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# Asymptotic stability of invasion fronts for reaction-diffusion equations

Louis Garenaux

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# THÈSE

En vue de l'obtention du

## DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par : *l'Université Toulouse 3 Paul Sabatier (UT3 Paul Sabatier)*

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Présentée et soutenue le *11 juillet 2022* par :

**Louis Garénaux**

**Stabilité asymptotique des fronts d'invasion dans les équations de  
réaction-diffusion**

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## UNIVERSITÉ TOULOUSE III – PAUL SABATIER

École doctorale **MITT**

Unité de recherche **Institut de Mathématiques de Toulouse**

Thèse présentée par **Louis GARÉNAUX**

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# Stabilité asymptotique des fronts d'invasion dans les équations de réaction-diffusion

Thèse dirigée par Grégory FAYE

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UNIVERSITÉ TOULOUSE III – PAUL SABATIER

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# Asymptotic stability of invasion fronts in reaction-diffusion equations

Thesis supervised by Grégory FAYE

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**Mots clés :** équations de réaction-diffusion, fronts de propagation, stabilité asymptotique non-linéaire, stabilité spectrale, estimées de résolvante ponctuelles, espaces à poids, fronts FKPP critiques

**Keywords:** reaction diffusion equations, propagation fronts, non-linear asymptotic stability, spectral stability, pointwise resolvent estimates, weighted spaces, critical FKPP fronts





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Le doute est une vraie et belle force. Veille simplement à ce qu'elle te pousse toujours vers l'avant.

---

Pierre Bottero

Mal nommer les choses, c'est ajouter au malheur de ce monde.

---

Albert Camus

Rien ne pousse à l'ombre des grands arbres.

---

Constantin Brancusi



---

**STABILITÉ ASYMPTOTIQUE DES FRONTS D'INVASION DANS LES ÉQUATIONS DE RÉACTION-DIFFUSION**
**Résumé**

Certaines équations aux dérivées partielles admettent des solutions en onde de propagation. Dans le cas particulier où le profil de l'onde connecte deux états constants distincts, on parle de front. Les équations de réaction-diffusion, introduites pour modéliser de nombreux phénomènes biologiques, laissent apparaître de telles solutions. Savoir lesquels de ces fronts sont stables est important, puisque ceux-ci attirent alors les solutions génériques, et décrivent donc leurs propriétés. Dans cette thèse, on s'intéresse à des fronts monostables, c'est-à-dire à des ondes connectant un état constant stable à un état constant instable. On obtient dans trois scénarios différents la stabilité asymptotique de fronts monostables.

Le chapitre 2 traite d'une équation de FKPP étendue. Cette EDP présente une dérivée d'ordre 4 dont le coefficient est choisi arbitrairement proche de 0. Dans ce contexte, nous obtenons à nouveau l'existence d'un front critique. De plus, nous étudions les propriétés spectrales de la linéarisation au voisinage de ce front. La description précise de ce spectre, et en particulier la situation à l'origine, a pour conséquence la stabilité asymptotique du front.

Le chapitre 3 s'intéresse à une situation où l'état constant à l'arrière d'un front FKPP devient également instable. Cette situation apparaît naturellement dans l'équation FKPP avec terme de réaction non-local, mais est plutôt étudiée ici pour un système. Lorsqu'une bifurcation de Turing déstabilise l'état constant à l'arrière, on montre que le front FKPP critique reste stable dans un repère en translation.

Le chapitre 4 étudie des équations de bilan, aussi appelées équations de réaction-advection. L'absence de laplacien, et la présence d'un terme quasi-linéaire, peuvent mener à des solutions discontinues en espace. Bien que le cadre semble différent, nos conclusions sont très similaires au cas FKPP. En particulier, nous obtenons la stabilité de fronts surcritiques.

**Mots clés :** équations de réaction-diffusion, fronts de propagation, stabilité asymptotique non-linéaire, stabilité spectrale, estimées de résolvante ponctuelles, espaces à poids, fronts FKPP critiques

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**ASYMPTOTIC STABILITY OF INVASION FRONTS IN REACTION-DIFFUSION EQUATIONS****Abstract**

Several Partial Differential Equations admit propagating waves as solutions. In case where the wave profile connects two distinct constant states, these waves are called fronts. They typically appear as solutions of reaction-diffusion equations, and model a large number of biological phenomena. To decide which of these solutions are the stable ones is fundamental, since the latter attract, and thus describe, generic solutions. In this PhD, we focus on monostable fronts, that are waves connecting a constant stable state to a constant unstable state. We obtain in three distinct situations the asymptotic stability of monostable fronts.

Chapter 2 deals with an extended FKPP equation. This PDE contains a fourth order space derivative, whose coefficient is chosen arbitrarily close to 0. In this context, we re-obtain the existence of a critical front. Furthermore, we study the spectral properties of the linearization near this front. The asymptotic stability of the latter is a consequence of the precise description of the spectrum, in particular of the situation at the origin.

Chapter 3 is about a situation where the constant state behind a FKPP front destabilizes. Although this phenomenon naturally appears in the FKPP equation with non-local reaction term, we rather investigate it for a system of equations. When a Turing bifurcation destabilizes the constant state behind the front, we show that the critical FKPP front stays asymptotically stable in a co-moving frame.

Chapter 4 studies balance laws, also known as reaction-advection equations. The absence of Laplacian, together with the presence of a quasi-linear term, may lead to space-discontinuous solutions. Despite the apparently distinct framework, our conclusions are very similar to the FKPP case. In particular, we obtain super-critical fronts stability.

**Keywords:** reaction diffusion equations, propagation fronts, non-linear asymptotic stability, spectral stability, pointwise resolvent estimates, weighted spaces, critical FKPP fronts

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# Chapitre 1

## Introduction

### Sommaire du chapitre

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### 1.1 Présentation du sujet

#### 1.1.1 Réaction-diffusion

Les équations de réaction-diffusion sont des équations paraboliques semi-linéaires, qui modélisent l'évolution d'une population d'individus au cours du temps et dans l'espace. Les premiers modèles considérés sont de la forme

$$\partial_t u = D\Delta u + R(u).$$

L'inconnue  $u(t, x) \in \mathbb{R}^n$  modélise les nombres d'individus  $u_1, \dots, u_n$  de chacune des espèces, à l'instant  $t \in (0, +\infty)$  et à la position  $x \in \Omega$ , où  $\Omega$  est un ouvert de  $\mathbb{R}^d$ . La dynamique de cette population est gouvernée par deux phénomènes. Premièrement, les individus se déplacent de façon homogène en espace, suivant un mouvement brownien. Lorsque le nombre d'individus devient grand, ce mouvement fait apparaître le terme de diffusion  $D\Delta u$ , où la matrice de diffusion  $D$  est diagonale – à coefficients diagonaux strictement positifs – et où le laplacien est défini par  $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_d}^2$ . Deuxièmement, le nombre d'individus en un point donné est susceptible de varier, notamment par des naissances, des morts, des compétitions entre espèces ou par une pression venant de l'environnement. Toujours dans la limite d'un grand nombre d'individus, ce phénomène fait apparaître le terme de réaction  $R(u)$ , où  $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$  est souvent une fonction polynomiale. Une dérivation formelle de ces équations est expliquée plus en détails dans [Roq13, chapitres 1 et 2].

Les premiers exemples de modèles de réaction-diffusion sont l'équation bistable, qu'on évoque en section 1.5.2, et l'équation de FKPP, dont on va largement parler dans la suite. Les problèmes que l'on considérera seront tous en une seule dimension d'espace, et sur des domaines non-bornés. Fixons dès à présent  $d = 1$  et  $\Omega = \mathbb{R}$ .

### 1.1.2 Stabilité asymptotique

L'équation présentée au paragraphe précédent s'écrit sous la forme d'une équation d'évolution

$$\partial_t u = F(u),$$

pour une certaine application  $F$ . Par analogie avec les équations aux dérivées ordinaires (EDO), on dit que  $\bar{u}$  est un équilibre si  $F(\bar{u}) = 0$ . Ceci assure que  $u(t) = \bar{u}$  est une solution constante de l'équation ci-dessus.

On dit de plus que l'équilibre  $\bar{u}$  est *stable* si pour tout  $\varepsilon > 0$ , il existe un  $\delta > 0$  tel que : si on se fixe une condition initiale  $u_0$  qui soit  $\delta$  proche de  $\bar{u}$ , alors la solution associée  $u(t)$  est  $\varepsilon$  proche de  $\bar{u}$  pour tout temps. Par opposition, on dit que l'équilibre  $\bar{u}$  est *instable* s'il n'est pas stable, c'est à dire s'il existe  $\varepsilon > 0$  tel que pour toute condition initiale  $u_0$ , la solution  $u(t)$  associée n'est pas  $\varepsilon$  proche de  $\bar{u}$  pour au moins un temps  $t$ . Enfin, on dit de l'équilibre  $\bar{u}$  qu'il est *asymptotiquement stable* s'il est stable, et qu'en plus d'être majorée par  $\varepsilon$ , la distance entre la solution  $u(t)$  et l'équilibre  $\bar{u}$  tend vers 0 quand  $t \rightarrow +\infty$ .

Pour les problèmes qui nous intéressent, l'inconnue  $u$  appartient à un espace de dimension infinie. Plusieurs choix sont donc possibles pour mesurer la distance entre  $u(t)$  et  $\bar{u}$ . On utilisera les espaces de Sobolev  $W^{k,p}(\mathbb{R})$ , et plus particulièrement  $L^\infty(\mathbb{R})$ . Dans ce contexte, un résultat générique sera de la forme suivante.

**Theorem A.** *Il existe  $C$  et  $\delta$  des constantes strictement positives telles que : si une condition initiale  $u_0$  s'écrit  $u_0 = \bar{u} + v_0$  avec  $\|v_0\|_Y \leq \delta$ , alors la solution de (1.1.2) avec condition initiale  $u_0$  existe pour tout temps, et s'écrit  $u(t) = \bar{u} + v(t)$ , où la perturbation  $v$  vérifie*

$$\|v(t)\|_X \leq C h(t) \|v_0\|_Y.$$

La fonction  $h : (0, +\infty) \rightarrow (0, +\infty)$  est appelée taux de décroissance, et on s'attend à ce que  $h(t) \rightarrow 0$  quand  $t \rightarrow +\infty$ . Les normes  $\|\cdot\|_X$  et  $\|\cdot\|_Y$  dépendront du problème considéré, il se trouve que nous utiliserons des espaces de Sobolev à poids :  $\|v(t)\|_X \stackrel{\text{def}}{=} \|v(t)\omega^{-1}\|_{L^\infty(\mathbb{R})}$ . On renvoie à la fin de la section 1.4.2 pour des définitions de ces poids.

### 1.1.3 Fronts d'invasion

Un exemple largement étudié, et tout à fait représentatif du contenu que l'on présentera, est l'équation de Fisher, Kolmogorov-Petrovsky-Piskunov (KPP)

$$\partial_t u = d \partial_{xx} u + ru \left(1 - \frac{u}{K}\right).$$

Cette équation scalaire ( $n = 1$ ) de réaction-diffusion a été introduite simultanément dans [Fis37] et [KPP37]. Elle modélise une population soumise à un taux de naissance  $r$  constant et à un taux de mort  $\frac{ru}{K}$  proportionnel à la taille de la population. Les constantes strictement positives  $d$  et  $K$  représentent respectivement le taux de diffusion de l'espèce, et la capacité d'accueil de l'environnement. Un changement d'échelle – en temps, en espace et de la solution – permet de

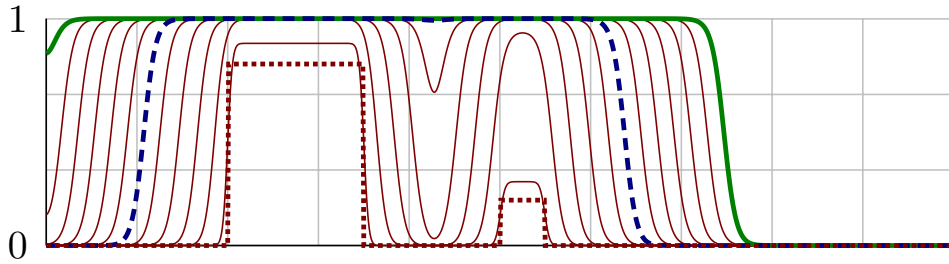


FIGURE 1.1 – Solution de l'équation de FKPP pour une condition initiale à support compact (en rouge avec petits pointillés). À  $t = 12$ , la solution est proche de la concaténation de deux fronts (en bleu avec grands pointillés). À  $t = 24$ , chacun de ces deux fronts s'est propagé à vitesse constante vers la droite ou la gauche (en vert avec trait plein). Des temps intermédiaires équidistants sont représentés en traits rouges fins, les abscisses varient entre 0 et 200.

se ramener à l'équation adimensionnée

$$\partial_t u = \partial_{xx} u + u(1 - u). \quad (1.1)$$

Cette EDP admet deux solutions constantes :  $u(t, x) = 1$  et  $u(t, x) = 0$ . Elles sont respectivement asymptotiquement stable et instable.<sup>1</sup> D'autre part, (1.1) admet des solutions en forme de fronts  $u(t, x) = q_c(x - ct)$  se déplaçant à vitesse  $c$  et avec un profil  $q_c$ .

**Theorem B** ([KPP37], [AW78]). *Pour tout  $c \geq 2$ , il existe une fonction  $q_c : \mathbb{R} \rightarrow (0, 1)$ , solution de*

$$0 = \partial_{yy} q + c \partial_y q + q(1 - q), \quad (1.2)$$

*qui soit de classe  $C^\infty$  et qui connecte les deux états constants à l'infini :*

$$\lim_{y \rightarrow -\infty} q_c(y) = 1, \quad \lim_{y \rightarrow +\infty} q_c(y) = 0.$$

L'équation (1.1) est précisément étudiée pour ces solutions en forme de front. En effet, lorsque la condition initiale est à support compact – ce qui peut représenter biologiquement un patch de population – la solution ressemble après un certain temps à la concaténation de deux fronts qui se propagent vers  $+\infty$  et  $-\infty$  à la vitesse la plus faible  $c_* \stackrel{\text{def}}{=} 2$ , voir la figure 1.1. De même, si la condition initiale a un comportement spatial proche du front  $q_c$  à l'infini – par exemple si  $u_0(x) \sim q_c(x)$  quand  $x \rightarrow +\infty$  – alors la solution ressemble après un certain temps à un front de vitesse  $c$ . Voir [RR18, section 1.1] pour une discussion plus précise.

Dans les deux cas, l'espace vide (état  $u = 0$ ) est progressivement envahi par l'espèce considérée (état  $u = 1$ ), on parle donc de fronts d'invasion.<sup>2</sup> D'un point de vue biologique, comprendre précisément des modèles comme (1.1) permet de prédire la vitesse d'invasion d'une espèce en fonction de la répartition initiale de la population, ou des coefficients  $d$ ,  $r$ , et  $K$ .

Pour finir, le théorème B montre également que chaque profil  $q_c$  est monotone, et unique à translation près.<sup>3</sup> De plus lorsque  $c \in (0, 2)$ , il existe encore des solutions de (1.2) connectant 1 à 0 mais celles-ci ne sont ni positives, ni monotones. Voir la figure 1.2. Au-delà de l'absence de

1. On peut rapidement le voir sur l'EDO associée :  $u' = u(1 - u)$ , d'inconnue  $u(t) \in \mathbb{R}$ . C'est encore vrai pour l'EDP.

2. On les appelle aussi fronts monostables, en référence à la stabilité des états constants 1 et 0.

3. Comme les coefficients de l'équation (1.1) ne dépendent pas de l'espace, les translatées  $(t, x) \mapsto u(t, x + h)$  d'une solution  $u$  sont encore solutions pour  $h \in \mathbb{R}$ .



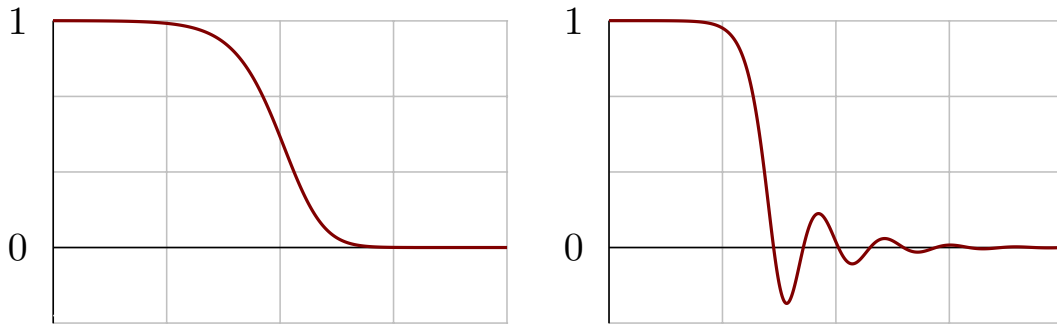


FIGURE 1.2 – Graphe des fronts de l'équation FKPP. À gauche le front critique  $x \mapsto q_*(x)$ . À droite un front sous-critique ( $c < c_*$ ) qui connecte 1 à 0 en oscillant au voisinage de 0. Ce changement de comportement est dû à l'apparition d'une valeur propre complexe pour la dynamique en espace.

sens biologique de ces fronts sous-critiques, nous verrons en sections 1.4.2 et 1.5.1 qu'il est peu probable de les observer numériquement.

#### 1.1.4 Problématique générale

De nombreux phénomènes biologiques mettent en jeu des fronts d'invasion, et peuvent être modélisés par des problèmes mathématiques plus ou moins complexes. Une question d'intérêt est de décrire les propriétés de la solution – par exemple une vitesse de propagation ou une fréquence d'oscillation – à partir des constantes présentes dans le modèle considéré. Pour répondre à ces questions, plusieurs approches peuvent être utilisées.

Pour certains problèmes – par exemple l'équation de FKPP – les propriétés de la solution peuvent être obtenues par principe de comparaison. Cette méthode robuste a l'avantage d'exploiter au mieux la structure du problème, et donc de décrire précisément quelles conditions initiales feront apparaître la vitesse de propagation étudiée. Ces méthodes s'appliquent en particulier aux équations scalaires, et à certains systèmes. Voir par exemple [NRR17; GL19].

Pour d'autres problèmes avec moins de structure, on peut adopter un point de vue dynamique : on met en évidence un équilibre, et on montre qu'il est asymptotiquement stable. De cette façon, on peut prédire la vitesse de propagation pour les conditions initiales « proches » de l'équilibre en question. Avec ce point de vue, la vitesse sélectionnée par le système est celle qui est la plus proche de la condition initiale.

Dans la suite, on adoptera la seconde démarche. Par exemple pour l'équation de FKPP, si l'on se place dans un repère en translation via le changement de variable  $y = x - ct$ , on voit que  $u(t, y) = q_c(y)$  est une solution indépendante du temps pour la dynamique translatée

$$\partial_t u = \partial_{yy} u + c \partial_y u + u(1 - u). \quad (1.3)$$

Chaque  $q_c$  est donc un équilibre, et on peut étudier sa stabilité. Cette question a été d'abord traitée d'un point de vue dynamique par Sattinger [Sat76] pour les fronts *surcritiques*, *i.e.* se déplaçant à vitesse  $c > 2$ . Le résultat obtenu est de la forme du théorème A en page 2, avec  $h(t) = e^{-\eta t}$  pour un  $\eta > 0$  dépendant de  $c$ . En quelques mots, l'utilisation d'espaces à poids permet de stabiliser les fronts surcritiques avec un trou spectral.

Le cas du front critique  $q_*$  – celui de plus petite vitesse  $c = c_* \stackrel{\text{def}}{=} 2$  – est plus subtil : la démarche de [Sat76] ne s'applique pas, car aucun poids ne permet d'obtenir de trou spectral. Ce

problème a d'abord été étudié par [EW91; Kir92; BK92], puis [Gal94; FH18; AS21]. Les résultats les plus récents et les plus aboutis obtiennent un résultat du type théorème A avec une norme  $\|v(t)\|_X = \|\omega^{-1}v(t)\|_{L^\infty(\mathbb{R})}$  ou  $\|v(t)\|_X = \|\omega^{-1}v(t)\|_{H^1(\mathbb{R})}$ , un taux de décroissance  $h(t) = (1+t)^{-3/2}$ , et une preuve que ce dernier est optimal.<sup>4</sup>

Remarquons que grâce au changement de repère  $y = x - ct$ , l'instabilité à l'avant du front a été absorbée. On parle de stabilité convective, pour désigner le fait que le front n'est stable que dans des repères en translation. Ceci sera particulièrement visible au niveau du spectre linéaire, *c.f.* section 1.4.2. Ainsi, le front et sa vitesse forment une structure qui absorbe naturellement l'instabilité de l'état à l'avant.

## 1.2 Démarche principale

Pour étudier la stabilité d'un équilibre  $\bar{u}$ , nous suivons comme annoncé plus haut une approche issue de la dynamique, qui peut être décomposée en trois étapes : étude spectrale, stabilité au niveau linéaire, stabilité au niveau non-linéaire. On décrit ici ces étapes, avant de présenter en section 1.3 le contenu de la thèse.

### 1.2.1 Linéarisation

On commence par supposer que la solution s'écrit sous la forme « équilibre plus perturbation » :

$$u(t, y) = \bar{u}(y) + v(t, y).$$

Puisque  $u$  et  $\bar{u}$  vérifient une dynamique de la forme (1.3), on récupère par soustraction une dynamique pour  $v$ , que l'on écrit comme

$$\partial_t v = \mathcal{A}v + N(v).$$

Ici,  $\mathcal{A}v$  est la partie linéaire en  $v$ , tandis que  $N(v)$  regroupe les termes non-linéaires restants. Si l'on arrive à montrer que la solution – on la note  $e^{t\mathcal{A}}p_0$  – du problème linéaire

$$\begin{cases} \partial_t p = \mathcal{A}p, \\ p(0, y) = p_0(y), \end{cases}$$

tend suffisamment vite vers 0 lorsque  $t \rightarrow +\infty$ , on utilise la formule de Duhamel

$$v(t) = e^{t\mathcal{A}}v_0 + \int_0^t e^{(t-s)\mathcal{A}}N(v(s))ds$$

couplée à un lemme de Gronwall pour montrer que la solution  $v$  du problème complet tend vers 0 lorsque  $t \rightarrow +\infty$ .

### 1.2.2 Estimées linéaires

D'après la section précédente, on peut commencer par étudier la stabilité asymptotique de l'équilibre  $p(t, x) = 0$  pour la dynamique linéaire  $\partial_t p = \mathcal{A}p$ . Pour résoudre ce problème plus

<sup>4</sup> Ici, et la plupart du temps dans la suite, le poids  $y \mapsto \omega(y)$  est une fonction lisse et strictement positive qui vaut  $q_*(y)$  pour les  $y \geq 1$ , et 1 pour les  $y \leq -1$ .

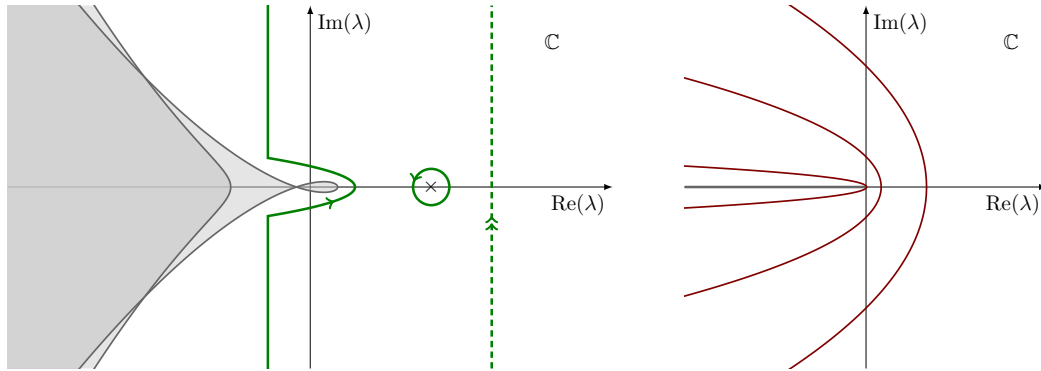


FIGURE 1.3 – À gauche : deux exemples de contours d’intégration pour la transformation inverse de Laplace. En pointillés avec deux flèches, le contour initial. En traits pleins avec une flèche, une déformation du contour, qui contourne une valeur propre instable, et passe à gauche de l’axe imaginaire. Le spectre d’un opérateur linéaire est représenté en gris. À droite : en gris le spectre de l’équation de la chaleur (*i.e.* la demi-droite  $(-\infty, 0]$ ). En rouge les contours utilisés en section 1.4.3, lorsque  $\rho$  prend les valeurs 1, 0.5 et 0.1 (de droite à gauche).

simple, on se repose sur une description du spectre de l’opérateur  $\mathcal{A}$ . De la même façon que pour les EDO, une localisation du spectre comme

$$\Sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\eta\}$$

se traduit par une estimation de la forme  $\|e^{t\mathcal{A}}p_0\|_{L^\infty(\mathbb{R})} \leq Ce^{-\tilde{\eta}t}\|p_0\|_{L^\infty(\mathbb{R})}$  pour des constantes  $C > 0$  et  $\tilde{\eta} < \eta$ . Cependant, les problèmes que l’on considérera n’auront pas de trou spectral comme ci-dessus : au contraire, une courbe de « spectre essentiel » viendra toucher l’axe imaginaire en 0, de sorte que

$$(-\infty, 0] \subset \Sigma(\mathcal{A}).$$

Pour récupérer une estimation linéaire précise, on utilisera plutôt une formule d’inversion de Laplace – voir [Paz83, section 1.7] ou [ABHN11, section 3.12] :

$$e^{t\mathcal{A}}p_0 = \frac{1}{2i\pi} \int_{\Lambda} e^{\lambda t} (\lambda - \mathcal{A})^{-1} p_0 d\lambda. \quad (1.4)$$

Celle-ci est valide pour un contour d’intégration rectiligne de la forme  $\Lambda_{\text{rect}} = \{\eta + i\xi : \xi \in \mathbb{R}\}$ , avec  $\eta > 0$  assez grand. Remarquons que dans cette formule, le temps n’apparaît que dans le terme  $e^{\lambda t}$ , dont le module est inférieur à  $e^{\operatorname{Re}(\lambda)t}$ . Ainsi, choisir un contour  $\Lambda$  comme ci-dessus, avec  $\eta > 0$ , permet difficilement d’obtenir mieux qu’une croissance exponentielle en temps.

Heureusement, il est possible de déplacer le contour  $\Lambda$  dans les régions où la résolvante  $(\lambda - \mathcal{A})^{-1}p_0$  est holomorphe en  $\lambda$ . Cette condition interdit à  $\Lambda$  d’intersecter  $\Sigma(\mathcal{A})$ , mais permet de minimiser la portion de courbe passant par  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geq 0\}$ , voir figure 1.3. Une idée importante, utilisée dans [Gal94; FH18], est de choisir un  $\Lambda$  dépendant du temps. Ceci permet de récupérer des estimations optimales. On présente cette démarche plus en détails en sections 1.4.4 et 1.4.3.

Le nouvel objectif est donc d’obtenir des estimations de la résolvante  $(\lambda - \mathcal{A})^{-1}p_0$  par rapport à  $\lambda \in \mathbb{C}$ , et de les insérer dans la formule (1.4) pour un choix judicieux de contour  $\Lambda$ .

### 1.2.3 Étude spectrale : localisation

D'après la section précédente, la dynamique en temps long du problème de départ est décrite par le spectre de  $\mathcal{A}$  et par sa résolvante  $(\lambda - \mathcal{A})^{-1}$ . On définit le premier dans cette section, et on présentera la seconde en section 1.4.3.

On renvoie au livre [KP13] pour plus de précisions sur les définitions présentées ici. Dans tous les problèmes que l'on considérera, la partie linéaire sera de la forme (1.5) suivante. Soit  $k$  un entier positif  $k \in \mathbb{N}_{\geq 0}$ , et pour  $0 \leq j \leq k-1$ , des fonctions continues  $a_j : \mathbb{R} \rightarrow \mathbb{R}$  avec des limites finies en  $\pm\infty$  :

$$a_j^\pm \stackrel{\text{def}}{=} \lim_{y \rightarrow \pm\infty} a_j(y),$$

Alors, l'opérateur linéaire  $\mathcal{A} : H^k(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  défini par

$$\mathcal{A}(y) \stackrel{\text{def}}{=} \partial_y^k + \sum_{j=0}^{k-1} a_j(y) \partial_y^j, \quad (1.5)$$

est fermé, de domaine dense  $H^k(\mathbb{R})$ . Dans la suite, on notera  $a_k(y) \stackrel{\text{def}}{=} 1$ . On définit le spectre de  $\mathcal{A}$  comme

$$\Sigma(\mathcal{A}) \stackrel{\text{def}}{=} \left\{ \lambda \in \mathbb{C} : \lambda - \mathcal{A} \text{ n'a pas d'inverse } L^2(\mathbb{R}) \rightarrow H^k(\mathbb{R}) \text{ borné} \right\}.$$

Pour  $\lambda \in \mathbb{C}$ , on dit que  $\lambda - \mathcal{A}$  est Fredholm si les deux entiers  $\dim \ker(\lambda - \mathcal{A})$  et  $\text{codim im}(\lambda - \mathcal{A})$  sont finis. Dans ce cas, on note

$$\text{Fred}(\lambda - \mathcal{A}) \stackrel{\text{def}}{=} \dim \ker(\lambda - \mathcal{A}) - \text{codim im}(\lambda - \mathcal{A})$$

son indice de Fredholm. Ceci permet de décomposer le spectre en deux parties. D'une part le spectre essentiel :<sup>5</sup>

$$\Sigma_{\text{ess}}(\mathcal{A}) \stackrel{\text{def}}{=} \{ \lambda \in \Sigma(\mathcal{A}) : \lambda - \mathcal{A} \text{ n'est pas Fredholm d'indice } 0 \},$$

d'autre part le spectre ponctuel

$$\Sigma_{\text{pt}}(\mathcal{A}) = \Sigma(\mathcal{A}) \setminus \Sigma_{\text{ess}}(\mathcal{A}).$$

Le spectre ponctuel est composé de valeurs propres isolées, de multiplicités finies. En ceci, il ressemble au spectre en dimension fini. Remarquons que la condition  $\text{Fred}(\lambda - \mathcal{A}) = 0$  se réécrit comme le théorème du rang.

On suppose de plus – ce sera vérifié dans la suite – que la convergence des coefficients se fait à vitesse exponentielle : il existe des constantes strictement positives  $\alpha$  et  $C$  telles que pour tout  $0 \leq j \leq k$

$$\sup_{y \geq 0} |a_j(y) - a_j^+| e^{\alpha|y|} \leq C, \quad \sup_{y \leq 0} |a_j(y) - a_j^-| e^{\alpha|y|} \leq C.$$

Dans ce cas, le spectre essentiel est simple à calculer : introduisons les deux opérateurs asymptotiques

$$\mathcal{A}^\pm \stackrel{\text{def}}{=} \sum_{i=0}^k a_i^\pm \partial_y^i.$$

Alors l'indice de Fredholm de  $\lambda - \mathcal{A}$  augmente (respectivement diminue) de 1 lorsque  $\lambda$  traverse

5. Dans la définition qui suit, comprendre «  $\text{Fred}(\lambda - \mathcal{A})$  est non nul ou non défini ».

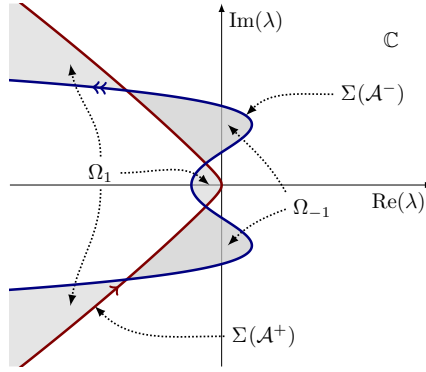


FIGURE 1.4 – Exemple de bordures de Fredholm, lorsque  $\mathcal{A}^- = -\partial_{xxxx} - 2\partial_{xx} + \partial_x - \frac{1}{2}$  et  $\mathcal{A}^+ = -\frac{1}{2}\partial_{xxxx} - \partial_{xxx} + \partial_{xx} + 2\partial_x$ . On a représenté  $\Sigma(\mathcal{A}^+)$  en rouge avec une flèche simple et  $\Sigma(\mathcal{A}^-)$  en bleu avec une flèche double. Les deux régions  $\Omega_1$  et  $\Omega_{-1}$  où l'indice de Fredholm est non nul (et vaut respectivement 1 et  $-1$ ) sont grisées : elles sont localisées « à gauche » des deux courbes.

la courbe de Fredholm  $\Sigma(\mathcal{A}^+)$  (respectivement  $\Sigma(\mathcal{A}^-)$ ) de droite à gauche. Remarquons qu'à l'aide de la transformation de Fourier, les spectre de ces opérateurs à coefficients constants s'expriment simplement à l'aide des symboles de  $\mathcal{A}^\pm$  :

$$\Sigma(\mathcal{A}^\pm) = \left\{ \lambda(\xi) = \sum_{j=0}^k (i\xi)^j a_j^\pm : \xi \in \mathbb{R} \right\}.$$

Ces deux courbes séparent le plan complexe en plusieurs régions, on décide si chacune fait partie du spectre essentiel en calculant l'indice de Fredholm correspondant. Pour les problèmes que l'on considérera,  $\text{Fred}(\lambda - \mathcal{A}) = 0$  lorsque  $\lambda$  est un réel suffisamment grand, ce qui permet de déduire  $\text{Fred}(\lambda - \mathcal{A})$  dans chaque région. En pratique, le spectre essentiel est localisé « à gauche » des deux courbes  $\Sigma(\mathcal{A}^\pm)$ .

Au contraire, le spectre ponctuel est plus complexe à borner. Un outil important, valable lorsque  $k = 2$ , est la théorie de Sturm-Liouville que l'on présente brièvement en section 1.4.1.

### 1.3 Contenu de la thèse

Comme expliqué plus haut, les fronts d'invasion semblent apparaître dans de nombreux problèmes différents. On présente maintenant trois situations pour lesquelles la structure de l'équation de FKPP est modifiée ou détériorée. Dans chaque cas, on obtient la stabilité asymptotique en temps de fronts monostables. Ceci laisse à penser que cette propriété est robuste relativement au changement d'équation, c'est-à-dire stable structurellement.

#### 1.3.1 Stabilité spectrale pour l'équation de FKPP étendue

Tout d'abord, le chapitre 2 contient un article coécrit avec Montie Avery, doctorant à l'université du Minnesota. Nous nous intéressons à l'équation scalaire

$$\partial_t u = -\delta^2 \partial_{xxxx} u + \partial_{xx} u + u(1 - u), \quad (1.6)$$

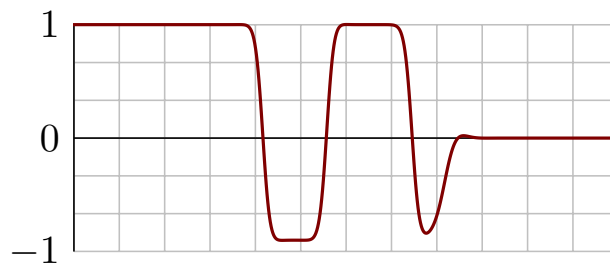


FIGURE 1.5 – Motif oscillants pour l'équation de FKPP étendue avec terme de réaction  $R(u) = u(1-u)(u+0.9)$ , avec  $\delta = \frac{1}{3}$  et une condition initiale de type Heaviside. Dans un premier temps, on voit apparaître un front connectant 1 à 0. Comme celui-ci prend des valeurs négatives, il est remplacé dans un deuxième temps par la succession d'un front bistable entre 1 et  $-0.9$  et d'un front monostable entre  $-0.9$  et 0. Le même phénomène se reproduit, et crée des motifs oscillant. Dans un troisième temps à l'arrière du front, les fronts bistables se déplacent, de sorte que l'état constant  $-0.9$  disparaît.

qu'on appellera équation de FKPP étendue. Elle a été introduite par [DS88; CER87] comme un modèle jouet pour des fronts qui laissent derrière eux des motifs oscillants. Des mécanismes similaires sont attendus dans le cas de liquides nématiques ou dans des cellules de Rayleigh-Bénard – voir l'article de Dee et van Saarloos ci-dessus, ainsi que ses références.

Ainsi, à la valeur critique  $\delta_* \stackrel{\text{def}}{=} \frac{1}{\sqrt{12}}$ , une transition des fronts de type FKPP – monotones et positifs – vers des ondes qui changent de signe est attendue, et peut être vérifiée numériquement dans un cas de réaction bistable, voir la figure 1.5. En effet, le problème biharmonique  $\partial_t u = -\partial_{xxxx} u$  ne vérifie pas de principe du maximum fort : pour une condition initiale strictement positive, la solution prend en général des valeurs négatives pour des temps arbitrairement petits. Voir [FGG08] et [GG09] et leurs références pour une étude du problème biharmonique.

L'équation (1.6) sera plus tard étudiée dans [RD98], où elle est obtenue comme une équation d'amplitude pour un système de réaction-diffusion, dans lequel une bifurcation à deux paramètres (aussi appelée bifurcation de co-dimension 2) se produit. Dans ce scénario, l'ansatz qui mène normalement à une équation de Ginzburg-Landau n'est pas utilisable,<sup>6</sup> la référence en question compare les équations de Ginzburg-Landau et (1.6), ainsi que leurs domaines de validité respectifs.

Une question naturelle semble donc d'étudier l'existence et la stabilité de fronts de propagations  $x \mapsto q_c(\delta, x)$  de l'équation (1.6). Dans cette direction, une étude perturbative  $0 < \delta \ll 1$  a été menée par [RW01]. Ils concluent à l'existence de fronts pour chaque vitesse strictement positive, et à la stabilité asymptotique des fronts critiques et surcritiques, *i.e.* ceux de vitesse  $c \geq c_*(\delta)$ . L'existence repose sur un argument de perturbation géométrique singulière – voir par exemple [Jon95]. La stabilité est obtenue par des estimations d'énergies, et peut être réécrite comme le théorème A, où  $X$  est de la forme  $L^\infty(\omega)$  pour un poids vérifiant  $\omega(x) = \mathcal{O}(q'_c(\delta, x))$  quand  $x \rightarrow +\infty$ , et où  $h$  tend vers 0 sans taux explicite.

L'étude [AG20] présentée au chapitre 2 résout la question de ce taux de décroissance. Nous obtenons de nouveau l'existence et la stabilité du front critique pour un régime perturbé  $0 < \delta \ll 1$ , cette fois avec un taux de décroissance optimal. Pour cela, nous faisons une étude du spectre au niveau linéaire, et montrons que la situation spectrale à  $\delta = 0$  n'est pas détériorée lorsque  $0 < \delta \ll 1$ . La stabilité au niveau spectral est donc robuste quand l'équation est perturbée par

6. Cela revient à faire un développement en puissance de  $e^{i\xi x}$  avec  $\xi = 0$ .

une dérivée d'ordre supérieur. On obtient ainsi un résultat de la forme suivante.

**Theorem C.** *Lorsque  $0 < \delta \ll 1$ , l'équation (1.6) admet un front monostable se déplaçant à la vitesse critique  $c_*(\delta)$ . De plus, le spectre linéaire au voisinage du front est marginalement stable :*

1. Une seule courbe de spectre touche l'axe imaginaire en 0,
2. Il n'y a pas de spectre ponctuel de partie réelle positive,
3. La situation en  $\lambda = 0$  est la même que pour FKPP : il n'y a pas de résonance.

Ceci permet ensuite d'appliquer les résultats récents de [AS21 ; AS22] : un front critique, solution d'un problème parabolique semi-linéaire scalaire avec une dérivée principale d'ordre arbitraire, et vérifiant les hypothèses spectrales mentionnées ci-dessus, est asymptotiquement stable avec taux optimal.<sup>7</sup> On récupère donc directement un résultat de la forme du théorème A avec  $h(t) = (1 + t)^{-3/2}$ .

### 1.3.2 Stabilité non-linéaire convective d'un front bi-instable

Une autre classe d'équation qui laisse apparaître des fronts avec motifs périodiques à l'arrière sont les équations avec terme de réaction non-local, dont l'archétype est l'équation scalaire de FKPP non-locale :

$$\partial_t u = \partial_{xx} u + \mu u(1 - \phi * u). \quad (1.7)$$

Ici  $\phi \in L^1(\mathbb{R})$  est un noyau dont la transformée de Fourier prend des valeurs négatives,<sup>8</sup> et  $u * v$  désigne la convolution en espace. Ce modèle est étudié pour sa pertinence biologique : il modélise l'évolution d'une espèce qui interagit non seulement avec son environnement proche, mais également avec l'environnement éloigné. Lorsque la force  $\mu$  du terme de réaction<sup>9</sup> dépasse une valeur critique  $\mu_c$ , le comportement de cette équation change drastiquement.

Lorsque  $\mu < \mu_c$ , les seuls états d'équilibre constants sont 0 et 1, et il existe des fronts d'invasion monotones, voir [BNPR09]. De plus, il n'existe pas de front qui ne soit pas monotone, voir [FZ11]. Pour ce régime, le comportement de (1.7) est donc proche de l'équation de FKPP classique : l'état stable 1 envahit l'état instable 0. Si au contraire  $\mu > \mu_c$ , des états d'équilibre périodiques apparaissent au voisinage de 1, voir [HR14] ; il existe des fronts connectant ces états périodiques à l'état constant 1, voir [FH15] pour une étude perturbative. Ainsi, dans ce régime, l'état 1 devient instable et est envahi par un motif périodique stable.

Une question encore ouverte quand  $\mu > \mu_c$ , est celle de l'existence et de la stabilité d'un front connectant les deux équilibres instables 1 et 0, voir [NPT11 ; HR14]. Pour discuter la stabilité en évitant la question de l'existence, jugée trop difficile, on étudie plutôt le modèle suivant,<sup>10</sup> dans lequel  $\mu$  est pris strictement positif.

$$\begin{cases} \partial_t u = \partial_{xx} u + u(1 - u^2) + v, \\ \partial_t v = -(1 + \partial_{xx})^2 v + \mu v - v^3 - \gamma v(1 - u). \end{cases} \quad (1.8)$$

Celui-ci a le premier avantage d'admettre un front d'invasion explicite  $Q_* = (q_*, 0)^T$ , où  $q_*$  est le front critique pour l'équation de FKPP scalaire. De plus, ce système sépare distinctement le

<sup>7</sup>. Ajoutons qu'un retard logarithmique est présent pour certaines conditions initiales [AS22].

<sup>8</sup>. Cette hypothèse technique est nécessaire pour voir apparaître des oscillations en espace.

<sup>9</sup>. Via un changement d'échelle, on peut aussi voir  $\mu$  comme la portée en espace du noyau  $\phi$ .

<sup>10</sup>. Ce modèle, ou des variantes proches, ont déjà été étudiés dans plusieurs travaux, c.f. [GSU04 ; GS07 ; BGS09 ; FHSS21]

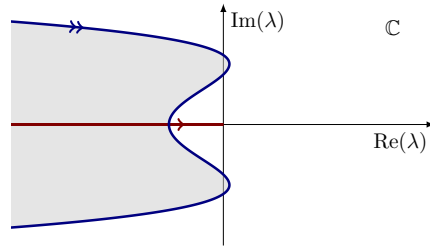


FIGURE 1.6 – Situation spectrale schématique pour le système (1.8). Pour  $x \rightarrow +\infty$ , le spectre ressemble à celui de l'équation de la chaleur : une courbe de spectre vient toucher l'origine. Pour  $x \rightarrow -\infty$ , le spectre ressemble à celui de l'équation de Swift-Hohenberg : lorsque  $\mu > 0$ , une courbe passe par les deux points instables  $\mu \pm i$ . Pour l'équation (1.7), une courbe de spectre similaire traverse l'axe imaginaire lorsque  $\mu > \mu_c$ .

mécanisme de propagation et l'instabilité au voisinage de l'état d'équilibre 1. En effet, on est en présence d'un couplage entre l'équation de FKPP (1.1) et l'équation de Swift-Hohenberg

$$\partial_t v = -(1 + \partial_{xx})^2 v + \mu v - v^3. \quad (1.9)$$

Cette dernière est obtenue dans [SH77, voir leurs équations (19) à (21)]<sup>11</sup> comme une simplification d'une équation de Bénard au voisinage d'une instabilité convective. L'équation (1.9) laisse apparaître des fronts connectant un motif périodique stable à un état constant instable – voir [EW91] – c'est-à-dire un phénomène très proche de l'instabilité de l'état 1 dans l'équation non-locale (1.7). Ainsi, bien que le système (1.8) soit plus simple par la disparition du terme non-local, il a un comportement très proche de (1.7). En effet, lorsque  $\mu$  devient positif, une courbe de spectre traverse l'axe imaginaire,<sup>12</sup> créant les motifs périodiques. C'est précisément le même phénomène qui est à l'œuvre pour le problème non-local, voir la figure 1.6.

Le chapitre 3 présente l'article [Gar21], dans lequel j'obtiens la stabilité asymptotique du front critique  $Q_*$ , avec un taux de décroissance  $h(t) = (1+t)^{-3/2}$ . Pour cela, j'utilise la démarche en trois étapes présentée en section 1.2. Les deuxième et troisième étapes sont très proches de ce qui a déjà été fait pour la stabilité du front FKPP critique, voir par exemple la section 1.4.3 ci-après. La première étape est moins évidente que dans le cas de l'équation scalaire (1.1), et reprend un argument déjà utilisé dans [BGS09].

**Theorem D.** *Lorsque  $0 < \mu \ll 1$ , et en écrivant la solution de (1.8) comme  $(u(t, x), v(t, x))^T = Q_*(x - c_* t) + P(t, x - c_* t)$ , on a le résultat de stabilité suivant. Si la perturbation initiale  $P_0$  est suffisamment petite, alors la perturbation au temps  $t$  vérifie :*

$$\|P(t)\|_{X_1} \leq C \frac{1}{(1+t)^{3/2}} \|P_0\|_Y, \quad \|P(t)\|_{X_2} \leq C \sqrt{\mu},$$

pour des espaces de Sobolev à poids  $X_1, X_2$  et  $Y$ .

On peut donc conclure que la stabilité d'un front d'invasion monostable critique est une propriété robuste par rapport à l'(in)stabilité de l'état d'équilibre à l'arrière du front.

11. Voir également [SH08].

12. On peut déjà le voir en linéarisant (1.8) au voisinage de 0.



### 1.3.3 Un problème de réaction-advection

Enfin, le chapitre 4 présente un article en cours de rédaction avec L. Miguel Rodrigues, professeur à l'université de Rennes 1. Nous nous intéressons à des équations hyperboliques scalaires d'ordre 1, dont un exemple typique est

$$\partial_t u + \frac{1}{2} \partial_x (u^2) = ru(1-u), \quad (1.10)$$

avec  $r > 0$ . Lorsque  $r = 0$ , on retrouve l'équation de Burgers sans viscosité [Bur48], utilisée pour décrire la dynamique des gaz, et plus récemment le trafic routier. L'absence de laplacien, et la présence du terme quasi-linéaire  $u \partial_x u$  ont pour effet de créer en temps fini des discontinuités – appelées chocs – dans une solution initialement lisse.

Lorsque  $r$  devient non-nul, l'équation cesse d'être une loi de conservation, mais est toujours obtenue par un bilan physique. On parle donc de « balance law », que l'on propose de traduire ici par « équation de bilan ». Comme pour les problèmes précédent, l'équation (1.10) admet les équilibres  $u(t, x) = 1$  et  $u(t, x) = 0$ , qui sont respectivement asymptotiquement stable et instable.

S'il existe des fronts continus connectant ces deux équilibres, la présence de chocs autorise une plus grande variété d'ondes, comme par exemple des impulsions, *i.e.* des profils connectant 0 à 1. Voir [Sin96] pour des exemples explicites.

Une classification des ondes bistables – *i.e.* connectant deux états stables<sup>13</sup> – a été récemment menée dans les articles [DR20; DR21] lorsque le terme de flux  $\frac{1}{2}u^2$  n'est plus forcément convexe ou concave. Les mécanismes conduisant à l'instabilité y sont listés et étudiés, des preuves de stabilité sont obtenues lorsque ces mécanismes sont absents. Mentionnons en particulier qu'en présence d'un « point sonique »<sup>14</sup>, le spectre est stabilisé lorsque l'on impose de la régularité – plutôt que de la localisation – aux perturbations.

Comme montré dans [MS97], la dynamique en temps long des équations de bilan est gouvernée par ces ondes bistables, mais aussi par le comportement au voisinage des équilibres instables. On se propose dans le chapitre 4 d'étudier l'existence et la stabilité des fronts monostables continus. Nos conclusions sont – dans le cas particulier de l'équation (1.10) – l'existence d'un front monostable  $\underline{u}_c$  pour chaque vitesse  $c$  strictement plus grande que la vitesse critique  $c_* \stackrel{\text{def}}{=} 1$ , ainsi que la stabilité de  $\underline{u}_c$  dans divers espaces à poids. Le résultat principal est donc de la forme du théorème A, avec  $X = W^{1,\infty}(\omega)$ , et un taux de décroissance comme  $h(t) = 1$ ,  $h(t) = (1+t)^{-\eta}$  ou  $h(t) = e^{-\eta t}$  suivant le poids considéré ( $\eta > 0$ ). Les poids utilisés sont suffisamment forts pour que les solutions restent continues en espace.

**Theorem E.** *Pour chaque  $c > 1$ , il existe un front continu  $\underline{u}_c$  pour (1.10). De plus,  $\underline{u}_c$  attire les conditions initiales qui lui sont exponentiellement proches (lorsque  $x \rightarrow +\infty$ ).*

## 1.4 Techniques, outils

Nous continuons cette introduction par une description de certains aspects techniques qui seront utilisés dans les chapitres à venir. Ainsi, on espère rendre plus précise la démarche expliquée en section 1.2.

<sup>13</sup>. Il faut donc remplacer le terme de droite dans (1.10) par  $u(1-u)(u-1/3)$ , voir la section 1.5.2 pour une discussion sur l'équation bistable en réaction-diffusion.

<sup>14</sup>. Pour l'exemple précis (1.10), c'est une valeur de  $x$  telle que  $\underline{u}(x) = 0$ . En général, c'est un endroit où la dynamique linéaire d'une perturbation  $v$  ressemble à  $\partial_t u = (x - x_0) \partial_x v + rv$ .

### 1.4.1 Sturm Liouville

Avec les mêmes notations que dans la section 1.2.3, on peut caractériser précisément le spectre ponctuel de  $\mathcal{A}$  lorsque  $m = 2$ .

**Theorem F.** Soit  $\mathcal{A} = \partial_{yy} + a_1(y)\partial_y + a_0(y)$  avec  $a_j$  des fonctions réelles convergeant exponentiellement vite vers des constantes  $a_j^\pm$ . Alors le spectre ponctuel de  $\mathcal{A}$  est composé d'un nombre fini  $n \in \mathbb{N}_{\geq 0}$  de valeurs propres  $\lambda_k$ . Celles-ci sont réelles, simples, et s'ordonnent comme

$$\max(a_0^+, a_0^-) < \lambda_{n-1} < \dots < \lambda_0.$$

De plus, la fonction propre associée à  $\lambda_k$  s'annule exactement  $k$  fois sur  $\mathbb{R}$ .

Pour les équations bistable et FKPP, pour lesquelles les coefficients sont indépendants de l'espace, les translatés du front  $x \mapsto \underline{u}_c(t, x + h)$  sont également des solutions ( $h \in \mathbb{R}$ ). En dérivant l'équation (1.2) par rapport à  $h$ , on retrouve  $0 = \mathcal{A}q'_c$ , c'est-à-dire que  $\lambda = 0$  est valeur propre de  $\mathcal{A}$ . Puisque les fronts sont monotones par construction,  $q'_c$  ne s'annule pas. Ainsi, lorsque 0 fait partie du spectre ponctuel,  $\lambda_0 = 0$  est le seul élément du spectre ponctuel de partie réelle positive.

Ce résultat s'adapte lorsque les coefficients de  $\mathcal{A}$  sont périodiques, et que l'on regarde le spectre relativement à  $L^2_{\text{per}}(0, 1)$ , ou lorsque le domaine spatial  $\Omega$  est borné, c.f. [KP13, section 2.3]. Bien que très précis, ces résultats ont le défaut de n'être applicable que pour un petit nombre de problèmes. Des extensions aux cas de systèmes on été obtenues, et le cas d'un domaine multidimensionnel ( $\Omega \subset \mathbb{R}^d$ ) est à l'étude, voir [Bec19] et ses références.

### 1.4.2 Stabilisation par des poids exponentiels

Pour étudier la stabilité d'une onde progressive, il est naturel de se placer dans un repère en translation. Comme on va le voir ici, l'utilisation de poids exponentiels dans un tel repère stabilise la dynamique. Pour simplifier l'explication, on effectue les calculs au voisinage d'un équilibre constant instable :

$$\partial_t u = ru,$$

avec  $r > 0$ . Les solutions sont données par  $u(t, x) = e^{rt}u_0(x)$ , on observe une croissance exponentielle. Plaçons nous dans un repère en translation à vitesse  $c$ , par le changement de variable  $u(t, x) = v(t, x - ct)$ . Alors  $v$  vérifie

$$\partial_t v = c\partial_y v + rv. \quad (1.11)$$

Plutôt que de simplement supposer  $v(t) \in L^2(\mathbb{R})$ , on fait l'hypothèse que  $v$  a un supplément de localisation en  $+\infty$  :

$$v(t, y) = e^{-\kappa y} w(t, y), \quad w(t) \in L^2(\mathbb{R}),$$

avec  $\kappa > 0$ . Alors la perturbation à poids  $w$  vérifie

$$\partial_t w = c\partial_y w + (r - c\kappa)w, \quad (1.12)$$

que l'on peut résoudre explicitement comme  $w(t, y) = e^{(r-c\kappa)t}w_0(y + ct)$  : on observe maintenant une décroissance exponentielle si  $c\kappa > r$ . Ceci se traduit visuellement par une stabilisation du spectre de l'opérateur linéaire, voir figure 1.7.

Il est également possible de gagner une décroissance plus faible en utilisant des poids polynomiaux plutôt qu'exponentiels. Après avoir utilisé le poids exponentiel optimal  $c\kappa = r$

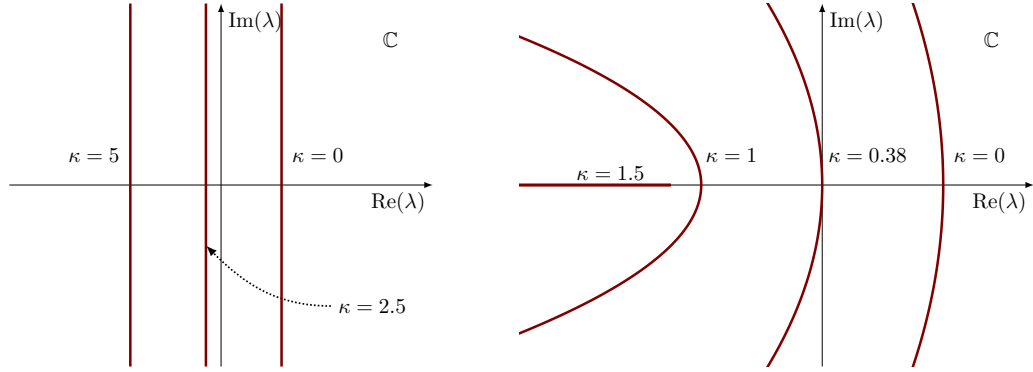


FIGURE 1.7 – À gauche : spectre de l'opérateur  $c\partial_x + (r - c\kappa)$  pour  $c = \frac{1}{2}$ ,  $r = 1$ , et différentes valeurs de  $\kappa$ . À droite : spectre de l'opérateur  $d\partial_{xx} + (c - 2d\kappa)\partial_x + (r - c\kappa + d\kappa^2)$  pour  $d = 4$ ,  $c = 6$ ,  $r = 4$  et différentes valeurs de  $\kappa$ .

pour supprimer la croissance exponentielle, on est ramené à un problème de la forme

$$\partial_t w = c\partial_y w,$$

de sorte que la solution  $w(t, y) = w_0(y + ct)$  vérifie pour  $y \geq 0$  et  $\eta \geq 0$  :

$$|w(t, y)| = \frac{1}{|1 + y + ct|^\eta} |(1 + y + ct)^\eta w_0(y + ct)| \leq \frac{1}{(1 + ct)^\eta} \|x \mapsto (1 + x)^\eta w_0(x)\|_{L^\infty(\mathbb{R})}.$$

Plutôt qu'utiliser un poids exponentiel sur-optimal  $c\kappa > r$ , on a simplement supposé que  $w$  possède des moments. On a échangé cette localisation en espace contre de la décroissance en temps.

Il est rapide d'adapter les calculs en présence d'un laplacien. Si  $u$  vérifie

$$\partial_t u = d\partial_{xx} u + ru,$$

alors les mêmes changements de variables amènent à

$$\partial_t w = d\partial_{yy} w + (c - 2d\kappa)\partial_y w + (r - c\kappa + d\kappa^2)w.$$

Pour ce problème, prendre  $\kappa$  grand n'est d'aucune utilité : l'étalement causé par le laplacien détruit la localisation. Au contraire, à localisation  $\kappa$  fixée, augmenter la vitesse  $c$  permet de stabiliser le problème. Autrement dit, si l'observateur se déplace suffisamment vite vers la droite, il ne voit pas l'instabilité de l'état constant.<sup>15</sup> Ainsi, pour chaque vitesse et coefficient  $d$ , il existe un choix de poids optimal, qui atteint la valeur  $r_{\min} \stackrel{\text{def}}{=} \min_{\kappa \in \mathbb{R}} d\kappa^2 - c_*\kappa + r$ .

Remarquons que la vitesse critique est définie comme  $c_* = 2\sqrt{rd}$ , de sorte que  $r_{\min} = 0$  est atteint en  $\kappa = \sqrt{\frac{r}{d}}$ . Le meilleur choix de  $\kappa$  n'apporte donc pas de décroissance exponentielle, et annule aussi le terme de transport  $c_* - 2d\kappa$ . Il n'est donc pas possible d'utiliser des poids polynomiaux comme ci-dessus. Par ailleurs, pour une vitesse sous-critique  $c < 2\sqrt{rd}$ , le coefficient  $r_{\min}$  est strictement positif, de sorte qu'une croissance exponentielle sera présente quel que soit le poids choisi. Ainsi, pour ces problèmes, le spectre intersecte toujours  $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > 0\}$  et on dit

<sup>15</sup>. Pour le problème simplifié que l'on regarde, cela n'empêche pas  $u(t, x)$  de croître infiniment lorsque  $t \rightarrow +\infty$ . Pour l'équation complète de FKPP, cette croissance sature à 1, et crée le motif du front.

que le *spectre absolu* est instable *c.f.* [KP13]. Il n'est donc pas possible d'obtenir la stabilité des fronts sous-critiques.

Dans les problèmes que l'on considérera, l'instabilité sera seulement présente dans la région  $y \geq 0$ , et il ne sera pas nécessaire d'utiliser de poids pour les  $y \leq 0$ . On utilisera donc plutôt les notations suivantes. Pour chaque  $\kappa \geq 0$ , soit  $\omega_\kappa$  une fonction  $\mathbb{R} \rightarrow (0, +\infty)$ , de classe  $C^\infty$ , et qui vérifie

$$\omega_\kappa(x) = \begin{cases} e^{-\kappa x} & \text{si } 1 \leq x, \\ 1 & \text{si } x \leq -1. \end{cases}$$

De même, pour chaque  $\eta \geq 0$ , soit  $\rho_\eta$  une fonction  $\mathbb{R} \rightarrow (0, +\infty)$ , de classe  $C^\infty$ , et qui vérifie

$$\rho_\eta(x) = \begin{cases} \langle x \rangle^{-\eta} & \text{si } 1 \leq x, \\ 1 & \text{si } x \leq -1. \end{cases}$$

Ici, on a utilisé la notation  $\langle x \rangle \stackrel{\text{def}}{=} \sqrt{1+x^2}$ . Ces poids exponentiels et polynomiaux suffiront dans la plupart des cas à décrire les comportements souhaités. Dans les chapitres 2 et 3, il sera également nécessaire de stabiliser la dynamique du côté de  $-\infty$ . Pour des réels  $\kappa_1, \kappa_2$ , on utilisera donc des fonctions  $\omega_{\kappa_1, \kappa_2} : \mathbb{R} \rightarrow (0, +\infty)$ , de classe  $C^\infty$ , vérifiant

$$\omega_{\kappa_1, \kappa_2}(x) = \begin{cases} e^{-\kappa_1 x} & \text{si } 1 \leq x, \\ e^{-\kappa_2 x} & \text{si } x \leq -1. \end{cases}$$

Par ailleurs, on aura besoin au chapitre 3 d'utiliser des espaces de Sobolev uniformément localisés. On consigne donc ici la notation

$$\rho_{u,1}(x) \stackrel{\text{def}}{=} \langle x \rangle^{-2} = \frac{1}{1+x^2}.$$

Remarquons pour finir que l'opérateur linéaire à poids s'obtient comme une conjugaison de l'opérateur initial. Par exemple, le passage de (1.11) à (1.12) se fait via le calcul

$$e^{\kappa y} (c \partial_y + r) (e^{-\kappa y} w) = (c \partial_y + (r - c\kappa)) w.$$

Plus généralement dans les chapitres suivants, on notera  $\mathcal{A}$  l'opérateur linéaire du problème initial, et  $\mathcal{L} \stackrel{\text{def}}{=} \omega^{-1} \mathcal{A} \omega$  l'opérateur conjugué, correspondant à la dynamique dans un espace à poids. On renvoie au lemme B.1.1 en annexe pour une expression explicite de ces opérateurs conjugués dans un cas simple.

### 1.4.3 Estimées résolvente ponctuelles : exemple

Pour clarifier les deux dernières étapes de la démarche présentée en section 1.2, on présente ici les estimations résolvantes sur l'exemple de l'équation de la chaleur scalaire en une dimension :  $p(t, y) \in \mathbb{R}$  et  $y \in \mathbb{R}$ , et

$$\partial_t p = \partial_{yy} p, \quad p(0, \cdot) = p_0.$$

Ces idées ont été initialement développées par Howard et Zumbrun *c.f.* [HZ98; HZ00] pour des équations de Burgers avec viscosité. Elles ont été adaptées par [FH18] pour l'équation de FKPP, et plus récemment dans un cadre discret en espace, pour montrer la stabilité de schémas numériques, voir [Coe21; CF21].

Pour « factoriser » la condition initiale, on cherche les solutions sous la forme d'un noyau intégral

$$p(t, y) = \int_{\mathbb{R}} \mathcal{G}(t, y, z) p_0(z) dz.$$

Le noyau  $\mathcal{G}$  est appelé fonction de Green temporelle.<sup>16</sup> Il vérifie le problème temporel fondamental

$$\partial_t \mathcal{G} = \partial_{yy} \mathcal{G}, \quad \mathcal{G}(0, \cdot, z) = \delta_z,$$

où  $\delta_z$  est la distribution de Dirac en  $y = z$ . On passe du domaine temporel au domaine spectral via la transformation de Laplace : pour  $\lambda$  complexe, soit

$$\check{u}(\lambda) \stackrel{\text{def}}{=} (\mathfrak{L}u)(\lambda) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-\lambda t} u(t) dt.$$

Pour  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ , le noyau de Green spectral  $G \stackrel{\text{def}}{=} \mathfrak{L}\mathcal{G}$  vérifie alors le problème spectral fondamental

$$(\lambda - \partial_{yy})G(\lambda, \cdot, z) = \delta_z, \quad (1.13)$$

et on remarque qu'il est un noyau pour la résolvante :

$$(\lambda - \partial_{yy})^{-1} p_0(y) = \int_{\mathbb{R}} G(\lambda, y, z) p_0(z) dz.$$

Ainsi, plutôt que d'obtenir des bornes sur l'opérateur résolvante de la forme

$$\|(\lambda - \partial_{yy})^{-1} p_0\|_{L^2(\mathbb{R})} \leq h(\lambda) \|p_0\|_{L^2(\mathbb{R})},$$

on se contente de bornes ponctuelles en espace : on contrôle le noyau  $G$  comme :

$$|G(\lambda, y, z)| \leq h(\lambda, y, z).$$

Cette démarche a le premier avantage de choisir a posteriori l'espace des perturbations que l'on autorise –  $L^2(\mathbb{R})$  ou autre – en fonction de la borne obtenue. De plus, ceci permet d'obtenir des majorations plus fine, en jouant sur les régions espace-temps.

On peut se convaincre que pour  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ , la seule solution bornée de (1.13) est

$$G(\lambda, y, z) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|y-z|}.$$

En effet, si on se contente de regarder ce problème pour  $y \in (z, +\infty)$ , on est réduit à résoudre une ODE, dont les solutions sont les  $y \mapsto ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y}$ , avec  $a$  et  $b$  des réels. Ceci est encore vrai pour  $y \in (-\infty, z)$ . On obtient donc une solution de (1.13) en recollant ces solutions exponentielles, le delta de Dirac apparaît naturellement si ce recollement est  $\mathcal{C}^0$  mais pas  $\mathcal{C}^1$  en  $y = z$ . Puisque  $G$  est un noyau pour la résolvante, on veut le construire aussi localisé en espace que possible. On se restreint à trouver  $a$  et  $b$  réels tels que

$$G(\lambda, y, z) = \begin{cases} ae^{\sqrt{\lambda}(y-z)} & \text{si } y < z, \\ be^{-\sqrt{\lambda}(y-z)} & \text{si } z < y. \end{cases}$$

<sup>16</sup>. Notons que son expression est connue pour cet exemple :  $\mathcal{G}(t, y, z) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{|y-z|^2}{4t}\right)$ .

Les conditions de saut en  $y = z$  imposent  $a = b = \frac{1}{2\sqrt{\lambda}}$ , c'est-à-dire l'expression annoncée plus haut.

Plutôt que d'utiliser l'expression explicite de  $G$  – qui ne sera pas disponible dans les cas qui nous intéressent – on se contente d'utiliser une majoration de  $G$ . Dans la suite, on suppose seulement que pour tout  $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ , pour tous réels  $y$  et  $z$ , on a

$$|G(\lambda, y, z)| \leq \frac{1}{2|\lambda|^{1/2}} e^{-\operatorname{Re}(\sqrt{\lambda})|y-z|},$$

et que  $G$  est holomorphe par rapport à  $\sqrt{\lambda}$ . De ces deux hypothèses, on déduit qu'il existe une fonction holomorphe  $\sqrt{\lambda} \mapsto H(\sqrt{\lambda}, y, z)$ , bornée uniformément par rapport à  $(y, z) \in \mathbb{R}^2$ , et telle que  $G(\lambda, y, z) = H(\sqrt{\lambda}, y, z) \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|y-z|}$ . On insère cette expression dans la formule d'inversion de Laplace (1.4), avec un contour parabolique :

$$\Lambda \stackrel{\text{def}}{=} \left\{ \lambda(\xi) = (\rho + i\xi)^2 : \xi \in \mathbb{R} \right\},$$

où  $\rho$  est à choisir strictement positif. On calcule alors

$$\mathcal{G}(t, y, z) = \frac{1}{2i\pi} \int_{\Lambda} e^{\lambda t} G(\lambda, y, z) d\lambda = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(\rho+i\xi)^2 t - (\rho+i\xi)|y-z|} H(\rho + i\xi, y, z) d\xi.$$

Puis on majore cette intégrale – à variable d'intégration réelle – par

$$|\mathcal{G}(t, y, z)| \leq \frac{C}{2\pi} \int_{\mathbb{R}} e^{\rho^2 t - \rho|y-z| - \xi^2 t} d\xi \leq \frac{C}{\sqrt{4\pi t}} e^{\rho^2 t - \rho|y-z|}.$$

Ici, on a utilisé le calcul classique  $\int_{\mathbb{R}} e^{-\xi^2} d\xi = \sqrt{\pi}$ , et  $C$  est une borne de la fonction  $H$ , dans notre cas :  $C = 1$ . On peut alors choisir  $\rho \stackrel{\text{def}}{=} \frac{|y-z|}{2t}$  pour retrouver

$$|\mathcal{G}(t, y, z)| \leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{|y-z|^2}{4t}}.$$

Ainsi, on obtient une majoration de la perturbation  $p$  au cours du temps, à partir d'une majoration de la fonction de Green spatiale. Ce type d'estimation sur  $G$  peut être obtenue en résolvant l'EDO (1.13). Remarquons que le choix de  $\rho$  permet de coupler temps et espace, de sorte que le contour  $\Lambda$  donne lieu à une estimation bornée en temps, malgré le fait que  $\Lambda$  intersecte  $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$ , voir la figure 1.3.

#### 1.4.4 Fonction de Evans : FKPP et équation de la chaleur

Pour l'équation de FKPP, la localisation du spectre est très similaire à celle de l'équation de la chaleur.<sup>17</sup> Cependant, la décroissance linéaire est en  $t^{-3/2}$  pour la première, au lieu du  $t^{-1/2}$  obtenu pour la seconde dans le paragraphe précédent. Cette différence est importante, puisqu'une décroissance intégrable à l'infini simplifie considérablement le passage du linéaire au non-linéaire. Elle est à relier à la situation spectrale en  $\lambda = 0$ .

Pour l'équation de la chaleur, on a vu apparaître une singularité au moment du calcul  $a = b = \frac{1}{2\sqrt{\lambda}}$  – voir la section 1.4.3. En effet, les deux solutions  $\varphi_+(y) = e^{-\sqrt{\lambda}y}$  et  $\varphi_-(y) = e^{\sqrt{\lambda}y}$

<sup>17</sup>. On peut comparer le spectre de l'équation de la chaleur – en partie gauche de la figure 1.3 – à celui du front critique de FKPP – en partie centrale de la figure 2.2, voir page 34.

que l'on recolle deviennent colinéaires lorsque  $\lambda$  tend vers 0,<sup>18</sup> donnant lieu à des solutions constantes pour le problème aux valeurs propres :  $\partial_{yy}\varphi = 0$ . Au contraire pour l'équation de FKPP, les solutions que l'on recolle ne deviennent pas colinéaires lorsque  $\lambda$  tend vers 0, de sorte qu'il n'y a pas de singularité dans  $G$  : les solutions du problème aux valeurs propres sont non-bornées en espace. Quand  $\lambda$  est proche de 0, on obtient donc une description de la fonction de Green comme

$$G^{\text{kpp}}(\lambda, y, z) = H(\sqrt{\lambda}, y, z)e^{-\sqrt{\lambda}|y-z|},$$

pour une fonction  $H$  bornée et holomorphe. Un calcul rapide semble annoncer qu'un gain de  $\sqrt{\lambda}$  se convertit dans la formule d'inversion de Laplace en un gain de  $t^{-1/2}$ . Si on fait une étude plus fine de la fonction  $H$ , on récupère en fait un gain de  $t^{-1}$  en payant de la localisation en espace. On peut heuristiquement expliquer ce gain de la façon suivante. Pour  $y$  et  $z$  distincts et fixés, on a

$$e^{-\sqrt{\lambda}|y-z|} = \pm \partial_y \left( \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}|y-z|} \right).$$

Donc, puisque la transformation de Laplace commute avec les dérivées en espace, on s'attend à récupérer au niveau temporel :

$$|\mathcal{G}^{\text{kpp}}(t, y, z)| \approx |\partial_y \mathcal{G}^{\text{heat}}(t, y, z)| \leq \frac{|y-z|}{\sqrt{\pi} 8t^{3/2}} e^{-\frac{|y-z|}{4t}}.$$

Le fait que les solutions recollées ne deviennent pas colinéaires en  $\lambda = 0$  a pu être démontré dans [FH18] pour l'équation de FKPP. En quelques mots : la solution du problème spectral  $y \mapsto \varphi_-(\lambda, y)$  choisie pour son comportement en  $-\infty$  est, à un poids connu près, la dérivée du front :  $\varphi_-(0, y) = q'_*(y)\omega(y)$ . La solution  $y \mapsto \varphi_+(\lambda, y)$  que l'on utilise en  $+\infty$  a un comportement comme  $e^{-\sqrt{\lambda}y}$ , et reste donc bornée lorsque  $\lambda \rightarrow 0$ . Comme on sait par ailleurs que  $q'_*(y)\omega(y) \sim y$  pour  $y \rightarrow +\infty$ , on conclut que  $\varphi_-(0, y) \neq \varphi_+(0, y)$ . On propose de retrouver le comportement  $q'_*(y) \sim ye^{-y}$  au chapitre A.

Le lien qu'on évoque entre les singularités de  $G$  et les solutions du problème aux valeurs propres est rendu précis par la fonction de Evans. On renvoie à [KP13] pour une construction générale.

### 1.4.5 Équations d'amplitude

Lorsque l'on fait varier les paramètres de certains systèmes, les solutions peuvent brusquement changer de comportement. L'exemple le plus simple est sans doute l'EDO

$$y'(t) = \mu y(t), \quad y(0) = y_0 \in \mathbb{R},$$

dont les solutions  $y(t) = e^{\mu t} y_0$  sont bornées<sup>19</sup> lorsque  $\mu \leq 0$ , et non bornées lorsque  $\mu > 0$ .

Un autre exemple, sans doute plus pertinent, est fourni par l'EDO en deux dimension

$$\begin{cases} x'(t) = \mu x - y - xy^2 - x^3, \\ y'(t) = \mu y + x - x^2 y - y^3. \end{cases}$$

Le passage en coordonnées polaires  $(x, y) = (r \cos(\theta), r \sin(\theta))$  simplifie la dynamique, puisqu'on a

<sup>18</sup>. Au sens où les deux vecteurs  $(\varphi^\pm(y), \partial_y \varphi^\pm(y))^T$  deviennent colinéaires lorsque  $\lambda \rightarrow 0$ , pour chaque  $y$  fixé.

<sup>19</sup>. On se restreint ici à  $t \geq 0$ .

alors :

$$\begin{cases} r'(t) = \mu r - r^3, \\ \theta'(t) = 1. \end{cases}$$

Pour  $\mu \leq 0$ , le seul point d'équilibre est  $r = 0$ , il est stable. Au contraire pour  $\mu > 0$ , les équilibres sont  $r = -\sqrt{\mu}$ ,  $r = 0$  et  $r = \sqrt{\mu}$ , qui sont respectivement stable, instable et stable. Si on revient à la dynamique de  $(x, y)$ , on observe alors que  $(x, y) = (0, 0)$  est stable pour  $\mu \leq 0$ , mais instable pour  $\mu > 0$  : il est remplacé par le cycle stable  $(x(t), y(t)) = (\sqrt{\mu} \cos(t), \sqrt{\mu} \sin(t))$ . On est en présence d'une bifurcation de Hopf, voir les livres [HI11] et [SU17] pour plus de détails.

Pour des problèmes plus complexes, l'expression explicite des solutions en fonction du paramètre  $\mu$  n'est pas disponible. Rappelons par exemple l'équation de Swift-Hohenberg (1.9) :

$$\partial_t u = -(1 + \partial_{xx})^2 u + \mu u - u^3.$$

Lorsque  $\mu < 0$ , l'équilibre 0 est asymptotiquement stable. Pour  $\mu > 0$ , les simulations numériques laissent apparaître des fronts de propagation modulés en espace.<sup>20</sup> Ceci laisse supposer que les solutions sont proches de

$$U(t, x) \stackrel{\text{def}}{=} \varepsilon \operatorname{Re} \left( e^{ix} A(\varepsilon^2 t, \varepsilon x) \right),$$

pour un profil complexe  $A$  à déterminer, et où on a noté  $\varepsilon \stackrel{\text{def}}{=} \sqrt{\mu}$ . Pour rendre ceci plus précis, on se place dans la limite où  $\mu$  tend vers 0 en étant positif, de sorte que les solutions se développent en puissance de  $\varepsilon$  :

$$u(t, x) = U(t, x) + \mathcal{O}(\varepsilon^2).$$

La forme particulière de  $U$  a en fait été choisie pour annuler les termes d'ordre  $\varepsilon$  dans l'erreur de résolution : en notant

$$E(t, x) \stackrel{\text{def}}{=} \partial_t u + (1 + \partial_{xx})^2 u - \mu u + u^3,$$

on a déjà  $E(t, x) = \mathcal{O}(\varepsilon^2)$ . En poussant le développement de la solution un cran plus loin

$$u(t, x) = U(t, x) + \tilde{U}(t, x) + \mathcal{O}(\varepsilon^3), \quad (1.14)$$

et en choisissant soigneusement  $\tilde{U}$ ,<sup>21</sup> on obtient

$$E(t, x) = \varepsilon^3 \left( \partial_T A - 4\partial_{XX} A - A + 3A|A|^2 \right) + \mathcal{O}(\varepsilon^4).$$

Annuler le terme en  $\varepsilon^3$  devient alors une condition que doit satisfaire  $A$  pour que l'ansatz choisi soit pertinent. On appelle alors équation d'amplitude – ou parfois équation de modulation – la condition

$$\partial_T A = 4\partial_{XX} A + A - 3A|A|^2. \quad (1.15)$$

Dans le cas présent, c'est une équation de Ginzburg-Landau, elle admet des solutions sous forme de fronts :  $A(T, X) = Q(X - cT)$ . Une fois cette équation obtenue, on peut insérer dans (1.14) la solution  $A$  de l'équation (1.15). Il est alors naturel d'étudier le régime de temps pour lequel  $E(t, x)$  reste petit, ceci a été fait dans [CE90]. On renvoie également à [MS95], qui transfère une propriété de l'équation d'amplitude (1.15) au problème original (1.9) comme par exemple l'existence d'un attracteur global.

Mentionnons que de nombreuses équations sont obtenues comme des équations d'amplitude :

<sup>20</sup>. On peut aussi les prédire à partir de la situation spectrale autour de  $u(t, x) = 0$  : les modes de Fourier les plus instables sont  $x \mapsto e^{\pm ix}$

<sup>21</sup>. L'expression à choisir s'écrit en fonction de  $A$  et de ses dérivées.



c'est le cas par exemple des équations de Schrödinger et de Kuramoto-Sivashinsky, voir [Mie02, sections 2.6 et 2.3]. Les nombreuses équations de la mécanique des fluides sont également obtenues de cette façon à partir des équations de Navier-Stokes.

## 1.5 Problèmes connexes

### 1.5.1 Existence globale de solutions

On discute l'existence et l'unicité de solutions aux équations de réaction-diffusion. On rappelle que les domaines spatiaux sont non-bornés, par exemple  $\Omega = \mathbb{R}^d$ .

Les équations paraboliques linéaires, avec conditions initiales raisonnables, sont bien posées pour tout temps  $t \in (0, +\infty)$ . Voir [Paz83, sections 7.2 et 7.3] ou [Hen81] pour la théorie dans les espaces de Sobolev. Voir également [Fri13] pour le cas extrême d'une condition initiale  $u_0(x) = \mathcal{O}(e^{\alpha|x|^2})$ .

Lorsqu'on ajoute une partie non-linéaire, l'existence de solutions en temps long n'est plus assurée. Fujita a ainsi montré que pour  $p < 1 + \frac{2}{d}$ , les solutions de l'équation scalaire

$$\partial_t u = \Delta u + u^p$$

explosent en temps fini lorsque la condition initiale est positive, non identiquement nulle, et de taille arbitraire : voir [Fuj66] et l'introduction de [Alf17].

Dans l'équation de FKPP, l'exposant  $p = 2$  de la non-linéarité est strictement en dessous de l'exposant de Fujita, et ce phénomène d'explosion est susceptible d'arriver pour des conditions initiales prenant des valeurs négatives. Ainsi, tout résultat de stabilité d'un front critique ou surcritique  $c \geq c_*$  suppose – parfois au travers d'une condition sur la taille des perturbations – que la donnée initiale vérifie  $u_0(x) \geq 0$  pour tout  $x \in \mathbb{R}$ . De même bien qu'on ne connaisse pas le comportement en temps des solutions au voisinage des fronts sous-critiques  $c < c_*$ , on peut conjecturer une explosion en temps fini de ces solutions, à cause des valeurs négatives qu'elles prennent. Voir la figure 1.2.

### 1.5.2 Stabilité orbitale

Dans de nombreux problèmes physiques, les coefficients de l'équation considérée sont indépendant de l'espace. Ainsi, les translatées d'une solution  $(t, x) \mapsto u(t, x + \psi)$  sont également solutions pour chaque  $\psi \in \mathbb{R}$ . Dans un tel scénario, un front n'est jamais isolé, et la solution associée à une condition initiale proche de  $q_c(x)$  se rapprochera en temps long de  $(t, x) \mapsto q_c(x - ct + \psi)$  pour un  $\psi$  potentiellement non nul mais dépendant de la condition initiale. On parle de stabilité orbitale, car la solution dérive le long de l'orbite de l'équilibre. Ce phénomène est par exemple présent pour l'équation bistable

$$\partial_t u = \partial_{xx} u + u(1-u)(u-1/3).$$

Pour de tels problèmes, il est connu que la phase est associée à la dérivée du front  $q'_c$ , c'est à dire à la fonction propre correspondant à la valeur propre 0. Quand  $\lambda = 0$  est une valeur propre isolée, il est possible de construire la phase de sorte que la solution soit orthogonale à  $q'_c$ , voir par exemple [Sat76]. On mentionne [JNRZ13] pour une étude similaire dans le cas d'ondes périodiques dans les équations de réaction-diffusion. Citons également [MT14] pour un résultat de stabilité orbitale par rapport à la vitesse, dans le cadre d'une équation de Korteweg-de Vries (KdV).

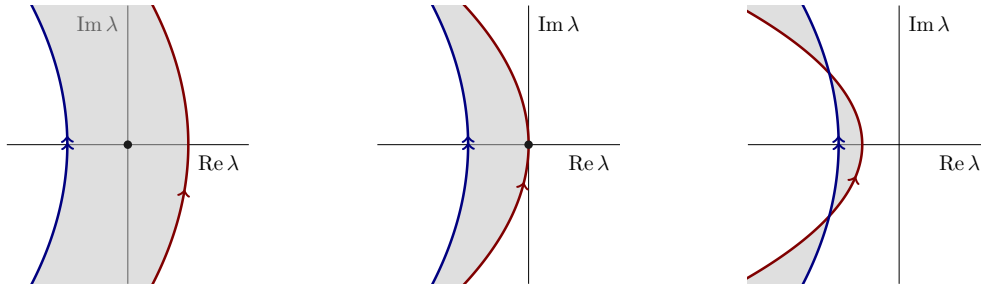


FIGURE 1.8 – Spectre de la linéarisation au voisinage d’un front surcritique dans différents espaces à poids : sans poids (gauche), avec poids optimal (centre), avec poids sur-optimal (droite). Dans ces espaces et lorsque  $x \rightarrow +\infty$ , la dérivée du front est respectivement : exponentiellement décroissante ; bornée ; non-bornée. On la représente par un point à l’origine lorsqu’elle correspond à une solution bornée du problème aux valeurs propres.

Dans le cas de l’équation de FKPP, on n’observe pas de stabilité orbitale, bien que les translatsés des fronts soient également solutions. En effet,  $\lambda = 0$  n’est pas une valeur propre isolée pour ce problème : elle fait partie du spectre essentiel. On a vu qu’il est possible de se placer dans un espace à poids pour stabiliser ce dernier. Cependant, dans un tel espace, la solution de  $\mathcal{A}u = 0$  n’est plus localisée, et la valeur propre en 0 est effacée. Ainsi, la stabilisation des fronts FKPP passe par le choix d’une topologie dans laquelle deux translatsés d’un même front sont toujours éloignés.

### 1.5.3 Fronts FKPP surcritiques

On a déjà mentionné que Sattinger obtient dans [Sat76] la stabilité asymptotique des fronts FKPP surcritiques, en utilisant un poids suffisamment fort pour stabiliser le spectre, avec un trou spectral. Il est légitime de se demander ce qui se passe lorsque le poids exponentiel est choisi optimal, de sorte que le spectre est encore tangent à l’axe imaginaire, voir la figure 1.8. Dans ce cadre, la dynamique admet encore un terme de transport non-nul, et il est possible d’abandonner des poids polynomiaux en espace pour récupérer de la décroissance en temps  $t^{-\eta}$  avec  $\eta \geq 0$ . Cette approche semble suffisante pour obtenir de la stabilité asymptotique, et ressemble à celle utilisée dans le chapitre 4. Notons que la fonction de Evans  $D$  ne s’annule pas en 0 pour ce problème, on peut le voir des deux façons suivantes :

- Rappelons que  $D$  est indépendante du poids utilisé, et on utilise un poids  $\tilde{\omega}$  plus fort que le poids optimal  $\omega_c$ , de façon à décoller le spectre de l’axe imaginaire. Puisque  $\lambda = 0$  est alors en dehors du spectre essentiel,  $D(0) = 0$  est équivalent à ce que 0 soit valeur propre isolée. Par Sturm-Liouville, on déduit que  $\lambda = 0$  admet au plus une fonction propre : la dérivée du front  $y \mapsto q'_c(y)\tilde{\omega}(y)$ . Or, un calcul rapide montre que  $q'_c\tilde{\omega}$  est non borné en  $+\infty$ , et n’est donc pas une fonction propre. On en déduit que  $D(0) \neq 0$ .
- Par définition,  $D(0) = \det(q'_c(0), \varphi_+(0, 0))$ . Or, la construction du front  $q_c$  – voir [Sat76, section 6] – assure que son comportement en  $+\infty$  est générique :  $q'_c(y) = a\varphi_+(0, y) + b\psi_+(0, y)$ , avec  $a$  et  $b$  des réels non nuls. Ainsi,  $q'_c$  et  $\varphi_+$  ne sont pas colinéaires.

On mentionne [Kap94] pour une étude où la fonction de Evans s’annule en 0 et où le spectre essentiel est tangent à l’axe imaginaire.



# Marginal stability for the extended KPP equation

## Outline of the chapter

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## 2.1 Introduction

### 2.1.1 Background and main results

The extended Fisher-KPP equation

$$u_t = -\delta^2 u_{xxxx} + u_{xx} + f(u), \quad f(0) = f(1) = 0, \quad \delta \in \mathbb{R} \quad (2.1)$$

is a fundamental model for understanding the dynamics of invasion fronts in systems without comparison principles [DS88], and may further be derived as an amplitude equation near certain co-dimension 2 bifurcations in reaction-diffusion systems [RD98]. Indeed, while rigorous results on front propagation from steep initial data are typically limited to equations with comparison principles, the *marginal stability conjecture* predicts that invasion speeds in spatially extended systems are universally predicted by marginal spectral stability of an associated invasion front [Saa03]. In the current setting, such invasion fronts solve the traveling wave equation

$$0 = -\delta^2 q'''' + q'' + cq' + f(q), \quad q(-\infty) = 1, \quad q(\infty) = 0. \quad (2.2)$$

The review paper [Saa03] presents many examples in which this conjectured behavior is observed in systems without comparison principles through numerical simulations, physical experiments, and formal asymptotic analysis. The lack of a comparison principle is essential to much of the

interesting dynamics explored in [Saa03], in which invasion fronts select features of periodic patterns generated in their wake. Concurrent to the present work, Avery and Scheel gave a rigorous proof of the marginal stability conjecture for unpatterned invasion in higher order parabolic systems, identifying precise spectral criteria which lead to selection of critical pulled fronts [AS22]. The present work establishes that these spectral assumptions hold for (2.1) for  $\delta$  sufficiently small, thereby establishing front selection in the absence of comparison principles and making progress towards understanding the dynamics of pattern forming fronts explored in [Saa03].

Here we assume  $f$  is of Fisher-KPP type:  $f(0) = f(1) = 0$ ,  $f'(0) > 0$ ,  $f'(1) < 0$ , and for instance  $f''(u) < 0$  for all  $u \in (0, 1)$ ; see Section 2.1.2 for comments on this last assumption. In this case, the marginal stability conjecture predicts that strongly localized initial data in (2.1) propagate with the *linear spreading speed*  $c_*(\delta)$ , a distinguished speed for which solutions to the linearization

$$u_t = -\delta^2 u_{xxxx} + u_{xx} + cu_x + f'(0)u$$

generically grow exponentially pointwise for  $c < c_*(\delta)$  but decay for  $c > c_*(\delta)$ . The linear spreading speed may be more precisely characterized by the location of simple pinched double roots of the associated dispersion relation; see below for details. Our first result establishes the existence of a critical front traveling with the linear spreading speed, which was previously proved by Rottschäfer and Wayne using geometric singular perturbation theory [RW01].

**Theorem 2.1.1** (Existence of the critical front). *There exists  $\delta_0 > 0$  such that for all  $\delta \in (-\delta_0, \delta_0)$ , there exists  $c = c_*(\delta)$  and a smooth traveling front  $q_*$  solving (2.2), such that*

$$q_*(x; \delta) = (\mu(\delta) + x)e^{-\eta_*(\delta)x} + O(e^{-(\eta_*(\delta) + \eta)x}), \quad x \rightarrow +\infty$$

for some  $\eta > 0$ , where  $\delta \mapsto \eta_*(\delta) \in C^1(-\delta_0, \delta_0)$ , with  $\eta_*(\delta) = \sqrt{f'(0)} + O(\delta^2)$  when  $\delta \rightarrow 0$ , and  $\delta \mapsto \mu(\delta)$  is continuous, with  $\mu(0) = 1$ . Moreover,  $q_*(\cdot; \delta)$  depends continuously on  $\delta$ , uniformly in space.

Our proof is based on a far-field/core decomposition, relying only on basic Fredholm properties of the linearization about the critical front for  $\delta = 0$  together with explicit preconditioners which regularize the singular perturbation. We believe our methods have further utility in describing bifurcations from pushed to pulled front propagation as well as analyzing invasion fronts in nonlocal equations; see Section 2.1.2 below. We also mention that the existence of both invasion fronts and fronts connecting two stable states in fourth order parabolic equations, including the extended Fisher-KPP equation with  $\delta$  not necessarily small, was established in [BHV01] using topological arguments.

Perturbations  $v(t, x - c_*(\delta)t) = u(t, x) - q_*(x - c_*(\delta)t; \delta)$  of the critical front in (2.1) solve

$$v_t = \mathcal{A}(\delta)v + f(q_* + v) - f(q_*) - f'(q_*)v, \quad (2.3)$$

where  $\mathcal{A}(\delta) : H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is the linearization about the critical front, defined through

$$\mathcal{A}(\delta) := -\delta^2 \partial_x^4 + \partial_x^2 + c_* \partial_x + f'(q_*(x; \delta)). \quad (2.4)$$

The essential spectrum of the linearization  $\mathcal{A}(\delta)$  is unstable due to the instability of the background state  $u \equiv 0$ . Hence, to establish a stability result, one restricts to perturbations with prescribed exponential localization. The optimal exponential weight here matches the decay

rate of the critical front; we therefore define

$$\omega_*(x; \delta) = \begin{cases} e^{\eta_*(\delta)x}, & x \geq 1, \\ 1, & x \leq -1, \end{cases} \quad (2.5)$$

so that the conjugate operator  $\mathcal{L}(\delta) = \omega_*(\cdot; \delta)A(\delta)\omega_*(\cdot; \delta^{-1}) : H^4(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  describes the linearized dynamics of perturbations in this weighted space. The essential spectrum of  $\mathcal{L}(\delta)$  is marginally stable, touching the imaginary axis only at the origin; see Figure 2.1 and Lemma 2.2.1 for details. Our main result establishes spectral stability for  $\mathcal{L}(\delta)$  as required by the marginal stability conjecture in light of [AS22].

**Theorem 2.1.2** (Spectral stability). *There exists a  $\delta_0 > 0$  such that for all  $\delta \in (-\delta_0, \delta_0)$  the operator  $\mathcal{L}(\delta)$  has no eigenvalues  $\lambda$  with  $\operatorname{Re} \lambda \geq 0$ , and there does not exist a bounded solution to  $\mathcal{L}(\delta)u = 0$ .*

Together with Lemmas 2.2.1 and 2.2.2 which control the essential spectrum, Theorem 2.1.2 says that the critical front is marginally spectrally stable, in the sense that

- the essential spectrum of  $\mathcal{L}(\delta)$  is contained entirely in the left half plane except for a single branch which touches the imaginary axis at the origin;
- the linearization  $\mathcal{L}(\delta)$  has no unstable eigenvalues, or eigenvalues on the imaginary axis;
- there is no “embedded eigenvalue” (more precisely, no resonance pole of the Evans function) in the essential spectrum at  $\lambda = 0$ .

The results in [AS21] therefore imply nonlinear stability of the critical front against localized perturbations, with sharp decay rates and precise characterization of the leading order asymptotics. To state these, we first define for  $r \in \mathbb{R}$  a smooth positive one-sided algebraic weight  $\rho_r$  which satisfies

$$\rho_r(x) = \begin{cases} 1, & x \leq -1, \\ (1+x^2)^{r/2}, & x \geq 1. \end{cases}$$

We then have the following nonlinear stability results.

**Corollary 2.1.3** (Nonlinear stability). *Let  $r > \frac{3}{2}$  and  $\delta_0 > 0$  as in Theorem 2.1.2. There exist constants  $\varepsilon > 0$  and  $C > 0$  such that if  $\delta \in (-\delta_0, \delta_0)$  and  $\|\omega_*(\cdot; \delta)\rho_r v_0\|_{H^1} < \varepsilon$ , then*

$$\|\omega_*(\cdot; \delta)\rho_{-r}v(t, \cdot)\|_{H^1} \leq \frac{C\varepsilon}{(1+t)^{3/2}},$$

where  $v$  is the solution to (2.3) with initial data  $v_0$ . Furthermore if  $r > \frac{5}{2}$ , then there exists a real number  $\alpha_* = \alpha_*(\omega_*(\cdot; \delta)\rho_r v_0)$ , depending smoothly on  $\omega_*(\cdot; \delta)\rho_r v_0$  in  $H^1(\mathbb{R})$ , such that for  $t > 1$ ,

$$\|\rho_{-r}\omega_*(\cdot; \delta)(v(t, \cdot) - \alpha_* t^{-3/2} q'_*(\cdot; \delta))\|_{H^1} \leq \frac{C\varepsilon}{(1+t)^2}.$$

Crucial to this sharp  $t^{-3/2}$  decay compared to the standard diffusive decay rate  $t^{-1/2}$  is the lack of an embedded eigenvalue of the linearization at  $\lambda = 0$ , as captured here in Theorem 2.1.2. Nonlinear stability of the critical front for  $\delta \neq 0$  was obtained in [RW01] via weighted energy estimates, but without a precise characterization of the decay rate, while the  $t^{-3/2}$  decay rate obtained here is sharp in light of the asymptotics given in Corollary 2.1.3.

Finally, we mention that the spectral assumptions claimed in Theorem 2.1.2, Lemma 2.2.1, and Lemma 2.2.2 lead to stability against weakly localized perturbations, up to a logarithmic shift, see [AS22].

**Corollary 2.1.4** (Front selection). *Fix  $r > 2$ . For any  $\varepsilon > 0$  there exists a class of initial data  $\mathcal{U}_\varepsilon$ , including some nontrivial data supported on a half-line, such that for any  $u_0 \in \mathcal{U}_\varepsilon$ , we have*

$$\sup_{x \in \mathbb{R}} |\rho_{-1}(x) \omega_*(x; \delta) [u(x + \sigma(t), t) - q_*(x; \delta)]| < \varepsilon,$$

where  $u$  is the solution to (2.1) with initial data  $u_0$ , and

$$\sigma(t) = c_*(\delta)t - \frac{3}{2\eta_*(\delta)} \log t + x_\infty(u_0)$$

for some  $x_\infty(u_0) \in \mathbb{R}$ . Moreover,  $\mathcal{U}_\varepsilon$  is open in the topology induced by the norm  $\|f\| = \|\rho_r \omega_*(\cdot; \delta) f\|_{L^\infty}$ .

This result confirms that open classes of steep initial data propagate with the linear spreading speed  $c_*(\delta)$ , up to a universal logarithmic delay, as predicted by the marginal stability conjecture [Saa03]. In the classical Fisher-KPP equation,  $\delta = 0$ , analogous convergence results for non-negative steep data may be shown using comparison principles [AW78; HNRR13; NRR17; Lau85] or probabilistic methods [Bra78; Bra83]. We believe Corollary 2.1.4 represents an important step in extending results on front selection beyond equations with comparison principles and toward pattern forming systems.

*Remark 2.1.5.* While there are some limited results available on front propagation in some non-local equation without comparison principles (see e.g. [BHR17; BHR20]), these results rely on exploiting the specific structure of the equation to reduce the consideration to a scalar Fisher-KPP type equation to which comparison principle arguments may be applied. To the best of our knowledge, Corollary 2.1.4 is the first result establishing the marginal stability conjecture in which the comparison principle does not play any role in the nonlinear argument, and the first result for an equation higher than second order in space.

## 2.1.2 Remarks

**Assumptions on  $f$ .** Since we prove our results by perturbing from the classical Fisher-KPP equation, our results hold for any smooth nonlinearity  $f$  which satisfies  $f(0) = f(1) = 0$ ,  $f'(0) > 0$ ,  $f'(1) < 0$ , and for which existence and spectral stability of the critical front hold for the classical Fisher-KPP equation with this reaction term. In particular, this is implied by the assumption  $f''(u) < 0$  for  $u \in (0, 1)$  [Sat76, Theorem 5.5], which we state in the introduction. This can be weakened, for instance, to the assumption that  $0 < f(u) \leq f'(0)u$  for  $u \in (0, 1)$ ; see e.g. [AW78].

**General approach – preconditioning.** Our approach to regularizing the singular perturbation is based on preconditioning with an appropriately chosen operator. To illustrate the main idea, briefly consider the eigenvalue problem for the unweighted linearization,  $(\mathcal{A}(\delta) - \lambda)u = 0$ . Applying  $(1 - \delta^2 \partial_x^2)^{-1}$  to  $\mathcal{A}(\delta) - \lambda$ , we obtain

$$\begin{aligned} (1 - \delta^2 \partial_x^2)^{-1} (\mathcal{A}(\delta) - \lambda) &= (1 - \delta^2 \partial_x^2)^{-1} [(1 - \delta^2 \partial_x^2) \partial_x^2 + c_*(\delta) \partial_x + f'(q_*) - \lambda] \\ &= \partial_x^2 + (1 - \delta^2 \partial_x^2)^{-1} (c_*(\delta) \partial_x + f'(q_*) - \lambda) \\ &= \partial_x^2 + c_*(\delta) \partial_x + f'(q_*(\cdot; \delta)) - \lambda + T(\delta) (c_*(\delta) \partial_x + f'(q_*(\cdot; \delta)) - \lambda), \end{aligned}$$

where  $T(\delta) = (1 - \delta^2 \partial_x^2)^{-1} - 1$ . Once we prove that the terms involving  $T(\delta)$  are continuous in  $\delta$ , the eigenvalue problem becomes essentially a regular perturbation of the classical Fisher-KPP

linearization, at  $\delta = 0$ . We prove the necessary estimates on the preconditioners using direct Fourier analysis in Section 2.2.3. This approach is inspired by that used to construct oblique stripe solutions in a quenched Swift-Hohenberg equation in [GS18].

**Stability to less localized perturbations.** We note that under the spectral stability conditions we prove here, in addition to Corollary 2.1.3, one also immediately obtains stability under less localized perturbations from [AS21, Theorems 3 and 4], with a prescribed decay rate which is slower than  $t^{-3/2}$ . See for details.

**Geometric vs. functional analytic point of view.** We remark here that one should also be able to prove the spectral stability results obtained here using geometric dynamical systems methods, in particular geometric singular perturbation theory in the sense of Fenichel [Fen79] together with the gap lemma [GZ98; KS98], which is used to extend the Evans function into the essential spectrum. An attractive feature of our approach here is that it is quite self contained, ultimately relying mostly on basic Fredholm theory and Fourier analysis. We also remark that in principle the functional analytic methods could be adapted, together with the approach to linear stability through obtaining resolvent estimates via far-field/core decompositions in [AS21], to problems in stability of critical fronts in nonlocal equations, since these methods do not rely as heavily on the presence of an underlying phase space. Some of the relevant Fredholm theory for nonlocal operators has been developed in [FS14; FS15].

**Natural range for  $\delta$ .** In this chapter, we have restricted to small  $\delta$ . However, we believe that similar results should hold true for larger values of this parameter. While the existence of fronts is established in [BHV01] for all speeds  $c > 0$  and  $\delta \in \mathbb{R}$ , we do not have access to explicit decay when  $x \rightarrow +\infty$  for this fronts, which seems necessary to establish precise stability. Monotonicity of the front would imply such a precise decay by use of Ikehara's theorem, see chapter A. An important value is  $\bar{\delta} = 1/\sqrt{12f'(0)}$ , at which the dispersion relation admits a triple root, and the essential spectrum of the linearized operator becomes tangent to the imaginary axis. Nonlinear stability at or above this value of  $\delta$  is therefore fundamentally outside the scope of [AS21].

**Supercritical and subcritical fronts.** If we consider a supercritical front, traveling with speed  $c > c_*(\delta)$  and constructed in [RW01], one can simplify the argument of Theorem 2.1.2 to prove that the linearization about such a front has no unstable point spectrum. For these fronts, one can use an exponential weight to push the essential spectrum entirely into the left half plane, and thereby with the analogue of Theorem 2.1.2 obtain stability of supercritical fronts with an exponential decay rate using standard semigroup methods (see e.g. [Hen81]). Subcritical fronts, with  $c < c_*(\delta)$ , have unstable absolute spectrum, meaning in particular that the essential spectrum of the linearization about any of these fronts is unstable in any exponentially weighted space. A modified version of our proof of Theorem 2.1.1 should also give existence of these supercritical and subcritical fronts using functional analytic methods, although we do not give the details here.

**Additional notation.** For  $r > 0$ , we let  $B(0, r)$  denote the ball of radius  $r$  centered at the origin in the complex plane. We will use the notation  $\langle u, v \rangle = \int_{\mathbb{R}} u \bar{v} dx$  to denote the standard inner product on  $L^2(\mathbb{R}, \mathbb{C})$ . We let  $\mathcal{B}(X, Y)$  denote the space of bounded linear operator between two Banach spaces  $X$  and  $Y$ .

**Outline.** The remainder of this chapter is organized as follows. In Section 2.2, we compute some preliminary information needed for our analysis (the linear spreading speed in (2.2) and the cokernel of  $\mathcal{L}(0)$ ) and prove some necessary estimates on our preconditioner. In Section 2.3, we use explicit preconditioners and a far-field/core decomposition to prove Theorem 2.1.1, establishing existence of the critical front. In the remaining sections, we complete the proof of Theorem 2.1.2 using a decomposition of the point spectrum into three regions, see Figure 2.3. In Section 2.4 we define a functional analytic analogue  $E$  of the Evans function near  $\lambda = 0$ , through a Liapunov-Schmidt reduction of the eigenproblem. Then, we use it together with knowledge of



the spectrum of  $\mathcal{L}(0)$  to prove that  $\mathcal{L}(\delta)$  has no resonance at the origin or unstable eigenvalues near the origin, when  $\delta$  is small. In Section 2.5, we rule out unstable eigenvalues away from the origin using both a perturbation argument and ellipticity of the operator.

## 2.2 Preliminaries

### 2.2.1 Exponential weights

In addition to the critical weight (2.5) which we use to shift the essential spectrum out of the right half plane, we will need further exponential weights to recover Fredholm properties of  $\mathcal{L}(\delta)$  and related operators for our far-field/core analysis. For  $\eta_{\pm} \in \mathbb{R}$ , we define a smooth positive weight function  $\omega_{\eta_-, \eta_+}$  satisfying

$$\omega_{\eta_-, \eta_+} = \begin{cases} e^{\eta_- x}, & x \leq -1, \\ e^{\eta_+ x}, & x \geq 1. \end{cases}$$

If  $\eta_- = 0$  and  $\eta_+ = \eta$ , then we write  $\omega_{\eta_-, \eta_+} = \omega_{\eta}$ . If  $\eta_- = \eta_+ = \eta$ , we choose  $\omega_{\eta, \eta}(x) = e^{\eta x}$ .

Given an integer  $m$ , we define the exponentially weighted Sobolev space  $H_{\eta_-, \eta_+}^m(\mathbb{R})$  through the norm

$$\|f\|_{H_{\eta_-, \eta_+}^m} = \|\omega_{\eta_-, \eta_+} f\|_{H^m}.$$

We note that for  $\eta > 0$  we have  $H_{0, \eta}^m(\mathbb{R}) = H^m(\mathbb{R}) \cap H_{\eta, \eta}^m(\mathbb{R})$  as well as the following equivalence of norms

$$\|f\|_{H_{0, \eta}^m} \sim \|f\|_{H^m} + \|f\|_{H_{\eta, \eta}^m}. \quad (2.6)$$

This characterization of the one-sided weighted spaces is useful in obtaining estimates on operators defined by Fourier multipliers on these spaces, and we make use of this below in Section 2.2.3.

### 2.2.2 Linear spreading speed and essential spectrum

The linear spreading speed, marking the transition from pointwise growth to pointwise decay in the linearization about  $u \equiv 0$ , is characterized here by the location of simple pinched double roots of the *dispersion relation*

$$d_c^+(\lambda, \nu) = -\delta^2 \nu^4 + \nu^2 + c\nu + f'(0) - \lambda; \quad (2.7)$$

see [HS14b] for background.

**Lemma 2.2.1** (Linear spreading speed). *There exists  $\delta_0 > 0$  such that for  $\delta \in (-\delta_0, \delta_0)$ , there exists a critical speed  $c_* = c_*(\delta)$ , and an exponent  $\eta = \eta_*(\delta) > 0$  for the critical weight such that the right dispersion relation (2.7) satisfies the following properties.*

(i) *Simple pinched double root: for  $\lambda, \nu$  near  $0 \in \mathbb{C}$ :*

$$d_{c_*}^+(\lambda, -\eta_* + \nu) = \nu^2 \sqrt{1 - 12\delta^2 f'(0)} - \lambda + O(\nu^3), \quad (2.8)$$

with  $\sqrt{1 - 12\delta^2 f'(0)} > 0$ .

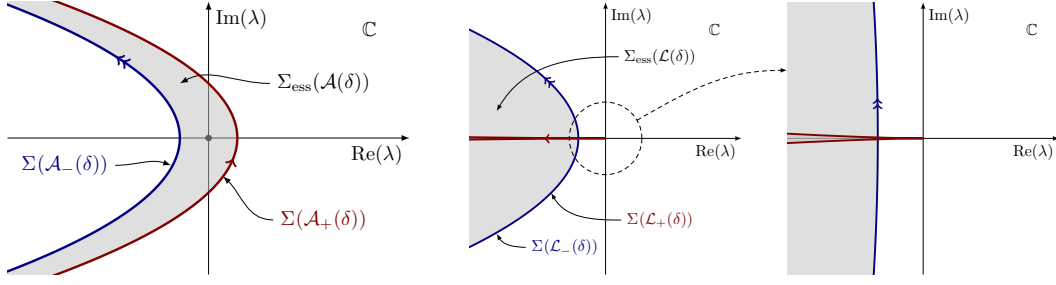


Figure 2.1 – Left: the essential spectrum of the unweighted operator  $\mathcal{A}(\delta)$  is bounded by the Fredholm borders in blue and red. Middle and right: overview and zoom near the origin of the essential spectrum of the weighted operator  $\mathcal{L}(\delta)$ .

(ii) *Minimal critical spectrum:* if  $d_{c_*}^+(i\kappa, -\eta_* + ik) = 0$  for some  $\kappa, k \in \mathbb{R}$ , then  $\kappa = k = 0$ .

(iii) *No unstable essential spectrum:* if  $d_{c_*}^+(\lambda, -\eta_* + ik) = 0$  for some  $k \in \mathbb{R}$  and  $\lambda \in \mathbb{C}$ , then  $\text{Re } \lambda \leq 0$ .

We prove Lemma 2.2.1 below, but first we explain how this lemma determines the essential spectrum for  $\mathcal{L}(\delta)$ , the linearization about the critical front in the exponentially weighted space with critical weight determined by this lemma. The operator  $\mathcal{L}(\delta)$  has the precise form

$$\mathcal{L}(\delta) = \omega_*(\cdot; \delta) \mathcal{A}(\delta) \omega_*(\cdot; \delta)^{-1} = -\delta^2 \partial_x^4 + \delta^2 a_3 \partial_x^3 + (1 + \delta^2 a_2) \partial_x^2 + a_1 \partial_x + a_0,$$

where the coefficients  $a_i(x; \delta)$  converge to limits  $a_i^\pm(\delta)$  exponentially quickly when  $x \rightarrow \pm\infty$ , and are defined using the local notation  $\zeta(x) := 1/\omega_*(x; \delta)$  by the following expressions:

$$a_3 = -4 \frac{\zeta'}{\zeta}, \quad a_2 = -6 \frac{\zeta''}{\zeta}, \quad a_1 = c_* + 2 \frac{\zeta'}{\zeta} - 4\delta^2 \frac{\zeta'''}{\zeta}, \quad a_0 = f'(q_*) + c_* \frac{\zeta'}{\zeta} + \frac{\zeta''}{\zeta} - \delta^2 \frac{\zeta''''}{\zeta}. \quad (2.9)$$

We note that  $\zeta^{(k)}(x)/\zeta(x) = (-\eta_*)^k$  for  $x \geq 1$ .

For such a linear operator, the essential spectrum is delimited by the two Fredholm borders, which are defined using the asymptotic dispersion relations. More precisely, the boundaries of the essential spectrum of  $\mathcal{L}$  are determined by the essential spectrum of the limiting operators  $\mathcal{L}_\pm$ , obtained by sending  $x \rightarrow \pm\infty$  [KP13; FS03]. From the construction of  $c_*$ ,  $\eta_*$  (see the proof of Lemma 2.2.1 below), we have at  $+\infty$ :

$$\mathcal{L}_+(\delta) = -\delta^2 \partial_x^4 + 4\eta_* \delta^2 \partial_x^3 + (1 - 6\delta^2 \eta_*^2) \partial_x^2. \quad (2.10)$$

The spectrum of this constant coefficient operator is, via Fourier transform, readily seen to be marginally stable; see the red curves of Figure 2.1. Notice that for  $\delta$  small,  $\eta = \eta_*(\delta)$  is the only reasonable value for which  $\mathcal{L}_+$  has a non positive zeroth order term; any other choice of  $\eta(\delta)$  will lead to spectral instability for  $\mathcal{L}(\delta)$ . At  $-\infty$ , there is no contribution from  $\omega_*(\cdot; \delta)$ , hence

$$\mathcal{L}_- = \mathcal{A}_- = -\delta^2 \partial_x^4 + \partial_x^2 + c_* \partial_x + f'(1)$$

has a stable spectrum, with spectral gap  $f'(1) < 0$ . Via the Fourier transform, this spectrum is determined by the asymptotic dispersion relation

$$d_{c_*}^-(\lambda, \nu) = -\delta^2 \nu^4 + \nu^2 + c_*(\delta) \nu + f'(1) - \lambda.$$

**Lemma 2.2.2** (Stability on the left). *If  $d_{c_*}^-(\lambda, ik) = 0$  for some  $k \in \mathbb{R}$ , then  $\operatorname{Re} \lambda < 0$ .*

Lemmas 2.2.1 and 2.2.2 together with Palmer's theorem [Pal84; Pal88] imply that the essential spectrum of  $\mathcal{L}(\delta)$  is marginally stable, touching the imaginary axis only at the origin [KP13; FS03]; see Figure 2.1.

*Proof of Lemma 2.2.1.* We first look for  $c_*, \eta_* > 0$  which satisfy (2.8). The polynomial  $\nu \mapsto d_c^+(\lambda, \nu)$  at  $\lambda = 0$  admits  $-\eta$  as a double root if and only if  $(c, \eta)$  satisfies

$$\begin{cases} 0 = d_c^+(0, -\eta) = -\delta^2 \eta^4 + \eta^2 - c\eta + f'(0), \\ 0 = \partial_\nu d_c^+(0, -\eta) = 4\delta^2 \eta^3 - 2\eta + c. \end{cases}$$

We remove  $c$  from the first equation by using the second one, and find a quadratic equation satisfied by  $\eta^2$ , which has roots  $\pm\eta_1, \pm\eta_2$  where

$$\eta_1 := \frac{1}{|\delta|\sqrt{6}} \sqrt{1 + \sqrt{1 - 12\delta^2 f'(0)}} \sim \frac{1}{|\delta|\sqrt{3}}, \quad \eta_2 := \frac{1}{|\delta|\sqrt{6}} \sqrt{1 - \sqrt{1 - 12\delta^2 f'(0)}} \sim \sqrt{f'(0)}. \quad (2.11)$$

where the asymptotics hold for  $\delta \rightarrow 0$ . The choice  $\eta_* = \eta_2$  and  $c_* = 2\eta_* - 4\delta^2 \eta_*^3$  leads to

$$c_*(\delta) = 2\sqrt{f'(0)} - \delta^2 f'(0)^{3/2} + O(\delta^4).$$

The other double roots do not determine linear spreading speeds, as they are not *pinched*. Recall that a root  $\nu(\lambda)$  is said to be pinched at  $\lambda_0$  if it has multiplicity 2 there, and if its associated continuation  $\nu_1(\lambda)$  and  $\nu_2(\lambda)$  satisfies  $\operatorname{Re}(\nu_1(\lambda)) < 0 < \operatorname{Re}(\nu_2(\lambda))$  when  $\operatorname{Re}(\lambda) \in (0, +\infty)$  is large enough. This condition ensures that at  $\lambda_0$ , no separation can be made between unstable and stable spatial modes, thus leading to absolute spectrum there. We refer to [HS14b, section 4.1] for details on pinched double roots. We also point to [KP13, section 3.2] for the definition of absolute spectrum.

We now fix  $\delta_0 = 1/\sqrt{12f'(0)}$ . Then for  $|\delta| < \delta_0$ , we obtain using the expression of  $\eta_*(\delta) = \eta_2$  that:

$$\frac{\partial_\nu^2 d_{c_*}^+(0, -\eta_*)}{2!} = 1 - 6\delta^2 \eta_*^2 = \sqrt{1 - 12\delta^2 f'(0)} > 0.$$

Hence  $(\lambda, \nu) = (0, -\eta_*)$  is a simple double root and (2.8) is proved. Such an expansion together with the lack of unstable essential spectrum ensures that this root is pinched; see [HS14b, Lemma 4.4]. Alternatively, Lemma 2.4.2 below directly proves that the root is pinched.

We briefly mention the regularity of  $\eta_*$  with respect to  $\delta$ . Pushing up the development (2.11) in powers of  $\delta$ , one may directly see that  $\delta \mapsto \eta_*(\delta)$  is  $\mathcal{C}^1$  for  $\delta \in (0, \delta_0)$ . The same development ensures that  $\eta_*'(0) = 0$ , so that  $\eta_*$  is in fact  $\mathcal{C}^1$  on  $(-\delta_0, \delta_0)$ .

We now check the two remaining conditions in Lemma 2.2.1. We equate the polynomial  $\nu \mapsto d_{c_*}^+(\lambda, \nu)$  with its Taylor series centered at the double root  $\nu = -\eta_*$  to obtain

$$\operatorname{Re} d_{c_*}^+(\lambda, -\eta_* + ik) = -\operatorname{Re} \lambda + \operatorname{Re} \left( \sum_{j=0}^4 (ik)^j \frac{\partial_\nu^j d_{c_*}^+(\lambda, -\eta_*)}{j!} \right) = -\operatorname{Re} \lambda - \delta^2 k^4 - (1 - 6\delta^2 \eta_*^2) k^2 \leq 0 \quad (2.12)$$

if  $\operatorname{Re} \lambda \geq 0$ , since from (2.11), we have  $1 - 6\delta^2 \eta_*^2 = \sqrt{1 - 12\delta^2 f'(0)} \geq 0$ . This proves hypothesis (iii). In case where  $\operatorname{Re}(\lambda) = 0$ , the inequality in (2.12) is an equality if and only if  $k = 0$ , for which we have  $d_{c_*}^+(\lambda, -\eta_*) = \operatorname{Im}(\lambda)$ . Hence, hypothesis (ii) is proved.  $\square$

### 2.2.3 Preconditioner estimates

Here we prove the estimates we will need on our preconditioner  $(1 - \delta^2 \partial_x^2)^{-1}$ , by directly examining its Fourier symbol.

**Lemma 2.2.3.** *Fix  $\eta > 0$  sufficiently small, and fix an integer  $m$ . Then there exist constants  $\delta_0 > 0$  and  $C = C(\delta_0, \eta)$  such that if  $|\delta| < \delta_0$ ,*

$$\|(1 - \delta^2 \partial_x^2)^{-1}\|_{L^2_{0,\eta} \rightarrow L^2_{0,\eta}} \leq C, \quad (2.13)$$

$$\|(1 - \delta^2 \partial_x^2)^{-1}\|_{H^m_{0,\eta} \rightarrow H^{m+1}_{0,\eta}} \leq \frac{C}{|\delta|}. \quad (2.14)$$

*Proof.* By (2.6), it suffices to prove the estimates separately for  $L^2$  and for  $L^2_{\eta,\eta}$  with  $\eta > 0$  small. Since multiplication by  $e^{\eta \cdot}$  is an isomorphism from  $L^2_{\eta,\eta}(\mathbb{R})$  to  $L^2(\mathbb{R})$ , to prove estimates for  $(1 - \delta^2 \partial_x^2)^{-1}$  on  $L^2_{\eta,\eta}$ , it suffices to consider the inverse of the conjugate operator

$$e^{\eta \cdot} (1 - \delta^2 \partial_x^2) e^{-\eta \cdot} = 1 - \delta^2 (\partial_x - \eta)^2$$

acting on  $L^2(\mathbb{R})$ . This is the advantage of using (2.6) to separate estimates on  $L^2_{0,\eta}(\mathbb{R})$  into estimates on  $L^2(\mathbb{R})$  and  $L^2_{\eta,\eta}(\mathbb{R})$ : the conjugate operator arising from studying  $(1 - \delta^2 \partial_x^2)$  on  $L^2_{\eta,\eta}(\mathbb{R})$  has constant coefficients since the weight is a fixed exponential function, and so we can directly estimate its inverse using the Fourier transform. In the following, we will use the standard notation  $\langle k \rangle = (1 + k^2)^{1/2}$  several times.

Fix  $\eta \geq 0$ . By Plancherel's theorem,

$$\|(1 - \delta^2 (\partial_x - \eta)^2)^{-1} f\|_{L^2} = \left\| \frac{1}{1 - \delta^2 (i \cdot - \eta)^2} \hat{f}(\cdot) \right\|_{L^2} \leq \sup_{k \in \mathbb{R}} \left| \frac{1}{1 - \delta^2 (ik - \eta)^2} \right| \|\hat{f}\|_{L^2}.$$

Let  $\delta_0 = \min(1, 1/(\sqrt{2}\eta))$ , so that  $\delta_0^2 \eta^2 \leq 1/2$ , and hence if  $|\delta| < \delta_0$ ,

$$1 + \delta^2 (k^2 - \eta^2) = 1 - \delta^2 \eta^2 + \delta^2 k^2 \geq \frac{1}{2} + \delta^2 k^2. \quad (2.15)$$

Then for any  $\delta$  with  $|\delta| < \delta_0$ , we have

$$\left| \frac{1}{1 - \delta^2 (ik - \eta)^2} \right|^2 = \frac{1}{(1 + \delta^2 (k^2 - \eta^2))^2 + 4k^2 \delta^4 \eta^2} \leq \frac{1}{(1 + \delta^2 (k^2 - \eta^2))^2} \leq \frac{1}{(\frac{1}{2} + \delta^2 k^2)^2} \leq C,$$

with  $C$  depending only on  $\delta_0$  and  $\eta$ , and so

$$\|(1 - \delta^2 \partial_x^2)^{-1}\|_{L^2_{\eta,\eta} \rightarrow L^2_{\eta,\eta}} \leq C.$$

Since this holds for any fixed  $0 \leq \eta < 1$ , in particular also for  $\eta = 0$ , we obtain (2.13) by combining these estimates with (2.6).

Now we prove (2.14), again by obtaining bounds on the Fourier symbol of the inverse of the conjugate operator for  $\eta > 0$  and  $\eta = 0$ . By Plancherel's theorem, for any fixed  $0 \leq \eta < 1$ , we have

$$\|(1 - \delta^2 (\partial_x^2 - \eta))^2)^{-1} f\|_{H^{m+1}} = \left\| \frac{1}{1 - \delta^2 (i \cdot - \eta)^2} \langle \cdot \rangle^{m+1} \hat{f}(\cdot) \right\|_{L^2} \leq \sup_{k \in \mathbb{R}} \left| \frac{1}{1 - \delta^2 (ik - \eta)^2} \langle k \rangle \right| \|\hat{f}\|_{H^m}.$$

Again, let  $\delta_0 = \min(1, 1/(\sqrt{2}\eta))$ . Then, by (2.15), we have

$$\left| \frac{1}{1 - \delta^2(ik - \eta)^2} \langle k \rangle \right|^2 = \frac{\delta^2 + \delta^2 k^2}{(1 + \delta^2(k^2 - \eta^2))^2 + 4k^2\delta^4\eta^2} \frac{1}{\delta^2} \leq \frac{\delta^2 + \delta^2 k^2}{\left(\frac{1}{2} + \delta^2 k^2\right)^2} \frac{1}{\delta^2} \leq \frac{C}{\delta^2},$$

from which we obtain

$$\|(1 - \delta^2(\partial_x^2 - \eta))^{-1} f\|_{H^{m+1}} \leq \frac{C}{|\delta|} \|f\|_{H^m}.$$

Since this holds for  $\eta \geq 0$ , we obtain (2.14) from the equivalence of norms (2.6).  $\square$

We now state and prove the estimates we will need on the difference between the preconditioner and the identity,  $T(\delta) = (1 - \delta^2\partial_x^2)^{-1} - 1$ .

**Lemma 2.2.4.** *Fix  $\eta > 0$  sufficiently small. There exists a constant  $\delta_0$  such that the mapping  $\delta \mapsto T(\delta)$  is continuous from  $(-\delta_0, \delta_0)$  to  $\mathcal{B}(H_{0,\eta}^1, L_{0,\eta}^2)$ , the space of bounded linear operators from  $H_{0,\eta}^1(\mathbb{R})$  to  $L_{0,\eta}^2(\mathbb{R})$  with the operator norm topology.*

*Proof.* As in the proof of Lemma 2.2.3, it suffices to establish continuity in  $\delta$  of the conjugate operator  $T_\eta(\delta) := (1 - \delta^2(\partial_x - \eta)^2)^{-1} - 1$  on  $L^2(\mathbb{R})$  for  $\eta \geq 0$  sufficiently small. For  $\delta$  nonzero, we write

$$1 - \delta^2(\partial_x - \eta)^2 = \delta^2 \left( \frac{1}{\delta^2} - (\partial_x - \eta)^2 \right).$$

By standard spectral theory, we therefore see that  $T_\eta(\delta)$  is continuous in  $\delta$  provided  $\delta$  is nonzero and  $1/\delta^2$  is in the resolvent set of the operator  $(\partial_x - \eta)^2$ . Computing the spectrum of this operator with the Fourier transform, one readily finds that there exists a  $\delta_1$  depending on  $\eta$  such that this continuity holds for  $0 < \delta < \delta_1$ .

We now establish continuity at  $\delta = 0$  via direct estimates on the Fourier multiplier.

$$\hat{T}_\eta(\delta, k) = \frac{\delta^2(ik - \eta)^2}{1 - \delta^2(ik - \eta)^2}.$$

Since we are proving continuity of  $T_\eta(\delta)$  from  $H^1$  to  $L^2$ , we gain a helpful factor of  $\langle k \rangle$  — that is, it suffices to estimate  $|\hat{T}_\eta(\delta, k)|/\langle k \rangle$ . By (2.15), for  $|\delta| < \delta_0 := \min\{\delta_1, 1/\sqrt{2}\eta\}$  we have

$$\begin{aligned} \left| \hat{T}_\eta(\delta, k) \frac{1}{\langle k \rangle} \right| &= \left| \frac{\delta^2(ik - \eta)^2}{1 - \delta^2(ik - \eta)^2} \frac{1}{\langle k \rangle} \right| = \frac{\delta^2 \sqrt{(\eta^2 - k^2)^2 + 4k^2\eta^2}}{\sqrt{(1 - \delta^2(\eta^2 - k^2))^2 + 4\delta^4 k^2 \eta^2}} \frac{|\delta|}{(\delta^2 + \delta^2 k^2)^{1/2}} \\ &\leq |\delta| \left( \frac{\delta^4 \eta^4 + \delta^4 k^4 + 2\delta^4 k^2 \eta^2}{\left[ \left( \frac{1}{2} + \delta^2 k^2 \right)^2 + 4\delta^4 k^2 \eta^2 \right] (\delta^2 + \delta^2 k^2)} \right)^{1/2}, \end{aligned}$$

using (2.15) in the denominator. We now split the factor in the parenthesis, first estimating

$$\frac{\delta^4 \eta^4 + 2\delta^4 k^2 \eta^2}{\left[ \left( \frac{1}{2} + \delta^2 k^2 \right)^2 + 4\delta^4 k^2 \eta^2 \right] (\delta^2 + \delta^2 k^2)} \leq \frac{\delta^4 \eta^4 + 2\delta^4 k^2 \eta^2}{\frac{1}{4}(\delta^2 + \delta^2 k^2)} = \frac{\delta^2 \eta^4 + 2\delta^2 k^2 \eta^2}{\frac{1}{4}(1 + k^2)} \leq C,$$

where  $C$  depends only on  $\delta_0$  and  $\eta$ . For the remaining term, we have

$$\frac{\delta^4 k^4}{\left[\left(\frac{1}{2} + \delta^2 k^2\right)^2 + 4\delta^4 k^2 \eta^2\right](\delta^2 + \delta^2 k^2)} \leq \frac{\delta^4 k^4}{\left(\frac{1}{2} + \delta^2 k^2\right)^2 (\delta^2 k^2)} = \frac{\delta^2 k^2}{\left(\frac{1}{2} + \delta^2 k^2\right)^2} \leq C,$$

again with constant  $C$  only depending on  $\eta$  and  $\delta_0$ . From this estimate on the Fourier symbol together with Plancherel's theorem, we obtain

$$\|T_\eta(\delta)\|_{H^1 \rightarrow L^2} \leq C|\delta|,$$

for  $|\delta| < \delta_0$ , and so in particular  $\delta \mapsto T_\eta(\delta)$  is continuous at  $\delta = 0$ , which completes the proof of the lemma.  $\square$

### 2.2.4 Fredholm properties at $\delta = 0$

We will further need the Fredholm properties of  $\mathcal{L}(0)$ , which is the linearization in the weighted space of the classical FKPP problem  $\delta = 0$ . The classical Fisher-KPP front at  $\delta = 0$  may be constructed via simple phase plane methods (see [Sat76]). We denote this front by  $q_0$ , it satisfies the following asymptotics,

$$q_0(x) = (ax + b)e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x}), \quad x \rightarrow \infty, \quad (2.16)$$

where  $a \neq 0$  and  $b \in \mathbb{R}$  are constants (see e.g. [Gal94] for asymptotics of the critical Fisher-KPP front). Choosing a suitable translate  $x \mapsto q_0(x + h)$  that we still denote by  $q_0$ , we can assume that (2.16) holds with  $a = 1$ , we further note  $\mu_0$  the corresponding value of  $b$ . Direct computations lead to  $h := \frac{1}{\eta_*(0)} \log(a)$  and  $\mu_0 := \frac{ah+b}{e^{\eta_*(0)h}} = \frac{1}{\eta_*(0)} \log(a) + \frac{b}{a}$ . In the following two lemmas, we describe the kernel, the cokernel and the range of  $\mathcal{L}(0)$ . They will both be needed for the existence of the critical front  $q_*(\cdot; \delta)$  in Section 2.3, and for the control of small eigenvalues in Section 2.4. Since we only focus on properties at  $\delta = 0$  in this section, we will sometimes denote  $\eta_*(0)$  and  $c_*(0)$  simply by  $\eta_*$  and  $c_*$  in this section.

**Lemma 2.2.5.** *For  $\eta > 0$ , the operator  $\mathcal{L}(0) : H_{0,\eta}^2(\mathbb{R}) \rightarrow L_{0,\eta}^2(\mathbb{R})$  is Fredholm with index  $-1$ , with trivial kernel and with cokernel spanned by  $\varphi(x) = (\omega_*(x; 0))^{-1} e^{c_*(0)x} q_0'(x)$ .*

*Proof.* Recall that the asymptotic operators are given by  $\mathcal{L}_+(0) = \partial_x^2$  and  $\mathcal{L}_-(0) = \partial_x^2 + c_*(0)\partial_x + f'(1)$ . For  $\eta > 0$ , define the conjugate operator:

$$\mathcal{L}_\eta(0) = \omega_{0,\eta} \mathcal{L}(0) \omega_{0,\eta}^{-1} : H^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

with asymptotic operators  $\mathcal{L}_{\eta,+} = (\partial_x - \eta)^2$  and  $\mathcal{L}_{\eta,-} = \mathcal{L}_-$ . Since the multiplication  $\omega_{0,\eta} : L_{0,\eta}^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is an isomorphism, the Fredholm indices satisfy

$$\text{Fred } \mathcal{L}_\eta(0) = 0 + \text{Fred } \mathcal{L}(0) + 0 = \text{Fred } \mathcal{L}(0).$$

Then the conjugate operator is defined on a unweighted space, and its Fredholm borders are the two oriented curves  $\sigma(\mathcal{L}_{\eta,+}) = \{-k^2 + i2\eta k + \eta^2 : k \in \mathbb{R}\}$  and  $\sigma(\mathcal{L}_{\eta,-}) = \{-k^2 + ic_*(0)k + f'(1) : k \in \mathbb{R}\}$ , which are away from  $0 \in \mathbb{C}$ ; see Figure 2.2. This ensures that  $\mathcal{L}_\eta(0)$  is Fredholm, we now compute its index  $\text{Fred}(\mathcal{L}_\eta(0) - \lambda)$  at  $\lambda = 0$ . For  $\lambda$  to the right of the essential spectrum, we use Palmer's

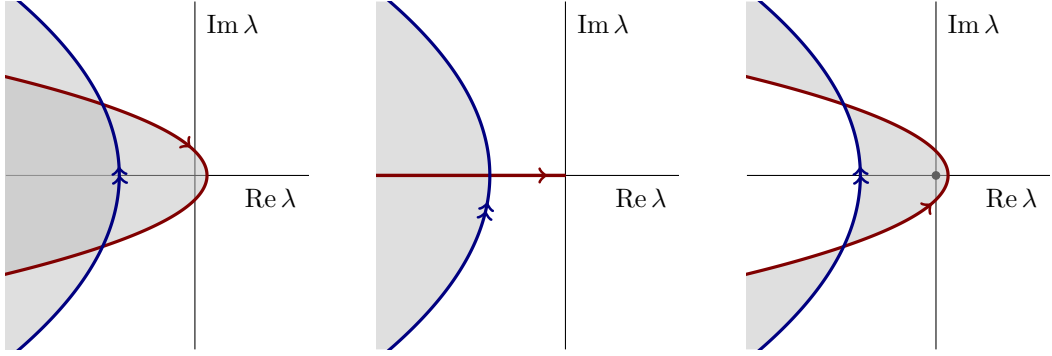


Figure 2.2 – Essential spectrum of  $\mathcal{L}_\eta(0)$  for  $\eta < 0$  at left,  $\eta = 0$  at middle, and  $\eta > 0$  at right. For  $\eta > 0$ , the positive Fredholm border has reverse orientation so that  $\text{Fred}(\mathcal{L}_\eta) = -1$ , while  $\text{Fred}(\mathcal{L}_\eta) = 1$  for  $\eta < 0$ .

theorem to compute the Fredholm index from the Morse indices (see e.g. [KP13; FS03]),

$$\text{Fred}(\mathcal{L}_\eta(0) - \lambda) = \dim E_-^u(\lambda) - \dim E_+^u(\lambda) = 2 - 2 = 0.$$

where  $E_\pm^u$  are the unstable eigenspaces at  $\pm\infty$ . To prove that these spaces share the same dimension, one can take  $|\lambda|$  large enough and use a standard normalization; see [FH18, proof of Lemma 3.1]. Then the index decreases to  $-1$  when  $\lambda$  crosses  $\sigma(\mathcal{L}_{\eta,+})$ , since the latter curve has reverse orientation, see [KP13]. Hence at  $\lambda = 0$ , we have shown that  $\text{Fred} \mathcal{L}(0) = \text{Fred} \mathcal{L}_\eta(0) = -1$ .

To compute the kernel, we note that

$$u \in \ker \mathcal{L}(0) \quad \text{if and only if} \quad u \in H_{0,\eta}^2(\mathbb{R}) \text{ and } \mathcal{A}(0)\omega_*(\cdot;0)^{-1}u = 0, \quad (2.17)$$

with  $\mathcal{A}(0) = \partial_x^2 + c_*(0)\partial_x + f'(q_0(x))$ . Studying the asymptotic growth of the ODE  $\mathcal{A}(0)u = 0$ , one can construct a basis of solutions  $\{q'_0, \zeta\}$ , with exponential behavior at  $-\infty$ :  $\zeta(x) \sim \exp((-\sqrt{f'(0)} - \alpha)x)$  and  $q'_0(x) \sim \exp((-\sqrt{f'(0)} + \alpha)x)$  with  $\alpha = \sqrt{f'(0) - f'(1)} > \sqrt{f'(0)} > 0$ . See [FH18, proof of Lemma 2.2] for a similar construction. Furthermore, the derivative of the front has weak exponential decay at  $+\infty$ :  $q'_0(x) \sim x\omega_*(x;0)^{-1}$ . Hence, neither  $\zeta$  nor  $q'_0$  are sufficiently localized to satisfy the right hand condition in (2.17), so that  $\ker \mathcal{L}(0) = \{0\}$ . In particular, since  $\mathcal{L}(0)$  has Fredholm index  $-1$ , this implies that  $\ker(\mathcal{L}(0)^*)$  is one-dimensional.

Finally, it is easily computed that  $\tilde{\mathcal{A}}(0) := \exp(\frac{c_*}{2}\cdot)\mathcal{A}(0)\exp(-\frac{c_*}{2}\cdot)$  is self-adjoint with respect to the  $L^2(\mathbb{R})$ -inner product  $\langle \cdot, \cdot \rangle$ , so that for  $v \in H_{0,-\eta}^2(\mathbb{R})$  and  $u \in H_{0,\eta}^2(\mathbb{R})$ :

$$\begin{aligned} \langle u, \mathcal{L}^*(0)v \rangle &= \langle \mathcal{L}(0)u, v \rangle = \langle \mathcal{A}(0)(\omega_*^{-1}u), \omega_*v \rangle = \langle \tilde{\mathcal{A}}(0)(e^{\frac{c_*}{2}\cdot}\omega_*^{-1}u), e^{-\frac{c_*}{2}\cdot}\omega_*v \rangle \\ &= \langle e^{c_*\cdot}\omega_*^{-1}u, \mathcal{A}(0)(e^{-c_*\cdot}\omega_*v) \rangle, \end{aligned}$$

which ensures that  $v \in \ker(\mathcal{L}(0)^*)$  if and only if  $v \in H_{0,-\eta}^2(\mathbb{R})$  and  $\mathcal{A}(0)e^{-c_*(0)\cdot}\omega_*(\cdot;0)v = 0$ . For  $x \rightarrow -\infty$ ,  $\omega_*(x)^{-1}e^{c_*x}\zeta(x) \sim \exp((\sqrt{f'(0)} - \alpha)x)$  is not bounded, hence  $\ker(\mathcal{L}(0)^*) = \text{Span}(\omega_*^{-1}e^{c_*\cdot}q'_0)$ . On the first hand, notice that  $\zeta$  does not contribute to  $\ker(\mathcal{L}(0)^*)$ : when  $x \rightarrow -\infty$

$$\omega_*(x;0)^{-1}e^{c_*(0)x}\zeta(x) \sim \exp((\sqrt{f'(0)} - \alpha)x)$$

is unbounded, thus the left hand side does not belong to  $H_{0,-\eta}^2(\mathbb{R})$ . On the other hand,  $\mathcal{L}(0)^*\varphi = 0$ ,

where

$$\varphi(x) := \omega_*(x; 0)^{-1} e^{c_*(0)x} q'_0(x) \quad (2.18)$$

belongs to  $H_{0,-\eta}^2(\mathbb{R})$ , as we prove now. Using the fact that  $c_*(0) = 2\eta_*(0)$  from the proof of Lemma 2.2.1, we see that  $\varphi$  grows as a polynomial when  $x \rightarrow +\infty$ :

$$\varphi(x) = \omega_*(x; 0)^{-1} e^{c_*(0)x} q'_0(x) \sim e^{-\eta_*(0)x} e^{c_*(0)x} x e^{-\eta_*(0)x} \sim x, \quad x \rightarrow \infty.$$

Furthermore,  $\varphi(x)$  is exponentially localized as  $x \rightarrow -\infty$ , since both  $e^{c_*(0)x}$  and  $q'_0(x)$  are separately exponentially localized there, and  $\omega(x; 0)^{-1} \equiv 1$  for  $x \leq 1$ . It follows that  $\varphi \in H_{0,-\eta}^2(\mathbb{R})$ , as claimed. Since  $\ker(\mathcal{L}(0)^*)$  is one dimensional,  $\mathcal{L}^*(0)\varphi = 0$ , and  $\varphi \in H_{0,-\eta}^2(\mathbb{R})$ , we conclude that  $\ker(\mathcal{L}(0)^*) = \text{Span}(\varphi)$ , as desired.  $\square$

Now that the kernel of  $\mathcal{L}(0) : H_{0,\eta}^2(\mathbb{R}) \rightarrow L_{0,\eta}^2(\mathbb{R})$  is fully described, we can give a useful description of its range. Let  $P : L_{0,\eta}^2(\mathbb{R}) \rightarrow \text{im}(\mathcal{L}(0))$  denote the orthogonal projection onto  $\text{im} \mathcal{L}(0)$  with respect to the  $L_{0,\eta}^2(\mathbb{R})$ -inner product.

**Lemma 2.2.6.** *For  $\eta > 0$  small enough, the range of  $\mathcal{L}(0) : H_{0,\eta}^2(\mathbb{R}) \rightarrow L_{0,\eta}^2(\mathbb{R})$  is*

$$\text{im}(\mathcal{L}(0)) = \{u \in L_{0,\eta}^2(\mathbb{R}) : \langle u, \varphi \rangle = 0\},$$

where  $\varphi$  is defined in the above Lemma 2.2.5 and  $\langle \cdot, \cdot \rangle$  denotes the  $L^2(\mathbb{R})$  inner product.

*Proof.* Assume that  $u \in \text{im}(\mathcal{L}(0))$ , so that  $u = \mathcal{L}(0)\tilde{u}$  with  $\tilde{u} \in H_{0,\eta}^2(\mathbb{R})$ . Then  $\langle u, \varphi \rangle = \langle \tilde{u}, \mathcal{L}(0)^*\varphi \rangle = 0$ . To prove the reverse inclusion, write  $u \in L_{0,\eta}^2(\mathbb{R})$  as

$$u = Pu + (1 - P)u.$$

From Lemma 2.2.5,  $\mathcal{L}(0)$  is Fredholm, hence its range is closed and  $P$  is well defined. Furthermore,  $\text{Fred} \mathcal{L}(0) = -1$  and  $\ker \mathcal{L}(0) = \{0\}$ , so that  $1 - P$  has a one dimensional range:

$$(1 - P)u = \alpha(u)\psi,$$

with  $\psi \in L_{0,\eta}^2(\mathbb{R})$  fixed, and  $\alpha : H_{0,\eta}^2(\mathbb{R}) \rightarrow \mathbb{R}$  linear. Assuming that  $\langle u, \varphi \rangle = 0$ , we obtain

$$0 = \langle Pu, \varphi \rangle + \langle (1 - P)u, \varphi \rangle = \langle \tilde{u}, \mathcal{L}(0)^*\varphi \rangle + \alpha(u)\langle \psi, \varphi \rangle = \alpha(u)\langle \psi, \varphi \rangle, \quad (2.19)$$

for some  $\tilde{u} \in H_{0,\eta}^2(\mathbb{R})$ . Hence either  $\alpha(u) = 0$  or  $\langle \psi, \varphi \rangle = 0$ . If  $\langle \psi, \varphi \rangle = 0$ , then for all  $v \in H_{0,\eta}^2(\mathbb{R})$ , we would have  $\langle v, \varphi \rangle = \langle \tilde{v}, \mathcal{L}(0)^*\varphi \rangle + \alpha(v)\langle \psi, \varphi \rangle = 0$ , which is to say that  $\varphi = 0$  and is a contradiction. Hence from (2.19) we conclude  $\alpha(u) = 0$ , so that  $u = Pu \in \text{im}(\mathcal{L}(0))$ .  $\square$

## 2.2.5 Implicit function theorem

Our proofs of Theorems 2.1.1 and 2.1.2 rely on applying the implicit function theorem to suitably constructed maps. However, due to the singularly perturbed structure of (2.1), even after applying our regularization procedure, the associated maps to which we wish to apply the implicit function theorem are not  $\mathcal{C}^1$  in  $\delta$ , but instead merely continuous. We therefore rely on the following version of the implicit function theorem which requires only continuity in



the parameter. This essentially follows, for instance, from the material in [Hen81], but since the precise form we use is not stated there, we state it precisely here and adapt the proof from [Hen81] to suit our version.

**Lemma 2.2.7** (Implicit function theorem with continuity in parameters). *Let  $X$  and  $Y$  be Banach spaces, and let  $\Delta$  be a metric space. Let  $F : X \times \Delta \rightarrow Y$  be continuous. Suppose that for each fixed  $\delta \in \Delta$ ,  $x \mapsto F(x, \delta)$  is continuously differentiable, and further that the map  $(x, \delta) \mapsto D_x F(x, \delta) : X \times \Delta \rightarrow \mathcal{B}(X, Y)$  is continuous. Suppose that for some  $(x_0, \delta_0) \in X \times \Delta$ , we have  $F(x_0, \delta_0) = 0$  and  $D_x F(x_0, \delta_0)$  has a bounded inverse. Then there exists a neighborhood  $U \times V \subset X \times \Delta$  containing  $(x_0, \delta_0)$  and a function  $x_*(\delta)$  such that  $F(x, \delta) = 0$  for  $(x, \delta) \in U \times V$  if and only if  $x = x_*(\delta)$ . Moreover, the map  $\delta \mapsto x_*(\delta) : V \rightarrow U$  is continuous, with  $x_*(\delta_0) = x_0$ .*

*Proof.* Let  $L = (D_x F(x_0, \delta_0))^{-1}$ , and define  $G : X \times \Delta \rightarrow X$  by

$$G(x, \delta) = x - LF(x, \delta). \quad (2.20)$$

We compute

$$\begin{aligned} D_x G(x, \delta) &= I - L(D_x F(x, \delta)) = I - L(D_x F(x_0, \delta_0)) + L(D_x F(x_0, \delta_0) - D_x F(x, \delta)) \\ &= L(D_x F(x_0, \delta_0) - D_x F(x, \delta)), \end{aligned}$$

since  $L(D_x F(x_0, \delta_0)) = I$  by construction. By continuity of  $D_x F$  and boundedness of  $L$ , there exists  $0 < \theta < 1$  and a neighborhood  $U \times V$  of  $(x_0, \delta_0)$  such that for any  $(x, \delta) \in U \times V$ , we have

$$\|D_x G(x, \delta)\|_{X \rightarrow Y} = \|L(D_x F(x_0, \delta_0) - D_x F(x, \delta))\|_{X \rightarrow Y} \leq \theta.$$

Hence for each  $\delta \in V$ , the map  $x \mapsto G(x, \delta) : \bar{U} \rightarrow Y$  is a contraction, and so by the Banach fixed point theorem, for each  $\delta \in V$  there exists  $x_*(\delta)$  such that  $G(x_*(\delta), \delta) = x_*(\delta)$ . It follows from invertibility of  $L$  that  $F(x_*(\delta), \delta) = 0$ , and from continuity of  $G$  it follows that  $\delta \mapsto x_*(\delta)$  is continuous, as desired.  $\square$

## 2.3 Existence of the critical front – proof of Theorem 2.1.1

Our approach is to capture the weak exponential decay at  $+\infty$  implied by the pinched double root by solving (2.2) with an ansatz

$$q(x; \delta) = \chi_-(x) + w(x) + \chi_+(x)(\mu + x)e^{-\eta_*(\delta)x}, \quad \mu \in \mathbb{R} \quad (2.21)$$

where  $\chi_+$  is a smooth positive cutoff function satisfying

$$\chi_+(x) = \begin{cases} 1, & x \geq 3, \\ 0, & x \leq 2, \end{cases} \quad (2.22)$$

and  $\chi_-(x) = \chi_+(-x)$ . For brevity, we denote by  $\psi(\mu, \delta)$  the function

$$\psi(x; \mu, \delta) = (\mu + x)e^{-\eta_*(\delta)x}.$$

We will require  $w$  to be exponentially localized, with a decay rate faster than  $e^{-\eta_*(\delta)x}$  — this localized piece is the *core* of the solution, while  $\chi_-$  and  $\chi_+ \psi$  capture the *far-field* behavior. Similar far-field/core decompositions have been used to construct heteroclinic solutions to pattern-

forming systems in [AGGMS19; GS18]. Inserting the ansatz (2.21) into the traveling wave equation (2.2), we get an equation

$$\mathcal{A}_+(\delta)(\chi_- + w + \chi_+ \psi(\mu, \delta)) + N(\chi_- + w + \chi_+ \psi(\mu, \delta)) = 0, \quad (2.23)$$

where  $\mathcal{A}_+(\delta) = -\delta^2 \partial_x^4 + \partial_x^2 + c_*(\delta) \partial_x + f'(0)$ , and  $N(q) = f(q) - f'(0)q$ . Since we want to require  $w$  to decay faster than the front itself, we first let  $v = \omega_* w$ , so that (2.23) becomes

$$0 = F(v; \mu, \delta) := \mathcal{S}(\delta)v + \omega_*(\cdot; \delta) \mathcal{A}_+(\delta)(\chi_- + \chi_+ \psi) + \omega_*(\cdot; \delta) N(\chi_- + \omega_*(\cdot; \delta)^{-1}v + \chi_+ \psi),$$

where  $\mathcal{S}(\delta) = \omega_*(\cdot; \delta) \mathcal{A}_+(\delta) \omega_*(\cdot; \delta)^{-1}$  is the conjugate operator

$$\mathcal{S}(\delta) = -\delta^2 \partial_x^4 + \delta^2 a_3(x; \delta) \partial_x^3 + (1 + \delta^2 a_2(x; \delta)) \partial_x^2 + a_1(x; \delta) \partial_x + \tilde{a}_0(x; \delta), \quad (2.24)$$

where the coefficients  $a_i$  are given in (2.9) for  $i = 1, 2$  or  $3$  while

$$\tilde{a}_0 = f'(0) + \omega_*(\cdot; \delta) (c_* \partial_x + \partial_x^2 - \delta^2 \partial_x^4) \omega_*(\cdot; \delta)^{-1},$$

since we are linearizing about the unstable state  $u \equiv 0$  rather than the front itself, which we are in the process of constructing.

Since  $\omega_*(x; \delta) = 1$  on the support of  $\chi_-$  and  $\omega_*(x; \delta) = e^{\eta_*(\delta)x}$  on the support of  $\chi_+$ , we simplify  $F$  to

$$F(v; \mu, \delta) = \mathcal{S}(\delta)v + \mathcal{A}_+(\delta)\chi_- + \mathcal{S}(\delta)[(\mu + \cdot)\chi_+] + \omega_*(\cdot; \delta) N(\chi_- + \omega_*(\cdot; \delta)^{-1}v + \chi_+ \psi).$$

Then, we extract from  $N$  terms that are linear in  $v$ , together with residual terms that are  $v$ -independent. We write

$$\omega_*(\cdot; \delta) N(\chi_- + \omega_*(\cdot; \delta)^{-1}v + \chi_+ \psi) = \mathcal{N}(v; \mu, \delta) + Q(\mu, \delta)v + R(\mu, \delta)$$

where

$$\mathcal{N}(v; \mu, \delta) = \omega_*(\cdot; \delta) \left[ f(\chi_+ + \omega_*(\cdot; \delta)^{-1}v + \chi_+ \psi) - f(\chi_- + \chi_+ \psi) - f'(\chi_- + \chi_+ \psi) \omega_*(\cdot; \delta)^{-1}v \right], \quad (2.25)$$

and

$$Q(\mu, \delta)v = (f'(\chi_- + \chi_+ \psi) - f'(0))v, \quad R(\mu, \delta) = \omega_*(\cdot; \delta) [f(\chi_- + \chi_+ \psi) - f'(0)(\chi_- + \chi_+ \psi)].$$

Altogether,  $F$  decomposes as the sum of a linear term, a residual term, and a nonlinear term:

$$F(v; \mu, \delta) = [\mathcal{S}(\delta) + Q(\mu, \delta)]v + \mathcal{R}(\mu, \delta) + \mathcal{N}(v; \mu, \delta), \quad (2.26)$$

where  $\mathcal{N}(v; \mu, \delta)$  is given by (2.25), and

$$\mathcal{R}(\mu, \delta) = R(\mu, \delta) + \mathcal{A}_+(\delta)\chi_- + \mathcal{S}(\delta)[(\mu + \cdot)\chi_+].$$

At  $\delta = 0$ , equation (2.2) (or equivalently  $F(v; \mu, 0) = 0$ ) is the traveling wave equation for the Fisher-KPP equation, and thus we have  $F(v_0; \mu_0, 0) = 0$  where the front  $q_0$  and the constant  $\mu_0$  are defined immediately after equation (2.16), and where

$$v_0 = \omega_*(\cdot; 0)q_0 - \chi_- - \chi_+ \omega_*(\cdot; 0)\psi(\mu_0, 0),$$

is exponentially localized.

To regularize the singular perturbation and enforce exponential localization of  $v$ , we consider

$$G(v; \mu, \delta) = (1 - \delta^2 \partial_x^2)^{-1} F(v; \mu, \delta),$$

as a nonlinear function  $G : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \rightarrow L_{0,\eta}^2(\mathbb{R})$ , for  $\eta > 0$  sufficiently small.

**Lemma 2.3.1.** *Fix  $\eta > 0$  sufficiently small. There exists  $\delta_0 > 0$  such that the map*

$$(v, \mu, \delta) \mapsto G(v, \mu, \delta) : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \times (-\delta_0, \delta_0) \rightarrow L_{0,\eta}^2(\mathbb{R})$$

*is well-defined, continuously differentiable in  $v$  and  $\mu$ , and continuous in  $\delta$ . Moreover, the map*

$$(v; \mu, \delta) \mapsto \partial_{(v,\mu)} G(v; \mu, \delta) : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \times (-\delta_0, \delta_0) \rightarrow \mathcal{B}(H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R}, L_{0,\eta}^2(\mathbb{R}))$$

*is continuous.*

*Proof.* We use (2.26) to write  $G$  as

$$G(v; \mu, \delta) = (1 - \delta^2 \partial_x^2)^{-1} \mathcal{S}(\delta)v + (1 - \delta^2 \partial_x^2)^{-1} [Q(\mu, \delta)v + \mathcal{R}(\mu, \delta) + \mathcal{N}(v; \mu, \delta)]. \quad (2.27)$$

Using the fact that  $f$  is smooth and that  $H_{0,\eta}^2(\mathbb{R})$  is a Banach algebra, one readily finds by Taylor expanding  $f$  where it appears in  $\mathcal{N}$  and  $Q$  that if  $v \in H_{0,\eta}^2(\mathbb{R})$ , then

$$\|Q(\mu, \delta)v + \mathcal{N}(v; \mu, \delta)\|_{L_{0,\eta}^2} < \infty.$$

We group the terms  $\mathcal{A}_+(\delta)\chi_-$  and  $R(\mu, \delta)$  together, observing that since derivatives of  $\chi_-$  are compactly supported, for  $x \leq -3$  we have  $\mathcal{A}_+(\delta)\chi_- = f'(0)\chi_-$ , and so writing

$$\mathcal{A}_+(\delta)\chi_- + R(\mu, \delta) = (\mathcal{A}_+(\delta)\chi_- - f'(0)\chi_-) + \omega_*[f(\chi_- + \chi_+\psi) - f'(0)\chi_+\psi],$$

the term  $(\mathcal{A}_+(\delta)\chi_- - f'(0)\chi_-)$  is compactly supported, while for the other term we have by Taylor expansion

$$\|\omega_*[f(\chi_- + \chi_+\psi) - f'(0)\chi_+\psi]\|_{L_{0,\eta}^2} \leq C\|\omega_*(\chi_+\psi)\|_{L_{0,\eta}^2} < \infty.$$

The remaining term  $\mathcal{S}(\delta)[(\mu + \cdot)\chi_+]$  in  $\mathcal{R}(\mu, \delta)$  is strongly localized by the choice of the far-field ansatz:  $\chi_-(x)$  is identically zero for  $x$  large, and for  $x$  large the fact that  $(\eta_*(\delta), c_*(\delta))$  solve the system (2.2.2) implies that every term in  $\mathcal{S}(\delta)$  has at least two derivatives in it, so  $\mathcal{S}(\delta)(\mu + \cdot) \equiv 0$  on the support of  $\chi_+$ , and the only terms that remain are compactly supported commutator terms. Hence

$$\|\mathcal{R}(\mu, \delta)\|_{L_{0,\eta}^2} \leq \|R(\mu, \delta) + \mathcal{A}_+(\delta)\chi_-\|_{L_{0,\eta}^2} + \|\mathcal{S}(\delta)[(\mu + \cdot)\chi_+]\|_{L_{0,\eta}^2} < \infty.$$

Together with (2.13) of Lemma (2.2.3), this implies that the second term of (2.27) is in  $L_{0,\eta}^2(\mathbb{R})$ , and so to check that  $G$  is well-defined, it only remains to estimate the first term in (2.27). For this term, we use the specific form of  $\mathcal{S}(\delta)$ , given in (2.24), to write

$$(1 - \delta^2 \partial_x^2)^{-1} \mathcal{S}(\delta) = \partial_x^2 + \delta^2(1 - \delta^2 \partial_x^2)^{-1} [a_3 \partial_x^3 + a_2 \partial_x^2] + (1 - \delta^2 \partial_x^2)^{-1} (a_1 \partial_x + \tilde{a}_0). \quad (2.28)$$

Since  $a_3$  and  $a_2$  are smooth, constant outside of fixed compact set, and bounded uniformly in  $\delta$ ,

we have

$$\|a_3 \partial_x^3 + a_2 \partial_x^2\|_{H_{0,\eta}^2 \rightarrow H_{0,\eta}^{-1}} \leq C.$$

Combining this with estimate (2.14) of Lemma 2.2.3, we obtain

$$\|\delta^2(1 - \delta^2 \partial_x^2)^{-1}(a_3 \partial_x^3 + a_2 \partial_x^2)\|_{H_{0,\eta}^2 \rightarrow L_{0,\eta}^2} \leq C|\delta|. \quad (2.29)$$

The other terms in (2.28) are readily seen to be uniformly bounded in  $\delta$  as operators from  $H_{0,\eta}^2(\mathbb{R})$  to  $L_{0,\eta}^2(\mathbb{R})$  for  $\delta$  sufficiently small, from which we conclude that  $G$  is well-defined.

Since the nonlinearity  $f$  is smooth, differentiability in  $v$  follows readily from the fact that  $H_{0,\eta}^2(\mathbb{R})$  is a Banach algebra whose norm controls the  $L^\infty$  norm. Differentiability in  $\mu$  is also readily attainable from smoothness of  $f$  and the exponential localization of our ansatz. The preconditioner plays little role in these arguments — when treating the residual terms or the nonlinearity, we do not need to use the preconditioner at all to obtain smoothness in  $v$  and  $\mu$ .

The residual terms as well as the nonlinearity are also readily seen to be continuous in  $\delta$ . The main subtlety is to handle the term  $(1 - \delta^2 \partial_x^2)^{-1} \mathcal{S}(\delta)$ , which we write as

$$(1 - \delta^2)^{-1} \mathcal{S}(\delta) = (\partial_x^2 + a_1 \partial_x + \tilde{a}_0) + \delta^2(1 - \delta^2 \partial_x^2)^{-1}(a_3 \partial_x^3 + a_2 \partial_x^2) + T(\delta)(a_1 \partial_x + \tilde{a}_0),$$

where  $T(\delta) = (1 - \delta^2 \partial_x^2)^{-1} - 1$ . The operator  $\partial_x^2 + a_1(x, \delta) \partial_x + \tilde{a}_0(x, \delta)$  is continuous in  $\delta$  from  $H_{0,\eta}^2$  to  $L_{0,\eta}^2$ , since the coefficients are smooth and uniformly bounded in  $\delta$ . The second term is continuous in  $\delta$  by (2.29), and the last term is continuous in  $\delta$  by Lemma 2.2.4. Continuity of  $\partial_{(v,\mu)} G(v; \mu, \delta)$  proceeds analogously.  $\square$

With the appropriate regularity of  $G$  in hand, we now aim to solve near  $(v_0, \mu_0, 0)$  using the implicit function theorem. The linearization about this solution in  $v$  is given by

$$\partial_v G(v_0; \mu_0, 0) = \mathcal{S}(0) + Q(\mu_0, 0) + \partial_v \mathcal{N}(v_0; \mu_0, 0) = \mathcal{S}(0) + f'(q_0) - f'(0) = \mathcal{L}(0).$$

From Lemma 2.2.5,  $\partial_v G(v_0; \mu_0, 0)$  is Fredholm with index  $-1$ , so that the joint linearization  $\partial_{(v,\mu)} G(v_0; \mu_0, 0)$  is Fredholm index 0 by the Fredholm bordering lemma [SS08, Lemma 4.4]. We show that in fact the joint linearization has full range, and hence is invertible. We do this by computing an appropriate projection on the cokernel  $\mathcal{L}(0)^* = \text{Span}(\varphi)$ , and so we first record useful asymptotics of  $\varphi$ .

**Lemma 2.3.2.** *The function  $\varphi$  defined in Lemma 2.2.5 satisfies*

$$\lim_{x \rightarrow \infty} \varphi'(x) = -\eta_*(0) \neq 0. \quad (2.30)$$

*Proof.* We start by extracting asymptotics of  $q'_0$  and  $q''_0$ , and then use the formula  $\varphi(x) = \omega_*(x; 0)^{-1} e^{c_*(0)x} q'_0(x)$  to obtain the corresponding asymptotics of  $\varphi'$ . The asymptotics of  $q'_0$  and  $q''_0$  may be formally obtained by differentiating the asymptotics  $q_0(x) \sim (\mu_0 + x)e^{-\eta_*(0)x}$ . However, strictly speaking these asymptotics do not necessarily control the behavior of the derivatives, and so we rigorously justify this computation below.

Since the Fisher-KPP equation is translation invariant,  $p_0 := q'_0$  solves the linearized equation

$$(\partial_x^2 + c_*(0) \partial_x + f'(q_0(x)))p_0 = 0.$$

To make this equation autonomous, we couple it to the equation  $q_0'' + c_*(0)q_0' + f(q_0) = 0$ , and write the corresponding system in 4 “unknowns”  $(Q_1, Q_2, Q_3, Q_4) = (q_0, q_0', p_0, p_0')$  as

$$\begin{pmatrix} Q_1' \\ Q_2' \\ Q_3' \\ Q_4' \end{pmatrix} = \begin{pmatrix} Q_2 \\ -c_*(0)Q_2 - f(Q_1) \\ Q_4 \\ -c_*(0)Q_4 - f'(Q_1)Q_3 \end{pmatrix}. \quad (2.31)$$

Notice that  $(Q_1^0, Q_2^0, Q_3^0, Q_4^0) := (q_0, q_0', q_0', q_0'')$  solves this system by construction. Furthermore, the phase plane construction of  $q_0$  implies that  $q_0(x), q_0'(x) \rightarrow 0$  as  $x \rightarrow \infty$ , which in turn by the equation  $q_0'' + c_0 q_0' + f(q_0) = 0$  implies that also  $q_0''(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Hence the solution  $(Q_1^0, Q_2^0, Q_3^0, Q_4^0)$  lies in the stable manifold of the origin. The linearization of (2.31) at the origin is

$$\begin{pmatrix} Q_1' \\ Q_2' \\ Q_3' \\ Q_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\eta_*(0)^2 & -2\eta_*(0) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\eta_*(0)^2 & -2\eta_*(0) \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \quad (2.32)$$

The block matrix  $\begin{pmatrix} 0 & 1 \\ -\eta_*(0)^2 & -2\eta_*(0) \end{pmatrix}$  has a repeated eigenvalue  $-\eta_*(0)$ , which forms a Jordan block with eigenvector  $e_0 = \begin{pmatrix} 1 \\ -\eta_*(0) \end{pmatrix}$  and generalized eigenvector  $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . The expressions of  $e_0$  and  $e_1$  can be either computed directly, or hinted from the companion structure of the block matrix. Solutions to (2.31) in the stable manifold of the origin therefore have expansions

$$\begin{pmatrix} Q_1(x) \\ Q_2(x) \end{pmatrix} = c_1 e_0 e^{-\eta_*(0)x} + c_2 (e_0 x + e_1) e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x}),$$

$$\begin{pmatrix} Q_3(x) \\ Q_4(x) \end{pmatrix} = c_3 e_0 e^{-\eta_*(0)x} + c_4 (e_0 x + e_1) e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x}). \quad (2.33)$$

In particular, since  $(Q_1^0, Q_2^0, Q_3^0, Q_4^0) = (q_0, q_0', q_0', q_0'')$  is such a solution, we see that  $q_0'$  and  $q_0''$  have expansions of the form

$$q_0'(x) = (\alpha x + \beta) e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x}), \quad q_0''(x) = (\tilde{\alpha} x + \tilde{\beta}) e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x})$$

for some constants  $\alpha, \beta, \tilde{\alpha}$ , and  $\tilde{\beta}$ . The coefficients  $c_1$  and  $c_2$  are fixed by the first line of (2.33), and the fact that  $Q_1^0(x) = q_0(x) \sim (\mu_0 + x) e^{-\eta_*(0)x}$ . Using the expansions of  $q_0'$  and  $q_0''$  together with the second, third and fourth lines of (2.33) allow to compute first  $\alpha, \beta$  then  $c_3, c_4$  and finally  $\tilde{\alpha}, \tilde{\beta}$ . Altogether, we obtain the following asymptotic, that coincide with a formal differentiation of  $q_0$ :

$$q_0'(x) = (1 - \eta_*(0)\mu_0 - \eta_*(0)x) e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x}),$$

$$q_0''(x) = (\eta_*(0)^2 \mu_0 - 2\eta_*(0) + \eta_*(0)^2 x) e^{-\eta_*(0)x} + O(x^2 e^{-2\eta_*(0)x}).$$

Note for  $x \geq 1$ , we have  $\varphi(x) = e^{-\eta_*(0)x} e^{c_0 x} q_0'(x) = e^{\eta_*(0)x} q_0'(x)$ , using the fact that  $c_0 = 2\eta_*(0)$ . Hence for  $x \gg 1$ , we have after a short computation

$$\varphi'(x) = \eta_*(0) e^{\eta_*(0)x} q_0'(x) + e^{\eta_*(0)x} q_0''(x) = -\eta_*(0) + O(x^2 e^{-\eta_*(0)x}),$$

as desired.  $\square$

We now use these asymptotics of  $\varphi'$  to show that  $\partial_{(v,\mu)}G(v_0; \mu_0, 0)$  is invertible.

**Lemma 2.3.3.** *The joint linearization  $\partial_{(v,\mu)}G(v_0; \mu_0, 0) : H_{0,\eta}^2(\mathbb{R}) \times \mathbb{R} \rightarrow L_{0,\eta}^2(\mathbb{R})$  is invertible.*

*Proof.* To show that  $\partial_{(v,\mu)}G(v_0; \mu_0, 0)$  is invertible, we show that  $\partial_\mu G(v_0; \mu_0, 0)$  is linearly independent from the range of  $\mathcal{L}(0)$ . More precisely,  $\partial_\mu G(v_0; \mu_0, 0)$ , by the strictest definition, is a linear operator from  $\mathbb{R}$  to  $L_{0,\eta}^2(\mathbb{R})$ , but this linear operator may be naturally identified with  $\partial_\mu G(v_0; \mu_0, 0) \cdot 1 \in L_{0,\eta}^2(\mathbb{R})$ , and we show that this function does not lie in the range of  $\mathcal{L}(0)$ . From Lemma 2.2.6, it is enough to obtain  $\langle \partial_\mu G(v_0; \mu_0, 0), \varphi \rangle \neq 0$ . After a short computation, one finds

$$\partial_\mu G(v_0; \mu_0, 0) = \mathcal{S}(0)\chi_+ + (f'(q_0) - f'(0))\chi_+ = \mathcal{L}(0)\chi_+.$$

We compute  $\langle \mathcal{L}(0)\chi_+, \varphi \rangle$  via integration by parts, with the goal being to move  $\mathcal{L}(0)$  onto the other side of the inner product as its adjoint and exploit the fact that  $\mathcal{L}(0)^*\varphi = 0$ . However, we must be careful since  $\varphi$  and  $\chi_+$  are not localized at  $\infty$ , and in fact there is one boundary term from integration by parts which does not vanish. We see this by writing

$$\int_{\mathbb{R}} \chi_+'' \varphi dx = - \int_{\mathbb{R}} \chi_+' \varphi' dx = \int_{\mathbb{R}} \chi_+ \varphi'' - [\chi_+' \varphi']_{-\infty}^{\infty} = \langle \chi_+, \varphi'' \rangle - \varphi'(+\infty) = \langle \chi_+, \varphi'' \rangle + \eta_*(0),$$

We use Lemma 2.3.2 in the final equality to replace  $\varphi'(\infty)$  with  $-\eta_*(0)$ . Recalling that  $\mathcal{L}(0) = \partial_x^2 + f'(q_*)$  for  $x \geq 1$ , we obtain

$$\langle \mathcal{L}(0)\chi_+, \varphi \rangle = \langle \chi_+, \mathcal{L}(0)^*\varphi \rangle + \eta_*(0) = \eta_*(0) = \frac{c_*(0)}{2} > 0,$$

which concludes the proof.  $\square$

*Proof of Theorem 2.1.1.* Since  $G(v_0; \mu_0, 0) = 0$ ,  $\partial_{(v,\mu)}G(v_0; \mu_0, 0)$  is invertible by Lemma 2.3.3, and  $G$  has the requisite regularity by Lemma 2.3.1, by the implicit function theorem (Lemma 2.2.7) there exist  $v(\delta) \in H_{0,\eta}^2(\mathbb{R})$  and  $\mu(\delta) \in \mathbb{R}$  depending continuously on  $\delta$  near  $\delta = 0$  such that  $G(v(\delta); \mu(\delta), \delta) = 0$ . By construction of  $G$ , this implies that

$$q_*(x; \delta) := \chi_-(x) + \omega_*(x; \delta)^{-1}v(x; \delta) + \chi_+(x)(\mu(\delta) + x)e^{-\eta_*(\delta)x}$$

solves (2.2). The claim that  $q_*(\cdot, \delta) \rightarrow q_*(\cdot; 0) = q_0$  uniformly in space follows from the form of this ansatz, together with the fact that  $H_{0,\eta}^2(\mathbb{R})$  is continuously embedded in  $L^\infty(\mathbb{R})$ .  $\square$

## 2.4 Small eigenvalues

Having established existence of the critical front, we are now ready to study the point spectrum of the linearization about the front. Here we show that there is no eigenvalue in a neighborhood of the origin, and in particular no resonance embedded in the essential spectrum at the origin. For this, we follow [PS10]: apply a Lyapunov-Schmidt reduction to construct a scalar function which vanishes at the eigenvalues, in a similar manner to the Evans function.

Throughout this section, we set  $\Omega(\delta) := \{0\} \cup (\mathbb{C} \setminus \sigma_{\text{ess}}(\mathcal{L}_\delta))$ , and restrict to  $\lambda \in \Omega(\delta)$ . Then  $\lambda$  is off the negative real axis, so that the principal value of  $\gamma := \sqrt{\lambda}$  is defined by  $\text{Re } \gamma \geq 0$ .

**Proposition 2.4.1.** *There exists  $\delta_0, \gamma_0 > 0$  and a function  $E : (-\delta_0, \delta_0) \times B(0, \gamma_0) \rightarrow \mathbb{C}$ , continuous in  $\delta$  and analytic in  $\gamma$  such that for all  $\gamma^2 \in \Omega(\delta)$ , the eigenvalue problem*

$$(\mathcal{L}(\delta) - \gamma^2)u = 0 \quad (2.34)$$

*admits a bounded solution  $u$  if and only if  $E(\delta, \gamma) = 0$ . Furthermore,  $E(0, 0) \neq 0$ . In particular, there exists  $\gamma_1, \delta_1 > 0$  such that for all  $\delta \in (-\delta_1, \delta_1)$ ,  $\mathcal{L}(\delta)$  has no eigenvalues on  $B(0, \gamma_1^2) \cap \Omega(\delta_1)$ .*

For any fixed  $\delta \neq 0$ , notice that (2.34) is a linear, non degenerate ODE with smooth (i.e.  $C^\infty$  with respect to  $x \in \mathbb{R}$ ) coefficients, so that any solution  $u$  belongs to  $C^\infty(\mathbb{R})$ . Furthermore, such a solution admits exponential expansions at  $\pm\infty$  (see the proof of Lemma 2.4.2 hereafter), so that when  $\gamma^2$  is to the right of the essential spectrum,  $u$  is bounded if and only if it lies in  $H^4(\mathbb{R})$ , which is to say it is an eigenfunction. We will therefore consider bounded solutions from this point forward: for  $\gamma^2$  to the right of the essential spectrum, they correspond with eigenfunctions, while at  $\gamma = 0$  they capture resonances of  $\mathcal{L}(\delta)$ .

We first show that a bounded solution of (2.34) decomposes into two parts: a uniformly localized part, and a slowly decaying part, whose rate is  $\gamma$ -close to 0.

**Lemma 2.4.2.** *Near  $(\delta, \gamma) = (0, 0)$ , the roots of the polynomial  $v \mapsto d_{c_*}^+(\gamma^2, -\eta_* + v)$  satisfy:*

$$v_1 = -\frac{1}{|\delta|} + O(1), \quad v_2 = -\gamma + O(\delta\gamma + \gamma^2), \quad v_3 = \gamma + O(\delta\gamma + \gamma^2), \quad v_4 = \frac{1}{|\delta|} + O(1),$$

where each  $O$  is taken as  $\delta$  and  $\gamma$  goes to 0.

*In particular, there exists  $\delta_0 > 0$ ,  $\gamma_0 > 0$  and  $\eta > 0$  such that for all  $\delta \in (-\delta_0, \delta_0)$ , and  $\gamma \in B(0, \gamma_0)$  with  $\gamma^2 \in \Omega(\delta)$ , a bounded and smooth solution  $u$  of (2.34) decompose as*

$$u(x) = w(x) + \beta \chi_+(x) e^{v_2 x}, \quad (2.35)$$

where  $w \in H_{0, \eta}^2(\mathbb{R})$  and  $\beta \in \mathbb{C}$ . In this decomposition,  $\chi_+$  is the cutoff function (2.22).

*Proof.* The claimed expansions of the four roots is purely technical and is postponed to the end of the proof. Rewrite (2.34) as a first order ODE in  $\mathbb{R}^4$ :

$$\partial_x U = M(x; \delta, \gamma)U.$$

where  $U = (u, u', u'', u^{(3)})^T$ . The matrix  $M$  converges towards  $M_\pm(\delta, \gamma)$  when  $x \rightarrow \pm\infty$ , with an exponential rate which is independent of  $\delta$  and  $\gamma$ . The eigenvalues of this asymptotic matrices  $M_\pm$  are the roots of the dispersion relations  $d_{c_*}^\pm(\gamma^2, -\eta_* + \cdot)$ . It is standard that with such a convergence rate, these eigenvalues determine the behavior of  $U$  at  $\pm\infty$ ; see for example [FH18, proof of Lemma 2.2].

More precisely, the behavior at  $+\infty$  is the following. For  $\gamma \neq 0$ , the four roots are distinct, so that the exponential behavior is ensured:  $U(x) \sim \sum_{i=1}^4 c_i(U) e^{v_i x}$  when  $x \rightarrow +\infty$ , with  $c_i(U, \delta, \gamma)$  are vectors that does not depend on  $x$ . As  $\gamma^2 \notin \sigma_{\text{ess}}(\mathcal{L}(\delta))$ , the two small roots satisfy  $\text{Re } v_2(\delta, \gamma) < 0 < \text{Re } v_3(\delta, \gamma)$ , so that a bounded  $U$  has exactly the claimed form. At  $\gamma = 0$ , the two small roots merge to form a Jordan block. The proof in the above reference adapts, and we have the following expansion:  $U(x) \sim c_1(U) e^{v_1 x} + c_2(U) + c_3(U)x + c_4(U) e^{v_4 x}$  when  $x \rightarrow +\infty$ . The terms  $c_2(U) + c_3(U)x$  correspond to the 2-dimensional center space at  $\gamma = 0$ , with the weakly growing term  $c_3(U)x$  a consequence of the presence of the Jordan block. Once again the claimed decomposition is satisfied.

At  $-\infty$ , the four roots of  $d_{c_*}^-(\gamma^2, -\eta_* + \cdot)$  are distinct, and bounded away from 0 with spectral gap uniform in  $(\delta, \gamma)$ . Then the expansion  $U(x) \sim \sum_{i=1}^4 c_i^-(U) e^{v_i^- x}$  holds at  $x \rightarrow -\infty$ , so that any

bounded  $U$  lies in  $H^2(\mathbb{R}_-)$ . Hence the claimed decomposition holds. For an alternative argument not relying on the dynamical systems view of exponential expansions, see Remark 2.4.7.

We now establish the expansions of the roots by applying the implicit function theorem to  $d_{c_*}^+$ . From the choice of  $\eta_*$  (see also (2.10)) we have

$$g_0(\delta, \gamma, \nu) := d_{c_*}^+(\gamma^2, -\eta_* + \nu) = -\delta^2 \nu^4 + 4\eta_* \delta^2 \nu^3 + (1 - 6\delta^2 \eta_*^2) \nu^2 - \gamma^2.$$

To avoid any  $\delta$  singularity, we get rid of the  $\delta^2$  in the dominant term by changing variables  $\mu := \nu|\delta|$ :

$$g_1(\delta, \gamma, \mu) := \delta^2 g_0\left(\gamma, \delta, \frac{\mu}{|\delta|}\right) = -\mu^4 + 4\eta_* |\delta| \mu^3 + (1 - 6\delta^2 \eta_*^2) \mu^2 - \gamma^2 \delta^2.$$

At  $(\delta, \gamma) = (0, 0)$ , this reduces to  $g_1(0, 0, \mu) = -\mu^2(\mu - 1)(\mu + 1)$ . Applying the implicit function theorem to the simple root  $-1$ , we construct a root  $\mu_1(\delta, \gamma)$  for  $g_1(\delta, \gamma, \cdot)$  whose derivatives can be computed iteratively by differentiating the relation  $g_1(\delta, \gamma, \mu_1(\delta, \gamma)) = 0$ . One can show by induction that any pure derivative in  $\gamma$  is null:  $\partial_\gamma^k \mu_1(0, 0) = 0$  for  $k \in \mathbb{N}^*$ . This ensures that the Taylor expansion has the form

$$\mu_1(\delta, \gamma) = -1 - \delta \frac{\partial_\delta g_1(0, 0, -1)}{\partial_\mu g_1(0, 0, -1)} - \gamma \frac{\partial_\gamma g_1(0, 0, -1)}{\partial_\mu g_1(0, 0, -1)} + O(\delta^2 + \delta\gamma) = -1 + O(\delta).$$

Coming back to the original variable, we define  $\nu_1(\delta, \gamma) = \mu_1(\delta, \gamma)/|\delta|$ , which satisfies the claimed expansion. The same steps can be applied to define  $\mu_4(\delta, \gamma) = 1 + O(\delta)$ , which in turn leads to  $\nu_4(\delta, \gamma)$  as claimed.

To unfold the double root at  $\mu = 0$ , we change variables once again to  $\nu = \gamma\sigma$ :

$$g_2(\delta, \gamma, \sigma) = \frac{g_0(\delta, \gamma, \gamma\sigma)}{\gamma^2} = -\delta^2 \gamma^2 \sigma^4 + 4\eta_* \delta^2 \gamma \sigma^3 + (1 - 6\delta^2 \eta_*^2) \sigma^2 - 1.$$

At  $(\delta, \gamma) = (0, 0)$ , this reduces to  $g_2(0, 0, \sigma) = (\sigma - 1)(\sigma + 1)$ . Applying the implicit function theorem once again gives rise to

$$\sigma_2(\delta, \gamma) = -1 + O(\delta + \gamma), \quad \sigma_3(\delta, \gamma) = 1 + O(\delta + \gamma),$$

which in turns leads to the claimed estimates on  $\nu_2(\delta, \gamma) = \gamma\sigma_2(\delta, \gamma)$  and  $\nu_3(\delta, \gamma) = \gamma\sigma_3(\delta, \gamma)$ .  $\square$

As in the existence of the critical front, our problem is singular at  $\delta = 0$ . Hence, we apply the same preconditioner: when  $\delta$  is small, (2.34) is equivalent to

$$(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2) u = 0.$$

We now use the decomposition of Lemma 2.4.2 to separate out the localized part of our problem from the far-field behavior, which will allow us to make use of the Fredholm properties on weighted spaces of Section 2.2.4. In the following, for  $\delta \in (-\delta_0, \delta_0)$  and  $\gamma \in B(0, \gamma_0)$  we let

$$A(\delta, \gamma, \eta) = \{w + \beta \chi_+ e^{\nu_2(\delta, \gamma) \cdot} : w \in H_{0, \eta}^2(\mathbb{R}), \beta \in \mathbb{R}\},$$

denote the set where the ansatz obtained above holds.

**Lemma 2.4.3.** *There exist positive constants  $\delta_0, \gamma_0$  and  $\eta$  such that if  $\delta \in (-\delta_0, \delta_0)$ ,  $\gamma \in B(0, \gamma_0)$  and  $u \in A(\delta, \gamma, \eta)$ , then  $(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2) u \in L_{0, \eta}^2(\mathbb{R})$ .*



*Proof.* First,  $(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)w$  belongs to  $L^2_{0,\eta}(\mathbb{R})$  by the choice of the preconditioner, using the same regularization effect we observed in (2.28). Then, as  $\chi_+$  is smooth, vanishes on  $(-\infty, 2)$  and is constant on  $(3, +\infty)$ , it only remains to show that  $(\mathcal{L}(\delta) - \gamma^2)e^{v_2 \cdot} \in L^2_{0,\eta}(\mathbb{R}_+)$ . For  $x \geq 1$ , almost all coefficients of  $\mathcal{L}$  are constants, see (2.9), hence we compute

$$(\mathcal{L}(\delta) - \gamma^2)e^{v_2 x} = (\mathcal{L}(\delta) - \mathcal{L}_+(\delta))e^{v_2 x} + (\mathcal{L}_+(\delta) - \gamma^2)e^{v_2 x} = (f'(q_*(x; \delta)) - f'(0))e^{v_2 x} + d_{c_*}^+(\gamma^2, v_2 - \eta_*)e^{v_2 x},$$

Recall that the polynomial  $v \mapsto d_{c_*}^+(\lambda, -\eta_* + v)$  is the dispersion relation of  $\mathcal{L}_+(\delta) - \lambda$ , i.e. that  $d_{c_*}^+(\gamma^2, -\eta_* + \partial_x) = \mathcal{L}_+(\delta) - \gamma^2$ , and that  $\mathcal{L}_+(\delta)$  is the asymptotic operator (2.10). From the definition of  $v_2(\delta, \gamma)$ ,  $d_{c_*}^+(\gamma^2, -\eta_* + v_2) = 0$ , hence for  $x \geq 1$ :

$$(\mathcal{L}(\delta) - \gamma^2)e^{v_2 x} = (f'(q_*(x)) - f'(0))e^{v_2 x} = f''(0)q_*(x; \delta)e^{v_2 x} + \mathcal{O}(e^{v_2 x} q_*(x; \delta)^2).$$

The right hand side belongs to  $L^2_{0,\eta}(\mathbb{R})$  as long as  $\eta$  satisfies

$$-\eta_* + \operatorname{Re} v_2(\delta, \gamma) < -\eta. \quad (2.36)$$

We can take a smaller  $\gamma_0$  than in Lemma 2.4.2, so that  $\sup_{\delta, \gamma} \{-\eta_*(\delta) + \operatorname{Re} v_2(\delta, \gamma)\} < 0$ , which then allows to fix  $\eta > 0$  so that (2.36) is satisfied for all  $\delta \in (-\delta_0, \delta_0)$  and  $\gamma \in B(0, \gamma_0)$ . This concludes the proof.  $\square$

We can now use Lemma 2.2.6 to decompose our problem into a part which belongs to  $\operatorname{im} \mathcal{L}(0)$  and a complementary part. Recall that  $P : L^2_{0,\eta}(\mathbb{R}) \rightarrow \operatorname{im}(\mathcal{L}(0))$  and that  $\varphi$  allows to describe  $\operatorname{im}(\mathcal{L}(0))$ . Fix  $\delta \in (-\delta_0, \delta_0)$ , and  $\gamma \in B(0, \gamma_0) \cap \Omega(\delta)$ . If  $u$  is a bounded solution of (2.34) then  $(w, \beta) \in H^2_{0,\eta}(\mathbb{R}) \times \mathbb{C}$  defined in Lemma 2.4.2 solves:

$$\begin{cases} P(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)(w + \beta h) = 0, \\ \langle (1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)(w + \beta h), \varphi \rangle = 0, \end{cases} \quad (2.37)$$

where  $h(x) = \chi_+(x)e^{v_2(\gamma)x}$ . Reciprocally, if  $(w, \beta) \in H^2_{0,\eta}(\mathbb{R}) \times \mathbb{C}$  satisfies (2.37), then  $u = w + \beta h$  is bounded and satisfies (2.34). We write the first equation as

$$0 = \mathcal{F}(w, \beta; \gamma, \delta) := P(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)(w + \beta h). \quad (2.38)$$

and solve it with the implicit function theorem. We will then use the second equation to define  $E(\delta, \gamma)$ .

**Lemma 2.4.4.** *For  $\eta > 0$  sufficiently small, the map*

$$\mathcal{F} : H^2_{0,\eta}(\mathbb{R}) \times \mathbb{C} \times B(0, \gamma_0) \times (-\delta_0, \delta_0) \rightarrow \operatorname{Im} \mathcal{L}(0)$$

*is continuously differentiable in  $w$  and  $\beta$ , analytic in  $\gamma$ , and continuous in  $\delta$ . Moreover, the map*

$$(w, \beta; \gamma, \delta) \mapsto \partial_w \mathcal{F}(w, \beta; \gamma, \delta) : H^2_{0,\eta}(\mathbb{R}) \times \mathbb{C} \times B(0, \gamma_0) \times (-\delta_0, \delta_0) \rightarrow \mathcal{B}(H^2_{0,\eta}(\mathbb{R}), \operatorname{Im} \mathcal{L}(0))$$

*is continuous.*

*Proof.* Note that  $\mathcal{F}$  is linear in  $w$  and  $\beta$ , so differentiability is automatic provided the linear part in  $w$  is well defined, which is guaranteed here by Lemma 2.4.3. For the continuity of

$\partial_w \mathcal{F}(w, \beta; \gamma, \delta)$ , we write

$$(1 - \delta^2 \partial_x^2)^{-1} \mathcal{L}(\delta) = \partial_x^2 + a_1 \partial_x + a_0 + \delta^2 (1 - \delta^2 \partial_x^2)^{-1} (a_3 \partial_x^3 + a_2 \partial_x^2) + T(\delta)(a_1 \partial_x + a_0).$$

We see by Lemmas 2.2.3 and 2.2.4 that  $(1 - \delta^2 \partial_x^2)^{-1} \mathcal{L}(\delta)$  is a well-defined family of bounded operators from  $H_{0,\eta}^2$  to  $L_{0,\eta}^2$ , depending continuously on  $\delta$ . This is of course preserved when we compose with the projection  $P$ . We write the other term in the linearization in  $w$  as

$$\gamma^2 (1 - \delta^2 \partial_x^2)^{-1} w = \gamma^2 w + \gamma^2 T(\delta)w,$$

which is again continuous in  $\gamma$  and  $\delta$  as a bounded linear operator from  $H_{0,\eta}^2$  to  $L_{0,\eta}^2$  by Lemma 2.2.4. Hence  $\partial_w \mathcal{F}$  is continuous in its three last variables. Analyticity of  $\mathcal{F}$  in  $\gamma$  follows as in [PS10, Proposition 5.11]. For the continuity of  $\mathcal{F}$  with respect to  $\delta$ , it only remains to look at the terms associated to  $h$ . We rewrite

$$(\mathcal{L}(\delta) - \gamma^2)h = [\mathcal{L}(\delta), \chi_+] e^{v_2(\delta, \gamma) \cdot} + \chi_+ (\mathcal{L}(\delta) - \mathcal{L}_+(\delta)) e^{v_2(\delta, \gamma) \cdot},$$

using the fact that  $(\mathcal{L}_+(\delta) - \gamma^2) e^{v_2(\delta, \gamma) \cdot} = 0$ , and where  $[\mathcal{L}(\delta), \chi_+] = \mathcal{L}(\delta)(\chi_+ \cdot) - \chi_+ \mathcal{L}(\delta)$  is the commutator between these operators. In this form, we recognize that  $[\mathcal{L}(\delta), \chi_+]$  and  $(\mathcal{L}(\delta) - \mathcal{L}_+(\delta))$  are both differential operators with exponentially localized coefficients, with rate uniform in  $\delta$  for  $\delta$  small. By Lemma 2.4.2,  $e^{v_2(\delta, \gamma)x}$  is continuous in  $\gamma$  and  $\delta$  for each fixed  $x$ , and the uniform localization of  $[\mathcal{L}(\delta), \chi_+]$  and  $(\mathcal{L}(\delta) - \mathcal{L}_+(\delta))$  guarantees that these terms are continuous in  $\delta$  in  $L_{0,\eta}^2$  for  $\eta$  small. In fact, since  $h$  is a smooth function, we see that  $\delta \mapsto (\mathcal{L}(\delta) - \gamma^2)h$  is in particular continuous from  $(-\delta_1, \delta_1)$  to  $H_{0,\eta}^1$ . Taking into account the preconditioner, we write

$$(1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2)h = (\mathcal{L}(\delta) - \gamma^2)h + T(\delta)(\mathcal{L}(\delta) - \gamma^2)h.$$

By Lemma 2.2.4, this term is continuous in  $\delta$ , as desired.  $\square$

**Corollary 2.4.5.** *For  $\gamma, \delta$  sufficiently small, and for  $\beta \in \mathbb{C}$ , there is a family of solutions  $w$  to (2.38) which have the form*

$$w(\beta; \gamma, \delta) = \beta \tilde{w}(\gamma, \delta). \quad (2.39)$$

Moreover, any solution to (2.38) with  $\gamma, \delta$  small has this form.

*Proof.* We begin with the trivial solution  $\mathcal{F}(0, 0; 0, 0) = 0$ . The linearization in  $w$  about this trivial solution is  $\partial_w \mathcal{F}(0, 0; 0, 0) = P\mathcal{L}(0)$ , which is invertible by Lemmas 2.2.5 and 2.2.6. Together with Lemma 2.4.4, this implies that we can solve near this trivial solution with the implicit function theorem (Lemma 2.2.7), obtaining a unique solution  $w(\beta; \gamma, \delta)$  in a neighborhood  $\mathcal{U}$  of  $(0, 0; 0, 0)$ . Since (2.38) is linear in  $w$  and  $\beta$ , by uniqueness any solution in this neighborhood can be written as

$$w(\beta; \gamma, \delta) = \beta \tilde{w}(\gamma, \delta)$$

for some function  $\tilde{w}(\gamma, \delta) \in H_{0,\eta}^2(\mathbb{R})$ .

To see this, note that if for some fixed  $\gamma, \delta$  small we have another solution  $(w_0, \beta_0)$  to (2.38) which does not a priori have this form, by dividing by a sufficiently large constant  $K(\|w_0\|_{H_{0,\eta}^2}, \beta)$  we get another solution which belongs to the neighborhood  $\mathcal{U}$  where we have solved with the implicit function theorem, and so we conclude that

$$\frac{w_0}{K(\|w_0\|_{H_{0,\eta}^2}, \beta)} = \frac{\beta_0}{K(\|w_0\|_{H_{0,\eta}^2}, \beta)} \tilde{w}(\gamma, \delta),$$

and hence the solution  $(w_0, \beta_0)$  in fact has the form (2.39), as claimed.  $\square$

Having solved the first equation in (2.37) with the implicit function theorem, we now insert this solution  $w(\beta; \gamma, \delta) = \beta \tilde{w}(\gamma, \delta)$  into the second equation, so that (2.37) has a solution if and only if

$$0 = E(\delta, \gamma) := \langle (1 - \delta^2 \partial_x^2)^{-1} (\mathcal{L}(\delta) - \gamma^2) (\tilde{w}(\gamma, \delta) + h), \varphi \rangle. \quad (2.40)$$

Note that we have been able to eliminate the  $\beta$  dependence in this equation, since all terms in this equation are linear in  $\beta$  by Corollary 2.4.5. Since the projection  $P$  played no role in the proof of Lemma 2.4.4, the same argument shows that  $E$  is continuous in both of its arguments.

**Lemma 2.4.6.** *The function  $E : (-\delta_0, \delta_0) \times B(0, \gamma_0) \rightarrow \mathbb{C}$  is continuous in both arguments, and analytic in  $\gamma$  for fixed  $\delta$ .*

*Proof of Proposition 2.4.1.* It only remains to prove that  $E(0, 0) \neq 0$ . From (2.40), we see that

$$E(0, 0) = \langle \mathcal{L}(0) (\tilde{w}(0, 0) + \chi_+), \varphi \rangle.$$

Since  $\tilde{w}(0, 0) \in H_{0, \eta}^2(\mathbb{R})$ , we have

$$\langle \mathcal{L}(0) \tilde{w}(0, 0), \varphi \rangle = \langle \tilde{w}(0, 0), \mathcal{L}(0)^* \varphi \rangle = 0.$$

Hence we obtain

$$E(0, 0) = \langle \mathcal{L}(0) \chi_+, \varphi \rangle = \eta_*(0) \neq 0,$$

by the computation in the proof of Lemma 2.3.3.  $\square$

*Remark 2.4.7.* Rather than using the spatial dynamics approach to exponential expansions outlined in the proof of Lemma 2.4.2 to show directly that eigenfunctions have the form (2.35), one can instead show that for  $\delta \neq 0$ ,  $(\mathcal{L}(\delta) - \gamma^2) : H^4(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is invertible if  $E(\delta, \gamma) \neq 0$  and  $\gamma^2$  is to the right of the essential spectrum of  $\mathcal{L}(\delta)$ , using an argument adapted from [PS10]. Indeed, if  $\gamma^2$  is to the right of the essential spectrum, then  $\mathcal{L}(\delta) - \gamma^2$  is Fredholm with index 0 on  $L^2(\mathbb{R})$ , and in particular has closed range, so to invert this operator on  $L^2(\mathbb{R})$ , it suffices to solve  $(\mathcal{L}(\delta) - \gamma^2)u = g$  for  $g$  in the dense subspace  $L_{0, \eta}^2(\mathbb{R})$ . The fact that the range of  $\mathcal{L}(\delta) - \gamma^2$  is closed then implies  $\mathcal{L}(\delta) - \gamma^2$  is surjective, and hence invertible since it is Fredholm of index 0. The open mapping theorem then implies the inverse is bounded, so  $\gamma^2$  will be in the resolvent set of  $\mathcal{L}(\delta)$ . To solve  $(\mathcal{L}(\delta) - \gamma^2)u = g$  for  $g \in L_{0, \eta}^2(\mathbb{R})$ , one looks for solutions in the form (2.35), and finds that  $(w, \beta)$  solve the system (2.37) but with  $(0, 0)^T$  on the right hand side replaced by  $(Pg, \langle g, \varphi \rangle)^T$ . We can always solve the first equation with the implicit function theorem, and we can solve the second equation precisely when  $E(\delta, \gamma) \neq 0$ , as claimed. At  $\gamma = 0$  we lose Fredholm properties on  $L^2(\mathbb{R})$ , but the fact that  $E(\delta, 0) \neq 0$  implies there is no solution to  $\mathcal{L}(\delta)u = 0$  of the form  $u = w + \beta \chi_+$  for  $w$  exponentially localized, and this is actually all that is needed in [AS21] to prove nonlinear stability. One could additionally use a modified far-field/core decomposition at  $\gamma = 0$  to prove that all bounded solutions to  $\mathcal{L}(\delta)u = 0$  have the form  $u = w + \beta \chi_+$ .

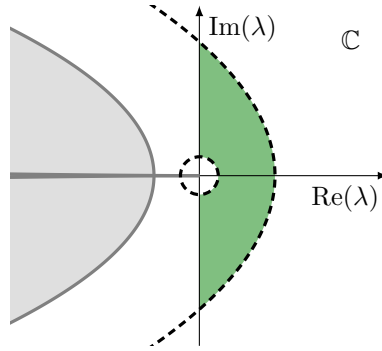


Figure 2.3 – Three regions for the study of the point spectrum. The function  $E(\delta, \gamma)$  from Section 2.4 rules out point spectrum in the dashed ball centered at the origin, together with a potential eigenvalue at the origin. Proposition 2.5.1 excludes point spectrum to the right of the dashed curve. Finally, the green region to the right contains no point spectrum provided  $\delta$  is small enough, see Proposition 2.5.2.

## 2.5 Large and intermediate eigenvalues – proof of Theorem 2.1.2

Here, we conclude the study of the point spectrum. We first exclude any large unstable point spectrum, using mostly that the operator is sectorial.

**Proposition 2.5.1.** *There exists a compact set  $K \subset \mathbb{C}$  such that for all  $\delta$  small, any eigenvalue  $\lambda$  of  $\mathcal{L}(\delta)$  with  $\operatorname{Re} \lambda \geq 0$  lies in  $K$ . More precisely, an eigenvalue  $\lambda$  satisfies:*

$$\operatorname{Re} \lambda \leq \|b(\cdot; \delta)\|_{\infty}, \quad |\operatorname{Im} \lambda| \leq c_* \sqrt{\|b(\cdot; \delta)\|_{\infty} - \operatorname{Re} \lambda},$$

where  $b(\cdot; \delta) = f'(q_*(\cdot; \delta))$  is uniformly bounded.

*Proof.* We work with  $\delta \in (-\delta_0, \delta_0)$ , with  $\delta_0$  small enough so that Theorem 2.1.1 applies. Assume that  $\lambda \in \mathbb{C}$  and  $\psi \in H^4(\mathbb{R})$  satisfy  $\mathcal{L}(\delta)\psi = \lambda\psi$ . Coming back to the unweighted operator  $\mathcal{A}(\delta) = \omega_*^{-1}\mathcal{L}(\delta)\omega_*(\cdot; \delta)$ , defined by (2.4), we obtain

$$\mathcal{A}(\delta)\phi = \lambda\phi \tag{2.41}$$

with  $\phi = \omega_*(\cdot; \delta)^{-1}\psi \in H^4(\mathbb{R})$ . Up to a scalar multiplication, we can assume that  $\|\phi\|_{L^2(\mathbb{R})} = 1$ . Now we take the  $L^2(\mathbb{R})$ -inner product of (2.41) with  $\phi$ , and pass into Fourier space, to obtain that:

$$\int_{\mathbb{R}} (-\delta^2 \xi^4 - \xi^2 + i c_* \xi) |\hat{\phi}(\xi)|^2 d\xi + \int_{\mathbb{R}} f'(q_*(x)) |\phi(x)|^2 dx = \lambda, \tag{2.42}$$

where  $\mathcal{F}u = \hat{u}$  denotes the Fourier transform of a function  $u$ . We let  $I_0 = \int_{\mathbb{R}} f'(q_*(x)) |\phi(x)|^2 dx$  denote the 0th order term. Then, real and imaginary parts of equation (2.42) give

$$\operatorname{Re} \lambda - I_0 = - \int_{\mathbb{R}} (\delta^2 \xi^4 + \xi^2) |\hat{\phi}(\xi)|^2 d\xi, \quad \operatorname{Im} \lambda = c_* \int_{\mathbb{R}} \xi |\hat{\phi}(\xi)|^2 d\xi.$$

Hence by the Cauchy-Schwartz inequality:

$$0 \leq (\operatorname{Im} \lambda)^2 \leq c_*^2 \|\hat{\phi}\|_{L^2}^2 \int_{\mathbb{R}} \xi^2 |\hat{\phi}(\xi)|^2 d\xi \leq c_*^2 \int_{\mathbb{R}} (\delta^2 \xi^4 + \xi^2) |\hat{\phi}(\xi)|^2 d\xi = c_*^2 (I_0 - \operatorname{Re} \lambda). \quad (2.43)$$

Note that  $b(\cdot; \delta)$  is uniformly bounded with respect to  $\delta \in (-\delta_0, \delta_0)$ , since this holds for  $q_*(\cdot; \delta)$  from Theorem 2.1.1, and since  $f'$  is continuous.

Observe that  $|I_0| \leq \|b\|_{\infty} \|\phi\|_{L^2}^2$ . Inserting this into (2.43) leads to the claimed bounds on  $\operatorname{Re} \lambda$  and  $\operatorname{Im} \lambda$ . These bounds together with the requirement  $\operatorname{Re} \lambda \geq 0$  define a compact set  $K$ .  $\square$

We now conclude the proof of Theorem 2.1.2 by excluding the possibility of any eigenvalues in the intermediate region; see Figure 2.3.

**Proposition 2.5.2.** *For each  $\delta_0 > 0$  sufficiently small, there exists  $r(\delta_0) > 0$  with  $r(\delta_0) \rightarrow 0$  as  $\delta_0 \rightarrow 0$  such that for all  $\delta$  with  $|\delta| < \delta_0$ , the operator  $\mathcal{L}(\delta)$  has no eigenvalues in  $\{\operatorname{Re} \lambda \geq 0\} \setminus B(0, r(\delta_0))$ .*

*Proof.* Suppose to the contrary that there exists a sequence  $\delta_n \rightarrow 0$  with corresponding eigenvalues  $\lambda_n$  bounded away from the origin, with  $\operatorname{Re} \lambda_n \geq 0$ , and with eigenfunctions  $u_n$ . We normalize the eigenfunctions so that  $\|u_n\|_{H^2} = 1$  for all  $n$ . By Proposition 2.5.1, these eigenvalues all belong to the compact set  $K$ . By compactness, we extract a subsequence along which  $\lambda_n \rightarrow \lambda_{\infty}$  for some  $\lambda_{\infty}$  with  $\lambda_{\infty} \in K$ , and  $\lambda_{\infty} \neq 0$ , since the sequence was bounded away from the origin. We now show that in this limit,  $\lambda_{\infty}$  is an eigenvalue for  $\mathcal{L}(0)$  with  $\operatorname{Re} \lambda \geq 0$ , contradicting the spectral stability of this operator.

These eigenfunctions solve  $(\mathcal{L}(\delta_n) - \lambda_n)u_n = 0$ . We precondition by applying  $(1 - \delta_n^2 \partial_x^2)^{-1}$  to both sides of this equation, obtaining

$$\left[ \partial_x^2 + a_1(\cdot, \delta_n) \partial_x + a_0(\cdot, \delta_n) + E_1(\delta_n) + E_2(\delta_n) - \lambda_n - \lambda_n T(\delta_n) \right] u_n = 0, \quad (2.44)$$

where

$$E_1(\delta_n) = \delta_n^2 (1 - \delta_n^2 \partial_x^2)^{-1} (a_3(\cdot, \delta_n) \partial_x^3 + a_2(\cdot, \delta_n) \partial_x^2), \quad E_2(\delta_n) = T(\delta_n) (a_1(\cdot, \delta_n) \partial_x + a_0(\cdot, \delta_n)).$$

We relate this to the KPP linearization  $\mathcal{L}(0)$  by rewriting (2.44) as

$$(\mathcal{L}(0) - \lambda_{\infty})u_n = -E_1(\delta_n)u_n - E_2(\delta_n)u_n + E_3(\delta_n)u_n + (\lambda_n - \lambda_{\infty})u_n + \lambda_n T(\delta_n)u_n =: f_n$$

where

$$E_3(\delta_n) = (a_1(\cdot, 0) - a_1(\cdot, \delta_n)) \partial_x + (a_0(\cdot, 0) - a_0(\cdot, \delta_n)).$$

It follows from Lemma 2.2.3 and the fact that the coefficients  $a_j(\cdot; \delta)$  are uniformly bounded in  $\delta$  that

$$\|E_1(\delta_n)u_n\|_{L^2} \leq C \delta_n \|u_n\|_{H^2} = C \delta_n.$$

Similarly, by Lemma 2.2.4 we see that  $E_2(\delta_n)u_n \rightarrow 0$  and  $\lambda_n T(\delta_n)u_n \rightarrow 0$  in  $L^2$  as  $n \rightarrow \infty$ , since  $\lambda_n$  and  $u_n$  are uniformly bounded in  $n$ . Lastly, by the construction of the exponential weights, the fact that  $q_*(\cdot; \delta)$  converges uniformly to  $q_*(\cdot; 0)$  as  $\delta \rightarrow 0$  by Theorem 2.1.1, and the assumption that  $\|u_n\|_{H^2}$  is uniformly bounded, we see that also  $E_3(\delta_n)u_n \rightarrow 0$  in  $L^2$  as  $n \rightarrow \infty$ . Hence  $f_n$  converges to zero in  $L^2$  as  $n \rightarrow \infty$ .

Since  $\lambda_\infty$  is not in the spectrum of  $\mathcal{L}(0)$ , we can invert  $(\mathcal{L}(0) - \lambda_\infty)$  to write

$$u_n = (\mathcal{L}(0) - \lambda_\infty)^{-1} f_n,$$

from which we observe that  $u_n \rightarrow 0$  in  $H^2(\mathbb{R})$  as  $n \rightarrow \infty$  by boundedness of the resolvent operator. This is a contradiction since we have normalized  $u_n$  so that  $\|u_n\|_{H^2} = 1$ .  $\square$

*Proof of Theorem 2.1.2.* By Proposition 2.4.1, there exist  $\gamma_1, \delta_1 > 0$  so that for all  $\delta \in (-\delta_1, \delta_1)$ ,  $\mathcal{L}(\delta)$  has no eigenvalues in  $B(0, \gamma_1^2)$ , and also has no resonance at  $\lambda = 0$ . By Proposition 2.5.2, there exists a  $\delta_0 > 0$  so for all  $\delta \in (-\delta_0, \delta_0)$ ,  $\mathcal{L}(\delta)$  has no eigenvalues in  $\{\operatorname{Re} \lambda \geq 0\} \setminus B(0, \frac{\gamma_1^2}{2})$ . Hence for all  $\delta \in (-\delta_0, \delta_0)$ ,  $\mathcal{L}(\delta)$  has no eigenvalues in  $\{\operatorname{Re} \lambda \geq 0\}$ , as desired.  $\square$



# Nonlinear asymptotic stability of a bi-unstable front

## Outline of the chapter

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## 3.1 Introduction

The aim of the present work is to study the asymptotic stability of a heteroclinic structure – often called a front – that connects two unstable constant states. Fronts, together with other patterns such as periodic equilibrium, naturally appear as solutions of reaction-diffusion equations. Among those special solutions, the stable ones are preserved over time and thus are likely to be observed.

One of the simplest reaction-diffusion equation is the well known KPP equation, first studied by [KPP37; Fis37]:

$$\partial_t u = d \partial_{xx} u + \alpha u(1 - u). \quad (3.1)$$

Here and in the following we work with  $t > 0$  and  $x \in \mathbb{R}$ . The unknown  $u(t, x) \in \mathbb{R}$  is scalar, and represents the density of a spatially extended population, subject to a Brownian-like displacement, demographic effects and competition. Coefficients  $d > 0$  and  $\alpha > 0$  are respectively the diffusion coefficient and the growth rate.

Equation (3.1) admits two constant equilibria  $u(t, x) = 1$  and  $u(t, x) = 0$ , the former is stable with respect to time, the latter is unstable. For each speed  $c$  larger than the critical speed



$c_* := 2\sqrt{d\alpha}$ , the above equation also admits monotone traveling fronts [AW78], *i.e.* solutions of the form  $u(t, x) = q_c(x - ct)$ , with profiles  $q_c$  that connect the stable equilibrium to the unstable one:

$$\lim_{y \rightarrow -\infty} q_c(y) = 1, \quad \lim_{y \rightarrow +\infty} q_c(y) = 0.$$

Despite the instability of the constant state 0, these propagating solutions are *convectively stable*, which is to say stable in a comoving frame.

Another interesting equation is the KPP equation with a non-local interaction term [FG89; Gou00; GVA06]:

$$\partial_t u = d\partial_{xx}u + \mu u(1 - \phi * u). \quad (3.2)$$

Here the notation  $\phi * u$  stands for the convolution with respect to space. The kernel of interaction  $\phi \in L^1(\mathbb{R})$  is fixed and satisfies  $\|\phi\|_{L^1(\mathbb{R})} = 1$ . We assume that its Fourier transform takes negative values. It is standard to further assume that  $\phi$  is positive, symmetric, with finite second order moment. The growth rate is denoted by  $\mu$ , it can also be interpreted, after scaling, as the spatial range of  $\phi$ .

Due to the non-local reaction term, much less is known about equation (3.2). When  $\mu$  is smaller than a critical value  $\mu_c > 0$ , there exists exactly two constant equilibria  $u(t, x) = 1$ , and  $u(t, x) = 0$ , the only fronts are monotone and connect 1 to 0 [BNPR09] [FZ11]. On the contrary when  $\mu > \mu_c$ , periodic equilibria appear around  $u(t, x) = 1$  [HR14], and there exists modulated traveling fronts connecting them to the unstable state  $u(t, x) = 0$  [FH15]. When  $\mu$  is close to  $\mu_c$ , this secondary front propagates at speed  $\mathcal{O}(\sqrt{\mu - \mu_c})$ , that is much slower than the KPP speed  $2\sqrt{d\mu}$ . Hence, when set in a frame that travels at speed  $c_* = 2\sqrt{d\mu}$ , it is expected that the punctually unstable state  $u(t, x) = 1$  becomes convectively stable. In this setting, numerical investigations suggest that some KPP front connecting 1 to 0 remains and is stable, despite the fact that both its endpoints are unstable [NPT11].

Many tools and proofs do not transfer from (3.1) to (3.2), so that the existence and stability of a KPP front for equation (3.2) is still a largely open question [NPT11; HR14; FH15]. To take a first step in this direction, the present work restricts to the following simpler system that share similar behaviors, and was already introduced in [GSU04; GS07; FH15]:

$$\begin{cases} \partial_t u = d\partial_{xx}u + \alpha u(1 - u^2) + \beta v, \\ \partial_t v = -(\partial_{xx} + 1)^2 v + v(\mu - \sigma v^2) - \gamma v(1 - u). \end{cases} \quad (3.3)$$

Here and in the following,  $d, \alpha, \mu, \sigma, \gamma$  are positive constants, and  $\beta \in \mathbb{R}$  is non-zero. System (3.3) is a coupling between a KPP equation (3.1) – we refer to remark 3.1.1 hereafter – and a scalar Swift-Hohenberg (SH) equation

$$\partial_t v = -(\partial_{xx} + 1)^2 v + \mu v - v^3. \quad (3.4)$$

The latter is a simple model for periodic pattern formation [SH77; CH93], that admits one constant equilibrium  $v(t, x) = 0$ . It is stable when  $\mu < 0$ , and destabilize when  $\mu$  becomes positive. More precisely, at linear level, a curve of essential spectrum crosses the imaginary axis at non-zero Fourier frequencies  $\xi = \pm 1$ . All periodic modes  $x \mapsto e^{i\xi x}$  with frequency  $\xi$  close to  $\pm 1$  become unstable at linear level, which we refer to as a Turing instability. For equation (3.4), this instability is balanced by nonlinear terms, so that when starting near  $v(t, x) = 0$ , a bounded periodic pattern appears and invades space at speed  $\sqrt{\mu}$  [EW91]. We refer to Figure 3.1 for an illustration of a typical solution of (3.3).

Notice that the same phenomenon appears in (3.2): by linearizing near  $u = 1$ , it yields

$$\partial_t u = d \partial_{xx} u - \mu \hat{\phi} * u,$$

so that the linear essential spectrum crosses the imaginary axis when  $d\xi^2 + \mu \hat{\phi}(\xi) < 0$ , that is when  $\mu > \mu_c := \inf \left\{ \frac{-d\xi^2}{\hat{\phi}(\xi)} : \xi \in \mathbb{R} \text{ such that } \hat{\phi}(\xi) < 0 \right\}$ , we further refer to [GVA06].

*Remark 3.1.1.* The first equation in (3.3) differs from (3.1) in that the monostable reaction term  $u(1-u)$  has been replaced by a bistable one  $f(u) := u(1-u)(u+1)$ . Since the reaction  $f$  is still of KPP type, this is not a consequent change: we simply consider the KPP front that connects 1 to 0, and do not consider the additional equilibrium  $-1$ . It would be interesting to investigate whether the current work adapts to a general KPP-type reaction term  $f$ . We recall to the reader that  $f$  is said to be of KPP type if its two consecutive zeros  $u = 1$  and  $u = 0$  are respectively stable and unstable  $f'(1) < 0 < f'(0)$ , and if  $f$  lies below its tangent: for all  $u \in (0, 1)$ ,  $f(u) \leq u f'(0)$ .

A crucial advantage of (3.3) compared to (3.2) is the known existence of monotone propagating fronts that connect the two states  $(u, v) = (1, 0)$  and  $(u, v) = (0, 0)$ , namely

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = Q_c(x - ct) := \begin{pmatrix} q_c(x - ct) \\ 0 \end{pmatrix}.$$

In the following, we will restrict to the slowest of them, that is  $c = c_* = 2\sqrt{d\alpha}$ , and will refer to  $Q_* := Q_{c_*}$  as the *critical front*. Our conclusion is that when the state behind the front destabilizes, the critical front  $Q_*$  remains asymptotically stable in weighted spaces, with the same decay rate as in the  $\mu < 0$  case. Here is a simplified version of the main result, we refer to Theorem 3.2.1 below for a complete statement.

**Theorem 3.1.2.** *There exists positive constants  $\mu_0$ ,  $\delta$  and  $C$ , a Sobolev weighted norm  $\|\cdot\|$ , and weights  $\omega_1$ ,  $\omega_2$  such that the following holds. If  $\mu \in (0, \mu_0)$  and if  $P_0 : \mathbb{R} \rightarrow \mathbb{R}^2$  satisfies*

$$\|P_0\| \leq \delta,$$

*then the solution to (3.3) with initial condition  $(u(0), v(0))^T = Q_* + P_0$  exists for all times  $t > 0$ , decomposes as  $(u(t, x), v(t, x))^T = Q_*(x - c_*t) + P(t, x - c_*t)$ , and satisfies*

$$\|\omega_1^{-1} P(t)\|_{L^\infty(\mathbb{R})} \leq C \frac{\|P_0\|}{(1+t)^{3/2}}, \quad \|\omega_2^{-1} P(t)\|_{L^\infty(\mathbb{R})} \leq C(\sqrt{\mu} + \|P_0\|).$$

For the simpler equation (3.1), the decay rate  $t^{-3/2}$  is known to be optimal for the stability of the critical front, see [Gal94; FH18; AS21] and corresponds to an enhanced diffusive decay, due to the absence of resonance in the linear spectrum at the origin. Stability of *supercritical* fronts – that is  $c > c_*$  – is also known with exponential decay rates [Sat76], for in this case, the use of strong enough weighted spaces allows to stabilize the spectrum with gap.

Multiple situations where a bistable front – *i.e.* a front that connects two stable states – undergoes a Turing bifurcation were already investigated. When bifurcation occurs behind the front, [SS01] showed that no modulated front appears: the Turing pattern travels at slow speed  $\mathcal{O}(\sqrt{\mu})$ , hence is left behind the front. Nevertheless, the front survives after bifurcation, its nonlinear stability is showed in [BGS09] for a general second-order setting. We also mention [GS07] for a quadratic coupling  $\beta v^2$  instead of a linear one. The signed term therein allows to obtain *a priori* estimates by applying comparison principle techniques to simplify the system.

To handle the system (3.3), we will mainly follow the approach used in [BGS09]. However,

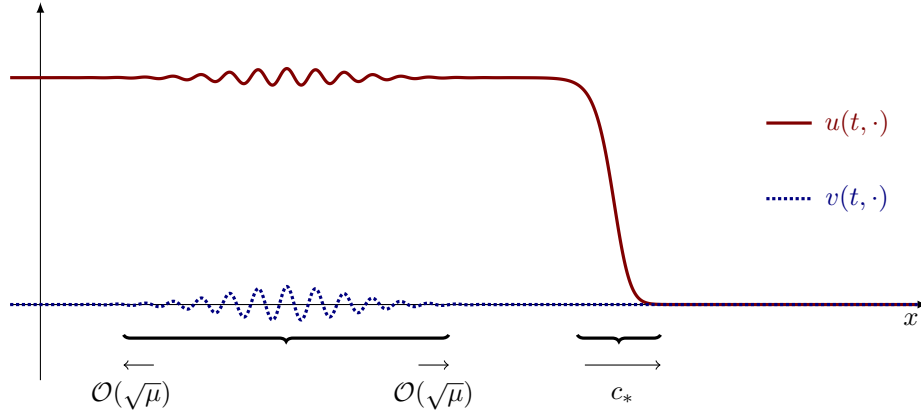


Figure 3.1 – Illustration of a typical solution  $(u, v)$ , for  $\mu$  positive and small. The front – right brace – has fixed shape and moves to the right at constant speed  $c_*$ . The Turing pattern – left brace – continuously expands at slow speed  $\mathcal{O}(\sqrt{\mu})$ , towards left and right. Both states  $(0, 0)$  and  $(1, 0)$  – respectively ahead and behind the front – are pointwise unstable, but convectively stable in the critical weighted space.

adequate tools will be used to handle the instability of the invaded state  $(0, 0)$ . On the one hand, the essential spectrum is unstable in the present situation, even in the case  $\mu < 0$ . To tackle this issue, we work in exponentially weighted spaces, and use a Green’s function-based approach described in [FH18] to obtain linear estimates, see subsection 3.2.2 for details. We stress that since the front is critical, the optimal choice of weight only allows to marginally stabilize the spectrum, in the sense that it lies at the left of the imaginary axis but touches it. On the other hand, theorem 3.1.2 is not stated up to a shift of the front, in contrast with the orbital stability of bistable fronts. This simpler behavior is expected, and follows from the fact that the translation eigenvalue  $0 \in \mathbb{C}$  does not contribute to a zero of the so-called Evans function, due to the corresponding eigenfunction  $q'_*$  having only weak exponential decay:  $q'_*(x) \sim (ax + b)e^{-\frac{c_*}{2}x}$  when  $x \rightarrow +\infty$ , see [FH18; AS21].

We also mention that the linear coupling term  $\beta v$  forces us to work with the same weights on the KPP and SH components, see further Remark 3.3.2, so that  $\omega_1$  and  $\omega_2$  affect both  $u$  and  $v$  components. This typically happens at  $+\infty$ : conjugation by the critical KPP weight, although necessary to stabilize the KPP spectrum, may destabilize the SH part of the spectrum. We counter this effect by imposing a lower bound on  $\gamma$ , which stabilizes the SH spectrum at  $+\infty$ . A situation where it is not possible to stabilize both components at the same time – referred to as remnant instability – was investigated for a similar system in [FHSS21].

In the supercritical case  $c > c_*$ , it is possible to obtain spectral gap for the essential spectrum, and to remove the translation eigenvalue, by use of a strong enough weight [Sat76]. In such a case, the situation is simpler than the bistable case: exponential stability without a phase can be obtained by adapting the proofs that follow, or the ones in [BGS09].

A problem that is not investigated here is the case of a Turing bifurcation ahead of a critical monostable front, whereas its equivalent for bistable front has been studied. Existence and linear stability of modulated fronts – *i.e.* coherent structures that link a stable state behind to Turing pattern ahead – in a general second-order bistable setting was shown in [SS01]. For a system that resembles (3.3), nonlinear stability of such structure was obtained in [GSU04]. When bifurcation occurs both ahead and behind a bistable front, spectral stability of either traveling or

standing pulses is showed for a general system [SS00] and [SS04], while nonlinear stability of a traveling pulse for a precise system is showed in [GSU04]. As mentioned above, the case of Turing patterns ahead of a monostable front remains open, and would be an interesting direction to investigate.

Another natural question is the existence and stability of a structure that successively connects the periodic pattern, the constant state 1 and the constant state 0. We remark that such a solution depends on time in any moving frame, so that existence itself is an open interesting question.

At last, an extension of the above result to the original non-local equation (3.2) would be of primary interest.

The rest of this chapter goes as follows. In section 3.2, we introduce some notations, state our main result in its complete form, and present the main ideas of its proof. The argument can be split into two parts, which are respectively treated in section 3.3 and 3.4. The former is dedicated to the decay estimate that corresponds to the unbounded weight  $\omega_1$ , while the latter shows the bound corresponding to the KPP-like weight  $\omega_2$ .

## 3.2 Main result

### 3.2.1 Notations

We investigate now the stability of the traveling front  $Q_* = (q_*, 0)^T$ . Insert the following ansatz in (3.3):

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = Q_*(\tilde{x}) + P(t, \tilde{x}),$$

with  $\tilde{x} = x - c_*t$ . Then the perturbation  $P = (p_1, p_2)^T$  satisfies

$$\partial_t P = \mathcal{A}P + N(P) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{A}^{\text{kpp}} & \beta \\ 0 & \mathcal{A}^{\text{sh}} \end{pmatrix} P + \begin{pmatrix} N_1(P) \\ N_2(P) \end{pmatrix},$$

where the linear operator  $\mathcal{A} : L^2(\mathbb{R})^2 \rightarrow L^2(\mathbb{R})^2$  is closed, has dense domain  $H^2(\mathbb{R}) \times H^4(\mathbb{R})$  and is defined by:

$$\mathcal{A}^{\text{kpp}} = d\partial_{xx} + c_*\partial_x + \alpha(1 - 3q_*(\tilde{x})), \quad \mathcal{A}^{\text{sh}} = -(1 + \partial_{xx})^2 + c_*\partial_x + \mu - \gamma(1 - q_*(\tilde{x})), \quad (3.5)$$

while remaining nonlinear terms are expressed as:

$$N_1(P) = -\alpha(3q_*(\tilde{x})p_1^2 + p_1^3), \quad N_2(P) = \gamma p_1 p_2 - \sigma p_2^3. \quad (3.6)$$

When  $x \rightarrow \pm\infty$ , linear operators converge to differential operators with constant coefficients  $\mathcal{A}^{\text{kpp},\pm}$  and  $\mathcal{A}^{\text{sh},\pm}$ . In the following, we may denote  $\nu \mapsto \mathcal{A}^{\text{kpp},\pm}(\nu)$  the associated symbols, often called *dispersion relations*, that is for  $\nu \in \mathbb{C}$ ,

$$\begin{aligned} \mathcal{A}^{\text{kpp},+}(\nu) &:= d\nu^2 + c_*\nu + \alpha, & \mathcal{A}^{\text{kpp},-}(\nu) &:= d\nu^2 + c_*\nu - 2\alpha, \\ \mathcal{A}^{\text{sh},+}(\nu) &:= -\nu^4 - 2\nu^2 - 1 + c_*\nu + \mu - \gamma, & \mathcal{A}^{\text{sh},-}(\nu) &:= -\nu^4 - 2\nu^2 - 1 + c_*\nu + \mu. \end{aligned}$$

The essential spectrum of  $\mathcal{A}$  contains the four curves  $\mathcal{A}^{k,\pm}(i\mathbb{R})$ , with  $k \in \{\text{kpp}, \text{sh}\}$ . Thus, it is unstable see Figure 3.2, so that we do not hope for an asymptotic stability result against general perturbation  $P \in H^2(\mathbb{R}) \times H^4(\mathbb{R})$ . Instead, we restrict to weighted perturbations  $P(t, \tilde{x}) =$

$\omega(\tilde{x})U(t, \tilde{x})$ , with  $U \in H^2(\mathbb{R}) \times H^4(\mathbb{R})$ . As we shall see in further Remark 3.3.2, the coupling term  $\beta v$  imposes the use of a scalar weight. Namely  $\omega_* \stackrel{\text{def}}{=} \omega_{\text{kpp}} \omega_{\text{sh}}$ , with smooth positive functions  $\omega_{\text{kpp}} : \mathbb{R} \rightarrow \mathbb{R}$  and  $\omega_{\text{sh}} : \mathbb{R} \rightarrow \mathbb{R}$  that satisfy

$$\omega_{\text{kpp}}(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \leq -1, \\ e^{-\frac{c_*}{2d}x} & \text{if } x \geq 1, \end{cases} \quad \omega_{\text{sh}}(x) \stackrel{\text{def}}{=} \begin{cases} e^{\theta x} & \text{if } x \leq -1, \\ 1 & \text{if } x \geq 1, \end{cases} \quad (3.7)$$

where the exponent  $\theta < 0$  is small and depends only on  $\alpha$ , and  $d$ , see Proposition 3.3.1. The weight  $\omega_{\text{kpp}}$  decays exponentially to 0 at  $+\infty$ , which allows to marginally stabilize the KPP spectrum, while  $\omega_{\text{sh}}(x)$  diverges exponentially to  $+\infty$  when  $x \rightarrow -\infty$  which allows to stabilize the SH spectrum. Hence, we insert the new ansatz

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = Q_*(\tilde{x}) + \omega_*(\tilde{x})U(t, \tilde{x}) \quad (3.8)$$

in (3.3) to obtain that  $U = (u_1, u_2)^T$  satisfies:

$$\partial_t U = \mathcal{L}U + \mathcal{N}(U) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{L}^{\text{kpp}} & \beta \\ 0 & \mathcal{L}^{\text{sh}} \end{pmatrix} U + \begin{pmatrix} \mathcal{N}_1(U) \\ \mathcal{N}_2(U) \end{pmatrix}, \quad (3.9)$$

where all linear terms are closed, densely defined operators. They are obtained as conjugation of the unweighted linear operators,

$$\mathcal{L}^k = \omega_*^{-1} \mathcal{A}^k \omega_*$$

when  $k \in \{\text{kpp}, \text{sh}\}$  and nonlinear terms express according to  $\mathcal{N}_k(U) = \omega_*^{-1} N_k(\omega_* U)$ . It is a standard computation to show that the symbols associated to the asymptotic operators  $\mathcal{L}^{\text{kpp}, \pm}$  and  $\mathcal{L}^{\text{sh}, \pm}$  – obtained by replacing coefficients by their limits  $x \rightarrow \pm\infty$  – are simply given by

$$\mathcal{L}^{k,+}(v) = \mathcal{A}^{k,+} \left( v - \frac{c_*}{2d} \right), \quad \mathcal{L}^{k,-}(v) = \mathcal{A}^{k,-}(v + \theta),$$

when  $k \in \{\text{kpp}, \text{sh}\}$ .

At linear level, it is expected that  $U$  will decay in time at rate  $t^{-3/2}$ , if the exponent  $\theta$  is well chosen, that is if the following spectral assumption on  $\mathcal{L}$  holds:

$$\Sigma_{\text{ess}}(\mathcal{L}) \subset \{\text{Re } \lambda \leq -\eta\} \cup \{-\xi^2 : \xi \in \mathbb{R}\}. \quad (H_1)$$

Such strong diffusive decay  $t^{-3/2}$  comes from the fact that the translation eigenmode  $q'_*$  in weighted spaces is unbounded at  $x \rightarrow +\infty$ , and thus does not contribute to a resonance at the origin  $0 \in \mathbb{C}$ , see [AS21; FH18].

We point out that due to the unbounded weight  $\omega_{\text{sh}}$ , the previous ansatz (3.8) does not really control perturbations behind the front: they are authorized to grow if they move to the left of the front. As a matter of fact, the use of unbounded weight makes nonlinear terms  $U \mapsto \mathcal{N}_i(U)$  non-Lipschitz, since their coefficients are unbounded with respect to space. This is a common problem that was already dealt with in [BGS09] using the upcoming second ansatz, more details are given in section 3.2.2.

To control sharply perturbations behind the front, we use a second ansatz, where perturbations are measured in the steady frame, since they account for the Turing pattern that does not

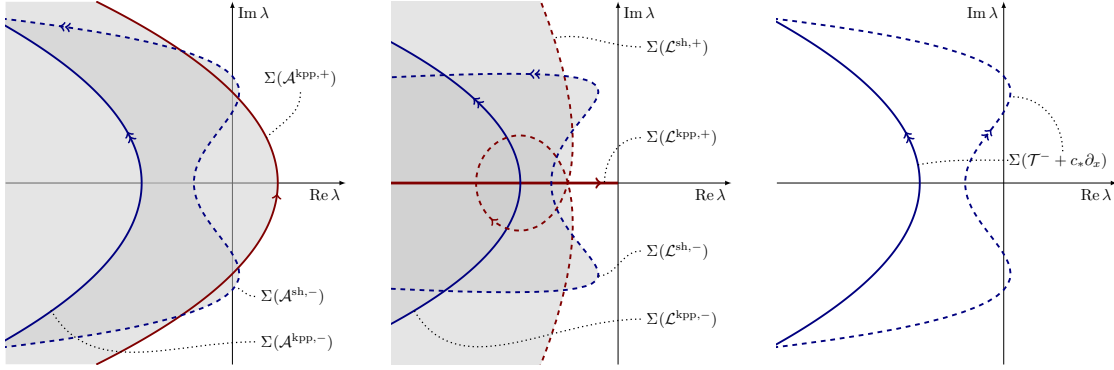


Figure 3.2 – Essential spectrum of linear operators. Fredholm borders corresponding to KPP (respectively SH) component are plain (respectively dashed). Curves in red with single arrow (respectively blue with double arrow) correspond to  $+\infty$  (respectively  $-\infty$ ) borders. Left:  $\Sigma_{\text{ess}}(\mathcal{A})$  with two unstable curves,  $\Sigma(\mathcal{A}^{\text{sh},+})$  is out of picture to the left. Center:  $\Sigma_{\text{ess}}(\mathcal{L})$  with only marginally stable spectrum. Right:  $\Sigma_{\text{ess}}(\mathcal{T}^- + c_* \partial_x)$  with one unstable curve, which corresponds to Turing instability. The essential spectrum of  $\mathcal{T}^-$  is real and obtained from the one represented by application of  $z \mapsto \text{Re } z$ . Notice that the spectrum of  $\mathcal{T}$  can not be marginally stabilized, due to the lack of a first order derivative.

travel at front speed  $c_*$ :

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = Q_*(\tilde{x}) + \omega(\tilde{x}) V(t, x). \quad (3.10)$$

Here again  $\tilde{x} = x - c_* t$ , and we have noted

$$\omega \stackrel{\text{def}}{=} \rho_{-1} \omega_{\text{kpp}},$$

where the polynomial weight  $\rho_{-1}$  is a smooth positive function defined in chapter 1: for all  $\eta \in \mathbb{R}$ , let  $\rho_\eta$  be a smooth positive function that satisfies

$$\rho_\eta(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x \leq -1, \\ \langle x \rangle^{-\eta} & \text{if } x \geq 1, \end{cases} \quad (3.11)$$

where  $\langle x \rangle \stackrel{\text{def}}{=} \sqrt{1 + x^2}$ . We emphasize that although the perturbation  $V$  is set in the laboratory frame  $x$ , the weight is set in the comoving frame  $\tilde{x}$ , since we are ultimately investigating the stability of the front in the comoving frame. The use of a weight  $\rho_{-1}$  is needed so that the following Proposition 3.4.4 holds. By inserting (3.10) into (3.3), we obtain that

$$\omega \partial_t V - c_* (\partial_x \omega) V = \mathcal{A}(\omega V) - c_* \partial_x (\omega V) + N(\omega V),$$

which rewrites as

$$\begin{aligned} \partial_t V &= \mathcal{T}V + \mathcal{Q}(V), \\ &= \mathcal{T}^- V + \mathcal{Q}^-(V) + \mathcal{S}(\tilde{x}, V), \end{aligned} \quad (3.12)$$

where linear operators express as:

$$\mathcal{T} \stackrel{\text{def}}{=} \omega(\tilde{x})^{-1}(\mathcal{A} - c_* \partial_x) \omega(\tilde{x}) + c_* \frac{\partial_x \omega}{\omega}(\tilde{x}) \quad \mathcal{T}^- = \lim_{\tilde{x} \rightarrow -\infty} \mathcal{T}(\tilde{x}),$$

while nonlinear terms are defined by  $\mathcal{Q}(V) = \omega(\tilde{x})^{-1} N(\omega(\tilde{x})V)$ , and  $\mathcal{Q}^-(V) = \lim_{\tilde{x} \rightarrow -\infty} \mathcal{Q}(V)$ . We are left with an error term

$$\mathcal{S}(V) \stackrel{\text{def}}{=} \mathcal{T}V - \mathcal{T}^-V + \mathcal{Q}(V) - \mathcal{Q}^-(V).$$

Since  $\omega(\tilde{x}) = 1$  for  $\tilde{x} \leq -1$ , we decompose  $\mathcal{T}^-$  as:

$$\mathcal{T}^- = \begin{pmatrix} d\partial_{xx} - 2\alpha & \beta \\ 0 & -(1 + \partial_{xx})^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{T}_0 + \mathcal{T}_\mu. \quad (3.13)$$

and  $\mathcal{Q}^-$  as

$$\mathcal{Q}^-(V) = \begin{pmatrix} -3\alpha v_1^2 \\ \gamma v_1 v_2 \end{pmatrix} + \begin{pmatrix} -\alpha v_1^3 \\ -\sigma v_2^3 \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{Q}_2 + \mathcal{Q}_3. \quad (3.14)$$

Thus we get the following expression for the source term:

$$\begin{aligned} \mathcal{S}(\cdot, V) &= \left[ \omega^{-1}(\mathcal{A} - c_* \partial_x) \omega - (\mathcal{A} - c_* \partial_x) + c_* \frac{\partial_x \omega}{\omega} + \begin{pmatrix} 3\alpha(1 - q_*^2) & 0 \\ 0 & -\gamma(1 - q_*) \end{pmatrix} \right] V \\ &\quad + (\omega - 1) \begin{pmatrix} -3\alpha q_* v_1^2 \\ \gamma v_1 v_2 \end{pmatrix} + (\omega^2 - 1) \begin{pmatrix} -\alpha v_1^3 \\ -\sigma v_2^3 \end{pmatrix}. \end{aligned} \quad (3.15)$$

To control perturbation  $V$ , we will compute an amplitude equation, which reveals to be a Ginzburg-Landau equation with real coefficients

$$\partial_T A = 4\partial_{XX} A + A + b_{GL} A |A|^2. \quad (\text{GL})$$

An important property for our result to hold is that the Turing bifurcation is *supercritical*:

$$\text{In } (\text{GL}), \text{ the coefficient } b_{GL} \text{ is negative.} \quad (H_2)$$

It will allow to control the Turing pattern behind the front. Computation of the coefficient  $b_{GL}$  is done in appendix 3.5, we refer to equation (3.47) and the lines that immediately follow for a definition of  $b_{GL}$ . See also lemma 3.5.2 there.

The perturbations  $U$  and  $V$  are linked through

$$V(t, x) = \frac{\omega_{\text{sh}}(\tilde{x})}{\rho_{-1}(\tilde{x})} U(t, \tilde{x}). \quad (3.16)$$

Let  $r > 0$  be a real number. To measure spatial localization and regularity, we use Sobolev spaces, and introduce the following norm:

$$\|U\|_{X_m} \stackrel{\text{def}}{=} \|U\|_{W^{2,\infty}(\mathbb{R}) \times W^{4,\infty}(\mathbb{R})} + \|\rho_{-2-m} U\|_{L^\infty(\mathbb{R})}.$$

We can now state our main result.

**Theorem 3.2.1.** *There exists an open, nonempty set of parameters  $\Omega$  such that for  $(\alpha, \beta, d, \sigma, \gamma)$  in  $\Omega$ , both hypothesis  $(H_1)$  and  $(H_2)$  holds. For such a choice of parameters, and for  $r > 0$ , the following*

holds. There exists positive constants  $C$ ,  $\mu_0$ , and  $\delta$  such that for all  $\mu \in (0, \mu_0]$ , if

$$U_0 \in \left\{ U \in W^{2,\infty}(\mathbb{R}) \times W^{4,\infty}(\mathbb{R}) : \omega_{sh} U \in W^{1,\infty}(\mathbb{R}_-) \text{ and } \rho_{-2-m} U \in L^\infty(\mathbb{R}_+) \right\},$$

and if

$$\|V_0\|_{W^{1,\infty}(\mathbb{R})} + \|U_0\|_{X_m} \leq \delta,$$

where  $V_0 := \frac{\omega_{sh} U_0}{\rho_{-1}}$ , then the solution of (3.3) with initial condition  $Q_* + \omega_{kpp} \omega_{sh} U_0 = Q_* + \rho_{-1} \omega_{kpp} V_0$  exists for all time, and writes as

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = Q_*(\tilde{x}) + \omega_{kpp}(\tilde{x}) \omega_{sh}(\tilde{x}) U(t, \tilde{x}) = Q_*(\tilde{x}) + \rho_{-1}(\tilde{x}) \omega_{kpp}(\tilde{x}) V(t, x).$$

For all  $t > 0$ , the weighted perturbations satisfy:

$$\left\| \frac{U(t, \cdot)}{\rho_{-1}} \right\|_{L^\infty(\mathbb{R})} \leq C \frac{\|U_0 \rho_{-2-m}\|_{L^\infty(\mathbb{R})}}{(1+t)^{3/2}}, \quad \|V(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C(\sqrt{\mu} + \delta).$$

Here and in the following, the exponent  $m$  plays almost no role in the argument as long as it is positive. To keep notations light, we will fix it to 1 in the proofs, and refer to remark 3.3.17 for the  $m > 0$  case. In fact, we can refine the decay of the perturbation  $U$ .

**Corollary 3.2.2.** *Let  $1 < p \leq +\infty$ , let  $(\alpha, \beta, d, \sigma, \gamma) \in \Omega$ , with  $\Omega$  as in Theorem 3.2.1, and let  $m > 0$ . There exists positive constants  $C$ ,  $\mu_0$ ,  $\delta$  and  $\eta$  such that if*

$$\|V_0\|_{W^{1,\infty}(\mathbb{R})} + \|U_0\|_{X_m} \leq \delta, \quad \|U_0 \rho_{-2-m}\|_{L^p(\mathbb{R})} \leq \delta,$$

then with the same notations as in Theorem 3.2.1, each component of the weighted perturbation  $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  satisfies:

$$\left\| \frac{u_1(t, \cdot)}{\rho_{-1}} \right\|_{L^p(\mathbb{R})} \leq C \frac{\|U_0 \rho_{-2-m}\|_{L^p(\mathbb{R})}}{(1+t)^{\frac{3}{2} - \frac{1}{2p}}}, \quad \|u_2(t, \cdot)\|_{L^p(\mathbb{R})} \leq C e^{-\eta t} \|U_0\|_{L^p(\mathbb{R})}.$$

### 3.2.2 Sketch of the proof

First, the dynamic for fully weighted perturbation  $U$  writes as:

$$\partial_t U = \mathcal{L}U + \mathcal{N}(U).$$

The linear operator  $\mathcal{L}$  has marginally stable spectrum and the nonlinear part  $\mathcal{N}$  is unbounded with respect to  $x \in \mathbb{R}$ , due to  $|\omega_{sh}(x)| \rightarrow \infty$  when  $x \rightarrow -\infty$ . At linear level, the spectral situation is exactly the one of the critical KPP front: stable essential spectrum except for one curve touching the origin, no unstable point spectrum and no resonance at the origin – see Propositions 3.3.1 and 3.3.3. Hence, an algebraic decay  $t^{-3/2}$  is expected, and is obtained following the approach from [FH18] in the scalar KPP case. In the present situation, we will strongly rely on the fact that the linear dynamic is triangular, the main ideas go as follows. We first look at the solution to the linear Cauchy problem

$$\partial_t p = \mathcal{L}p, \quad p(0, \cdot) = p_0, \quad (3.17)$$



and assume that it is expressed through a kernel:  $p(t, x) = \int_{\mathbb{R}} \mathcal{G}_t(x, y) p_0(y) dy$ . The matrix valued function  $\mathcal{G}(\cdot, y)$  has to satisfy the linear problem (3.17), with a Dirac delta initial condition:  $\mathcal{G}_0(x, y) = \delta_y(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Remark that  $\mathcal{G}$  is an upper-triangular matrix, due to the triangular structure of  $\mathcal{L}$  and  $\mathcal{G}_0$ . We note it

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}^{\text{kpp}} & \mathcal{G}^{\text{co}} \\ 0 & \mathcal{G}^{\text{sh}} \end{pmatrix},$$

where  $\mathcal{G}^k$  solves  $\partial_t \mathcal{G}_t = \mathcal{L}^k \mathcal{G}_t$  when  $k \in \{\text{kpp}, \text{sh}\}$ , and  $\mathcal{G}_t^{\text{co}}$  accounts for the coupling terms. We then apply Laplace transform to obtain the spectral Green kernel  $G_\lambda(x, y) = \int_0^{+\infty} e^{-\lambda t} \mathcal{G}_t(x, y) dt$ , that satisfies the *fundamental eigenproblem*

$$(\lambda - \mathcal{L})G_\lambda = \delta_y,$$

for suitable  $\lambda \in \mathbb{C}$ . As above, it is an upper-triangular matrix that writes

$$G = \begin{pmatrix} G^{\text{kpp}} & G^{\text{co}} \\ 0 & G^{\text{sh}} \end{pmatrix},$$

where each  $G^k$  is the Laplace transform of  $\mathcal{G}^k$  for  $k \in \{\text{kpp}, \text{sh}, \text{co}\}$ . Hence, they satisfy

$$\begin{aligned} (\lambda - \mathcal{L}^{\text{kpp}})G_\lambda^{\text{kpp}}(\cdot, y) &= \delta_y, & (\lambda - \mathcal{L}^{\text{sh}})G_\lambda^{\text{sh}}(\cdot, y) &= \delta_y, \\ (\lambda - \mathcal{L}^{\text{kpp}})G_\lambda^{\text{co}}(\cdot, y) &= \beta G_\lambda^{\text{sh}}(\cdot, y). \end{aligned}$$

The homogeneous eigenproblems  $(\lambda - \mathcal{L}^k)\phi = 0$  with  $k \in \{\text{kpp}, \text{sh}\}$  are simply ODEs. It happens – see [HZ98, section 3] – that their solutions admit exponential behaviors at  $\pm\infty$ :

$$\phi(x) = e^{\nu(\lambda)x} (1 + e^{-r|x|} \kappa(x)),$$

where  $\nu \in \mathbb{C}$  is a solution of the dispersion relations  $\lambda - \mathcal{L}^{k,\pm}(\nu) = 0$ ; the real number  $r > 0$  does not depend on  $\nu$  nor  $\lambda$ ;  $\kappa$  is a bounded function on the half-line  $\mathbb{R}_\pm$ . Such solutions can be concatenated to construct  $G_\lambda^k(\cdot, y) \in L^2(\mathbb{R})$ . Then, the coupled spectral green function expresses as the  $L^2(\mathbb{R})$ -inner product

$$G_\lambda^{\text{co}}(x, y) = \langle G_\lambda^{\text{kpp}}(x, \cdot), \beta G_\lambda^{\text{sh}}(\cdot, y) \rangle.$$

This leads to bound of  $G_\lambda$  with respect to  $\lambda$ , which are converted into temporal decay for  $\mathcal{G}_t$  through the inverse Laplace transform:

$$\mathcal{G}_t(x, y) = \frac{1}{2i\pi} \int_{\Lambda} e^{\lambda t} G_\lambda(x, y) d\lambda,$$

where  $\Lambda$  is a contour in the resolvent set  $\mathbb{C} \setminus \Sigma(\mathcal{L})$ , that can be chosen as a continuous deformation of a sectorial contour, see [Dav02, section 3].

To obtain bound at nonlinear level, we then use Duhamel's formula:

$$U(t, x) = \int_{\mathbb{R}} \mathcal{G}_t(x, y) U_0(y) dy + \int_0^t \int_{\mathbb{R}} \mathcal{G}_{t-\tau}(x, y) \mathcal{N}(U(\tau, y)) dy d\tau.$$

The fact that  $\mathcal{N}(U)$  is unbounded is a major issue. In [BGS09], the unbounded weight  $\omega_{\text{sh}}$  is absorbed by transforming nonlinear terms into linear ones: writing  $\omega_{\text{sh}}^{-1} (\omega_{\text{sh}} u)^k = (\omega_{\text{sh}} u)^{k-1} u$  for

an integer  $k \geq 2$ , and showing that  $\|\omega_{\text{sh}} u\|_{L^\infty(\mathbb{R})}$  is bounded with respect to time. We follow the same line. The material presented above is detailed in section 3.3, and leads to the following proposition, the proof of which can be found in section 3.3.5.

**Proposition 3.2.3** (*V* bounded implies decay of *U*). *Assume hypothesis (H<sub>1</sub>) holds, fix  $m > 0$  and  $1 < p \leq +\infty$ . There exists positive constants  $\delta, \mu_0, C_{\text{stab}}$  and  $\eta$  such that for all  $0 < \mu < \mu_0$ , the following holds. Fix a positive constant  $t_V$ , an initial condition  $U_0$  such that  $U_0 \rho_{-2-m} \in L^p(\mathbb{R})$ , and assume that for all  $0 \leq t \leq t_V$ , the solution to (3.12) with initial condition  $V_0 := \frac{\omega_{\text{sh}} U_0}{\rho_{-1}}$  satisfies*

$$\|V(t)\|_{L^\infty(\mathbb{R})} \leq \delta.$$

Then the solution *U* for the Cauchy problem (3.9) with initial condition  $U_0$  is defined for all  $0 \leq t \leq t_V$ , and satisfies

$$\left\| \frac{u_1(t)}{\rho_{-1}} \right\|_{L^p(\mathbb{R})} \leq C_{\text{stab}} \frac{\|U_0 \rho_{-2-m}\|_{L^p(\mathbb{R})}}{(1+t)^{\frac{3}{2} - \frac{1}{2p}}}, \quad \|u_2(t)\|_{L^p(\mathbb{R})} \leq C_{\text{stab}} e^{-\eta t} \|U_0\|_{L^p(\mathbb{R})}.$$

Furthermore,  $C_{\text{stab}}$  depends neither on  $t_V$  or  $\sup_{0 \leq t \leq t_V} \|V(t)\|_{L^\infty(\mathbb{R})}$ .

Second, we turn to  $\frac{\omega_{\text{sh}}}{\rho_{-1}} U = V$ , and show that it is bounded in time,<sup>1</sup> remind that

$$\partial_t V = \mathcal{T}V + \mathcal{Q}(V) = \mathcal{T}^- V + \mathcal{Q}^-(V) + \mathcal{S}(\bar{x}, V).$$

We show that  $\mathcal{S}$  is sufficiently localized in space so that we can extract a  $\omega_{\text{sh}}$  from it. This allows to control  $\mathcal{S}(V)$  by  $U/\rho_{-1}$  – see (3.16) – which ensures decay in time as done in [GS07, below Proposition 3.3] and [BGS09, Lemma 4.1]. Hence after a suitable time,  $V$  is driven by the dynamic at  $-\infty$ , which allows to get rid of the marginally stable curve of essential spectrum at  $+\infty$ . Since  $\mathcal{T}^-$  undergoes a Turing bifurcation, we follow the steps of [Sch94c; BGS09], that mostly relies on the use of *mode-filters*, see [Sch94a].

We first show that periodic patterns are naturally selected: after a time  $\frac{T}{\mu}$ , perturbation  $V$  has at first order an oscillating profile. This is commonly referred to as the *approximation property*. For such profiles, using multi-scale analysis, dynamic of the whole system reduces to an amplitude equation, which in our case appears to be the Ginzburg-Landau (GL) equation: setting  $\varepsilon = \sqrt{\mu}$ , if  $V = \psi(A) + \mathcal{O}(\varepsilon^2)$ , with  $\psi(\varepsilon, A)(t, x) \stackrel{\text{def}}{=} \varepsilon A(\varepsilon^2 t, \varepsilon x) e^{ix} \zeta_c + c.c.$ ,<sup>2</sup> with a suitable  $\zeta_c \in \mathbb{R}^2$ , then  $A(T, X) \in \mathbb{C}$  satisfies

$$\partial_T A = 4\partial_{XX} A + A + b_{GL} A |A|^2.$$

We refer to [Mie02; MS95] for the derivation of amplitude equation. For suitable values of the parameter  $\gamma > 0$ , the coefficient  $b_{GL} \in \mathbb{R}$  is negative, as shown in section 3.5. Hence the Turing bifurcation is supercritical, and (GL) is known to have a bounded global attractor – see [MS95, Theorem 3.4] – which ensures that  $A$  is bounded with respect to time. In case where the bifurcation is subcritical, a quintic term is often add to recover a precise behavior, we do not explore this line. To conclude that  $V$  stays small for all time, we use the *approximation property*: if  $V$  is close to  $\psi(A)$  at time  $t_0 > 0$ , then the solution of the whole system emanating from  $V_0$  is defined upon time  $t_0 + T/\mu$ , and remains close to  $\psi(A)$ . This last step can be applied as many

1. The presence of an extra  $\frac{1}{\rho_{-1}}$  is necessary so that source term  $\mathcal{S}(\bar{x}, V)$  decay in time, see Proposition 3.4.4.

2. Here and in the following, we note  $c.c$  for the complex conjugate:  $z + c.c \stackrel{\text{def}}{=} z + \bar{z} = 2 \operatorname{Re}(z)$ .

time as needed, without deterioration of constants. All these arguments are made precise in section 3.4. They lead to the next proposition, which is proved on page 94.

**Proposition 3.2.4** (Decay of  $U$  implies  $V$  bounded). *Assume hypothesis  $(H_2)$  holds, let  $m > 0$  and  $r_0 > 0$ . Then there exists a positive constant  $\mu_0$  such that for all  $\mu \in (0, \mu_0]$ , the following holds. Assume that*

$$\|V_0\|_{W^{1,\infty}(\mathbb{R})} + \|U_0\|_{X_m} \leq r_0 \sqrt{\mu},$$

where  $U_0 := \frac{\rho_{-1} V_0}{\omega_{sh}}$ , and that  $C_1$  and  $t_U$  are positive constants such that for all  $0 \leq t \leq t_U$ , the solution to (3.9) with initial condition  $U_0$  satisfies:

$$\left\| \frac{U(t)}{\rho_{-1}} \right\|_{L^\infty(\mathbb{R})} \leq C_1 \frac{\|U_0 \rho_{-2-m}\|_{L^\infty(\mathbb{R})}}{(1+t)^{3/2}}.$$

Then there exists  $C_2 > 0$  – that does not depend on  $r_0$ ,  $t_U$ ,  $\mu_0$  or  $C_1$  – such that for all  $0 \leq t \leq t_U$ , the solution  $V(t)$  is defined and satisfies

$$\|V(t)\|_{L^\infty(\mathbb{R})} \leq C_2 \sqrt{\mu} (1 + r_0).$$

Furthermore there exists a constant  $T$  such that if  $\frac{T}{\mu} \leq t \leq t_U$ , then the above bound improves to  $\|V(t)\|_{L^\infty(\mathbb{R})} \leq C_2 \sqrt{\mu}$ .

Finally, we combine Propositions 3.2.3 and 3.2.4 to prove Theorem 3.2.1. It may seem unclear how to use jointly those two propositions. The important point is that  $C_{\text{stab}}$  is independent of the bound on  $V$ . It reads as follows.

*Proof. Theorem 3.2.1.* First apply Lemma 3.5.2 to obtain the existence of  $\Omega$  that allows to fulfill both hypothesis  $(H_1)$  and  $(H_2)$ .

Let  $\delta$  be fixed by Proposition 3.2.3, let  $\mu_0$  and  $r_0$  be positive constants to be fixed later, and let  $\rho_{-1} V_0 = \omega_{sh} U_0$  such that:

$$U_0 \rho_{-2-m} \in L^\infty(\mathbb{R}), \quad \|V_0\|_{W^{1,\infty}(\mathbb{R})} + \|U_0\|_{X_m} \leq r_0 \sqrt{\mu}.$$

Since  $\mathcal{T}$  is sectorial, and  $V \mapsto \mathcal{Q}(V)$  is locally Lipschitz, the solution  $t \in I \mapsto V(t) \in L^\infty(\mathbb{R})$  to equation (3.12) is uniquely defined on a open, nonempty, maximal set  $I$  – see [Hen81]. Hence there exists  $t_0$  and  $C$  positive constants such that

$$\|V(t)\|_{L^\infty(\mathbb{R})} \leq C$$

for all  $0 \leq t \leq t_0$ . Applying Proposition 3.2.3, this ensures that for all  $0 \leq t \leq t_0$ , the perturbation  $U/\rho_{-1}$  is uniquely defined in  $L^\infty(\mathbb{R})$ , and satisfies

$$N(t) \stackrel{\text{def}}{=} (1+t)^{3/2} \left\| \frac{U(t)}{\rho_{-1}} \right\|_{L^\infty(\mathbb{R})} \leq C_{\text{stab}} \|U_0 \rho_{-2-m}\|_{L^\infty(\mathbb{R})}. \quad (3.18)$$

Consider  $t_1$  the first time where this inequality may fail:

$$t_1 \stackrel{\text{def}}{=} \inf E, \quad E \stackrel{\text{def}}{=} \{t > 0 : N(t) > C_{\text{stab}} \|U_0 \rho_{-2-m}\|_{L^\infty(\mathbb{R})}\}.$$

Assuming by contradiction that  $E$  is nonempty, remark that  $0 < t_0 \leq t_1 < +\infty$ . From Proposition

3.2.4,  $V$  is defined for all times  $0 \leq t \leq t_1$ , with bound

$$\|V(t)\|_{L^\infty(\mathbb{R})} \leq C_2 \sqrt{\mu} (1 + K_0), \quad 0 \leq t \leq t_1$$

However, by continuity of  $t \mapsto \|V(t)\|_{L^\infty(\mathbb{R})}$ , it is possible to define  $V$  for larger times: there exists  $t_2 > t_1$  such that  $V$  is defined up to time  $t_2$ , with deteriorated bound

$$\|V(t)\|_{L^\infty(\mathbb{R})} \leq 2C_2 \sqrt{\mu} (1 + r_0), \quad 0 \leq t \leq t_2.$$

By choosing both  $\mu_0$  and  $\sqrt{\mu_0} r_0$  small enough in the first place, we can assume  $2C_2 \sqrt{\mu} (1 + r_0) \leq \delta$  so that Proposition 3.2.3 applies again, and we recover (3.18) for times  $0 \leq t \leq t_2$ . This is a contradiction with the definition of  $t_1$ , hence we conclude that  $E$  is empty, and that (3.18) holds for all times. Applying Proposition 3.2.4, we recover the claimed bound on  $V$ . Assuming now that  $\|\rho_{-2-m} U_0\|_{L^p} \leq \delta$ , we can finally apply Proposition 3.2.3 to recover  $L^p(\mathbb{R})$  estimates on  $U$  for all times.  $\square$

### 3.3 Decay of perturbations in fully weighted space

#### 3.3.1 Essential and point spectrum

We say that a scalar differential operator  $\mathcal{D}(x) = \sum_{j=0}^n a_j(x) \partial_x^j$  is *exponentially asymptotic* if it converges with uniform exponential rate at  $\pm\infty$ : there exists  $r > 0$  such that for all  $0 \leq j \leq n$ ,

$$\lim_{x \rightarrow \pm\infty} e^{r|x|} |a_j(x) - a_j^\pm| = 0.$$

We note  $\Sigma(\mathcal{D})$  the spectrum of  $\mathcal{D}$ , i.e. the set of complex numbers  $\lambda$  such that  $\lambda - \mathcal{D} : H^n(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is not bounded invertible. To study the spectrum, we decompose it into two distinct parts. We say that  $\lambda - \mathcal{D}$  is Fredholm if its Fredholm index

$$\text{Fred}(\lambda - \mathcal{D}) \stackrel{\text{def}}{=} \dim \ker(\lambda - \mathcal{D}) - \text{codim im}(\lambda - \mathcal{D})$$

is defined and finite. Note in particular that a Fredholm operator has closed range, for the co-dimension of its range to be defined. We define the point spectrum as

$$\Sigma_{\text{pt}}(\mathcal{D}) \stackrel{\text{def}}{=} \{\lambda \in \Sigma(\mathcal{D}) : \text{Fred}(\lambda - \mathcal{D}) = 0\},$$

it corresponds to the subset in which the rank-nullity theorem holds. Then, the essential spectrum is the complementary set:

$$\Sigma_{\text{ess}}(\mathcal{D}) \stackrel{\text{def}}{=} \{\lambda \in \Sigma(\mathcal{D}) : \text{Fred}(\lambda - \mathcal{D}) \text{ is non defined or nonzero}\}.$$

With such definitions, the asymptotic operator with piece-wise constant coefficients

$$\mathcal{D}^\infty(x) \stackrel{\text{def}}{=} \begin{cases} \mathcal{D}^+ \stackrel{\text{def}}{=} \sum_{j=0}^n a_j^+ \partial_x^j & \text{if } x > 0, \\ \mathcal{D}^- \stackrel{\text{def}}{=} \sum_{j=0}^n a_j^- \partial_x^j & \text{if } x < 0, \end{cases}$$

share the same essential spectrum as  $\mathcal{D}$ , see [KP13, p 40]. If both  $\mathcal{D}^+$  and  $\mathcal{D}^-$  are elliptic – in one dimensional space, it comes down to both  $(-1)^{n/2} a_n^+$  and  $(-1)^{n/2} a_n^-$  being positive – then

$\Sigma_{\text{ess}}(\mathcal{D}) = \Sigma_{\text{ess}}(\mathcal{D}^\infty)$  is located to the left of the asymptotic Fredholm borders

$$\Sigma(\mathcal{D}^+) \cup \Sigma(\mathcal{D}^-) = \left\{ \sum_{j=0}^n a_j^+(i\xi)^j : \xi \in \mathbb{R} \right\} \cup \left\{ \sum_{j=0}^n a_j^-(i\xi)^j : \xi \in \mathbb{R} \right\} \subset \mathbb{C},$$

see again [KP13, Theorem 3.1.13].<sup>3</sup>

For a matrix operator:  $\mathcal{D} = \begin{pmatrix} \mathcal{D}_{1,1} & \mathcal{D}_{1,2} \\ \mathcal{D}_{2,1} & \mathcal{D}_{2,2} \end{pmatrix}$ , the asymptotic spectra  $\Sigma(\mathcal{D}^\pm)$  are still obtained through Fourier transform. The operator  $\lambda - \mathcal{D}^+$  is invertible if and only if the matrix  $\lambda - \hat{\mathcal{D}}^+(\xi)$  is invertible for all  $\xi \in \mathbb{R}$  – since Fourier transform is an isometry – which is equivalent to  $\xi \mapsto \det(\lambda - \hat{\mathcal{D}}^+(\xi))$  never vanishing. In our special case  $\mathcal{D} = \mathcal{L}$ , the matrix  $\mathcal{L}^+$  is triangular, hence the determinant is nonzero if both diagonal coefficients maintain away from zero. We conclude that

$$\Sigma(\mathcal{L}^+) = \Sigma(\mathcal{L}^{\text{kpp},+}) \cup \Sigma(\mathcal{L}^{\text{sh},+}),$$

and the same goes at  $-\infty$ . Once again, the essential spectrum of  $\mathcal{L}$  is located to the left of  $\Sigma(\mathcal{L}^+) \cup \Sigma(\mathcal{L}^-)$ .

**Proposition 3.3.1** (Marginally stable essential spectrum). *Fix  $\alpha, d$  positive. Then there exists  $\mu_0 > 0$  and  $\theta < 0$  such that the monotone weight  $\omega_* = \omega_{\text{kpp}} \omega_{\text{sh}}$  with*

$$\omega_{\text{kpp}}(x) = \begin{cases} 1 & \text{if } x \leq -1, \\ e^{-\frac{c_*}{2d}x} & \text{if } x \geq 1, \end{cases} \quad \omega_{\text{sh}}(x) = \begin{cases} e^{\theta x} & \text{if } x \leq -1, \\ 1 & \text{if } x \geq 1, \end{cases}$$

satisfies the following for all  $0 < \mu < \mu_0$ . For all  $\gamma > \gamma_{\text{rem}}$ , with  $\gamma_{\text{rem}}$  as in [FHSS21]:

$$\gamma_{\text{rem}} \stackrel{\text{def}}{=} 8 \left( \frac{\alpha}{d} \right)^2 + 4 \frac{\alpha}{d} - 2\alpha + \mu_0,$$

the operator  $\mathcal{L} = \omega_*^{-1} \mathcal{A} \omega_*$  has marginally stable essential spectrum. More precisely, the Fredholm curve corresponding to the essential KPP spectrum touches the imaginary axis only at  $\lambda = 0$ :

$$\Sigma(\mathcal{L}^{\text{kpp},+}) = \{-\xi^2 : \xi \in \mathbb{R}\},$$

while the three other Fredholm borders have spectral gap: there exists  $\eta > 0$  depending only on  $d, \alpha$  and  $\gamma$  such that for all  $\mu \in (0, \mu_0)$ ,

$$\Sigma(\mathcal{L}^{\text{kpp},-}) \cup \Sigma(\mathcal{L}^{\text{sh},+}) \cup \Sigma(\mathcal{L}^{\text{sh},-}) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq -3\eta\}. \quad (3.19)$$

In particular, hypothesis  $(H_1)$  is fulfilled.

*Proof.* The asymptotic operators  $\mathcal{L}^{\text{kpp},\pm}$  and  $\mathcal{L}^{\text{sh},\pm}$  are obtained as conjugation of  $\mathcal{A}^{\text{kpp},\pm}$  and  $\mathcal{A}^{\text{sh},\pm}$  with pure exponential weight  $e^{\theta x}$  or  $e^{-\frac{c_*}{2d}x}$ . Direct computations show that  $\mathcal{L}^{\text{kpp},+} = \mathcal{A}^{\text{kpp},+}(-\frac{c_*}{2d} + \partial_x)$ , where we identify the operator  $\mathcal{A}^{\text{kpp},+}$  with its symbol  $\mathcal{A}^{\text{kpp},+}(X) = dX^2 + c_*X + \alpha$

<sup>3</sup> If  $\mathcal{D}^\pm$  are elliptic, then both  $\lambda - \mathcal{D}^\pm$  are invertible for  $\text{Re } \lambda$  large enough. This is equivalent to have  $\frac{\eta}{2}$  stable (respectively unstable) spatial eigenvalues for  $\lambda - \mathcal{D}^+$  (respectively  $\lambda - \mathcal{D}^-$ ) and ensures that the region  $S_1$  that contains  $(\eta, +\infty)$  – for  $\eta \in \mathbb{R}$  large enough – does not belong to  $\Sigma_{\text{ess}}(\mathcal{D}^\infty)$ .

evaluated at  $X = \partial_x$ . Hence, the Fredholm border is given by

$$\Sigma(\mathcal{L}^{\text{kpp},+}) = \left\{ \mathcal{A}^{\text{kpp},+} \left( -\frac{c_*}{2d} + i\xi \right) : \xi \in \mathbb{R} \right\} = \left\{ -\xi^2 : \xi \in \mathbb{R} \right\}.$$

Using similar notations for the three other curves, we get

$$\text{Re}(\mathcal{A}^{\text{kpp},-}(\theta + i\xi)) = \text{Re}(d(\theta + i\xi)^2 + c_*(\theta + i\xi) - 2\alpha) \leq -2\alpha + d\theta^2 + c_*\theta < -\alpha$$

for  $\theta \in (-\theta_0, \theta_0)$ , where  $\theta_0 > 0$  only depends on  $\alpha$  and  $d$ . Similarly with  $\mathcal{A}^{\text{sh},-}(X) = -(X^2 + 1)^2 + cX + \mu$ , a direct computation shows that  $\text{Re}(\mathcal{A}^{\text{sh},-}(\theta + i\xi))$  is maximal at  $\pm\xi_0 = \pm\sqrt{1 + 3\theta^2}$ , hence:

$$\text{Re}(\mathcal{A}^{\text{sh},-}(\theta + i\xi)) \leq -\theta^4 - 2\theta^2 + c_*\theta + \mu + \xi_0^4 - 1 = \mu + c_*\theta + 4\theta^2 + 8\theta^4$$

We assume that  $\theta < 0$  is close enough to 0 so that  $\frac{c_*}{2} + 4\theta + 8\theta^3 \geq 0$ . Further assuming that  $\eta > 0$  is small enough so that  $\mu_0 \stackrel{\text{def}}{=} -3\eta + \frac{c_*}{2}|\theta|$  is positive, we conclude that for all  $\mu \in (0, \mu_0]$ ,

$$\text{Re}(\mathcal{A}^{\text{sh},-}(\theta + i\xi)) \leq \mu + \frac{c_*}{2}\theta + \theta \left( \frac{c_*}{2} + 4\theta + 8\theta^3 \right) \leq \mu + \frac{c_*}{2}\theta \leq -3\eta.$$

Finally for  $\mathcal{A}^{\text{sh},+}(X) = \mathcal{A}^{\text{sh},-}(X) - \gamma$ , the same calculation with  $\zeta_0 \stackrel{\text{def}}{=} \sqrt{1 + 3(\frac{c_*}{2d})^2}$  shows that

$$\text{Re}(\mathcal{A}^{\text{sh},+}(-\frac{c_*}{2d} + i\xi)) \leq -\left(\frac{c_*}{2d}\right)^4 - 2\left(\frac{c_*}{2d}\right)^2 - \frac{c_*^2}{2d} + \mu_0 - \gamma + \zeta_0^4 - 1 = \gamma_{\text{rem}} - \gamma.$$

Hence for  $\gamma > \gamma_{\text{rem}}$ , there exists  $\eta > 0$  such that  $\text{Re}(\mathcal{A}^{\text{sh},+}(-\frac{c_*}{2} + i\xi)) < -3\eta$  as claimed.  $\square$

*Remark 3.3.2.* We examine the case of distinct weights on KPP and SH component. Note  $\Omega(x) = \text{Diag}(\omega_1(x), \omega_2(x))$ , and replace ansatz (3.8) by  $(u, v)^T(t, x) = Q_*(\bar{x}) + \Omega(\bar{x})U(t, \bar{x})$ , such that perturbation  $U$  is driven at linear level by

$$\partial_t U = \begin{pmatrix} \omega_1^{-1} \mathcal{A}^{\text{kpp}} \omega_1 & \frac{\omega_2}{\omega_1} \beta \\ 0 & \omega_2^{-1} \mathcal{A}^{\text{sh}} \omega_2 \end{pmatrix} U.$$

For our study to be relevant, we need  $\frac{\omega_2(x)}{\omega_1(x)}$  to be bounded with respect to  $x$ . The computations done in Proposition 3.3.3 shows that both  $\omega_1(x) \leq e^{-\frac{c_*}{2d}x}$  for  $x \geq 1$  and  $\omega_2(x) \geq e^{\theta x}$  for  $x \leq -1$  are necessary to obtain marginal spectral stability. With the condition  $\omega_2(x) \leq C\omega_1(x)$  for  $x \in \mathbb{R}$ , we are left with  $\omega_1 = \omega_2$ .

**Proposition 3.3.3** (Stable point spectrum). *With the same assumptions on  $\mu_0, \mu, \alpha, d, \gamma, \eta$  as in previous Proposition 3.3.1, the operator  $\mathcal{L}$  has no eigenvalue in the following set:*

$$\Omega_{\text{eig}} \stackrel{\text{def}}{=} \{ \lambda \in \mathbb{C} : \text{Re } \lambda > -3\eta \} \setminus (-\infty, 0).$$

*Proof.* We assume by contradiction that there exists  $\lambda \in \Omega_{\text{eig}}$  and a nonzero  $\phi = (\phi^{\text{kpp}}, \phi^{\text{sh}}) \in H^2(\mathbb{R}, \mathbb{R}) \times H^4(\mathbb{R}, \mathbb{R})$  such that

$$(\lambda - \mathcal{L})\phi = 0.$$

In particular, the second eigenproblem is decoupled:  $(\lambda - \mathcal{L}^{\text{sh}})\phi^{\text{sh}} = 0$ . We show it has no other

solution than  $\phi^{\text{sh}} = 0$  in  $H^4(\mathbb{R})$ , which will imply that the first eigenproblem  $(\lambda - \mathcal{L}^{\text{kpp}})\phi^{\text{kpp}} = \beta\phi^{\text{sh}}$  admits no other solution than  $\phi^{\text{kpp}} = 0$ . This is a contradiction.

Both  $\mathcal{L}^{\text{sh}}$  and  $\mathcal{A}^{\text{sh}}$  are not easy to work with, because they respectively have non-constant coefficients or unstable essential spectrum. Instead, we use

$$\mathcal{B}^{\text{sh}} \stackrel{\text{def}}{=} e^{-\theta x} \mathcal{A}^{\text{sh}} e^{\theta x} = e^{-\theta x} \omega_* \mathcal{L}^{\text{sh}} \omega_*^{-1} e^{\theta x} \stackrel{\text{def}}{=} b_0(x) + \sum_{j=1}^4 b_j \partial_x^j,$$

with  $\theta$  and  $\omega_*(x)$  defined in the above Proposition 3.3.1. Then  $\varphi \stackrel{\text{def}}{=} e^{-\theta x} \omega_* \phi^{\text{sh}}$  lies in  $H^4(\mathbb{R})$  due to  $e^{-\theta x} \omega_*(x) \leq 1$ ,<sup>4</sup> and satisfies  $(\lambda - \mathcal{B}^{\text{sh}})\varphi = 0$ . Take the  $L^2(\mathbb{R}, \mathbb{C})$ -inner product with  $\varphi$ :

$$\lambda \|\varphi\|_{L^2}^2 = \langle b_0(x)\varphi, \varphi \rangle + \sum_{j=1}^4 b_j \langle \partial_x^j \varphi, \varphi \rangle.$$

Then  $\langle \partial_x^j \varphi, \varphi \rangle = (-1)^j \overline{\langle \partial_x^j \varphi, \varphi \rangle}$ , so that the real part of the above equality writes:

$$\text{Re}(\lambda) \|\varphi\|_{L^2}^2 = \int_{\mathbb{R}} b_0 |\varphi|^2 dx - b_2 \|\partial_x \varphi\|_{L^2}^2 + b_4 \|\partial_x^2 \varphi\|_{L^2}^2 \leq -3\eta \|\varphi\|_{L^2}^2. \quad (3.20)$$

The inequality is obtained using that  $b_2 = 2(1 + 3\theta^2) > 0$ , that  $b_4 = -1 < 0$  and finally that

$$b_0(x) = \mu - \gamma(1 - q_*(x)) + c_*\theta - (\theta^2 + 1)^2 \leq \mu + c_*\theta \leq \mu + c_*\theta + 4\theta^2 + 8\theta^4 \leq -3\eta.$$

Recall we have chosen  $\text{Re } \lambda > -3\eta$ , hence (3.20) implies  $\varphi = 0$  and then  $\phi^{\text{sh}} = 0$  as claimed.

Now the first eigenproblem writes  $(\lambda - \mathcal{L}^{\text{kpp}})\phi^{\text{kpp}} = 0$ , hence  $\omega_* \phi^{\text{kpp}} \in H^2(\mathbb{R})$  is an eigenfunction for  $\mathcal{A}^{\text{kpp}}$ . Using Sturm-Liouville theory – see e.g. [KP13, p 33.] – eigenvalues of  $\mathcal{A}^{\text{kpp}}$  are real and non-positive, since  $q'_* \in H^2(\mathbb{R})$  does not vanish and satisfies  $\mathcal{A}^{\text{kpp}} q'_* = 0$ . Hence  $\mathcal{A}^{\text{kpp}}$  has no eigenvalues outside of  $(-\infty, 0]$ , and so does  $\mathcal{L}^{\text{kpp}}$ . Since  $\frac{1}{\omega_*(x)} q'_*(x) \sim ax + b$  when  $x \rightarrow +\infty$ , the derivative of the front does not contribute to an eigenfunction for  $\mathcal{L}^{\text{kpp}}$ , hence it has no eigenvalue outside  $(-\infty, 0)$ . The first eigenproblem imposes either  $\lambda \in (-\infty, 0)$  or  $\phi^{\text{kpp}} = 0$ , which is a contradiction, and complete the proof.  $\square$

### 3.3.2 Construction of decaying ODE solutions

Here we solve the linear non-autonomous ODE  $(\lambda - \mathcal{L}^{\text{sh}})\phi = 0$ , with unknown  $\phi$ . Such an ODE writes as

$$\partial_x^4 \phi + a_3(x) \partial_x^3 \phi + \dots + a_0(x) \phi - \lambda \phi = 0,$$

with coefficients  $a_i$  that converge exponentially to constants at  $\pm\infty$ . The upcoming Lemma states that the solutions to this equation express at first order using the solutions of the asymptotic ODE, i.e. replacing the  $a_i(x)$  by their limits  $a_i^+$  or  $a_i^-$ . It is standard to set  $\Phi := (\phi, \partial_x \phi, \dots, \partial_x^3 \phi)^T$  and

$$A^{\text{sh}}(\lambda, x) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda - a_0(x) & -a_1(x) & -a_2(x) & -a_3(x) \end{pmatrix},$$

4. Recall that  $-\frac{c_*}{2} \leq \theta$  when  $\mu$  is small enough.

so that the above ODE writes as  $\partial_x \Phi = A^{\text{sh}}(\lambda, x)\Phi$ . Due to its special structure, the matrix  $A^{\text{sh}}$  is sometimes called the *companion matrix* associated to the polynomial  $X^4 + a_3(x)X^3 + \dots + a_0 - \lambda$ . Direct computations show that the latter is precisely the characteristic polynomial of  $A^{\text{sh}}$ , and that  $(1, v, v^2, v^3)^T$  is an eigenvector of  $A^{\text{sh}}$  when  $v$  is a root of the above polynomial.

**Lemma 3.3.4.** *Let  $A : t \in \mathbb{R} \mapsto A(t) \in \mathcal{M}_n(\mathbb{C})$  be a continuous, matrix-valued function, that converges at exponential speed towards  $A_\infty$  when  $t \rightarrow +\infty$ . Let  $\alpha > 0$  that satisfies  $\|A(t) - A_\infty\| \leq Ce^{-\alpha t}$  for  $t \geq 0$ .<sup>5</sup> Let  $(v_i)_{1 \leq i \leq n}$  be a basis of  $\mathbb{C}^n$ . Then there exists  $(y_i)_{1 \leq i \leq n}$  a basis of solutions for the ODE*

$$y'(t) = A(t)y(t), \quad (3.21)$$

and  $\kappa_i \in L^\infty(0, +\infty)^n$  such that for  $t \geq 0$  we have

$$y_i(t) = e^{tA_\infty}(v_i + e^{-\alpha t} \kappa_i(t)).$$

Furthermore, if  $\lambda \mapsto A(t, \lambda)$ ,  $A_\infty$  and  $v$  are holomorphic with respect to an extra parameter  $\lambda$ , then  $\lambda \mapsto \kappa(\lambda, \cdot)$  is also holomorphic.

We delay the proof to the later section 3.8, and apply this Lemma to the matrix  $A^{\text{sh}}(\lambda, x)$ , for  $\lambda$  to the right of  $\Sigma(\mathcal{L}^{\text{sh}})$ . There, eigenvalues of  $A^{\text{sh}, \pm}$  are distinct and holomorphic, so that there exists a basis  $(v_i^{\text{sh}, \pm}(\lambda))_i$  of holomorphic eigenvectors of  $A^{\text{sh}, \pm}$ :

$$v_i^{\text{sh}, \pm}(\lambda) := \left( 1 \quad v_i^{\text{sh}, \pm}(\lambda) \quad (v_i^{\text{sh}, \pm}(\lambda))^2 \quad (v_i^{\text{sh}, \pm}(\lambda))^3 \right)^T.$$

Using Lemma 3.3.4 allows to define two basis of solutions  $(\phi_i^{\text{sh}, +}(\lambda, \cdot))_{1 \leq i \leq 4}$  and  $(\phi_i^{\text{sh}, -}(\lambda, \cdot))_{1 \leq i \leq 4}$  for the ODE  $(\lambda - \mathcal{L}^{\text{sh}})\phi = 0$ , with exponential behavior at  $+\infty$  or  $-\infty$ :

$$\phi_i^{\text{sh}, \pm} = e^{v_i^{\text{sh}, \pm} x} (v_i^{\text{sh}, \pm} + e^{-r|x|} \kappa_i^{\text{sh}, \pm}(x)), \quad x \in \mathbb{R}_\pm,$$

where  $v_i^{\text{sh}, \pm}$  is the eigenvalue associated to  $v_i^{\text{sh}, \pm}$ , and  $\kappa_i^{\text{sh}, \pm} \in L^\infty(\mathbb{R}_\pm)$ . We use similar notations for  $\mathcal{L}^{\text{kpp}}$ .

**Lemma 3.3.5.** *The dispersion relation  $\lambda - \mathcal{L}^{\text{kpp}, \pm}(v) = 0$  and  $\lambda - \mathcal{L}^{\text{sh}, \pm}(v) = 0$  are respectively second order and fourth order polynomial in  $v$ . Their respective roots  $v_i^{\text{kpp}, \pm}(\lambda)$  and  $v_i^{\text{sh}, \pm}(\lambda)$  have the following localization:*

- (1) (spectral gap away from the spectrum) There exists  $\kappa_2 > 0$  such that for all  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda \geq -2\eta$ ,

$$|\text{Re } v| \geq \kappa_2,$$

where  $v$  stands for  $v_i^{\text{sh}, \pm}(\lambda)$  or  $v_i^{\text{kpp}, -}(\lambda)$ .

- (2) (pinched double root at the origin) Let  $I = (-\infty, 0) \subset \mathbb{C}$ , and  $V$  be a neighborhood of  $0 \in \mathbb{C}$ . There exists  $C$  positive such that for  $\lambda \in V \setminus I$ , and  $\lambda \rightarrow 0$ ,

$$v_1^{\text{kpp}, +}(\lambda) \sim -C\sqrt{\lambda}, \quad v_2^{\text{kpp}, +}(\lambda) \sim C\sqrt{\lambda}.$$

- (3) (elliptic operators) There exists  $R, C, \theta$  positive constant such that for  $\lambda \in \mathbb{C}$  with  $|\lambda| \geq R$  and

5. Here  $\|\cdot\|$  is any operator norm.



$$\operatorname{Re} \lambda \geq -\theta |\operatorname{Im} \lambda|,$$

$$|\operatorname{Re} v_i^{\text{kpp},\pm}(\lambda)| \geq C|\lambda|^{1/2}, \quad |\operatorname{Re} v_i^{\text{sh},\pm}(\lambda)| \geq C|\lambda|^{1/4}.$$

*Proof.* Points (1) and (2) are easily obtained. Point (3) relies on a scaling argument, see [FH18, Lemma 3.1] for  $v_i^{\text{kpp},\pm}$ , the case of  $v_i^{\text{sh},\pm}$  adapts as follow. First set  $\psi(x) \stackrel{\text{def}}{=} \phi(x|\lambda|^{1/4})$ , and apply Lemma 3.3.4 to construct solutions that are close to solutions of  $\lambda + |\lambda|\partial_x^4 \psi = 0$ . In particular, the asymptotic matrix eigenvalues for  $\psi$  are  $\tilde{v}(\lambda) \stackrel{\text{def}}{=} z\sqrt[4]{\lambda}/|\lambda|$ , where  $z \in \{e^{i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{-i\frac{3\pi}{4}}\}$ . Notice that  $|\operatorname{Re} z| = 1/\sqrt{2}$ , hence we get  $|\operatorname{Re} \tilde{v}| \geq C$ , by restricting to  $\lambda$  of the form  $re^{i\theta}$ , with  $|\theta| \leq 3\pi/4$ . Reverting back to the original variable, the asymptotic matrix eigenvalues for  $\phi$  satisfy  $|\operatorname{Re} v| \geq C|\lambda|^{1/4}$ .  $\square$

In the following, we will compute determinant of several matrices, whose columns depend on  $\phi_i^{\text{sh},\pm}$  or  $\phi_i^{\text{kpp},\pm}$ . Suppose we are given  $n$  scalar functions  $\phi_1, \dots, \phi_n$  depending on the space variable  $x$ . Then, we write

$$\operatorname{Det}(\phi_1, \dots, \phi_n) \stackrel{\text{def}}{=} \det \begin{pmatrix} \phi_1 & \cdots & \phi_n \\ \vdots & & \vdots \\ \partial_x^{n-1} \phi_1 & \cdots & \partial_x^{n-1} \phi_n \end{pmatrix} = \det \left( \left( \partial_x^{j-1} \phi_i \right)_{1 \leq i, j \leq n} \right). \quad (3.22)$$

where “det” stands for the determinant. We also introduce the wronskian function composed of all decaying solutions:

$$\mathcal{W}^{\text{kpp}} \stackrel{\text{def}}{=} \operatorname{Det}(\phi_1^{\text{kpp},+}, \phi_2^{\text{kpp},-}), \quad \mathcal{W}^{\text{sh}} \stackrel{\text{def}}{=} \operatorname{Det}(\phi_1^{\text{sh},+}, \phi_2^{\text{sh},+}, \phi_3^{\text{sh},-}, \phi_4^{\text{sh},-}).$$

For each  $\lambda \in \mathbb{C}$ , the function  $y \mapsto \mathcal{W}^{\text{kpp}}(\lambda, y)$  is either identically zero or does not vanish, see Lemma B.1.2. The particular value  $\mathcal{W}^{\text{kpp}}(\lambda, 0)$  is often called the Evans function. When  $\lambda$  is restricted to the right of  $\Sigma_{\text{ess}}(\mathcal{L}^{\text{kpp}})$ , the ODE solutions  $\phi_1^{\text{kpp},+}$  and  $\phi_2^{\text{kpp},-}$  are decaying at  $x \rightarrow \pm\infty$ . It is classical to show that in this region,  $\mathcal{W}^{\text{kpp}}(\lambda, 0)$  vanishes exactly when  $\lambda$  is an eigenvalue for  $\mathcal{L}^{\text{kpp}}$ , with same multiplicity. The same goes for  $\mathcal{W}^{\text{sh}}$ .

### 3.3.3 Construction and estimations of the kernel for the resolvent operator

Here, we construct and control the Green function  $G_\lambda(x, y)$ , when  $\lambda$  is outside  $\Sigma(\mathcal{L})$ . To describe the different behaviors at  $\pm\infty$ , we first introduce two exponential weights. For each two real  $\kappa_1, \kappa_2$ , let  $\omega_{\kappa_1, \kappa_2} : \mathbb{R} \rightarrow \mathbb{R}$  be a positive,  $C^\infty$  function that satisfies

$$\omega_{\kappa_1, \kappa_2}(x) = \begin{cases} e^{\kappa_1 x} & \text{if } x \leq -1, \\ e^{\kappa_2 x} & \text{if } x \geq 1. \end{cases}$$

We now state bound for  $G_\lambda^{\text{sh}}$ . Since  $\mathcal{L}^{\text{sh}}$  is a fourth order operator, the proof is lengthier than for a second order operator. To keep the exposition simple, the proof has been moved to section 3.6. Note that [HH05] presents similar computations in a viscous shock front context.

**Proposition 3.3.6.** *Let  $\Omega \subset \mathbb{C}$  be a compact set to the right of  $\Sigma(\mathcal{L}_\theta^{\text{sh},-}) \cup \Sigma(\mathcal{L}^{\text{sh},+})$ . Then for each  $(x, y) \in \mathbb{R}^2$ , the map  $\lambda \mapsto G_\lambda^{\text{sh}}(x, y)$  is holomorphic from  $\Omega$  to  $\mathbb{C}$ , and there exists  $C, \kappa_2 > 0$  such that the*

following holds. For all  $(x, y) \in \mathbb{R}^2$ , and  $\lambda \in \Omega$ :

$$|G_\lambda^{sh}(x, y)| \leq C e^{-\kappa_2|x-y|}.$$

Furthermore, if  $x \geq 0$  or  $y \geq 0$ , then we have:

$$|G_\lambda^{sh}(x, y)| \leq C e^{-\kappa_2|x-y|} \omega_{\kappa_2,0}(x) \omega_{\kappa_2,0}(y). \quad (3.23)$$

We now turn to the Green function  $G_\lambda^{kpp}(x, y)$ , which corresponds to a single KPP equation. Remark that such a result was already proved in [FH18, Lemma 3.2]. We rewrite it in a more condensed form that separates the behaviors at  $+\infty$  and  $-\infty$ . See the further Remark 3.3.13.

**Proposition 3.3.7.** *There exists a (small) positive constant  $M_s$  such that the following holds. Let  $\Omega_s := B(0, M_s) \setminus (-\infty, 0]$ , where  $B(0, M_s) \subset \mathbb{C}$  is a ball centered at the origin, and where the real negative axis  $(-\infty, 0]$  corresponds to the absolute spectrum of  $\mathcal{L}^{kpp}$ . Then for all  $(x, y) \in \mathbb{R}^2$ ,  $\lambda \mapsto G_\lambda^{kpp}(x, y)$  is holomorphic  $\Omega_s \rightarrow \mathbb{C}$ . Furthermore, there exists positive constants  $\kappa_1, C$  such that for all  $\lambda \in \Omega_s$ , and  $x \geq 0$  or  $y \geq 0$ ,*

$$|G_\lambda^{kpp}(x, y)| \leq C e^{-\operatorname{Re} \sqrt{\lambda}|x-y|} \omega_{\kappa_1,0}(x) \omega_{\kappa_1,0}(y).$$

If on the contrary,  $\lambda \in \Omega_s$ ,  $x \leq 0$  and  $y \leq 0$ , then

$$|G_\lambda^{kpp}(x, y)| \leq C e^{-\kappa_1|x-y|}. \quad (3.24)$$

*Proof.* We use the decaying ODE solution  $\phi_2^{kpp,-}$ ,  $\phi_1^{kpp,+}$  constructed above. To keep notation as light as possible, we write them  $\phi^-$  and  $\phi^+$  respectively. Similarly, we note  $\psi^- = \phi_1^{kpp,-}$  and  $\psi^+ = \phi_2^{kpp,+}$  the growing ODE solutions. Remind that both  $(\phi^-, \psi^-)$  and  $(\phi^+, \psi^+)$  are basis for solution of the ODE  $(\lambda - \mathcal{L}^{kpp})\varphi = 0$ . The Green function expresses as

$$G_\lambda^{kpp}(x, y) = \frac{1}{\mathcal{W}^{kpp}(\lambda, y)} \begin{cases} \phi^-(y)\phi^+(x) & \text{if } y < x, \\ \phi^-(x)\phi^+(y) & \text{if } x < y. \end{cases}$$

We begin by the case  $x \geq 0$  or  $y \geq 0$ . For  $\lambda$  to the right of  $\Sigma(\mathcal{L}^{kpp,-})$  and close enough to the origin, the spatial eigenvalues are such that the exponential obtained in [FH18, Lemma 3.2] are smaller than  $e^{-|x-y|\operatorname{Re} \sqrt{\lambda}}$ . Then each  $\mathcal{O}$  leads to our weights  $\omega_{\kappa_1,0}(x)\omega_{\kappa_1,0}(y)$ . The case  $x \leq 0$  and  $y \leq 0$  is in [FH18, Lemma 3.2].  $\square$

In the following, we will also need to bound the derivatives of the spectral Green kernel. Since our problem is parabolic, to differentiate the semigroup must gains us extra temporal decay. At spectral level, it converts into extra power of  $\sqrt{\lambda}$ . Due to  $G_\lambda^{kpp}$  being a kernel, Dirac delta correction terms appear when differentiating more than the order of  $\mathcal{L}^{kpp}$ . However, they will be easily absorbed at temporal level.

**Proposition 3.3.8.** *There exists a positive constant  $M_s$  such that, with  $\Omega_s := B(0, M_s) \setminus (-\infty, 0]$ , the following holds. For any integer  $j \in \mathbb{N}_{\geq 1}$ . There exists a sum of Dirac delta derivatives:*

$$P_j(\delta_{x=y}) = \sum_{k=0}^{j-2} a_k^j(\lambda, y) \delta_{x=y}^{(k)}$$

with coefficients  $a_k^j$  holomorphic with respect to  $\lambda \in \Omega_s$  and bounded with respect to  $y \in \mathbb{R}$ , such that  $\lambda \mapsto \partial_x^j G_\lambda^{\text{kpp}}(x, y) - P_j(\delta_{x=y})$  is holomorphic  $\Omega_s \rightarrow \mathbb{C}$ . Furthermore, there exists positive constants  $\kappa_1, C$  such that for all  $\lambda \in \Omega_s$ , if  $x \geq 0$  or  $y \geq 0$ , then

$$\left| \partial_x^j G_\lambda^{\text{kpp}}(x, y) - P_j(\delta_{x=y}) \right| \leq C |\lambda|^{j/2} e^{-\text{Re} \sqrt{\lambda} |x-y|} \omega_{\kappa_1, 0}(x) \omega_{\kappa_1, 0}(y).$$

If on the contrary,  $\lambda \in \Omega_s$ ,  $x \leq 0$  and  $y \leq 0$ , then

$$\left| \partial_x^j G_\lambda^{\text{kpp}}(x, y) - P_j(\delta_{x=y}) \right| \leq C |\lambda|^{j/2} e^{-\kappa_1 |x-y|}.$$

*Proof.* For the case  $j = 1$  no correction terms appear, and the proof is close to the  $j = 0$  case [FH18]. In the following lines, we briefly mention the differences between those two cases. We compute

$$\partial_x G_\lambda^{\text{kpp}}(\cdot, y) = \frac{1}{\mathcal{W}_\lambda^{\text{kpp}}(y)} \begin{cases} \phi^-(y) \partial_x \phi^+ & \text{on } (y, +\infty), \\ \phi^+(y) \partial_x \phi^- & \text{on } (-\infty, y). \end{cases}$$

The two cases  $x > y$  and  $x < y$  are symmetric, we restrict to the former. When  $\lambda \in \Omega_s$  with  $M_s$  small enough, there exists a positive constant  $\alpha$ , and functions  $\kappa_j(\sqrt{\lambda}) \in L^\infty(0, +\infty)$  such that for  $j \in \{0, 1\}$  and  $x \geq 0$ ,

$$\partial_x^j \phi^+(x) = (-\sqrt{\lambda})^j e^{-\sqrt{\lambda} x} (1 + e^{-\alpha |x|} \kappa_j(\sqrt{\lambda}, x)).$$

Hence, the bound of  $G_\lambda^{\text{kpp}}(\cdot, y)$  in the case  $j = 0$  straightforwardly extend to the  $j = 1$  case, with an extra factor  $|\lambda|^{1/2}$ . When  $x \leq 0$ , we project onto the  $(\phi^-, \psi^-)$  basis:

$$\partial_x \phi^+(x) = \frac{\text{Det}(\partial_x \phi^+, \psi^-)}{\text{Det}(\phi^-, \psi^-)} \phi^-(x) + \frac{\text{Det}(\phi^-, \partial_x \phi^+)}{\text{Det}(\phi^-, \psi^-)} \psi^-(x),$$

we refer to the above mentioned reference for more details, or to the proof of Proposition 3.3.6, see section 3.6. Since both  $\phi^-$  and  $\psi^-$  are bounded by  $e^{\kappa_1 x}$ , and using that  $\text{Det}(\phi^-, \psi^-) > 0$  uniformly with respect to  $\lambda$ , we get to  $|\partial_x \phi^+(x)| \leq C |\lambda|^{1/2} e^{\kappa_1 x}$ . This conclude the case  $j = 1$ .

When  $j = 2$ , we compute

$$\partial_x^2 G_\lambda^{\text{kpp}}(\cdot, y) = \delta_y + \frac{1}{\mathcal{W}_\lambda^{\text{kpp}}(y)} \begin{cases} \phi^-(y) \partial_x^2 \phi^+ & \text{on } (y, +\infty), \\ \phi^+(y) \partial_x^2 \phi^- & \text{on } (-\infty, y), \end{cases}$$

and similar computations as above show the claimed estimate for  $\partial_x^2 G_\lambda^{\text{kpp}}(x, y) - \delta_y$ .

For  $j \geq 2$ , the correction terms can be computed recursively:

$$P_{j+1}(\delta_{x=y}) = \delta_{x=y} \frac{\phi^- \partial_y^{j+1} \phi^+ - \phi^+ \partial_y^{j+1} \phi^-}{\mathcal{W}_\lambda(y)} + \sum_{k=1}^{j-1} a_k^j(y) \delta_{x=y}^{(k)}$$

Then,  $\partial_x^j G_\lambda^{\text{kpp}}(x, y) - P_j(\delta_{x=y})$  is handled as above.  $\square$

We now obtain a statement on  $G_\lambda^{\text{co}}$ , similar to the above Proposition 3.3.7. We use the structure of the equation satisfied by  $G_\lambda^{\text{co}}$  to directly transfer the spatial decay from  $G_\lambda^{\text{kpp}}$  to  $G_\lambda^{\text{co}}$ .

**Proposition 3.3.9.** *There exists a positive constant  $M_s$  such that, with  $\Omega_s := B(0, M_s) \setminus (-\infty, 0]$ , the map  $\lambda \mapsto G_\lambda^{\text{co}}(x, y)$  is holomorphic  $\Omega_s \rightarrow \mathbb{C}$ . Furthermore, there exists positive constants  $\kappa_3$  and  $C$  such that for all  $\lambda \in \Omega_s$ , if  $x \geq 0$  or  $y \geq 0$ , then:*

$$|G_\lambda^{\text{co}}(x, y)| \leq C e^{-\text{Re} \sqrt{\lambda} |x-y|} \omega_{\kappa_3, 0}(x) \omega_{\kappa_3, 0}(y). \quad (3.25)$$

If on the contrary,  $\lambda \in \Omega_s$ ,  $x \leq 0$  and  $y \leq 0$ , then

$$|G_\lambda^{\text{co}}(x, y)| \leq C e^{-\kappa_3 |x-y|}.$$

*Proof.* Recall that

$$(\lambda - \mathcal{L}^{\text{kpp}}) G_\lambda^{\text{co}} = \beta G_\lambda^{\text{sh}}. \quad (3.26)$$

Choosing  $M_s$  sufficiently small ensures that  $\lambda$  is located to the right of  $\Sigma(\mathcal{L}^{\text{kpp}})$ , so that the operator  $\lambda - \mathcal{L}^{\text{kpp}}$  is invertible. For such  $\lambda$ , we get

$$G_\lambda^{\text{co}}(x, y) = \beta \int_{\mathbb{R}} G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau.$$

Indeed, we have constructed  $G_\lambda^{\text{kpp}}(x, y)$  as the solution of the fundamental problem associated with (3.26), i.e. replace the right hand side by a Dirac delta. When the Dirac delta is replaced by any source term, it give birth to a convolution-like solution as above.

We first bound the above expression of  $G_\lambda^{\text{co}}$  when  $x \leq 0$  and  $y \leq 0$ . On the first hand, when  $\tau \leq 0$ , the estimate (3.24) on  $G_\lambda^{\text{kpp}}$  is available, so that using the bound on  $G_\lambda^{\text{sh}}$  from Proposition 3.3.6, and relying on Lemma B.1.3, we obtain that

$$\left| \int_{-\infty}^0 G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C e^{-\kappa_3 |x-y|},$$

with  $\kappa_3 \stackrel{\text{def}}{=} \frac{1}{2} \min(\kappa_1, \kappa_2)$ . On the other hand when  $\tau \geq 0$ , we rather bound  $G_\lambda^{\text{kpp}}$  with the first estimate of Proposition 3.3.7 and neglect the  $e^{-\text{Re} \sqrt{\lambda} |x-y|}$  factor:

$$\left| \int_0^{+\infty} G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C \int_0^{+\infty} \omega_{\kappa_1, 0}(x) e^{-\kappa_2 |\tau-y|} d\tau.$$

From the signs of  $y$  and  $\tau$ , we observe that  $|\tau - y| = \tau - y = \tau + |y|$ , and conclude that

$$\left| \int_0^{+\infty} G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C e^{-\kappa_1 |x| - \kappa_2 |y|} \int_0^{+\infty} e^{-\kappa_2 \tau} d\tau \leq C e^{-\kappa_3 |x-y|}.$$

Altogether, when  $x \leq 0$  and  $y \leq 0$  the claimed bound on  $G_\lambda^{\text{co}}(x, y)$  is shown.

We now turn to the case where  $x \geq 0$  or  $y \geq 0$ . On the first hand, when  $\tau \geq 0$  the weights  $\omega_{\kappa_i, 0}$  can be factored out directly:

$$\left| \int_0^{+\infty} G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C e^{-\text{Re} \sqrt{\lambda} |x-y|} \omega_{\kappa_1, 0}(x) \omega_{\kappa_2, 0}(y) \int_0^{+\infty} E_1(x, y, \tau) d\tau,$$

with

$$E_1(x, y, \tau) \stackrel{\text{def}}{=} \exp\left(\text{Re} \sqrt{\lambda} |x - y| - \text{Re} \sqrt{\lambda} |x - \tau| - \kappa_2 |\tau - y|\right).$$

Using  $|x - y| \leq |x - \tau| + |\tau - y|$ , and assuming that  $M_s$  is so small that  $\operatorname{Re} \sqrt{\lambda} \leq \frac{1}{2} \kappa_2$ , we obtain that

$$E_1(x, y, \tau) \leq \exp\left((\operatorname{Re} \sqrt{\lambda} - \kappa_2)|\tau - y|\right) \leq e^{-\frac{\kappa_2}{2}|\tau - y|}. \quad (3.27)$$

In particular,  $\int_0^{+\infty} E_1(x, y, \tau) d\tau \leq \frac{2}{\kappa_2}$ . On the other hand, when  $\tau \leq 0$  we discuss each case separately. Let us begin with the case where  $x \geq 0$  and  $y \geq 0$ . There, the weights  $\omega_{\kappa_1, 0}(x)$  and  $\omega_{\kappa_2, 0}(y)$  are bounded from below by a constant  $C > 0$ , and thus does not contribute to the bound (3.25). From Propositions 3.3.6 and 3.3.7, where we neglect the terms  $\omega_{\kappa_2, 0}(\tau)$  and  $\omega_{\kappa_1, 0}(\tau)$ , we obtain that

$$\left| \int_{-\infty}^0 G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C e^{-\operatorname{Re} \sqrt{\lambda}|x - y|} \int_{-\infty}^0 E_1(x, y, \tau) d\tau.$$

From (3.27), we conclude that  $\int_{-\infty}^0 E_1(x, y, \tau) d\tau \leq \frac{2}{\kappa_2}$ . Second, we treat the case where  $y \leq 0 \leq x$ . We now retain the term  $\omega_{\kappa_1, 0}(\tau)$ , to obtain that

$$\left| \int_{-\infty}^0 G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C e^{-\operatorname{Re} \sqrt{\lambda}|x - y|} \int_{-\infty}^0 E_1(x, y, \tau) e^{-\kappa_1|\tau|} d\tau.$$

Recall that  $\kappa_3 = \frac{1}{2} \min(\kappa_1, \kappa_2)$ . From the previous bound (3.27) on  $E_1$ , and from the triangular inequality  $|\tau - y| + |\tau| \geq |y|$ , we get

$$E_1(x, y, \tau) e^{-\kappa_1|\tau|} \leq e^{-\kappa_3|y|} e^{-\kappa_3|\tau|}.$$

In particular  $\int_{-\infty}^0 E_1(x, y, \tau) e^{-\kappa_1|\tau|} d\tau \leq \omega_{\kappa_3, 0}(y) \frac{1}{\kappa_3}$ . Finally, we deal with the  $x \leq 0 \leq y$  case. Here,  $G_\lambda^{\text{kpp}}$  is bounded with its second estimate. Retaining the  $\omega_{\kappa_2, 0}(\tau)$  term, we compute as above that

$$\left| \int_{-\infty}^0 G_\lambda^{\text{kpp}}(x, \tau) G_\lambda^{\text{sh}}(\tau, y) d\tau \right| \leq C e^{-\operatorname{Re} \sqrt{\lambda}|x - y|} \int_{-\infty}^0 E_2(x, y, \tau) e^{-\kappa_2|\tau|} d\tau,$$

where

$$E_2(x, y, \tau) \stackrel{\text{def}}{=} \exp\left(\operatorname{Re} \sqrt{\lambda}|x - y| - \kappa_1|x - \tau| - \kappa_2|\tau - y|\right).$$

From the triangular inequality  $|x - y| \leq |x - \tau| + |\tau - y|$ , and assuming that  $M_s$  is small enough, we obtain that

$$E_2(x, y, \tau) \leq \exp\left(-\frac{\kappa_1}{2}|x - \tau| - \frac{\kappa_2}{2}|\tau - y|\right).$$

Together with the triangular inequality  $|x - \tau| + |\tau| \geq |x|$ , we conclude that

$$\int_{-\infty}^0 E_2(x, y, \tau) e^{-\kappa_2|\tau|} d\tau \leq \int_{-\infty}^0 e^{-\kappa_3|x|} e^{-\kappa_3|\tau - y|} d\tau \leq \omega_{\kappa_3, 0}(x) \frac{2}{\kappa_3}.$$

The three sub-cases that compose the case where  $x \geq 0$  and  $y \geq 0$  are now down, and the proof is complete.  $\square$

Now that we have controled  $G_\lambda$  when  $\lambda$  lies near the origin, we turn to bounds when  $\lambda$  is far from the origin. This is needed to transfer spectral behavior into temporal decay. Indeed,  $G_\lambda$  will be integrated along a spectral contour that surround the spectrum and hence goes to infinity.

**Proposition 3.3.10.** *There exists a (large) positive constant  $M_1$  and a constant  $C > 0$  such that: for*

all  $\lambda$  to the right of  $\Sigma(\mathcal{L})$ , with  $|\lambda| \geq M_I$ , and all  $(x, y) \in \mathbb{R}^2$ , we have

$$\left| G_\lambda^{kpp}(x, y) \right| \leq \frac{C}{|\lambda|^{1/2}} e^{-|\lambda|^{1/2}|x-y|}, \quad \left| G_\lambda^{sh}(x, y) \right| \leq \frac{C}{|\lambda|^{3/4}} e^{-|\lambda|^{1/4}|x-y|}. \quad (3.28)$$

The first estimate still holds if  $G^{kpp}$  is replaced by  $G^{co}$ . In particular, for  $i \in \{kpp, sh, co\}$

$$\|G_\lambda^i\|_{L^\infty(\mathbb{R}_x, L^1(\mathbb{R}_y))} \leq \frac{C}{|\lambda|}, \quad \|G_\lambda^i\|_{L^\infty(\mathbb{R}_y, L^1(\mathbb{R}_x))} \leq \frac{C}{|\lambda|}.$$

*Proof.* The last claimed inequalities are easily obtained from the first ones, and implies that

$$\left\| x \mapsto \int_{\mathbb{R}} G_\lambda^i(x, y) v(y) dy \right\|_{L^p(\mathbb{R})} \leq \frac{C}{|\lambda|} \|v\|_{L^p(\mathbb{R})}$$

using complex interpolation. The first estimates are obtained through the scaling argument began in the proof of Lemma 3.3.5. The second ones are necessary to obtain small time estimates on  $\int_\Gamma e^{\lambda t} G_\lambda d\lambda$ , with  $\Gamma$  being a sectorial contour.  $\square$

We conclude this section with a statement about the intermediate bounded region

$$\{\lambda \in \mathbb{C} \setminus \Sigma(\mathcal{L}) : |\lambda| \in [M_s, M_I]\}.$$

**Proposition 3.3.11.** *Let  $\Omega \subset \mathbb{C} \setminus \Sigma(\mathcal{L})$  be a compact set. Then for all  $(x, y) \in \mathbb{R}^2$ , the map  $\lambda \mapsto G_\lambda(x, y)$  is holomorphic from  $\Omega$  to the set of complex  $2 \times 2$  matrices, and there exists positive constants  $C$  and  $\kappa$  such that*

$$|G_\lambda^i(x, y)| \leq C e^{-\kappa|x-y|}, \quad i \in \{kpp, sh, co\}.$$

*Proof.* When  $i = sh$ , we refer to proposition 3.3.6. The case  $i = kpp$  shows along similar and simpler lines: recall that

$$G_\lambda^{kpp}(x, y) = \frac{1}{\mathcal{W}^{kpp}(\lambda, y)} \begin{cases} \phi^-(y)\phi^+(x) & \text{if } y < x, \\ \phi^-(x)\phi^+(y) & \text{if } x < y. \end{cases}$$

Due to  $\Omega$  being away from  $\Sigma_{pt}(\mathcal{L}^{kpp})$ , the Evans function  $|\mathcal{W}^{kpp}(\lambda)|$  is bounded from below by a positive constant, uniformly with respect to  $\lambda$ . Due to  $\Omega$  being away from  $\Sigma_{ess}(\mathcal{L}^{kpp})$ , the functions  $\phi^\pm$  and  $\psi^\mp$  have exponential behavior  $e^{\mp\kappa \cdot}$ , with constant  $\kappa$  uniform with respect to  $\lambda$ .

For the last case  $i = co$ , we combine the two previous ones and rely on Lemma B.1.3. For all  $(x, y) \in \mathbb{R}^2$ ,

$$|G_\lambda^{co}(x, y)| \leq |\beta| \int_{\mathbb{R}} e^{-\kappa|x-\tau|} e^{\kappa|\tau-y|} d\tau \leq C e^{-\frac{\kappa}{2}|x-y|},$$

and the proof is complete.  $\square$

### 3.3.4 Linear dynamic

We now convert spatial localization of spectral Green functions  $G_\lambda$  into temporal decay in appropriate spaces, using the inverse Laplace transform and adequate integration contours.

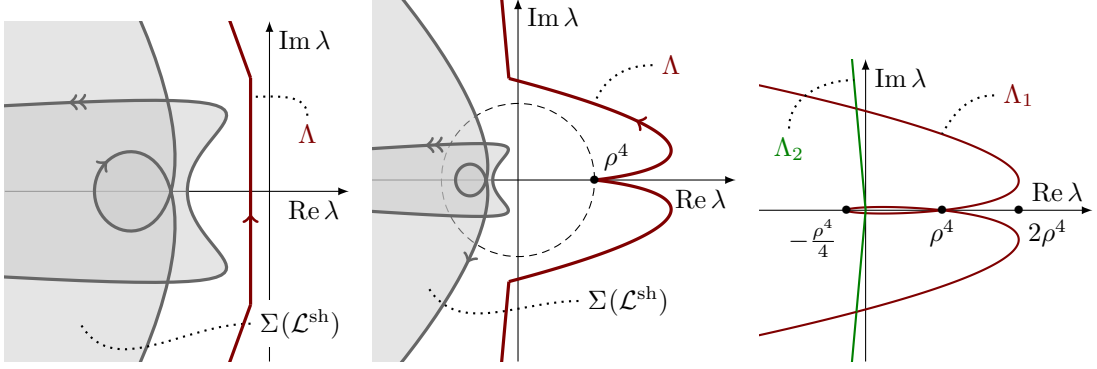


Figure 3.3 – Spectral contour used to obtain estimates on  $\mathcal{G}_t^{\text{sh}}$  when  $t \rightarrow +\infty$ . Left figure: the contour used to obtain exponential decay for large time is located in the region  $\{\text{Re } \lambda < 0\}$ . Center figure: the contour  $\Lambda$  used to obtain the Gaussian scaling lays outside of a (large) ball of radius  $\rho^4$ , and remains to the right of the essential spectrum. Right figure: in the region  $\{\text{Re } \lambda \geq 0\}$ , the contour  $\Lambda$  follows a parabolic contour of order 4. In the region  $\{\text{Re } \lambda \leq 0\}$ , it follows a sectorial contour up to  $\text{Im } \lambda \rightarrow \pm\infty$ .

**Proposition 3.3.12.** *There exists positive constants  $C$ ,  $\eta$  and  $K$  such that:*

$$|\mathcal{G}_t^{\text{sh}}(x, y)| \leq C e^{-\eta t} e^{-\kappa_2 |x-y|}, \quad t \geq 1,$$

$$|\mathcal{G}_t^{\text{sh}}(x, y)| \leq \frac{C}{t^{1/4}} \exp\left(-\kappa_2 \frac{|x-y|^{4/3}}{t^{1/3}}\right), \quad |x-y| \geq Kt, \quad t \geq 1.$$

In particular, for any  $1 \leq p \leq +\infty$  and  $t \geq 0$ , we have

$$\|\mathcal{G}_t^{\text{sh}} \cdot w\|_{L^p(\mathbb{R})} \leq C e^{-\eta t} \|w\|_{L^p(\mathbb{R})}, \quad (3.29)$$

where we have noted  $(\mathcal{G}_t^{\text{sh}} \cdot w)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \mathcal{G}_t^{\text{sh}}(x, y) w(y) dy$ .

*Proof.* Since  $\mathcal{L}^{\text{sh}}$  is sectorial with spectral gap  $\eta$  from Propositions 3.3.1 and 3.3.3, the exponential decay in time of  $\mathcal{G}_t^{\text{sh}}(x, y)$  follows from the above bound on  $\mathcal{G}_\lambda^{\text{sh}}$ : we use an inverse Laplace transform that follows two spectral contours. When  $|\lambda| \geq M_I$ , Proposition 3.3.10 is applied to a sectorial contour in the region  $\Omega_\delta \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C} : \text{Arg}(\lambda) \leq \pi/2 + \delta\}$ . Here  $\delta > 0$  is chosen small enough so that  $\Omega_\delta \cap \Sigma(\mathcal{L}^{\text{sh}}) = \emptyset$ . When  $|\lambda| \leq M_I$ , Proposition 3.3.6 is applied on a contour of the form  $\left\{ \lambda(\xi) \stackrel{\text{def}}{=} -\frac{\eta}{2} + i\xi : \xi \in [-\xi_0, \xi_0] \right\}$ , see Figure 3.3, left picture.

We claim that the Gaussian scaling follows from direct but long computations, with a large parabolic contour of the form  $\Lambda \stackrel{\text{def}}{=} \left\{ \lambda(\xi) \stackrel{\text{def}}{=} (\rho + z\xi)^4 : \xi \in \mathbb{R} \right\}$ , where  $\rho \stackrel{\text{def}}{=} C \left( \frac{|x-y|}{t} \right)^{1/3}$  for a large enough constant  $C$ , and where  $z \stackrel{\text{def}}{=} e^{i\pi/4}$  satisfies  $z^4 = -1$ , see Figure 3.3, center and right pictures.

When  $t \geq 1$ , the bound (3.29) directly follows from the first pointwise bound. When  $t \leq 1$ , it is a consequence of standard parabolic theory.  $\square$

*Remark 3.3.13.* We now state a similar result for  $\mathcal{G}_t^{\text{kpp}}$  and  $\mathcal{G}_t^{\text{co}}$ . Remark that in [FH18, Proposition 4.1], the situation is similar except that the weight  $\omega_*$  was exponentially decaying both at  $+\infty$  and  $-\infty$ . In our setting, since  $\omega_*(x) \rightarrow +\infty$  when  $x \rightarrow -\infty$ , we can not absorb any supplementary

polynomial during the nonlinear argument. Hence, we need to refine the aforementioned result with respect to the polynomial weight. We use the spatial decay  $\omega_{\kappa_1,0}(y)$  obtained for both  $G_\lambda^{\text{kpp}}$  and  $G_\lambda^{\text{co}}$  in the above Propositions 3.3.7 and 3.3.9. This allows to bypass the weight at  $-\infty$  during nonlinear argument.

**Proposition 3.3.14.** *Recall that  $\rho_{-1}$  is defined by (3.11). Fix  $j \in \{0, 1, 2\}$  and restrict to  $t > 1$ . Then there exists positive constants  $C$ ,  $K$  and  $\kappa_1$  such that the following two pointwise estimate hold. If  $|x - y| \geq Kt$ , then*

$$\left| \partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y) \right| \leq \frac{C}{t^{\frac{j+1}{2}}} \left( 1 + \frac{|x-y|}{\sqrt{t}} \right)^j e^{-\kappa_1 \frac{|x-y|^2}{t}} h(x, y).$$

If on the contrary  $|x - y| \leq Kt$ , then

$$\left| \partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y) \right| \leq \frac{1 + |x-y|^{1+[j/2]}}{t^{3/2+[j/2]}} e^{-\kappa_1 \frac{|x-y|^2}{t}} h(x, y).$$

In the above expressions,  $(x, y) \mapsto h(x, y)/\rho_{-2}(y)$  is integrable in  $y$  uniformly in  $x$ :

$$0 \leq h(x, y) \leq C \begin{cases} \omega_{\kappa_1,0}(x) \omega_{\kappa_1,0}(y) & \text{if } x \geq 0 \text{ or } y \geq 0, \\ e^{-\kappa_1|x-y|} & \text{if } x \leq 0 \text{ and } y \leq 0. \end{cases}$$

In particular, for  $j \in \{0, 1, 2\}$ , for all  $1 \leq q \leq p \leq +\infty$ , and all  $t \geq 1$ ,

$$\left\| \frac{\partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w}{\rho_{-1}} \right\|_{L^p(\mathbb{R})} \leq \frac{C}{t^{\frac{3}{2} - \frac{1}{2p}}} \|\rho_{-3} w\|_{L^q(\mathbb{R})}. \quad (3.30)$$

All the above still hold when  $\mathcal{G}_t^{\text{kpp}}$  and  $\kappa_1$  are replaced by  $\mathcal{G}_t^{\text{co}}$  and  $\kappa_3$  respectively.

*Proof.* We first investigate the case  $j = 0$ . We recall that [FH18, Proposition 4.1] proves that there exists  $K > 0$  such that:

- if  $|x - y| \geq Kt$ , then  $\left| \mathcal{G}_t^{\text{kpp}}(x, y) \right| \leq \frac{C}{\sqrt{t}} e^{-\kappa \frac{|x-y|^2}{t}}$ ,
- if  $|x - y| \leq Kt$ , then  $\left| \mathcal{G}_t^{\text{kpp}}(x, y) \right| \leq C \frac{1+|x-y|}{t^{3/2}} e^{-\kappa \frac{|x-y|^2}{t}}$ .

Having kept the extra exponential localization  $-\omega_{\kappa_1,0}(x)\omega_{\kappa_1,0}(y)$  when  $x \geq 0$  or  $y \geq 0$  – in our estimates of  $G_\lambda^{\text{kpp}}(x, y)$ , the proof of [FH18] straightforwardly adapts with this extra decay. The same goes for the other case  $x \leq 0$  and  $y \leq 0$ .

In case where  $j > 0$ , derivatives pass through the inverse Laplace transform formula, and we can choose a sectorial contour  $\Lambda$  before differentiation:

$$\partial_x^j \mathcal{G}_t^{\text{kpp}}(\cdot, y) = \frac{1}{2i\pi} \int_\Lambda e^{\lambda t} \partial_x^j G_\lambda^{\text{kpp}}(\cdot, y) d\lambda.$$

We begin by showing the claimed estimate when correction terms are not present, that is we consider

$$\partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y) - \frac{1}{2\pi} \int_\Lambda e^{\lambda t} P_j(\delta_{x=y}) d\lambda$$

instead of  $\partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y)$ . We first treat the long time scenario:  $|x - y| \leq Kt$ . Using estimate from Proposition 3.3.8, there exists a function  $H$  holomorphic with respect to  $\lambda$  outside  $I \stackrel{\text{def}}{=} (-\infty, 0]$ ,



such that  $\|H(\lambda, \cdot, \cdot)\|_{L^\infty(\mathbb{R}_x, L^1(\mathbb{R}_y))} < +\infty$  and

$$\partial_x^j G_\lambda^{\text{kpp}}(x, y) - P_j(\delta_{x=y}) = \sqrt{\lambda}^j e^{-\sqrt{\lambda}|x-y|} H(\lambda, x, y).$$

We set  $\rho = \kappa \frac{|x-y|}{t}$  with  $\kappa > 0$  to be fixed later. Close to the origin, we use a parabolic contour:

$$\Lambda_1(\rho) = \left\{ \lambda(\xi) = (\rho + i\xi)^2 : \xi \in (-\xi_0, \xi_0) \right\},$$

and obtain with  $\tilde{H}(\xi, x, y, t) \stackrel{\text{def}}{=} \frac{1}{\pi} e^{i(2\rho\xi t - \xi|x-y|)} H_{\lambda(\xi)}(x, y)$  that

$$\begin{aligned} \frac{1}{2i\pi} \int_{\Lambda_1} e^{\lambda t} (\partial_x^j G_\lambda^{\text{kpp}}(\cdot, y) - P_j(\delta_{x=y})) d\lambda &= e^{\rho^2 t - \rho|x-y|} \int_{-\xi_0}^{\xi_0} e^{-\xi^2 t} \tilde{H}(\xi, x, y, t) (\rho + i\xi)^{j+1} d\xi, \\ &\stackrel{\text{def}}{=} e^{\rho^2 t - \rho|x-y|} \int_{-\xi_0}^{\xi_0} z(\xi, x, y, t) d\xi. \end{aligned}$$

Due to  $\mathcal{L}^{\text{kpp}}$  having real coefficients, we have  $G_\lambda^{\text{kpp}} = \overline{G_{\bar{\lambda}}^{\text{kpp}}}$ , and the same goes for  $H$ . Since  $\lambda(-\xi) = \overline{\lambda(\xi)}$  we deduce that in the above equation, the integrand  $z$  satisfies  $z(-\xi, x, y, t) = \overline{z(\xi, x, y, t)}$ . As a consequence, we compute

$$\begin{aligned} \frac{1}{4i\pi} \int_{\Lambda_1} e^{\lambda t} (\partial_x^j G_\lambda^{\text{kpp}}(\cdot, y) - P_j(\delta_{x=y})) d\lambda &= e^{\rho^2 t - \rho|x-y|} \int_{-\xi_0}^{\xi_0} e^{-\xi^2 t} \text{Re}(z(\xi, x, y, t)) d\xi, \\ &= e^{(\kappa^2 - \kappa) \frac{|x-y|^2}{t}} \int_{-\xi_0}^{\xi_0} e^{-\xi^2 t} \left[ \text{Re} \tilde{H}(\xi, x, y, t) \text{Re}(\rho + i\xi)^{j+1} \right. \\ &\quad \left. - \text{Im} \tilde{H}(\xi, x, y, t) \text{Im}(\rho + i\xi)^{j+1} \right] d\xi. \end{aligned}$$

We first develop  $(\rho + i\xi)^{j+1} = \sum_{k=0}^{j+1} \binom{j+1}{k} (i\xi)^k \rho^{j+1-k}$ , and notice that  $\rho \leq \kappa K$ . Then for  $k$  even, we bound

$$\rho^{j+1-k} \leq \kappa^{1+j-k} K^{j-\lfloor j/2 \rfloor - k/2} \left( \frac{|x-y|}{t} \right)^{1+\lfloor j/2 \rfloor - k/2},$$

with  $j - \lfloor j/2 \rfloor - k/2 \geq 0$ . Finally, each power of  $\xi$  provide a power of  $1/\sqrt{t}$  when integrated:

$$0 \leq \int_{-\xi_0}^{\xi_0} \xi^k e^{-\xi^2 t} d\xi \leq \frac{1}{t^{(k+1)/2}} \int_{\mathbb{R}} (\xi \sqrt{t})^k e^{-\xi^2 t} \sqrt{t} d\xi \leq \frac{C_k}{t^{(k+1)/2}}.$$

Hence, using that  $\text{Re} \tilde{H}$  is bounded, we compute that

$$\left| \int_{-\xi_0}^{\xi_0} e^{-\xi^2 t} \text{Re} \tilde{H}(\xi, x, y, t) \text{Re}(\rho + i\xi)^{j+1} d\xi \right| \leq C_j \frac{1 + |x-y|^{1+\lfloor j/2 \rfloor}}{t^{3/2+\lfloor j/2 \rfloor}}.$$

The terms with  $k$  odd, corresponding to the imaginary part of  $(\rho + i\xi)^{j+1}$ , are bounded similarly:

$$\rho^{j+1-k} \leq \kappa^{j+1-k} K^{j-\lfloor j/2 \rfloor - \frac{k-1}{2}} \left( \frac{|x-y|}{t} \right)^{1+\lfloor j/2 \rfloor - \frac{k+1}{2}}.$$

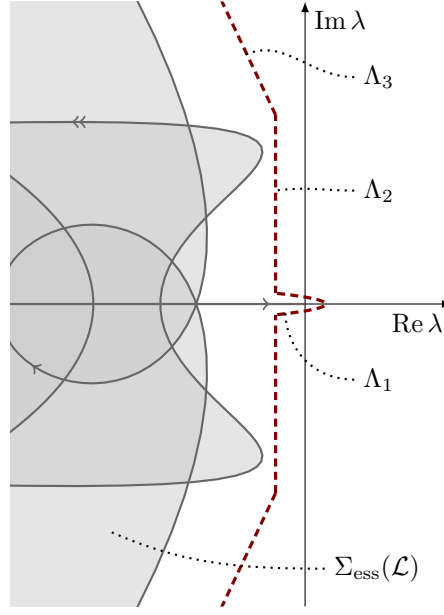


Figure 3.4 – Spectral contour used to bound  $\mathcal{G}_t^{\text{kpp}}$  and  $\mathcal{G}_t^{\text{co}}$ . The essential spectrum of the full operator is pictured in gray. The contour  $\Lambda$  decomposes into three part. Close to the origin, a parabolic contour  $\Lambda_1$  that avoid the negative real axis; far from the origin, a sectorial contour  $\Lambda_3$  that goes up to  $\text{Im } \lambda \rightarrow \pm\infty$ ; a contour  $\Lambda_2$  in the stable region  $\{\text{Re } \lambda \leq -\eta\}$ , that connects the other two.

Since  $\xi \mapsto \text{Im}(\tilde{H}(\xi))$  is odd, it provide an extra power of  $\xi$ :  $|\text{Im} \tilde{H}(\xi)| \leq \xi C$ . Hence we compute that

$$\left| \int_{-\xi_0}^{\xi_0} e^{-\xi^2 t} \text{Im} \tilde{H}(\xi, x, y, t) \text{Im}(\rho + i\xi)^{j+1} d\xi \right| \leq C_j \frac{1 + |x - y|^{1+\lfloor j/2 \rfloor}}{t^{3/2+\lfloor j/2 \rfloor}}.$$

To extend the contour  $\Lambda_1$  up to  $\text{Im}(\lambda) \rightarrow \pm\infty$ , we use a sectorial contour  $\Lambda_3$  when  $\lambda$  is large, together with the bound from Proposition 3.3.10. To connect this sectorial contours with  $\Lambda_1$ , we use a bounded path  $\Lambda_2$  in the stable region  $\{\text{Re}(\lambda) < 0\}$ , together with the bound from Proposition 3.3.11, see Figure 3.4. Those two supplementary contribution decay exponentially in time, we refer to [FH18] for precise computations.

In the short time scenario:  $|x - y| \geq Kt$ , we assume that  $K$  is large enough so that the parabolic contour  $\Lambda_1$  is far enough from the origin, which in turn guarantees that the large  $\lambda$  estimates from Proposition 3.3.10 applies. The computations are then similar to above, with  $(\rho + i\xi)^{j+1}$  replaced by

$$(\rho + i\xi)^j = \sum_{k=0}^j \binom{j}{k} (i\xi)^k \rho^{j-k}.$$

We now use  $\rho = \kappa \frac{|x-y|}{t}$ , to obtain that the  $k$ -th term in the above sum contribute as

$$\binom{j}{k} (i\xi)^k \rho^{j-k} = C \sqrt{t} (\xi \sqrt{t})^k \times \frac{1}{t^{\frac{j+1}{2}}} \left( \frac{|x-y|}{\sqrt{t}} \right)^{j-k},$$

which leads to the claimed pointwise estimate when correction terms are not taken into account.

We now show that correction terms does not contribute for  $j \leq 2$ , which will conclude the proof of the pointwise estimate. When  $j = 1$ , the sum that define  $P_1$  is empty – see Proposition 3.3.8 – thus  $P_1(\delta_{x=y}) = 0$ . For  $j = 2$ , there is only one coefficient  $a_{0,2} = 1$ . Thus the corresponding contribution writes

$$\frac{1}{2i\pi} \int_{\Lambda} e^{\lambda t} P_2(\delta_{x=y}) d\lambda = \frac{\delta_{x=y}}{2i\pi} \int_{\Lambda} e^{\lambda t} d\lambda$$

Remark now that the integral on the right hand side is zero, since  $\Lambda$  is sectorial:

$$\int_{\mathbb{R}} e^{\lambda(\xi)t} \lambda'(\xi) d\xi = \lim_{R \rightarrow +\infty} \left[ e^{\lambda(\xi)t} \right]_{-R}^R = 0.$$

Turning now to the  $L^q - L^p$  estimate (3.30), we use the extra exponential localization we have gained in  $h$ . In the rest of the proof, we denote  $k \stackrel{\text{def}}{=} \lfloor \frac{j}{2} \rfloor$  with  $j \in \{0, 1, 2\}$ . We begin with the case where  $(p, q) = (+\infty, +\infty)$ , and decompose  $\rho_k(x) \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w(x) = \int_{\mathbb{R}} \rho_k(x) \mathcal{G}_t^{\text{kpp}}(x, y) w(y) dy$  into two parts.

To handle the first part, that corresponds to  $|x - y| \geq Kt$ , we use the Gaussian kernel  $\exp\left(-\kappa_1 \frac{|x-y|^2}{t}\right)$  to absorb powers of  $\frac{|x-y|}{\sqrt{t}}$ , and gain time decay from  $K \leq \frac{1}{\sqrt{t}} \frac{|x-y|}{\sqrt{t}}$ . Neglecting the contribution from  $h$ , it reads:

$$\begin{aligned} \int_{|x-y| \geq Kt} \left| \partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y) w(y) \right| dy &\leq C \frac{\|w\|_{L^\infty(\mathbb{R})}}{t^{\frac{j+1}{2}}} \int_{|x-y| \geq Kt} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^j e^{-\kappa_1 \frac{|x-y|^2}{t}} dy, \\ &\leq C \frac{\|w\|_{L^\infty(\mathbb{R})}}{t^{\frac{j+3}{2}}} \int_{|x-y| \geq Kt} \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{j+3} e^{-\kappa_1 \frac{|x-y|^2}{t}} \frac{1}{\sqrt{t}} dy, \\ &\leq C \frac{\|w\|_{L^\infty(\mathbb{R})}}{t^{3/2+k}} \int_{\mathbb{R}} (1+z)^{j+2} e^{-\kappa_1 z^2} dz. \end{aligned}$$

We recall that  $\rho_k \leq 1 \leq \rho_{-k}$  to conclude this part. The second part corresponds to  $|x - y| \leq Kt$ , we purposely deteriorate the time decay obtained at the point-wise level to absorb unbounded spatial weights, through  $\left(\frac{|x-y|}{t}\right)^k \leq C$ . Neglecting the Gaussian kernel, it leads to

$$\int_{|x-y| \leq Kt} \left| \rho_1(x) \partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y) w(y) \right| dy \leq C \frac{\|\rho_{-3} w\|_{L^\infty(\mathbb{R})}}{t^{3/2}} I_1(t, x),$$

where

$$I_1(t, x) \stackrel{\text{def}}{=} \int_{|x-y| \leq Kt} \frac{h(x, y)}{\rho_{-2}(y)} \frac{1 + |x - y|}{\rho_{-1}(x) \rho_{-1}(y)} dy.$$

We are now down to show that  $I_1(t, x)$  is bounded uniformly with respect to its variables. On the first hand, if  $x \leq 0$  and  $y \leq 0$ , the bound on  $h$  ensures that  $|x - y| h(x, y)$  is integrable in  $y$ , uniformly with respect to  $x$ . Using that  $\rho_{-1} \geq 1$ , we conclude that  $I_1(t, x) \leq C$ . On the second hand, if  $x \geq 0$  or  $y \geq 0$ , we rather handle the remaining power of  $|x - y|$  by using that

$$1 + |x - y| \leq 1 + |x| + |y| \leq (1 + |x|)(1 + |y|),$$

for all  $(x, y) \in \mathbb{R}^2$ . Now, when  $x \geq 0$ , it is direct to check that

$$\frac{h(x, y)}{\rho_{-2}(y)} \leq \frac{1}{1+y^2} \omega_{\tilde{\kappa}_1, 0}(y) \omega_{\kappa_1, 0}(x),$$

where  $\tilde{\kappa}_1 \stackrel{\text{def}}{=} \frac{1}{2} \kappa_1$ . Then, from

$$\frac{(1+|x|)\omega_{\kappa_1, 0}(x)}{\rho_{-1}(x)} \frac{(1+|y|)\omega_{\tilde{\kappa}_1, 0}(y)}{\rho_{-1}(y)} \leq C, \quad (3.31)$$

we conclude that  $I_1(t, x) \leq C \int_{\mathbb{R}} \frac{1}{1+y^2} dy \leq C$ . It is direct to check that the above two bounds are still valid when we only assume that  $y \geq 0$ . Altogether, we have shown that  $I_1(t, x) \leq C$ , which conclude the  $(p, q) = (+\infty, +\infty)$  case.

Remark that we could have conserved the optimal decay by strengthening the weight, that is showing

$$\left\| \rho_{1+k} \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{t^{3/2+k}} \|\rho_{-3-k} w\|_{L^\infty(\mathbb{R})},$$

Indeed, it is classical to show the existence of a constant  $C_k$  such that  $1+|x-y|^k \leq C_k(1+|x|^k+|y|^k) \leq C_k(1+|x|^k)(1+|y|^k)$  for all  $(x, y) \in \mathbb{R}^2$ .

We now turn to the  $(p, q) = (1, 1)$  case. Here, we need to handle an additional integral with respect to  $x$ . We fully exploit the Gaussian kernel, and loose a  $\sqrt{t}$  to control its  $L^1(\mathbb{R})$  norm. The argument closely follow the one above, we do not give full details. The bound of the second part reads

$$\iint_{|x-y| \leq Kt} \left| \rho_1(x) \mathcal{G}_t^{\text{kpp}}(x, y) w(y) \right| dy dx \leq \frac{C}{t} \int_{\mathbb{R}} I_2(t, y) |\rho_{-1}(y) w(y)| dy,$$

where

$$I_2(t, y) \stackrel{\text{def}}{=} \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-\kappa \frac{|x-y|^2}{t}} \frac{1+|x-y|}{\rho_{-1}(x) \rho_{-1}(y)} h(x, y) dx.$$

Following a similar approach as the one used to bound  $I_1$ , we conclude that

$$I_2(t, y) \leq C \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{-\kappa \frac{|x-y|^2}{t}} dx \leq C.$$

To control the first part, we neglect the contribution of  $h$ , and gain powers of  $\sqrt{t}$  from  $K \leq \frac{1}{\sqrt{t}} \frac{|x-y|}{\sqrt{t}}$ . It reads

$$\iint_{|x-y| \geq Kt} \left| \mathcal{G}_t^{\text{kpp}}(x, y) w(y) \right| dy dx \leq \frac{C}{t} \int_{\mathbb{R}} I_3(t, y) |w(y)| dy,$$

with

$$I_3(t, y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left( \frac{1+|x-y|}{\sqrt{t}} \right)^{j+2} e^{-\kappa \frac{|x-y|^2}{t}} \frac{1}{\sqrt{t}} dx = \int_{\mathbb{R}} (1+z)^{j+2} e^{-\kappa z^2} dz \leq C.$$

Together with  $\rho_1 \leq 1 \leq \rho_{-1}$ , we have proved that

$$\left\| \rho_1 \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{L^1(\mathbb{R})} \leq \frac{C}{t} \|\rho_{-3} w\|_{L^1(\mathbb{R})}.$$

Once again, we could have retained a stronger decay by strengthening both weights, we claim that

$$\left\| \rho_{3+k} \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{L^1(\mathbb{R})} \leq \frac{C}{t^{3/2+k}} \|\rho_{-3-k} w\|_{L^1(\mathbb{R})},$$

but do not prove it.

Finally, we treat the case  $(p, q) = (+\infty, 1)$  using similar arguments. The control of the first part reads

$$\begin{aligned} \int_{|x-y| \geq Kt} \left| \mathcal{G}_t^{\text{kpp}}(x, y) w(y) \right| dy &\leq \frac{C}{t^{3/2}} \int_{\mathbb{R}} \left( 1 + \frac{|x-y|}{\sqrt{t}} \right)^{j+2} e^{-\kappa_1 \frac{|x-y|^2}{t}} |w(y)| dy, \\ &\leq \frac{C}{t^{3/2}} \|w(y)\|_{L^1(\mathbb{R})}, \end{aligned}$$

due to the fact that the map  $z \in \mathbb{R} \mapsto (1+z)^2 e^{-\kappa_1 z^2}$  is bounded. To control the second part we neglect the Gaussian kernel, it reads

$$\begin{aligned} \int_{|x-y| \leq Kt} \left| \mathcal{G}_t^{\text{kpp}}(x, y) w(y) \right| dy &\leq \frac{C}{t^{3/2}} \int_{\mathbb{R}} \frac{1+|x-y|}{\rho_{-1}(x)\rho_{-1}(y)} h(x, y) |\rho_{-1}(y) w(y)| dy, \\ &\leq \frac{C}{t^{3/2}} \|w(y)\|_{L^1(\mathbb{R})}, \end{aligned}$$

using to the above bound (3.31). We have shown that

$$\left\| \rho_1 \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{t^{3/2}} \|\rho_{-3} w\|_{L^1(\mathbb{R})}.$$

We now obtain the general estimate when  $1 \leq q \leq p \leq +\infty$ . Using the  $(+\infty, +\infty)$  and  $(+\infty, 1)$  bound, we conclude from complex interpolation, see [BL76, Theorem 5.1.1], that for any  $p \in [1, +\infty]$ ,

$$\left\| \rho_1 \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{L^\infty(\mathbb{R})} \leq \frac{C}{t^{3/2}} \|\rho_{-3} w\|_{L^p(\mathbb{R})}.$$

Now using the  $(+\infty, \tilde{p})$  and the  $(1, 1)$  estimates, complex interpolation leads to a  $(p, q)$  estimate, with

$$\frac{1}{p} \stackrel{\text{def}}{=} \frac{1-\theta}{\infty} + \frac{\theta}{1}, \quad \frac{1}{q} \stackrel{\text{def}}{=} \frac{1-\theta}{\tilde{p}} + \frac{\theta}{1}, \quad \theta \in [0, 1].$$

We fix  $\theta \stackrel{\text{def}}{=} \frac{1}{p}$  and  $\tilde{p} \stackrel{\text{def}}{=} \frac{(p-1)q}{p-q} \in [1, +\infty]$ . For this estimate, the temporal decay is  $t^{\frac{3}{2}(1-\theta)} t^\theta = t\sqrt{t}^{1-\theta} = t^{\frac{3}{2}-\frac{1}{2p}}$ . Hence

$$\left\| \rho_1 \partial_x^j \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{L^p(\mathbb{R})} \leq \frac{C}{t^{\frac{3}{2}-\frac{1}{2p}}} \|\rho_{-3} w\|_{L^q(\mathbb{R})},$$

and the proof is complete.  $\square$

*Remark 3.3.15.* Using standard parabolic regularity, we obtain for  $0 < t \leq 1$  and  $1 \leq p \leq +\infty$  that:

$$\left\| \mathcal{G}_t^{\text{kpp}} \cdot w \right\|_{W^{j,p}(\mathbb{R})} \leq C \|w\|_{W^{j,p}(\mathbb{R})}.$$

Then from  $\inf(1, \frac{1}{t}) \leq \frac{2}{1+t}$ , estimate (3.30) holds true for  $t \geq 0$ ,  $p = q$ , and  $\frac{1}{t}$  replaced by  $\frac{1}{1+t}$ .

*Remark 3.3.16.* When  $p = +\infty$ , inequality (3.30) is the optimal  $t^{-3/2}$  decay. It does not seem possible to obtain estimates with  $p < q$ , since  $\mathcal{G}_t(x, y)$  can not absorb both integral in  $x$  and  $y$ .

*Remark 3.3.17.* In the above proof, the term  $\rho_{-3}$  is arbitrary, and could be replaced by  $\rho_{-1}h$  with  $\int_0^{+\infty} \frac{dy}{h(y)} < +\infty$ . Thus the statements above, and the ones that follow, stay true if one writes  $\rho_{-2-m}$  with  $m > 0$  instead of  $\rho_{-3}$ . However, it should be stressed that constants  $C$  will depend on  $m$ , with the asymptotic  $C \rightarrow +\infty$  when  $m \rightarrow 0$ .

### 3.3.5 Nonlinear dynamic

Here we finally prove Proposition 3.2.3, i.e. that  $U = (u_1, u_2)^\top$  decays in time provided  $V = (v_1, v_2)^\top$  is bounded. Throughout this section, we will assume that there exists  $C_\mu > 0$  such that for all  $0 \leq t \leq t_V$ ,

$$\left\| \frac{\omega_{\text{sh}}}{\rho_{-1}} U(t) \right\|_{L^\infty(\mathbb{R})} = \|V(t)\|_{L^\infty(\mathbb{R})} \leq C_\mu. \quad (3.32)$$

To lighten notations, we may note  $C_\mu$  instead of  $CC_\mu$  when  $C$  is a constant not depending on  $C_\mu$ . Remind that  $V(t)$  and  $\frac{\omega_{\text{sh}}}{\rho_{-1}} U(t)$  differ only by a change of frame, so that the first equality is automatically satisfied.

**Proposition 3.3.18.** *For all  $1 < p \leq +\infty$ , there exists positive constants  $C_{\text{stab}}, \eta, \delta$ , such that if  $C_\mu \leq \delta$ , then the solution  $U$  of (3.9) emanating from initial condition  $U_0$  – with  $U_0 \rho_{-3} \in L^p(\mathbb{R})$  – is defined for  $0 \leq t \leq t_V$ , and satisfies*

$$\left\| \frac{u_1(t, \cdot)}{\rho_{-1}} \right\|_{L^p(\mathbb{R})} \leq C_{\text{stab}} \frac{\|U_0 \rho_{-3}\|_{L^p(\mathbb{R})}}{(1+t)^{\frac{3}{2} - \frac{1}{2p}}}, \quad \|u_2(t, \cdot)\|_{L^p(\mathbb{R})} \leq C_{\text{stab}} e^{-\frac{\eta}{2}t} \|U_0\|_{L^p(\mathbb{R})}.$$

Furthermore,  $C_{\text{stab}}$  is independent of  $C_\mu$ .

*Proof.* In this proof, we abbreviate  $K_p \stackrel{\text{def}}{=} \|U_0 \rho_{-3}\|_{L^p(\mathbb{R})}$ . We follow the same argument as in [BGS09, Lemma 3.2], which is adapted to the nonlinear system after conjugation by an unbounded weight.

We fix  $0 < \tilde{\eta} < \eta$  where  $\eta$  is given by the exponential decay of  $u_2$  at linear level: see (3.29) in Proposition 3.3.12. For  $1 < p \leq +\infty$ , we note  $\sigma(p) \stackrel{\text{def}}{=} \frac{3}{2} - \frac{1}{2p} > 1$ . Then we note

$$\Theta_1(t) \stackrel{\text{def}}{=} \sup_{s \in (0, t)} (1+s)^{\sigma(p)} \left\| \frac{u_1(s)}{\rho_{-1}} \right\|_{L^p(\mathbb{R})}, \quad \Theta_2(t) \stackrel{\text{def}}{=} \sup_{s \in (0, t)} e^{\tilde{\eta}s} \|u_2(s)\|_{L^p(\mathbb{R})},$$

together with  $\Theta(t) = \max(\Theta_1(t), \Theta_2(t))$ , and show that  $\Theta$  is bounded in time.

The key point here is to control unbounded nonlinear terms, recall that

$$\begin{aligned} \mathcal{N}_1(U) &= -3\alpha q_* \omega_{\text{kpp}} \omega_{\text{sh}} u_1^2 - \alpha \omega_{\text{kpp}}^2 \omega_{\text{sh}}^2 u_1^3, \\ \mathcal{N}_2(U) &= \gamma \omega_{\text{kpp}} \omega_{\text{sh}} u_1 u_2 - \sigma \omega_{\text{kpp}}^2 \omega_{\text{sh}}^2 u_2^3. \end{aligned}$$

Using from (3.32) that  $\frac{\omega_{\text{sh}} U}{\rho_{-1}}$  is bounded, that both  $\omega_{\text{kpp}}$  and  $q_*$  are bounded, and that  $\rho_{-1} \geq 1$ , we obtain the following bounds

$$|\mathcal{N}_1(U)| \leq |3\alpha q_*| \left| \frac{\omega_{\text{sh}} u_1}{\rho_{-1}} \right| |\rho_{-1} \omega_{\text{kpp}} u_1| + |\alpha \omega_{\text{kpp}}| \left| \frac{\omega_{\text{sh}} u_1}{\rho_{-1}} \right|^2 |\rho_{-2} \omega_{\text{kpp}} u_1| \leq C_\mu |\rho_{-2} \omega_{\text{kpp}} u_1|, \quad (3.33)$$

$$|\mathcal{N}_2(U)| \leq C \left| \frac{\omega_{\text{sh}} u_1}{\rho_{-1}} \right| |\rho_{-1} \omega_{\text{kpp}} u_2| + C \left| \frac{\omega_{\text{sh}} u_2}{\rho_{-1}} \right|^2 |\rho_{-2} \omega_{\text{kpp}} u_2| \leq C_\mu |\rho_{-2} \omega_{\text{kpp}} u_2|.$$

Remark that although we are left with a single power of  $u_1$  or  $u_2$ , the unbounded term  $\omega_{\text{sh}}$  has been absorbed. In particular,  $U \mapsto \mathcal{N}(U)$  is globally Lipschitz, hence the solution  $U$  is defined in  $L^p(\mathbb{R})$  for times  $0 \leq t \leq T$ , see [Hen81]. We can now use the  $\omega_{\text{kpp}}$  factor in the right hand sides of (3.33), to gain as many polynomial weight as needed:  $\omega_{\text{kpp}} \leq C\rho_6$ . This allows to absorb the  $\rho_{-2}$  factor in (3.33) and to gain an extra  $\rho_{-3}$  that will be needed to apply linear estimates (3.30). It reads:

$$|\mathcal{N}_1(s, y)| \leq \frac{C_\mu}{\rho_{-3}(y)(1+s)^{\sigma(p)}} \left| \frac{(1+s)^{\sigma(p)}}{\rho_{-1}(y)} u_1(s, y) \right|,$$

hence

$$\|\rho_{-3} \mathcal{N}_1(s)\|_{L^p(\mathbb{R})} \leq \frac{C_\mu}{(1+s)^{\sigma(p)}} \Theta_1(t).$$

Similarly we get to

$$\|\rho_{-3} \mathcal{N}_2(U(s))\|_{L^p} \leq C_\mu e^{-\bar{\eta}s} \|e^{\bar{\eta}s} u_2(s, \cdot)\|_{L^p} \leq C_\mu e^{-\bar{\eta}s} \Theta_2(t).$$

The Duhamel's formula decomposes into

$$\begin{aligned} u_2(t, x) &= \int_{\mathbb{R}} \mathcal{G}_t^{\text{sh}}(x, y) u_2^0(y) dy + \int_0^t \int_{\mathbb{R}} \mathcal{G}_{t-s}^{\text{sh}}(x, y) \mathcal{N}_2(s, y) dy ds. \\ u_1(t, x) &= \int_{\mathbb{R}} \mathcal{G}_t^{\text{kpp}} u_1^0 + \mathcal{G}_t^{\text{co}} u_2^0 dy + \int_0^t \int_{\mathbb{R}} \mathcal{G}_{t-s}^{\text{kpp}} \mathcal{N}_1 + \mathcal{G}_{t-s}^{\text{co}} \mathcal{N}_2 dy ds. \end{aligned}$$

Take the  $L^p(\mathbb{R})$  norm of Duhamel's formula on  $u_2$ , and pass the norm into the integral using Minkowski integral inequality to obtain that

$$\|u_2(t)\|_{L^p(\mathbb{R})} \leq \|\mathcal{G}_t^{\text{sh}} \cdot u_2^0\|_{L^p} + \int_0^t \|\mathcal{G}_{t-s}^{\text{sh}} \cdot \mathcal{N}_2(s, \cdot)\|_{L^p} ds.$$

Now using the linear estimate (3.29) from Proposition 3.3.12 and the above nonlinear estimate, we obtain:

$$\|e^{\bar{\eta}t} u_2(t)\|_{L^p(\mathbb{R})} \leq C \|u_2^0\|_{L^p(\mathbb{R})} + C_\mu \Theta_2(t) \int_0^t e^{-(\eta-\bar{\eta})(t-s)} ds,$$

which after integration and taking the supremum on  $t \in (0, \tau)$  reads

$$\Theta_2(\tau) \leq \frac{C \|u_2^0\|_{L^p(\mathbb{R})}}{1 - \frac{C_\mu}{\eta - \bar{\eta}}},$$

assuming that the denominator is positive. By imposing  $C_\mu \leq \delta \leq \frac{\eta - \bar{\eta}}{2}$ , this condition is fulfilled and we recover  $\Theta_2(t) \leq 2C \|u_2^0\|_{L^p(\mathbb{R})}$  for  $0 \leq t \leq T$ . Remind that  $C$  does not depend on  $C_\mu$ . This is the claimed exponential temporal decay for  $u_2$ . Turning now towards Duhamel's formula for  $u_1$ , we apply linear estimate (3.30) with  $p = q$  – see also Remark 3.3.15 – together with both above

nonlinear estimates, to obtain that:

$$\begin{aligned} (1+t)^{\sigma(p)} \left\| \frac{u_1(t, \cdot)}{\rho_{-1}} \right\|_{L^p} &\leq C \|U_0 \rho_{-3}\|_{L^p(\mathbb{R})} \\ &\quad + \int_0^t \frac{(1+t)^{\sigma(p)}}{(1+t-s)^{\sigma(p)}} \left( \|\rho_{-3} \mathcal{N}_1(s, \cdot)\|_{L^p(\mathbb{R})} + \|\rho_{-3} \mathcal{N}_2(s, \cdot)\|_{L^p(\mathbb{R})} \right) ds, \\ &\leq CK_p + C_\mu \Theta(t) \int_0^t \left( \frac{(1+t)}{(1+t-s)(1+s)} \right)^{\sigma(p)} ds. \end{aligned}$$

Standard computations on integral – see Lemma B.1.5 and remark that  $\sigma(p) > 1$  – lead to

$$\Theta_1(t) \leq CK_p + C_\mu \Theta(t).$$

Summing the above  $\Theta_2(t) \leq 2CK_p$ , we get to  $\Theta(t) \leq \frac{CK_p}{1-C_\mu}$  as soon as  $1 - C_\mu > 0$ . By assuming that  $C_\mu \leq \delta \leq \frac{1}{2}$ , we get to  $\Theta(t) \leq CK_p$ , which implies the claimed temporal decay for  $u_1$ . Remark that  $C_{\text{stab}} \stackrel{\text{def}}{=} C$  does not depends on  $C_\mu$ .  $\square$

We conclude this section by a refinement of Proposition 3.3.18.

**Corollary 3.3.19.** *There exists positive constants  $C_{\text{stab}}$ ,  $\eta$ ,  $\delta$ , such that if  $C_\mu \leq \delta$ , then the solution  $U$  of (3.9) emanating from initial condition  $U_0$  – with  $U_0 \rho_{-3} \in W^{2,p}(\mathbb{R}) \times W^{4,p}(\mathbb{R})$  – is defined for  $0 \leq t \leq t_V$ , and satisfies*

$$\left\| \frac{U(t)}{\rho_{-1}} \right\|_{W^{2,\infty}(\mathbb{R}) \times W^{4,\infty}(\mathbb{R})} \leq C \frac{\|U_0\|_{X_1}}{(1+t)^{\frac{3}{2}}}.$$

Furthermore,  $C_{\text{stab}}$  is independent of  $C_\mu$ .

*Proof.* The proof goes along similar lines as the one of Proposition 3.3.18. Writing the Duhamel formula for  $u_1$  and differentiating it, we remark that derivatives only act on Green's kernel:

$$\begin{aligned} \partial_x^j u_1(t, x) &= \int_{\mathbb{R}} \partial_x^j \mathcal{G}_t^{\text{kpp}}(x, y) u_1^0(y) + \partial_x^j \mathcal{G}_t^{\text{co}}(x, y) u_2^0(y) dy \\ &\quad + \int_0^t \int_{\mathbb{R}} (\partial_x^j \mathcal{G}_{t-s}^{\text{kpp}}) \mathcal{N}_1(u) + (\partial_x^j \mathcal{G}_{t-s}^{\text{co}}) \mathcal{N}_2(u) dy ds. \end{aligned}$$

To bound the right hand side, we can then combine linear decay from Proposition 3.3.14 – see also Remark 3.3.15 – and the decay of  $U(t)$  from Proposition 3.3.18. Control of nonlinear terms goes exactly as in the proof of the 3.3.18: the unbounded weight  $\omega_{\text{sh}}$  is absorbed by converting powers of  $U$  into powers of  $V$ , while the weight  $\omega_{\text{kpp}}$  allows to gain enough powers of  $\rho_*$  so that linear decay holds.

To show decay of  $\partial_x^k u_2$ , we use the same approach. Since  $\mathcal{L}^{\text{sh}}$  is sectorial with spectral gap, the linear estimate  $\|\mathcal{G}_t^{\text{sh}} u_2\|_{W^{4,\infty}(\mathbb{R})} \leq C e^{-\eta t} \|u_2\|_{W^{4,\infty}(\mathbb{R})}$  directly follows from a sectorial contour.  $\square$



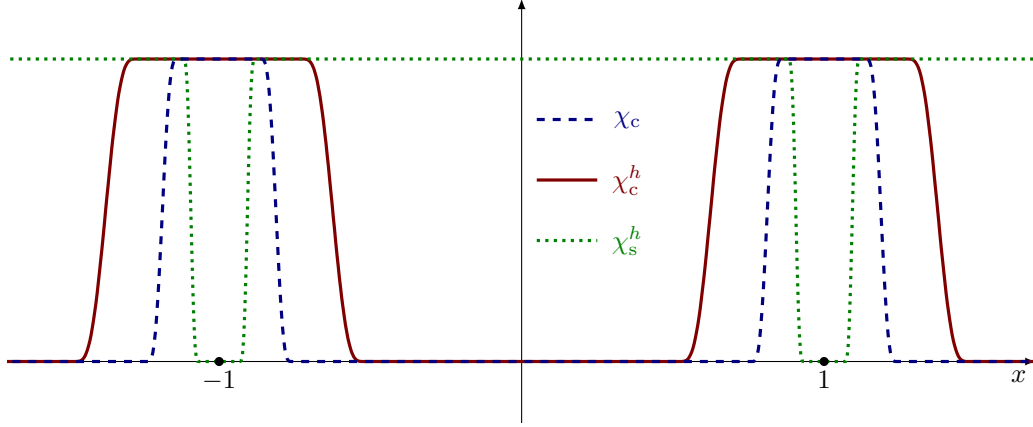


Figure 3.5 – Cut-off functions used to separate the critical frequencies from the stable ones.  $\chi_c$  is represented with a dashed blue line,  $\chi_c^h$  is represented with a red line, while  $\chi_s^h$  is represented with a dotted green line. In particular,  $\chi_c \cdot (\chi_c * \chi_c) = 0$  see Lemma 3.4.6, and  $\chi_c \cdot \chi_c^h = \chi_c$ .

### 3.4 Perturbations in partially weighted space are bounded in time

#### 3.4.1 Mode filters, linear dynamic

Since part of the spectrum of  $\mathcal{T}^-$  is unstable, the dynamic for  $V$  is unstable at linear level. We count on the nonlinear terms to control  $V$  for large bounded times  $t \leq \frac{T}{\mu}$ . To do so, we mostly follow the approach of [Sch94c]. We separate the *critical* from the *stable* modes in our solution: the first ones grow or are bounded at linear level, while the second ones decay exponentially, uniformly in  $\mu$ . For each positive integer  $n \in \mathbb{N}_{\geq 1}$ , denote the following closed subset of  $\mathbb{R}$ :

$$I_c(n) := \left[-1 - \frac{1}{2^n}, -1 + \frac{1}{2^n}\right] \cup \left[1 - \frac{1}{2^n}, 1 + \frac{1}{2^n}\right].$$

Let us use a smooth, positive cut-off function  $\chi_c : \mathbb{R} \rightarrow \mathbb{R}$  that select critical frequencies, that are  $\xi$  near  $\pm 1$ :

$$\chi_c(\xi) = \begin{cases} 1 & \text{if } \xi \in I_c(3), \\ 0 & \text{if } \xi \notin I_c(2), \end{cases} \quad (3.34)$$

see Figure 3.5.

The coefficients  $1 - 2^{-3}$ ,  $1 + 2^{-3}$  are chosen close enough to 1 so that quadratic terms do not interact with linear ones, see the proof of Lemma 3.4.6 below. Since  $\mathcal{T}^-$  is a differential operator with constant coefficients (3.13), its action after Fourier transform is exactly multiplication by the matrix

$$\hat{\mathcal{T}}^-(\xi) := \begin{pmatrix} -d\xi^2 - 2\alpha & \beta \\ 0 & -(1 - \xi^2)^2 + \mu \end{pmatrix}.$$

The latter has two eigenvalues

$$\lambda_c(\xi) = -(1 - \xi^2) + \mu, \quad \lambda_s(\xi) = -d\xi^2 - 2\alpha,$$

with associated eigenvectors  $\zeta_i$ , for  $i \in \{s, c\}$ . In Fourier space, recall that the parallel projections of a vector  $\hat{V}$  onto each of the eigenspaces are obtained by  $\langle \hat{V}, \zeta_i^*(\xi) \rangle \zeta_i(\xi)$ . In physical space, we will separate critical from stable frequencies using:

$$\Pi_c V = \mathcal{F}^{-1} \left( \xi \mapsto \chi_c(\xi) \langle \hat{V}(\xi), \zeta_c^*(\xi) \rangle \zeta_c(\xi) \right), \quad \Pi_s V \stackrel{\text{def}}{=} V - \Pi_c V$$

Remark that  $\chi_c^2 \neq \chi_c$ , so that neither  $\Pi_c$  or  $\Pi_s$  are projections. We introduce two new cut-off functions  $\chi_c^h$  and  $\chi_s^h$  that satisfy

$$\chi_c^h(\xi) = \begin{cases} 1 & \text{if } \xi \in I_c(2), \\ 0 & \text{if } \xi \notin I_c(1), \end{cases} \quad \chi_s^h(\xi) = \begin{cases} 1 & \text{if } \xi \notin I_c(3), \\ 0 & \text{if } \xi \in I_c(4), \end{cases}$$

so that denoting

$$\begin{aligned} \Pi_c^h V &:= \mathcal{F}^{-1} \left( \chi_c^h \langle \hat{V}(\xi), \zeta_c^*(\xi) \rangle \zeta_c(\xi) \right), \\ \Pi_s^h V &:= \mathcal{F}^{-1} \left( \chi_s^h \langle \hat{V}(\xi), \zeta_c^*(\xi) \rangle \zeta_c(\xi) \right) + \mathcal{F}^{-1} \left( \langle \hat{V}(\xi), \zeta_s^*(\xi) \rangle \zeta_s(\xi) \right), \end{aligned}$$

we have  $\Pi_i^h \Pi_j = \Pi_j$  for  $i \in \{c, s\}$ . Ultimately, we will use scalar “projections” onto half of the critical Fourier modes, to select only frequencies close to  $\xi = 1$ :

$$\pi_1^h V = \mathcal{F}^{-1} \left( \xi \mapsto \chi_c^h(\xi) \mathbf{1}_{\xi > 0} \frac{1}{\langle \zeta_c(1), \zeta_c^*(\xi) \rangle} \langle \hat{V}(\xi), \zeta_c^*(\xi) \rangle \right). \quad (3.35)$$

or close to  $\xi = -1$ :

$$\pi_{-1}^h V = \mathcal{F}^{-1} \left( \xi \mapsto \chi_c^h(\xi) \mathbf{1}_{\xi < 0} \frac{1}{\langle \zeta_c(1), \zeta_c^*(\xi) \rangle} \langle \hat{V}(\xi), \zeta_c^*(\xi) \rangle \right).$$

remark that if  $V$  is real valued, then  $\pi_{-1}^h V$  and  $\pi_1^h V$  are complex conjugate. Such decompositions are well-behaved with Fourier transform. However, we need to measure objects with  $L^\infty(\mathbb{R})$  norms, since the pattern we want to study behave as the solution of a bistable equation. To combine those two constraint, we use the so-called uniformly-localized space, see for example [Sch94a]. We first define the weight

$$\rho_{u,1}(x) \stackrel{\text{def}}{=} \frac{1}{1+x^2}.$$

Then, we say that  $u \in L_{u,1}^2(\mathbb{R})$  if

$$\|u\|_{L_{u,1}^2(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{y \in \mathbb{R}} \|\rho_{u,1}(\cdot - y) u\|_{L^2(\mathbb{R})} < +\infty.$$

Similarly, we define  $H_{u,1}^s(\mathbb{R})$  and its norm by  $\|u\|_{H_{u,1}^s(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{y \in \mathbb{R}} \|\rho_{u,1}(\cdot - y) u\|_{H^s(\mathbb{R})}$  when  $s \geq 0$ . We will use the following injections to link  $H_{u,1}^s(\mathbb{R})$  with  $W^{k,p}(\mathbb{R})$ , the latter being used to estimate Green’s kernel.

**Lemma 3.4.1.** *Let  $s \in \mathbb{R}$  with  $s \geq 1$ . Then the following injections are continuous:  $W^{s,\infty}(\mathbb{R}) \subset H_{u,1}^s(\mathbb{R}) \subset L^\infty(\mathbb{R}) \subset L_{u,1}^2(\mathbb{R})$ .*

*Proof.* From one hand the injection  $H^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$  leads to

$$\|u\|_{H_{u,1}^s(\mathbb{R})} \geq \sup_{y \in \mathbb{R}} \|u(\cdot - y)\rho_{u,1}\|_{L^\infty(-1,1)} \geq \frac{1}{2} \sup_{y \in \mathbb{R}} \|u(\cdot - y)\|_{L^\infty(-1,1)} = \frac{1}{2} \|u\|_{L^\infty(\mathbb{R})},$$

It corresponds to the second injection. From the other hand,  $\|u\rho_{u,1}\|_{H^s(\mathbb{R})} \leq \|u\|_{W^{s,\infty}(\mathbb{R})}\|\rho_{u,1}\|_{H^s(\mathbb{R})}$  leads to the first and third injection.  $\square$

As announced, we will need to estimate operators in Fourier space. We will use the following Lemma, see [Sch94b, Section 3.1] and [Sch94c, Lemma 5] for proofs.

**Lemma 3.4.2.** *Let  $d, n$  be positive integers, and  $M$  an operator that acts in Fourier space as a point-wise linear application:  $\hat{M}(\xi) \in \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^n)$  with  $Mu \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\xi \mapsto \hat{M}(\xi)\hat{u}(\xi))$ . Then for  $q, s \geq 0$ ,*

$$\|Mu\|_{H_{u,1}^s(\mathbb{R})^d} \leq C(s, q) \left\| \xi \mapsto \langle \xi \rangle^{\frac{s-q}{2}} \hat{M}(\xi) \right\|_{\mathcal{C}_b^2(\mathbb{R}, \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^n))} \|u\|_{H_{u,1}^q(\mathbb{R})^n},$$

with a positive constant  $C(s, q)$  independent of  $u$  and  $M$ . Furthermore if  $\alpha$  and  $\xi_0$  are reals, then

$$\|x \mapsto e^{i\xi_0 x} Mu(\alpha x)\|_{H_{u,1}^s(\mathbb{R})^d} \leq C \|x \mapsto e^{i\xi_0 x} u(\alpha x)\|_{H_{u,1}^q(\mathbb{R})^n},$$

where the above constant satisfies

$$C \leq C(s, q) \left\| \xi \mapsto \langle \xi \rangle^{\frac{s-q}{2}} \hat{M}\left(\frac{\xi - \xi_0}{\alpha}\right) \right\|_{\mathcal{C}_b^2(\mathbb{R}, \mathfrak{L}(\mathbb{R}^d, \mathbb{R}^n))}.$$

We are now able to state and prove linear estimates.

**Lemma 3.4.3.** *Let  $T$  and  $\mu_0$  be positive constants. Then there exists  $C > 0$  and  $\kappa > 0$  such that for all  $0 < \mu < \mu_0$ , the following holds. For  $s \geq 0$ , and  $0 \leq t \leq \frac{T}{\mu}$ ,*

$$\|e^{tT^-} \Pi_c^h V\|_{H_{u,1}^s(\mathbb{R})} \leq C \|V\|_{H_{u,1}^s(\mathbb{R})},$$

while for any  $t \geq 0$ ,

$$\|e^{tT^-} \Pi_s^h V\|_{H_{u,1}^s(\mathbb{R})} \leq C e^{-\kappa t} \|V\|_{H_{u,1}^s(\mathbb{R})}.$$

Both estimates still holds when  $\Pi_i^h$  are replaced by  $\Pi_i$ .

*Proof.* Since  $T^-$  has constant coefficients – see (3.13) – it acts in Fourier space through multiplication. Hence we rely on multiplier theory, see [MS95]. Since at fixed Fourier parameter  $\xi \in \mathbb{R}$ , the eigenvalues of matrix  $\hat{T}^-(\xi)$  satisfy  $\text{Re } \lambda_j(\xi) \leq \mu$ , we obtain using Lemma 3.4.2 that for  $t \leq \frac{T}{\mu}$ :

$$\|e^{tT^-} \Pi_c^h V\|_{H_{u,1}^s(\mathbb{R})} \leq \left\| \exp(t\hat{T}^-) \hat{\Pi}_c^h \right\|_{\mathcal{C}^2(\mathbb{R}, \mathcal{M}_2(\mathbb{R}))} \|V\|_{H_{u,1}^s(\mathbb{R})} \leq C e^{2t\mu} \|V\|_{H_{u,1}^s(\mathbb{R})},$$

with  $\mathcal{M}_2(\mathbb{R})$  the set of  $2 \times 2$  matrices. Using that  $e^{2t\mu} \leq e^{2T}$ , we get the claimed estimate if  $C$  is large enough. The case of  $e^{tT^-} \Pi_s^h$  adapts easily: for  $t \geq 0$  and  $\xi$  in the support of  $\chi_s$ , the eigenvalues of  $\hat{T}^-(\xi)$  satisfy  $\text{Re } \lambda_j(\xi) \leq -2\kappa$ . To see that  $\Pi_i^h$  may be replaced by  $\Pi_i$ , simply use that  $e^{tT^-} \Pi_i = e^{tT^-} \Pi_i^h \Pi_i$  and that  $\|\Pi_i V\|_{H_{u,1}^s(\mathbb{R})} \leq \|V\|_{H_{u,1}^s(\mathbb{R})}$  to obtain the result.  $\square$

### 3.4.2 Nonlinear dynamic: shadowing the global attractor of the Ginzburg-Landau equation

Following [GS07], we drive the perturbation using the dynamic at  $-\infty$ , to the cost of an extra source term  $\mathcal{S}$ . We first show that this term defined by (3.15) is sufficiently localized in space so that we can extract a  $\|\frac{V}{\omega_{\text{sh}}(\cdot - c_* t)}\|_{L^\infty} = \|\frac{U}{\rho_{-1}}\|_{L^\infty}$  from it. In the rest of this section, we assume that there exists  $C_1$  and  $t_U$  positive constants such that for all  $0 \leq t \leq t_U$ ,

$$\left\| \frac{U(t)}{\rho_{-1}} \right\|_{L^\infty(\mathbb{R})} \leq C_1 \frac{\|U_0\|_{X_1}}{(1+t)^{3/2}}. \quad (3.36)$$

As above, we will note  $C_1$  instead of  $CC_1$  when  $C$  is a constant not depending on  $C_1$ . All arguments below rely on the fact that  $U$  does not blow up in finite time. Hence in the rest of this section, we always restrict – even when it is not clearly stated – to times  $0 \leq t \leq t_U$ . Furthermore, we recall the notation  $\mu = \varepsilon^2$ , with  $\varepsilon > 0$ .

We first obtain decay of derivatives of  $U$ , and then control the source term. Recall that

$$\|U\|_{X_1} \stackrel{\text{def}}{=} \|U\|_{W^{2,\infty}(\mathbb{R}) \times W^{4,\infty}(\mathbb{R})} + \|\rho_{-3}U\|_{L^\infty(\mathbb{R})}.$$

**Proposition 3.4.4.** *There exists positive constants  $\delta$  and  $C$  such that for all  $0 \leq t \leq t_U$ ,*

$$\|\mathcal{S}(\cdot - c_* t, V(t, \cdot))\|_{H_{\text{ul}}^1(\mathbb{R})} \leq C \left\| \frac{V(t, \cdot)}{\omega_{\text{sh}}(\cdot - c_* t)} \right\|_{W^{2,\infty}(\mathbb{R}) \times W^{4,\infty}(\mathbb{R})} \leq C \frac{\|U_0\|_{X_1}}{(1+t)^{3/2}},$$

where  $U_0 := \frac{\rho_{-1}V(0, \cdot)}{\omega_{\text{sh}}}$ .

*Proof.* From Lemma 3.4.1, it is enough to bound  $\|\mathcal{S}(\tilde{x}, V(t))\|_{W^{1,\infty}(\mathbb{R})}$ . For  $\tilde{x} \leq -1$ , we have  $\omega(\tilde{x}) = 1$ , hence

$$\mathcal{S}(\tilde{x}, V) = (1 - q_*(\tilde{x})) \begin{pmatrix} 3\alpha(1 + q_*(\tilde{x}))v_1 \\ -\gamma v_2 \end{pmatrix}.$$

We show that  $\|\omega_{\text{sh}}(1 - q_*)\|_{W^{1,\infty}(-\infty, 0)}$  is finite. The equilibrium point  $(q, q') = (1, 0)$  for the front equation is a saddle, with one positive eigenvalue  $\kappa = (-1 + \sqrt{3})\sqrt{\frac{\alpha}{d}}$ . Basic ODE dynamic then ensures that

$$\sup_{x \leq 0} |q_*(x) - 1|e^{-\kappa x} < +\infty, \quad \sup_{x \leq 0} |q'_*(x)|e^{-\kappa x} < +\infty.$$

For  $\tilde{x} \leq -1$  and  $|\theta|$  small enough,  $\omega_{\text{sh}}(\tilde{x}) = e^{\theta \tilde{x}} \leq e^{-\kappa \tilde{x}}$ , which leads to

$$\|\mathcal{S}(\cdot - c_* t, V(t, \cdot))\|_{W^{1,\infty}(-\infty, 0)} \leq C \left\| \frac{V(t)}{\omega_{\text{sh}}(\cdot - c_* t)} \right\|_{W^{1,\infty}(-\infty, 0)}.$$

Now for  $\tilde{x} \geq 1$ , recall the expression of the source term (3.15). For linear terms, we use the commutator to gain one derivative: the operator

$$\omega^{-1}(\mathcal{A} - c_* \partial_x) \omega - (\mathcal{A} - c_* \partial_x) = \omega^{-1}[\mathcal{A} - c_* \partial_x, \omega]$$

exhibit at most third order derivatives, so that it maps  $W^{2,\infty}(\mathbb{R}) \times W^{4,\infty}(\mathbb{R})$  onto  $W^{1,\infty}(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})$ . Furthermore  $\|\omega^{-1} \partial_x \omega\|_{W^{1,\infty}(0, +\infty)} < \infty$ . Altogether, it leads to

$$\|\mathcal{S}(\cdot, V)\|_{W^{1,\infty}(0, +\infty)} \leq C \|V\|_{W^{2,\infty} \times W^{4,\infty}(0, +\infty)} + C \|V\|_{W^{1,\infty}(0, +\infty)}^2 + C \|V\|_{W^{1,\infty}(0, +\infty)}^3.$$

Remark that  $\omega_{\text{sh}}(\bar{x}) = 1$ , hence the first claimed estimate is shown. Then  $\frac{V(t, \bar{x})}{\omega_{\text{sh}}(\bar{x})} = \frac{U(t, \bar{x})}{\rho_{-1}(\bar{x})}$ , and Proposition 3.3.19 ensures the second claimed estimate. The proof is complete.  $\square$

*Remark 3.4.5.* Since  $t \mapsto (1+t)^{-3/2}$  is integrable at 0, the above control of the source term implies that solutions to (3.12) with initial condition at  $t = 0$  are defined and continuous on an open interval.

We now state nonlinear estimate, which relies on mode filters, see section 3.4.1.

**Lemma 3.4.6** (Non-linear estimates). *Recall the expression of  $\mathcal{Q}^-$  in (3.14). There exists  $C > 0$  such that for all  $V \in H_{u,1}^1(\mathbb{R})$ ,*

$$\|\Pi_s \mathcal{Q}^-(\varepsilon \Pi_c V + \varepsilon^2 \Pi_s V)\|_{H_{u,1}^1(\mathbb{R})} \leq C \varepsilon^2 (\|\Pi_c V\|_{H_{u,1}^1(\mathbb{R})} + \|\Pi_s V\|_{H_{u,1}^1(\mathbb{R})})^2,$$

while

$$\|\Pi_c \mathcal{Q}^-(\varepsilon \Pi_c V + \varepsilon^2 \Pi_s V)\|_{H_{u,1}^1(\mathbb{R})} \leq C \varepsilon^3 (\|\Pi_c V\|_{H_{u,1}^1(\mathbb{R})} + \|\Pi_s V\|_{H_{u,1}^1(\mathbb{R})})^2.$$

*Proof.* The first estimate is immediate. The second one comes from [Sch94a], and proves as follows. In Fourier space, multiplication becomes convolution. Since  $\text{Supp}(\chi_c) = [-5/4, -3/4] \cup [3/4, 5/4]$  and  $\text{Supp}(\chi_c * \chi_c) = [-5/2, -3/2] \cup [-1/2, 1/2] \cup [3/2, 5/2]$  do not intersect, we deduce that

$$\Pi_c(\Pi_c V_1 \times \Pi_c V_2) = \mathcal{F}^{-1}(\chi_c \times (\chi_c \hat{V}_1 * \chi_c \hat{V}_2)) = 0.$$

Hence lowest order quadratic terms vanish when  $\Pi_c$  is applied, leaving  $\varepsilon^3$  terms at leading order.  $\square$

We now build up the tools we will need for the proof of proposition 3.2.4. This proof is done at the end of the present section, that is on page 94. Due to the constant state  $(1, 0)$  being unstable, it is uneasy to prove existence of solutions for all times  $t \geq 0$ . The general approach is the one highlighted in [Sch94c]. We chose initial conditions of size  $r_0 \varepsilon$ , with  $r_0$  not necessarily small, and prove that after a large time  $t_1 \sim \varepsilon^{-2}$ , the solution is again of size  $r_0 \varepsilon$ , provided that  $r_0$  is large enough. Hence, the same procedure can be applied again, to define solution for all times  $t \in [0, t_U]$ .

As a first step, we show in the following Lemma that after a large enough period of time, the critical and stable frequencies of  $V$  separate, that is the stable component is smaller than the critical one. Furthermore, the latter is attracted to a scaled oscillating pattern.

**Lemma 3.4.7** (Attractivity). *Let  $r_0 > 0$ . Then there exists positive constants  $T_{\text{att}}$  and  $\varepsilon_0$ , a constant  $C_{\text{att}} > 0$  depending on  $C_1$  such that for all  $0 < \varepsilon < \varepsilon_0$  the following holds. If  $V_0$  satisfies*

$$\|V_0\|_{H_{u,1}^1(\mathbb{R})} + \|U_0\|_{X_1} \leq r_0 \varepsilon, \quad (3.37)$$

where we have noted  $U_0 := \frac{\rho_{-1} V_0}{\omega_{\text{sh}}}$ , then the solution  $V$  to (3.12) with initial condition  $V_0$  exists for all time  $0 \leq t \leq \max\left(\frac{T_{\text{att}}}{\varepsilon^2}, t_U\right)$ , and decomposes as  $V = V_c + V_s$ , with  $\Pi_i^h V_i = V_i$ . When  $0 \leq t \leq \max\left(\frac{T_{\text{att}}}{\varepsilon}, t_U\right)$ , it satisfies

$$\|V_i(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{att}} r_0 \varepsilon, \quad i \in \{c, s\},$$

while for times  $\frac{T_{\text{att}}}{\varepsilon^2} \leq t \leq \max\left(\frac{T_{\text{att}}}{\varepsilon^2}, t_U\right)$ , we have

$$\|V_c(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{att}} r_0 \varepsilon, \quad \|V_s(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{att}} r_0 \varepsilon^2.$$

Finally, at time  $T_{\text{att}}/\varepsilon^2$ , the critical part is – after scaling – an oscillating function of order  $\varepsilon$ : denoting

$$A_0(X) \stackrel{\text{def}}{=} \frac{1}{\varepsilon} e^{-i\frac{X}{\varepsilon}} \pi_1^h V_c \left( \frac{T_{\text{att}}}{\varepsilon^2}, \frac{X}{\varepsilon} \right), \quad (3.38)$$

we have  $\|A_0\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{att}} r_0$ .

*Proof.* Since  $\mathcal{T}^-$  is sectorial,  $V \mapsto \mathcal{Q}^-(V)$  is locally Lipschitz and  $t \mapsto \mathcal{S}(\tilde{x}, V(t, x))$  is integrable at 0, solution  $V$  to (3.12) with initial condition  $V_0$  is uniquely defined as long as it does not blow up. Hence the estimates that follow ensure that  $V$  is defined up to time  $\max(T_{\text{att}}/\varepsilon^2, t_U)$ . In the following, we implicitly work with the bound  $t \leq t_U$ .

To construct  $V$ , it is enough to solve the following system, with initial condition  $V_i(0) = \Pi_i V(0)$ :

$$\begin{cases} \partial_t V_c = \mathcal{T}^- V_c + \Pi_c \mathcal{Q}^-(V_c + V_s) + \Pi_c \mathcal{S}(\tilde{x}, V_c + V_s), \\ \partial_t V_s = \mathcal{T}^- V_s + \Pi_s \mathcal{Q}^-(V_c + V_s) + \Pi_s \mathcal{S}(\tilde{x}, V_c + V_s), \end{cases} \quad (3.39)$$

and then set  $V = V_c + V_s$ . A solution  $(V_c, V_s)$  of the decoupled system (3.39) is not guaranteed to write as  $V_i = \Pi_i V$ , however the critical-stable separation still holds, so that Lemma 3.4.3 applies. On the first hand,

$$\|e^{t\mathcal{T}^-} V_s\|_{H_{u,1}^1(\mathbb{R})} = \|e^{t\mathcal{T}^-} \Pi_s^h V_s\|_{H_{u,1}^1(\mathbb{R})} \leq e^{-\kappa t} \|V_s\|_{H_{u,1}^1(\mathbb{R})}.$$

On the other hand there exists a constant  $C$  that does not depend on  $\varepsilon$  such that for bounded times  $t \leq \frac{T_{\text{att}}}{\varepsilon^2}$ :

$$\|e^{t\mathcal{T}^-} V_c\|_{H_{u,1}^1(\mathbb{R})} \leq C e^{T_{\text{att}}} \|V_c\|_{H_{u,1}^1(\mathbb{R})} \leq C \|V_c\|_{H_{u,1}^1(\mathbb{R})}.$$

We introduce the local notations  $W_i \stackrel{\text{def}}{=} \varepsilon^{-1} V_i$  and  $\Theta_i(t) \stackrel{\text{def}}{=} \sup_{0 \leq \tau \leq t} \|W_i(\tau)\|_{H_{u,1}^1(\mathbb{R})}$ . Using Duhamel formula, nonlinear estimate from Lemma 3.4.6, and finally Proposition 3.4.4 to obtain integrability of  $\mathcal{S}$  with respect to time, we see that

$$\begin{aligned} \|W_c(t)\|_{H_{u,1}^1(\mathbb{R})} &\leq C \|W_c(0)\|_{H_{u,1}^1(\mathbb{R})} + C_1 \frac{\|U_0\|_{X_1}}{\varepsilon} \int_0^t \frac{d\tau}{(1+\tau)^{3/2}} + C\varepsilon \int_0^t (\Theta_c(t) + \Theta_s(t))^2 d\tau, \\ &\leq C_1 r_0 + C\varepsilon t (\Theta_c(t) + \Theta_s(t))^2. \end{aligned}$$

In a similar way, standard integral computations – see Lemma B.1.5 – ensure that

$$\begin{aligned} \|W_s(t)\|_{H_{u,1}^1(\mathbb{R})} &\leq C e^{-\kappa t} \|W_s(0)\|_{H_{u,1}^1(\mathbb{R})} + C_1 \frac{\|U_0\|_{X_1}}{\varepsilon} \int_0^t \frac{e^{-\kappa(t-\tau)} d\tau}{(1+\tau)^{3/2}} \\ &\quad + C\varepsilon (\Theta_c(t) + \Theta_s(t))^2 \int_0^t e^{-\kappa(t-\tau)} d\tau, \\ &\leq \frac{C_1 r_0}{(1+t)^{3/2}} + C\varepsilon (\Theta_c(t) + \Theta_s(t))^2. \end{aligned} \quad (3.40)$$

Assume that  $T_{\text{att}}$  is so small that  $C_1 r_0 \times C T_{\text{att}} \leq \frac{1}{8}$ , and that  $\varepsilon_0$  is so small that  $C_1 r_0 \times C\varepsilon \leq \frac{1}{8}$ . Take the supremum with respect to  $t \in (0, \varepsilon^{-1} T_{\text{att}})$  in the two above equations to obtain – with  $\Theta := \max(\Theta_c, \Theta_s)$  – that

$$\Theta(t) \leq C_1 r_0 + \frac{1}{8C_1 r_0} \Theta(t)^2.$$

Applying a standard nonlinear argument – see Lemma B.1.6 – we recover  $\Theta(T_{\text{att}}/\varepsilon) \leq 2C_1 r_0$ ,

which is the first claimed estimate. Now for  $\varepsilon^{-2/3}T_{\text{att}} \leq t \leq \varepsilon^{-1}T_{\text{att}}$ , we have  $\frac{1}{(1+t)^{3/2}} \leq C\varepsilon$ , hence  $\|W_s(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_1 r_0 \varepsilon$  from (3.40).

This little improvement will allow us to propagate estimates until time  $T_{\text{att}}/\varepsilon^2$ . We set our new initial time at  $\varepsilon^{-3/4}T_{\text{att}}$ , that is in between  $\varepsilon^{-2/3}T_{\text{att}}$  and  $\varepsilon^{-1}T_{\text{att}}$ : for  $t \geq 0$ , we now use the new local notations

$$W_c(t) := \varepsilon^{-1}V_c(t + \varepsilon^{-3/4}T_{\text{att}}), \quad W_s(t) := \varepsilon^{-2}V_c(t + \varepsilon^{-3/4}T_{\text{att}}),$$

together with  $\Theta_i(t) \stackrel{\text{def}}{=} \sup_{0 \leq \tau \leq t} \|W_i(\tau)\|_{H_{u,1}^1(\mathbb{R})}$ . Remark that from the above, we know that

$$\|W_c(0)\|_{H_{u,1}^1(\mathbb{R})} + \|W_s(0)\|_{H_{u,1}^1(\mathbb{R})} \leq C_1 r_0.$$

Then Duhamel formulae, nonlinear estimate from Lemma 3.4.6 and integrability of  $\mathcal{S}$  from proposition 3.4.4 ensure that for  $0 \leq t \leq T_{\text{att}}/\varepsilon^2$ ,

$$\begin{aligned} \|W_c(t)\|_{H_{u,1}^1(\mathbb{R})} &\leq C\|W_c(0)\|_{H_{u,1}^1(\mathbb{R})} + C_1 \frac{\|U_0\|_{X_1}}{\varepsilon} \int_0^t \frac{d\tau}{(1 + T_{\text{att}}/\varepsilon^{3/4} + \tau)^{3/2}} \\ &\quad + C\varepsilon^2 t (\Theta_c(t) + \Theta_s(t))^2, \\ &\leq C_1 r_0 + CT_{\text{att}} (\Theta_c(t) + \Theta_s(t))^2. \end{aligned}$$

Turning to the estimate on  $W_s$ , we see on the first hand that  $(1 + \varepsilon^{-3/4}T_{\text{att}} + \tau)^{-3/2} \leq C\varepsilon^{9/8}$  when  $\tau \geq 0$ . On the second hand, we know from the first part of our argument that  $\|W_s(t)\|_{H_{u,1}^1(\mathbb{R})}$  stays bounded for times  $0 \leq t \leq \varepsilon^{-1}T_{\text{att}} - \varepsilon^{-3/4}T_{\text{att}}$ , so that we can restrict the present part to times  $t \geq \varepsilon^{-1}T_{\text{att}} - \varepsilon^{-3/4}T_{\text{att}}$ . For such times, we deduce as above that

$$\begin{aligned} \|W_s(t)\|_{H_{u,1}^1(\mathbb{R})} &\leq Ce^{-\kappa t} \|W_s(0)\|_{H_{u,1}^1(\mathbb{R})} + C_1 \frac{\|U_0\|_{X_1}}{\varepsilon^2} \int_0^t \frac{e^{-\kappa(t-\tau)} d\tau}{(1 + T_{\text{att}}/\varepsilon^{3/4} + \tau)^{3/2}} \\ &\quad + C(\Theta_c(t) + \Theta_s(t))^2 \int_0^t e^{-\kappa(t-\tau)} d\tau, \\ &\leq C_1 r_0 \left( \exp\left(-\kappa T_{\text{att}} \frac{1 - \varepsilon^{1/12}}{\varepsilon^{3/4}}\right) + \varepsilon^{1/8} \right) + C(\Theta_c(t) + \Theta_s(t))^2, \\ &\leq C_1 r_0 \varepsilon^{1/8} + C(\Theta_c(t) + \Theta_s(t))^2, \end{aligned}$$

provided that  $\varepsilon_0$  is small enough. By taking  $T_{\text{att}}$  and  $\varepsilon_0$  eventually smaller, we conclude as above that  $\Theta(T_{\text{att}}/\varepsilon^2) \leq 2C_1 r_0$ , which is the second claimed estimate. To bound  $A_0$ , we use the scaled estimate in Lemma 3.4.2

$$\left\| X \mapsto e^{-i\frac{X}{\varepsilon}} M u \left( \frac{X}{\varepsilon} \right) \right\|_{H_{u,1}^1(\mathbb{R})} \leq \|\xi \mapsto \langle \xi \rangle^{1/2} \hat{M}(\varepsilon\xi + 1)\|_{\mathcal{C}_b^2(\mathbb{R}, \mathcal{L}(\mathbb{R}^2, \mathbb{R}))} \left\| X \mapsto e^{-i\frac{X}{\varepsilon}} u \left( \frac{X}{\varepsilon} \right) \right\|_{L_{u,1}^2(\mathbb{R})},$$

with  $M = \pi_1^h$  and  $u = V_c(\frac{T_{\text{apr}}}{\varepsilon^2})$ . To estimate  $\hat{M}(\varepsilon\xi + 1)$ , we can restrict to the  $\mathcal{C}_b^0$  norm, since derivative gains us power of  $\varepsilon$ . Recall that

$$\hat{M}(\xi)\hat{u} = \chi_c^h(\xi) \langle \zeta_c(1), \zeta_c^*(\xi) \rangle^{-1} \langle \hat{u}, \zeta_c^*(\xi) \rangle$$

so that Cauchy-Schwartz leads to  $\|\hat{M}(\xi)\|_{\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)} \leq |\chi_c^h(\xi) \zeta_c^*(\xi) \langle \zeta_c(1), \zeta_c^*(\xi) \rangle^{-1}| \leq C|\chi_c^h(\xi)|$ . Here

we can assume that the support of  $\chi_c$  is so small that  $\langle \zeta_c(1), \zeta_c^*(\xi) \rangle$  does not vanish by continuity. We emphasize that this support is still independent of  $\mu$ , so that the separation of frequencies comes with spectral gap. Using the support of  $\chi_c^h$ , we obtain

$$\|\xi \mapsto \langle \xi \rangle^{1/2} \hat{M}(\varepsilon \xi + 1)\|_{C_b^0(\mathbb{R}, \mathcal{L}(\mathbb{R}^2, \mathbb{R}))} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Now rescaling the  $L_{u,1}^2(\mathbb{R})$  norm, we get

$$\|X \mapsto e^{-i\frac{X}{\varepsilon}} u(X/\varepsilon)\|_{L_{u,1}^2(\mathbb{R})} \leq \sqrt{\varepsilon} \|e^{-ix} u\|_{L_{u,1}^2(\mathbb{R})} \leq \sqrt{\varepsilon} \|u\|_{L_{u,1}^2(\mathbb{R})}.$$

Hence we have shown that  $\|A_0\|_{H_{u,1}^1(\mathbb{R})} \leq C\varepsilon^{-1} \|V_c(\varepsilon^{-2} T_{\text{apr}})\|_{L_{u,1}^2(\mathbb{R})} \leq C_1 r_0$ .  $\square$

From the previous proposition, we see that existence of  $V$  is uneasy to prove when

$$t \geq t_0 := \frac{T_{\text{att}}}{\varepsilon^2},$$

due to the growth of critical modes  $\|e^{tT^-} \Pi_c V_0\|_{H_{u,1}^1(\mathbb{R})} \geq C e^{\varepsilon^2 t} \|\Pi_c V_0\|_{H_{u,1}^1(\mathbb{R})}$ . For this times on,  $V$  does not correspond to a perturbation of the constant state  $(1, 0)$ . However, (3.38) suggests that its leading order terms  $V_c$  resembles after scaling to an oscillating profile, which is expected to be stable for the original equation (3.12). In the following, we linearize around a suitable oscillating profile, and show that the error is defined and stay small for times  $t \geq t_0$ .

As a second step, we decompose  $V$  as

$$V(t, x) = \psi(A)(t, x) + R(t, x),$$

where  $\psi(A)$  is the suitably chosen ansatz, that account for oscillating Turing pattern, expected to be stable, and where the error is  $R := V - \psi(A)$ . Namely,  $T, X \mapsto A(T, X)$  is the solution to the Ginzburg Landau equation (GL) with initial condition  $A(t_0) = A_0$  – see (3.38) – and

$$\psi(A) := \varepsilon \psi_c(\varepsilon, A) + \varepsilon^2 \psi_s(\varepsilon, A),$$

where the critical and stable part of  $\psi$  are defined, when  $t \geq t_0$ , by

$$\psi_c(\varepsilon, A)(t, x) = (e^{ix} A(\varepsilon^2 t, \varepsilon x) + c.c.) \zeta_c, \quad (3.41)$$

$$\psi_s(\varepsilon, A)(t, x) = |A(\varepsilon^2 t, \varepsilon x)|^2 \zeta_0 + e^{ix} \partial_X A(\varepsilon^2 t, \varepsilon x) \zeta_1 + e^{2ix} A(\varepsilon^2 t, \varepsilon x)^2 \zeta_2 + c.c.$$

All  $\zeta_i$  and  $\zeta_c$  are two-dimensional vectors with explicit expressions, see section 3.5. Recall that here and in the following, we denote  $c.c$  the complex conjugate:  $z + c.c := z + \bar{z} = 2 \operatorname{Re} z$ . The choice of  $\psi_s$  is made clearer in section 3.5, it ensures that the error of approximation  $R \stackrel{\text{def}}{=} V - \psi(\varepsilon, A)$  stays small when time evolves.

The benefit of the above decomposition is that the oscillating part is defined of all times  $t \geq t_0$ , as shown in the next Lemma.

**Lemma 3.4.8** (Global attractor for Ginzburg-Landau). *There exists positive constants  $C$  and  $C_{GL}$  such that for all  $A_0 \in H_{u,1}^1(\mathbb{R})$ , the solution  $T, X \mapsto A(T, X)$  of the Ginzburg-Landau equation (GL) with initial condition  $A(T_0) = A_0$  satisfies for all  $T \geq T_0$ :*

$$\|A(T)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{GL} + C e^{-(T-T_0)} \|A_0\|_{H_{u,1}^1(\mathbb{R})}.$$



*Proof.* This result is exactly [Sch94c, Theorem 7], we give the general plan of proof, and refer to [CE90, Appendix A] for precise computations. Let  $A$  be the solution of the Ginzburg Landau equation (GL) with initial condition  $A_0$ . In the following we fix  $T_0 \stackrel{\text{def}}{=} 0$ . For each fixed  $y \in \mathbb{R}$ , let

$$I_y(T) := \langle A, A \rangle_{L^2(\rho_{u,1}(\cdot - y))} := \int_{\mathbb{R}} \rho_{u,1}(X - y) A(X) \overline{A(X)} dX.$$

Then from the equation satisfied by  $A$ ,

$$\partial_t I_y = 2 \operatorname{Re} \left\langle A, 4\partial_{XX}A + A - b_{GL}A|A|^2 \right\rangle_{L^2(\rho_{u,1}(\cdot - y))}$$

Since  $|A|^4 \geq m|A|^2$  when  $|A|^2 \geq m$ , it follow from the computations made in [CE90, Proposition 4.3] that there exists a constant  $C_{GL}$  that only depends on the coefficients of (GL) such that

$$\partial_t I_y \leq C_{GL} - I_y.$$

A standard Gronwall Lemma ensures that  $I_y(T) \leq C_{GL} + e^{-T}I_y(0)$ . Taking the supremum over all  $y \in \mathbb{R}$  leads to  $\|A(T)\|_{L^2_{u,1}(\mathbb{R})} \leq C_{GL} + e^{-T}\|A_0\|_{L^2_{u,1}(\mathbb{R})}$ . Turning to the  $H^1_{u,1}(\mathbb{R})$  bound, we see that the equation satisfied by  $\partial_X A$  writes

$$\partial_T \partial_X A = 4\partial_{XX} \partial_X A + \partial_X A - b_{GL}(|A|^2 \partial_X A + 2A \operatorname{Re}(\bar{A} \partial_X A)). \quad (3.42)$$

In particular, it does not contain a term of the form  $-\partial_X A |\partial_X A|^p$  for some  $p \geq 1$ , so that the above procedure do not directly adapt.

However for any constant  $m \in \mathbb{R}$ , and thanks to [CE90, Lemma 4.1], there exists a positive constant  $C$ , such that

$$\langle \partial_X A, (\partial_{XX} + 1 + m)\partial_X A \rangle_{L^2_{u,1}(\rho(\cdot - y))} \leq C \langle A, A \rangle_{L^2(\rho(\cdot - y))}.$$

Hence, with  $J_y(T) := \langle \partial_X A, \partial_X A \rangle_{L^2(\rho_{u,1}(\cdot - y))}$ , and due to the third term in the right hand side of (3.42) being non-negative,

$$\partial_T J_y(T) \leq C \|A\|_{L^2(\rho(\cdot - y))} - m J_y.$$

Specifying to  $m = 1$ , a Gronwall Lemma ensures the existence of a positive constant  $C_{GL}$  such that  $J_y(t) \leq C_{GL} + C e^{-T}(I_y(T) + J_y(t))$ . Taking the supremum over all  $y \in \mathbb{R}$  leads to the claim bound.  $\square$

We now conclude the second step by showing that the error  $R$  stay small for a large enough period of time, which is to say that  $\psi(A)$  approximate  $V$  accurately.

**Lemma 3.4.9** (Approximation). *Let  $T_{appr}$  be a positive constant, and assume that  $\varepsilon^{-2}T_{att} =: t_0 \leq t_U$ . Let  $A_0 \in H^1_{u,1}(\mathbb{R})$  and let  $V$  be a solution of (3.12) defined for times  $0 < t \leq t_0$ , such that*

$$\|R(t_0)\|_{H^1_{u,1}(\mathbb{R})} = \|V(t_0) - \psi(\varepsilon, A_0)\|_{H^1_{u,1}(\mathbb{R})} \leq C \varepsilon^{5/4}.$$

*Denote  $A$  the solution of (GL) with initial condition  $A_0$  at  $t = t_0$ , and  $R(t) \stackrel{\text{def}}{=} V(t) - \psi(\varepsilon, A(t))$  as long as it is defined. Then there exists a positive constant  $C_{appr}$  that depends on  $C$ , such that for all times  $t \in [t_0, t_U]$ , the error decomposes as  $R(t) = R_c(t) + R_s(t)$ , with  $\Pi_i^h R_i(t) = R_i(t)$ . If furthermore,*

$t_0 \leq t \leq \max(t_0 + \varepsilon^{-2/3}T_{\text{appr}}, t_U)$ , then

$$\|R_i(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{appr}}\varepsilon^{5/4}, \quad i \in \{c, s\},$$

while if  $t_0 + \frac{T_{\text{appr}}}{\varepsilon^{2/3}} \leq t \leq t_0 + \frac{T_{\text{appr}}}{\varepsilon^2} \leq t_U$ , then

$$\|R_c(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{appr}}\varepsilon^{5/4}, \quad \|R_s(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{appr}}\varepsilon^{9/4}.$$

As a consequence,  $V$  is defined for all times  $0 < t \leq \max(t_0 + \varepsilon^{-2}T_{\text{appr}}, t_U)$ .

*Proof.* Insert  $R(t) = V(t) - \psi(\varepsilon, A(t))$  in (3.12) to obtain that for  $t \geq t_0$ ,  $R$  satisfies

$$\partial_t R = T^- R + \mathcal{Q}^-(R) + \mathcal{S}(t) + \text{Res}(\psi) + F(\psi, R),$$

where the residual term is

$$\text{Res}(\psi) \stackrel{\text{def}}{=} -\partial_t \psi + T^- \psi + \mathcal{Q}^-(\psi),$$

and where  $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$F(V_1, V_2) := \mathcal{Q}^-(V_1 + V_2) - \mathcal{Q}^-(V_1) - \mathcal{Q}^-(V_2).$$

We decompose Duhamel formula into a critical and stable part:

$$\Pi_i R(t) = e^{(t-t_0)T^-} \Pi_i R(t_0) + \int_{t_0}^t e^{(t-\tau)T^-} \Pi_i (F(\psi, R) + \mathcal{Q}^-(R) + \mathcal{S} + \text{Res}(\psi)) d\tau,$$

with  $i \in \{c, s\}$ . To close a nonlinear argument, we follow [Sch94c]. Since  $A$  satisfies (GL), the leading order terms in the residual  $\text{Res}(\psi(\varepsilon, A))$  vanish. The bounds from lemma 3.5.1 below ensures that for times  $t_0 \leq t \leq t_0 + \varepsilon^{-2}T_{\text{appr}}$ ,

$$\left\| \int_{t_0}^t e^{(t-\tau)T^-} \Pi_s \text{Res}(\psi)(\tau) d\tau \right\|_{H_{u,1}^1(\mathbb{R})} \leq C\varepsilon^{9/4},$$

$$\left\| \int_{t_0}^t e^{(t-\tau)T^-} \Pi_c \text{Res}(\psi)(\tau) d\tau \right\|_{H_{u,1}^1(\mathbb{R})} \leq C(t-t_0)\varepsilon^{7/2} \leq C\varepsilon^{5/4}.$$

There is no source term in the above reference, we group it together with residual terms. Using the decay of  $\mathcal{S}$  from Proposition 3.4.4, we conclude that  $\|\mathcal{S}(t, \cdot)\|_{H_{u,1}^1(\mathbb{R})} \leq C\|U_0\|_{X_1} \varepsilon^3 \leq Cr_0 \varepsilon^4$  when

$\frac{T_{\text{att}}}{\varepsilon^2} \leq t \leq t_U$ . This implies that

$$\left\| \int_{t_0}^t e^{(t-\tau)T^-} \Pi_s \mathcal{S}(\tau) d\tau \right\|_{H_{u,1}^1(\mathbb{R})} \leq Cr_0 \varepsilon^4 \int_{t_0}^t e^{-\kappa(t-\tau)} d\tau \leq C\varepsilon^4,$$

$$\left\| \int_{t_0}^t e^{(t-\tau)T^-} \Pi_c \mathcal{S}(\tau) d\tau \right\|_{H_{u,1}^1(\mathbb{R})} \leq C(t-t_0)r_0 \varepsilon^4.$$

Hence as long as  $t_0 \leq t \leq t_0 + \varepsilon^{-2}T_{\text{appr}}$ , the estimates on  $\mathcal{S}$  and  $\text{Res}(\psi)$  have same powers of  $\varepsilon$ . Since these two terms are independent of  $V$ , they play the same role in the nonlinear argument and can be grouped together. Thus the proof of [Sch94c, Lemma 11] adapts in the present case,

through a similar plan as for lemma 3.4.7.  $\square$

We now turn to the proof of proposition 3.2.4: we show that  $V$  remains bounded in time.

*Proof. Proposition 3.2.4.* Recall that we restrict to times such that  $t \leq t_U$ , so that estimate (3.36) holds through all the proof.

We show that there exists positive constants  $r_0, \varepsilon_0$  such that for all initial condition  $\|V_0\|_{H_{u,1}^1(\mathbb{R})} \leq r_0 \varepsilon$ , the solution  $V(t)$  is defined up to time  $t_1 := \varepsilon^{-2}(T_{\text{apr}} + T_{\text{att}})$  and satisfies  $\|V(t_1)\|_{H_{u,1}^1(\mathbb{R})} \leq r_0 \varepsilon$ . Thus, this procedure can be applied as many time as needed to define  $V(t)$  for all times  $t \in [0, t_U]$ .

Let  $r_0$  and  $\varepsilon_0$  be positive constants to be fixed later, and let  $V_0$  such that  $\|V_0\|_{H_{u,1}^1(\mathbb{R})} \leq r_0 \varepsilon$ . Waiting a time  $t_0 := \varepsilon^{-2}T_{\text{att}}$ , see Lemma 3.4.7,  $V$  now decomposes as  $V = V_c + V_s$ , with the leading order  $V_c$  writing as:

$$\hat{V}_c(t_0, \xi) = \mathcal{F}(x \mapsto \varepsilon e^{ix} A_0(\varepsilon x))(\xi) \zeta_c(\xi) + c.c.,$$

i.e.  $V_c(t_0)$  is obtained as the modulation of a large profile  $x \mapsto A_0(\varepsilon x)$ , with  $\|A_0\|_{H_{u,1}^1(\mathbb{R})} \leq Cr_0$  from Lemma 3.4.7. We propagate this ansatz by decomposing  $V(t, x) = \psi(\varepsilon, A)(t, x) + R(t, x)$ , see (3.41). Applying Lemma 3.4.8, we conclude that  $\|A(\varepsilon^2 t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{GL} + Cr_0 e^{-\varepsilon^2(t-t_0)}$  for all times  $t \geq t_0$ . However,  $A$  and  $\psi(\varepsilon, A)$  does not depend on the same space variable, and the scaling  $\varepsilon x \rightarrow x$  in  $L_{u,1}^2(\mathbb{R})$ -norms makes a  $\varepsilon^{-1/2}$  appear. To handle this, we use the injections from lemma 3.4.1, and then scale the space variable in  $L^\infty(\mathbb{R})$ -norms. With  $t \geq t_0$ , it reads:

$$\begin{aligned} \|\psi_c(\varepsilon, A)(t)\|_{H_{u,1}^1(\mathbb{R})} &\leq C \|x \mapsto A(\varepsilon^2 t, \varepsilon x)\|_{H_{u,1}^1(\mathbb{R})}, \\ &\leq C \|x \mapsto A(\varepsilon^2 t, \varepsilon x)\|_{L_{u,1}^2(\mathbb{R})} + C\varepsilon \|x \mapsto \partial_X A(\varepsilon^2 t, \varepsilon x)\|_{L_{u,1}^2(\mathbb{R})}, \\ &\leq C \|x \mapsto A(\varepsilon^2 t, \varepsilon x)\|_{L^\infty(\mathbb{R})} + C\varepsilon \|x \mapsto \partial_X A(\varepsilon^2 t, \varepsilon x)\|_{L_{u,1}^2(\mathbb{R})}, \\ &\leq C \|A(\varepsilon^2 t)\|_{L^\infty(\mathbb{R})} + C\sqrt{\varepsilon} \|\partial_X A(\varepsilon^2 t)\|_{L_{u,1}^2(\mathbb{R})}, \\ &\leq C \|A(\varepsilon^2 t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{GL} + Cr_0 e^{-\varepsilon^2(t-t_0)}. \end{aligned}$$

Similar estimates lead to  $\|\psi_s(\varepsilon, A)(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{GL} + Cr_0 e^{-\varepsilon^2 t}$ . Now for  $t \geq t_0$  we can estimate

$$\begin{aligned} \|V(t)\|_{H_{u,1}^1(\mathbb{R})} &\leq \|\psi(\varepsilon, A)(t)\|_{H_{u,1}^1(\mathbb{R})} + \|V(t) - \psi(\varepsilon, A)(t)\|_{H_{u,1}^1(\mathbb{R})}, \\ &\leq \varepsilon (C_{GL} + Cr_0 e^{-\varepsilon^2(t-t_0)}) + \|R(t)\|_{H_{u,1}^1(\mathbb{R})}, \end{aligned} \quad (3.43)$$

and we now turn toward the bound on the error  $R(t)$ . We initially decompose it as the sum of  $R_c(t_0) \stackrel{\text{def}}{=} V_c(t_0) - \varepsilon \psi_c(A_0)$  and  $R_s(t_0) \stackrel{\text{def}}{=} V_s(t_0) - \varepsilon^2 \psi_s(A_0)$ .

Since  $R_c(t_0)$  has compact support in Fourier space, we can gain one derivative, and treat separately the behavior close to 1 and  $-1$ :

$$\|R_c(t_0)\|_{H_{u,1}^1(\mathbb{R})} \leq C \|\pi_1^h R_c(t_0)\|_{L_{u,1}^2(\mathbb{R})} + C \|\pi_{-1}^h R_c(t_0)\|_{L_{u,1}^2(\mathbb{R})}.$$

From  $\pi_{-1}^h R_c(t_0) = \pi_1^h \overline{R_c(t_0)}$ , it is enough to bound only one of the two terms. Recall that from the definition (3.38) of  $A_0$ ,

$$\pi_1^h V_c(t_0, x) = \varepsilon e^{ix} A_0(\varepsilon x),$$

while definitions of  $\psi_c$  (3.41) and  $\pi_1^h$  (3.35) ensure that

$$\pi_1^h(\varepsilon\psi_c(A_0))(x) = \mathcal{F}^{-1}\left(\chi_c^h \mathbf{1}_{\xi>0} \mathcal{F}(\varepsilon e^{iy} A_0(\varepsilon y))\right) = \varepsilon e^{ix} \mathcal{F}^{-1}(\chi_0 \mathcal{F}(y \mapsto A_0(\varepsilon y))).$$

where  $\chi_0(\xi) := \chi_c^h(1 + \xi) \mathbf{1}_{1+\xi>0}$  is supported around  $\xi = 0$ . Hence

$$\begin{aligned} \pi_1^h R_c(t_0, x) &= \pi_1^h(V_c(t_0) - \varepsilon\psi_c(A_0))(x) = \varepsilon e^{ix} \mathcal{F}^{-1}((1 - \chi_0) \mathcal{F}(A_0(\varepsilon \cdot))), \\ &= \varepsilon e^{ix} (MA_0)(\varepsilon x), \end{aligned}$$

where the multiplier  $M$  is defined by  $\hat{M}(\xi) = 1 - \chi_0(\varepsilon\xi)$ . We first rescale space, and apply lemma 3.4.2, to obtain that

$$\begin{aligned} \|\pi_1^h R_c(t_0)\|_{L_{u,1}^2(\mathbb{R})} &\leq \frac{1}{\sqrt{\varepsilon}} \left\| x \mapsto \varepsilon e^{i\frac{x}{\varepsilon}} (MA_0)(x) \right\|_{L_{u,1}^2(\mathbb{R})}, \\ &\leq \sqrt{\varepsilon} \left\| \frac{1 - \chi_0(\varepsilon\xi)}{\langle \xi \rangle} \right\|_{C_b^2(\mathbb{R})} \|A_0\|_{H_{u,1}^1(\mathbb{R})}. \end{aligned} \quad (3.44)$$

Due to  $\chi_0(\varepsilon\xi) = \chi_c^h(\varepsilon\xi + 1) \mathbf{1}_{\varepsilon\xi+1>0} = 1$  when  $\xi \in [\frac{-1}{4\varepsilon}, \frac{1}{4\varepsilon}]$ , we conclude that

$$\frac{1 - \chi_0(\varepsilon\xi)}{\langle \xi \rangle} \leq \langle \xi \rangle^{-1} \mathbf{1}_{|\xi| \geq \frac{1}{4\varepsilon}} \leq \langle 1/(4\varepsilon) \rangle^{-1} \leq 4\varepsilon.$$

The first two derivatives of the left hand side bound similarly, so that inserting into (3.44), and using the bound on  $A_0$  from Lemma 3.4.7 we obtain that  $\|\pi_1^h R_c(t_0)\|_{L_{u,1}^2(\mathbb{R})} \leq \varepsilon^{3/2} C_{\text{att}} r_0$ .

Furthermore from Lemma 3.4.7:

$$\begin{aligned} \|R_s(t_0)\|_{H_{u,1}^1(\mathbb{R})} &\leq \|V_s(t_0)\|_{H_{u,1}^1(\mathbb{R})} + \varepsilon^2 \|\psi_s(A_0)\|_{H_{u,1}^1(\mathbb{R})}, \\ &\leq C_{\text{att}} r_0 \varepsilon^2 + C_{\text{att}} \varepsilon^2 \|x \mapsto A_0(\varepsilon x)\|_{H_{u,1}^2(\mathbb{R})}. \end{aligned}$$

Using that  $x \mapsto A_0(\varepsilon x)$  has compact support in Fourier space, we can gain two derivatives, inject into  $L^\infty(\mathbb{R})$ , scale, and finally inject back into  $H_{u,1}^1(\mathbb{R})$ . It reads:  $\|x \mapsto A_0(\varepsilon x)\|_{H_{u,1}^2(\mathbb{R})} \leq C \|A_0\|_{H_{u,1}^1(\mathbb{R})}$ .

Altogether, we have shown that  $\|R(t_0)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{att}} r_0 \varepsilon^{5/4}$ , so that lemma 3.4.9 applies with a constant  $T_{\text{apr}}$  that we are about to choose. There exists a positive constant  $C_{\text{apr}}$  that depends on  $C_{\text{att}} r_0$  such that for all times  $t \in [t_0, t_1]$ , where  $t_1 := t_0 + \varepsilon^{-2} T_{\text{apr}}$ :

$$\|R(t)\|_{H_{u,1}^1(\mathbb{R})} \leq C_{\text{apr}} \varepsilon^{3/2}.$$

Injecting into (3.43) leads when  $t \in [t_0, t_1]$  to

$$\|V(t)\|_{H_{u,1}^1(\mathbb{R})} \leq \varepsilon \left( C_{GL} + C r_0 e^{-\varepsilon^2(t-t_0)} + C_{\text{apr}} \varepsilon^{1/4} \right),$$

where  $C$  does not depend on  $r_0$ . We now fix  $r_0 \geq 3C_{GL}$ , take  $T_{\text{apr}}$  so large that  $C r_0 e^{-T_{\text{apr}}} \leq C_{GL}$  and  $\varepsilon_0$  so small that  $C_{\text{apr}} \varepsilon^{1/4} \leq C_{GL}$  to obtain that

$$\|V(t_1)\|_{H_{u,1}^1(\mathbb{R})} \leq 3C_{GL} \varepsilon \leq r_0 \varepsilon.$$

We can apply the above procedure with successive initial condition  $V(t_k)$  where  $t_k := k\varepsilon^{-2}(T_{\text{att}} + T_{\text{apr}})$ , and conclude that the bound  $\|V(t_k)\|_{H_{u,1}^1(\mathbb{R})} \leq 3C_{GL}\sqrt{\mu}$  holds for all  $k \in \mathbb{N}_{\geq 1}$ . It is a by-product of the above procedure that for  $k \in \mathbb{N}_{\geq 1}$ , and all intermediate times  $t \in [t_k, t_{k+1}]$ ,

$$\|V(t)\|_{H_{u,1}^1(\mathbb{R})} \leq 2C_{GL} + C\|V(t_k)\|_{H_{u,1}^1(\mathbb{R})}.$$

In particular for all times  $t \geq t_1$ , the bound  $\|V(t)\|_{H_{u,1}^1(\mathbb{R})} \leq 3C_{GL}\sqrt{\mu}$  still holds.  $\square$

### 3.5 Ginzburg-Landau equation

Here we derive the Ginzburg-Landau amplitude equation from the dynamic at  $-\infty$ . We study the following system:

$$\partial_t V = \mathcal{T}_0 V + \mathcal{T}_\mu V + \mathcal{N}_2(V) + \mathcal{N}_3(V), \quad (3.45)$$

where the linear terms are given by:

$$\mathcal{T}_0 = \begin{pmatrix} d\partial_x^2 - 2\alpha & \beta \\ 0 & -(1 + \partial_x^2)^2 \end{pmatrix}, \quad \mathcal{T}_\mu = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix},$$

while the quadratic and cubic terms write as:

$$\mathcal{N}_2(V) = \begin{pmatrix} -3\alpha v_1^2 \\ \gamma v_1 v_2 \end{pmatrix} \quad \mathcal{N}_3(V) = -\begin{pmatrix} \alpha v_1^3 \\ \sigma v_2^3 \end{pmatrix}.$$

In the following, we note  $M(X)$  the symbol of the constant coefficient operator  $\mathcal{T}_0$ , so that  $\mathcal{T}_0 = M(\partial_x)$ . We also note  $\varepsilon = \sqrt{\mu}$ . Assume that the solution to such system writes according to the following ansatz:

$$\psi(t, x) \stackrel{\text{def}}{=} \varepsilon(e^{ix} A(\varepsilon^2 t, \varepsilon x) + c.c.)\zeta_c + \varepsilon^2(A_0\zeta_0 + e^{ix}A_1\zeta_1 + e^{2ix}A_2\zeta_2 + c.c.).$$

Here and in the following  $c.c.$  is the complex conjugate:  $z + c.c.$  stands for  $z + \bar{z} = 2\text{Re}z$ . The amplitude  $A \in \mathbb{C}$  is the main unknown, while each  $A_i(\varepsilon^2 t, \varepsilon x)$  is an amplitude we will fix later. The vector  $\zeta_c$  is chosen to be an eigenvector for matrix  $M(i)$ , associated to eigenvalue 0, while the vectors  $\zeta_i$  will be fixed later. We will further use a vector  $\zeta_c^*$  in the kernel of  $M(i)^*$ , normalized so that  $\langle \zeta_c, \zeta_c^* \rangle = 1$ . Finally, let  $\zeta_s$  and  $\zeta_s^*$  be two vectors such that for all  $V \in \mathbb{R}^2$ ,

$$V = \langle V, \zeta_c^* \rangle \zeta_c + \langle V, \zeta_s^* \rangle \zeta_s. \quad (3.46)$$

These vectors are uniquely determined by the conditions  $\langle \zeta_i, \zeta_j^* \rangle = \mathbf{1}_{i=j}$ , when  $i, j \in \{c, s\}$ . We will further need their expressions:

$$\zeta_c = \begin{pmatrix} \beta \\ d + 2\alpha \end{pmatrix}, \quad \zeta_c^* := \begin{pmatrix} 0 \\ \frac{1}{d+2\alpha} \end{pmatrix}, \quad \zeta_s := \begin{pmatrix} \frac{1}{d+2\alpha} \\ 0 \end{pmatrix}, \quad \zeta_s^* := \begin{pmatrix} d + 2\alpha \\ -\beta \end{pmatrix}.$$

**Lemma 3.5.1.** *Let  $(T, X) \mapsto A(T, X)$  be a solution to (GL), and let*

$$\text{Res}(\psi) := -\partial_t \psi + \mathcal{T}_0 \psi + \mathcal{T}_\mu \psi + \mathcal{N}_2(\psi) + \mathcal{N}_3(\psi).$$

Then there exists amplitudes  $A_i$  that depend on  $A$ , vectors  $\zeta_i$  and a constant  $C$  such that

$$\|II_s \text{Res}(\psi)\|_{H_{u,1}^1(\mathbb{R})} \leq C\varepsilon^2\sqrt{\varepsilon}, \quad \|II_c \text{Res}(\psi)\|_{H_{u,1}^1(\mathbb{R})} \leq C\varepsilon^3\sqrt{\varepsilon}.$$

*Proof.* We inject the ansatz in system (3.45), and then identify terms of order  $\varepsilon^l e^{ikx}$  for  $l \in \mathbb{N}_{\geq 1}$  and  $k \in \mathbb{Z}$ . As usual for such computation, the  $\varepsilon$  order vanishes. Then  $\varepsilon^2 e^{ikx}$  terms will determine amplitudes  $A_k$  and vectors  $\zeta_k$ . Finally, the  $\varepsilon^3 e^{ikx}$  equation will lead to Ginzburg-Landau equation. We may neglect all terms of order  $\varepsilon^3 e^{2ix}$  and  $\varepsilon^3 e^0$ , since they corresponds to stable frequencies, and will be damped anyway. We easily obtain:

$$\partial_t \psi = \varepsilon^3 (e^{ix} \partial_T A \zeta_c + c.c)$$

with slow time variable  $T = \varepsilon^2 t$ . Similarly, we note  $X = \varepsilon x$  the large space variable. To compute  $\mathcal{T}_0 \psi$ , we use a formal Taylor expansion, see Lemma B.1.1:

$$\begin{aligned} \mathcal{T}_0(\varepsilon e^{ix} A \zeta_c) &= \varepsilon e^{ix} M(i + \partial_x) A \zeta_c, \\ &= \varepsilon e^{ix} \left( A M(i) \zeta_c + \varepsilon \partial_X A M'(i) \zeta_c + \frac{1}{2} \varepsilon^2 \partial_{XX} A M''(i) \zeta_c \right) + \mathcal{O}(\varepsilon^4). \end{aligned}$$

The computation can be easily adapted for  $\mathcal{T}_0(\varepsilon^2 e^{2ikx} A_k \zeta_k)$ , for  $k \in \{0, 1, 2\}$ . The  $\mathcal{T}_\mu(\psi)$  term is easily developed, and we are left with the nonlinear terms. We note  $B : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a symmetric bilinear application that satisfies  $B(V, V) = \mathcal{N}_2(V)$ . It is given by:

$$B(V, W) = \begin{pmatrix} -3\alpha v_1 w_1 \\ \frac{\gamma}{2}(v_1 w_2 + v_2 w_1) \end{pmatrix}.$$

Hence, we obtain:

$$\begin{aligned} \mathcal{N}_2(\psi) &= \varepsilon^2 \left( e^{2ix} A^2 \mathcal{N}_2(\zeta_c) + c.c + 2\varepsilon^2 |A|^2 \mathcal{N}_2(\zeta_c) \right) + \varepsilon^3 (e^{ix} 2AA_0 B(\zeta_c, \zeta_0) + c.c) \\ &\quad + \varepsilon^3 (e^{ix} 2\bar{A}A_2 B(\zeta_c, \zeta_2) + c.c) + \mathcal{O}_{|k| \neq 1}(\varepsilon^3 e^{ikx} + \varepsilon^4). \end{aligned}$$

Similarly, the cubic term develop as:

$$\mathcal{N}_3(\psi) = \varepsilon^3 \left( 3e^{ix} A |A|^2 \mathcal{N}_3(\zeta_c) + c.c \right) + \mathcal{O}_{|k| \neq 1}(\varepsilon^3 e^{ikx} + \varepsilon^4).$$

Collecting all the above calculus, we see that

$$\text{Res}(\psi) = \varepsilon^2 \left( E_0 + (e^{ix} E_1 + c.c) + (e^{2ix} E_2 + c.c) \right) + \varepsilon^3 \left( e^{ix} E_3 + c.c \right) + \mathcal{O}_{|k| \neq 1}(\varepsilon^3 e^{ikx} + \varepsilon^4),$$

where

$$\begin{aligned} E_0 &:= A_0 M(0) \zeta_0 + 2|A|^2 \mathcal{N}_2(\zeta_c), \\ E_1 &:= \partial_X A M'(i) \zeta_c + A_1 M(i) \zeta_1, & E_2 &:= A_2 M(2i) \zeta_2 + A^2 \mathcal{N}_2(\zeta_c). \end{aligned}$$

Due to both matrix  $M(0)$  and  $M(2i)$  being invertible, we can impose:

$$\begin{aligned} A_1 &\stackrel{\text{def}}{=} \partial_X A, & A_0 &\stackrel{\text{def}}{=} |A|^2, & A_2 &\stackrel{\text{def}}{=} A^2, \\ \zeta_0 &\stackrel{\text{def}}{=} -2M(0)^{-1} \mathcal{N}_2(\zeta_c), & \zeta_2 &\stackrel{\text{def}}{=} -M(2i)^{-1} \mathcal{N}_2(\zeta_c). \end{aligned}$$

We also fix  $\zeta_1$  such that  $M'(i) \zeta_c + M(i) \zeta_1 = 0$ , which is possible due to  $M'(i) \zeta_c \in \text{im}(M(i))$ . With

these choices, we have  $E_0 = E_1 = E_2 = 0$ , and the expression of  $E_3$  is given by

$$E_3 := -\partial_T A \zeta_c + \partial_{XX} A (M''(i)\zeta_c + M'(i)\zeta_1) + A \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \zeta_c \\ + A|A|^2 (2B(\zeta_c, \zeta_0) + 2B(\zeta_c, \zeta_2) + 3\mathcal{N}_3(\zeta_c)).$$

We claim that the condition  $\langle E_3, \zeta_c^* \rangle = 0$  corresponds to  $A$  being a solution to the scalar Ginzburg-Landau equation (GL). It is easily seen that

$$\langle M''(i)\zeta_c + M'(i)\zeta_1, \zeta_c^* \rangle = 4, \quad \left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \zeta_c, \zeta_c^* \right\rangle = 1,$$

we delay the explicit computation of the coefficient  $b_{GL}$  in front of cubic term, remark that it is a real number.

To obtain precise bounds on  $\Pi_i \text{Res}(\psi)$ , we refer to [Sch94c, Lemma 14], and give here a general idea of the proof. Remark that for all  $\xi \in \mathbb{R}$ ,  $\langle \zeta_s, \zeta_c^*(0) \rangle = 0$ , so that  $\Pi_c(\zeta_s) = 0$ . In general, from the decomposition (3.46), we see that for any  $V \in \mathbb{R}^2$ ,

$$\Pi_c V = \Pi_c (\langle V, \zeta_c^* \rangle \zeta_c).$$

In particular,  $E_3$  does not contribute to  $\Pi_c \psi$ . Due to  $\Pi_c$  being localized near  $\xi = \pm 1$  in Fourier space, the  $\varepsilon^3 e^{ikx}$  with  $|k| \neq 1$  does not contribute either to  $\Pi_c \psi$ . We conclude that leading order terms in the expression of  $\Pi_c \psi$  are of the form  $\varepsilon^4 e^{ix} \tilde{A}(\varepsilon^2 t, \varepsilon x) \zeta_c$ . Hence

$$\|\Pi_c \text{Res}(\psi)(t)\|_{H_{u,1}^1(\mathbb{R})} \leq \varepsilon^4 \|x \mapsto \tilde{A}(\varepsilon^2 t, \varepsilon x)\|_{H_{u,1}^1(\mathbb{R})} \leq \varepsilon^3 \sqrt{\varepsilon} \|\tilde{A}(\varepsilon^2 t)\|_{H_{u,1}^1(\mathbb{R})} \leq C\varepsilon^3 \sqrt{\varepsilon}.$$

In the case of  $\Pi_s \text{Res}(\psi)$ , leading order terms are of order  $\varepsilon^3$ , so that scaling the space variable lead to the claimed bound.  $\square$

We now explicitly compute the coefficient  $b_{GL}$  in front of cubic terms. Elementary computations lead to

$$M(0)^{-1} = \frac{-1}{2\alpha} \begin{pmatrix} 1 & \beta \\ 0 & 2\alpha \end{pmatrix} \quad M(2i)^{-1} = \frac{-1}{9(4d+2\alpha)} \begin{pmatrix} 9 & \beta \\ 0 & 4d+2\alpha \end{pmatrix} \quad \mathcal{N}_2(\zeta_c) = \begin{pmatrix} -3\alpha\beta^2 \\ \gamma\beta(d+2\alpha) \end{pmatrix},$$

from which we deduce that:

$$\zeta_0 = \begin{pmatrix} \beta^2(\gamma \frac{d+2\alpha}{\alpha} - 3) \\ 2\gamma\beta(d+2\alpha) \end{pmatrix}, \quad \zeta_2 = \frac{1}{9(4d+2\alpha)} \begin{pmatrix} \beta^2(\gamma(d+2\alpha) - 27\alpha) \\ \gamma\beta(d+2\alpha)(4d+2\alpha) \end{pmatrix}.$$

Then, it follows

$$\mathcal{N}_3(\zeta_c) = -\begin{pmatrix} \alpha\beta^3 \\ \sigma(d+2\alpha)^3 \end{pmatrix}, \quad B(\zeta_c, \zeta_0) = \begin{pmatrix} -3\alpha\beta^3(\gamma \frac{d+2\alpha}{\alpha} - 3) \\ \beta^2 \frac{\gamma}{2}(d+2\alpha)(2\gamma + \gamma \frac{d+2\alpha}{\alpha} - 3) \end{pmatrix}, \\ B(\zeta_c, \zeta_2) = \frac{1}{9(4d+2\alpha)} \begin{pmatrix} -3\alpha\beta^3(\gamma(d+2\alpha) - 27\alpha) \\ \beta^2 \frac{\gamma}{2}(d+2\alpha)(\gamma(5d+4\alpha) - 27\alpha) \end{pmatrix}.$$

We can finally compute that:

$$\begin{aligned} b_{GL} &:= \langle 2B(\zeta_c, \zeta_0) + 2B(\zeta_c, \zeta_2) + 3\mathcal{N}_3(\zeta_c, \zeta_c^*), \zeta_c^* \rangle, \\ &= \gamma^2 \beta^2 \left( \frac{19}{9} + (d+2\alpha) \left( \frac{1}{\alpha} + \frac{1}{9(4d+2\alpha)} \right) \right) \\ &\quad - 3\gamma \beta^2 \left( 1 + \frac{\alpha}{4d+2\alpha} \right) - 3\sigma(d+2\alpha)^2. \end{aligned} \quad (3.47)$$

We see that  $b_{GL}$  is a degree 2 polynomial in  $\gamma$  which we note  $P(\gamma)$ . Altogether, we obtain the following Ginzburg-Landau equation:

$$\partial_T A = 4\partial_{XX} A + A + P(\gamma)A|A|^2. \quad (\text{GL})$$

Since  $P$  admits two roots with distinct sign, there exists  $\gamma_{GL} > 0$  such that for all  $\gamma \in (0, \gamma_{GL})$ , we have  $P(\gamma) < 0$ , i.e. hypothesis  $(H_2)$  is fulfilled. Recall that  $\alpha, d, \sigma > 0$ . We compute:

$$\gamma_{GL} = \frac{3(4d+3\alpha)}{2a(4d+2\alpha)} + \sqrt{\left( \frac{3(4d+3\alpha)}{2a(4d+2\alpha)} \right)^2 + 3\sigma \frac{(d+2\alpha)^2}{a\beta^2}}, \quad a \stackrel{\text{def}}{=} \frac{19}{9} + (d+2\alpha) \left( \frac{1}{\alpha} + \frac{1}{9(4d+2\alpha)} \right).$$

**Lemma 3.5.2.** *There exists an open, nonempty set of parameters  $\Omega$  such that for all  $(\alpha, \beta, d, \sigma) \in \Omega$ , the ordering  $\gamma_{\text{rem}} < \gamma_{GL}$  holds. In particular for  $\gamma \in (\gamma_{\text{rem}}, \gamma_{GL})$ , both hypothesis  $(H_1)$  and  $(H_2)$  holds true.*

*Proof.* The fact that  $\Omega$  is open comes from the continuity of both  $\gamma_{\text{rem}}$  and  $\gamma_{GL}$  with respect to  $\alpha, \beta, d, \sigma > 0$ . To see that  $\Omega \neq \emptyset$ , remark that  $\gamma_{GL} \rightarrow +\infty$  when either  $\beta \rightarrow 0$  or  $\sigma \rightarrow +\infty$ .  $\square$

*Remark 3.5.3.* The ansatz we propose here only develop up to order  $\varepsilon^2$ , while the information we extract is held by  $\varepsilon^3$  terms. If ones try to push the ansatz one order further:

$$\begin{aligned} V(t, x) &= \varepsilon(e^{ix} A \zeta_c + c.c) + \varepsilon^2(e^{ix} A_{1,1} \zeta_{1,1} + e^{2ix} A_{1,2} \zeta_{1,2} + c.c + A_{1,0} \zeta_{1,0}) \\ &\quad + \varepsilon^3(e^{ix} A_{2,1} \zeta_{2,1} + e^{2ix} A_{2,2} \zeta_{2,2} + e^{3ix} A_{2,3} \zeta_{2,3} + c.c + A_{2,0} \zeta_{2,0}), \end{aligned}$$

the only changes is the presence of an extra  $A_{2,1} \langle L(i) \zeta_{2,1}, \zeta_c^* \rangle$  term in (GL). By definition of  $\zeta_c^*$ , this term vanishes anyway.

## 3.6 Proof of Proposition 3.3.6

*Proof. Proposition 3.3.6.* Here, we write  $a(x, \lambda) \lesssim b(x, \lambda)$  to stand for  $a(x, \lambda) \leq Cb(x, \lambda)$  where  $C$  is a constant. Recall we have construct ODE solutions with exponential behavior. For  $0 \leq j \leq 3$ , we have

$$\partial_x^j \phi(\lambda, x) = e^{v x} \left( v^j + \mathcal{O}_{\mathbb{R}^\pm \times K}(e^{-\alpha|x|}) \right), \quad (3.48)$$

where  $\phi$  stands for  $\phi_i^{\text{sh}, \pm}$  and  $v$  for  $v_i^{\text{sh}, \pm}(\lambda)$ . Here, we noted  $g(\lambda, x) = \mathcal{O}_{\mathbb{R}^+}(f(x))$  if  $|g(\lambda, x)| \lesssim f(x)$  when  $x \geq 0$ , and if  $\lambda \mapsto g(\lambda, x)$  is holomorphic on  $K$  for almost all  $x \geq 0$ . The according notation holds for  $\mathcal{O}_{\mathbb{R}^-}$ . In particular,  $|\phi_i^{\text{sh}, \pm}(x)| \lesssim e^{x \text{Re}(v_i^\pm)}$  when  $x \in \mathbb{R}_\pm$ . In the following, we drop the ‘‘sh’’



exponent. Then, the Green function expresses using the decaying solutions:

$$G_\lambda^{\text{sh}}(x, y) = \begin{cases} \sum_{i=1}^2 b_i(\lambda, y) \phi_i^+(\lambda, x) & \text{if } y < x, \\ \sum_{i=3}^4 b_i(\lambda, y) \phi_i^-(\lambda, x) & \text{if } x < y, \end{cases}$$

where the coefficients  $b_i(\lambda, y)$  are determined by the jump condition:

$$\begin{pmatrix} \phi_1^+ & \phi_2^+ & \phi_3^- & \phi_4^- \\ \partial_x \phi_1^+ & \partial_x \phi_2^+ & \partial_x \phi_3^- & \partial_x \phi_4^- \\ \partial_{xx} \phi_1^+ & \partial_{xx} \phi_2^+ & \partial_{xx} \phi_3^- & \partial_{xx} \phi_4^- \\ \partial_{xxx} \phi_1^+ & \partial_{xxx} \phi_2^+ & \partial_{xxx} \phi_3^- & \partial_{xxx} \phi_4^- \end{pmatrix}(\lambda, y) \cdot \begin{pmatrix} b_1 \\ b_2 \\ -b_3 \\ -b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Recall notation (3.22) for determinants. We use Cramer's rule to compute the coefficients  $b_i(\lambda, y)$ :

$$b_1 = \frac{\text{Det}(\phi_2^+, \phi_3^-, \phi_4^-)}{\mathcal{W}^{\text{sh}}}, \quad b_2 = -\frac{\text{Det}(\phi_1^+, \phi_3^-, \phi_4^-)}{\mathcal{W}^{\text{sh}}}, \quad (3.49)$$

and

$$b_3 = \frac{\text{Det}(\phi_1^+, \phi_2^+, \phi_4^-)}{\mathcal{W}^{\text{sh}}}, \quad b_4 = -\frac{\text{Det}(\phi_1^+, \phi_2^+, \phi_3^-)}{\mathcal{W}^{\text{sh}}}.$$

It may happens that  $\nu_2^+ - \nu_1^+$  or  $\nu_4^- - \nu_3^-$  vanish, causing a singularity in  $(b_1, b_2)$  or  $(b_3, b_4)$ . However, this singularity can be erased in the expression of  $G_\lambda^{\text{sh}}$  since at first order  $b_1(\lambda, y) + b_2(\lambda, y) = \frac{1}{\mathcal{W}^{\text{sh}}} \text{Det}(\phi_2 - \phi_1, \phi_3, \phi_4) = \mathcal{O}_\lambda(1)$ . Recall that we assume the  $\nu_i^\pm$  are sorted by real part:  $\text{Re } \nu_1^+ \leq \text{Re } \nu_2^+ \leq \text{Re } \nu_3^+ \leq \text{Re } \nu_4^+$  (the same ordering holds for the  $\nu_i^-$ ), and that from Lemma 3.3.5 – (1), there exists  $\kappa_2 > 0$  such that:

$$\text{Re } \nu_{1,2}^+ \leq -\kappa_2, \quad \kappa_2 \leq \text{Re } \nu_{3,4}^+ \quad (3.50)$$

We now prove the claimed result, depending on the ordering between  $x, y$  and 0.

(i)  $y < 0 < x$ . We need to decompose  $\phi_2^+(y)$  into the  $(\phi_i^-)_{1 \leq i \leq 4}$  basis:  $\phi_2^+ = \sum_{i=1}^4 c_i \phi_i^-$ . We can differentiate three times this decomposition to obtain a Cramer system. Solving it leads to

$$\begin{aligned} \phi_2^+(y) &= \frac{\text{Det}(\phi_2^+, \phi_2^-, \phi_3^-, \phi_4^-)}{\text{Det}(\phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-)} \phi_1^-(y) + \frac{\text{Det}(\phi_1^-, \phi_2^+, \phi_3^-, \phi_4^-)}{\text{Det}(\phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-)} \phi_2^-(y) \\ &\quad + \frac{\text{Det}(\phi_1^-, \phi_2^-, \phi_2^+, \phi_4^-)}{\text{Det}(\phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-)} \phi_3^-(y) + \frac{\text{Det}(\phi_1^-, \phi_2^-, \phi_3^-, \phi_2^+)}{\text{Det}(\phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-)} \phi_4^-(y). \end{aligned}$$

Once again, the  $y$  dependence in each of the fraction can be dropped thanks to Lemma B.1.2. The numerator is holomorphic with respect to  $\lambda$ , hence bounded when  $\lambda \in K$ , and the denominator does not vanish.<sup>6</sup> In the following, we may simply write

$$c_1^-(\varphi) \stackrel{\text{def}}{=} \frac{\text{Det}(\varphi, \phi_2^-, \phi_3^-, \phi_4^-)}{\text{Det}(\phi_1^-, \phi_2^-, \phi_3^-, \phi_4^-)}$$

to note the first coefficient of  $\varphi$  when decomposed in the  $(\phi_i^-)_{1 \leq i \leq 4}$  basis. We extend this notation to other coefficients:  $c_i^-(\varphi)$ , and to the decomposition into  $(\phi_i^+)_{1 \leq i \leq 4}$  basis:  $c_i^+(\varphi)$ .

6. A vanishing determinant would implies that  $\lambda$  is an eigenvalue, which can not hold due to  $\text{Re } \lambda \geq -2\eta$ .

Using the above decomposition into the expression of  $G_\lambda^{\text{sh}}$ , we obtain:

$$\begin{aligned} \left| \text{Det}(\phi_2^+, \phi_3^-, \phi_4^-)(y) \right| &= \left| \sum_{i=1}^2 c_i^-(\phi_2^+) \text{Det}(\phi_i^-, \phi_3^-, \phi_4^-)(y) \right|, \\ &\lesssim e^{\text{Re}(\nu_1^- + \nu_3^- + \nu_4^-)y} + e^{\text{Re}(\nu_2^- + \nu_3^- + \nu_4^-)y}, \end{aligned} \quad (3.51)$$

since  $\lambda \mapsto \text{Det}(\phi_i^-, \phi_3^-, \phi_4^-)(0)$  is holomorphic on  $K$ . Using again Lemma B.1.2, we see that

$$\mathcal{W}^{\text{sh}}(\lambda, y) = e^{\text{Re}(\nu_1^- + \nu_2^- + \nu_3^- + \nu_4^-)y} \mathcal{W}^{\text{sh}}(\lambda, 0).$$

Using Proposition 3.3.3, we see that  $\mathcal{W}^{\text{sh}}(\lambda, 0) \gtrsim 1$  when  $\lambda$  lies in  $K$ . Collecting the above estimates (3.51), (3.48) and (3.50), we now see that

$$\left| \phi_1^+(x) \frac{\text{Det}(\phi_2^+, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \right| \lesssim e^{x \text{Re}(\nu_1^+) - y \text{Re}(\nu_2^-)} + e^{x \text{Re}(\nu_1^+) - y \text{Re}(\nu_1^-)} \lesssim e^{-\kappa_2(x-y)}.$$

The exact same approach for the second term leads to:

$$\left| \phi_2^+(x) \frac{\text{Det}(\phi_1^+, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \right| \lesssim e^{x \text{Re}(\nu_2^+) - y \text{Re}(\nu_2^-)} + e^{x \text{Re}(\nu_2^+) - y \text{Re}(\nu_1^-)} \lesssim e^{-\kappa_2(x-y)}.$$

The first case is now done. For the following ones, same tools are used. We nevertheless give the general plan of the proofs to clarify some points.

(ii)  $0 < y < x$ . We decompose both  $\phi_3^-(y)$  and  $\phi_4^-(y)$  into the  $(\phi_i^+)_{1 \leq i \leq 4}$  basis, and use the ordering (3.50) of the spatial eigenvalues to obtain

$$\begin{aligned} \left| \frac{\text{Det}(\phi_2^+, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \right| &\lesssim \sum_{\substack{i \neq j \\ i, j \neq 2}} \left| c_i^+(\phi_3^-) c_j^+(\phi_4^-) \frac{\text{Det}(\phi_2^+, \phi_i^+, \phi_j^+)(y)}{e^{\text{Re}(\nu_1^+ + \dots + \nu_4^+)y}} \right|, \\ &\lesssim e^{-y \text{Re}(\nu_1^+)} + e^{-y \text{Re}(\nu_3^+)} + e^{-y \text{Re}(\nu_4^+)}. \end{aligned}$$

The second and third term are treated directly:  $-y \text{Re}(\nu_{3,4}^+) \leq -y \kappa_2 \leq y \kappa_2$ . For the first term, we need to use the ordering of  $x$  and  $y$ . Since  $x - y \geq 0$ , we have  $\text{Re}(\nu_1^+)(x - y) \leq -\kappa_2(x - y)$ . Hence:

$$\left| \phi_1^+(x) \frac{\text{Det}(\phi_2^+, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \right| \lesssim e^{x \text{Re}(\nu_1^+) - y \text{Re}(\nu_1^+)} + e^{x \text{Re}(\nu_1^+) - y \text{Re}(\nu_{3,4}^+)} \lesssim e^{-\kappa_2(x-y)},$$

A similar argument leads to:

$$\left| \phi_2^+(x) \frac{\text{Det}(\phi_1^+, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \right| \lesssim e^{x \text{Re}(\nu_2^+) - y \text{Re}(\nu_2^+)} + e^{x \text{Re}(\nu_2^+) - y \text{Re}(\nu_{3,4}^+)} \lesssim e^{-\kappa_2(x-y)}.$$

(iii)  $y < x < 0$ . Here we need to decompose both  $\phi_1^+(x)$  and  $\phi_2^+(y)$ . Taken independently, both term  $\phi_1^+ b_1$  and  $\phi_2^+ b_2$  do not decay as claimed, due to the following asymmetric growing rates, which correspond to terms where the two projections are done on the same element

of  $(\phi_i^-)_{1 \leq i \leq 4}$ :

$$E_1(\lambda, x, y) \stackrel{\text{def}}{=} c_1^-(\phi_1^+)c_1^-(\phi_2^+)\phi_1^-(x) \frac{\text{Det}(\phi_1^-, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \sim e^{\nu_1^- x - \nu_2^- y},$$

and

$$E_2(\lambda, x, y) \stackrel{\text{def}}{=} c_2^-(\phi_1^+)c_2^-(\phi_2^+)\phi_2^-(x) \frac{\text{Det}(\phi_2^-, \phi_3^-, \phi_4^-)(y)}{\mathcal{W}^{\text{sh}}(y)} \sim e^{\nu_2^- x - \nu_1^- y}.$$

However, these terms appear both in  $\phi_1^+ b_1$  and  $\phi_2^+ b_2$ , so that they cancel out in the expression of  $G_\lambda^{\text{sh}}$ , recall from (3.49) that  $b_1$  and  $b_2$  are obtained with opposite sign in their expressions. We compute

$$\begin{aligned} |\phi_1^+ b_1 - E_1 - E_2|(x, y) &\lesssim \left( e^{x \text{Re}(\nu_1^-)} + e^{x \text{Re}(\nu_3^-)} + e^{x \text{Re}(\nu_4^-)} \right) e^{-y \text{Re}(\nu_1^-)} \\ &\quad + \left( e^{x \text{Re}(\nu_2^-)} + e^{x \text{Re}(\nu_3^-)} + e^{x \text{Re}(\nu_4^-)} \right) e^{-y \text{Re}(\nu_2^-)}, \\ &\lesssim e^{-\kappa_2(x-y)}. \end{aligned}$$

For the second inequality we have used the same method as in the previous case. From one hand  $(x-y)\text{Re}(\nu_1^+) \leq -\kappa_2(x-y)$  due to the ordering of  $x$  and  $y$ . From the other hand,  $x \text{Re}(\nu_3) \leq x \kappa_2 \leq -x \kappa_2$  coupled with  $-y \text{Re}(\nu_1^-) \leq -y \kappa_2$ . Similar computations lead to

$$|\phi_2^+ b_2 + E_1 + E_2|(x, y) \lesssim e^{-\kappa_2(x-y)}.$$

We conclude using triangular inequality:  $|\phi_1^+(x)b_1(y) + \phi_2^+(x)b_2(y)| \lesssim e^{-\kappa_2(x-y)}$ .

We are now done with all cases where  $y < x$ . The  $x < y$  cases are mirror versions of the three above cases. Computations can be adapted easily, we omit them. This completes the proof of the first stated inequality.

We now treat the second one (3.23). Assume first that  $x \geq 0$ . In that case, the term  $\omega_{\kappa_2, 0}(x)$  is bounded from below by a constant  $C > 0$ . In particular, this term plays no role in (3.23). Furthermore, we can restrict to the ordering  $y \leq 0 \leq x$ , so that:

$$|G_\lambda^{\text{sh}}(x, y)| \leq C e^{-\kappa_2|x-y|} = C e^{-\frac{\kappa_2}{2}(x-y)} e^{\frac{\kappa_2}{2}y} e^{-\frac{\kappa_2}{2}x} \leq e^{-\frac{\kappa_2}{2}|x-y|} \omega_{\kappa_2/2, 0}(y).$$

Up to a change of notation  $\tilde{\kappa}_2 = \frac{1}{2}\kappa_2$ , this is the claimed (3.23).

The case where  $y \geq 0$  is similar, and can be adapted from the latter paragraph. The proof is now complete.  $\square$

### 3.7 Three equilibrium points

It appears that for certain sets of parameters  $(d, \alpha, \beta, \gamma, \sigma)$ , system (3.3) admits nontrivial equilibrium  $(u_{eq}, v_{eq})$  with  $v_{eq} \sim \sqrt{\mu}$ . To keep dynamics as simple as possible, for example when working with numerical simulations, we can restrict to parameters that satisfy the following statement. This is by no means necessary to our study, since the main result is perturbative.

**Proposition 3.7.1** (Three equilibrium points). *Set  $\mu_0 < 1$  and  $\gamma > 0$ . Then for positive  $d, \alpha, \beta$ , there exists  $\sigma_0 > 0$  such that for all  $\sigma > \sigma_0$ , and  $0 \leq \mu < \mu_0$ , system (3.3) admits only the steady states  $(u, v) = (\pm 1, 0)$  and  $(u, v) = (0, 0)$ .*

*Proof.* Assume that  $(u, v) \in \mathbb{R}^2$  is a constant solution of (3.3). If  $v = 0$ , the only solutions are  $(\pm 1, 0)$  and  $(0, 0)$ . Else, the point  $(u, v)$  lies in the intersection of two curves:

$$\begin{cases} v = -\frac{\alpha}{\beta}u(1-u^2), \\ u = 1 - \frac{1}{\gamma}(\mu - 1 - \sigma v^2). \end{cases}$$

For  $v \leq 0$ , they do not intersect since they respectively ensures  $u \leq 1$  and  $u > 1$ .<sup>7</sup> For  $v > 0$ , we use the tangent line to the first curve at  $(u, v) = (1, 0)$ :

$$T = \left\{ \left( 1 + u, \frac{2\alpha}{\beta}u \right) : u \in \mathbb{R} \right\} = \left\{ \left( 1 + \frac{\beta}{2\alpha}v, v \right) : v \in \mathbb{R} \right\}.$$

Remark that the function  $u \mapsto -\frac{\alpha}{\beta}u(1-u^2)$  is convex for  $u > 1$ . Hence the two curves do not intersect for positive  $v$  provided the second curve do not intersect  $T$ , which reads:

$$1 + \frac{\beta}{2\alpha}v < 1 - \frac{1}{\gamma}(\mu - 1 - \sigma v^2) \quad \Leftrightarrow \quad \sigma v^2 - \frac{\gamma\beta}{2\alpha}v + 1 - \mu > 0.$$

It holds true for sufficiently large  $\sigma > 0$ . This complete the proof.  $\square$

### 3.8 Proof of Lemma 3.3.4

*Proof.* Lemma 3.3.4. We make the change of variable  $z(t) = e^{-tA_\infty}y(t)$ , so that

$$z' = (A(t) - A_\infty)z, \quad (3.52)$$

which writes as

$$z(t) = z(0) + \int_0^t R(s)z(s)ds = v - \int_t^{+\infty} R(s)z(s)ds \quad (3.53)$$

with  $\|R(t)\| = \|A(t) - A_\infty\| \leq Ce^{-\alpha t}$ , and  $v = z(0) + \int_0^{+\infty} R(s)z(s)ds$ . Hence for  $z$  bounded, equation (3.52) together with condition  $z(t) \rightarrow v$  when  $t \rightarrow +\infty$  is equivalent to (3.53). We now make the change of variable  $x(t) = e^{\frac{\alpha}{2}t}(z(t) - v)$ , that satisfies:

$$x(t) = (Kx)(t) \stackrel{\text{def}}{=} - \int_t^{+\infty} e^{-\frac{\alpha}{2}(t-s)} R(s)(x(s) + v)ds. \quad (3.54)$$

The operator  $K$  is a contraction from  $L^\infty(T, +\infty)^n$  to itself, provided  $T$  is large enough. Indeed, for  $t \geq T$  we have

$$\|Kx - K\tilde{x}\|(t) \leq \|x - \tilde{x}\|_{L^\infty(T, +\infty)} \int_t^{+\infty} e^{-\frac{\alpha}{2}(t-s) - \alpha s} ds \leq \frac{2e^{-\alpha T}}{\alpha} \|x - \tilde{x}\|_{L^\infty(T, +\infty)}.$$

The Picard fixed point theorem ensures the existence of a unique  $\kappa \in L^\infty(T, +\infty)$  solution of (3.54). Reverting back all changes of variable, we obtain a solution of (3.21) with

$$y(t) = e^{tA_\infty}(v + e^{-\frac{\alpha}{2}t}\kappa(t))$$

as claimed. By Cauchy-Lipschitz theorem, we flow this solution backward to define it on  $\mathbb{R}$ .

<sup>7</sup> The second inequality comes from  $\gamma > 0$  and  $\mu < 1$ .

We now assume that  $(v_i)_{1 \leq i \leq n}$  is a basis of  $\mathbb{R}^n$ . Since the family  $(e^{-tA_\infty} y_i(t))_i$  converges to  $(v_i)_i$  when  $t \rightarrow +\infty$ , we obtain that

$$\det(e^{-tA_\infty} y_1(t), \dots, e^{-tA_\infty} y_n(t)) = \det(e^{-tA_\infty}) \det(y_1(t), \dots, y_n(t))$$

is nonzero for  $t$  large enough, by continuity of the determinant. This ensures that  $(y_i)_i$  is a basis for the solutions of equation (3.21).

Finally, we suppose that  $A(t)$ ,  $A_\infty$  and  $v$  are holomorphic with respect to  $\lambda$ . Then the operator  $K_\lambda : L^\infty(T, +\infty)^n \rightarrow L^\infty(T, +\infty)^n$  defined by (3.54) is holomorphic with respect to  $\lambda$ , and so is  $\kappa$ . This concludes the proof.  $\square$

# Super-critical monostable fronts in scalar balance laws

## Outline of the chapter

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## 4.1 Introduction

In the present contribution, we continue the study, initiated in [DR20; DR21], of the large-time asymptotic behavior of solutions to first order scalar hyperbolic balance laws, in neighborhoods of traveling waves. Explicitly, let  $f$  and  $g$  be two smooth scalar functions on  $\mathbb{R}$  and consider the associated one-dimensional scalar balance law

$$\partial_t u + \partial_x(f(u)) = g(u). \quad (4.1)$$

A *traveling-wave* solution to (4.1) is a solution of the form  $u(t, x) = \underline{u}(x - ct)$ , for some speed  $c$  and some profile  $\underline{u}$ , and our goal is to understand the dynamics near such special solutions. Note that we include as special cases, standing waves that correspond to the case when  $c = 0$ , and constant equilibria that correspond to the case when the profile  $\underline{u}$  is constant.

As in [DR20; DR21], we allow profiles to be discontinuous, but, then, restrict, among possible weak solutions, to entropy-admissible solutions. In particular, the classical Kruřkov theory [Kru70] applies to the problems we study, and yields uniqueness for the notion of solution we use throughout. The classical motivation for the consideration of non smooth solutions is that even smooth initial data may lead to the formation of discontinuities in finite time. For more on the latter and other relevant elementary background on hyperbolic equations, we refer the reader to [Bre00].

A dramatic consequence of allowing discontinuous solutions is that Equation (4.1) may possess a tremendously huge number of traveling-wave solutions. Yet one of the main upshots of the analysis of [DR21] is that very few of those are stable. Let us describe the analysis underlying the claim in a few words. In the framework of piecewise-smooth *unweighted* topologies [DR21] provides

- a complete classification of non-degenerate traveling waves according to their spectral stability;
- proofs that for those waves spectral instability (resp. stability) yields dynamical nonlinear instability (resp. asymptotic stability).

The classification relies on the fact, proved in [DR21], that, in such a framework, spectral (and nonlinear) instability occur if and only if the profile exhibits at least one of the following features:

- an endstate  $u_\infty$  — that is, a limit of  $\underline{u}$  at  $+\infty$  or  $-\infty$  — such that  $g'(u_\infty) > 0$ ;
- a discontinuity point  $d_0$  at which  $\frac{[g(\underline{u})]_{d_0}}{[\underline{u}]_{d_0}} > 0$  (with  $[u]_x \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} (u(x + \varepsilon) - u(x - \varepsilon))$  denoting jump at  $x$ );
- a characteristic value  $u_\star$  — that is, a value  $u_\star$  of  $\underline{u}$  with  $f'(u_\star) = c$  — such that  $g'(u_\star) < 0$ .

Our goal is to extend the foregoing analysis by elucidating which of the instabilities due to an unstable endstate may be stabilized by working in a *weighted* topology. In particular we aim at describing the dynamics near monostable or even nullistable fronts — *i.e.* structures with exactly 1 or 0 asymptotic stable state. A general motivation for such kind of studies is that even though those waves are unstable in unweighted topologies they may play the role of an elementary block in the description of the large-time dynamics of compactly supported initial data. See for instance for the most classical model study. We expect the case of Equation (4.1) to be no exception. In this direction, we mention that, under rather stringent but natural assumptions on  $f$  and  $g$  — including the strict convexity of  $f$  and the strict dissipativity at infinity of  $g$  —, it has already been proved in [Sin96; MS97], by comparison principle arguments, that starting from an  $L^\infty$  initial datum constant near  $-\infty$  and near  $+\infty$ , the large-time dynamics is well captured in  $L^\infty$  topology by piecing together traveling waves (constants, fronts or periodic waves).

At the linearized level, all the instabilities due to an unstable endstate may be stabilized with a suitably designed weight. Yet only the cases when the spectral stabilization may be carried out with a weight decaying at infinity are relevant for the nonlinear stability analysis. One conclusion of our analysis is that, as may be guessed from heuristic arguments, endstate instabilities may be nonlinearly stabilized provided that unstable wave-packets, or, equivalently here, characteristics, travel outward the unstable infinity. Note in particular that unstable constant states cannot be stabilized in this way, since they are unstable at both infinities but with characteristics pointing outward at one infinity, and inward at the other one.

As a first step, we restrict in the rest of this chapter to continuous monostable profiles. To adapt this study to the case of a profile with one shock, it will be necessary to constantly adjust the position of the shock using a phase  $\psi(t)$ , so that the perturbed solution can be compared to the unperturbed one. This was done in [DR20; DR21] for bistable fronts using the Rankine-Hugoniot condition:

$$c + \psi'(t) = \frac{[f(u)]_{d_0+ct+\psi(t)}}{[u]_{d_0+ct+\psi(t)}}.$$

This procedure is expected to adapt to the monostable case, and is under investigation at the present time.

The organisation of this chapter goes as follow. In section 4.2, we show existence of continuous front solutions. In section 4.3, we state our main result: stability of these continuous fronts.

There, we also describe the perturbation dynamic, and introduce some notations. Finally, we prove theorem 4.3.1 in the remaining sections. In section 4.4 we stabilize the main non-autonomous dynamic using a fixed exponential weight. In section 4.5 we incorporate semi-linear and forcing terms into the argument.

## 4.2 Choice of speed, existence

Let  $u_-, u_+ \in \mathbb{R}$  be two consecutive simple zeros of  $g$ :

$$g(u_-) = 0 = g(u_+), \quad g'(u_-) < 0 < g'(u_+).$$

We look for  $c \in \mathbb{R}$  and a continuous  $\underline{u}$  such that the profile equation is satisfied,

$$(f'(\underline{u}) - c)\underline{u}' = g(\underline{u}), \quad (4.2)$$

and such that the profile connect the two equilibria  $u_-, u_+ \in \mathbb{R}$ :

$$\lim_{y \rightarrow -\infty} \underline{u}(y) = u_-, \quad \lim_{y \rightarrow +\infty} \underline{u}(y) = u_+. \quad (4.3)$$

In the following, we will often use the interval  $I \stackrel{\text{def}}{=} (u_-, u_+)$ . We further define the critical speed

$$c_* \stackrel{\text{def}}{=} \sup_{u \in I} f'(u).$$

**Lemma 4.2.1.** *Let  $K \in \mathbb{N}_{\geq 1}$ , assume that  $f \in \mathcal{C}^{K+1}(\mathbb{R})$  and that  $g \in \mathcal{C}^K(\mathbb{R})$ . Then, for all  $c > c_*$  there exists  $\underline{u} \in \mathcal{C}^K(\mathbb{R})$  a solution of the profile equation (4.2) that connect the two equilibria  $u_-$  and  $u_+$ , i.e. (4.3) is satisfied. Furthermore  $\underline{u}$  is monotonic.*

*Proof.* By choice of  $c$ , the first order Cauchy problem

$$\underline{u}' = \frac{g(\underline{u})}{f'(\underline{u}) - c} \quad (4.4)$$

admits a unique maximal solution for each initial condition

$$\underline{u}(x_0) = u_0 \in I. \quad (4.5)$$

The regularity of  $g$  and  $f'$  ensures that  $u \in \mathcal{C}^K(\mathbb{R})$ . Since both constants  $\underline{u} = u_-$  and  $\underline{u} = u_+$  satisfy (4.4), solutions to (4.4) – (4.5) are bounded:  $\underline{u}(x) \in I$ . Then  $\underline{u}$  is defined globally from standard Cauchy-Lipschitz theory.

In case where  $u_- \leq u_+$ , the stability condition on  $u_-$  and  $u_+$  imposes that  $g < 0$  on  $I$ . Together with the choice of  $c$ , (4.4) ensures that  $\underline{u}$  is increasing, and thus converging to  $u_+$  (resp.  $u_-$ ) when  $x \rightarrow +\infty$  (resp.  $x \rightarrow -\infty$ ). In case where  $u_+ \leq u_-$ , limits are obtained similarly. To conclude the proof, remark that the right hand side in (4.4) does not change sign, hence  $\underline{u}$  is monotonic.  $\square$

*Remark 4.2.2.* The above proof restrict to the case of *invasion* fronts. We briefly comment on the remaining cases.

- The case  $c < \tilde{c}_*$ , with  $\tilde{c}_* \stackrel{\text{def}}{=} \inf_{u \in I} f'(u)$ , is symmetric and obtained from both changes of variable  $\tilde{x} = -x$  and  $\tilde{f} = -f$ . The corresponding front travels slower than all perturbations,



and should therefore be considered as traveling to the left. Remark that it connects the unstable state  $u_+$  at  $-\infty$  to the stable state  $u_-$  at  $+\infty$ .

- The case where  $c \in f'(I)$  corresponds to the presence of a sonic point  $u_\star$  and would imply that  $g$  vanishes at  $u_\star \in I$ . This is prevented by our assumption on  $g$ .
- The limit case  $c = c_*$  (or equivalently  $c = \tilde{c}_*$ ) still need to be discussed.

### 4.3 Main result

To state our main result, we need to introduce spatial weights. Here we recall the notations from the introduction, see section 1.4.2. Note  $\langle x \rangle \stackrel{\text{def}}{=} (1 + x^2)^{1/2}$ . Then for  $\eta > 0$ , let  $\rho_\eta$  be a smooth, positive function such that

$$\rho_\eta(x) = \begin{cases} \langle x \rangle^{-\eta} & \text{if } x \geq 1, \\ 1 & \text{if } x \leq -1. \end{cases}$$

We will also use exponential weights. For  $\kappa > 0$ , let  $\omega_\kappa$  be a smooth, positive function such that

$$\omega_\kappa(x) = \begin{cases} e^{-\kappa x} & \text{if } x \geq 1, \\ 1 & \text{if } x \leq -1. \end{cases}$$

**Theorem 4.3.1.** *Assume  $g \in \mathcal{C}^3(\mathbb{R})$  and  $f \in \mathcal{C}^4(\mathbb{R})$ , and fix  $c > c_*$ . Let  $\phi$  such that  $e^\phi = |\underline{u} - u_+|$ . In particular  $\phi(y) \sim -\frac{g(u_+)}{c-f'(u_+)}y$  when  $y \rightarrow +\infty$ . Then there exists positive constants  $\delta, C$ , such that for all  $v_0 \in BUC^1(\mathbb{R})$  with*

$$\|v_0\|_{W^{1,\infty}(\mathbb{R})} \leq \delta,$$

*there exist a unique solution  $u \in \mathcal{C}^0((0, +\infty), BUC^1(\mathbb{R}))$  to (4.1) with initial condition  $u_0(x) = \underline{u}_c(x) + e^{\phi(x)}v_0(x)$ . Furthermore, it writes as  $u(t, x) = \underline{u}_c(x-ct) + e^{\phi(x-ct)}v(t, x-ct)$  with a weighted perturbation  $v$  that satisfies the following three estimates. For all  $t \geq 0$ ,*

$$\|v(t)\|_{W^{1,\infty}(\mathbb{R})} \leq C\|v_0\|_{W^{1,\infty}(\mathbb{R})}. \quad (4.6)$$

*Let  $\kappa$  be a small enough, non-negative constant. Then there exists positive constants  $\eta$  and  $C$  such that for all  $t \geq 0$ ,*

$$\|\omega_\kappa^{-1}v(t)\|_{W^{1,\infty}(\mathbb{R})} \leq Ce^{-\eta t}\|\omega_\kappa^{-1}v_0\|_{W^{1,\infty}(\mathbb{R})}.$$

*Let  $\eta$  and  $\eta'$  be non-negative constants such that  $0 \leq \eta' \leq \eta$ . Then there exist a positive constant  $C$  such that for all  $t \geq 0$ ,*

$$\|\rho_{\eta'}^{-1}v(t)\|_{W^{1,\infty}(\mathbb{R})} \leq \frac{C}{(1+t)^{\eta-\eta'}}\|\rho_{\eta'}^{-1}v_0\|_{W^{1,\infty}(\mathbb{R})}.$$

*Remark 4.3.2.*

- The first estimate is obtained by shifting the linear unstable spectrum to the left, up to the imaginary axis. This is done through the weight  $e^\phi$ , that acts in the  $x \geq 0$  region to stabilize the unstable state  $u_+$ .
- Assuming additional exponential localization of the initial condition is equivalent to use a stronger weight, with the effect of stabilizing spectrum with a gap. This is done in the second estimate.

— In the critical situation with no spectral gap, it is however possible to gain some decay by losing localization in space, as showed in the third estimate. It has to be note that the Dakto-Pazzy Theorem ensures that no polynomial decay can be obtained at linear level in the case where  $\eta = \eta'$ .

Let us point out that the space  $BUC^k(\mathbb{R})$ , defined as

$$BUC^k(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ u \in W^{k,\infty}(\mathbb{R}) : \text{each } u^{(j)} \text{ is bounded and uniformly continuous, } 0 \leq j \leq k \right\}$$

is strictly smaller than  $W^{k,\infty}(\mathbb{R})$ . However, it is dense in  $BUC^0(\mathbb{R})$ , whereas  $W^{k,\infty}$  is not dense in  $L^\infty(\mathbb{R})$ . Hence, this spaces provide densely defined operators, which will prevent any unnecessary difficulty in the following.

*Proof. (Theorem 4.3.1).* The construction of a suitable  $\phi$  that is non-negative, non-increasing, and satisfies  $\phi(y) \sim -\frac{g(u_+)}{c-f'(u_+)}y$ , when  $y \rightarrow +\infty$  is done in Lemma 4.4.2, see also remark 4.4.3. Hence, there there exists  $C > 0$  such that

$$\frac{1}{C}e^\phi \leq |\underline{u} - u_+| \leq Ce^\phi.$$

By a bounded modification of  $\phi$ , we can rather assume that  $e^\phi = |\underline{u} - u_+|$ . In the above statement of Theorem 4.3.1, this change of  $\phi$  corresponds to a multiplication of  $\|\cdot\|_{W^{1,\infty}(\mathbb{R})}$  by a constant, and thus only deteriorates the constant  $C$  in the three claimed estimates.

The first claimed estimate is shown in Proposition 4.5.1. The third one is proved in Proposition 4.5.2, in the case  $\eta' = 0$ . Other cases are directly obtained from Lemma B.2.2. The second estimate is shown in a simplified case, see (4.16) below. The complete case is easily obtained using the same arguments as for the other two estimates.  $\square$

Let us introduce some notations by describing the perturbation dynamic. First set the problem in a moving frame by noting  $y = x - ct \in \mathbb{R}$  the translated space variable. Then, linearize near the front  $\underline{u}$ , that is assume the solution of (4.1) writes

$$u(t, x) = \underline{u}(y) + p(t, y),$$

we obtain, subtracting the front equation, that the perturbation  $p$  satisfies

$$\partial_t p - \mathcal{A}_p p = N(p), \tag{4.7}$$

with transport-growth terms grouped in

$$\mathcal{A}_p \bar{p} \stackrel{\text{def}}{=} [c - f'(\underline{u} + p)] \partial_y \bar{p} + [g'(\underline{u}) - \partial_y(f'(\underline{u}))] \bar{p},$$

and remaining non-linear terms in

$$N(p) \stackrel{\text{def}}{=} [g(\underline{u} + p) - g(\underline{u}) - g'(\underline{u})p] + \underline{u}' [f'(\underline{u} + p) - f'(\underline{u}) - f''(\underline{u})p].$$

Finally, introduce a weight  $e^\phi$ , that is assume the solution of (4.1) rather writes as

$$u(t, x) = \underline{u}(y) + e^{\phi(y)} v(t, y). \tag{4.8}$$

where  $\phi$  satisfies the following technical assumption:<sup>1</sup>

$\phi$  is a non-positive, non-increasing, smooth function  $\mathbb{R} \rightarrow \mathbb{R}$ , with bounded derivatives.

There exists positive constants  $C, \kappa$  such that  $e^\phi \leq C\omega_\kappa$  and  $\phi'' \leq C\omega_\kappa$ . (H $_\phi$ )

Then the weighted perturbation  $v$  satisfies

$$\partial_t v - \mathcal{L}_v v = \mathcal{N}(v), \quad (4.9)$$

where  $\mathcal{L}_v$  and  $\mathcal{N}$  are obtained by conjugation:

$$\mathcal{L}_v = e^{-\phi} \mathcal{A}_{e^\phi v} e^\phi, \quad \mathcal{N}(v) = e^{-\phi} \mathcal{N}(e^\phi v).$$

More precisely, the expression of  $\mathcal{L}_v$  reads:

$$\begin{aligned} \mathcal{L}_v \tilde{v} &\stackrel{\text{def}}{=} [c - f'(\underline{u} + e^\phi v)] \partial_y \tilde{v} + [g'(\underline{u}) - \partial_y(f'(\underline{u})) + (c - f'(\underline{u} + e^\phi v))\phi'] \tilde{v}, \\ &\stackrel{\text{def}}{=} a_0(y, v(t, y)) \partial_y \tilde{v} + b_0(y, v(t, y)) \tilde{v}. \end{aligned} \quad (4.10)$$

We also need to describe the first derivative of the perturbation.

**Lemma 4.3.3.** *Assume  $g \in C^3(\mathbb{R})$  and  $f \in C^4(\mathbb{R})$ . Assume that  $H_\phi$  holds. Then, using ansatz (4.8), the derivative  $\partial_y v$  of the weighted perturbation satisfies*

$$(\partial_t - \mathcal{L}_v^{(1)})(\partial_y v) = \mathcal{R}(\partial_y v) + \mathcal{H},$$

with main terms defined by

$$\begin{aligned} \mathcal{L}_v^{(1)} &\stackrel{\text{def}}{=} \mathcal{L}_v - \underline{u}' f''(\underline{u} + e^\phi v), \\ &\stackrel{\text{def}}{=} a_1(y, v(t, y)) \partial_y + b_1(y, v(t, y)), \end{aligned}$$

and remaining terms being well localized in space: there exist positive constants  $C, \delta, \kappa$  such that for all  $v \in BUC^1(\mathbb{R})$  that satisfies  $\|v\|_{W^{1,\infty}(\mathbb{R})} \leq \delta$ ,

$$\|\omega_\kappa^{-1} \mathcal{R}(\partial_y v)\|_{L^\infty(\mathbb{R})} \leq C\delta \|\partial_y v\|_{L^\infty(\mathbb{R})}, \quad \|\omega_\kappa^{-1} \mathcal{H}\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{L^\infty(\mathbb{R})}.$$

*Proof.* First note that  $\underline{u}'$  and further derivatives of the front are localized enough so that there exist positive constants  $C, \kappa$  such that  $\underline{u}' \leq C\omega_\kappa$ . Fix a positive  $\delta$ . Differentiate (4.9) to see that  $\partial_y v$  satisfies

$$(\partial_t - \mathcal{L}_v)(\partial_y v) = [\partial_y, \mathcal{L}_v]v + \partial_y(\mathcal{N}(v)). \quad (4.11)$$

The commutator computes as

$$\begin{aligned} [\partial_y, \mathcal{L}_v]v &\stackrel{\text{def}}{=} \partial_y(\mathcal{L}_v v) - \mathcal{L}_v(\partial_y v), \\ &= -\underline{u}' f''(\underline{u} + e^\phi v) \partial_y v - e^\phi f''(\underline{u} + e^\phi v) (\partial_y v)^2 - \phi' e^\phi f''(\underline{u} + e^\phi v) v \partial_y v \\ &\quad + \partial_y(b_0(y, v))v. \end{aligned}$$

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1. One should have in mind a function with  $\phi(y) = -\kappa y$  for  $y \geq 1$ , and a positive fixed  $\kappa$ .

We compute that

$$\begin{aligned}\partial_y(b_0(y, v)) &= g'(\underline{u})\underline{u}' - \partial_{yy}(f'(\underline{u})) + \phi''(c - f'(\underline{u} + e^\phi v)) \\ &\quad - \phi' f''(\underline{u} + e^\phi v)(\underline{u}' + e^\phi \partial_y v + \phi e^\phi v), \\ &\stackrel{\text{def}}{=} \omega_\kappa (H_1 + H_2 \partial_y v + H_3 v).\end{aligned}$$

Furthermore, using Taylor expansions,  $\partial_y(\mathcal{N}(v))$  decomposes as:

$$\partial_y(\mathcal{N}(v)) = H_4 \omega_\kappa v^2 + H_5 \omega_\kappa v \partial_y v,$$

where  $\|y \mapsto H_i(y)\|_{L^\infty(\mathbb{R})}$  is bounded uniformly with respect to all  $v \in BUC^1(\mathbb{R})$  that satisfy  $\|v\|_{W^{1,\infty}(\mathbb{R})} \leq \delta$ , for  $i \in \{4, 5\}$ . Altogether, we obtain the claimed dynamic on  $\partial_y v$ , with forcing and remaining terms given by

$$\mathcal{H} \stackrel{\text{def}}{=} \omega_\kappa (H_1 v + H_3 v^2 + H_4 v^2),$$

$$\mathcal{R}(\partial_y v) \stackrel{\text{def}}{=} [\omega_\kappa H_2 + \omega_\kappa H_5 - \phi' e^\phi f''(\underline{u} + e^\phi v)] v \partial_y v - e^\phi f''(\underline{u} + e^\phi v) (\partial_y v)^2.$$

The bounds on  $\mathcal{R}$  and  $\mathcal{H}$  are immediate.  $\square$

## 4.4 Linear estimates

Here we obtain bounds for solutions of a linear problem, and study the full dynamic in section 4.5.

**Proposition 4.4.1.** *Let  $a \in C^1(\mathbb{R}_+, BUC^1(\mathbb{R}))$  and  $b \in C^0(\mathbb{R}_+, BUC^1(\mathbb{R}))$ , and assume there exists a positive  $a_0 \in \mathbb{R}$  such that  $a \geq a_0$ . Note*

$$\mathcal{L}_{a,b} \stackrel{\text{def}}{=} a(t, x) \partial_x + b(t, x)$$

the closed linear operator on  $BUC^0(\mathbb{R})$ , with dense domain  $BUC^1(\mathbb{R})$ . Furthermore, note  $\mathcal{S}_{a,b}(t, \tau)v_0$  the solution at time  $t$  of the non-autonomous Cauchy problem

$$\partial_t v = \mathcal{L}_{a,b} v, \quad v(\tau, \cdot) = v_0,$$

with initial condition  $v_0 \in BUC^0(\mathbb{R})$ . Then for all  $0 \leq \tau \leq t$ ,

$$\|\mathcal{S}_{a,b}(t, \tau)v_0\|_{L^\infty(\mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})} \exp\left(\int_\tau^t \sup_{\mathbb{R}} b(s, \cdot) ds\right).$$

*Proof.* See [DR20, Proposition 1.8].  $\square$

A direct consequence of the previous result is that for a fixed  $v \in BUC^0(\mathbb{R})$ , the spectrum of  $\mathcal{L}_v$  lies to the left of  $\sup_{y \in \mathbb{R}} b_0(y, v(y))$ . A suitable choice of  $\phi$  make this bound non-positive, as shown in the following result.

**Lemma 4.4.2.** *Fix  $c > c_*$ , and  $\eta$  such that  $0 \leq \eta < -g'(u_-)$ . Then, there exists  $\phi_\eta$  non-increasing and non-positive; constants  $m \in \mathbb{R}$  and  $\delta > 0$  such that*

1. For all  $v \in BUC^0(\mathbb{R})$  that satisfy  $\|v\|_{L^\infty(\mathbb{R})} \leq \delta$ , for all  $y \in \mathbb{R}$ , the growth terms in  $\mathcal{L}_v$  and  $\mathcal{L}_v^{(1)}$  are non-positive:

$$b_0(y, v(y)) \leq -\eta, \quad b_1(y, v(y)) \leq -\eta.$$

2. The weight  $e^{\phi_\eta}$  does not act at  $-\infty$ , and is exponentially decaying at  $+\infty$ :

$$\phi_\eta(y) = \begin{cases} 0 & \text{if } y \leq m, \\ -\kappa y + \mathcal{O}(1) & \text{when } y \rightarrow +\infty, \end{cases}$$

where  $\kappa \stackrel{\text{def}}{=} \frac{g'(u_+) + \eta}{\sigma - f'(u_+)} > 0$ . In particular, Lemma 4.3.3 applies.

*Proof.* Let  $\phi$  be a non-positive, non-increasing function to be fixed later. From the choice of  $c$  and continuity of  $f'$ , there exists positive constants  $\delta$  and  $\alpha_0$  such that for all  $u \in I + [-\delta, \delta]$  we have  $\sigma - f'(u) \geq \alpha_0$ . To control the  $b_i$  uniformly with respect to  $v$ , we define two functions  $y \mapsto \alpha(y)$  and  $y \mapsto \beta(y)$ . Let

$$M \stackrel{\text{def}}{=} \sup_{u \in I + [-\delta, \delta]} f''(u),$$

and

$$\beta \stackrel{\text{def}}{=} g'(\underline{u}) - \partial_y(f'(\underline{u})) + \max(0, -\underline{u}' M e^\phi),$$

together with

$$\alpha(y) \stackrel{\text{def}}{=} \inf_{v \in [-\delta, \delta]} (\sigma - f'(\underline{u}(y) + v e^{\phi(y)})).$$

From the above choice of  $\delta$  we have that  $\alpha(y) \geq \alpha_0 > 0$ . Hence for all  $v \in BUC^0(\mathbb{R})$  with  $\|v\|_{L^\infty(\mathbb{R})} \leq \delta$ , for all  $y \in \mathbb{R}$  and for  $i \in \{0, 1\}$ ,

$$b_i(y, v(y)) \leq \beta(y) + \alpha(y) \phi'(y).$$

Recall the usual notations  $z^- \stackrel{\text{def}}{=} -\min(0, z)$  and  $z^+ \stackrel{\text{def}}{=} \max(0, z)$  for the positive and negative parts of  $z$ , and define

$$\phi'_\eta(y) \stackrel{\text{def}}{=} -\frac{(\beta(y) + \eta)^+}{\alpha(y)}, \quad \phi_\eta(y) \stackrel{\text{def}}{=} \int_{-\infty}^y \phi'_\eta(y) dy.$$

Finally note  $z = \beta(y) + \eta$ , so that

$$b_i(y, v(y)) \leq z - z^+ - \eta = -z^- - \eta$$

is less than  $-\eta$ , which proves (1). For the second point, remark that

$$\lim_{y \rightarrow -\infty} \beta(y) + \eta = g'(u_-) + \eta < 0,$$

thus there exists  $m \in \mathbb{R}$  such that  $\phi'_\eta(y) = 0$  for  $y \leq m$ . The same goes for  $\phi_\eta$ . To obtain the correct behavior of  $\phi$  at  $+\infty$ , remark that

$$\lim_{y \rightarrow +\infty} \beta(y) + \eta = g'(u_+) + \eta > 0, \tag{4.12}$$

so that  $\phi'_\eta(y) \geq -\frac{g'(u_+) + \eta}{2\alpha_0}$  for  $y$  sufficiently large. This implies that  $\phi_\eta(y) \rightarrow -\infty$  when  $y \rightarrow +\infty$ , which in turn ensures that

$$\lim_{y \rightarrow +\infty} \alpha(y) = c - f'(u_+). \tag{4.13}$$

Finally, remark that both limits (4.12) and (4.13) occur at exponential speed,<sup>2</sup> so that there exists  $\theta > 0$  and  $h$  bounded such that

$$\phi'_\eta(y) = -\frac{g'(u_+) + \eta}{c - f'(u_+)} + e^{-\theta y} h(y).$$

Then with

$$\kappa \stackrel{\text{def}}{=} \frac{g'(u_+) + \eta}{c - f'(u_+)},$$

$\gamma(y) \stackrel{\text{def}}{=} m \frac{g'(u_+) + \eta}{c - f'(u_+)} + \int_m^y e^{-\theta y} h(y) dy$  is bounded and

$$\phi_\eta(y) = \int_m^y \phi'_\eta(y) dy = -\kappa y + \gamma(y).$$

Currently,  $\phi$  does not satisfy  $H_\phi$ , since it is not smooth. However, it is enough to replace the positive part  $(\cdot)^+$  in its definition by any smooth and nondecreasing approximation of  $(\cdot)^+$ . The other constraints of  $H_\phi$  are not altered by such change.  $\square$

*Remark 4.4.3.* When  $\eta = 0$ , the decay rate  $\kappa_0 = \frac{g'(u_+)}{c - f'(u_+)}$  corresponds exactly to the behavior of the front:  $\underline{u}(y) - u_+ \sim C e^{-\kappa_0 y}$  when  $y \rightarrow +\infty$ .

As mentioned above, exponentially localized terms will decay faster than non-localized ones. This is a consequence of the following lemma, that allow to convert spatial localization of the data into a stronger weight.

**Lemma 4.4.4.** *The family  $e^{\phi_\eta}$  describe a continuous range of exponential behaviors: the map  $\eta \mapsto \kappa_\eta$  is continuous and increasing on  $\eta \in [0, -g'(u_-)]$ . In particular for all  $\kappa \in [0, \frac{-g'(u_-)}{c - f'(u_+)})$ , there exists a unique  $\eta \in [0, -g'(u_-)]$  and  $C > 0$  such that*

$$\frac{1}{C} e^{\phi_0} \omega_\kappa \leq e^{\phi_\eta} \leq C e^{\phi_0} \omega_\kappa.$$

*Proof.* The proof is direct from the expression of  $\kappa$  in the above proof of Lemma 4.4.2.  $\square$

We are about to continue this section with proofs of linear estimates, let us introduce some notations. Let  $I \subset \mathbb{R}_+$  be an interval, that we denote  $I \stackrel{\text{def}}{=} (\tau, T)$  with  $0 \leq \tau \leq T \leq +\infty$ . Let  $v_0 \in BUC^1(\mathbb{R})$  be an initial condition, and let  $\tilde{v} \in \mathcal{C}^0(I, BUC^1(\mathbb{R}))$ . In the following, for  $t \in I$  we will denote  $\mathcal{S}_{\tilde{v}}(t, \tau)v_0$  the solution at time  $t$  of the linear Cauchy problem

$$\partial_t v = \mathcal{L}_{\tilde{v}} v, \quad v(\tau, \cdot) = v_0. \quad (4.14)$$

Finally when  $\eta = 0$ , we keep the simple notation  $\phi \stackrel{\text{def}}{=} \phi_0$ . We now combine Proposition 4.4.1 and both Lemmas 4.4.2, 4.4.4 to obtain the three following statements.

**Proposition 4.4.5** (Optimal weight). *Let  $\delta$  be fixed by Lemma 4.4.2, let  $v_0 \in BUC^0(\mathbb{R})$  and  $\tilde{v} \in \mathcal{C}^0(\mathbb{R}_+, BUC^0(\mathbb{R}))$  such that for all  $t \in I$ ,*

$$\|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \leq \delta.$$

<sup>2</sup>. This is a consequence of the front construction. Linearization of the profile equation when  $y \rightarrow +\infty$  leads to exponential behavior of  $\underline{u}$ .

Then for all  $t \in I$ , the following estimate holds:

$$\|\mathcal{S}_{\tilde{v}}(t, \tau)v_0\|_{L^\infty(\mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})}. \quad (4.15)$$

*Proof.* From the assumptions on  $\tilde{v}$ , Lemma 4.4.2 applies with  $\eta = 0$ . Hence (4.15) is a direct application of Proposition 4.4.1.  $\square$

**Proposition 4.4.6** (Exponential weights). *Let  $\kappa$  such that  $0 < \kappa \leq \frac{-g'(u_-)}{\sigma - f'(u_+)}$ . Then there exists positive constants  $\delta, \eta$  and  $C$  such that if  $v_0 \in BUC^0(\mathbb{R})$  and  $\tilde{v} \in C^0(\mathbb{R}_+, BUC^0(\mathbb{R}))$  satisfy*

$$\|v_0 \omega_\kappa^{-1}\|_{L^\infty(\mathbb{R})} < +\infty \quad \|\tilde{v}(t)\|_{L^\infty(\mathbb{R})} \leq \delta$$

for all  $t \in I$ , then

$$\|\omega_\kappa^{-1} \mathcal{S}_{\tilde{v}}(t, \tau)v_0\|_{L^\infty(\mathbb{R})} \leq C e^{-\eta(t-\tau)} \|\omega_\kappa^{-1} v_0\|_{L^\infty(\mathbb{R})}. \quad (4.16)$$

*Proof.* Fix  $\kappa > 0$  small enough so that Lemma 4.4.4 applies. Then there exists  $\eta > 0$  and  $C > 0$  such that

$$\frac{1}{C} e^{-(\phi_\eta - \phi)} \leq \omega_\kappa^{-1} \leq C e^{-(\phi_\eta - \phi)}.$$

Let us introduce  $w_0 \stackrel{\text{def}}{=} e^{-(\phi_\eta - \phi)} v_0$ , which is bounded from the assumption on  $v_0$ , and

$$w(t) \stackrel{\text{def}}{=} e^{-(\phi_\eta - \phi)} \mathcal{S}_{\tilde{v}}(t, \tau)v_0.$$

From the computation

$$e^{-(\phi_\eta - \phi)} \mathcal{L}_{\tilde{v}} e^{\phi_\eta - \phi} = e^{-\phi_\eta} \mathcal{A}_{e^{\phi_\eta \tilde{v}}} e^{\phi_\eta} \stackrel{\text{def}}{=} \mathcal{L}_{\tilde{v}, \eta},$$

we remark that  $w$  satisfies

$$\partial_t w = \mathcal{L}_{\tilde{v}, \eta} w, \quad w(\tau) = w_0,$$

Although  $\mathcal{L}_{\tilde{v}, \eta}$  is not equal to  $e^{-\phi_\eta} \mathcal{A}_{e^{\phi_\eta \tilde{v}}} e^{\phi_\eta}$ , which is the proper conjugation of  $\mathcal{A}_{\tilde{v}}$  by the weight  $e^{\phi_\eta}$ , they both satisfy Lemma 4.4.2, as its proof straightforwardly adapt for the second operator. Taking  $\delta$  small enough, we can assume that  $C\delta$  is so small that Lemma 4.4.2 applies with the  $\eta$  we use. We then apply Proposition 4.4.1 to the equation satisfied by  $w$ , to obtain that

$$\|w(t)\|_{L^\infty(\mathbb{R})} \leq e^{-\eta(t-\tau)} \|w_0\|_{L^\infty(\mathbb{R})}.$$

Since the weights  $\omega_\kappa$  and  $e^{\phi_\eta - \phi}$  are equivalent, we get that  $\|w_0\|_{L^\infty(\mathbb{R})} \leq C \|\omega_\kappa^{-1} v_0\|_{L^\infty(\mathbb{R})}$  and that  $\|\omega_\kappa^{-1} \mathcal{S}_{\tilde{v}}(t, \tau)v_0\|_{L^\infty(\mathbb{R})} \leq C \|w(t)\|_{L^\infty(\mathbb{R})}$ , which together with the above estimate conclude the proof.  $\square$

**Proposition 4.4.7** (Polynomial weights). *Let  $\eta > 0$ , let  $\delta$  be fixed by Lemma 4.4.2, and  $v_0 \in BUC^0(\mathbb{R})$  such that  $\|v_0\|_{L^\infty(\mathbb{R})} \leq \delta$ . Let  $\eta > 0$ . There exist a positive constant  $C$  such that for all  $t \in I$ , the solution  $\mathcal{S}(t, \tau)v_0$  of (4.14) satisfies*

$$\|\mathcal{S}(t, \tau)v_0\|_{L^\infty(\mathbb{R})} \leq C \frac{1}{(1+t-\tau)^\eta} \|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R})}. \quad (4.17)$$

*Proof.* We only show the first estimate on  $\mathcal{S}$ . The second one shows similarly, since  $\mathcal{L}^{(1)}$  and  $\mathcal{L}$  share the same structure.

In the following, we note  $v(t) = \mathcal{S}_{\tilde{v}}(t, \tau)v_0$ , and decompose it using linearity of  $\mathcal{S}_{\tilde{v}}$  as:

$$v(t) = \mathcal{S}_{\tilde{v}}(t, \tau)v_0 = \mathcal{S}_{\tilde{v}}(t, \tau)(v_0\mathbf{1}_{\mathbb{R}_-}) + \mathcal{S}_{\tilde{v}}(t, \tau)(v_0\mathbf{1}_{\mathbb{R}_+}).$$

Recall the definition of  $a_0$  and  $b_0$  from the expression (4.10) of the operator  $\mathcal{L}_{\tilde{v}}$ . Recall also that from assumption on  $\tilde{v}$ , there exists a positive constant  $\alpha_0$  such that  $a_0(y, \tilde{v}(t, y)) \geq \alpha_0$  – see the beginning of the proof of Lemma 4.4.2. Hence the characteristics point to the left, so that the problem can be restricted to the half space  $\mathbb{R}_+$  without boundary condition at  $y = 0$ . Noting  $\mathcal{S}_{\tilde{v}}^+$  the corresponding restriction of the evolution operator, we note:

$$v_l(t) \stackrel{\text{def}}{=} \mathcal{S}_{\tilde{v}}(t, \tau)(v_0\mathbf{1}_{\mathbb{R}_-}), \quad v_r(t) \stackrel{\text{def}}{=} \mathcal{S}_{\tilde{v}}^+(t, \tau)(v_0\mathbf{1}_{\mathbb{R}_+}).$$

Notice that  $v_l(t, y)$  is defined for all  $y \in \mathbb{R}$ , while  $v_r(t, y)$  only makes sense when  $y \geq 0$ . On the first hand,  $v_0\mathbf{1}_{\mathbb{R}_-}$  is exponentially localized, so that from (4.16) with  $\kappa > 0$  small enough, and the corresponding  $\eta > 0$ ,

$$\|v_l(t)\|_{L^\infty(\mathbb{R})} \leq \|\omega_\kappa^{-1}\mathcal{S}_{\tilde{v}}(t, \tau)(v_0\mathbf{1}_{\mathbb{R}_-})\|_{L^\infty(\mathbb{R})} \leq Ce^{-\eta(t-\tau)}\|v_0\|_{L^\infty(\mathbb{R})}. \quad (4.18)$$

On the other hand, when  $y \geq 0$  we see  $\mathcal{L}_v$  as a free transport perturbed by an exponentially localized term. Introduce  $\mathcal{T}_{\tilde{v}}(t, \tau)v_{0,r}$  the solution of the Cauchy problem

$$\partial_t v = a_0(y, \tilde{v}(t, y))\partial_y v, \quad v(\tau) = v_{0,r},$$

so that the Duhamel formulae for  $v_r$  writes

$$v_r(t) = \mathcal{T}_{\tilde{v}}(t, \tau)v_{0,r} + \int_\tau^t \mathcal{T}_{\tilde{v}}(t, s)(b_0(\cdot, v(s))v_r(s))ds. \quad (4.19)$$

Since  $a_0(y, \tilde{v}(t, y)) \geq \alpha_0 > 0$ , we can express  $\mathcal{T}_{\tilde{v}}$  using characteristics. Note  $X(t, y)$  the solution at time  $t$  of the following ODE:

$$\partial_t X = -a_0(X, \tilde{v}(t, X)), \quad X(\tau, y) = y.$$

Then, it is well known that

$$\mathcal{T}_{\tilde{v}}(t, \tau)v_{0,r} = v_{0,r}(X^{-1}(t, y)),$$

with  $X^{-1}$  being the inverse of the map  $y \mapsto X(t, y)$ . Integrate the bound  $a_0(X, v(t, X)) \geq \alpha_0$ , to deduce that  $X(t, y) \leq -\alpha_0(t - \tau) + y$ . Hence  $X^{-1}(t, y) \geq \alpha_0(t - \tau) + y$ , which leads to the following estimate on  $\mathcal{T}_{\tilde{v}}$ : for any  $\eta$  and  $0 \leq \tau \leq t$ ,

$$\begin{aligned} |\mathcal{T}_{\tilde{v}}(t, \tau)v_{0,r}(y)| &= \frac{1}{\langle X^{-1}(t, y) \rangle^\eta} \left| \langle X^{-1}(t, y) \rangle^\eta v_{0,r}(X^{-1}(t, y)) \right|, \\ &\leq \rho_\eta(y + \alpha_0(t - \tau)) \|\rho_\eta^{-1} v_{0,r}\|_{L^\infty(\mathbb{R}_+)}. \end{aligned}$$

Thus since  $y \geq 0$ ,

$$\|\mathcal{T}_{\tilde{v}}(t, \tau)v_{0,r}\|_{L^\infty(\mathbb{R}_+)} \leq \frac{C}{(1 + t - \tau)^\eta} \|\rho_\eta^{-1} v_{0,r}\|_{L^\infty(\mathbb{R}_+)}.$$

We now transfer this bound to  $v_r$ . Recall that  $b_0$  is exponentially localized, let  $\tilde{\kappa} > 0$  be a small enough constant such that  $b_0(\cdot, \tilde{v}) \leq C\omega_{\tilde{\kappa}}$ , and let  $\tilde{\eta}$  be the corresponding decay rate from Proposition 4.4.6. Then using the bound on  $\mathcal{T}_{\tilde{v}}$  into (4.19), together with estimate (4.16) with



$\mathcal{T}_{\tilde{v}}(t, s)$  instead of  $\mathcal{S}_{\tilde{v}}(t, s)$ , we get that

$$\begin{aligned} \|v_r(t)\|_{L^\infty(\mathbb{R}_+)} &\leq C \frac{\|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R}_+)}}{(1+t-\tau)^\eta} + \left\| \omega_{\tilde{\kappa}}^{-1} \int_\tau^t \mathcal{T}_{\tilde{v}}(t, s) b_0(\cdot, v(s)) v_r(s) ds \right\|_{L^\infty(\mathbb{R}_+)}, \\ &\leq C \frac{\|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R}_+)}}{(1+t-\tau)^\eta} + \int_\tau^t e^{-\tilde{\eta}(t-s)} \|\omega_{\tilde{\kappa}}^{-1} b_0(\cdot, v(s)) v_r(s)\|_{L^\infty(\mathbb{R})} ds, \\ &\leq C \frac{\|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R}_+)}}{(1+t-\tau)^\eta} + \int_\tau^t e^{-\tilde{\eta}(t-s)} \|v_r(s)\|_{L^\infty(\mathbb{R})} ds. \end{aligned}$$

Multiply on both sides by  $e^{\tilde{\eta}(t-\tau)}$  and apply a standard Gronwall argument to obtain that

$$\|v_r(t)\|_{L^\infty(\mathbb{R}_+)} \leq \frac{C}{(1+t-\tau)^\eta} \|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R}_+)}.$$

Adding up with (4.18), we recover the claimed estimate (4.17). The proof is complete.  $\square$

**Corollary 4.4.8.** *The above Propositions 4.4.5 to 4.4.7 still holds if  $\mathcal{L}_v$  is replaced by  $\mathcal{L}_v^{(1)}$  in (4.14). More precisely, there exist a positive constant  $\delta$  such that if  $v_0 \in BUC^1(\mathbb{R})$  and if  $\tilde{v} \in \mathcal{C}^0(\mathbb{R}_+, BUC^0(\mathbb{R}))$  satisfy*

$$\|\tilde{v}\|_{L^\infty(\mathbb{R})} \leq \delta,$$

then, denoting  $\mathcal{S}_{\tilde{v}}^{(1)}(t, \tau) \partial_y v_0$  the solution to the Cauchy problem

$$\partial_t v = \mathcal{L}_{\tilde{v}}^{(1)} v, \quad v(\tau, \cdot) = \partial_y v_0,$$

the following two estimates holds. For all  $\eta \geq 0$ , there exists a positive constant  $C$  such that for all  $0 \leq \tau \leq t$ ,

$$\|\mathcal{S}_{\tilde{v}}^{(1)}(t, \tau) v_0\|_{L^\infty(\mathbb{R})} \leq C \frac{1}{(1+t-\tau)^\eta} \|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R})}. \quad (4.20)$$

If  $\kappa > 0$  is small enough, then there exists positive constants  $\eta$  and  $C$  such that

$$\|\omega_\kappa^{-1} \mathcal{S}_{\tilde{v}}^{(1)}(t, \tau) \partial_y v_0\|_{L^\infty(\mathbb{R})} \leq C e^{-\eta(t-\tau)} \|\omega_\kappa^{-1} \partial_y v_0\|_{L^\infty(\mathbb{R})}. \quad (4.21)$$

*Proof.* Remark that  $a_1 = a_0$  and that from Lemma 4.4.2 point (1),  $b_1$  satisfies the same bound as  $b_0$ . Hence the proofs are identical.  $\square$

## 4.5 Dynamic for the complete problem

Here we simply incorporate  $\mathcal{N}$ ,  $\mathcal{R}$  and  $\mathcal{H}$  – see equation (4.9) and lemma 4.3.3 – into the argument.

**Proposition 4.5.1.** *There exists positive constants  $C$ ,  $\delta$ , such that for all  $v_0 \in BUC^1(\mathbb{R})$  satisfying  $\|v_0\|_{W^{1,\infty}(\mathbb{R})} \leq \delta$ , the solution of (4.9) with initial condition  $v(0, \cdot) = v_0$  is defined for all time, and satisfies:*

$$\|v(t)\|_{W^{1,\infty}(\mathbb{R})} \leq C \|v_0\|_{W^{1,\infty}(\mathbb{R})}.$$

*Proof.* Let  $\delta$  be imposed from Lemma 4.4.2, fix  $v_0$  with  $\|v_0\|_{W^{1,\infty}(\mathbb{R})} \leq \frac{\delta}{2}$ , and write (4.9) under

Duhamel form:

$$v(t) = \mathcal{S}_v(t, 0)v_0 + \int_0^t \mathcal{S}_v(t, \tau)\mathcal{N}(v(\tau))d\tau. \quad (4.22)$$

For some  $t_0 > 0$ , the solution  $v(t)$  is defined in the classical sense for all  $t \in [0, t_0]$  – see [Kru70] and [Bre00, chapter 6] for well-posedness of balance laws – and satisfies  $\|v(t)\|_{W^{1,\infty}(\mathbb{R})} \leq \delta$ .

Since  $\mathcal{N}(v) = e^{-\phi}N(e^{\phi}v)$  with  $N$  that is at least quadratic, there exists a positive constant  $C$  and  $\kappa > 0$  such that

$$\|\mathcal{N}(v)\|_{L^\infty(\mathbb{R})} \leq C\|\omega_\kappa v^2\|_{L^\infty(\mathbb{R})}.$$

When  $t \in [0, t_0]$ , Propositions 4.4.5 and 4.4.6 apply, there exists  $\eta > 0$  such that

$$\begin{aligned} \|v(t)\|_{L^\infty(\mathbb{R})} &\leq \|v_0\|_{L^\infty(\mathbb{R})} + C \int_0^t \|\omega_\kappa^{-1}\mathcal{S}_v(t, \tau)\mathcal{N}(v(\tau))\|_{L^\infty(\mathbb{R})} d\tau, \\ &\leq \|v_0\|_{L^\infty(\mathbb{R})} + C \int_0^t e^{-\eta(t-\tau)}\|v(\tau)^2\|_{L^\infty(\mathbb{R})} d\tau, \\ &\leq \|v_0\|_{L^\infty(\mathbb{R})} + C\delta \int_0^t e^{-\eta(t-\tau)}\|v(\tau)\|_{L^\infty(\mathbb{R})} d\tau. \end{aligned}$$

Multiply on both sides by  $e^{\eta t}$  and apply a standard Gronwall Lemma to get that

$$\begin{aligned} \|v(t)\|_{L^\infty(\mathbb{R})} &\leq \|v_0\|_{L^\infty(\mathbb{R})} \left( 1 + C\delta \int_0^t e^{(C\delta-\eta)(t-\tau)} d\tau \right), \\ &\leq \|v_0\|_{L^\infty(\mathbb{R})} \left( 1 + \frac{C\delta}{\eta - C\delta} \right). \end{aligned}$$

Take  $\delta$  small enough so that the right hand side above is smaller than  $\delta$ , to conclude that the above estimate apply for all  $t \geq 0$ .

We proceed similarly to obtain the bound on the derivative. Recall the Duhamel formula:

$$\partial_y v(t) = \mathcal{S}_v^{(1)}(t, 0)\partial_y v_0 + \int_0^t \mathcal{S}_v^{(1)}(t, \tau)(\mathcal{H}(\tau) + \mathcal{R}(\partial_y v(\tau)))d\tau.$$

As above, chose  $\kappa > 0$  small enough such that  $1 \leq \omega_\kappa^{-1} \leq Ce^{-\phi}$ . Then Corollary 4.4.8 ensures the existence of  $\eta$  such that:

$$\begin{aligned} \|\partial_y v(t)\|_{L^\infty(\mathbb{R})} &\leq \|\partial_y v_0\|_{L^\infty(\mathbb{R})} + \int_0^t \|\omega_\kappa^{-1}\mathcal{S}_v^{(1)}(t, \tau)(\mathcal{H}(\tau) + \mathcal{R}(\partial_y v(\tau)))\|_{L^\infty(\mathbb{R})} d\tau, \\ &\leq \|\partial_y v_0\|_{L^\infty(\mathbb{R})} + C \int_0^t e^{-\eta(t-\tau)}\|e^{-\phi}(\mathcal{H}(\tau) + \mathcal{R}(\partial_y v(\tau)))\|_{L^\infty(\mathbb{R})} d\tau. \end{aligned}$$

From the above proof,  $\|v(t)\|_{L^\infty(\mathbb{R})} \leq \delta$  for all times  $t \geq 0$ . Further assuming that for all  $\tau \in [0, t_0]$ ,  $\|\partial_y v(\tau)\|_{L^\infty(\mathbb{R})} \leq C_1\delta$ , we get from Lemma 4.3.3 that for  $t \leq t_0$ :

$$\|\partial_y v(t)\|_{L^\infty(\mathbb{R})} \leq \|\partial_y v_0\|_{L^\infty(\mathbb{R})} + \delta \int_0^\infty e^{-\eta\tau} d\tau + C_1\delta \int_0^t e^{-\eta(t-\tau)}\|\partial_y v\|_{L^\infty(\mathbb{R})} d\tau.$$

Multiplying both sides of the above inequality by  $e^{\eta t}$ , a Gronwall Lemma ensures that

$$\|\partial_y v(t)\|_{L^\infty(\mathbb{R})} \leq \left( \|\partial_y v_0\|_{L^\infty(\mathbb{R})} + C\delta \right) \left( 1 + \frac{C_1\delta}{\eta - C_1\delta} \right),$$

provided that  $\eta - C_1\delta > 0$ . By taking  $\delta$  sufficiently small, this condition is ensured, and there exists a positive constant  $C_2$  such that

$$\|\partial_y v(t)\|_{L^\infty(\mathbb{R})} \leq C_2\delta.$$

Furthermore, by taking  $\delta$  small enough, the constant  $C_2$  is independent of  $C_1$ , so that we can choose  $C_1 \stackrel{\text{def}}{=} C_2$  in the first place. This allows to let  $t_0$  go to  $\infty$ . Since the proof holds for any smaller  $\delta$  than the one immediately above, we can restrict to  $v_0$  such that  $\|v_0\|_{W^{1,\infty}(\mathbb{R})} = \delta$ , and the proof is complete.  $\square$

**Proposition 4.5.2.** *Let  $\eta > 0$ . There exists positive constants  $C, \delta$ , such that for all  $v_0 \in BUC^1(\mathbb{R})$  satisfying  $\|\rho_\eta^{-1} v_0\|_{W^{1,\infty}(\mathbb{R})} \leq \delta$ , the solution of (4.9) with initial condition  $v(0, \cdot) = v_0$  satisfies:*

$$\|v(t)\|_{W^{1,\infty}(\mathbb{R})} \leq \frac{C}{(1+t)^\eta} \|\rho_\eta^{-1} v_0\|_{W^{1,\infty}(\mathbb{R})}.$$

*Proof.* Let  $\delta$  be fixed by the previous proposition 4.5.1. Then  $v$  is defined globally in time, and is so small that the linear decay from Proposition 4.4.6 is available. The Duhamel formulae writes

$$v(t) = \mathcal{S}_v(t, 0)v_0 + \int_0^t \mathcal{S}_v(t, \tau) \mathcal{N}(v(\tau)) d\tau.$$

Estimate the initial condition term with (4.20), and the integral term with (4.16) to obtain that

$$\|v(t)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{(1+t)^\eta} \|\rho_\eta^{-1} v_0\|_{L^\infty(\mathbb{R})} + \int_0^t e^{-\tilde{\eta}(t-\tau)} \|\omega_{\tilde{\kappa}}^{-1} \mathcal{N}(v(\tau))\|_{L^\infty(\mathbb{R})} d\tau. \quad (4.23)$$

Using the extra localization of the non-linear term, we obtain for  $\tilde{\kappa} > 0$  small enough that there exists a positive constant  $C$  such that

$$\|\omega_{\tilde{\kappa}}^{-1} \mathcal{N}(v)\|_{L^\infty(\mathbb{R})} \leq C \|\omega_{\tilde{\kappa}}^{-1} e^\phi v^2\|_{L^\infty(\mathbb{R})} \leq C\delta \|v\|_{L^\infty(\mathbb{R})}.$$

Insert it into (4.23), multiply on both sides by  $e^{-\tilde{\eta}t}$  and apply a Gronwall Lemma to obtain the claimed estimate with  $L^\infty(\mathbb{R})$  norms. To convert these into  $W^{1,\infty}(\mathbb{R})$  norms, control the derivative  $\partial_y v$  using the estimates on  $\mathcal{S}_v^{(1)}$ , as done above: for  $\tilde{\kappa} > 0$  small enough such that  $1 \leq \omega_{\tilde{\kappa}}^{-1} \leq C e^{-\phi}$ , Corollary 4.4.8 ensures the existence of  $\tilde{\eta}$  such that we get:

$$\begin{aligned} \|\partial_y v(t)\|_{L^\infty(\mathbb{R})} &\leq C \frac{\|\rho_\eta^{-1} \partial_y v_0\|_{L^\infty(\mathbb{R})}}{(1+t)^\eta} + \int_0^t \|\omega_{\tilde{\kappa}}^{-1} \mathcal{S}_v^{(1)}(t, \tau) (\mathcal{H}(\tau) + \mathcal{R}(\partial_y v(\tau)))\|_{L^\infty(\mathbb{R})} d\tau, \\ &\leq C \frac{\|\rho_\eta^{-1} \partial_y v_0\|_{L^\infty(\mathbb{R})}}{(1+t)^\eta} + C \int_0^t e^{-\tilde{\eta}(t-\tau)} \|e^{-\phi} (\mathcal{H}(\tau) + \mathcal{R}(\partial_y v(\tau)))\|_{L^\infty(\mathbb{R})} d\tau. \end{aligned}$$

To control the forcing term  $\mathcal{H}$ , use Lemma 4.3.3 together with the above decay of  $\|v\|_{L^\infty(\mathbb{R})}$ .

Relying on lemma B.2.1 below, we obtain that

$$\|\partial_y v(t)\|_{L^\infty(\mathbb{R})} \leq C \frac{\|\rho_\eta^{-1} v_0\|_{W^{1,\infty}(\mathbb{R})}}{(1+t)^\eta} + C \int_0^t e^{-\tilde{\eta}(t-\tau)} \|e^{-\phi} \mathcal{R}(\partial_y v(\tau))\|_{L^\infty(\mathbb{R})} d\tau.$$

The residual term  $\mathcal{R}$  is treated as in the previous proof, and a Gronwall Lemma conclude this proof.  $\square$



# Conclusion

## Sommaire du chapitre

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Dans cette thèse, j'ai étudié la stabilité asymptotique de fronts de propagation. Pour chacun des problèmes considérés, des questions naturelles sont encore ouvertes, et sont discutées dans une première section. Plus généralement, ces trois années de thèses ont soulevé d'autres interrogations liées à des problèmes connexes, qui sont abordées dans une seconde section.

## 5.1 Questions reliées au contenu de la thèse

### Comportement de FKPP étendue pour $\delta = \delta_* = 1/\sqrt{12}$ et au-delà

Le chapitre 2 conclut à la stabilité du front le plus lent lorsque  $\delta$  est petit dans l'équation de FKPP étendue. En particulier, cette étude n'est plus valable pour le régime d'intérêt où  $\delta$  est proche de la valeur critique  $\delta_* = \frac{1}{\sqrt{12}}$ .

Pour ce régime, l'existence de fronts de propagation (pour chaque vitesse  $c > 0$ ) est connue grâce à des méthodes topologiques, *c.f.* [BHV01]. Cependant, on dispose de très peu d'informations à leur sujet. Mettons en avant que pour une valeur de  $\delta$  fixé, tous les fronts pourraient être oscillants (et le sont vraisemblablement), de sorte que la distinction entre front sur- et sous-critiques ne serait accessible que d'un point de vue de la stabilité spectrale. Dans ce cadre, il serait intéressant d'explicitier le comportement des fonctions de Green lorsque le coefficient de diffusion devient nul.

### Décrire les oscillations de Turing par les fonctions de Green

Dans le chapitre 3, une partie importante de la preuve est de mener en parallèle deux arguments de stabilité complémentaires. Même si cette approche fonctionne ici, elle est légèrement

inconfortable à mettre en place. Une façon de découpler la stabilité dans les différents espaces à poids serait de considérer pour la perturbation  $V$  la dynamique totale  $\partial_t V = \mathcal{T}V + \mathcal{Q}(V)$ , plutôt que de se restreindre à la dynamique en  $-\infty$ . Dans la preuve actuelle, cette restriction est nécessaire pour contrôler les oscillations instables : elle permet de comparer efficacement les termes linéaires et non-linéaires via la transformation de Fourier, voir le lemme 3.4.6. Adapter cette démarche à des opérateurs à coefficients non-constants – par exemple en utilisant la fonction de Green pour décrire les motifs périodiques – permettrait ainsi de simplifier la démarche globale. Le point clef semble être l’adaptation du lemme cité ci-dessus dans le domaine de Laplace.

## Front critique pour les équations de bilan

Lorsque la condition initiale est à support compact, on s’attend à observer un front qui se déplace seulement à la vitesse critique  $c_*$ . Si on se place dans un tel référentiel, il n’est pas possible de stabiliser la dynamique à l’infini avec un poids exponentiel. L’étude de la stabilité, ou de la stabilité asymptotique d’un tel objet est une question intéressante et naturelle.

## 5.2 Problèmes connexes

### Fronts monostable en cascade

De nombreuses situations biologiques mettent en jeu plusieurs fronts qui envahissent l’espace successivement : une première espèce envahit l’espace rapidement, sa présence permet à une autre espèce plus lente de se propager. On parle de cascades ou de terrasses.

On peut apercevoir un exemple de telle structure dans le schéma global de preuve du chapitre 3 : l’instabilité de l’état à l’arrière du front est contrôlée par l’existence d’un état stable, ici le motif périodique de Turing. Ainsi, les perturbations à l’arrière du front suivent une équation de Ginzburg-Landau – c.f. l’amplitude  $A$  de la section 3.4.2 – et forment vraisemblablement un second front derrière celui étudié, qui connecterait l’état constant et instable 1 à l’état stable et périodique de Turing.<sup>1</sup>

De telles cascades de fronts ont été étudiées lorsque tous les états connectés sont stables, et sont constants ou périodiques. Voir [GM20; GR20] et [LS15] pour l’existence de profils, et leur stabilité. L’article récent [GL19] étudie la situation de deux fronts monostables successifs, dans un cadre d’une compétition entre espèces. Dans un premier temps, une espèce faible et rapide envahit un espace vide. Dans un second temps, l’autre espèce plus lente mais plus forte, remplace la première. Une conclusion importante concerne les vitesses respectives d’invasion : lorsque les deux vitesses attendues sont proches, la présence de la première espèce a pour effet d’accélérer le déplacement de la seconde. Ce mécanisme est également identifié pour d’autres problèmes, voir [HS14a; FGH21].

On ne s’attend pas à ce que ce phénomène soit présent dans le système présenté au chapitre 3, car le second front se déplace très lentement, à vitesse  $\mathcal{O}(\sqrt{\mu})$ . Cependant, cette interaction entre fronts monostables n’est à ce jour pas encore décrite d’un point de vue dynamique. Avec Bastian Hilder, post-doctorant à l’université de Lund, nous essayons de comprendre ces interactions, en montrant la stabilité asymptotique d’un recollement de deux fronts monostables. Si cette approche est concluante, on peut espérer qu’elle s’applique à l’équation de FKPP avec terme de réaction non-local.

<sup>1</sup>. On note au passage que la formation d’un tel front de Turing – borné en espace – est nécessaire à l’obtention de la stabilité du front principal. Ceci est à mettre en parallèle avec la nécessité de l’état stable 1 dans la formation d’un front FKPP.

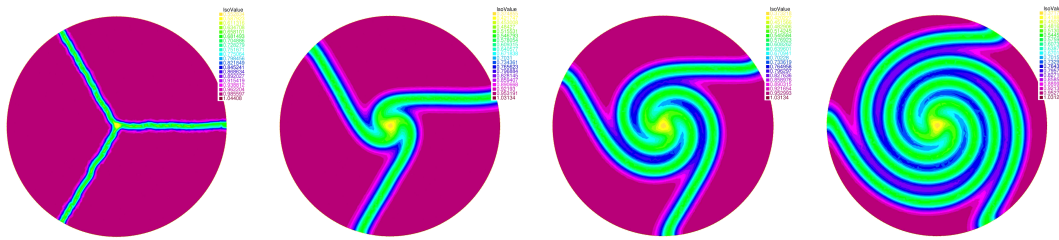


FIGURE 5.1 – Solution d’un système de compétition asymétrique avec réactions de type FKPP à trois espèces  $u, v, w$ , aux temps  $t = 0, 10, 20$  et  $30$ . On a représenté  $h \stackrel{\text{def}}{=} (u^2 + v^2 + w^2)^{1/2}$ , l’échelle de couleur va de  $h = 0.4$  en jaune à  $h = 1$  en rouge. Pour la condition initiale, chaque espèce occupe un tiers du domaine spatial.

## Spirales

Pour des problèmes en plusieurs dimensions d’espace, par exemple  $\Omega = \mathbb{R}^2$ , la plupart des phénomènes que l’on a évoqué sont conservés, les fronts sont simplement remplacés par des fronts planaires  $u(t, x, y) = q_c(x - ct)$  et se déplacent dans une direction privilégiée. Une situation intrinsèquement deux-dimensionnelle est celle d’une compétition entre trois espèces, suivant un modèle pierre-feuille-ciseaux. Dans ce cas, on voit apparaître une spirale d’Archimède à trois branches, voir figure 5.1. Étudier la stabilité d’un tel objet pour un système précis semble une question très intéressante. Le travail de [SS20] décrit de manière précise le spectre essentiel de telles spirales dans un cadre général, lorsque la vitesse radiale est dirigée vers l’extérieur de la spirale. Il serait une fois de plus intéressant de le confronter à l’exemple explicite mentionné plus haut. On remarque que l’étude du spectre ponctuel reste ouverte, et que l’analyse non-linéaire est à faire.

On mentionne [TVZ19; STVZ22] pour un résultat d’existence d’équilibres en forme de spirales, dans le cas où la compétition entre espèces est prise infini. Dans ce scénario, les espèces se ségrègent, et il est possible de décrire très précisément la frontière qui les sépare. Ainsi, Terracini Verzini et Zilio montrent qu’au voisinage de 0, la spirale a un profil logarithmique. Ceci contraste avec le comportement en spirales d’Archimède obtenu dans les simulations pour les  $x$  loin de 0.

## 5.3 Problèmes dispersifs

Les lignes qui précèdent donnent un bon aperçu de la stabilité d’ondes de propagation lorsque l’équation est diffusive ou dirigée par un terme de transport. Un phénomène qu’on ne décrit pas est le cas d’équations dispersives, comme par exemple l’équation de Schrödinger ou l’équation de KdV.

Nous projetons avec Björn de Rijk, Dr. à l’Institut Technologique de Karlsruhe (KIT) d’étudier dans un cadre dispersif la stabilité de constantes et d’ondes périodiques. Nous voulons comprendre quelles peuvent-être les interactions entre des termes diffusifs et des termes dissipatifs, comme cela a été fait dans [BPW94] pour une équation de Burgers-KdV à l’aide de semi-groupes renormalisés.

Pour ces équations, seule une décroissance non-intégrable est obtenue au niveau linéaire, contrairement aux situations que l’on a étudiées dans les chapitres précédents. Dans ce cas, il n’est possible de transférer la décroissance au niveau non-linéaire que si les termes  $u^p$  à ajouter ont un exposant  $p$  assez grand.

Il a été mis en évidence dans le travail [RS20] que la présence de transport aide à transférer



une décroissance diffusive au niveau non-linéaire. En quelques mots : si dans un système deux composantes  $u_i$  et  $u_j$  se déplacent à des vitesses différentes, alors les termes de la forme  $u_i u_j$  peuvent être ajoutés à l'argument non-linéaire (alors qu'ils ne peuvent pas l'être si les vitesses sont identiques).

Lorsque des termes dispersifs sont présents – par exemple  $\partial_{xxx}$  ou  $i\partial_{xx}$  – chaque fréquence est transportée à une vitesse qui lui est propre. On peut donc espérer retrouver un mécanisme similaire à celui de de Rijk et Schneider, permettant d'obtenir de la stabilité même en présence de termes non-linéaires critiques.

# Weak exponential decay

## Outline of the chapter

<b>A.1 Introduction</b>	<b>125</b>
<b>A.2 Result</b>	<b>125</b>

## A.1 Introduction

As mentioned in the above section 1.4.4, the presence or absence of a bounded solution to the eigenproblem at  $\lambda = 0$  strongly impacts the behavior of the dynamic for large times, since it contributes to a singularity of the spectral Green function. In both cases of the KPP and the extended KPP equations, the only potential eigenfunctions are the derivative of the fronts, which are unbounded in weighted spaces, due to their weak exponential decay  $q_*(x) \sim axe^{-\kappa_* x}$  with  $a \neq 0$ .

Here, we re-obtain this property from a version of Ikehara's theorem, which is a Tauberian result for the Laplace transform (here with respect to space). We assume that the dispersion relation has a double root  $-\kappa_*$ , and that no other root is located in between  $-\kappa_*$  and 0. Note that this result would apply to all monotonic fronts of the extended KPP equation, without restriction on  $\delta$ .

## A.2 Result

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^2$  function with two consecutive simple zeros

$$f(0) = 0 = f(1), \quad f'(1) < 0 < f'(0), \quad f(u) > 0 \text{ for all } u \in (0, 1).$$

Let  $n \in \mathbb{N}_{\geq 0}$  be an even number, and let  $P(X) = a_0 + a_1 X + \dots + a_n X^n$  be a polynomial, with  $a_0 = f'(0)$ . Assume that there exists  $\kappa_* > 0$  such that  $-\kappa_*$  is the closest-to-zero negative root of  $P$ , and that it is a double root: there exists a polynomial  $Q$  and a real  $\kappa_0 > \kappa_*$  such that

$$P(X) = (X + \kappa_*)^2 Q(X),$$

with  $Q(X) \neq 0$  for all  $X \in \{\lambda \in \mathbb{C} : -\kappa_0 \leq \operatorname{Re} \lambda \leq 0\}$ . Denote  $\tilde{P} = P - a_0$ , and assume that there exists  $y \mapsto \underline{u}(y)$  that satisfies

$$0 = \tilde{P}(\partial_y)\underline{u} + f(\underline{u}),$$

that connects the two constant states 1 and 0,

$$\lim_{y \rightarrow -\infty} \underline{u}(y) = 1, \quad \lim_{y \rightarrow +\infty} \underline{u}(y) = 0,$$

with the latter convergence occurring at generic exponential speed: there exists positive constant  $\kappa_1$  and  $C$  such that for all  $y \geq 0$ ,

$$\frac{1}{C} e^{-\kappa_0 y} \leq |\underline{u}(y) - 0| \leq C e^{-\kappa_1 y}. \quad (\text{A.1})$$

**Theorem A.2.1.** *Assume that  $f$  lies below its tangent at the origin:  $f(u) < u f'(0)$  for all  $u \in (0, 1)$ , and that  $y \mapsto \underline{u}(y)$  is monotonic. Then the front  $\underline{u}$  has weak exponential decay: there exists a non-zero constant  $\alpha$  such that when  $y \rightarrow +\infty$ ,*

$$\underline{u}(y) \sim \alpha y e^{-\kappa_* y}.$$

The two main ideas can be found in [CC04]: apply Ikehara's theorem, which is a Tauberian theorem for the Laplace transform, and use the monotonicity of  $\underline{u}$  to locate the singularity of its Laplace transform. We first recall a version of Hikehara's theorem that can be found in [ABHN11, chapter 4].

**Lemma A.2.2.** *Let  $\beta > -1$  and  $\lambda_0 \in \mathbb{C}$ . Assume there exists a holomorphic function  $H$  such that for all  $\lambda$  with  $\operatorname{Re} \lambda > \lambda_0$ :*

$$\check{u}(\lambda) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-\lambda y} u(y) dy = \frac{H(\lambda)}{(\lambda - \lambda_0)^{\beta+1}}.$$

Denote  $\Gamma$  the Euler Gamma function. Then when  $y \rightarrow +\infty$ ,

$$\frac{u(y)}{y^\beta e^{\lambda_0 y}} \rightarrow \frac{H(\lambda_0)}{\Gamma(\beta+1)}.$$

*Proof.* Note  $g(y) = \frac{u(y)}{y^\beta e^{\lambda_0 y}}$ , and apply [ABHN11, Theorem 4.2.8] with  $f \stackrel{\text{def}}{=} g$  and  $t \stackrel{\text{def}}{=} y$ , to obtain the claimed convergence in Cesàro's sense. Applying Theorem 4.1.2 - c therein, we obtain convergence in Abel's sense. Finally, the Theorem 4.2.14 therein applies using (A.1), so that the claimed convergence holds in the usual sense.  $\square$

*Proof. (Theorem A.2.1).* Recall that the front  $\underline{u}$  satisfies

$$P(\partial_y)\underline{u} = f'(0)\underline{u} - f(\underline{u}). \quad (\text{A.2})$$

Fix  $y_0 \in \mathbb{R}$ , and denote  $\check{u}(\lambda) \stackrel{\text{def}}{=} \mathfrak{L} u(\lambda) \stackrel{\text{def}}{=} \int_{y_0}^{+\infty} e^{-\lambda y} u(y) dy$  the Laplace transform of  $u$ . Then, with a few integration by parts, we obtain that for all  $\lambda \in \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \geq -\kappa_0\}$ ,

$$\mathfrak{L}(P(\partial_y)\underline{u}) = P(\lambda)\check{u}(\lambda) - S(\lambda, y_0) = I(\lambda, y_0), \quad (\text{A.3})$$

where we have defined

$$I(\lambda, y_0) \stackrel{\text{def}}{=} \int_{y_0}^{+\infty} [f'(0)\underline{u}(y) - f(\underline{u}(y))] e^{-\lambda y} dy,$$

and

$$S(\lambda, y_0) \stackrel{\text{def}}{=} \sum_{k=1}^n a_k \sum_{j=0}^{k-1} \lambda^j \partial_y^{k-1-j} u(y_0).$$

Rearranging, (A.3) writes

$$\check{u}(\lambda) = \frac{H(\lambda)}{(\lambda + \kappa_*)^2}, \quad (\text{A.4})$$

with  $H$  the function defined by

$$H(\lambda, y_0) \stackrel{\text{def}}{=} \frac{S(\lambda, y_0) + I(\lambda, y_0)}{Q(\lambda)}.$$

Remark that  $\lambda \mapsto H(\lambda, y_0)$  is holomorphic on  $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) > -\min(2\kappa_1, \kappa_0)\}$ : using (A.1) and a Taylor expansion of  $f$  we obtain that when  $y \rightarrow +\infty$ ,

$$f(\underline{u}(x)) - f'(0)\underline{u}(x) = \mathcal{O}(\underline{u}(x)^2) = \mathcal{O}(e^{-2\kappa_1 x}).$$

We claim that there exists a  $y_0 \in \mathbb{R}$  such that  $H(-\kappa_*)$  is non-zero. Indeed, since  $\underline{u}$  is monotonic and converges to 1 at  $-\infty$ , we have that  $I(\lambda, y_0)$  converges to the finite value  $\sum_{k=1}^n a_k \lambda^{k-1}$  when  $y_0 \rightarrow -\infty$ . On the contrary,  $f'(0)\underline{u}(y) - f(\underline{u}(y))$  converges to  $f'(0) > 0$ , so that  $I(\lambda, y_0) \rightarrow +\infty$  when  $y_0 \rightarrow -\infty$ . Hence the claim holds, we fix such a  $y_0$ .

Since  $\underline{u}$  is monotonic, we know from [Wid15, theorem5b p.58] that  $\check{u}(\lambda)$  can only have singularity on its axis of convergence. Then in view of (A.4) and using that  $2\kappa_1 > \kappa_1$ , we conclude that  $\kappa_1 = \kappa_*$ . Hence using continuity extension, equation (A.4) holds on  $\{\lambda \in \mathbb{C} : \text{Re}(\lambda) \geq \kappa_*\} \setminus \{\kappa_*\}$ . This allows to apply Lemma A.2.2: when  $y \rightarrow +\infty$  we have

$$\frac{\underline{u}(y)}{y e^{-\kappa_* y}} \rightarrow H(-\kappa_*) \neq 0.$$

□



# Appendix B

## Standard Lemmas

### B.1 Relative to chapter 3

**Lemma B.1.1.** Let  $M$  be a matrix with polynomial coefficients:  $M(X) = (P_{i,j}(X))_{1 \leq i,j \leq n}$ , we note  $d := \max_{i,j} \deg(P_{i,j})$  the greatest degree in  $M$ , and  $\mathcal{L}$  the associated differential operator:  $\mathcal{L} = M(\partial_x)$ . Then the conjugation of  $\mathcal{L}$  by the weight  $e^{\vartheta x}$  writes as:

$$e^{-\vartheta x} \mathcal{L} e^{\vartheta x} = M(\vartheta + \partial_x) = \sum_{k=0}^d \frac{1}{k!} M^{(k)}(\vartheta) \partial_x^k,$$

where  $M^{(k)}$  is the matrix whose coefficients are the polynomial's derivatives:  $(P_{i,j}^{(k)})_{1 \leq i,j \leq n}$ .

*Proof.* We first show the scalar case  $n = 1$ . By linearity, it is enough to consider the case  $\mathcal{L} = \partial_x^d$ , which corresponds to  $P(X) = X^d$ . It can be easily computed that  $e^{-\vartheta x} \partial_x (e^{\vartheta x}) = (\vartheta + \partial_x)$ , rising it to the power  $d$  reads:  $e^{-\vartheta x} \mathcal{L} (e^{\vartheta x}) = (\vartheta + \partial_x)^d = P(\vartheta + \partial_x)$ . To finish the scalar case, we do a Taylor expansion at order  $d$ . Such expansion is exact due to  $P$  being polynomial:

$$P(\vartheta + X) = \sum_{k=0}^d \frac{P^{(k)}(\vartheta)}{k!} X^k.$$

We then formally replace  $X$  by  $\partial_x$ . The case of a matrix of differential operator is straightforward, it suffices to apply the above to each component of the matrix.  $\square$

**Lemma B.1.2.** Let  $\mathcal{L} = \sum_{j=0}^n a_j(x) \partial_x^j$  be a differential operator with coefficients  $x \mapsto a_j(x)$ . Let  $\phi_1, \dots, \phi_n$  be solutions of the ODE

$$(\lambda - \mathcal{L})\phi = 0, \tag{B.1}$$

and note  $\Phi_i = (\phi_i, \dots, \partial_x^{n-1} \phi_i)^T$ . Then

$$\det(\Phi_1, \dots, \Phi_n)(\lambda, x) = \exp\left(-\int_0^x a_{n-1}(s) ds\right) \det(\Phi_1, \dots, \Phi_n)(\lambda, 0).$$

*Proof.* For the proof, we will note  $\Phi$  the  $n \times n$  matrix whose  $i$ -th column is  $\Phi_i$ . Remark that  $\mathcal{W}(\lambda, x) := \det(\Phi)(\lambda, x)$  satisfy:

$$\partial_x \mathcal{W} = \text{Tr}\left(\text{Com}(\Phi)^T \cdot \partial_x(\Phi)\right) = \text{Tr}\left(\text{Com}(\Phi)^T \cdot A \cdot \Phi\right),$$

where  $A(\lambda, x)$  is the  $n \times n$  matrix obtained if one vectorise the ODE (B.1). Since the trace is invariant under circular permutation, we recognize

$$\partial_x \mathcal{W} = \text{Tr}\left(A \cdot \Phi \cdot \text{Com}(\Phi)^T\right) = \det(\Phi) \text{Tr}(A).$$

This scalar ODE on  $\mathcal{W}$  is easily solved. Noticing that  $\text{Tr}(A(\lambda, x)) = -a_{n-1}(x)$  conclude the proof.  $\square$

**Lemma B.1.3.** *Let  $\kappa_1$  and  $\kappa_2$  be two positive constants, and let  $\kappa \stackrel{\text{def}}{=} \min(\kappa_1, \kappa_2)$ . Then for all  $(x, y) \in \mathbb{R}^2$ , we have the following bound on the convolution of two exponential distributions:*

$$\int_{\mathbb{R}} e^{-\kappa_1|x-\tau|} e^{-\kappa_2|\tau-y|} d\tau \leq \frac{2}{\kappa} e^{-\frac{\kappa}{2}|x-y|}.$$

*Proof.* It is enough to bound  $I(x, y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \exp\left(\frac{\kappa}{2}|x-y| - \kappa|x-\tau| - \kappa|\tau-y|\right) d\tau$ . From triangular inequality, it follows that

$$I(x, y) \leq \int_{\mathbb{R}} \exp\left(-\frac{\kappa}{2}(|x-\tau| + |\tau-y|)\right) d\tau \leq \int_{\mathbb{R}} \exp\left(-\kappa\left|\frac{x+y}{2} - \tau\right|\right) d\tau \leq \frac{2}{\kappa},$$

and the proof is complete.  $\square$

**Lemma B.1.4.** *Let  $p, q \in \mathbb{R} \cup \{+\infty\}$  such that  $1 \leq q \leq p \leq +\infty$ , and  $X, Y$  be two measurable spaces. Then the following injection is continuous:*

$$L^q(X, L^p(Y, \mathbb{R})) \subset L^p(Y, L^q(X, \mathbb{R})).$$

*Proof.* This is the well-known Minkowsky's integral inequality. We claim that it can be obtained also from interpolation between Lebesgue's spaces, see [BL76, Theorem 5.1.2].  $\square$

**Lemma B.1.5.** *Let  $\alpha, \beta > 1$  and  $\gamma > 0$  be reals. Then there exists  $C > 0$  such that for all  $t \geq 0$ :*

$$\int_0^t \frac{ds}{(1+s)^\alpha (1+t-s)^\beta} \leq \frac{C}{(1+t)^{\min(\alpha, \beta)}}, \quad \int_0^t \frac{e^{-\gamma(t-\tau)}}{(1+\tau)^\alpha} d\tau \leq \frac{C}{(1+t)^\alpha}. \quad (\text{B.2})$$

*Proof.* One can slightly adapt this statement when  $\alpha = 1$  or  $\beta = 1$ , see [Xin92]. First, remark that

$$I_{\alpha, \beta}(t) := \int_0^{t/2} \frac{ds}{(1+s)^\alpha (1+t-s)^\beta} \leq \frac{1}{(1+t/2)^\beta} \left[ \frac{1-\alpha}{(1+s)^{\alpha-1}} \right]_0^{t/2} \leq \frac{\alpha-1}{(1+t/2)^\beta}.$$

Hence, we cut the integral in (B.2) at  $t/2$ , and the change of variable  $z = t - s$  leads to

$$\int_0^t \frac{ds}{(1+s)^\alpha (1+t-s)^\beta} = I_{\alpha, \beta}(t) + I_{\beta, \alpha}(t) \leq \frac{C}{(1+t/2)^{\min(\alpha, \beta)}} \leq \frac{C \cdot 2^{\min(\alpha, \beta)}}{(1+t)^{\min(\alpha, \beta)}}.$$

Which is the first claimed estimate. The second one follows from  $e^{-\gamma(t-\tau)} \leq \frac{C}{(1+\tau)^\alpha}$ .  $\square$

**Lemma B.1.6.** *Let  $b$  be a positive constant, and note  $a_0(b) = \frac{1}{4b} > 0$ . Then for all  $0 < a < a_0$ , the following holds. If  $t \mapsto x(t)$  is a positive continuous function that satisfy  $x(t) \leq a + bx(t)^2$  and such that  $x(0) \leq 2a$ , then  $x(t) \leq 2a$  for all  $t \geq 0$ .*

*Proof.* Introduce  $f(x) = a + bx^2 - x$ . Then  $f(2a) = a(4ba - 1) < 0$  when  $a$  is smaller than  $a_0$ , while  $f(0) = a > 0$ . Assume  $x(0) \leq 2a$ . Then the connected component of  $f^{-1}(\mathbb{R}_+)$  that contains  $x(0)$  is include in  $(-\infty, 2a)$ . Since  $f(x(t)) \geq 0$  by construction, the continuity of  $x$  implies  $x(t) \leq 2a$  for all  $t > 0$ . This conclude the proof.  $\square$

## B.2 Relative to chapter 4

**Lemma B.2.1.** *Let  $\alpha$  and  $\eta$  be real positive constants. Then there exist a positive constant  $C$  such that for all  $t > 0$ ,*

$$\int_0^t e^{-\eta(t-\tau)} \frac{1}{(1+\tau)^\alpha} d\tau \leq \frac{C}{(1+t)^\alpha}.$$

*Proof.* Cut the integral at  $\tau = t/2$  to obtain the result. From one hand,

$$\int_0^{t/2} e^{-\eta(t-\tau)} \frac{1}{(1+\tau)^\alpha} d\tau \leq e^{-\frac{\eta}{2}t} \frac{t}{2} \leq \frac{2}{\eta} e^{-\frac{\eta}{4}t}.$$

From the other hand,

$$\int_{t/2}^t e^{-\eta(t-\tau)} \frac{1}{(1+\tau)^\alpha} d\tau \leq \frac{1}{(1+t/2)^\alpha} \int_{t/2}^t e^{-\eta(t-\tau)} d\tau \leq \frac{2^\alpha}{(1+t)^\alpha} \frac{1}{\eta}.$$

Sum the two above estimates to conclude the proof.  $\square$

**Lemma B.2.2.** *Let  $\alpha, \beta, \gamma$  be positive constants such that  $\alpha + \beta \leq \gamma$ . Then there exists a positive constant  $C$  such that for all  $x \geq 0$  and  $y \geq 0$ ,  $\langle x \rangle^\alpha \langle y \rangle^\beta \leq C \langle x + y \rangle^\gamma$ .*

*Proof.* If  $x \leq 1$  or  $y \leq 1$ , the bound is trivial. When  $x \geq 1$  and  $y \geq 1$ , take the inequality to the power  $\frac{1}{\gamma}$ , to restrict to the case  $\gamma = 1$ . Then both  $\alpha \leq 1$  and  $\beta \leq 1$ , so that

$$(1+x^2)^\alpha (1+y^2)^\beta \leq (1+x^{2\alpha})(1+y^{2\beta}) \leq 1+x^{2\alpha} + y^{2\beta} + \max(x^2, y^2)^{\alpha+\beta} \leq 2(1+x^2 + y^2).$$

Using that  $xy$  is positive, the left hand side is bounded by  $2\langle x + y \rangle^2$ .  $\square$





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Résumé

Certaines équations aux dérivées partielles admettent des solutions en onde de propagation. Dans le cas particulier où le profil de l'onde connecte deux états constants distincts, on parle de front. Les équations de réaction-diffusion, introduites pour modéliser de nombreux phénomènes biologiques, laissent apparaître de telles solutions. Savoir lesquels de ces fronts sont stables est important, puisque ceux-ci attirent alors les solutions génériques, et décrivent donc leurs propriétés. Dans cette thèse, on s'intéresse à des fronts monostables, c'est-à-dire à des ondes connectant un état constant stable à un état constant instable. On obtient dans trois scénarios différents la stabilité asymptotique de fronts monostables.

Le chapitre 2 traite d'une équation de FKPP étendue. Cette EDP présente une dérivée d'ordre 4 dont le coefficient est choisi arbitrairement proche de 0. Dans ce contexte, nous obtenons à nouveau l'existence d'un front critique. De plus, nous étudions les propriétés spectrales de la linéarisation au voisinage de ce front. La description précise de ce spectre, et en particulier la situation à l'origine, a pour conséquence la stabilité asymptotique du front.

Le chapitre 3 s'intéresse à une situation où l'état constant à l'arrière d'un front FKPP devient également instable. Cette situation apparaît naturellement dans l'équation FKPP avec terme de réaction non-local, mais est plutôt étudiée ici pour un système. Lorsqu'une bifurcation de Turing déstabilise l'état constant à l'arrière, on montre que le front FKPP critique reste stable dans un repère en translation.

Le chapitre 4 étudie des équations de bilan, aussi appelées équations de réaction-advection. L'absence de laplacien, et la présence d'un terme quasi-linéaire, peuvent mener à des solutions discontinues en espace. Bien que le cadre semble différent, nos conclusions sont très similaires au cas FKPP. En particulier, nous obtenons la stabilité de fronts surcritiques.

**Mots clés :** équations de réaction-diffusion, fronts de propagation, stabilité asymptotique non-linéaire, stabilité spectrale, estimées de résolvante ponctuelles, espaces à poids, fronts FKPP critiques

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ASYMPTOTIC STABILITY OF INVASION FRONTS IN REACTION-DIFFUSION EQUATIONS

Abstract

Several Partial Differential Equations admit propagating waves as solutions. In case where the wave profile connects two distinct constant states, these waves are called fronts. They typically appear as solutions of reaction-diffusion equations, and model a large number of biological phenomena. To decide which of these solutions are the stable ones is fundamental, since the latter attract, and thus describe, generic solutions. In this PhD, we focus on monostable fronts, that are waves connecting a constant stable state to a constant unstable state. We obtain in three distinct situations the asymptotic stability of monostable fronts.

Chapter 2 deals with an extended FKPP equation. This PDE contains a fourth order space derivative, whose coefficient is chosen arbitrarily close to 0. In this context, we re-obtain the existence of a critical front. Furthermore, we study the spectral properties of the linearization near this front. The asymptotic stability of the latter is a consequence of the precise description of the spectrum, in particular of the situation at the origin.

Chapter 3 is about a situation where the constant state behind a FKPP front destabilizes. Although this phenomenon naturally appears in the FKPP equation with non-local reaction term, we rather investigate it for a system of equations. When a Turing bifurcation destabilizes the constant state behind the front, we show that the critical FKPP front stays asymptotically stable in a co-moving frame.

Chapter 4 studies balance laws, also known as reaction-advection equations. The absence of Laplacian, together with the presence of a quasi-linear term, may lead to space-discontinuous solutions. Despite the apparently distinct framework, our conclusions are very similar to the FKPP case. In particular, we obtain super-critical fronts stability.

**Keywords:** reaction diffusion equations, propagation fronts, non-linear asymptotic stability, spectral stability, pointwise resolvent estimates, weighted spaces, critical FKPP fronts

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