modules over the distribution algebras.

Remark 11.2.4. Suppose that G is a uniform pro-p-group and $\phi : \mathbb{Z}_p^d \to G$ is an analytic chart given by a basis of the group (e.g. Example 14.1.5). Using this chart, one can define for any $h \in \mathbb{Q}_{>0}$ affinoid groups $\mathbb{G}^{(h)}$ in such a way that the p^h-analytic functions on $\mathbb{Z}_p^d$ coincide with the rigid functions on $\mathbb{G}^{(h)}$ after taking pullbacks by ϕ. In this situation, the $\mathbb{G}^{(h)}$-analytic vectors are those whose orbit map is p^h-analytic. Given $h > 0$, the group $\mathbb{G}^{(h^+)} := \bigcup_{h'>h} \mathbb{G}^{(h)}$ is a Stein group. In the definition of derived analytic vectors of equations (11.2.2) and (11.2.3) we will take $\mathbb{G} = \mathbb{G}^{(h)}$ and $\mathring{\mathbb{G}} = \mathbb{G}^{(h^+)}$. The locally analytic functions on G are

$$C^{la}(G, K) = \varinjlim_{h \to +\infty} C(\mathbb{G}^{(h)}, K) = \varinjlim_{h \to +\infty} C(\mathbb{G}^{(h^+)}, K).$$

Now let $\mathcal{D}^{la}(G, K) = C^{la}(G, K)^\vee$ be the algebra of locally analytic distributions. We point out that the analogous statement of Theorem 11.2.3 does not hold in general for locally analytic representations and $\mathcal{D}^{la}(G, K)$-modules. In the particular case of locally analytic representations on LB spaces of compact type, this is nevertheless true, and was already known by [ST02].

11.2.3. Comparison theorems in cohomology

We finish this introduction by describing the main applications of Theorems 11.2.1 and 11.2.3 to the study of the cohomology of continuous representations.

For $C \in D(K_\blacksquare[G])$, we define the solid group cohomology of C to be the complex

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C).$$

Let $\mathfrak{g}$ be the Lie algebra of G and $U(\mathfrak{g})$ its universal enveloping algebra. Let $\mathring{\mathbb{G}}$ be a Stein group neighbourhood of G as in Theorem 11.2.3 and $\mathcal{D}(\mathring{\mathbb{G}}, K)$ the distribution algebra of $\mathring{\mathbb{G}}$. If in addition C is $\mathring{\mathbb{G}}$-analytic, we define the $\mathring{\mathbb{G}}$-analytic cohomology of C as $R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathring{\mathbb{G}},K)}(K, C)$, and its Lie algebra cohomology to be $R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)$. Using Bar resolutions and Theorem 11.2.3, one verifies that these definitions recover the usual continuous, analytic, and Lie algebra cohomology groups.

Our first new result compares continuous cohomology of a solid representation and the continuous cohomology of its locally analytic vectors. This result can be seen as a p-adic analogue of a theorem of P. Blanc [Bla79] and G. D. Mostow [Mos61] in the archimedean setting, which compares continuous and differentiable cohomology of a real Lie group G.

Let $C \in D(K_\blacksquare[G])$, we define the derived locally analytic vectors of C as the homotopic colimit

$$C^{Rla} = \operatorname*{hocolim}_{\mathbb{G}} C^{R\mathbb{G}-an}$$

where $\mathbb{G}$ runs over all the affinoid neighbourhoods of G. We say that C is derived locally analytic if $C^{Rla} = C$. If V is a Banach representation, then $H^0(V^{Rla})$ coincides with the locally analytic vectors of V in the sense of [Eme17, Definition 3.5.3]. We have the following theorem.

Theorem 11.2.5 (Theorem 15.2.1)**.** *Let $C \in D(K_\blacksquare[G])$ and let C^{Rla} be the complex of derived locally analytic vectors of C. Then*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) \cong R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C^{Rla}).$$

In particular, if $V \in \mathrm{Solid}(K_\blacksquare[G])$ *then, setting* $V^{R^i la} := H^i(V^{Rla})$ *for* $i \geq 0$ *there is a spectral sequence of solid* K*-vector spaces*

$$E_2^{i,j} := \underline{\mathrm{Ext}}^i_{K_\blacksquare[G]}(K, V^{R^j la}) \implies \underline{\mathrm{Ext}}^{i+j}_{K_\blacksquare[G]}(K, V).$$

We give an application in the classical context. Let V be a continuous representation of G on a complete locally convex K-vector space. Denote $H^i_{cont}(G, V)$ the usual continuous cohomology groups. These coincide with the underlying sets of the solid cohomology groups $\underline{\mathrm{Ext}}^i_{K_\blacksquare[G]}(K, V)$. We say that V has no higher locally analytic vectors if $V^{R^i la} = 0$ for all $i > 0$. This is the case for admissible representations (cf. [Pan20] or Proposition 14.5.3). One deduces the following corollary.

Corollary 11.2.6. *If* V *has no higher locally analytic vectors, then for all* $i \geq 0$*,*

$$H^i_{cont}(G, V) = H^i_{cont}(G, V^{la}).$$

Our last result concerns a generalisation of Lazard's comparison between continuous, (locally) analytic and Lie algebra cohomology from finite dimensional representations V to arbitrary solid derived (locally) analytic representations. We have the following theorem.

Theorem 11.2.7 (Continuous vs. analytic vs. Lie algebra cohomology, Theorem 15.2.3)**.** *Let* $C \in D(K_\blacksquare[G])$ *be a derived* $\mathring{\mathbb{G}}$*-analytic complex. Then*[3]

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) \cong R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathring{\mathbb{G}}, K)}(K, C) \cong (R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C))^G.$$

11.3. Organisation of the paper

In Section 12 we review very briefly the theory of condensed mathematics and solid abelian groups. We recall the notion of analytic ring and give some examples that will be used throughout the text.

In Section 13 we develop the theory of solid K-vector spaces following the appendix of [Bos21]. Most of the results exposed in this section will be presented in the forthcoming work [CS]. We review in particular the main properties of classical vector spaces: Banach, Fréchet, *LB* and *LF* spaces. Our main original result is Theorem 11.2.1 generalising the classical anti-equivalence between *LB* spaces of compact type and nuclear Fréchet vector spaces.

In Section 14 we introduce the different analytic neighbourhoods $\mathbb{G}^{(h)}$ of our p-adic Lie group G. We begin in §14.1 by introducing spaces of analytic functions, following [Eme17] closely. Then, we define the algebras of distributions as the duals of the spaces of analytic functions. We also introduce another class of distribution algebras, used already in [ST03], which are more adapted to the coordinates of the Iwasawa algebra. In §14.3, we define the notion of analytic and derived analytic representation, we prove Theorem 11.2.3, except for a technical lemma whose proof is postponed to Section 15. We finish the section with some applications to locally analytic and admissible representations.

Finally, in Section 15, we recall (Theorem 15.3.1) a lemma of Serre used by Lazard to construct finite free resolutions of the trivial representation when G is a uniform pro-p-group. We use this result, as well as its enhancement due to Kohlhaase (Theorem 15.3.2) to prove the technical lemma necessary for Theorem 11.2.3. We state Theorems 11.2.5 and 11.2.7 in §15.2 and give a proof in §15.4. We conclude with some formal consequences, namely by showing a solid version of Hochschild-Serre and proving a duality between group homology and cohomology.

[3] The Lie algebra cohomology $R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)$ lands naturally in the derived category of smooth representations of G on solid K-vector spaces. Since K is of characteristic 0, taking G-invariants in this category is exact and the superscript G means the composition with this functor, cf. Remark 15.2.4.

12. Recollections in condensed mathematics

First, we review some elementary notions in condensed mathematics: we recall the definitions of condensed sets, solid abelian groups and analytic rings. In the future we will be only interested in the categories of solid modules, and modules of analytic rings over $\mathbb{Z}_{\blacksquare}$.

12.0.1. Condensed objects

In their recent work [Sch19] and [Sch20], Clausen and Scholze have introduced the new world of condensed mathematics, which aims to be the good framework where algebra and topology live together. Roughly speaking, a condensed set/group/ring is a sheaf in sets/groups/rings in the pro-étale site of a geometric point $*_{\text{proét}}$. It is equivalent to the category of profinite sets and continuous maps, with coverings given by finitely many continuous maps which are jointly surjective.

Definition 12.0.1 ([Sch19, Definitions 2.1 and 2.11])**.** A condensed set/group/ring/... $\mathcal{F}$ is a sheaf over $*_{\text{proét}}$ with values in Sets, Groups, Rings,... We denote by Cond the category of condensed sets and Cond(Ab) that of condensed abelian groups. If R is a condensed ring, we denote by Cond(R) the category of condensed R-modules[1].

There is a functor from topological spaces to condensed sets whose restriction to the category of compactly generated Hausdorff topological spaces is fully faithful. First recall that a topological space X is compactly generated if a map $X \to Y$ to another topological space Y is continuous if and only if the composition $S \to X \to Y$ is continuous for all maps $S \to X$ from a compact Hausdorff space. A compact Hausdorff space can be written as a quotient of a profinite set. Indeed, if S is a compact Hausdorff space, let S_{dis} denote the underlying set with the discrete topology, and βS_{dis} its Stone-Čech compactification. Then βS_{dis} is profinite and the natural map $\beta S_{dis} \to S$ is a surjective map of compact Hausdorff spaces. In particular, we can test if a topological space is compactly generated by restricting to the profinite sets.

Let π denote the category of topological spaces, we define the functor $(\_) : \pi \to \text{Cond}$ mapping a topological space T to the condensed set

$$T : S \mapsto \underline{T}(S) = \text{Cont}(S, T)$$

where $\text{Cont}(S, T)$ is the set of continuous functions from S to T. Recall that a condensed set X is called quasi-compact if there is a profinite set S and a surjective map $\underline{S} \to X$. Similarly, a condensed set X is quasi-separated if for any pair of profinite sets S and S' over X the fiber product $\underline{S} \times_X \underline{S'}$ is quasi-compact. From now on we identify a profinite set S with the condensed set $\underline{S}$. Given a condensed set X and a profinite set S we define $\underline{\text{Cont}}(S, X)$ to be the condensed set whose value at a profinite set S' is

$$\underline{\text{Cont}}(S, X)(S') = \text{Hom}_{CondSet}(S \times S', X) = X(S \times S').$$

We have the following result.

[1] There are set theoretical issues with this definition as $*_{\text{proét}}$ is not small, what [Sch19] does is to cut-off by a large cardinal κ, considering the category of sheaves of κ-small profinite sets $*_{\text{proét}}$ and take the direct limit. Hence the category of condensed sets is not the category of sheaves of a site.

Proposition 12.0.2 ([Sch20, Proposition 1.2]). *Consider the functior $T \mapsto \underline{T}$ from $\mathbb{T}$ to* Cond

1. *The functor has a left adjoint $X \mapsto X(*)_{\text{top}}$ sending any condensed set X to the set $X(*)$ equipped with the quotient topology arising from the map*

$$\bigsqcup_{S, a \in X(S)} S \to X(*)$$

with S profinite.

2. *Restricted to compactly generated topological spaces, the functor is fully faithful.*

3. *The functor induces an equivalence between the category of compact Hausdorff spaces and qcqs condensed sets.*

4. *The functor induces a fully faithful functor from the category of compactly generated weak Hausdorff spaces, to quasi-separated condensed sets. The category of quasi-separated condensed sets is equivalent to the ind-category of compact Hausdorff spaces "$\varinjlim_i T_i$" where all transition maps $T_i \to T_j$ are closed immersions. If $X_0 \to X_1 \to \cdots$ is a sequence of compact Hausdorff spaces with closed immersions and $X = \varinjlim_n X_n$ as topological spaces, then the map*

$$\varinjlim_n \underline{X_n} \to \underline{X}$$

is an isomorphism of condensed sets.

Among the class of profinite sets there is the special class of extremally disconnected sets, which are the projective objects in the category $*_{\text{proét}}$. Moreover, all of them are retractions of a Stone-Čech compactification of a discrete set. Let E*xtdis* denote the full subcategory of extremally disconnected sets. The condensed sets can be defined using only this kind of profinites

Proposition 12.0.3 ([Sch19, Prop. 2.7]). *Consider the site of extremally disconnected sets with covers given by finite families of jointly surjective maps. Its category of sheaves is equivalent to the category of condensed sets via the restriction from profinite sets. Hence, a condensed set is a functor $X : \text{Extdis} \to \text{Set}$ such that $X(\emptyset) = *$ and $X(S_1 \sqcup S_2) = X(S_1) \times X(S_2)$.*

Extremally disconnected sets play a similar role as points do for locally ringed spaces. Namely, if $F \to G$ is a map of condensed sets, it is injective (resp. surjective) if and only if it is so after evaluating at all extremally disconnected sets.

The inclusion Cond(Ab) $\to$ Cond admits a left adjoint $T \mapsto \mathbb{Z}[T]$, where $\mathbb{Z}[T]$ is the sheafification of $S \mapsto \mathbb{Z}[T(S)]$. Let S be an extremally disconnected set, Proposition 12.0.3 implies that the object $\mathbb{Z}[S]$ is projective in the category Cond(Ab). We have a tensor product in Cond(Ab) given by the sheafification of the usual tensor product at the level of points. We also have an internal $\underline{\text{Hom}}$ in Cond(Ab) defined as

$$\underline{\text{Hom}}(M, N)(S) = \text{Hom}(M \otimes \mathbb{Z}[S], N)$$

for S extremally disconnected. If R is a condensed ring we write $R[S] := R \otimes \mathbb{Z}[S]$. They form a family of compact projective generators of Cond(R). Then, if X is a condensed abelian group and S is a profinite set we have $\underline{\text{Cont}}(S, X) = \underline{\text{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S], X)$. All the nice properties of the category of condensed abelian groups are summarised in the following theorem

Theorem 12.0.4 ([Sch19, Theo. 2.2]). *The category of condensed abelian groups is an abelian category which satisfies the Grothendieck axioms $(AB3)$, $(AB4)$, $(AB5)$, $(AB6)$, $(AB3^*)$ and $(AB4^*)$: all limits $(AB3^*)$ and colimits $(AB3)$ exist, arbitrary products $(AB4^*)$, arbitrary direct sums $(AB4)$ and*

filtered colimits (AB5) *are exact, and* (AB6)*: for all index sets J and filtered categories I_j, $j \in J$, with functor $i \mapsto M_i$, from I_j to condensed abelian groups, the natural map*

$$\varinjlim_{(i_j \in I_j)_{j\in J}} \prod_{j\in J} M_{i_j} \to \prod_{j\in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism. Moreover, the category of condensed abelian groups is generated by compact projective objects given by $\mathbb{Z}[S]$ with S an extremally disconnected set.

12.0.2. Analytic rings

Next, we recall the notion of analytic ring

Definition 12.0.5 ([Sch19] Def. 7.1, 7.4 and [Sch20] Def. 6.12)**.** A pre-analytic ring $(\mathcal{A}, \mathcal{M})$ is the data of a condensed ring $\mathcal{A}$ (called the underlying condensed ring of the analytic ring) equipped with a functor

$$\text{Extdis} \to \text{Cond}(\mathcal{A}) \quad S \mapsto \mathcal{M}[S],$$

called the functor of measures[2] of $(\mathcal{A}, \mathcal{M})$, that sends finite disjoint union into products, and a natural transformation of functors $S \to \mathcal{M}[S]$.

A pre-analytic ring is said to be analytic, if for any complex $C : \ldots \to C_i \to \ldots \to C_1 \to C_0 \to 0$ of $\mathcal{A}$-modules such that each C_i is a direct sum of objects of the form $\mathcal{M}[T]$ for varying extremally disconnected sets T, the map

$$R\underline{\text{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\underline{\text{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$

is an isomorphism for all extremally disconnected sets S. An analytic ring $(\mathcal{A}, \mathcal{M})$ is normalized if $\mathcal{A} \to \mathcal{M}[*]$ is an isomorphism.

Example 12.0.6.

1. Solid modules ([Sch19, Theorem 5.8]). We define the analytic ring $\mathbb{Z}_\blacksquare$ to have underlying ring $\mathbb{Z}$, and functor of measures mapping an extremally disconnected $S = \varprojlim_i S_i$, written as an inverse limit of finite sets, to the condensed abelian group $\mathbb{Z}_\blacksquare[S] := \varprojlim_i \mathbb{Z}[S_i]$.

2. ([Sch19, Theorem 8.1]). More generally, let A be a discrete commutative algebra, and let $S = \varprojlim_i S_i$ be an extremally disconnected set. We have an analytic ring $A_\blacksquare$ with underlying condensed ring A, and with functor of measures given by

$$A_\blacksquare[S] = \varinjlim_{B \subset A} \varprojlim_i B[S_i]$$

   where B runs over all the $\mathbb{Z}$-algebras of finite type in A.

3. Let p be a prime number, K a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ its valuation ring. Let $S = \varprojlim_i S_i$ be an extremally disconnected set. We have analytic structures for the rings $\mathcal{O}_K$ and K, denoted $\mathcal{O}_{K,\blacksquare}$ and $K_\blacksquare$ respectively, given by

$$\mathcal{O}_{K,\blacksquare}[S] := \varprojlim_i \mathcal{O}_K[S_i] \text{ and } K_\blacksquare[S] := \mathcal{O}_{K,\blacksquare}[S][\frac{1}{p}] = K \otimes_{\mathcal{O}_K} \mathcal{O}_{K,\blacksquare}[S].$$

   Note that $\mathcal{O}_K[S]$ is profinite for all S.

[2] This terminology is justified as follows. If $(\mathcal{A}, \mathcal{M})$ is an analytic ring, P is an $(\mathcal{A}, \mathcal{M})$-module, S is an extremally disconnected set, $f \in \text{Cont}(S, P)$ and $\mu \in \mathcal{M}[S]$ then, using the isomorphism $\text{Hom}_{\mathcal{A}}(\mathcal{M}[S], P) = \text{Cont}(S, P)$, one can evaluate f at μ and define $\int f \cdot \mu = f(\mu) \in P$, which allows to see μ as a linear functional on the space of functions $f \in \text{Cont}(S, P)$.

4. ([And21, Theorem 1.5]) Let (A, A^+) be a Huber pair. Andreychev defines an analytic ring $(A, A^+)_\blacksquare$ associated to (A, A^+), whose underlying ring is $\underline{A}$, and with functor of measures

$$(A, A^+)_\blacksquare[S] = \varinjlim_{B \to A^+, M} \varprojlim_i \underline{M}[S_i],$$

where the colimit is taken over all the finitely generated subrings $B \subset A^+$ and all the quasi-finitely generated B-submodules M of A. If $A^+ = A^\circ$ we simply write $A_\blacksquare$ for $(A, A^\circ)_\blacksquare$.

5. A particular example of the previous case is the analytic ring associated to the Tate algebra $(K\langle T\rangle, O_K\langle T\rangle)$, namely, the analytic ring $K\langle T\rangle_\blacksquare$ whose functor of measures is given by

$$K\langle T\rangle_\blacksquare[S] = K\langle T\rangle \otimes_{O_K\langle T\rangle} (\varprojlim_{s\in\mathbb{N}, i\in I} (O_K/p^s O_K[T])[S_i]).$$

In the previous equation, $O_K/p^s O_K[T]$ is the polynomial ring over $O_K/p^s O_K$ in the variable T.

The following theorem explains the importance of the analytic rings

Theorem 12.0.7 ([Sch19, Prop. 7.5])**.** *Let $(\mathcal{A}, \mathcal{M})$ be an analytic ring.*

1. *The full subcategory*

$$\mathrm{Cond}(\mathcal{A}, \mathcal{M}) \subset \mathrm{Cond}(\mathcal{A})$$

of all $\mathcal{A}$-modules M such that for all extremally disconnected set S, the map

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}[S], M) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}[S], M)$$

is an isomorphism, is an abelian subcategory stable under all limits, colimits and extensions. Objects of the form $\mathcal{M}[S]$, where S is an extremally disconnected profinite set, constitute a family of compact projective generators of $\mathrm{Cond}(\mathcal{A}, \mathcal{M})$*. The inclusion functor admits a left adjoint*

$$\mathrm{Cond}(\mathcal{A}) \to \mathrm{Cond}(\mathcal{A}, \mathcal{M}), \quad M \mapsto M \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$$

which is the unique colimit preserving extension of the functor given by $\mathcal{A}[S] \mapsto \mathcal{M}[S]$. Finally, if $\mathcal{A}$ is commutative, there is a unique symmetric monoidal tensor product $- \otimes_{(\mathcal{A},\mathcal{M})} -$ on $\mathrm{Cond}(\mathcal{A}, \mathcal{M})$ *making the functor $- \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$ symmetric monoidal.*

2. *The functor of derived categories*

$$D(\mathrm{Cond}(\mathcal{A}, \mathcal{M})) \to D(\mathrm{Cond}(\mathcal{A}))$$

is fully faithful and its essential image is stable under all limits and colimits and given by those $C \in D(\mathrm{Cond}(\mathcal{A}))$ for which the map

$$R\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}[S], C)$$

is an isomorphism for all extremally disconnected set S. In that case, the map

$$R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{M}[S], C) \to R\underline{\mathrm{Hom}}_{\mathcal{A}}(\mathcal{A}[S], C)$$

is also an isomorphism.

An object $C \in D(\mathrm{Cond}(\mathcal{A}))$ lies in $D(\mathrm{Cond}(\mathcal{A}, \mathcal{M}))$ if and only if for each $n \in \mathbb{Z}$, the cohomology group $H^n(C)$ lies in $\mathrm{Cond}(\mathcal{A}, \mathcal{M})$*. The inclusion functor $D(\mathrm{Cond}(\mathcal{A}, \mathcal{M})) \subset D(\mathrm{Cond}(\mathcal{A}))$ admits a left adjoint*

$$D(\mathrm{Cond}(\mathcal{A})) \to D(\mathrm{Cond}(\mathcal{A}, \mathcal{M})), \quad C \mapsto C \otimes^L_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$$

which is the left derived functor of $\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$. Finally, if $\mathcal{A}$ is commutative, there is a unique symmetric monoidal tensor product $- \otimes^L_{(\mathcal{A},\mathcal{M})} -$ on $D(\mathrm{Cond}(\mathcal{A}, \mathcal{M}))$ making the functor $- \otimes^L_{\mathcal{A}} (\mathcal{A}, \mathcal{M})$ symmetric monoidal.

Remark 12.0.8. The functor $-\otimes^L_{(\mathcal{A},\mathcal{M})}-$ is the derived functor of $-\otimes_{(\mathcal{A},\mathcal{M})}-$ if and only if $\mathcal{M}[S']\otimes^L_{(\mathcal{A},\mathcal{M})}\mathcal{M}[S''] = \mathcal{A}[S\times S']\otimes^L_{\mathcal{A}}(\mathcal{A},\mathcal{M})$ sits in degree 0, cf. [Sch19, Warning 7.6]. This is the case for the analytic rings of Example 12.0.6, and in fact for all analytic ring over $\mathbb{Z}_\blacksquare$.

In the following we shall write $D(\mathcal{A},\mathcal{M})$ for the derived category $D(\mathrm{Cond}(\mathcal{A},\mathcal{M}))$. The functor $-\otimes^L_{\mathcal{A}}(\mathcal{A},\mathcal{M})$ should be thought as a completion with respect to the measures $\mathcal{M}$.

One of the main theorems of [Sch19] is the proof that $\mathbb{Z}_\blacksquare$ is an analytic ring, it has as input several non trivial computations of Ext-groups of locally compact abelian groups. The category $\mathrm{Solid}_{\mathbb{Z}}$ is called the category of solid abelian groups.

Let p be a prime number, then $\mathbb{Z}_p$ is a solid abelian group (being an inverse limit of discrete abelian groups) and for S extremally disconnected we have (cf. [Sch19, Proposition 7.9])

$$\mathbb{Z}_{p,\blacksquare}[S] = \mathbb{Z}_p\otimes^L_{\mathbb{Z}_\blacksquare}\mathbb{Z}_\blacksquare[S] = \mathbb{Z}_p\otimes_{\mathbb{Z}_\blacksquare}\mathbb{Z}_\blacksquare[S]. \tag{12.0.1}$$

Definition 12.0.9. Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ its ring of integers, we denote by $\mathrm{Solid}(\mathcal{O}_K)$ (resp. $\mathrm{Solid}(K)$) the category of solid $\mathcal{O}_K$-modules (resp. the category of solid K-vector spaces) and $D(\mathcal{O}_{K,\blacksquare})$, $D(K_\blacksquare)$ their respective derived categories. Given a solid algebra $A\in\mathrm{Solid}(\mathcal{O}_K)$ we let $D(A)$ denote the derived category of solid A-modules. If $(\mathcal{A},\mathcal{M})$ is an analytic ring over $\mathbb{Z}_\blacksquare$, we let $\mathrm{Solid}(\mathcal{A},\mathcal{M})$ denote the category of $(\mathcal{A},\mathcal{M})$-solid modules.

Remark 12.0.10. An object in $\mathrm{Solid}(\mathcal{O}_K)$ (resp. $\mathrm{Solid}(K)$) is the same as an object in $\mathrm{Solid}(\mathbb{Z})$ endowed with an action of $\mathcal{O}_K$ (resp. K). Indeed, this follows directly from Theorem 12.0.7 and (12.0.1).

12.0.3. Analytic rings attached to Tate algebras

In the case of a Tate algebra $(\mathcal{A},\mathcal{M}) = \mathcal{O}_K\langle T_1,\dots,T_d\rangle_\blacksquare$ we have the following key description of the completion functor which follows from the observations of [Sch19, Lecture VIII].

Proposition 12.0.11 ([And21, Proposition 3.13])**.** *Let $\mathcal{O}_K\langle\underline{T}\rangle := \mathcal{O}_K\langle T_1,\dots,T_d\rangle$. We denote $\mathcal{O}_K\langle\underline{T}\rangle^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_K\langle\underline{T}\rangle,\mathcal{O}_K)$. For any $C,W\in D(\mathcal{O}_{K,\blacksquare})$, there is a functorial isomorphism*

$$R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(W, C\otimes^L_{\mathcal{O}_{K,\blacksquare}}\mathcal{O}_K\langle\underline{T}\rangle_\blacksquare) = R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(W\otimes^L_{\mathcal{O}_{K,\blacksquare}}\mathcal{O}_K\langle\underline{T}\rangle^\vee, C).$$

Proof. By [And21, Proposition 3.13], for any $C\in D(\mathbb{Z}_\blacksquare)$, there is an isomorphism

$$C\otimes^L_{\mathbb{Z}_\blacksquare}\mathbb{Z}[T]_\blacksquare = R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\frac{\mathbb{Z}((T^{-1}))}{T\mathbb{Z}[T]}, C).$$

In particular, if $C\in D(\mathcal{O}_{K,\blacksquare})$, from [And21, Lemma 4.7] we have $\mathcal{O}_K\langle T\rangle_\blacksquare = \mathcal{O}_{K,\blacksquare}\otimes^L_{\mathbb{Z}_\blacksquare}\mathbb{Z}[T]_\blacksquare$ as analytic rings, one gets

$$C\otimes^L_{\mathcal{O}_{K,\blacksquare}}\mathcal{O}_K\langle T\rangle_\blacksquare = R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\frac{\mathcal{O}_K((T^{-1}))}{T\mathcal{O}_K[T]}, C).$$

Let $W\in D(\mathcal{O}_K)$. Applying the functor $R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(W,-)$ and the adjunction between $\otimes^L$ and $R\underline{\mathrm{Hom}}$, we get

$$R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(W, C\otimes^L_{\mathcal{O}_{K,\blacksquare}}\mathcal{O}_K\langle T\rangle_\blacksquare) = R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(W\otimes^L_{\mathcal{O}_{K,\blacksquare}}\frac{\mathcal{O}_K((T^{-1}))}{T\mathcal{O}_K[T]}, C).$$

Noting that $\frac{\mathcal{O}_K((T^{-1}))}{T\mathcal{O}_K[T]} = \mathcal{O}_K\langle T\rangle^\vee$ as $\mathcal{O}_K\langle T\rangle$-modules, this finishes the proof for the case of one variable.

The case of several variables is treated by induction noticing that if A is a discrete algebra of finite type over $\mathbb{Z}$ then

$$R\underline{\mathrm{Hom}}_A(W, C\otimes^L_{A_\blacksquare}A[T]_\blacksquare) = R\underline{\mathrm{Hom}}_A(W\otimes^L_{A_\blacksquare}\frac{A((T^{-1}))}{TA[T]}, C)$$

for $C,W\in\mathcal{D}(A_\blacksquare)$, which follows easily from [And21, Proposition 3.13]. □

Remark 12.0.12. Proposition 12.0.11 can be seen as an instance of the six functor formalism of [Sch19, Theorem 8.2].

13. Non-archimedean condensed functional analysis

The main purpose of this section is to state a duality between two classes of solid vector spaces over a finite extension of $\mathbb{Q}_p$, namely Fréchet spaces and LS spaces (to be defined below), generalising the duality between Banach spaces and Smith spaces (c.f. [Sch20, Theorem 3.8]), and the duality between nuclear Fréchet spaces and LB spaces of compact type (see, e.g., [ST02, Theorem 1.3]). For doing so, we will use many results on the theory of condensed non-archimedean functional analysis developed by Clausen and Scholze. Since results haven't yet appeared in the literature, we give a detailed account with proofs included. The reader should be aware that many of the results in this chapter must be attributed to Clausen and Scholze [CS], and the only original results are those concerning duality, principally Theorem 13.3.13.

13.1. Banach and Smith spaces

Let K and $\mathcal{O}_K$ be as in the previous section, and let ϖ be a uniformiser of $\mathcal{O}_K$. In this paragraph we focus our attention in the category of solid K-vector spaces (or $K_\blacksquare$-vector spaces), i.e. the category $\mathrm{Solid}(K)$. We start with some basic concepts

Definition 13.1.1.

1. A solid K-Banach space (or simply a Banach space) is a solid K-vector space V admitting a ϖ-adically complete $\mathcal{O}_{K,\blacksquare}$-module $V^0 \subset V$ such that
   a) $V = V^0 \otimes_{\mathcal{O}_{K,\blacksquare}} K$.
   b) $V^0/\varpi^s V^0$ is discrete for all $s \in \mathbb{N}$.

   We say that V^0 is a lattice of V.

2. A solid K-Smith space (or simply a Smith space) is a solid K-vector space W admitting a profinite $\mathcal{O}_{K,\blacksquare}$-submodule W^0 such that $W = W^0 \otimes_{\mathcal{O}_{K,\blacksquare}} K$. We say that W^0 is a lattice of W.

Remark 13.1.2. Over $\mathbb{R}$, a Smith space ([Sch20, Def. 3.6]) is a complete locally convex topological $\mathbb{R}$-vector space W containing a compact absolutely convex subset $C \subset V$ such that $V = \bigcup_{a>0} aC$. As its p-adic analogue, we define a K-Smith space to be a topological K-vector space W containing a compact $\mathcal{O}_K$-module $W^0 \subset W$ such that $W = W^0[\frac{1}{\varpi}]$.

Proposition 13.1.3. *The functor $V \mapsto V(*)_{\mathrm{top}}$ induces an equivalence of categories between solid and classical K-Banach spaces (resp. between solid and classical K-Smith spaces).*

Proof. Let V be a Banach space over K in the classical sense, and let $V^0 \subset V$ be the unit ball. Then V^0 is endowed with the ϖ-adic topology, i.e. it is the inverse limit $V^0 = \varprojlim_s V^0/\varpi^s V^0$, with $V^0/\varpi^s V^0$ discrete for all $s \in \mathbb{N}$. By Proposition 12.0.2 we know that

$$\underline{V} = \varinjlim_{n \in N} (\varprojlim_{s \in \mathbb{N}} \underline{V^0/\varpi^s V^0}) \varpi^{-n} = \underline{V^0} \otimes_{\mathcal{O}_{K,\blacksquare}} K,$$

so that $\underline{V}$ is a solid K-Banach space. Conversely, if V is a solid Banach space then $V(*)_{\mathrm{top}}$ is clearly a K-Banach space in the classical sense.

Let W be a (classical) K-Smith space and $W^0 \subset W$ a lattice. As W^0 is compact, it is profinite and Proposition 12.0.2 implies that $\underline{W} = \underline{W^0} \otimes_{O_{K,\blacksquare}} K$ is a solid Smith space as in our previous definition. The converse is clear. □

Proposition 13.1.4. *There is a natural functor*

$$\mathcal{LC}_K \to \mathrm{Solid}(K) \ : \ V \mapsto \underline{V}$$

from the category of complete locally convex K-vector spaces to solid K-vector spaces.

Proof. This follows from the fact that any complete locally convex K-vector space can be written as a cofiltered limit of Banach spaces [Sch02, Chap. I, §4], and the adjunction of Proposition 12.0.2. □

From now, all the complete locally convex K-vector spaces will be considered as solid K-vector unless otherwise specified. The following result shows that solid Banach and Smith spaces over K have orthonormal basis.

Lemma 13.1.5.

1. *A solid K-vector space is Banach if and only if it is of the form*

$$\widehat{\bigoplus}_{i\in I} K := (\varprojlim_{s} (\bigoplus_{i\in I} O_K/\varpi^s))[\frac{1}{\varpi}],$$

   for some index set I.

2. *A solid K-vector space is Smith if and only if it is of the form*

$$(\prod_{i\in I} O_K)[\frac{1}{\varpi}],$$

   for some index set I. In particular, by [Sch19, Corollary 5.5] and Theorem 12.0.7, Smith spaces form a family of compact projective generators of the category Solid(K) *of solid K-vector spaces.*

Proof. By definition, an object of the form $\widehat{\bigoplus}_{i\in I} K$ is a solid K-Banach space. Conversely, let V be a $K_{\blacksquare}$-Banach space and $V^0 \subset V$ a lattice. By Proposition 13.1.3 $V(*)_{\mathrm{top}}$ is a Banach space with unit ball $V^0(*)_{\mathrm{top}} \subset V(*)_{\mathrm{top}}$ and $\underline{V(*)_{\mathrm{top}}} = V$. But then $V^0(*)_{\mathrm{top}}$ has an orthonormal O_K-basis by taking any lift of a $O_K/\varpi O_K$-basis of $\overline{V^0/\varpi}$. This proves (1).

To prove (2), let W be a Smith space and $W^0 \subset W$ a lattice. Since W^0 is a profinite $O_{K,\blacksquare}$-module, $W^0 = \underline{W^0(*)_{\mathrm{top}}}$ with $W^0(*)_{\mathrm{top}}$ a profinite O_K-module in the usual sense. Moreover, $W^0(*)_{\mathrm{top}}$ is flat and the topological Nakayama's lemma implies that $W^0(*)_{\mathrm{top}}$ must be of the form $\prod_{i\in I} O_K$ ([SGA70, Exposé VII$_B$, 0.3.8]). □

We will need the following useful proposition

Proposition 13.1.6 ([CS])**.** *Let V be a solid K-vector space. The following statements are equivalent*

1. *V is a Smith space*

2. *V is quasi-separated, and there is a compact O_K-submodule $M \subset V$ such that $V = M[\frac{1}{\varpi}]$.*

Moreover, the class of Smith spaces is stable under extensions, closed subobjects and quotients by closed subobjects.

Proof. The equivalence between (1) and (2) follows immediately from the characterisation of Smith spaces of Lemma 13.1.5. The fact that Smith spaces are stable under extensions follows from the fact that they are projective, and hence every extension splits. The stability under closed subobjects and quotients follows from the description of a Smith space as in (2) of the equivalence. □

The next lemma provides a anti-equivalence between Banach and Smith spaces for solid K-vector spaces, c.f. [Smi52] (or also [Sch20, Theorem 3.8]) for the analogous statement over the real or complex numbers.

Lemma 13.1.7 ([CS]). *The assignment $V \mapsto V^\vee$ induces an anti-equivalence between K-Banach spaces and K-Smith spaces. More precisely, the following holds.*

1. $\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus}_i \mathcal{O}_K, \mathcal{O}_K) = \prod_i \mathcal{O}_K$ *and* $\underline{\mathrm{Hom}}_K(\widehat{\bigoplus}_{i\in I} K, K) = (\prod_{i\in I} \mathcal{O}_K)[\frac{1}{p}]$.
2. $\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\prod_{i\in I} \mathcal{O}_K, \mathcal{O}_K) = \widehat{\bigoplus}_{i\in I} \mathcal{O}_K$ *and* $\underline{\mathrm{Hom}}_K((\prod_{i\in I} \mathcal{O}_K)[\frac{1}{p}], K) = \widehat{\bigoplus}_{i\in I} K$.

Proof. To prove (1), notice that

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus_i} \mathcal{O}_K, \mathcal{O}_K) &= \varprojlim_s \underline{\mathrm{Hom}}_{\mathcal{O}_K/\varpi^s}(\bigoplus_i \mathcal{O}_K/\varpi^s, \mathcal{O}_K/\varpi^s) \\ &= \varprojlim_s \prod_i \mathcal{O}_K/\varpi^s \\ &= \prod_i \mathcal{O}_K.\end{aligned}$$

To prove the second equality it is enough to show that

$$\underline{\mathrm{Hom}}_K(\widehat{\bigoplus_i} K, K) = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\bigoplus_i} \mathcal{O}_K, \mathcal{O}_K)[\frac{1}{\varpi}].$$

Let S be an extremally disconnected set, by adjunction it is enough to show that

$$\mathrm{Hom}_K(\widehat{\bigoplus_i} K, \underline{\mathrm{Cont}}(S, K)) = \mathrm{Hom}_{\mathcal{O}_K}(\widehat{\bigoplus_i} \mathcal{O}_K, \underline{\mathrm{Cont}}(S, \mathcal{O}_K))[\frac{1}{\varpi}].$$

But all the solid spaces involved arise as the condensed set associated to a compactly generated Hausdorff topological space, the claim follows from the fact that lattices are mapped to lattices for continuous maps of classical Banach spaces.

Part (2) follows from the fact that an object of the form $\prod_{i\in I} \mathcal{O}_K$ is a retraction of a compact projective generator $\mathcal{O}_{K,\blacksquare}[S]$ for S extremally disconnected, and the fact that

$$\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[S], \mathcal{O}_K) = \underline{\mathrm{Cont}}(S, \mathcal{O}_K).$$

This finishes the proof of the Lemma. □

Remark 13.1.8. Part (2) of the previous lemma also holds with $R\underline{\mathrm{Hom}}$ since $\prod_{i\in I} \mathcal{O}_K$ is a projective $\mathcal{O}_{K,\blacksquare}$-module. The same proof of the first assertion of (1) can also be adapted to show that $R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\widehat{\oplus}_i \mathcal{O}_K, \mathcal{O}_K) = \prod_i \mathcal{O}_K$. The authors ignore how to calculate $R\underline{\mathrm{Hom}}_K(\widehat{\bigoplus}_{i\in I} K, K)$.

We now study the behaviour of the tensor product.

Proposition 13.1.9. *Let $V = \prod_{i\in I} \mathcal{O}_K$ and $W = \prod_{j\in J} \mathcal{O}_K$. Then*

$$V \otimes^L_{\mathcal{O}_{K,\blacksquare}} W = \prod_{(i,j)\in I\times J} \mathcal{O}_K.$$

Proof. See [Sch19, Proposition 6.3]. □

The following useful result shows that the solid tensor product coincides with the projective tensor product of K-Banach spaces.

Lemma 13.1.10 ([CS]). *Let V and V' be classical Banach spaces over K, and let $V\widehat{\otimes}_K V'$ denote its projective tensor product. Then $\underline{V\widehat{\otimes}_K V'} = \underline{V} \otimes_{K_\blacksquare} \underline{V'}$.*

Proof. Fix an isomorphism $V = \widehat{\bigoplus}_{i\in I} K$. As a convergent series $\sum_i a_i$ has only countably many terms different from 0, we can write

$$\underline{V} = \varinjlim_{I'\subset I} \widehat{\bigoplus}_{i\in I'} K$$

where I' runs over all the countable subsets of I. Therefore, we can assume that $V \cong V' \cong \widehat{\bigoplus}_{n\in\mathbb{N}} K$. Let $\mathscr{S}$ denote the direct set of functions $f : \mathbb{N} \to \mathbb{Z}$ such that $f(n) \to +\infty$ as $n \to +\infty$, endowed with the order $f \preceq g$ iff $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Thus, we can write

$$\widehat{\bigoplus}_{n\in\mathbb{N}} K = \varinjlim_{f\in\mathscr{S}} \prod_{n\in\mathbb{N}} O_K \varpi^{f(n)}. \tag{13.1.1}$$

Indeed, by evaluating at an extremally disconnected set S, $\underline{V}(S) = \mathrm{Cont}(S, \underline{V}) = \widehat{\bigoplus}_{n\in\mathbb{N}} \mathrm{Cont}(S, K)$ has a natural Banach space structure, for which a function $\phi : S \to \underline{V}$ can be written in a unique way as a sum $\phi = \sum_n \phi_n$ with $\phi_n \in \mathrm{Cont}(S, K)$, such that $|\phi_n| \to 0$ as $n \to \infty$. Then, from (13.1.1) and Proposition 13.1.9 we deduce that

$$\underline{V} \otimes_{K_\blacksquare} \underline{V'} = \varinjlim_{f,g\in\mathscr{S}} \prod_{n\in\mathbb{N}} O_K \varpi^{f(n)} \otimes_{O_{K,\blacksquare}} \prod_{m\in\mathbb{N}} O_K \varpi^{g(m)} = \varinjlim_{f,g\in\mathscr{S}} \prod_{n,m\in\mathbb{N}\times\mathbb{N}} O_K \varpi^{f(n)+g(m)}.$$

Given $f, g \in \mathscr{S}$ define the function $h_{f,g} : \mathbb{N}\times\mathbb{N} \to \mathbb{Z}$ as $h_{f,g}(n,m) = f(n) + g(m)$. Let $\mathscr{S}'$ be the direct set of functions $h : \mathbb{N}\times\mathbb{N} \to \mathbb{Z}$ such that $h(n,m) \to +\infty$ as $\max\{n,m\} \to +\infty$. Then the set $\{h_{f,g}\}_{f,g\in\mathscr{S}}$ is a cofinal family in $\mathscr{S}'$. Indeed, given $h : \mathbb{N}\times\mathbb{N} \to \mathbb{Z}$, if we define $f(n) = \frac{1}{2}\min_m h(n,m)$ and $g(m) = \frac{1}{2}\min_n h(n,m)$, then $h \preceq h_{f,g}$. Therefore,

$$\varinjlim_{f,g\in\mathscr{S}} \prod_{n,m\in\mathbb{N}\times\mathbb{N}} O_K \varpi^{h_{f,g}(n,m)} = \varinjlim_{h\in\mathscr{S}'} \prod_{n,m} O_K \varpi^{h(n,m)} = \widehat{\bigoplus}_{n,m} K.$$

□

Let us recall the concept of a nuclear solid K-vector space.

Definition 13.1.11 ([Sch20, Definition 13.10]). Let $V \in \mathrm{Solid}(K)$. We say that V is nuclear if, for all extremally disconnected set S, we have

$$\underline{\mathrm{Hom}}_K(K_\blacksquare[S], V) = \underline{\mathrm{Hom}}_K(K_\blacksquare[S], K) \otimes_{K_\blacksquare} V$$

Remark 13.1.12. We warn the reader that this notion of nuclearity differs from the classical one, say in [Sch02]. Indeed, if a Banach space is nuclear in the classical sense then it is finite dimensional (cf. *loc. cit.* §19). On the other hand, solid K-Banach spaces are always nuclear in the condensed sense.

Corollary 13.1.13 ([CS]). *Let V be a Banach space over K, then V is a nuclear K-vector space.*

Proof. The result follows from Lemma 13.1.10 by applying the $\otimes$ and $\underline{\mathrm{Hom}}$ adjuntion, and the duality between Banach and Smith spaces. □

Corollary 13.1.14. *Let $V \in \mathrm{Solid}(K)$ be a Banach space and $W \in \mathrm{Solid}(K)$ be Smith. Then*

$$\underline{\mathrm{Hom}}_K(V, W) = V^\vee \otimes_{K_\blacksquare} W.$$

$$\underline{\mathrm{Hom}}_K(W, V) = W^\vee \otimes_{K_\blacksquare} V.$$

Proof. The second equality follows from nuclearity of V. For the first equality, tensor-Hom adjunction gives

$$\underline{\mathrm{Hom}}_K(V, W) = \underline{\mathrm{Hom}}_K(V \otimes_{K_\blacksquare} W^\vee, K),$$

and the results follows immediately from the description of the tensor product of two Banach spaces (Lemma 13.1.10) and duality between Banach and Smith spaces (Lemma 13.1.7). □

We finish this section with an elementary lemma that will be needed later.

Lemma 13.1.15.

1. *Let V be a solid K-vector space such that the maximal quasi-separated quotient V^{qs} is zero. Then $\underline{\mathrm{Hom}}(V, K) = 0$.*
2. *A map of Banach spaces $V \to V'$ is injective (resp. with dense image) if and only if its dual $V'^\vee \to V^\vee$ has dense image (resp. is injective).*

Proof. Let S be a extremally disconnected set, then

$$\underline{\mathrm{Hom}}_K(V, K)(S) = \mathrm{Hom}_K(V \otimes K_\blacksquare[S], K) = \mathrm{Hom}_K(V, \underline{\mathrm{Cont}}(S, K)).$$

But $\underline{\mathrm{Cont}}(S, K)$ is a Banach space. Then, by adjunction (Proposition 12.0.2), we get

$$\underline{\mathrm{Hom}}_K(V, K)(S) = \mathrm{Hom}_K(V(*)_{\mathrm{top}}, \mathrm{Cont}(S, K)).$$

Since $V^{qs} = 0$, the maximal Hausdorff quotient of $V(*)_{\mathrm{top}}$ is zero. This implies that

$$\mathrm{Hom}_K(V(*)_{\mathrm{top}}, \mathrm{Cont}(S, K)) = 0$$

proving (1).

To prove (2), let $f : V \to V'$ be a map of Banach spaces. Suppose that f has not dense image and let $\overline{f(V)} \subset V'$ be the closure of its image. Then $V'/\overline{f(V)}$ is a non zero Banach space and we have a short exact sequence

$$0 \to \overline{f(V)} \to V' \to V'/f(\overline{V}) \to 0$$

which splits as any Banach space over a local field is orthonormalizable. Taking duals we get a short exact sequence

$$0 \to (V'/\overline{f(V)})^\vee \to V'^\vee \to \overline{f(V)}^\vee \to 0.$$

Since $f^\vee : V'^\vee \to V^\vee$ factors through $\overline{f(V)}^\vee$, the map $f^\vee$ is not injective. Conversely, suppose that the map $f^\vee$ is not injective, then its kernel $\ker f^\vee$ is a closed subspace of $V'^\vee$ which is a Smith space by Proposition 13.1.6. Since the quotient $V'^\vee/\ker(f^\vee)$ is also a Smith space, there is a retraction $r : V'^\vee \to \ker(f^\vee)$. Taking duals one sees that the composition $V \to V' \to \ker(f^\vee)^\vee$ is zero and that the last map is surjective (because of r), this implies that f has not dense image.

Finally, suppose that $f : V \to V'$ is injective. If $f^\vee : V'^\vee \to V^\vee$ does not have dense image, $\overline{f(V'^\vee)} \subset V^\vee$ is a closed Smith subspace and its quotient $V^\vee/\overline{f(V'^\vee)}$ is a non zero Smith space. Taking duals we get a short exact sequence

$$0 \to (V^\vee/\overline{f(V'^\vee)})^\vee \to V \to \overline{f(V'^\vee)}^\vee \to 0.$$

But $f : V \to V'$ factors through $\overline{f(V'^\vee)}^\vee$, this is a contradiction with the injectivity of f. Conversely, Suppose that $f^\vee : V'^\vee \to V^\vee$ has dense image, consider the quotient

$$0 \to V'^\vee \to V^\vee \to Q \to 0$$

Taking duals one obtains an exact sequence

$$0 \to \underline{\mathrm{Hom}}_K(Q, K) \to V \to V'.$$

But part (1) implies that $\underline{\mathrm{Hom}}_K(Q, K) = 0$, proving that f is injective. □

13.2. Quasi-separated solid K-vector spaces

We shall use the following results throughout the text without explicit mention. They are due to Clausen and Scholze, and explained to us by Guido Bosco in the study group of La Tourette.

Proposition 13.2.1 ([CS]). *Let V be a solid K-vector space. The following are equivalent,*

1. *V is quasi-separated.*
2. *V is equal to the filtered colimit of its Smith subspaces.*

Proof. Let V be a quasi-separated $K_\blacksquare$-vector space, let W_1, W_2 be Smith subspaces of V. As V is quasi-separated, $W_1 \cap W_2$ is a closed Smith subspace of W_1, and the sum $W_1 + W_2 \subset V$ is isomorphic to $(W_1 \oplus W_2)/W_1 \cap W_2$. This shows that the Smith subspaces of V form a direct system, let V_0 denote their colimit. We claim that $V/V_0 = 0$, let W' be a Smith space and $f : W' \to V/V_0$ a map of solid $K_\blacksquare$-vector spaces. As W' is projective, there is a lift $f' : W' \to V$. But $\ker f' \subset W'$ is a closed Smith subspace since V is quasi-separated. This implies that f' factors through V_0 and that $f = 0$. Since the Smith spaces form a family of compact projective generators of $K_\blacksquare$-vector spaces, one must have $V/V_0 = 0$ proving (1) $\Rightarrow$ (2).

Conversely, let $V = \varinjlim_{i \in I} V_i$ be a vector space written as a filtered colimit of Smith spaces by injective transition maps. Let S_1, S_2 be two profinite sets and $f_j : S_j \to V$ be two maps for $j = 1, 2$. As the S_i are profinite, there exists $i \in I$ such that f_j factors through V_i for $j = 1, 2$. Then, as the map $V_i \to V$ is injective, one has

$$S_1 \times_V S_2 = S_1 \times_{V_i} S_2.$$

The implication (2) $\Rightarrow$ (1) follows as a Smith space is quasi-separated. □

Lemma 13.2.2 ([CS]). *A quasi-separated K-solid space is flat. In other words, if V is a quasi-separated $K_\blacksquare$-vector space, then $- \otimes_{K_\blacksquare} V = - \otimes^L_{K_\blacksquare} V$.*

Proof. Let V be a quasi-separated $K_\blacksquare$-vector space. Since filtered colimits are exact in the category of condensed abelian groups, and the solid tensor product commutes with colimits, by Proposition 13.2.1 it is enough to prove the lemma for $V = \prod_I \mathcal{O}_K[\frac{1}{p}]$ a Smith space. Let $W \in \mathrm{Solid}(K)$. We want to show that $V \otimes^L_{K_\blacksquare} W$ is concentrated in degree zero. As the Smith spaces are compact projective generators, W can be written as a quotient $0 \to W'' \to W' \to W \to 0$ where W' is a direct sum of Smith spaces. Then we are reduced to showing that $0 \to W'' \otimes_{K_\blacksquare} V \to W' \otimes_{K_\blacksquare} V$ is injective. Since W'' is quasi-separated, by Proposition 13.2.1, it can be written as filtered colimit of its Smith subspaces. Therefore, by compacity of Smith spaces, the arrow $W'' \to W'$ is a filtered colimit of injections of Smith spaces. It is hence enough to show that if $\prod_{J_1} \mathcal{O}_K[\frac{1}{p}] \to \prod_{J_2} \mathcal{O}_K[\frac{1}{p}]$ is an injective map, then

$$\prod_{J_1} \mathcal{O}_K[\frac{1}{p}] \otimes_{K_\blacksquare} V \to \prod_{J_2} \mathcal{O}_K[\frac{1}{p}] \otimes_{K_\blacksquare} V$$

is injective. This follows from the tensor product of two Smith Spaces (Proposition 13.1.9)

$$(\prod_I \mathcal{O}_K[\frac{1}{p}]) \otimes_{K_\blacksquare} (\prod_J \mathcal{O}_K[\frac{1}{p}]) = \prod_{I\times J} \mathcal{O}_K[\frac{1}{p}].$$

This finishes the proof. □

13.3. Fréchet and *LS* spaces

Our next goal is to extend the duality between Banach and Smith spaces to a larger class of solid K-vector spaces. We need a definition

Definition 13.3.1.

1. A solid Fréchet space (or simply a Fréchet space) is a solid K-vector space which can be written as a countable cofiltered limit of Banach spaces.

2. A solid *LS* (resp. *LB*, resp. *LF*) space is a solid K-vector space which can be written as a countable filtered colimit of Smith (resp. Banach, resp. Fréchet) spaces with injective transition maps.

Lemma 13.3.2.

1. *The functor $V \mapsto V(*)_{\mathrm{top}}$ induces an equivalence of categories between solid and classical Fréchet spaces and $V = \underline{V(*)_{\mathrm{top}}}$.*

2. *An LS space is quasi-separated. Conversely, a quasi-separated $K_\blacksquare$-vector space W is an LS space if and only if it is countably compactly generated, i.e. for every surjection $\bigoplus_{i\in I} P_i \to W$ by direct sums of Smith spaces, there is a countable index subset $I_0 \subset I$ such that $\bigoplus_{i\in I_0} P_i \to W$ is surjective.*

Proof. Part (1) follows from the fact that a classical Fréchet space is complete for a countable family of seminorms (i.e. it can be written as a countable cofiltered limit of Banach spaces), Proposition 12.0.2 (1), and Proposition 13.1.3. For part (2), the fact that an *LS* space is quasi-separated follows from Proposition 13.2.1. Let W be quasi-separated $K_\blacksquare$-vector space. Assume it is countably compactly generated. Write $W = \varinjlim_{W'\subset W} W'$ as a the colimit of its Smith subspaces. As W is quasi-separated, the sum of two Smith subspaces is Smith, so the colimit is filtered. By hypothesis, there are countably many W' such that $W = \varinjlim_{s\in\mathbb{N}} W'_s$. Moreover, we can assume that $W_0 \subset W_1 \subset \cdots$. This proves that W is an *LS* space. Conversely, let W be an *LS* space and let $\bigoplus_{i\in I} P_i \to W$ be a surjective map with P_i Smith. The image P'_i of P_i in W is a Smith space since W is quasi-separated, hence $W = \sum_i P'_i$. Thus, without loss of generality we can assume that $\varinjlim_{i\in I} P'_i$ is filtered and equal to W. Let $W = \varinjlim_{s\in\mathbb{N}} W_s$ be a presentation as a countable colimit of Smith spaces by injective transition maps. By compactness of the Smith spaces, for all s there exists i_s such that $M_n \subset P'_{i_s} \subset M$. We can assume that $P'_{i_s} \subset P'_{i_{s+1}}$ for all $s \in \mathbb{N}$. Thus, $\bigoplus_{s\in\mathbb{N}} P_{i_s} \to W$ is surjective, this finishes the lemma. □

The following lemma says that we can always choose a presentation of a Fréchet spaces as an inverse limit of Banach spaces with dense transition maps.

Lemma 13.3.3. *Let V be a solid Fréchet space, then we can write $V = \varprojlim_{n\in\mathbb{N}} V_n$ with V_n Banach spaces such that $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ has dense image for all $n \in \mathbb{N}$. Conversely, let $\{V_n\}_{n\in N}$ be a cofiltered limit of Banach spaces such that $V_{n+1}(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ has dense image, and let $V = \varprojlim V_n$ be its inverse limit. Then $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ has dense image for all $n \in \mathbb{N}$.*

Proof. Let $V = \varprojlim_{n\in\mathbb{N}} V_n$ be a presentation of the Fréchet space as an inverse limit of Banach spaces. Changing V_n by the completion of the image of V, we obtain a desired presentation with $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ of dense image. Conversely, let $\{V_n\}_{n\in\mathbb{N}}$ be an inverse system of Banach spaces with maps $V_{n+1} \to V_n$ of dense image, let $V = \varprojlim_n V_n$ be a Fréchet space, we want to show that the image of $V(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$ is dense for all $n \in \mathbb{N}$. Fix $n_0 \in \mathbb{N}$, let $w \in V_{n_0}(*)_{\mathrm{top}}$ and $1 > \epsilon > 0$. Let $|\cdot|_n$ denote the norm of $V_n(*)_{\mathrm{top}}$, without loss of generality we assume that $|\cdot|_n \leq |\cdot|_{n+1}$. By density of the transition maps $\phi_n^{n+1} : V_{n+1}(*)_{\mathrm{top}} \to V_n(*)_{\mathrm{top}}$, there exists $v_{n_0+1} \in V_{n+1}(*)_{\mathrm{top}}$ such that $|\phi_{n_0}^{n_0+1}(v_{n_0+1}) - w| \leq \epsilon$. By induction, for all $n \geq n_0 + 1$ we can find $v_n \in V_n(*)_{\mathrm{top}}$ such that $|\phi_n^{n+1}(v_{n+1} - v_n)| \leq \epsilon^n$. Let $n \geq n_0 + 1$ be fixed and let $k \geq 0$, then by construction the sequence $\{\phi_n^{n+k}(v_{n+k})\}$ converges in $V_n(*)_{\mathrm{top}}$ to an element v'_n. Moreover, it is immediate to check that $\phi_n^{n+1}(v'_{n+1}) = v_n$ so that $v' = (v'_n) \in V(*)_{\mathrm{top}}$, and $|\phi_{n_0}^{n_0+1}(v'_{n_0+1} - w)| \leq \epsilon$. This proves the lemma. □

13.3.1. Properties of Fréchet spaces

We now present some basic properties of Fréchet spaces, most of the results in the context of condensed mathematics are due to Clausen and Scholze [CS].

Lemma 13.3.4 (Topological Mittag-Leffler [CS])**.** *Let $V = \varprojlim_n V_n$ be Fréchet space written as an inverse limit of Banach spaces with dense transition maps. Then*

$$R^j \varprojlim_n V_n = 0$$

for all $j > 0$. In particular, $V = R\underline{\mathrm{Hom}}_K(V^\vee, K)$.

Proof. See [Bos21, Lemma A.18]. □

Lemma 13.3.5 ([CS])**.** *Let $(V_n)_{n\in\mathbb{N}}$ and $(W_m)_{m\in\mathbb{N}}$ be countable families of Banach spaces.*

1. *We have*
$$(\prod_n V_n) \otimes^L_{K_\blacksquare} (\prod_m W_m) = \prod_{n,m} V_n \otimes_{K_\blacksquare} W_m.$$
2. *More generally, if $V = \varprojlim_n V_n$ and $W = \varprojlim_m W_m$ are Fréchet spaces written as inverse limits of Banach spaces by dense transition maps, one has*
$$V \otimes^L_{K_\blacksquare} W = \varprojlim_{n,m} V_n \otimes_{K_\blacksquare} W_m.$$

Proof. Property (AB6) of Theorem 12.0.4 and Proposition 13.2.1 imply that products of quasi-separated solid K-vector spaces are quasi-separated. Then, by Lemma 13.2.2, all the derived tensor products in the statements are already concentrated in degree 0. By Lemma 13.3.4 we have a short exact sequences

$$0 \to \varprojlim_n V_n \to \prod_n V_n \to \prod_n V_n \to 0$$
$$0 \to \varprojlim_m W_m \to \prod_m W_m \to \prod_m W_m \to 0.$$

Then (2) follows from (1) by taking the tensor product of the above sequences.

For (1), suppose that the statement is true for all V_n and W_n posseding countable orthonormal basis. Let $V_n \cong \widehat{\bigoplus}_{I_n} K$ and $W_m \cong \widehat{\bigoplus}_{J_m} K$ for all $n, m \in N$. We write $V_n = \varinjlim_{I'_n \subset I_n} \widehat{\bigoplus}_{I'_n} K$ and $W_m =$

$\varinjlim_{J'_m \subset J_m} \widehat{\bigoplus}_{J'_m} K$ with I'_n and J'_m running among all the countable subsets of I_n and J_m respectively. Then

$$\begin{aligned}
(\prod_n V_n) \otimes_{K_\blacksquare} (\prod_m W_m) &= [\prod_n (\varinjlim_{I'_n \subset I_n} \widehat{\bigoplus}_{I'_n} K)] \otimes_{K_\blacksquare} [\prod_m (\varinjlim_{J'_m \subset J_m} \widehat{\bigoplus}_{J'_m} K)] \\
&= \varinjlim_{\substack{\forall (n,m) \in \mathbb{N}\times\mathbb{N} \\ I'_n \times J'_m \subset I_n \times J_m}} [\prod_n (\widehat{\bigoplus}_{I'_n} K)] \otimes_{K_\blacksquare} [\prod_m (\widehat{\bigoplus}_{J'_m} K)] \\
&= \varinjlim_{\substack{\forall (n,m) \in \mathbb{N}\times\mathbb{N} \\ I'_n \times J'_m \subset I_n \times J_m}} \prod_{n,m} (\widehat{\bigoplus}_{I'_n \times J'_m} K) \\
&= \prod_{n,m} (\varinjlim_{I'_n \times J'_m \subset I_n \times J_m} \widehat{\bigoplus}_{I'_n \times J'_m} K) \\
&= \prod_{n,m} (V_n \otimes_{K_\blacksquare} W_m).
\end{aligned}$$

Hence, we are left to prove (1) for $W_m = V_n = \widehat{\bigoplus}_{\mathbb{N}} K$ for all $n, m \in \mathbb{N}$. Let $\mathscr{S}$ be the filtered set of functions $f : \mathbb{N} \to \mathbb{Z}$ such that $f(k) \to +\infty$ as $k \to +\infty$. For all $n, m \in \mathbb{N}$ we can write

$$\begin{aligned}
V_n = \widehat{\bigoplus}_{\mathbb{N}} K = \varinjlim_{f_n \in \mathscr{S}} \prod_{k \in \mathbb{N}} \mathcal{O}_K \varpi^{f_n(k)} \\
W_m = \widehat{\bigoplus}_{\mathbb{N}} K = \varinjlim_{g_m \in \mathscr{S}} \prod_{s \in \mathbb{N}} \mathcal{O}_K \varpi^{g_m(s)}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
(\prod_n V_n) \otimes_{K_\blacksquare} (\prod_m W_m) &= [\varinjlim_{\forall n,\, f_n \in \mathscr{S}} \prod_n \prod_k \mathcal{O}_K \varpi^{f_n(k)}] \otimes_{K_\blacksquare} [\varinjlim_{\forall m,\, g_m \in \mathscr{S}} \prod_m \prod_s \mathcal{O}_K \varpi^{g_m(s)}] \\
&= [\varinjlim_{\substack{\forall (n,m) \\ f_n, g_m \in \mathscr{S}}} \prod_{n,m} (\prod_k \mathcal{O}_K \varpi^{f_n(k)}) \times (\prod_s \mathcal{O}_K \varpi^{g_m(s)})].
\end{aligned}$$

Given $f, g \in \mathscr{S}$ we define $h_{f,g} : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ as $h_{f,g}(k, s) = f(k) + g(s)$. Let $\mathscr{S}'$ be the set of functions $h : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ such that $h(n, m) \to \infty$ as $\min\{n, m\} \to \infty$. Then the family of functions $\{h_{f,g}\}_{f,g \in \mathscr{S}}$ is cofinal in $\mathscr{S}'$ (see the proof of Lemma 13.1.10). One obtains

$$\begin{aligned}
(\prod_n V_n) \otimes_{K_\blacksquare} (\prod_m W_m) &= \varinjlim_{\substack{\forall (n,m) \\ h_{n,m} \in \mathscr{S}'}} \prod_{n,m} \prod_{k,s} \mathcal{O}_K \varpi^{h_{n,m}(k,s)} \\
&= \prod_{n,m} [\varinjlim_{h_{n,m} \in \mathscr{S}'} \prod_{k,s} \mathcal{O}_K \varpi^{h_{n,m}(k,s)}] \\
&= \prod_{n,m} \widehat{\bigoplus}_{k,s} K \\
&= \prod_{n,m} (V_n \otimes_{K_\blacksquare} W_m),
\end{aligned}$$

this finishes the proof. □

Proposition 13.3.6 ([CS]). *A Fréchet space is a nuclear $K_\blacksquare$-vector space.*

Proof. Let $V = \varprojlim_n V_n$ be a Fréchet space written as an inverse limit of Banach spaces with dense transition maps. Let S be an extremally disconnected set, then by Corollary 13.1.13

$$\begin{aligned}\underline{\mathrm{Hom}}_{K_\blacksquare}(K_\blacksquare[S], V) &= \varprojlim_n \underline{\mathrm{Hom}}_{K_\blacksquare}(K_\blacksquare[S], V_n)\\ &= \varprojlim_n (\underline{\mathrm{Hom}}_{K_\blacksquare}(K_\blacksquare[S], K) \otimes_{K_\blacksquare} V_n)\\ &= \underline{\mathrm{Hom}}_{K_\blacksquare}(K_\blacksquare[S], K) \otimes_{K_\blacksquare} (\varprojlim_n V_n)\\ &= \underline{\mathrm{Hom}}_{K_\blacksquare}(K_\blacksquare[S], K) \otimes_{K_\blacksquare} V,\end{aligned}$$

this finishes the proof. □

The following two lemmas describe the maps between LF, Fréchet and Banach spaces

Lemma 13.3.7. *Let $V = \varprojlim_n V_n$ be a Fréchet space written as a countable cofiltered limit of Banach spaces with projection maps $V \to V_n$ of dense image. Let W be a Banach space. Then any continuous linear map $f : V \to W$ factors through some V_n. More generally, we have $\underline{\mathrm{Hom}}_K(V, W) = \varinjlim_n \underline{\mathrm{Hom}}_K(V_n, W)$.*

Proof. First, evaluating $\underline{\mathrm{Hom}}_K(V, W)$ at an extremally disconnected set, using adjunction and the nuclearity of W, one reduces to showing that $\mathrm{Hom}_K(V, W) = \varinjlim_n \mathrm{Hom}_K(V_n, W)$. Since all the spaces involved come from compactly generated topological K-vector spaces, we might assume that V and W are classical Fréchet and Banach spaces respectively. Let $|\cdot|_n$ be the seminorm of V given by V_n, without loss of generality we may assume that $|\cdot|_n \leq |\cdot|_{n+1}$. We denote the norm of W by $\|\cdot\|$. The map f factors through V_n if and only if it is continuous with respect to the seminorm $|\cdot|_n$. Suppose that f does not factor through any n, then there exist sequences of vectors $(v_{n,m})_m$ in V for all n such that

$$|v_{n,m}|_n \xrightarrow{m\to\infty} 0 \text{ and } \|f(v_{n,m})\| \geq 1 \quad \forall m.$$

Moreover, we may assume that $|v_{n,n}|_n < \frac{1}{n}$. Then the sequence $(v_{n,n})_n$ converges to 0 in V but $\|f(v_{n,n})\| \geq 1$ for all n, which is a contradiction with the continuity of f. □

Lemma 13.3.8. *Let $W = \varinjlim_n W_n$ and $W' = \varinjlim_m W'_m$ be LF spaces presented as a filtered colimit of Fréchet spaces by injective transition maps. Then*

$$\underline{\mathrm{Hom}}_K(W, W') = \varprojlim_n \varinjlim_m \underline{\mathrm{Hom}}_K(W_n, W'_m).$$

Proof. First observe that formally

$$\underline{\mathrm{Hom}}_K(W, W') = \varprojlim_n \underline{\mathrm{Hom}}_K(W_n, W'),$$

so we can assume that $W = W_0$ is a Fréchet space. Let S be an extremally disconnected set, then by nuclearity of W'

$$\underline{\mathrm{Hom}}_K(W, \varinjlim_m W'_m)(S) = \mathrm{Hom}_K(W \otimes_K K[S], \varinjlim_m W'_m) = \mathrm{Hom}_K(W, \varinjlim_m W'_m \otimes_{K_\blacksquare} \underline{\mathrm{Cont}}(S, K)),$$

which shows that one can reduce to proving

$$\mathrm{Hom}(W, \varinjlim_m W'_m) = \varinjlim_m \mathrm{Hom}(W, W'_m).$$

So let

$$W \to \varinjlim_m W'_m$$

be a map of solid K-vector spaces. Evaluating at an extremally disconnected set S, we get a map

$$f : W(S) \to \varinjlim_m W'_m(S)$$

between a (classical) Fréchet space and a (classical) LF space. We claim that this map factors through some m. Indeed, this follows from [Sch02, Corollary 8.9], but we also give a direct argument. Assume not. Then there exists a sequence $(x_m)_{m\geq 1}$ in $W(S)$ such that $f(x_m) \notin W'_m(S)$. Multiplying x_m by big powers of p, we can assume $x_m \to 0$ as $m \to +\infty$. This translates into the existence of a map

$$\mathbb{N} \cup \{\infty\} \to \varinjlim_m W'_m(S), \quad m \mapsto f(m), \infty \mapsto 0$$

from the profinite set $\mathbb{N} \cup \{\infty\}$ into an LF-space. Since $\mathbb{N} \cup \{\infty\}$ is profinite, this maps must factorise through some m, which is a contradiction. This shows that, for each extremally disconnected set S, there exists a smallest $n(S) \in \mathbb{N}$ such that the map

$$W(S) \to \varinjlim_m W'_m(S)$$

factors through $W(S) \to W'_{n(S)}(S) \to \varinjlim_m W'_m(S)$. We conclude the proof by showing that the $n(S)'s$ are uniformly bounded. We argue again by contradiction. Assume that there are extremally disconnected sets $S_1, S_2, \ldots$ such that $n(S_i) \to +\infty$ as $i \to +\infty$. Let $S = \prod_i S_i$, which is a profinite set. Let $\widetilde{S}$ be an extremally disconnected set surjecting to S. Let $i \in \mathbb{N}$ be such that $n(S_i) > n(\widetilde{S})$ and let $S_i \to \widetilde{S}$ be a section of the surjection $\widetilde{S} \to S \to S_i$. Then the map $W(S_i) \to \varinjlim_m W'_m(S_i)$ factors through

$$W(S_i) \to W(\widetilde{S}) \to W'_{n(\widetilde{S})}(\widetilde{S}) \to W'_{n(\widetilde{S})}(S_i),$$

which is a contradiction. This finishes the proof.

□

13.3.2. Spaces of compact type

Before proving the duality between Fréchet and LS spaces let us recall the definition of (classical) nuclear Fréchet space and a LB space of compact type. Due to the fact that Fréchet spaces are always nuclear in the world of solid K-vector spaces (Proposition 13.3.6), we will say that a classical nuclear Fréchet space is a Fréchet space of compact type. We recall the definition of trace class maps and compact maps for $K_\blacksquare$-vector spaces.

Definition 13.3.9 ([Sch20, Definition 13.11])**.**

1. A trace class map of Smith spaces is a K-linear map $f : Q_1 \to Q_2$ such that there is a map $g : K \to Q_1^\vee \otimes_K Q_2$ such that f is the composition $Q_1 \xrightarrow{1\otimes g} Q_1 \otimes Q_1^\vee \otimes Q_2 \to Q_2$.
2. A map of Banach spaces is compact if its dual is a trace class map.

Definition 13.3.10.

1. A Fréchet space V is of compact type if it has a presentation $V = \varprojlim_n V_n$ as an inverse limit of Banach spaces where the maps $V_{n+1} \to V_n$ are compact.
2. A LS space is of compact type if it admits a presentation $W = \varinjlim_{n\in\mathbb{N}} W_n$ by injective trace class maps of Smith spaces.

3. Let W be a Smith space with lattice $W^0 \subset W$. We denote by W^B the Banach space whose underlying space is
$$W^B = \varprojlim_{s\in\mathbb{N}} (\underline{W^0(*)_{\mathrm{dis}}}/\varpi^s)[\frac{1}{\varpi}].$$
In other words, W^B is the Banach space structure attached to the underlying set $W(*)$ with unit ball $W^0(*)$. Note that there is a natural injective map with dense image $W^B \to W$.

4. Let V be a Banach space, we denote $V^S = (V^{\vee,B})^\vee$ and call this space the "Smith completion of V". Note that there is an injective map with dense image $V \to V^S$.

The following two results will be very useful later when studying spaces of compact type. The reader can compare them to the ideas appearing in [Sch02, §16] (see, e.g., the discussion after [Sch02, Proposition 16.5]).

Lemma 13.3.11.

1. *A map of Smith spaces $W \to W'$ is trace class if and only if it factors as $W \to W'^B \to W'$. Dually, a map of Banach spaces $V \to V'$ is of compact type if and only of it can be extended to $V \to V^S \to V'$.*

2. *Let $f : V \to W$ be a map from a Banach space to a Smith space, then f extends to a commutative diagram*
$$\begin{array}{ccc} V & \longrightarrow & V^S \\ \downarrow & & \downarrow \\ W^B & \longrightarrow & W. \end{array}$$

Proof. (1) The factorization for a morphism of Banach spaces follows from the one of Smith spaces. Let $f : W \to W'$ be a map of Smith spaces and suppose that it factors through W'^B. Then f belongs to $\mathrm{Hom}(W, W'^B) = W^\vee \otimes_{K_\blacksquare} W'^B$ as W'^B is Banach. This shows that f if a trace class map. Conversely, let $f : W \to W'$ be trace class, then there is $g : K \to W^\vee \otimes W'$ such that f factors as a composition $W \xrightarrow{1\otimes g} W \otimes W^\vee \otimes W' \to W'$. But taking ϖ-completions of the underlying discrete objects, the map g factors through $K \to W^\vee \otimes W'^B$, so that f factors through W'^B as expected.

(2) Clearly f factors as $V \to W^B \to W$. Taking duals we see that $f^\vee : W^\vee \to V^\vee$ factors as $W^\vee \to V^{\vee,B} \to V$, taking duals again one gets the factorization $f : V \to V^S \to W$. □

Corollary 13.3.12.

1. *Let $V = \varprojlim_n V_n$ be a Fréchet space of compact type. Then we can write $V = \varprojlim_{n\in N} V_n^S$ as an inverse limit of Smith spaces with trace class transition maps. Conversely, any such vector space is a Fréchet space of compact type.*

2. *Let $W = \varinjlim_n W_n$ be an LS space of compact type, then $W = \varinjlim_n W_n^B$ can be written as a filtered colimit of Banach spaces with injective compact maps. Conversely, a colimit of Banach spaces by injective compact maps is an LS space of compact type. In particular, being a LB or a LS space of compact type is equivalent.*

13.3.3. Duality

We conclude with the main result of this chapter.

Theorem 13.3.13.

1. *The functor $V \mapsto \underline{\mathrm{Hom}}_K(V, K)$ induces an antiequivalence between Fréchet and LS spaces such that $\underline{\mathrm{Hom}}_K(V, V') = \underline{\mathrm{Hom}}_K(V'^\vee, V^\vee)$, extending the one between Banach and Smith spaces. Moreover, V is Fréchet of compact type if and only if $V^\vee$ is an LS space of compact type.*

2. *Let $V = \varprojlim_{n\in N} V_n$ be a Fréchet space and $W_m = \varinjlim_{n\in\mathbb{N}} W_n$ an LS space. Then*

$$\underline{\mathrm{Hom}}_K(W, V) = W^\vee \otimes_{K_\blacksquare} V \text{ and } \underline{\mathrm{Hom}}_K(V, W) = V^\vee \otimes_{K_\blacksquare} W.$$

In particular, if V and V' are Fréchet spaces (resp. LS spaces) then

$$(V \otimes_{K_\blacksquare} V')^\vee = V^\vee \otimes_{K_\blacksquare} V'^\vee.$$

Proof. (1) Let V be a Fréchet space and let $V = \varprojlim_n V_n$ be a presentation as an inverse limit of Banach spaces with transition maps of dense image. Let S by an extremally disconnected set, we want to compute

$$\underline{\mathrm{Hom}}_K(V, K)(S) = \mathrm{Hom}_K(V \otimes_{O_{K,\blacksquare}} O_{K,\blacksquare}[S], K) = \mathrm{Hom}_K(\varprojlim_n V_n, \underline{\mathrm{Cont}}(S, K)).$$

By Lemma 13.3.7 we have

$$\mathrm{Hom}_K(\varprojlim_n V_n, \underline{\mathrm{Cont}}(S, K)) = \varinjlim_n \mathrm{Hom}_K(V_n, \underline{\mathrm{Cont}}(S, K)) = \varinjlim_n \mathrm{Hom}_K(V_n \otimes_{O_{K,\blacksquare}} O_{K,\blacksquare}[S], K).$$

In other words, we have a natural isomorphism

$$\underline{\mathrm{Hom}}_K(V, K) = \varinjlim_n V_n^\vee.$$

By Lemma 13.1.15 (2), the transition maps $V_n^\vee \to V_{n+1}^\vee$ are injective, proving that $V^\vee$ is an *LS* space.

Conversely, let W be an *LS* space and $W = \varinjlim_n W_n$ a presentation as a colimit of Smith spaces with injective transition maps. Then if follows formally that

$$\underline{\mathrm{Hom}}_K(W, K) = \varprojlim_n W_n^\vee.$$

By Lemma 13.1.15 (2) again, the transition maps $W_{n+1}^\vee \to W_n^\vee$ have dense image. It is clear from the construction that $(V^\vee)^\vee = V$ and $(W^\vee)^\vee = W$ for V Fréchet and W an *LS* space. This implies formally that $\mathrm{Hom}_K(V, V') = \mathrm{Hom}_K(V'^\vee, V^\vee)$, which gives the antiequivalence between Fréchet and *LS* spaces. Finally, by Corollary 13.3.12 and the previous computation, the duality restricts to Fréchet and *LS* spaces of compact type.

We now extend the equality of homomorphisms to the internal $\underline{\mathrm{Hom}}$. Let V and V' be Fréchet spaces and S an extremally disconnected set. We have

$$\begin{aligned}
\underline{\mathrm{Hom}}_K(V, V')(S) &= \mathrm{Hom}_K(V \otimes_{K_\blacksquare} K_\blacksquare[S], V') \\
&= \mathrm{Hom}_K(V, \underline{\mathrm{Cont}}(S, K) \otimes_{K_\blacksquare} V') \\
&= \mathrm{Hom}_K((\underline{\mathrm{Cont}}(S, K) \otimes_{K_\blacksquare} V')^\vee, V^\vee) \\
&= \mathrm{Hom}_K(V'^\vee \otimes_{K_\blacksquare} K_\blacksquare[S], V^\vee) \\
&= \underline{\mathrm{Hom}}_K(V'^\vee, V^\vee)(S).
\end{aligned}$$

The second equality follows from adjunction and nuclearity of V'. The third equality is the duality between Fréchet and LS spaces. The fourth equality follows from the compatibility of the tensor product and duality between Fréchet and LS spaces of part (2).

(2) Let $V = \varprojlim_n V_n$ and $W = \varinjlim_m W_m$ be a Fréchet and an LS space respectively. We can write

$$\begin{aligned}\underline{\mathrm{Hom}}_K(W, V) &= \varprojlim_m \underline{\mathrm{Hom}}_K(W_m, V) \\ &= \varprojlim_m W_m^\vee \otimes_{K_\blacksquare} V \\ &= W^\vee \otimes_{K_\blacksquare} V\end{aligned}$$

where the first equality is formal, the second equality follows from nuclearity of Fréchet spaces, and the third equality from the tensor product of Fréchet spaces (Proposition 13.3.5). Dually, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_K(V, W) &= \underline{\mathrm{Hom}}_K(V \otimes_{K_\blacksquare} W^\vee, K) \\ &= \underline{\mathrm{Hom}}_K(\varprojlim_{n,m} V_n \otimes_{K_\blacksquare} W_m^\vee, K) \\ &= \varinjlim_{n,m} \underline{\mathrm{Hom}}_K(V_n \otimes_{K_\blacksquare} W_m^\vee, K) \\ &= \varinjlim_{n,m} V_n^\vee \otimes_{K_\blacksquare} W_m \\ &= V^\vee \otimes_{K_\blacksquare} W,\end{aligned}$$

where the first equality follows from self duality of LS spaces and the tensor-Hom adjunction, the second equality from the tensor product of Fréchet spaces, the third equality from Lemma 13.3.7, and the last two equalities from Corollary 13.1.14 and the commutativity between tensor product and colimits.

□

14. Representation Theory

Let G be a compact p-adic Lie group. In this section we translate the theory of analytic and locally analytic representations of G from the classical framework to condensed mathematics. We hope that this new point of view could simplify some proofs and provide a better understanding of the theory. Our main sources of inspiration are the works of Lazard [Laz65], Schneider-Teitelbaum [ST02, ST03] and Emerton [Eme17].

We begin with the introduction of different algebras of distributions, each one serves to a particular purpose in the theory. Namely, there are distribution algebras arising from affinoid groups which appear naturally in the analytification functor of Proposition 12.0.11, distribution algebras which are localizations of the Iwasawa algebra, and distribution algebras algebras attached to Stein analytic groups relating the previous two.

Next, we introduce the category of solid G-modules, it is a generalisation of the category of non-archimedean topological spaces endowed with a continuous action of G (Banach, Fréchet, LB, LF, ...).

We continue with the definition of analytic representations for the affinoid and Stein analytic groups of Definition 14.1.4. We recall how the analytic vectors for a Banach representation are defined, and how it serves as motivation for the derived analytic vectors of solid G-modules. We prove the main Theorem 14.3.9 which, roughly speaking, says that being analytic for a Stein analytic group as in Definition 14.1.4 is the same as being a module over its distribution algebra.

We end with an application to locally analytic representations, reproving a theorem of Schneider-Teitelbaum describing a duality between locally analytic representations on LB spaces of compact type, and Fréchet modules of compact type over the algebra of locally analytic distributions, cf. Proposition 14.4.2. We state a conjectural generalisation to a duality between locally analytic bounded complexes of LB compact type and bounded Fréchet complexes of compact type endowed with an action of the algebra of locally analytic distributions.

14.1. Function spaces and distribution algebras

In the following paragraph we define different classes of spaces of functions and distributions that will be used throughout this text. These are algebras already appearing in the literature ([ST03, §4], [Eme17, §5]) that we introduce in a way adapted to our interests and purposes.

Let G be a compact p-adic Lie group of dimension d.

Definition 14.1.1. The Iwasawa algebra of G is defined as the solid ring

$$O_{K,\blacksquare}[G] = \varprojlim_{H \trianglelefteq G} O_K[G/H] \in \mathrm{Solid}(O_K),$$

$$K_{\blacksquare}[G] = O_{K,\blacksquare}[G][1/p] = (\varprojlim_{H \trianglelefteq G} O_K[G/H])[1/p] \in \mathrm{Solid}(K),$$

where H runs over all the open and normal subgroups of G.

Remark 14.1.2. Classically, the Iwasawa algebra is denoted by $\Lambda(G, O_K)$ and $\Lambda(G, K)$. We decide to adopt the new notations from the theory of condensed mathematics to best fit our results in this language.

Remark 14.1.3. According to the notations of Example 12.0.6, the Iwasawa algebra corresponds precisely to the evaluation at G of the functor of measures of the analytic rings $(O_K, O_K)_\blacksquare$ and $(K, O_K)_\blacksquare$ which, in turn, are commonly denoted by $O_{K,\blacksquare}$ and $K_\blacksquare$ respectively.

In order to define the space of locally analytic functions attached to G, we need to work with coordinates locally around the identity. By [DdSMS99, Cor. 8.34], there exists an open normal subgroup G_0 of G which is a uniform pro-p group [1]. Such a group can be equipped with a valuation w and an ordered basis $g_1, \dots, g_d \in G_0$. By [Laz65, III 3.1.3, Proposition], after shrinking G_0 if necessary, we can assume that $w(g_1) = \dots = w(g_d) = w_0 > 1$ is an integer. This basis induces charts

$$\phi : \mathbb{Z}_p^d \to G_0, \quad (x_1, \dots, x_d) \mapsto g_1^{x_1} \cdot \dots \cdot g_d^{x_d}$$

such that ϕ is an homeomorphism between G_0 and $\mathbb{Z}_p^d$ with $w(g_1^{x_1} \cdot \dots \cdot g_d^{x_d}) = w_0 + \min_{1 \le i \le d} v_p(x_i)$, and such that the map $\psi : G_0 \times G_0 \to G_0$, $(g, h) \mapsto gh^{-1}$, defining the group structure of G_0, is given by an analytic function $\psi : \mathbb{Z}_p^d \times \mathbb{Z}_p^d \to \mathbb{Z}_p^d$ with coefficients in $\mathbb{Z}_p$. By further shrinking G_0 if necessary we can also assume that conjugation on G_0 by any element of $g \in G$ is given by a family of power series with bounded coefficients.

Let $r = p^{-s} > 0$ for $s \in \mathbb{Q}$, and $\mathbb{D}^d_{\mathbb{Q}_p}(r) \subset \mathbb{A}^d_{\mathbb{Q}_p}$ the affinoid polydisc of radius r. If $s \in \mathbb{Z}$ then $\mathbb{D}^d_{\mathbb{Q}_p}(r)$ is the affinoid space defined by the algebra $\mathbb{Q}_p\langle \frac{T_1}{p^s}, \dots \frac{T_d}{p^s} \rangle$, where $T_1, \dots, T_d$ are the coordinates of $\mathbb{A}^d_{\mathbb{Q}_p}$. We let $\mathring{\mathbb{D}}^d(r) = \bigcup_{r' < r} \mathbb{D}^d(r')$ denote the open polydisc of radius r.

Definition 14.1.4. We define the following rigid analytic groups

1. $\mathbb{G}_0 = (\mathbb{D}^d_{\mathbb{Q}_p}(1), \psi)$; the affinoid group defined by the group law ψ of G_0.
2. For any $h \in \mathbb{Q}_{\ge 0}$, the affinoid groups $\mathbb{G}_h = (\mathbb{D}^d_{\mathbb{Q}_p}(p^{-h}), \psi)$ of radius p^{-h}. We also denote $\mathbb{G}_0^{(h)} = G_0 \mathbb{G}_h \subset \mathbb{G}_0$.
3. For any $h \in \mathbb{Q}_{\ge 0}$ the Stein groups $\mathbb{G}_{h^+} = \cup_{h' > h} \mathbb{G}_{h'}$ and $\mathbb{G}_0^{(h^+)} = G_0 \mathbb{G}_{h^+} = \cup_{h' > h} \mathbb{G}_0^{(h)}$.

Example 14.1.5. If $G = \mathrm{GL}_2(\mathbb{Z}_p)$, an example of a uniform pro-p-subgroup G_0 is

$$G_0 = \begin{pmatrix} 1 + p^n\mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p^n\mathbb{Z}_p \end{pmatrix}$$

for $n \ge 2$ if $p = 2$ and $n \ge 1$ if $p > 2$. Let's take $p > 2$ and $n = 1$. In this case $\mathbb{G}_h$ is the rigid analytic group

$$\mathbb{G}_h = \begin{pmatrix} 1 + \mathbb{D}^1_{\mathbb{Q}_p}(p^{-h-1}) & \mathbb{D}^1_{\mathbb{Q}_p}(p^{-h-1}) \\ \mathbb{D}^1_{\mathbb{Q}_p}(p^{-h-1}) & 1 + \mathbb{D}^1_{\mathbb{Q}_p}(p^{-h-1}) \end{pmatrix} = \begin{pmatrix} 1 + p^{h+1}\mathbb{D}^1_{\mathbb{Q}_p}(1) & p^{h+1}\mathbb{D}^1_{\mathbb{Q}_p}(1) \\ p^{h+1}\mathbb{D}^1_{\mathbb{Q}_p}(1) & 1 + p^{h+1}\mathbb{D}^1_{\mathbb{Q}_p}(1) \end{pmatrix},$$

whereas the Stein group $\mathbb{G}_{h^+}$ is equal to

$$\mathbb{G}_{h^+} = \begin{pmatrix} 1 + p^{h+1}\mathring{\mathbb{D}}^1_{\mathbb{Q}_p}(1) & p^{h+1}\mathring{\mathbb{D}}^1_{\mathbb{Q}_p}(1) \\ p^{h+1}\mathring{\mathbb{D}}^1_{\mathbb{Q}_p}(1) & 1 + p^{h+1}\mathring{\mathbb{D}}^1_{\mathbb{Q}_p}(1) \end{pmatrix}.$$

The following lemma says that the rigid spaces of Definition 14.1.4 are indeed rigid analytic groups.

Lemma 14.1.6. *The affinoid $\mathbb{G}_h$ is an open normal subgroup of $\mathbb{G}_0$ stable by conjugation of G.*

[1] Recall that a uniform pro-p group H is a pro-p group which is finitely generated, torsion free and powerful, i.e., $[H, H] \subseteq H^p$ if $p > 2$ or $[H, H] \subseteq H^4$ if $p = 2$.

Proof. By simplicity we suppose that $h \in \mathbb{N}$. The map

$$\psi : \mathbb{G}_0 \times \mathbb{G}_0 \to \mathbb{G}_0 \quad : \quad (x, y) \mapsto xy^{-1}$$

is defined by a family of power series $(Q_1(X, Y), \ldots, Q_d(X, Y))$ with integral coefficients satisfying the group axioms. In particular, one has $Q_i(0, 0) = 0$. The inclusion $\mathbb{G}_h \to \mathbb{G}_0$ is given by the map $\mathbb{Q}_p\langle T_1, \ldots, T_d\rangle \to \mathbb{Q}_p\langle \frac{T_1}{p^h}, \ldots, \frac{T_d}{p^h}\rangle$. Thus, the image of $\frac{T_i}{p^h}$ by the multiplication map ψ is equal to

$$\frac{1}{p^h} Q_i(X, Y) = \frac{1}{p^h} \sum_{(\alpha,\beta)\neq 0} a_{\alpha,\beta} X^\alpha Y^\beta = \sum_{(\alpha,\beta)\neq 0} a_{\alpha,\beta} p^{h(|\alpha|+|\beta|-1)} \left(\frac{X}{p^h}\right)^\alpha \left(\frac{Y}{p^h}\right)^\beta .$$

This shows that ψ restricts to a map $\mathbb{G}_h \times \mathbb{G}_h \to \mathbb{G}_h$, proving that $\mathbb{G}_h$ is a subgroup of $\mathbb{G}_0$. A similar argument shows that $\mathbb{G}_h$ is normal in $\mathbb{G}_0$ and that it is stable by the conjugation of elements of G. □

14.1.1. Analytic distributions

Classically, the analytic vectors of Banach representations are defined via the affinoid algebras of the analytic groups of Definition 14.1.4, see [Eme17, §3]. In order to develop properly this theory for solid K-vector spaces we shall need to introduce some notation for the algebras of the analytic groups, as well as for their analytic distributions. Recall once more that we see all complete locally convex K-vector spaces as solid objects.

Definition 14.1.7.

1. We consider the following spaces of functions
   i. $C(\mathbb{G}_0^{(h)}, \mathcal{O}_K) := \mathcal{O}^+(\mathbb{G}_0^{(h)}) \otimes_{\mathbb{Z}_p} \mathcal{O}_K$; the power bounded analytic functions of the affinoid group.
   ii. $C(\mathbb{G}_0^{(h)}, K) := \mathcal{O}(\mathbb{G}_0^{(h)}) \otimes_{\mathbb{Q}_p} K$; the regular functions of the affinoid group.
   iii. $C(\mathbb{G}_0^{(h^+)}, K) := \varprojlim_{h'>h^+} C(\mathbb{G}_0^{(h)}, K)$; the regular functions of the Stein group.
2. We define the following spaces of distributions
   i. $\mathcal{D}^{(h)}(G_0, \mathcal{O}_K) := \underline{\mathrm{Hom}}_{\mathcal{O}_K}(C(\mathbb{G}_0^{(h)}, \mathcal{O}_K), \mathcal{O}_K)$.
   ii. $\mathcal{D}^{(h)}(G_0, K) := \underline{\mathrm{Hom}}_{K}(C(\mathbb{G}_0^{(h)}, K), K)$.
   iii. $\mathcal{D}^{(h^+)}(G_0, K) := \underline{\mathrm{Hom}}_{K}(C(\mathbb{G}_0^{(h^+)}, K), K)$.

Remark 14.1.8. Let $h \in \mathbb{N}$, the algebra $C(\mathbb{G}_0^{(h)}, K)$ is the space of functions on G_0 with values in K which are analytic of radius p^{-h}. More precicely, using the coordinates $g_1, \ldots, g_d \in G_0$, and identifying G_0 with $\mathbb{Z}_p^d$, $C(\mathbb{G}_0^{(h)}, K)$ is the space of functions $f : \mathbb{Z}_p^d \to K$ whose restriction at cosets $x + p^h\mathbb{Z}_p^d$ is given by a convergent power series with coefficients in K.

Remark 14.1.9. Observe that, by Theorem 13.3.13, one has $C(\mathbb{G}_0^{(h)}, K) = \mathcal{D}^{(h)}(G_0, K)^\vee$ as well as $\mathcal{D}^{(h^+)}(G_0, K) = \varinjlim_{h'>h} \mathcal{D}^{(h')}(G_0, K)$.

Lemma 14.1.10. *The solid $\mathcal{O}_K$-module $\mathcal{D}^{(h)}(G_0, \mathcal{O}_K)$ has a natural structure of associative unital $\mathcal{O}_K$-algebra induced by the multiplication map $\mathbb{G}_0^{(h)} \times \mathbb{G}_0^{(h)} \to \mathbb{G}_0^{(h)}$. In particular, $\mathcal{D}^{(h)}(G_0, K)$ and $\mathcal{D}^{(h^+)}(G_0, K)$ are associative unital $\mathcal{O}_K$-algebras.*

Proof. The multiplication $\mathbb{G}_0^{(h)} \times \mathbb{G}_0^{(h)} \to \mathbb{G}_0^{(h)}$ defines a comultiplication map

$$\nabla : C(\mathbb{G}_0^{(h)}, \mathcal{O}_K) \to C(\mathbb{G}_0^{(h)}, \mathcal{O}_K) \otimes_{\mathcal{O}_{K,\blacksquare}} C(\mathbb{G}_0^{(h)}, \mathcal{O}_K).$$

As $C(\mathbb{G}_0^{(h)}, \mathcal{O}_K)$ is an orthonormalizable Banach $\mathcal{O}_K$-module, taking the dual of ∇ one obtains a map $\mathcal{D}^{(h)}(G_0, \mathcal{O}_K) \otimes_{\mathcal{O}_{K,\blacksquare}} \mathcal{D}^{(h)}(G_0, \mathcal{O}_K) \to \mathcal{D}^{(h)}(G_0, \mathcal{O}_K)$ which is easily seen to be the convolution product. □

14.1.2. Localizations of the Iwasawa algebra

Recall that we have fixed an open normal subgroup G_0 which is a uniform pro-p-group with basis $g_1, \dots, g_d$ of constant valuation $w_0 > 1$. Let $\mathbf{b}_i = [g_i] - 1 \in \mathcal{O}_{K,\blacksquare}[G_0](*)$, one has [ST03, §4]

$$\mathcal{O}_{K,\blacksquare}[G_0] = \prod_{\alpha \in \mathbb{N}^d} \mathcal{O}_K \mathbf{b}^\alpha$$

where $\mathbf{b}^\alpha = \mathbf{b}_1^{\alpha_1} \cdots \mathbf{b}_d^{\alpha_d}$ for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$.

Remark 14.1.11. Taking Mahler expansion of continuous functions on $\phi : G_0 \cong \mathbb{Z}_p^d$, an explicit computation of finite differences shows that the elements $\mathbf{b}^\alpha$ correspond to the dual basis of the Mahler basis $\binom{x}{\alpha}$ i.e., $\mathbf{b}^\alpha(f) = c_\alpha$ for any continuous function $f \in \mathrm{Cont}(G_0, K)$ such that $\phi^*(f) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \binom{x}{\alpha}$.

We now introduce a second family of distribution algebras using the basis $(\mathbf{b}_i)_{1 \leq i \leq d}$, these can be thought of as localizations of the Iwasawa algebra $K_\blacksquare[G_0]$.

Definition 14.1.12. Let $h > 0$ be rational. We define the condensed rings $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)$ and $\mathcal{D}_{(h)}(G_0, K)$ so that, for any extremaly disconnected set S, one has

- $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)(S) = \{\sum_{\alpha \in \mathbb{N}^d} a_\alpha \mathbf{b}^\alpha \ : \ \sup_\alpha \{|a_\alpha| p^{-\frac{p^{-h}}{p-1}|\alpha|}\} \leq 1, \quad a_\alpha \in \mathrm{Cont}(S, \mathcal{O}_K)\}$,
- $\mathcal{D}_{(h)}(G_0, K)(S) = \{\sum_{\alpha \in \mathbb{N}^d} a_\alpha \mathbf{b}^\alpha \ : \ \sup_\alpha \{|a_\alpha| p^{-\frac{p^{-h}}{p-1}|\alpha|}\} < +\infty, \quad a_\alpha \in \mathrm{Cont}(S, K)\}$,

Remark 14.1.13. The condensed module $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)$ is in fact a profinite $\mathcal{O}_K$-module provided $b(h) = \frac{p^{-h}}{p-1} \in v_p(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. Indeed, if $b(h)$ is the valuation of an element of K, we have an isomorphism of profinite $\mathcal{O}_K$-modules

$$\mathcal{D}_{(h)}(G_0, \mathcal{O}_K) = \prod_{\alpha \in \mathbb{N}^d} \mathcal{O}_K \left(\frac{\mathbf{b}_1}{p^{b(h)}}\right)^{\alpha_1} \dots \left(\frac{\mathbf{b}_d}{p^{b(h)}}\right)^{\alpha_d}.$$

Remark 14.1.14. The distribution algebras $\mathcal{D}_{(h)}(G_0, K)$ are variations of the (p-adic completions of the) rings $\hat{A}^{(m)}$ of [Eme17, §5.2], adapted to the Iwasawa algebra instead of the enveloping algebra of Lie G.

Lemma 14.1.15. *The multiplication map of $K_\blacksquare[G_0]$ extends uniquely to a multiplication map of $\mathcal{D}_{(h)}(G_0, \mathcal{O}_K)$.*

Proof. This follows directly from Proposition 4.2 of [ST03]. □

One can describe the analytic distributions of Definition 14.1.7 in terms of the the basis $\mathbf{b}_i$ of the Iwasawa algebra.

Proposition 14.1.16. *Let S be an extremally disconnected set, then*

$$\mathcal{D}^{(h)}(G_0, K)(S) = \left\{ \sum_{\alpha \in \mathbb{N}^d} a_\alpha \mathbf{b}^\alpha \ : \ \sup_\alpha \{|a_\alpha| p^{-\frac{p^{-h}|\alpha| - s(\alpha)}{p-1}}\} < +\infty, \quad a_\alpha \in \mathrm{Cont}(S, K) \right\},$$

where $s(\alpha) = \sum_{1 \leq i \leq d} s(\alpha_i)$ and $s(\alpha_i)$ is the sum of the p-adic digits of α_i. In particular, one has

$$\mathcal{D}^{(h^+)}(G_0, K)(S) = \left\{ \sum_{\alpha \in \mathbb{N}^d} a_\alpha \mathbf{b}^\alpha \ : \ \sup_\alpha \{|a_\alpha| p^{-\frac{p^{-h}|\alpha| - s(\alpha)}{p-1}}\} < +\infty \text{ for some } h' > h, \quad a_\alpha \in \mathrm{Cont}(S, K) \right\}.$$

Proof. Let $\phi : \mathbb{Z}_p^d \to G_0$ be the chart defined by the basis $g_1, \dots, g_d$. By a theorem of Amice (c.f. [**?**, III.(1.3.8)]), a continuous function $f : G_0 \to K$ is h-analytic (i.e. belongs to $C(\mathbb{G}_0^{(h)}, K)$) if and only if

$$v(c_\alpha) - \frac{p^{-h}|\alpha| - s(\alpha)}{p-1} \to +\infty$$

whenever $\alpha \to +\infty$, where $\phi^* f(g) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \binom{x}{\alpha}$. After dualizing this gives the claimed result as the algebra of analytic distributions is attached to its underlying topological space. □

Remark 14.1.17. Using the formula

$$v_p(\alpha!) = \frac{|\alpha| - s(\alpha)}{p-1},$$

one can rewrite the above condition on the valuation of the coefficients as

$$\sup_\alpha \{|a_\alpha \alpha!|\, p^{-a(h)|\alpha|}\} < +\infty,$$

where $a(h) = \frac{p^{-h}-1}{p-1}$.

Corollary 14.1.18. *There is an isomorphism of solid O_K-algebras*

$$\mathcal{D}^{(h^+)}(G_0, K) \cong \varinjlim_{h'>h} \mathcal{D}_{(h')}(G_0, K).$$

Proof. Let $h'' > h'$, then one can write

$$p^{-\frac{p^{-h'}|\alpha| - s(\alpha)}{p-1}} = p^{-\frac{p^{-h''}}{p-1}|\alpha|} p^{-\frac{(p^{-h'} - p^{-h''})|\alpha| - s(\alpha)}{p-1}}.$$

Since $p^{-h'} - p^{-h''} > 0$, we see that $(p^{-h'} - p^{-h''})|\alpha| - s(\alpha) \to +\infty$ as $|\alpha| \to +\infty$. This implies that for any $h'' > h'$ we have

$$\mathcal{D}_{(h'')}(G_0, K) \subseteq \mathcal{D}^{(h')}(G_0, K) \subset \mathcal{D}_{(h')}(G_0, K).$$

Taking limits as $h' \to h^+$ and $h'' \to h^+$ one obtains the corollary. □

14.1.3. Distribution algebras over G

The algebras we have defined can be extended to the whole compact group G in a obvious way. Indeed, by Lemma 14.1.6 the spaces $\mathcal{D}^{(h)}(G_0, K)$ and $\mathcal{D}_{(h)}(G_0, K)$ admit an action of G extending the inner action of $K_\blacksquare[G_0]$. Let us define the distributions

$$\mathcal{D}^{(h)}(G, K) = K_\blacksquare[G] \otimes_{K_\blacksquare[G_0]} \mathcal{D}^{(h)}(G_0, K),$$

$$\mathcal{D}^{(h^+)}(G, K) = \varinjlim_{h'>h} \mathcal{D}^{(h')}(G, K)$$

$$\mathcal{D}_{(h)}(G, K) = K_\blacksquare[G] \otimes_{K_\blacksquare[G_0]} \mathcal{D}_{(h)}(G_0, K),$$

where in the tensor products we see $\mathcal{D}_{(h)}(G_0, K)$ as a left $K_\blacksquare[G_0]$-module. They are unital associative algebras admitting $K_\blacksquare[G]$ as a dense subspace. Notice that even though the distributions algebras over G depend on the choice of the open normal subgroup $G_0 \subset G$, the projective systems $\{\mathcal{D}^{(h)}(G, K)\}_{h>0}$ and $\{\mathcal{D}_{(h)}(G, K)\}_{h>0}$ do not. We define the algebra of locally analytic distributions of G as the Fréchet algebra

$$\mathcal{D}^{la}(G, K) = \varprojlim_{h \to \infty} \mathcal{D}^{(h^+)}(G, K) = \varprojlim_{h \to \infty} \mathcal{D}^{(h)}(G, K).$$

For future reference let us introduce algebras of analytic functions over G. Let $h \geq 0$ be a rational number and $\mathbb{G}_0^{(h)}$ the rigid analytic groups of Definition 14.1.4, recall that they are stable under conjugation by G. Let $\mathbb{G}^{(h)}$ be the rigid analytic group given by $G\mathbb{G}_0^{(h)}$, i.e. if $s_1, \dots, s_n$ are representatives of the cosets G/G_0 then

$$\mathbb{G}^{(h)} = \bigsqcup_{i=1}^{n} s_i \mathbb{G}_0^{(h)}.$$

We also define

$$\mathbb{G}^{(h^+)} = \bigcup_{h'>h} \mathbb{G}^{(h)}.$$

Let $C(\mathbb{G}^{(h)}, K) = \mathscr{O}(\mathbb{G}^{(h)}) \otimes_{\mathbb{Q}_p} K$ be the affinoid algebra of analytic functions of G over K of radius p^{-h}, it is immediate to check that $\mathcal{D}^{(h)}(G, K) = \underline{\mathrm{Hom}}_K(C(\mathbb{G}^{(h)}, K), K)$. We define $C(\mathbb{G}^{(h)}, \mathcal{O}_K)$ and $C(\mathbb{G}^{(h^+)}, K)$ in the obvious way.

14.2. Solid G-modules

Let G be a profinite group and $\mathcal{O}_{K,\blacksquare}[G]$ its Iwasawa algebra over $\mathcal{O}_K$.

Lemma 14.2.1. *Let V be a solid $\mathcal{O}_K$-module. The following are equivalent.*

1. *V has a structure of $\mathcal{O}_{K,\blacksquare}[G]$-module*
2. *For any extremally disconnected set S there is a functorial $\mathcal{O}_K$-linear action*

$$C(S, G) \times V(S) \to V(S).$$

In this case, we say that V is a solid G-module over $\mathcal{O}_K$. We denote the category of solid $\mathcal{O}_{K,\blacksquare}[G]$-modules by $\mathrm{Solid}(\mathcal{O}_{K,\blacksquare}[G])$ *and its derived category by* $D(\mathcal{O}_{K,\blacksquare}[G])$.

Proof. Let V be an $\mathcal{O}_{K,\blacksquare}[G]$-module and S an extremally disconnected set. There is a natural map of condensed sets $[\cdot] : G \to \mathcal{O}_{K,\blacksquare}[G]$. The action of $\mathcal{O}_{K,\blacksquare}[G]$ over V is provided by a linear map $\mathcal{O}_{K,\blacksquare}[G] \otimes_{\mathcal{O}_{K,\blacksquare}} V \to V$ satisfying the usual axioms. Composing with $[\cdot]$ and evaluating at S we obtain a map $C(S, G) \times V(S) \to V(S)$, it is easy to check that this provides a functorial action as in (2).

Conversely, to have such a functorial action is equivalent to having a map of condensed sets $G \times V \to V$ making the usual diagrams commutative. By adjunction of the functor $X \to \mathbb{Z}[X]$, this provides a linear map $\mathbb{Z}[G] \otimes_{\mathbb{Z}} V \to V$. As V is a solid $\mathcal{O}_K$-module it extends uniquely to a map $\mathcal{O}_{K,\blacksquare}[G] \otimes_{\mathcal{O}_{K,\blacksquare}} V \to V$ which is easily seen to satisfy the obvious diagrams of an $\mathcal{O}_{K,\blacksquare}[G]$-module. □

Let us extend the definition of the condensed set of "continuous functions" to complexes:

Definition 14.2.2. Let V be a solid $\mathcal{O}_K$-module and S a profinite set, we denote by $\underline{\mathrm{Cont}}(S, V)$ the $\mathcal{O}_K$-module $\underline{\mathrm{Hom}}(\mathbb{Z}[S], V) = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[S], V)$. More generally, for $C \in D(\mathcal{O}_{K,\blacksquare})$, we denote $\underline{\mathrm{Cont}}(S, C) := R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[S], C)$, which is consistent with the previous definition as $\mathcal{O}_{K,\blacksquare}[S]$ is a projective module.

Proposition 14.2.3. *The functor $V \mapsto \underline{\mathrm{Cont}}(G, V)$ is exact and factors through a functor* $\mathrm{Solid}(\mathcal{O}_K) \to \mathrm{Solid}(\mathcal{O}_{K,\blacksquare}[G^2])$ *induced by the left and right regular actions respectively. Moreover, it extends to an exact functor of derived categories $D(\mathcal{O}_{K,\blacksquare}) \to D(\mathcal{O}_{K,\blacksquare}[G^2])$.*

Proof. As G is profinite, $\mathcal{O}_{K,\blacksquare}[G]$ is a compact projective $\mathcal{O}_{K,\blacksquare}$-module. This makes the functor $\underline{\mathrm{Cont}}(G, V)$ exact. Thus, the second statement reduces to the first one. We prove the first statement.

Let S be an extremally disconnected set, then $\underline{\mathrm{Cont}}(G,V)(S) = \mathrm{Cont}(G\times S, V) = V(G\times S)$. Let us define a map

$$\mathrm{Cont}(S,G^2)\times V(G\times S)\to V(G\times S)$$

as in (2) of Lemma 14.2.1. Let $f=(f_1,f_2): S\to G^2$ and $v: G\times S\to V$ be objects in $\mathrm{Cont}(S,G^2)$ and $V(G\times S)$ respectively. We define the product $f\cdot v$ to be the composition

$$\begin{array}{ccccc} S\times G & \longrightarrow & S\times G & \longrightarrow & V \\ (s,g) & \longmapsto & (s, f_1(s)^{-1}gf_2(s)) & \longmapsto & v(s,f_1(s)^{-1}gf_2(s)). \end{array}$$

It is immediate so check that this endows $\underline{\mathrm{Cont}}(G,V)$ with a action of G^2 which is the left and right regular action on the first and second component respectively. □

Remark 14.2.4. If we suppose in addition that V is a $\mathcal{O}_{K,\blacksquare}[G]$-module, then $\underline{\mathrm{Cont}}(G,V)$ is naturally a $\mathcal{O}_{K,\blacksquare}[G^3]$-module. Namely, for S be an extremally disconnected set, $f_1,f_2,f_3\in\mathrm{Cont}(S,G)$ and $v\in V(G\times S)$, we have the action

$$[(f_1,f_2,f_3)\cdot v](g,s) = f_3(s)v(f_1(s)^{-1}gf_2(s),s).$$

This action induces an exact functor of derived categories $D(\mathcal{O}_{K,\blacksquare}[G])\to D(\mathcal{O}_{K,\blacksquare}[G^3])$.

Definition 14.2.5. Let V be a solid $G^n=\overbrace{G\times\cdots\times G}^{n}$-representation over $\mathcal{O}_K$. Given $I\subset\{1,2,\dots,n\}$ we denote by $\star_I$ the diagonal action of G on V induced by the embedding $\iota_I: G\to G^n$ in the components of I. We denote by $V_{\star_I}$ the module V endowed with the action $\star_I$. If $I=\emptyset$ we write $V_0:=V_\emptyset$ for the $\mathcal{O}_{K,\blacksquare}$-module V endowed with the trivial action of G.

The following proposition basically says that any action on a solid module is continuous (compare it with [Eme17, Definition 3.2.8]).

Proposition 14.2.6. *Let C be an object in $D(\mathcal{O}_{K,\blacksquare}[G])$. Then there is a natural quasi-isomorphism of $\mathcal{O}_{K,\blacksquare}[G]$-modules*

$$R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,\underline{\mathrm{Cont}}(G,C)_{\star_{1,3}})\xrightarrow{\sim} C, \tag{14.2.1}$$

where the action of $\mathcal{O}_{K,\blacksquare}[G]$ in the left-hand-side is via the $\star_2$-action. The inverse of this map is called the orbit map of C.

Proof. First, we claim that there exists a natural quasi-isomorphism $\underline{\mathrm{Cont}}(G,C)_{\star_{1,3}}\simeq\underline{\mathrm{Cont}}(G,C)_{\star_1}$ for $C\in D(\mathcal{O}_{K,\blacksquare}[G])$. Suppose that the previous is true, then we have

$$\begin{aligned} R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,\underline{\mathrm{Cont}}(G,C)_{\star_{1,3}}) &\simeq R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,\underline{\mathrm{Cont}}(G,C)_{\star_1}) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K,R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[G],C)_{\star_1}) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_K\otimes^L_{\mathcal{O}_{K,\blacksquare}[G]}\mathcal{O}_{K,\blacksquare}[G],C) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_K,C)=C. \end{aligned}$$

To prove the claim, it is enough to define a natural isomorphism $\underline{\mathrm{Cont}}(G,V)_{\star_{1,3}}\to\underline{\mathrm{Cont}}(G,V)_{\star_1}$ for $V\in\mathrm{Solid}(\mathcal{O}_{K,\blacksquare}[G])$. Let S be an extremally disconnected set, and take $v\in V(G\times S)$. Consider the inverse map $u: G\to G\ \ g\mapsto g^{-1}$ and the multiplication map $m_V: G\times V\to V$. Define $\psi_V(v)$ to be the composition

$$\psi_V(v): G\times S\xrightarrow{u\times v} G\times V\xrightarrow{m_V} V.$$

The application

$$\begin{aligned} \psi_V: V(G\times S)&\to V(G\times S) \\ v&\mapsto\psi_V(v) \end{aligned}$$

induces an isomorphism of solid $\mathcal{O}_K$-modules $\psi_V : \underline{\mathrm{Cont}}(G, V) \to \underline{\mathrm{Cont}}(G, V)$. It is easy to check that it transfers the $\star_{1,3}$-action to the $\star_1$-action and the $\star_2$-action to the $\star_{2,3}$-action. This proves the claim and that the isomorphism (14.2.1) is G-equivariant. □

Remark 14.2.7. The previous proof shows that if V is a $\mathcal{O}_{K,\blacksquare}[G]$-module arising from a topological space, the isomorphism $V \to (\underline{\mathrm{Cont}}(G, V)_{\star_{1,3}})^G$ is given by the usual orbit map.

14.3. Analytic representations

Let $h \geq 0$ and $\mathbb{G}^{(h)}$ the rigid analytic group of §14.1 extending the group law of G. We recall that $\mathbb{G}^{(h)}$ depends on the choice of an open normal uniform pro-p-subgroup $G_0 \subset G$. To motivate the forthcoming definitions of analytic vectors let us first recall how this works for Banach spaces, where we follow [Eme17, §3].

Let V be a K-Banach space endowed with a continuous action of G, the space of V-valued $\mathbb{G}^{(h)}$-analytic functions is by definition the p-adically complete tensor product $C(\mathbb{G}^{(h)}, V) := C(\mathbb{G}^{(h)}, K)\widehat{\otimes}_K V$. As V and $C(\mathbb{G}^{(h)}, K)$ are Banach spaces, the p-adically complete tensor product coincides with the solid tensor product $C(\mathbb{G}^{(h)}, K) \otimes_{K_\blacksquare} V$ (Lemma 13.1.10). This space has an action of G^2 given by the left and right regular actions of G, and an extra action of G induced by the one of V. Following the notation of Definition 14.2.5, the $\mathbb{G}^{(h)}$-analytic vectors of V is the Banach space

$$V^{\mathbb{G}^{(h)}-an} := (C(\mathbb{G}^{(h)}, V)_{\star_{1,3}})^G. \tag{14.3.1}$$

There is a natural map $V^{\mathbb{G}^{(h)}-an} \to V$ given by evaluating at $1 \in \mathbb{G}^{(h)}$, and V is $\mathbb{G}^{(h)}$-analytic if the previous arrow is an isomorphism.

To generalise the previous construction of analytic vectors to solid $K_\blacksquare[G]$-modules we need to rewrite (14.3.1) in a slightly different way. Consider the affinoid ring $(C(\mathbb{G}^{(h)}, K), C(\mathbb{G}^{(h)}, \mathcal{O}_K))$, it is a finite product of Tate power series rings in d-variables. In Example 12.0.6 (4) and (5), we saw how the affinoid ring provides a natural analytic ring that we denote as $C(\mathbb{G}^{(h)}, K)_\blacksquare$. We also denote by $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare$ the analytic ring attached to its subalgebra of power-bounded elements. Now, as V is a K-Banach vector space, one has

$$C(\mathbb{G}^{(h)}, V) = C(\mathbb{G}^{(h)}, K) \otimes_{K_\blacksquare} V = C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes_{K_\blacksquare} V,$$

where the last tensor product is the completion functor with respect to the measures of $C(\mathbb{G}^{(h)}, K)_\blacksquare$, see Theorem 12.0.7 and Proposition 12.0.11. Hence, we can write the $\mathbb{G}^{(h)}$-analytic vectors of V in the form

$$V^{\mathbb{G}^{(h)}-an} = \underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, (C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes_{K_\blacksquare} V)_{\star_{1,3}}).$$

In order to generalise the construction of analytic vectors we need some basic properties of the tensor $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_{K,\blacksquare}} -$.

Proposition 14.3.1. *Consider the functor $V \mapsto C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_{K,\blacksquare}} V$ for $V \in \mathrm{Solid}(\mathcal{O}_K)$. The following statements hold.*

1. *The functor is exact.*
2. *It induces an exact functor of derived categories $D(\mathcal{O}_{K_\blacksquare}) \to D(\mathcal{O}_{K,\blacksquare}[G^2])$ given by the left and right regular actions.*
3. *There is a functorial map $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes^L_{\mathcal{O}_K} C \to \underline{\mathrm{Cont}}(G, C)$ for $C \in D(\mathcal{O}_{K_\blacksquare})$ compatible with the left and right regular actions.*

Proof. Exactness follows from Proposition 12.0.11. Indeed, as $\mathbb{G}^{(h)}$ is a finite disjoint union of polydiscs one has

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes^L_{\mathcal{O}_{K,\blacksquare}} V = R\underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{D}^{(h)}(G, \mathcal{O}_K), V). \tag{14.3.2}$$

But $\mathcal{D}^{(h)}(G, \mathcal{O}_K)$ is a projective $\mathcal{O}_K$-module, this implies that

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes^L_{\mathcal{O}_{K,\blacksquare}} V = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{D}^{(h)}(G, \mathcal{O}_K), V) = C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_{K,\blacksquare}} V$$

is exact.

To prove (2), it is enough to show that $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_{K,\blacksquare}} V$ has natural left and right regular actions for $V \in \mathrm{Solid}(\mathcal{O}_K)$. Writing V as a quotient $P_1 \to P_0 \to V$ of objects of the form $P_i = \bigoplus_{I_i} \prod_{J_i} \mathcal{O}_K$, we have an exact sequence

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_K} P_1 \xrightarrow{f} C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_K} P_0 \to C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_K} V \to 0.$$

The functor $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_K} -$ commutes with colimits and, by Equation (14.3.2), it also commutes with products. Hence

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes P_i = \bigoplus_{I_i} \prod_{J_i} C(\mathbb{G}^{(h)}, \mathcal{O}_K)$$

and these modules are equipped with the natural left and right regular actions of G. Moreover, the map f is equivariant for these actions. We endow $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_K} V$ with the action induced by the quotient map. It is easy to check that this action is independent of the presentation of V, and that it is functorial.

For the last statement, it is enough to construct a functorial equivariant map

$$C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_K} V \to \underline{\mathrm{Cont}}(G, V)$$

for $V \in \mathrm{Solid}(\mathcal{O}_K)$. Recall that by definition $\underline{\mathrm{Cont}}(G, V) = \underline{\mathrm{Hom}}_{\mathcal{O}_K}(\mathcal{O}_{K,\blacksquare}[G], V)$. Similarly as before, we are reduced to constructing the map for an object of the form $P = \bigoplus_I \prod_J \mathcal{O}_K$. As both functors commute with colimits and products, one reduces to treat the case $P = \mathcal{O}_K$, for which we have the natural inclusion $C(\mathbb{G}^{(h)}, \mathcal{O}_K) \to \underline{\mathrm{Cont}}(G, \mathcal{O}_K)$ provided by $G \subset \mathbb{G}^{(h)}$, which is equivariant for the left and right regular actions of G. This ends the proof. □

Remark 14.3.2. It is clear that if V is a solid G-module then $C(\mathbb{G}^{(h)}, \mathcal{O}_K)_\blacksquare \otimes_{\mathcal{O}_{K,\blacksquare}} V$ can be endowed with an action of G^3.

Definition 14.3.3. Let $h \geq 0$

1. Let $V \in \mathrm{Solid}(K_\blacksquare[G])$, the space of $\mathbb{G}^{(h)}$-analytic vectors of V is the solid $K_\blacksquare[G]$-module

$$V^{\mathbb{G}^{(h)}-an} := \underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, (C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes_{K_\blacksquare} V)_{\star_{1,3}})$$

where the action of G is induced by the $\star_2$-action. Similarly, we define the $\mathbb{G}^{(h^+)}$-analytic vectors of V to be

$$V^{\mathbb{G}^{(h^+)}-an} := \varprojlim_{h'>h} V^{\mathbb{G}^{(h')}-an}.$$

2. Given a complex $C \in D(K_\blacksquare[G])$ we define the derived $\mathbb{G}^{(h)}$-analytic vectors of C as the complex in $D(K_\blacksquare[G])$

$$C^{R\mathbb{G}^{(h)}-an} := R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, (C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes_{K_\blacksquare}{}^L C)_{\star_{1,3}})$$

where the action of G is induced by the $\star_2$-action. Similarly, we define the derived $\mathbb{G}^{(h^+)}$-analytic vectors of C to be

$$C^{R\mathbb{G}^{(h^+)}-an} := R\varprojlim_{h'>h} C^{R\mathbb{G}^{(h')}-an}.$$

Remark 14.3.4. As $C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes_{K_\blacksquare} -$ is exact, $C \mapsto C^{R\mathbb{G}^{(h)}-an}$ is the right derived functor of $V \mapsto V^{\mathbb{G}^{(h)}-an}$. Similarly, $C \mapsto C^{R\mathbb{G}^{(h^+)}-an}$ is the right derived functor of $V \mapsto V^{\mathbb{G}^{(h^+)}-an}$.

Lemma 14.3.5. *Let $h \geq 0$ and $C \in D(K_\blacksquare[G])$. There is a natural morphism of objects in $D(K_\blacksquare[G])$*

$$C^{R\mathbb{G}^{(h)}-an} \to C.$$

Proof. Notice that we have a natural map

$$C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes_K^L C \to \underline{\mathrm{Cont}}(G, C)$$

which commutes with the three actions of G. Taking $\star_{1,3}$-invariant one gets the lemma by Proposition 14.2.6. □

Definition 14.3.6. Let $h \geq 0$

1. A solid $K_\blacksquare[G]$-module V is called $\mathbb{G}^{(h)}$-analytic if the natural map $V^{\mathbb{G}^{(h)}-an} \to V$ is an isomorphism. Similarly, it is called $\mathbb{G}^{(h^+)}$-analytic if $V^{\mathbb{G}^{(h^+)}-an} \to V$ is an isomorphism.

2. A complex $C \in D(K_\blacksquare[G])$ is called derived $\mathbb{G}^{(h)}$-analytic if the natural map $C^{R\mathbb{G}^{(h)}-an} \to C$ is a quasi-isomorphism. Similarly, it is derived $\mathbb{G}^{(h^+)}$-analytic if the map $C^{R\mathbb{G}^{(h^+)}-an} \to C$ is a quasi-isomorphism.

So far we have introduced two definitions of analytic vectors depending on whether we choose the radius to be closed or open. It turns out that to actually have a theory in terms of distribution algebras, we need to work with the Stein analytic groups $\mathbb{G}^{(h^+)}$. We will need the following Lemma (to be proved in §15.3, see Corollary 15.3.5).

Lemma 14.3.7. *One has $\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{(h^+)}(G, K) = \mathcal{D}^{(h^+)}(G, K)$.*

An immediate consequence of the previous result is the following fully-faithfulness property.

Corollary 14.3.8. *The category* $\mathrm{Solid}(\mathcal{D}^{(h^+)}(G, K))$ *(resp. $D(\mathcal{D}^{(h^+)}(G, K))$) is a full subcategory of* $\mathrm{Solid}(K_\blacksquare[G])$ *(resp. $D(K_\blacksquare[G])$). In other words, if $V, V' \in \mathrm{Solid}(\mathcal{D}^{(h^+)}(G, K))$ and $C, C' \in D(\mathcal{D}^{(h^+)}(G, K))$ then*

$$\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(V, V') = \underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(V, V')$$
$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(C, C') = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(C, C').$$

Proof. It follows from the usual extension of scalars:

$$\begin{aligned}
R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(C, C') &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} C, C') \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} (\mathcal{D}^{(h^+)}(G, K) \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C), C') \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}((\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{(h^+)}(G, K)) \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C, C') \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G, K) \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C, C') \\
&= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(C, C').
\end{aligned}$$

□

We can now state the main theorem of this section.

Theorem 14.3.9. *Let $W \in D(K_\blacksquare)$ and $C \in D(K_\blacksquare[G])$. The following holds.*

1. *There are a natural isomorphisms of $K_\blacksquare[G]$-modules*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h)}(G,K)\otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h)}-an})$$
$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h^+)}-an})$$

   The $K_\blacksquare[G]$-module structure of the terms inside the $R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}$ in the LHS are the left multiplication on the distribution algebras and the action of C. The G-action of the LHS $R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(-,-)$ is induced by the right multiplication on the distribution algebras.

2. *The category of $\mathbb{G}^{(h^+)}$-analytic representations of G is equal to* $\mathrm{Solid}(\mathcal{D}^{(h^+)}(G,K))$. *In other words, a $K_\blacksquare[G]$-module V is $\mathbb{G}^{(h^+)}$-analytic if and only if the action of $K_\blacksquare[G]$ extends to an action of $\mathcal{D}^{(h^+)}(G,K)$.*

3. *Furthermore, a complex $C \in D(K_\blacksquare[G])$ is derived $\mathbb{G}^{(h^+)}$-analytic if and only if for all $n \in \mathbb{Z}$ the cohomology groups $H^n(C)$ are $\mathbb{G}^{(h^+)}$-analytic. Equivalently, C is derived $\mathbb{G}^{(h^+)}$-analytic if and only if it belongs to the essential image of $D(\mathcal{D}^{(h^+)}(G,K))$.*

Proof. (1) Let $W \in D(K_\blacksquare)$ and $C \in D(K_\blacksquare[G])$, by Proposition 12.0.11 there is a natural quasi-isomorphism

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h)}(G,K)\otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C(\mathbb{G}^{(h)},K)_\blacksquare \otimes^L_K C). \tag{14.3.3}$$

It is easy to verify that the left and right regular actions of the RHS are translated in the left and right multiplication of the distributions in the LHS. Indeed, one can reduce to $W = \prod_i \mathcal{O}_K[\frac{1}{p}]$ and $C = K_\blacksquare[G]$ in which case it is straightforward. Then, the $\star_{1,3}$-action in the RHS translates in the left multiplication on the distributions and the action on C in the LHS. Taking $R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K,-)$ in (14.3.3) one gets

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h)}(G,K)\otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h)}-an}).$$

Taking derived inverse limits and using that $R\underline{\mathrm{Hom}}$ commutes with colimits in the first factor and limits in the second factor, one gets

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K)\otimes^L_{K_\blacksquare} W, C) = R\underline{\mathrm{Hom}}_K(W, C^{R\mathbb{G}^{(h^+)}-an}).$$

(2) Consider the pre-analytic ring $(K_\blacksquare[G], \mathcal{M}^{(h^+)})$ such that for any extremally disconnected S one has

$$\mathcal{M}^{(h^+)}(S) = \mathcal{D}^{(h^+)}(G,K)\otimes_{K_\blacksquare} K_\blacksquare[S].$$

Corollary 14.3.8 implies that it is in fact an analytic ring. Indeed, let $P^\bullet$ be a complex of $K_\blacksquare[G]$-modules concentrated in positive homological degrees whose terms are direct sums of $\mathcal{M}^{(h^+)}[S_i]$ for $\{S_i\}_{i\in I}$ a family of profinite sets. Let S be a profinite set, then

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{M}^{(h^+)}[S], P^\bullet) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{M}^{(h^+)}[S], P^\bullet) \\ &= R\underline{\mathrm{Hom}}_K(K_\blacksquare[S], P^\bullet). \end{aligned}$$

Moreover, the category of solid $(K_\blacksquare[G], \mathcal{M}^{(h^+)})$-modules is equal to the category of solid $\mathcal{D}^{(h^+)}(G,K)$-modules. More precisely, by Theorem 12.0.7, a family of compact projective generators of $\mathrm{Solid}(K_\blacksquare[G], \mathcal{M}^{(h^+)})$ is given by $\mathcal{M}^{(h^+)}(S)$ for S extremally disconnected, which are naturally $\mathcal{D}^{(h^+)}(G,K)$-modules, and any $K_\blacksquare[G]$-linear map between these objects is automatically $\mathcal{D}^{(h^+)}(G,K)$-linear again by Corollary 14.3.8 again. Hence, part (1) and Theorem 12.0.7 imply that a $K_\blacksquare[G]$-module V is $\mathbb{G}^{(h^+)}$-analytic if and only if it is a $\mathcal{D}^{(h^+)}(G,K)$-module.

(3) Theorem 12.0.7 and the same argument as before tells us that a complex $C \in D(K_\blacksquare[G])$ is derived $\mathbb{G}^{(h^+)}$-analytic if and only if it belongs to the essential image of $D(\mathcal{D}^{(h^+)}(G,K))$, if and only if for all $n \in \mathbb{Z}$ the module $H^n(C)$ is a $\mathcal{D}^{(h^+)}(G,K)$-module, finishing the proof. □

14.4. Locally analytic representations

We finish this section with some applications of Theorem 14.3.9 to the theory of locally analytic representations. Let us begin with the definition of the locally analytic vectors.

Definition 14.4.1.

1. Let V be a solid $K_\blacksquare[G]$-module, the space of locally analytic vectors of V is the solid $K_\blacksquare[G]$-module
$$V^{la} := \varinjlim_{h\to\infty} V^{\mathbb{G}^{(h)}-an} = \varinjlim_{h\to\infty} V^{\mathbb{G}^{(h^+)}-an}.$$
We say that V is locally analytic if the natural map $V^{la} \to V$ is an isomorphism.

2. Let $C \in D(K_\blacksquare[G])$, the derived locally analytic vectors of C is the complex
$$C^{Rla} := \operatorname*{hocolim}_{h\to\infty} C^{R\mathbb{G}^{(h)}-an} = \operatorname*{hocolim}_{h\to\infty} C^{R\mathbb{G}^{(h^+)}-an}.$$
We say that C is derived locally analytic if the natural map $C^{Rla} \to C$ is a quasi-isomorphism.

In §14.1.3 we defined the algebra of locally analytic distributions of G as the Fréchet algebra of compact type
$$\mathcal{D}^{la}(G,K) = \varprojlim_{h\to\infty} \mathcal{D}^{(h)}(G,K) = \varprojlim_{h\to\infty} \mathcal{D}^{(h^+)}(G,K).$$
Since a locally analytic representation V is a (homotopic) colimit of $\mathbb{G}^{(h^+)}$−analytic representations, Theorem 14.3.9 implies that V is naturally a $\mathcal{D}^{la}(G,K)$-module. Furthermore, this structure is unique as $\mathcal{D}^{la}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{la}(G,K) = \mathcal{D}^{la}(G,K)$, see Corollary 15.3.5. Nevertheless, not all the $\mathcal{D}^{la}(G,K)$-modules are locally analytic representations of G, e.g. $\mathcal{D}^{la}(G,K)$ is not a locally analytic representation as it cannot be written as a colimit of $\mathcal{D}^{(h^+)}(G,K)$-modules. Indeed, if $\mathcal{D}^{la}(G,K)$ was locally analytic then $1 \in \mathcal{D}^{la}(G,K)$ would be analytic for certain group $\mathbb{G}^{(h^+)}$, this would provide a section of the map $\mathcal{D}^{la}(G,K) \to \mathcal{D}^{(h^+)}(G,K)$ which is a contradiction as it is never surjective. In the following propositions we will try to give some conditions for a $\mathcal{D}^{la}(G,K)$-module to be a locally analytic representation of G.

Proposition 14.4.2. *Let V be a Banach $K_\blacksquare[G]$-module. The following are equivalent.*

1. *the $K_\blacksquare[G]$-module structure of V extends to $\mathcal{D}^{la}(G,K)$.*
2. *the $K_\blacksquare[G]$-module structure of $V^\vee$ extends to $\mathcal{D}^{la}(G,K)$.*
3. *V is $\mathbb{G}^{(h^+)}$-analytic for some $h \geq 0$.*

Proof. Formally, (1) and (2) are equivalent as the dual of a solid $\mathcal{D}^{la}(G,K)$ module is naturally a solid $\mathcal{D}^{la}(G,K)$-module, (3) implies (1) is clear from the previous discussion. Let us show that (1) implies (3). Suppose that V is a $\mathcal{D}^{la}(G,K)$-module. Consider the multiplication map $m_V : \mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V \to V$. As V is Banach and $\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V$ is a Fréchet space, Lemma 13.3.7 implies that there exists $h \geq 0$ such that m_V factors as
$$\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V \to \mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare} V \to V.$$
It is immediate to check that this endows V with an structure of $\mathcal{D}^{(h)}(G,K)$-module. By Theorem 14.3.9 V is $\mathbb{G}^{(h'^+)}$-analytic for $h' > h$. □

Proposition 14.4.3 (Schneider-Teitelbaum)**.** *Let V be an LS space of compact type, cf. Definition 13.3.10. The following are equivalent*

1. *the $K_\blacksquare[G]$-module structure of V extends to $\mathcal{D}^{la}(G,K)$-module.*
2. *the $K_\blacksquare[G]$-module structure of $V^\vee$ extends to $\mathcal{D}^{la}(G,K)$-module.*
3. *V is a locally analytic representation of G.*

Proof. By Theorem 13.3.13, (1) and (2) are equivalent. It is also clear that (3) implies (1). Suppose that V is a $\mathcal{D}^{la}(G,K)$-module which is an LS space of compact type. The multiplication map $m_V : \mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V \to V$ gives an element of $\mathrm{Hom}(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V, V)$. Let $V = \varinjlim_n V_n$ be a presentation as a colimit of Smith spaces by injective transition maps, and let V_n^B be the underlying Banach space of V_n (cf. Definition 13.3.10). As V is of compact type we have $V = \varinjlim_n V_n^B$. Therefore

$$\begin{aligned}
\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V, V) &= \varprojlim_n \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V_n^B, V)\\
&= \varprojlim_n \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K), (V_n^B)^\vee \otimes_{K_\blacksquare} V)\\
&= \varprojlim_n C^{la}(G,K)\otimes_{K_\blacksquare} (V_n^B)^\vee \otimes_{K_\blacksquare} V\\
&= \varprojlim_n \varinjlim_m C^{la}(G,K)\otimes_{K_\blacksquare} (V_n^B)^\vee \otimes_{K_\blacksquare} V_m^B,
\end{aligned}$$

where the first equality is formal, the second follows from the fact that V is nuclear, and the third equality follows from Theorem 13.3.13. This shows that given $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that the map $m_V : \mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V_n^B \to V$ factors through $m_V : \mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V_n^B \to V_m^B$. By lemma 13.3.7 there exists $h \geq 0$ such that m_V factors as $\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V_n^B \to \mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare} V_n^B \to V_m^B$. Equivalently, there is $h \geq 0$ (maybe different) such that m_V factors as $\mathcal{D}^{la}(G,K)\otimes_{K_\blacksquare} V_n \to \mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare} V_n \to V_m$. Let V_n' be the image of $\mathcal{D}^{(h)}(G,K)\otimes_{K_\blacksquare} V_n \to V_m$, it is a Smith space endowed with an action of $\mathcal{D}^{(h)}(G,K)$ extending the one of $\mathcal{D}^{la}(G,K)$. It is immediate to see that $V = \varinjlim_n V_n'$, this shows that V is written as a colimit of $\mathcal{D}^{(h'^+)}(G,K)$-modules (for $h' > h$), which implies that it is a locally analytic representation of G by Theorem 14.3.9. □

One may wonder whether the previous propositions can be extended to the bounded derived category. One way to attack the problem is to use the derived enhancement of Lemma 13.3.7, which we do not know whether is true or not

Conjecture 1. *Let $V = \varprojlim_n V_n$ be a Fréchet space of compact type written as a limit of Banach spaces with compact and dense transition maps, then*

$$R\underline{\mathrm{Hom}}_K(V,K) = \operatorname{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n,K)$$

Corollary 14.4.4. *Suppose that Conjecture 1 is true. Let $V = \varprojlim_n V_n$ be a Fréchet space of compact type and B a Banach space, then*

1. *$R\underline{\mathrm{Hom}}_K(V,B) = \operatorname{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n,B)$.*
2. *We have*
$$R\underline{\mathrm{Hom}}_K(V,B) = V^\vee \otimes_{K_\blacksquare} B.$$
In particular $V^\vee$ is the derived dual of V.
3. *The duality Theorem 13.3.13 gives a derived duality between bounded complexes of Fréchet spaces of compact type and bounded complexes of LS spaces of compact type.*

4. *Let C be a bounded complex of Fréchet spaces of compact type and D a bounded complex of LS spaces of compact type. Then*

$$R\underline{\mathrm{Hom}}_K(C,D) = R\underline{\mathrm{Hom}}_K(C,K)\otimes^L_{K_\blacksquare} D$$

and

$$R\underline{\mathrm{Hom}}_K(D,C) = R\underline{\mathrm{Hom}}_K(D,K)\otimes^L_{K_\blacksquare} C.$$

5. *Let C and C' be bounded complexes of Fréchet spaces of compact type. Then*

$$R\underline{\mathrm{Hom}}_K(C,C') = R\underline{\mathrm{Hom}}_K(C'^\vee, C^\vee).$$

Proof. Part (1) follows from Conjecture 1 using the fact that $K_\blacksquare[S]\otimes^L_{K_\blacksquare} K_\blacksquare[S'] = K_\blacksquare[S\times S']$ for S and S' profinite sets, that

$$R\underline{\mathrm{Hom}}_K(V,K)(S) = R\mathrm{Hom}(V\otimes_{K_\blacksquare} K_\blacksquare[S],K) = R\mathrm{Hom}(V,\underline{\mathrm{Cont}}(S,K))$$

and that any Banach space over K is a direct sumand of a space of the form $\underline{\mathrm{Cont}}(S,K)$ for S a profinite set.

If V is Fréchet of compact type, we can write $V = \varprojlim_n V_n^S$ where V_n^S is the Smith completion of V_n (cf. Corollary 13.3.12). Then

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(V,B) &= \operatorname*{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n,B)\\ &= \operatorname*{hocolim}_n R\underline{\mathrm{Hom}}_K(V_n^S,B)\\ &= \operatorname*{hocolim}_n (V_n^S)^\vee\otimes_{K_\blacksquare} B\\ &= \varinjlim_n V_n^\vee\otimes_{K_\blacksquare} B = V^\vee\otimes_{K_\blacksquare} B. \end{aligned}$$

Part (3) follows immediately from part (2) and an easy induction via the stupid truncation. For part (4), notice that if C is a bounded Fréchet complex of compact type, and D is a bounded LS complex of compact type, then

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(C,D) &= R\underline{\mathrm{Hom}}_K(C\otimes^L_{K_\blacksquare} D^\vee,K)\\ &= C^\vee\otimes^L_{K_\blacksquare} D \end{aligned}$$

since $C\otimes^L_{K_\blacksquare} D^\vee$ is a bounded Fréchet compact type complex. Similarly one shows $R\underline{\mathrm{Hom}}_K(D,C) = D^\vee\otimes^L_K C$.

Finally, by a devisage using the stupid filtration, (5) is reduced to showing that if V and V' are Fréchet spaces of compact type then

$$R\underline{\mathrm{Hom}}_K(V,V') = R\underline{\mathrm{Hom}}_K(V'^\vee,V^\vee).$$

Writing $V' = \varprojlim_n V'_n$ one gets, by (1),

$$R\underline{\mathrm{Hom}}_K(V,V') = R\varprojlim_n R\underline{\mathrm{Hom}}_K(V,V'_n) = R\varprojlim_n(V^\vee\otimes_{K_\blacksquare} V'_n).$$

Dually, we have that

$$\begin{aligned} R\underline{\mathrm{Hom}}_K(V'^\vee,V^\vee) &= R\varprojlim_n R\underline{\mathrm{Hom}}_K(V'^{,\vee}_n,V^\vee)\\ &= R\varprojlim_n(V'_n\otimes_{K_\blacksquare} V^\vee) \end{aligned}$$

which gives the desired equality. □

Under the assumption of the previous Conjecture we could improve Propositions 14.4.2 and 14.4.3 to bounded derived complexes:

Proposition 14.4.5. *Suppose that Conjecture 1 holds. Let $C \in D(\mathcal{D}^{la}(G,K)_\blacksquare)^b$ be a bounded solid $\mathcal{D}^{la}(G,K)$-module. Assume that either*

(a) C is quasi-isomorphic to a bounded complex of K-Banach spaces as $K_\blacksquare$-complex.

(b) C is quasi-isomorphic to a bounded complex of LS spaces of compact type as $K_\blacksquare$-complex.

Then C is a derived locally analytic representation of G. In particular there is a derived duality between locally analytic complexes quasi-isomorphic to bounded complexes of LS spaces of compact type, and $\mathcal{D}^{la}(G,K)$-complexes quasi-isomorphic to bounded complexes of Frechét spaces of compact type.

Proof. We will prove in Corollary 15.3.5 that

$$\mathcal{D}^{la}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{la}(G,K) = \mathcal{D}^{la}(G,K). \tag{14.4.1}$$

Let C be a complex in $D(\mathcal{D}^{la}(G,K))$ which is quasi-isomorphic to a bounded complex of Banach spaces as $K_\blacksquare$-complex. By Corollary 14.4.4 (1) we have

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K), C) = \operatorname*{hocolim}_{h\to\infty} R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G,K), C).$$

Taking $K_\blacksquare[G]$-invariants one gets that

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{la}(G,K), C) = \operatorname*{hocolim}_{h\to\infty} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G,K), C).$$

But (14.4.1) implies that the LHS is equal to C while Theorem 14.3.9 implies that the RHS is equal to $\operatorname{hocolim}_h C^{R\mathbb{G}^{(h^+)}}$. This shows that C is derived locally analytic.

Now suppose that C is quasi-isomorphic to a bounded complex of *LS* spaces of compact type. Then from Corollary 14.4.4 (3) and (4) one has

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K), C) = C^{la}(G,K) \otimes^L_{K_\blacksquare} C = \operatorname*{hocolim}_{h\to\infty} C(\mathbb{G}^{(h)}, K)_\blacksquare \otimes^L_{K_\blacksquare} C.$$

Taking $K_\blacksquare[G]$-invariants one gets again by (14.4.1) that $C = C^{Rla}$, i.e. that C is a derived locally analytic representation of G. □

Remark 14.4.6. In the situation (a) of the previous proposition, one can show in addition that C is derived $\mathbb{G}^{(h^+)}$-analytic for some $h > 0$. Indeed, it is enough to prove that for all $n \in \mathbb{Z}$ there is $h > 0$ such that $H^n(C)$ is derived $\mathbb{G}^{(h^+)}$-analytic. As C is bounded, we are left to show that if V is a $\mathcal{D}^{la}(G,K)$-module which is a quotient of two K-Banach spaces (not necessarily $\mathcal{D}^{la}(G,K)$-modules), then V is already a $\mathcal{D}^{(h^+)}(G,K)$-module for some $h > 0$. Let $m_V : \mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare} V \to V$ be the multiplication map, by Corollary 14.4.4 (4) one has that

$$R\underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G,K) \otimes_{K_\blacksquare} V, V) = \operatorname*{hocolim}_{h\to\infty} R\underline{\mathrm{Hom}}_K(\mathcal{D}(\mathbb{G}^{(h)}, K) \otimes_{K_\blacksquare} V, V).$$

Thus, m_V factors as $m_V : \mathcal{D}(\mathbb{G}^{(h)}, K) \otimes_{K_\blacksquare} V \to V$ for some $h > 0$. After taking $h' > h$ one shows that the map $\mathcal{D}(\mathbb{G}^{(h'^+)}, K) \otimes_{K_\blacksquare} V \to V$ is in fact an action of $\mathcal{D}(\mathbb{G}^{(h'^+)}, K)$, proving that V is derived $\mathbb{G}^{(h^+)}$-analytic as desired.

14.5. Admissible representations

Before writing down the statements of the cohomological comparison results, let us show how the theory developed till now together with some algebraic facts about the Iwasawa and the distribution algebras, provide a description of the locally analytic vectors of an admissible representation in terms of its dual. All results in this sections were already known ([ST03], [Pan20]).

Definition 14.5.1. A Banach representation V of G is admissible if its dual is a finite module over the Iwasawa algebra. Equivalently, if V admits a closed immersion into a finite direct sum of $\underline{\mathrm{Cont}}(G, K)$.

We recall the following important results of the classical Iwasawa and distribution algebras

Theorem 14.5.2 (Lazard, Schneider-Teitelbaum)**.** *The following holds*

1. *The (classical) Iwasawa algebra $K_\blacksquare[G](*)$ is a coherent algebra.*
2. *The (classical) distribution algebras $\mathcal{D}^{(h^+)}(G, K)(*)$ are flat over $\mathcal{O}_{K,\blacksquare}[G](*)$ algebraically.*
3. *Let $h' > h$, then $\mathcal{D}^{(h'^+)}(G, K)(*)$ is a flat algebra over $\mathcal{D}^{(h^+)}(G, K)(*)$ algebraically.*
4. *The (classical) locally analytic distribution algebra $\mathcal{D}^{la}(G, K)(*)$ is faithfully flat over $K_\blacksquare[G]$ algebraically.*

Proof. Part (1) is [ST03, Rem. 4.6], part (2) is [ST03, Prop. 4.7], part (3) is [ST03, Theo. 4.9] and part (4) is [ST03, Theo. 4.11]. □

Proposition 14.5.3. *Let V be a Banach G-representation, then*

$$V^{R\mathbb{G}^{(h^+)}-an} = R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} V^\vee, K).$$

In particular, if V is admissible then

$$V^{R\mathbb{G}^{(h^+)}-an} = V^{\mathbb{G}^{(h^+)}-an} = \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G, K) \otimes_{K_\blacksquare[G]} V^\vee, K).$$

Furthermore, $V^{la} = \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G, K) \otimes_{K_\blacksquare[G]} V^\vee, K)$.

Proof. By Theorem 14.3.9 we have

$$\begin{aligned} V^{R\mathbb{G}^{(h^+)}-an} &= R\underline{\mathrm{Hom}}_{K[G]}(\mathcal{D}^{(h^+)}(G, K), V) \\ &= R\underline{\mathrm{Hom}}_{K[G]}(\mathcal{D}^{(h^+)}(G, K), R\underline{\mathrm{Hom}}_K(V^\vee, K)) \\ &= R\underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} V^\vee, K). \end{aligned}$$

Moreover, if V is admissible then $V^\vee$ is a finite $K_\blacksquare[G]$-module. By flatness of the distribution algebra one gets that $\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} V^\vee = \mathcal{D}^{(h^+)}(G, K) \otimes_{K_\blacksquare[G]} V^\vee$ is concentrated in degree 0 (we warn that the flatness is only algebraic, and that we use the fact that $V^\vee$ is a finite module over the Iwasawa algebra), this implies the second claim. For the last statement, one gets by Lemma 13.3.7 that

$$\begin{aligned} V^{la} &= \varinjlim_{h\to\infty} V^{\mathbb{G}^{(h^+)}-an} \\ &= \varinjlim_{h\to\infty} \underline{\mathrm{Hom}}_K(\mathcal{D}^{(h^+)}(G, K) \otimes_{K_\blacksquare[G]} V^\vee, K) \\ &= \underline{\mathrm{Hom}}_K(\mathcal{D}^{la}(G, K) \otimes_{K_\blacksquare[G]} V^\vee, K). \end{aligned}$$

□

Corollary 14.5.4. *Let V be an admissible Banach G-representation, then $V^{la} \subset V$ is a dense subspace.*

Proof. As the category of admissible representations is an abelian category, and a closed subrepresentation of an admissible representation is admissible, one is left to prove that the functor $V \mapsto V^{la}$ is non zero. This follows from Proposition 14.5.3 and the fact that $\mathcal{D}^{la}(G, K)$ is faithfully flat over $K_\blacksquare[G]$. □

15. Cohomology

In this last chapter we present our main applications to group cohomology. We obtain in particular

1. An isomorphism between the continuous cohomology and the cohomology of the derived locally analytic vectors for solid representations. This can be seen as a p-adic version of theorems of P. Blanc and G. D. Mostow for real Lie groups, cf. [Bla79, Mos61].

2. A comparison theorem, for a locally analytic G-representation V, between its continuous and its locally analytic cohomology. This generalises the classical result of Lazard [Laz65] for finite dimensional $\mathbb{Q}_p$-representations to arbitrary solid K-vector spaces.

3. A comparison theorem between locally analytic cohomology and Lie algebra cohomology. This recovers and generalises a result of Tamme [Tam15] (cf. also [HKN11] and [Lec12]), which in turns was a generalisation of the other main result of Lazard for finite dimensional representations.

The proof of (1) is an immediate consequence of our main Theorem 14.3.9 and a result of Kohlhaase (Theorem 15.3.2).

The key input for the comparison result in (2) is the existence of a finite free resolution of the trivial representation, as a module over the Iwasawa algebra of a small neighbourhood of 1 in G. This is an application of a lemma of Serre used by Lazard in [Laz65, V Def. 2.2.2]. We shall also need a version of the lemma proved by Koohlhase for distribution algebras, cf. [Koh11, Theo. 4.4]. Once one has this lemma at hand, the proof of (2) is rather formal using the machinery developed throughout this text.

Finally, to show (3) we follow the proof of Tamme constructing a resolution of the trivial representation in terms of the de Rham complex of the analytic groups. Then, we apply the same formal computation as before.

15.1. Continuous, analytic and locally analytic cohomology

In this section we define different cohomology groups, they correspond to continuous, analytic and locally analytic cohomology in the literature. Indeed, using the Bar resolutions and Proposition 12.0.11, one verifies that whenever V is a solid representation coming from a "classical space" (i.e. a Banach, Fréchet, LB or LF space) our definitions coincide with the usual ones.

In the following we will use the conventions of Section 14.1. In particular, we fix a compact p-adic Lie group G and an open normal uniform pro-p-group $G_0 \subset G$, we denote the h-analytic neighbourhood of G as $\mathbb{G}^{(h)}$, and define its open h-analytic neighbourhood as $\mathbb{G}^{(h^+)} = \bigcup_{h'>h} \mathbb{G}^{(h')}$. Recall from Lemma 14.2.1 that a solid $\mathcal{O}_K$-module V is an $\mathcal{O}_K$-linear G-representation if and only if it is a module over the Iwasawa algebra $\mathcal{O}_{K,\blacksquare}[G]$. If V is in addition a $\mathbb{G}^{(h^+)}$-analytic representation, by the main Theorem 14.3.9, V is naturally equipped with a $\mathcal{D}^{(h^+)}(G,K)$-module structure.

Definition 15.1.1.

1. Let $C \in D(\mathcal{O}_{K,\blacksquare}[G])$. We define the continuous group cohomology of C as $R\underline{\mathrm{Hom}}_{\mathcal{O}_{K,\blacksquare}[G]}(\mathcal{O}_K, C)$.

2. Let $C \in D(\mathcal{O}_{K,\blacksquare}[G])$ be a derived $\mathbb{G}^{(h^+)}$-analytic representation. We define the $\mathbb{G}^{(h^+)}$-analytic cohomology of C as $R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C)$.

3. Let $C \in D(\mathcal{O}_{K,\blacksquare}[G])$ be a derived locally analytic representation. We define the locally analytic cohomology of C as $\operatorname*{hocolim}_{h\to\infty} R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C)$.

15.2. Comparison results

Next we state the main theorems of this section, which will be proved in §15.4.

Theorem 15.2.1. *Let $C \in D(K_\blacksquare[G])$, then*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C^{Rla}).$$

Remark 15.2.2. More concretely, we will show that

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C^{R\mathbb{G}^{(h^+)-an}}).$$

for $h >> 0$.

Theorem 15.2.3 (Continuous vs. analytic vs. Lie algebra cohomology)**.** *Let $C \in D(K_\blacksquare[G])$ be a derived $\mathbb{G}^{(h^+)}$-analytic representation. Then*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) \cong R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) \cong (R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C))^G.$$

Remark 15.2.4. The RHS term of the equation above means the following: if C is a derived $\mathbb{G}^{(h^+)}$-analytic complex, then there is an open normal subgroup $H \subset G$ such that

$$R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C),$$

and the group G/H acts on the previous cohomology complex. As we are working in characteristic 0 and G/H is a finite group, taking invariants in the category of solid $K[G/H]$-modules is exact and one can form the complex

$$R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)^G := R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)^{G/H}.$$

15.3. Key lemmas

In the following we will work with complexes with equal terms but different differential maps. To make explicit the differentials we use the following notation: let C be a (homological) complex of $\mathcal{O}_{K,\blacksquare}$-modules with i-th term C_i and i-th differential $d_i : C_i \to C_{i-1}$, we note

$$C = [\cdots \to C_{i+1} \to C_i \to C_{i-1} \to \cdots ; d_\bullet].$$

15.3.1. Iwasawa and distribution algebras

The following result is the main input for our calculations.

Theorem 15.3.1 (Lazard-Serre)**.** *Let G_0 a uniform pro-p group of dimension d. Then there exists a projective resolution of the trivial module $\mathbb{Z}_p$ of the form*

$$P := [0 \to \mathbb{Z}_{p,\blacksquare}[G_0]^{\binom{d}{d}} \to \cdots \to \mathbb{Z}_{p,\blacksquare}[G_0]^{\binom{d}{i}} \to \cdots \to \mathbb{Z}_{p,\blacksquare}[G_0]^{\binom{d}{0}}; \alpha_\bullet].$$

Proof. We briefly sketch how the complex P is constructed from [Laz65, Définition 2.2.2.1, Lemme 2.1.1]. Let $g_1, \dots, g_d \in G_0$ be a basis of the group and $b_i = [g_i] - 1 \in \mathbb{Z}_{p,\blacksquare}[G_0](*)$. The valuation of G_0 defines a filtration in $\mathbb{Z}_{p,\blacksquare}[G_0](*)$ whose graded algebra $\mathrm{gr}^\bullet(\mathbb{Z}_{p,\blacksquare}[G_0](*))$ is isomorphic to $\mathbb{F}_p[\pi][\overline{b}_1, \dots, \overline{b}_d]$, where $\mathbb{F}[\pi] = \mathrm{gr}^\bullet(\mathbb{Z}_p)$ is the graduation of $\mathbb{Z}_p$ for the filtration induced by (p). Then, the Koszul complex of $\mathbb{F}[p][\overline{b}_1, \dots, \overline{b}_d]$ with respect to the regular sequence $(\overline{b}_1, \dots, \overline{b}_d)$ can be lifted by approximations to the complex P of the theorem. Furthermore, the proof also lifts a chain homotopy $\tilde{s}_\bullet$ between the identity and the augmentation map $\epsilon : K[\overline{b}_1, \dots, \overline{b}_d] \to \mathbb{F}_p[\pi]$, to a chain homotopy $s_\bullet$ between the identity and the augmentation map $\epsilon : \mathbb{Z}_{p,\blacksquare}[G_0] \to \mathbb{Z}_p$. □

Theorem 15.3.2 (Kohlhaase)**.** *Let G_0 a uniform pro-p group of dimension d and $h > 0$. Then*

$$\mathcal{O}_K \otimes^L_{\mathcal{O}_{K,\blacksquare}[G_0]} \mathcal{D}_{(h)}(G_0, K) = K.$$

More precisely, the differentials $\alpha_i : \mathcal{O}_{K,\blacksquare}[G_0]^{\binom{d}{i}} \to \mathcal{O}_{K,\blacksquare}[G_0]^{\binom{d}{i-1}}$ of the resolution given by Theorem 15.3.1 extend to maps $\alpha_i : \mathcal{D}_{(h)}(G_0, K)^{\binom{d}{i}} \to \mathcal{D}_{(h)}(G_0, K)^{\binom{d}{i-1}}$, inducing a resolution of the trivial module K of the form

$$P_{(h)} := [0 \to \mathcal{D}_{(h)}(G_0, K)^{\binom{d}{d}} \to \cdots \to \mathcal{D}_{(h)}(G_0, K)^{\binom{d}{i}} \to \cdots \to \mathcal{D}_{(h)}(G_0, K)^{\binom{d}{0}}; \alpha_\bullet].$$

Proof. This is essentially [Koh11, Theorem 4.4]. Let $g_1, \dots, g_d \in G_0$ be a basis and $b_i \in \mathbb{Z}_{p,\blacksquare}[G_0](*)$. The idea of the proof is to show that the differentials $\alpha_\bullet$ and the chain homotopy $s_\bullet$ of Theorem 15.3.1 are continuous with respect to the norms $|\sum_\alpha a_\alpha \mathbf{b}^\alpha|_r = \sup_\alpha |a_\alpha| r^{|\alpha|}$ for $\frac{1}{p} < r < 1$. Thus, the differentials $\alpha_\bullet$ and the chain homotopy $s_\bullet$ extend to the weak completion of these norms (i.e. the completion with respect to a radius r seen as a subspace in the completion of a slightly bigger radius $r < r' < 1$). Note that the distribution algebras constructed in this way are precisely the algebras $\mathcal{D}_{(h)}(G_0, K)$ of Definition 14.1.12. □

Remark 15.3.3. By definition $\mathcal{D}_{(h)}(G, K) := K_\blacksquare[G] \otimes_{K_\blacksquare[G_0]} \mathcal{D}_{(h)}(G_0, K)$. Therefore $\mathcal{D}_{(h)}(G, K) \otimes_{K_\blacksquare[G]} K = \mathcal{D}_{(h)}(G_0, K) \otimes_{K_\blacksquare[G_0]} K = K$, as K is of characteristic 0. This implies that

$$\mathcal{D}_{(h)}(G, K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G, K) = K_\blacksquare[G] \otimes^L_{K_\blacksquare[G_0]} (\mathcal{D}_{(h)}(G_0, K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G_0, K)).$$

With the help of the previous theorem we can compute the following derived tensor product

Proposition 15.3.4. *We have*

$$\mathcal{D}_{(h)}(G, K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G, K) = \mathcal{D}_{(h)}(G, K).$$

Proof. First, we reduce to the case when G is a uniform pro-p-group by Remark 15.3.3. By Theorem 15.3.1 we can write

$$[0 \to K_\blacksquare[G] \to \cdots \to K_\blacksquare[G]^d \to K_\blacksquare[G]; \alpha] \simeq K.$$

Tensoring with $\mathcal{D}_{(h)}(G, K)$ over K we get

$$[0 \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G, K) \to \cdots K_\blacksquare[G]^d \otimes_K \mathcal{D}_{(h)}(G, K) \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G, K); \alpha \otimes 1] \simeq \mathcal{D}_{(h)}(G, K).$$

The quasi-isomorphism above is of $K_\blacksquare[G]$-modules for the diagonal action of G in the terms of the complex.

Let $\iota : K_\blacksquare[G] \to K_\blacksquare[G]$ be the antipode, i.e. the map induced by the inverse of the group, and denote in the same way its extension to the distribution algebra $\mathcal{D}_{(h)}(G, K)$. Consider the composition

$$K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G, K) \xrightarrow{(1\otimes\iota)\otimes 1} K_\blacksquare[G] \otimes_K K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G, K) \xrightarrow{1\otimes m} K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G, K),$$

where m is the left multiplicaiton of $K_\blacksquare[G]$ on the distribution algebra. This map defines a G-equivariant isomorphism

$$\phi : K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K) \cong K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K)_0$$

where the action of G in the image is left multiplication on $K_\blacksquare[G]$ and trivial on $\mathcal{D}_K(G,K)_0$. Notice that ϕ can be extended naturally to a G-equivariant isomorphism

$$\phi : \mathcal{D}_{(h)}(G,K) \otimes_K \mathcal{D}_{(h)}(G,K) \cong \mathcal{D}_{(h)}(G,K) \otimes_K \mathcal{D}_{(h)}(G,K)_0.$$

We define the complex

$$[0 \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K)_0 \to \cdots \to K_\blacksquare[G] \otimes_K \mathcal{D}_{(h)}(G,K)_0; \beta_\bullet] \tag{15.3.1}$$

to be the complex whose differentials are given by $\beta_\bullet = \phi \circ \alpha_\bullet \circ \phi^{-1}$. Notice that the maps β extend to respective complex with terms direct sums of $\mathcal{D}_{(h)}(G,K) \otimes_K \mathcal{D}_{(h)}(G,K)_0$.

Using this complex one can easily compute the derived tensor product by replacing the right $\mathcal{D}_{(h)}(G,K)$ with (15.3.1):

$$\begin{aligned}
\mathcal{D}_{(h)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}_{(h)}(G,K) &\simeq [\cdots \to \mathcal{D}_{(h)}(G,K) \otimes_{K_\blacksquare[G]} (K_\blacksquare[G]^{\binom{d}{i}} \otimes_K \mathcal{D}_{(h)}(G,K)_0) \to \cdots ; 1 \otimes \beta] \\
&= [\ldots \to \mathcal{D}_{(h)}(G,K)^{\binom{d}{i}} \otimes_K \mathcal{D}_{(h)}(G,K)_0 \to \cdots ; \beta] \\
&\simeq [\cdots \to \mathcal{D}_{(h)}(G,K)^{\binom{d}{i}} \otimes_K \mathcal{D}_{(h)}(G,K) \to \cdots ; \alpha \otimes 1] \\
&\simeq \mathcal{D}_{(h)}(G,K),
\end{aligned}$$

In the above sequence of isomorphisms, the first quasi-isomorphism follows from the observation that the action of G on the complex (15.3.1) representing $\mathcal{D}_{(h)}(G,K)$ is trivial on the factor $\mathcal{D}_{(h)}(G,K)_0$. The second step is trivial. The third one follows by applying ϕ^{-1}. The fourth quasi-isomorphism follows from Theorem 15.3.2. This finishes the proof.

□

Corollary 15.3.5. *We have*

$$\begin{aligned}
\mathcal{D}^{(h^+)}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{(h^+)}(G,K) &= \mathcal{D}^{(h^+)}(G,K) \\
\mathcal{D}^{la}(G,K) \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{la}(G,K) &= \mathcal{D}^{la}(G,K).
\end{aligned}$$

Proof. This follows from the previous proposition and the fact that $\mathcal{D}^{(h^+)}(G,K)$ can be written as a colimit of distribution algebras $\mathcal{D}_{(h')}(G,K)$, cf. Corollary 14.1.18. The case of $\mathcal{D}^{la}(G,K)$ follows from the same proof of Proposition 15.3.4 knowing that the complex of Theorem 15.3.2 extends to $\mathcal{D}^{la}(G,K)$. □

15.3.2. Enveloping and distribution algebras

Let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G and $U(\mathfrak{g})$ its enveloping algebra. Let $\mathbb{G}_h$ be a rigid analytic group of Definition 14.1.4) and let $G_h = \mathbb{G}_h(\mathbb{Q}_p)$ be its rational points. Note that $G_h \subset G$ is an open compact subgroup. We denote by $\mathcal{D}(\mathbb{G}_h, K)$ the distribution algebra of $\mathbb{G}_{(h)}$-analytic functions, i.e. the dual of $C(\mathbb{G}_h, K)$. We also denote $\mathcal{D}(\mathbb{G}_{h^+}, K) = \varinjlim_{h'>h} \mathcal{D}(\mathbb{G}_{h'}, K)$, in other words, the distribution algebra of the rigid analytic group defined by the open unit polydisc $\mathbb{G}_{h^+} = \bigcup_{h'>h} \mathbb{G}_{h'}$. We assume that $\mathbb{G}_{h^+}(\mathbb{Q}_p) = G_h$.

Proposition 15.3.6 (Tamme)**.** *Keep the above notation, and let*

$$CE(\mathfrak{g}) := [0 \to U(\mathfrak{g}) \otimes \wedge^d \mathfrak{g} \to \cdots \to U(\mathfrak{g}) \otimes \mathfrak{g} \to U(\mathfrak{g}); d]$$

be the Chevalley-Eilenberg complex resolving the trivial representation K. Then $\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes^L_{U(\mathfrak{g})} K = K$. Moreover, the complex $\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes^L_{U(\mathfrak{g})} CE(\mathfrak{g})$ is the dual of the global sections of the de Rham complex of $\mathbb{G}_{h^+}$.

Proof. Let $[\Omega^\bullet_{\mathbb{G}_{h^+}}, d]$ be the de Rham complex of $\mathbb{G}_{h^+}$, notice that the global sections of $\Omega^i_{\mathbb{G}_{h^+}}$ are equal to $C(\mathbb{G}_{h^+}, K) \otimes_K \bigwedge^i(\mathfrak{g}^\vee)$. As $\mathbb{G}_{h^+}$ is a open polydisc, the Poincaré lemma holds and the global sections of the de Rham complex is

$$C(\mathbb{G}_{h^+}, K) \xrightarrow{d} C(\mathbb{G}_{h^+}, K) \otimes_K \mathfrak{g}^\vee \xrightarrow{d} \cdots \xrightarrow{d} C(\mathbb{G}_{h^+}, K) \otimes_K \bigwedge^d \mathfrak{g}^\vee \to 0, \tag{15.3.2}$$

which is quasi-isomorphic to K via the inclusion of the constant functions $K \subset C(\mathbb{G}_{h^+}, K)$. It is easy to show that the dual of (15.3.2) is equal to $\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes_{U(\mathfrak{g})} CE(\mathfrak{g})$. Finally, the fact that $\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes_{U(\mathfrak{g})} CE(\mathfrak{g})$ is a projective resolution of K as $\mathcal{D}(\mathbb{G}_{h^+}, K)$-module follows from the exactness of (15.3.2) and the duality Theorem 13.3.13. □

Remark 15.3.7. The same proof of Proposition 15.3.4 applies in this situation. In particular, one can show that $\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes^L_{U(\mathfrak{g})} \mathcal{D}(\mathbb{G}_{h^+}, K) = \mathcal{D}(\mathbb{G}_{h^+}, K)$ and that $\mathrm{Solid}(\mathcal{D}(\mathbb{G}_{h^+}, K))$ is a full subcategory of the category of solid $U(\mathfrak{g})$-modules.

15.4. Proofs

Proof of Theorem 15.2.1. Let $C \in D(K_\blacksquare[G])$. By Theorem 14.3.9 we have

$$C^{R\mathbb{G}^{(h^+)}-an} = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G, K), C).$$

We now compute

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C^{R\mathbb{G}^{(h^+)}-an}) &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathcal{D}^{(h^+)}(G, K), C)) \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K \otimes^L_{K_\blacksquare[G]} \mathcal{D}^{(h^+)}(G, K), C) \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C), \end{aligned}$$

where the second equality is the tensor-Hom adjunction, and the third one follows from Theorem 15.3.2. □

Proof of Theorem 15.2.3. Let C be a derived $\mathbb{G}^{(h^+)}$-analytic representation of G. Theorem 14.3.9 says that C is a $\mathcal{D}^{(h^+)}(G, K)$-module. By Theorem 15.3.2 one has

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(\mathcal{D}^{(h^+)}(G, K) \otimes^L_{K_\blacksquare[G]} K, C) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C). \end{aligned}$$

On the other hand, since $\mathcal{D}^{(h^+)}(G, K) = O_{K,\blacksquare}[G] \otimes_{O_{K,\blacksquare}[G_h]} \mathcal{D}(\mathbb{G}_{h^+}, K)$, one has

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{G}_{h^+},K)}(K, C)^{G/G_h}.$$

By Proposition 15.3.6 we get

$$\begin{aligned} R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C) &= R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{G}_{h^+},K)}(\mathcal{D}(\mathbb{G}_{h^+}, K) \otimes^L_{U(\mathfrak{g})} K, C) \\ &= R\underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{G}_{h^+},K)}(K, C). \end{aligned}$$

Putting all together we obtain

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = R\underline{\mathrm{Hom}}_{U(\mathfrak{g})}(K, C)^G$$

as we wanted. □

15.5. Further Applications

We conclude by stating and showing some formal consequences that might be of interest for applications.

15.5.1. Shapiro's lemma and Hochschild-Serre

Let G be a compact p-adic Lie group of dimension d and H a closed subgroup of dimension e. One can find an open uniform pro-p-group $G_0 \subset G$ satisfying the following conditions:

1. $H_0 := H \cap G_0$ is an uniform pro-p-group.
2. There are charts $\phi_{G_0} : \mathbb{Z}_p^d \to G_0$ and $\phi_{H_0} : \mathbb{Z}_p^e \to H_0$ such that $\phi_{G_0} \circ \iota_e = \phi_{H_0}$, where $\iota_e : \mathbb{Z}_p^e \to \mathbb{Z}_p^d$ is the inclusion in the last e-components.

Indeed, taking $\mathfrak{g}_0 \subset \mathfrak{g}$ a small enough lattice as in §5.2 [Eme17] and $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$, one can take $G_0 := \exp(\mathfrak{g}_0)$ and $H_0 := \exp(\mathfrak{h}_0)$. The profinite groups H_0 and G_0 allow us to define compatible rigid analytic neighbourhoods $\mathbb{H}^{(h^+)}$ and $\mathbb{G}^{(h^+)}$ of H and G respectively, with $\mathbb{G}^{(h^+)}/\mathbb{H}^{(h^+)}$ is a finite disjoint union of open polydiscs of dimension $d - e$, and such that

$$C(\mathbb{G}^{(h^+)}, K) = C(\mathbb{G}^{(h^+)}/\mathbb{H}^{(h^+)}, K) \otimes_{K_\blacksquare} C(\mathbb{H}^{(h^+)}, K).$$

In other words, if $\mathcal{D}^{(h^+)}(G/H, K)$ denotes the dual of $C(\mathbb{G}^{(h^+)}/\mathbb{H}^{(h)}, K)$, we have an isomorphism of right $\mathcal{D}^{(h^+)}(H, K)$-modules

$$\mathcal{D}^{(h^+)}(G, K) = \mathcal{D}^{(h^+)}(G/H, K) \otimes_{K_\blacksquare} \mathcal{D}^{(h^+)}(H, K).$$

One has a similar description as left $\mathcal{D}^{(h^+)}(H, K)$-modules.

Definition 15.5.1. For $C \in D(K_\blacksquare[H])$ we define the solid induction and coinduction of C from H to G as

$$\mathrm{ind}_H^G(C) := K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} C,$$

$$\mathrm{coind}_H^G(C) := R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K_\blacksquare[G], C),$$

where the action of G is given by left multiplication on $K_\blacksquare[G]$ for the induction, and by right multiplication on $K_\blacksquare[G]$ for the coinduction. If C is derived $\mathbb{H}^{(h^+)}$-analytic, define the analytic induction and coinduction as

$$h\text{-}\mathrm{ind}_H^G(C) := \mathcal{D}^{(h^+)}(G, K) \otimes^L_{\mathcal{D}^{(h^+)}(H,K)} C,$$

$$h\text{-}\mathrm{coind}_H^G(C) := R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(\mathcal{D}^{(h^+)}(G, K), C).$$

Proposition 15.5.2. *(Shapiro's lemma) Let $C \in D(K_\blacksquare[G])$, $C' \in D(K_\blacksquare[H])$. Then ind_H^G (resp. coind_H^G) is the left (resp. right) adjoint of the restriction map $D(K_\blacksquare[G]) \to D(K_\blacksquare[H])$. In other words,*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(\mathrm{ind}_H^G(C'), C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C', C),$$

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(C, \mathrm{coind}_H^G(C')) = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C, C').$$

Analogously, if C and C' are derived $\mathbb{G}^{(h^+)}$-analytic and $\mathbb{H}^{(h^+)}$-analytic representations, then

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(h\text{-}\mathrm{ind}_H^G(C'), C) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(C', C),$$

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(C, h\text{-}\mathrm{coind}_H^G(C')) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(C, C').$$

Proof. The first statement follows formally:

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} C, C') = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C, R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G], C')) = R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(C, C').$$

The rest of the statements are proved in a similar way. □

Proposition 15.5.3. *(Hochschild-Serre) Let $H \subset G$ be a normal closed subgroup and $C \in D(K_\blacksquare[G])$. Then*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G/H]}(K, R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C)).$$

If C is derived $\mathbb{G}^{(h^+)}$-analytic, then

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G/H,K)}(K, R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(H,K)}(K, C)).$$

Proof. By Shapiro's lemma we have

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} K, C).$$

Applying the functor $R\underline{\mathrm{Hom}}_{K_\blacksquare[G/H]}(K, -)$ to both sides, using $K_\blacksquare[G] \otimes^L_{K_\blacksquare[H]} K = K_\blacksquare[G/H]$ and the usual adjunction one obtains

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G/H]}(K, R\underline{\mathrm{Hom}}_{K_\blacksquare[H]}(K, C)) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C),$$

as desired. The rest of the statements are proved in a similar way. □

15.5.2. Homology and duality

Definition 15.5.4. Let $C \in D(K_\blacksquare[G])$. We define the solid group homology of C as

$$K \otimes^L_{K_\blacksquare[G]} C.$$

Analogously, if C is derived $\mathbb{G}^{(h^+)}$-analytic, define its $\mathbb{G}^{(h^+)}$-analytic homology as

$$K \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C.$$

We have the following formal duality between homology and cohomology.

Lemma 15.5.5. *Let $C \in D(K_\blacksquare[G])$. Then*

$$R\underline{\mathrm{Hom}}_K(K \otimes^L_{K_\blacksquare[G]} C, K) = R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, R\underline{\mathrm{Hom}}_K(C, K)).$$

If C is $\mathbb{G}^{(h^+)}$-analytic, then

$$R\underline{\mathrm{Hom}}_K(K \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C, K) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, R\underline{\mathrm{Hom}}_K(C, K)).$$

Let $K(\chi) = \bigwedge^d \mathfrak{g}^\vee$ denote the determinant of the dual adjoint representation of G. Using Lazard-Serre's Theorem 15.3.1 one easily deduces that $R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, K_\blacksquare[G])$, endowed with the right multiplication of G, is a character concentrated in degree $-d$. Moreover, using the de Rham complex of $\mathbb{G}^{(h^+)}$ one can even prove that

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, K_\blacksquare[G]) = R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, \mathcal{D}^{(h^+)}(G, K)) = K(\chi)[-d]. \tag{15.5.1}$$

The following theorem relates cohomology and homology in a more interesting way.

Theorem 15.5.6. *Let $C \in D(K_\blacksquare[G])$. Then there is a natural quasi-isomorphism*

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) = K(\chi)[-d] \otimes^L_{K_\blacksquare[G]} C.$$

Furthermore, if C is derived $\mathbb{G}^{(h^+)}$-analytic, we have

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^{(h^+)}(G,K)}(K, C) = K(\chi)[-d] \otimes^L_{\mathcal{D}^{(h^+)}(G,K)} C.$$

Proof. First observe that, given any G-equivariant map $\alpha : K_\blacksquare[G]_{\star_1} \to K_\blacksquare[G]_{\star_1}$, one has a commutative diagram

$$\begin{array}{ccc} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, C) & \xleftarrow{\sim} & \underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, K_\blacksquare[G]) \otimes^L_{K_\blacksquare[G]} C \\ \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \alpha^* \otimes 1} \\ R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, C) & \xleftarrow{\sim} & \underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, K_\blacksquare[G]) \otimes^L_{K_\blacksquare[G]} C, \end{array} \tag{15.5.2}$$

where the action of G on the $R\underline{\mathrm{Hom}}$'s of the right hand side terms is induced by the $\star_2$-action on $K_\blacksquare[G]$. Notice that there is a natural identification of right $K_\blacksquare[G]$-modules

$$R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, K_\blacksquare[G]) = K_\blacksquare[G]_{\star_2}.$$

Recall that, by Theorem 15.3.1, we have a projective resolution $K \simeq [K_\blacksquare[G]^{\binom{d}{\bullet}}_{\star_1}; \alpha_\bullet]$. We obtain

$$\begin{aligned} R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, C) &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}([K_\blacksquare[G]^{\binom{d}{\bullet}}_{\star_1}; \alpha_\bullet], C) \\ &= [\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K_\blacksquare[G]_{\star_1}, K_\blacksquare[G])^{\binom{d}{\bullet}}; \alpha^*_\bullet \otimes 1] \otimes^L_{K_\blacksquare[G]} C \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}([K_\blacksquare[G]^{\binom{d}{\bullet}}_{\star_1}; \alpha_\bullet], K_\blacksquare[G]) \otimes^L_{K_\blacksquare[G]} C \\ &= R\underline{\mathrm{Hom}}_{K_\blacksquare[G]}(K, K_\blacksquare[G]) \otimes^L_{K_\blacksquare[G]} C \\ &= K(\chi)[-d] \otimes^L_{K_\blacksquare[G]} C, \end{aligned}$$

where the second equality follows by (15.5.2), and the last one by (15.5.1). The statement for $\mathbb{G}^{(h^+)}$-analytic cohomology is proven in the same way. □

Part IV.

Locally analytic completed cohomology of Shimura varieties and overconvergent BGG maps

16. Introduction

Let p be a prime number and $C = \mathbb{C}_p$ the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, we let $\mathcal{O}_C$ denote the ring of integers of C. The main subject of this article is the study of the Hodge-Tate theory of the proétale cohomology of Shimura varieties. Indeed, we generalize Lue Pan's description of the locally analytic completed cohomology of modular curves in terms of a sheaf of locally analytic functions [Pan20], to arbitrary Shimura varieties. The most important technical tool is a slightly more general version of the axiomatic Sen theory à la Berger-Colmez [BC08], which can be applied to locally analytic representations of p-adic Lie groups.

16.1. The main results

The Calegari-Emerton conjecture

As a motivation we recall the definition of Emerton's completed cohomology for Shimura varieties and the Calegari-Emerton conjectures. Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ and $(\mathbf{G}, X)$ a Shimura datum, i.e. X is a $\mathbf{G}(\mathbb{R})$-conjugacy class of cocharacters $h : \mathrm{Res}^{\mathbb{C}}_{\mathbb{R}}(\mathbb{G}_m) \to \mathbf{G}_{\mathbb{R}}$ satisfying certain axioms (cf. [Del79, 2.1.1] or [Mil05, Def. 5.5]). For simplicity, we will suppose that the center of $\mathbf{G}$ has no non-split subtorus which is split over $\mathbb{R}$. Let $\mathbb{A}_{\mathbb{Q}}^{\infty,p}$ denote the prime-to-p finite adèles of $\mathbb{Q}$ and fix from now on a neat compact open subgroup $K^p \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$. Given a compact open subgroup $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ we let $\mathrm{Sh}_{K^pK_p}(\mathbb{C})$ denote the level K^pK_p complex analytic Shimura variety

$$\mathrm{Sh}_{K^pK_p}(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times X)/K^pK_p$$

where $\mathbf{G}(\mathbb{Q})$ acts diagonally and K^pK_p only acts on $\mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty})$. The complex analytic Shimura varieties have algebraic models over the reflex field E of the Shimura datum, we denote this scheme by $\mathrm{Sh}_{K^pK_p}$. For any inclusion of open compact subgroups $K'_p \subset K_p$ of $\mathbf{G}(\mathbb{Q}_p)$ we have a finite étale map of Schemes $\mathrm{Sh}_{K^pK'_p} \to \mathrm{Sh}_{K^pK_p}$, if K'_p is normal in K_p then this map is Galois with group K_p/K'_p.

Let $\mathrm{Sh}_{K^pK_p,C}$ be the base change to C. Emerton's completed cohomology groups are defined as

$$\widehat{H}^{\bullet}(K^p, \mathbb{Z}_p) := \varprojlim_{s\to\infty} \varinjlim_{K_p\to 1} H^{\bullet}_{\text{ét}}(\mathrm{Sh}_{K^pK_p,C}, \mathbb{Z}/p^s\mathbb{Z}).$$

Fixing an isomorphism of fields $C \cong \mathbb{C}$, the Artin's comparison theorem of étale and Betti cohomology (cf. [Art68]) provides an isomorphism of cohomology groups

$$H^{\bullet}_{\text{ét}}(\mathrm{Sh}_{K^pK_p,C}, \mathbb{Z}/p^s\mathbb{Z}) \cong H^{\bullet}_{\text{Betti}}(\mathrm{Sh}_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s\mathbb{Z}),$$

so that

$$\widehat{H}^{\bullet}(K^p, \mathbb{Z}_p) = \varprojlim_{s\to\infty} \varinjlim_{K_p\to 1} H^{\bullet}_{\text{Betti}}(\mathrm{Sh}_{K^pK_p}(\mathbb{C}), \mathbb{Z}/p^s\mathbb{Z}).$$

This is Emerton's original definition of completed cohomology, which can be extended to general locally symmetric spaces, see [Eme06]. There is also a version with compact supports, which we denote by $\widehat{H}^{\bullet}_c(K^p, \mathbb{Z}_p)$. The spaces $\widehat{H}^{\bullet}_{(c)}(K^p, \mathbb{Z}_p)$ are p-adically complete representations of $\mathbf{G}(\mathbb{Q}_p)\times G_E$, where G_E is the absolute Galois group of E. Most of the Calegari-Emerton conjecture for Shimura varieties can be stated as follows:

Conjecture 2. *The completed cohomology groups $\widehat{H}^\bullet(K^p,\mathbb{Z}_p)$ and $\widehat{H}^\bullet_c(K^p,\mathbb{Z}_p)$ are concentrated in degrees $[0,d]$, where d is the dimension of the Shimura varieties.*

The first major step towards this conjecture was made by Scholze in [Sch15], where he proves the vanishing for the compactly supported cohomology of Shimura varieties of Hodge-type. Scholze's proof uses the primitive comparison theorem to write down $\widehat{H}^\bullet_c(K^p,\mathbb{Z}_p)\widehat{\otimes}\mathcal{O}_C$ in terms of the analytic cohomology of the perfectoid Shimura variety, then it uses Grothendieck's dimension bound for sheaf cohomology to deduce the vanishing. It turns our that using the perfectoid toroidal compactifications at infinite level of Pilloni-Stroh [PS16], and the pro-Kummer-étale theory of Diao-Lan-Liu-Zhu [DLLZ19], the same strategy of Scholze can be applied to prove the vanishing of $\widehat{H}^\bullet(K^p,\mathbb{Z}_p)$ for Shimura varieties of Hodge-type.

The next and most recent advance towards this conjecture is due to Hansen-Johansson [HJ20]. Restricted to Shimura varieties, they proved the conjecture for groups $\mathbf{G}$ such that $\mathbf{G}^{der}$ admits a connected Shimura datum of pre-abelian type. A sketch of their proof is the following: first, they extend Scholze's theorem of perfectoidness of infinite level Shimura varieties to Shimura varieties of pre-abelian type. Then, using Scholze's proof for Hodge type one obtains the vanishing of $\widehat{H}^\bullet_c(K^p,\mathbb{Z}_p)$. Finally, they perform a careful topological study of the boundary of the Borel-Serre compactification of the Shimura varieties to obtain the vanishing for $\widehat{H}^\bullet(K^p,\mathbb{Z}_p)$.

As it can be noticed, the common strategy is showing that the infinite level Shimura varieties are perfectoid, then the proétale or pro-Kummer-étale cohomology and the primitive comparison theorem will do the job for translating the completed cohomologies in terms of the analytic cohomology of a perfectoid space of cohomological dimension $\leq d = \dim \mathrm{Sh}$. Unfortunately, we still do not know if any infinite level-at-p Shimura variety is perfectoid. Nonetheless, using the theory of locally analytic vectors and the admissibility of the completed cohomology proven by Emerton, we can actually prove a rational version of Conjecture 2.

Theorem 16.1.1. *The rational Calegari-Emerton conjecture holds for any Shimura variety. In other words, the cohomology groups $\widehat{H}^i(K^p,\mathbb{Q}_p)=\widehat{H}^i(K^p,\mathbb{Z}_p)[\frac{1}{p}]$ and $\widehat{H}^i_c(K^p,\mathbb{Q}_p)=\widehat{H}^i_c(K^p,\mathbb{Z}_p)[\frac{1}{p}]$ vanish for $i>d$.*

The strategy of the proof is very similar to the other ones: first, since taking locally analytic vectors of admissible representations is exact, we can study the locally analytic vectors of the completed cohomology. In fact, it turns out that $\widehat{H}^\bullet_{(c)}(K^p,\mathbb{Q}_p)$ is a complete Banach space over $\mathbb{Q}_p$ endowed with an unitary action of $\mathbf{G}(\mathbb{Q}_p)\times G_E$, whose restriction to $\mathbf{G}(\mathbb{Q}_p)$ is admissible. Then, to prove the conjecture it is enough to prove that $\widehat{H}^{\bullet,la}_{(c)}(K^p,\mathbb{Q}_p)^{la}$ is concentrated in degrees $[0,d]$; here V^{la} means the space of locally analytic vectors for the action of $\mathbf{G}(\mathbb{Q}_p)$. One can translate the process of taking locally analytic vectors as taking proétale cohomology over the rigid analytic Shimura variety, of the local system $C^{la}(K_p,\mathbb{Q}_p)$ of locally analytic functions of K_p for any $K_p\subset\mathbf{G}(\mathbb{Q}_p)$ small enough. The primitive comparison theorem implies that the C-scalar extension of the locally analytic completed cohomology can be computed as the pro-Kummer-étale cohomology over $\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p,C}$ of $C^{la}(K_p,\widehat{\mathcal{O}})$, where $\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p,C}$ is some toroidal compactification of the Shimura variety, and $\widehat{\mathcal{O}}$ is the completed structural sheaf of the pro-Kummer-étale site. Then, one uses relative Sen theory to compute the projection of sites of the sheaf $C^{la}(K_p,\widehat{\mathcal{O}})$ from the pro-Kummer-étale to the analytic site; it boils down that this is concentrated in degree 0. Taking colimits as $K_p\to 1$, one gets that the completed cohomology can be computed as the analytic cohomology of some sheaf over the analytic site of $\mathrm{Sh}^{\mathrm{tor}}_{K^p,C}$, the infinite level Shimura variety. This proves the vanishing since the analytification of $\mathrm{Sh}^{\mathrm{tor}}_{K^p,C}$ has cohomological dimension $\leq d$.

In order to write down more precise statements let us fix some other notation. Let $L/\mathbb{Q}_p$ be a finite extension for which the group $\mathbf{G}$ is split, and let $\mu:\mathbb{G}_m\to\mathbf{G}_{\mathbb{C}}$ be the Hodge cocharacter, i.e. the restriction of $h:\mathbb{G}_m\times\mathbb{G}_m\to\mathbf{G}_{\mathbb{C}}$ to the first component. We fix a conjugate of μ which is defined over L. We let $E\to L$ be some fixed map of fields and let $\mathcal{S}h_{K^pK_p,L}$ denote the analityfication of the Shimura

variety to an adic space over $\mathrm{Spa}(L, \mathcal{O}_L)$, cf. [Hub96]. By [Pin90], we have toroidal compactifications $\mathrm{Sh}^{\mathrm{tor}}_{K^pK_p}$ of $\mathrm{Sh}_{K^pK_p}$, we let $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ be its analytification over L. Once we fix a K_p, we can take toroidal compactifications for a decreasing sequence $\cdots \subset K_p'' \subset K_p' \subset K_p$ of compact open subgroups such that the map $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p'',L} \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p',L}$ is a finite Kummer-étale map of adic spaces, cf. [DLLZ19]. We let $\mathcal{S}h_{K^p,L} = \varprojlim_{K_p \to 1} \mathcal{S}h_{K^pK_p,L}$ be the infinite level Shimura variety seen as an object in $\mathcal{S}h_{K^p,L,\mathrm{proét}}$, and let $\mathcal{S}h^{\mathrm{tor}}_{K^p,L}$ be the inverse limit of the toroidal compactifications seen as an object in $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\mathrm{prokét}}$. The space $\mathcal{S}h^{\mathrm{tor}}_{K^p,L}$ is then a K_p-torsor over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$. We have the following theorem (Theorem 22.3.16)

Theorem 16.1.2. *Let $\widehat{\mathcal{O}}_{\mathcal{S}h}$ be the restriction of the completed sheaf of $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{prokét}}$ to $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$, and let $\mathcal{O}^{la}_{\mathcal{S}h}$ denote the subsheaf of locally analytic sections for the action of K_p. We have an isomorphism*[1]

$$R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathcal{O}^{la}_{\mathcal{S}h}).$$

For K_p let $\mathcal{I}_{K_p}$ denote the ideal sheaf of the boundary over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$, and let $\mathcal{I} = \varinjlim_{K_p \to 1} \mathcal{I}_{K_p}$ be its colimit seen as a sheaf over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$. Let $\mathcal{O}^{sm}_{\mathcal{S}h}$ denote the smooth vectors of $\mathcal{O}^{la}_{\mathcal{S}h}$, equivalently, the colimit of the structural sheaves of the finite level Shimura varieties. We have the following isomorphism

$$R\Gamma_{c,\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathcal{O}^{la}_{\mathcal{S}h}\widehat{\otimes}_{\mathcal{O}^{\mathrm{sm}}_{\mathcal{S}h}}\mathcal{I})$$

where the completed tensor product is as LB spaces[2] *see Definition 17.1.6.*

Remark 16.1.3. We also prove a version for the dual locally analytic completed cohomology, in that scenario $\mathcal{O}^{la}_{\mathcal{S}h}$ is replaced by a a sheaf of locally analytic distributions $\mathcal{D}^{la}_{\mathcal{S}h}$ over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$, see Theorem 22.3.16.

The overconvergent BGG maps

Next, we explain the main motivation that led us to the theory of Lue Pan of locally analytic vectors, namely the BGG decompositions. These are nothing but the Hodge-Tate structure of the pro-étale cohomology of $\overline{\mathbb{Q}}_p$-local systems of Shimura varieties. The BGG decompositons in the complex analytic setting are due to Faltings [Fal83], and for étale cohomology of Siegel varieties (or PEL Shimura varieties more generally) to Faltings-Chai [FC90]. Faltings's proof for the étale cohomology depends on the BGG spectral sequence for the de Rham cohomology and the "étale-de Rham" comparison theorem. Nevertheless, he also gave a purely local proof for modular curves in [Fal87]. It turns out that an analogous of Faltings's BGG method for de Rham cohomology also holds for the proétale cohomology via the π_{HT} period map. To illustrate the general statement, let us sketch the main steps of the BGG decompostion for the modular curves (also know as the Eichler-Shimura decompositions).

Let $\mathbf{G} = \mathrm{GL}_2$ and $\mathcal{Y}_{K^pK_p} = \mathcal{S}h_{K^pK_p,L}$ be the modular curve of level K^pK_p seen as an adic space over $\mathbb{Q}_p$. We let $\mathcal{X}_{K^pK_p}$ be its natural compactification ([DR73]). Let $E \to \mathcal{Y}_{K^pK_p}$ be the universal elliptic curve and $E^{sm} \to \mathcal{X}_{K^pK_p}$ its extension to a semiabelian scheme. Let ω_E denote the sheaf of invariant differentials of E over $\mathcal{Y}_{K^pK_p}$, we denote in the same way its canonical extension to $\mathcal{X}_{K^pK_p}$ (i.e. the sheaf of invariant differentials of E^{sm}). For an integer k we shall denote $\omega^k_E = \omega^{\otimes k}_E$. Let $T_pE = \varprojlim_n E[p^n]$ be the Tate module of E seen as a proétale local system of $\mathcal{Y}_{K^pK_p}$, we also let T_pE denote its natural extension to the pro-Kummer-étale site of $\mathcal{X}_{K^pK_p}$. Let $\mathbf{B} \subset \mathrm{GL}_2$ be the Borel subgroup of upper triangular matrices and $\mathbf{N}$ its unipotent radical. Let $\mathcal{X}_{K^p} = \varprojlim_{K_p \to 1} \mathcal{X}_{K^pK_p}$ be

[1] The upper script *la* in the LHS refers to the derived locally analytic vectors of the complex as explained in §17.2. Its cohomology groups are just the locally analytic vectors of the completed cohomologies since these are admissible representations.

[2] i.e. countable filtered colimits of Banach spaces by injective transition maps.

Scholze's perfectoid modular curve and $\pi_{\mathrm{HT}} : \mathcal{X}_{K^p} \to \mathscr{F}\ell = \mathbb{P}^1$ the Hode-Tate period map. The application π_{HT} is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and is defined via the Hodge-Tate exact sequence

$$0 \to \omega_E^{-1} \otimes_{\mathscr{O}_{\mathcal{X}}} \widehat{\mathscr{O}_{\mathcal{X}}}(1) \to T_p E \otimes_{\widehat{\mathbb{Z}}_p} \widehat{\mathscr{O}_{\mathcal{X}}} \to \omega_E \otimes_{\mathscr{O}_{\mathcal{X}}} \widehat{\mathscr{O}_{\mathcal{X}}} \to 0 \tag{16.1.1}$$

and the universal trivialization of the Tate module $\alpha : \widehat{\mathbb{Z}}_p^2 \cong T_p E$ over $\mathcal{X}_{K^p}$.

Given V a finite dimensional $\mathbf{B}$-representation, one can define $\mathcal{V} := B\backslash(\mathrm{GL}_2 \times V)$ a GL_2-equivariant vector bundle over $\mathscr{F}\ell$. Let $\mathbf{T} \subset \mathbf{B}$ be the diagonal and $(k_1, k_2) : \mathbf{T} \to \mathbb{G}_m$ a character, we define a line bundle $\mathcal{L}(k_1, k_2) := \mathbf{B}\backslash(\mathrm{GL}_2 \times (k_2, k_1))$; the convention is made in such a way that if $\lambda = (k_1, k_2)$ is dominant, then $\Gamma(\mathscr{F}\ell, \mathcal{L}(\kappa)) = V_\lambda$ is the irreducible representation of GL_2 of highest weight λ.

The sequence (16.1.1) encodes the pullbacks of GL_2-equivariant vector bundles of $\mathscr{F}\ell$ by π_{HT}. Indeed, let St denote the standard representation of GL_2, the $\mathbf{B}$-filtration of St is

$$0 \to (1, 0) \to \mathrm{St} \to (0, 1) \to 0. \tag{16.1.2}$$

Passing to GL_2-equivariant sheaves over $\mathscr{F}\ell$, one obtains a short exact sequence of GL_2-equivariant vector bundles

$$0 \to \mathcal{L}(0, 1) \to \mathrm{St} \otimes \mathscr{O}_{\mathscr{F}\ell} \to \mathcal{L}(1, 0) \to 0$$

whose pullback via π_{HT} is (16.1.1).

Let w_0 be the non trivial element of the Weyl group of GL_2, and let $C_{w_0} = \mathbf{B}\backslash\mathbf{B}w_0\mathbf{B}$ be the big Bruhat cell of $\mathscr{F}\ell$. One has a short exact sequence of $\mathbf{B}$-representations

$$0 \to V_\lambda \to \mathscr{O}(C_{w_0}) \otimes \lambda \to \mathscr{O}(C_{w_0}) \otimes (w_0 \cdot \lambda) \to 0 \tag{16.1.3}$$

where $w_0 \cdot \lambda$ is the dot action of the Weyl group. The sequence (16.1.3) is nothing but the $\mathbf{T}$-finite vectors of the algebraic dual of the BGG-resolution of $V_\lambda^\vee$ with respect to the opposite Borel, see [RC21a, Prop. 3.4.3] and §21.3 in this document. One of the main results of *loc. cit.* is that, taking the GL_2-equivariant associated sheaves of (16.1.3) and pulling back by π_{HT}, one obtains a short exact sequence

$$0 \to V_\lambda \otimes \widehat{\mathscr{O}_{\mathcal{X}}} \to \omega_E^{k_2 - k_1} \mathscr{O}\mathbb{C}_{\log}(k_1) \to \omega_E^{k_1 - k_2 + 2} \mathscr{O}\mathbb{C}_{\log}(k_2 - 1) \to 0$$

where $\mathscr{O}\mathbb{C}_{\log} = \mathrm{gr}^0 \mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$ is a period sheaf in the pro-Kummer-étale site, cf. [DLLZ18]. Projecting to the analytic site and taking cohomology one recovers Faltings's ES decomposition:

$$H^\bullet_{\text{proét}}(\mathcal{Y}_{K^p K_p, C}, V_\lambda) \otimes C = H^\bullet_{\mathrm{an}}(\mathcal{X}_{K^p K_p, C}, \omega_E^{k_2 - k_1}) \otimes C(k_1) \oplus H^{\bullet - 1}_{\mathrm{an}}(\mathcal{X}_{K^p K_p, C}, \omega_E^{k_1 - k_2 + 2}) \otimes C(k_2 - 1).$$

In general, let $\mathbf{P}_\mu \subset \mathbf{G}$ be the parabolic subgroup defined by $\mathbf{P}_\mu = \{x \in \mathbf{G} : \lim_{t \to 0} \mathrm{Ad}(\mu(t))(x) \text{ exists}\}$. Let $\mathbf{M}_\mu \subset \mathbf{P}_\mu$ denote a Levi factor. Let $\mathscr{F}\ell = \mathbf{P}_\mu\backslash\mathbf{G}_L$ be the flag variety and $\pi_{\mathrm{HT}} : \mathcal{S}h^{\mathrm{tor}}_{K^p, L} \to \mathscr{F}\ell$ the Hodge-Tate period map which is deduced from the p-adic Riemman Hilbert correspondence of [DLLZ18], see [BP21, §4.4.38]. Let $\mathbf{T} \subset \mathbf{M}_\mu$ be a maximal torus and for $\lambda \in X^*(\mathbf{T})$ a dominant weight of $\mathbf{M}_\mu$ (resp. $\mathbf{G}$) let W_λ (resp. V_λ) denote the irreducible representation of highest weight λ. Let w_0 denote the longest element of the Weyl group of $\mathbf{G}$, we write $\mathcal{W}(\lambda)$ for the $\mathbf{G}$-equivariant vector bundle over $\mathscr{F}\ell$ given as

$$\mathcal{W}(\lambda) = \mathbf{P}_\mu\backslash(\mathbf{G} \times W^\vee_{-w_0(\lambda)}).$$

The convention is chosen so that $\Gamma(\mathscr{F}\ell, \mathcal{W}(\lambda)) = V_\lambda$ if λ is dominant for $\mathbf{G}$. Given K_p, let $\pi_{K_p} : \mathcal{S}h^{\mathrm{tor}}_{K^p, C} \to \mathcal{S}h^{\mathrm{tor}}_{K^p K_p, C}$ be the natural map. Attached to $\mathcal{W}(\lambda)$, there is an automorphic vector bundle $\mathcal{M}(\lambda)$ over $\mathcal{S}h^{\mathrm{tor}}_{K^p K_p, L}$ satisfying the relation

$$\pi_{\mathrm{HT}}^*(\mathcal{W}(\lambda)) = \pi_{K_p}^* \mathcal{M}(\lambda) \otimes_{\widehat{\mathscr{O}}_{\mathcal{S}h}} \widehat{\mathscr{O}}_{\mathcal{S}h}(w_0(\lambda)(\mu)).$$

We have the following theorem (Theorem 22.2.6)

Theorem 16.1.4. *The dual* BGG *resolution of V_λ, denoted as* $\mathrm{BGG}(\lambda)^\vee$*, defines a* **G***-equivariant sheaf over $\mathscr{F}\ell$ whose pullback by π_{HT} is of the form*

$$[\mathcal{M}^\vee(-w_0(\lambda))\otimes\mathscr{O}\mathbb{C}_{\log}(\lambda(\mu))\to\cdots\to\bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k}}\mathcal{M}^\vee(-w_0(w\cdot\lambda))\otimes\mathscr{O}\mathbb{C}_{\log}(w\cdot\lambda(\mu))\to$$
$$\cdots\to\mathcal{M}^\vee(-w_0(w_0^{\mathbf{M}}\cdot\lambda))\otimes\mathscr{O}\mathbb{C}_{\log}(w_0^{\mathbf{M}}\cdot\lambda(\mu))]$$

where:

1. *${}^{\mathbf{M}}W$ is the set of minimal length representatives of the quotient of Weyl groups $W_{\mathbf{M}_\mu}\backslash W_{\mathbf{G}_L}$.*
2. *$w_0^{\mathbf{M}}$ is the longest element in ${}^{\mathbf{M}}W$.*

Furthermore, let $\nu_{K_p}:\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{prok\acute{e}t}}\to\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{k\acute{e}t}}$ be the natural projection of sites. Then, the projection $R\nu_{K_p,*}$ gives rise a quasi-isomorphism

$$R\nu_{K_p,*}(V_\lambda\otimes\widehat{\mathscr{O}})=\bigoplus_{w\in{}^{\mathbf{M}}W}\mathcal{M}^\vee(-w_0(w\cdot\lambda))\otimes C(w\cdot\lambda(\mu))[-\ell(w)].$$

In particular, projecting to the analytic site and taking global cohomology one obtains the BGG *decomposition*

$$H^\bullet_{\mathrm{pro\acute{e}t}}(\mathcal{S}h_{K^pK_p,C},V_\lambda)\otimes C=\bigoplus_{w\in{}^{\mathbf{M}}W}H^{\bullet-\ell(w)}_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C},\mathcal{M}^\vee(-w_0(w\cdot\lambda))\otimes C(w\cdot\lambda(\mu)).\tag{16.1.4}$$

Remark 16.1.5. The reason for the action $-w_0$ in the formulas of the previous theorem is due to the normalization of the automorphic vector bundles $\mathcal{M}(\kappa)$. If instead we would have chosen $\mathcal{V}_\kappa:=\mathcal{M}(-w_0(\kappa))=\mathbf{P}_\mu\backslash\mathbf{G}\times W_\kappa^\vee$ as normalization, we would get

$$H^\bullet_{\mathrm{pro\acute{e}t}}(\mathcal{S}h_{K^pK_p,C},V_\lambda)=\bigoplus_{w\in{}^{\mathbf{M}}W}H^{\bullet-\ell(w)}_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C},\mathcal{V}^\vee_{w\cdot\lambda})\otimes C(w\cdot\lambda(\mu)),$$

a formula which is closer to Faltings-Chai BGG decomposition [FC90, Theo. 6.2]. Note that in *loc. cit.* , V_λ corresponds to $\mathbb{V}(\lambda)^\vee$; this is because, for Siegel varieties $\mathcal{X}$, the standard representation St defines the Tate module T_pA of the universal abelian variety A via π_{HT}, and the attached vector bundle with connection of T_pA is (a Tate twist of) $\mathscr{H}^1_{dR}(A^\vee/\mathcal{X})$.

A natural question arises from Theorem 16.1.4, namely, is it possible to interpolate the BGG maps of (16.1.4)? There are instances of an affirmative answer to this question, let us recall some of them.

- The first work on this direction is due to Andreatta-Iovita-Stevens [AIS15] for the H^0 ES map of the modular curve. They constructed a map from a proétale H^1-cohomology group of modular symbols over $\mathcal{Y}_0(p^n)$ towards a space of overconvergent modular forms.
- In [CHJ17], Chojecki-Hansen-Johansson reinterpretates AIS work in terms of the perfectoid modular curve and the π_{HT} period map, they also generalize de H^0 ES map to Shimura curves.
- There is also the work of Barrera and Gao [BG21] on unitary Shimura curves over totally real fields which follows closely the construction of AIS.
- In the Siegel case, one has the construction due to Diao-Rosso-Wu [DRW21] of the H^0 BGG map. Their ideas follow those of CHJ and AIS.
- For higher BGG maps, one finds the previous work of the author [RC21a] for the H^1 ES map of modular curves. Some of the strategies of *loc. cit.* shall be used throughout this document. The main idea is that the BGG maps should be first constructed at the level of the flag variety, and then pulled back to the Shimura varieties via π_{HT}.

It was the recent work of Lue Pan [Pan20] the one which led this problem to the right direction. Pan's theory of locally analytic vectors of completed cohomology provided an exhausting description of the $\mathfrak{b} = \lambda$ isotypic part of the completed cohomology of a modular curve, in terms of overconvergent modular forms, where $\mathfrak{b} = \mathrm{Lie}\,\mathbf{B}$. He managed to discover a bridge between $\mathcal{D}$-modules over the flag variety, and pro-étale cohomology of modular curves (much like the one provided by the Borel embedding in the complex analytic situation), a relation which is non trivial since the sheaf of differentials of a perfectoid space is zero.

We keep writing $\mathbf{P} = \mathbf{P}_\mu$ and $\mathbf{M} = \mathbf{M}_\mu$. Let $\mathbf{B} \subset \mathbf{G}_L$ be a Borel subgroup contained in $\mathbf{P}_\mu$ and containing the maximal torus $\mathbf{T}$. Let $\mathscr{F}\ell = \bigsqcup_{w\in \mathrm{M}_W} C_w$ be the decomposition of $\mathscr{F}\ell$ in Bruhat cells, let $Z_w = \overline{C_w}$ be the Schubert varieties and $Z_i = \bigcup_{\ell(w)=d-i} Z_w$. We have a filtration of closed subspaces of $\mathscr{F}\ell$

$$\emptyset \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0 = \mathscr{F}\ell\,.$$

The complement of this filtration $U_i = \mathscr{F}\ell \setminus Z_{d-i}$ gives rise to an open filtration of $\mathscr{F}\ell$ with graded pieces $U_i \setminus U_{i+1} = \bigsqcup_{\ell(w)=i} C_w$.

The inverse image by π_{HT} of the U_i's induce an open filtration of $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$, we let $j_i : \pi^{-1}_{\mathrm{HT}}(U_i) \subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$ and $j_w : \pi^{-1}_{\mathrm{HT}}(C_w) \subset \mathcal{S}h^{\mathrm{tor}}_{K^p,C}$. Let $\mathscr{F}$ be a sheaf over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$, we define the following overconvergent cohomologies with partial compact supports (see §18)

$$R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F}) := R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, j_{w,!}j_w^{-1}\mathscr{F}), \tag{16.1.5}$$

where $j_{w,!}$ is the extension by zero functor. We also denote $H^\bullet_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F})$ for the cohomology groups of $R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F})$. The filtration of $\mathscr{F}\ell$ induces a decreasing filtration of the sheaf $\mathscr{F}$ with $\mathrm{Fil}^i\,\mathscr{F} = j_{i,!}j_i^{-1}\mathscr{F}$ and graded pieces $\bigoplus_{\ell(w)=i} j_{w,!}j_w^{-1}\mathscr{F}$. Taking cohomology, one obtains a "higher Coleman theory" spectral sequence

$$E_1^{p,q} = \bigoplus_{\ell(w)=p} H^{p+q}_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F}) \Rightarrow H^{p+q}_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F}).$$

For all $w \in W$ one can define a sheaf of overconvergent automorphic forms $C(\mathcal{M}^\dagger_{w,\mathrm{dR}})$ in overconvergent neighbourhoods of $\pi^{-1}_{\mathrm{HT}}(C_w)$ (see §23.1). They contain naturally all the sheaves of overconvergent automorphic forms of higher Coleman theory of [BP21]. Let us sketch the definition of the sheaf $C(\mathcal{M}^\dagger_{w,\mathrm{dR}})$. First, one constructs a decreasing family of overconvergent neighbourhoods $\{C_w(\epsilon)\}_{\epsilon>0}$ of C_w. Over $C_w(\epsilon)$ one constructs a trivialization $s : C_w(\epsilon) \to \mathcal{M}_{\mathscr{F}\ell}$ of the $\mathbf{M}_\mu$-torsor $\mathcal{M}_{\mathscr{F}\ell} := \mathbf{N}_\mu\backslash\mathbf{G}_L$. Let $\mathcal{M}^\dagger_\mu = \{\mathcal{M}_\mu(\gamma)\}_{\gamma>0}$ be a basis of open affinoid subgroups of 1 in the analytification of $\mathbf{M}_\mu$, using the trivialization s one can construct a trivial $\mathcal{M}^\dagger$-torsor (i.e. a inverse system of $\mathcal{M}(\gamma)$-torsors for $\gamma > 0$) which we denote by $\mathcal{M}^\dagger_{w,\mathscr{F}\ell}$. We let $C(\mathcal{M}^\dagger_{\mathscr{F}\ell})$ denote the $\mathscr{O}_{C_w(\epsilon)}$-sheaf of functions of $\mathcal{M}^\dagger_{w,\mathscr{F}\ell}$, which is just the colimit of the sheaf of functions of the $\mathcal{M}(\gamma)$-torsors. Then, locally on $C_w(\epsilon)$, $C(\mathcal{M}^\dagger_{\mathscr{F}\ell})$ can be written as a colimit of locally analytic K_p-equivariant sheaves. Moreover, it has an horizontal action $\mathfrak{m}_{\star_1}$ of $\mathfrak{m} = \mathrm{Lie}\,\mathbf{M}_\mu$ induced by the compatible (left regular) $\mathcal{M}(\gamma)$-actions of the torsors. One defines the sheaf of overconvergent modular forms as

$$C(\mathcal{M}^\dagger_{w,\mathrm{dR}}) = (\pi^{-1}_{\mathrm{HT}}(C(\mathcal{M}^\dagger_{w,\mathscr{F}\ell}))\widehat{\otimes}_{\pi^{-1}_{\mathrm{HT}}(\mathscr{O}_{\mathscr{F}\ell})}\widehat{\mathscr{O}}_{Sh})^{K_p-sm}.$$

Let $\mathfrak{b} = \mathrm{Lie}\,\mathbf{B}$ and let $\lambda : \mathfrak{b} \to C$ be a character. The interpolations of the BGG maps are constructed by taking the λ-isotypic part of the locally analytic completed cohomology as follows (see Theorems 23.2.1)

Theorem 16.1.6. *Let $\mathscr{O}^{la}_{Sh}$ be the sheaf of locally analytic sections at infinite level and $\mathfrak{b}_{\mathbf{M}} = \mathrm{Lie}\,\mathbf{M}_\mu \cap \mathbf{B}$ the Lie algebra of the Borel subgroup of $\mathbf{M}_\mu$. Let ρ and $\rho_{\mathbf{M}}$ be a half of the sum of the positive roots of $\mathbf{G}_L$ and $\mathbf{M}$ respectively, set $\rho^{\mathbf{M}} = \rho - \rho_{\mathbf{M}}$.*

1. *The derived* $\mathfrak{b} = \lambda$ *isotypic part of the locally analytic completed cohomology can be computed as*
$$(R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C)^{R\mathfrak{b}=\lambda} = R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\text{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{\mathcal{S}h})).$$

2. *Let* $\epsilon >> 0$*, then* $R\text{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{\mathcal{S}h})|_{\pi^{-1}_{\text{HT}}(C_w(\epsilon))}$ *is concentrated in cohomological degrees* $[0, d-\ell(w)]$*. Furthermore, we have a highest vector map*
$$\Upsilon_w : R\text{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{\mathcal{S}h})|_{\pi^{-1}_{\text{HT}}(C_w(\epsilon))} \to C(\mathcal{M}^{\dagger}_{w,\text{dR}})^{\mathfrak{b}_{\mathbf{M}},\star_1 = w\cdot\lambda+2\rho^{\mathbf{M}}}[\ell(w)-d],$$
surjective in $H^{d-\ell(w)}$*-cohomology.*

3. *The open Bruhat filtration of* $\mathscr{F}\ell$ *induces a spectral sequence*
$$E_1^{p,q} = \bigoplus_{\substack{w\in{}^{\mathbf{M}}W \\ \ell(w)=p}} H^{p+q}_{c,w}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\text{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{\mathcal{S}h})) \Rightarrow H^{p+q}((R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C)^{R\mathfrak{b}=\lambda}).$$
In addition, the map Υ_w *induces an overconvergent* BGG *map*
$$\text{BGG}_w : R\Gamma_{c,w}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\text{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{\mathcal{S}h})) \to R\Gamma_{c,w}(\mathcal{S}h^{\text{tor}}_{K^p,C}, C(\mathcal{M}^{\dagger}_{w,\text{dR}})^{\mathfrak{b}_{\mathbf{M}},\star_1 = w\cdot\lambda+2\rho^{\mathbf{M}}})[\ell(w)-d].$$

4. *Let* $\mathscr{I} \subset \mathscr{O}^{sm}_{\mathcal{S}h}$ *be the ideal defining the cusps for any finite level. Analogous statements hold for cohomology with compact supports after exchanging* $\mathscr{O}^{la}_{\mathcal{S}h}$ *by* $\mathscr{O}^{la}_{\mathcal{S}h}\widehat{\otimes}_{\mathscr{O}^{sm}_{\mathcal{S}h}}\mathscr{I}$ *and* $C(\mathcal{M}^{\dagger}_{w,\text{dR}})$ *by* $C(\mathcal{M}^{\dagger}_{w,\text{dR}})\widehat{\otimes}_{\mathscr{O}^{sm}_{\mathcal{S}h}}\mathscr{I}$*.*

Remark 16.1.7. We also prove a version of this theorem for the dual locally analytic completed cohomology, see Theorem 23.2.2. It involves a sheaf of "differential operators" over $\mathcal{S}h^{\text{tor}}_{K^p,C}$ and the duals of the overconvergent autormorphic sheaves.

16.2. An overview of the paper

The main new tool we use is a generalization of relative Sen theory of Berger-Colmez [BC08] to locally analytic representations of p-adic Lie groups. The theory of locally analytic representations is briefly reviewed in Section 17; in order to use some comparison results between continuous and Lie algebra cohomology, we follow the approach of solid locally analytic representations introduced by J. Rodrigues Jacinto and the author in [RJRC21]. In Section 18 we prove some devisage of cohomology with partial compact supports which will be use later on to relate the cohomologies (16.1.5) with the overconvergent cohomologies of higher Coleman theory [BP21]. In Section 19 we generalize the Sen theory axioms of Berger-Colmez allowing higher dimensional Sen traces. We prove that Sen theory can be apply to locally analytic representations.

Following the ideas of [BC16] and [Pan20], we study the Hodge-Tate cohomology of Shimura varieties, namely, the proétale cohomology of $\widehat{\mathscr{O}}$-modules. Given a compact p-adic Lie group G, and a pro-Kummer-étale G-torsor $\widetilde{\mathcal{X}} \to \mathcal{X}$, we use Sen theory to construct a universal Sen bundle which is nothing but the p-adic Simpson correspondance of the G-torsor $\widetilde{\mathcal{X}}$. We prove that the projection from the pro-Kummer-étale site to the analytic site can be computed as the G-invariant vectors of the Koszul complex defined by the Sen bundle on $\widetilde{X}$, in other words, the G-invariant vectors of the Higgs bundle over $\widetilde{X}$ obtained by the p-adic Simpson correspondance. This is the content of Chapter 20.

Chapter 21 is dedicated to some computations over the flag variety which will be used later on in the construction of the overconvergent BGG maps. In Chapter 22 we set up the theory of Shimura varieties as in [DLLZ18]; using the p-adic Riemman-Hilbert correspondances of *loc. cit.* , we briefly explain rephrasing [BP21, §4.4] how an abstract π_{HT} period map can be constructed at the level of

diamonds. We compute the Sen bundle in terms of a **G**-equivariant sheaf over the flag variety and use the theory of §20 to obtain Theorem 16.1.2, we deduce Theorem 16.1.1 as a corollary. We also prove the classical p-adic BGG decompositions of Theorem 16.1.4.

Finally, in Chapter 23, we define the sheaves of overconvergent automorphic forms $C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$. Then, we use the machinery of §20 to study the derived λ-isotypic part of the locally analytic completed cohomology for the action of $\mathfrak{b}$, in terms of the derived λ-isotypic part of the sheaf $\mathscr{O}^{la}_{\mathcal{S}h}$. This last is studied in terms of $\widetilde{\mathcal{D}}$-modules over $\mathscr{F}\ell$, with $\widetilde{\mathcal{D}}$ the sheaf of universal twisted differential operators. By studying the derived λ-isotypic part of these $\widetilde{\mathcal{D}}$-modules in the overconvergent neighbourhoods $C_w(\epsilon)$ of C_w, we deduce Theorem 16.1.6.

17. Locally analytic representations of p-adic Lie groups

The main objective of this chapter is to introduce the tools in non-archimedean functional analysis and representation theory that will be used in the rest of the document. Our approach to these subjects "lives" in between two worlds: the classical one, as in [Sch02] or [Eme17], and the recently developed condensed mathematical world of Clausen and Scholze [CS], cf. Appendix *A* of [Bos21]. In fact, we will be only interested in classical topological spaces (i.e. Banach, Fréchet, *LB* and *LF* spaces), but the use of condensed mathematics helps us to treat the theory algebraically without topological issues. The principal reason to use condensed mathematics in this work is to apply the theory of solid locally analytic representations of Joaquín Rodrigues Jacinto and the author [RJRC21]. More precisely, we need the notion of derived locally analytic vectors, whose "underlying set" was already studied in [Pan20]. We also need some comparison results for group cohomology in the solid setting, generalizing previous work of Lazard [Laz65], Schneider-Teitelbaum [ST02,ST03], Tamme [Tam15], *et al.* Moreover, the condensed framework allows us to work in a derived category, which will be necessary for the correct statement of the theorems.

From now on we fix p a prime number, we let $\mathbb{Q}_p$ be the field of p-adic numbers and let $C = \mathbb{C}_p$ be the p-adic completion of an algebraic closure of $\mathbb{Q}_p$.

17.1. Classical definitions

Let K be a non-archimedean extension of $\mathbb{Q}_p$, i.e., a field extension endowed with a multiplicative valuation $|\cdot| : K \to \mathbb{R}_{\geq 0}$ extending the p-adic valuation of $\mathbb{Q}_p$, for which K is complete. The classical approach to the theory of non-archimedean functional analysis is via locally convex vector spaces over K, see for example [Sch02]. Throughout this section we will be only interested in the subclass of *LF* spaces, which contains all Banach and Fréchet spaces, and all the countable filtered colimits (with injective transition maps) between them, cf. [Eme17]. We wont be concern in topological questions like cokernels of maps of these objects since condensed mathematics ([Sch19], [CS]) already solves this problem.

One of the main purposes of this section is to recall how to work formally with *LF* spaces, in particular, how the continuous homomorphisms are defined, and how to compute the completed tensor product of two *LF* spaces. The second main goal is to define locally analytic representations and the analytic distribution algebras, we will follow [RJRC21, §3 and 4] for the notations.

17.1.1. Non-archimedean functional analysis

Definition 17.1.1. 1. A Banach space is a topological K-vector space V such that there exists an open O_K-submodule $V^0 \subset V$ endowed with the p-adic topology, which is complete and separated. We say that V^0 is a lattice of V.

2. A Fréchet space is a topological K-vector space which can be written as a (topological) countable cofiltered limit of Banach spaces.

3. Given V and V' two Fréchet spaces over K, we denote by $\mathrm{Hom}_K(V, V')$ the space of continuous maps from V to V'

In nature, one often encounters locally convex vector spaces which are written in terms of Banach and Fréchet spaces. The next definition provides a formal way to work with those arising as countable colimits:

Definition 17.1.2. 1. Let $f : V \to W$ be a map between Banach spaces. We say that f is compact if for some lattices $V^0 \subset V$ and $W^0 \subset W$ such that $f(V^0) \subset W^0$, and for all $n \geq 0$, the image of V^0 in W^0/p^n is isomorphic to a finite $\mathcal{O}_K/p^n$-module[1].

2. An *LB* space is a countable filtered system $\{V_n\}_{n\in\mathbb{N}}$ of Banach spaces with injective transition maps. By an abuse of notation we write $V = \varinjlim_n V_n$ for the *LB*-space.

3. An *LF* space is a countable filtrered system $\{F_n\}_{n\in\mathbb{N}}$ of Fréchet spaces with injective transition maps. By an abuse of notation we write $F = \varinjlim_n F_n$ for the *LF* space.

4. Let $F = \varinjlim_n F_n$ and $F' = \varinjlim_m F'_m$ be *LF* spaces. We define the set of continuous homomorphisms from F to F' to be

$$\mathrm{Hom}_K(F, F') := \varprojlim_n \varinjlim_m \mathrm{Hom}_K(F_n, F_m).$$

Remark 17.1.3. Notice that a Banach space is a Fréchet space, and that an *LB* space is an *LF* space. The definition of *LF* spaces and the Hom_K set above is justified by [Sch02, Cor. 8.9] and [RJRC21, Lem. 3.25].

Definition 17.1.4. 1. A Fréchet space V is called of compact type (or a compact Fréchet space) if it is written as cofiltered limit of Banach spaces with compact transition maps.

2. An *LB* space is of compact type (or a compact *LB* space) if it is written as a filtered colimit of Banach spaces by compact transition maps.

Remark 17.1.5. In the classical literature, a compact Fréchet space is called a nuclear Fréchet space. We have changed this nomination due to the fact that Fréchet spaces are always nuclear in the sense of solid K-vector spaces, see [Bos21, Cor. A.22].

Definition 17.1.6. 1. Let V and W be Banach spaces with lattices V^0 and W^0 respectively. We define the projective tensor product of Banach spaces as

$$V\widehat{\otimes}_K W = \varprojlim_n ((V^0 \otimes_{\mathcal{O}_K} W^0)/p^n)[\frac{1}{p}].$$

2. Let $V = \varprojlim_n V_n$ and $W = \varprojlim_m W_m$ be Fréchet spaces. Their projective tensor product is the Fréchet space

$$V\widehat{\otimes}_K W = \varprojlim_{n,m} V_n\widehat{\otimes}_K W_m.$$

3. Finally, let $F = \varinjlim_n F_n$ and $F' = \varinjlim_m F'_m$ be *LF* spaces, we define their tensor product as

$$F\widehat{\otimes}_K F' := \varinjlim_{n,m} F_n\widehat{\otimes}_K F'_m.$$

Remark 17.1.7. It is important to highlight that the projective tensor product of Fréchet spaces matches with the solid tensor product, see [Bos21, Prop. A.25].

[1] This is not the usual definition of compactness found in the literature, e.g. [Sch02, §12], but it is equivalent if K is a finite extension of $\mathbb{Q}_p$.

17.1.2. A quick review on compact p-adic Lie groups

Let G be a compact p-adic Lie group and $\mathfrak{g} = \mathrm{Lie}\, G$ its Lie algebra. Let $C(G, \mathbb{Z}_p)$ be the space of continuous functions of G with values in $\mathbb{Z}_p$ endowed with the p-adic topology. Let $\mathbb{Z}_p[[G]] = \mathrm{Hom}_{\mathbb{Z}_p}(C(G, \mathbb{Z}_p), \mathbb{Z}_p)$ be the algebra of continuous distributions of G endowed with the weak topology. In other words, $\mathbb{Z}_p[[G]]$ is the Iwasawa algebra of G over $\mathbb{Z}_p$.

In order to define locally analytic functions on G we need to fix some coordinates locally around the identity. By [DdSMS99, Cor. 8.34], there exists a compact open subgroup $G_0 \subset G$ which is a uniform pro-p-group. By a theorem of Lazard [Laz65, Prop. III. 3.1.3], the group G_0 admits a discrete valuation with rational values $w : G_0 \to \mathbb{Q}_{>0}$, and a basis $g_1, \dots, g_d \in G_0$ with $d = \dim G$ inducing an homeomorphism

$$\phi : \mathbb{Z}_p^d \to G_0, \quad (x_1, \dots, x_d) \mapsto g_1^{x_1} \cdots g_d^{x_d}, \tag{17.1.1}$$

such that the pullback by ϕ of the multiplication law $(g, h) \mapsto gh^{-1}$ in G_0 is given by power series with coefficients in $\mathbb{Z}_p$:

$$\phi^{-1}(\phi(x_1, \dots, x_d)\phi(y_1, \dots y_d)^{-1}) = (F_1(\underline{x}, \underline{y}), \dots, F_d(\underline{x}, \underline{y})) \text{ with } F_i(\underline{X}, \underline{Y}) \in \mathbb{Z}_p\langle \underline{X}, \underline{Y}\rangle. \tag{17.1.2}$$

Furthermore, after shrinking G_0 if necessary, we can assume that $w(g_i) = r > 1$ for all $i = 1, \dots, d$.

From now on let us fix a uniform open normal subgroup G_0 of G, and a chart $\phi : \mathbb{Z}_p^d \to G_0$ as in (17.1.1). The chart ϕ defines an affinoid group whose $\mathbb{Q}_p$-points are equal to G_0:

Definition 17.1.8. Let $\mathbb{G}_0$ be the affinoid group whose underlying adic space is a polydisc $\mathbb{D}^d_{\mathbb{Q}_p} = \mathrm{Spa}(\mathbb{Q}_p\langle T_1, \dots, T_d\rangle, \mathbb{Z}_p\langle T_1, \dots, T_d\rangle)$, and whose multiplicative law $(g, h) \mapsto gh^{-1}$ is provided by the map

$$\Theta : \mathscr{O}(\mathbb{G}_0) \to \mathscr{O}(\mathbb{G}_0 \times \mathbb{G}_0) \tag{17.1.3}$$

defined by the power series F_i of (17.1.2).

By taking smaller radius in $\mathbb{G}_0$, one can defines a family of affinoid groups which encodes the infinitesimal p-adic Lie group structure of G_0.

Lemma 17.1.9. *Let $h > 0$ be rational, and let $\mathbb{G}_{0,h}$ be the affinoid subspace of $\mathbb{G}_0$ given by*

$$\mathbb{G}_{0,h} = \mathbb{G}_0\langle \frac{T_1}{p^h}, \dots, \frac{T_d}{p^h}\rangle.$$

Then $\mathbb{G}_{0,h}$ is a normal subgroup of $\mathbb{G}_0$.

Proof. To prove that $\mathbb{G}_{0,h}$ is a subgroup, it is enough to show that the map Θ of (17.1.3) sends T_i/p^h to $\mathscr{O}^+(\mathbb{G}_{0,h} \times \mathbb{G}_{0,h})$. But the power series F_i does not have constant term, therefore $\Theta(T_i/p^h) = F_i/p^h$ is an element in $\mathscr{O}^+(\mathbb{G}_{0,h} \times \mathbb{G}_{0,h})$. The fact that $\mathbb{G}_{0,h}$ is a normal subgroup of $\mathbb{G}_0$ is shown in a similar way considering the morphism $\mathbb{G}_0 \times \mathbb{G}_0 \to \mathbb{G}_0$ given by conjugation: $(g, h) \mapsto ghg^{-1}$. □

Definition 17.1.10. Let $h \geq 0$ be rational, we define the following rigid analytic neighbourhoods of G and their spaces of functions:

1. The affinoid group $\mathbb{G}^{(h)} = G\mathbb{G}_{0,(h)}$, which is a finite disjoint union of polydiscs of radius p^{-h}. We let
$$C(\mathbb{G}^{(h)}, K) = \mathscr{O}(\mathbb{G}^{(h)})\widehat{\otimes}_{\mathbb{Q}_p} K$$
be the space of rigid analytic functions of $\mathbb{G}^{(h)}$ over K, and
$$\mathcal{D}(\mathbb{G}^{(h)}, K) = \mathrm{Hom}_{\mathbb{Q}_p}(C(\mathbb{G}^{(h)}, \mathbb{Q}_p), K)$$
its weak dual.

   We call $C(\mathbb{G}^{(h)}, K)$ (resp. $\mathcal{D}(\mathbb{G}^{(h)}, K)$) the space of K-valued $\mathbb{G}^{(h)}$-analytic functions (resp. $\mathbb{G}^{(h)}$-analytic distributions) of G.

2. The Stein group $\mathbb{G}^{(h^+)} = \bigcup_{h'\to h^+} \mathbb{G}^{(h')}$, which is a disjoint union of open polydiscs of radius p^{-h}. We let
$$C(\mathbb{G}^{(h^+)}, K) := \varprojlim_{h'\to h^+} C(\mathbb{G}^{(h')}, K)$$
be the space of rigid analytic functions of $\mathbb{G}^{(h^+)}$, which is naturally a compact Fréchet space. We also let
$$\mathcal{D}(\mathbb{G}^{(h^+)}, K) = \varinjlim_{h'\to h^+} \mathcal{D}(\mathbb{G}^{(h^+)}, K)$$
be the dual of $C(\mathbb{G}^{(h^+)}, K)$, which is an *LB* space of compact type, see [ST02, Cor. 1.4] and [RJRC21, Theo. 3.30] for the duality between compact Fréchet spaces and *LB* spaces of compact type.

   We call $C(\mathbb{G}^{(h^+)}, K)$ (resp. $\mathcal{D}(\mathbb{G}^{(h^+)}, K)$) the space of K-valued $\mathbb{G}^{(h^+)}$-analytic functions (resp. $\mathbb{G}^{(h^+)}$-analytic distributions) of G.

3. We define the space of K-valued locally analytic functions of G as the *LB* space of compact type
$$C^{la}(G, K) := \varinjlim_{h\to\infty} C(\mathbb{G}^{(h)}, K) = \varinjlim_{h\to\infty} C(\mathbb{G}^{(h^+)}, K).$$
   Dually, we define the space of K-valued locally analytic distributions of G as the compact Fréchet space
$$\mathcal{D}^{la}(G, K) = \varprojlim_{h\to\infty} \mathcal{D}(\mathbb{G}^{(h)}, K) = \varprojlim_{h\to\infty} \mathcal{D}(\mathbb{G}^{(h^+)}, K).$$

Remark 17.1.11. The rigid analytic groups $\mathbb{G}^{(h)}$ and $\mathbb{G}^{(h^+)}$ depend on the uniform pro-p-group $G_0 \subset G$ and the chart ϕ. The same is true for their spaces of analytic functions and distributions. Nonetheless, the (co)filtered systems of analytic functions and distributions as $h \to \infty$ are independent of those choices.

The previous algebras of distributions are adapted to the rigid analytic neigbbourhoods of G. However, only the family $\mathcal{D}(\mathbb{G}^{(h^+)}, K)$ is well adapted to the study of locally analytic vectors of G-representations. The only "difficulty" with these distribution algebras appears if one wants to stay in the category of *LF* spaces. In fact, the $\mathbb{G}^{(h^+)}$-analytic vectors of *LF* representations are inverse limits of *LF* spaces, which in turn have a perfect sense as solid K-vector spaces, but that are no easy to track as topological K-vector spaces. A way to avoid this, and keep working with *LF* representations, is to use a different class of distributions algebras adapted to the chart ϕ.

Let $\phi : \mathbb{Z}_p^d \to G_0$ be a chart defined by a basis $g_1, \dots, g_d$. Let $b_i = g_i - 1 \in \mathbb{Z}_p[[G]]$ be elements in the Iwasawa algebra. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ we denote $\mathbf{b}^\alpha := b_1^{\alpha_1} \cdots b_d^{\alpha_d}$.

Proposition 17.1.12 ([ST03, §4]). *We have an isomorphism of profinite $\mathbb{Z}_p$-modules*
$$\mathbb{Z}_p[[G_0]] = \prod_{\alpha\in\mathbb{N}^d} \mathbb{Z}_p \mathbf{b}^\alpha.$$

Definition 17.1.13. Let $h > 0$. We define the space of distributions $\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$ as
$$\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p) = \{\sum_\alpha a_\alpha \mathbf{b}^\alpha \in \prod_{\alpha\in\mathbb{N}^d} \mathbb{Q}_p \mathbf{b}^\alpha : \sup_{\alpha\in\mathbb{N}} \{|a_\alpha| p^{-\frac{p^{-h}}{p-1}|\alpha|}\} < 1\}.$$

Remark 17.1.14. Let $b(h) = \frac{p^{-h}}{p-1}$. If K is a finite extension of $\mathbb{Q}_p$ and $b(h)$ is a valuation in K, then one has
$$\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} O_K = \prod_{\alpha\in\mathbb{N}^d} O_K \frac{\mathbf{b}^\alpha}{p^{b(h)|\alpha|}}.$$

This implies that, if $b(h) \in \mathbb{Q}$, then $\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$ is a profinite $\mathbb{Z}_p$-module. Furthermore, by Proposition 4.2 of [ST03] the space $\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$ is a profinite algebra admitting $\mathbb{Z}_p[[G_0]]$ as a dense subalgebra. We set $\mathcal{D}_{(h)}(G_0, \mathbb{Q}_p) := \mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)[\frac{1}{p}]$ endowed with the weak topology.

From now on we will assume that $b(h)$ is rational whenever we work with the algebra $\mathcal{D}_{(h)}(G_0, \mathbb{Q}_p)$.

Definition 17.1.15. 1. Let $h > 0$, we define the distribution algebras

$$\mathcal{D}(G_{(h)}, \mathbb{Q}_p) := \mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_0]]} \mathcal{D}_{(h)}(G_0, \mathbb{Q}_p)$$
$$\mathcal{D}(G_{(h)}, \mathbb{Z}_p) := \mathbb{Z}_p[[G]] \otimes_{\mathbb{Z}_p[[G_0]]} \mathcal{D}_{(h)}(G_0, \mathbb{Z}_p).$$

We let $C(G_{(h)}, \mathbb{Q}_p)$ denote the dual of $\mathcal{D}(G_{(h)}, \mathbb{Q}_p)$ seen as a Banach space over $\mathbb{Q}_p$.

2. Let K be a non archimedean extension of $\mathbb{Q}_p$. We define the distribution algebra

$$\mathcal{D}(G_{(h)}, \mathcal{O}_K) = \varprojlim_n (\mathcal{O}_K/p^n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{D}(G_{(h)}, \mathbb{Z}_p)/p^n)$$

$$\mathcal{D}(G_{(h)}, K) = \mathcal{D}(G_{(h)}, \mathcal{O}_K)[\frac{1}{p}],$$

where we see $\mathcal{D}(G_{(h)}, \mathbb{Z}_p)/p^n$ as a profinite $\mathbb{Z}_p$-module and $\mathcal{O}_K/p^n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{D}(G_{(h)}, \mathbb{Z}_p)/p^n$ as an Ind-profinite $\mathbb{Z}_p$-module. We let $C(G_{(h)}, K)$ denote the strong dual of $\mathcal{D}(G_{(h)}, K)$.

3. We define the Iwasawa algebra over $\mathcal{O}_K$ (resp. K) to be

$$\mathcal{O}_K[[G]] := \varprojlim_n \mathcal{O}_K/p^n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}_p[[G]]$$

$$K[[G]] := \mathcal{O}_K[[G]]\frac{1}{p}.$$

Remark 17.1.16. 1. The symbol $G_{(h)}$ for the previous distributions algebras is purely notational; there is not an underlying rigid analytic group whose distribution algebra is equal to $\mathcal{D}(G_{(h)}, K)$. By an abuse of notation, we call $C(G_{(h)}, K)$ the space of "analytic functions of $G_{(h)}$".

2. The definition of $\mathcal{D}(G_{(h)}, K)$ is made in such a way that matches with the solid tensor product $\mathcal{D}_{(h)}(G, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p,\blacksquare} K$. Furthermore, the K-Banach space $C(G_{(h)}, K)$ is equal to the projective tensor product $C(G_{(h)}, \mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Q}_p} K$. Similarly for the definition of the Iwasawa algebra over K.

The relation between the previous three families of distribution algebras is described in the following proposition

Proposition 17.1.17 ([RJRC21, Cor. 4.18])**.** *Let $h > 0$. We have a natural isomorphism of LB spaces of compact type*

$$\mathcal{D}(\mathbb{G}^{(h^+)}, K) = \varinjlim_{h' \to h^+} \mathcal{D}(\mathbb{G}^{(h')}, K) = \varinjlim_{h' \to h^+} \mathcal{D}(G_{(h')}, K).$$

Finally, let us recall the notion of a continuous G-representation in LF spaces. Again, the reader can check that it matches with the notion of a solid action of G on an LF space, or with the classical notion of continuous representation of [Eme17], cf. [RJRC21, Lem. 4.19].

Definition 17.1.18. 1. A continuous Fréchet representation of G is a Fréchet space V endowed with a continuous K-linear action

$$G \times V \to V.$$

2. A continuous LF representation of G is an LF space F endowed with a K-linear action of G such that there is a presentation of LF space $F = \varinjlim F_n$, where F_n is a continuous Fréchet representation of G.

3. A morphism of continuous LF representations of G is a G-equivariant continuous map of LF spaces.

17.1.3. Classical locally analytic representations

Next, we recall the definition notion of analytic representations on LF spaces, cf. [Eme17]. First, we introduce some spaces of analytic functions:

Definition 17.1.19. Let F be an LF space.

1. The space of continuous functions of G with values in F is the LF space
$$C(G, F) := C(G, K)\widehat{\otimes}_K F.$$

2. Let $h > 0$ and $\mathbb{G}^{(h)}$ denote the h-affinoid neighbourhood of G. The space of $\mathbb{G}^{(h)}$-analytic functions with values in F is the LF space
$$C(\mathbb{G}^{(h)}, F) := C(\mathbb{G}^{(h)}, K)\widehat{\otimes}_K F.$$

3. Let $h > 0$ and $\mathbb{G}^{(h^+)}$ the h-Stein analytic neighbourhood of G. The space of $\mathbb{G}^{(h^+)}$-analytic functions with values in F is the Proj-LF space
$$C(\mathbb{G}^{(h^+)}, F) := \varprojlim_{h' \to h'} C(\mathbb{G}^{(h')}, F).$$
Notice that if V is a Féchet G-representation then $C(\mathbb{G}^{(h^+)}, V)$ is a Fréchet space as well.

4. Let $h > 0$, the space of $G_{(h)}$-analytic functions with values in F is the LF space
$$C(G_{(h)}, F) := C(G_{(h)}, K)\widehat{\otimes}_K F.$$

Lemma 17.1.20 ([RJRC21, Prop. 4.21 and 4.26])**.** *Let F be an LF representation of G and $C(F)$ a space of functions of the Definition 17.1.19. Then $C(F)$ is endowed with a natural action of $G \times G \times G$ given by*
$$(g_1, g_2, g_3) \cdot \phi(h) = g_3 \cdot \phi(g_1^{-1} h g_2).$$

Remark 17.1.21. The Definition 17.1.19 is motivated by the the duality between the distributions and the spaces of functions. In fact, if $\mathcal{D}$ denotes the Iwasawa algebra, or any of the analytic distributions defined above, the spaces of functions $C(F)$ of the previous definition coincide with the underlying topological space of the internal Hom in solid K-vector spaces:
$$C(F) = \underline{\mathrm{Hom}}_K(\mathcal{D}, F)(*).$$

Notation 1. Let F be a representation of $G^n = \overbrace{G \times \cdots \times G}^{n\text{-times}}$. Let $I \subset \{1, \ldots, n\}$ be a subset, we denote by $F_{\star_I}$ the representation G given by restricting F to the I-diagonal embedding of G. If $I = \emptyset$ we denote by $F_0 = F_\emptyset$ the space F endowed with the trivial action of G. In the situation of Lemma 17.1.20, we call $\star_1$ and $\star_2$ the left and right regular action respectively.

Definition 17.1.22. Let F be a continuous LF representation of G.

1. Let $\mathcal{G}$ denote $\mathbb{G}^{(h)}$, $\mathbb{G}^{(h^+)}$ or $G_{(h)}$. We define the $\mathcal{G}$-analytic vectors of F to be the LF space (resp. Proj-LF space if $\mathcal{G} = \mathbb{G}^{(h^+)}$)
$$F^{\mathcal{G}-an} := (C(\mathcal{G}, F)_{\star_{1,3}})^G$$
endowed with the $\star_2$-action of G.

2. We define the space of locally analytic vectors of F to be the LF space

$$F^{la} = \varinjlim_{h\to\infty} F^{\mathbb{G}^{(h)}-an} = \varinjlim_{h\to\infty} F^{\mathbb{G}^{(h^+)}} = \varinjlim_{h\to\infty} F^{G_{(h)}-an}.$$

In fact, we have that $F^{la} = (C^{la}(G,K)\widehat{\otimes}_K F)^G_{\star_{1,3}}$.

Lemma 17.1.23. *Let F be a continuous LF representation of G. Let $\mathcal{G}$ denote $\mathbb{G}^{(h)}$, $\mathbb{G}^{(h^+)}$ or $G_{(h)}$. There is a natural inclusion*

$$F^{\mathcal{G}-an} \to F.$$

Proof. This follows from the fact that $(C(G,F)_{\star_{1,3}})^G = F$ and that we have continuous inclusions $C(\mathcal{G},F) \subset C(G,F)$, see Proposition 4.26 of [RJRC21] for a more general statement. □

Definition 17.1.24. Let F be an LF representation of G, let $h > 0$ and let $\mathcal{G}$ denote $\mathbb{G}^{(h)}$, $\mathbb{G}^{(h^+)}$ or $G_{(h)}$. We say that F is $\mathcal{G}$-analytic if the natural map $F^{\mathcal{G}-an} \to F$ is an isomorphism of LF spaces. We say that F is locally anlaytic if the natural map $F^{la} \to F$ is an isomorphism of LF spaces.

17.2. Derived locally analytic representations

The notion of derived locally analytic vectors for Banach representations was firstly considered by Lue Pan in [Pan20]. In fact, given a Banach representation V of G, Pan studies the cohomology groups $\mathrm{Hom}_G(K, C^{la}(K,V)_{\star_{1,3}})$ in a set-theoretical level (i.e. without topology). In the joint work with Joaquín Rodrigues Jacinto [RJRC21], we develop this idea in the condensed framework obtaining a more conceptual understanding of locally analytic representations in terms of distribution algebras. The goal of this section is to briefly introduce this notion for LF representations, and to state how the locally analyticity condition is translated in terms of the distribution algebras defined in §17.1.1. We shall also define the notion of locally analyticity with respect to the action of the Lie algebra of G.

17.2.1. Locally analytic representations of G

Definition 17.2.1. Let $F^\bullet$ be a complex of LF representations of G. Let $h > 0$ and $\mathcal{G}$ denote $\mathbb{G}^{(h)}$, $\mathbb{G}^{(h^+)}$ or $G_{(h)}$.

1. The derived $\mathcal{G}$-analytic vectors of $F^\bullet$ is the complex

$$F^{\bullet,R\mathcal{G}-an} := R\mathrm{Hom}_G(K, C(\mathcal{G}, F^\bullet)_{\star_{1,3}})$$

endowed with the $\star_2$-action of G. We say that $F^\bullet$ is derived $\mathcal{G}$-analytic if the natural map $F^{\bullet,R\mathcal{G}-an} \to F^\bullet$ is a quasi-isomorphism.

2. The derived locally analytic vectors of $F^\bullet$ is the colimit

$$F^{\bullet,Rla} := \varinjlim_{h\to\infty} F^{\bullet,R\mathbb{G}^{(h)}-an} = \varinjlim_{h\to\infty} F^{\bullet,R\mathbb{G}^{(h^+)}-an} = \varinjlim_{h\to\infty} F^{\bullet,RG_{(h)}-an}.$$

3. We say that $F^\bullet$ is derived $\mathcal{G}$-analytic if the natural map $F^{\bullet,R\mathcal{G}} \to F^\bullet$ is a quasi-isomorphism. We say that $F^\bullet$ is derived locally analytic if $F^{\bullet,Rla} \to F^\bullet$ is a quasi-isomorphism.

Remark 17.2.2. If $\mathcal{G} = \mathbb{G}^{(h)}$ or $G_{(h)}$, then $F^{\bullet,R\mathcal{G}-an}$ is represented by a complex of LF spaces. Indeed, by a lemma of Lazard and Serre, K admits a finite free resolution as $\mathbb{Z}_p[[G]]$-module provided G is small enough (see [Laz65, Définition 2.2.2.1]). Then, the map $F^{\bullet,R\mathcal{G}-an} \to F^\bullet$ being a quasi-isomorphism means that the cone is strictly acyclic (equivalently, acyclic as a solid K-vector space). In the case of $\mathcal{G} = \mathbb{G}^{(h^+)}$, the complex of derived $\mathbb{G}^{(h^+)}$-analytic vectors is no longer represented by LF spaces in general, so the meaning of $F^{\bullet,R\mathcal{G}-an} \to F^\bullet$ being a quasi-isomorphism must be considered in the category of solid K-vector spaces.

The main result relating the derived analytic vectors and the distribution algebras is the following theorem

Theorem 17.2.3 ([RJRC21, Theo. 4.34]). *Let $h > 0$ and $\mathcal{G}$ denote $\mathbb{G}^{(h)}$, $\mathbb{G}^{(h^+)}$ or $G_{(h)}$. Let $F^\bullet$ be a complex of LF representations of G. There are natural quasi-isomorphisms of G-representations*

$$F^{\bullet,R\mathcal{G}-an} = R\mathrm{Hom}_G(\mathcal{D}(\mathcal{G},K),F^\bullet), \tag{17.2.1}$$

where $\mathcal{D}(\mathcal{G},K)$ is seen as a G-module via the left multiplication of G. The G module structure of the RHS is induced by the right multiplication of G on the distributions.

Furthermore, let $\mathcal{G}$ denote $\mathbb{G}^{(h^+)}$ or $G_{(h)}$, then $F^\bullet$ is derived $\mathcal{G}$-analytic if and only if $F^\bullet$ is a complex of continuous $\mathcal{D}(\mathcal{G},K)$-modules.

Remark 17.2.4. The previous theorem says, in particular, that if F is a $\mathcal{D}(G_{(h)},K)$-module for some $h > 0$, then $F^{RG_{(h')}-an} = F$ for all $h' \geq h$, i.e. it is derived $G_{(h')}$-analytic for all $h' \geq h$.

Example 17.2.5.

1. One of the most important examples is the case of admissible representations. Suppose that K is a finite extension of $\mathbb{Q}_p$. We say that a unitary Banach representation V of G is admissible if its continuous dual $V^\vee$ is a finite module over the Iwasawa algebra $K[[G]] = \mathcal{O}_K[[G]][\frac{1}{p}]$. If V is admissible, one has that

$$V^{RG_{(h)}-an} = R\mathrm{Hom}_G(K, R\mathrm{Hom}_K(\mathcal{D}(G_{(h)},K),V)).$$

But $\mathcal{D}(G_{(h)},K)$ is a projective solid K-vector space, this fact and the duality for Banach and Smith spaces (see. [RJRC21, §3]) imply that

$$V^{RG_{(h)}-an} = R\mathrm{Hom}_K(\mathcal{D}(G_{(h)},K)\otimes^L_{K[[G]]} V^\vee, K).$$

But the theory of distributions of Schneider-Teitelbaum [ST03] says that the algebra $\mathcal{D}(G_{(h)},K)$ is flat (algebraically) over $K[[G]]$. Since $V^\vee$ is a finite $K[[G]]$-module one gets that

$$V^{RG_{(h)}-an} = R\mathrm{Hom}_K(\mathcal{D}(G_{(h)},K)\otimes^L_{K[[G]]} V^\vee, K) = \mathrm{Hom}_K(\mathcal{D}(G_{(h)},K)\otimes_{K[[G]]} V^\vee, K).$$

In particular, $V^{RG_{(h)}-an} = V^{G_{(h)}-an}[0]$ is concentrated in degree 0. Taking colimits as $h \to \infty$ one obtains the formula

$$V^{la} = \mathrm{Hom}_K(\mathcal{D}^{la}(G,K)\otimes_{K[[G]]} V^\vee, K).$$

2. It is easy to see that taking derived locally analytic vectors is an idempotent functor. Indeed, let $F^\bullet$ be a complex of LF representations of G. Then

$$\begin{aligned}(F^{\bullet,Rla})^{Rla} &= \varinjlim_{h,h'\to\infty} (F^{RG_{(h')}-an})^{RG_{(h)}-an}\\ &= \varinjlim_{h\to\infty} (F^{RG_{(h)}-an})^{RG_{(h)}-an}\\ &= \varinjlim_{h\to\infty} F^{RG_{(h)}-an}\\ &= F^{Rla},\end{aligned}$$

where in the first equality we use that taking derived locally analytic vectors commute with colimits, and in the third equality we use Theorem 17.2.3 to get that $(F^{RG_{(h)}-an})^{RG_{(h)}-an} = F^{RG_{(h)}-an}$.

3. The most basic spaces which only have locally analytic vectors in higher level are $K[[G]]$ and $\mathcal{D}^{la}(G,K)$. Indeed, one can show that

$$(K[[G]])^{Rla} = (\mathcal{D}^{la}(G,K))^{Rla} = C^{la}(G,K)\otimes\chi[-\dim G],$$

where $\chi = \det \mathrm{Lie}(G)^\vee$.

From Theorem 17.2.3 we also obtain the following well known result giving a criterion when a Banach representation is locally analytic:

Corollary 17.2.6. *Let V be a Banach representation of G. Then V is locally analytic if and only if for any G-stable lattice $V^0 \subset V$ (eq. for some G-stable lattice) the action of G on V^0/p factors through a finite quotient. Furthermore, if this is the case, V is $\mathbb{G}^{(h)}$-analytic for some $h > 0$ (eq. $\mathbb{G}^{(h^+)}$ or $G_{(h)}$-analytic for maybe a different $h > 0$).*

Proof. Let V be a locally analytic Banach representation. Then $V = V^{la} = \varinjlim_h V^{\mathbb{G}^{(h)}-an}$ as *LB* spaces. By [Sch02, Cor. 8.9] we must have $V = V^{\mathbb{G}^{(h)}-an}$ for some $h > 0$ (cf. [RJRC21, Lem. 3.23]). This implies that V admits a G closed immersion

$$V \hookrightarrow C(\mathbb{G}^{(h)}, V)_{\star_2} = C(\mathbb{G}^{(h)}, K)_{\star_2}\widehat{\otimes}V_0.$$

Then, it is easy to see that G acts through a finite quotient on $C(\mathbb{G}^{(h)}, \mathcal{O}_K)/p$, which implies the necessary condition.

Conversely, let $V^0 \subset V$ be any G-stable lattice of V, and let $G' \subset G$ be an open normal subgroup such that G' acts trivially on V^0/p. Without loss of generality, we can assume that $G_0 \subset G'$. By Theorem 17.2.3, it suffices to show that there is $h > 0$ such that the action of G_0 on V^0 extends naturally to a continuous action of $\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$. Let $g_1, \dots, g_d \in G_0$ be a basis and $b_i \in \mathbb{Z}_p[[G_0]]$ the topological basis of the Iwasawa algebra. Let $f(\mathbf{b}) = \sum_\alpha a_\alpha \mathbf{b}^\alpha \in \mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$. By definition we have $\sup_\alpha(|a_\alpha| p^{-\frac{p^{-h}}{p-1}|\alpha|}) < 1$. But the condition on V^0 implies that $\|b_i\| \leq p^{-1}$ as operators of V^0. This shows that the multiplication of $f(\mathbf{b})$ on V^0 is well defined for h big enough, which provides the action of $\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$ on V^0. □

17.2.2. Locally analytic representations of $\mathfrak{g}$

In the applications to the flag variety, we will work with sheaves that are not G-equivariant, but which admit an equivariant action of the Lie algebra of G. To define properly the locally analytic condition in this case, we use a variant of the distribution algebras of §17.1.1, which were already considered by Emerton in [Eme17]. Let $\mathfrak{g} = \operatorname{Lie} G$ and let $\mathfrak{g}^0 \subset \mathfrak{g}$ be a $\mathbb{Z}_p$-lattice satisfying $[\mathfrak{g}^0, \mathfrak{g}^0] \subset p\mathfrak{g}^0$. Let $Y_1, \dots, Y_d$ be a $\mathbb{Z}_p$-basis of $\mathfrak{g}^0$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Given $\alpha \in \mathbb{N}^d$ we denote $\underline{Y}^\alpha := Y_1^{\alpha_1} \cdots Y_d^{\alpha_d}$, by the Poicaré-Birkhoff-Witt theorem one has that

$$U(\mathfrak{g}) = \bigoplus_{\alpha \in \mathbb{N}^d} \mathbb{Q}_p \underline{Y}^\alpha.$$

Definition 17.2.7. Let $h > 0$ be rational, we define the following distribution algebras

$$\mathcal{D}(\mathfrak{g}_h, \mathbb{Z}_p) = \{\sum_{\alpha \in \mathbb{N}^d} a_\alpha \underline{Y}^\alpha : \sup_\alpha\{|a_\alpha| p^{h|\alpha|}\} \leq 1\}$$

and $\mathcal{D}(\mathfrak{g}_h, \mathbb{Q}_p) = \mathcal{D}(\mathfrak{g}_h, \mathbb{Z}_p)[\frac{1}{p}]$. We also set

$$\mathcal{D}(\mathfrak{g}_h, K) := \varprojlim_n (\mathcal{O}_K/p^n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathcal{D}(\mathfrak{g}_h, \mathbb{Z}_p)/p^n)[\frac{1}{p}]$$

Remark 17.2.8. Similarly as for the distribution algebras $\mathcal{D}_{(h)}(G_0, \mathbb{Z}_p)$, the algebra $\mathcal{D}(\mathfrak{g}_h, \mathbb{Z}_p)$ is profinite. Indeed, if h is rational and K is a finite extension of $\mathbb{Q}_p$ having an element p^h of p-adic valuation $|p^h| = p^{-h}$, then we have

$$\mathcal{D}(\mathfrak{g}_h, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_K = \prod_{\alpha \in \mathbb{N}^d} \mathcal{O}_K p^{h|\alpha|} \underline{Y}^\alpha.$$

Inspired by Theorem 17.2.3 we make the following definition:

Definition 17.2.9. Let $F^\bullet$ be an *LF* complex of $\mathfrak{g}$-modules.

1. The derived $\mathfrak{g}_h$-analytic vectors of $F^\bullet$ is the complex
$$F^{\bullet,R\mathfrak{g}_h-an} := R\mathrm{Hom}_{\mathfrak{g}}(\mathcal{D}(\mathfrak{g}_h, K), F^\bullet).$$
We say that $F^\bullet$ is derived $\mathfrak{g}_h$-analytic if the natural map $F^{\bullet R\mathfrak{g}_h-an} \to F^\bullet$ is a quasi-isomorphism.
2. The derived locally analytic vectors of $F^\bullet$ is the complex
$$F^{\bullet,Rla} := \varinjlim_{h\to\infty} F^{\bullet,R\mathfrak{g}_h-an}.$$
We say that $F^\bullet$ is derived locally analytic if the natural map $F^{\bullet,Rla} \to F^\bullet$ is a quasi-isomorphism.

The same proof of Theorem 17.2.3 ([RC21a, Theo. 4.34]) implies the following version with the Lie algebra.

Theorem 17.2.10. *Let $F^\bullet$ be an LF complex of $\mathfrak{g}$-modules. Then $F^\bullet$ is derived $\mathfrak{g}_h$-analytic if and only if $F^\bullet$ is a complex of $\mathcal{D}(\mathfrak{g}_h, K)$-modules.*

One easily deduces the following analogous of Corollary 17.2.6.

Corollary 17.2.11. *Let V be a Banach space endowed with an action of $\mathfrak{g}$. Then V is a locally analytic $\mathfrak{g}$-module, even $\mathfrak{g}_h$ analytic for some $h > 0$.*

Finally, let us define the locally analytic distribution algebra of $\mathfrak{g}$ as the Fréchet-Stein algebra $\mathcal{D}^{la}(\mathfrak{g}, K) = \varprojlim_{h\to\infty} \mathcal{D}(\mathfrak{g}_h, K)$. We also denote $C(\mathfrak{g}_h, K) := \mathrm{Hom}_K(\mathcal{D}(\mathfrak{g}_h, K), K)$ and $C^{la}(\mathfrak{g}, K) = \mathrm{Hom}_K(\mathcal{D}^{la}(\mathfrak{g}, K), K) = \varinjlim_{h\to\infty} C(\mathfrak{g}_h, K)$. The space $C^{la}(\mathfrak{g}, K)$ is nothing but the space of germs at the identity of $C^{la}(G, K)$.

Definition 17.2.12. A $G^\dagger$-representation is a direct system $V = \varinjlim_n V_n$ of G_n-modules with equivariant transition maps, where $G_n \to 1$ as $n \to \infty$.

Remark 17.2.13. A $\mathfrak{g}$-locally analytic representation is a $G^\dagger$-representation since the $\mathfrak{g}_h$-analyticity condition implies that the action of $\mathfrak{g}$ can be integrated to some neighbourhood $G' \subset G$ of 1.

17.2.3. Cohomological comparison theorems

As an application of Theorem 17.2.3 one has the following comparison results in group cohomology.

Theorem 17.2.14 ([RJRC21, 5.2])**.** *Let $F^\bullet$ be a complex of LF representations of G, then there is a natural quasi-isomorphism of cohomology complexes*
$$R\mathrm{Hom}_G(K, F^\bullet) = R\mathrm{Hom}_G(K, F^{\bullet,Rla}). \tag{17.2.2}$$
In particular, if F is an LF representation and $F^{Rla} = F^{la}[0]$, we have that
$$R\mathrm{Hom}_G(K, F) = R\mathrm{Hom}_G(K, F^{la}).$$

Remark 17.2.15. The proof of the theorem also shows an analytic version: let $h > 0$ and let $\mathcal{G}$ denote $\mathbb{G}^{(h^+)}$ or $G_{(h)}$, then
$$R\mathrm{Hom}_G(K, F^\bullet) = R\mathrm{Hom}_G(K, F^{\bullet,R\mathcal{G}-an}).$$
In fact, one of the main technical ingredients is the equality $\mathcal{D}(G_{(h)}, K)\otimes^L_{K[[G]]_\blacksquare}\mathcal{D}(G_{(h)}, K) = \mathcal{D}(G_{(h)}, K)$, where the thensor product is taken in the category of solid K-vector spaces. The previous theorem follows formally from this, and the description of $F^{\bullet,RG_{(h)}-an}$ in terms of $\mathcal{D}(G_{(h)}, K)$.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Another application of Theorem 17.2.3 is the comparison of cohomologies for locally analytic representations.

Theorem 17.2.16 ([RJRC21, Theo. 5.4]). *Let $\mathcal{G}$ denote $\mathbb{G}^{(h^+)}$ or G_h and let $F^\bullet$ be an LF complex of derived $\mathcal{G}$-analytic representations. Then there are quasi-isomorphisms of cohomology complexes*

$$R\mathrm{Hom}_G(K, F^\bullet) = R\mathrm{Hom}_{\mathcal{D}(\mathcal{G},K)}(K, F^\bullet) = R\mathrm{Hom}_{U(\mathfrak{g})}(K, F^\bullet)^G.$$

In particular, if F is an locally analytic LF representation of G then

$$R\mathrm{Hom}_G(K, F) = R\mathrm{Hom}_{U(\mathfrak{g})}(\mathfrak{g}, F)^G.$$

Remark 17.2.17. The invariants $R\mathrm{Hom}_{U(\mathfrak{g})}(K, F^\bullet)^G$ are well defined as the Lie algebra cohomology complex $R\mathrm{Hom}_{U(\mathfrak{g})}(K, F^\bullet)$ lands in the category of smooth K-representations of G. Since K is of characteristic 0 and G is compact, taking invariants is an exact functor for smooth representations.

Another key ingredient in the proof of the previous theorem is the flatness of the distribution algebras $\mathcal{D}(G_{(h)}, K)$ over the Iwasawa algebra, and the flatness of $\mathcal{D}(\mathfrak{g}_h, K)$ over $(\mathfrak{g})$. In fact, we only need to know that tensoring with the distribution algebras preserves the trivial representation. We have the following version of the previous theorems for the enveloping algebra:

Theorem 17.2.18. *Let $F^\bullet$ be an LF module over $U(\mathfrak{g})$. Then*

$$R\mathrm{Hom}_{U(\mathfrak{g})}(K, F^\bullet) = R\mathrm{Hom}_{U(\mathfrak{g})}(K, F^{\bullet,Rla}) = R\mathrm{Hom}_{\mathcal{D}(\mathfrak{g}_h,K)}(K, F^{\bullet,R\mathfrak{g}_h-an}).$$

17.3. Locally analytic sheaves over rigid spaces

In this section we define a notion of *LF* sheaf over adic space. We impose some quasi-coherent conditions for the sheaves to be acyclic on affinoid spaces. Then, we give an ad-hoc definition of a locally analytic equivariant action of a p-adic Lie group G on *LF* sheaves.

17.3.1. *LF* sheaves over adic spaces

Definition 17.3.1. Let X be a adic space over $\mathrm{Spa}(K, O_K)$.

1. An *LF* sheaf of $\mathscr{O}_{X,\mathrm{an}}$-modules is a topological $\mathscr{O}_{X,\mathrm{an}}$-sheaf $\mathscr{F}$ such that there exists a basis $\mathcal{B}$ of X_{an} by quasi-compact open subspaces such that $\mathscr{F}(U)$ is an *LF* space for all $U \in \mathcal{B}$.

2. An *LF* sheaf $\mathscr{F}$ of $\mathscr{O}_{X,\mathrm{an}}$-modules is said *quasi-coherent* if there exists an open cover $\{U_i\}_{i\in I}$ of X_{an} by affinoids, such that for all $i \in I$ and $V \subset U_i$ an open affinoid we have $\mathscr{F}(V) = \mathscr{F}(U)\widehat{\otimes}_{\mathscr{O}_{X,\mathrm{an}}(U)}\mathscr{O}_{X,\mathrm{an}}(V)$.

3. A quasi-coherent Banach sheaf $\mathscr{F}$ of $\mathscr{O}_{X,\mathrm{an}}$-modules is called *locally projective* if there is a covering $\{U_i\}_{i\in I}$ of X_{an} by affinoids as in (2) satisfying the following condition: $\mathscr{F}(U_i)$ is a projective Banach $\mathscr{O}_{X,\mathrm{an}}(U_i)$-module, i.e., a direct sumand of an ON $\mathscr{O}_{X,\mathrm{an}}(U_i)$-module.

4. A quasi-coherent Fréchet sheaf $\mathscr{F}$ on X is said *squarable* if there exists an open cover $\{U_i\}_{i\in I}$ of X_{an} by affinoids as in (2) satisfying the following condition: $\mathscr{F}(U_i) = \varprojlim_n B_{i,n}$ with dense transition maps, where the $B_{i,n}$ are projective Banach $\mathscr{O}_{X,\mathrm{an}}(U_i)$-modules. We say that the cover $\{U_i\}_{i\in I}$ *squares* $\mathscr{F}$ and that $\mathscr{F}$ is *squared* on U_i.

Lemma 17.3.2. *Let $\mathscr{F}$ be a quasi-coherent Fréchet sheaf over X and let $U \subset X$ be an open affinoid such that $\mathscr{F}$ is squared on U. Then $R\Gamma_{\mathrm{an}}(U, \mathscr{F}) = \mathscr{F}(U)$.*

Proof. Let $\mathscr{F}(U) = \varprojlim_n B_n$ be a presentation of $\mathscr{F}(U)$ as an inverse limit of projective Banach $\mathscr{O}_{X,\mathrm{an}}(U)$-modules with dense transition maps. Let $\mathscr{B}_n$ be the sheaf over U mapping $V \subset U$ to $\mathscr{B}_n(V) := B_n \widehat{\otimes}_{\mathscr{O}_{X,\mathrm{an}}(U)} \mathscr{O}_{X,\mathrm{an}}(V)$. Then $\mathscr{B}_n$ is a locally projective Banach sheaf over U. Formally one gets that

$$R\Gamma_{\mathrm{an}}(U, \mathscr{F}) = R\varprojlim_n R\Gamma_{\mathrm{an}}(U, \mathscr{B}_n).$$

Thus, by topological Mittag-Leffler [SGA70, Remarque 13.2.4], it is enough to show that $R\Gamma_{\mathrm{an}}(U, \mathscr{B}_n) = R\Gamma_{\mathrm{an}}(U, \mathscr{B}_n) = B_n$. But B_n is a direct summand of an orthonormalizable Banach $\mathscr{O}_{X,\mathrm{an}}(U)$-module. Hence it suffices to show that

$$R\Gamma_{\mathrm{an}}(U, \widehat{\bigoplus_{i\in I}} \mathscr{O}_{X,\mathrm{an}}) = \widehat{\bigoplus_{i\in I}} \mathscr{O}_{X,\mathrm{an}}(U)$$

which follows from Tate's acyclicity theorem. □

17.3.2. Locally analytic sheaves over adic spaces

Definition 17.3.3 ([Sch18, Def. 2.1])**.** Let X be an adic space and G a profinite group. An action of G on X is said to be continuous if for any open affinoid $U = \mathrm{Spa}(A, A^+) \subset X$ there exists an open subgroup $G' \subset G$ stabilizing U such that the action morphism $G' \times A \to A$ is continuous.

Lemma 17.3.4. *Let X be a locally noetherian adic space over $\mathrm{Spa}(K, O_K)$ endowed with a continuous action of G. Let $U = \mathrm{Spa}(A, A^+) \subset X$ be an open affinoid and G_U the stabilizer of U in G. Then the action of G_U on A is locally analytic.*

Proof. Let $A_0 \subset A^+$ be a ring of definition. As X is locally noetherian over $\mathrm{Spa}(K, O_K)$, we can assume that A_0 is topologically finitely generated over O_K by $f_1, \dots, f_s \in A^+$. Moreover, without loss of generality we can assume that G_U acts on A_0. By Corollary 17.2.6, it is enough to show that G_U acts through a finite quotient on A_0/p. But A_0/p is a finitely generated algebra over O_K/p, generated by $f_1, \dots, f_s \mod p$. As the action of G_U on A_0/p is continuous, we know that there exists an open subgroup $G' \subset G_U$ fixing $f_1, \dots, f_s \mod p$. This proves the lemma. □

Let X be a locally noetherian adic space over $\mathrm{Spa}(K, O_K)$ endowed with a continous action of G. By Lemma 17.3.4, the action of G on the structural sheaf of X is locally analytic after taking sections in open quasi-compact subspaces of X. This leads to the following definition:

Definition 17.3.5. Let $\mathscr{F}$ be a G-equivariant LF sheaf over X_{an}. We say that $\mathscr{F}$ is locally analytic if for any quasi-compact open subset $U \subset X$ the LF space $\mathscr{F}(U)$ is a locally analytic G_U-representation.

Remark 17.3.6. Notice that for a general LF sheaf over X one needs to check the locally analyticity condition in a base for the analytic topology. If in addition $\mathscr{F}$ is quasi-coherent, the locally analyticity only needs to be checked on an affinoid covering as in Definition 17.3.1 (2).

Definition 17.3.7. Let $\mathfrak{g} = \mathrm{Lie}\, G$.

1. A $\mathfrak{g}$-equivariant LF sheaf over X_{an} is an LF sheaf of $\mathscr{O}_{X,\mathrm{an}}$-modules endowed with an infinitesimal action of $\mathfrak{g}$ such that for any $f \in \mathscr{O}_{X,\mathrm{an}}$, $v \in \mathscr{F}$ and $Y \in \mathfrak{g}$, one has
$$Y \cdot (fv) = (Y \cdot f)v + f(Y \cdot f).$$
2. A $\mathfrak{g}$-equivariant sheaf $\mathscr{F}$ over X_{an} is said locally analytic if there exists a basis of affinoid neighbourhoods $\{U_i\}_{i\in I}$ of X such that $\mathscr{F}(U_i)$ is a locally analytic $\mathfrak{g}$-module for all i, in the sense of Definition 17.2.9.

Let X be a locally noetherian fs log adic space over $\mathrm{Spa}(K, O_K)$, see [DLLZ19] for the definition of a log adic space and the (pro-)Kummer-étale site. Let G be a compact p-adic Lie group and $\widetilde{X} \to X$ a G-pro-Kummer-étale torsor. For a closed subgroup $H \subset G$ the quotient $\widetilde{X}/H$ is a profinite-Kummer-étale covering of X, if H is open it is actually finite Kummer-étale over X. For a log adic space Y we let $Y_{\text{két,qcqs}}$ denote the site of qcqs Kummer-étale adic spaces over Y. By construction ([DLLZ19, §5.1]) one has

$$(\widetilde{X}/H)_{\text{két,qcqs}} = 2\text{-}\varprojlim_{H\subset G'} (\widetilde{X}/G')_{\text{két,qcqs}}, \tag{17.3.1}$$

where G' runs over all the compact open subgroups of G containing H. In particular, an object $U \in (\widetilde{X})_{\text{két,qcqs}}$ has a G'-equivariant action over $\widetilde{X}$ for some compact open subgroup $G' \subset G$. We will need the following definition later on:

Definition 17.3.8. Let $\widehat{\mathscr{O}}_X$ denote the completed structural sheaf over $X_{\text{prokét}}$.

1. The completed sheaf $\widehat{\mathscr{O}}_{\widetilde{X},\text{két}}$ is by definition the restriction of $\widehat{\mathscr{O}}_X$ to $\widetilde{X}_{ket}$. The sheaf of locally analytic sections of $\widehat{\mathscr{O}}_{\widetilde{X},\text{két}}$ is the sheaf sending $U \in \widetilde{X}_{\text{két,qcqs}}$ to

$$\mathscr{O}^{la}_{\widetilde{X},\text{két}}(U) = \widehat{\mathscr{O}}_X(U)^{G_U - la}$$

   where $G_U \subset G'$ is an open compact subgroup acting on U. We will denote by $\mathscr{O}^{la}_{\widetilde{X}}$ the restriction of $\mathscr{O}^{la}_{\widetilde{X},\text{két}}$ to the analytic site of $\widetilde{X}$.

2. We let $\mathscr{O}^{sm}_{\widetilde{X},\text{két}} \subset \widetilde{\mathscr{O}}_{\widetilde{X},\text{két}}$ denote the subsheaf of smooth section in qcqs open subsets. Let $\mathscr{O}^{sm}_{\widetilde{X}}$ be the restriction of $\mathscr{O}^{sm}_{\widetilde{X},\text{két}}$ to $\widetilde{X}_{\text{an}}$, equivalently, $\mathscr{O}^{sm}_{\widetilde{X}} = \varinjlim_{G'\subset G} \mathscr{O}_{\widetilde{X}/G',\text{an}}$ where G' runs over all the open subgroups of G.

3. An LF sheaf over $\widetilde{X}_{\text{két}}$ is a topological sheaf over $\widetilde{X}_{\text{két}}$ such that for all $U \in \widetilde{X}_{\text{két,qcqs}}$ the space $\mathscr{F}(U)$ is an LF space. We say that $\mathscr{F}$ is $G^\dagger$ equivariant if there exists a basis $\mathcal{B}$ of objects in $\widetilde{X}_{\text{két,qcqs}}$ such that for all $U \in \mathcal{B}$, the sheaf $\mathscr{F}|_U$ can be written as a colimit of G_n-equivariant LF sheaves $\mathscr{F}|_U = \varinjlim_n \mathscr{F}_n$ over U, where $G^\dagger = \{G_n\}_{n\in\mathbb{N}}$ is a decreasing basis of open compact subgroups of G stabilizing U. We refer to the equivariant action of the decreasing open subgroups $\{G_n\}$ as the $G^\dagger$-action.

4. A $G^\dagger$-equivariant LF sheaf is said locally analytic if for any $U \in \widetilde{X}_{\text{két}}$ the action of $G^\dagger$ on $\mathscr{F}(U)$ is locally analytic.

Remark 17.3.9. By (17.3.1), the equivariant action over an object $U \in \widetilde{X}_{\text{két}}$ is only well defined infinitesimally closed to 1. Namely, if G_1 and G_2 are two compact open subgroups of G action equivariantly on U, then there exists $G_3 \subset G_1 \cap G_3$ compact open such that the action of G_1 and G_2 agree on G_3. Therefore, only the action of $G^\dagger$ is well defined in objects of $\widetilde{X}_{\text{két,qcqs}}$, in other words, there is not a well defined stabilizer G_U. In particular, as the locally analyticity condition only depends on $G^\dagger$, parts (1) and (3) of the previous definition make sense.

18. Cohomology of a filtered space

In this chapter we recall the notions of cohomology with compact and closed supports of a topological space. Then, given a topological space filtered by finitely many open subsets, we attach two spectral sequences which heuristically are dual in their construction. The results presented here are technical, and will be used in the construction of the BGG maps of §23. As a better reference we send to [Sta20, Tag 01DW].

18.1. Cohomology with closed and compact supports

Let X be a topological space, $\iota : Z \subset X$ a closed subspace and $j : U = X\backslash Z \subset X$. For a topological space Y, let Ab_Y denote the category of abelian sheaves over Y, and $D(\mathrm{Ab}_Y)$ its derived category. We denote by $\mathrm{Ab} = \mathrm{Ab}_*$ the category of abelian groups. The inclusions $\iota : Z \subset X$ and $j : U \subset X$ induce pairs of adjoint functors

$$\iota^{-1} : D(\mathrm{Ab}_X) \rightleftarrows D(\mathrm{Ab}_Z) : \iota_*, \quad j^{-1} : D(\mathrm{Ab}_X) \rightleftarrows D(\mathrm{Ab}_U) : Rj_*$$

called the restriction and the (derived) pushforward respectively. Moreover, the functor ι_* (resp. the functor j^{-1}) has a right adjoint (resp. a left adjoint)

$$\iota_* : D(\mathrm{Ab}_Z) \rightleftarrows D(\mathrm{Ab}_X) : R\iota^!, \quad j_! : D(\mathrm{Ab}_U) \rightleftarrows D(\mathrm{Ab}_X) : j^{-1},$$

where $R\iota^!$ is the derived sections with supports at Z (resp. $j_!$ is the extension by zero). The definition of $j_!$ is standard, let us explain how $R\iota^!$ is constructed. Consider the functor of "sections with closed supports in Z"

$$\Gamma_Z(X, -) : \mathrm{Ab}_X \longrightarrow \mathrm{Ab}$$

$$\mathscr{F} \longmapsto \Gamma_Z(X, \mathscr{F}) := \{s \in \mathscr{F}(X) : \mathrm{supp}(s) \subset Z\},$$

where $\mathrm{supp}(s)$ is the support of the section s. One defines the functor of "sections with support in Z" to be the functor $\iota^! : \mathrm{Ab}_X \to \mathrm{Ab}_Z$ sending a sheaf $\mathscr{F}$ to

$$V \subset Z \mapsto \iota^!\mathscr{F}(V) = \Gamma_V(U, \mathscr{F}),$$

for $U \subset X$ any open subset such that $U \cap Z = V$. The functor $R\iota^!$ is then the right derived functor of $\iota^!$. It is not hard to see that given an object $M \in D(\mathrm{Ab}_X)$ we have fundamental exact triangles

$$j_!j^{-1}M \to M \to \iota_*\iota^{-1}M \xrightarrow{+}$$
$$\iota_*R\iota^!M \to M \to Rj_*j^{-1}M \xrightarrow{+}.$$

Definition 18.1.1. Let X be a topological space and $\iota_C : C \subset X$ a locally closed subspace. Let $j : U \subset X$ be an open subset such that $k : C \subset U$ is closed. The lower and upper shriek functors are given by

$$\iota_! = j_! \circ k_* : D(\mathrm{Ab}_C) \to D(\mathrm{Ab}_X), \quad R\iota^! = Rk^! \circ j^{-1} : D(\mathrm{Ab}_X) \to D(\mathrm{Ab}_Z).$$

In this situation we have pairs of adjoint functors

$$\iota_C^{-1} : D(\mathrm{Ab}_X) \rightleftarrows D(\mathrm{Ab}_C) : R\iota_{C,*}, \quad \iota_{C,!} : D(\mathrm{Ab}_C) \rightleftarrows D(\mathrm{Ab}_X) : R\iota_C^!.$$

We can now define the cohomology with compact and closed supports in a locally closed subspace C of X.

Definition 18.1.2. Let X be a topological space and $\iota_C : C \subset X$ a locally closed subspace. Let $M \in D(\mathrm{Ab}_X)$.

1. We define the cohomology of M with compact supports on C to be the complex
$$R\Gamma_{c,C}(X, M) := R\Gamma(X, \iota_{C,!}\iota_C^{-1}M).$$
2. We define the cohomology of M with closed supports on C to be the complex
$$R\Gamma_C(X, M) := R\Gamma(X, R\iota_{C,*}R\iota_C^{!}M).$$

We also write $H^i_{c,C}(X, -) = H^i(R\Gamma_{c,C}(X, -))$ and $H^i_C(X, -) = H^i(R\Gamma_C(X, -))$ for $i \in \mathbb{Z}$.

The cohomologies with compact and closed supports satisfy the following properties:

Proposition 18.1.3. *Let X be a topological space and $\iota_C : C \subset X$ a locally closed subspace. The following hold*

1. *Let $Y \subset X$ be a locally closed subspace containing an open neighbourhood of C in X. Then $R\Gamma_C(X, -) = R\Gamma_C(Y, -)$. Similarly, if Y contains $\overline{C}$ then $R\Gamma_{c,C}(X, -) = R\Gamma_{c,C}(Y, -)$.*
2. *Let $C = C_1 \bigsqcup C_2$ be a disjoint union of locally closed subsets of X such that there exists an open $U \subset X$ containing both C_1 and C_2 as closed subspaces. We have*
$$R\Gamma_{c,C}(X, -) = R\Gamma_{c,C_1}(X, -) \oplus R\Gamma_{c,C_2}(X, -)$$
$$R\Gamma_C(X, -) = R\Gamma_{C_1}(X, -) \oplus R\Gamma_{C_2}(X, -).$$
3. *Let $C' \subset C \subset X$ be another locally closed subspace. If $C' \subset C$ is closed we have functorial restriction/correstriction maps*
$$R\Gamma_{c,C}(X, -) \to R\Gamma_{c,C'}(X, -)$$
$$R\Gamma_{C'}(X, -) \to R\Gamma_C(X, -).$$
If $C' \subset C$ is open we have functorial correstriction/restriction maps
$$R\Gamma_{c,C'}(X, -) \to R\Gamma_{c,C}(X, -)$$
$$R\Gamma_C(X, -) \to R\Gamma_{C'}(X, -).$$

Proof. In the following we write $\mathrm{In}(V, W) : V \subset W$ for the inclusion map.

1. Let $\mathscr{F} \in \mathrm{Ab}_X$, if Y contains a neighbourhood of C, we can write $C \subset U \subset Y$ with U an open subspace of X. Then
$$\begin{aligned} R\mathrm{In}(C, X)_*R\mathrm{In}(C, X)^! &= R\mathrm{In}(U, X)_*R\mathrm{In}(C, U)_*R\mathrm{In}(C, U)^!\mathrm{In}(U, X)^{-1}. \\ R\mathrm{In}(C, Y)_*R\mathrm{In}(C, Y)^! &= R\mathrm{In}(U, Y)_*R\mathrm{In}(C, U)_*R\mathrm{In}(C, U)^!\mathrm{In}(U, Y)^{-1}. \end{aligned}$$
Taking cohomology one gets
$$R\Gamma_C(X, \mathscr{F}) = R\Gamma_C(U, \mathscr{F}|_U) = R\Gamma_C(Y, \mathscr{F}|_Y).$$
Similarly, if Y contains $\overline{C}$, one has
$$\begin{aligned} \mathrm{In}(C, X)_!\mathrm{In}(C, X)^{-1} &= \mathrm{In}(\overline{C}, X)_*\mathrm{In}(C, \overline{C})_!\mathrm{In}(C, X)^{-1} \\ \mathrm{In}(C, Y)_!\mathrm{In}(C, Y)^{-1} &= \mathrm{In}(\overline{C}, Y)_*\mathrm{In}(C, \overline{C})_!\mathrm{In}(C, Y)^{-1} \end{aligned}$$
Taking cohomology we find
$$R\Gamma_{c,C}(X, \mathscr{F}) = R\Gamma_{c,C}(\overline{C}, \mathscr{F}|_{\overline{C}}) = R\Gamma_{c,C}(Y, \mathscr{F}|_Y).$$

2. The inclusions $\mathrm{In}(C_i, U)$ (with $i = 1, 2$) and $\mathrm{In}(C, U)$ are closed immersions. Thus

$$\mathrm{In}(C,X)_!\mathrm{In}(C,X)^{-1} = \mathrm{In}(U,X)_!\mathrm{In}(C_1,U)_!\mathrm{In}(C_1,X)^{-1} \oplus \mathrm{In}(U,X)_!\mathrm{In}(C_2,U)_!\mathrm{In}(C_2,X)^{-1}$$
$$R\mathrm{In}(C,X)_*R\mathrm{In}(C,X)^! = R\mathrm{In}(C_1,X)_*R\mathrm{In}(C_1,U)^!\mathrm{In}(U,X)^{-1} \oplus R\mathrm{In}(C_2,X)_*R\mathrm{In}(C_2,U)^!\mathrm{In}(U,X)^{-1},$$

taking cohomology one gets (2).

3. Let us first suppose that $\mathrm{In}(C', C)$ is a closed immersion. Then $\mathrm{In}(C', C)_! = \mathrm{In}(C', C)_* = R\mathrm{In}(C', C)_*$, and given $\mathscr{F} \in \mathrm{Ab}_X$ we have a natural maps

$$\mathrm{Res} : \mathrm{In}(C,X)^{-1}\mathscr{F} \to \mathrm{In}(C',C)_*\mathrm{In}(C',X)^{-1}\mathscr{F}$$
$$\mathrm{Cor} : \mathrm{In}(C',C)_*R\mathrm{In}(C',X)^!\mathscr{F} \to R\mathrm{In}(C,X)^!\mathscr{F}.$$

Composing with $\mathrm{In}(C, X)_!$ and $R\mathrm{In}(C, X)_*$ respectively, one obtains maps

$$\mathrm{Res} : \mathrm{In}(C,X)_!\mathrm{In}(C,X)^{-1}\mathscr{F} \to \mathrm{In}(C',X)_!\mathrm{In}(C',X)^{-1}\mathscr{F}$$
$$\mathrm{Cor} : R\mathrm{In}(C',X)_*R\mathrm{In}(C',X)^!\mathscr{F} \to R\mathrm{In}(C,X)_*R\mathrm{In}(C,X)^!\mathscr{F}.$$

Taking cohomology one obtains the restriction/correstriction maps. The case when $C' \subset C$ is open is similar.

□

We will need the following lemma which says that the cohomologies defined above can be approximated by "smaller cohomologies" under certain overconvergent hypothesis.

Lemma 18.1.4. *Let X be a spectral space and $C \subset X$ a locally closed subspace. Suppose that the following hypothesis hold*

1. *There exists a family $\{U^\alpha\}_{\alpha\in\mathbb{Q}_{\geq 0}}$ of decreasing open neighbourhoods of C such that*
   (a) *$C = \bigcap_{\alpha>0} U^\alpha$ and C is closed in all U^α.*
   (b) *For all $\alpha > 0$ we have $\overline{U^\alpha} \cap U^0 = \bigcap_{\alpha'<\alpha} U^\alpha$.*
2. *There exists an increasing covering by qcqs open subspaces $U^0 = \bigcup_{\beta\in\mathbb{Q}_{\geq 0}} U^0_\beta$ satisfying:*
   (a) *The intersection $U^\alpha_\beta = U^\alpha \cap U^0_\beta$ is qcqs for all $\alpha, \beta \geq 0$.*
   (b) *The intersections $C_\beta := C \cap U^\alpha_\beta$ are qcqs subspaces of C.*
   (c) *For all $\alpha, \beta > 0$ we have $\overline{U^\alpha_\beta} = \bigcap_{\substack{\alpha'<\alpha\\ \beta'>\beta}} U^{\alpha'}_{\beta'}$.*

Given $\alpha, \beta > 0$ let us denote $U^\alpha_{\beta^-} = \bigcup_{\beta'<\beta} U^\alpha_{\beta'}$ and $U^{\alpha^+}_\beta = \bigcup_{\alpha'>\alpha} U^{\alpha'}_\beta$. Then, given a sheaf $\mathscr{F}$ on U^0, there are natural quasi-isomorphisms

$$R\Gamma_{c,C}(X,\mathscr{F}) = \varinjlim_{\alpha,\beta\to\infty} R\Gamma_{c,U^\alpha_{\beta^-}}(U^\alpha_{\beta+1},\mathscr{F})$$
$$R\Gamma_C(X,\mathscr{F}) = R\varprojlim_{\alpha,\beta\to\infty} R\Gamma_{\overline{U^{\alpha+1^+}_\beta}\cap U^\alpha_\beta}(U^\alpha_\beta,\mathscr{F}).$$

Proof. As notation, let us write $\mathrm{In}(Z, Y) : Z \subset Y$ for the inclusion maps. By conditions 2.(b) and 2.(c) we have

$$C = \bigcup_{\beta\to\infty} C_\beta = \bigcup_{\beta\to\infty} \overline{C_\beta}.$$

This implies that

$$\varinjlim_{\beta\to\infty} \mathrm{In}(C_\beta, X)_!\mathrm{In}(C_\beta, X)^{-1} \xrightarrow{\sim} \mathrm{In}(C, X)_!\mathrm{In}(C, X)^{-1}$$
$$R\mathrm{In}(C, X)_*R\mathrm{In}(C, X)^! \xrightarrow{\sim} R\varprojlim_{\beta\to\infty} R\mathrm{In}(C_\beta, X)_*R\mathrm{In}(C_\beta, X)^!$$

where the maps are the correstriction and restriction maps respectively. Indeed, as $C_\beta \subset C_{\beta'}$ is open for $\beta' > \beta$, these maps exist. Then, it suffices to check the equality after evaluating at a sheaf and for the non-derived functors, in such a case one can just go to the stalks and prove that the arrows above are isomorphisms. As X is a spectral space, it is qcqs and admits a basis of quasi-compact objects, this implies that taking global cohomology commutes with filtered colimits. Therefore, taking cohomology over X one finds

$$R\Gamma_{c,C}(X, \mathscr{F}) = \varinjlim_{\beta\to\infty} R\Gamma_{c,C_\beta}(U^\alpha, \mathscr{F}) \tag{18.1.1}$$
$$R\Gamma_C(X, \mathscr{F}) = R\varprojlim_{\beta\to\infty} R\Gamma_{C_\beta}(U^\alpha, \mathscr{F}) \tag{18.1.2}$$

for any $\alpha \geq 0$.

Next, we are going to approximate $R\Gamma_{c,C_\beta}(U^\alpha, \mathscr{F})$ and $R\Gamma_{C_\beta}(U^\alpha, \mathscr{F})$ with "smaller cohomologies", we split the problem in two cases:

(a) **Case of $R\Gamma_{C_\beta}$.** It is enough to show that there is an isomorphism commuting with the restriction maps along β

$$R\mathrm{In}(C_\beta, U^\alpha)_*R\mathrm{In}(C_\beta, U^\alpha)^! \to R\varprojlim_{\alpha'\to\infty} R\mathrm{In}(\overline{U_\beta^{\alpha'}} \cap U_\beta^\alpha, U^\alpha)_*R\mathrm{In}(\overline{U_\beta^{\alpha'}} \cap U_\beta^\alpha, U^\alpha)^!. \tag{18.1.3}$$

Indeed, taking cohomology over U^α we would get $R\Gamma_{C_\beta}(U^\alpha, \mathscr{F}) = R\varprojlim_{\alpha'\to\infty} R\Gamma_{\overline{U_\beta^{\alpha'}}\cap U_\beta^\alpha}(U_\beta^\alpha, \mathscr{F})$.

Since $C_\beta \subset \overline{U}_\beta^{\alpha''} \cap U_\beta^\alpha \subset \overline{U}_\beta^{\alpha'} \cap U_\beta^\alpha$ are closed immersions for $\alpha < \alpha' < \alpha''$, we have correstriction maps

$$R\mathrm{In}(C_\beta, U^\alpha)_*R\mathrm{In}(C_\beta, U^\alpha)^! \to R\mathrm{In}(\overline{U_\beta^{\alpha''}} \cap U_\beta^\alpha, U^\alpha)_*R\mathrm{In}(\overline{U_\beta^{\alpha'}} \cap U_\beta^\alpha, U^\alpha)^! \to$$
$$\to R\mathrm{In}(\overline{U_\beta^{\alpha'}} \cap U_\beta^\alpha, U^\alpha)_*R\mathrm{In}(\overline{U_\beta^{\alpha'}} \cap U_\beta^\alpha, U^\alpha)^!,$$

this induces the map (18.1.3). To show that it is an isomorphism it is enough to do it for the non-derived functors, and for $\mathscr{F}$ a sheaf over U_β^α. But then $\mathscr{F}$ applied to the LHS of (18.1.3) is the subsheaf of $\mathscr{F}$ of sections supported in C_β, and the RHS applied to $\mathscr{F}$ is the subsheaf of sections supported on $\bigcap_{\alpha'}(\overline{U_\beta^{\alpha'}} \cap U_\beta^\alpha) = C_\beta$, proving what we wanted.

(b) **Case of $R\Gamma_{c,C_\beta}$** It is enough to show that there is an isomorphism commuting with the restriction maps along β

$$\mathrm{In}(C_\beta, U^\alpha)_!\mathrm{In}(C_\beta, U^\alpha)^{-1} = \varinjlim_{\alpha'\to\infty} R\mathrm{In}(U^{\alpha'}, U^\alpha)_*\mathrm{In}(U_\beta^{\alpha'}, U^{\alpha'})_!\mathrm{In}(U_\beta^{\alpha'}, U^\alpha)^{-1}. \tag{18.1.4}$$

Indeed, taking cohomology over U^α one would get

$$R\Gamma_{c,C_\beta}(U^\alpha, \mathscr{F}) = \varinjlim_{\alpha'\to\infty} R\Gamma_{c,U_\beta^{\alpha'}}(U^{\alpha'}, \mathscr{F}).$$

Since $U^{\alpha''} \cap U_\beta^{\alpha'} = U_\beta^{\alpha''}$ for $\alpha' < \alpha''$, one has pullback maps on the RHS of (18.1.4)

$$\mathrm{In}(U_\beta^{\alpha'}, U^{\alpha'})_! \to R\mathrm{In}(U^{\alpha''}, U^{\alpha'})_* \mathrm{In}(U_\beta^{\alpha''}, U^{\alpha''})_!.$$

As $C_\beta \subset U_\beta^{\alpha'}$ is closed, we have restriction maps commuting with the direct system

$$R\mathrm{In}(U^{\alpha'}, U^{\alpha})_* \mathrm{In}(U_\beta^{\alpha'}, U^{\alpha'})_! \mathrm{In}(U_\beta^{\alpha'}, U^{\alpha})^{-1} \to \mathrm{In}(C_\beta, U^{\alpha})_! \mathrm{In}(C_\beta, U^{\alpha})^{-1}.$$

This proves the existence of the map (18.1.4). To show that it is an isomorphism one argues as for C_β in (a).

Using the previous computations, (18.1.1) and (18.1.2) we find

$$R\Gamma_{c,C}(X, \mathscr{F}) = \varinjlim_{\beta \to \infty} R\Gamma_{c,C_\beta}(U^\alpha, \mathscr{F}) = \varinjlim_{\alpha,\beta \to \infty} R\Gamma_{c,U_\beta^\alpha}(U_{\beta+1}^\alpha, \mathscr{F})$$

$$R\Gamma_C(X, \mathscr{F}) = R \varprojlim_{\beta \to \infty} R \varprojlim_{\beta \to \infty} R\Gamma_{C_\beta}(U^\alpha, \mathscr{F}) = R \varprojlim_{\beta,\alpha \to \infty} R\Gamma_{\overline{U_\beta^{\alpha+1} \cap U_\beta^\alpha}}(U_\beta^\alpha, \mathscr{F}).$$

Replacing U_β^α by $U_\beta^{\alpha^+}$ or $U_{\beta^-}^\alpha$ when necessary one gets the lemma. □

18.2. Spectral sequences attached to a filtered space

The cohomologies with closed and compact supports appear naturally when one tries to compute sheaf cohomology of a filtered space. In the following we shall denote by $\mathrm{In}(V, W)$ the inclusion map $V \subset W$.

Definition 18.2.1. Let X be a topological space, and $0 = U_{d+1} \subset U_d \subset \cdots \subset U_1 \subset U_0 = X$ a decreasing filtration consisting of open subspaces. Let us take $Z_i := X \backslash U_{d+1-i}$ for $i \in \mathbb{N}$ so that we also have a filtration $0 = Z_{d+1} \subset Z_d \cdots \subset Z_1 \subset Z_0 = X$ consisting of closed subspaces. Let us denote $C_k = Z_k \backslash Z_{k+1} = U_{d-k} \backslash U_{d+1-k}$.

1. There are natural correstriction morphisms of functors

   $$\mathrm{In}(U_{d+1}, X)_! \mathrm{In}(U_{d+1}, X)^{-1} \to \mathrm{In}(U_d, X)_! \mathrm{In}(U_d, X)^{-1} \to \cdots \to \mathrm{In}(U_0, X)_! \mathrm{In}(U_0, X)^{-1}$$

   fitting in distinguished triangles

   $$\mathrm{In}(U_{k+1}, X)_! \mathrm{In}(U_{k+1}, X)^{-1} \to \mathrm{In}(U_k, X)_! \mathrm{In}(U_k, X)^{-1} \to \mathrm{In}(C_{d-k}, X)_! \mathrm{In}(C_{d-k}, X)^{-1} \xrightarrow{+}.$$

   In particular, one has a spectral sequence of cohomologies with compact supports

   $$E_1^{p,q} = H_{c,C_{d-p}}^{p+q}(X, -) \Rightarrow H^{p+q}(X, -).$$

2. There are natural correstriction morphisms of functors

   $$R\mathrm{In}(Z_{d+1}, X)_* R\mathrm{In}(Z_{d+1}, X)^! \to R\mathrm{In}(Z_d, X)_* R\mathrm{In}(Z_d, X)^! \to \cdots \to R\mathrm{In}(Z_0, X)_* R\mathrm{In}(Z_0, X)^!$$

   fitting in distinguished triangles

   $$R\mathrm{In}(Z_{k+1}, X)_* R\mathrm{In}(Z_{k+1}, X)^! \to R\mathrm{In}(Z_k, X)_* R\mathrm{In}(Z_k, X)^! \to R\mathrm{In}(C_k, X)_* R\mathrm{In}(C_k, X)^! \xrightarrow{+}.$$

   In particular, one has a spectral sequence of cohomologies with closed supports

   $$E_1^{p,q} = H_{C_p}^{p+q}(X, -) \Rightarrow H^{p+q}(X, -).$$

19. Sen theory

Sen theory has shown to be a powerful tool in the Galois theory of p-adic fields. For example, it is used to compute Galois cohomology over period rings:

Proposition 19.0.1 ([Tat67, Prop. 8]). *Let $\mathbb{C}_p$ denote the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, and let $G_{\mathbb{Q}_p}$ denote the absolute Galois group. For $i \in \mathbb{Z}$ we let $\mathbb{C}_p(i)$ denote the i-th Tate twist. Then*

$$H^k(G_{\mathbb{Q}_p}, \mathbb{C}_p(i)) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{Q}_p & \text{if } i = 0 \text{ and } k = 0 \\ \mathbb{Q}_p \log \chi_{\mathrm{cyc}} & \text{if } i = 0 \text{ and } k = 1. \end{cases}$$

In [BC08], Berger-Colmez define an axiomatic framework where Sen theory can be applied. Using this formalism, different constructions attached to finite dimensional Galois representations become formally the same: the Sen module (relative to $\mathbb{Q}_p^{\mathrm{cyc}}$), the overconvergent (φ, Γ)-module (relative to $\widetilde{\mathbb{B}}^{\dagger}(\mathbb{Q}_p^{\mathrm{cyc}})$), the module $\mathbf{D}_{\mathrm{diff}}$ of differential equations (relative to $\mathbb{B}_{\mathrm{dR}}(\mathbb{Q}_p^{\mathrm{cyc}})$). Moreover, using Sen theory, Berger-Colmez describe in [BC16] the locally analytic vectors of completed Galois extensions of $\mathbb{Q}_p$ with group isomorphic to a p-adic Lie group.

The work of Lue Pan [Pan20] is another very important application of this tool. Inspired from the work of Berger-Colmez, Pan describes the p-adic Simpson correspondance of the $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant local systems of the modular curve in terms of the flag variety. Furthermore, using the strategy of [BC16] of approximating the space of locally analytic functions of (an open compact subgroup of) $\mathrm{GL}_2(\mathbb{Q}_p)$ by finite dimensional subrepresentations, he manages to use the axiomatic Sen theory to compute the p-adic Simpson correspondance of this "interpolation" of finite rank local systems.

The main goal of this chapter is to provide a more conceptual understanding of this interpolation process that occurs in the work of Pan and Berger-Colmez. More precisely, we will prove that the construction of the Sen module holds not only for finite rank representations of the profinite group Π (which is $G_{\mathbb{Q}_p}$ in classical Sen theory), but for a larger class of locally analytic representations (under some orthonormal assumptions). The theory developed here is not yet in its greater generality, but it will suffice for the purposes of this paper.

19.1. Colmez-Sen-Tate axioms

Let us introduce the terminology. Let (A, A^+) be a uniform affinoid $\mathbb{Q}_p$-algebra, given $B \subset A$ a closed subalgebra we denote $B^+ := B \cap A^+$. Let $d \geq 1$ be an integer, Π a profinite group and $\chi : \Pi \to \mathbb{Z}_p^d$ a surjective continuous character with kernel H. Given $\Pi' \subset \Pi$ an open subgroup and $H' := H \cap \Pi'$ we define the following objects:

- Let $N_{H'}$ be the normalizer of H' in Π.
- Let $\Gamma_{H'} = N_{H'}/H'$ and $C_{H'} \subset \Gamma_{H'}$ its center. By Lemma 3.1.1 of [BC08] the group $C_{H'}$ is open in $\Gamma_{H'}$.
- We let $n_1(H') \in \mathbb{N}$ be the smallest integer n such that $\chi(C_{H'})$ contains $p^n\mathbb{Z}_p^d$.

- More generally, given an open subgroup $C' \subset C_{H'}$ such that $\ker\chi \cap C' = 1$, we denote by $n(C') \in \mathbb{N}$ the smallest integer such that $p^n\mathbb{Z}_p^d \subset \chi(C')$.

- Let C' be as above. Let $e_1, \dots, e_d \in \mathbb{Z}_p^d$ be the standard basis, for $n \geq n(C')$ we let $\gamma_1^{(n)}, \dots, \gamma_d^{(n)}$ denote the inverse image of $p^n e_1, \dots, p^n e_d$ in C'. Thus, if $k \geq 0$, we have that $\gamma_i^{(n+k)} = (\gamma_i^{(n)})^{p^k}$. We shall write $\boldsymbol{\gamma}^{(n)} = (\gamma_1^{(n)}, \dots, \gamma_d^{(n)})$ and $\langle \boldsymbol{\gamma}^{(n)} \rangle \subset C'$ the generated subgroup.

Let us suppose that Π acts continuously on (A, A^+). Notice that the action of Π on A^+/p^s is smooth for any $s \geq 1$ since this last ring is discrete, we suppose in addition that $A^+/p^s = \varinjlim_{H' \subset H} A^{H',+}/p^s$ where H' runs over all the open subgroups of H. Let $I \subset \mathbb{Q}_{\geq 0}$ be a dense additive submonoid containing $\mathbb{N}$, suppose that there are topologically nilpotent units $\{\varpi^\epsilon\}_{\epsilon \in I_{>0}}$ in A^H such that

1. For any $x \in \mathrm{Spa}(A, A^+)$ we have $|\varpi^{\epsilon+\delta}|_x = |\varpi^\epsilon|_x |\varpi^\delta|_x$.

2. Let $\|\cdot\| : A \to \mathbb{R}$ be the norm making A^+ the unit ball and $\varpi^\epsilon A^+$ the ball of radius $p^{-\epsilon}$. Then $\|\cdot\|$ is a submultiplicative non-archimedean norm, i.e. it satisfies
   - $\|xy\| \leq \|x\|\|y\|$.
   - $\|x + y\| \leq \sup\{\|x\|, \|y\|\}$.

3. Π acts by isometries on $(A, \|\cdot\|)$. We have $\|p\| \leq \|\varpi^t\|$ and $\|p^{-1}\| \leq \|\varpi^{-1/t}\|$ for some $t > 0$.

Condition (1) implies that the elements ϖ^ϵ are multiplicative units for the norm $\|\cdot\|$. Moreover, the ideal of topologically nilpotent elements of A is equal to $A^{++} = \bigcup_{\epsilon \in I_{>0}} \varpi^\epsilon A^+$. From now on we always take $\epsilon \in I$. In the following we consider almost mathematics with respect to the sequence $\{\varpi^\epsilon\}_{\epsilon > 0}$.

Remark 19.1.1. In the main application of the paper the units ϖ^ϵ will be algebraic numbers over $\mathbb{Q}_p$ with p-adic valuation $|\varpi|^\epsilon = |p|^\epsilon$. We have decided to develop the theory in this slightly more general situation where ϖ^ϵ might not be algebraic over $\mathbb{Q}_p$, in order to include the framework of overconvergent (φ, Γ)-modules, where the elements ϖ^ϵ arise as Teichmüller lifts $[\pi^{\flat,\epsilon}]$ of p-power roots of $\pi = [\epsilon] - 1 \in \mathbb{Q}_p^{cyc,\flat}$, where $\epsilon = (\zeta_{p^n})_{n \geq 1}$ is a compatible sequence of p-power roots of unit, see [CC98].

Definition 19.1.2 (Colmez-Sen-Tate axioms)**.** We define the following axioms for the triple (A, Π, χ).

(CST0) *Almost purity*. For $H_1 \subset H_2 \subset H$ open subgroups, the trace map $\mathrm{Tr}_{H_2/H_1} : A^{+,H_1} \to A^{+,H_2}$ is almost surjective.

(CST1) *Tate's normalized traces*. There is $c_2 > 0$, for all open subgroup $H' \subset H$ an integer $n(H') \geq n_1(H')$, a sequence of closed subalgebras $(A_{H',n})_{n \geq n(H')}$ of $A^{H'}$, and for $n \geq n(H')$ $\mathbb{Q}_p$-linear maps $R_{H',n} : A^{H'} \to A_{H',n}$, satisfying the following conditions :

0. We can write $I = \bigcup_{n \in \mathbb{N}} I_n$ as colimit of additive submonoids such that for any H' and $n \geq n(H')$ we have $\{\varpi^\epsilon\}_{\epsilon \in I_n} \subset A_{H',n}$.
1. If $H_1 \subset H_2 \subset H$ are open subgroups, then $A_{H_2,n} = A_{H_1,n}^{H_2}$ and $R_{H_1,n}|_{A^{H_2}} = R_{H_2,n}$.
2. $R_{H',n}$ is an $A_{H',n}$-linear projection onto $A_{H',n}$. We let $X_{H',n}$ denote the kernel of $R_{H',n}$ and $X_{H',n}^+ := X_{H',n} \cap A^+$.
3. $gA_{H',n} = A_{gH'g^{-1},n}$ and $gR_{H',n}(x) = R_{gH'g^{-1},n}(gx)$ for all $g \in \Pi$ and $x \in A^{H'}$.
4. For all $n \geq n(H')$ and $x \in A^{+,H'}$, we have $R_{H',n}(x) \in \varpi^{-c_2} A_{H',n}^+$. In other words $\|R_{H',n}(x)\| \leq |\varpi^{-c_2}|\|x\|$ for $x \in A^{H'}$.
5. Given $x \in A^{H'}$ we have $\lim_{n \to \infty} R_{H',n}(x) = x$.
6. The action of $\Gamma_{H'}$ on the Banach algebra $A_{H',n}$ is locally analytic. Equivalently, there is an open subgroup $\Pi' \subset \Pi$ with $\Pi' \cap H = H'$ such that the action of Π' on $A_{H',n}^+/\varpi$ is trivial.

(CST2) *Bounds for the vanishing of cohomology.* There exists $c_3 > 0$, and for an open subgroup $\Pi' \subset \Pi$ an integer $n(\Pi') \geq n_1(H')$ such that if $n \geq n(\Pi')$, and $C' \subset C_H$ is an open subgroup satisfying $C' \cap \ker \chi = 1$ and $n(C') \leq n$, then for all $n(C') \leq m \leq n$ the cohomology groups $H^i(\langle \boldsymbol{\gamma}^{(m)} \rangle, \varpi^{\epsilon} X^+_{H,n})$ are ϖ^{c_3}-torsion for $i = 0, 1, 2$ and $\epsilon \in \pm I$.

In the application to rigid spaces we will have the following stronger axioms

(CST1*) *Decomposable traces.* Let $\Pi' \subset \Pi$ be an open subgroup and $H' = \Pi' \cap H$. There exists c_2 and an integer $n(H') \geq n_1(H')$ satisfying:

1. For $n \geq n(H')$ and $i = 1, \ldots, d$, there are closed $\mathbb{Q}_p$-subalgebras $A^i_{H',n}$ of $A^{H'}$, and $A^i_{H',n}$-linear projections $R^i_{H',n} : A^{H'} \to A^i_{H',n}$. We let $X^i_{H',n}$ denote the kernel of $R^i_{H',n}$ and $X^{i,+}_{H',n} = X^i_{H',n} \cap A^+$.
2. For $g \in C_{H'}$ we have that $gA^i_{H',n} = A^i_{H',n}$ and $gR^i_{H',n}(x) = R^i_{H',n}(gx)$ for all $x \in A^{H'}$ and all $i = 1, \ldots, d$.
3. For $x \in A^{+,H'}$ we have $R^i_{H',n}(x) \in \varpi^{-c_2} A^{i,+}_{H',n}$. In other words,
$$\|R_{H',n}(x)\| \leq |\varpi^{-c_2}| \|x\|.$$
4. Given a fixed H' and n, the maps $R^i_{H',n}$ commute for $i = 1, \ldots, d$, and their composition $R_{H',n} := R^d_{H',n} \circ \cdots \circ R^1_{H',n}$ satisfies the axiom (CST1).

(CST2*) *Strong bounds for the vanishing of cohomology.* There exists $c_3 > 0$, and for an open subgroup $\Pi' \subset \Pi$ an integer $n(\Pi') \geq n_1(H')$, such that if $n \geq n(\Pi')$, and $C' \subset C_H$ is an open subgroup with $C' \cap \ker \chi = 0$ and $n(C') \leq n$, then for all $n(C') \leq m \leq n$ we have:

- The multiplication map $\gamma^{(m)}_i - 1 : X^i_{H',n} \to X^i_{H',n}$ is invertible with $\|(\gamma^{(m)}_i - 1)^{-1}\| \leq |\varpi^{-c_3}|$, i.e. $(\gamma^{(m)}_i - 1)^{-1}(x) \in \varpi^{-c_3} X^{i,+}_{H',n}$ for $x \in X^{i,+}_{H',n}$.

Remark 19.1.3. The ϖ^{c_3}-torsion on (CST2) means the following: let $\boldsymbol{\gamma}^{(m)} = (\gamma^{(m)}_1, \ldots, \gamma^{(m)}_d)$ the cohomology $R\Gamma(\langle \boldsymbol{\gamma}^{(m)} \rangle, \varpi^{\epsilon} X^+_{H',n})$ is represented by a Koszul complex $\mathrm{Kos}(\boldsymbol{\gamma}^{(m)}, \varpi^{\epsilon} X^+_{H',n})$. Then $H^i(\boldsymbol{\gamma}^{(m)}, X_{H',n}) = 0$ for $i = 0, 1, 2$ and if β is a i-cocycle for $\varpi^{\epsilon} X^+_{H',n}$ for $i = 0, 1, 2$, the exists a $(i-1)$-cochain β' of $\varpi^{\epsilon - c_3} X_{H',n}$ such that $d(\beta') = \beta$. The condition for $i = 2$ guarantees that we can a lift a 1-cocycle β of $X^+_{H',n}/\varpi^{\epsilon}$ to a 1-cocycle β' of $X^+_{H',n}$ which agrees with β modulo $\varpi^{\epsilon - c_3}$.

Remark 19.1.4. The axioms (CST0), (CST1) and (CST2) above are generalizations of the axioms (TS1), (TS2) and (TS3) of [BC08] respectively. There is a subtle difference between (CST1) and (TS2), which are the additional properties (0) and (6). Condition (6) hold in the context of classical arithmetic Sen theory and for overconvergent (φ, Γ)-modules, it arises from the intuition that one is decompleting the algebra $A^{H'}$ by its locally analytic vectors for $\Gamma_{H'}$. The condition (0) says that the topologically nilpotent units for which the almost setting is defined are locally analytic. The axioms (CST1*) and (CST2*) are stronger generalizations of (TS2) and (TS3) which we will encounter in the geometric applications, see Example 19.1.7.

Lemma 19.1.5. *Suppose that (CST1*) holds. Then (CST2*) implies (CST2). Moreover, under (CST2*) the group cohomology $R\Gamma(\langle \boldsymbol{\gamma}^{(m)} \rangle, \varpi^{\epsilon} X^+_{H',n})$ is ϖ^{c_3}-torsion for $n(C') \leq m \leq n$.*

Proof. With out loss of generality let us take $\epsilon = 0$, the argument for arbitrary ϵ is the same. For $\alpha_1, \ldots, \alpha_d \in \{\pm\}$ set $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. We define
$$X^{\alpha}_{H',n} := (\prod_{\alpha_i = +} R^i_{H',n})(\prod_{\alpha_j = -} (1 - R^j_{H',n}))A^{H'}.$$

Let $(+) = (+, \dots, +)$, by (CST1*) (4) we have $A_{H',n} = X^{(+)}_{H',n}$, and $X_{H',n} = \bigoplus_{\alpha \neq (+)} X^{\alpha}_{H',n}$ as $C_{H'}$-modules. Notice that if $\alpha_i = -$, then $X^{\alpha}_{H',n} \subset X^{i}_{H',n}$ and the restriction of $\gamma_i^{(m)} - 1$ to $X^{\alpha}_{H',n}$ is still an isomorphism. Without loss of generality we can take $\alpha_1 = -$. Define the following maps for $0 \leq i \leq d-1$

$$h^i : \bigwedge^{i} (X^{\alpha}_{H',n})^{\oplus d} \to \bigwedge^{i+1} (X^{\alpha}_{H',n})^{\oplus d}$$
$$(x_{\iota_1,\dots,\iota_i})_{\iota_1<\dots<\iota_i} \mapsto \begin{cases} 0 \text{ if } \iota_1 = 1 \\ ((\gamma_1 - 1)^{-1} x_{1,\iota_1,\dots,\iota_i})_{\iota_1<\dots<\iota_i} \text{ otherwise.} \end{cases}$$

The group cohomology $R\Gamma(\langle \boldsymbol{\gamma}^{(m)} \rangle, X^{\alpha}_{H',n})$ is represented by the Koszul complex $\mathrm{Kos}(\boldsymbol{\gamma}^{(m)}, X^{\alpha}_{H',n})$. A direct computation shows that the map $h^\bullet$ is a chain homotopy between the identity and 0 on $\mathrm{Kos}(\boldsymbol{\gamma}^{(m)}, X^{\alpha}_{H',n})$, in particular $R\Gamma(\langle \boldsymbol{\gamma}^{(m)} \rangle, X^{\alpha}_{H',n}) = 0$. To see that $R\Gamma(\langle \boldsymbol{\gamma}^{(m)} \rangle, X^{\alpha,+}_{H',n})$ is ϖ^{c_3} torsion, notice that the homotopy $h^\bullet$ is bounded by $\|\varpi^{-c_3}\|$ by (CST2*). This proves the lemma. □

Definition 19.1.6. A Sen theory (in characteristic 0) is a triple (A, Π, χ) as above satisfying (CST0)-(CST2). If in addition we can take c_2 and c_3 arbitrarily small as $n \to \infty$, and (A, Π, χ) satisfies (CST1*) (resp. (CST1*) and (CST2*)) we say that (A, Π, χ) is a decomposable Sen theory (resp. a strongly decomposable Sen theory).

Example 19.1.7. The most important example for this paper is given by products of perfectoid torus and discs. Let $\mathbb{T}_{\mathbb{C}_p} := \mathrm{Spa}(\mathbb{C}_p\langle T^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_p}\langle T^{\pm 1}\rangle)$ and $\mathbb{D}_{\mathbb{C}_p} := \mathrm{Spa}(\mathbb{C}_p\langle S\rangle, \mathcal{O}_{\mathbb{C}_p}\langle S\rangle)$, we denote $\mathbb{S}^{(e,d-e)}_{\mathbb{C}_p} := \mathbb{T}^{e}_{\mathbb{C}_p} \times \mathbb{D}^{d-e}_{\mathbb{C}_p}$. Let $\mathbb{T}_{\mathbb{C}_p,n}$ and $\mathbb{D}_{\mathbb{C}_p,n}$ be the finite (Kummer-)étale covers of the torus and the disc defined by taking a p^n-th root of T and S respectively. Let $\mathbb{T}_{\mathbb{C}_p,\infty} = \varprojlim_n \mathbb{T}_{\mathbb{C}_p,n}$ and $\mathbb{D}_{\mathbb{C}_p,\infty} = \varprojlim_n \mathbb{D}_{\mathbb{C}_p,n}$ denote the perfectoid torus and perfectoid unit disc. We denote $\mathbb{S}^{(e,d-e)}_{\mathbb{C}_p,n} = \mathbb{T}^{e}_{\mathbb{C}_p,n} \times \mathbb{D}^{d-e}_{\mathbb{C}_p,n}$ and $\mathbb{S}^{(e,d-e)}_{\mathbb{C}_p,\infty} := \varprojlim_n \mathbb{S}^{(e,d-e)}_{\mathbb{C}_p,n} = \mathbb{T}^{e}_{\mathbb{C}_p,\infty} \times \mathbb{D}^{d-e}_{\mathbb{C}_p,\infty}$.

We set $\Pi = \mathbb{Z}_p(1)^d$, $A = \mathscr{O}(\mathbb{S}^{(e,d-e)}_{\mathbb{C}_p,\infty})$ and let $\chi : \Pi \xrightarrow{\sim} \mathbb{Z}_p^d$ be the isomorphism provided by a compatible system of p-th power roots of unit $(\zeta_{p^n})_{n\in\mathbb{N}}$. Let $f_1, \dots, f_d$, be the standard basis of $\mathbb{Z}_p^d$ so that

$$f_i T_j^{\frac{1}{p^n}} = \zeta_{p^n}^{\delta_{i,j}} T_j^{\frac{1}{p^n}}$$
$$f_i S_j^{\frac{1}{p^n}} = \zeta_{p^n}^{\delta_{i,j}} S_j^{\frac{1}{p^n}}$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. Then (A, Π, χ) is a strongly decomposable Sen theory. Indeed, for $n \in \mathbb{N}$ define

$$A^i_n = \begin{cases} \mathbb{C}_p\langle T_1^{\pm\frac{1}{p^\infty}}, \dots, T_i^{\pm\frac{1}{p^n}}, \dots, T_e^{\pm\frac{1}{p^\infty}}, \underline{S}^{\frac{1}{p^\infty}} \rangle \text{ if } 1 \leq i \leq e \\ \mathbb{C}_p\langle \underline{T}^{\pm\frac{1}{p^\infty}}, S_{e+1}^{\frac{1}{p^\infty}}, \dots, S_i^{\frac{1}{p^n}}, \dots, S_d^{\frac{1}{p^\infty}} \rangle \text{ otherwise.} \end{cases}$$

One has normalized Sen traces $R^i_n : A \to A^i_n$ which are given as the unique continuous extension to A of the normalized traces

$$R^i_{n,m} = \frac{1}{p^{n-m}} \mathrm{Tr}_{A^i_n/A^i_m} : A^i_n \to A^i_m.$$

Therefore, if $n > m$ and $1 \leq i \leq e$ one gets

$$(f_i^{p^m}) \cdot T_i^{\pm\frac{1}{p^n}} - T_i^{\pm\frac{1}{p^n}} = (\zeta_{p^{n-m}}^{\pm 1} - 1) T_i^{\pm\frac{1}{p^n}},$$

whence $f_i^{p^m} - 1$ is invertible on $X^i_n := \ker R^i_n$ and bounded by $|\zeta_{p^{n-m}} - 1|$. A similar property holds for the S_j's. Now, defining $A_n = \mathscr{O}(\mathbb{S}^{(e,d-e)}_{\mathbb{C}_p,n})$, and $R_n = R^d_n \circ \cdots \circ R^1_n$, one immediately verifies (CST1*) and (CST2*).

19.2. Relative locally analytic representations

We keep the conventions of the triple (A, Π, χ) as in the beginning of the previous section. In the next paragraph we will give an ad-hoc definition of a relative locally analytic representation over a Sen theory. The motivation is provided by Corollary 17.2.6, saying that, a continuous action of a compact p-adic Lie group G on a Banach space V is locally analytic if and only if there is a G-stable lattice $V^0 \subset V$ such that G acts through a finite quotient on V^0/p.

To adapt the devisages of [BC08] we need to consider continuous 1-cocycles of infinite rank A-modules. In other words, we will want to consider continuous maps from Π to some $\mathrm{Aut}(V)$, where V is an ON Banach A-module. In order to endow $\mathrm{Aut}(V)$ with the good topology (i.e. the one which appears naturally when considered as a condensed set) let us show the following lemma:

Lemma 19.2.1. *Let $V = \widehat{\bigoplus}_I Av_i$, $W = \widehat{\bigoplus}_J Aw_j$ be ON Banach A-modules, and let $v_i^\vee : V \to A$ be the projection onto the i-th component (resp. $w_j^\vee : W \to A$). Then*

$$\begin{aligned} \mathrm{Hom}_A(V, W) = \quad & (\prod_I \widehat{\bigoplus}_J A^+ w_j \otimes v_i^\vee)[\frac{1}{p}] \\ \psi \mapsto \quad & \sum_I \sum_J \langle w_j^\vee, \psi(v_i)\rangle w_j \otimes v_i^\vee. \end{aligned} \tag{19.2.1}$$

We endow the A-module $\mathrm{Hom}_A(V, W)$ with the compact open topology, equivalently, with the natural topology of the RHS of the equation (19.2.1).

Proof. We have

$$\begin{aligned} \mathrm{Hom}_A(V, W) &= \mathrm{Hom}_A(\widehat{\bigoplus}_I A^+ v_i, \widehat{\bigoplus}_J A^+ w_j)[\frac{1}{p}] \\ &= (\varprojlim_{s\in\mathbb{N}} \mathrm{Hom}_A(\bigoplus_I A^+/p^s v_i, \bigoplus_J A^+/p^s w_j))[\frac{1}{p}] \\ &= (\varprojlim_{s\in\mathbb{N}} \prod_I \mathrm{Hom}_A(A^+/p^s v_i, \bigoplus_J A^+/p^s w_j))[\frac{1}{p}] \\ &= (\varprojlim_{s\in\mathbb{N}} \prod_I \bigoplus_J A^+/p^s w_j \otimes v_i^\vee)[\frac{1}{p}] \\ &= (\prod_I \widehat{\bigoplus}_J A^+ w_j \otimes v_i^\vee)[\frac{1}{p}]. \end{aligned}$$

□

Definition 19.2.2. Let V be an ON Banach A-module. We endow the group $\mathrm{Aut}_A(V)$ with the subspace topology of the inclusion

$$\begin{aligned} \mathrm{Aut}_A(V) &\to \mathrm{End}_A(V) \times \mathrm{End}_A(V) \\ \psi &\mapsto (\psi, \psi^{-1}). \end{aligned}$$

One immediately verifies the following proposition

Proposition 19.2.3. *The group $\mathrm{Aut}_A(V)$ is a topological group, i.e. the multiplication and the inverse maps are continuous. The inclusion $\mathrm{Aut}_A(V) \to \mathrm{End}_A(V) \times \mathrm{End}_A(V)$ is closed and equal to the inverse image of $(\mathrm{id}_V, \mathrm{id}_V)$ of the multiplication map*

$$\begin{aligned} \mathrm{End}_A(V) \times \mathrm{End}_A(V) \to \quad & \mathrm{End}_A(V) \times \mathrm{End}_A(V) \\ (\psi, \psi') \mapsto \quad & (\psi \circ \psi', \psi' \circ \psi). \end{aligned} \tag{19.2.2}$$

In particular, a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $\mathrm{Aut}_A(V)$ converges to an element $\psi \in \mathrm{Aut}_A(V)$ if and only if $\psi_n \to \psi$ and $\psi_n^{-1} \to \psi^{-1}$ in $\mathrm{End}_A(V)$ as $n \to \infty$.

Proof. It is easy to show that the composition map

$$\begin{aligned} \mathrm{End}_A(V) \times \mathrm{End}_A(V) &\to \mathrm{End}_A(V) \\ (\psi, \psi') &\mapsto \psi \circ \psi' \end{aligned}$$

is continuous for the compact open topology of $\mathrm{End}_A(V)$, e.g. by computing the matrix composition in a presentation $\mathrm{End}_A(V) \cong (\prod_I \widehat{\bigoplus}_I A^+)[\frac{1}{p}]$, or by knowing that this map extends to a map of condensed sets. Thus, it is enough to show that $\mathrm{Aut}_A(V)$ is identified with the pre-image of $(\mathrm{id}_V, \mathrm{id}_V)$ via the map (19.2.2), which is clear by definition. □

The following lemma will be useful to construct invertible elements in $\mathrm{Aut}_A(V)$.

Lemma 19.2.4. *Let $M \in \mathrm{End}_A(V)$ be an endomorphism whose norm operator satisfies $\|M\| \leq |\varpi^\epsilon|$ for some $\epsilon > 0$ and some lattice $V^0 \subset V$. Then $1 - M \in \mathrm{Aut}_A(V)$ and its inverse is given by the convergent series $(1-M)^{-1} = \sum_{n=0}^{\infty} M^n$.*

Proof. Write $V = \widehat{\bigoplus}_I A$ so that $\mathrm{End}_A(V) \cong (\prod_I \widehat{\bigoplus}_I A^+)[\frac{1}{p}]$. By Proposition 19.2.3, it is enough to show that $\sum_{n=0}^{\infty} M^n$ converges in $\mathrm{End}_A(V)$, and that the sequence $((1-M)\sum_{n=0}^{m} M^n)_{m\in\mathbb{N}}$ converges to id_V. But by hypothesis $M' = \frac{1}{\varpi^\epsilon} M$ is an operator of V^0, thus $\sum_{n=0}^{\infty} M^n = \sum_{n=0}^{\infty} \varpi^{\epsilon n} M'$ converges as $\mathrm{End}_{A^+}(V^0) \cong \prod_I \widehat{\bigoplus}_I A^+$ is p-adically complete, and both ϖ^ϵ and p are topologically nilpotent units of A. One shows in a similar way that the sequence $((1-M)\sum_{n=0}^{m} M^n)_{m\in\mathbb{N}}$ converges to id_V finishing the proof. □

Given an index set I let us denote $\mathrm{GL}_I(A)$ the topological group $\mathrm{Aut}_A(\widehat{\bigoplus}_I A) \subset (\prod_I \widehat{\bigoplus}_I A^+)[\frac{1}{p}]$. It has a natural action of Π on the coefficients. Whence, ON Banach A-semilinear representations of "rank I" are equivalent to 1-cocycles of Π on $\mathrm{GL}_I(A)$. We denote by e_i the standard basis of $\widehat{\bigoplus}_I A$.

Definition 19.2.5. An ON Banach A-semilinear representation $\rho : \Pi \times V \to V$ is said relative locally analytic if there exists a basis $\{v_i\}_{i\in I}$ generating a lattice V^0 such that:

- There is $\Pi' \subset \Pi$ an open subgroup stabilizing V^0 and $\epsilon > 0$ such that the action of Π' on $\{v_i \bmod \varpi^\epsilon\}_{i\in I}$ is trivial.

We say that $\{v_i\}_{i\in I}$ is a relative locally analytic basis of V.

The previous definition can be rewritten in terms of 1-cocycles.

Definition 19.2.6. Let V be an ON Banach A-module and ρ an A-semilinear action of Π on V. Let $\mathbf{v} = \{v_i\}_{i\in I}$ be an ON basis of V, let $\Upsilon : \widehat{\bigoplus}_{i\in I} A \to V$ denote the A-linear isomorphism provided by the basis $\mathbf{v}$, and let $\sigma_{\mathbf{v}}$ be the A-semilinear action of Π fixing $\mathbf{v}$. We define the 1-cocycle of ρ attached to $(V, \mathbf{v})$ to be the continuous map $U : \Pi \to \mathrm{GL}_I(A)$ given by

$$g \mapsto \Upsilon^{-1} \circ \rho(g) \circ \sigma_{\mathbf{v}}(g)^{-1} \circ \Upsilon.$$

An ON basis $\mathbf{v}$ of V is relative locally analytic if and only if there exists $\epsilon > 0$ and $\Pi' \subset \Pi$ an open subgroup such that the 1-cocycle $U|_{\Pi'}$ has values in $\mathrm{GL}_I(A^+)$ and is trivial modulo ϖ^ϵ. We say that U is a locally analytic 1-cocycle.

The following lemma says that composing by matrices in $\mathrm{Aut}_A(V)$ which are closed enough to 1 preserves relative locally analytic basis.

Lemma 19.2.7. *Let V be an ON locally analytic representation of Π and $\mathbf{v} = \{v_i\}_{i\in I}$ a relative locally analytic basis, let V^0 be the lattice spanned by $\{v_i\}$. Let $\psi \in \mathrm{End}_A(V)$ be an operator such that $\|1 - \psi\| \leq |\varpi^\epsilon|$ for some $\epsilon > 0$. Then $\psi(\mathbf{v}) = \{\psi(v_i)\}_{i\in I}$ is a relative locally analytic basis of V.*

Proof. Let $\Pi' \subset \Pi$ be an open subgroup stabilizing V^0, and let $\epsilon' > 0$ such that the action of Π' on $\{v_i \mod \varpi^{\epsilon'}\}_{i\in I}$ is trivial. Let $\epsilon'' = \min\{\epsilon, \epsilon'\}$, then $\psi(v_i) \equiv v_i \mod \varpi^{\epsilon''}$ and Π' acts on $\{\psi(v_i) \mod \varpi^{\epsilon''}\}_{i\in I}$ trivially. This proves the lemma. □

Example 19.2.8.
1. Let $\Pi = G$ be a compact p-adic Lie group and W be a Banach locally analytic representation over $\mathbb{Q}_p$. Then, by Corollary 17.2.6, $W\widehat{\otimes}_{\mathbb{Q}_p}A$ is a relative locally analytic representation of Π.
2. Slightly more generally, suppose that Π admits by quotient $\Pi \to G$ a compact p-adic Lie group. Let W be a Banach locally analytic representation of G over $\mathbb{Q}_p$. Then $W\widehat{\otimes}_{\mathbb{Q}_p}A$ is a relative locally analytic representation of Π. This is the situation we will face in the application to Shimura varieties.

Let us finish this section with an ad hoc generalization of relative locally analytic representations to LF spaces.

Definition 19.2.9.
1. A squarable Fréchet A-module is a topological A-module F admitting a presentation $F = \varprojlim V_n$ with V_n ON Banach A-modules and dense transition maps. A squarable LF A-module is a countable filtered colimit of squarable Fréchet A-modules by injective transition maps.
2. A relative pro-locally analytic Fréchet A-semilinear representation of Π is a continuous A-semilinear representation of Π on a squarable Frechét A-module F, which can be written as $F = \varprojlim_n V_n$ with V_n relative locally analytic A-Banach representations of Π. We say that F is locally analytic if the ϵ of Definition 19.2.5 can be chosen uniform for all the V_n's. A relative locally analytic LF representation of Π is a countable filtered colimit of relative locally analytic Fréchet representations.

19.3. The Sen functor

Let (A, Π, χ) be a Sen theory. Our next goal is to define the Sen functor, which is nothing but a derived functor of locally analytic vectors, and to show that the Sen functor has a very good behaviour for relative locally analytic ON Banach representations of Π. The strategy is to generalize the devisage of [BC08] from finite rank A modules to ON Banach A-modules.

Definition 19.3.1 (The Sen functor)**.**
1. Let F be a relative locally analytic LF representation of Π, and $H' \subset H$ an open subgroup. We define the Sen module of F to be
$$S_{H'}(F) := (F^{H'})^{\Gamma_{H'}-la}.$$
2. Let $C^\bullet$ be a complex of relative locally analytic LF representations of Π, and $H' \subset H$ an open subgroup. We define the derived Sen module of $C^\bullet$ to be
$$RS_{H'}(C^\bullet) := R\Gamma(H', C^\bullet)^{R\Gamma_{H'}-la}.$$

Remark 19.3.2. The Sen functor in [BC08] is denoted as $D_{H'}$, since we are already using D and $\mathcal{D}$ for derived categories and distributions we prefer to use the letter S for the Sen functor.

Let us state the main theorem of this section, cf. Proposition 3.3.1 of [BC08].

Theorem 19.3.3. *Let (A, Π, χ) be a Sen theory. Let V be a relative locally analytic ON Banach representation of Π and $\mathbf{v} = \{v_i\}_{i\in I}$ a relative locally analytic basis. Let $U : \Pi \to \mathrm{GL}_I(A)$ be the locally analytic 1-cocycle induced by $\mathbf{v}$ (see Definition 19.2.6). Let $s > 2c_2 + 2c_3$ and let $\Pi' \subset \Pi$ be an open normal subgroup such that $U|_{\Pi'} \equiv 1 \mod \varpi^s$. Let $H' = \Pi' \cap H$ and $n \geq n(H')$. The following holds*

1. *V contains a unique ON Banach $A_{H',n}$-submodule $S_{H',n}(V)$, and there is a basis $\mathbf{v}' = \{v'_i\}_{i\in I}$ of $S_{H',n}(V)$ such that:*
   (a) *The $A_{H',n}$-module $S_{H',n}(V)$ is fixed by H' and stable by Π. Moreover, $S_{H',n}(V)$ is a locally analytic representation of $\Gamma_{H'} = \Pi/H'$.*
   (b) *We have $A\widehat{\otimes}_{A_{H',n}} S_{H',n}(V) = V$ as an A-semilinear representation of Π. The matrix M of base change from $\mathbf{v}$ to $\mathbf{v}'$ is trivial modulo $\varpi^{c_3+c_2}$.*
   (c) *Let U' denote the 1-cocycle with respect to the basis $\mathbf{v}'$. For $\gamma \in \Pi'/H'$, the matrix $U'_\gamma \in \mathrm{GL}_I(A_{H',n})$ is trivial modulo $\varpi^{c_3+c_2}$.*
2. *Suppose in addition that (A,Π,χ) is strongly decomposable. Let F be a relative locally analytic LF representation of Π. Then*

$$RS_{H'}(F) = S_{H'}(F) = \varinjlim_n S_{H',n}(F),$$

in other words, the derived Sen functor is concentrated in degree 0.

From now on we suppose that (A,Π,χ) is a Sen theory, i.e., that it satisfies the axioms (CST0)-(CST2) of Definition 19.1.2. In order to prove Theorem 19.3.3 we need a series of technical lemmas. We first start with a devisage which is nothing but almost étale descent.

Lemma 19.3.4. *Let $H' \subset H$ be an open subgroup and V a relative locally analytic ON A-Banach representation of Π. Let $\mathbf{v} = \{v_i\}_{i\in I}$ be a relative locally analytic basis generating a lattice V^0. Let $r > 0$ and $\Pi' \subset \Pi$ an open subgroup with $H' = \Pi' \cap H$, suppose that Π' acts trivially on $\mathbf{v}$ modulo ϖ^r. The following hold.*

1. *Let $0 < a < r$, there is a basis $\{v'_i\}_{i\in I}$ of $V^{H'}$ contained in $V^{0,H'}$ such that $v_i \equiv v'_i \mod \varpi^{r-a}$ and $V^{0,H'}/\varpi^{r-a} =^{ae} \bigoplus_{i\in I}(A^{+,H'}/\varpi^{r-a})v'_i$ as Π'/H'-module.*
2. *For all $s \geq 0$ we have $R\mathrm{Hom}_{H'}(1, V^0/\varpi^s) =^{ae} (V^0/\varpi^s)^{H'} =^{ae} V^{0,H'}/\varpi^s$. Taking derived inverse limits we have $R\mathrm{Hom}_{H'}(1, V^0) =^{ae} V^{0,H'}$.*

Proof. First, we claim that $R\mathrm{Hom}_{H'}(1, V^0/\varpi^s) =^{ae} (V^0/\varpi^s)^{H'}$ for all $s > 0$. By taking short exact sequences

$$0 \to V^0/\varpi^{rn} \xrightarrow{\times\varpi^r} V^0/\varpi^{r(n+1)} \to V^0/\varpi^r \to 0$$

it is enough to take $s = r$. By hypothesis, we have an isomorphism of semilinear H'-representations provided by the basis $\{v_i\}_{i\in I}$

$$V^0/\varpi^r \cong \bigoplus_I A^+/\varpi^r.$$

Then, it suffices to show that $R\mathrm{Hom}_{H'}(1, A^+/\varpi^r) =^{ae} A^{+,H'}/\varpi^r$. By hypothesis we can write $A^+/\varpi^r = \varinjlim_{H''\subset H'} A^{+,H''}/\varpi^r$ where H'' runs over all the open normal subgroups of H'. Then

$$R\mathrm{Hom}_{H'}(1, A^+/\varpi^r) = \varinjlim_{H''\subset H'} R\mathrm{Hom}_{H'/H''}(1, A^{+,H''}/\varpi^r).$$

Let $\epsilon > 0$, by (CTS0) there exists $\alpha \in A^{+,H''}$ such that $\mathrm{Tr}^{H''}_{H'}(\alpha) = \varpi^\epsilon$. Let $A^{+,H''}[H'/H'']$ be the semilinear group ring of H'/H'', and $\varepsilon : A^{+,H''}[H'/H''] \to A^{+,H''}$ the augmentation map, which is a morphism of left $A^{+,H''}[H'/H'']$-modules. Then $\varpi^\epsilon\varepsilon$ admits a section $h_\epsilon : A^{+,H''} \to A^{+,H''}[H'/H'']$ given by $x \mapsto \sum_{g\in H'/H''} xg(\alpha)\cdot g$. This proves that $A^{+,H''}$ is almost $A^{+,H''}[H''/H']$-projective, which implies the claim. Taking derived inverse limits we see that $R\mathrm{Hom}_{H'}(1, V^0)$ is almost concentrated in degree 0. Finally, taking cohomology of the short exact sequences for $s > 0$

$$0 \to V^0 \xrightarrow{\varpi^s} V^0 \to V^0/\varpi^s \to 0$$

one gets (2).

To prove (1), let $0 < a < r$ and let $\varpi^a v_i' \in V^{0,H'}$ be a lift of $(\varpi^a v_i \mod \varpi^r) \in V^{0,H'}/\varpi^r$. Then $v_i' \in V^{0,H'}$ and $\mathbf{v}' = \{v_i'\}_{i\in I}$ is an ON basis of $V^{H'}$ such that $v_i' \equiv v_i \mod \varpi^{r-a}$. This proves the lemma. □

Lemma 19.3.5. *Let $\delta, a, b \in \mathbb{R}_{>0}$ such that $a \geq c_2 + c_3 + \delta$ and $b \geq \sup\{a + c_2, 2c_2 + 2c_3 + \delta\}$. Let $H' \subset H$ be an open subgroup, $n \geq n(H)$, and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$ a sequence of linearly independent elements in $C_{H'}$, let $\langle\boldsymbol{\gamma}\rangle$ be the subgroup generated by the γ_i's. Let $(U_1, \ldots, U_d)$ be a 1-cocycle of $\langle\boldsymbol{\gamma}\rangle$ in $\mathrm{GL}_I(A^{+,H'})$ satisfying*

i. $U_i = 1 + U_{i,1} + U_{i,2}$ where $U_{i,1} \in \prod_I \widehat{\bigoplus}_I A^+_{H',n}$ and $U_{i,2} \in \prod_I \widehat{\bigoplus}_I A^{+,H'}$.

ii. $U_{i,1} \equiv 0 \mod \varpi^a$ and $U_{i,2} \equiv 0 \mod \varpi^b$.

Then there exists $M \in \mathrm{GL}_I(A^{+,H'})$ with $M \equiv 1 \mod \varpi^{b-c_2-c_3}$ such that

i. $M^{-1}U_i\gamma_i(M) = 1 + V_{i,1} + V_{i,2}$ with $V_{i,1} \in \prod_I \widehat{\bigoplus}_I A^+_{H',n}$ and $V_{i,2} \in \prod_I \widehat{\bigoplus}_I A^{+,H'}$.

ii. We have $V_{i,1} \equiv 0 \mod \varpi^a$ and $V_{i,2} \equiv 0 \mod \varpi^{b+\delta}$.

Proof. Let $R_{H',n} : A^{H'} \to A_{H',n}$ be the projection map and $X_{H',n}$ its kernel. Since we have the decomposition $A^{H'} = A_{H',n} \oplus X_{H',n}$, the following space decomposes via $R_{H',n}$:

$$(\prod_I \widehat{\bigoplus}_I A^+)[\frac{1}{p}] = (\prod_I \widehat{\bigoplus}_I A^{H',+})[\frac{1}{p}] \oplus (\prod_I \widehat{\bigoplus}_I X^+_{H',n})[\frac{1}{p}].$$

Then, using the bound of (CST1), we can write $U_{i,2} = R_{H',n}(U_{i,2}) + W_i$ with $W_i \in \prod_i \widehat{\bigoplus} X^+_{H',n}$ and $W_i \equiv 0 \mod \varpi^{b-c_2}$. The cocycle condition of $(U_j)_{j=1}^d$ is equivalent to the equality

$$\begin{aligned} 0 &= U_j\gamma_j(U_i) - U_i\gamma_i(U_j) \\ &= U_{j,1} + U_{j.2} + \gamma_j(U_{i,1}) + \gamma_j(U_{i,2}) - U_{i,1} - U_{i,2} - \gamma_i(U_{j,1}) - \gamma_i(U_{j,2}) + Q_1 + Q_2 \end{aligned}$$

for all $1 \leq i, j \leq d$, with $Q_1 \in \prod_I \widehat{\bigoplus}_I A^+_{H',n}[\frac{1}{p}]$, and $Q_2 \in \varpi^{a+b} \prod_I \widehat{\bigoplus}_I A^{+,H'}$. Applying $1 - R_{H',n}$ we find

$$0 = W_j + \gamma_j(W_i) - W_i - \gamma_i(W_j) + R_{H',n}(Q_2)$$

where $R_{H',n}(Q_2) \equiv 0 \mod \varpi^{a+b-c_2}$. Therefore, $(W_i)_{i=1}^d$ defines a 1-cocycle of $\prod_I \bigoplus_I (X^+_{H',n}/\varpi^{a+b-c_2})$. By (CST2), there exists a 1-cocycle $(W_j')_{j=1}^d$ in $\prod_I \widehat{\bigoplus}_I X^+_{H',n}$ such that $(W_j')_{j=1}^d \equiv (W_j)_{j=1}^d \mod \varpi^{a+b-c_2-c_3}$. In particular, $(W_j')_{j=1}^d \equiv (W_j)_{j=1}^d \equiv 0 \mod \varpi^{b-c_2}$. Again by (CST2), there exists $M_0 \in \prod_I \widehat{\bigoplus} X^+_{H',n}$ such that $W_j' = M_0 - \gamma_j M_0$ for $j = 1, \ldots, d$ and $M_0 \equiv 0 \mod \varpi^{b-c_2-c_3}$. Taking $M = 1 + M_0$ we get the lemma. □

Corollary 19.3.6. *Let $\delta > 0$ and $b \geq 2c_2 + 2c_3 + \delta$. Let $H' \subset H$ be an open subgroup and $U_1, \ldots, U_d \in \mathrm{GL}_I(A^{+,H'})$ a 1-cocycle verifying $U_j \equiv 1 \mod \varpi^b$ for $j = 1, \ldots, d$. Then there exists $M \in \mathrm{GL}_I(A^{+,H'})$ with $M \equiv 1 \mod \varpi^{b-c_3-c_2}$ such that*

$$M^{-1}U_j\gamma_j(M) \in \mathrm{GL}_I(A^+_{H',n}) \text{ for } j = 1, \ldots, d.$$

Proof. By the previous lemma there exists $M^{(1)} \in \mathrm{GL}_I(A^{+,H'})$ with $M^{(1)} \equiv 1 \mod \varpi^{b-c_2-c_3}$ such that

$$U_i^{(1)} := M^{(1),-1}U_i\gamma_i(M^{(1)}) \in \mathrm{GL}_I(A^+_{H',n}) \mod \varpi^{b+\delta}.$$

Let $k \in \mathbb{N}_{\geq 1}$, by induction we can find matrices $M^{(k)} \in \mathrm{GL}_I(A^{+,H'})$ with $M^{(k)} \equiv 1 \mod \varpi^{b+\delta(k-1)-c_2-c_3}$ with

$$U_i^{(k)} := M^{(k),-1}U_i^{(k-1)}\gamma_i(M^{(k)}) \in \mathrm{GL}_I(A^+_{H',n}) \mod \varpi^{b+\delta k}.$$

Taking $k \to \infty$, and $M := M^{(1)}M^{(2)}\cdots$ one sees that the 1-cocycle $(U_i')_{i=1}^d := (M^{-1}U_i\gamma_i(M))$ takes values in $\mathrm{GL}_I(A^+_{H',n})$, and that $(U_i')_{i=1}^d \equiv 1 \mod \varpi^{b-c_2-c_3}$. □

Lemma 19.3.7. *Let $H' \subset H$ be an open subgroup, $n \geq n(H')$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$ a sequence of linearly independent elements of $C_{H'}$, and $B \in \mathrm{GL}_I(A^{H'})$. Suppose that we are given with $V_{1,j}, V_{2,j} \in \mathrm{GL}_I(A^+_{H',n})$ with $V_{1,j} \equiv V_{2,j} \equiv 1 \mod \varpi^{c_3+\epsilon}$ for some $\epsilon > 0$, and that $\gamma_j(B) = V_{j,1} B V_{j,2}$. Then $B \in \mathrm{GL}_I(A_{H',n})$.*

Proof. Consider $C = B - R_{H',n}(B)$, then $\gamma_j(C) = V_{j,1} C V_{j,2}$. We have

$$\gamma_j(C) - C = (V_{j,1} - 1) C V_{j,2} + V_{j,1} C (V_{j,2} - 1) - (V_{j,1} - 1) C (V_{j,2} - 1).$$

Then $C \in \varpi^r \prod_I \widehat{\bigoplus}_I A^{+,H'}$ implies $\gamma_i(C) - C \in \varpi^{r+c_3+\epsilon} \prod_I \widehat{\bigoplus}_I A^{+,H'}$ for $i = 1, \dots, d$. On the other hand, (CST2) provides an isomorphism ι between $X_{H',n}$ and the 1-cocycles $Z^1(X_{H',n}) \subset X^d_{H',n}$ such that $\iota^{-1}(Z^1(\varpi^{r+c_3+\epsilon} X^+_{H',n})) \subset \varpi^{r+\epsilon} X^+_{H',n}$. Therefore, $\gamma_j(C) - C \in \varpi^{r+c_3+\epsilon} \prod_I \widehat{\bigoplus}_I A^{+,H'}$ for $j = 1, \dots, d$ implies $C \in \varpi^{r+\epsilon} \prod_I \widehat{\bigoplus}_I A^{+,H'}$. On deduces that $C = 0$ and that $B = R_{H',n}(B) \in \mathrm{GL}_I(A^+_{H',n})$. □

Proof of Theorem 19.3.3 (1). Let $U : \Pi \to \mathrm{GL}_I(A^+)$ be the 1-cocycle defined by the basis $\{v_i\}_I$. By hypothesis $U|_{\Pi'} \equiv 1 \mod \varpi^s$ with $s > 2c_2 + 2c_3$. Let $\epsilon > 0$ such that $s' := s - \epsilon > 2c_2 + 2c_3$. By Lemma 19.3.4 we have $R\Gamma(H', V) = V^{H'}$, and there exists a matrix $M' \in \mathrm{GL}_I(A^+)$ with $M' \equiv 1 \mod \varpi^{s'}$ such that the cocycle $U'_g := M'^{-1} U_g g(M')$ is trivial over H'.

Then, U' is a 1-cocycle over $\mathrm{GL}_I(A^{+,H'})$ satisfying $U'|_{\Pi'} \equiv 1 \mod \varpi^{s'}$. Let $n(H') \leq m \leq n$ and $\boldsymbol{\gamma} = (\gamma_1^{(m)}, \dots, \gamma_d^{(m)})$ be a pre-image of $(p^m e_i)_{i=1}^d$ via $\chi : C_{\Pi'} \to \mathbb{Z}_p^d$. Let $\delta > 0$ be such that $s' \geq 2c_2 + 2c_3 + \delta$, by Corollary 19.3.6 there exists $M' \in \mathrm{GL}_I(A^{+,H'})$ with $M'' \equiv 1 \mod \varpi^{s'-c_3-c_2}$ such that

$$M''^{-1} U'_{\gamma_j} \gamma_j(M'') \in \mathrm{GL}_I(A^+_{H',n}) \text{ for } j = 1, \dots, d.$$

Define $U''_g := M''^{-1} U'_g g(M'')$, and let us show that U'' is a 1-cocycle of Π in $\mathrm{GL}_I(A^+_{H',n})$. Let $g \in \Pi/H'$, as $\gamma_j \in C_{H'}$ for all $j = 1, \dots, d$ we see that

$$\begin{aligned} U''_{g\gamma_j} &= U''_{\gamma_j g} \\ U''_g g(U''_{\gamma_j}) &= U''_{\gamma_j} \gamma_j(U''_g). \end{aligned}$$

Thus, $\gamma_j(U''_g) = U''^{-1}_{\gamma_j} U''_g g(U''_{\gamma_j})$. But $U''^{-1}_{\gamma_j}, g(U''_{\gamma_j}) \in \mathrm{GL}_I(A^+_{H',n})$ are congruent to 1 modulo $\varpi^{s'-c_3-c_2}$, and $s' - c_3 - c_2 > c_3 + c_2$. By lemma 19.3.7 we have $U''_g \in \mathrm{GL}_I(A_{H',n})$ proving that U'' is a 1-cocycle in $\mathrm{GL}_I(A^+_{H',n})$ whose restriction to Π' is congruent to 1 modulo $\varpi^{c_3+c_2}$. Setting $M := M'M''$ and $\psi : V \to V$ the associated isomorphism of A-modules, let $\{v'_i\} := \psi(\{v_i\})$, $S_{H',n}(V)$ be the ON $A_{H',n}$-module spanned by $\{v'_i\}_I$ and $S_{H',n}(V^0)$ the lattice generated. Then $S_{H',n}(V) \subset V$ is stable by Π and the action factors through $\Gamma_{H'} = \Pi/H'$. Furthermore, by construction we have an isomorphism of semilinear A-representations of Π

$$A \widehat{\otimes}_{A_{H',n}} S_{H',n}(V) = V.$$

It is left to show that $S_{H',n}(V)$ is a locally analytic representation of $\Gamma_{H'}$. But this follows from Corollary 17.2.6, the fact that the elements $\gamma_1, \dots, \gamma_d$ act trivially on the basis $\{v'_i\}_I \mod \varpi^{c_3+c_2}$, and that the action on the Banach algebra $A_{H',n}$ is already locally analytic. □

We still need an additional technical lemma for proving part (2). Roughly speaking, it says that the Koszul complexes of the spaces $X_{H',n}$ for the action of $C_{H'}$ kill the locally analytic representations. In the rest of the section we suppose that (A, Π, χ) is a strongly decomposable Sen theory.

Lemma 19.3.8. *Let $C' \subset C_{H'}$ be torsion free and $n \geq n(C')$. Let $n(C) \leq m \leq k \leq n$ and $\boldsymbol{\gamma} = (\gamma_i^{(k)})_{i=1}^d$ the inverse image of $(p^k e_i)_{i=1}^d$ via $\chi : C' \to \mathbb{Z}_p^d$. Suppose that $\|\gamma_i^{(k)} - 1\|_{A_{H',m}} < |\varpi^{c_3}|$ for all i. Let V be a locally analytic ON $A_{H',m}$-Banach representation of $\Gamma_{H'}$ such that $\|\gamma_i^{(k)} - 1\|_V < |\varpi^{c_3}|$ for $i = 1, \dots, d$. Then*

$$R\Gamma(\langle \boldsymbol{\gamma} \rangle, X_{H',n} \widehat{\otimes}_{A_{H',m}} V) = 0.$$

Proof. For simplicity let us write $A_m = A_{H',m}$. Given $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \{\pm\}^d$ consider the decomposition of Lemma 19.1.5

$$A^{H'} = \bigoplus_{\boldsymbol{\alpha}\in\{\pm\}^d} X^{\boldsymbol{\alpha}}_{H',n}$$

so that

$$A_n = X^{(+)}_{H',n} \text{ and } X_{H',n} = \bigoplus_{\boldsymbol{\alpha}\neq(+)} X^{\boldsymbol{\alpha}}_{H',n}.$$

Then, it suffices to show that for all $\alpha \neq (+)$ one has

$$R\Gamma(\langle\boldsymbol{\gamma}\rangle, X^{\alpha}_{H',n}\widehat{\otimes}_{A_m} V) = 0.$$

Without loss of generality we can assume that $\alpha_1 = -$. Set $\gamma = \gamma_1^{(k)}$. By the proof of Lemma 19.1.5 it is enough to show that $\gamma - 1$ is invertible as an operator of the tensor product.

First, let us assume that V is isomorphic to $V = A_m\widehat{\otimes}_{\mathbb{Q}_p} W$ with W a locally analytic Banach $\Gamma_{H'}$-representation over $\mathbb{Q}_p$ for which $\|\gamma-1\|_{A_m\widehat{\otimes}V} < |\varpi^{c_3}|$. Then, we have an explicit inverse over $X\widehat{\otimes}_{A_m} V = X\widehat{\otimes}_{\mathbb{Q}_p} W$:

$$(\gamma-1)^{-1} = -\sum_{i=0}^{\infty} \gamma^{-1}(\gamma^{-1}-1)^{-(i+1)} \otimes (\gamma-1)^i \tag{19.3.1}$$

which converges by the hypothesis on the bounds of $\gamma - 1$.

Write $G = \Gamma_{H'}$, and let us use the notation of §17.1.1 for the analytic group neighbourhoods of G and their distribution algebras. Then A_m is a $G_{(h)}$-analytic representation for some $h > 0$. Furthermore, by the bound of $\|\gamma - 1\|$ on A_m, we can assume that $\|\gamma - 1\|_{C(G_{(h)},A_m)_{\star_1}} < |\varpi^{c_3}|$.

For a general $G_{(h)}$-analytic representation V we have a closed embedding provided by the orbit map

$$o_V : V \to C(G_{(h)}, \mathbb{Q}_p)_{\star_2}\widehat{\otimes}_{\mathbb{Q}_p} V_0$$

where V_0 has the trivial action of G. The image of o_V is identified with the $\star_{1,3}$-invariants. Write $V = A_m\widehat{\otimes}_{\mathbb{Q}_p} W$ as the extension of scalars of a Banach space over $\mathbb{Q}_p$. Rewriting $C(G_{(h)}, \mathbb{Q}_p)_{\star_2}\widehat{\otimes}_{\mathbb{Q}_p} V_0 = C(G_{(h)}, A_m)_{\star_2} \otimes_{\mathbb{Q}_p} W$, and composing with the isomorphism

$$\begin{aligned} C(G_{(h)}, A_m)_{\star_2} &\cong C(G_{(h)}, A_m)_{\star_{1,3}} \\ f &\mapsto (g \mapsto gf(g^{-1})). \end{aligned}$$

the orbit o_V identifies V with a closed A_m-module of

$$C(G_{(h)}, A_m)_{\star_{1,3}} \otimes_{\mathbb{Q}_p} W_0 = C(G_{(h)}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\mathbb{Q}_p} V_\sigma, \tag{19.3.2}$$

where V_σ is the semilinear G-representation provided by the extension of scalars $V = A_m\widehat{\otimes}_{\mathbb{Q}_p} W$. The evaluation at 1 gives a retraction of (19.3.2) onto V, this implies that the quotient

$$Q := (C(G_{(h)}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\mathbb{Q}_p} V_\sigma)/o_V(V)$$

is an ON A_m-Banach representation of G which is $G_{(h)}$-analytic with the same bounds for $\gamma - 1$ that V. Repeating this procedure we find a resolution of V

$$0 \to V \to A_m\widehat{\otimes}_{\mathbb{Q}_p} W^1 \to A_m\widehat{\otimes}_{\mathbb{Q}_p} W^2 \to \cdots =: A_m\widehat{\otimes}_{\mathbb{Q}_p} W^\bullet$$

by representations of the form $A_m \otimes_{\mathbb{Q}_p} W^\ell$, with W^ℓ a locally analytic representation of G for which $\|\gamma - 1\|_{A_m\widehat{\otimes}W^\ell} < |\varpi^{c_3}|$ for all $\ell \geq 1$. This implies that

$$R\Gamma(\langle\boldsymbol{\gamma}\rangle, X\widehat{\otimes}_{A_m} V) \simeq R\Gamma(\langle\boldsymbol{\gamma}\rangle, X\widehat{\otimes}_{\mathbb{Q}_p} W^\bullet) = 0$$

where in the last equality we use the first case considered above. This finishes the lemma. □

Proof of Theorem 19.3.3 (2). Write $G = \Gamma_{H'}$, and consider the notations of distribution algebras of §17.1.1. First, let V be an ON relative locally analytic representation of Π over A. By part (1) we already know that $V^{H'} = R\Gamma(H', V)$, and that $V^{H'} = A^{H'}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V)$ for some $m >> 0$ depending only on the analyticity condition of V. Thus, we can write

$$V^{H'} = \varinjlim_n X_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V) \oplus \varinjlim_n A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V).$$

As $S_{H',m}(V)$ and $A_{H',n}$ are locally analytic representations of G, Theorem 17.2.3 implies that

$$(\varinjlim_n A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V))^{RG-la} = \varinjlim_n A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V).$$

Even more, we know that for a fixed n there is $h > 0$ only depending on V such that

$$(A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V))^{RG_{(h)}-an} = A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V).$$

We want to show that

$$(\varinjlim_n X_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V))^{RG-la} = 0.$$

Indeed, it is enough to show that for any $h > 0$ there exists $n \geq 0$ such that

$$R\Gamma(\langle\boldsymbol{\gamma}\rangle, X_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V)\widehat{\otimes}_{\mathbb{Q}_p} C(G_{(h)}, \mathbb{Q}_p)_{\star_1})) = 0 \tag{19.3.3}$$

with the notations as in Lemma 19.3. But this follows by *loc. cit.* and the fact that c_3 can be choosen arbitrarily small provided $n >> 0$.

We have proven that if V is a relative locally analytic ON A-Banach representation of Π then

$$RS_{H'}(V) = S_{H'}(V) = \mathrm{colim}_n S_{H',n}(V).$$

Let $F = \varprojlim_k V_k$ be a squarable relative locally analytic Fréchet representation of Π. Then the V_n are ON Banach relative locally analytic representations of Π with an uniform ϵ as in Definition 19.2.9, and the transition maps $V_{k+1} \to V_k$ are dense. Using part (1) for the V_k's, there exists $m \in \mathbb{N}$ such that we can write

$$F = \varinjlim_n (\varprojlim_k X_{H,n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V_k) \oplus \varprojlim_k A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V_k)).$$

Using Theorem 17.2.3 and (19.3.3) one obtains that

$$RS_{H'}(F) = S_{H'}(F) = \varinjlim_n \varprojlim_k A_{H',n}\widehat{\otimes}_{A_{H',m}} S_{H',m}(V_k). \tag{19.3.4}$$

Furthermore, by construction of the Sen functors $S_{H',n}(-)$, the transition maps of the projective limits of (19.3.4) are dense for all n and k big enough.

Finally, the case of a locally analytic LF representation follows formally from the Fréchet case since $RS_{H'}(-)$ commutes with filtered colimits. □

19.4. Group cohomology via Sen theory

Throughout this section (A, Π, χ) will denote a strongly decomposable Sen theory. We finish this chapter with some formal consequences of Theorems 17.2.3 and 19.3.3 regarding the group cohomology of relative locally analytic representations of Π.

Corollary 19.4.1. *Let F be a relative locally analytic LF A-representation of Π. Let $\Pi' \subset \Pi$ be an open subgroup and $H' = \Pi' \cap H$. Let $\mathfrak{n}_{H'} := \mathrm{Lie}\,\Pi/H'$ Then*

$$R\Gamma(\Pi', F) = (R\Gamma(\mathfrak{n}_{H'}, S_{H'}(F)))^{\Pi'/H'}.$$

Proof. By Hochschild-Serre, almost purity (Lemma 19.3.4) and Theorems 17.2.14 and 17.2.16 we have that

$$\begin{aligned} R\Gamma(\Pi', F) &= R\Gamma(\Pi'/H', F^{H'}) \\ &= R\Gamma(\Pi'/H', S_{H'}(F)) \\ &= R\Gamma(\mathfrak{n}_{H'}, S_{H'}(F))^{\Pi'/H'}. \end{aligned}$$

□

We will consider a last hypothesis which holds in the Hodge-Tate situation, i.e., for the Sen theory of the sheaf $\widehat{\mathcal{O}}_X$, where X is a log smooth adic space over $\mathrm{Spa}(K, O_K)$. In the arithmetic case over $\mathbb{Q}_p$, this hypothesis is nothing but the Ax-Sen-Tate theorem.

(AST) Let $\Pi' \subset \Pi'$ be an open subgroup and $H' = \Pi' \cap H$. A Sen theory (A, Π, χ) satisfies the Ax-Sen-Tate property if the following conditions hold:

i. $A_{H',n} = A^{\Pi_{H',n}}$, where $\Pi_{H',n}$ is the inverse image of $p^n\mathbb{Z}_p$ via $\chi : \Pi' \to \mathbb{Z}_p^d$ for all $n >> 0$.

ii. The traces $R_{H',n} : A^{H'} \to A_{H',n}$ are constructed from normalized traces

$$\begin{aligned} R^m_{H',n} : A_{H',m} &\to A_{H',n} \\ x &\mapsto \frac{1}{p^{m-n}} \sum_{g \in \Pi_{H',m}/\Pi_{H',n}} g(x). \end{aligned}$$

Remark 19.4.2. Notice that this axiom makes the proof of Lemma 19.3.8 slightly easier, as the displayed equation (19.3.1) can be applied directly to $X_{H',n}\widehat{\otimes}_{A_{H',m}} V$ since the action of $\langle \gamma \rangle$ on $A_{H',m}$ is trivial.

A Sen theory satisfying the Ax-Sen-Tate axiom can be endowed with a Sen operator as follows.

Definition 19.4.3. Suppose that (A, Π, χ) satisfies (AST). Let F be a relative locally analytic *LF* A-representation of Π. Let $H' \subset H$ be an open subgroup. The Sen operator of F is the A-linear map

$$\mathrm{Sen}(F) : F \to F \otimes_{\mathbb{Q}_p} \mathfrak{n}_{H'}^\vee$$

given by the A-extension of scalars of the derivations

$$S_{H'}(F) \to S_{H'}(F) \otimes_{\mathbb{Q}_p} \mathfrak{n}_{H'}^\vee.$$

In the rest of the section we suppose that (A, Π, χ) satisfies the Ax-Sen-Tate axiom. The following results describe some cohomological properties of A-semilinear Π-representations in terms of their Sen operators. We fix $\Pi' \subset \Pi$ an open subgroup and set $H' = \Pi' \cap H$.

Lemma 19.4.4. *Let $C^\bullet$ be a complex of relative locally analytic LF A-representations of Π with Π-equivariant differential maps. Suppose that the Sen map $C^\bullet \to C^\bullet \otimes_{\mathbb{Q}_p} \mathfrak{n}_{H'}^\vee$ is homotopically equivalent to 0 as topological $A[[\Pi]]$-modules. Then $H^i(S_{H'}(C^\bullet))$ is a trivial $\mathfrak{n}_{H'}$-module for all $i \in \mathbb{Z}$ and*

$$R\Gamma(\Pi', C^\bullet) = \bigoplus_{j=0}^{d} (S_{H'}(C^\bullet) \otimes \bigwedge^j \mathfrak{n}_{H'}^\vee)^{\Pi'/H'}[-j].$$

In particular, for all $i \in \mathbb{Z}$ we have

$$R\Gamma(\Pi', H^j(C^\bullet)) = \bigoplus_{j=0}^{d} (H^j(S_{H'}(C^\bullet)) \otimes \bigwedge^j \mathfrak{n}_{H'}^\vee)^{\Pi'/H'}[-j]. \tag{19.4.1}$$

Proof. By Corollary 19.4.1 we have

$$R\Gamma(\Pi', C^\bullet) \quad = \quad R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet))^{\Pi'/H'}.$$

But the cohomology $R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet))$ is represented by the total complex of the Koszul bicomplex $\mathrm{Kos}(\mathfrak{n}_{H'}^\vee, S_{H'}(C))^{\bullet,\bullet}$ whose terms are

$$\mathrm{Kos}(\mathfrak{n}_{H'}^\vee, S_{H'}(C)^\bullet)^{k,l} = S_{H'}(C)^k \otimes \bigwedge^l \mathfrak{n}_{H'}^\vee.$$

Then, the homotopy between 0 and the map $\mathrm{Sen}(C^\bullet) : C^\bullet \to C^\bullet \otimes \mathfrak{n}^\vee$ provides a homotopy between 0 and the morphism of the row complexes

$$\mathrm{Kos}(\mathfrak{n}_{H'}^\vee, C)^{\bullet,l} \to \mathrm{Kos}(\mathfrak{n}_{H'}^\vee, C)^{\bullet,l+1}$$

which is just a twist of $\mathrm{Sen}(C^\bullet)$. Taking total complexes and locally analytic vectors this provides a homotopy between 0 and the identity of the complex $R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet))$. Furthermore, it provides an explicit splitting

$$R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet)) = \bigoplus_{j=0}^{d} S_{H'}(C^\bullet) \otimes \bigwedge^j \mathfrak{n}_{H'}^\vee[-j].$$

This shows in particular that the cohomology groups of $S_{H'}(C^\bullet)$ have a trivial action of $\mathfrak{n}_H$, so that taking Π'/H'-invariants is well defined as derived objects. This finishes the lemma. □

Definition 19.4.5. Let $C^\bullet$ be a complex as in Lemma 19.4.4. Suppose that the Sen map $\mathrm{Sen}(C^\bullet) : C^\bullet \to C \otimes \mathfrak{n}_{H'}^\vee$ is homotopically equivalent to zero. We denote

$$(C^\bullet)^{\Pi'} := (S_{H'}(C^\bullet))^{\Pi'/H'}.$$

Nonetheless, by the previous lemma we have that $H^i(C^\bullet)^{\Pi'} = H^i(S_{H'}(C^\bullet))^{\Pi'/H'})$ for all $i \in \mathbb{Z}$.

Remark 19.4.6. The previous definition is provisory for this paper. In general, Theorem 19.3.3 should hold in a larger category of "relative locally analytic" solid modules over an analytic ring attached to A, and its content should be that the Sen functor is exact. Then, Lemma 19.4.4 should generalize to the statement that taking Π'-invariants for relative locally analytic solid A-modules with trivial Sen operators is exact. In that context, $\mathrm{Kos}(\mathfrak{n}_{H'}^\vee, C^\bullet)^{\Pi'}$ would have an honest meaning without passing through the Sen module.

Proposition 19.4.7. *Let $C^\bullet$ be a bounded complex of relative locally analytic LF representations of Π. Then*

$$R\Gamma(\Pi', C^\bullet) = R\Gamma(\mathfrak{n}_{H'}, C^\bullet)^{\Pi'}.$$

Remark 19.4.8. The cohomology group $R\Gamma(\mathfrak{n}_{H'}, C^\bullet)$ is represented by the total complex of the Koszul bicomplex $\mathrm{Kos}(\mathfrak{n}_{H'}^\vee, C^\bullet)$. The Π-invariants of the total complex are well defined by Lemma 19.4.4.

Proof. By Corollary 19.4.1 we have

$$R\Gamma(\Pi', C^\bullet) = R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet))^{\Pi'/H'}.$$

The complex $R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C)^\bullet)$ is represented by $\mathrm{Kos}(\mathfrak{n}_{H'}^\vee, S_{H'}(C))^{\bullet,\bullet}$. On the other hand, Theorem 19.3.3 implies that we can write $S_{H'}(C^\bullet) = \varinjlim_n S_{H',n}(C^\bullet)$ as a colimit of analytic subspaces defined over $A_{H',n}$. Let $A_{H',\infty} := \varinjlim_n A_{H',n}$, the theorem says in addition that

$$S_{H'}(C^\bullet)\widehat{\otimes}_{A_{H',\infty}} A := \varinjlim_n S_{H',n}(C^\bullet)\widehat{\otimes}_{A_{H',n}} A = C^\bullet.$$

Then, one gets that

$$R\Gamma(\mathfrak{n}_{H'}, C^\bullet) = R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet))\widehat{\otimes}_{A_{H',\infty}} A.$$

It is easy to construct an explicit homotopy between the identity and the Sen operator map

$$R\Gamma(\mathfrak{n}_{H'}, C^\bullet) \to R\Gamma(\mathfrak{n}_{H'}, C^\bullet) \otimes_{\mathbb{Q}_p} \mathfrak{n}_{H'}^\vee.$$

By Lemma 19.4.4, we know that

$$R\Gamma(\mathfrak{n}_{H'}, C^\bullet)^{\Pi'} = R\Gamma(\mathfrak{n}_{H'}, S_{H'}(C^\bullet))^{\Pi'/H'},$$

this proves the proposition. □

20. Hodge-Tate theory over rigid spaces

Let (K, K^+) be a complete non-archimedean discretely valued extension of $\mathbb{Q}_p$, let C be the p-adic completion of an algebraic closure of K. Let X be an fs log smooth adic space over (K, K^+), for $? \in \{\text{an}, \text{ét}, \text{két}, \text{proét}, \text{prokét}\}$ we let $X_?$ denote the corresponding site over X. We write X_C for the C-extension of scalars of X. Throughout this chapter we denote by $\widehat{\mathscr{O}}^{(+)}$ the completed (bounded) structural sheaf of $X_{\text{prokét}}$, we also let $\mathscr{O}_?^{(+)}$ denote the uncompleted (bounded) structural sheaves over $X_?$. We refer to [DLLZ19] for the formalism of the Kummer-étale site. We let $\mathbb{B}_{\text{dR}}^{(+)}$ and $\mathscr{O}\mathbb{B}_{\text{dR,log}}^{(+)}$ be the de (log) Rham period sheaves over $X_{\text{prokét}}$. We also consider the Hodge-Tate period sheaf $\mathscr{O}\mathbb{C}_{\text{log}} = \text{gr}^0(\mathscr{O}\mathbb{B}_{\text{dR,log}})$, see [DLLZ18].

The main goal of this chapter is to apply the Sen theory formalism of §19 to Hodge-Tate cohomology in rigid analytic geometry. More precisely, let G be a compact p-adic Lie group and $\widetilde{X}$ a pro-Kummer-étale G-torsor of X. Let V be a locally analytic representation of G over $\mathbb{Q}_p$, we can consider the local system $\underline{V}$ of X defined by the torsor $\widetilde{X}$. We want to prove the following theorem

Theorem 20.0.1. *There exists a natural $\widehat{\mathscr{O}}$-Higgs bundle attached to V, namely, there is a $\widehat{\mathscr{O}}$-linear map*

$$\theta_X(V) : \underline{V}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}(1) \to \underline{V}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}} \otimes_{\mathscr{O}} \Omega_X^1(\log)$$

with $\theta_X(V) \wedge \theta_X(V) = 0$ satisfying the following properties.

1. *The formation $V \mapsto \theta_X(V)$ is functorial on V.*
2. *The operator θ_V only depends on the Sen bundle*

$$\text{Sen}_X : \widehat{\mathscr{O}}(1) \to \underline{\text{Lie}\, G}\widehat{\otimes}\widehat{\mathscr{O}} \otimes \Omega_X^1(\log),$$

   where the RHS acts by derivations on V.

3. *Let $\nu : X_{C,\text{prokét}} \to X_{C,\text{két}}$ be the projection of sites. Then one has*

$$R\nu_*(\underline{V}\widehat{\otimes}\widehat{\mathscr{O}}) = \nu_*(R\Gamma(\theta_V, \underline{V}\widehat{\otimes}\widehat{\mathscr{O}})),$$

   in particular, $R^i\nu_*(\underline{V}\widehat{\otimes}\widehat{\mathscr{O}}) = \nu_* H^i(\theta_V, \underline{V}\widehat{\otimes}\widehat{\mathscr{O}})$ for all $i \in \mathbb{Z}$.

In §20.1 we construct the Sen bundle Sen_X using the results of §19. Roughly speaking, we first perform local computations depending only on Kummer-étale charts of X, where we essentially reduce to the Sen theory of a product of tori and discs as in Example 19.1.7. The next task is to show that these local constructions of the Sen bundle glue to Sen_X, where an explicit change of variables is made.

Finally in §20.2, we use the results of the previous section to prove Theorem 20.0.1. Then, under certain non-degeneracy hypothesis of Sen_X, we take $V = C^{la}(G, \mathbb{Q}_p)$ the space of locally analytic functions of G and compute the projection $R\nu_* C^{la}(G, \widehat{\mathscr{O}}_X)$ from the pro-Kummer-étale to the Kummer-étale site. We finish this section with another proof of the p-adic Simpson correspondance, as stated in [LZ17, DLLZ18, Wan21] using our formalism.

20.1. The Sen bundle

Many of the ideas of this section come from the work of Lue Pan [Pan20]. The method of Sen was already used in [LZ17] to prove the p-adic Simpson correspondance for local systems, which is at the same time an application of Kedlaya-Liu decompletions in [KL19].

20.1.1. The set-up

We keep the notation of the beginning of the chapter, namely (K, K^+) is a complete non-archimedean discretely valued extension of $\mathbb{Q}_p$, and C the p-adic completion of an algebraic closure. We let X be an fs log smooth adic space over (K, K^+), let G be a compact p-adic Lie group and $\widetilde{X} \to X$ a pro-Kummer-étale G-torsor. All the fiber products are as fs log adic spaces in the sense of [DLLZ19, Prop. 2.3.27], we highlight that it may differ from the fiber product of usual adic spaces.

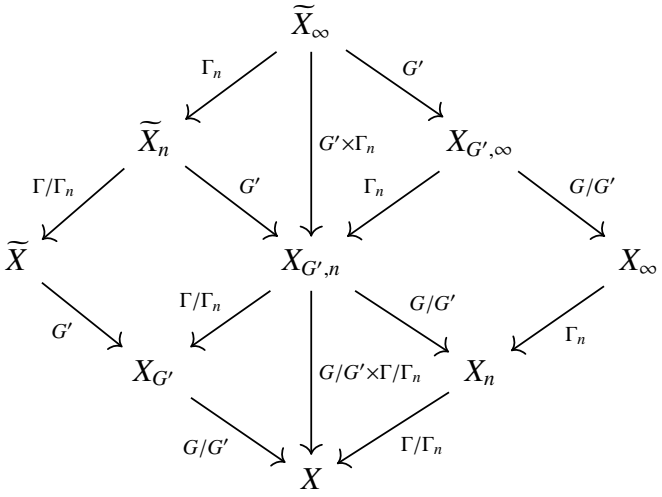
In the following we will suppose that X is affinoid and that it has logarithmic coordinates, i.e. that there exists a Kummer-étale map $\psi : X \to \mathbb{S}_K^{(e,d-e)}$ which factors as a finite composition of rational localizations and finite étale maps, and that the log structure of X is the pullback of the log structure of $\mathbb{S}^{(e,d-e)}$ defined by the normal crossing divisor $S_{e+1} \cdots S_d = 0$, cf. Example 19.1.7. We let $\mathbb{S}_{K,\infty}^{(e,d-e)}$ be the relative perfectoid product of tori and polydiscs over $\mathbb{S}_K^{(e,d-e)}$, and let Γ denote its Galois group. Notice that $\Gamma = \mathbb{Z}_p(1)^d$ is a p-adic Lie group isomorphic to $\mathbb{Z}_p^d$ after fixing a compatible system of p-th power roots of 1. We let $\Gamma_n = p^n\Gamma$ and let $\mathbb{S}_{K,n}^{(e,d-e)} = \mathbb{S}_{K,\infty}^{(e,d-e)}/\Gamma_n$.

Remark 20.1.1. In [DLLZ19, Definition 5.3.1] a log affinoid perfectoid is modeled in n-divisible monoids for all $n \in \mathbb{N}$. Thus, $\mathbb{D}_{C,\infty}$ is not log affinoid following this convention as we have not taken n-th roots of S for $(n, p) = 1$. Nevertheless, since we are only interested in p-adic Hodge theory, all the abelian sheaves we work with are p-adically complete in some sense (more precisely, all these sheaves are solid $\mathbb{Z}_p$-modules). In particular, when taking pro-Kummer-étale cohomology it is enough to trivialize at log affinoid perfectoid objects modeled in p-divisible monoids, so $\mathbb{D}_{C,\infty}$ suffices for computations.

Given an open subgroup $G' \subset G$ we let $X_{G'} = \widetilde{X}/G'$. The space $X_{G'}$ is finite Kummer-étale over X and if G' is normal it is Galois with group G/G'. We have a presentation as objects in $X_{\text{prokét}}$

$$\widetilde{X} = \varprojlim_{G' \subset G} X_{G'}.$$

Given $n \in \mathbb{N} \cup \{\infty\}$ and $G' \subset G$ we let $X_n = X \times_{\psi, \mathbb{S}_K^{(e,d-e)}} \mathbb{S}_{K,n}^{(e,d-e)}$ and $X_{G',n} = X_{G'} \times_X X_n$. We also denote $\widetilde{X}_n = \widetilde{X} \times_X X_n$ and $\widetilde{X}_\infty = \widetilde{X} \times_X X_\infty$. The following diagram illustrates the relation between these spaces as profinite-Kummer-étale objects in $X_{\text{prokét}}$



(20.1.1)

We make the following hypothesis which says, rougly speaking, that the only ramification at the boundary is given by taking p-th power roots:

(HYP) For any open subgroup $G' \subset G$ the map $X_{G'} \to X$ is, locally étale, modeled in a morphism of fs monoids $Q \to P$ such that the kernel and the cokernel of $Q^{gp} \to P^{gp}$ are finite p-groups.

Remark 20.1.2. As $\widetilde{X}$ is a G-torsor and G is a p-adic Lie group, there exists $G' \subset G$ an open subgroup such that for all open subgroup $G'' \subset G'$ the map $X_{G''} \to X_{G'}$ satisfies (HYP). In fact, this holds if G is a uniform pro-p-group, and follows by the description of the Kummer-étale fundamental group of a log geometric point, see [DLLZ19, Coro. 4.4.22].

We fix the following notation for the global functions of the C-scalar extension of the previous spaces for $G' \subset G$ and $n \in \mathbb{N} \cup \{\infty\}$:

$$\widetilde{B}_n = \widehat{\mathscr{O}}_X(\widetilde{X}_{C,n}), \quad B_{G',n} := \widehat{\mathscr{O}}(X_{C,G',n}), \quad B_{G'} := \widehat{\mathscr{O}}(X_{C,G'}),$$

similarly for the spaces of bounded functions. Notice that, as $\widehat{\mathscr{O}}$ is a pro-Kummer-étale sheaf, we have

$$(\widetilde{B}_\infty)^{\Gamma_n} = \widetilde{B}_n \text{ and } (\widetilde{B}_\infty)^{G'} = B_{G',\infty}.$$

If $G' \subset G$ is open and $n \in \mathbb{N}$ we also have

$$B_{G',n} = \widehat{\mathscr{O}}(X_{C,G',n}) = \mathscr{O}_{\text{két}}(X_{C,G',n}) = \mathscr{O}_{\text{an}}(X_{C,G',n}).$$

The Abhyankar's lemma tells us that the map $X_{G'} \to X$ becomes étale after adding enough ramification at the cusps[1]:

Lemma 20.1.3 ([DLLZ19, Lem. 4.2.2.]). *Let $G' \subset G$ be an open subgroup, then there exists $n \geq 0$ such that for all $m \geq n$ the map $X_{G',m} \to X_m$ is finite étale.*

One deduces the following consequence from [DLLZ19, Lem. 6.1.9] or [Sch13a, Lem. 4.5]

Proposition 20.1.4. *The space $X_{G',\infty}$ is affinoid perfectoid and the map $\widetilde{X}_\infty \to X_{G',\infty}$ is profinite-étale. Furthermore, we have the following relations between the rings of functions*

i $B^+_{G',\infty} = (\varinjlim_n B^+_{G,n})^{\wedge -p}$ *is an integral perfectoid ring.*

ii $\widetilde{B}^+_\infty = (\varinjlim_{G',n} B^+_{G',n})^{\wedge -p} = (\varinjlim_{G'} B^+_{G',\infty})^{\wedge -p} = (\varinjlim_n \widetilde{B}^+_n)^{\wedge -p}$ *is an integral perfectoid ring.*

iii *The ring $\widetilde{B}_\infty$ is Galois over $B_{G',\infty}$ of group G'.*

We saw in Example 19.1.7 that the tower $\{\mathbb{S}^{(e,d-e)}_{C,n}\}_{n\in\mathbb{N}}$ of products of tori and polydiscs gives rise a strongly decomposable Sen theory. Our next task is to show that their pullback to $X_{\text{prokét}}$ satisfy the Colmez-Tate-Sen axioms

Proposition 20.1.5. *The triple $(\widetilde{B}_\infty, G \times \Gamma, \text{pr}_2)$ is a strongly decomposable Sen theory.*

Proof. By Proposition 20.1.4 the triple $(\widetilde{B}_\infty, G\times\Gamma, \text{pr}_2)$ satisfies the almost purity condition (CST0). It is left to see that the pullbaks of the traces of Example 19.1.7 via $\psi : X \to \mathbb{S}^{(e,d-e)}_C$ satisfy (CST1*) and (CST2*) with c_2 and c_3 arbitrarily small. Given $G' \subset G$ an open subgroup we let $n(G')$ be the smallest

[1] In fact, this is a consequence of [DLLZ19, Prop. 4.2.1] after refining the argument of Lemma 4.2.3 in *loc. cit.* allowing only p-th power ramification.

integer n such that $X_{G',m} \to X_m$ is étale for all $m \geq n$, by Lemma 20.1.3 we know that $n(G') < \infty$. Let $n \geq n(G')$ and $1 \leq i \leq d$, recall that

$$A_n = \mathscr{O}(\mathbb{S}_{C,n}^{(e,d-e)}) = C\langle \underline{T}^{\pm \frac{1}{p^n}}, \underline{S}^{\frac{1}{p^n}} \rangle$$
$$A = \mathscr{O}(\mathbb{S}_{C,\infty}^{(e,d-e)}) = C\langle \underline{T}^{\pm \frac{1}{p^\infty}}, \underline{S}^{\frac{1}{p^\infty}} \rangle$$
$$A_n^i = \begin{cases} C\langle T_1^{\pm \frac{1}{p^\infty}}, \dots, T_i^{\pm \frac{1}{p^n}}, \dots, T_e^{\pm \frac{1}{p^\infty}}, \underline{S}^{\frac{1}{p^\infty}} \rangle \text{ if } 1 \leq i \leq e \\ C\langle \underline{T}^{\pm \frac{1}{p^\infty}}, S_{e+1}^{\frac{1}{p^\infty}}, \dots, S_i^{\frac{1}{p^n}}, \dots, S_d^{\frac{1}{p^\infty}} \rangle \text{ otherwise.} \end{cases}$$

Let $\{\gamma_i\}_{i=1}^d$ be the standard basis of Γ, given by fixing a compatible sequence of p-th power roots of unity. Let us define

$$B_{G',n}^i := (B_{G',n} \widehat{\otimes}_{A_n} A_n^i)^{\wedge -u}$$

where the tensor product is as Banach algebras, and the u-completion is nothing but the p-adic completion with unit ball the integral closure of $B_{G',n}^+ \widehat{\otimes}_{A_n^+} A_n^{i,+}$.

We claim that the traces $R_n^i : A \to A_n^i$ extend to $R_{G',n}^i : B_{G',\infty} \to B_{G',n}^i$ with $\|R_{G',i}\| \leq |p^{-c_2}|$,for c_2 arbitrarily small as $n \to \infty$. Indeed, we have that

$$B_{G',\infty} = (B_{G',n} \widehat{\otimes}_{A_n} A)^{\wedge -u},$$

and R_n^i extends to $B_{G',n}^i$-linear maps

$$R_{G',n}^i : B_{G',n} \widehat{\otimes}_{A_n} A \to B_{G',n} \widehat{\otimes}_{A_n} A_n^i \to B_{G',n}^i$$

such that $\|R_{G',n}\| \leq |p^{-c_2}|$. We have to show that the image of $(B_{G',n} \widehat{\otimes}_{A_n^+} A^+)^+$ is bounded in $B_{G',n}^i$. But Lemma 4.5 of [Sch13a] imply that, given $\epsilon > 0$, there exists $n \geq n(G')$ such that for all $m \geq n$, the cokernel of the map $B_{G',m}^+ \widehat{\otimes}_{A_m^+} A^+ \to B_{G',\infty}^+$ is killed by p^ϵ, this implies the claim and that c_2 can be taken arbitrary small as $n \to \infty$.

Finally, we have to show that for $n(G) \leq m \leq n$ the map $\gamma_i^{p^m-1} - 1$ over $X_{G',n}^i := \ker R_{G',n}$ is invertible, with inverse bounded by $|p^{-c_3}|$, and that we can take c_3 arbitrarily small when $n \to \infty$. Indeed, this follows by the same argument as before and the analogous property for the Sen theory $(A, \Gamma, \mathrm{id}_\Gamma)$. □

As a consequence of Theorem 19.3.3, we have the following corollary:

Corollary 20.1.6. *The Sen functor $V \mapsto S_{G'}(V)$ restricts to the category of locally analytic representations of G'. More precisely, given F an LF locally analytic representation of G' over $\mathbb{Q}_p$, the space $\widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} F$ is a relative locally analytic representation of $\widetilde{B}_\infty$, and*

$$RS_{G'}(\widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} F) = ((\widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} F)^{G' \times 1})^{R\Gamma - la} = ((\widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} F)^{G' \times 1})^{\Gamma - la}.$$

In particular, we have a $G' \times \Gamma$-equivariant action of the Sen operators

$$\mathrm{Sen}(F) : \widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} F \to (\mathrm{Lie}\,\Gamma)^\vee \otimes_{\mathbb{Q}_p} (\widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} F). \tag{20.1.2}$$

Remark 20.1.7. As the action of Γ on F and $\mathrm{Lie}\,\Gamma$ is trivial, the Sen operator factors through

$$\mathrm{Sen}(F) : \widetilde{B} \widehat{\otimes}_C F \to (\mathrm{Lie}\,\Gamma)^\vee \otimes_{\mathbb{Q}_p} (\widetilde{B} \widehat{\otimes}_C F)$$

Remark 20.1.8. By taking colimits as $G' \to 1$, we can define the Sen module of a locally analytic representation of $\mathfrak{g}$ (see Definition 17.2.9). Indeed, let F be an *LF* locally analytic representation of $\mathfrak{g}$, and write $F = \mathrm{colim}_{h \to \infty} F_h$ as a colimit of $\mathfrak{g}_h$-analytic representations. Let $G^\dagger = \{G' \subset G\}$ be the overconvergent neighbourhood of 1 in G, see Definition 17.3.8. Since F is a $\mathfrak{g}$-locally analytic

representation it admits a natural action of $G^\dagger$, which is nothing but the integration of $\mathfrak{g}$ to an action of some $G' \subset G$ on each F_h. Then we define the Sen module of F as the $\widetilde{B}_\infty^{G-sm,\Gamma-la}$-$LF$ space

$$S_{\mathfrak{g}}(F) := \varinjlim_{h,G'\subset G} S_{G'}(F_h).$$

One has that $\widetilde{B}_\infty \widehat{\otimes}_C F = \widetilde{B}\widehat{\otimes}_{\widetilde{B}_\infty^{G-sm,\Gamma-la}} S_{\mathfrak{g}}(F)$. The construction of Sen operators is the same as for G-analytic representations. The cohomological computations of §19.4 hold if one takes the smooth $G^\dagger$-vectors:

$$R\Gamma(G^\dagger \times \Gamma, (\widetilde{B}_\infty \widehat{\otimes}_C F)) = (R\Gamma(\mathfrak{n}^0, \widetilde{B}\widehat{\otimes}F))^{G^\dagger - sm},$$

where we identify all the $\mathfrak{n}_{G'}$ for $G' \subset G$ with $\mathfrak{n}^0 := \operatorname{Lie}\Gamma$.

20.1.2. Glueing

In this paragraph we show that the Sen operators (20.1.2) factor through the $\widehat{\mathscr{O}}$-linear action of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$, and that they glue to a map of pro-Kummer-étale sheaves over $X_{\text{prokét}}$

$$\operatorname{Sen}_X : \Omega^1_X(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}, \tag{20.1.3}$$

where $\Omega^1_X(\log)$ is the sheaf of log differential forms over X, and $\mathfrak{g}$ is seen as a pro-Kummer-étale local system via the G-torsor $\widetilde{X} \to X$ and the adjoint action of G. Let $C^{la}(\mathfrak{g}, C)$ denote the germs of locally analytic functions of G at 1, cf. §17.2.2. We need a lemma

Lemma 20.1.9. *Let F be an LF locally analytic representation of $\mathfrak{g}$ over $\mathbb{Q}_p$ (cf. Definition 17.2.9). Then F admits a closed immersion via the orbit map*

$$F \to C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_2} \widehat{\otimes}_{\mathbb{Q}_p} F_0,$$

where $C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_2}$ is endowed with the right regular action and $F_0 = F$ with the trivial action.

Proof. This follows formally by writing F as a colimit of analytic representations and Theorems 17.2.3 and 17.2.10. □

The following is a local version of the theorem we want to prove

Proposition 20.1.10. *Let F be a locally analytic representation of $\mathfrak{g}$. Then the Sen action*

$$\operatorname{Sen}(F) : \widetilde{B}\widehat{\otimes}_{\mathbb{Q}_p} F \to (\operatorname{Lie}\Gamma)^\vee \otimes_{\mathbb{Q}_p} (\widetilde{B}\widehat{\otimes}_{\mathbb{Q}_p} F)$$

factors through a Galois equivariant map $\operatorname{Sen} \in (\operatorname{Lie}\Gamma)^\vee \otimes_{\mathbb{Q}_p} (\widetilde{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathfrak{g})$ *making the following diagram commutative*

$$\begin{array}{ccc} \widetilde{B}\widehat{\otimes}_{\mathbb{Q}_p} F & \xrightarrow{\operatorname{Sen}\otimes 1} & (\operatorname{Lie}\Gamma)^\vee \otimes_{\mathbb{Q}_p} (\widetilde{B}\widehat{\otimes}_{\mathbb{Q}_p}\mathfrak{g}) \otimes_{\mathbb{Q}_p} F \\ & \searrow^{\operatorname{Sen}(F)} & \downarrow{\scriptstyle 1\otimes\partial} \\ & & (\operatorname{Lie}\Gamma)^\vee \otimes_{\mathbb{Q}_p} (\widetilde{B}\widehat{\otimes}_{\mathbb{Q}_p} F), \end{array}$$

where $\partial : \mathfrak{g} \otimes_{\mathbb{Q}_p} F \to F$ is the derivation map.

Proof. By Lemma 20.1.9 and the functoriality of the Sen operators, it is enough to consider $F = C^{la}(\mathfrak{g}, \mathbb{Q}_p)$. Then, the Sen map is a $G^\dagger \times \Gamma$-equivariant map

$$\operatorname{Sen}(C^{la}(\mathfrak{g}, \mathbb{Q}_p)) : \widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_2} \to (\operatorname{Lie}\Gamma)^\vee \otimes_{\mathbb{Q}_p} \widetilde{B}_\infty \widehat{\otimes}_{\mathbb{Q}_p} C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_2}.$$

Moreover, the Sen operators satisfy the Leibniz rule by construction, and are invariant under the $\star_1$-action of $\mathfrak{g}$. Therefore, they must factor through $\star_1$-invariant derivations $\operatorname{Sen} : \operatorname{Lie}\Gamma \to (\widetilde{B}_\infty)^\Gamma \otimes_{\mathbb{Q}_p} \mathfrak{g} = \widetilde{B} \otimes_{\mathbb{Q}_p} \mathfrak{g}$ where $\mathfrak{g}$ acts via the $\star_2$-action on $C^{la}(\mathfrak{g}, \mathbb{Q}_p)$. The proposition follows. □

Remark 20.1.11. The previous proposition implies that, in order to compute the Sen operators, it is only necessary to consider a faithful representation of $\mathfrak{g}$.

Our next task is to understand the behaviour of the Sen operators in families after changing the chart ψ, and show that they glue to a morphism (20.1.3). The idea of the proof is to use explicit local coordinates for the Sen modules attached to $C^{la}(\mathfrak{g}, \mathbb{Q}_p)$ from two different charts, and compute directly the Sen operators.

Case $G = \Gamma$

In a first approximation we consider the case when $G = \Gamma \cong \mathbb{Z}_p(1)^d$ is a torsor arising from a perfectoid chart. We have to introduce some notation. Let X be an fs log smooth affinoid space over $\mathrm{Spa}(K, K^+)$, and $\psi_1, \psi_2 : X \to \mathbb{S}_K^{(e,d-e)}$ two étale charts which factor as compositions of rational localizations and finite étale maps. Let $X_{\infty,0} := \mathbb{S}_{K,\infty}^{(e,d-e)} \times_{\mathbb{S}_K^{(e,d-e)},\psi_1} X$ and $X_{0,\infty} := X \times_{\psi_2, \mathbb{S}_K^{(e,d-e)}} \mathbb{S}_{K,\infty}^{(e,d-e)}$, and let Γ_1 and Γ_2 denote the Galois groups of $X_{\infty,0}$ and $X_{0,\infty}$ over X respectively. We let $X_{\infty,\infty} := X_{\infty,0} \times_X X_{0,\infty}$ be the pro-Kummer-étale torsor of group $\Gamma_1 \times \Gamma_2$. Let $B_{\infty,0} := \widehat{\mathscr{O}}_X(X_{C,\infty,0})$, $B_{0,\infty} := \widehat{\mathscr{O}}_X(X_{C,0,\infty})$ and $B_{\infty,\infty} := \widehat{\mathscr{O}}_X(X_{C,\infty,\infty})$. One can then consider the Sen module associated to $C^{la}(\mathrm{Lie}\,\Gamma_1, \mathbb{Q}_p)_{\star_1}$ and $C^{la}(\mathrm{Lie}\,\Gamma_2, \mathbb{Q}_p)_{\star_1}$ respectively. Furthermore, we have by definition

$$S_{\mathrm{Lie}\,\Gamma_1}(C^{la}(\mathrm{Lie}\,\Gamma_1, B_{\infty,\infty})_{\star_{1,3}}) = B_{\infty,\infty}^{\Gamma_1\times\Gamma_2-la} = S_{\mathrm{Lie}\,\Gamma_2}(C^{la}(\mathrm{Lie}\,\Gamma_2, B_{\infty,\infty})_{\star_{1,3}}).$$

Theorem 19.3.3 also says that the orbit maps provide $B_{\infty,\infty}$-linear isomorphisms

$$C^{la}(\mathrm{Lie}\,\Gamma_1, B_{\infty,\infty}) \cong B_{\infty,\infty} \widehat{\otimes}_{B_{\infty,\infty}^{\Gamma_1\times\Gamma_2-sm}} B_{\infty,\infty}^{\Gamma_1\times\Gamma_2-la} \cong C^{la}(\mathrm{Lie}\,\Gamma_2, B_{\infty,\infty}).$$

Let us fix a compatible system of p-th power roots of unity $(\zeta_{p^n})_{n\in N}$ and let $\gamma_{1,(1)}, \dots \gamma_{d,(1)} \in \Gamma_1$ and $\gamma_{1,(2)}, \dots, \gamma_{d,(2)} \in \Gamma_2$ denote the standard basis. We let $\theta_{j,(i)} \in \mathrm{Lie}\,\Gamma_i$ denote the derivative in the direction $\gamma_{j,(i)}$ for $j = 1, \dots, d$ and $i = 1, 2$.

Let $T_{1,(1)}, \dots, T_{e,(1)}, S_{e+1,(1)}, \dots S_{d,(1)} \in \mathscr{O}_{\mathrm{an}}(X)$ and $T_{1,(2)}, \dots, T_{e,(2)}, S_{e+1,(2)}, \dots S_{d,(2)} \in \mathscr{O}_{\mathrm{an}}(X)$ be the coordinates given by the charts ψ_1 and ψ_2 respectively. We have two basis for the log differentials of X

$$\Omega^1_X(\log) = \bigoplus_{j=1}^{e} \mathscr{O}_{\mathrm{an}}\, \mathrm{dlog}\, T_{j,(i)} \oplus \bigoplus_{j=e+1}^{d} \mathrm{dlog}\, S_{j,(i)}.$$

Let $\partial_{j,(i)}$ denote the dual basis of $\{\mathrm{dlog}\, T_{j,(i)}, \mathrm{dlog}\, S_{j,(i)}\}$ for $i = 1, 2$, and

$$\partial_{j,(1)} = \sum_{k=1}^{d} a_{j,k} \partial_{k,(2)} \tag{20.1.4}$$

with $a_{j,k} \in \mathscr{O}_{\mathrm{an}}(X)$ the base change matrix.

Proposition 20.1.12. *The following hold*

1. *We have $\theta_{j,(1)} = -\sum_{k=1}^{d} a_{j,k}\theta_{k,(2)}$.*
2. *There are elements $z_1, \dots, z_d \in B_{\infty,\infty}^{\Gamma_1\times\Gamma_2-la}$ such that*

$$\theta_{j,(2)}(z_k) = \delta_{j,k} \text{ and } \theta_{j,(1)}(z_k) = a_{j,k}.$$

Proof. Consider the Faltings extension

$$0 \to \widehat{\mathscr{O}}(1) \to \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}} \to \Omega^1_X(\log) \otimes_{\mathscr{O}_{\mathrm{an}}} \widehat{\mathscr{O}} \to 1$$

and the elements

$$Y_{j,(1)} = \begin{cases} \log(\frac{[T^\flat_{j,(1)}]}{T_{j,(1)}}) & \text{if } 1 \leq j \leq e \\ \log(\frac{[S^\flat_{j,(1)}]}{S_{j,(1)}}) & \text{if } e+1 \leq j \leq d \end{cases}$$

in $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(X_{\infty,0})$ (resp. $Y_{j,(2)}$ for $X_{0,\infty}$). Let $\varepsilon = (\zeta_{p^n})_{n\in N}$ and $t = \log[\varepsilon] \in B^+_{dR}$. Then

$$\gamma_{j,(1)}Y_{k,(1)} = Y_{k,(1)} + \delta_{j,k}t \text{ and } dY_{k,(1)} = \begin{cases} -\operatorname{dlog} T_{k,(1)} & \text{if } 1 \leq k \leq e \\ -\operatorname{dlog} S_{k,(1)} & \text{if } e+1 \leq k \leq d \end{cases} \tag{20.1.5}$$

(resp. for $Y_{k,(2)}$). Let us consider the reduction of the variables $Y_{k,(i)}$ in $\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(X_{C,\infty,\infty})$. Then $\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(X_{C,\infty,\infty})$ is a faithful representation of both Γ_1 and Γ_2, and it determines the action of the Sen operators $\theta_{j,(1)}$ and $\theta_{j,(2)}$. By definition, we have $\theta_{j,(1)}(t) = \theta_{j,(2)}(t) = 0$ and

$$\theta_{j,(1)}(Y_{k,(1)}) = \delta_{j,k} = \theta_{j,(2)}(Y_{k,(2)}).$$

Therefore, by (20.1.4) and (20.1.5) one obtains

$$\theta_{j,(1)} = \sum_{k=1}^{d} -a_{j,k}\theta_{k,(2)},$$

this proves (1).

To prove (2), notice that the elements $t, Y_{j,(2)} \in \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(X_{C,\infty,\infty})$ provide an isomorphism

$$B_{\infty,\infty}t \oplus \bigoplus_{j=1}^{d} B_{\infty,\infty}Y_{j,(2)} = \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(X_{C,\infty,\infty}). \tag{20.1.6}$$

Taking Γ_2-invariants, and a lift $Y'_{k,(2)} \in \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(X_{C,\infty,0})$ of $dY_{k,(2)}$ (e.g. $Y'_{k,(2)} = \sum_\ell b_{k,\ell}Y_{\ell,(1)}$ with $(b_{k,j})_{k,\ell}$ the inverse of $(a_{k,\ell})_{k,\ell}$), one can write

$$Y'_{k,(2)} = Y_{k,(2)} - z_k t \text{ with } z_k \in B_{\infty,\infty}.$$

The invariance of $Y'_{k,(2)}$ by the action of Γ_2 implies the relation

$$\theta_{j,(2)}(z_k) = \delta_{j,k}.$$

Finally, it is clear that we can choose the z_k's to be Γ_1-locally analytic (eg. with $Y'_{k,(2)} = \sum_\ell b_{k,\ell}Y_{\ell,(1)}$ as before), and part (1) implies that $\theta_{j,(1)}(z_k) = a_{j,k}$. This finishes the proposition. □

Corollary 20.1.13. *Let $M > 0$, $n = n(M) \in \mathbb{N}$ and $z_{k,M} \in B^{p^n\Gamma_1\times p^n\Gamma_2-sm}_{\infty,\infty}$ be an element such that $|z_k - z_{k,M}| \leq |p^M|$ for all $k = 1,\dots,d$. Then we can write*

$$B^{\Gamma_1\times\Gamma_2-la}_{\infty,\infty} = \varinjlim_{M\to\infty} B^{p^n\Gamma_1-an,p^n\Gamma_2}_{\infty,\infty}\langle z_k - z_{k,M} :\ 1 \leq k \leq d\rangle. \tag{20.1.7}$$

as an LB space.

Proof. It is clear that the RHS is included in the LHS as the elements z_k and $z_{k,M}$ are analytic for some subgroup $p^m\Gamma_1 \times p^m\Gamma_2$. Conversely, given $f \in B^{p^m\Gamma_1\times p^m\Gamma_2-an}_{\infty,\infty}$, the Sen operators $\theta_{k,(2)}$ satisfy

$$|\theta_{k,(2)}(f)|_{p^m\Gamma_1\times p^m\Gamma_2-an} \leq C|f|_{p^m\Gamma_1\times p^m\Gamma_2-an}$$

where C is a constant that only depends on m, and the norm $|\cdot|_{p^m\Gamma_1\times p^m\Gamma_2-an}$ is the norm of the Banach space $B^{p^m\Gamma_1\times p^m\Gamma_2-an}_{\infty,\infty}$. Taking M such that $Cp^{-M} << 1$, one can consider the series for $j = 1,\dots,d$

$$D^i_{\theta_{(2)}}(f) := \sum_{k\in\mathbb{N}}(-1)^k\frac{(z_i - z_{i,M})^k}{k!}(\theta_{i,(2)})^k(f).$$

Then $D^i_{\theta_{(2)}}(f)$ is a $\Gamma_1 \times \Gamma_2$-locally analytic vector and one has the bound for $M >> 0$

$$|D^i_{\theta_{(2)}}(f)| \leq |f|.$$

Furthermore, a direct computation shows that $\theta_{i,(2)}(D^i_{\theta^{(2)}}(f)) = 0$ proving that it is $\langle \gamma_{i,(2)} \rangle$-smooth. Define

$$D_{\theta_{(2)}}(f) := D^d_{\theta_{(2)}} \circ \cdots \circ D^2_{\theta_{(2)}} \circ D^1_{\theta_{(2)}}(f).$$

Then, as the Sen operators commute with each other, we have that $D_{\theta_{(2)}}(f) \in B^{p^{m'}\Gamma_1 - an \times p^{m'}\Gamma_2}_{\infty,\infty}$ for $m' >> 0$ depending only on the radius of analyticity of f.

For $\alpha \in \mathbb{N}^d$ define

$$f_\alpha := D_{\theta_{(2)}}\left(\frac{\theta_{\alpha,(2)}(f)}{\alpha!}\right),$$

where $\theta_{\alpha,(2)} := (\theta_{d,(2)})^{\alpha_d} \circ \cdots \circ (\theta_{1,(2)})^{\alpha_1}$. A direct computation gives that

$$f = \sum_{\alpha \in \mathbb{N}^d} f_\alpha (\mathbf{z} - \mathbf{z}_M)^\alpha \tag{20.1.8}$$

where $(\mathbf{z}-\mathbf{z}_M)^\alpha := (z_1 - z_{1,M})^{\alpha_1} \cdots (z_d - z_{d,M})^{\alpha_d}$. One checks that (20.1.8) is an element of $B^{p^n\Gamma_1 - an, p^n\Gamma_2}_{\infty,\infty}\langle z_k - z_{k,M} : 1 \leq k \leq d \rangle$ for $M, n >> 0$ depending only on the radius of analyticity of f. □

Notation 2. We shall write the direct limit (20.1.7) as

$$B^{\Gamma_1 \times \Gamma_2 - la}_{\infty,\infty} = B^{\Gamma_1 - la, \Gamma_2 - sm}_{\infty,\infty}\{\mathbf{z} - \mathbf{z}_M\},$$

knowing that $z_{k,M}$ varies as $M \to \infty$.

The proof of Proposition 20.1.12 can be adapted to the case when ψ_2 is not necessarily a chart. We let $\psi_2 : X \to \mathbb{S}_K^{(e',d'-e')}$ be a map of adic spaces (not necessarily étale), and keep the notations introduced above for the induced coordinates over X.

Proposition 20.1.14. *Let $(b_{i,j})_{i,j} \in \mathscr{O}_{\mathrm{an}}(X)^{d' \times d}$ be the $d' \times d$ transformation matrix from $\psi_2^* \Omega^1_{\mathbb{S}_K^{(e',d'-e')}}(\log) \to \Omega^1_X(\log) = \psi_1^* \Omega^1_{\mathbb{S}_K^{(e,d-e)}}(\log)$ induced by the* dlog *differentials of the coordinates of ψ_2 and ψ_1.*

1. *Let $(a_{i,j})_{i,j} \in \mathscr{O}_{\mathrm{an}}(X)^{d \times d'}$ be the dual matrix of $(b_{i,j})_{i,j}$, and consider the Sen operators*

$$\mathrm{Sen} : \mathrm{Lie}\,\Gamma_1 \to B_{\infty,\infty} \otimes \mathrm{Lie}\,\Gamma_2.$$

   Then $\mathrm{Sen}(\theta_{i,(1)}) = -\sum_j a_{i,j}\theta_{j,(2)}$.

2. *Moreover, there are elements*

$$z_1, \ldots, z_{d'} \in B^{\Gamma_1 \times \Gamma_2 - la}_{\infty,\infty} = S_{\mathrm{Lie}\,\Gamma_2}(C^{la}(\mathrm{Lie}\,\Gamma_2, B_{\infty,\infty})_{\star_{1,3}})$$

   such that

$$\theta_{i,(2)}(z_j) = \delta_{i,j} \text{ and } \theta_{i,(1)}(z_j) = a_{i,j}.$$

Proof. We have a map of Faltings's extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathscr{O}}_X(1) & \longrightarrow & \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log_X} & \longrightarrow & \widehat{\mathscr{O}}_X \otimes \Omega^1_X(\log) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \psi_2^*} & & \uparrow{\scriptstyle \psi_2^*} & & \uparrow{\scriptstyle \psi_2^*} & & \\
0 & \longrightarrow & \widehat{\mathscr{O}}_{\mathbb{S}_K^{(e',d'-e')}}(1) & \longrightarrow & \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log_{\mathbb{S}_K^{(e',d'-e')}}} & \longrightarrow & \widehat{\mathscr{O}}_{\mathbb{S}_K^{(e',d'-e')}} \otimes \Omega^1_{\mathbb{S}_K^{(e',d'-e')}}(\log) & \longrightarrow & 0.
\end{array}$$

Then, since $\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log_{\mathbb{S}_K^{(e',d'-e')}}}$ is a faithful representation of Γ_2, the proof of (1) of Proposition 20.1.12 implies that

$$\mathrm{Sen}(\theta_{i,(1)}) = -\sum_j a_{i,j}\theta_{j,(2)}$$

as wanted. Furthermore, one can construct the elements z_i as in (2) of Proposition 20.1.12 by performing the same argument with the pullback of the Faltings extension of $\mathbb{S}_K^{(e',d'-e')}$. □

General G

We want to prove the following theorem.

Theorem 20.1.15. *Let X be an fs log smooth adic space over $\mathrm{Spa}(K, K^+)$ with log structure given by normal crossing divisors, G a p-adic Lie group and $\widetilde{X} \to X$ a pro-Kummer-étale G-torsor. Then the Sen operators given by local charts of X glue to a morphism of $\widehat{\mathscr{O}}_X$-vector bundles over $X_{\mathrm{prokét}}$*

$$\mathrm{Sen}_X : \Omega^1_X(\log)^\vee \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) \to \underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_X.$$

Moreover, let H be a p-adic Lie group, Y be another adic space as above and $\widetilde{Y} \to Y'$ an H-torsor. Let $H \to G$ be a morphism of groups and suppose that we have a commutative diagram compatible with the actions of the groups

$$\begin{array}{ccc} \widetilde{Y} & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}. \tag{20.1.9}$$

Then the following square is commutative

$$\begin{array}{ccc} f^*\Omega^1_X(\log)^\vee \otimes_{\mathscr{O}_Y} (1)\widehat{\mathscr{O}}_Y & \xrightarrow{f^*\mathrm{Sen}_X} & f^*\underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_Y \\ \uparrow & & \uparrow \\ \Omega^1_Y(\log)^\vee \otimes_{\mathscr{O}_Y} \widehat{\mathscr{O}}_Y & \xrightarrow{\mathrm{Sen}_Y} & \underline{\mathfrak{h}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_Y \end{array} \tag{20.1.10}$$

Remark 20.1.16. The Sen operator can be written equivalently as a map

$$\theta_X : \widehat{\mathscr{O}}_X(1) \to \underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}_p}} \widehat{\mathscr{O}}_X \otimes \Omega^1_X(\log).$$

Note that by construction of the Sen operator one has $\theta_X \wedge \theta_X = 0$, this proves the first two points of Theorem 20.0.1.

First, we formally reduce the statement to the case when X is affinoid admiting a chart $\psi : X \to \mathbb{S}_K^{(e,d-e)}$.

Lemma 20.1.17. *Suppose that Theorem 20.1.15 holds for fs log affinoid spaces admitting a chart to $\mathbb{S}_K^{(e,d-e)}$. Then it holds for an arbitrary X.*

Proof. Let $U_\bullet$ be an hypercover of X by objects in $X_{\text{ét}}$ satisfying the hypothesis of the lemma. Let $\widetilde{U}_\bullet := U_\bullet \times_X \widetilde{X}$ be the G-torsor over $U_\bullet$. By hypothesis on the existence and functoriality of Sen for the affinoids, we have a map of $\widehat{\mathscr{O}}_{U_\bullet}$-sheaves on $U_{\bullet,\mathrm{prokét}}$

$$\mathrm{Sen}_{U_\bullet} : \Omega^1_{U_\bullet}(\log)^\vee \otimes_{\mathscr{O}_{U_\bullet}} \widehat{\mathscr{O}}_{U_\bullet}(1) \to \underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_{U_\bullet}.$$

By glueing we obtain the desired map Sen_X. Functoriality of Sen_X for arbitrary adic spaces is proven in the same way. □

Thus, we can assume that X and Y are fs log smooth affinoid spaces over $\mathrm{Spa}(K, K^+)$ admitting charts to $\mathbb{S}_K^{(e,d-e)}$ for some $e, d \in \mathbb{N}$. We first show the existence of the map Sen_X. Let us keep the notation of §20.1.1, and let us introduce some more:

- We denote $\widetilde{X}_{\infty,0} = X_{\infty,0} \times_X \widetilde{X}$, $\widetilde{X}_{0,\infty} = \widetilde{X} \times_X X_{0,\infty}$ and $\widetilde{X}_{\infty,\infty} = X_{\infty,0} \times_X \widetilde{X} \times_X X_{0,\infty}$.
- We denote $\widetilde{B}_{\infty,0} := \widehat{\mathscr{O}}_X(\widetilde{X}_{C,\infty,0})$, $\widetilde{B}_{0,\infty} = \widehat{\mathscr{O}}_X(\widetilde{X}_{C,0,\infty})$ and $\widetilde{B}_{\infty,\infty} = \widehat{\mathscr{O}}_X(\widetilde{X}_{C,\infty,\infty})$.

We have the following proposition

Proposition 20.1.18. *Let $\theta_{i,(1)}$ and $\theta_{i,(2)}$ denote the Sen operators arising from the charts ψ_1 and ψ_2 respectively. Let $A = (a_{i,j})$ denote the base change matrix from ψ_2 to ψ_1 of $\Omega^1_X(\log)^\vee$ corresponding to the stantard basis of $\Omega^1_{\mathbb{S}_C^{(e,d-e)}}(\log)$ (cf. (20.1.4)). Then the following diagram commutes*

$$\begin{array}{ccc} \widetilde{B} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\,\Gamma_1 & \xrightarrow{\theta_{(1)}} & \widetilde{B} \otimes_{\mathbb{Q}_p} \mathfrak{g} \\ {\scriptstyle -A}\downarrow & \nearrow_{\theta_{(2)}} & \\ \widetilde{B} \otimes_{\mathbb{Q}_p} \mathrm{Lie}\,\Gamma_2 & & \end{array}$$

where we identify $\mathrm{Lie}\,\Gamma_1 \cong \mathrm{Lie}\,\Gamma_2 \cong \mathbb{Q}_p(1)^d$ via their standard basis. In particular, the local Sen operators glue to Sen_X.

Remark 20.1.19. The reason to the $-A$ in the proposition is the -1 of Proposition 20.1.12, which is a consequence of (20.1.5). Notice that a priori this is a statement over the C-extension of X. Nonetheless, the Sen operators are Galois equivariant and $\mathrm{Gal}(C/K)$ acts via the cyclotomic character on $\mathrm{Lie}\,\Gamma_i$.

Proof. By Theorem 19.3.3, the orbit maps provide $\mathrm{Lie}(\Gamma_1 \times G \times \Gamma_2)$-equivariant and $\widetilde{B}_{\infty,\infty}$-linear isomorphisms

$$C^{la}(\mathrm{Lie}\,\Gamma_1 \times \mathfrak{g}, \widetilde{B}_{\infty,\infty})_{\star_2,\theta_{(2)}} = \widetilde{B}_{\infty,\infty} \widehat{\otimes}_{\widetilde{B}^{\Gamma_1 \times G \times \Gamma_2 - sm}_{\infty,\infty}} \widetilde{B}^{\Gamma_1 \times G \times \Gamma_2 - la}_{\infty,\infty} = C^{la}(\mathfrak{g} \times \mathrm{Lie}\,\Gamma_2, \widetilde{B}_{\infty,\infty})_{\star_2,\theta_{(1)}}. \tag{20.1.11}$$

We also have $\widetilde{B}_{\infty,\infty}$-linear isomorphisms

$$\widetilde{B}_{\infty,\infty} \widehat{\otimes}_{\widetilde{B}^{\Gamma_1 \times G - sm}_{\infty,0}} \widetilde{B}^{\Gamma_1 \times G - la}_{\infty,0} = C^{la}(\mathfrak{g}, \widetilde{B}_{\infty,\infty}) = \widetilde{B}_{\infty,\infty} \widehat{\otimes}_{\widetilde{B}^{G \times \Gamma_2 - sm}_{0,\infty}} \widetilde{B}^{G \times \Gamma_2 - la}_{0,\infty}. \tag{20.1.12}$$

Recall that, by definition, the Sen operators are constructed as the derivations of $\mathrm{Lie}\,\Gamma_1$ and $\mathrm{Lie}\,\Gamma_2$ in the spaces of locally analytic vectors, the idea of the proof is to use the above isomorphisms to compare both.

By Proposition 20.1.12, there are $\Gamma_1 \times \Gamma_2$-locally analytic elements $z_1, \ldots, z_d$ in $B_{\infty,\infty}$ such that

$$\theta_i^{(2)}(z_j) = \delta_{i,j} \text{ and } \theta_i^{(1)}(z_j) = a_{i,j}. \tag{20.1.13}$$

By the proof of Corollary 20.1.13, we can find elements $z_{i,M} \in B^{\Gamma_1 \times \Gamma_2 - sm}_{\infty,\infty}$ with $|z_i - z_{i,M}| < |p^M|$ as $M \to \infty$, such that

$$\widetilde{B}^{\Gamma_1 \times G \times \Gamma_2 - la}_{\infty,\infty} = \widetilde{B}^{\Gamma_1 \times G - la, \Gamma_2 - sm}_{\infty,\infty}\{\mathbf{z} - \mathbf{z}_M\}.$$

Let $f \in C^{la}(\mathfrak{g}, \mathbb{Q}_p) \subset C^{la}(\mathfrak{g}, \widetilde{B}_{\infty,\infty})$. We want to show that

$$\theta_{i,(1)}(f) = \sum_j -a_{i,j}\theta_{j,(2)}(f).$$

By (20.1.12), there are $r_k \in \widetilde{B}_{0,\infty}$ and $v_k \in \widetilde{B}_{0,\infty}^{G\times\Gamma_2-la}$ such that

$$f(g) = \sum_k r_k g \cdot v_k.$$

Notice that by definition of the Sen action

$$\theta_{j,(2)}(f) = \sum_k r_k g \cdot \theta_{j,(2)}(v_k).$$

There exists $M >> 0$ such that

$$v_k = \sum_{\alpha\in\mathbb{N}^d} \frac{1}{\alpha!} v_{\alpha,k}(\mathbf{z}-\mathbf{z}_M)^\alpha$$

with $v_{\alpha,k} = D_{\theta_{(2)}}(\theta_{\alpha,(2)}(v_k))$ as in Corollary 20.1.13. Moreover, $v_{\alpha,k} \in B_{\infty,\infty}^{\Gamma_1\times G-la,\Gamma_2-sm}$ for all k and α. We deduce that

$$f(g) = \sum_k r_k \sum_{\alpha\in\mathbb{N}^d} \frac{1}{\alpha!} g \cdot v_{\alpha,k}(\mathbf{z}-\mathbf{z}_M)^\alpha.$$

By definition of θ_1 one gets that

$$\theta_{i,(1)}(f)(g) = \sum_k \sum_{\alpha\in\mathbf{N}^d} \frac{r_k(\mathbf{z}-\mathbf{z}_M)^\alpha}{\alpha!} g \cdot \theta_{i,(1)}(v_{\alpha,k}).$$

But by definition

$$v_{\alpha,k} = \sum_{\alpha\in\mathbb{N}^d} \frac{(-1)^{|\alpha|}}{\alpha!} (\mathbf{z}-\mathbf{z}_M)^\alpha \theta_{\alpha,(2)}(v_k).$$

By (20.1.13), and the fact that the $\theta_{\alpha,(2)}(v_k)$ are Γ_1-smooth, the previous implies that

$$\theta_{i,(1)}(v_{\alpha,k}) = -\sum_{j=1}^d a_{i,j} v_{\alpha+1_j,k}.$$

A direct computation shows that

$$\begin{aligned}
\theta_{i,(1)}(f)(g) &= -\sum_{j=1}^d a_{i,j} \sum_k \sum_{\alpha\in\mathbf{N}^d} \frac{r_k(\mathbf{z}-\mathbf{z}_M)^\alpha}{\alpha!} g \cdot v_{\alpha+1_j,k} \\
&= -\sum_{j=1}^d a_{i,j} \sum_k r_k g \cdot \theta_{j,(2)}(v_k) \\
&= -\sum_{j=1}^d a_{i,j} \theta_{j,(2)}(f)(g)
\end{aligned}$$

as wanted. □

Finally, we prove the functoriality with respect to the group. We first prove the version when $X = Y$.

Lemma 20.1.20. *In Theorem 19.3.3, suppose that $X = Y$ is affinoid, then the commutativity of (20.1.10) holds.*

Proof. Let $\widetilde{X}_G \to X$ and $\widetilde{X}_H \to X$ denote the G and H-torsors respectively, and let $f : \widetilde{X}_H \to \widetilde{X}_G$ be the equivariant map. Let X_∞ be the Γ-torsor given by a system of coordinates, let $\widetilde{B}_\infty$ and $\widetilde{A}_\infty$ be the rings of functions of $\widetilde{X}_G \times_X X_\infty$ and $\widetilde{X}_H \times_X X_\infty$ respectively. Then f induces a map

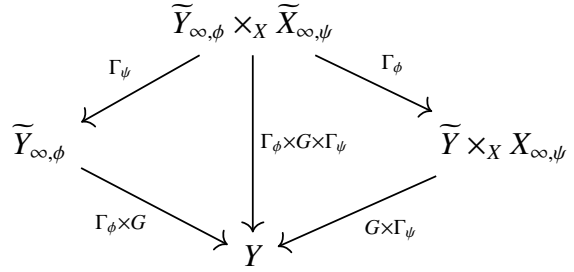
$$f^* : C^{la}(\mathfrak{g}, \widetilde{B}_\infty) \to C^{la}(\mathfrak{h}, \widetilde{A}_\infty)$$

equivariant for the action Γ, and compatible with the homomorphism $H \to G$. The lemma follows by the definition of the Sen operators as derivations of $\mathrm{Lie}\,\Gamma$ in the Sen modules. □

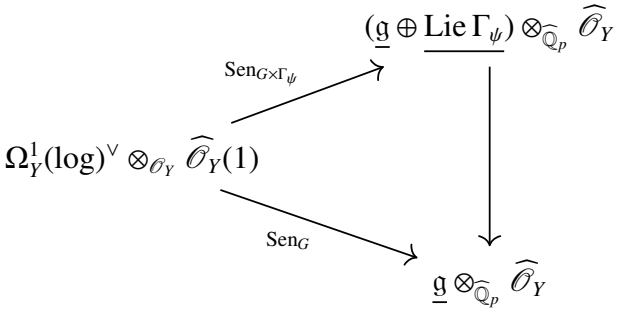
Next, we prove the version when the square (20.1.9) is cartesian

Lemma 20.1.21. *In Theorem 19.3.3, suppose that X and Y are affinoids admitting charts and that $\widetilde{Y} = Y \times_X \widetilde{X}$. Then (20.1.10) is commutative.*

Proof. Let ψ_X and ϕ_Y be system of coordinates of X and Y respectively, and let $X_{\infty,\psi}$ and $Y_{\infty,\phi}$ denote the Γ_X and Γ_Y torsors induced by the perfectoid coordinates. Let $\widetilde{Y}_{\infty,\phi} = Y_{\infty,\phi} \times_Y \widetilde{Y}$ and $\widetilde{X}_{\infty,\phi} = \widetilde{X} \times_X X_{\infty,\psi}$, consider the following diagram of torsors



By Lemma (20.1.20) we have a commutative diagram of Sen morphisms



But $\underline{\mathrm{Lie}\,\Gamma_\psi} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_Y = f^*\Omega^1_X(\log)^\vee \otimes_{\mathscr{O}_Y} \widehat{\mathscr{O}}_Y(1)$. We claim that the projection of $\mathrm{Sen}_{G\times\Gamma_\psi}$ onto $f^*\Omega^1_X(\log)^\vee \otimes_{\mathscr{O}_Y} \widehat{\mathscr{O}}_Y(1)$ is the pushforward of tangent vectors. Indeed, this follows from the commutativity of Faltings's extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathscr{O}}_Y(1) & \longrightarrow & \mathrm{gr}^1 \mathscr{OB}^+_{\mathrm{dR},\log_Y} & \longrightarrow & \Omega^1_Y \otimes_{\mathscr{O}_Y} \widehat{\mathscr{O}}_Y & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle f^*} & & \\ 0 & \longrightarrow & \widehat{\mathscr{O}}_Y(1) & \longrightarrow & f^*\mathrm{gr}^1 \mathscr{OB}^+_{\mathrm{dR},\log_X} & \longrightarrow & f^*\Omega^1_X \otimes_{\mathscr{O}_Y} \widehat{\mathscr{O}}_Y & \longrightarrow & 0, \end{array}$$

from Proposition 20.1.14, and from the proof of Proposition 20.1.18. □

Proposition 20.1.22. *In Theorem 19.3.3, suppose that X and Y are affinoid admitting charts. Then (20.1.10) is commutative.*

Proof. This follows from Lemmas 20.1.20 and 20.1.21 after writing the diagram (20.1.9) as a composition $\widetilde{Y} \to Y \times_X \widetilde{X} \to \widetilde{X}$. □

The Sen morphism encodes the directions of perfectoidness of systems of coordinates of X. We have the following conjecture, which is a generalization of a theorem of Sen saying that a p-adic Galois representation of a finite extension of $\mathbb{Q}_p$ has vanishing Sen operator if and only if it is potentially unramified, see Corollary 3.32 of [FO].

Conjecture 3. *The Sen morphism $\mathrm{Sen}_X : \Omega^1_X(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_X$ is a $\widehat{\mathscr{O}}_X$-subbundle (i.e. it is injective with locally free quotient) if and only if $\widetilde{X}_C$ is a perfectoid space.*

As we will show in §22, the pro(Kummer)étale torsors defining the infinite level Shimura varieties satisfy this subbundle condition. The proof of this fact never uses the perfectoidness of the Shimura variety, but the p-adic Riemman-Hilbert correspondance of [DLLZ18].

Notation 3. From now on, we will write θ_X for the Sen operator Sen_X (or θ is X is clear from the context). Given F a locally analytic LF $\mathfrak{g}$-representation over $\mathbb{Q}_p$, we let $\theta_X(F)$ denote the Sen operator of the $G^\dagger$ equivariant $\widehat{\mathscr{O}}_{\widetilde{X}}$-sheaf $F\widehat{\otimes}_{\mathbb{Q}_p}\widehat{\mathscr{O}}_{\widetilde{X}}$.

20.2. Application to proétale cohomology

We keep the previous notations, i.e., X is an fs log smooth adic space over K with log structure given by normal crossing divisors, and $\widetilde{X}$ a pro-Kummer-étale G-torsor over X. In this last section we apply the globalization of the Sen operators of §20.1 and the computations of group cohomology via Sen theory of §19.4 to finally prove part (3) of Theorem 20.0.1. In the process, we also prove a version of this theorem with coefficients over the completed structural sheaf of the boundary divisor, this will be used in §22 for the cohomology with compact supports. We finish by explaining the relation between the Sen operator θ_X, and the p-adic Simpson correspondance of [LZ17, DLLZ18, Wan21].

20.2.1. Pro-Kummer-étale cohomology of relative locally analytic $\widehat{\mathscr{O}}$-modules

Let X and $\widetilde{X} \to X$ be as before. Let $D \subset X$ be the boundary divisor. Étale locally on X, D can be written as a disjoint union of irreducible components $D = \bigcup_{a\in I} D_a$, where the finite intersections of the D_a's are smooth. If this holds locally analytic over X we say that D is a strict normal crossing divisor, for simplicity let us assume that this is the case. Given $J \subset I$ a finite subset we let $D_J = \bigcap_{a\in I} D_a$ and $\iota_J : D_J \subset X$, we will simply write $\widehat{\mathscr{O}}^{(+)}_{D_J}$ for the sheaf $\iota_{J,*}\widehat{\mathscr{O}}^{(+)}_{D_J}$ over $X_{\mathrm{prokét}}$.

Consider the following scenario. Let W be a locally noetherian adic space over (K, K^+) endowed with an action of G. Let $\pi_W^\Diamond : \widetilde{X}^\Diamond \to W^\Diamond$ be a G-equivariant map of diamonds over $\mathrm{Spd}\, K$. In particular, as the map of diamonds respect the untilts, we have a map of ringed sites

$$\pi_W : (\widetilde{X}_{\mathrm{prokét}}, \widehat{\mathscr{O}}_{\widetilde{X}}) \to (W_{\mathrm{an}}, \mathscr{O}_{W,\mathrm{an}}).$$

Definition 20.2.1. 1. Let $\mathscr{F}$ be a squarrable LF sheaf over W (see Definition 17.3.1). We define the pullback $\pi_W^*(\mathscr{F})$ to be the $\widehat{\mathscr{O}}_{\widetilde{X}}$-sheaf mapping a log affinoid perfectoid $U = \mathrm{Spa}(R, R^+)$ the LF space

$$\Gamma(U, \pi_W^{-1}(\mathscr{F})\widehat{\otimes}_{\pi_W^{-1}(\mathscr{O}_W)}\mathscr{O}_U).$$

In particular, if $\mathscr{F}$ is of the form $\varinjlim_n \varprojlim_m \mathscr{B}_n$ with $\mathscr{B}_{n,m}$ ON Banach sheaves, we have that

$$\pi_w^*(\mathscr{F}) = \varinjlim_n \varprojlim_m \pi_W^{-1}(\mathscr{B}_{n,m})\widehat{\otimes}_{\pi_W^{-1}(\mathscr{O}_W)}\widehat{\mathscr{O}}_{\widetilde{X}},$$

where the last tensor product is given by the projective tensor product (or just the solid tensor product) of Banach spaces on affinoid perfectoids.

2. Let $G' \subset G$ be an open subgroup, we denote by $\nu_{G'} : X_{C,G',\mathrm{proét}} \to X_{C,G',\mathrm{két}}$ the projection of sites. Let $G^\dagger = \{G' \subset G\}$ denote the overconvergent neighbourhood of 1 in G. Let $\mathscr{F}$ be a $G^\dagger$-equivariant squarrable LF sheaf over W which can be written, locally on W_{an}, as a colimit of G'-equivariant squarrable LF sheaves for $G' \to 1$, say $\mathscr{F} = \varinjlim_{G'\to 1} \mathscr{F}_{G'}$. We define the following projection living in $\widetilde{X}_{\mathrm{két}}$:

$$R\nu_{\infty,*}\pi_W^*(\mathscr{F}) := \varinjlim_{G'\to 1} R\nu_{G',*}\pi_W^*(\mathscr{F}_{G'}).$$

Recall some notations from Definition 17.3.8. Let $? \in \{\text{két}, \text{an}\}$ and $\widehat{\mathscr{O}}_{\widetilde{X}_{C,?}}$ the restriction of $\widehat{\mathscr{O}}_X$ to $\widetilde{X}_{C,?}$. The sections of the sheaves $\widehat{\mathscr{O}}_{\widetilde{X},?}$ at qcqs objects admit an action of $G^\dagger$, we let $\mathscr{O}^{la}_{\widetilde{X}_{C,?}}$ denote the subsheaf of locally analytic sections. We also define $\mathscr{O}^{sm}_{\widetilde{X}_{C,?}}$ to be the subsheaf of smooth sections. Note that $\mathscr{O}^{sm}_{\widetilde{X}_{C,?}} = \varinjlim_{G' \to 1} \mathscr{O}_{X_{C,G',?}}$.

Theorem 20.2.2. *Let $\mathscr{F}$ be a squarrable LF sheaf over W endowed with a locally analytic action of $\mathfrak{g}$. . Assume that, locally on W_{an}, $\mathscr{F}$ is of the form $\mathscr{F} = \varinjlim_n \mathscr{F}_n$ with $\mathscr{F}_n$ squarrable Fréchet and $\mathscr{F}_n = \varprojlim_m \mathscr{B}_{n,m}$ is a presentation with dense transition maps, where the $\mathscr{B}_{n,m}$ are projective Banach sheaves endowed with a $\mathfrak{g}$-equivariant action over W which is analytic for a uniform radius of analyticity as $m \to \infty$ and n is fixed. Let $\lambda : \widetilde{X}_{C,\text{két}} \to \widetilde{X}_{C,\text{an}}$ be the projection, then*

$$R\nu_{\infty,*}(\pi_W^*\mathscr{F}) = R\Gamma(\theta_X, (\pi_W^*\mathscr{F})|_{\widetilde{X}_{\text{két}}})^{G^\dagger - sm} \tag{20.2.1}$$

$$R(\lambda \circ \nu_\infty)_*(\pi_W^*\mathscr{F}) = R\Gamma(\theta_X, (\pi_W^*\mathscr{F})|_{\widetilde{X}_{\text{an}}})^{G^\dagger - sm}. \tag{20.2.2}$$

Moreover, for $J \subset I$ a finite subset we have

$$R\nu_{\infty,*}(\pi_W^*\mathscr{F}\widehat{\otimes}_{\widehat{\mathscr{O}}_X}\widehat{\mathscr{O}}_{D_J}) = R\Gamma(\theta_X, (\pi_W^*\mathscr{F})|_{\widetilde{X}_{\text{két}}})^{G^\dagger - sm}\widehat{\otimes}^L_{\mathscr{O}^{sm}_{\widetilde{X},\text{két}}}\mathscr{O}^{sm}_{D_J,\text{két}}$$

$$R(\lambda \circ \nu_\infty)_*(\pi_W^*\mathscr{F}\widehat{\otimes}_{\widehat{\mathscr{O}}_X}\widehat{\mathscr{O}}_{D_J}) = R\Gamma(\theta_X, (\pi_W^*\mathscr{F})|_{\widetilde{X}_{\text{an}}})^{G^\dagger - sm}\widehat{\otimes}^L_{\mathscr{O}^{sm}_{\widetilde{X},\text{két}}}\mathscr{O}^{sm}_{D_J,\text{an}}.$$

where $\mathscr{O}^{sm}_{D_J,?} = \varinjlim_{G' \to 1} \mathscr{O}_{D_J \times_X X_{G',?}}$.

Proof. The statement is local on the analytic topology of $\widetilde{X}$. Then, we can assume without lose of generality that X is affinoid and that it admits a chart $\psi : X \to \mathbb{S}_K^{(e,d-e)}$ and that $\mathscr{F}$ can be written as a colimit of squarrable Fréchet spaces. Since $X_?$ for $? \in \{\text{két}, \text{an}\}$ admits a basis consisting in qcqs objects, the derived pushforwards $R\nu_{G',*}$ commute with limits and filtered colimits. Then,

$$R\nu_{G',*}(\pi_W^*\mathscr{F}) = \varinjlim_n \varprojlim_m R\nu_{G',*}(\pi_W^*\mathscr{B}_{n,m}).$$

Let X_∞ denote the Γ-torsor obtained by taking the pullback of the perfectoid coordinates. Then, for an affinoid $U \in X_{C,G',\text{két}}$ we have by Proposition 19.4.7

$$\begin{aligned} R\Gamma_{\text{prokét}}(U, \pi_W^*\mathscr{B}_{n,m}) &= R\Gamma(G' \times \Gamma, \pi_W^*\mathscr{B}_{n,m}(\widetilde{X} \times_{X_{G'}} U \times_X X_\infty)) \\ &= R\Gamma(\theta_X, \pi_W^*\mathscr{B}_{n,m}(\widetilde{X} \times_{X_{G'}} U))^{G'}. \end{aligned}$$

Now, let $\widetilde{U} \in \widetilde{X}_{\text{két}}$ be a qcqs object arising from an affinoid $U_{G'} \in X_{G'}$. For $G'' \subset G'$ let $U_{G''}$ denote the pullback to $X_{G''}$. Then we have that

$$\begin{aligned} R\nu_{\infty,*}(\pi_W^*\mathscr{F})(\widetilde{U}) &= \varinjlim_{G'',n} R\Gamma_{\text{prokét}}(U_{G''}, \pi_W^*\mathscr{F}_n) \\ &= \varinjlim_{G'',n}\varprojlim_m R\Gamma_{\text{prokét}}(U_{G''}, \pi_W^*\mathscr{B}_{n,m}) \\ &= \varinjlim_{G'',n}\varprojlim_m R\Gamma(\theta_X, \pi_W^*\mathscr{B}_{n,m}(\widetilde{U}))^{G''} \\ &= \varinjlim_{G'',n}(R\Gamma(\theta_X, \pi_W^*\mathscr{F}_n)(\widetilde{U}))^{G''} \\ &= \varinjlim_n R\Gamma(\theta_X, \pi_W^*\mathscr{F}_n)(\widetilde{U})^{G^\dagger - sm} \\ &= R\Gamma(\theta_X, \pi_W^*\mathscr{F})(\widetilde{U})^{G^\dagger - sm}. \end{aligned}$$

This proves (20.2.1). Notice that the same computation holds for a qcqs object $\widetilde{U} \in \widetilde{X}_{\mathrm{an}}$ arising as the pullback of an open affinoid subspace $U \subset X_{G'}$, this implies (20.2.2).

Finally, the statement for $\pi_W^*\mathscr{F}\widehat{\otimes}_{\widehat{\mathscr{O}}_X}\widehat{\mathscr{O}}_{D_J}$ can be reduced to the case $\mathscr{F} = C^{la}(\mathfrak{g}, \mathscr{O}_W)_{\star_{1,3}}$ by first reducing to the ON Banach case, and then using the same strategy of Lemma 19.3.8. This last situation will be handle in Theorem 20.2.4. □

Remark 20.2.3. The condition imposed on the sheaf $\mathscr{F}$ is ad hoc, but it suffices for the applications of this article. The proof of Lemma 19.3.8 suggests that there should be a category of "relative locally analytic $\widehat{\mathscr{O}}_X$ sheaves over X", generated by the sheaves of locally analytic functions of pro-Kummer-étale G-torsors where G is a p-adic Lie group. Then, the (rational) p-adic Simpson correspondance could be seen as the construction of a natural Sen operator $\theta_X(\mathscr{F})$ for $\mathscr{F}$ a sheaf in this category.

20.2.2. Locally analytic vectors of $\widehat{\mathscr{O}}$

We make the following non-degeneracy hypothesis for the Sen operators

(BUN) The Sen morphism $\theta_X : \Omega^1_X(\log)^\vee \otimes_{\mathscr{O}_X} \widehat{\mathscr{O}}_X(1) \to \underline{\mathfrak{g}} \otimes_{\widehat{\mathbb{Q}}_p} \widehat{\mathscr{O}}_X$ is a subbundle. In other words, θ_X is injective and its cokernel is an $\widehat{\mathscr{O}}_X$-vector bundle.

Theorem 20.2.4. *Let $\nu : X_{C,\mathrm{prok\acute{e}t}} \to X_{C,\mathrm{k\acute{e}t}}$ and $\lambda : \widetilde{X}_{C,\mathrm{k\acute{e}t}} \to \widetilde{X}_{C,\mathrm{an}}$ denote the projection of sites. Then*

$$R\nu_{\infty,*}(C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}_X) = \mathscr{O}^{la}_{\widetilde{X}_C,\mathrm{k\acute{e}t}}$$
$$R(\lambda\circ\nu_\infty)_*(C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}_X) = \mathscr{O}^{la}_{\widetilde{X}_C,\mathrm{an}}.$$

Moreover, let $J \subset I$, then

$$R\nu_{\infty,*}(C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}_{D_J}) = \mathscr{O}^{la}_{\widetilde{X}_C,\mathrm{k\acute{e}t}}\widehat{\otimes}_{\mathscr{O}^{sm}_{\widetilde{X}_C,\mathrm{k\acute{e}t}}}\mathscr{O}^{sm}_{D_J,\mathrm{k\acute{e}t}}$$
$$R(\lambda\circ\nu_\infty)_*(C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}_{D_J}) = \mathscr{O}^{la}_{\widetilde{X}_C,\mathrm{an}}\widehat{\otimes}_{\mathscr{O}^{sm}_{\widetilde{X}_C,\mathrm{an}}}\mathscr{O}^{sm}_{D_J,\mathrm{an}}.$$

Proof. The statement is local on $\widetilde{X}_?$ for $? \in \{\mathrm{k\acute{e}t}, \mathrm{an}\}$, so we can assume that X is affinoid and that it admits a chart $\psi : X \to \mathbb{S}_K^{(e,d-e)}$. Let $\widetilde{U} \in \widetilde{X}_{C,\mathrm{k\acute{e}t}}$ be a qcqs object arising from an affinoid $U \in X_{C,G',\mathrm{k\acute{e}t}}$ with $G' \subset G$. For $G'' \subset G'$ we shall denote $U_{G''} := X_{C,G''} \times_{X_{C,G'}} U$. Let $J \subset \{e+1, \dots, d\}$ and consider D_J the divisor defined by $(S_a : a \in J)$. Without loss of generality we can assume that $J = \{k+1, \dots, d\}$. Let X_∞ be the pullback over X of the perfectoid coordinates of $\mathbb{S}_K^{(e,d-e)}$, and let us denote $\widetilde{U}_\infty := \widetilde{U} \times_X X_\infty$. Then, since $\widetilde{U}_\infty$ is perfectoid, we have that

$$R\Gamma_{\mathrm{prok\acute{e}t}}(U, C^{la}(G', \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}_{D_J}) = R\Gamma(G'\times\Gamma, C^{la}(G', \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\widehat{\mathbb{Q}}_p}\widehat{\mathscr{O}}_{D_J}(\widetilde{U}_\infty)).$$

Therefore

$$\begin{aligned} R\nu_{\infty,*}(C^{la}(\mathfrak{g}, \mathbb{Q}_p)_{\star_1}\widehat{\otimes}\widehat{\mathscr{O}}_{D_J})(\widetilde{U}) &= \varinjlim_{G''\to 1} R\Gamma_{\mathrm{pro\acute{e}t}}(U_{G''}, C^{la}(G'', \mathbb{Q}_p)_{\star_1}\widehat{\otimes}\widehat{\mathscr{O}}_{D_J}) \\ &= \varinjlim_{G''\to 1} R\Gamma(G'\times\Gamma, C^{la}(G', \mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\mathbb{Q}_p}\widehat{\mathscr{O}}_{D_J}(\widetilde{U}_\infty)). \end{aligned}$$

But the triple $(\widehat{\mathscr{O}}_{D_J}(\widetilde{U}_\infty), G'\times\Gamma_{J^c}, \mathrm{pr}_{J^c})$ is a Sen theory, where $\Gamma_{J^c} \subset \Gamma$ correspond to the J^c-th components with $J^c = \{e+1, \dots, d\}\backslash J$. Indeed, the Sen traces of Example 19.1.7 specializes to Sen traces of the boundary $(S_a = 0 : a \in J)$, and the proof of Proposition 20.1.5 still holds in this situation. However, D_J also has monodromy given by the action of Γ_J, the J-th components of Γ, and $\mathrm{Lie}\,\Gamma_J$

acts via the reduction of the Sen map to the boundary: $\theta_X : \mathrm{Lie}\,\Gamma \to \Omega^1_x(\log)\widehat{\otimes}\widehat{\mathscr{O}}_{D_J}$. Therefore, by Proposition 19.4.7 one finds that

$$\begin{aligned} R\Gamma(G'\times\Gamma, C^{la}(G'',\mathbb{Q}_p)_{\star_1}\widehat{\otimes}_{\mathbb{Q}_p}\widehat{\mathscr{O}}_{D_J}(\widetilde{U}_\infty)) &= R\Gamma(\theta_X, C^{la}(G'',\widehat{\mathscr{O}}_{D_J}(\widetilde{U}))_{\star_{1,3}})^{G''} \\ &= R\Gamma(\theta_X, C^{la}(G'',\widehat{\mathscr{O}}_X(\widetilde{U}))_{\star_{1,3}})^{G''}\widehat{\otimes}_{\mathscr{O}_X(U_{G''})}\mathscr{O}_{D_J}(U_{G''}). \end{aligned}$$

For the second equality one uses the description of $R\Gamma(\theta_X, C^{la}(G'',\widehat{\mathscr{O}}_X)_{\star_{1,3}})^{G''}$ as the Γ-invariants of the Lie Γ-Koszul complex of the Sen module of $C^{la}(G'',\widehat{\mathscr{O}}_X)$, see Definition 19.4.5. Hence, one deduces

$$R\nu_{\infty,*}(C^{la}(\mathfrak{g},\mathbb{Q}_p)_{\star_1}\widehat{\otimes}\widehat{\mathscr{O}}_{D_J})(\widetilde{U}) = R\Gamma(\theta_X, C^{la}(\mathfrak{g},\mathbb{Q}_p)_{\star_1}\widehat{\otimes}\widehat{\mathscr{O}}_X(\widetilde{U}))^{G^\dagger-sm}\widehat{\otimes}_{\widehat{\mathscr{O}}^{sm}_{\widetilde{X}}(\widetilde{U})}\mathscr{O}^{sm}_{D_J}(\widetilde{U}).$$

This proves the final part of Theorem 20.2.2.

Finally, if the hypothesis (BUN) holds, we can suppose that the image of θ_X admits a complement given by a subspace $V \subset \mathfrak{g}$. In particular, the Poincaré lemma implies that

$$R\Gamma(\theta_X, C^{la}(\mathfrak{g},\mathbb{Q}_p)_{\star_1}\widehat{\otimes}\widehat{\mathscr{O}}_X(\widetilde{U})) = C^{la}(\mathfrak{g},\widehat{\mathscr{O}}_X(\widetilde{U}))^{\theta_X=0}_{\star_{1,3}}.$$

But then, the $G^\dagger$-smooth vectors of $C^{la}(\mathfrak{g},\widehat{\mathscr{O}}_X(\widetilde{U}))^{\theta_X=0}_{\star_{1,3}}$ are nothing but the $G^\dagger$-smooth and Γ-invariant vectors of $C^{la}(\mathfrak{g},\widehat{\mathscr{O}}_X(\widetilde{U}_\infty))_{\star_{1,3}}$, which are precisely the locally analytic vectors $\mathscr{O}^{la}_{\widetilde{X},\text{két}}(\widetilde{U})$. Notice that the previous argument does not change if we take instead $\widetilde{U} \in \widetilde{X}_{C,\text{an}}$, this ends the proof of the theorem. □

20.2.3. The p-adic Simpson correspondance

In this paragraph we explain the relation between the Sen operator and the p-adic Simpson correspondance of [LZ17, DLLZ18, Wan21]. Let us recall (part) of one of the main theorems of [LZ17, DLLZ18].

Theorem 20.2.5 ([LZ17, Theo. 2.1] and [DLLZ18, Theo. 3.2.4])**.** *Let X be an fs log smooth adic space over $\mathrm{Spa}(K,K^+)$ and let $\mathbb{L}$ be a pro-Kummer-étale $\mathbb{Q}_p$-local system admitting a lisse lattice $\mathbb{L}^0 \subset \mathbb{L}$. Let $\mathscr{O}\mathbb{C}_{\log} = \mathrm{gr}^0\mathscr{O}\mathbb{B}_{\mathrm{dR},\log}$ be the Hodge-Tate period sheaf, and $\nu : X_{C,\text{prokét}} \to X_{C,\text{két}}$ be the projection of sites. Then*

$$\mathcal{H}(\mathbb{L}) := R\nu_*(\mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log})$$

is a $\mathrm{Gal}(C/K)$-equivariant log *Higgs bundle concentrated in degree* 0*. Let θ denote the Higgs operator of $\mathcal{H}(\mathbb{L})$. Then one has*

$$R\nu_*\mathbb{L}\widehat{\otimes}\widehat{\mathscr{O}}_X = R\Gamma(\theta,\mathcal{H}(\mathbb{L})).$$

It turns out that using the theory we have developed so far we can deduce the previous theorem. In fact, let us suppose without loss of generality that $\mathbb{L}^0$ is of rank n. Define the $\mathrm{GL}_n(\mathbb{Z}_p)$-torsor

$$\widetilde{X} := \mathrm{Isom}(\mathbb{Z}_p^n,\mathbb{L}^0).$$

Thus, $\mathbb{L}$ is constructed from the standard representation of GL_n via the torsor $\widetilde{X}$. In particular, by Theorem 20.1.15, $\mathbb{L}\otimes\widehat{\mathscr{O}}_X$ has a Sen operator θ_X. On the other hand, the sheaf $\mathscr{O}\mathbb{C}_{\log}$ can be written as

$$\mathscr{O}\mathbb{C}_{\log} = \varinjlim_{k\to\infty}\mathrm{Sym}^k(\frac{1}{t}\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}),$$

where $\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ is the Faltings extension and $t = \log([\epsilon]) \in B^+_{\mathrm{dR}}$. But we have proven that the Sen operator acts on $\frac{1}{t}\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$ by

$$\theta_X : \frac{1}{t}\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}\otimes\Omega^1_X(\log)^\vee\otimes\widehat{\mathscr{O}}_X(1) \to \Omega^1_X(\log)\otimes\Omega^1_X(\log)^\vee\otimes\widehat{\mathscr{O}}_X \xrightarrow{\mathrm{Tr}} \widehat{\mathscr{O}}_X \subset \frac{1}{t}\mathrm{gr}^1\mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}$$

where the second arrow is the trace map. Thus, by Theorem 20.2.2 one gets that

$$R\nu_*(\mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log}) = \nu_* R\Gamma(\theta_X, \mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log}).$$

Hence, we are left to show that $R\Gamma(\theta_X, \mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log})$ is concentrated in degree 0, and that it admits a reduction to a vector bundle. We need the following lemma, which is essentially Lemma 2.15 of [LZ17].

Lemma 20.2.6. *The image of $\theta_X : \Omega^1_X(\log)^\vee\otimes\widehat{\mathscr{O}}_X(1) \to \mathfrak{gl}_n\otimes\widehat{\mathscr{O}}_X$ is contained in a nilpotent subalgebra.*

Proof. It is enough to prove that the action of θ_X on $\mathbb{L}$ is nilpotent. But the coefficients of its characteristic polynomial are given by

$$\sigma_i : \Omega^1_X(\log)^\vee\otimes\widehat{\mathscr{O}}_X(1) \to \mathrm{End}_{\widehat{\mathscr{O}}_X}(\bigwedge^i \mathbb{L}\otimes\widehat{\mathscr{O}}_X) \xrightarrow{\mathrm{Tr}} \widehat{\mathscr{O}}_X,$$

and the σ_i are Galois equivariant. This forces $\sigma_i = 0$ proving that θ_X is nilpotent. □

Now, knowing that the action of θ_X is nilpotent on $\mathbb{L}$, one can show by taking perfectoid coordinates with Galois group Γ, that the action of a basis of $\mathrm{Lie}\,\Gamma$ on $\mathbb{L}\widehat{\otimes}\mathscr{O}\mathbb{C}_{\log}$ can be integrated. Therefore, $R\Gamma(\theta_X, \mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log}) = (\mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log})^{\theta_X=0}$ and

$$\mathcal{H}(\mathbb{L}) = \nu_*(\mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log})^{\theta_X=0}.$$

The fact that $\mathcal{H}(\mathbb{L})$ is a vector bundle can be deduced by a more careful study of the Sen module $S_\psi(\mathbb{L}\otimes\mathscr{O}\mathbb{C}_{\log})$ in local coordinates ψ.

Finally, let us mention the relation with the work of Wang [Wan21]. Let X be a rigid analytic space over $\mathbb{C}_p$ admitting a liftable good reduction $\mathfrak{X}$ over $O_{\mathbb{C}_p}$ (this means that $\mathfrak{X}$ admits a lifting over A_{inf}/ξ^2 where $\xi = ([\epsilon]-1)/([\epsilon^{\frac{1}{p}}]-1)$). We have the following theorem

Theorem 20.2.7 ([Wan21, Theo. 5.3])**.** *Let $\mathscr{O}\mathbb{C}^\dagger_{\log}$ denote the overconvergent Hodge-Tate period sheaf of Wang. Let $a \geq 1/(p-1)$ and $\nu : X_{\text{prokét}} \to X_{\text{két}}$ be the projection of sites. Then the functor*

$$\mathcal{H}(\mathcal{L}) := \nu_*(\mathcal{L}\otimes\mathscr{O}\mathbb{C}^\dagger_{\log})$$

induces an equivalence from the category of a-small generalized representations to the category of a-small Higgs bundles.

Remark 20.2.8. An a-small generalized representation of rank l is a locally free $\widehat{\mathscr{O}}_X$-module $\mathcal{L}$ admitting a lattice $\mathcal{L}^0$ such that there is $b > a + \mathrm{val}(\rho_k)$ with $\mathcal{L}^0/p^b =^{ae} (\mathscr{O}^+_X/p^b)^l$ (ρ_K is an element in $\mathfrak{m}_{\mathbb{C}_p}$ depending on the ramification of a discretely valued subfield).

The way how Wang constructs the sheaf $\mathscr{O}\mathbb{C}^\dagger_{\log}$ is by considering a particular lattice of the Faltings extension provided by the lifting of $\mathfrak{X}$ to A_2, cf. [Wan21, Coro. 2.19]. Locally on coordinates, the ring $\mathscr{O}\mathbb{C}^\dagger_{\log}$ is nothing but the completion of a polynomial algebra to an overconvergent polydisc of radius $|\rho_k|$ (cf. [Wan21, Theo. 2.27]). The a-smallness condition is a finite rank version of the relative locally analytic condition of Definition 19.2.5, where one imposes a fixed radius of analyticity. Then, the decompletion used by Wang in [Wan21, §3.1] is the integral version of the decompletion provided by Berger-Colmez axiomatic Sen theory [BC08].

21. The flag variety

In the application to Shimura varieties, in order to compute the geometric Sen operator and the Hodge-Tate cohomology of locally analytic local systems, it is important to first study the analogous problem over the flag variety.

We begin with a set up of the terminology of reductive groups and their representation theory. We recall the Bruhat decomposition of partial flag varieties, and the equivalence between $\mathbf{G}$-equivariant vector bundles of $\mathrm{FL} = \mathbf{P}\backslash\mathbf{G}$ and $\mathbf{P}$-representations, where $\mathbf{P}$ is a parabolic subgroup of $\mathbf{G}$. Then, we recall some properties of the category $\mathcal{O}$ and its relative version $\mathcal{O}^{\mathfrak{p}}$. We sketch how the BGG complexes of finite dimensional representations of $\mathbf{G}$ are constructed, we follow [Hum08]. Finally, we define the sheaf of twisted differential operators of Beilinson-Bernstein [BB81] and we study the Lie algebra cohomology of two $\mathfrak{g}$-equivariant sheaves over the analytification $\mathscr{F}\ell$ of the flag variety, namely, the sheaf $C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})$ and its dual $\mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})$.

21.1. Conventions

Let K be a field of characteristic 0 and $\mathbf{G}$ a split reductive group over K. In the applications K will be a finite extension of $\mathbb{Q}_p$ or $\mathbb{C}_p$. Let $\mathbf{B} \subset \mathbf{G}$ be a Borel subgroup and $\mathbf{N}^+$ its unipotent radical, let $\mathbf{T} \subset \mathbf{B}$ be a fixed maximal torus. We denote by $\overline{\mathbf{B}}$ and $\mathbf{N}^-$ the oposite Borel and its unipotent radical respectively.

Let $X^*(\mathbf{T})$ be the group of characters of $\mathbf{T}$, and $\Phi \subset X^*(\mathbf{T})$ the roots of $(\mathbf{G}, \mathbf{T})$. We let $\Phi^{\pm}$ denote the positive and negative roots. Let $\mathfrak{g}$, $\mathfrak{b}$, $\overline{\mathfrak{b}}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{h}$ be the Lie algebras of $\mathbf{G}$, $\mathbf{B}$, $\overline{\mathbf{B}}$, $\mathbf{N}^+$, $\mathbf{N}^-$ and $\mathbf{T}$ respectively. Let $\mathfrak{h}^*$ denote the K-dual of $\mathfrak{h}$, we identify in the natural way $X^*(\mathbf{T}) \otimes_{\mathbb{Z}} K = \mathfrak{h}^*$, so that a "weight" is an element in $\mathfrak{h}^*$, and an "algebraic weight" is an element in $X^*(\mathbf{T})$. By definition, the adjoint representation of $\mathbf{G}$ has a weight decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{n}_\alpha$, with each $\mathfrak{n}_\alpha$ a 1-dimensional K-vector space. We have $\mathfrak{n}^{\pm} = \bigoplus_{\alpha\in\Phi^{\pm}} \mathfrak{n}_\alpha$. Let $X^*(\mathbf{T})_{\mathbb{Q}} := X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $X^*(\mathbf{T})^+_{\mathbb{Q}}$ the cone of positive weights, we denote $X^*(\mathbf{T})^+ := X^*(\mathbf{T})^+_{\mathbb{Q}} \cap X^*(\mathbf{T})$.

Let W be the Weyl group of $\mathbf{G}$, for each $\alpha \in \Phi$ we let $s_\alpha \in W$ denote the simple reflection mapping $\alpha \mapsto -\alpha$. The Weyl group has a length function $\ell : W \to [0, |\Phi^+|]$ whose value at w is defined as the minimal length of a presentation as product of simple reflections. The Weyl group acts on weights by the formula $w(\kappa)(t) = \kappa(w^{-1}tw)$ for $w \in W$ and $\kappa \in \mathfrak{h}^*$. The length of an element $w \in W$ is also equal to $\ell(w) = |w(\Phi^-) \cap \Phi^+|$. Let $w_0 \in W$ denote the longest element of length $|\Phi^+|$, it has the property that $w_0(\Phi^-) = \Phi^+$ or, equivalently, that $w_0\mathbf{N}^+{w_0}^{-1} = \mathbf{N}^-$, in particular $w_0^2 = 1$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$ be a half of the sum of the positive roots, we define the dot action of W on the weight space as $w \cdot \kappa := w(\kappa + \rho) - \rho$.

Let $\mathbf{B} \subset \mathbf{P} \subset \mathbf{G}$ be a parabolic subgroup and $\mathbf{N}$ its unipotent radical. Let $\mathbf{M} \subset \mathbf{P}$ be a Levi factor containing $\mathbf{T}$. Let $\overline{\mathbf{P}}$ be the opposite Parabolic with respect to $\mathbf{M}$ and $\overline{\mathbf{N}}$ the unipotent radical. We denote $\mathfrak{p}$, $\overline{\mathfrak{p}}$, $\mathfrak{n}$, $\overline{\mathfrak{n}}$ and $\mathfrak{m}$ for the Lie algebras of $\mathbf{P}$, $\overline{\mathbf{P}}$, $\mathbf{N}$, $\overline{\mathbf{N}}$ and $\mathbf{M}$ respectively. Let $\Phi_{\mathbf{M}} \subset \Phi$ denote the root system of $(\mathbf{M}, \mathbf{T})$, let $\Phi^{\pm}_{\mathbf{M}}$ be their positive and negative roots, and write $\Phi^{\pm,\mathbf{M}} = \Phi^{\pm}\backslash\Phi^{\pm}_{\mathbf{M}}$. Let $\mathbf{X}^*(\mathbf{T})^+_{\mathbf{M},\mathbb{Q}}$ be the cone of positive weights of $\mathbf{M}$ and $X^+(\mathbf{T})^+_{\mathbf{M}} := X^*(\mathbf{T})^+_{\mathbf{M},\mathbb{Q}} \cap X^*(\mathbf{T})$, notice that $X^*(\mathbf{T})^+_{\mathbb{Q}} \subset X^*(\mathbf{T})^+_{\mathbb{Q},\mathbf{M}}$. Let $W_{\mathbf{M}}$ be the Weyl group of $\mathbf{M}$ and $w_{0,\mathbf{M}}$ its longest element. The quotient $W_{\mathbf{M}}\backslash W$ has a natural set of representatives ${}^{\mathbf{M}}W \subset W$ (called the Kostant representatives) which are those of minimal length. An element w in the Weyl group belongs to ${}^{\mathbf{M}}W$ if and only if $\Phi^+_{\mathbf{M}} \subset w(\Phi^+)$. The set ${}^{\mathbf{M}}W$ has an involution given by $w \mapsto w_{0,\mathbf{M}}ww_0$, it satisfies the property $\ell(w_{0,\mathbf{M}}ww_0) = |\Phi^{+,\mathbf{M}}| - \ell(w)$. In particular,

$w_0^{\mathbf{M}} := w_{0,\mathbf{M}} w_0 \in {}^{\mathbf{M}}W$ is the longest element with $\ell(w_0^{\mathbf{M}}) = \Phi^{+,\mathbf{M}}$. We let $\rho_{\mathbf{M}} := \frac{1}{2}\sum_{\alpha\in\Phi^+_{\mathbf{M}}} \alpha$ be a half of the sum of the positive roots of $\mathbf{M}$, and write $\rho^{\mathbf{M}} = \rho - \rho_{\mathbf{M}}$.

21.1.1. The Bruhat decomposition

Let $\mathrm{FL} := \mathbf{P}\backslash\mathbf{G}$ be the partial flag variety. The Bruhat decomposition of FL is the decomposition in the locally closed subschemes given by its $\mathbf{B}$-orbits

$$\mathrm{FL} = \bigsqcup_{w\in{}^{\mathbf{M}}W} C_w,$$

where $C_w = \mathbf{P}\backslash\mathbf{P}w\mathbf{B}$ is the w-Bruhat cell. Let $d = \dim_K \mathrm{FL} = |\Phi^{-,\mathbf{M}}|$, we let $Z_w := \overline{C_w}$ be the schematic closure of C_w. The scheme Z_w is written in terms of Bruhat cells as $Z_w = \bigsqcup_{v\leq w} C_v$, with $\leq$ being the Bruhat order of the Weyl group. The Bruhat stratification of FL is the decreasing filtration of closed subschemes

$$\emptyset = Z_{d+1} \subset Z_d \subset \cdots \subset Z_0 = \mathrm{FL}$$

where

$$Z_i = \bigcup_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=d-i}} Z_w = \bigsqcup_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)\leq d-i}} C_w.$$

Notice that $Z_i\backslash Z_{i+1} = \bigsqcup_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=d-i}} C_w$.

Given $\alpha \in \Phi$ we let $\mathbf{N}_\alpha \subset \mathbf{G}$ denote the root subgroup of α so that $\mathfrak{n}_\alpha = \mathrm{Lie}\,\mathbf{N}_\alpha$. As a group scheme, $\mathbf{N}_\alpha$ is isomorphic to the additive group $\mathbb{G}_{a,K}$. The Bruhat cells have coordinates in terms of root subgroups:

Lemma 21.1.1. *The projection map $\mathbf{G} \to FL$ induces an isomorphism of schemes over K*

$$w(\overline{\mathbf{N}}_w \cap \mathbf{N}^+) \cong w \prod_{\alpha\in\Phi^+\cap w^{-1}\Phi^{-,\mathbf{M}}} \mathbf{N}_\alpha \xrightarrow{\sim} C_w$$

where $\overline{\mathbf{N}}_w = w^{-1}\overline{\mathbf{N}}w$.

Proof. By definition, C_w is the $\mathbf{B}$-orbit of w. Since $w\mathbf{T}w^{-1} = \mathbf{T}$, C_w is equal to the $\mathbf{N}^+$-orbit of w. The group $\mathbf{N}^+$ is equal as a scheme to the product of its root subgroups in any order, one can write

$$\mathbf{N}^+ = (\mathbf{P}_w \cap \mathbf{N}^+) \times (\overline{\mathbf{N}}_w \cap \mathbf{N}^+)$$

with $\mathbf{P}_w = w^{-1}\mathbf{P}w$. The lemma follows. □

We will also consider the complement of the Bruhat stratification, namely, for $w \in {}^{\mathbf{M}}W$ we let $Y_w = \bigsqcup_{v\geq w} C_v$. The subscheme Y_w is open in FL containing C_w as a closed subspace. Let $Y_i = \bigcup_{\ell(w)=i} Y_w = \bigcup_{\ell(w)\geq i} C_i$, we have a filtration by open subspaces $\emptyset = Y_{d+1} \subset Y_d \subset \cdots \subset Y_0 = \mathrm{FL}$.

21.2. G-equivariant vector bundles over the flag variety

Given $\mathbf{H}$ an algebraic group over K, we denote by $\mathrm{Rep}_K^{\mathrm{Alg}}(\mathbf{H})$ the category of finite dimensional K-linear representations of $\mathbf{H}$. Let X be a scheme over K endowed with an action of $\mathbf{H}$, we denote by $\mathbf{H}\text{-VB}_X$ the category of $\mathbf{H}$-equivariant vector bundles (VB) over X.

Let $\mathrm{FL}^+ = \mathbf{B}\backslash\mathbf{G}$ be the full flag variety of $\mathbf{G}$. We have the following well known result

Proposition 21.2.1. *There is a natural equivalence of categories*

$$\begin{array}{rcl} \mathrm{Rep}_K^{\mathrm{Alg}}(\mathbf{B}) & \longleftrightarrow & \mathbf{G}\text{-}\mathrm{VB}_{\mathrm{FL}^+} \\ V & \rightsquigarrow & \mathbf{G}\times^{\mathbf{B}} V \\ \mathcal{V}|_{[1]} & \leftsquigarrow & \mathcal{V} \end{array}$$

where $\mathbf{G}\times^{\mathbf{B}} V$ *is the quotient* $\mathbf{B}\backslash(\mathbf{G}\times V)$ *by the diagonal action, and* $\mathcal{V}|_{[1]}$ *is its fiber at* 1.

Remark 21.2.2. Let $\mathrm{Rep}_K^{\mathrm{LAlg}}(\mathbf{B})$ be the category of Ind-algebraic representations of $\mathbf{B}$. The functor of Proposition 21.2.1 extends naturally to a functor from $\mathrm{Rep}_K^{\mathrm{LAlg}}(\mathbf{B})$ to the category of $\mathbf{G}$-equivariant quasi-coherent sheaves over FL^+.

The projection map $\mathbf{B}\to\mathbf{T}$ induces a fully faithful functor $\mathrm{Rep}_K^{\mathrm{Alg}}(\mathbf{T})\to\mathrm{Rep}_K^{\mathrm{Alg}}(\mathbf{B})$, composing with the functor of Proposition 21.2.1 one has the following construction

Definition 21.2.3. Let $\kappa\in X^*(\mathbf{T})$ be an algebraic weight, we denote by $\mathcal{L}(\kappa)$ the $\mathbf{G}$-equivariant line bundle over FL^+given by $\mathbf{G}\times^{\mathbf{B}}(w_0(\kappa)K)$.

Remark 21.2.4. Following this conventions, the global sections of $\mathcal{L}(\kappa)$ are the functions $f:\mathbf{G}\to\mathbf{A}^1$ such that $f(bg)=w_0(\kappa)(b)f(g)$. In fact, an element $f\in\Gamma(\mathrm{FL}^+\,\mathcal{L}(\kappa))$ can be identified with a function $g\mapsto(g,f(g))\in\mathbf{G}\times w_0(\kappa)K$ for $g\in\mathbf{G}$. This function must descent to the quotient $\mathbf{G}\times^{\mathbf{B}} w_0(\kappa)K$, in other words it must satisfy

$$(bg,f(bg))\sim b(g,f(g))=(bg,w_0(\kappa)(b)f(g)),$$

for $b\in\mathbf{B}$, i.e. $f(bg)=w_0(\kappa)(b)f(g)$. Hence, if $\kappa\in X^*(\mathbf{T})^+$ the global sections $\Gamma(\mathrm{FL}^+,\mathcal{L}(\kappa))=V_\kappa$ is isomorphic to the irreducible representation of highest weight κ.

We can construct the sheaves $\mathcal{L}(\kappa)$ in the following way. Let $\mathbf{T}_{\mathrm{FL}^+}:=\mathbf{N}^+\backslash\mathbf{G}$ be the natural $\mathbf{T}$-torsor over FL^+, and denote $\tilde\pi^+:\mathbf{T}_{\mathrm{FL}^+}\to\mathrm{FL}^+$ the projection map. Then $\tilde\pi^+_*\mathscr{O}_{\mathbf{T}_{\mathrm{FL}^+}}$ is endowed with a left regular action of $\mathbf{T}$ given as $t\star_1 f(g)=f(t^{-1}g)$. Therefore, there is a natural isomorphism

$$\mathcal{L}(\kappa)=\tilde\pi^+_*\mathscr{O}_{\mathbf{T}_{\mathrm{FL}^+}}[-w_0(\kappa)_{\star_1}].$$

The coherent cohomology of the $\mathbf{G}$-equivariant line bundles over FL^+ is perfectly known by the Borel-Weil-Bott Theorem:

Theorem 21.2.5 ([Jan03, Cor. 5.5])**.** *Let* $\kappa\in X^*(\mathbf{T})$.

1. *If there is no* $w\in W$ *such that* $w\cdot\kappa$ *is dominant then* $R\Gamma(\mathrm{FL}^+,\mathcal{L}(\kappa))=0$.
2. *If there is* $w\in W$ *such that* $w\cdot\kappa$ *is dominant, then such a* w *is unique and we have* $R\Gamma(\mathrm{FL}^+,\mathcal{L}(\kappa))=V_{w\cdot\kappa}[-\ell(w)]$, *where* $V_{w\cdot\kappa}$ *is the irreducible representation of* $\mathbf{G}$ *of highest weight* $w\cdot\kappa$.

Let $\mathbf{P}\subset\mathbf{G}$ be the parabolic subgroup, $\mathbf{N}\subset\mathbf{P}$ its unipotent radical and $\mathbf{M}$ a Levi factor containing $\mathbf{T}$. Let $\mathrm{FL}=\mathbf{P}\backslash\mathbf{G}$ be the partial flag variety. The previous constructions and statements can be generalized to this case. Indeed, there is a natural equivalence of categories

$$\begin{array}{rcl} \mathrm{Rep}_K^{\mathrm{Alg}}(\mathbf{P}) & \longleftrightarrow & \mathbf{G}\text{-}\mathrm{VB}_{\mathrm{FL}} \\ V & \rightsquigarrow & \mathbf{G}\times^{\mathbf{P}} V \\ \mathcal{V}|_{[1]} & \leftsquigarrow & \mathcal{V}. \end{array}$$

Definition 21.2.6. Let $\kappa\in X^*(\mathbf{T})^+_{\mathbf{M}}$ be a dominant weight for $\mathbf{M}$.

1. We denote by W_κ the irreducible representation of $\mathbf{M}$ of highest weight κ.

2. We let $\mathcal{W}(\kappa)$ be the $\mathbf{G}$-equivariant VB over FL given by

$$\mathcal{W}(\kappa) = \mathbf{G} \times^{\mathbf{P}} W_{w_0^{\mathbf{M}}(\kappa)} = \mathbf{G} \times^{\mathbf{P}} W^{\vee}_{-w_0(\kappa)}.$$

Remark 21.2.7. Similarly as in Remark 21.2.4, the sheaf $\mathcal{W}(\kappa)$ is chosen in such a way that $\Gamma(\mathrm{FL}, \mathcal{W}(\kappa)) = V_\kappa$. Moreover, let $\pi : \mathrm{FL}^+ \to \mathrm{FL}$ be the projection of flag varieties, one has that $\pi_* \mathcal{L}(\kappa) = \mathcal{W}(\kappa)$. We also have the following alternative construction of $\mathcal{W}(\kappa)$: let $\mathbf{M}_{\mathrm{FL}} = \mathbf{N}\backslash\mathbf{P}$ be the natural $\mathbf{M}$-torsor over FL and $\tilde{\pi} : \mathbf{M}_{\mathrm{FL}} \to \mathrm{FL}$ its natural projection. The sheaf $\tilde{\pi}_* \mathscr{O}_{\mathbf{M}_{\mathrm{FL}}}$ has a natural left regular action of $\mathbf{M}$ given as $m \star_1 f(g) = f(m^{-1}g)$. Then, there is a natural isomorphism

$$\mathcal{W}(\kappa) = \tilde{\pi}_* \mathscr{O}_{\mathbf{M}_{\mathrm{FL}}}[-w_0(\kappa)_{\star_1}]$$

where the isotypic part in the RHS is with respect to the Borel subgroup $\mathbf{B} \cap \mathbf{M} \subset \mathbf{M}$.

One gets the following corollary of Theorem 21.2.5

Corollary 21.2.8. *Let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}}$.*

1. *If there is no $w \in {}^{\mathbf{M}}W$ such that $w \cdot \kappa$ is dominant with respect to $\mathbf{G}$, then $R\Gamma(\mathrm{FL}, \mathcal{W}(\kappa)) = 0$.*
2. *If there is $w \in {}^{\mathbf{M}}W$ such that $w \cdot \kappa$ is dominant with respect to $\mathbf{G}$, then such a w is unique and $R\Gamma(\mathrm{FL}, \mathcal{W}(\kappa)) = V_{w\cdot\kappa}[-\ell(w)]$.*

Proof. Let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}}$ and consider the projection of flag varieties $\pi : \mathrm{FL}^+ \to \mathrm{FL}$. Then π is a locally trivial fibration with fiber $\mathrm{FL}_{\mathbf{M}} = (\mathbf{M} \cap \mathbf{B})\backslash\mathbf{M}$ the flag variety of $\mathbf{M}$. By Theorem 21.2.5 we have

$$R\pi_* \mathcal{L}(\kappa) = \mathcal{W}(\kappa).$$

Therefore, we get that

$$R\Gamma(\mathrm{FL}, \mathcal{W}(\kappa)) = R\Gamma(\mathrm{FL}^+, \mathcal{L}(\kappa)).$$

Again, by the Borel-Weil-Bott theorem, if there is no $w \in W$ such that $w \cdot \kappa$ is dominant for $\mathbf{G}$ then the cohomology complex vanishes. Conversely, suppose that there is $w \in W$ with $w \cdot \kappa$ dominant, it is enough to show that $w \in {}^{\mathbf{M}}W$. Indeed, we can write uniquely $w = w'w''$ with $w' \in W_{\mathbf{M}}$ and $w'' \in {}^{\mathbf{M}}W$. Then $w'' \cdot \kappa \in X^*(\mathbf{T})^+_{\mathbf{M}}$, and $w' \cdot \lambda = w' \cdot_{\mathbf{M}} \lambda$ for $\lambda \in X^*(\mathbf{T})$. Thus, by the Borel-Weil-Bott theorem again, we must have $w' = 1$ and $w \in {}^{\mathbf{M}}W$. The corollary follows. □

21.3. The dual BGG resolution

In this section we introduce the relative BGG complexes that will be used later on to compute the Hodge-Tate decompositions of Shimura varieties. For sake of completeness, we begin with a brief introduction of the category $\mathcal{O}$ constructed from the data $(\mathfrak{g}, \mathfrak{b})$ of the Lie algebras of $\mathbf{G}$ and $\mathbf{B}$. We state some important properties of this category, and sketch the construction of the BGG complexes. Then, we introduce the relative category $\mathcal{O}^{\mathfrak{p}}$ as a full subcategory of $\mathcal{O}$ satisfying some finiteness properties with respect to $\mathfrak{m} = \mathrm{Lie}\,\mathbf{M}$. We finish with a more geometric construction of the dual BGG resolution via the de Rham complex of the flag variety. We follow [Hum08].

Given a Lie algebra $\mathfrak{k}$ we let $U(\mathfrak{k})$ denote its enveloping algebra over K, and we let $\mathcal{Z}(\mathfrak{k})$ denote the center of $U(\mathfrak{k})$.

Definition 21.3.1. The BGG category $\mathcal{O}$ is the full subcategory of $\mathrm{Mod}(U(\mathfrak{g}))$ whose objects are the modules M satisfying:

$\mathcal{O}1$ M is a finitely generated $U(\mathfrak{g})$-module.

$\mathcal{O}2$ M is $\mathfrak{h}$-semisimple, that is, it has a weight decomposition

$$M = \bigoplus_{\lambda \in \mathfrak{h}} M^{\lambda}.$$

$\mathcal{O}3$ M is locally $\mathfrak{n}^+$-finite: for every $v \in M$ the space $U(\mathfrak{n}^+)v$ is finite dimensional.

The following axioms are deduced from the previous ones

$\mathcal{O}4$ All weight spaces of M are finite dimensional.

$\mathcal{O}5$ The set of weights $\Pi(M)$ of M is contained in the union of finitely many sets of the form $\lambda - \Gamma$, where Γ is the additive monoid generated by Φ^+.

It turns out that the category $\mathcal{O}$ is an abelian category satisfying very nice properties:

Theorem 21.3.2 (Ch. §1 [Hum08])**.** *The category $\mathcal{O}$ is an abelian category which satisfies:*

(a) *$\mathcal{O}$ is a noetherian category, i.e. every module is a noetherian $U(\mathfrak{g})$-module.*

(b) *$\mathcal{O}$ is closed under submodules, quotients, and finite direct sums.*

(c) *If $M \in \mathcal{O}$ and L is a finite dimensional $U(\mathfrak{g})$-module, then $L \otimes M$ also lies in $\mathcal{O}$.*

(d) *If $M \in \mathcal{O}$, then M is $\mathcal{Z}(\mathfrak{g})$-finite.*

(e) *If $M \in \mathcal{O}$, then M is finitely generated as $U(\mathfrak{n}^-)$-module.*

Among the objects in $\mathcal{O}$, the Verma modules are the first examples not arising from algebraic representations.

Definition 21.3.3. Let $\lambda \in \mathfrak{h}^*$. The Verma module of weight λ is the induced representation

$$\mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} K(\lambda).$$

We let $L(\lambda)$ denote the irreducible quotient of $\mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$.

Let $\lambda \in \mathfrak{h}^*$ and $v^+ \in \mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ the highest weight vector. The Poincaré-Birkhoff-Witt theorem implies that $\mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ is a free $U(\mathfrak{n}^-)$-module of rank one with generator v^+. The action of $\mathcal{Z}(\mathfrak{g})$ on v^+ is given by a character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to K$, called the infinitesimal character, this implies that $\mathcal{Z}(\mathfrak{g})$ acts on $\mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ via χ_λ as well. Moreover, let $\xi : \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{h})$ be the Harisch-Chandra homomorphism, then one has

$$\chi_\lambda = \lambda \circ \xi.$$

The Harisch-Chandra map is an isomorphism onto $U(\mathfrak{h})^{W,\cdot}$, where W acts via the dot action. As a consequence, one obtains that $\chi_{w\cdot\lambda} = \chi_\lambda$ for all $\lambda \in \mathfrak{h}^*$ and $w \in W$.

For constructing the BGG decomposition we need to recall the relative left standard resolution.

Definition 21.3.4. For $0 \leq k \leq |\Phi^+| = e$ set $D_k = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \bigwedge^k \mathfrak{g}/\mathfrak{b}$. The relative left standard resolution is the complex

$$\mathrm{DR}^\vee(\mathfrak{g}, \mathfrak{b}) := [D_e \xrightarrow{\partial_d} \cdots \to D_1 \xrightarrow{\partial_1} D_0]$$

with differentials

$$\begin{aligned} \partial_k(u \otimes \xi_1 \wedge \cdots \wedge \xi_k) = &\sum_{i=1}^{k} (-1)^{i+1} u z_i \otimes \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k \\ &+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} u \otimes \overline{[z_i, z_j]} \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k, \end{aligned} \tag{21.3.1}$$

where z_i is any lift of $\xi_i \in \mathfrak{g}/\mathfrak{b}$ to $\mathfrak{g}$.

Remark 21.3.5. It will be useful to write the terms of $\mathrm{DR}^\vee(\mathfrak{g},\mathfrak{b})$ in a different order, namely,

$$D_k = \bigwedge^k \mathfrak{b}\backslash\mathfrak{g} \otimes_{U(\mathfrak{b})} U(\mathfrak{g})$$

with differentials

$$\begin{aligned}\partial_k(\xi_k \wedge \cdots \wedge \xi_1 \otimes u) = &\sum_{i=1}^{k}(-1)^{i+1}\xi_k \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \xi_1 \otimes z_i u \\ &+ \sum_{1\le i<j\le k}(-1)^{i+j}\xi_k \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_1 \wedge \widehat{[z_j, z_i]} \otimes u.\end{aligned}$$

This is the relative right standard resolution, using the involution mapping $X \mapsto -X$ for $X \in \mathfrak{g}$, it can be seen as a left $U(\mathfrak{g})$-complex which is naturally isomorphic to the standard resolution of Definition 21.3.4.

Proposition 21.3.6 (Ch. §9 [Hum08]). *Let $\epsilon : D_0 \to K$ be the augmentation map. Then ϵ induces a quasi-isomorphism $\mathrm{DR}^\vee(\mathfrak{g},\mathfrak{b}) \simeq K$.*

Definition 21.3.7. Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight, the BGG-resolution of V_λ is the complex $\mathrm{BGG}(\mathfrak{g},\mathfrak{b},\lambda)$ obtained by taking the (generalized) χ_λ-eigenspace with respect to $\mathcal{Z}(\mathfrak{g})$ of the complex $V_\lambda \otimes_K \mathrm{DR}^\vee(\mathfrak{g},\mathfrak{b})$, where the action of $\mathfrak{g}$ in the latter is the diagonal action.

Theorem 21.3.8 (Bernstein-Gelfand-Gelfand). *The complex $\mathrm{BGG}(\mathfrak{g},\mathfrak{b},\lambda)$ is a direct summand of $V_\lambda \otimes_K \mathrm{DR}^\vee(\mathfrak{g},\mathfrak{b})$ and a resolution of V_λ. Moreover, it is of the form*

$$\mathrm{BGG}(\mathfrak{g},\mathfrak{b},\lambda) = \left[\mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(w_0\cdot\lambda) \to \cdots \to \bigoplus_{\ell(w)=k} \mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(w\cdot\lambda) \to \cdots \to \mathrm{Ver}_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)\right].$$

Let us now introduce the relative category $\mathcal{O}^{\mathfrak{p}}$ and the relative BGG resolution.

Definition 21.3.9. The category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $\mathrm{Mod}(U(\mathfrak{g}))$ of objects M satisfying

- $\mathcal{O}^{\mathfrak{p}}1$ M is a finitely generated $U(\mathfrak{g})$-module.
- $\mathcal{O}^{\mathfrak{p}}2$ As $U(\mathfrak{m})$-module, M is a semisimple representation whose irreducible factors are finite dimensional $\mathfrak{m}$-modules.
- $\mathcal{O}^{\mathfrak{p}}3$ M is locally $\mathfrak{n}$-finite.

One can also describe the category $\mathcal{O}^{\mathfrak{p}}$ in terms of $\mathcal{O}$:

Proposition 21.3.10 (Ch. §9.2 [Hum08]). *Let $M \in \mathcal{O}$ has set of weights $\Pi(M)$. The following are equivalent.*

(a) *M is locally $\mathfrak{n}_{\mathbf{M}}^- = \mathfrak{m} \cap \mathfrak{n}^-$-finite.*

(b) *For all $w \in W_{\mathbf{M}}$ and $\mu \in \Pi(M)$, we have $\dim_K M^\mu = \dim_K M^{w(\mu)}$.*

(c) *The set $\Pi(M)$ is stable under $W_{\mathbf{M}}$.*

(d) *$M \in \mathcal{O}^{\mathfrak{p}}$.*

Furthermore, the following holds

1. *$\mathcal{O}^{\mathfrak{p}}$ is closed under finite direct sums, submodules, quotients and extensions in $\mathcal{O}$.*

2. *$\mathcal{O}^{\mathfrak{p}}$ is stable under tensor of finitely dimensional representations of $U(\mathfrak{g})$.*

3. *If $\lambda \in X^*(\mathbf{T})$, and the simple module $L(\lambda)$ lies in $\mathcal{O}^{\mathfrak{p}}$, then $\lambda \in X^*(\mathbf{T})^+_{\mathbf{M}}$*

The relative version of the Verma modules is the following:

Definition 21.3.11. Let $\lambda \in X^*(\mathbf{T})^+_{\mathbf{M}}$ and let W_λ be the irreducible representation of $\mathbf{M}$ of highest weight λ. We define the parabolic Verma module as the induction

$$\mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W_\lambda.$$

In the relative case we also have a standard resolution, namely, for $0 \leq k \leq |\Phi^+_{\mathbf{P}}| = d$ let $D^{\mathfrak{g}}_k = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bigwedge^k \mathfrak{g}/\mathfrak{p}$, then the de relative standard complex is

$$\mathrm{DR}^\vee(\mathfrak{g}, \mathfrak{p}) = [D^{\mathfrak{p}}_d \xrightarrow{\partial_d} \cdots \to D^{\mathfrak{p}}_0]$$

with differentials given by the formula (21.3.1).

Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight of $\mathbf{G}$ and V_λ its irreducible representation of highest weight λ. The relative BGG complex of weight λ is the complex obtained from $V_\lambda \otimes_K \mathrm{DR}^\vee(\mathfrak{g}, \mathfrak{p})$ by taking (generalized) χ_λ-eigenspaces. We have the following theorem of Lepowsky and Rocha

Theorem 21.3.12 ([Lep77, RC80]). *Let $\lambda \in X^*(\mathbf{T})^+$. Then the* BGG *complex* $\mathrm{BGG}(\mathfrak{g}, \mathfrak{p}, \lambda)$ *is direct summand of* $\mathrm{DR}^\vee(\mathfrak{g}, \mathfrak{p})$ *and a resolution of V_λ. Moreover, it has the form*

$$\mathrm{BGG}(\mathfrak{g}, \mathfrak{p}, \lambda) = \left[\mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(w_0^{\mathbf{M}} \cdot \lambda) \to \cdots \to \bigoplus_{\substack{w \in {}^{\mathbf{M}}W \\ \ell(w)=k}} \mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(w \cdot \lambda) \to \cdots \to \mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(\lambda) \right].$$

In the applications to the Hodge-Tate decomposition of Shimura varieties, we shall use the admissible dual of the relative BGG resolution.

Definition 21.3.13. Let $M \in \mathcal{O}$, the admissible dual of M is the subvector space $M^\vee \subset \mathrm{Hom}_K(M, K)$ of $\mathfrak{h}$-finite vectors. In other words, if $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$ is the weight decomposition of M, then $M^\vee = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathrm{Hom}_K(M^\lambda, K)$. We endow $M^\vee$ with the $\mathfrak{g}$-action

$$(Xf)(m) = f(-Xm)$$

for $X \in \mathfrak{g}$, $f \in M^\vee$ and $m \in M$.

Lemma 21.3.14. *Let $M \in \mathcal{O}$ and let $M^\vee$ be its admissible dual, then the action of $\mathfrak{n}^-$ on $M^\vee$ is locally finite. Moreover, the functor $M \mapsto M^\vee$ is exact.*

Proof. The exactness of the functor is clear as the weight spaces of M are finite dimensional. The finiteness of the action of $\mathfrak{n}^-$ follows from $\mathcal{O}5$. □

Remark 21.3.15. It is not hard to prove that $M^\vee$ belongs to the category $\mathcal{O}$ of the opposite Borel $\overline{\mathfrak{b}}$, say $\overline{\mathcal{O}}$. Moreover, the functor $M \mapsto M^\vee$ induces an antiequivalence of categories $\mathcal{O}^{op} \to \overline{\mathcal{O}}$.

Proposition 21.3.16. *Let $M \in \mathcal{O}$ and suppose that the weight decomposition of M is algebraic, i.e. $M^\lambda \neq 0$ implies $\lambda \in X^*(\mathbf{T})$. Then the action of $\overline{\mathfrak{b}}$ on $M^\vee$ integrates to an Ind-algebraic action of $\overline{\mathbf{B}} = \mathbf{N}^-\mathbf{T}$.*

Proof. As $\mathfrak{n}^-$ is a split unipotent algebra over a field of characteristic 0, a locally finite representation of $\mathfrak{n}^-$ integrates to an Ind-algebraic representation of $\mathbf{N}^-$. The fact that the weights of M are algebraic implies that the action can be actually extended to $\overline{\mathbf{B}}$. □

Corollary 21.3.17. *Let $M \in \mathcal{O}^{\mathfrak{p}}$ and suppose that the $\mathfrak{m}$-decomposition of M is algebraic. Then the action of $\overline{\mathfrak{p}}$ on $M^\vee$ integrates to an action of $\overline{\mathbf{P}}$.*

Proof. By Proposition 21.3.16, the action of $\overline{n}$ extends to $\overline{\mathbf{N}}$. As the $\mathfrak{m}$-decomposition is algebraic one verifies that the action of $\overline{\mathfrak{p}}$ on $M^\vee$ can be extend to an action of $\overline{\mathbf{P}}$. □

Definition 21.3.18. Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight of $\mathbf{G}$. The relative dual BGG resolution of $V_\lambda^\vee$ is the $(\overline{\mathbf{P}}, \mathfrak{g})$-resolution $V_\lambda^\vee \to \mathrm{BGG}^\vee(\mathfrak{g}, \mathfrak{p}, \lambda)$, where

$$\mathrm{BGG}^\vee(\mathfrak{g}, \mathfrak{p}, \lambda) = \left[\mathrm{Ver}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)^\vee \to \cdots \to \bigoplus_{\substack{w \in {}^{\mathbf{M}}W \\ \ell(w)=k}} \mathrm{Ver}_{\mathfrak{p}}^{\mathfrak{g}}(w \cdot \lambda)^\vee \to \mathrm{Ver}_{\mathfrak{p}}^{\mathfrak{g}}(w_0^{\mathbf{M}} \cdot \lambda)^\vee \right].$$

The dual BGG resolution can be constructed in a slightly more geometric way. Let $\mathrm{FL} = \mathbf{P}\backslash\mathbf{G}$ be the flag variety, and consider the de Rham complex

$$\mathrm{DR}(\mathrm{FL}) = [\mathscr{O}_{\mathrm{FL}} \xrightarrow{d} \Omega^1_{\mathrm{FL}} \to \cdots \to \Omega^d_{\mathrm{FL}}].$$

We have the following proposition

Proposition 21.3.19. *Consider the big cell $C^1 = \mathbf{P}\backslash\mathbf{P}\overline{\mathbf{P}} = \mathbf{P}\backslash\mathbf{P}\overline{\mathbf{N}} \subset \mathrm{FL}$. Then*

$$\mathrm{DR}(\mathrm{FL})(C^1) = (\mathrm{DR}^\vee(\mathfrak{g}, \mathfrak{p}))^\vee.$$

Proof. By Remark 21.3.5 we have

$$\mathrm{DR}^\vee(\mathfrak{g}, \mathfrak{p}) = [D_0 \to \cdots \to D_d]$$

with $D_k = \bigwedge^k \mathfrak{p}\backslash\mathfrak{g} \otimes_{U(\mathfrak{b})} U(\mathfrak{g})$. On the other hand, the global sections of Ω^K_{FL} over C^1 are

$$\Omega^k_{\mathrm{FL}}(C^1) = \mathscr{O}_{\mathrm{FL}}(C^1) \otimes_K \bigwedge^k (\mathfrak{p}\backslash\mathfrak{g})^\vee$$

where we see an element $\omega \in (\mathfrak{p}\backslash\mathfrak{g})^\vee$ as the differential form acting on a vector field $\mathfrak{x}_x \in T_{\mathrm{FL}}(C^1)$ as

$$\omega(\mathfrak{x})(x) = \omega(\overline{n}(x)\mathfrak{x}_x\overline{n}^{-1}(x)),$$

with $\overline{n}(x) \in \overline{\mathbf{N}}$ being the unique representative of $x \in C^1$. In other words, given $x \in C^1$ the fiber of tangent space T_{FL} at x is given by $\overline{n}(x)^{-1}\mathfrak{p}\overline{n}(x)\backslash\mathfrak{g}$, thus a vector field $\mathfrak{x} \in T_{\mathrm{FL}}(C^1)$ satisfies $\overline{n}(x)\mathfrak{x}_x\overline{n}^{-1}(x) \in \mathfrak{p}\backslash\mathfrak{g}$ and the pairing $\omega(\mathfrak{x})$ is well defined. More explicitly, we have a natural isomorphism given by right derivations $\mathscr{O}_{C^1} \otimes \overline{\mathfrak{n}} \cong T_{\mathrm{FL}}|_{C^1}$. Let $X \in \overline{\mathfrak{n}} \cong \mathfrak{p}\backslash\mathfrak{g}$ and define the vector bundle over C^1

$$\mathfrak{x}_x := \overline{n}(x)^{-1}X\overline{n}(x) \in \mathscr{O}_{C^1} \otimes \overline{\mathfrak{n}},$$

then $\omega(\mathfrak{x})(x) = \omega(\overline{n}(x)\mathfrak{x}\overline{n}(x)^{-1}) = \omega(X)$ is constant over C^1.

Consider the pairing for $0 \le k \le d$,

$$\begin{aligned} \langle \cdot, \cdot \rangle_k : \Omega^k_{\mathrm{FL}}(C^1) \times D_k &\to K \\ (f \otimes \omega, \xi_k \wedge \cdots \wedge \xi_1 \otimes u) &\mapsto \omega(\xi_k \wedge \cdots \wedge \xi_1)(u \star_2 f)(1), \end{aligned}$$

where $u \star_2 f$ is the right regular action. Then $\langle \cdot, \cdot \rangle_k$ is a $\mathfrak{g}$-equivariant perfect pairing for the admissible dual:

$$\Omega^k_{\mathrm{FL}}(C^1) = D_k^\vee.$$

One easily checks that the the dual of ∂_k is the differential $d : \Omega^k_{\mathrm{FL}}(C^1) \to \Omega^{k+1}_{\mathrm{FL}}(C^1)$, this shows the proposition. □

Corollary 21.3.20. *Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight. The dual BGG resolution $\mathrm{BGG}^\vee(\mathfrak{g}, \mathfrak{p}, \lambda)$ of $V_\lambda^\vee$ is equal to the $\mathcal{Z}(\mathfrak{g}) = \chi_{-w_0(\lambda)}$-eigenspace of the de Rham complex $V_\lambda^\vee \otimes_K \mathrm{DR}(\mathrm{FL})(C^1)$. Equivalently, let $\underline{V_\lambda^\vee} \otimes_K \mathscr{O}_{\mathrm{FL}}$ be the $\mathbf{G}$-equivariant vector bundle with integrable connexion ∇ over FL defined by $V_\lambda^\vee$. Then*

$$\mathrm{BGG}^\vee(\mathfrak{g}, \mathfrak{p}, \lambda) = \mathrm{DR}(\underline{V_\lambda^\vee} \otimes_K \mathscr{O}_{\mathrm{FL}}, \nabla)(C^1)^{\mathcal{Z}(\mathfrak{g})=\chi_{-w_0(\lambda)}}.$$

Proof. This follows by Proposition 21.3.19 and the construction of the BGG complex in Definition 21.3.11. □

Remark 21.3.21. Let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}}$, as $\overline{\mathbf{P}}$-module $\mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(\kappa)^\vee$ is isomorphic to $W_\kappa^\vee \otimes \mathscr{O}(C^1)$. More precisely, we have an isomorphism of $\overline{\mathfrak{p}}$-modules $\mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(\kappa) \cong \mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(1) \otimes W_\kappa$, taking duals one has $\mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(\kappa)^\vee \cong W_\kappa^\vee \otimes \mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(1)^\vee$ as $\overline{\mathbf{P}}$-modules. On the other hand, there is a natural $\mathfrak{g}$-equivariant pairing

$$\langle \cdot, \cdot \rangle : \mathscr{O}(C^1) \otimes \mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(1) \to K$$

given by $\langle f, u \rangle = \iota(u) \star_2 f(1)$ where $\iota : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the involution sending $X \mapsto -X$ for $X \in \mathfrak{g}$. This pairing induces an isomorphism of $\mathfrak{g}$-modules $\mathrm{Ver}^{\mathfrak{g}}_{\mathfrak{p}}(1)^\vee = \mathscr{O}(C^1)$ which integrates to an isomorphism of $(\overline{\mathbf{P}}, \mathfrak{g})$-modules.

Now, let $\mathscr{O}(C^1)^{\leq 1} \subset \mathscr{O}(C^1)$ be the subspace of polynomials of degree ≤ 1 given by the coordinates of the root groups $\mathbf{N}_\alpha$. For $n \in \mathbb{N}$, $\mathscr{O}(C^1)^{\leq n} = \mathrm{Sym}^n \mathscr{O}(C^1)^{\leq 1}$ is the space of polynomials of degree $\leq n$, whence $\mathscr{O}(C^1) = \varinjlim_n \mathrm{Sym}^n \mathscr{O}(C^1)^{\leq 1}$. We have a short exact sequence of $\overline{\mathbf{P}}$-modules

$$0 \to K \to \mathscr{O}(C^1)^{\leq 1} \to \overline{\mathfrak{n}}^\vee \to 0,$$

the action of $\overline{\mathfrak{p}}$ is trivial on K (as it its just given by derivations on $\mathscr{O}(C^1)$), the sequence is split for the action of the Levi $\mathfrak{m}$, $\overline{\mathfrak{n}}$ acts on $\mathscr{O}(C^1)^{\leq 1}$ through the projection onto $\overline{\mathfrak{n}}^\vee$ and the derivation map by considering $\overline{\mathfrak{n}}^\vee \subset \mathscr{O}(C^1)$. If $\mathfrak{n}$ is abelian, the action of $\overline{\mathfrak{n}}$ on $\mathscr{O}(C^1)^{\leq 1}$ factors through the trace map

$$\overline{\mathfrak{n}} \otimes \mathscr{O}(C^1)^{\leq 1} \to \overline{\mathfrak{n}} \otimes \overline{\mathfrak{n}}^\vee \xrightarrow{\mathrm{Tr}} K \subset \mathscr{O}(C^1)^{\leq 1}.$$

Summarizing, the dual BGG resolution of $V_\lambda^\vee$ is isomorphic as $\overline{\mathbf{P}}$-module to

$$\mathrm{BGG}^\vee(\mathfrak{g}, \mathfrak{p}, \lambda) = [W_\lambda^\vee \otimes_K \mathscr{O}(C^1) \to \cdots \to \bigoplus_{\substack{w \in {}^{\mathbf{M}}W \\ \ell(w)=k}} W_{w\cdot\lambda}^\vee \otimes_K \mathscr{O}(C^1) \to \cdots W_{w_0^{\mathbf{M}}\cdot\lambda}^\vee \otimes_K \mathscr{O}(C^1)].$$

We are actually interested in the dual BGG resolution of $V_\lambda^\vee$ seen as a $(\mathbf{P}, \mathfrak{g})$-module, for this, we need to understand the combinatorics of the Weyl groups and the Kostant representatives when $\mathbf{M}$ is considered as a quotient of $\overline{\mathbf{P}}$. First of all, we want to keep the same Borel subgroup of the Levi, namely, $\mathbf{B}_{\mathbf{M}} = \mathbf{M} \cap \mathbf{B}$. Thus, we will take $\widetilde{\mathbf{B}} := w_0^{\mathbf{M}} \mathbf{B} w_0^{\mathbf{M},-1} \subset \overline{\mathbf{P}}$ as a Borel subgroup. Then, the possitive roots with respect to $\widetilde{\mathbf{B}}$ are $\widetilde{\Phi}^+ = \Phi^{-,\mathbf{M}} \bigsqcup \Phi^+_{\mathbf{M}}$. Let ${}^{\mathbf{M}}W^{op}$ denote the Kostant representatives with respect to the opposite parabolic, one verifies that

$${}^{\mathbf{M}}W^{op} = \{w_{0,\mathbf{M}} w w_{0,\mathbf{M}} : w \in {}^{\mathbf{M}}W\}.$$

The dot action of an element $w \in W$ with respect to $\widetilde{\mathbf{B}}$ is given by

$$w \cdot' \lambda = w(\lambda + \rho_{\mathbf{M}} - \rho^{\mathbf{M}}) - \rho_{\mathbf{M}} + \rho^{\mathbf{M}}.$$

Let V_λ be the irreducible representation of $\mathbf{G}$ of highest weight λ with respect to $\mathbf{B}$. Then, the highest weight with respect to $\widetilde{\mathbf{B}}$ is equal to $w_0^{\mathbf{M}}(\lambda)$. Finally, the open Bruhat cell of $\mathrm{FL}^{\mathrm{std}} = \overline{\mathbf{P}} \backslash \mathbf{G}$ associated to $\mathbf{P}$ is equal to $\overline{C}^1 = \overline{\mathbf{P}} \backslash \overline{\mathbf{P}} \mathbf{P} = \overline{\mathbf{P}} \backslash \overline{\mathbf{P}} \mathbf{N}$. One has the following corollary

Corollary 21.3.22. *There is a quasi isomorphism of* $(\mathbf{P}, \mathfrak{g})$*-modules* $V_\lambda^\vee \simeq \mathrm{BGG}^\vee(\mathfrak{g}, \overline{\mathfrak{p}}, w_0^{\mathbf{M}}(\lambda))$*. Moreover, we have that*

$$\mathrm{BGG}^\vee(\mathfrak{g}, \overline{\mathfrak{p}}, w_0^{\mathbf{M}}(\lambda)) = [W_{-w_0(\lambda)} \otimes \mathscr{O}(\overline{C}^1) \to \cdots \to \bigoplus_{\substack{w \in {}^{\mathbf{M}}W \\ \ell(w)=k}} W_{w\cdot(-w_0(\lambda))} \otimes \mathscr{O}(\overline{C}^1) \to \cdots W_{w_0^{\mathbf{M}}\cdot(-w_0(\lambda))} \otimes \mathscr{O}(\overline{C}^1)].$$

Proof. We have $W_\kappa^\vee = W_{-w_{0,\mathbf{M}}(\kappa)}$. Thus, by Remark 21.3.21 it is enough to show that

$$-w_{0,\mathbf{M}}(w_{0,\mathbf{M}} w w_{0,\mathbf{M}} \cdot' w_0^{\mathbf{M}}(\lambda)) = w \cdot (-w_0(\lambda)).$$

This follows from

$$\begin{aligned} -w_{0,\mathbf{M}}(w_{0,\mathbf{M}} w w_{0,\mathbf{M}} \cdot' w_0^{\mathbf{M}}(\lambda)) &= -w_{0,\mathbf{M}}(w_{0,\mathbf{M}} w w_0(\lambda) + w_{0,\mathbf{M}} w(-\rho_{\mathbf{M}} - \rho^{\mathbf{M}}) - \rho_{\mathbf{M}} + \rho^{\mathbf{M}}) \\ &= w(-w_0(\lambda)) + w(\rho) + -\rho_{\mathbf{M}} - \rho^{\mathbf{M}} \\ &= w(-w_0(\lambda) + \rho) - \rho \\ &= w \cdot (-w_0(\lambda)). \end{aligned}$$

□

21.4. $\widetilde{\mathcal{D}}$-modules over the flag variety

In this section, we study the quasi-coherent sheaf of germs at 1 of locally analytic functions of $\mathbf{G}$ with values in $\mathscr{O}_{\mathscr{F}\ell}$. Later on, in Chapters 22 and 23, we shall translate some Lie algebra cohomology computations of this sheaf into their analogue for the locally analytic vectors of the completed structural sheaf of the infinite level Shimura variety.

21.4.1. The sheaf of twisted differential operators

Definition 21.4.1. We define the following $\mathbf{G}$-equivariant subbundles of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$

1. We let $\mathfrak{p}^0$ denote the subbundle of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ consisting on the vector fields $\mathfrak{X} \in \mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ such that $\mathfrak{X}_x \in \mathfrak{p}_x = \mathrm{Lie}\, \mathbf{P}_x$ for $x \in \mathrm{FL}$, where $\mathbf{P}_x$ is the Parabolic defined by the point $x \in \mathrm{FL}$.
2. We let $\mathfrak{n}^0$ denote the subbundle of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ consisting on the vector fields $\mathfrak{X} \in \mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ such that $\mathfrak{X}_x \in \mathfrak{n}_x = \mathrm{Lie}\, \mathbf{N}_x$ for $x \in \mathrm{FL}$, where $\mathbf{N}_x$ is the unipotent radical of $\mathbf{P}_x$.
3. We let $\mathfrak{m}^0$ denote the quotient $\mathfrak{p}^0/\mathfrak{n}^0$.

Remark 21.4.2. The sheaves $\mathfrak{p}^0$ and $\mathfrak{n}^0$ can be constructed via the functor of Proposition 21.2.1. Indeed, the Lie algebras $\mathfrak{p}$ and $\mathfrak{n}$ are sub $\mathbf{P}$-representations of $\mathfrak{g}$ for the adjoint action of $\mathbf{P}$. Then

$$\mathfrak{p}^0 = \mathbf{G} \times^{\mathbf{P}} \mathfrak{p} \text{ and } \mathfrak{n}^0 = \mathbf{G} \times^{\mathbf{P}} \mathfrak{n}.$$

Note that the quotient $(\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g})/\mathfrak{p}^0$ is naturally isomorphic to the tangent space of FL.

The sheaf $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ has a natural structure of Lie algebroid over FL with bracket

$$[f \otimes X, f' \otimes X'] = f(X \star_2 f') \otimes X' - f'(X' \star_2 f) \otimes X + ff' \otimes [X, X'].$$

Lemma 21.4.3. *The subbundles* $\mathfrak{p}^0$ *and* $\mathfrak{n}^0$ *are ideals of* $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$*.*

Proof. Notice that the action of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ on $\mathscr{O}_{\mathrm{FL}}$ by derivations

$$(f \otimes X) \star_2 f' = f(X \star_2 f')$$

factors through a morphism of Lie algebroids

$$\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g} \to T_{\mathrm{FL}} \tag{21.4.1}$$

with T_{FL} the tangent bundle of FL. But the kernel of (21.4.1) is $\mathfrak{p}^0$, this implies that $\mathfrak{p}^0$ is an ideal of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$.

On the other hand, the left regular action of $\mathbf{G}$ on $\mathbf{N}\backslash\mathbf{G}$ induces a morphism of Lie algebroids

$$\mathscr{O}_{\mathbf{N}\backslash\mathbf{G}} \otimes \mathfrak{g} \to T_{\mathbf{N}\backslash\mathbf{G}}. \tag{21.4.2}$$

But $\mathfrak{n}^0$ can be recovered as the $\mathbf{M}$-invariants of the kernel of (21.4.2) for the left regular action. This implies that $\mathfrak{n}^0$ is an ideal of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$. □

Corollary 21.4.4. *The sheaf of differential operators $\mathcal{D}$ of* FL *is naturally isomorphic to the quotient*

$$\mathscr{O}_{\mathrm{FL}} \otimes U(\mathfrak{g})/(\mathscr{O}_{\mathrm{FL}} \otimes U(\mathfrak{g}))\mathfrak{p}^0.$$

Proof. This follows by Lemma 21.4.3 and the fact that the map (21.4.1) is surjective. □

Definition 21.4.5. We define the sheaf of universal twisted differential operators of FL to be the algebra

$$\widetilde{\mathcal{D}} = \mathscr{O}_{\mathrm{FL}} \otimes U(\mathfrak{g})/(\mathscr{O}_{\mathrm{FL}} \otimes U(\mathfrak{g}))\mathfrak{n}^0.$$

Remark 21.4.6. Notice that $\widetilde{\mathcal{D}}$ can be recovered as the $\mathbf{M}$-invariant sections for the left regular action of the sheaf of differential operators of $\mathbf{N}\backslash\mathbf{G}$.

Let $U(\mathfrak{m}^0)$ be the enveloping algebra of $\mathfrak{m}^0$ seen as a Lie algebra over $\mathscr{O}_{\mathrm{FL}}$, it is contained naturally in $\widetilde{\mathcal{D}}$. If $\mathbf{P} = \mathbf{B}$ is a Borel subgroup, then $\mathfrak{m}^0 = \mathfrak{h}^0$ is the horizontal Cartan subalgebra of $\mathscr{O}_{\mathrm{FL}} \otimes \mathfrak{g}$ and one has $U(\mathfrak{h}) = \Gamma(\mathrm{FL}, U(\mathfrak{h}^0))$. In general, we have the following result which is a consequence of a theorem of Kostant [Kos63, Theo. 0.13]

Proposition 21.4.7. *Let $\mathcal{Z}(\mathfrak{m})$ be the center of the enveloping algebra of $\mathfrak{m}$, and let $E \subset U(\mathfrak{g})$ be the vector space generated by all the powers x^k of nilpotent elements $x \in \mathfrak{m}$. Let $\mathcal{Z}(\mathfrak{m}^0)$ and $\mathscr{E}$ be the* $\mathbf{G}$*-equivariant vector bundles over* FL *obtained from the adjoint action of* $\mathbf{M}$*. Then we have a decomposition of* $\mathbf{G}$*-equivariant $\mathcal{Z}(\mathfrak{m}^0)$-modules*

$$U(\mathfrak{m}^0) = \mathcal{Z}(\mathfrak{m}^0) \otimes_{\mathscr{O}_{\mathrm{FL}}} \mathscr{E}.$$

In particular, one has $\Gamma(\mathrm{FL}, U(\mathfrak{m}^0)) = \mathcal{Z}(\mathfrak{m}) \otimes \Gamma(\mathrm{FL}, \mathscr{E})$.

Remark 21.4.8. Kostant's theorem also provides the decomposition of $\mathscr{E}$ as direct sum of $\mathbf{G}$-equivariant vector bundles, as well as the degrees of the isotypic parts of the symmetric algebra $\mathrm{Sym}^*(\mathfrak{m}^0) = \mathrm{gr}^* U(\mathfrak{m}^0)$.

Remark 21.4.9. In the application to Shimura varieties, the previous proposition will translate to the fact that there exists an horizontal action of $\mathcal{Z}(\mathfrak{m})$ on the locally analytic completed cohomology (actually one would have an action of the global sections of $U(\mathfrak{m}^0)$). As for the case of the modular curve of Lue Pan, this action will be related with the Hodge-Tate weights of the Galois representation.

Remark 21.4.10. The algebra $\widetilde{\mathcal{D}}$ is difficult to study directly; it is an algebraic interpolation of distribution algebras attached to the $\mathbf{G}$-equivariant vector bundles over FL arising from $\mathbf{M}$-representations. A way to overcome this problem is to work with smaller twisted distribution algebras as in the theory of Beilinson-Bernstein [BB81]. More precisely, let $\lambda : U(\mathfrak{m}) \to K$ be a character and define $\mathcal{D}_\lambda := \lambda \otimes_{U(\mathfrak{m}^0)} \widetilde{\mathcal{D}}$, then $\mathcal{D}_\lambda$ is a twisted sheaf of differential operators which has been studied by Soergel [Soe90], Holland and Polo [HP96], *et. al*, proving for example a localization theorem for antidominant regular weights.

21.4.2. Locally analytic sheaves over the flag variety

From now on we assume that $\mathbf{G}$ is a reductive group over $\mathbb{Q}_p$ and that $K/\mathbb{Q}_p$ is a complete nonarchimedean extension for which $\mathbf{G}$ is split. We let $\mathscr{F}\ell$ be the analytification of $\mathrm{FL} = \mathbf{P}\backslash\mathbf{G}_K$ as an adic space over $\mathrm{Spa}(K, O_K)$. We assume in addition that $\mathfrak{n} = \mathrm{Lie}\,\mathbf{N}$ is abelian (this shall be the case for Shimura varieties). In the rest of the chapter we will only work in the analytic site of adic spaces.

Notation 4. Let $\mathbf{H}$ be an affine algebraic group over $\mathrm{Spec}\,K$, we denote in calligraphic font $\mathcal{H}$ its analytification to an adic space over $\mathrm{Spa}(K, O_K)$. Suppose that $\mathbf{H}$ has an integral model $\mathbf{H}^0$ over $\mathrm{Spec}\,O_K$. We denote by $\mathcal{H}^0 \subset \mathcal{H}$ the open subgroup whose value at an affinoid pair (R, R^+) is

$$\mathcal{H}^0(R, R^+) = \mathbf{H}^0(R^+).$$

Let $\epsilon > 0$ be a rational number, we let $\mathcal{H}(\epsilon)$ denote the open subgroup of $\mathcal{H}$ whose value at (R, R^+) is

$$\mathcal{H}(\epsilon) = \ker(\mathbf{H}^0(R^+) \to \mathbf{H}^0(R^+/p^\epsilon)),$$

where $p^\epsilon \in \mathbb{C}_p$ is an element with valuation $|p^\epsilon| = |p|^\epsilon$.

Let $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}$, and let $C^{la}(\mathfrak{g}, K)$ be the germs at 1 of locally analytic functions of G with values in K. We consider the following sheaf over $\mathscr{F}\ell_{\mathrm{an}}$

$$C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) := C^{la}(\mathfrak{g}, K)\widehat{\otimes}_K \mathscr{O}_{\mathscr{F}\ell}.$$

Equivalently, let $\iota : \mathscr{F}\ell \to \mathscr{F}\ell \times \mathcal{G}$ be the unit section, then $C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) = \iota^{-1}\mathscr{O}_{\mathscr{F}\ell \times \mathcal{G}}$.

Dually, let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a lattice such that $[\mathfrak{g}_0, \mathfrak{g}_0] \subset p\mathfrak{g}_0$. Consider the following completion of the enveloping algebra

$$\mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) = \varprojlim_{h\to\infty} \mathcal{D}(\mathfrak{g}_h, O_{\mathscr{F}\ell}) = \varprojlim_{h\to\infty} \mathcal{D}(\mathfrak{g}_h, \mathbb{Q}_p)\widehat{\otimes}_{\mathbb{Q}_p}\mathscr{O}_{\mathscr{F}\ell},$$

where $\mathcal{D}(\mathfrak{g}_h, \mathbb{Q}_p)$ are the analytic distribution algebras of Definition 17.2.7.

Definition 21.4.11. Let $\mathfrak{n}^0 \subset \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}$ be the subbundle of Definition 21.4.1. We define the locally analytic twisted sheaf of differential operators to be the algebra

$$\widetilde{\mathcal{D}}^{la} := \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes_{U(\mathfrak{n}_0)} 1 = 1 \otimes_{U(\mathfrak{n}^0)} \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}).$$

For $h > 0$ let us also define $\widetilde{\mathcal{D}}^{h-an} := \mathcal{D}(\mathfrak{g}_h, \mathscr{O}_{\mathscr{F}\ell}) \otimes_{U(\mathfrak{n}^0)} 1$. In particular $\widetilde{\mathcal{D}}^{la} = \varprojlim_h \widetilde{\mathcal{D}}^{h-an}$.

Let $\mathfrak{b}$ be the Lie algebra of the Borel subgroup, and $\lambda \in \mathfrak{h}^*$. The goal of the following sections is to provide a better description of the $\widetilde{\mathcal{D}}^{la}$-modules

$$\begin{aligned} R\Gamma(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes K(-\lambda)_{\star_2}) \\ R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes^L_{U(\mathfrak{b}_{\star_2})} \lambda). \end{aligned} \tag{21.4.3}$$

Remark 21.4.12. Unfortunately, we are not able to give a complete answer to this question, and only some bounds in terms of the Bruhat stratification are obtained, as well as the highest weight subquotients that one expects to appear in the Hodge-Tate decompositions. In order to provide a detailed description of (21.4.3), it would be useful to translate this problem in terms of twisted Harish-Chandra sheaves and Whittaker modules, cf. [MS14]. Indeed, in this theory one has avaliable an explicit description of the irreducible objects, and any twisted Harish-Chandra sheaf is an extension of those.

As a first step to describe (21.4.3), let us show that the $\mathfrak{n}^0$-cohomology is concentrated in degree 0.

Proposition 21.4.13. *We have* $R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})) = C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0}$. *Dually,* $R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})) = \widetilde{\mathcal{D}}^{la} \otimes_{\mathscr{O}_{\mathscr{F}\ell}} \det \mathfrak{n}^{0,\vee}[-d]$.

Moreover, let $V \subset \mathscr{F}\ell$ *be an affinoid such that* $\mathfrak{n}^0|_V$ *is trivial and has a complement in* $\mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}$ *given by a parabolic subgroup* $\mathfrak{p}' \subset \mathfrak{g}_K$. *Then*

$$\mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) = \varprojlim_{h\to\infty} \mathcal{D}(\mathfrak{n}^0_h)\widehat{\otimes}_K \mathcal{D}(\mathfrak{p}'_h) \quad \text{and} \quad C^{la}(\mathfrak{g}, \mathscr{O}_V) = \varinjlim_{h\to\infty} \mathcal{D}(\mathfrak{n}^0_h)^\vee \widehat{\otimes}_K \mathcal{D}(\mathfrak{p}'_h)^\vee, \tag{21.4.4}$$

where the sheaves $\mathcal{D}(\mathfrak{n}^0_h)$ *and* $\mathcal{D}(\mathfrak{p}'_h)$ *are analytic distributions algebras of* $\mathfrak{n}^0$ *and* $\mathfrak{p}'$ *respectively, obtained by completing the enveloping algebras with respect to some lattices* $\mathfrak{n}^0_0$ *and* $\mathfrak{p}'_0$ *of* $\mathfrak{n}^0$ *and* $\mathfrak{p}'$ *respectively. The duals of* (21.4.4) *are with respect to* $\mathscr{O}_V$.

Proof. Let us first prove (21.4.4). Let $\mathfrak{n}^0_0 \subset \mathfrak{n}^0$ be a $\mathscr{O}^+_V$-lattice given by a fixed basis $\{\widetilde{X}_1, \dots, \widetilde{X}_d\}$. Since $\mathfrak{n}$ is abelian and $\mathfrak{n}^0$ is $\mathscr{O}_{\mathscr{F}\ell}$-linear, we have $[\mathfrak{n}^0, \mathfrak{n}^0] = 0$. Thus, we can define the distribution algebras (which are nothing but completions of the symmetric algebra of $\mathfrak{n}^0$)

$$\mathcal{D}(\mathfrak{n}^0_h) := \{\sum_{\alpha\in\mathbb{N}^d} a_\alpha \underline{\widetilde{X}}^\alpha : \ a_\alpha \in \mathscr{O}_V \text{ and } \sup_{\alpha\in\mathbb{N}}\{|a_\alpha| p^{h|\alpha|}\} < \infty\}.$$

Equivalently we can define

$$\mathcal{D}(\mathfrak{n}^0_h) = \prod_{\alpha\in\mathbb{N}} \left(\mathscr{O}^+_V \frac{\widetilde{X}^\alpha}{p^{h|\alpha|}} \right) [\frac{1}{p}]$$

endowed with the product topology. On the other hand, let $\mathfrak{p}'_0 \subset \mathfrak{p}'$ be a lattice satisfying $[\mathfrak{p}'_0, \mathfrak{p}'_0] \subset p\mathfrak{p}'_0$. Then we can define distribution algebras $\mathcal{D}(\mathfrak{p}'_h)$ as in Definition 17.2.7. Let $\mathcal{D}^{la}(\mathfrak{g}, K) = C^{la}(\mathfrak{g}, K)^\vee$, by the Poincaré-Birkhoff-Witt theorem we get that

$$\mathcal{D}^{la}(\mathfrak{g}, K)\widehat{\otimes}\mathscr{O}_V = \varprojlim_{h\to\infty} \mathcal{D}(\mathfrak{n}^0_h)\widehat{\otimes}_K \mathcal{D}(\mathfrak{p}'_h).$$

Taking $\mathscr{O}_V$-duals one finds that

$$C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) = \varinjlim_{h\to\infty} \mathcal{D}(\mathfrak{n}^0_h)^\vee \widehat{\otimes}_K \mathcal{D}(\mathfrak{p}'_h)^\vee. \tag{21.4.5}$$

Finally, let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a lattice satisfying $[\mathfrak{g}_0, \mathfrak{g}_0] \subset p\mathfrak{g}_0$. One also has

$$C^{la}(\mathfrak{g}, \mathscr{O}_V) = \varinjlim_{h\to\infty} \mathcal{D}(\mathfrak{g}^+_h)^\vee \widehat{\otimes}_K \mathscr{O}_V. \tag{21.4.6}$$

Since the direct systems of lattices $\{p^h\mathfrak{n}^0_0 \oplus p^h\mathscr{O}^+_V \otimes \mathfrak{p}'_0\}_{h>0}$ and $\{p^h\mathscr{O}^+_V \otimes \mathfrak{g}_0\}_{h>0}$ are cofinal in each other, one has an isomorphism between the direct systems of the colimits (21.4.5) and (21.4.6). Taking $\mathfrak{n}^0$-cohomology one obtains that

$$\begin{aligned} R\Gamma(\mathfrak{n}^0, \mathcal{D}(\mathfrak{n}^0_h)^\vee \widehat{\otimes}_K \mathcal{D}(\mathfrak{p}'_h)^\vee) &= R\mathrm{Hom}_{\mathscr{O}_V}(\mathscr{O}_V \otimes^L_{U(\mathfrak{n}^0)} \mathcal{D}(\mathfrak{n}^0_h)\widehat{\otimes}_K \mathcal{D}(\mathfrak{p}'_h), \mathscr{O}_V) \\ &\cong \mathcal{D}(\mathfrak{p}'_h)^\vee \widehat{\otimes}_K \mathscr{O}_V. \end{aligned}$$

By taking limits as $h \to \infty$ one has

$$R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_V)) = C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{1,3}=0}.$$

Dually, we have

$$\begin{aligned} R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})) &= (1 \otimes^L_{U(\mathfrak{n}^0)} \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_V)) \otimes \det \mathfrak{n}^{0,\vee} \\ &= \widetilde{\mathcal{D}}^{la} \otimes \det \mathfrak{n}^{0,\vee}[-d]. \end{aligned}$$

□

Remark 21.4.14. The Poincaré-Birkhoff-Witt theorem also implies that

$$R\Gamma(\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes K(-\lambda)) = C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{b}_{\star_2}=\lambda}$$
$$\mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes^{L}_{U(\mathfrak{b})} \lambda = \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes_{U(\mathfrak{b})} \lambda.$$

In particular, the difficulty of computing (21.4.3) arises from the relation between the derivations of $\mathfrak{n}^0$ and $\mathfrak{b}$.

21.4.3. $\mathfrak{b}$-cohomology

The next step is the study of the $\mathfrak{b}_{\star_2}$-cohomology of $C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0}$ and the $\mathfrak{b}_{\star_2}$-homology of $\widetilde{\mathcal{D}}^{la} \otimes \det \mathfrak{n}^{0,\vee}$. The heuristics behind the computations are the following: the Lie algebra $\mathfrak{b}$ is "vertical" while $\mathfrak{n}^0$ is "horizontal" over $\mathscr{F}\ell$. This means that $\mathfrak{b}$ is constant and acts by derivations on $\mathscr{O}_{\mathscr{F}\ell}$, while $\mathfrak{n}^0$ is $\mathscr{O}_{\mathscr{F}\ell}$-linear and moves according to the points of $\mathscr{F}\ell$. Therefore, when comparing the cohomologies of $\mathfrak{n}^0$ and $\mathfrak{b}$, one expects that their difference depend on the relative position of $\mathfrak{n}^0$ with respect to $\mathfrak{b}$. This last is parametrized precisely by the Bruhat stratification of $\mathscr{F}\ell$.

Fix integral models $\mathbf{H}^0$ over O_K of the algebraic groups $\mathbf{H} = \mathbf{G}_K, \mathbf{P}, \mathbf{M}$, etc. Let us recall that $\mathbf{N}^+ \subset \mathbf{B}$ and $\mathbf{N}^- \subset \overline{\mathbf{B}}$ are the unipotent radicals of the Borel subgroups, while $\mathbf{N} \subset \mathbf{P}$ and $\overline{\mathbf{N}} \subset \overline{\mathbf{P}}$ are the unipotent radicals of the parabolics[1]. Given $\alpha \in \Phi$, we denote by $\mathbf{N}_\alpha \subset \mathbf{G}_K$ the α-root subgroup so that $\mathrm{Lie}\, \mathbf{N}_\alpha = \mathfrak{n}_\alpha$. Let $w \in {}^{\mathbf{M}}W$ and $j_w : C_w \subset \mathscr{F}\ell$ be the w-Bruhat cell, we have an isomorphism of adic spaces

$$C_w \cong w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w).$$

Definition 21.4.15. For $\epsilon > 0$ define the following family of overconvergent neighbourhoods of C_w:

$$C_w(\epsilon) = C_w \cdot (\mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w) \cong w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)(\mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w),$$

where $\mathcal{N}^-(\epsilon) \subset \mathcal{N}^-$ is the affinoid polydisc of radius $p^{-\epsilon}$. Given $\delta > 0$ let $\mathcal{N}^+(-\delta) \subset \mathcal{N}^+$ be the affinoid polydisc of radius p^{δ}. We define the following affinoid subsapces

$$C_w(\infty, \delta) = w(\mathcal{N}^+(-\delta) \cap w^{-1}\overline{\mathcal{N}}w)$$
$$C_w(\epsilon, \delta) = w(\mathcal{N}^+(-\delta) \cap w^{-1}\overline{\mathcal{N}}w)(\mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w).$$

We also let $\mathscr{I}_w$ denote the ideal $\mathscr{O}_{C_w(\epsilon)}$ defining C_w.

Remark 21.4.16. The family of affinoids $\{C_w(\epsilon, \delta)\}_{\epsilon,\delta>0}$ satisfies the hypothesis of Lemma 18.1.4. This will be used later on to relate certain cohomologies with compact or closed supports over the infinite level Shimura variety, with the cohomologies with supports of higher Coleman theory.

The following definition is motivated from the relation between $\mathfrak{n}^0$ and $\mathfrak{n}^+$ in the neighbourhoods $C_w(\epsilon)$.

Definition 21.4.17. Let us identify $C_w(\epsilon) \cong w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)(\mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w)$, and write $x = wn^+(x)n^-(x)$ with $n^+(x) \in \mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w$ and $n^-(x) \in \mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w$ for $x \in C_w(\epsilon)$. Let $\alpha \in \Phi$ and $X_\alpha \in \mathfrak{n}_\alpha$ a basis, we define the following vector fields over $C_w(\epsilon)$

$$\mathfrak{X}_\alpha(x) := (wn^+(x))^{-1} X_\alpha wn^+(x)$$
$$\widetilde{\mathfrak{X}}_\alpha(x) := (wn^+(x)n^-(x))^{-1} X_\alpha wn^+(x)n^-(x).$$

Since $n^+(x) \in \mathcal{N}^+$, one easily verifies that

$$\mathscr{O}_{C_1(\epsilon)} \otimes \mathfrak{n}^+ = \bigoplus_{\alpha \in w(\Phi^+)} \mathscr{O}_{C_1(\epsilon)} \mathfrak{X}_\alpha \quad \text{and} \quad \mathfrak{n}^0|_{C_1(\epsilon)} = \bigoplus_{\alpha \in \Phi^{+,\mathbf{M}}} \mathscr{O}_{C_w(\epsilon)} \widetilde{\mathfrak{X}}_\alpha.$$

Let us define the following subbundles of $\mathscr{O}_{C_1(\epsilon)} \otimes \mathfrak{b}$:

[1] To justify this non standard notation, note that $\mathrm{Lie}\, \mathbf{N}^+ = \mathfrak{n}^+$ and that $\mathrm{Lie}\, \mathbf{N} = \mathfrak{n}$.

1. $\mathfrak{n}_w^+ := \bigoplus_{\alpha\in\Phi^+,\mathbf{M}\cap w(\Phi^+)} \mathscr{O}_{C_w(\epsilon)}\mathfrak{X}_\alpha$.
2. $\overline{\mathfrak{n}}_w^+ := \bigoplus_{\alpha\in\Phi^-,\mathbf{M}\cap w(\Phi^+)} \mathscr{O}_{C_w(\epsilon)}\mathfrak{X}_\alpha$
3. Let $\mathfrak{b}_\mathbf{M}$ be the Borel subalgebra of $\mathfrak{m}$, we let $\mathfrak{b}_{\mathbf{M},w}$ be the subbundle of $\mathscr{O}_{C_w(\epsilon)}\otimes\mathfrak{b}$ whose fiber at x is
$$\mathfrak{b}_{\mathbf{M},w}(x) = (wn^+(x))^{-1}\mathfrak{b}_\mathbf{M} wn^+(x).$$
4. We will denote $\mathfrak{p}_w^+ := \mathfrak{n}_w^+ \oplus \mathfrak{b}_{\mathbf{M},w}$ and $\overline{\mathfrak{p}}_w^+ := \overline{\mathfrak{n}}_w^+ \oplus \mathfrak{b}_{\mathbf{M},w}$.

Remark 21.4.18. Note that $\mathscr{O}_{C_w(\epsilon)}\otimes\mathfrak{b} = \mathfrak{n}_w^+ \oplus \mathfrak{b}_{\mathbf{M},w} \oplus \overline{\mathfrak{n}}_w^+$. Moreover, since $n^+(x) \in \mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w$ one has $\overline{\mathfrak{n}}_w^+ = \mathscr{O}_{C_w(\epsilon)}\otimes(\mathfrak{n}^+ \cap w^{-1}\overline{\mathfrak{n}}w)$.

Lemma 21.4.19. *We have $\mathfrak{n}_w^+ \subset \mathfrak{n}^0 + \mathscr{I}_w\otimes\mathfrak{g}$ and $\mathfrak{p}_w^+ \subset \mathfrak{p}^0 \oplus \mathscr{I}_w\otimes\mathfrak{g}$. Let $j_w : C_w \subset C_w(\epsilon)$ be the inclusion, then $j_w^*\mathfrak{n}_w^+ = (\mathscr{O}_{C_w}\otimes\mathfrak{n}^+)\cap j_w^*\mathfrak{n}^0$ and $j_w^*\mathfrak{p}_w^+ = (\mathscr{O}_{C_w}\otimes\mathfrak{b})\cap j_w^*\mathfrak{p}^0$.*

Proof. We can write
$$\begin{aligned}\mathfrak{X}_\alpha &= n^-(x)\widetilde{\mathfrak{X}}_\alpha(n^-(x))^{-1}\\ &= \widetilde{\mathfrak{X}}_\alpha + [\log n^-(x), \widetilde{X}_\alpha] + O((\log n^-(x))^2),\end{aligned}$$
where log is the inverse of the exponential map $\exp\mathfrak{g}\to\mathcal{G}$. But the coefficients of $\log n^-(x)$ in any basis of $\mathfrak{g}$ generate the ideal $\mathscr{I}_w$, this implies the first statement of the lemma.

Taking pullbacks to C_w one sees that $\mathfrak{X}_\alpha \equiv \widetilde{\mathfrak{X}}_\alpha \mod \mathscr{I}_w$ for all $\alpha\in\Phi$. The second part of the lemma follows by definition of $\mathfrak{n}_w^+$ and $\mathfrak{p}_w^+$ and the fact that $\Phi_\mathbf{M}^+ \subset w(\Phi^+)$. □

Using the previous lemma one deduces the following vanishing of $\mathfrak{b}$-(co)homology.

Lemma 21.4.20. *The following hold*

1. *Let $\epsilon > 0$, the complex $R\Gamma(\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{1,3}}=0}\otimes K(-\lambda))$ is concentrated in degrees $[0, d-\ell(w)]$. Similarly, the complex $(\widetilde{\mathcal{D}}^{la}|_{C_w(\epsilon)}\otimes\det\mathfrak{n}^{0,\vee})\otimes^L_{U(\mathfrak{b}_{\star_2})}\lambda$ is concentrated in degrees $[-d+\ell(w), 0]$.*
2. *The cohomology group $H^{d-\ell(w)}(\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{1,3}}=0})$ has support in C_w. Similarly, the homology group $H_{d-\ell(w)}(\mathfrak{b}_{\star_2}, \widetilde{\mathcal{D}}^{la}|_{C_w(\epsilon)}\otimes\det\mathfrak{n}^{0,\vee}\otimes K(-\lambda)_{\star_2})$ has support in C_w.*

Proof. Let us consider
$$\mathfrak{n}_w^{0,+} := \bigoplus_{\alpha\in\Phi^+,\mathbf{M}\cap w(\Phi^+)} \mathscr{O}_{C_w(\epsilon)}\widetilde{\mathfrak{X}}_\alpha \text{ and } \mathfrak{n}_w^{0,-} := \bigoplus_{\alpha\in\Phi^+,\mathbf{M}\cap w(\Phi^-)} \mathscr{O}_{C_w(\epsilon)}\widetilde{\mathfrak{X}}_\alpha$$
so that $\mathfrak{n}^0|_{C_w(\epsilon)} = \mathfrak{n}_w^{0,+}\oplus\mathfrak{n}_w^{0,-}$. Write $W = \bigoplus_{\alpha\in(\Phi^-,\mathbf{M}\cup\Phi_\mathbf{M})\cap w(\Phi^-)} \mathscr{O}_{C_w(\epsilon)}\mathfrak{X}_\alpha \subset \mathscr{O}_{C_1(\epsilon)}\otimes\mathfrak{g}$, then
$$\mathscr{O}_{C_w(\epsilon)}\otimes\mathfrak{g} = \mathfrak{n}_w^{0,-}\oplus W\oplus\mathscr{O}_{C_w(\epsilon)}\otimes\mathfrak{b}.$$
The Poincaré-Birkhoff-Witt theorem implies that
$$\begin{aligned}\mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) &= \varprojlim_{h\to\infty} \mathcal{D}(n_{w,h}^{0,-})\widehat{\otimes}_{\mathscr{O}_{C_w(\epsilon)}}\mathcal{D}(W_h)\widehat{\otimes}_K\mathcal{D}(\mathfrak{b}_h)\\ C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) &= \varinjlim_{h\to\infty} (\mathcal{D}(n_{w,h}^{0,-})\widehat{\otimes}_{\mathscr{O}_{C_w(\epsilon)}}\mathcal{D}(W_h)\widehat{\otimes}_K\mathcal{D}(\mathfrak{b}_h))^\vee\end{aligned}$$

where we take completions with respect to the lattices provided by the basis $\widetilde{\mathfrak{x}}_\alpha$ and $\mathfrak{x}_\alpha$, and duals with respect to $\mathscr{O}_{C_w(\epsilon)}$. Therefore

$$\begin{aligned}
R\Gamma(\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{1,3}}=0} \otimes K(-\lambda)) &= R\Gamma(\mathfrak{b}_{\star_2} \times \mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) \otimes K(-\lambda)) \\
&= \varinjlim_{h\to\infty} R\Gamma(\mathfrak{b}_{\star_2} \times \mathfrak{n}^0_{\star_{1,3}}, (\mathcal{D}(n^{0,-}_{w,h})\widehat{\otimes}\mathcal{D}(W_h)\widehat{\otimes}\mathcal{D}(\mathfrak{b}_h))^\vee \otimes K(-\lambda)) \\
&= \varinjlim_{h\to\infty} R\Gamma(\mathfrak{n}^{0,+}_{w,\star_{1,3}}, (1 \otimes_{\mathcal{D}(n^{0,-}_{w,h})} \mathcal{D}(n^{0,-}_{w,h})\widehat{\otimes}\mathcal{D}(W_h)\widehat{\otimes}\mathcal{D}(\mathfrak{b}_h) \otimes_{\mathcal{D}(\mathfrak{b}_h)} K(\lambda))^\vee) \\
&= R\Gamma(\mathfrak{n}^{0,+}_{w,\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^{0,-}_{w,\star_{1,3}}=0,\ \mathfrak{b}_{\star_2}=\lambda}).
\end{aligned} \tag{21.4.7}$$

Similarly, one finds that

$$(\widetilde{\mathcal{D}}^{la}|_{C_w(\epsilon)} \otimes \det \mathfrak{n}^{0,\vee}) \otimes^L_{U(\mathfrak{b}_{\star_2})} \lambda = 1 \otimes^L_{U(\mathfrak{n}^{0,+}_w)} (1 \otimes_{U(\mathfrak{n}^{0,-}_w)} (\mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) \otimes \det \mathfrak{n}^{0,\vee}) \otimes_{U(\mathfrak{b}_{\star_2})} \lambda). \tag{21.4.8}$$

But $\mathfrak{n}^{0,+}_w$ is an $\mathscr{O}_{C_w(\epsilon)}$-vector bundle of rank $d-\ell(w)$. One obtains part (1) by taking the Koszul complex.

To prove part (2), recall that $Y_w = \bigsqcup_{v\geq w} C_v$ is an open subspace of $\mathscr{F}\ell$ containing C_w as a closed subspace. Lemma 1 of [BL03, §I] implies that $C_w(\epsilon) \subset Y_w$ for $\epsilon > 0$. Indeed, it implies that the opposite Schubert variety $X^w = \overline{C^w}$ with $C^w = \mathbf{P}\backslash\mathbf{P}w\overline{\mathbf{B}}$ is contained in Y_w, cf. Lemma 3.12 of [BP21]. This shows that $C_w(\epsilon)\backslash C_w \subset Y_w\backslash C_w$ is contained in an union of Bruhat cells C_v with $\ell(v) > \ell(w)$. In particular, the restriction of the complexes of part (1) to $C_w(\epsilon)\backslash C_w$ are concentrated in cohomological degrees $< d - \ell(w)$ (resp. homological degrees $< d - \ell(w)$). This proves the lemma. □

For computations, it will be convenient to take a change of variables which will move the actions of $\mathfrak{n}^0$ and $\mathfrak{b}$ to the same side. Consider the map

$$\begin{aligned}
\Psi : \mathscr{F}\ell \times \mathcal{G} &\to \mathscr{F}\ell \times \mathcal{G} \\
(x, g) &\mapsto (xg, g).
\end{aligned} \tag{21.4.9}$$

The following lemma describes how the $\star$-actions on $\mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}}$ change after applying Ψ.

Lemma 21.4.21. *Let $\mathfrak{X} \in \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}} \otimes_{\mathbb{Q}_p} \mathfrak{g}$, $X \in \mathfrak{g}$ and $f \in \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}}$. Define $\Psi_*(\mathfrak{X})_{(x,g)} = g^{-1}\mathfrak{X}_{(xg^{-1},g)}g$, then $\Psi_*(\mathfrak{X}) \in \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}} \otimes_{\mathscr{O}_{\mathscr{F}\ell}} \mathfrak{n}^0$ if and only if $\mathfrak{X} \in \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}} \otimes_{\mathscr{O}_{\mathscr{F}\ell}} \mathfrak{n}^0$. Moreover*

1. $\Psi_*(\mathfrak{X} \star_{1,3} f) = -\Psi_*(\mathfrak{X}) \star_2 \Psi_*(f)$,
2. $\Psi_*(X \star_2 f) = X \star_{2,3} \Psi_*(f)$.

Proof. Let $\mathfrak{X}$ be a vector field in $\mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}} \otimes \mathfrak{g}$ satisfying $\mathfrak{X}_{(x,g)} \in \mathfrak{n}^0(x) = \mathfrak{n}_x$ for all $(x, g) \in X \times \mathcal{G}$. Then,

$$\Psi_*(\mathfrak{X})_{(x,g)} = g^{-1}\mathfrak{X}_{(xg^{-1},g)}g \in g^{-1}\mathfrak{n}_{xg^{-1}}g = \mathfrak{n}_x,$$

proving that $\Psi_*(\mathfrak{X}) \in \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}} \otimes_{\mathscr{O}_{\mathscr{F}\ell}} \mathfrak{n}^0$. The converse is proven in the same way.

Finally, one directly computes (1) and (2):

$$\begin{aligned}
\Psi_*(\mathfrak{X} \star_{1,3} f)(x, g) &= (\mathfrak{X} \star_{1,3} f)(xg^{-1}, g) \\
&= \frac{d}{dt}|_{t=0} f(xg^{-1} \exp(t\mathfrak{X}_{(xg^{-1},g)}), \exp(-t\mathfrak{X}_{(xg^{-1},g)})g) \\
&= \frac{d}{dt}|_{t=0} \Psi_*(f)(x, \exp(-t\mathfrak{X}_{(xg^{-1},g)})g) \\
&= \frac{d}{dt}|_{t=0} \Psi_*(f)(x, g\exp(-t\Psi_*(\mathfrak{X})_{(x,g)})) \\
&= -\Psi_*(\mathfrak{X}) \star_2 \Psi_*(f)(x, g).
\end{aligned}$$

$$\begin{aligned}\Psi_*(X\star_2 f)(x,g) &= (X\star_2 f)(xg^{-1},g)\\ &= \frac{d}{dt}|_{t=0} f(xg^{-1}, g\exp(tX))\\ &= \frac{d}{dt}|_{t=0} f(x\exp(tX)(g\exp(tX))^{-1}, g\exp(tX))\\ &= \frac{d}{dt}|_{t=0}\Psi_*(f)(x\exp(tX), g\exp(tX))\\ &= X\star_{2,3}\Psi_*(f)(x,g)\end{aligned}$$

□

Corollary 21.4.22. *We have* $\Psi_*(\mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}}^{\mathfrak{n}^0_{\star_{1,3}}=0}) = \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}}^{\mathfrak{n}^0_{\star_2}=0} = \mathscr{O}_{\mathscr{F}\ell\times\mathcal{G}}^{\mathfrak{n}^0_{\star_{2,3}}=0}$.

Proof. The first equality follows directly from the previous Lemma. The second holds since $\mathfrak{n}^0$ is $\mathscr{O}_{\mathscr{F}\ell}$-linear. □

What the previous lemma achieves is to transport the left regular action of $\mathfrak{n}^0$ to the right regular action. Notice that the subspace $\mathscr{F}\ell\times 1\subset\mathscr{F}\ell\times\mathcal{G}$ is invariant under Ψ. Thus, we have induced quasi-isomorphisms

$$\begin{aligned}R\Gamma(\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g},\mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0}\otimes K(-\lambda)) &\xrightarrow{\Psi_*} R\Gamma(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g},\mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{2,3}}=0}\otimes K(-\lambda))\\ (\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee})\otimes^L_{U(\mathfrak{b}_{\star_2})}\lambda &\xrightarrow{\Psi_*} (\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee})\otimes^L_{U(\mathfrak{b}_{\star_{2,3}})}\lambda.\end{aligned}$$

In the equations (21.4.7) and (21.4.8) we have first computed $\mathfrak{b}$-cohomology and then $\mathfrak{n}^0$-cohomology. If we reverse the order of computations, one deduces the following lemma

Lemma 21.4.23. *Let $\epsilon>0$, we have quasi-isomorphisms*

$$\begin{aligned}&R\Gamma(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g},\mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0}\otimes K(-\lambda)) =\\ &\quad[C^{la}(\mathfrak{g},\mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0,\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda}\xrightarrow{d}(C^{la}(\mathfrak{g},\mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0}\otimes\mathfrak{n}_w^{+,\vee})^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda}\xrightarrow{d}\\ &\qquad\cdots\xrightarrow{d}(C^{la}(\mathfrak{g},\mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0}\otimes\bigwedge^{d-\ell(w)}\mathfrak{n}_w^{+,\vee})^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda}]\end{aligned}$$

concentrated in cohomological degrees $[0, d-\ell(w)]$*, and*

$$\begin{aligned}&(\widetilde{\mathcal{D}}^{la}|_{C_w(\epsilon)}\otimes\det\mathfrak{n}^{0,\vee})\otimes_{U(\mathfrak{b}_{\star_{2,3}})}\lambda =\\ &[(\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee}\otimes\bigwedge^{d-\ell(w)}\mathfrak{n}_w^+)\otimes_{U(\overline{\mathfrak{p}}^+_{w,\star_{2,3}})}\lambda\xrightarrow{d}(\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee}\otimes\bigwedge^{d-\ell(w)-1}\mathfrak{n}_w^+)\otimes_{U(\overline{\mathfrak{p}}^+_{w,\star_{2,3}})}\lambda\xrightarrow{d}\\ &\qquad\cdots\xrightarrow{d}(\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee})\otimes_{U(\overline{\mathfrak{p}}^+_{w,\star_{2,3}})}\lambda\end{aligned}$$

concentrated in cohomological degrees $[\ell(w)-d, 0]$.

Remark 21.4.24. The duality between Lie algebra homology and cohomology provides and isomorphism of sheaves

$$\mathrm{Hom}_{\mathscr{O}_{C_w(\epsilon)}}(H_{d-\ell(w)}(\mathfrak{b}_{\star_2},\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee}\otimes K(-\lambda)),\mathscr{O}_{C_w(\epsilon)}) = H^{d-\ell}(\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g},\mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{1,3}}=0}\otimes\det\mathfrak{n}^0\otimes K(-\lambda)). \tag{21.4.10}$$

Thus, instead of studying the homology of $\widetilde{\mathcal{D}}^{la}\otimes\det\mathfrak{n}^{0,\vee}$ we can study the cohomology of $C^{la}(\mathfrak{g},\mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{1,3}=0}}\otimes\mathfrak{n}^0$.

By Lemma 21.4.19, the differentials $d^i : V_i \to V_{i+1}$ of the Koszul complexes of Lemma 21.4.23 have image in $\mathscr{I}_w V_{i+1} \cap \ker d^{i+1}$. The highest weight vector subquotients of $C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})$ are constructed as follows:

Proposition 21.4.25. *We have natural surjective maps*

$$H^{d-\ell(w)}(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}))^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes K(-\lambda)) \twoheadrightarrow (C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0, \overline{\mathfrak{p}}^+_{w,\star_{2,3}}=w^{-1}(w\cdot\lambda+2\rho^{\mathbf{M}})}) \tag{21.4.11}$$

$$H^{d-\ell(w)}(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \det \mathfrak{n}^0 \otimes K(-\lambda)) \twoheadrightarrow (C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0})^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=w^{-1}(w\cdot\lambda)} \tag{21.4.12}$$

where $\rho^{\mathbf{M}} = \rho - \rho_{\mathbf{M}}$ is a half of the sum of the roots of $\Phi^{+,\mathbf{M}}$.

Proof. By Lemma 21.4.23 we have

$$H^{d-\ell(w)}(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes K(-\lambda)) = \operatorname{coker}[(C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \bigwedge^{d-\ell(w)-1} \mathfrak{n}_w^{+,\vee})) \xrightarrow{d} (C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \bigwedge^{d-\ell(w)} \mathfrak{n}_w^{+,\vee}))]^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda}.$$

By Lemma 21.4.19, $\mathfrak{n}_w^+ \subset \mathfrak{n}^0 + \mathscr{I}_w \otimes \mathfrak{g}$, this implies that the image of d is contained in $C^{la}(\mathfrak{g}, \mathscr{I}_w)^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \bigwedge^{d-\ell(w)} \mathfrak{n}_w^{+,\vee}$ and that the cohomology admits a surjective map

$$H^{d-\ell(w)}(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes K(-\lambda)) \twoheadrightarrow (C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \bigwedge^{d-\ell(w)} \mathfrak{n}_w^{+,\vee})^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda}.$$

Note that the algebra $\overline{\mathfrak{p}}_w^+ \cap \mathscr{O}_{C_w} \otimes \mathfrak{n}^+$ acts nilpotently on the basis $\{\mathfrak{X}_\alpha : \alpha \in w(\Phi^+)\}$ of $\mathscr{O}_{C_w} \otimes \mathfrak{n}^+$ via the adjoint representation. Consider the short exact sequence

$$0 \to \overline{\mathfrak{p}}_w^+ \cap \mathscr{O}_{C_w} \otimes \mathfrak{n}^+ \to \mathscr{O}_{C_w} \otimes \mathfrak{n}^+ \to \mathfrak{n}_w^+ \to 0,$$

then $\bigwedge^{d-\ell(w)} \mathfrak{n}_w^{+,\vee}$ is a rank 1 subbundle of $\mathscr{O}_{C_w} \otimes \bigwedge^{d-\ell(w)} \mathfrak{n}^{+,\vee}$ stable under the action of $\overline{\mathfrak{p}}_w^+$, and has a generator killed by $\overline{\mathfrak{p}}_w^+ \cap \mathscr{O}_{C_w} \otimes \mathfrak{n}^+$, namely $\bigwedge_{\alpha \in \Phi^{+,\mathbf{M}} \cap w(\Phi^+)} \mathfrak{X}_\alpha^\vee$. Furthermore, $\bigwedge^{\ell(w)} \mathfrak{n}_w^{+,\vee}$ has weight $-\chi = -\sum_{\alpha \in w^{-1}(\Phi^{+,\mathbf{M}}) \cap \Phi^+} \alpha$. This shows that

$$(C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \bigwedge^{d-\ell(w)} \mathfrak{n}_w^{+,\vee})^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda} = (C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0, \overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda+\chi}.$$

But $\chi = w^{-1}(w\rho + \rho - 2\rho_{\mathbf{M}})$ and

$$\begin{aligned} \lambda + \chi &= w^{-1}(w\lambda + w\rho - \rho + 2\rho^{\mathbf{M}}) \\ &= w^{-1}(w \cdot \lambda + 2\rho^{\mathbf{M}}) \end{aligned}$$

where $\rho^{\mathbf{M}} = \rho - \rho_{\mathbf{M}}$. This proves (21.4.11).

To prove (2), by the same argument as before, we have a surjective map

$$H^{d-\ell(w)}(\mathfrak{b}_{\star_{2,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \det \mathfrak{n}^0 \otimes K(-\lambda)) \to (C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0} \otimes \det \mathfrak{n}^0)^{\overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda+\chi} \tag{21.4.13}$$

But $\det \mathfrak{n}^0|_{C_w(\epsilon)} \cong \mathscr{O}_{C_w(\epsilon)} v$ where $v : C_w(\epsilon) \to \det \mathfrak{n}$ is the constant function mapping to a fixed non-zero vector. Therefore, for $X \in \mathfrak{b}$ one has $X \star_2 v = 2w^{-1}(\rho^{\mathbf{M}})(X)$. Then, the RHS of (21.4.13) is equal to

$$C^{la}(\mathfrak{g}, \mathscr{O}_{C_w})^{\mathfrak{n}^0_{\star_{2,3}}=0, \overline{\mathfrak{p}}^+_{w,\star_{2,3}}=\lambda+\chi-2w^{-1}(\rho^{\mathbf{M}})}.$$

Since $\lambda + \chi + -2w^{-1}\rho^{\mathbf{M}} = w^{-1}(w \cdot \lambda)$ we are done. □

Taking pullbacks of (21.4.11) and (21.4.12) by $\Psi : \mathscr{F}\ell \times \mathcal{G} \to \mathscr{F}\ell \times \mathcal{G}$, one obtains maps of complexes

$$\begin{aligned} R\Gamma(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) \otimes K(-\lambda)) \to \\ C^{la}(\mathfrak{g}, \mathscr{O}_{\Psi^{-1}(C_w)})^{\mathfrak{n}^0_{\star_{1,3}}=0, \mathfrak{b}_{\star_2}=w^{-1}(w\cdot\lambda+2\rho^{\mathbf{M}})}[\ell(w)-d] \end{aligned} \tag{21.4.14}$$

$$\begin{aligned} R\Gamma(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) \otimes \det \mathfrak{n}^0 \otimes K(-\lambda)) \to \\ (C^{la}(\mathfrak{g}, \mathscr{O}_{\Psi^{-1}(C_w)}))^{\mathfrak{n}^0_{\star_{1,3}}=0, \mathfrak{b}_{\star_2}=w^{-1}(w\cdot\lambda)}[\ell(w)-d], \end{aligned}$$

where $\Psi^{-1}(C_w) \subset \mathscr{F}\ell \times \mathcal{G}$ is a twisted Bruhat cell.

Remark 21.4.26. The maps (21.4.11) and (21.4.12) are independent of ϵ; they factor through the colimit as $\epsilon \to \infty$. One has an isomorphism of inverse sistems

$$\{\Psi^{-1}(C_w(\epsilon,\delta) \times \mathcal{G}(\gamma))\}_{\epsilon,\gamma} = \{C_w(\epsilon,\delta) \times \mathcal{G}(\gamma)\}_{\epsilon,\gamma}$$

for any $\delta > 0$. This shows that the arrows (21.4.14) are well defined.

The RHS of the equations (21.4.14) can be computed as functions of an overconvergent torsor over $C_w(\epsilon)$. We need some notation, let $\mathcal{M}_{\mathscr{F}\ell} := \mathcal{N}\backslash\mathcal{G}_K$ be the natural $\mathcal{M}$-torsor over $\mathscr{F}\ell$, we define the following overconvergent torsors

Definition 21.4.27. 1. For $\gamma > 0$ consider $\mathcal{M}(\gamma)$-torsor over $C_w(\epsilon)$

$$\begin{aligned} \mathrm{pr}_w : \mathcal{M}_{w,\mathscr{F}\ell}(\gamma) &:= \mathcal{N}(\gamma)\backslash\mathcal{P}(\gamma)w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)(\mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w) \\ &\to C_w(\epsilon) = \mathcal{P}(\gamma)\backslash\mathcal{P}(\gamma)w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)(\mathcal{N}^-(\epsilon) \cap w^{-1}\overline{\mathcal{N}}w). \end{aligned}$$

   Let $\mathcal{M}^\dagger = \{\mathcal{M}(\gamma)\}_{\gamma\to\infty}$ be the overconvergent neighbourhood of 1 in $\mathcal{M}$. We let $\mathcal{M}^\dagger_{w,\mathscr{F}\ell} := \{\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)\}_{\gamma\to\infty}$ denote the overconvergent torsor over $C_w(\epsilon)$.

2. We set $C(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)) := \mathrm{pr}_{w,*}(\mathscr{O}_{\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)})$. Taking colimits as $\gamma \to \infty$ we define $C(\mathcal{M}^\dagger_{w,\mathscr{F}\ell}) = \varinjlim_{\gamma\to\infty} C(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma))$.

3. We set $\mathcal{D}(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)) := C(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma))^\vee$, where the dual is as sheaves over $\mathscr{O}_{C_w(\epsilon)}$. Taking limits as $\gamma \to \infty$ we define $\mathcal{D}(\mathcal{M}^\dagger_{w,\mathscr{F}\ell}) = \varprojlim_{\gamma\to\infty} \mathcal{D}(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma))$.

Remark 21.4.28. Note that, locally on affinoids of $C_w(\epsilon)$, the torsor $\mathcal{M}(\gamma)$ admits a right overconvergent action of $\mathcal{G}^\dagger$, and a left action of $\mathcal{M}(\gamma)$. These translate into a locally analytic right (resp. left) regular action of $\mathfrak{g}$ (resp. $\mathfrak{m}$) on the sheaves $C(\mathcal{M}^\dagger_{w,\mathscr{F}\ell})$ and $\mathcal{D}(\mathcal{M}^\dagger_{w,\mathscr{F}\ell})$. We denote these actions by $\mathfrak{g}_{\star_2}$ and $\mathfrak{m}_{\star_1}$ respectively.

Proposition 21.4.29. *Let* $\mathfrak{b}_{\mathbf{M}} = \mathrm{Lie}\,\mathbf{M} \cap \mathbf{B}$ *be the Lie algebra of the Borel subgroup of* $\mathbf{M}$. *We have a* $\mathfrak{g}$*-equivariant isomorphism of sheaves over* $C_w(\epsilon)$

$$C^{la}(\mathfrak{g}, \mathscr{O}_{\Psi^{-1}(C_w)})^{\mathfrak{n}^0_{\star_{1,3}}=0, \mathfrak{b}_{\star_2}=w^{-1}\chi} = C(\mathcal{M}^\dagger_{w,\mathscr{F}\ell})^{\mathfrak{b}_{\mathbf{M}},\star_1=\chi}, \tag{21.4.15}$$

where the first term is endowed with the $\mathfrak{g}_{\star_{1,3}}$*-action, and the second term with the* $\mathfrak{g}_{\star_2}$*-action.*

Proof. Let $\mathcal{G}^\dagger = \{\mathcal{G}(\gamma)\}_{\gamma>0}$ be the overconvergent neighbourhood of the identity on $\mathcal{G}$ and let $C^\dagger_w = \{C_w(\epsilon)\}_\epsilon$. Over the affinoid subspaces $C_w(\epsilon,\delta) \subset C_w(\epsilon)$ we have an action of $\mathcal{G}(\gamma)$ for some $\gamma > 0$, this is not true over the whole $C_w(\epsilon)$ in general (take $w = w_0^{\mathbf{M}}$ to be the longest element, then $C_w\mathcal{G}(\gamma) = \mathscr{F}\ell$ for any $\gamma > 0$). By an abuse of language we will say that $C^\dagger_w$ has a action of $\mathcal{G}^\dagger$. In the following we will freely take quotients by subgroups of $C^\dagger_w \times \mathcal{G}^\dagger$, knowing that the real meaning is taking quotients on affinoid subspaces by affinoid subgroups, and then taking the inverse system they define.

We have the following diagram of overconvergent spaces (i.e. maps of inverse systems of spaces)

$$\begin{array}{ccccc} C_w^\dagger \times \mathcal{G}^\dagger & \xrightarrow{\Psi} & C_w^\dagger \times \mathcal{G}^\dagger & \xleftarrow{\iota_w} & C_w \times \mathcal{G}^\dagger \\ & & & & \downarrow{\scriptstyle \pi_w} \\ & & & & C_w^\dagger \end{array}$$

where ι_w is the Zariski closed immersion and $\pi_w(x, g) = xg^{-1}$. Recall that, by Lemma 21.4.21, the actions of $\mathfrak{n}^0_{\star_{1,3}}$ and $\mathfrak{g}_{\star_2}$ are transformed to the actions of $\mathfrak{n}^0_{\star_{2,3}}$ and $\mathfrak{g}_{\star_{2,3}}$ after taking the pushforward by Ψ. Let $\mathcal{N}^{0,\dagger} \subset (C_w^\dagger \times \mathcal{G}^\dagger)$ be the subgroup whose fiber over $x \in C_w^\dagger$ is $\mathcal{N}_x^\dagger$, the unipotent group fixing x (in other words, $\mathcal{N}^{0,\dagger}$ is the image of $\mathfrak{n}^0$ via the exponential map). Then, taking $\mathfrak{n}^0$ invariants is equivalent to taking the following quotients of spaces

$$\begin{array}{ccccc} \mathcal{N}^{0,\dagger}\backslash C_w^\dagger \times \mathcal{G}^\dagger & \xrightarrow{\Psi} & C_w^\dagger \times \mathcal{G}^\dagger/\mathcal{N}^{0,\dagger} & \xleftarrow{\iota_w} & C_w \times \mathcal{G}^\dagger/\mathcal{N}^{0,\dagger} \\ & & & & \downarrow{\scriptstyle \pi_w} \\ & & & & C_w^\dagger \end{array}$$

The new map π_w is well defined since $x\mathcal{N}_x^\dagger = x$ for all $x \in \mathscr{F}\ell$. Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{N}^{0,\dagger}\backslash C_w^\dagger \times \mathcal{G}^\dagger & \xleftarrow{\Psi^{-1}\circ\iota_w} & C_w \times \mathcal{G}^\dagger/\mathcal{N}^{0,\dagger} \\ & \searrow^{\mathrm{pr}_1} & \downarrow{\scriptstyle \pi_w} \\ & & C_w^\dagger. \end{array}$$

In the quotient $(C_w(\infty, \delta) \times \mathcal{G}^\dagger)/\mathcal{N}^{0,\dagger}$, the action of $\mathfrak{n}_w^+$ is already trivial because $\mathfrak{n}_w^+ = \mathfrak{n}^0 \cap \mathscr{O}_{C_w} \otimes \mathfrak{b}$. We are left to study the action of $\overline{\mathfrak{p}}_w^+ = \overline{\mathfrak{n}}_w^+ \oplus \mathfrak{b}_{\mathbf{M},w}$.

We have an isomorphism of affine spaces $C_w = w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)$. The map π_w factors as

$$(C_w \times \mathcal{G}^\dagger)/\mathcal{N}^{0,+} \xrightarrow{\widetilde{\pi}_w} \mathcal{N}^\dagger\backslash w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)\mathcal{G}^\dagger \to \mathcal{P}^\dagger\backslash w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)\mathcal{G}^\dagger = C_w^\dagger,$$

where $\widetilde{\pi}_w(wn^+(x), g) = wn^+(x)g^{-1}$ for $x \in C_w$ and $g \in \mathcal{G}^\dagger$. Since $\overline{\mathfrak{n}}_w^+ = \mathscr{O}_{C_w} \otimes (\mathfrak{n}^+ \cap w^{-1}\overline{\mathfrak{n}}w)$, taking $\overline{\mathfrak{n}}^+_{w,\star_{2,3}}$-invariants is equivalent to taking the right diagonal quotient by the action of the overconvergent subgroup $\overline{\mathcal{N}}_w^{+,\dagger}$ of neighbourhoods of 1 in $\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w$. It is easy to see that this quotient is naturally identified with $\widetilde{\pi}_w$. But

$$\mathcal{N}^\dagger\backslash w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)\mathcal{G}^\dagger = \mathcal{M}^\dagger_{w,\mathscr{F}\ell}$$

by definition.

It remains to take the λ-isotypic part for the action of $\mathfrak{b}_{\mathbf{M},w}$. As $w \in {}^{\mathbf{M}}W$ we have $w(\Phi^+) \supset \Phi^+_{\mathbf{M}}$, in other words we have that $\mathfrak{b}_{\mathbf{M}} \subset w\mathfrak{b}w^{-1}$. Let $\mathcal{B}_{\mathbf{M}}$ denote the Borel subgroup of $\mathcal{M}$, and let $\mathcal{B}^\dagger_{\mathbf{M}}$ be the overconvergent subgroup at 1. For $m \in w^{-1}\mathcal{B}^\dagger_{\mathbf{M}}w$, the $\star_{2,3}$ action commutes with $\widetilde{\pi}_w$ as follows[2]:

$$\widetilde{\pi}_w((wx, g) \star_{2,3} m) = \widetilde{\pi}_{\mathrm{HT}}(wm^{-1}xm, gm) = wm^{-1}xg^{-1} = wm^{-1}w^{-1}wxg^{-1}.$$

In other words, the $\star_{2,3}$ action of $\mathfrak{b}_{\mathbf{M},w}$ becomes the left multiplication by $\mathfrak{b}_{\mathbf{M}}$ on the $\mathcal{M}^\dagger$ torsor $\mathcal{M}^\dagger_{w,\mathscr{F}\ell}$. The previous computation shows that the $\mathfrak{b}_{\mathbf{M},\mathbf{w},\star_{2,3}} = w^{-1}\chi$ isotypic component of the algebra of functions of $(C_w^\dagger \times \mathcal{G}^\dagger)/\langle \mathcal{N}^{0,\dagger}, \overline{\mathcal{N}}_w^{+,\dagger}\rangle$ is identified with the space

$$C^{la}(\mathfrak{g}, \mathscr{O}_{\Psi^{-1}(C_w)})^{\mathfrak{n}^0_{\star_{1,3}}=0, \mathfrak{b}_{\star_2}=w^{-1}\chi} \cong C(\mathcal{M}^\dagger_{w,\mathscr{F}\ell})^{\mathfrak{b}_{\mathbf{M},\star_1}=\chi}.$$

[2]To define the action properly, one has to work with the normalizer of $\overline{\mathcal{N}}_w^+$ in $w^{-1}\mathcal{B}^\dagger_{\mathbf{M}}w$. However, as $w^{-1}\mathfrak{b}_{\mathbf{M}}w = \mathfrak{h} \rtimes w^{-1}\mathfrak{n}^+_{\mathbf{M}}w$, we can find a filtration $\mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \cdots \mathfrak{k}_l = w^{-1}\mathfrak{b}_{\mathbf{M}}w$ with $\mathfrak{k}_1 = \overline{\mathfrak{n}}_w^+$ and $\mathfrak{k}_l$ an ideal in $\mathfrak{k}_{l+1}$. This allows us to make the quotients step by step.

It is also clear from the construction that the $\star_{1,3}$ action of $\mathfrak{g}$ is transformed to the $\mathfrak{g}_{\star_2}$ action of $C(\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell})^{\mathfrak{b}_{\mathbf{M},\star_1}=\chi}$. This finishes the proof. □

Remark 21.4.30. The sheaf $C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})$ can be endowed with an equivariant action of $\mathcal{G}$, namely, for $g \in \mathcal{G}$ and $f \in C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell})$ the action $g \star_{1,2,3} f$ is well defined. The sheaf $R\mathrm{Hom}(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}))$ is clearly a $\mathcal{B}$-equivariant sheaf over $\mathscr{F}\ell$ under $\star_{1,2,3}$; the subbundle $\mathfrak{n}^0 \subset \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}$ is an ideal, and $\mathcal{B}$ acts on $\mathfrak{b}$ by the adjoint action. It turns out that the sheaf $C(\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell})$ can be endowed with a $\mathcal{B}$-equivariant structure such that the maps (21.4.15) are $\mathcal{B}$-equivariant for all w.

More precisely, let us write $C_w \cong w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)$, given a point $x \in C_w$ we denote $x = wn^+(x)$. For an element $b \in \mathcal{B}$ we have $xb = wn^+(xb)$, this endows $(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)$ with an action of $\mathcal{B}$. Then, for $wn^+(x)g \in \mathcal{M}^{\dagger}_{w,\mathscr{F}\ell} = \mathcal{N}^{\dagger}\backslash w(\mathcal{N}^+ \cap w^{-1}\overline{\mathcal{N}}w)\mathcal{G}^{\dagger}$, we have

$$(wn^+(x)g) \star_2 b = wn^+(xb)(b^{-1}gb).$$

This action is well defined by Proposition 21.4.29.

We can finally state the main theorem of this section

Theorem 21.4.31. *Let $w \in {}^{\mathbf{M}}W$ be a Kostant representative and $C_w \subset \mathscr{F}\ell$ the w-Bruhat cell. Let $\epsilon > 0$ and $C_w(\epsilon)$ the ϵ-neighbourhood of the Bruhat cell.*

1. *The restrictions to $C_w(\epsilon)$ of the complexes $R\Gamma(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes K(-\lambda))$ and $R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes^L_{U(\mathfrak{b}_{\star_2})} \lambda)$ are concentrated in degrees $[0, d-\ell(w)]$ and $[\ell(w), d]$ respectively. Moreover, the cohomology group $H^{d-\ell(w)}$ (resp. $H^{\ell(w)}$) has support in C_w.*

2. *There are $(\mathcal{B}, \mathfrak{g})$-equivariant (dual) highest weight vector maps*

$$\Upsilon_w : R\Gamma(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_w(\epsilon)}) \otimes K(-\lambda)) \to C(\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell})^{\mathfrak{b}_{\mathbf{M},\star_1}=w\cdot\lambda+2\rho^{\mathbf{M}}}[\ell(w) - d]$$
$$\Upsilon^{\vee}_w : (w \cdot \lambda) \otimes_{U(\mathfrak{b}_{\mathbf{M},\star_1})} \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell})[-\ell(w)] \to R\Gamma(\mathfrak{n}^0_{\star_{1,3}}, \mathcal{D}^{la}(\mathfrak{g}, \mathscr{O}_{\mathscr{F}\ell}) \otimes^L_{U(\mathfrak{b}_{\star_2})} \lambda)$$

which are surjective (resp. injective) in degree $d - \ell(w)$ (resp. degree $\ell(w)$).

Proof. Part (1) was already proven in Lemma 21.4.23. Part (2) is a consequence of Propositions 21.4.25 and 21.4.29 and the duality between Lie algebra homology and cohomology (21.4.10). □

21.5. An example: GL_2

In the following we provide an example of the previous computations on Lie algebra cohomology of sheaves over the flag variety of GL_2. We will see that the descriptions obtained from this point of view are equivalent to the explicit computations of Lue Pan in [Pan20, §5].

Let $\mathbf{B} \subset \mathrm{GL}_2$ be the Borel subgroup of upper triangular matrices, and $\mathbf{N}$ its unipotent radical. We let $\mathbf{T} \subset \mathbf{B}$ be the diagonal torus, $\overline{\mathbf{B}}$ be the opposite Borel and $\overline{\mathbf{N}}$ its unipotent radical. The flag variety is isomorphic to $\mathbb{P}^1$ via $g \mapsto [0:1] \cdot g$, where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $[x:y] \cdot g = [ax + cy : bx + dy]$. The Weyl group of GL_2 has two elements: $W = \{1, w_0\}$, where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The Bruhat decomposition of $\mathbb{P}^1$ is

$$\mathbb{P}^1 = C_1 \sqcup C_{w_0} = [0:1] \sqcup [1:0]\mathbf{N} \cong * \sqcup \mathbb{A}^1.$$

Let $\mathfrak{g} = \mathrm{Lie}\,\mathrm{GL}_2$ be the Lie algebra, let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra, $\mathfrak{n} = \mathrm{Lie}\,\mathbf{N}$ and $\overline{\mathfrak{n}} = \mathrm{Lie}\,\overline{\mathbf{N}}$. Then

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

Let us denote

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathfrak{b} = K \cdot H \oplus K \cdot Z$, $\mathfrak{n} = K \cdot X$ and $\overline{\mathfrak{n}} = K \cdot Y$. One has the relations $[Z, \mathfrak{g}] = 0$, $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$.

From now on we see $\mathbb{P}^1$ as an adic space over $\mathrm{Spa}(K)$, and denote $\infty = [0 : 1]$. Let $\lambda : \mathfrak{b} \to K$ be a character written as $\lambda = (k_1, k_2)$. We are interested in the $\widetilde{\mathcal{D}}^{la}$-module

$$R\mathrm{Hom}(\mathfrak{n}^0_{\star_{1,3}} \times \mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}, \mathscr{O}_{\mathbb{P}^1}) \otimes (-\lambda)_{\star_2}). \tag{21.5.1}$$

Theorem 21.4.31 tells us that the restriction of (21.5.1) to C_{w_0} is simply

$$C^{la}(\mathfrak{g}, \mathbb{P}^1)^{\mathfrak{n}^0_{\star_{1,3}}, \mathfrak{b}_{\star_2} = \lambda} \cong C(\mathcal{M}^{\dagger}_{w_0, \mathbb{P}^1})^{\mathfrak{b}_{\star_1} = (k_2, k_1)}.$$

It is left to compute (21.5.1) in overconvergent neighbourhoods of ∞. Let $\epsilon > 0$ and consider the neighbourhood of ∞:

$$C_1(\epsilon) := \{[t : 1] : |t| \leq p^{-\epsilon}\} = [0 : 1] \begin{pmatrix} 1 & 0 \\ p^{\epsilon}\mathbb{D}^1_K & 1 \end{pmatrix}.$$

By Corollary 21.4.22 it is enough to compute

$$R\mathrm{Hom}_{\mathfrak{b}_{\star_{2,3}}}(\lambda, C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}} = 0}). \tag{21.5.2}$$

By Lemma 21.4.23, the cohomology (21.5.2) is quasi-isomorphic to the complex

$$C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}} = 0, \mathfrak{h} = \lambda} \xrightarrow{dX} C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}} = 0, \mathfrak{h} = \lambda + 2\rho}. \tag{21.5.3}$$

To describe this complex exhaustively we need some coordinates. Let us write $\mathcal{G}^{\dagger} = \{\mathcal{G}(\gamma)\}_{\gamma \to \infty}$ for the overconvergent neighbourhood of 1 in GL_2. An element in $C_1(\epsilon) \times \mathcal{G}^{\dagger}$ can be written uniquely as (t, g) where

$$g = (1 + z) \begin{pmatrix} 1 + s & 0 \\ x & (1 + s)^{-1} \end{pmatrix} y(t) \quad \text{and} \quad y(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 + ty & y \\ -t^2 y & 1 - ty \end{pmatrix}.$$

Under this presentation, $C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{n}^0_{\star_{2,3}} = 0}$ is identified with $K\langle t/p^{\epsilon}\rangle\{s, z, x\}$, where

$$A\{T\} = \varinjlim_{\epsilon \to \infty} A\langle T/p^{\epsilon}\rangle$$

for an affinoid algebra A. Since Z is in the center of $\mathfrak{g}$, we can take $Z = \lambda(Z) = (\lambda + \rho)(Z)$ isotypic parts and replace (21.5.3) by

$$(1 + z)^{Z(\lambda)} K\langle t/p^{\epsilon}\rangle\{s, x\}^{H = \lambda(H)} \xrightarrow{dX} (1 + z)^{Z(\lambda)} K\langle t/p^{\epsilon}\rangle\{s, x\}^{H = \lambda(H) + 2}. \tag{21.5.4}$$

Without loss of generality we can take $Z(\lambda) = 0$. Next, we want to compute the action of H and X on the variables t, s and x.

Lemma 21.5.1. *We have*

1. $H \star_{2,3} t = 2t$, $H \star_{2,3} s = 1 + s$ *and* $H \star_{2,3} x = x$.
2. $X \star_{2,3} t = -t^2$, $X \star_{2,3} s = -(1 + s)t$ *and* $X \star_{2,3} x = \frac{t^2}{1+s} - tx$.

Proof. Notice that

$$[t:1]\begin{pmatrix}(1+\varepsilon) & 0\\ 0 & (1+\varepsilon)^{-1}\end{pmatrix} = [(1+\varepsilon)^2 t:1],$$
$$\begin{pmatrix}1+s & 0\\ x & (1+s)^{-1}\end{pmatrix}\begin{pmatrix}(1+\varepsilon) & 0\\ 0 & (1+\varepsilon)^{-1}\end{pmatrix} = \begin{pmatrix}(1+\varepsilon)(1+s) & 0\\ (1+\varepsilon)x & (1+\varepsilon)^{-1}(1+s)^{-1}\end{pmatrix}.$$

Deriving with respect to ε and evaluating at $\varepsilon = 0$ one gets (1).

To prove (2), note that

$$[t:1]\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix} = [\frac{t}{1+\varepsilon t}:1],$$

deriving and evaluating at $\varepsilon = 0$ one gets $X \star_{2,3} t = -t^2$. To compute the other two derivatives, we have to write the product

$$\begin{pmatrix}1+s & 0\\ x & (1+s)^{-1}\end{pmatrix}\begin{pmatrix}1 & \varepsilon\\ 0 & 1\end{pmatrix}$$

in the form

$$\begin{pmatrix}1+\tilde{s} & 0\\ \tilde{x} & (1+\tilde{s})^{-1}\end{pmatrix}\tilde{y}(t).$$

Then $X \star_{2,3} s = \frac{d}{d\varepsilon}\tilde{s}|_{\varepsilon=0}$ and $X \star_{2,3} x = \frac{d}{d\varepsilon}\tilde{x}|_{\varepsilon=0}$. Expanding both products we find the equations

$$\begin{pmatrix}a_{11} & a_{1,2}\\ a_{21} & a_{22}\end{pmatrix} : \begin{pmatrix}1+s & \varepsilon(1+s)\\ x & \varepsilon x + (1+s)^{-1}\end{pmatrix} = \begin{pmatrix}(1+\tilde{s})(1+t\tilde{y}) & (1+\tilde{s})\tilde{y}\\ \tilde{x}(1+t\tilde{y}) - t^2\tilde{y}(1+\tilde{s})^{-1} & \tilde{x}\tilde{y} + (1+\tilde{s})^{-1}(1-t\tilde{y})\end{pmatrix}$$

Dividing the a_{12} by the a_{11} equation one gets

$$\frac{\tilde{y}}{1+t\tilde{y}} = \varepsilon.$$

Deriving and evaluating at $\varepsilon = 0$, one finds by the chain rule that $(\frac{d}{d\varepsilon}\tilde{y})|_{\varepsilon=0} = 1$. Deriving and evaluating at $\varepsilon = 0$ in the equation a_{11}, one gets by the chain rule that $X \star_{2,3} s = -(1+s)t$. Finally, deriving with respect to ε and evaluating at $\varepsilon = 0$ in a_{21} one gets $X \star_{2,3} x = \frac{t^2}{1+s} - tx$. □

Corollary 21.5.2. *We have* $H \star_{2,3} \frac{t}{(1+s)^2} = H \star_{2,3} \frac{x}{1+s} = 0$, $X \star_{2,3} \frac{t}{(1+s)^2} = X \star_{2,3} \frac{x}{1+s} = \frac{t^2}{(1+s)^2}$.

Now, we have an equality $K\langle t/p^\epsilon\rangle\{s,x\} = K\langle\frac{t}{(1+s)^2p^\epsilon}\rangle\{s,\frac{x}{1+s}\}$. Thus,

$$K\langle t/p^\epsilon\rangle\{s,x\}^{H=\lambda(H)} = (1+s)^{\lambda(H)}K\langle\frac{t}{(1+s)^2p^\epsilon}\rangle\{\frac{x}{1+s}\}.$$

The complex (21.5.4) becomes

$$(1+s)^{\lambda(H)}K\langle\frac{t}{(1+s)^2p^\epsilon}\rangle\{\frac{x}{1+s}\} \xrightarrow{dX} (1+s)^{\lambda(H)+2}K\langle\frac{t}{(1+s)^2p^\epsilon}\rangle\{\frac{x}{1+s}\}. \tag{21.5.5}$$

We have the following proposition (cf. [Pan20, Prop. 5.2.10 and Prop. 5.2.12])

Proposition 21.5.3. *The following hold*

*(*ker *1) Suppose that* $\lambda(H) \notin \mathbb{N}$, *then* $\varinjlim_{\epsilon\to\infty} H^0(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{b}_{\star_2}=\lambda}) = 0$.

*(*ker *2) Suppose that* $\lambda(H) \in \mathbb{N}$, *then* $\varinjlim_{\epsilon\to\infty} H^0(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{b}_{\star_2}=\lambda}) = \Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)}\otimes_K C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=w_0(\lambda)}$, *where* $\mathcal{N}^\vee_{\infty/\mathbb{P}^1}$ *is the conormal bundle of* $\infty \subset \mathbb{P}^1_K$, *and* $\Psi : \mathbb{P}^1_K \times \mathrm{GL}_2 \to \mathbb{P}^1_K \times \mathrm{GL}_2$ *is the map* $(x,g) \mapsto (xg, g)$.

(coker 1) *The group* $\varinjlim_{\epsilon\to\infty} H^1(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{b}_{\star_2}=\lambda})$ *is an extension of the form*

$$0 \to \mathcal{M}_\lambda \to \varinjlim_{\epsilon\to\infty} H^1(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{b}_{\star_2}=\lambda}) \to C(\mathcal{M}^\dagger_{1,\mathscr{F}\ell})^{\mathfrak{b}_{\star_1}=\lambda+2\rho} \to 0.$$

(coker 2) *Suppose that* $\lambda(H) \notin \mathbb{N}$ *and that*

$$\sup_{n,m}\{|\frac{1}{\lambda(H)-n}|^{1/(m+n)}, \sup_{1\le k\le n}\{|\left(\frac{m+1}{\lambda(H)-n+1}\right)\cdots\left(\frac{m+k}{\lambda(H)-n+k}\right)|\}^{1/(n+m)}\} < \infty.$$

Then $\mathcal{M}_\lambda = 0$.

(coker 3) *Suppose that* $\lambda(H) \in \mathbb{N}$. *Then* $\mathcal{M}_\lambda = \Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)+1} \otimes_K C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=w_0(\lambda)}$. *In particular, since* $\lambda(H) \neq -1$ *(i.e* $w_0\cdot\lambda \neq \lambda$*), we have a direct sum decomposition*

$$\varinjlim_{\epsilon\to\infty} H^1(\mathfrak{n}^0_{\star_{1,3}}, C^{la}(\mathfrak{g}, \mathscr{O}_{C_1(\epsilon)})^{\mathfrak{b}_{\star_2}=\lambda}) = C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=\lambda+2\rho} \oplus \Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)+1} \otimes_K C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=w_0(\lambda)}.$$

Proof. Given $f \in (1+s)^{\lambda(H)}K\langle\frac{t}{(1+s)^2p^\epsilon}\rangle\{\frac{x}{1+s}\}$ let us write

$$f(t,x,s) = (1+s)^{\lambda(H)}\sum_{n,m\ge 0} a_{n,m}\left(\frac{t}{(1+s)^2}\right)^n\left(\frac{x}{1+s}\right)^m$$

with $a_{n,m} \in K$ satisfying some convergence conditions. An explicit computation shows that

$$X\star_{2,3} f(t,x,s) = (1+s)^{\lambda(H)+2}\sum_{n,m\ge 0}((n-1-\lambda(H))a_{n-1,m}+(m+1)a_{n-2,m+1})\left(\frac{t}{(1+s)^2}\right)^n\left(\frac{x}{1+s}\right)^m. \quad (21.5.6)$$

Then, $f \in \ker dX$ if and only if

$$(n-\lambda(H))a_{n,m} + (m+1)a_{n-1,m+1} = 0 \quad \forall n,m \ge 0. \quad (21.5.7)$$

Suppose that $\lambda(H) \notin \mathbb{N}$, then $n-\lambda(H) \neq 0$ for all $n \ge 0$ and $a_{n,m} = \frac{m+1}{\lambda(H)-n}a_{n-1,m+1}$. Taking $n = 0$ this implies that $a_{0,m} = 0$ for all $m \ge 0$. By an inductive argument one has that $a_{n,m} = 0$ for all $n,m \ge 0$, this shows (ker 1).

Suppose that $\lambda(H) \in \mathbb{N}$. The same argument shows that for $n < \lambda$ one has $a_{n,m} = 0$ for all $m \ge 0$. Taking $n = \lambda$ in (21.5.7) the equality is trivial, so we move to $n > \lambda$. We have that

$$\begin{aligned} a_{n,m} &= \frac{m+1}{\lambda(H)-n}a_{n-1,m+1} = \left(\frac{m+1}{\lambda(H)-n}\right)\left(\frac{m+2}{\lambda(H)-n+1}\right)a_{n+2,m+2} = \cdots \\ \cdots &= \left(\frac{m+1}{\lambda(H)-n}\right)\cdots\left(\frac{m+k}{\lambda(H)-n+k-1}\right)\cdots\left(\frac{m+n-\lambda(H)}{-1}\right)a_{\lambda,m+n-\lambda} \\ &= (-1)^{n-\lambda(H)}\binom{m+n-\lambda(H)}{m}a_{\lambda,m+n-\lambda}. \end{aligned}$$

Therefore, if $f(s,t,x) \in \ker dX$ one has that

$$\begin{aligned} f(s,t,x) &= (1+s)^{\lambda(H)}\sum_{\substack{n\ge\lambda(H)\\ m\ge 0}}(-1)^{n-\lambda(H)}\binom{m+n-\lambda(H)}{m}a_{\lambda,m+n-\lambda}\left(\frac{t}{(1+s)^2}\right)^n\left(\frac{x}{1+s}\right)^m \\ &= (1+s)^{\lambda(H)}\left(\frac{t}{(1+s)^2}\right)^{\lambda(H)}\sum_{n,m\ge 0}(-1)^n\binom{n+m}{m}a_{\lambda,m+n}\left(\frac{t}{(1+s)^2}\right)^n\left(\frac{x}{1+s}\right)^m \\ &= (1+s)^{\lambda(H)}\left(\frac{t}{(1+s)^2}\right)^{\lambda(H)}\sum_{k\ge 0}a_{\lambda,k}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\left(\frac{t}{(1+s)^2}\right)^{k-m}\left(\frac{x}{1+s}\right)^m \\ &= (1+s)^{-\lambda(H)}t^{\lambda(H)}\sum_{k\ge 0}a_{\lambda,k}\left(\frac{x}{1+s}-\frac{t}{(1+s)^2}\right)^k. \end{aligned}$$

Consider the multiplication map of Proposition 21.4.29

$$\tilde{\pi}_1 : C_1(\epsilon) \times \overline{\mathcal{B}}(\gamma) \cong \overline{\mathcal{N}}(\epsilon) \times \overline{\mathcal{B}}(\gamma) \to \mathcal{N}\backslash\mathcal{NT}(\gamma)\overline{\mathcal{N}}(\epsilon)$$
$$(\overline{n}, \overline{b}) \mapsto \overline{n}\overline{b}^{-1}$$

for $\delta >> 0$. Explicitly, if $\overline{n} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $\overline{b} = (1+z)\begin{pmatrix} 1+s & 0 \\ x & (1+s)^{-1} \end{pmatrix}$ one has

$$\tilde{\pi}_1(\overline{n}, \overline{b}) = (1+z)^{-1}\begin{pmatrix} (1+s)^{-1} & 0 \\ 0 & 1+s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{t}{(1+s)^2} - \frac{x}{1+s} & 1 \end{pmatrix}.$$

Then, the conormal bundle of $\infty \subset \mathbb{P}^1_K$ in the coordinate t is given by $K \cdot t$, thus $(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)} = K \cdot t^{\lambda(H)}$. On the other hand, the space $C_1(\mathcal{M}^\dagger_{1,\mathscr{F}\ell})^{\mathfrak{h}_1 = w_0(\lambda)}$ on the coordinate $\frac{t}{(1+s)^2} - \frac{x}{(1+s)}$ is identified with the functions

$$f(z, s, t, x) = (1+z)^{\lambda(Z)}(1+s)^{-\lambda(H)} \sum_{k \geq 0} b_k \left(\frac{x}{1+s} - \frac{t}{(1+s)^2} \right)^k,$$

this proves (ker 2).

The point (coker 1) was already proven in Theorem 21.4.31, and can be seen again in the formula (21.5.6). Hence, we have that

$$X \star_{2,3} f(s, t, x) = (1+s)^{\lambda(H)} t \sum_{n,m \geq 0} (n - \lambda(H))a_{n,m} + (m+1)a_{n-1,m+1} \left(\frac{t}{(1+s)^2} \right)^n \left(\frac{x}{1+s} \right)^m.$$

Given $b_{n,m}$ we want to solve the recursive equation

$$b_{n,m} = (n - \lambda(H))a_{n,m} + (m+1)a_{n-1,m+1} \quad \forall n, m \geq 0. \tag{21.5.8}$$

Suppose that $\lambda(H) \notin \mathbb{N}$, then

$$a_{n,m} = \frac{1}{\lambda(H) - n}((m+1)a_{n-1,m+1} - b_{n,m}).$$

Taking $n = 0$ one sees that $a_{0,m} = -\frac{1}{\lambda(H)} b_{0,m}$ for all $m \geq 0$. An inductive argument shows that

$$a_{n,m} = -\frac{1}{\lambda(H) - n} b_{n,m} - \frac{1}{\lambda(H) - n} \sum_{k=1}^{n} \left(\frac{m+1}{\lambda(H) - n + 1} \right) \cdots \left(\frac{m+k}{\lambda(H) - n + k} \right) b_{n-k,m+k}.$$

The convergence condition of (coker 2) guarantees that the series

$$f(t, x, s) = (1+s)^{\lambda(H)} \sum_{n,m \geq 0} a_{n,m} \left(\frac{t}{(1+s)^2} \right)^n \left(\frac{x}{1+s} \right)^m$$

converges for $\epsilon >> 0$, proving that $\varinjlim_{\epsilon \to \infty} \mathcal{M}_\lambda(\epsilon) = 0$.

Finally, suppose that $\lambda(H) \in \mathbb{N}$. The previous computation shows that we can solve the equation (21.5.8) uniquely for all $n < \lambda(H)$. Therefore, we may assume that $b_{n,m} = 0$ for $n < \lambda(H)$. The input $n = \lambda(H)$ in (21.5.8) would imply $b_{\lambda(H),m} = 0$ for all $m \geq 0$, and if this holds we can solve the equation for all $n > \lambda(H)$. Therefore, $\mathcal{M}_\lambda$ is represented by the power series

$$\begin{aligned} f(s, t, x) &= (1+s)^{\lambda(H)} t \left(\frac{t}{(1+s)^2} \right)^{\lambda(H)} \sum_{m \geq 0} b_m \left(\frac{x}{1+s} \right)^m \\ &= (1+s)^{-\lambda(H)} t^{\lambda(H)+1} \sum_{m \geq 0} b_m \left(\frac{x}{1+s} \right)^m. \end{aligned}$$

But taking the multiplication map as before

$$\tilde{\pi}_1 : C_1 \times \overline{\mathcal{B}}(\gamma) \cong 1 \times \overline{\mathcal{B}}(\gamma) \to \mathcal{N}\backslash\mathcal{NT}(\gamma)\overline{\mathcal{N}}(\gamma)$$
$$(1, \overline{b}) \mapsto \overline{b}^{-1},$$

this space is identified with $\Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)+1} \otimes_K C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=w_0(\lambda)}$ as wanted. □

Remark 21.5.4. The $\star_{1,3}$ action is trivial on $\Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})$, but $\mathfrak{b}_{\star_2}$ acts via 2ρ. Therefore, the highest weight of $\Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)} \otimes_K C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=w_0(\lambda)}$ is $w_0(\lambda) + 2\lambda(H)\rho = \lambda$. However, $\Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})^{\otimes\lambda(H)} \otimes_K C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=w_0(\lambda)} \neq C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_{\star_1}=\lambda}$ as $G^\dagger$-equivariant sheaves over $C_1(\epsilon)$. Indeed, the horizontal action $\mathfrak{b}^0_{\star_3}$ differs on both sheaves as it is trivial on $\Psi^*(\mathcal{N}^\vee_{\infty/\mathbb{P}^1})$, and it is different on $C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_1=\lambda}$ and $C(\mathcal{M}^\dagger_{\mathscr{F}\ell,1})^{\mathfrak{b}_1=w_0(\lambda)}$ unless $\lambda(H) = 0$, i.e. $w_0(\lambda) = \lambda$.

22. Shimura varieties

In this chapter, we apply the machinery of Sen theory of §20 to Shimura varieties. We describe the locally analytic vectors of the completed cohomology in terms of the locally analytic vectors of the completed structural sheaf at infinite level. As a first application, we deduce that the rational completed cohomology is concentrated in degrees $[0, d]$, where d is the dimension of the Shimura variety, proving in this way a rational version of the Calegary-Emerton conjectures [CE12] for *any* Shimura variety. If the Shimura datum if of abelian type, this follows from Theorem 5.3 of [Sch15] as the infinite level Shimura variety is perfectoid (see [She17]), in fact, the Calegari-Emerton conjectures hold with integral coefficients in this situation. In [HJ20], Hansen-Johansson prove many cases of the conjectures reducing to pre-abelian Shimura varieties. What is remarkable in our proof is that it never uses the perfectoidness of the Shimura variety; the important input is the non vanishing of the Sen operators.

We begin in §22.1 with the general set up of Shimura varieties, we will follow the conventions of §5 [DLLZ18]. In §22.1.1, we define the infinite level Shimura variety and recall the construction of the Hodge-Tate period map of diamonds as in §4.4 of [BP21]. In §22.2 we make the preparations for proving the main theorem of the chapter, namely, in §22.2.1 we show that the Sen bundle of a toroidal compactification of a Shimura variety is identified with the pullback via π_{HT} of the $\mathbf{G}$-equivariant subbundle $\mathfrak{n}^0 \subset \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}$ of §21.4. As an application, we give a purely local proof of the classical Hodge-Tate decompositions of Shimura varieties in §22.2.2; the idea is to use Faltings's BGG method and the π_{HT} period map. Finally, in §22.3, we recall the definition of Emerton's completed cohomology and how it can be computed in terms of pro-étale cohomology. We prove the main theorem of the chapter (Theorem 22.3.16), and in §22.3.1 we compute the arithmetic Sen operator in terms of the horizontal action $\theta_{\mathfrak{m}}$ arising from the center of $U(\mathfrak{m}^0)$, cf. Proposition 21.4.7.

In the following we will omit the subscript in the structural sheaves of the Shimura varieties unless otherwise specified, namely, we denote by $\mathscr{O}^{(+)}$ and $\hat{\mathscr{O}}^{(+)}$ the uncompleted and completed structural sheaves of the Shimura varieties in the pro(-Kummer-)étale site. We also let $\mathscr{O}_?^{(+)}$ for $? \in \{\mathrm{an}, \text{ét}, \text{két}\}$ denote the structural sheaves of the Shimura varieties in the analytic, étale and Kummer-étale site respectively.

22.1. The set-up

Let $(\mathbf{G}, X)$ be a Shimura datum, that is, $\mathbf{G}$ is a reductive group over $\mathbb{Q}$ and X a $\mathbf{G}(\mathbb{R})$-conjugacy class of cocharacters

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{R}}.$$

satisfying the axioms ([Del79, 2.1.1] or [Mil05, Def. 5.5]) :

SV1. For all $h \in X$, the Hodge structure on $\mathrm{Lie}(\mathbf{G}_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of type $\{(-1,1),(0,0),(1,-1)\}$.

SV2. For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{R}}$.

SV3. $\mathbf{G}^{\mathrm{ad}}$ has no simple $\mathbb{Q}$-factors with compact real points.

Let $E/\mathbb{Q}$ be the reflex field of $(\mathbf{G}, X)$, and for $K \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty})$ a neat compact open subgroup, let Sh_K denote the canonical model of the Shimura variety of level K over $\mathrm{Spec}\, E$. Recall that the $\mathbb{C}$-points of

Sh_K are equal to

$$\mathrm{Sh}_K(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash(X \times \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty}))/K.$$

From now on we will fix $K^p \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$ a neat compact subgroup of level prime to p. Given $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ a compact open subgroup we let $\mathrm{Sh}_{K^pK_p}$ denote the Shimura variety of level K^pK_p. We will be interested in the tower

$$\{\mathrm{Sh}_{K^pK_p}\}_{K_p \subset \mathbf{G}(\mathbb{Q}_p)}.$$

Let $\mathbf{Z}$ be the center of $\mathbf{G}$ and $\mathbf{Z}_c \subset \mathbf{Z}$ the maximal $\mathbb{Q}$-anisotropic torus which is $\mathbb{R}$-split. Given two levels $K' \subset K \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty})$ with K' normal in K, the map of Shimura varieties $\mathrm{Sh}_{K'} \to \mathrm{Sh}_K$ is finite étale with Galois group isomorphic to $K/(K', K \cap \overline{\mathbf{Z}(\mathbb{Q})})$, where $\overline{\mathbf{Z}(\mathbb{Q})}$ is the closure of $\mathbf{Z}(\mathbb{Q})$ in $\mathbf{Z}(\mathbb{A}_{\mathbb{Q}}^{\infty})$, cf [Del79, §2.19]. Since K is neat, $K \cap \overline{\mathbf{Z}(\mathbb{Q})} \subset \mathbf{Z}_c(\mathbb{A}_{\mathbb{Q}}^{\infty})$. In particular, the infinite-at-p level Shimura variety $\mathrm{Sh}_{K^p} := \varprojlim_{K_p} \mathrm{Sh}_{K^pK_p}$ is a Galois cover of $\mathrm{Sh}_{K^pK_p}$ of group

$$\widetilde{K}_p := K^pK_p/(K^p, K^pK_p \cap \overline{\mathbf{Z}(\mathbb{Q})}).$$

Thus, if $K'_p \subset K_p$ is normal, the Galois cover $\mathrm{Sh}_{K^pK'_p} \to \mathrm{Sh}_{K^pK_p}$ has group $\widetilde{K}_p/\widetilde{K}'_p$. Let us denote $\widetilde{\mathfrak{g}} = \mathrm{Lie}\,\widetilde{K}_p$.

Let $\mathbf{G}^c$ denote the quotient of $\mathbf{G}$ by $\mathbf{Z}_c$, and let $\mathfrak{g}^c = \mathrm{Lie}\,\mathbf{G}^c_{\mathbb{Q}_p}$. From our previous discussion, there is a map $\widetilde{\mathfrak{g}} \to \mathfrak{g}^c$, the obstruction for this map being an isomorphism depends on the Leopoldt's conjecture, in fact, this map is an isomorphism of Lie algebras if and only if the image of $\mathbf{Z}(\mathbb{Q})$ in $\mathbf{Z}(\mathbb{Q}_p)$ generates an open subgroup. Given a subgroup $\mathbf{H}$ of $\mathbf{G}$ we denote by $\mathbf{H}^c$ its image in $\mathbf{G}^c$, similarly for the subgroups $K \subset \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^{\infty})$, K^p and K_p.

Let $\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ be the Hodge cocharacter, i.e. the restriction of h to the first factor of $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}})_{\mathbb{C}} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. Associated to μ, we have two parabolic subgroups $\mathbf{P}^{\mathrm{std}}_{\mu} = \{g \in \mathbf{G}_{\mathbb{C}} : \lim_{t\to\infty} \mathrm{Ad}(\mu(t))g \text{ exists}\}$ and $\mathbf{P}_{\mu} = \{g \in \mathbf{G}_{\mathbb{C}} : \lim_{t\to 0} \mathrm{Ad}(\mu(t))g \text{ exists}\}$, we denote by $\mathbf{N}_{\mu} \subset \mathbf{P}_{\mu}$ the unipotent radical. Let $\mathrm{FL}^{\mathrm{std}} = \mathbf{G}_{\mathbb{C}}/\mathbf{P}^{\mathrm{std}}_{\mu}$ and $\mathrm{FL} = \mathbf{P}_{\mu}\backslash\mathbf{G}_{\mathbb{C}}$ denote the flag varieties. We recall that the conjugacy class of μ is defined over E, so the same holds true for $\mathrm{FL}^{\mathrm{std}}$ and FL. Note that we have a surjective map $\widetilde{K}_p \to \widetilde{K}_p$ whose kernel is in the center of $\widetilde{K}_p$ for K_p small enough.

Given a Shimura variety Sh_K, we denote by Sh^*_K its minimal compactification, and given an auxiliary cone decomposition Σ we shall denote by $\mathrm{Sh}^{\mathrm{tor}}_K$ its toroidal compactification as in [Pin90] (see [FC90] for an algebraic construction in the Siegel case). We can and will assume that the toroidal compactification is projective smooth and that the boundary divisor is a strict normal crossings divisor. Even if the toroidal compactification depends on the cone decomposition, we shall omit the subscript Σ in the notation; this will not be important in our results as the coherent cohomology is unchanged after refining the cone decomposition (see [Har90, §2] and [Lan17, Prop. 7.5]). Besides, the Kummer-étale cohomology only depends on the open Shimura variety by a purity theorem [DLLZ19, Theo. 4.6.1]. Given $K' \subset K$, we can always modify the toroidal compactification of $\mathrm{Sh}^{\mathrm{tor}}_{K'}$ for the map $\mathrm{Sh}^{\mathrm{tor}}_{K'} \to \mathrm{Sh}^{\mathrm{tor}}_K$ to be finite Kummer-étale (e.g. by the Abhyankar's lemma [DLLZ19, Prop. 4.2.1]). Furthermore, after fixing K_p, one can take toroidal compactifications adapted to a decreasing sequence of subgroups $\cdots \subset K^2_p \subset K^1_p \subset K_p$, in the sense that the maps of the tower $\{\mathrm{Sh}^{\mathrm{tor}}_{K^pK^n_p}\}_{K^n_p \subset K_p}$ are finite Kummer-étale.

Let $L/\mathbb{Q}_p$ be a finite extension containing the reflex field E and such that $\mathbf{G}$ is split over L. We let $\mathcal{S}h_{K,L}$ denote the p-adic analytification of $\mathrm{Sh}_{K,L} := \mathrm{Sh}_K \times_{\mathrm{Spec}\,E} \mathrm{Spec}\,L$ to an adic space over $\mathrm{Spa}(L, \mathcal{O}_L)$, cf. [Hub96]. We denote by $\mathcal{S}h^*_{K,L}$ and $\mathcal{S}h^{\mathrm{tor}}_{K,L}$ the analytification of the minimal and toroidal compactifications. As is explained in [LS18, §3], given $V \in \mathrm{Rep}_L(\mathbf{G}^c)$ a finite dimensional representation, we can attach an automorphic étale local system $V_{\text{ét}}$ over $\mathcal{S}h_{K,L}$. By purity of torsion local systems [DLLZ19, Prop. 4.2.1], $V_{\text{ét}}$ has a natural extension to a Kummer étale local system over $\mathcal{S}h^{\mathrm{tor}}_{K,L}$ which we denote by $V_{\text{két}}$. Let $\mathcal{S}h^{\mathrm{tor}}_{K^p,L} := \varprojlim_{K'_p \subset K_p} \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ be the toroidal infinite level Shimura variety, seen as an object in $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\text{prokét}}$. Essentially by definition, $V_{\text{két}}$ is the local system attached to

the $\widetilde{K}_p$-representation V and the $\widetilde{K}_p$-torsor $\pi_{K_p} : \mathcal{S}h^{\mathrm{tor}}_{K^p,L} \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$. From now on we will see the flag varieties as schemes over L, and we shall denote by $\mathscr{F}\ell^{\mathrm{std}}$ and $\mathscr{F}\ell$ their analytifications.

By the p-adic Riemman-Hilbert correspondance [DLLZ18, Theo. 5.3.1], there is a filtered vector bundle with integral log connection $(V_{\mathrm{dR}}, \nabla)$ defined over $\mathcal{S}h^{\mathrm{tor}}_{K,L}$, and a natural isomorphism

$$V_{\text{két}} \otimes_{\widehat{L}} \mathscr{O}\mathbb{B}_{\mathrm{dR},\log} = V_{\mathrm{dR}} \otimes_{\mathscr{O}_{\mathcal{S}h^{\mathrm{tor}}_{K,L}}} \mathscr{O}\mathbb{B}_{\mathrm{dR},\log} \tag{22.1.1}$$

as sheaves over the pro-Kummer-étale site of $\mathcal{S}h^{\mathrm{tor}}_{K,L}$, compatible with filtrations and the log connection. We say that the triple $(V_{\mathrm{dR}}, \nabla, \mathrm{Fil}^\bullet)$ is a filtered integrable connection. By GAGA [Con06], V_{dR} defines a filtered integrable connection over the algebraic Shimura variety $\mathrm{Sh}^{\mathrm{tor}}_{K,L}$.

By definition, the formation $V \rightsquigarrow V_{\text{két}}$ preserves the monoidal structure of $\mathrm{Rep}_L(V)$ so that the same holds true for $V \rightsquigarrow V_{\mathrm{dR}}$. By the Tannakian formalism [DM82], there is a $\mathbf{G}^c_L$-torsor $\mathbf{G}^c_{\mathrm{dR}}$ over $\mathrm{Sh}^{\mathrm{tor}}_{K,L}$ parametrizing all the vector bundles V_{dR}. By [DLLZ18, Theo. 5.3.1], the Hodge cocharacter μ defines the decreasing filtration of V_{dR}, applying the Tannakian formalism again we obtain a right $\mathbf{P}^{\mathrm{std},c}_\mu$ torsor $\mathbf{P}^{\mathrm{std},c}_{\mu,\mathrm{dR}}$ over $\mathrm{Sh}^{\mathrm{tor}}_{K,L}$, as well as an inclusion of torsors $\mathbf{P}^{\mathrm{std},c}_{\mu,\mathrm{dR}} \to \mathbf{G}^c_{\mathrm{dR}}$. In particular, we have a $\mathbf{G}^c_L$-equivariant morphism $\pi_{\mathrm{dR}} : \mathbf{G}^c_{\mathrm{dR}} \to \mathrm{FL}^{\mathrm{std}}_L$. Let $\mathbf{M}^c_\mu \subset \mathbf{P}^{\mathrm{std},c}_\mu$ be the Levi factor, we define the modular torsor $\mathbf{M}^c_{\mu,\mathrm{dR}}$ to be the pushout $\mathbf{P}^{\mathrm{std},c}_{\mu,\mathrm{dR}} \times^{\mathbf{P}^{\mathrm{std},c}_\mu} \mathbf{M}^c_\mu$. We denote by $\mathcal{G}^c_{\mathrm{dR}}$, $\mathcal{P}^{\mathrm{std},c}_{\mu,\mathrm{dR}}$ and $\mathcal{M}^c_{\mu,\mathrm{dR}}$ the analytification of the torsors seen as spaces over $\mathcal{S}h^{\mathrm{tor}}_{K,L}$.

Remark 22.1.1. It will be convenient for us to see $\mathbf{M}^c_{\mu,\mathrm{dR}}$ as a left $\mathbf{M}^c_\mu$-torsor. This can be easily done by defining the left action $m \cdot x := xm^{-1}$ for $m \in \mathbf{M}^c_\mu$ and $x \in \mathbf{M}^c_{\mu,\mathrm{dR}}$.

Definition 22.1.2. Let $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}_L$ be a maximal torus and a Borel subgroup respectively. Suppose that $\mathbf{T} \subset \mathbf{M}_\mu$ and $\mathbf{B} \subset \mathbf{P}_\mu$, we let $\mathbf{B}_{\mathbf{M}} = \mathbf{B} \cap \mathbf{M}_\mu$ denote the Borel subgroup of the Levi. Let w_0 and $w_{0,\mathbf{M}}$ be the longest elements of the Weyl group of $\mathbf{G}^c_L$ and $\mathbf{M}^c_\mu$ respectively. Let $\kappa \in X^*(\mathbf{T}^c)^+_{\mathbf{M}}$ be a dominant weight for $\mathbf{M}^c_\mu$, and let W_κ denote the irreducible representation of highest weight κ. We define the automorphic sheaf over $\mathcal{S}h_{K^pK_p,L}$ of weight κ to be

$$\mathcal{M}(\kappa) := \mathbf{M}^c_{\mu,\mathrm{dR}} \times^{\mathbf{M}^c_\mu} W^\vee_{-w_0(\kappa)},$$

where we see $\mathbf{M}^c_\mu$ as a left torsor.

Remark 22.1.3. Let us explain the convention of the automorphic sheaves in terms of the functions of the torsor. Let $t : \mathbf{M}^c_{\mu,\mathrm{dR}} \to \mathrm{Sh}_{K,L}$ be the natural projection of the torsor. If $\mathbf{M}^c_{\mu,\mathrm{dR}}$ is seen as a left torsor, the sheaf $\mathcal{M}(\kappa)$ is given as (see Remark 21.2.7)

$$\mathcal{M}(\kappa) = t_* \mathscr{O}_{\mathbf{M}^c_{\mu,\mathrm{dR}}}[-w_0(\kappa)_{\star_1}].$$

In other words, its sections in affine subschemes $U \subset \mathrm{Sh}_{K,L}$ are equal to

$$\mathcal{M}(\kappa)(U) = \{f : t^{-1}(U) \to \mathbb{A}^1 \mid f(b^{-1}m) = w_0(\kappa)(b^{-1})f(m) \text{ for all } b \in \mathbf{B}^c_{\mathbf{M}_\mu}\}.$$

Equivalently, if we see $\mathbf{M}^c_{\mu,\mathrm{dR}}$ as a right torsor we have

$$\mathcal{M}(\kappa)(U) = \{f : t^{-1}(U) \to \mathbb{A}^1 \mid f(mb) = w_0(\kappa)(b^{-1})f(m) \text{ for all } b \in \mathbf{B}^c_{\mathbf{M}_\mu}\}.$$

In the notation of [BP21, §4.1.1] we have $\mathcal{M}(\kappa) = \mathcal{V}^\vee_{-w_0(\kappa)}$.

22.1.1. Infinite level Shimura varieties

Let $\mathcal{S}h_{K^p,L} = \varprojlim_{K_p \subset \mathbf{G}(\mathbb{Q}_p)} \mathcal{S}h_{K^pK_p,L}$ be the infinite level Shimura variety. Let us fix $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ a compact open subgroup and $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ a smooth projective toroidal compactification as in the previous section. Let $\mathcal{S}h^{\mathrm{tor}}_{K^p,L} = \varprojlim_{K'_p \subset K_p} \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ be the infinite level toroidal Shimura variety. We highlight

that, while $\mathcal{S}h_{K^p,L}$ has an action of $\mathbf{G}(\mathbb{Q}_p)$, the toridal compactification $\mathcal{S}h^{\mathrm{tor}}_{K^p,L}$ only has an action of K_p. Nevertheless, the projective system $\mathcal{S}h^{\mathrm{tor},\infty}_{K^p,L} := \{\mathcal{S}h^{\mathrm{tor},\Sigma}_{K^p,L}\}_\Sigma$ of all the toroidal compactifications (induced by cone decompositions Σ) has a natural action of $\mathbf{G}(\mathbb{Q}_p)$ via Hecke correspondances.

In [Sch15], Scholze proved that for Hodge-type Shimura varieties, the inverse limit $\mathcal{S}h^*_{K^p,L} := \varprojlim_{K_p} \mathcal{S}h^*_{K^pK_p,L}$ of the minimal compactifications of the Shimura varieties has a natural structure of a perfectoid space[1]. In [PS16], Pilloni an Stroh have shown that the same holds true for Siegel varieties and the toroidal compactifications, using [Lan19] one deduces the same result for Shimura varieties of Hodge type. Furthermore, in the situation of Hodge-type Shimura varieties, Caraiani-Scholze [CS17] have constructed a $\mathbf{G}(\mathbb{Q}_p)$-equivariant Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{S}h_{K^p,L} \to \mathscr{F}\ell$$

which is affine in a precise sense (see [Sch15, Theo. III.3.18] and Proposition 22.1.8 down below). They also proved that pullbacks of $\mathbf{G}$-equivariant vector bundles of $\mathscr{F}\ell$ induced from the Levi $\mathbf{M}^c_\mu$ are naturally isomorphic to $\mathbf{G}(\mathbb{Q}_p)$-equivariant vector bundles of $\mathcal{S}h_{K^p,L}$ obtained from $\mathbf{M}^c_{\mu,\mathrm{dR}}$, see [CS17, Prop. 2.3.9]. As is pointed out in [BP21, §4.4], for general Shimura varieties one can construct a Hodge-Tate period map of diamonds over $\mathrm{Spd}\, L$

$$\pi_{\mathrm{HT}} : \mathcal{S}h^{\mathrm{tor},\diamond}_{K^p,L} \to \mathscr{F}\ell^\diamond \tag{22.1.2}$$

which pullbacks $\mathbf{G}$-equivariant vector bundles of $\mathscr{F}\ell$ to K_p-equivariant $\widehat{\mathscr{O}}$-vector bundles over $\mathcal{S}h^{\mathrm{tor}}_{K^p,L,\mathrm{prok\acute{e}t}}$.

Let us explain how (22.1.2) is defined. Let $V \in \mathrm{Rep}_L \mathbf{G}^c$ be a finite dimensional representation, $V_{\mathrm{k\acute{e}t}}$ its Kummer étale local system over $\mathcal{S}h^{\mathrm{tor}}_{K,L}$ and $(V_{\mathrm{dR}}, \nabla, \mathrm{Fil}^\bullet)$ its associated filtered integral connexion via the p-adic Riemman-Hilbert correspondance. We have a natural isomorphism

$$V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \mathscr{O}\mathbb{B}_{\mathrm{dR,log}} = V_{\mathrm{dR}} \otimes_{\mathscr{O}} \mathscr{O}\mathbb{B}_{\mathrm{dR,log}}$$

compatible with the filtrations and the log-connexion. This isomorphism defines two $\mathbb{B}^+_{\mathrm{dR}}$ lattices in $V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \mathbb{B}_{\mathrm{dR}}$, namely, $\mathbb{M} = V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \mathbb{B}^+_{\mathrm{dR}}$ and $\mathbb{M}_0 = (V_{\mathrm{dR}} \otimes_{\mathscr{O}} \mathscr{O}\mathbb{B}^+_{\mathrm{dR,log}})^{\nabla=0}$. We define $\mathrm{Fil}^i \mathbb{M} := \xi^i \mathbb{M}$ and $\mathrm{Fil}^i \mathbb{M}_0 := \xi^i \mathbb{M}_0$ for $i \in \mathbb{Z}$, where ξ is a local generator of $\theta : \mathbb{B}^+_{\mathrm{dR}} \to \widehat{\mathscr{O}}$. These two lattices are related in the following way:

Proposition 22.1.4 ([Sch13a, Prop. 7.9]). *The lattices $\mathbb{M}$ and $\mathbb{M}_0$ satisfy*

$$(\mathbb{M} \cap \mathrm{Fil}^i \mathbb{M}_0)/(\mathbb{M} \cap \mathrm{Fil}^{i+1} \mathbb{M}_0) = \mathrm{Fil}^{-i} V_{\mathrm{dR}} \otimes_{\mathscr{O}_{\mathcal{S}h^{\mathrm{tor}}_{K,L}}} \widehat{\mathscr{O}}_{\mathcal{S}h^{\mathrm{tor}}_{K,L}}$$

for all $i \in \mathbb{Z}$.

With this two filtrations we can define the increasing Hodge-Tate filtration of $V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \widehat{\mathscr{O}}_{\mathcal{S}h^{\mathrm{tor}}_{K,L}}$ to be:

$$\mathrm{Fil}_j(V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \widehat{\mathscr{O}}_{\mathcal{S}h^{\mathrm{tor}}_{K,L}}) = (\mathbb{M} \cap \mathrm{Fil}^{-j} \mathbb{M}_0)/(\mathrm{Fil}^1 \mathbb{M} \cap \mathrm{Fil}^{-j} \mathbb{M}_0) \tag{22.1.3}$$

with graded pieces

$$\mathrm{gr}_j(V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \widehat{\mathscr{O}}_{\mathcal{S}h^{\mathrm{tor}}_{K,L}}) = \mathrm{gr}^j(V_{\mathrm{dR}}) \otimes_{\mathscr{O}} \widehat{\mathscr{O}}(-j). \tag{22.1.4}$$

Let $\mathbf{G}^c_{\mathrm{HT}}$ be the K_p-equivariant $\mathbf{G}^c_L$-torsor over $\mathcal{S}h^{\mathrm{tor},\diamond}_{K^p,L}$ obtained by the formation of $V \rightsquigarrow V_{\mathrm{k\acute{e}t}} \otimes_{\widehat{L}} \widehat{\mathscr{O}}$. By the functoriality of the p-adic Riemman-Hilbert correspondance, the formation of the Hodge-Tate filtration is compatible with the Tannakian formalism, and we have defined a K_p-equivariant $\mathbf{P}^c_\mu$-torsor $\mathbf{P}^c_{\mu,\mathrm{HT}}$ over $\mathcal{S}h^{\mathrm{tor},\diamond}_{K^p,L}$ endowed with an equivariant map to $\mathbf{G}^c_{\mathrm{HT}}$. The inclusion $\mathbf{P}^c_{\mu,\mathrm{HT}} \subset \mathbf{G}^c_{\mathrm{HT}}$ defines a period map $\mathbf{G}^c_{\mathrm{HT}} \to \mathscr{F}\ell^\diamond$. But $\mathcal{S}h^{\mathrm{tor},\diamond}_{K^p,L}$ has a natural K_p-equivariant section to $\mathbf{G}^c_{\mathrm{HT}}$ as it already trivializes all the local systems $V_{\mathrm{k\acute{e}t}}$. One obtains the following corollary, which is a consequence of [DLLZ18, Theo. 5.3.1]

[1] To be precise, he proved that the image of $\mathcal{S}h^*_{K^pK_p,L}$ in a perfectoid Siegel variety is strongly Zariski closed, so perfectoid. However, by [HJ20, Prop. 5.14] the former is also represented by a perfectoid space.

Corollary 22.1.5. *The Hodge-Tate filtration of $\mathbf{G}^c_{\mathrm{HT}}$ induces a K_p-equivariant Hodge-Tate period map of diamonds over* $\operatorname{Spd} L$

$$\pi_{\mathrm{HT}} : \mathcal{S}h^{\mathrm{tor},\diamond}_{K^p,L} \to \mathscr{F}\ell^{\diamond}$$

satisfying the following properties:

1. *The pullback $\mathbf{M}^c_{\mu,\mathrm{HT}}$ of the $\mathbf{M}^c_\mu$-torsor $\mathbf{N}^c_\mu\backslash\mathbf{G}^c \to \mathscr{F}\ell$ via π_{HT} is canonically isomorphic to a Tate twist of the pullback of $\mathbf{M}^{c,}_{\mu,\mathrm{dR}}$ to $\mathcal{S}h^{\mathrm{tor},\diamond}_{K^p,L}$. More precisely, let $\pi_{K_p} : \mathcal{S}h^{\mathrm{tor}}_{K^p,L} \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ be the projection and $\mu : \mathbb{G}_m \to \mathbf{G}_L$ the Hodge cocharacter, then*[2]

$$\mathbf{M}^c_{\mu,\mathrm{HT}} = \pi^*_{K_p}(\mathbf{M}^{c,\diamond}_{\mu,\mathrm{dR}}) \times^{\mu} \widehat{\mathbb{Z}}_p(-1)^{\times}. \tag{22.1.5}$$

   Given a $\mathbf{G}^c$-equivariant vector bundle $\mathcal{W}$ over $\mathscr{F}\ell$, we will confuse $\pi^*_{\mathrm{HT}}(\mathcal{W})$ with the $\widehat{\mathscr{O}}$-vector bundle over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\mathrm{prok\acute{e}t}}$ that it defines.

2. *The formation of π_{HT} is compatible with Hecke correspondances at p. In other words, the maps π_{HT} glue to a $\mathbf{G}(\mathbb{Q}_p)$-equivariant map*

$$\pi_{\mathrm{HT}} : \mathcal{S}h^{\mathrm{tor},\infty,\diamond}_{K^p,L} \to \mathscr{F}\ell^{\diamond}\,.$$

Remark 22.1.6. In the previous construction of π_{HT}, we have considered vector bundles over diamonds, a concept which is not very clear as diamonds are quotients of perfectoid spaces in characteristic p. Nevertheless, if X is a diamond over $\operatorname{Spd} L$, the structural map $X \to \operatorname{Spd} L$ defines an untilt of X over $\operatorname{Spa} L$, and in particular a structural sheaf in the proétale site: $X^\sharp = (X_{\mathrm{pro\acute{e}t}}, \widehat{\mathscr{O}}_{X^\sharp})$. Then, for an algebraic group $\mathbf{H}$ over L, a $\mathbf{H}$ torsor over $X^\diamond/\operatorname{Spd} L$ is a functor $V \mapsto \mathcal{V}$ from algebraic representations of $\mathbf{H}$ to $\widehat{\mathscr{O}}_{X^\sharp}$-vector bundles over $X^\sharp$ preserving duals and tensor products.

Notice that, since π_{HT} is a map of diamonds over $\operatorname{Spd} L$, we have an induced morphism in the untilts:

$$\pi_{\mathrm{HT}} : (\mathcal{S}h^{\mathrm{tor}}_{K^p,L,\mathrm{prok\acute{e}t}}, \widehat{\mathscr{O}}) \to (\mathscr{F}\ell_{\mathrm{an}}, \mathscr{O}_{\mathscr{F}\ell}),$$

which pullbacks the $\mathbf{M}^c_\mu$-torsor $\mathbf{M}^c_{\mu,\mathrm{HT}}$ to (a Tate twist of) the torsor $\mathbf{M}_{\mu,\mathrm{dR}}$. From now on, when we mention the π_{HT} morphism we always refer to this last map of ringed sites, unless otherwise specified.

Remark 22.1.7. We explain in more detail how the Tate twist (22.1.5) translates in the automorphic vector bundles. Let $\kappa \in X^*(\mathbf{T}^c)^+_{\mathbf{M}^c_\mu}$ be a positive weight for $\mathbf{M}^c_\mu$, and let $\mathcal{W}(\kappa)$ and $\mathcal{M}(\kappa)$ be the associated vector bundles of weight κ over $\mathscr{F}\ell$ and $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ (see Definitions 21.2.6 and 22.1.2). First, forgetting the Galois action, we have K_p-equivariant isomorphisms $\pi^*_{\mathrm{HT}}(\mathcal{W}(\kappa)) = \pi^*_{K_p}(\mathcal{M}(\kappa))$ preserving tensor products and duals[3]. To find the correct Tate twist one can argue as follows: let $\lambda \in X^*(\mathbf{T}^c)^+$ be a dominant weight for $\mathbf{G}^c_L$ and V_λ the irreducible representation of highest weight λ. The Hodge-Tate filtration of $V_{\lambda,\mathrm{k\acute{e}t}}\otimes\widehat{\mathscr{O}}$ is induced from the $\mathbf{P}^c$-filtration of V_λ by taking the corresponding $\mathbf{G}$-equivariant vector bundles over $\mathscr{F}\ell$ and their pullbacks via π_{HT}. Moreover, by (22.1.3) the Hodge-Tate filtration is concentrated in degrees $[-\lambda(\mu), -w_0(\lambda)(\mu)]$ since the Hodge filtration of $V_{\lambda,\mathrm{dR}}$ is concentrated in degrees $[w_0(\lambda)(\mu), \lambda(\mu)]$ by definition. The representation V_λ admits W_λ as subrepresentation. Hence, since $\mathcal{W}(w_0^{\mathbf{M}}(\lambda)) = \mathbf{N}^c_\mu\backslash(\mathbf{G}^c \times W_\lambda)$, one has that

$$\pi^*_{\mathrm{HT}}(\mathcal{W}(w_0^{\mathbf{M}}(\lambda))) = \mathrm{Fil}_{-\lambda(\mu)}(V_{\mathrm{k\acute{e}t}} \otimes \widehat{\mathscr{O}}) = \mathcal{M}(w_0^{\mathbf{M}}(\lambda)) \otimes \widehat{\mathscr{O}}(\lambda(\mu)).$$

Therefore, for all $\kappa \in X^*(\mathbf{T}^c)^+_{\mathbf{M}^c_\mu}$ we have

$$\pi^*_{\mathrm{HT}}(\mathcal{W}(\kappa)) = \mathcal{M}(\kappa) \otimes \widehat{\mathscr{O}}(w_0(\kappa)(\mu)).$$

[2] Let G be a group acting on the left on two sheaves X and Y, by definition $X \times^G Y = G\backslash(X \times Y)$.

[3] This is why the convention taken in this paper and the convention of [BP21] differ, cf. Remark 22.1.3.

Following [BP21], affiness of the Hodge-Tate period map provides some vanishing theorems for the cohomology with partial supports of overconvergent automorphic sheaves. We will use the following proposition in Chapter 23.

Proposition 22.1.8 ([BP21, Prop. 4.4.53]). *Let $(\mathbf{G}, X)$ be an abelian type Shimura datum (eg. of Hodge type). Let $\mathcal{S}h^*_{K^p,L} = \varprojlim_{K'_p \subset K_p} \mathcal{S}h^*_{K^pK'_p,L}$ be the minimal compactification of the perfectoid Shimura variety. Then the Hodge-Tate period map $\pi_{HT} : \mathcal{S}h^{\mathrm{tor},\Diamond}_{K^p,L} \to \mathscr{F}\ell^\Diamond$ factors through $\mathcal{S}h^{*,\Diamond}_{K^p,L}$. Furthermore, there exists an affinoid cover $\{V_i\}_{i\in I}$ of $\mathscr{F}\ell$ such that, for each $i \in I$, there exists a level K_p and an affinoid $U_i \subset \mathcal{S}h^*_{K^pK_p,L}$ satisfying*

$$\pi_{\mathrm{HT}}^{-1}(V_i) = \pi_{K_p}^{-1}(U_i).$$

22.2. Classical BGG decompositions of Shimura varieties

Let $C = \mathbb{C}_p$ be the p-adic completion of an algebraic closure of L. Let $\mathfrak{g} = \mathrm{Lie}\,\mathbf{G}_{\mathbb{Q}_p}$, $\widetilde{\mathfrak{g}} = \mathrm{Lie}\,\widetilde{K}_p$ and $\mathfrak{g}^c = \mathrm{Lie}\,\mathbf{G}_{\mathbb{Q}_p}$. Let $\widetilde{\mathfrak{z}} := \ker(\widetilde{\mathfrak{g}} \to \mathfrak{g}^c)$, recall that, by the discussion at the beginning of §22.1, the obstruction for $\widetilde{\mathfrak{z}}$ being 0 depends on the closure of $\mathbf{Z}_c(\mathbb{Q})$ in $\mathbf{Z}_c(\mathbb{Q}_p)$ being open. For simplicity in the exposition, we will assume in the rest of the chapter that $\widetilde{\mathfrak{z}} = 0$, the proofs and the main theorems will hold in the general situation under some minor changes, only Theorem 22.3.18 requires an extra argument which we will explain in §22.3.1.

From now on we shall write $\mathbf{P}$, $\mathbf{N}$ and $\mathbf{M}$ for the groups $\mathbf{P}^c_\mu$, $\mathbf{N}^c_\mu$ and $\mathbf{M}^c_\mu$ respectively. We will use the same group theoretical conventions of §21, namely, we let $\mathfrak{p}$, $\mathfrak{n}$ and $\mathfrak{m}$ denote the Lie algebras of $\mathbf{P}$, $\mathbf{N}$ and $\mathbf{M}$ respectively. We will denote by $\overline{\mathbf{P}}$ and $\overline{\mathbf{N}}$ the opposite parabolic and its unipotent radical, and by $\overline{\mathfrak{p}}$ and $\overline{\mathfrak{n}}$ their Lie algebras. We fix $\mathbf{B} \subset \mathbf{P}$ a Borel subgroup and $\mathbf{T} \subset \mathbf{B}$ a maximal torus contained in $\mathbf{M}$, let $\mathbf{N}^+$ be the unipotent radical of $\mathbf{B}$. We write $\mathfrak{b}$, $\mathfrak{n}^+$ and $\mathfrak{h}$ for the Lie algebras of $\mathbf{B}$, $\mathbf{N}^+$ and $\mathbf{T}$ respectively. We also let $\mathscr{O}^{(+)}$ (resp. $\widehat{\mathscr{O}}^{(+)}$) denote the uncompleted (resp. completed) structural sheaves over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\mathrm{prokét}}$. We write $\Omega^1(\log)$ for the log differentials of $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ (which are preserved by pullbacks of Kummer-étale maps), and let $\theta_{Sh} : \Omega^1(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \mathfrak{g}^c_{\mathrm{két}} \otimes \widehat{\mathscr{O}}$ be the Sen operator.

22.2.1. The Sen bundle of a Shimura variety

In this paragraph we compute the Sen operator of a Shimura variety. Let $\mathfrak{n}^0 \subset \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}^c$ be the subbundle given by $\mathfrak{n}^0 = \mathcal{G}_L \times^{\mathcal{P}} \mathfrak{n}$, in other words, the subbundle whose fiber at $x \in \mathscr{F}\ell$ is $\mathrm{Lie}\,\mathcal{N}_x = \mathrm{Lie}\, x^{-1}\mathcal{N}x$. We also define $\mathfrak{p}^0 = \mathcal{G}_L \times^{\mathcal{P}} \mathfrak{p}$ and $\mathfrak{m}^0 = \mathfrak{p}^0/\mathfrak{n}^0$. Let $\pi_{\mathrm{HT}} : \mathcal{S}h^{\mathrm{tor}}_{K^p,L} \to \mathscr{F}\ell_L$ be the Hodge-Tate period map and $\pi_{K_p} : \mathcal{S}h^{\mathrm{tor}}_{K^p,L} \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ the natural projection.

Theorem 22.2.1. *The pullback by π_{HT} of $\mathfrak{n}^0 \subset \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{g}^c$ is naturally isomorphic to the pullback by π_{K_p} of the Sen operator $\theta_{Sh} : \Omega^1(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \mathfrak{g}^c_{\mathrm{két}} \otimes \widehat{\mathscr{O}}$, for any $K_p \subset \mathbf{G}(\mathbb{Q}_p)$.*

The strategy to prove this theorem is to construct the Faltings extension via π_{HT}. More precisely, we will show the following

Theorem 22.2.2. *Let $\overline{C}^1 = \overline{\mathbf{P}}\backslash\overline{\mathbf{P}}\mathbf{P} = \overline{\mathbf{P}}\backslash\overline{\mathbf{P}}\mathbf{N}$ be the big cell of $\mathrm{FL}^{\mathrm{std}}$ and $\mathscr{O}(\overline{C}^1)$ its ring of regular functions seen as a $\mathbf{P}$-module. , cf. Remark 21.3.21 for a more explicit description of this representation. Let $\underline{\mathscr{O}(\overline{C}^1)}$ denote the associated $\mathbf{G}^c$-equivariant quasi-coherent sheaf over $\mathscr{F}\ell$. There is a natural isomorphism of sheaves over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\mathrm{prokét}}$*

$$\pi^*_{\mathrm{HT}}\left(\underline{\mathscr{O}(\overline{C}^1)}\right) = \mathscr{O}\mathbb{C}_{\log}$$

Proof. Let $\mathfrak{g}^c_{\text{két}}$ be the Kummer-étale local system attached to the adjoint representation, and $\mathfrak{g}^c_{\text{dR}}$ its associated filtered integrable connexion. The Hodge filtration of $\mathfrak{g}^c_{\text{dR}}$ is concentrated in degrees $[-1, 1]$ by (SV1). Moreover, by (22.1.4) we know that

$$\begin{aligned} \mathrm{gr}^{-1}(\mathfrak{g}^c_{\text{dR}}) \otimes \widehat{\mathscr{O}}(1) &= \pi^*_{\text{HT}}(\mathfrak{n}^0), \\ \mathrm{gr}^{0}(\mathfrak{g}^c_{\text{dR}}) \otimes \widehat{\mathscr{O}} &= \pi^*_{\text{HT}}(\mathfrak{m}^0), \\ \mathrm{gr}^{1}(\mathfrak{g}^c_{\text{dR}}) \otimes \widehat{\mathscr{O}}(-1) &= \pi^*_{\text{HT}}(\mathfrak{n}^{0,\vee}), \end{aligned}$$

where in the last equality we have used the Killing form to identify $\mathfrak{g}^c/\mathfrak{p}$ with $\mathfrak{n}^\vee$. Let $\mathbb{M} = \mathfrak{g}^c_{\text{két}} \otimes \mathbb{B}^+_{\text{dR}}$ and $\mathbb{M}_0 = (\mathfrak{g}_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}})^{\nabla=0}$, and consider the exact sequence

$$0 \to \mathbb{M} \cap \mathbb{M}_0 \to \mathrm{Fil}^0(\mathfrak{g}^c_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}}) \to \mathrm{Fil}^0(\mathfrak{g}^c_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}} \otimes \Omega^1(\log)) \to \cdots.$$

Taking 0-th graded pieces we obtain a short exact sequence

$$0 \to \frac{\mathbb{M} \cap \mathbb{M}_0}{\mathrm{Fil}^1\, \mathbb{M} \cap \mathbb{M}_0} \to \mathrm{gr}^0(\mathfrak{g}^c_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}}) \to \mathrm{gr}^0(\mathfrak{g}^c_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}} \otimes \Omega^1(\log)) \to 0.$$

By Proposition 22.1.4 and the Hodge-Tate filtration (22.1.3), one finds that

$$\begin{aligned} \frac{\mathbb{M} \cap \mathbb{M}_0}{\mathrm{Fil}^1\, \mathbb{M} \cap \mathbb{M}_0} &= \pi^*_{\text{HT}}(\mathfrak{p}^0) \\ \mathrm{gr}^0(\mathfrak{g}^c_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}}) &= \mathrm{gr}^{-1}(\mathfrak{g}^c_{\text{dR}}) \otimes \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\text{dR,log}} \oplus \mathrm{gr}^0(\mathfrak{g}^c_{\text{dR}}) \otimes \widehat{\mathscr{O}} \\ &= \pi^*_{\text{HT}}(\mathfrak{n}^0) \otimes \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\text{dR,log}}(-1) \oplus \pi^*_{\text{HT}}(\mathfrak{m}^0) \\ \mathrm{gr}^0(\mathfrak{g}^c_{\text{dR}} \otimes \mathscr{O}\mathbb{B}^+_{\text{dR,log}} \otimes \Omega^1(\log)) &= \mathrm{gr}^{-1}(\mathfrak{g}^c_{\text{dR}}) \otimes \widehat{\mathscr{O}} \otimes \Omega^1(\log) \\ &= \pi^*_{\text{HT}}(\mathfrak{n}^0) \otimes \widehat{\mathscr{O}}(-1) \otimes \Omega^1(\log). \end{aligned}$$

Therefore, we obtain a short exact sequence

$$0 \to \pi^*_{\text{HT}}(\mathfrak{p}^0) \to \pi^*_{\text{HT}}(\mathfrak{n}^0)(-1) \otimes \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\text{dR,log}} \oplus \pi^*_{\text{HT}}(\mathfrak{m}^0) \xrightarrow{\overline{\nabla} \oplus \mathrm{KS}} \pi^*_{\text{HT}}(\mathfrak{n}^0)(-1) \otimes \Omega^1(\log) \to 0,$$

where $\overline{\nabla}$ is the tensor of $\overline{\nabla} : \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\text{dR,log}} \to \Omega^1(\log) \otimes \widehat{\mathscr{O}}$ with $\pi^*_{\text{HT}}(\mathfrak{n}^0)(-1)$, and KS is the $\widehat{\mathscr{O}}$-extension of scalars of the Kodaira-Spencer map

$$\mathrm{KS} : \mathrm{gr}^0(\mathfrak{g}^c_{\text{dR}}) \to \mathrm{gr}^{-1}(\mathfrak{g}^c_{\text{dR}}) \otimes \Omega^1(\log).$$

This extension defines a class $\eta \in \mathrm{Ext}^1_{\widehat{\mathscr{O}}}(\pi^*_{\text{HT}}(\mathfrak{n}^0)(-1) \otimes \Omega^1(\log), \pi^*_{\text{HT}}(\mathfrak{p}^0))$. Tensoring with the identity of $\pi^*_{\text{HT}}(\mathfrak{n}^{0,\vee})$ we obtain a class $\eta \otimes \mathrm{id}_{\mathfrak{n}^{0,\vee}} \in \mathrm{Ext}^1_{\widehat{\mathscr{O}}}(\pi^*_{\text{HT}}(\mathfrak{n}^0 \otimes \mathfrak{n}^{0,\vee})(-1) \otimes \Omega^1(\log), \pi^*_{\text{HT}}(\mathfrak{p}^0 \otimes \mathfrak{n}^{0,\vee}))$. We see $\mathfrak{n}^\vee$ as a subspace of $\mathscr{O}(\overline{C}^1)$ via the exponential map $\exp : \mathfrak{n} \to \mathbf{N}$, and the identification $\overline{C}^1 \cong \mathbf{N}$. Let $\mathscr{O}(\overline{C}^1)^{\leq 1}$ be the space of polynomials of degree ≤ 1. We have $\mathbf{P}$-equivariant maps $1 \hookrightarrow \mathfrak{n} \otimes \mathfrak{n}^\vee$, and $\mathfrak{n}^\vee \otimes \mathfrak{p} \twoheadrightarrow \mathscr{O}(\overline{C}^1)^{\leq 1}$, where the first is the dual of the trace map $\mathfrak{n} \otimes \mathfrak{n}^\vee \to 1$, and the second is given by derivations of $\mathfrak{p}$. Taking pushout and pullback diagrams of $\eta \otimes \mathrm{id}_{\mathfrak{n}^{0,\vee}}$, one obtains a class $\tilde{\eta} \in \mathrm{Ext}^1_{\widehat{\mathscr{O}}}\left(\widehat{\mathscr{O}}(-1) \otimes \Omega^1(\log), \pi^*_{\text{HT}}\left(\underline{\mathscr{O}(\overline{C}^1)^{\leq 1}}\right)\right)$. Following the explicit pushout and pullback extensions, one finds that $\tilde{\eta}$ is the extension

$$0 \to \pi^*_{\text{HT}}\left(\underline{\mathscr{O}(\overline{C}^1)^{\leq 1}}\right) \xrightarrow{(\alpha,\beta)} \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\text{dR,log}}(-1) \oplus \pi^*_{\text{HT}}(\mathfrak{n}^{0,\vee}) \xrightarrow{(\overline{\nabla}, \widetilde{\mathrm{KS}})} \Omega^1(\log) \otimes \widehat{\mathscr{O}}(-1) \to 0,$$

where $\widetilde{\mathrm{KS}}$ is the $\widehat{\mathscr{O}}(-1)$-extension of scalars of the Kodaira-Spencer isomorphism

$$\widetilde{\mathrm{KS}} : \mathrm{gr}^1(\mathfrak{g}^c_{\text{dR}}) \xrightarrow{\sim} \Omega^1(\log).$$

This implies that the map α is an isomorphism. Taking symmetric powers and direct limits one gets that

$$\begin{aligned}\pi_{\mathrm{HT}}^*\left(\underline{\mathscr{O}(\overline{C}^1)}\right) &= \varinjlim_n \mathrm{Sym}^n \pi_{\mathrm{HT}}^*\left(\underline{\mathscr{O}(\overline{C}^1)^{\leq 1}}\right)\\ &= \varinjlim_n \mathrm{Sym}^n \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(-1)\\ &= \mathscr{O}\mathbb{C}_{\log}\end{aligned}$$

as wanted. □

Remark 22.2.3. The previous proof works in the general situation where $\widetilde{X} \to X$ is a pro-Kummer-étale G^0-torsor, with $G^0 \subset \mathbf{G}(\mathbb{Q}_p)$ an open compact subgroup of the $\mathbb{Q}_p$-points of a reductive group, under the following conditions:

1. The G^0-torsor $\widetilde{X} \to X$ is de Rham, and the Hodge-Tate filtration is induced by a minuscule cocharacter $\mu : \mathbb{G}_m \to \mathbf{G}_{\mathbb{C}_p}$.
2. The Kodaira-Spencer map $\mathrm{KS} : \pi_{\mathrm{HT}}^*(\mathfrak{n}^{0,\vee}) \to \Omega^1(\log) \otimes \widehat{\mathscr{O}}(-1)$ is an isomorphism.

Proof of Theorem 22.2.1. Let $V \in \mathrm{Rep}_L \mathbf{G}^c$, since $\mathrm{gr}_n(V_{\text{két}} \otimes \widehat{\mathscr{O}}) = \mathrm{gr}^n(V_{\mathrm{dR}}) \otimes \widehat{\mathscr{O}}(-n)$ for all $n \in \mathbb{Z}$, the geometric Sen action of the graded pieces $\mathrm{gr}_n(V_{\text{két}} \otimes \widehat{\mathscr{O}})$ must be trivial. Hence, it sends $\mathrm{Fil}_{i+1}(V_{\text{két}} \otimes \widehat{\mathscr{O}})$ to $\mathrm{Fil}_i(V_{\text{két}} \otimes \widehat{\mathscr{O}})$. By the Tannakian formalism, this implies that the Sen operator factors through

$$\theta_{Sh} : \Omega^1(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \pi_{\mathrm{HT}}^*(\mathfrak{n}^0) \subset \mathfrak{g}^c_{\text{két}} \otimes \widehat{\mathscr{O}}.$$

But $\pi_{\mathrm{HT}}^*\left(\underline{\mathscr{O}(\overline{C}^1)}\right) = \mathscr{O}\mathbb{C}_{\log}$ by Theorem 22.2.2, and we know that the Sen action on the Faltings extension

$$0 \to \widehat{\mathscr{O}} \to \mathrm{gr}^1 \mathscr{O}\mathbb{B}^+_{\mathrm{dR},\log}(-1) \to \widehat{\mathscr{O}}(-1) \otimes \Omega^1(\log) \to 0$$

maps $\widehat{\mathscr{O}}$ to 0 and factors through the trace map $\Omega^1(\log)^\vee \otimes \Omega^1(\log) \to \mathscr{O}$. On the other hand, we have a short exact sequence

$$0 \to L \to \mathscr{O}(\overline{C}^1)^{\leq 1} \to \mathfrak{n}^\vee \to 0,$$

and the action of $\mathfrak{n}$ by derivations on $\mathscr{O}(\overline{C}^1)^{\leq 1}$ kills L, and factors through the trace map $\mathfrak{n} \otimes \mathfrak{n}^\vee \to L \subset \mathscr{O}(\overline{C}^1)^{\leq 1}$. Since $\mathrm{gr}^1(\mathscr{O}\mathbb{B}_{\mathrm{dR},\log})(-1) = \pi_{\mathrm{HT}}^*\left(\underline{\mathscr{O}(\overline{C}^1)^{\leq 1}}\right)$, the Sen map induces an isomorphism

$$\theta_{Sh} : \Omega^1(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \pi_{\mathrm{HT}}^*(\mathfrak{n}^0)$$

as wanted. □

Remark 22.2.4. A careful proofreading of the constructions shows that the dual of $\theta_{Sh} : \Omega^1(\log)^\vee \otimes \widehat{\mathscr{O}}(1) \to \pi_{\mathrm{HT}}^*(\mathfrak{n}^0)$ is the composition of $\pi_{\mathrm{HT}}^*(\mathfrak{n}^{0,\vee}) = \mathrm{gr}_1(\mathfrak{g}^c_{\text{két}} \otimes \widehat{\mathscr{O}}) \tilde{\to} \mathrm{gr}^1(\mathfrak{g}^c_{\mathrm{dR}}) \otimes \widehat{\mathscr{O}}(-1)$, and the $\widehat{\mathscr{O}}(-1)$-extension of scalars of the Kodaira-Spencer isomorphism $\widetilde{\mathrm{KS}} : \mathrm{gr}^1(\mathfrak{g}^c_{\mathrm{dR}}) \tilde{\to} \Omega^1(\log)$.

We have shown that the condition (BUN) of §20.2.2 holds in our situation, so we can refer to θ_{Sh} as the Sen bundle. One obtains the following corollary from Theorem 20.2.4

Corollary 22.2.5. *Let $\nu_{K_p} : \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\text{prokét}} \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\text{két}}$ and $\lambda : \mathcal{S}h^{\mathrm{tor}}_{K^p,C,\text{két}} \to \mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$ be the projection of sites. Let $\nu_{\infty,*} = \varinjlim_{K_p} \nu_{K_p,*}$ be the colimit of the pushforwards of finite levels. We have a natural Galois equivariant isomorphism of sheaves over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$*

$$R(\lambda \circ \nu_\infty)_*(C^{la}(\mathfrak{g}^c, \widehat{\mathscr{O}})) = \mathscr{O}^{la}_{\mathcal{S}h},$$

where $\mathscr{O}^{la}_{\mathcal{S}h}$ is the subsheaf of locally analytic sections of the completed structural sheaf at infinite level, see Definition 17.3.8.

22.2.2. Classical Hodge-Tate decompositions via π_{HT}

We keep the notation of §21, namely, for $\lambda \in X^*(\mathbf{T})^+$ (resp. $\lambda \in X^*(\mathbf{T})^+_{\mathbf{M}}$) we let V_λ (resp. W_λ) denote the irreducible representation of $\mathbf{G}$ (resp. of $\mathbf{M}$) of highest weight λ. Let $\mathcal{W}(\kappa) = \mathbf{G}^c_L \times^{\mathbf{P}} W_{w_0^{\mathbf{M}}(\kappa)}$ and $\mathcal{M}(\kappa) = \mathbf{M}^c_{\mu,\mathrm{dR}} \times^{\mathbf{M}} W_{w_0^{\mathbf{M}}(\kappa)}$ denote the associated vector bundles of weight κ over $\mathscr{F}\ell_L$ and $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ respectively. Let $\pi_{\mathrm{HT}} : \mathcal{S}h^{\mathrm{tor}}_{K^p,L} \to \mathscr{F}\ell$ be the Hodge-Tate period map, we shall confuse $\widetilde{K}_p$-equivariant sheaves over $\mathcal{S}h^{\mathrm{tor}}_{K^p,L,\mathrm{prokét}}$ with the pro-Kummer-étale sheaf it defines over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$.

The goal of this section is to describe the Hodge-Tate decompositions of the local systems $V^\vee_{\lambda,\mathrm{két}}$, more precisely, we will prove the following theorem:

Theorem 22.2.6. *Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight, and $C_{w_0^{\mathbf{M}}} = \mathbf{P}\backslash\mathbf{P}w_0^{\mathbf{M}}\mathbf{N} \subset \mathrm{FL}$ be the big cell of the flag variety. Let $\mathrm{BGG}^\vee(\lambda)$ the dual BGG complex of $V^\vee_\lambda$ as $(\mathfrak{g}^c, \mathbf{P})$-module, see Definition 21.3.18. By Corollary 21.3.22 we have an isomorphism of $\mathbf{P}$-representations*

$$\mathrm{BGG}^\vee(\lambda) = [W_{-w_0(\lambda)} \otimes \mathscr{O}(\overline{C}^1) \to \cdots \to \bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k}} W_{w\cdot(-w_0(\lambda))} \otimes \mathscr{O}(\overline{C}^1) \to \cdots \to W_{w_0^{\mathbf{M}}\cdot(-w_0(\lambda))} \otimes \mathscr{O}(\overline{C}^1)]. \tag{22.2.1}$$

Let $\underline{\mathrm{BGG}^\vee(\lambda)}$ be the associated $\mathbf{G}^c$-equivariant quasi-coherent complex over $\mathscr{F}\ell$. The following hold:

1. *We have a quasi-isomorphism $V^\vee_{\lambda,\mathrm{két}} \otimes \widehat{\mathscr{O}}[0] \simeq \pi^*_{HT}(\underline{\mathrm{BGG}^\vee(\lambda)})$ of complexes over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\mathrm{prokét}}$. The $(d-k)$-th term of $\pi^*_{HT}(\underline{\mathrm{BGG}^\vee(\lambda)})$ is of the form*

$$\bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k}} \mathcal{M}(-w_0(w\cdot\lambda)) \otimes \omega \otimes_{\mathscr{O}} \mathscr{O}\mathbb{C}_{\log}(w\cdot\lambda(\mu^{-1}) - d).$$

   where $\omega = \Omega^d(\log)$. Furthermore, let $\lambda \circ \nu_{K_p} : \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{prokét}} \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{an}}$ be the projection of sites, then

$$R(\lambda\circ\nu_{K_p})_*(V^\vee_{\lambda,\mathrm{két}} \otimes \widehat{\mathscr{O}}) = \bigoplus_{w\in{}^{\mathbf{M}}W} \mathcal{M}(-w_0(w\cdot\lambda)) \otimes \omega \otimes \mathbb{Z}_p(w\cdot\lambda(\mu^{-1}) - d)[\ell(w) - d].$$

   By taking cohomology over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{an}}$, and using the primitive comparison theorem, one gets

$$R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^pK_p,C}, V^\vee_{\lambda,\mathrm{ét}})\otimes_L C = \bigoplus_{w\in{}^{\mathbf{M}}W} R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega)\otimes C(w\cdot\lambda(\mu^{-1})-d)[\ell(w)-d].$$

2. *We have a quasi-isomorphism $j_!V^\vee_{\lambda,\mathrm{ét}}\widehat{\otimes}\widehat{\mathscr{O}}[0] \simeq j_!\widehat{\mathbb{Z}}_p\widehat{\otimes}\pi^*_{\mathrm{HT}}(\underline{\mathrm{BGG}^\vee(\lambda)})$. Furthermore,*

$$R(\lambda\circ\nu_{K_p})_*(j_!V^\vee_{\lambda,\mathrm{ét}}\widehat{\otimes}\widehat{\mathscr{O}}) = \bigoplus_{w\in{}^{\mathbf{M}}W} \mathcal{M}(-w_0(w\cdot\lambda)) \otimes \omega_{\mathrm{cusp}} \otimes \mathbb{Z}_p(w\cdot\lambda(\mu^{-1}) - d)[\ell(w) - d],$$

   where $\omega_{\mathrm{cusp}} = \Omega^d_{\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}}$. In particular, taking cohomology over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C,\mathrm{an}}$ one gets

$$R\Gamma_{\mathrm{proét},c}(\mathcal{S}h_{K^pK_p,C}, V^\vee_{\lambda,\mathrm{ét}})\otimes_L C = \bigoplus_{w\in{}^{\mathbf{M}}W} R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega_{\mathrm{cusp}})\otimes C(w\cdot\lambda(\mu^{-1})-d)[\ell(w)-d].$$

Proof of Theorem 22.2.6. By Theorem 22.2.2 we know that $\pi^*_{\mathrm{HT}}(\underline{\mathscr{O}(\overline{C}^1)}) = \mathscr{O}\mathbb{C}_{\log}$, and by Corollary 22.1.5 that $\pi^*_{\mathrm{HT}}(\mathcal{W}(\kappa)) = \mathcal{M}(\kappa) \otimes_{\mathscr{O}} \widehat{\mathscr{O}}(w_0(\kappa)(\mu))$. Moreover, $W_{w_{0,\mathbf{M}}ww_0\cdot(-w_0(\lambda))} = W^\vee_{w\cdot\lambda+2\rho^{\mathbf{M}}} = W^\vee_{w\cdot\lambda}\otimes\mathfrak{n}^\vee$. Then, the BGG complex can be written as

$$[W^\vee_{w_0^{\mathbf{M}}\cdot\lambda}\otimes\mathfrak{n}^\vee\otimes\mathscr{O}(\overline{C}^1) \to \cdots \to \bigoplus_{w\in{}^{\mathbf{M}}W\ell(w)=d-k} W^\vee_{w\cdot\lambda}\otimes\mathfrak{n}^\vee\otimes\mathscr{O}(\overline{C}^1) \to \cdots W^\vee_\lambda\otimes\mathfrak{n}^\vee\otimes\mathscr{O}(\overline{C}^1)].$$

Since $\pi_{\mathrm{HT}}^*(\mathfrak{n}^0) = \omega\otimes\widehat{\mathscr{O}}(-d)$ and $\mathcal{W}(\kappa) = \mathbf{G}^c\times^{\mathbf{P}} W^\vee_{-w_0(\kappa)}$, the previous equation shows that $\pi_{\mathrm{HT}}^*(\underline{\mathrm{BGG}^\vee(\lambda)})$ is of the shape described in the theorem. By the projection formula $R(\lambda\circ\nu_{K_p})_*(\mathscr{O}\mathbb{C}_{\log}) = \mathscr{O}_{\text{két}}$ one finds that $R(\lambda\circ\nu_{K_p})_*\pi_{\mathrm{HT}}^*(\underline{\mathrm{BGG}^\vee(\lambda)})$ has $(d-k)$-th term

$$\bigoplus_{\substack{w\in{}^{\mathbf{P}}W\\ \ell(w)=k}} \mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega\otimes\mathbb{Z}_p(w\cdot\lambda(\mu^{-1})-d).$$

It is left to see that the connecting map

$$\bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k}} \mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega\otimes\mathbb{Z}_p(w\cdot\lambda(\mu^{-1})-d)\to \bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k-1}} \mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega\otimes\mathbb{Z}_p(w\cdot\lambda(\mu^{-1})-d)$$

is zero. For this, it suffices to see that the composition

$$\bigoplus_{\substack{w\in{}^{\mathbf{M}}W\otimes\mathfrak{n}^{0,\vee}\\ \ell(w)=k}} \mathcal{W}(-w_0(w\cdot\lambda))\to(\underline{\mathrm{BGG}^\vee(\lambda)})^{d-k}\to(\underline{\mathrm{BGG}^\vee(\lambda)})^{d-k+1},$$

is zero, but it is the associated $\mathbf{G}^c$-equivariant morphism over $\mathscr{F}\ell$ of the $\mathbf{P}$-equivariant map

$$\bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k}} W^\vee_{w\cdot\lambda}\otimes\mathfrak{n}^\vee\to(\mathrm{BGG}^\vee(\lambda))^{d-k}\to(\mathrm{BGG}^\vee(\lambda))^{d-k+1}$$

which is 0. This finishes the proof of part (1).

For part (2), let $D=\bigcup_{a\in I} D_a$ be the decomposition of the cusps as union of their irreducible components. We can take our toroidal compactification in such a way that that any finite intersection of the D_a is smooth. For $J\subset I$ we set $D_J=\bigcap_{a\in J} D_a$, and denote $\iota_J: D_J\subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$. Consider the long exact sequence

$$0\to j_!\widehat{\mathbb{Z}}_p\to\widehat{\mathbb{Z}}_p\to\bigoplus_a \iota_{a,*}\widehat{\mathbb{Z}}_p\to\cdots\to\bigoplus_{|J|=l}\iota_{J,*}\widehat{\mathbb{Z}}_p\to\cdots\to\iota_{I,*}\widehat{\mathbb{Z}}_p\to 0$$

and its completed tensor with $\pi^*_{\mathrm{HT},K_p}(\underline{\mathrm{BGG}^\vee(\lambda)})$. It is enough to show that for all $J\subset I$ one has

$$R(\lambda\circ\nu_{K_p})_*(\iota_{J,*}\widehat{\mathscr{O}}_{D_J}\widehat{\otimes}_{\widehat{\mathscr{O}}}\pi^*_{\mathrm{HT}}(\underline{\mathrm{BGG}^\vee(\kappa)}))=\bigoplus_{w\in{}^{\mathbf{M}}W}\mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega\otimes_{\mathscr{O}_{\mathrm{an}}}\iota_{J,*}\mathscr{O}_{D_J,\mathrm{an}}(w\cdot\lambda(\mu^{-1})-d)[\ell(w)-d].$$

By Theorem 22.2.2, it is enough to prove that

$$R(\lambda\circ\nu_{K_p})_*(\iota_{J,*}\widehat{\mathscr{O}}_{D_J}\otimes_{\widehat{\mathscr{O}}}\mathscr{O}\mathbb{C}_{\log})=\iota_{J,*}\mathscr{O}_{D_J,\mathrm{an}},$$

but this follows from the projection formula [DLLZ18, Prop. 3.3.3]. □

Remark 22.2.7. The previous theorem applied to V_λ tells us that

$$R(\lambda\circ\nu_{K_p})_*(V_{\lambda,\text{ét}}\otimes\widehat{\mathscr{O}})=\bigoplus_{w\in{}^{\mathbf{M}}W}\mathcal{M}^\vee(-w_0(w\cdot\lambda))\otimes\mathbb{Z}_p(w\cdot\lambda(\mu))[-\ell(w)]. \tag{22.2.2}$$

Indeed, one has $V_\lambda = V^\vee_{-w_0(\lambda)}$, so that the dual BGG complex for V_λ is isomorphic to

$$[W_\lambda\otimes\mathscr{O}(\overline{C}^1)\to\cdots\to\bigoplus_{\substack{w\in{}^{\mathbf{M}}W\\ \ell(w)=k}}W_{w\cdot\lambda}\otimes\mathscr{O}(\overline{C}^1)\to\cdots\to W_{w_0^{\mathbf{M}}\cdot\lambda}\otimes\mathscr{O}(\overline{C}^1)].$$

We can write $W_{w\cdot\lambda}=W_{-w_0(-w_0(w\cdot\lambda))}$. Recall that $\mathcal{W}(\kappa)$ the VB over $\mathscr{F}\ell$ attached to $W^\vee_{-w_0(\kappa)}$, thus the VB attached to $W_{w\cdot\lambda}$ is $\mathcal{W}^\vee(-w_0(w\cdot\lambda))$. But Corollary 22.1.5 implies that

$$\pi^*_{\mathrm{HT}}(\mathcal{W}^\vee(-w_0(w\cdot\lambda)))=\mathcal{M}^\vee(-w_0(w\cdot\lambda))\otimes\mathscr{O}\mathbb{C}_{\log}(w\cdot\lambda(\mu)),$$

taking projections to the analytic site one obtains (22.2.2).

22.3. Locally analytic vectors of completed cohomology

We let $j : \mathcal{S}h_{K^pK_p,L} \subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ denote the open immersion of Shimura varieties. In this section we make explicit the relation between the locally analytic vectors of the completed cohomology, and the sheaf $\mathscr{O}^{la}_{Sh}$. First, let us recall the definition of Emerton's completed cohomology, we follow [Eme06].

Definition 22.3.1. Let $i \in \mathbb{Z}$ and Λ a p-adically complete ring. We define the i-th completed cohomology group of $\{\mathrm{Sh}_{K^pK_p,C}\}_{K_p}$ to be

$$\widetilde{H}^i(K^p, \Lambda) := \varprojlim_s \varinjlim_{K_p} H^i_{\text{ét}}(\mathrm{Sh}_{K^pK_p,C}, \Lambda/p^s\Lambda).$$

We also define the completed cohomology with compact supports[4]

$$\widetilde{H}^i_c(K^p, \Lambda) := \varprojlim_s \varinjlim_{K_p} H^i_{\text{ét},c}(\mathrm{Sh}_{K^pK_p,C}, \Lambda/p^s).$$

One can compute the completed cohomology with Λ/p^s-coefficients using the infinite level toroidal compactification as follows.

Proposition 22.3.2. *We have*

$$\widetilde{H}^i(K^p, \Lambda/p^s) = H^i_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^p}, \Lambda/p^s)$$
$$\widetilde{H}^i_c(K^p, \Lambda/p^s) = H^i_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^p}, j_!\Lambda/p^s).$$

Proof. By, GAGA for étale cohomology with torsion coefficients [Hub96], putiry of torsion local systems [DLLZ19, Theo. 4.6.1], and Lemma 4.5.3 of *loc. cit.* we have

$$R\Gamma_{\text{ét}}(\mathrm{Sh}_{K^pK_p,C}, \Lambda/p^s) = R\Gamma_{\text{két}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \Lambda/p^s)$$
$$R\Gamma_{\text{ét},c}(\mathrm{Sh}_{K^pK_p,C}, \Lambda/p^s) = R\Gamma_{\text{két}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, j_!\Lambda/p^s).$$

On the other hand, by Lemma 3.16 of [Sch13a] we have that

$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \Lambda/p^s) = \varinjlim_{K_p} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \Lambda/p^s)$$
$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, j_!\Lambda/p^s) = \varinjlim_{K_p} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, j_!\Lambda/p^s).$$

This proves the proposition. □

We want to show that the completed cohomology with $\mathbb{Z}_p$ coefficients can also be computed as the pro-Kummer-étale cohomology with $\widehat{\mathbb{Z}}_p$-coefficients of $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$. Since $\widehat{\mathbb{Z}}_p = R\varprojlim_s \mathbb{Z}/p^s$ and $j_!\widehat{\mathbb{Z}}_p = R\varprojlim_s j_!\mathbb{Z}/p^s$, one has a short exact sequence

$$0 \to R^1\varprojlim_s(H^{i-1}_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \mathbb{Z}/p^s)) \to H^i_{\text{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \widehat{\mathbb{Z}}_p) \to \widetilde{H}^i(K^p, \mathbb{Z}_p) \to 0,$$

(resp. for cohomology with compact supports). Therefore, we only need to show that the $R^1\varprojlim_s$ appearing above vanishes. This is a consequence of admissibility of completed cohomology:

[4] The maps between Shimura varieties are finite, hence there are natural pullback maps between the cohomologies with compact support.

Theorem 22.3.3 (Emerton). *Let $s \in \mathbb{N}$ and Λ denote $\mathbb{Z}_p$ or $\mathbb{Z}/p^s$. The complexes*

$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\Lambda}) \text{ and } R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, j_!\widehat{\Lambda})$$

are represented by a bounded complex of admissible $\widetilde{K}_p$-representations with terms isomorphic to finitely many copies of $C(\widetilde{K}_p, \Lambda)$. In particular, $H^i_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\Lambda})$ and $H^i_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, j_!\widehat{\Lambda})$ are admissible $\Lambda[[\widetilde{K}_p]]$-modules.

Proof. Let us first show the case of torsion coefficients. By Shaphiro's lemma and a Hochshild-Serre spectral sequence, there is a natural quasi-isomorphism

$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \mathbb{Z}/p^s) = R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, C(\widetilde{K}_p, \mathbb{Z}/p^s)) \tag{22.3.1}$$

(resp. for $j_!\mathbb{Z}/p^s$). By purity of torsion local systems, and GAGA for étale cohomology, the RHS of 22.3.1 is equal to

$$R\Gamma_{\text{ét}}(\mathcal{S}h_{K^pK_p,C}, C(\widetilde{K}_p, \mathbb{Z}/p^s)) = R\Gamma_{\text{ét}}(\text{Sh}_{K^pK_p,C}, C(\widetilde{K}_p, \mathbb{Z}/p^s))$$

(resp. for $j_!$ and cohomology with compact supports). Fix an isomorphism $C \cong \mathbb{C}$, by Artin's comparison theorem [Art68] we have

$$R\Gamma_{\text{ét}}(\text{Sh}_{K^pK_p,C}, C(\widetilde{K}_p, \mathbb{Z}/p^s)) = R\Gamma_{\text{Betti}}(\text{Sh}_{K^pK_p}(\mathbb{C}_p), C(\widetilde{K}_p, \mathbb{Z}/p^s))$$

(resp. for cohomology with compact supports).

Let $\text{Sh}_{K^pK_p}(\mathbb{C})^{\text{BS}}$ be a Borel-Serre compactification of $\text{Sh}_{K^pK_p}(\mathbb{C})$ (cf. [BS73]), it is a compact CW complex which is homotopocally equivalent to $\text{Sh}_{K^pK_p}(\mathbb{C})$. Let $S_\bullet$ be a finite simplicial resolution of $\text{Sh}_{K^pK_p}(\mathbb{C})^{\text{BS}}$, then

$$\begin{aligned} R\Gamma_{\text{Betti}}(\text{Sh}_{K^pK_p}(\mathbb{C}), C(\widetilde{K}_p, \mathbb{Z}/p^s)) &= R\Gamma_{\text{Betti}}(\text{Sh}_{K^pK_p}(\mathbb{C})^{\text{BS}}, C(\widetilde{K}_p, \mathbb{Z}/p^s)) \\ &= \text{Hom}^\bullet(\mathbb{Z}[S_\bullet], C(\widetilde{K}_p, \mathbb{Z}/p^s)) \end{aligned}$$

is a bounded complex whose terms are finite direct sums of $C(\widetilde{K}_p, \mathbb{Z}/p^s)$, in particular a bounded complex of admissible $\mathbb{Z}/p^s[[\widetilde{K}_p]]$-modules. Taking derived limits as $s \to \infty$, one has

$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\mathbb{Z}}_p) = \text{Hom}^\bullet(\mathbb{Z}[S_\bullet], C(\widetilde{K}_p, \mathbb{Z}_p))$$

proving that it is a bounded complex of admissible $\mathbb{Z}_p[[\widetilde{K}_p]]$-modules.

It remains to prove that $R\Gamma_{\text{Betti},c}(\text{Sh}_{K^pK_p}(\mathbb{C}), C(\widetilde{K}_p, \mathbb{Z}/p^s))$ is admissible. By Poincaré duality, cohomology with compact supports with coefficients $C(\widetilde{K}_p, \mathbb{Z}/p^s)$ is dual to cohomology with coefficients $\mathbb{Z}/p^s[[\widetilde{K}_p]]$, hence

$$\begin{aligned} R\Gamma_{\text{Betti},c}(\text{Sh}_{K^pK_p}(\mathbb{C}), C(\widetilde{K}_p, \mathbb{Z}/p^s)) &= R\Gamma_{\text{Betti},c}(\text{Sh}_{K^pK_p}(\mathbb{C})^{\text{BS}}, C(\widetilde{K}_p, \mathbb{Z}/p^s)) \\ &= R\text{Hom}_{\mathbb{Z}/p^s}(\text{Hom}^\bullet(\mathbb{Z}[S_\bullet], \mathbb{Z}/p^s[[\widetilde{K}_p]]), \mathbb{Z}/p^s)[-2d] \\ &= \mathbb{Z}[S_\bullet] \otimes C(\widetilde{K}_p, \mathbb{Z}/p^s)[-2d], \end{aligned}$$

this shows that the completed cohomology complex with compact supports and $\mathbb{Z}/p^s$-coefficients is bounded and admissible. Taking derived inverse limits as $s \to \infty$ one finds that

$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, j_!\widehat{\mathbb{Z}}_p) = \mathbb{Z}[S_\bullet] \otimes C(\widetilde{K}_p, \mathbb{Z}_p)[-2d],$$

proving that it is a bounded complex of admissible $\mathbb{Z}_p[[\widetilde{K}_p]]$-modules. □

Corollary 22.3.4. *We have isomorphisms*

$$\begin{aligned} H^i_{\text{prokét}}(\mathcal{Sh}^{\text{tor}}_{K^p,C}, \widehat{\mathbb{Z}}_p) &= \widetilde{H}^i(K^p, \mathbb{Z}_p) \\ H^i_{\text{prokét}}(\mathcal{Sh}^{\text{tor}}_{K^p,C}, j_!\widehat{\mathbb{Z}}_p) &= \widetilde{H}^i_c(K^p, \mathbb{Z}_p). \end{aligned}$$

Proof. This is a consequence of Proposition 1.2.12 of [Eme06] knowing that the cohomology complexes are represented by bounded complexes of admissible $\mathbb{Z}_p[[\widetilde{K}_p]]$-modules. □

The admissibility of completed cohomology implies the vanishing of higher locally analytic vectors, cf. [ST03, Theo. 7.1], [Pan20, Theo. 2.2.3] and [RJRC21, Prop. 4.43].

Corollary 22.3.5. *Let $i \in \mathbb{Z}$, we have*

$$\begin{aligned} H^i(R\Gamma_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{R\widetilde{K}_p - la}) &= H^i_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{\widetilde{K}_p - la} \\ H^i(R\Gamma_{\text{proét},c}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{R\widetilde{K}_p - la}) &= H^i_{\text{proét},c}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{\widetilde{K}_p - la} \end{aligned}$$

Remark 22.3.6. Since the property of locally analycity is local on the group, the cohomology groups of the previous corollary are independent of $\widetilde{K}_p$. We call these complexes the locally analytic completed cohomologies and we denote them by $R\Gamma_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}$ and $R\Gamma_{\text{proét},c}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}$.

We also need a version of this theorem relating the dual of the completed cohomology.

Proposition 22.3.7. *Consider the dual completed cohomologies*

$$\begin{aligned} R\Gamma_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p)^\vee &:= R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p), \mathbb{Z}_p) \\ R\Gamma_{\text{proét},c}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p)^\vee &:= R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_{\text{proét},c}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p), \mathbb{Z}_p) \end{aligned} \tag{22.3.2}$$

The complexes (22.3.2) are represented by bounded complexes of coadmissible $\mathbb{Z}_p[[\widetilde{K}_p]]$-modules with finite free terms. In particular, the cohomology groups of (22.3.2) are coadmissible $\mathbb{Z}_p[[\widetilde{K}_p]]$-modules and are in duality as topological $\mathbb{Z}_p$-modules with the completed cohomology groups. In other words, if $H^{i,\vee}_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p)$ denotes the i-th cohomology group of $R\Gamma_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p)^\vee$, one has a perfect pairing

$$H^i_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \mathbb{Q}_p/\mathbb{Z}_p) \times H^{2d-i,\vee}_{\text{proét}}(\mathcal{Sh}_{K^p,C}, \widehat{\mathbb{Z}}_p) \to \mathbb{Q}_p/\mathbb{Z}_p$$

(resp. for the cohomology with compact supports).

Proof. The proposition follows from the proof of Theorem 22.3.3 and the equality

$$R\text{Hom}_{\mathbb{Z}_p}(C(\widetilde{K}_p, \mathbb{Z}_p), \mathbb{Z}_p) = \mathbb{Z}_p[[\widetilde{K}_p]].$$

This last equality follows from the fact that $C(\widetilde{K}_p, \mathbb{Z}_p)$ has a ON basis as a p-adically complete $\mathbb{Z}_p$-module and that

$$\begin{aligned} R\text{Hom}_{\mathbb{Z}_p}(\widehat{\bigoplus_i}\mathbb{Z}_p, \mathbb{Z}_p) &= R\varprojlim_s R\text{Hom}_{\mathbb{Z}_p}(\widehat{\bigoplus_i}\mathbb{Z}_p, \mathbb{Z}/p^s) \\ &= R\varprojlim_s R\text{Hom}_{\mathbb{Z}/p^s}(\bigoplus_i \mathbb{Z}/p^s, \mathbb{Z}/p^s) \\ &= R\varprojlim_s \prod_i \mathbb{Z}/p^s \\ &= \prod_i \mathbb{Z}_p. \end{aligned}$$

□

Remark 22.3.8. By Poincaré duality, the dual of the completed cohomology is equal to the inverse limit of the finite level cohomologies $R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \widehat{\mathbb{Z}}_p)$ via the traces induced by the finite étale maps $\mathcal{S}h_{K^pK'_p,C} \to \mathcal{S}h_{K^pK_p,C}$ for $K'_p \subset K_p$ (resp. for the cohomology with compact supports).

Let us define $R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^\vee := R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Z}}_p)^\vee[\frac{1}{p}]$ (resp. for the cohomology with compact supports). One has the following corollary

Corollary 22.3.9. *Let $\mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p)$ denote the locally analytic distribution algebra of $\widetilde{K}_p$. We have*

$$H^i(\mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p) \otimes^L_{\mathbb{Q}_p[[\widetilde{K}_p]]} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^\vee) = \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p[[\widetilde{K}_p]]} H^{i,\vee}_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)$$
$$H^i(\mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p) \otimes^L_{\mathbb{Q}_p[[\widetilde{K}_p]]} R\Gamma_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^\vee) = \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p[[\widetilde{K}_p]]} H^{i,\vee}_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)$$

Proof. This follows from the previous proposition and the fact that $\mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p)$ is flat over $\mathbb{Q}_p[[\widetilde{K}_p]] := \mathbb{Z}_p[[\widetilde{K}_p]][\frac{1}{p}]$, cf. [ST03, Theo. 5.2]. □

Definition 22.3.10. The complexes of the previous corollary are called the dual locally analytic completed cohomologies. We denote them as $R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}$ and $R\Gamma_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}$, and their cohomology groups are denoted as $H^*_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}$ and $H^*_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}$ respectively.

Corollary 22.3.11. *We have a duality between locally analytic and dual locally analytic completed cohomologies*

$$(H^i_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la})^\vee = H^{2d-i}_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}$$
$$(H^i_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la})^\vee = H^{2d-i}_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}.$$

Our next goal is to relate the locally analytic vectors of completed cohomology with the sheaf of locally analytic vectors at infinite level. In order to do this, let us first relate the C-extension of scalars of the completed cohomology with the proétale cohomology of $\widehat{\mathscr{O}}$. The following is essentially Theorem IV.2.1 of [Sch15].

Proposition 22.3.12. *There are natural almost quasi-isomorphisms*

$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Z}}_p)\widehat{\otimes}\mathcal{O}_C =^{ae} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\mathscr{O}}^+)$$
$$R\Gamma_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Z}}_p)\widehat{\otimes}\mathcal{O}_C =^{ae} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, j_!\widehat{\mathbb{Z}}_p\widehat{\otimes}\widehat{\mathscr{O}}^+).$$

Remark 22.3.13. If $\mathcal{S}h^{\text{tor}}_{K^p,C}$ is perfectoid (e.g. for Shimura varieties of abelian type), the pro-Kummer-étale cohomology of the RHS terms of the proposition can be computed in the analytic site, and the restriction of $j_!\widehat{\mathbb{Z}}_p\widehat{\otimes}\widehat{\mathscr{O}}^+$ to the analytic site is the intersection of the ideal defining the boundary and $\widehat{\mathscr{O}}^+|_{\mathcal{S}h^{\text{tor}}_{K^p,C,\text{an}}}$.

Proof. Let $s \geq 1$, by purity of torsion local systems we have

$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \mathbb{Z}_p/p^s) = R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, \mathbb{Z}_p/p^s).$$

The primitive comparison theorem ([Sch13a, Theo. 5.1] and [DLLZ19, Theo. 6.2.1]) implies that

$$R\Gamma_{\text{két}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, \mathbb{Z}/p^s) \otimes \mathcal{O}_C =^{ae} R\Gamma_{\text{két}}(\mathcal{S}h^{\text{tor}}_{K^pK_p}, \mathscr{O}^+/p^s).$$

Taking inductive limits as $K_p \to 1$, and derived inverse limits as $s \to \infty$, one obtains the first almost equality.

For the cohomology with compact supports we argue as in Lemma 3.15 of [Sch13b]. Let $D_{K_p} \subset \mathcal{S}h^{\text{tor}}_{K^pK_p}, C$ denote the boundary divisor, and write $D_{K_p} = \bigcup_{a\in I} D_a$ as a union of irreducible divisors.

For $J \subset I$ we let $D_J := \bigcap_{a\in J} D_a$ endowed with the pullback log structure of $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$. Let $\iota_J : D_J \to \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$ denote the strict closed immersion. Then, we have a long exact sequence

$$0 \to j_!\mathbb{Z}/p^s \to \mathbb{Z}/p^s \to \bigoplus_{a\in I} \iota_a\iota_{a,*}\mathbb{Z}/p^s \to \cdots \to \bigoplus_{|J|=k} \iota_{J,*}\mathbb{Z}/p^s \to \cdots \to \iota_{I,*}\mathbb{Z}/p^s \to 0. \tag{22.3.3}$$

By Lemma 3.14 of [Sch13b] we have $\iota_{J,*}\mathbb{Z}/p^s \otimes \mathscr{O}^+/p^s = \iota_{J,*}\mathscr{O}^+_{D_J}/p^s$ for all $J \subset I$. Tensoring (22.3.3) with $\widehat{\mathscr{O}}^+$ and taking inverse limits as $s \to \infty$, we obtain a long exact sequence

$$0 \to j_!\widehat{\mathbb{Z}}_p\widehat{\otimes}\widehat{\mathscr{O}}^+ \to \widehat{\mathscr{O}}^+ \to \bigoplus_{a} \iota_{a,*}\widehat{\mathscr{O}}^+_{D_a} \to \cdots \to \iota_{I*}\widehat{\mathscr{O}}^+_{D_I} \to 0. \tag{22.3.4}$$

But for any $J \subset I$ we have

$$R\Gamma_{\mathrm{prokét}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \iota_{J,*}\widehat{\mathscr{O}}^+_{D_J}) = R\Gamma_{\mathrm{prokét}}(D_J, \widehat{\mathscr{O}}^+_J) =^{ae} R\Gamma_{\mathrm{prokét}}(D_J, \widehat{\mathbb{Z}}_p)\widehat{\otimes}\mathcal{O}_C,$$

where in the last equality we use the primitive comparison theorem as the D_J are proper log smooth over $\mathrm{Spa}(C, \mathcal{O}_C)$. The proposition for the cohomology with compact supports follows by taking cohomology of the sequence (22.3.4). □

Before stating the main theorem of this section, we need to define a sheaf of locally analytic distributions over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$.

Definition 22.3.14. We define the sheaf $\widetilde{\mathcal{D}}^{la}_{\mathcal{S}h}$ of locally analytic distributions over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$ to be the sheaf mapping a qcqs open $\widetilde{U}$ to

$$\widetilde{\mathcal{D}}^{la}_{\mathcal{S}h}(\widetilde{U}) = (1 \otimes_{U(\mathfrak{n}^0)} \mathcal{D}^{la}(\widetilde{K}_p, \mathscr{O}(\widetilde{U})))^{\widetilde{K}_{p,\star_{1,3}}},$$

where $\widetilde{U} = \pi^{-1}_{K_p}(U)$ is the pullback of a qcqs open subspace $U \subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$.

Remark 22.3.15. The previous sheaf is well defined, i.e. it is independent of K_p. Indeed, if $K'_p \subset K_p$ there is a natural isomorphism

$$(\mathcal{D}^{la}(\widetilde{K}'_p, \mathscr{O}(\widetilde{U})) \otimes_{U(\mathfrak{n}^0)} 1)^{\widetilde{K}'_{p,\star_{1,3}}} = (\mathcal{D}^{la}(\widetilde{K}_p, \mathscr{O}(\widetilde{U})) \otimes_{U(\mathfrak{n}^0)} 1)^{\widetilde{K}_{p,\star_{1,3}}}$$

given by the trace $f \mapsto \sum_{g\in\widetilde{K}_p/\widetilde{K}'_p} f \otimes g$.

Theorem 22.3.16. *Let $\mathscr{O}^{sm}_{\mathcal{S}h}$ be the uncompleted structural sheaf at infinite level and $\mathscr{I}^{sm} \subset \mathscr{O}^{sm}_{\mathcal{S}h}$ the ideal defining the boundary of $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$ for all K_p. There are natural quasi-isomorphisms for the locally analytic completed cohomology*

$$R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{O}^{la}_{\mathcal{S}h})$$
$$R\Gamma_{\mathrm{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{O}^{la}_{\mathcal{S}h}\widehat{\otimes}_{\mathscr{O}^{sm}_{\mathcal{S}h}}\mathscr{I}^{sm}).$$

Similarly, there are natural quasi-isomorphisms for the dual locally analytic completed cohomology

$$R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}\widehat{\otimes}C = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \widetilde{\mathcal{D}}^{la}_{\mathcal{S}h} \otimes \omega \otimes_{\mathscr{O}^{sm}_{\mathcal{S}h}} \mathscr{I}^{sm})[d]$$
$$R\Gamma_{\mathrm{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee}\widehat{\otimes}C = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \widetilde{\mathcal{D}}^{la}_{\mathcal{S}h} \otimes \omega)[d]$$

where $\omega = \det \Omega^1(\log)$.

Proof for the completed cohomology. Let us first show the non compact support case. By Shapiro's lemma one has

$$\begin{aligned} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la} &= R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, C^{la}(\widetilde{K}_p, \mathbb{Q}_p)) \\ &= \varinjlim_{K_p \to 1} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, C^{la}(\widetilde{K}_p, \mathbb{Q}_p)), \end{aligned}$$

where all the transition maps in the colimit are isomorphisms. Let $\nu_{K_p} : \mathcal{S}h^{\text{tor}}_{K^pK_p,C,\text{prokét}} \to \mathcal{S}h^{\text{tor}}_{K^pK_p,C,\text{két}}$ and $\lambda : \mathcal{S}h^{\text{tor}}_{K^p,C,\text{két}} \to \mathcal{S}h^{\text{tor}}_{K^p,C,\text{an}}$ be the projection of sites. Let us write $\nu_{\infty,*} = \varinjlim_{K_p} \nu_{K_p,*}$. By Corollary 22.2.5 one has that[5]

$$R(\lambda \circ \nu_\infty)_* C^{la}(\mathfrak{g}^c, \widehat{\mathscr{O}}) = \mathscr{O}^{la}_{\mathcal{S}h}.$$

The primitive comparison theorem implies that

$$\begin{aligned} (R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\mathbb{Q}}_p)\widehat{\otimes}C)^{la} &= \varinjlim_{K_p \to 1} R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, C^{la}(\widetilde{K}_p, \widehat{\mathscr{O}})) \\ &= \varinjlim_{K_p \to 1} R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, R(\lambda \circ \nu_{K_p})_*(C^{la}(\widetilde{K}_p, \widehat{\mathscr{O}}))) \\ &= R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R(\lambda \circ \nu_\infty)_*(C^{la}(\mathfrak{g}^c, \widehat{\mathscr{O}}))) \\ &= R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \mathscr{O}^{la}_{\mathcal{S}h}). \end{aligned}$$

This proves the case of non compact supports. For the case of compact supports, one argues as in Proposition 22.3.12 to reduce to the equalities

$$R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \widehat{\mathscr{O}}_{D_J})^{la} = R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, \mathscr{O}^{la}_{\mathcal{S}h} \otimes_{\mathscr{O}^{sm}_{\mathcal{S}h}} \mathscr{O}^{sm}_{D_J}), \tag{22.3.5}$$

where $\widehat{\mathscr{O}}^{sm}_{D_J}$ is the sheaf of smooth sections of $\widehat{\mathscr{O}}|_{\pi^{-1}_{K_p}(D_J)}$. But then, by Theorem 20.2.4 one has

$$R(\lambda \circ \nu_\infty)_*(C^{la}(\mathfrak{g}^c, \widehat{\mathscr{O}}_{D_J})) = \mathscr{O}^{la}_{\mathcal{S}h} \otimes_{\mathscr{O}^{sm}_{\mathcal{S}h}} \mathscr{O}^{sm}_{D_J},$$

taking analytic cohomology over $\mathcal{S}h^{\text{tor}}_{K^p,C}$ one gets (22.3.5). □

Proof for the dual completed cohomology. First, note that the proof of Theorem 22.3.3 implies

$$\begin{aligned} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee} &= R\Gamma_{\text{proét},c}(\mathcal{S}h_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p))[2d] \\ R\Gamma_{\text{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee} &= R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p))[2d]. \end{aligned}$$

The primitive comparison theorem gives

$$\begin{aligned} R\Gamma_{\text{proét},c}(\mathcal{S}h_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p))\widehat{\otimes}C &= R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, j_!\widehat{\mathbb{Z}}_p\widehat{\otimes}\mathcal{D}^{la}(\widetilde{K}_p, \widehat{\mathscr{O}})) \\ R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p))\widehat{\otimes}C &= R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \widehat{\mathscr{O}})). \end{aligned}$$

Writing $\mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p) = R\varprojlim_{h\to\infty} \mathcal{D}^{h^+-an}(\widetilde{K}_p, \mathbb{Q}_p)$ one finds that

$$R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p)) = R\varprojlim_{h\to\infty} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, \mathcal{D}^{h^+-an}(\widetilde{K}_p, \mathbb{Q}_p)).$$

(resp. for the cohomology with compact supports). But the sheaf $\mathcal{D}^{h^+-an}(\widetilde{K}_p, \widehat{\mathscr{O}})$ is the pullback via π_{HT} of a squarrable locally analytic LB-sheaf over $\mathscr{F}\ell$. By Theorem 20.2.2 we know that

$$\begin{aligned} R(\lambda \circ \nu_\infty)_*(\mathcal{D}^{h^+-an}(\widetilde{K}_p, \widehat{\mathscr{O}})) &= (R\Gamma(\mathfrak{n}^0, \mathcal{D}^{h^+-an}(\widetilde{K}_p, \widehat{\mathscr{O}}|_{\mathcal{S}h^{\text{tor}}_{K^p,C,\text{an}}})))^{\widetilde{K}_p-sm} \\ &= (1 \otimes_{U(\mathfrak{n}^0)} \mathcal{D}^{h^+-an}(\widetilde{K}_p, \widehat{\mathscr{O}}|_{\mathcal{S}h^{\text{tor}}_{K^p,C,\text{an}}}))^{\widetilde{K}_p-sm} \otimes \omega. \end{aligned}$$

[5] Recall that we have made the assumption Lie $\widetilde{K}_p = \mathfrak{g}^c$.

Let $U \subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$ be an affinoid subspace admitting coordinates to a product of tori and polydiscs, let $\widetilde{U}$ be its pullback to infinite level. Then

$$\begin{aligned} R\Gamma_{\mathrm{prokét}}(U, \mathcal{D}^{la}(\widetilde{K}_p, \widehat{\mathscr{O}})) &= \varprojlim_{h\to\infty} (1 \otimes_{U(\mathfrak{n}^0)} \mathcal{D}^{h^+-an}(\widetilde{K}_p, \widehat{\mathscr{O}}(\widetilde{U})))^{\widetilde{K}_p} \otimes \omega(U) \\ &= (1 \otimes_{U(\mathfrak{n}^0)} \mathcal{D}^{la}(\widetilde{K}_p, \widehat{\mathscr{O}}(\widetilde{U})))^{\widetilde{K}_p} \otimes \omega(U) \\ &= \widetilde{\mathcal{D}}^{la}_{\mathcal{S}h}(\widetilde{U}) \otimes \omega(U). \end{aligned}$$

Taking Čech cohomology one obtains that

$$R\Gamma_{\mathrm{proét}}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \mathcal{D}^{la}(\widetilde{K}_p, \widehat{\mathscr{O}})) = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \widetilde{\mathcal{D}}^{la}_{\mathcal{S}h} \otimes \omega),$$

proving the case of the dual locally analytic completed cohomology with compact supports. To prove the case of the dual locally analytic completed cohomology, one argues as for the locally analytic completed cohomology with compact supports, reducing to a computation in terms of the structural sheaf of the smooth intersections of the boundary divisor, where one can apply Theorem 20.2.2 again. □

Corollary 22.3.17. *The rational completed cohomologies $\widetilde{H}^i(K^p, \mathbb{Q}_p)$ and $\widetilde{H}^i_c(K^p, \mathbb{Q}_p)$ vanish for $i > d$.*

Proof. The argument is essentially the same of [Sch15, Theo. 5.3]. By Theorem 22.3.16, the locally analytic vectors of the C-scalar extension of the completed cohomology complexes can be computed as the analytic cohomology of sheaves over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$. But $|\mathcal{S}h^{\mathrm{tor}}_{K^p,C}|$ has cohomological dimension $\leq d$ being the inverse limit by qcqs maps of noetherian spaces of dimension $\leq d$. Then, the vanishing for the locally analytic completed cohomology follows by Grothendieck's bound for the cohomology of noetherian spaces. Finally, as the completed cohomology groups are admissible by Theorem 22.3.3, their locally analytic vectors are dense (dually, $\mathcal{D}^{la}(\widetilde{K}_p, \mathbb{Q}_p)$ is fully faithful over $\mathbb{Q}_p[[\widetilde{K}_p]]$), which implies the corollary. □

22.3.1. The arithmetic Sen operator via $\theta_{\mathfrak{m}}$

We end this chapter with the computation of the arithmetic Sen operator for $\mathscr{O}^{la}_{\mathcal{S}h}$, cf. [Pan20, §5.1]. Let $\widetilde{\mathfrak{g}} = \mathrm{Lie}\,\widetilde{K}_p$, we no longer assume that $\widetilde{\mathfrak{g}} = \mathfrak{g}^c$, we let $\widetilde{\mathfrak{z}} = \ker(\widetilde{\mathfrak{g}} \to \mathfrak{g}^c)$. Theorem 22.3.16 tells us that $R\Gamma_{\mathrm{prokét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathscr{O}})^{la} = R\Gamma_{\mathrm{an}}(\mathcal{S}h_{K^p,C}, \mathscr{O}^{la}_{\mathcal{S}h})$. Moreover, the action of $\widetilde{\mathfrak{g}}$ on the former term is computed via its action on the sheaf $\mathscr{O}^{la}_{\mathcal{S}h}$. By construction, the sheaf $\mathscr{O}^{la}_{\mathcal{S}h}$ also admits an action of the Lie algebra $\mathscr{O}_{\mathscr{H}\ell} \otimes \mathfrak{g}$ which is trivial when restricted to $\mathfrak{n}^0$. In particular, $\mathscr{O}^{la}_{\mathcal{S}h}$ carries an horizontal action of $\mathfrak{m}^0 = \mathfrak{p}^0/\mathfrak{n}^0$ and afortriori an action of $\mathcal{Z}(\mathfrak{m})$, this last extends to the locally analytic completed cohomology[6]. Let $\mu : \mathbb{G}_m \to \mathbf{G}$ be the Hodge cocharacter, since $\mathbf{M}$ is the centralizer of μ in $\mathbf{G}^c_L$, one has a natural operator $\theta_\mu \in \mathcal{Z}(\mathfrak{m})$ given as $\theta_\mu = \mathrm{Lie}\,\mu(1)$.

We have the following theorem

Theorem 22.3.18. *The cohomologies $R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C$ and $R\Gamma_{\mathrm{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C$ admit an arithmetic Sen operator θ_L. More precisely, the locally analytic cohomology is a relative locally analytic representation of $Gal_L = Gal(C/L)$ in the sense of Definition 19.2.5, whence it admits a C-linear Sen operator in the sense of Definition 19.4.3. Moreover, we have that $\theta_L = \theta_\mu$.*

Proof. Let us first show the existence of the arithmetic Sen operator. We only treat the case without compact supports, the other being similar. We have that

$$R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathscr{O}})^{la} = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{O}^{la}_{\mathcal{S}h}).$$

[6] This action arises as the $\theta_{\mathfrak{m},\star_2}$-action of $C^{la}(\widetilde{\mathfrak{g}}, \widehat{\mathscr{O}})$ which is equal to $-\theta_{\mathfrak{m},\star_1}$.

The LHS can be computed as the colimit along all the Čech complexes of the RHS with respect to hypercovers of $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$ by qcqs open subspaces. Hence, it suffices to show that locally on $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$ the sheaf $\mathscr{O}^{la}_{\mathcal{S}h}$ is a relative locally analytic representation of Gal_L. For this, note that in order to apply the geometric Sen theory formalism of §20, it is enough to extend scalars to L^{cyc}, namely, we need enough ramification for the Colmez-Sen-Tate axioms of §19.1 to hold. Therefore, if $\mathscr{O}^{la}_{\mathcal{S}h,L^{cyc}}$ denote the locally analytic sections of the completed structural sheaf restricted to $\mathcal{S}h^{tor}_{K^p,L^{cyc}}$, one has that $\mathscr{O}^{la}_{\mathcal{S}h,\mathrm{an}}(\widetilde{U}_C) = C\widehat{\otimes}_{L^{cyc}}\mathscr{O}^{la}_{\mathcal{S}h,L^{cyc}}(\widetilde{U})$ for $\widetilde{U} \subset \mathcal{S}h^{\mathrm{tor}}_{K^p,L^{cyc}}$ a qcqs open subspace. Then, it is enough to show that locally on $\mathcal{S}h^{\mathrm{tor}}_{K^p,L^{cyc}}$, the sheaf $\mathscr{O}^{la}_{\mathcal{S}h,L^{cyc}}$ is relative locally analytic for the action of $\Gamma_L = \mathrm{Gal}(L^{cyc}/L)$.

Let $U \subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ be an open affinoid which admits a chart $\psi : U \to \mathbb{S}_L^{(e,d-e)}$ to a product of tori and polydiscs. Let $\widetilde{U} \subset \mathcal{S}h^{\mathrm{tor}}_{K^p,L^{cyc}}$ denote the pullback of U to $\mathcal{S}h^{\mathrm{tor}}_{K^p,L^{cyc}}$, $U_\infty = U \times_{\mathbb{S}_L^{(e,d-e)}} \mathbb{S}_{L,\infty}^{(e,d-e)}$, and $\widetilde{U}_\infty = \widetilde{U} \times_U U_\infty$. Let us write $\widetilde{K}_p^\dagger = \{\widetilde{K}'_p\}_{\widetilde{K}'_p \subset \widetilde{K}_p}$ for the overconvergent neighbourhood of 1 in $\widetilde{K}_p$, and $\Gamma = \mathrm{Gal}(U_\infty/U)$. By definition, the Sen module attached to $C^{la}(\widetilde{\mathfrak{g}}, \mathbb{Q}_p)$ is the space $S_{\widetilde{\mathfrak{g}}}(C^{la}) := \widehat{\mathscr{O}}(\widetilde{U}_\infty)^{\widetilde{K}_p^\dagger - la, \Gamma - la}$ of $\widetilde{K}_p^\dagger \times \Gamma$-locally analytic functions of $\widehat{\mathscr{O}}(\widetilde{U}_\infty)$. Furthermore, Γ_L acts on Γ via multiplication by the cyclotomic character, which gives us an action of $\widetilde{K}_p^\dagger \times (\Gamma_L \ltimes \Gamma)$ on $S_{\widetilde{\mathfrak{g}}}(C^{la})$. But $S_{\widetilde{\mathfrak{g}}}(C^{la})$ admits a closed $\widetilde{K}_p^\dagger \times (\Gamma_L \ltimes \Gamma)$-equivariant L^{cyc}-semilinear closed embedding

$$S_{\widetilde{\mathfrak{g}}}(C^{la}) \hookrightarrow C^{la}(\widetilde{\mathfrak{g}} \times \Gamma, L^{cyc})\widehat{\otimes}_L V_0,$$

where V_0 is some LB space over L, and Γ_L acts as

$$(\sigma \star f)(g, \gamma) = \sigma(f(g, \sigma^{-1}(\gamma))).$$

Taking a Banach subspace $V_n \subset V_0$ and a radius of analyticity $h > 0$, the space $C^{h-an}(\widetilde{\mathfrak{g}} \times \Gamma, L^{cyc})\widehat{\otimes}V_n$ is Banach. It is clear that the reduction modulo p^ϵ of a lattice is isomorphic to $\bigoplus_I \mathcal{O}_{L^{cyc}}/p^\epsilon$ for some $\epsilon > 0$. Therefore, the devisage Lemma 19.3.5 holds and we can apply the decompletion given by Sen theory for the pair (L^{cyc}, Γ_L). In particular, there exists a C-linear arithmetic Sen operator θ_L acting on $\mathscr{O}^{la}_{\mathcal{S}h}$ as wanted.

It is left to show that $\theta_L = \theta_\mu$. Let $\widehat{\mathscr{O}}_{\mathcal{S}h}$ denote the restriction of the completed structural sheaf to $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$. Since the action of θ_L on $\mathscr{O}^{sm}_{\mathcal{S}h}$ is trivial (this sheaf is just the C-scalar extension of the finite level structural sheaves), the arithmetic Sen operator extends to a $\widehat{\mathscr{O}}_{\mathcal{S}h}$-linear derivation on $C^{la}(\widetilde{\mathfrak{g}}, \widehat{\mathscr{O}}_{\mathcal{S}h})^{\mathfrak{n}^0_{\star_{1,3}}=0} = \widehat{\mathscr{O}}_{\mathcal{S}h}\widehat{\otimes}_{\mathscr{O}^{sm}_{\mathcal{S}h}}\mathscr{O}^{la}_{\mathcal{S}h}$. In addition, θ_L is a right $\mathfrak{g}$-invariant derivation by functoriality of its construction. This implies that θ_L factors through a map of sheaves on $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L,\mathrm{prokét}}$

$$\theta_L : \widehat{L} \to \pi^*_{\mathrm{HT}}(\mathfrak{n}^0)\backslash(\widehat{\mathscr{O}} \otimes \widetilde{\mathfrak{g}}).$$

But $\pi^*_{\mathrm{HT}}(\mathfrak{n}^0)\backslash(\widehat{\mathscr{O}} \otimes \widetilde{\mathfrak{g}})$ fits in a short exact sequence

$$0 \to \pi^*_{HT}(\mathfrak{m}^0 \oplus \mathscr{O}_{\mathscr{F}\ell} \otimes \widetilde{\mathfrak{z}}) \to \pi^*_{\mathrm{HT}}(\mathfrak{n}^0)\backslash(\widehat{\mathscr{O}} \otimes \widetilde{\mathfrak{g}}) \to \pi^*_{\mathrm{HT}}(\mathfrak{n}^{0,\vee}) \to 0,$$

where $\pi^*_{HT}(\mathfrak{m}^0)$ has Hodge-Tate weight 0 and $\pi^*_{\mathrm{HT}}(\mathfrak{n}^{0,\vee})$ has Hodge-Tate weight 1. Besides, $\pi^*_{\mathrm{HT}}(\mathscr{O}_{\mathscr{F}\ell}\otimes\widetilde{\mathfrak{z}})$ has Hodge-Tate weight 0 since $\mathbf{G}$ acts trivially on $\widetilde{\mathfrak{z}}$.

This implies that θ_L factors through $\theta_L : \widehat{\mathbb{Q}}_p \to \pi^*_{\mathrm{HT}}(\mathfrak{m}^0 \oplus \mathscr{O}_{\mathscr{F}\ell} \otimes \mathfrak{z})$, this last acting on $\mathscr{O}^{la}_{\mathcal{S}h}$ via $\star_1$. Let Z_0 be a faithful representation of $\mathbf{G}_L/\mathbf{G}_L^{\mathrm{der}}$ whose action of $\mathfrak{g}$ factors through $\widetilde{\mathfrak{g}}$. Then Z_0 is a $\widetilde{K}_p$ -module for K_p small enough. Let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}}$, to compute the image of θ_L, it is enough to know the Hodge-Tate weight of $\pi^*_{\mathrm{HT}}(\mathcal{W}(\kappa)) \oplus Z_{0,\text{két}} \otimes \widehat{\mathscr{O}}$, let us first focus on $\mathcal{W}(\kappa)$. By Corollary 22.1.5 we know that

$$\pi^*_{\mathrm{HT}}\mathcal{W}(\kappa) = \mathcal{M}(w_0^{\mathbf{M}}(\kappa) \otimes \widehat{\mathscr{O}}(w_0(\kappa)(\mu)).$$

But $\theta_{\mathfrak{m},\star_1}$ acts on $\mathcal{W}(\kappa)$ via the left regular action on the sections. By definition, the sections of $\mathcal{W}(\kappa)$ are the functions $f : \mathbf{M}_{\mathscr{F}\ell} \to \mathbf{A}^1$ such that $f(b^{-1}x) = w_0(\kappa)(b^{-1})f(x)$ for $b \in \mathbf{B}_{\mathbf{M}}$. This implies that the action of θ_L on $\pi^*_{\mathrm{HT}}(\mathcal{W}(\kappa))$ is equal to $-\theta_{\mu,\star_1} = \theta_{\mu,\star_2} = \theta_\mu$.

To compute the Sen action of $Z_{0,\text{két}} \otimes \widehat{\mathscr{O}}$, by functoriality we can reduce to the case of a torus $\mathbf{T}$ after taking a special point of $(\mathbf{G}, X)$ as in [Del79, §2.2.4]. Let E be the reflex field and $\text{res}_E : \text{Gal}_E \to \mathbb{G}_m(\mathbb{A}_E^\infty)$ the arithmetic reciprocity map (i.e. res_E maps Frobenius of unramified primes over ℓ to uniformizers in the places dividing ℓ). In this case, the action of $\sigma \in \text{Gal}_L$ on

$$\mathcal{S}h_{K^p,C} = \mathbf{T}(\mathbb{A}_{\mathbb{Q}}^\infty)/K^p\overline{\mathbf{T}(\mathbb{Q})}$$

is by right multiplication of $N_{E/\mathbb{Q}}(\mu(\text{res}_E^{-1}(\sigma)))$, where $N_{E/\mathbb{Q}} : \mathbf{T}(\mathbb{A}_E^\infty) \to \mathbf{T}(\mathbb{A}_{\mathbb{Q}}^\infty)$ is the norm map, see §2.2.3 of *loc. cit.* Let $\mathfrak{p}$ be a place over p and $L = E_{\mathfrak{p}}$, the reciprocity map res_E is compatible with the local reciprocity map $\text{res}_L : \text{Gal}_L \to L^\times$, and the completed cohomology is nothing but the continuous functions of $|\mathcal{S}h_{K^p,C}|$ to $\mathbb{Q}_p$. Let $f : |\mathcal{S}h_{K^p,C}| \to C$ be a locally analytic function, and let $\sigma \in \text{Gal}_L$, we have that

$$\sigma(f)(x) = f(\sigma^{-1}(x)) = f(xN_{L/\mathbb{Q}_p}(\mu(\text{res}_L(\sigma)))).$$

Thus, if f satisfies $f(xt) = tf(x)$ for $t \in K^pK_p/(K^p, K^pK_p \cap \overline{\mathbf{T}(\mathbb{Q})}) = \widetilde{K}_p$, one has $\sigma(f) = N_{L/\mathbb{Q}_p}(\text{res}_L(\sigma))f$ for $\sigma \in \text{Gal}_L$ close enough to 1. But the representation $N_{L/\mathbb{Q}_p} \circ \text{res}_L : \text{Gal}_L \to \mathbb{Q}_p^\times$ has Hodge-Tate weight 1, this proves that $\theta_L = \theta_\mu$ as wanted. □

23. Overconvergent Hodge-Tate decompositions

This last chapter concerns one of the main applications of the theory we have developed so far, namely, the definition of the overconvergent BGG maps for the locally analytic completed cohomology. We start in §23.1 with the definition of sheaves of overconvergent automorphic forms, following essentially the same construction of [BP21]. We define cohomologies with closed and compact supports of these sheaves at infinite level, and relate these cohomology complexes with those of finite level appearing in higher Coleman theory. Then, in §23.2, using the computations of §21.4.3 we construct the overconvergent BGG maps for the derived $\mathfrak{b} = \lambda$ isotypic part of the locally analytic cohomology, generalizing in this way previous works of Andreatta-Iovita-Stevens [AIS15], Chojecki-Hansen-Johansson [CHJ17], Barrera-Gao [BG21], Diao-Rosso-Wu [DRW21], and the author [RC21a]. We prove that these overconvergent BGG maps are compatible with the classical BGG decompositions of §22.2.

Throughout this chapter we will assume for simplicity that $\mathrm{Lie}\,\widetilde{K}_p = \mathfrak{g}^c$, in the case this equality does not hold, the main theorems remain true under some minor modifications that we left to the careful reader.

23.1. Sheaves of overconvergent automorphic forms

The objective of this section is to introduce the sheaves of overconvergent automorphic forms appearing in the description of $\mathfrak{b}$-isotypic parts of the locally analytic completed cohomology. We will see that these sheaves include the sheaves of overconvergent automorphic forms of higher Coleman theory of [BP21]. We keep the notation of §21 and 22.

Let $\mu : \mathbf{G}_m \to \mathbf{G}_L$ be a fixed Hodge cocharacter and $\mathbf{P} = \mathbf{P}^c_\mu \subset \mathbf{G}^c_L$ the parabolic subgroup in $\mathbf{G}^c_L$ defined by μ. Let $\mathbf{M}$ be the centralizer of μ in $\mathbf{G}^c_L$ and $\mathbf{N}$ the unipotent radical of $\mathbf{P}$. Let $\mathrm{FL} = \mathbf{P}\backslash\mathbf{G}_L$ denote the flag variety over L and $\mathscr{F}\ell$ its analytification to an adic space over $\mathrm{Spa}(L, \mathcal{O}_L)$. Let $\mathcal{M}_{\mathscr{F}\ell} := \mathcal{N}\backslash\mathcal{G}^c_L \to \mathscr{F}\ell$ be the natural $\mathcal{M}$-torsor over the flag variety and $\mathcal{M}_{\mathrm{dR}} = \mathcal{M}^c_{\mu,\mathrm{dR}}$ the analytification of the automorphic $\mathbf{M}$-torsor over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$ parametrizing automorphic vector bundles. By Corollary 22.1.5 one has an isomorphism of $\widetilde{K}_p$-equivariant torsors over $\mathcal{S}h^{\mathrm{tor}}_{K^p,L}$

$$\pi^*_{\mathrm{HT}}(\mathcal{M}_{\mathscr{F}\ell}) = \pi^*_{K_p}(\mathcal{M}_{\mathrm{dR}}) \times^{\mu} \widehat{\mathbb{Z}}_p(-1)^{\times}. \tag{23.1.1}$$

The equality (23.1.1) translates $\mathbf{G}^c_L$-equivariant vector bundles over $\mathscr{F}\ell$ arising from finite dimensional representations of $\mathbf{M}$, to automorphic vector bundles over the Shimura variety. Indeed, let $\kappa \in X^*(\mathbf{T})^+_{\mathbf{M}}$ be a dominant weight for the Levi subgroup and W_κ the irreducible representation of $\mathbf{M}$ of highest weight κ, let $\mathcal{W}(\kappa) = \mathbf{G}^c_L \times^{\mathbf{P}} W_{w_0^{\mathbf{M}}(\kappa)}$ be the $\mathbf{G}^c_L$-equivariant sheaf over $\mathscr{F}\ell$ and $\mathcal{M}_{K_p}(\kappa) = \mathcal{M}_{\mathrm{dR}} \times^{\mathcal{M}} W_{w_0^{\mathbf{M}}(\kappa)}$ the automorphic VB over $\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,L}$. We have a natural $\widetilde{K}_p$-equivariant isomorphism of sheaves over $\mathcal{S}h^{\mathrm{tor}}_{K^p,L,an}$

$$\pi^*_{\mathrm{HT}}(\mathcal{W}(\kappa))^{\widetilde{K}_p^{\dagger}-sm} = \varinjlim_{K_p \to 1} \mathcal{M}_{K_p}(\kappa) \otimes \mathbb{Z}_p(w_0(\kappa)(\mu)). \tag{23.1.2}$$

The construction of overconvergent automorphic sheaves has as input the reduction of the torsor $\mathcal{M}_{dR}$ to a torsor over some overconvergent neighbourhood of 1 in $\mathcal{M}$. This reduction is provided by

p-adic Hodge theory, namely, in the case of modular curves or Siegel varieties, the reduction of the torsors are obtained via the Hodge Tate exact sequence of the universal abelian variety, cf. [AIS15] and [AIP15]. In general, the reductions of the automorphic torsor are constructed from the flag variety and the π_{HT} period map as is shown in the next section, see also [BP21, §4.6].

23.1.1. Construction of the overconvergent modular sheaves

Definition 23.1.1. Let $\mathcal{H}$ be an analytic group over a complete non archimedean field F and $\mathcal{H}^\dagger = \{\mathcal{H}_n\}_{n\in\mathbb{N}}$ be a decreasing family of open subgroups of $\mathcal{H}$. Let X be an adic space over F. A $\mathcal{H}^\dagger$-torsor over $X_?$ with $? \in \{\text{an}, \text{ét}, \text{két}\}$ is a decreasing sequence $\mathcal{H}_X^\dagger = \{\mathcal{H}_{X,n}\}$ of $\mathcal{H}_n$-torsors such that the maps $\mathcal{H}_{X,n+1} \to \mathcal{H}_{X,n}$ are $\mathcal{H}_{X,n+1}$-equivariant for all $n \in \mathbb{N}$.

Let $U \subset \mathscr{Fl}$ be an open subspace stable by $\widetilde{K}_p$, and let $\mathcal{M}^\dagger = \{\mathcal{M}(\gamma)\}_{\gamma>0}$ be a basis of overconvergent neighbourhoods of 1 in $\mathcal{M}$. Suppose that we are given with a reduction of $\mathcal{M}_{\mathscr{Fl}}|_U$ to a $\widetilde{K}_p^\dagger$-equivariant $\mathcal{M}^\dagger$-torsor $\mathcal{M}^\dagger_{\mathscr{Fl},U} = \{\mathcal{M}_{\mathscr{Fl},U}(\gamma)\}_{\gamma>0}$. In other words, we have a decreasing sequence of $\widetilde{K}_p(\gamma)$-equivariant torsors $\mathcal{M}_{\mathscr{Fl},U}(\gamma)$ with $\widetilde{K}_p(\gamma) \to 1$ as $\gamma \to \infty$. The overconvergent space $\mathcal{M}^\dagger_{\mathscr{Fl},U}$ defines a locally analytic $\widetilde{K}_p^\dagger$-equivariant *LB* $\mathscr{O}_U$-algebra which we denote as $C(\mathcal{M}^\dagger_{\mathscr{Fl},U})$; it is given as the colimit of all the algebras of functions $C(\mathcal{M}_{\mathscr{Fl},U}(\gamma))$ of the $\mathcal{M}(\gamma)$-torsors converging to $\mathcal{M}^\dagger_{\mathscr{Fl},U}$.

Definition 23.1.2. Let $(U, \mathcal{M}^\dagger_{\mathscr{Fl},U})$ be as above, and set $V = \pi_{\mathrm{HT}}^{-1}(U)$. Let $\mathscr{O}^{sm}_{\mathcal{S}h} = \varinjlim_{K_p\to 1} \mathscr{O}_{\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}}$,an be the sheaf of $\widetilde{K}_p$-smooth functions of $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$, and $\mathscr{O}^{sm}_V$ its restriction to V. The sheaf of overconvergent automorphic forms defined by $(U, \mathcal{M}^\dagger_{\mathscr{Fl},U})$ is the *LB* $\mathscr{O}^{sm}_V$-sheaf given as

$$C(\mathcal{M}^\dagger_{\mathrm{dR},V}) := \pi^*_{\mathrm{HT}}(C(\mathcal{M}^\dagger_{\mathscr{Fl},U}))^{\widetilde{K}_p^\dagger - sm}. \tag{23.1.3}$$

We can also make a dual construction which is more adapted to the dual locally analytic completed cohomology. Indeed, for any $\widetilde{K}'_p \subset \widetilde{K}_p$ small enough and any $\gamma >> 0$ consider the space $\mathcal{M}_{\mathscr{Fl},U}(\gamma)\widetilde{K}'_p \subset \mathcal{M}_{\mathscr{Fl}}|_U$. Let $C(\mathcal{M}_{\mathscr{Fl},U}(\gamma)\widetilde{K}'_p)$ be the $\mathscr{O}_U$-algebra of functions of $\mathcal{M}_{\mathscr{Fl},U}(\gamma)\widetilde{K}'_p$ and $\mathcal{D}(\mathcal{M}_{\mathscr{Fl},U}(\gamma)\widetilde{K}'_p)$ its $\mathscr{O}_U$-dual. We denote

$$\mathcal{D}(\mathcal{M}^\dagger_{\mathscr{Fl},U}\widetilde{K}'_p) := \varprojlim_{\gamma\to\infty} \mathcal{D}(\mathcal{M}_{\mathscr{Fl},U}(\gamma)\widetilde{K}'_p).$$

The spaces $\mathcal{M}_{\mathscr{Fl},U}(\gamma)\widetilde{K}'_p$ have a natural $\widetilde{K}'_p$-equivariant action over U, so that $\mathcal{D}(\mathcal{M}^\dagger_{\mathscr{Fl},U}\widetilde{K}'_p)$ is a $\widetilde{K}'_p$-equivariant *LF* $\mathscr{O}_U$-sheaf over U.

Definition 23.1.3. Keep the previous notation, and let $V = \pi_{\mathrm{HT}}^{-1}(U)$. We define the sheaf of dual automorphic forms over V to be the inverse limit

$$\mathcal{D}(\mathcal{M}^\dagger_{\mathrm{dR},V}) := \varprojlim_{\widetilde{K}'_p\subset\widetilde{K}_p} \pi^*_{\mathrm{HT}}(\mathcal{D}(\mathcal{M}^\dagger_{\mathscr{Fl},U}\widetilde{K}'_p))^{\widetilde{K}'_p}$$

where the transition maps are given by trace maps.

Remark 23.1.4. Let $\widehat{\mathscr{O}}_{\mathcal{S}h}$ be the completed structural sheaf of $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$. The sheaves $C(\mathcal{M}^\dagger_{\mathscr{Fl},U})$ and $\mathcal{D}(\mathcal{M}^\dagger_{\mathscr{Fl},U})$ are $\widetilde{\mathcal{D}}^{la}$-modules in a natural way, being constructed from open subspaces of $\mathcal{M}_{\mathscr{Fl}}$. In particular, Theorems 19.3.3 and 20.2.2 imply that

$$\widehat{\mathscr{O}}_{\mathcal{S}h}|_V \widehat{\otimes}_{\mathscr{O}^{sm}_V} C(\mathcal{M}^\dagger_{\mathrm{dR},V}) = \pi^*_{\mathrm{HT}}(C(\mathcal{M}_{\mathscr{Fl},U})).$$

Dually, for any $\widetilde{K}'_p \subset \widetilde{K}_p$-small enough one has

$$\widehat{\mathscr{O}}_{\mathcal{S}h}|_V \widehat{\otimes}_{\mathscr{O}^{\widetilde{K}'_p}_{\mathcal{S}h}} \mathcal{D}(\mathcal{M}^\dagger_{\mathrm{dR},V}\widetilde{K}'_p)^{\widetilde{K}'_p} = \pi^*_{\mathrm{HT}}(\mathcal{D}(\mathcal{M}^\dagger_{\mathscr{Fl},U}\widetilde{K}'_p)).$$

Taking limits as $\widetilde{K}'_p \to 1$ one obtains

$$\widehat{\mathscr{O}}_{\mathcal{S}h}|_V \widehat{\otimes}_{\mathscr{O}_V^{sm}} \mathcal{D}(\mathcal{M}^\dagger_{\mathrm{dR},V}) := \varprojlim_{\widetilde{K}'_p \to 1} (\widehat{\mathscr{O}}_{\mathcal{S}h}|_V \widehat{\otimes}_{\mathscr{O}_{\mathcal{S}h}^{\widetilde{K}'_p}} \mathcal{D}(\mathcal{M}^\dagger_{\mathrm{dR},V} \widetilde{K}'_p)^{\widetilde{K}'_p}) = \pi^*_{\mathrm{HT}}(\mathcal{D}(\mathcal{M}^\dagger_{\mathscr{F}\ell,U})).$$

where the transition maps in the limit are trace maps.

Example 23.1.5. Let W be the Weyl group of $\mathbf{G}_L^c$ and $w \in {}^{\mathbf{M}}W$ be a Kostant representative. Let $C_w \subset \mathscr{F}\ell$ be the w-Bruhat cell, and for $\epsilon > 0$ let $C_w(\epsilon)$ the ϵ-overconvergent neighbourhood of C_w as in Definition 21.4.15. For $\gamma > 0$ consider the $\mathcal{M}(\gamma)$-torsor $\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)$ over $C_w(\epsilon)$ of Definition 21.4.27, set $\mathcal{M}^\dagger_{w,\mathscr{F}\ell} := \{\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)\}_{\gamma>0}$. We have sheaves of overconvergent automorphic forms over $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))$ which we denote by $C(\mathcal{M}^\dagger_{w,\mathrm{dR}})$ and $\mathcal{D}(\mathcal{M}^\dagger_{w,\mathrm{dR}})$.

Let $\delta > 0$ and consider the open polydisc $C_w(\infty, \delta) \subset C_w$ of radius p^δ, cf. Definition 21.4.15. For $\gamma > 0$, we will also consider the following sheaves over $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon, \delta))$:

$$C(\mathcal{M}_{w,\mathrm{dR}}(\gamma)) := \pi^*_{\mathrm{HT}}(C(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)))^{\widetilde{K}_p^\dagger - sm} \text{ and } \mathcal{D}(\mathcal{M}_{w,\mathrm{dR}}(\gamma)) := \pi^*_{\mathrm{HT}}(\mathcal{D}(\mathcal{M}_{w,\mathscr{F}\ell}(\gamma)))^{\widetilde{K}_p^\dagger - sm}.$$

Notice that $C(\mathcal{M}^\dagger_{w,\mathrm{dR}}) = \varinjlim_\gamma C(\mathcal{M}_{w,\mathrm{dR}}(\gamma))$ while $\mathcal{D}(\mathcal{M}^\dagger_{w,\mathrm{dR}}) = \varprojlim_{\gamma \to 1} \mathcal{D}(\mathcal{M}_{w,\mathrm{dR}}(\gamma))^{\widetilde{K}_p(\gamma)}$, where the transition maps are trace maps and $\widetilde{K}_p(\gamma) \to 1$ as $\gamma \to \infty$.

23.1.2. Relation with higher Coleman theory

In the following we relate the cohomology of the sheaves of overconvergent automorphic forms constructed above, with the overconvergent cohomology classes of higher Coleman theory, cf. [BP21, §6.3]. We define the following cohomologies with compact and closed supports, cf. §18.

Definition 23.1.6. Let $w \in {}^{\mathbf{M}}W$ and let $C_w \subset \mathscr{F}\ell$ be the w-Bruhat cell. In particular, we define the w-overconvergent cohomologies of automorphic forms as[1]

$$R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, C(\mathcal{M}^\dagger_{w,\mathrm{dR}})) := R\Gamma_{c,\pi_{\mathrm{HT}}^{-1}(C_w)}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}, C(\mathcal{M}^\dagger_{w,\mathrm{dR}}))$$
$$R\Gamma_{w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathcal{D}(\mathcal{M}^\dagger_{w,\mathrm{dR}})) := R\Gamma_{\pi_{\mathrm{HT}}^{-1}(C_w)}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathcal{D}(\mathcal{M}^\dagger_{w,\mathrm{dR}})).$$

We want to use Lemma 18.1.4 to obtain a devisage of the cohomology complexes of the previous definition, in terms of "smaller" overconvergent cohomology complexes arising from finite level. In order to make this precise we need to introduce some more notation. The following are essentially the overconvergent cohomologies in families considered in [BP21].

Definition 23.1.7. Let $C_w(\overline{\epsilon + 1^+}, \delta) = \overline{\bigcup_{\epsilon' > \epsilon+1} C_w(\epsilon', \delta)} \cap C_w(\epsilon, \delta)$ and $C_w(\epsilon, \delta^-) = \bigcup_{\delta > \delta'} C_w(\epsilon, \delta')$. Let $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ be a compact open subgroup fixing $C_w(\epsilon + 1, \delta)$ and $C_w(\epsilon, \delta)$. Given a $\widetilde{K}_p$-invariant open subspace $U \subset \mathscr{F}\ell$ we let $\pi_{\mathrm{HT}}^{-1}(U)/\widetilde{K}_p$ denote its quotient seen as an open subspace of $\mathcal{S}h^{\mathrm{tor}}_{K^p K_p, C}$.

1. Let $\gamma > 0$ and let $\mathcal{M}_{w,\mathscr{F}\ell}(\gamma) \subset \mathcal{M}_{\mathscr{F}\ell}$ be the reduction of the natural $\mathcal{M}$ torsor over $\mathscr{F}\ell$ to a $\mathcal{M}(\gamma)$-torsor over $C_w(\epsilon)$. We let

$$C(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma)) := C(\mathcal{M}_{w,\mathrm{dR}}(\gamma)\widetilde{K}_p)^{\widetilde{K}_p}$$

   be the sheaf over $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon, \delta))/\widetilde{K}_p$ of $\widetilde{K}_p$-invariant sections.

2. Dually, we define $\mathcal{D}(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma)) := \mathcal{D}(\mathcal{M}_{w,\mathrm{dR}}(\gamma)\widetilde{K}_p)^{\widetilde{K}_p}$ seen as a sheaf over $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon, \delta))/\widetilde{K}_p$.

[1] More generally, for a sheaf $\mathscr{F}$ over $\mathcal{S}h^{\mathrm{tor}}_{K^p,C,\mathrm{an}}$ we define $R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F})$ and $R\Gamma_w(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathscr{F})$ in a similar way.

3. Let $\epsilon, \delta, \gamma > 0$, we define the following overconvergent cohomologies with compact and closed supports

$$R\Gamma_{c,w}^{\epsilon,\delta,\gamma}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, C(\mathcal{M}^{\dagger}_{w,K_p,\mathrm{dR}})) := R\Gamma_{c,\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta^-))/\widetilde{K}_p}(\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta+1))/\widetilde{K}_p, C(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma)))$$
$$R\Gamma_{w}^{\epsilon,\delta,\gamma}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \mathcal{D}(\mathcal{M}^{\dagger}_{w,K_p,\mathrm{dR}})) := R\Gamma_{\pi^{-1}(C_w(\overline{\epsilon+1^+},\delta))/\widetilde{K}_p}(\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta))/\widetilde{K}_p, \mathcal{D}(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma))).$$

Remark 23.1.8. The sheaves

$$C(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma^-)) = \varinjlim_{\gamma>\gamma'} C(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma')) \text{ and } \mathcal{D}(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma^+)) = \varinjlim_{\gamma'>\gamma} \mathcal{D}(\mathcal{M}_{w,K_p,\mathrm{dR}}(\gamma'))$$

are squarrable *LB* sheaves over $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta))/\widetilde{K}_p$. Then, by [BP21, Theo. 4.1.8][2], the cohomology complexes $R\Gamma_{w}^{\epsilon,\delta,\gamma}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}, \mathscr{F})$ with $\mathscr{F} = C(\mathcal{M}^{\dagger}_{w,K_p,\mathrm{dR}}), \mathcal{D}(\mathcal{M}^{\dagger}_{w,K_p,\mathrm{dR}})$ are independent of the toroidal compactification.

Proposition 23.1.9. *For $\epsilon, \delta, \gamma > 0$ let $K_p(\epsilon,\delta,\gamma) \subset \mathbf{G}(\mathbb{Q}_p)$ be an open compact subgroup fixing $C_w(\epsilon,\delta)$, $C_w(\epsilon+1,\delta)$ and $\mathcal{M}_{w,\mathscr{H}}(\gamma)$, such that $K_p(\epsilon,\delta,\gamma) \to 1$ as $\gamma \to \infty$ for ϵ and δ fixed.*

1. *We have natural quasi-isomorphisms*

$$R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) = \varinjlim_{\epsilon,\delta,\gamma} R\Gamma_{c,w}^{\epsilon,\delta,\gamma}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p(\epsilon,\delta,\gamma),C}, C(\mathcal{M}^{\dagger}_{w,K_p(\epsilon,\delta,\gamma),\mathrm{dR}})) \tag{23.1.4}$$
$$R\Gamma_{w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) = R\varprojlim_{\epsilon,\delta,\gamma} R\Gamma_{w}^{\epsilon,\delta,\gamma}(\mathcal{S}h^{\mathrm{tor}}_{K^pK_p(\epsilon,\delta,\gamma),C}, \mathcal{D}(\mathcal{M}^{\dagger}_{w,K_p(\epsilon,\delta,\gamma),\mathrm{dR}})) \tag{23.1.5}$$

   where the transition maps for ϵ and δ are correstriction and restriction maps as in Lemma 18.1.4, and the transition maps for γ are the inclusions (resp. trace) maps.

2. *Let $\mathscr{J} \subset \mathscr{O}^{\mathrm{sm}}_{\mathcal{S}h}$ be the ideal defining the cusps at any level, and let $\mathscr{J}_{K_p} \subset \mathscr{J}$ be its K_p-invariant sections. Then (23.1.4) (resp. (23.1.5)) also holds for $C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})\widehat{\otimes}_{\mathscr{O}^{sm}_{\mathrm{Sh}}}\mathscr{J}$ and $C(\mathcal{M}^{\dagger}_{w,K_p,\mathrm{dR}})\widehat{\otimes}_{\mathscr{O}_{\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}}}\mathscr{J}_{K_p}$ (resp. $\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})\widehat{\otimes}_{\mathscr{O}^{sm}_{\mathcal{S}h}}\mathscr{J}$ and $\mathcal{D}(\mathcal{M}^{\dagger}_{w,K_p,\mathrm{dR}})\widehat{\otimes}_{\mathscr{O}_{\mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}}}\mathscr{J}_{K_p}$).*

Proof. We will only prove (1), the proof of (2) being similar. By Lemma 18.1.4 we know that

$$R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) = \varinjlim_{\epsilon,\delta} R\Gamma_{c,\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta^-))}(\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta+1)), C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}}))$$
$$R\Gamma_{w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) = R\varprojlim_{\epsilon,\delta} R\Gamma_{\pi_{\mathrm{HT}}^{-1}(C_w(\overline{\epsilon+1},\delta))}(\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon,\delta)), \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})).$$

The cohomology complexes of the RHS are represented by a cone of cohomologies on qcqs locally closed subspaces of $\mathcal{S}h^{\mathrm{tor}}_{K^p,C}$. To deal with the case of $C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$, notice that we have the following presentation of qcqs subspaces at infinite level as inverse limit of finite level ones

$$\pi^{-1}(C_w(\epsilon,\delta+1)) = \varprojlim_{K_p} R\Gamma(\pi^{-1}(C_w(\epsilon,\delta+1))/\widetilde{K}_p$$
$$\pi^{-1}(C_w(\epsilon,\delta+1)\backslash C_w(\epsilon,\delta^-)) = \varprojlim_{K_p} \pi^{-1}(C_w(\epsilon,\delta+1)\backslash C_w(\epsilon,\delta^-))/\widetilde{K}_p.$$

Moreover, we have that

$$C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}}) = \varinjlim_{\gamma} C(\mathcal{M}_{w,K_p(\gamma),\mathrm{dR}}(\gamma))$$

[2] Actually, by its references in [Har90].

where $K_p(\gamma) \to 1$ as $\gamma \to \infty$. This formally implies that, for $X = \pi^{-1}(C_w(\epsilon, \delta + 1))$ or $\pi^{-1}(C_w(\epsilon, \delta + 1) \backslash C_w(\epsilon, \delta^-))$, one has

$$R\Gamma_{\mathrm{an}}(X, C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) = \varinjlim_{\gamma} R\Gamma(X/\widetilde{K}_p(\gamma), C(\mathcal{M}_{w,K_p(\gamma),\mathrm{dR}}(\gamma))),$$

this proves (23.1.4).

To prove the case of $\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$, it is enough to show the following lemma

Lemma 23.1.10. *Let $U \subset \mathcal{S}h^{\mathrm{tor}}_{K^pK_p,C}$ be a qcqs open subspace, let $\widetilde{U}$ be its pullback to infinite level, and for $K'_p \subset K_p$ let $U_{K'_p}$ be its pullback to level K'_p. Then*

$$R\Gamma(\widetilde{U}, \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) = R\varprojlim_{\gamma} R\Gamma(U_{K_p(\gamma)}, \mathcal{D}(\mathcal{M}_{w,K_p(\gamma),\mathrm{dR}}(\gamma))) \tag{23.1.6}$$

where $K_p(\gamma) \to 1$ as $\gamma \to \infty$.

Proof. Notice that $\widetilde{U}$ is qcqs of sheaf-cohomological dimension $\leq d$ and that the derived limit is countably filtered. Let $\widetilde{V}_\bullet$ be a finite hypercover of $\widetilde{U}$, and without loss of generality assume that it arises as the pullback of a finite hypercover $V_\bullet$ of U. For $K'_p \subset K_p$ let $V_{\bullet,K'_p}$ be its pullback to $U_{K'_p}$. We have that

$$\begin{aligned} R\Gamma(\widetilde{V}_\bullet, \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})) &= R\varprojlim_{K'_p} R\Gamma(V_{\bullet,K_p}, \pi^*_{\mathrm{HT}}(\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell}\widetilde{K}'_p))^{\widetilde{K}'_p}) \\ &= R\varprojlim_{K'_p,\gamma} R\Gamma(V_{\bullet,K_p}, \pi^*_{\mathrm{HT}}(\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell}(\gamma)\widetilde{K}'_p))^{\widetilde{K}'_p})) \\ &= R\varprojlim_{\gamma} R\Gamma(V_{\bullet,K_p(\gamma)}, \mathcal{D}(\mathcal{M}_{w,K_p(\gamma),\mathrm{dR}}(\gamma))) \end{aligned}$$

where the first and second equalities are given by Definition 23.1.3 and the last equality by Definition 23.1.7. The maps with respect to K'_p are isomorphisms induced by traces. Finally, by Theorems 19.3.3 and 20.2.2, one deduces that the sheaves $\mathcal{D}(\mathcal{M}_{w,K_p(\gamma),\mathrm{dR}}(\gamma))$ over $U_{K_p(\gamma)}$ are squarrable LF sheaves (eg. look over affinoids admitting charts). Hence, the hypercohomologies over $V_{\bullet,K_p(\gamma)}$ already compute the cohomology complex of the RHS of (23.1.6). Taking colimits along all the $\widetilde{V}_\bullet$ one obtains the equality of (23.1.6) proving the lemma. □

□

Remark 23.1.11. The sheaves $C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$ and $\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$ are endowed with a natural action of $\mathfrak{m} = \mathrm{Lie}\,\mathbf{M}$ arising from the $\mathcal{M}^{\dagger}$-action on the torsor $\mathcal{M}^{\dagger}_{w,\mathscr{F}\ell}$. We denote this action by $\mathfrak{m}_{\star_1}$.

23.2. The overconvergent BGG maps

We can finally state the main theorems of this chapter. Let $\mathfrak{b} \subset \mathfrak{g}^c$ be a Borel subalgebra and $\mathfrak{b}_{\mathbf{M}} = \mathfrak{b} \cap \mathfrak{m}$ the Borel subalgebra of $\mathfrak{m}$. Let $\mathfrak{h} \subset \mathfrak{b}$ be the diagonal torus. We let $\rho^{\mathbf{M}} = \rho - \rho_{\mathbf{M}}$ be a half of the sum of the roots of $\mathfrak{n} \subset \mathfrak{p}$.

Theorem 23.2.1 (BGG maps for completed cohomology)**.** *Consider the Bruhat filtration of $\mathscr{F}\ell$ by open subspaces $\emptyset = Y_{d+1} \subset Y_d \subset \cdots \subset Y_0 = \mathscr{F}\ell$ with graded pieces $Y_k \backslash Y_{k+1} = \bigsqcup_{\substack{w \in {}^{\mathbf{M}}W \\ \ell(w)=k}} C_w$. Let $\lambda \in \mathfrak{h}^*_C$ be a weight.*

1. *The derived $\mathfrak{b} = \lambda$ isotypic part of the locally analytic completed cohomology can be computed as*

$$(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C)^{R\mathfrak{b}=\lambda} = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, R\mathrm{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{Sh})).$$

2. *Let $\epsilon > 0$, then $R\mathrm{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{Sh})|_{\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))}$ is concentrated in cohomological degrees $[0, d-\ell(w)]$. Furthermore, we have a highest weight vector map*

$$\Upsilon_w : R\mathrm{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{Sh})|_{\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))} \to C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})^{\mathfrak{b}_{\mathbf{M},\star_1}=w\cdot\lambda+2\rho^{\mathbf{M}}}[\ell(w)-d].$$

surjective in $H^{d-\ell(w)}$-cohomology.

3. *The open Bruhat filtration of $\mathscr{F}\ell$ induces a spectral sequence*

$$E_1^{p,q} = \bigoplus_{\substack{w\in{}^{\mathbf{M}}W \\ \ell(w)=p}} H^{p+q}_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, R\mathrm{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{Sh})) \Rightarrow H^{p+q}((R\Gamma_{\mathrm{proét}}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C)^{R\mathfrak{b}=\lambda}).$$

In addition, the map Υ_w induces an overconvergent BGG *map*

$$\mathrm{BGG}_w : R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, R\mathrm{Hom}_{\mathfrak{b}}(\lambda, \mathscr{O}^{la}_{Sh})) \to R\Gamma_{c,w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})^{\mathfrak{b}_{\mathbf{M},\star_1}=w\cdot\lambda+2\rho^{\mathbf{M}}})[\ell(w)-d].$$

4. *Let $\mathscr{J} \subset \mathscr{O}^{sm}_{Sh}$ be the ideal defining the cusps for any finite level. Analogous statements hold for cohomology with compact supports after exchanging $\mathscr{O}^{la}_{Sh}$ by $\mathscr{O}^{la}\widehat{\otimes}_{\mathscr{O}^{sm}_{Sh}}\mathscr{J}$ and $C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$ by $C(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})\widehat{\otimes}_{\mathscr{O}^{sm}_{Sh}}\mathscr{J}$.*

Theorem 23.2.2 (BGG maps for dual completed cohomology)**.** *Consider the Bruhat filtration of $\mathscr{F}\ell$ by closed subspaces $\emptyset = Z_{d+1} \subset Z_d \subset \cdots \subset Z_0 = \mathscr{F}\ell$ with graded pieces $Z_k \backslash Z_{k+1} = \bigsqcup_{\substack{w\in{}^{\mathbf{M}}W \\ \ell(w)=d-k}} C_w$. Let $\lambda \in \mathfrak{h}^*_C$ be a weight.*

1. *The derived $\mathfrak{b} = \lambda$ co-isotypic part of the dual locally analytic completed cohomology can be computed as*

$$R\Gamma_{\mathrm{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee} \otimes^L_{U(\mathfrak{b})} \lambda = R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, (\widetilde{\mathcal{D}}^{la}_{Sh} \otimes \omega) \otimes^L_{U(\mathfrak{b})} \lambda)[d]$$

with $\omega = \Omega^d(\log)$.

2. *Let $\epsilon > 0$, then $((\widetilde{\mathcal{D}}^{la}_{Sh} \otimes \omega) \otimes^L_{U(\mathfrak{b})} \lambda)|_{\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))}$ is concentraded in cohomological degrees $[\ell(w)-d, 0]$. Furthermore, we have a dual highest weight vector map*

$$\Upsilon^{\vee}_w : (w\cdot\lambda) \otimes_{U(\mathfrak{b}_{\mathbf{M},\star_1})} \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})[d-\ell(w)] \to ((\widetilde{\mathcal{D}}^{la}_{Sh} \otimes \omega) \otimes^L_{U(\mathfrak{b})} \lambda)|_{\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))}$$

which is injective for $H^{\ell(w)-d}$-cohomology.

3. *The closed Bruhat filtration of $\mathscr{F}\ell$ induces a spectral sequence*

$$E_1^{p,q} = \bigoplus_{\substack{w\in{}^{\mathbf{M}}W \\ \ell(w)=d-p}} H^{p+q}_{w}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, (\widetilde{\mathcal{D}}^{la}_{Sh} \otimes \omega) \otimes^L_{U(\mathfrak{b})} \lambda) \Rightarrow H^{p+q+d}(R\Gamma_{\mathrm{proét},c}(\mathcal{S}h_{K^p,C}, \widehat{\mathbb{Q}}_p)^{la,\vee} \otimes^L_{U(\mathfrak{b})} \lambda)$$

In addition, the map $\Upsilon^{\vee}_w$ induces a dual BGG *map*

$$\mathrm{BGG}^{\vee}_w : R\Gamma_w(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, (w\cdot\lambda) \otimes_{U(\mathfrak{b}_{\mathbf{M},\star_1})} \mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}}))[d-\ell(w)] \to R\Gamma_w(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, (\widetilde{\mathcal{D}}^{la}_{Sh} \otimes \omega) \otimes^L_{U(\mathfrak{b})} \lambda).$$

4. *Analogous statements hold for the dual of the completed cohomology after exchanging $\widetilde{\mathcal{D}}^{la}_{Sh}$ by $\widetilde{\mathcal{D}}^{la}_{Sh}\widehat{\otimes}_{\mathscr{O}^{sm}_{Sh}}\mathscr{J}$ and $\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})$ by $\mathcal{D}(\mathcal{M}^{\dagger}_{w,\mathrm{dR}})\widehat{\otimes}_{\mathscr{O}^{sm}_{Sh}}\mathscr{J}$.*

Proofs of Theorems 23.2.1 and 23.2.2. They follow from Theorems 20.2.2 and 21.4.31, and the construction of the overconvergent modular sheaves in Definitions 23.1.2 and 23.1.3 and Example 23.1.5. Indeed, the proof of Theorem 22.3.3 implies that

$$(R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C},\widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C)^{R\mathfrak{b}=\lambda} = R\Gamma_{\text{prokét}}(\mathcal{S}h^{\text{tor}}_{K^pK_p,C}, C^{la}(\widetilde{K}_p,\widehat{\mathscr{O}})^{\mathfrak{b}_{\star_2}=\lambda}).$$

Projecting the RHS to the analytic site of $\mathcal{S}h^{\text{tor}}_{K^pK_p,C}$ and taking colimits as $K_p \to 1$, one has by Theorem 20.2.2

$$\begin{aligned}(R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C},\widehat{\mathbb{Q}}_p)^{la}\widehat{\otimes}C)^{R\mathfrak{b}=\lambda} &= R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\Gamma(\mathfrak{n}^0_{\star_{1,3}}), C^{la}(\mathfrak{g}^c,\widehat{\mathscr{O}}_{\mathcal{S}h})^{\mathfrak{b}_{\star_2}=\lambda,\widetilde{K}_p^{\dagger}-sm})\\ &= R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\Gamma(\mathfrak{n}^0_{\star_{1,3}}\times\mathfrak{b}_{\star_2}, C^{la}(\mathfrak{g}^c,\widehat{\mathscr{O}}_{\mathcal{S}h})^{\mathfrak{b}_{\star_2}=\lambda})^{\widetilde{K}_p^{\dagger}-sm})\\ &= R\Gamma_{\text{an}}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\text{Hom}_{\mathfrak{b}}(\lambda,\mathscr{O}^{la}_{\mathcal{S}h})).\end{aligned}$$

An analogous argument holds for the dual completed cohomology. This proves (1).

Part (2) follows from the Definitions 23.1.2 and 23.1.3, and from the construction of the highest weight vector maps over $\mathscr{F}\ell$ in Theorem 21.4.31, after applying the pullback to $\mathcal{S}h^{\text{tor}}_{K^p,C}$, and the projection to the analytic site as in Theorem 20.2.2. Notice that $\mathfrak{n}^0$ act trivially on the sheaves involved, so that the Sen operator vanishes and taking $\widetilde{K}_p$-smooth vectors is exact.

Part (3) is an application of the spectral sequence of a filtered space as in Definition 18.2.1. Finally, part (4) follows from the previous steps and by the second part of Theorem 20.2.2. □

Remark 23.2.3. A more careful study of the $\widetilde{\mathcal{D}}^{la}$-module $R\text{Hom}_{\mathfrak{b}}(\lambda, C^{la}(\mathfrak{g}^c,\mathscr{O}_{\mathscr{F}\ell})^{\mathfrak{n}^0_{\star_{1,3}}=0})$ of §21.4.3 should provide a better description of the λ isotypic part of the locally analytic completed cohomology in terms of overconvergent autormophic forms. More precisely, the cohomologies

$$R\Gamma_{c,w}(\mathcal{S}h^{\text{tor}}_{K^p,C}, R\text{Hom}_{\mathfrak{b}}(\lambda,\mathscr{O}^{la}_{\mathcal{S}h})) \text{ for } w\in {}^{\mathbf{M}}W$$

should admit a filtration purely in terms of overconvergent automorphic cohomology classes, this would prove that the eigenvariety arising from completed cohomology is "the same" as the eigenvariety arising from overconvergent modular forms (modulo some degeneracy of an spectral sequence). Furthermore, one would expect that, generically in λ, the only cohomology classes appearing in the locally analytic completed cohomology are those occurring via the overconvergent BGG maps. See Proposition 21.5.3 for the example of GL_2.

23.2.1. Relation with classical Hodge-Tate decompositions

Let us explain in what sense Theorems 23.2.1 and 23.2.2 interpolate the classical BGG (or Hodge-Tate) decompositions of §22.2.2. Let $C^{alg}(\mathfrak{g}^c, L)$ be the germs at 1 of the locally algebraic functions of $\widetilde{K}_p$. Let $H_\alpha \in \mathfrak{h}$ be the coroot of $\alpha\in\Phi$. As $\mathfrak{g}^c\times\mathfrak{g}^c$-module via the left and right regular action respectively, the representation $C^{alg}(\mathfrak{g}^c,L)$ is isomorphic to

$$\bigoplus_{\substack{\lambda\in\mathfrak{h}^*\\ \langle\lambda,H_\alpha\rangle\geq 0\ \forall\ \alpha\in\Phi^+}} V_\lambda^\vee\otimes V_\lambda.$$

Let $\lambda\in\mathfrak{h}^*$ be a dominant weight. Taking λ-isotypic parts for the right regular action of the inclusion $C^{alg}(\mathfrak{g}^c,L)\subset C^{la}(\mathfrak{g}^c,L)$ one gets that

$$V_\lambda^\vee \subset C^{la}(\mathfrak{g}^c,L)^{\mathfrak{b}_{\star_2}=\lambda}. \tag{23.2.1}$$

Taking pro-étale cohomology for the $\widetilde{K}_p^{\dagger}$-local systems defined by (23.2.1) one gets a map

$$\varinjlim_{K_p} R\Gamma_{\text{proét}}(\mathcal{S}h_{K^pK_p,C}, V_\lambda^\vee)\to R\Gamma_{\text{proét}}(\mathcal{S}h_{K^p,C},\widehat{L})^{R\mathfrak{b}_{\star_2}=\lambda}.$$

Tensoring with C, we obtain by Theorem 23.2.1 a map

$$R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, R(\lambda\circ\nu_\infty)_*(V_\lambda^\vee\otimes\widehat{\mathscr{O}}))\to R\Gamma_{\mathrm{an}}(\mathcal{S}h^{\mathrm{tor}}_{K^p,C}, R\mathrm{Hom}_{\mathfrak{b}}(\lambda,\mathscr{O}^{la}_{\mathcal{S}h})).$$

But now, Theorem 22.2.6 implies that

$$R(\lambda\circ\nu_\infty)_*(V_\lambda^\vee\otimes\widehat{\mathscr{O}})=\bigoplus_{w\in{}^{\mathbf{M}}W}\mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega\otimes C(w\cdot\lambda(\mu^{-1})-d)[\ell(w)-d],\tag{23.2.2}$$

where $\mathcal{M}(\kappa)=\varinjlim_{K_p}\mathcal{M}_{K_p}(\kappa)$ is the colimit of all the automorphic sheaves of finite level attached to $W^\vee_{-w_0(\kappa)}$. Following (2) of Theorem 23.2.1, let us restrict to $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))$. We can write

$$\pi_{K_p}^*(\mathcal{M}_{K_p}(-w_0(w\cdot\lambda))\otimes\omega)\otimes\widehat{\mathscr{O}}(w\cdot\lambda(\mu^{-1})-d))=\pi_{\mathrm{HT}}^*(\mathcal{W}(-w_0(w\cdot\lambda))\otimes\mathfrak{n}^{0,\vee}),$$

and this last sheaf is attached to $W^\vee_{w\cdot\lambda+2\rho^{\mathbf{M}}}$ via the torsor $\mathbf{M}_{\mathscr{F}\ell}$ over $\mathscr{F}\ell$. More precisely, $\mathcal{W}(-w_0(w\cdot\lambda))\otimes\mathfrak{n}^{0,\vee}$ is equal to the sheaf $\mathscr{O}_{\mathbf{M}_{\mathscr{F}\ell}}[(w\cdot\lambda+2\rho^{\mathbf{M}})_{\mathbf{B}_{\mathbf{M},\star_1}}]$. This induces a natural $\widetilde{K}_p^\dagger$-equivariant map over $C_w(\epsilon)$

$$\mathcal{W}(-w_0(w\cdot\lambda))\otimes\mathfrak{n}^{0,\vee}\to C(\mathcal{M}^\dagger_{w,\mathscr{F}\ell})^{\mathfrak{b}_{\mathbf{M},\star_1}=w\cdot\lambda+2\rho^{\mathbf{M}}}.$$

Taking pullbacks by π_{HT} and projections to the analytic site, one obtains a commutative diagram of sheaves over $\pi_{\mathrm{HT}}^{-1}(C_w(\epsilon))$

$$\begin{array}{ccc} R\mathrm{Hom}_{\mathfrak{b}}(\lambda,\mathscr{O}^{la}_{\mathcal{S}h}) & \xrightarrow{\Upsilon_w} & C(\mathcal{M}^\dagger_{w,\mathrm{dR}})^{\mathfrak{b}_{\mathbf{M},\star_1}=w\cdot\lambda+2\rho^{\mathbf{M}}}[\ell(w)-d] \\ \uparrow & & \uparrow \\ R(\lambda\circ\nu_\infty)_*(V_\lambda^\vee\otimes\widehat{\mathscr{O}}) & \xrightarrow{\mathrm{pr}_w} & \mathcal{M}(-w_0(w\cdot\lambda))\otimes\omega\otimes C(w\cdot\lambda(\mu^{-1})-d)[\ell(w)-d]. \end{array}$$

Taking $R\Gamma_{c,w}$ cohomology one obtains maps compatibles with the spectral sequence of Theorem 23.2.1 (3), proving that *loc. cit.* is an interpolation of the BGG decompositions as expected. One can make analogous constructions for the λ co-isotypic part of the dual locally analytic completed cohomology.

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