Autour de l'équation de Schrödinger-Langevin et du système d'Euler amorti

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Autour de l'équation de Schrödinger-Langevin et du système d'Euler isotherme amorti

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Chapitre 1

Introduction

On introduit dans ce chapitre les contextes mathématique et physique de cette thèse, en essayant de mettre en avant le type de résultats que l'on est en mesure d'attendre pour l'équation de Schrödinger-Langevin : notions de solution, comportement en temps long des solutions sous certaines hypothèses de régularité (on parle de "if theorem"), passage à la limite vis-à-vis de certains paramètres de l'équation.

On commence par donner certaines notations et conventions qui seront utilisées et adoptées tout au long de ce manuscrit, en particulier pour l'écriture des systèmes issus de la mécanique des fluides compressibles, et on énonce certains lemmes d'analyse fonctionnelle importants pour l'étude de l'équation de Schrödinger logarithmique.
On présente ensuite le contexte des équations de Schrödinger non-linéaires, en particulier les spécificités introduites par la présence d'une non-linéarité logarithmique : invariance d'échelle, non-positivité de l'énergie, propagation de solutions gaussiennes.
Enfin, on introduit certaines notions de solutions faibles pour des systèmes de mécanique des fluides compressibles de type Euler ou Navier-Stokes qui vérifient certaines estimations a priori (comme l'estimation d'énergie ou l'estimation de BD-entropy), ce qui permet de fournir un cadre d'étude rigoureux pour l'équation de Schrödinger-Langevin.

1.1 Notations

Pour la variable de temps t, les espaces considérés seront des intervalles $I \subset \mathbb{R}$, l'ensemble des réels positifs $\mathbb{R}_+$ ou l'ensemble $\mathbb{R}$ lui-même. Pour la variable d'espace x, on considérera généralement l'espace des réels $\mathbb{R}^d$ de dimension $d \in \mathbb{N}^*$ ou le tore $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$.

De plus, la notation générique C désignera une constante strictement positive et uniforme vis-à-vis des différents paramètres pertinents du problème, que l'on précisera parfois explicitement en écrivant $C = C_{a_1,\dots,a_n}$ ou $C = C(a_1, \dots, a_n)$.

On notera indistinctement $\overline{\psi}$ ou ψ^* le conjugué complexe de la fonction ψ. On exprime également la moyenne d'une fonction ψ sur $\Omega = \mathbb{R}^d$ ou $\mathbb{T}^d$ par la notation

$$\langle \psi \rangle := \int_\Omega \psi(x) dx.$$

1.1.1 Opérateurs différentiels

Soit $u, v \in \mathbb{R}^d$ deux champs de vecteurs et $\sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$, $\tau = (\tau_{ij})_{1 \leq i,j \leq d}$ deux champs tensoriels définis sur un ouvert $\Omega \subset \mathbb{R}^d$ régulier. Premièrement, en notant $u_1, \ldots, u_d$ les coordonnées de u, on appelle respectivement *divergence*, *gradient* et *laplacien* de u les fonctions suivantes :

$$\operatorname{div}(u) = \sum_{i=1}^d \frac{\partial u_i}{\partial x_i}, \quad \nabla u = \left(\frac{\partial u_i}{\partial x_j} \right)_{1 \leq i,j \leq d} \quad \text{et} \quad \Delta u = \operatorname{div}(\nabla u).$$

On appelle également *produit tensoriel* de u et v le tenseur donné par :

$$u \otimes v = (u_i v_j)_{1 \leq i,j \leq d}.$$

En outre, on appelle *divergence* de σ le vecteur donné par :

$$\operatorname{div}(\sigma) = \left(\sum_{j=1}^d \frac{\partial \sigma_{ij}}{\partial x_j} \right)_{1 \leq i \leq d}.$$

Enfin, on appelle *produit scalaire* de σ et τ la fonction réelle :

$$\sigma : \tau = \sum_{1 \leq i,j \leq d} \sigma_{ij} \tau_{ij}.$$

À noter que par définition, on a $\sigma : \tau = \sigma^\top : \tau^\top$, où $\sigma^\top = (\sigma_{ji})_{1 \leq i,j \leq d}$ désigne la transposée du tenseur σ, et la norme associée à ce produit scalaire est simplement notée $|\cdot|$, de sorte que $|\sigma|^2 = \sigma : \sigma$. On rappelle également quelques identités utiles du calcul différentiel :

Lemme 1. *Soit ρ un scalaire et u, v, w trois vecteurs suffisamment réguliers sur Ω, alors on a :*

- $(u \otimes v) w = (v \cdot w) u$,
- $\operatorname{div}(u \otimes v) = (\operatorname{div}\ v) u + (v \cdot \nabla) u$,
- $\operatorname{div}(\rho u) = \nabla \rho \cdot u + \rho \operatorname{div}\ u$,
- $\operatorname{div}(\rho u \otimes v) = (\nabla \rho \cdot v) u + \rho (v \cdot \nabla) u + \rho \operatorname{div}(v) u$.

1.1.2 Espaces de Sobolev et transformation de Fourier

Sur $\mathbb{R}^d$, on considèrera les espaces réguliers usuels, à savoir :

- les espaces de Lebesgue pour $p \geq 1$,

$$L^p(\mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{C} \;\middle|\; \|u\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |u(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}$$

  et

$$L^\infty(\mathbb{R}^d) := \left\{ u : \mathbb{R}^d \to \mathbb{C} \;\middle|\; \|u\|_{L^\infty(\mathbb{R}^d)} := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |u(x)| < \infty \right\},$$

- les espaces de Sobolev pour $s \in \mathbb{R}$,

$$H^s(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \;\middle|\; \|u\|_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

  où on note indifféremment $\hat{u}$ ou $\mathcal{F}u$ la transformée de Fourier de u, pour $\xi \in \mathbb{R}^d$:

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x) dx.$$

Sur le tore $\mathbb{T}^d$, pour une fonction périodique $u \in L^2(\mathbb{T}^d)$ on note u_n pour $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ ses coefficients de Fourier définis par

$$u_n = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-in\cdot x} dx,$$

et on définit l'espace de Sobolev $H^s(\mathbb{T}^d)$ comme l'espace de Hilbert associé à la norme

$$\|u\|_{H^s} = \left(\sum_{n \in \mathbb{Z}^d} \left(1 + |n|^2\right)^s |u_n|^2 \right)^{\frac{1}{2}},$$

où $|n|^2 = n_1^2 + \cdots + n_d^2$.

On utilisera également la notation $\mathcal{C}_0^\infty(I \times \Omega)$ pour désigner l'espace des fonctions infiniment dérivables à support compact de $I \times \Omega$, où I désigne un intervalle de $\mathbb{R}$ et $\Omega = \mathbb{R}^d$ ou $\mathbb{T}^d$.

On énonce maintenant plusieurs lemmes classiques d'analyse fonctionnelle, dans une version parfois simplifiée, qui seront utilisés au cours de ce manuscrit.

Lemme 2. ***(Inégalité de Gagliardo-Nirenberg)****, voir [57, Théorème 1 p. 277].*
Soit $2 \leq p < \frac{2d}{(d-2)_+}$ *et* $1 \leq q, r \leq \infty$*, il existe* $C = C(d,p,q,r) > 0$ *tel que pour tout* $f \in H^1(\mathbb{R}^d)$ *et pour tout* $\theta \in [0,1]$ *tel que*

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{d}\right)\theta + \frac{1-\theta}{q},$$

on a

$$\|f\|_{L^p(\mathbb{R}^d)} \leq C\|\nabla f\|^{\theta}_{L^r(\mathbb{R}^d)}\|f\|^{1-\theta}_{L^q(\mathbb{R}^d)}.$$

Lemme 3. ***(Inégalité de Sobolev logarithmique)****, voir [94, Théorème 8.14].*
Soit $f \in H^1(\mathbb{R}^d)\backslash\{0\}$ *et* $\alpha > 0$*, alors*

$$\int_{\mathbb{R}^d} |f(x)|^2 \log|f(x)|^2 dx \leq \frac{\alpha^2}{\pi}\|\nabla f\|^2_{L^2(\mathbb{R}^d)} + \left(\log\|f\|^2_{L^2(\mathbb{R}^d)} - d(1+\log\alpha)\right)\|f\|^2_{L^2(\mathbb{R}^d)},$$

et on a égalité si et seulement si f *est un multiple de* $x \mapsto \exp\left(-\pi|x|^2/2\alpha^2\right)$ *à translation près.*

Lemme 4. ***(Inégalité de Csiszár–Kullback)****, voir [3, Théorème 8.2.7].*
Soit f*,* $g \geq 0$ *tel que* $\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} g$*, alors*

$$\|f-g\|_{L^1(\mathbb{R}^d)} \leq 2\|f\|_{L^1(\mathbb{R}^d)}\int_{\mathbb{R}^d} f(x)\log\left(\frac{f(x)}{g(x)}\right)dx.$$

Lemme 5. ***(Critère de de la Vallée-Poussin)****, voir [105, Chapitre II-5] ou [53].*
Soit $(\rho_n)_{n\in\mathbb{N}}$ *une suite bornée de* $L^1(\mathbb{R}^d)$*. On suppose qu'il existe une fonction mesurable positive* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *vérifiant*

$$\frac{\varphi(x)}{x} \underset{n\to\infty}{\longrightarrow} +\infty,$$

tel que

$$\sup_n \int_{\mathbb{R}^d} \varphi(\rho_n) < \infty.$$

Alors la famille $(\rho_n)_n$ *est uniformément intégrable, c'est-à-dire que*

$$\sup_n \int_{|\rho_n|>a} |\rho_n| \underset{a\to+\infty}{\longrightarrow} 0.$$

Remarque. Dans notre cadre d'application, on prendra $\varphi(x) = x|\log(x)|$.

Lemme 6. ***(Théorème de Dunford-Pettis)****, voir [30, Théorème IV-29] ou [55].*
Soit $(\rho_n)_{n\in\mathbb{N}}$ *une suite bornée de* $L^1(\mathbb{R}^d)$*, alors les deux propriétés suivantes sont équivalentes :*

(i) La famille $(\rho_n)_n$ *est relativement compacte pour la topologie faible* L^1*, c'est-à-dire qu'il*

existe une fonction $\rho \in L^1(\mathbb{R}^d)$ tel que, à extraction près,

$$\int_{\mathbb{R}^d} \rho_n \phi \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \rho\phi, \quad \text{pour tout } \phi \in L^\infty(\mathbb{R}^d).$$

(ii) La famille $(\rho_n)_n$ est uniformément intégrable (cf Lemma 5) et tendue, c'est-à-dire que

$$\sup_n \int_{|x|>R} |\rho_n| \underset{R\to\infty}{\longrightarrow} 0.$$

1.2 Contexte et motivation de la thèse

1.2.1 Équations de Schrödinger : du linéaire au non-linéaire

Le but de cette partie est avant tout de servir d'introduction aux équations de Schrödinger sur l'espace $\mathbb{R}^d$: le cadre explicite du cas linéaire permet facilement de présenter diverses propriétés de symétrie et d'invariance de ces équations, de mettre en lumière certaines lois de conservation et d'introduire des ensembles de solutions particulières. Ces différentes propriétés serviront par la suite de fil rouge pour la présentation des équations de Schrödinger non-linéaires avec non-linéarité puissance ou logarithmique, puis pour l'équation qui nous a principalement intéressé au cours de cette thèse : l'équation de Schrödinger-Langevin.

Équation de Schrödinger linéaire

La mécanique quantique, développée dans les années 1920, est un domaine de la physique théorique ayant pour but de décrire les phénomènes physiques opérant à l'échelle atomique. En 1925, le physicien autrichien Erwin Schrödinger introduit l'*équation de Schrödinger* afin de décrire l'évolution d'une particule au cours du temps, au même titre que le *principe fondamental de la dynamique* introduit par Newton en 1687 permet de décrire le mouvement d'un objet à l'échelle humaine. L'état d'une particule est décrit par une fonction de l'espace et du temps appelée *fonction d'onde* et notée $\psi(t,x)$, avec $t \in \mathbb{R}$ et $x \in \mathbb{R}^d$ (où $d \geq 1$ désigne la dimension de l'espace considéré). Cette description est probabiliste : pour une région de l'espace $\Omega \subset \mathbb{R}^d$, la probabilité de présence de la particule dans cette région au temps t est donnée par l'intégrale $\int_\Omega |\psi(t,x)|^2 dx$ de la densité de probabilité $|\psi(t,\cdot)|^2$ sur Ω. En particulier, cette approche probabiliste implique une première loi de conservation au cours du temps, celle de la norme $L^2(\mathbb{R}^d)$: pour tout temps $t \geq 0$,

$$\|\psi(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\psi(t,x)|^2 dx = \|\psi_0\|^2_{L^2(\mathbb{R}^d)}, \tag{1.2.1}$$

où $\psi_0 = \psi(0,\cdot)$ désigne la *position initiale* de la particule au temps $t = 0$.

Cette équation devait satisfaire plusieurs principes physiques :

- un **principe de causalité** : l'état initial ψ_0 doit déterminer de manière unique l'ensemble des états futurs $\{t > 0 \mid \psi(t,\cdot)\}$. D'un point de vue mathématique, cette condition demande l'existence d'une théorie de Cauchy bien posée, à savoir existence et unicité d'une solution globale pour toute condition initiale $\psi_0 \in L^2(\mathbb{R}^d)$.
- un **principe de superposition** : si ψ et $\tilde{\psi}$ décrivent toutes deux un état d'une particule quantique, alors $\alpha\psi + \beta\tilde{\psi}$ doit également décrire un état quantique (avec α et β deux

constantes), ce qui implique la *linéarité* de l'équation de Schrödinger.

- un **principe de correspondance** : il doit y avoir un continuum entre la mécanique quantique et la mécanique classique, c'est-à-dire que l'on doit retrouver certaines propriétés de la mécanique classique dans une certaine limite de l'équation de Schrödinger. Ce principe amène notamment l'introduction de la *constante de Planck réduite* $\hbar = h/2\pi$, avec $h \sim 6,626 \times 10^{-34}$J s, constante universelle qui décrit l'ordre de grandeur des phénomènes quantiques mais qui peut être considérée comme nulle à l'échelle humaine.

L'équation de Schrödinger linéaire peut alors s'écrire, pour une particule de masse m,

$$\boxed{i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\Delta\psi = V\psi,} \tag{1.2.2}$$

où $V \in \mathbb{R}$ désigne un potentiel réel décrivant l'environnement de la particule. On considérera en particulier dans la suite deux types de potentiels, le cas de l'équation de Schrödinger *libre* avec $V = 0$, et celui du *potentiel harmonique* $V(x) = \frac{\omega^2}{2}|x|^2$, dans lequel $\omega \in \mathbb{R}$ est une grandeur homogène à une fréquence. De même, on prendra généralement la normalisation de la constante de Planck $\hbar = 1$ et de la masse $m = 1$.

Comme la plupart des *équations aux dérivées partielles* (EDP), l'équation de Schrödinger linéaire possède certaines symétries, qui impliquent des lois de conservation via le théorème de Noether. Par exemple, la **symétrie par rotation** $\psi \mapsto e^{i\theta}\psi$ pour $\theta \in \mathbb{R}$ implique la *conservation de la masse* (1.2.1) évoquée supra. De même qu'en mécanique classique, la **symétrie par translation en temps** $\psi(t,x) \mapsto \psi(t-t_0,x)$ implique la *conservation de l'énergie*

$$E(t) = E_{\text{kin}}(t) + E_p(t) := \int_{\mathbb{R}^d} |\nabla\psi(t,x)|^2 dx + \int_{\mathbb{R}^d} V(x)|\psi(t,x)|^2 dx, \tag{1.2.3}$$

où E_{kin} désigne l'énergie cinétique liée au mouvement de la particule et E_p son énergie potentielle. Dans le cas de l'équation de Schrödinger libre ($V = 0$), l'absence de potentiel implique de nombreuses autres lois de conservation, dont on citera par exemple :

- la **symétrie par translation dans l'espace** $\psi(t,x) \mapsto \psi(t,x-x_0)$ d'une solution ψ de (1.2.2), qui implique la conservation du moment

$$J(t) := \int_{\mathbb{R}^d} \text{Im}\left(\overline{\psi}(t,x)\nabla\psi(t,x)\right) dx = J(0). \tag{1.2.4}$$

- le **principe d'invariance Galiléenne**, qui indique que si ψ est solution de (1.2.2), alors pour tout $v \in \mathbb{R}^d$,

$$(t,x) \mapsto \psi(t,x)e^{ix\cdot v}e^{it|v|^2/2}$$

est également solution de (1.2.2), ce qui implique notamment la *conservation du centre de masse*

$$\int_{\mathbb{R}^d} x|\psi(t,x)|^2 dx - tJ(t)$$

faisant intervenir le moment d'ordre 1 de la densité $|\psi|^2$.

L'étude mathématique de l'équation de Schrödinger commence bien évidemment par l'élaboration d'une théorie de Cauchy bien posée pour (1.2.2), à savoir que l'on veut montrer que l'on a **existence et unicité locale** (en temps) d'une solution ψ pour tout donnée initiale ψ_0 d'un espace bien choisi, avec **continuité par rapport aux données initiales** en montrant que l'application $\psi_0 \mapsto \psi$ est continue. Dans le cadre des EDPs linéaires sur $\mathbb{R}^d$, la transformation de Fourier permet de résoudre très efficacement le problème de Cauchy : pour l'équation de Schrödinger libre, l'unique solution de (1.2.2) avec donnée initiale $\psi(0,\cdot) = \psi_0 \in L^2(\mathbb{R}^d)$ est donnée par la formule

$$\psi(t,x) = \frac{1}{(2i\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} \psi_0(y)dy \tag{1.2.5}$$

pour $t > 0$.

On peut faire plusieurs remarques importantes à partir de cette formule explicite :

- l'expression (1.2.5) étant vraie pour tout $t > 0$, on a en fait **existence globale** (en temps) de la solution. De plus, on observe la décroissance de la norme L^∞ :

  $$\|\psi(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{(2\pi t)^{d/2}} \|\psi_0\|_{L^1(\mathbb{R}^d)}.$$

  Cette estimation est caractéristique du fait que l'équation de Schrödinger libre appartient à la famille des EDPs dispersives : les solutions "s'étalent" dans l'espace au cours du temps et tendent vers 0 quand $t \to \infty$.

- si on prend une condition initiale gaussienne $\psi_0(x) = e^{-z\frac{|x|^2}{2}}$, la solution ψ évolue également sous la forme d'une fonction gaussienne au cours du temps, via l'expression

  $$\psi(t,x) = \frac{1}{(1+itz)^{d/2}} \exp\left(-\frac{z}{1+itz}\frac{|x|^2}{2}\right).$$

  Il est également intéressant de noter qu'une gaussienne est également propagée en gaussienne dans le cas d'un potentiel harmonique $V(x) = \frac{\omega^2}{2}x^2$.

- si on définit la famille d'opérateurs $\left(e^{it\Delta/2}\right)_{t\geq 0}$ par $\psi(t,.) = e^{it\Delta/2}\psi_0$ via la formule (1.2.5) pour tout temps $t > 0$ et $e^{it\Delta/2}_{|t=0} = \mathrm{Id}$, cette famille définit un *semi-groupe unitaire* sur les espaces de Hilbert $H^s(\mathbb{R}^d)$.

L'évolution des normes de Lebesgue et de Sobolev des solutions d'EDPs dispersives est une question omniprésente dans l'étude moderne de ces EDPs, et constitue le prolongement logique de la mise en lumière des lois de conservation associées à ces équations : nous avons vu en effet que dans le cas de l'équation de Schrödinger libre, la norme L^2 d'une solution de (1.2.2) était conservée là où la norme L^∞ décroissait en $t^{-d/2}$ vers 0 pour une condition initiale $\psi_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. On peut donc se demander comment évoluent les normes L^p pour $2 < p < \infty$ de ces solutions. Une première réponse provient de formules d'interpolation, qui impliquent que pour $1 \leq q \leq 2$ et p défini par $1/p + 1/q = 1$, on a

$$\|e^{it\Delta/2}\psi_0\|_{L^p(\mathbb{R}^d)} \leq \frac{C_d}{t^{\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}}\|\psi_0\|_{L^q(\mathbb{R}^d)},$$

où $C_d > 0$ est une constante dépendant uniquement de la dimension $d \geq 1$. D'autres types d'estimations classiques dans l'étude des EDPs dispersives, appelées *estimations de Strichartz*, sont particulièrement utiles pour l'étude des équations de Schrödinger non-linéaires (c'est-à-dire quand on remplace le potentiel linéaire V par une fonction $F(\psi, \overline{\psi})$ dans l'équation (1.2.2)), dont nous allons maintenant donner un bref aperçu.

Équation de Schrödinger avec non-linéarité puissance

L'équation de Schrödinger non-linéaire (NLS) avec non-linéarité puissance,

$$\boxed{i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda|\psi|^{\gamma-1}\psi,} \tag{1.2.6}$$

où $\lambda \in \mathbb{R}^*$ et $\gamma > 1$, apparaît dans de nombreux domaines de la physique moderne tels que la physique quantique, l'océanographie ou l'électrodynamique. Par exemple, elle peut être vue comme une limite de type champ moyen $N \to \infty$ pour une fonction d'onde $\psi(t, x_1, \ldots, x_N)$ d'un problème d'interaction à N particules régi par l'équation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\sum_{k=1}^{N}\Delta_{x_j}\psi = \frac{1}{N}\sum_{1\leq j<k\leq N} r^{-d}V\left(\frac{x_j - x_k}{r}\right)\psi,$$

avec une longueur typique d'interaction $r = N^{-\beta}$ ($\beta > 0$), ou comme une équation d'enveloppe pour la propagation de vagues d'eau satisfaisant l'équation de Korteweg-de Vries

$$\partial_t u + \partial_x^3 u + \varepsilon u\partial_x u = 0,$$

dans la limite $\varepsilon \to 0$ (voir [89]).

Cette EDP non-linéaire conserve la plupart des propriétés de symétrie de l'équation de Schrödinger libre : invariance par rotation, par translation en espace et en temps, ou encore l'invariance galiléenne. Outre la conservation de la masse (1.2.1) et du moment (1.2.4), l'énergie associée à (1.2.6) est également constante au cours du temps :

$$E(t) := \|\nabla\psi(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \frac{2\lambda}{\gamma+1}\int_{\mathbb{R}^d} |\psi(t,x)|^{\gamma+1}dx = E(0). \tag{1.2.7}$$

De plus, (1.2.6) est équivalente à sa formulation intégrale via la formule de Duhamel

$$\psi(t,x) = e^{it\Delta/2}\psi_0 - i\lambda\int_0^t e^{i(t-s)\Delta/2}\left(|\psi|^{\alpha-1}\psi\right)(s,x)ds, \quad t>0,\ x\in\mathbb{R}^d.$$

Pour étudier (1.2.6) sur l'espace $\mathbb{R}^d$, on doit alors séparer deux cas :

- le cas *défocalisant* $\lambda > 0$: l'énergie (1.2.7) est alors positive (*ie* $E(t)\geq 0$), et en particulier la norme H^1 de ψ est bornée. On a alors une théorie de Cauchy locale bien posée dans H^1, avec la restriction $1<\gamma<1+4/d$ pour $d\geq 3$, en utilisant des arguments classiques de point fixe sur la formule de Duhamel et grâce aux estimations de Strichartz mentionnées plus haut. Cette existence et unicité locale en temps se propage de façon globale grâce à la conservation de l'énergie (1.2.7) en une solution $\psi\in L^\infty\Big(\mathbb{R};H^1(\mathbb{R}^d)\Big)$.
  Concernant la dynamique en temps long, toute solution disperse (c'est-à-dire que toute solution converge localement vers 0 en espace quand $t\to\infty$) d'une façon dépendant fortement du paramètre γ : par exemple, pour $\gamma > 1+4/d$, toute solution possède un comportement asymptotique linéaire,

  $$\exists\psi_+\in H^1(\mathbb{R}^d), \quad \|\psi(t,.) - e^{it\Delta/2}\psi_+\|_{H^1(\mathbb{R}^d)} \to 0 \quad \text{quand } t\to 0. \tag{1.2.8}$$

  À l'inverse, quand $\gamma\leq 1+2/d$, la dynamique en temps long ne peut pas être comparée à celle du cas linéaire, c'est-à-dire que si il existe $\psi_+\in L^2(\mathbb{R}^d)$ tel que

  $$\|\psi(t,\cdot) - e^{it\Delta/2}\psi_+\|_{L^2(\mathbb{R}^d)} \to 0,$$

  alors nécessairement $\psi_0=\psi_+=0$ (voir [15]). Ces types de résultats sont rigoureusement formulés dans le cadre de la théorie du *scattering*, et on renvoie le lecteur à [38] et ses références.

- le cas *focalisant* $\lambda < 0$: l'énergie n'est plus nécessairement positive, et une compétition peut s'opérer entre les termes d'énergie cinétique $\|\nabla\psi\|^2_{L^2}$ et d'énergie potentielle $\|\psi\|^{\gamma+1}_{L^{\gamma+1}}$. L'existence locale d'une solution, issue de la formule de Duhamel comme pour le cas défocalisant, peut se propager en existence globale dans le cas d'une non-linéarité ou de conditions initiales suffisamment petites. Plus précisément, on a existence (et unicité) d'une solution globale H^1 pour les cas suivants :

(i) $\gamma < 1 + \frac{4}{d}$,

(ii) $\gamma = 1 + \frac{4}{d}$ et $\|\psi_0\|_{L^2}$ suffisamment petit,

(iii) $1 + \frac{2}{d} < \gamma < 1 + \frac{4}{(d-2)_+}$ et $\|\psi_0\|_{H^1}$ suffisamment petit.

De nombreux autres résultats de ce type peuvent être trouvés dans [114]. Hors de ces cas, on peut observer des *phénomènes de blow-up*, qui correspondent généralement à des explosions de norme H^1 de solution en temps fini, et que l'on peut énoncer ainsi : il existe un temps $T^* > 0$ tel que

$$\lim_{t \to T^*} \|\nabla \psi\|_{L^2(\mathbb{R}^d)} = \infty.$$

Ce genre de résultat est généralement obtenu via un calcul de l'*identité du viriel*, et consiste alors à montrer que la quantité

$$t \mapsto \frac{d^2}{dt^2} \left(\int_{\mathbb{R}^d} |x|^2 |\psi(t,x)|^2 dx \right)$$

est inférieure à une constante négative (voir [69]). Dans le cas où l'on a existence globale, les solutions possèdent un profil asymptotique linéaire (1.2.8).

L'étude des solutions particulières de (1.2.6) constitue également un point important dans la compréhension du comportement de cette EDP. Contrairement au cas linéaire, une donnée initiale gaussienne n'évolue pas en gaussienne au cours du temps. En revanche, (1.2.6) possède des solutions appelées *ondes solitaires* ou *solitons* (ou encore *solitary waves* en anglais) qui s'écrivent sous la forme

$$\psi(t,x) = e^{i\omega t} \phi(x), \tag{1.2.9}$$

c'est-à-dire comme le produit d'une oscillation indépendante de l'espace $e^{i\omega t}$ et d'un profil constant $\phi(x)$. Il est alors assez facile de voir que (1.2.9) est solution de (1.2.6) si et seulement si ϕ est solution de l'équation elliptique non-linéaire

$$-\omega \phi + \frac{1}{2} \Delta \phi = \lambda |\phi|^{\gamma - 1} \phi. \tag{1.2.10}$$

Pour $\omega \leq 0$, l'équation (1.2.10) n'admet pas de solution non triviale (voir [85] et [2]). Ce résultat est également vrai pour $\omega > 0$ et $\lambda > 0$: en multipliant (1.2.10) par $\overline{\phi}$, en intégrant en espace et en prenant la partie réelle, on obtient en effet que

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 + \omega \int_{\mathbb{R}^d} |\phi|^2 + \lambda \int_{\mathbb{R}^d} |\phi|^{\gamma+1} = 0.$$

On peut donc se placer dans le cas focalisant $\lambda < 0$ avec $\omega > 0$ et $\gamma < 1 + 4/d$ afin d'obtenir l'existence d'une solution. En dimension $d = 1$, l'unique solution réelle positive non triviale de (1.2.10) s'écrit

$$\phi_\omega(x) = \left(\frac{\omega(\gamma+1)}{-2\lambda}\right)^{\frac{1}{\gamma-1}} \cosh\left(\frac{\gamma-1}{2}\sqrt{\omega}x\right)^{-\frac{2}{\gamma-1}}.$$

Cette solution est paire ; elle décroît exponentiellement vite à l'infini, et on peut montrer que toute solution (à valeurs complexes) de (1.2.10) peut s'écrire sous la forme $e^{i\theta_0}\phi_\omega(x - x_0)$, pour θ_0, $x_0 \in \mathbb{R}$ (voir par exemple [38, Théorème 8.1.6]). En dimension supérieure $d \geq 2$, on peut également montrer l'existence d'une unique solution positive et à symétrie radiale de (1.2.10) appelée *ground state*, que l'on note de même ϕ_ω. On peut dès lors écrire toute solution positive de (1.2.10) sous la forme $\phi_\omega(x-x_0)$ pour $x_0 \in \mathbb{R}^d$. Cette fonction s'obtient généralement par des méthodes variationelles (voir par exemple [18], [19] ou [109]) : en effet, ϕ_ω est l'unique minimiseur de l'énergie $E(t)$ sous la condition $\|\psi\|_{L^2} = \|\phi_\omega\|_{L^2}$ et modulo les invariances par translation en espace, en temps et invariance de jauge (multiplication par une constante de module 1). Par le principe d'invariance Galiléenne, on obtient alors un ensemble de solutions de (1.2.6) appelées *ondes progressives* (ou *traveling waves* en anglais) qui s'expriment sous la forme

$$\psi_v(t,x) = e^{i\omega t}e^{iv\cdot(x-vt)/2}\phi_\omega(x - vt) \tag{1.2.11}$$

pour toute constante $v \in \mathbb{R}^d$.

Une question naturelle à ce stade est celle de la stabilité du *ground state* : une petite perturbation $e^{iv\cdot x}\phi_\omega(x)$ avec $|v| \ll 1$ reste-t-elle suffisamment "proche" de ϕ_ω pour des temps arbitrairement grands ? Le terme "proche" doit évidemment dépendre de l'espace d'existence des solutions considérées ($H^1(\mathbb{R}^d)$ dans notre cas), mais doit également prendre en compte les invariances de l'équation. En effet, il est clair que les solutions de type ondes progressives (1.2.11) s'éloignent de ϕ_ω en norme H^1 pour des vecteurs v arbitrairement petits en raison des termes de phase, mais restent cependant sur l'orbite

$$\mathcal{O}(\phi_\omega) = \left\{e^{i\theta}\phi_\omega(. - y) \;\middle|\; \theta \in \mathbb{R},\ y \in \mathbb{R}^d\right\}.$$

Dans le cadre des équations de Schrödinger, la bonne notion de stabilité est donc celle de la *stabilité orbitale*, et peut s'énoncer ainsi : pour tout $\varepsilon > 0$, il existe $\eta > 0$ tel que,

$$\text{si } \|\psi_0 - \phi_\omega\|_{H^1} \leq \eta, \quad \text{alors } \sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\ \mathbb{R}^d} \|\psi(t,.) - e^{i\theta}\phi_\omega(. - y)\|_{H^1} \leq \varepsilon,$$

où ψ correspond à la solution de (1.2.6) avec donnée initiale $\psi_0 \in H^1(\mathbb{R}^d)$. Dans le cas sous-critique $\gamma < 1 + 4/d$, il a été démontré par Cazenave et Lions [40] que le *ground state* est orbitalement stable dans $H^1(\mathbb{R}^d)$.

Une remarque importante est que la plupart des résultats cités précédemment concernant l'existence, l'unicité et le comportement en temps long des solutions de l'équation de Schrödinger avec non-linéarité puissance sont obtenus en considérant la non-linéarité $\lambda|\psi|^{\gamma-1}\psi$ comme une quantité négligeable, par exemple en travaillant sur des temps courts ou quand la solution ψ est petite, et donc de considérer l'équation (1.2.6) comme une perturbation de l'équation linéaire associée. Ce genre de raisonnement, très efficace dans le cadre de non-linéarité suffisamment régulière en 0 (typiquement localement lipschitzienne) et vérifiant $F(0) = 0$, n'est plus applicable dans le cadre de l'équation de Schrödinger logarithmique, équation qui fait l'objet de la section suivante.

Équation de Schrödinger logarithmique

L'équation de Schrödinger logarithmique

$$\boxed{i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi\log(|\psi|^2)} \tag{1.2.12}$$

a été introduite par Białynicki-Birula et Mycielski en 1976 [20] afin de satisfaire une propriété de tensorisation : si ψ est solution d'une équation de Schrödinger avec non-linéarité $\lambda\psi F(|\psi|^2)$ où $F : \mathbb{R}_+ \to \mathbb{C}$, et ayant pour condition initiale un produit tensoriel

$$\psi_0(x) = \prod_{j=1}^{d} \psi_{0,j}(x_j), \quad x \in \mathbb{R}^d,$$

alors on veut que ψ puisse se décomposer comme un produit $\psi = \prod_{j=1}^{d} \psi_j$ où chaque ψ_j satisfait une équation de Schrödinger non-linéaire

$$i\partial_t\psi_j + \frac{1}{2}\partial_{x_j}^2\psi_j = \psi_j f_j(|\psi_j|^2),$$

où les $(f_j)_{1\leq j\leq d}$ sont des fonction réelles vérifiant l'équation fonctionnelle

$$\sum_{j=1}^{d} f_j(x_j) = F\left(\prod_{j=1}^{d} x_j\right) \quad \text{pour tout } x_1, \ldots, x_d > 0.$$

La non-linéarité logarithmique est alors la seule fonction vérifiant une telle propriété. Cette équation non-linéaire a depuis été utilisée pour modéliser de nombreux phénomènes physiques, notamment en optique quantique ([32], [87]), en physique nucléaire [73], pour des phénomènes de transport et de diffusion ([50], [72]), pour des systèmes quantiques ouverts ([75], [117]), en théorie de la gravité quantique [121] ou encore en théorie de la superfluidité et de la condensation de Bose-Einstein [8].

D'un point de vue mathématique, l'étude de cette équation remonte à la fin des années 70 par T. Cazenave et A. Haraux avec [39] puis [37], mais peu d'autres résultats ont ont été établis à son propos jusqu'à très récemment. On citera en particulier l'article de P. d'Avenia, E. Montefusco, et M. Squassina de 2014 [49], celui de A. H. Ardila de 2016 [5] et celui de R. Carles et I. Gallagher de 2018 [36]. Les non-linéarités logarithmiques se retrouvent également dans d'autres équations aux dérivées partielles non-linéaires ayant une importance physique, comme en optique quantique avec l'équation Korteweg-de Vries logarithmique et de Kadomtsev-Petviashvili logarithmique ([80], [113], [112]), ou bien encore en théorie quantique des champs, ainsi qu'en cosmologie avec l'équation de Klein-Gordon logarithmique ([17], [70], [107]).

Tout comme (1.2.6), l'équation de Schrödinger logarithmique conserve la plupart des symétries de l'équation de Schrödinger libre, comme celles de l'invariance par translation en espace et en temps, l'invariance galiléenne et l'invariance par transformation de jauge. Elle vérifie également les propriétés usuelles de conservation de la masse (1.2.1), de conservation du moment (1.2.4), et l'énergie associée

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi(t,x)|^2 dx + \lambda \int_{\mathbb{R}^d} |\psi(t,x)|^2 \left(\log\left(|\psi(t,x)|^2\right) - 1\right) dx$$

est formellement conservée pour tout $t \geq 0$. En retranchant la norme L^2 de ψ au carré, qui est constante, on peut redéfinir l'énergie de (1.2.12) par la formule

$$E(t) := \frac{1}{2}\|\nabla\psi(t,.)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |\psi(t,x)|^2 \log(|\psi(t,x)|^2) dx = E(0). \tag{1.2.13}$$

En revanche, cette équation possède plusieurs propriétés singulières qui la rendent remarquable dans l'étude des équations de Schrödinger non-linéaires. Tout d'abord, il est intéressant de noter que quel que soit le signe de $\lambda \in \mathbb{R}^*$, l'énergie (1.2.13) n'a pas de signe défini, et l'on n'a donc pas de borne dans H^1 a priori. Ensuite, cette équation possède une invariance d'échelle : si ψ est solution de l'équation (1.2.12), alors pour tout $\kappa > 0$,

$$\psi_\kappa(t,\cdot) := \kappa\psi(t,\cdot)e^{-2it\lambda\log\kappa}$$

est également solution de (1.2.12) pour tout temps $t \geq 0$ avec donnée initiale $\kappa\psi(0,.)$. Cette propriété, analogue au cas linéaire, est en revanche particulièrement inhabituelle dans le cadre non-linéaire, et implique que la taille de la solution n'impacte sa dynamique que par un terme d'oscillations. De plus, en dérivant l'expression précédente par rapport à $\kappa > 0$, on obtient que

$$\frac{d}{d\kappa}\psi_\kappa(t,.) = (1 - 2it\lambda)\psi(t,\cdot)e^{-2it\lambda\log\kappa}$$

n'a pas de limite en $\kappa \to 0$ pour $t > 0$, et donc que le flot $\varphi_t(\psi_0) \mapsto \psi(t, \cdot)$ ne peut être $\mathcal{C}^1$ quel que soit les espaces de fonctions considérés pour ψ et ψ_0.

Concernant le problème de Cauchy, de même que pour la non-linéarité puissance, on doit distinguer deux cas :

- pour le cas focalisant $\lambda < 0$, T. Cazenave et A. Haraux ont montré dans [39] l'existence et l'unicité d'une solution globale $\psi \in \mathcal{C}^0(\mathbb{R}; W)$ dans l'espace d'énergie

  $$W(\mathbb{R}^d) := \left\{ u \in H^1(\mathbb{R}^d) \;\middle|\; |u|^2 \log |u|^2 \in L^1(\mathbb{R}^d) \right\},$$

  avec condition initiale $\psi_0 \in W$. Leur preuve est basée sur une régularisation de la non-linéarité et des arguments de compacité, et utilise en particulier le fait que la partie négative de l'énergie $\lambda \int_{|\psi|>1} |\psi|^2 \log |\psi|^2$ est contrôlée par la norme H^1 de ψ par inégalité de Gagliardo-Nirenberg et conservation de la masse.
- pour le cas défocalisant $\lambda > 0$, R. Carles et I. Gallagher ont montré l'existence et la propagation de régularité pour une solution pour $\psi \in L^\infty_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^d) \cap \mathcal{F}^\alpha(\mathbb{R}^d))$ où on définit l'espace à poids

  $$\mathcal{F}^\alpha(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d) \;\middle|\; x \mapsto (1 + |x|^2)^{\frac{\alpha}{2}} u \in L^2(\mathbb{R}^d) \right\}$$

  pour tout $0 < \alpha \leq 1$. La preuve passe également par une régularisation du logarithme en 0 et un passage à la limite par argument de compacité, mais cette fois la partie négative de l'énergie $\lambda \int_{|\psi|<1} |\psi|^2 \log |\psi|^2$ nécessite un contrôle par le moment d'ordre $\alpha > 0$ de ψ via une inégalité d'interpolation, d'où l'hypothèse de régularité supplémentaire dans $\mathcal{F}^\alpha(\mathbb{R}^d)$ pour la solution ψ.

Il est également intéressant de noter que les auteurs précédemment cités utilisent la propriété algébrique du logarithme suivante, démontrée par T. Cazenave et A. Haraux dans [39],

$$\left| \operatorname{Im} \left(\left(z \log |z|^2 - z' \log |z'|^2 \right) (\overline{z} - \overline{z'}) \right) \right| \leq 4 |z - z'|^2, \quad \forall z, z' \in \mathbb{C},$$

que ce soit pour montrer la monotonie de l'opérateur $A\psi = -i(\Delta\psi - \lambda\psi \log |\psi|^2)$ (dans le cas focalisant) ou pour montrer l'unicité de la solution par un argument de type Gronwall.

Avant d'aborder l'étude du comportement asymptotique des solutions de (1.2.12) pour $t \to \infty$, il est essentiel de remarquer que l'équation de Schrödinger logarithmique partage une autre caractéristique avec le cas linéaire : celle de la propagation des solutions gaussiennes. Cette propriété, déjà identifiée dans l'article de Białynicki-Birula et Mycielski [20], a été mise en avant dans [36] de la façon suivante : en prenant le cas particulier de la dimension $d = 1$ et en injectant une gaussienne centrée de la forme

$$\varphi(t,x) = b(t)e^{-a(t)\frac{x^2}{2}}$$

dans (1.2.12), on obtient par identification polynomiale un système d'équations différentielles couplées sur a et b :

$$i\dot{b} - \frac{ab}{2} = \lambda b \log|b|^2 \quad \text{et} \quad i\dot{a} - a^2 = 2\lambda \operatorname{Re}(a).$$

La première équation permet d'exprimer b en fonction de a par une méthode de variation de la constante. La deuxième équation porte uniquement sur la fonction a, et par changements de variables successifs $a = -i\dot{w}/w$ et $w = re^{i\theta}$, et en notant $\alpha_0 = \operatorname{Re}(a(0))$ et $\beta_0 = \operatorname{Im}(a(0))$, on obtient finalement une équation différentielle non-linéaire d'ordre 2 sur r,

$$\ddot{r} = \frac{\alpha_0^2}{r^3} + \frac{2\lambda\alpha_0}{r}, \quad r(0) = 1,\ \dot{r}(0) = -\beta_0, \tag{1.2.14}$$

et l'on peut ensuite exprimer l'évolution des fonctions a et b (et donc celui de la gaussienne φ) en fonction de r et de sa dérivée $\dot{r}$. Cette propriété se généralise pour toute dimension $d \geq 1$ par propriété de tensorisation, en résolvant d équations découplées de la forme (1.2.14). Elle apporte ainsi un ensemble de solutions particulières à (1.2.12) pour toute dimension, ce qui constitue une description à la fois rare et précieuse pour l'étude d'une EDP non-linéaire, autant sur le plan théorique que numérique.

Nous allons maintenant voir que ces solutions gaussiennes jouent un rôle important dans la dynamique globale de l'équation de Schrödinger logarithmique. Pour le cas d'une non-linéarité défocalisante $\lambda < 0$, il a été démontré par T. Cazenave dans [37] qu'aucune solution ne disperse : en supposant l'existence d'un réel $k > 0$ tel que

$$L_k := \left\{ u \in W(\mathbb{R}^d) \;\middle|\; \|u\|_{L^2(\mathbb{R}^d)} = 1,\ E(u) \leq k \right\} \neq \emptyset,$$

l'auteur montre que

$$\inf_{u \in L_k,\ 1 \leq p \leq \infty} \|u\|_{L^p(\mathbb{R}^d)} > 0,$$

ce qui implique le résultat par conservation de la masse et de l'énergie. Comme pour le cas d'une non-linéarité puissance, (1.2.12) possède des solutions de type ondes solitaires de la forme

$$\psi(t,x) = e^{i\omega t}\phi_\omega(x) := e^{i\omega t} e^{\frac{d}{2} - \frac{\omega}{2\lambda}} e^{\lambda |x|^2},$$

où ϕ_ω est l'unique solution radiale (à translation près) $C^2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$ strictement positive de l'équation elliptique non-linéaire associée

$$-\omega\phi + \Delta\phi = \lambda\phi \log|\phi|^2. \tag{1.2.15}$$

Comme ces fonctions $(\phi_\omega)_{\omega\in\mathbb{R}}$ font partie de l'ensemble des solutions particulières gaussiennes décrites précédemment (et correspondent au cas où $a(t) = \lambda$ pour tout temps $t \geq 0$), elles sont appelées *Gaussons*. Il a été prouvé par A. Ardila dans [5], dans la continuité du travail de T. Cazenave [37] dans le cas radial, que ces Gaussons sont orbitalement stables, en s'appuyant sur l'inégalité de Sobolev logarithmique (Lemme 3) et sur la même minimisation sous contrainte de l'énergie $E(t)$ que pour le cas avec non-linéarité puissance.
De plus, il a été montré récemment par G. Ferriere dans [61] que les solutions de l'équation différentielle (1.2.14) sont périodiques dans le cas $\lambda < 0$, ce qui implique l'existence de solutions gaussiennes périodiques en temps appelées *breathers*. De fait, motivé par les résultats numériques de [13] sur l'interaction entre deux Gaussons, l'auteur montre également dans [63] l'existence de structures plus complexes, appelées *multigaussons* et *multibreathers* (et qui correspondent respectivement à une somme de Gaussons et de gaussiennes).

A l'inverse, dans le cas d'une non-linéarité logarithmique défocalisante, R. Carles et I. Gallagher ont montré dans [36] que toute solution disperse. Les solutions de (1.2.14) possèdent le même taux de dispersion

$$r(t) \sim 2t\sqrt{\lambda \log t}$$

quand $t \to \infty$, ce qui implique que la norme L^∞ de toute gaussienne solution tend vers 0 en $\left(t\sqrt{\log t}\right)^{-\frac{d}{2}}$. De fait, la contribution issue de la partie linéaire de (1.2.12) dans l'équation différentielle (1.2.14) n'influence pas la dynamique de l'équation en temps grand, et en considérant l'équation simplifiée

$$\ddot{\tau} = \frac{2\lambda}{\tau}, \quad \tau(0) = 1, \ \dot{\tau}(0) = 1,$$

les auteurs ont montré l'existence d'une dynamique universelle pour toute solution de (1.2.12) dans $H^1 \cap \mathcal{F}(H^1)$. En effectuant un scaling en espace $y = x/\tau(t)$ par le taux de dispersion τ, qui s'écrit

$$\psi(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \frac{\|\psi_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}} \exp\left(i\frac{\dot{\tau}}{\tau}\frac{|x|^2}{2}\right)$$

avec $\gamma(y) = e^{-|y|^2/2}$, l'énergie renormalisée

$$\mathcal{E}(t) := \frac{1}{\tau(t)^2}\|\nabla_y(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |v(t,y)|^2 \log\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy$$

est positive grâce au lemme de Csiszár-Kullback (cf Lemme 4) et uniformément bornée en temps, ce qui entraîne la convergence L^1 faible de $|v|^2$ vers la gaussienne γ^2, ainsi que la convergence de ces moments d'ordre 1 et 2 :

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} |v(t,y)|^2 dy \longrightarrow \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} \gamma(y)^2 dy \quad \text{pour } t \to \infty.$$

La preuve de la convergence L^1 faible repose notamment sur l'analyse d'une équation de type de Fokker-Planck avec potentiel quadratique, satisfaite par la densité $\rho = |v|^2$, et sur la présence de termes sources négligeables quand $t \to \infty$ dans cette équation.
Il est intéressant de noter que ces résultats impliquent un taux de dispersion en $\left(t\sqrt{\log t}\right)^{-\frac{d}{2}}$ plus rapide que pour le cas de Schrödinger libre ou avec non-linéarité puissance en $t^{-\frac{d}{2}}$. Cet effet, dû à la non-linéarité logarithmique, est encore présent dans le cas d'une double non-linéarité

$$i\partial_t \psi + \frac{1}{2}\Delta\psi = \lambda\psi \log(|\psi|^2) + \sigma\psi|\psi|^{\gamma-1}, \quad \sigma > 0,\ 1 < \gamma < 1 + 4/(d-2)_+,$$

comme souligné par les auteurs dans [36], ce qui indique de manière surprenante que la non-linéarité logarithmique, qui agit comme un potentiel à puits infini quand la solution tend vers 0, joue un rôle "plus fort" que les non-linéarités usuelles. De plus, les auteurs soulignent un autre phénomène singulier : celui de la croissance des normes de Sobolev

$$(\log t)^{\frac{s}{2}} \lesssim \|\psi(t,\cdot)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim (\log t)^{\frac{s}{2}}$$

pour tout $0 < s \leq 1$ et $\psi_0 \neq 0$, et où la notation $\dot{H}^s(\mathbb{R}^d)$ désigne les espaces de Sobolev homogènes standards. En revanche, peu de résultats sont connus pour des régularités plus grandes (H^s avec $s > 1$), car les équations sur les dérivées spatiales de la fonction ψ font intervenir des termes en $1/|\psi|^n$ avec $n \in \mathbb{N}$, issus des dérivations successives de la non-linéarité logarithmique, qu'il est difficile de contrôler. Pour le cas spécifique de H^2, un argument dû à Kato (voir par exemple [38, Section 5.3]) qui consiste à exprimer $\Delta\psi$ comme une fonction de $\partial_t\psi$ et de la non-linéarité permet de conclure quant à la propagation d'une régularité $\psi_0 \in H^2 \cap \mathcal{F}(H^\alpha)$ pour $\alpha > 0$, mais ce type de résultat reste encore un problème ouvert à l'heure actuelle pour des espaces de régularité supérieure H^s avec $s > 2$.

1.2.2 Mécanique des fluides compressibles

Dans cette partie, on présente d'une part certains résultats de mécanique des fluides compressibles, comme l'existence de solutions faibles (au sens des distributions, solutions faibles entropiques ou solutions dissipatives), et on esquisse d'autre part certaines preuves d'existence classique, qui passe généralement par des méthodes de compacité.
On explicite dans un premier temps le lien entre différents systèmes d'Euler-Korteweg et les équations de Schrödinger via la transformation de Madelung. On évoque ensuite différents résultats concernant les systèmes de type Navier-Stokes-Korteweg, en présentant notamment l'importance théorique du cas d'une loi de pression isotherme via la modélisation de ces équations d'un point de vue de théorie cinétique quantique. Enfin, on introduit et on explique l'influence de l'ajout d'un terme de dissipation dans les systèmes précédemment cités, notamment dans le cadre isotherme, qui justifie naturellement l'étude de l'équation de Schrödinger-Langevin, appelée également système d'Euler-Korteweg isotherme amorti.

Système d'Euler-Korteweg et transformation de Madelung

En effectuant la transformation de Madelung [99] $\psi = \sqrt{\rho}e^{iS/\hbar}$, ou de façon plus rigoureuse le changement de variable

$$\rho = |\psi|^2 \quad \text{et} \quad \rho u = \hbar \, \mathrm{Im}\,(\psi^* \nabla \psi)$$

dans l'équation de Schrödinger

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = F(|\psi|^2)\psi,$$

où $\hbar > 0$ et $F : \mathbb{R}_+ \to \mathbb{R}$, on obtient formellement le système d'Euler-Korteweg :

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, & (1.2.16a) \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla P(\rho) = \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), & (1.2.16b) \end{cases}$$

où $\nabla P(\rho) := \rho\nabla F(\rho)$ désigne le gradient de la loi de pression $\rho \mapsto P(\rho)$. Ces équations forment un système de mécanique des fluides compressibles : l'équation (1.2.16a) est appelée *équation de continuité*, et l'équation (1.2.16b) désigne l'*équation de bilan de la quantité de mouvement* ou *équation du moment* et est constituée d'un terme classique de convection $\mathrm{div}\,(\rho u \otimes u)$, du terme de pression $\nabla P(\rho)$ et d'un terme de capillarité quantique $\hbar^2\rho\nabla\left(\Delta\sqrt{\rho}/\sqrt{\rho}\right)$ dit de Korteweg [86] provenant de la partie linéaire $i\hbar\partial_t\psi + \hbar^2\Delta\psi/2$ de l'équation de Schrödinger.

D'un point de vue physique, cette formulation a d'importantes conséquences : elle est à la base de la mécanique Bohmienne, aussi appelée théorie de Broglie-Bohm, qui est une interprétation de la physique quantique formulée en 1952 par le physicien David Bohm et qui repose sur la théorie de l'onde-pilote développée par Louis de Broglie en 1927. En introduisant la notion de variables cachées, elle propose alors une approche réaliste et déterministe de la mécanique quantique, à l'inverse de la vision probabiliste adoptée au congrès Solvay en 1927 : il n'y a pas de frontière entre les mondes classique et quantique, et le passage de l'un à l'autre se fait de manière continue. Mathématiquement, les trajectoires quantiques convergent vers les trajectoires classiques lorsque l'on fait tendre la constante de Planck renormalisée $\hbar$ vers 0, et on obtient alors formellement les équations d'Euler compressible de la mécanique des fluides classique. Cette interprétation est le reflet de la notion d'onde-pilote proposée par de Broglie : une particule est à la fois une onde et un corpuscule. Le mouvement de ce corpuscule est alors guidé par les équations de mécanique des fluides vérifiées par l'onde. L'étude du terme non-linéaire dispersif

$$\hbar^2 \rho \nabla \left(\Delta \sqrt{\rho} / \sqrt{\rho} \right) = \frac{\hbar^2}{4} \mathrm{div} \left(\rho \nabla^2 \log \rho \right) = \frac{\hbar^2}{4} \Delta \nabla \rho - \hbar^2 \mathrm{div} \left(\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} \right) \tag{1.2.17}$$

d'ordre 3, également appelé *potentiel Bohmien*, est alors au cœur de cette approche. On citera le livre de référence de D. Dürr et S. Teufel [56], ou encore celui de Robert E. Wyatt [115], pour une introduction à ce domaine.

L'utilisation de la transformation de Madelung est délicate dans un contexte mathématique rigoureux, comme indiqué par R. Carles, R. Danchin et J.-C. Saut dans [35]. En effet, la division par ρ pour passer des équations de Schrödinger à l'équation d'Euler-Korteweg (1.2.16) implique une étude précise des domaines $\{\rho > 0\}$ et $\{\rho = 0\}$ lorsque l'on cherche à transposer des estimations provenant de la fonction d'onde ψ du cadre régulier de l'équation de Schrödinger à celui du cadre hyperbolique de (1.2.16). La présence de termes fortement non-linéaires (*ie* qui ne peuvent être vus comme des perturbations d'une équation linéaire sous-jacente) de la mécanique des fluides quantique implique naturellement l'étude d'existence de solutions au sens faible, en multipliant généralement les équations (1.2.16a) et (1.2.16b) par des fonctions tests $\mathcal{C}_0^\infty$ puis en faisant passer les différents opérateurs de dérivation sur ces fonctions très régulières par des intégrations par parties. Par exemple, pour le système (1.2.16), une notion classique de *solution faible* s'énonce ainsi : pour la donnée de conditions initiales bien préparées (généralement $(\rho_0, (\rho u)_0) \in L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$) et d'un temps d'existence $T > 0$, on dit que $(\rho, \rho u)$ est solution faible de (1.2.16) sur $[0, T[$, s'il existe des fonctions intégrables $\sqrt{\rho}$ et Λ tel que, en définissant $\rho := \sqrt{\rho}^2$ et $\rho u = \sqrt{\rho}\Lambda$, ces fonctions vérifient :

(i) La régularité globale

$$\sqrt{\rho} \in L^\infty([0, T[\,; H^1(\mathbb{R}^d)), \quad \Lambda \in L^2([0, T[\,; L^2(\mathbb{R}^d)),$$

avec la *condition de compatibilité*

$$\sqrt{\rho} \geq 0 \text{ p.p. sur } (0,\infty) \times \mathbb{R}^d, \quad \Lambda = 0 \text{ p.p. sur } \{\rho = 0\},$$

qui permet de donner un sens à l'écriture $u = \Lambda/\sqrt{\rho}1_{\rho>0}$ et d'assurer que la densité $\rho = |\psi|^2$ soit bien positive. Ce cadre fonctionnel constitue la régularité minimale afin que les intégrales de la formulation faible qui suivent soient bien définies.

(ii) Les équations (1.2.16a) et (1.2.16b) au sens des distributions, c'est-à-dire que pour tout $\eta \in \mathcal{C}_0^\infty([0,T[\times \mathbb{R}^d)$,

$$\int_0^T \int_{\mathbb{R}^d} (\rho \partial_t \eta + \rho u \cdot \nabla \eta) dx dt + \int_{\mathbb{R}^d} \rho_0 \eta(0) dx = 0,$$

et pour tout $\zeta \in \mathcal{C}_0^\infty([0,T[\times \mathbb{R}^d; \mathbb{R}^d)$,

$$\int_0^T \int_{\mathbb{R}^d} \left(\rho u \cdot \partial_t \zeta + \Lambda \otimes \Lambda : \nabla \zeta + P(\rho) \mathrm{div}(\zeta) + \hbar^2 \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} : \nabla \zeta - \frac{\hbar^2}{4} \rho \Delta \mathrm{div} \zeta \right) dx dt$$
$$+ \int_{\mathbb{R}^d} (\rho u)_0 \zeta(0) dx = 0.$$

(iii) Une *condition d'écoulement irrotationnel généralisé* : pour presque tout $t \geq 0$,

$$\nabla \wedge \rho u = 2 \nabla \sqrt{\rho} \wedge \Lambda$$

au sens des distributions. Cette dernière condition signifie que, dans le cas suffisamment régulier où l'on peut écrire la vélocité u, on obtient que $\rho \nabla \wedge u = 0$, c'est-à-dire que le champ u est irrotationnel par rapport à la mesure ρdx.

À noter que l'on utilise en général la dernière expression de (1.2.17) pour le potentiel bohmien dans les formulations faibles, et que ces expressions sont bien équivalentes quand la fonction ρ est suffisamment régulière.

Le système d'Euler-Korteweg possède bien évidemment plusieurs lois de conservation et propriétés d'invariance, analogues à celles des équations de Schrödinger associées : on citera en particulier la conservation de la masse pour tout temps $t \geq 0$,

$$\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|\rho_0\|_{L^1(\mathbb{R}^d)},$$

qui est une réécriture de (1.2.1), ou encore la conservation de l'énergie associée au système

(1.2.16),

$$E(t) = E_{\text{kin}}(t) + E_p(t) := \frac{1}{2}\int_{\mathbb{R}^d} \rho|u|^2 + \int_{\mathbb{R}^d}\left(\frac{\hbar^2}{2}|\nabla\sqrt{\rho}| + H(\rho)\right) = E(0),$$

où E_{kin} et E_p désignent respectivement les termes d'énergie cinétique et d'énergie potentielle, et l'énergie interne $H(\rho)$ est relié à la pression $P(\rho)$ via l'expression

$$H(\rho) = \int_1^\rho \frac{P(\sigma)}{\sigma^2} d\sigma.$$

Quand une solution faible possède en plus de la définition donnée précédemment une énergie bornée (pour presque tout temps, au vu de la régularité des fonctions ρ et ρu), on parle alors de *solution faible d'énergie finie*. Dans [4], P. Antonelli et P. Marcati ont montré l'existence de solutions faibles d'énergie finie pour le système d'Euler-Korteweg (1.2.16) avec loi de *pression barotropique* $P(\rho) = \rho^\gamma$, avec $\gamma > 1$. Leur approche est basée sur le lien entre ce système et l'équation de Schrödinger avec non-linéarité puissance (1.2.6) pour la constante de non-linéarité $\lambda = (\gamma-1)/\gamma$, en utilisant notamment l'existence de solutions régulières pour des données initiales $\psi_0 \in H^1(\mathbb{R}^d)$ ainsi que les estimations de Strichartz associées. Pour le cas d'une loi de *pression isotherme* $P(\rho) = \rho$, qui correspond cette fois-ci au cas de la non-linéarité logarithmique pour l'équation de Schrödinger (1.2.12), R. Carles, K. Carrapatoso et M. Hillairet ont montré dans [34] l'existence de solutions faibles d'énergie finie $(\rho, \rho u)$ pour des données initiales $\rho_0 = |\psi_0|^2$ et $(\rho u)_0 = \hbar \operatorname{Im}\left(\overline{\psi_0}\nabla\psi_0\right)$ et $\psi_0 \in H^1(\mathbb{R}^d) \cap \mathcal{F}^\alpha(\mathbb{R}^d)$ pour $0 < \alpha \leq 1$. De même, leur méthode repose essentiellement sur la transformation de Madelung et la théorie de Cauchy développée dans [36] pour l'équation de Schrödinger logarithmique (1.2.12).

Système de Navier-Stokes-Korteweg et limites visqueuses

Un autre système de mécanique des fluides quantique a été l'objet de nombreuses études dans la littérature mathématique : celui de Navier-Stokes-Korteweg, qui peut s'écrire

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & (1.2.18a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + \nu\operatorname{div}(\rho\mathbb{D}u), & (1.2.18b) \end{cases}$$

avec les conventions précédentes. Ce système correspond au système d'Euler-Korteweg auquel on a ajouté un terme de viscosité $\nu\operatorname{div}(\rho\mathbb{D}u)$ avec la partie symétrique du gradient de vitesse $\mathbb{D}u = (\nabla u + \nabla u^\top)/2$ et la constante de viscosité $\nu > 0$. Il est intéressant de noter que ce dernier terme n'a pas d'équivalent du côté de l'équation de Schrödinger : le terme de viscosité invalide la transformation de Madelung pour le système de Navier-Stokes-Korteweg. On ne peut donc pas utiliser cette correspondance pour montrer certains résultats d'existence de solutions comme nous l'avons présenté avec le système d'Euler-Korteweg dans la section précédente. Comme expliqué dans [108], une méthode classique pour obtenir l'existence de solutions faibles pour ce genre de système est alors la suivante :

(1) On établit tout d'abord des *estimations a priori* de type estimation d'énergie pour des solutions suffisamment régulières du système.

(2) On introduit ensuite des termes de viscosité dans ce système, par exemple des termes de dérivée d'ordre supérieur $\varepsilon_1 \Delta\rho$ à droite de l'équation de continuité (1.2.18a) et $\varepsilon_2 \Delta(\rho u)$ à droite de l'équation du moment (1.2.18b) avec ε_1, $\varepsilon_2 > 0$. Ces termes impliquent généralement des estimations a priori plus fortes sur ce système approché (pour cet exemple en $\varepsilon_1 \int |\nabla\rho|^2$ et $\varepsilon_2 \int |\nabla(\rho u)|^2$ dans l'estimation d'énergie), pour lequel on peut alors facilement montrer l'existence de solutions faibles. Typiquement, dans notre exemple, l'équation sur la continuité devient une équation de type parabolique

$$\partial_t \rho + \operatorname{div}(\rho u) = \varepsilon_1 \Delta\rho$$

pour laquelle il est relativement aisé d'obtenir des résultats d'existence de solutions à forte régularité. Dans la formulation faible, le terme de viscosité $\varepsilon_1 \Delta\rho$ n'implique pas de véritable contrainte de formulation puisque l'on peut faire passer les différentes dérivées sur la fonction test $\eta \in \mathcal{C}_0^\infty([0, T[\times \mathbb{T}^d)$ comme suit :

$$\int_0^T \int_{\mathbb{T}^d} (\rho \partial_t \eta + (\rho u) \cdot \nabla\eta + \rho\Delta\eta) dx dt + \int_{\mathbb{T}^d} \rho_0 \eta(0) dx = 0.$$

À noter que les termes de bord se compensent dans les intégrations par parties par périodicité des fonctions en la variable d'espace x. Cette étape génère donc une suite de solutions assez régulières $(\rho_{\varepsilon_1,\varepsilon_2}, (\rho u)_{\varepsilon_1,\varepsilon_2})_{\varepsilon_1,\varepsilon_2}$ du système approché.

(3) Enfin, la partie la plus difficile est en général de montrer que les estimations de la partie (1), nécessairement uniformes en ε_1 et ε_2, permettent d'extraire une sous-suite de $(\rho_{\varepsilon_1,\varepsilon_2}, (\rho u)_{\varepsilon_1,\varepsilon_2})_{\varepsilon_1,\varepsilon_2}$ qui converge via des arguments de compacité vers une solution faible $(\rho, \rho u)$ du système de départ. Cette limite peut s'effectuer en deux temps ($\varepsilon_1 \to 0$ puis $\varepsilon_2 \to 0$ ou inversement) ou simultanément en ε_1 et ε_2. En outre, il est en général plus simple d'obtenir des critères de compacité sur le tore $\mathbb{T}^d$, qui est compact, que sur l'espace $\mathbb{R}^d$ tout entier. C'est pourquoi la plupart des résultats d'existence en mécanique des fluides compressibles sont énoncés sur $\mathbb{T}^d$. La dimension d peut également jouer un rôle important dans les diverses estimations dues aux propriétés d'inclusions des espaces de Sobolev (généralement $1 \leq d \leq 3$).

Il est évident que le cas des viscosités présentées ici est un exemple parmi tant d'autres. La littérature de mécanique des fluides compressibles regorge de termes de régularisation introduits pour ce type de système (on citera notamment le livre de E. Feireisl [59]). De même, plusieurs types d'estimation a priori existent également pour ce type de système, et sont plus ou moins sensibles à l'ajout de certains de ces termes de régularisation : on utilisera par la suite l'estimation de BD-entropy due à D. Bresch et B. Desjardin [25]

$$\int_{\mathbb{T}^d} \left(\frac{1}{2}\rho|u+\nabla\log\rho|^2 + \frac{\rho^\gamma}{\gamma-1}\right) + \int_0^t \int_{\mathbb{T}^d} \left(\frac{1}{\gamma}|\nabla\rho^{\frac{\gamma}{2}}|^2 + \rho|\mathbb{A}u|^2\right) dxds \leq C(\rho_0,(\rho u)_0,\gamma),$$

où $\mathbb{A}u = (\nabla u - \nabla u^\top)/2$, dans le cas non-quantique $\hbar = 0$ d'une loi de pression barotropique $P(\rho) = \rho^\gamma$ avec $\gamma > 1$, qui n'est plus valide par exemple quand on ajoute un terme de viscosité artificiel dans l'équation de continuité (terme en ε_1 dans l'exemple précédent). On peut également évoquer l'estimation de Mellet-Vasseur

$$\int_{\mathbb{T}^d} \rho|u|^2 \log\left(1+|u|^2\right)(t,x)dx \leq C(\rho_0,(\rho u)_0,\gamma)$$

pour ce même jeu de paramètre, due à A. Mellet et A. Vasseur [101], qui n'est plus valide quand on ajoute un terme de capillarité de type Korteweg dans l'équation du moment.

L'existence de solutions faibles au système de Navier-Stokes-Korteweg est un problème délicat, dont la résolution dans le cas de données initiales arbitrairement grandes pour une loi de pression barotropique n'a été résolu que très récemment dans [88] par A. Vasseur et I. Lacroix-Violet en introduisant le concept solution renormalisée. Un autre des résultats les plus notables de ces dernières années est l'article de A. Vasseur et C. Yu [111], dans lequel les auteurs montrent l'existence de solutions faibles (au sens classique) pour ce système en présence de termes d'amortissement du type $r_0 u$ et $r_1\rho|u|^2 u$ dans l'équation du moment (1.2.18b) en dimension $d = 3$ d'espace. Ce résultat leur a d'ailleurs permis de montrer l'existence de solutions au système de Navier-Stokes compressible avec viscosité dégénérée dans [110] par un passage à la limite $\hbar \to 0$ dans le système décrit précédemment. On peut également mentionner l'article de A. Jüngel [84] où l'auteur obtient un résultat d'existence faible pour des fonctions test de la forme $\rho\eta$, ou encore l'article de M. Gisclon et I. Lacroix-Violet [68] qui traite l'existence de solutions faibles dans le cas d'une loi de pression singulière. Dans le cadre d'une pression isotherme $P(\rho) = \rho$, R. Carles, K. Carrapatoso et M. Hillairet ont également démontré dans [34] l'existence de solutions faibles d'un système renormalisé en imposant des conditions de régularité plus fortes sur la variable $\sqrt{\rho}$.

Il est intéressant de constater que, comme mentionné dans [24], la pertinence physique du système de Navier-Stokes-Korteweg dans le cas d'une loi de pression barotropique est assez discutable, en particulier lorsque la densité est proche de 0. En revanche, S. Brull et F. Méhats ont dérivé dans [31] un modèle issu de la théorie cinétique quantique, initié par P. Degond et C. Ringhofer dans [52], faisant intervenir des lois de pression isotherme et des termes de viscosité de type Navier-Stokes. Plus précisément, par une approche analogue à la méthode de Chapman-Enskog, appliquée au cadre de la mécanique quantique, les auteurs écrivent les équations d'hydrodynamique

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.2.19a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + \mathbb{P}) + \rho \nabla V = 0, & (1.2.19b) \end{cases}$$

où V correspond à un potentiel, et le tenseur de pression $\mathbb{P}$ s'écrit sous la forme

$$\mathbb{P} = \frac{1}{(2\pi\hbar)^3} \int_{\mathbb{R}^d} (p-u)(p-u) f(t,x,p) dp,$$

où la fonction $f(t,x,p)$ correspond à une fonction de densité de distribution. Ce système d'équations est un système ouvert en raison de l'indétermination de l'opérateur $\mathbb{P}$. Cependant à l'équilibre, on peut écrire $f \sim f^{eq}(\rho, \rho u)$ (cf [51]) et on obtient alors un système fermé qui correspond au système d'Euler-Korteweg isotherme décrit dans la section précédente. Quand on développe cet équilibre à l'ordre 1 en écrivant $f \sim f^{eq}(\rho, \rho u) + \nu f_1(\rho, \rho u)$ pour $\nu > 0$ petit, on obtient également un système fermé qui correspond au système Navier-Stokes-Korteweg isotherme. Cette perturbation du système d'Euler justifierait l'étude d'une limite visqueuse $\nu \to 0$ dans ce cadre, limite qu'il est en général difficile voire impossible d'obtenir pour des notions classiques de solutions faibles de mécanique des fluides compressibles quantique.

Pour remédier à ce problème, on peut s'intéresser à d'autres notions de solutions faibles qui seraient plus adaptées à la justification rigoureuse des limites visqueuses singulières. C'est le cas des solutions dites *dissipatives*, introduites par P.-L. Lions [96], qui sont basées sur l'utilisation de fonctionnelles particulières appelées *entropies relatives*, que l'on écrit

$$\mathcal{E}(U|V) : X \times Y \mapsto \mathbb{R}_+,$$

où X est un espace de Banach de *solutions faibles* U, et $Y \subset X$ est un espace de Banach de *solutions fortes* V inclues dans cet espace de solutions faibles du système de mécanique des fluides

$$\frac{d}{dt} U(t) = \mathcal{A}(t, U(t)), \; t > 0, \; U(0) = U_0,$$

où $\mathcal{A}$ désigne un générateur non-linéaire. Une entropie relative doit alors vérifier trois propriétés :

- **Propriété de distance.** Pour tout couple $(U, V) \in X \times Y$,

$$\mathcal{E}(U|V) \geq 0 \quad \text{et} \quad \mathcal{E}(U|V) = 0 \text{ si et seulement si } U = V.$$

- **Fonctionnelle de Lyapounov.** Si V est une *solution d'équilibre*, c'est-à-dire si $\mathcal{A}(t, V(t)) = 0$ pour tout $t \geq 0$, alors $V \in Y$ et

$$\frac{d}{dt}\mathcal{E}(U|V)(t) \leq 0.$$

- **Inégalité de Gronwall.** Pour tout couple $(U, V) \in X \times Y$, pour presque tout $t \geq 0$,

$$\mathcal{E}(U|V)(t) - \mathcal{E}(U|V)(0) \leq \int_0^t \mathcal{E}(U|V)(s)ds.$$

Les solutions dissipatives constituent alors des solutions U (suffisamment régulières) satisfaisant l'inégalité de Gronwall pour toute solution forte V du système d'origine. Elles vérifient un *principe d'unicité faible-forte*, ce qui signifie que la solution faible dissipative et la solution forte coïncident tant que cette dernière existe. Par conséquent, la solution forte est unique dans la classe des solutions faibles, ce qui est une conséquence directe de l'inégalité de Gronwall.

Dans leur récent article [27], D. Bresch, M. Gisclon, et I. Lacroix-Violet ont montré l'existence de solutions dissipatives pour ces deux systèmes avec une loi de pression barotropique. Ils ont ainsi fait le lien entre les solutions faibles du système de Navier-Stokes-Korteweg obtenues par A. Vasseur et I. Lacroix-Violet et le système d'Euler-Korteweg. Afin de construire une entropie relative, ils considèrent un système de type Euler-Korteweg *augmenté*, qui s'écrit, en notant $\overline{v} = \frac{\hbar}{2}\nabla \log \rho$,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.2.20a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \lambda \nabla \rho^\gamma = \dfrac{\hbar}{2}\operatorname{div}(\rho \nabla \overline{v}), & (1.2.20b) \\ \partial_t(\rho \overline{v}) + \operatorname{div}(\rho \overline{v} \otimes u) + \dfrac{\hbar}{2}\operatorname{div}\left(\rho \nabla u^\top\right) = 0. & (1.2.20c) \end{cases}$$

avec $\gamma > 1$. Comme une solution faible du système originel est également solution faible du système augmenté, les auteurs récupèrent, grâce au résultat d'existence de [88], une solution faible $(\rho_\nu, (\rho u)_\nu, (\rho \overline{v})_\nu)$ du système de Navier-Stokes-Korteweg augmenté

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.2.21a) \\ \partial_t(\rho w) + \operatorname{div}(\rho w \otimes u) + \lambda \nabla \rho^\gamma = \dfrac{\sqrt{\hbar^2 - \nu^2}}{2}\operatorname{div}(\rho \nabla v) + \dfrac{\nu}{2}\operatorname{div}(\rho \nabla w), & (1.2.21b) \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes u) + \operatorname{div}\left(\rho \nabla u^\top\right) = 0. & (1.2.21c) \end{cases}$$

où $w = u + \frac{\nu}{2}\nabla \log \rho$ et $v = \log \rho$. En se basant sur l'estimation de BD-entropy de (1.2.21), les auteurs construisent ensuite une entropie relative (cf [27]) dont l'inégalité de Gronwall associée converge bien vers celle de l'entropie relative du système d'Euler-Korteweg

$$\mathcal{E}_{ELK}(\rho, u, \overline{v}|R, U, \overline{V})(t) = \frac{1}{2}\int_{\mathbb{T}^d} \rho(|\overline{v} - \overline{V}|^2 + |u - U|^2) + \int_{\mathbb{T}^d} H(\rho|R),$$

où

$$H(\rho|R) = H(\rho) - H(R) - H'(R)(\rho - R)$$

et $H(\rho) = \frac{1}{\gamma-1}\rho^\gamma$, et on obtient donc une solution dissipative de (1.2.20). Il est important de noter que dans la définition de solution dissipative, dans la mesure où les systèmes de Navier-Stokes-Korteweg et Euler-Korteweg ne possèdent pas de solution au sens fort, les auteurs doivent introduire un terme d'erreur $\mathscr{E}(R, U)$ qui tient compte de cette pathologie. Une description plus précise sera abordée au Chapitre 1.3.3.

Équation de Schrödinger-Langevin ou système d'Euler-Korteweg isotherme avec amortissement

L'équation de Schrödinger-Langevin logarithmique (qu'on appellera plus simplement équation de Schrödinger-Langevin dans la suite) apparaît pour la première fois dans la littérature physique en 1985 dans une note de A.B. Nassar [104], et s'écrit formellement sous la forme d'une équation de Schrödinger non-linéaire :

$$\boxed{i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{\mu}{2i}\hbar\psi\log\left(\frac{\psi}{\psi^*}\right).} \tag{1.2.22}$$

Cette équation possède deux non-linéarités de type logarithmique : la première dépend du module de ψ et fait alors écho aux récents travaux de Białynicki-Birula et Mycielski [20] qui ont déjà été évoqués plus haut, et la deuxième fait intervenir l'argument de la fonction d'onde ψ sous la forme d'un potentiel dit de Langevin, introduit par M D. Kostin en 1972 dans ses travaux sur les systèmes dissipatifs quantiques [48]. À noter que dans son article de 1985, Nassar introduit également un champ électrique externe que nous ne considérerons pas dans la suite. De même, nous rappelons que nous adoptons la convention de normalisation $m = 1$. L'auteur évoque alors dans la suite le lien via la transformation de Madelung entre l'équation (1.2.22) et le système de mécanique des fluides compressibles de type Euler-Korteweg

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, & (1.2.23a) \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \lambda\nabla\rho + \mu\rho u = \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). & (1.2.23b) \end{cases}$$

Cette équation, introduite afin de proposer une interprétation stochastique de la mécanique quantique dans le contexte de la mécanique Bohmienne, a récemment eu un regain d'intérêt dans la communauté physique, notamment en mécanique quantique, afin de décrire la mesure continue de la position d'une particule quantique (on renvoie au livre de A.B. Nassar et S. Miret-Artés [103] ainsi qu'aux articles [118] et [102]) ; également en cosmologie et en mécanique statistique, notamment via les travaux de P.-H. Chavanis [45], [46] et [47]. Ce dernier indique notamment que la constante $\lambda = 2k_B T/\hbar$ correspond à un coefficient de friction quantique dont les deux signes, positif et négatif, sont d'intérêt (où k_B désigne la constante de Boltzmann et T est une température effective). À l'inverse, le coefficient de friction réel μ est supposé strictement positif. On adoptera donc dans la suite la convention suivante :

$$\boxed{\lambda \in \mathbb{R}^* \quad \text{et} \quad \mu > 0.}$$

L'équation de Schrödinger-Langevin (1.2.22) n'avait pas encore reçu de développement mathématique rigoureux jusqu'à maintenant à notre connaissance, en particulier en présence d'une non-linéarité logarithmique $\lambda\psi \log|\psi|^2$. Le potentiel de type Langevin $\frac{\mu}{2i}\psi\log(\psi/\psi^*)$ et son lien avec le terme de dissipation $\mu\rho u$ en mécanique des fluides est en revanche connu et étudié depuis plusieurs années, notamment depuis les travaux de A. Jüngel [81] ou [83]. Son caractère mal posé, du fait de la présence d'un logarithme complexe multivalué ainsi qu'un comportement indéterminé quand la fonction d'onde ψ est proche de 0, fait de son traitement rigoureux un problème délicat. On renvoie à l'introduction de [4] et à ses références pour une description plus détaillée des différentes restrictions mathématiques liées à sa présence dans le cadre d'une loi de pression barotropique $\lambda\nabla\rho^\gamma$ avec $\gamma > 1$ à la place d'une loi de pression isotherme $\lambda\nabla\rho$ dans l'équation (1.2.23b). À noter également que dans [98], J. L. López, José Luis et J. Montejo-Gámez ont montré l'existence locale de solutions dans un domaine borné pour l'équation de Schrödinger-Lagenvin dans le cas $\lambda = 0$, avec de fortes hypothèses de régularité sur les conditions initiales.

Formellement, cette équation préserve certaines lois de conservation des équations de Schrödinger décrites précédemment, en particulier la conservation de la masse (1.2.1) ainsi que les invariances de translation en espace ou en temps. De même, comme pour l'équation de Schrödinger logarithmique, l'équation de Schrödinger-Langevin possède une propriété d'invariance d'échelle : formellement, si ψ est solution de (1.2.22) avec condition initiale $\psi(0,.) = \psi_0$, alors

$$\kappa\psi \exp\left(-2i\frac{\lambda}{\mu}\log\kappa\left(1 - e^{\mu t}\right)\right)$$

est également solution de (1.2.22) pour tout $\kappa > 0$ avec condition initiale $\kappa\psi_0$. En revanche, le terme de Langevin $\frac{\mu}{2i}\psi\log(\psi/\psi^*)$ implique une dissipation de l'énergie, que l'on peut écrire formellement en formulation fluide :

$$\frac{1}{2}\int_{\mathbb{R}^d} \rho|u|^2 dx + \int_{\mathbb{R}^d} \left(|\nabla\sqrt{\rho}|^2 + \lambda\rho\log\rho\right) dx + \mu\int_0^t \int_{\mathbb{R}^d} \rho|u|^2 dx ds \leq C(\psi_0). \tag{1.2.24}$$

De même, les propriétés d'invariance Galiléenne et de conservation du moment ne sont pas vérifiées par (1.2.22). Cependant un calcul simple (cf Chapitre 2.2) permet de montrer formellement la convergence du centre de masse quand $t \to \infty$:

$$\int_{\mathbb{R}^d} x\rho(t,x)dx \to \int_{\mathbb{R}^d} x\rho_0(x)dx - \frac{1}{\mu}\int_{\mathbb{R}^d} (\rho u)_0(x)dx.$$

Cette équation possède de nombreuses autres propriétés qui rendent son étude mathématique intéressante. Nous allons voir dans le prochain chapitre que la propriété de propagation des solutions gaussiennes est encore vérifiée, comme dans le cas linéaire ou logarithmique. Qui plus est, il est assez inédit pour une équation de Schrödinger non-linéaire de posséder un terme de dissipation d'énergie mais qui préserve la conservation de la masse : cela signifie que le terme de Langevin "respecte" en quelque sorte la symétrie de l'équation de Schrödinger.

1.3 Résultats de la thèse

1.3.1 Dynamique en temps long de l'équation de Schrödinger-Langevin sur $\mathbb{R}^d$

Article de référence : **Dynamics of the Schrödinger-Langevin equation**, paru à *Nonlinearity* [41].

Nous allons maintenant présenter certains résultats obtenus dans cette thèse qui font l'objet du Chapitre 2. Motivés par les remarques de la littérature physique, notamment les articles de C. Zander, A. R. Plastino et J. Díaz-Alonso [118] et de S.V. Mousavi et S. Miret-Artés [102], on montre que toute donnée initiale gaussienne est propagée en une solution gaussienne de l'équation (1.2.22) sur l'espace $\mathbb{R}^d$ (le cadre gaussien très régulier permet de s'affranchir des problèmes de définition de l'équation (1.2.22) et d'avoir une théorie de Cauchy globale). De plus, on peut décrire explicitement le comportement à l'infini de ces solutions gaussiennes, ce qui fait l'objet du théorème suivant :

Théorème 1. *Voir [41, Theorem 2.1].*
Soit $\lambda \in \mathbb{R}^*$, on considère la donnée initiale gaussienne

$$\psi_0(t,x) = b_0 \exp\left(-\frac{1}{2}\sum_{j=1}^{d} a_{0j} x_j^2\right)$$

avec b_0, $a_{0,j} \in \mathbb{C}$, $\alpha_{0,j} = \operatorname{Re} a_{0,j} > 0$. Alors l'unique solution ψ de (1.2.22) est donnée par l'expression gaussienne

$$\psi(t,x) = b_0 \prod_{j=1}^{d} \frac{1}{\sqrt{r_j(t)}} \exp\left(i\phi_j(t) - \alpha_{0j}\frac{x_j^2}{2r_j^2(t)} + i\frac{\dot r_j(t)}{r_j(t)}\frac{x_j^2}{2}\right)$$

pour les fonctions réelles ϕ_j et r_j, telles que, quand $t \to \infty$,

- *si $\lambda < 0$,*

$$r_j(t) \to \sqrt{\frac{\alpha_{0j}}{-2\lambda}} \quad et \quad \dot r_j(t) \to 0,$$

de sorte que la solution ψ tend vers un profil gaussien universel, ie

$$|\psi(t,x)| \underset{t\to+\infty}{\longrightarrow} Ce^{\lambda|x|^2} \quad pour\ tout\ x \in \mathbb{R}^d,$$

où $C = C(\lambda, \alpha_0, b_0, d) > 0$,

- *et si $\lambda > 0$,*

$$r_j(t) \sim 2\sqrt{\frac{\lambda\alpha_{0j}}{\mu}t} \quad et \quad \dot r_j(t) \sim \sqrt{\frac{\lambda\alpha_{0j}}{\mu t}},$$

de sorte que

$$\|\psi(t,\cdot)\|_{L^\infty} = \hat{C} t^{-d/4} \quad \text{et} \quad \|\nabla\psi(t,\cdot)\|_{L^2} \sim \tilde{C}\frac{1}{\sqrt{t}}.$$

où $\hat{C} = \hat{C}(\alpha_0, b_0, \mu, \lambda, d)$ *et* $\tilde{C} = \tilde{C}(\alpha_0, b_0, \mu, \lambda, d) > 0$.

À noter que l'on peut également exprimer les termes de phase ϕ_j en fonction de la variable r_j, via l'expression

$$\phi_j(t) = \phi_{j,0} e^{-\mu t} + \int_0^t e^{-\mu(t-s)} \left(i\frac{\dot{r}_j(s)}{2r_j(s)} - \frac{\alpha_{0,j}}{r_j(s)^2} - 2\lambda \log|b_0| - \lambda \log r_j(s) \right) ds.$$

Pour résumer, le théorème précédent indique que

- dans le cas focalisant $\lambda < 0$, toute solution gaussienne tend ponctuellement vers une gaussienne limite $Ce^{\lambda|x|^2}$, où la constante C ne dépend que de la masse initiale de la gaussienne $\|\psi_0\|_{L^1(\mathbb{R}^d)}$. Nous allons voir dans la suite que cette gaussienne définit en fait un *ground state* (également appelé *Gausson* dans le cadre gaussien) de façon analogue à l'équation de Schrödinger logarithmique.
- dans le cas défocalisant $\lambda > 0$, toute solution gaussienne disperse en $t^{-d/4}$, qui est un taux de dispersion bien plus lent que dans le cas usuel linéaire de Schrödinger libre (1.2.2) ou Schrödinger avec non-linéarité puissance (1.2.6) (qui est en $t^{-d/2}$) ou que l'équation de Schrödinger logarithmique (1.2.12) (en $(t\sqrt{\log t})^{-d/2}$). En revanche, contrairement au cas logarithmique, les normes de Sobolev homogènes de ces solutions gaussiennes décroissent au cours du temps.

La preuve de ce théorème, qu'il suffit d'effectuer en dimension $d = 1$ par une propriété de tensorisation des solutions gaussiennes (cf Remarque 2.24 du Chapitre 2), repose sur des changements de variable analogues au cas de l'équation de Schrödinger logarithmique de [36] et par l'étude de l'équation différentielle non-linéaire d'ordre 2,

$$\boxed{\ddot{r} = \frac{\alpha_0^2}{r^3} + \frac{2\lambda\alpha_0}{r} - \mu\dot{r},} \tag{1.3.1}$$

avec $r(0) = 1$ et $\dot{r}(0) = -\beta_0$. Pour les deux signes de λ, cette étude repose sur la compréhension fine du comportement global de l'équation. Le comportement des solutions est illustré par la Figure 1.1 pour le cas $\lambda < 0$ et la Figure 1.2 pour $\lambda > 0$, pour différentes conditions initiales $\dot{r}(0) = -5, -1, 0, 1, 5$.

Le but est alors d'utiliser les résultats obtenus sur ces fonctions gaussiennes afin de décrire le comportement asymptotique de solutions plus générales de (1.2.22). Comme nous n'avons pas formulé de théorie de Cauchy pour cette équation, on suppose l'existence de *solutions faibles*

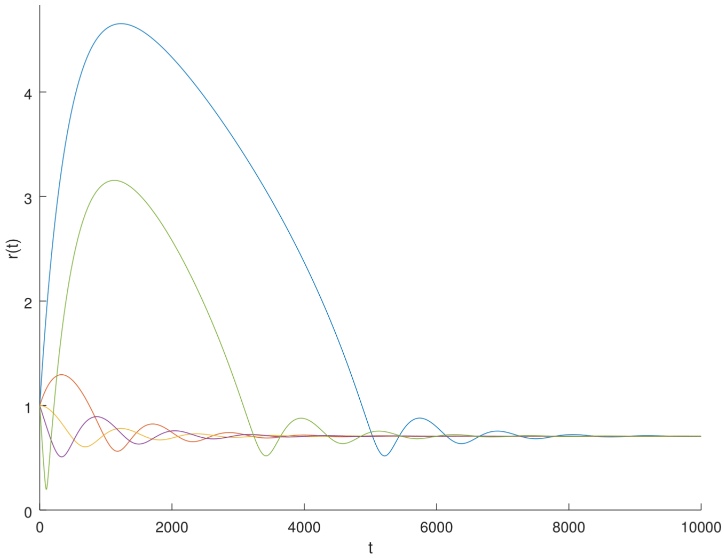

FIGURE 1.1 – Dynamique de l'équation (1.3.1) dans le cas focalisant $\lambda < 0$.

d'énergie finie pour le système d'équations de mécanique des fluides compressibles (1.2.23) (on renvoie à la Définition 2.2 du Chapitre 2 pour l'énoncé exact de cette hypothèse), qui fournit un cadre d'étude plus rigoureux de cette équation. Comme il est difficile de connaître la notion optimale de solution faible pour des problèmes de Cauchy en mécanique des fluides, nous faisons de plus certaines hypothèses de régularité sur nos solutions : on parle de *résultats de rigidité* (cf [108] et les références associées).

Dans le cadre d'une non-linéarité focalisante $\lambda < 0$, on suppose de plus que la densité ρ vérifie les hypothèses de régularité

$$\sqrt{\rho} \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^2(\mathbb{R}^d) \cap W(\mathbb{R}^d)),$$

et qu'elle possède un moment d'odre 2 uniformément borné,

$$\sup_{t \geq 0} \int_{\mathbb{R}^d} |x|^2 \rho(t,x) dx < \infty.$$

À noter que cette dernière hypothèse est à la fois satisfaite par l'ensemble des solutions gaussiennes de (1.2.22) (cf Remarque 2.17 du Chapitre 2) et par des simulations numériques (voir Figure 2.4). En effet, trois types de scénarios apparaissent généralement pour la dynamique de la densité ρ, grâce au fameux *lemme de concentration-compacité* du à P.-L. Lions [95] : soit la solution oscille ou explose en temps fini, ce qui est empêché par l'estimation d'énergie (1.2.24), soit la solution diverge simultanément vers $+\infty$ et $-\infty$ de façon symétrique, ce qui ne peut être

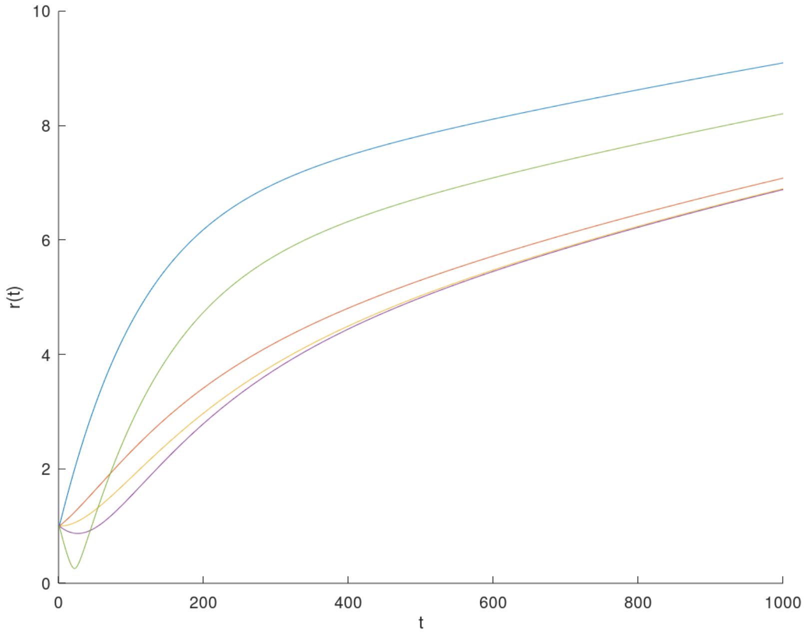

FIGURE 1.2 – Dynamique de l'équation (1.3.1) dans le cas défocalisant $\lambda > 0$.

exclu par la convergence du centre de masse $\int x\rho$ que nous avons déjà évoqués. Cette hypothèse de *tension* de la densité ρ empêche donc ce dernier scénario d'arriver, et nous renvoyons à la Remarque 2.7 du Chapitre 2 pour une discussion plus précise sur les difficultés rencontrées pour démontrer cette hypothèse. On peut alors énoncer le théorème suivant :

Théorème 2. *Voir [41, Theorem 2.6].*
Pour $\lambda < 0$, et sous les hypothèses de régularité présentées ci-dessus, en supposant que $\rho_0 \neq 0$, on a que

$$\rho(t,\cdot) \underset{t\to+\infty}{\rightharpoonup} c_\lambda e^{\lambda|x-x_\infty|^2} \quad \textit{faiblement dans } L^1(\mathbb{R}^d),$$

où $c_\lambda := \|\rho_0\|_{L^1(\mathbb{R}^d)}(-\lambda/\pi)^{d/2}$ et x_∞ est uniquement déterminé par

$$x_\infty\|\rho_0\|_{L^1(\mathbb{R}^d)} = \lim_{t\to\infty}\int_{\mathbb{R}^d} x\rho(t,x)dx = \int_{\mathbb{R}^d} x\rho_0(x)dx - \frac{1}{\mu}\int_{\mathbb{R}^d}\rho_0(x)u_0(x)dx.$$

Ce théorème implique donc la convergence, dans un sens faible, de la densité de toute solution de (1.2.22) vers un profil gaussien limite. Ce phénomène, singulier en mécanique des fluides compressibles ou en théorie des EDPs dispersives, est en revanche bien plus connu et compris en théorie cinétique, où les solutions convergent vers des Maxwelliennes, qui correspondent également à des fonctions gaussiennes. Sa démonstration repose notamment sur une étude du

Gausson $\phi_\omega(x) := e^{\frac{\omega}{2\lambda}+d} e^{\lambda|x|^2}$ solution de l'équation elliptique non-linéaire

$$-\frac{1}{2}\Delta f + \omega f + \lambda f \log|f|^2 = 0$$

analogue à celle effectuée dans [49], et l'utilisation du *principe d'invariance de LaSalle* [90] issu de la théorie des systèmes dynamiques : on remarque en effet que la dissipation liée au terme μ correspond à l'intégrale en temps de l'énergie cinétique dans (1.2.24), et donc que

$$\dot{E}_c(t) + \dot{E}_p(t) = -2\mu E_c(t) \leq 0, \tag{1.3.2}$$

ce qui implique que l'ensemble-limite de toute solution de (1.2.24) est réduit à l'ensemble

$$\Omega = \left\{(\rho, u) \;\middle|\; \dot{E}_c(\rho, u) + \dot{E}_p(\rho, u) = 0\right\} = \{(\rho, u) \mid E_c(\rho, u) = 0\}.$$

Ce résultat de convergence vers le Gausson limite est également confirmé par des simulations numériques, voir Figure 2.1.

Pour le cas de la non-linéarité défocalisante ($\lambda > 0$), on reprend l'idée de [36], et en supposant l'hypothèse supplémentaire de régularité

$$\sqrt{\rho} \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d)),$$

on peut énoncer le théorème suivant :

Théorème 3. *Voir [41, Theorem 2.10].*
Pour $\lambda > 0$, sous les hypothèses d'existence et de régularité décrites précédemment, on note $\tilde{\rho}$ la fonction renormalisée par le changement de variable

$$\rho(t, x) = \frac{1}{\tau(t)^{d/2}} \tilde{\rho}\left(t, \frac{x}{\tau(t)}\right),$$

où τ désigne l'unique solution $\mathcal{C}^\infty([0,\infty))$ de l'équation différentielle

$$\ddot{\tau} = \frac{2\lambda}{\tau} - \mu\dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0,$$

qui satisfait, quand $t \to +\infty$,

$$\tau(t) \sim 2\sqrt{\frac{\lambda t}{\mu}}.$$

Alors

$$\tilde{\rho}(t, \cdot) \underset{t\to+\infty}{\rightharpoonup} c_0 e^{-|x|^2} \quad \textit{faiblement dans } L^1(\mathbb{R}^d),$$

où $c_0 := \|\rho_0\|_{L^1(\mathbb{R}^d)} \pi^{-d/2}$.

Ce théorème implique également une convergence faible vers une gaussienne limite via une renormalisation par le taux de dispersion τ, car toute solution disperse vers 0, ce qui justifie le nom de non-linéarité "défocalisante". La preuve, qui repose sur le changement de variable décrit dans le théorème, se fait par l'analyse du système de mécanique des fluides renormalisé

$$\begin{cases} \partial_t \varrho + \dfrac{1}{\tau^2} \mathrm{div} \mathcal{J} = 0, & (1.3.3a) \\ \partial_t \mathcal{J} + \dfrac{1}{\tau^2} \mathrm{div} \left(\dfrac{\mathcal{J} \otimes \mathcal{J}}{\varrho} \right) + \lambda \nabla \varrho + 2\lambda y \varrho + \mu \mathcal{J} = \dfrac{1}{2\tau^2} \varrho \nabla \left(\dfrac{\Delta \sqrt{\varrho}}{\sqrt{\varrho}} \right). & (1.3.3b) \end{cases}$$

Formellement, en injectant l'équation de conservation de la masse dans celle du moment et en laissant $\tau(t) \to \infty$ dans le système précédent, on obtient une équation fermée sur ϱ de type Fokker-Planck

$$\partial_s \varrho = L \varrho,$$

où $L := \lambda \Delta + 2\lambda \mathrm{div}(y \cdot)$, équation pour laquelle il est connu que les solutions convergent en temps grand vers des solutions gaussiennes. À noter que contrairement à l'équation elliptique du cas focalisant, l'équation précédente n'est pas invariante par translation en espace, et par conséquent, la limite du théorème est bien unique (et pas seulement à translation près). On peut noter que la preuve repose également sur des théorèmes classiques d'analyse fonctionnelle, notamment pour montrer que l'entropie associée à cette renormalisation

$$\mathcal{E}_{\mathrm{ent}}(t) = \int_{\mathbb{R}^d} \varrho \log \left(\frac{\varrho}{\Gamma} \right) dx,$$

avec $\Gamma := e^{-|x|^2} = \gamma^2$, est bien positive par l'inégalité de Csiszár–Kullback (cf Lemme 4)

$$\mathcal{E}_{\mathrm{ent}}(t) \geq \frac{1}{2\|\Gamma\|_{L^1}} \|\varrho(t) - \Gamma\|_{L^1}^2 \geq 0,$$

mais aussi sur les théorèmes de De la Vallée-Poussin (cf Lemme 5) et de Dunford-Pettis (cf Lemme 6). Ce résultat est encore une fois appuyé par des simulations numériques (voir Figure 2.2 au Chapitre 2).

1.3.2 Stabilité d'ondes planes pour l'équation de Schrödinger logarithmique et Schrödinger-Langevin sur $\mathbb{T}^d$

Article de référence : **Around plane waves solutions of the Schrödinger-Langevin equation**, avec Erwan Faou, à paraître à *SIAM J. Math. Analysis* [44].

Les hypothèses de régularité exigées dans la section précédente sont directement liées au manque de compacité sur l'espace $\mathbb{R}^d$. Une approche classique dans l'étude des EDPs, notamment en mécanique des fluides compressibles, est de se placer dans une géométrie compacte, celle du tore $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. Les résultats d'existence de solution sont alors généralement plus faciles à établir. En revanche le comportement des solutions est bien différent, en particulier dans le cadre des EDPs dispersives. Dans le cas des équations de Schrödinger, l'étude des solutions gaussiennes sur $\mathbb{R}^d$ est alors remplacée par celle des ondes planes

$$\nu_m(t,x) = \rho e^{im\cdot x-\omega t}, \quad x \in \mathbb{T}^d, \tag{1.3.4}$$

avec $\rho > 0$, $m \in \mathbb{Z}^d$ et $\omega \in \mathbb{R}$, solutions déjà évoquées supra, et qui font alors bien partie des espaces de régularité classique (espaces de Lebesgue L^p et espaces de Sobolev H^s). Dans un article paru en 2013 [58], E. Faou, L. Gauckler et C. Lubich ont montré la stabilité orbitale *générique* des solutions de type onde plane (1.3.4) avec $\omega = |m|^2 + 2\lambda\rho^2$ de l'équation de Schrödinger cubique

$$i\partial_t\psi + \Delta\psi = \lambda|\psi|^2\psi, \tag{1.3.5}$$

dans le cas de petites perturbations autour de ces solutions dans des espaces de forte régularité $H^s(\mathbb{T}^d)$ avec $s \geq s_0$ pour s_0 suffisamment grand. Leur preuve est notamment basée sur des arguments classiques issus de la théorie des formes normales de Birkhoff dans le cadre des EDPs hamiltoniennes développé par D. Bambusi et B. Grébert [11], [71], et nécessite une restriction sur la taille des solutions dans le cas focalisant du type $1 + 2\lambda\rho^2 > 0$, et des conditions de non-résonances sur les valeurs propres du linéarisé, satisfaites pour presque tout ρ.

Dans le cas de l'équation de Schrödinger logarithmique

$$i\partial_t\psi + \Delta\psi = \lambda\psi \log|\psi|^2,$$

pour une donnée initiale constituée d'un seul mode de Fourier $\nu(0,x) = \rho e^{im\cdot x}$, cette équation possède également une unique solution de type onde plane $\nu_m(t,x) = \rho e^{i(m\cdot x-\omega t)}$ localisée sur le mode de Fourier m, avec la fréquence

$$\omega = |m|^2 + 2\lambda \log \rho.$$

Nous avons montré avec E. Faou dans [44], par une méthode de linéarisation analogue à [58] autour de ces ondes planes, la stabilité en temps grand des ondes planes solutions de (1.2.12) dans ces mêmes espaces de Sobolev H^s à forte régularité, pour presque tout $\lambda > -\frac{1}{2}$. Plus précisément, on a le théorème suivant :

Théorème 4. *Voir [44, Theorem 2].*
Soit $m \in \mathbb{Z}^d$ fixé. On suppose $-\frac{1}{2} < \lambda_- < \lambda_+$, $\rho > 0$, $\theta_0 \in \mathbb{R}$ et $N > 1$ fixés arbitrairement. Alors il existe $s_0 > 0$, $C \geq 1$ et un ensemble de mesure pleine Λ contenu dans l'intervalle $[\lambda_-, \lambda_+]$ tel que pour tout $s \geq s_0$ et pour tout $\lambda \in \Lambda$, il existe $\varepsilon_0 > 0$ tel qu'on ait le résultat suivant : si la donnée initiale ψ_0 est telle que

$$\|e^{-im.x}\psi_0 - \langle e^{-im.x}\psi_0\rangle\|_{H^s} = \varepsilon \leq \varepsilon_0, \quad \textit{et} \quad \|\psi^0\|_{L^2} = \rho,$$

alors la solution $\psi(t)$ de l'équation de Schrödinger logarithmique avec $\psi(0,x) = \psi_0(x)$ satisfait

$$\|e^{-im.x}\psi(t,\cdot) - \rho e^{i\theta(t)}\|_{H^s} \leq C\varepsilon \quad \textit{pour} \quad t \leq \varepsilon^{-N},$$

où $\theta(t)$ vérifie

$$|\dot{\theta} + |m|^2 - 2\lambda \log \rho| \leq C\varepsilon^2.$$

On peut faire plusieurs remarques sur ce théorème :

- comme $t \in \left[0, \varepsilon^{-N}\right]$ avec $\varepsilon \leq \varepsilon_0 \ll 1$ et que l'entier N peut être choisi arbitrairement grand, on a bien un résultat de stabilité en temps grand, qui fournit de plus un résultat d'existence de solutions à l'équation de Schrödinger logarithmique (1.2.12) dans le cadre des espaces de Sobolev à forte régularité H^s.
- par conservation de la norme L^2 de la solution ψ de (1.2.12), on a bien un résultat de stabilité orbitale du type

$$\inf_{\theta \in \mathbb{R}} \|e^{-im.x}\psi(t,\cdot) - e^{i\theta}\nu_m(0,\cdot)\|_{H^s} \leq C\varepsilon, \quad \text{pour} \quad t \leq \varepsilon^{-N}.$$

- l'hypothèse sur λ, qui fait intervenir un ensemble de mesure pleine Λ, découle des théorèmes classiques sur les formes normales de Birkhoff appliquées aux EDPs hamiltoniennes (on citera par exemple, en plus de [58], les résultats de [10] ou [12]).

La démonstration s'appuie sur les nombreux invariants de l'équation (1.2.12) : l'invariance galiléenne permet de se ramener au cas $m = 0$ de l'onde plane constante ν_0, et il suffit de montrer le résultat pour $\rho = 1$ grâce à l'invariance d'échelle. En décomposant la solution ψ sur la base de Fourier $\psi(t,x) = \sum_{j\in\mathbb{Z}^d} \psi_j(t) e^{ij\cdot x}$, et par un changement de variable

$$\psi = e^{i\theta}(a + w)$$

avec $\langle w \rangle = 0$ et $a > 0$, on est ramené à l'étude de la fonction $w = \sum_{j\neq 0} w_j(t) e^{ij\cdot x}$ par conservation de la norme L^2 de ψ et l'égalité de Parseval, qui peut alors s'écrire

$$a = \sqrt{1 - \|w\|^2_{L^2(\mathbb{T}^d)}}.$$

À noter que les équations sur les $(w_j)_{j\neq 0}$ permettent également de contrôler le terme de phase θ. En développant les termes logarithmiques en série formelle autour de la constante 1, on remarque alors qu'au niveau linéaire, l'équation sur le mode w_j est couplée à celle sur $\overline{w}_{-j}$ et inversement. En notant $W_n = (w_j, \overline{w}_{-j})^T$ pour $n = |j|^2 \geq 1$, on a alors une équation qui peut s'écrire sous la forme

$$i\dot{W}_n = A_n W_n + \mathcal{O}(\|W\|^2_{L^2}),$$

avec

$$A_n = \begin{pmatrix} n+\lambda & \lambda \\ -\lambda & -n-\lambda \end{pmatrix},$$

matrice symplectique hermitienne et de conditionnement inférieur à 2 (et donc uniforme en n), dont les valeurs propres $\pm\sqrt{n^2 + 2\lambda n}$ satisfont la condition de non-résonance de Bambusi [9] pour presque tout λ, et on peut alors appliquer un théorème analogue au Théorème 7.2 de [71] qui conduit au résultat. Encore une fois, ce résultat est corroboré par des simulations numériques en dimension $d = 1$: dans la Figure 3.1 du Chapitre 3, où l'on trace le module de la solution perturbée ψ, ou dans la Figure 3.3, où l'on trace l'évolution des actions $(|\psi_j|)_{j\in\mathbb{Z}}$ de la solution ψ.

Nous avons également également étudié le cas de l'équation de Schrödinger-Langevin

$$i\partial_t\psi + \Delta\psi = \lambda\psi\log|\psi|^2 + \frac{\mu}{2i}\psi\log\left(\frac{\psi}{\psi^*}\right),$$

où la situation est alors bien différente : on perd la structure Hamiltonienne réelle de (1.2.12) et on ne peut donc pas appliquer la théorie des formes normales. De plus, le terme d'amortissement en μ change également la dynamique des ondes planes : les solutions stationnaire de l'équation sont alors les solutions constantes de la forme

$$\nu = \rho e^{-2i\lambda\log\rho/\mu},$$

où $\rho > 0$. Pour $s > \frac{d}{2}$, l'espace de Sobolev $H^s(\mathbb{T}^d)$ est une algèbre, et on note C_s la constante liée à l'inégalité : pour tout u, $v \in H^s(\mathbb{T}^d)$,

$$\|uv\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s} .$$

On définit alors le sous-ensemble

$$\mathcal{U}_s = \left\{ u = e^{i\theta}(a+w) \,\middle|\, (a,\theta,w) \in \mathbb{R}_+ \times \mathbb{T} \times H^s(\mathbb{T}^d),\ a>0,\ \langle w \rangle = 0, \quad \left\| \frac{w}{a} \right\|_{H^s} < \frac{1}{C_s} \right\}$$

sur lequel sont bien définies de façon analytique les non-linéarités logarithmiques de l'équation de Schrödinger-Langevin $u(t) \mapsto \log |u(t)|^2 = \lambda u(t)(\log u(t) + \log u^*(t))$ et $u(t) \mapsto \log(u(t)) - \log(u^*(t))$. On peut alors énoncer le théorème suivant, qui indique que toute solution suffisamment proche de la constante ν pour $\rho > 0$ converge exponentiellement vite vers cette constante dans $H^s(\mathbb{T}^d)$, et sous la condition $\lambda > -\frac{1}{2}$:

Théorème 5. *Voir [44, Theorem 1].*
Soit $s > \frac{d}{2}$, $\lambda > -\frac{1}{2}$, $\mu > 0$ et $\rho > 0$. Alors il existe $\varepsilon_0 > 0$ tel que, pour toute donnée initiale $\psi_0 \in \mathcal{U}_s$ vérifiant

$$\|\psi_0 - \langle \psi_0 \rangle\|_{H^s(\mathbb{T}^d)} \leq \varepsilon_0 \quad \text{et} \quad \|\psi_0\|_{L^2(\mathbb{T}^d)} = \rho,$$

la solution ψ de l'équation de Schrödinger-Langevin, avec $\psi(0,.) = \psi_0 \in \mathcal{U}_s$, satisfait pour tout $t \geq 0$,

$$\|\psi(t,\cdot) - \rho e^{-2i\lambda \log \rho/\mu}\|_{H^s(\mathbb{T}^d)} \leq C e^{-\alpha t}(1+\beta t),$$

où $C = C(s,d,\rho,\varepsilon_0) > 0$, et où $\alpha > 0$ et $\beta \in \{0,1\}$ dépendent de μ et λ de la façon suivante :

(i) Si $\mu < 2\sqrt{1+2\lambda}$ alors $\alpha = \frac{\mu}{2}$ et $\beta = 0$.

(ii) Si $\mu = 2\sqrt{1+2\lambda}$ alors $\alpha = \frac{\mu}{2}$ et $\beta = 1$.

(iii) Si $\mu > 2\sqrt{1+2\lambda}$ alors

$$\alpha = \frac{\mu}{2} - \left(\frac{\mu^2}{4} - 1 - 2\lambda \right)^{\frac{1}{2}} \in \left(0, \frac{\mu}{2}\right),$$

et s'il existe un entier $n \geq 2$ tel que $\mu^2 = 4n^2 + 8\lambda n$ on a $\beta = 1$, et si ce n'est pas le cas, alors $\beta = 0$.

On peut alors faire plusieurs remarques :

- contrairement au cas de l'équation de Schrödinger logarithmique, on a un résultat dans $H^s(\mathbb{T}^d)$ avec une borne inférieure explicite pour $s > \frac{d}{2}$, et non pas pour $s \geq s_0$ avec s_0 inconnu. De même, le résultat est vrai pour tout $\lambda > -\frac{1}{2}$, et non pas sur un ensemble de mesure pleine.
- Il n'y a pas de restriction sur ρ, la solution onde plane peut être arbitrairement grande.

- Pour λ fixé et $\mu \to \infty$, le taux d'amortissement
$$\alpha = \frac{2\lambda + 1}{\mu} + \mathcal{O}\left(\frac{1}{\mu^3}\right)$$
tend vers 0. Ainsi, un coefficient d'amortissement μ plus élevé implique une convergence plus lente vers l'équilibre. Cet effet fait écho à des comportements connus dans d'autres contextes, on renvoie notamment à [74] et aux références associées. Cependant, à notre connaissance, ce type d'effet n'a encore jamais été observé dans le cas d'un modèle conservatif issu de la mécanique quantique.

Le début de la preuve suit la même trame que pour le cas de l'équation de Schrödinger logarithmique : par invariance d'échelle, on peut se ramener au cas de la constante $\nu = 1$ autour de laquelle on développe les fonctions logarithmiques de (1.2.22) en série formelle. Au niveau linéaire, on observe toujours un couplage entre les modes de Fourier w_j et $\overline{w}_{-j}$ de la solution $\psi = e^{i\theta}(a + w)$ (on peut encore exprimer a et θ comme des fonctions des $(w_j, \overline{w}_j)_{j \in \mathbb{Z}^d}$), mais cette fois la matrice A_n s'écrit

$$A_n = \begin{pmatrix} n + \lambda + \frac{\mu}{2i} & \lambda - \frac{\mu}{2i} \\ -\lambda - \frac{\mu}{2i} & -n - \lambda + \frac{\mu}{2i} \end{pmatrix}.$$

Les valeurs propres de A_n s'écrivent

$$\frac{\mu}{2i} \pm \sqrt{n^2 + 2\lambda n - \frac{\mu^2}{4}},$$

et on doit alors distinguer trois cas en fonction du mode $n = |j|^2 \geq 1$ pour étudier la dynamique de l'équation linéaire, qui correspondent aux trois points du théorème. On notera en particulier que la présence du β dans le théorème ci-dessus correspond à la présence de matrice de Jordan lors de la trigonalisation de la matrice A_n liée à la présence d'une valeur propre double. On conclut alors la preuve via des changements de variables successifs vis-à-vis des paramètres de l'équation linéaire et différents contrôles des normes de Sobolev des quantités impliquées. Les tracés de l'évolution du module de la solution $|\psi(t, .)|$ sur la Figure 3.2 et ceux des actions de ψ pour les cas $\lambda = 0.5$ et $\mu = 2$ (Figure 3.4) ou $\mu = 8$ (Figure 3.5) viennent appuyer les différents résultats du théorème.

1.3.3 Existence de solutions dissipatives au système d'Euler-Korteweg isotherme avec amortissement

Article de référence : **Global dissipative solutions of the defocusing isothermal Euler-Langevin-Korteweg equations**, paru à *Asymptotic Analysis* [42].

Les résultats de la section précédente impliquent l'existence de solutions fortes et en temps grand pour des solutions proches d'ondes planes. Pour obtenir des résultats d'existence dans un cadre général de solutions, le problème est plus délicat en raison du terme de dissipation en μ. On doit alors se baser sur la formulation de mécanique des fluides de l'équation de Schrödinger-Langevin (1.2.23). Notre première approche a été d'essayer de reproduire celle développée par P. Antonelli et P. Marcati dans [4] dans le cas d'une loi de pression barotropique, où les auteurs construisent une solution faible $(\rho, \rho u)$ du système

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.3.6a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma + \mu \rho u = \dfrac{\hbar^2}{2} \rho \nabla \left(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right), & (1.3.6b) \end{cases}$$

à l'aide d'une construction de type méthode de splitting : l'existence d'une solution ψ dans le cas $\mu = 0$ avec condition initiale ψ_0 suffisamment régulière fournit une solution sur un petit intervalle de temps $[0, \tau[$, puis cette solution au temps τ est modifiée afin de tenir compte du terme $\mu \rho u$ dans l'équation (1.3.6b). On vérifie alors que la nouvelle fonction incrémentée ψ^τ est suffisamment régulière, puis on réapplique ce procédé sur l'intervalle de temps $[\tau, 2\tau[$, et ainsi de suite. On montre enfin que ce schéma itératif est consistant, c'est-à-dire que si $(\psi^\tau)_\tau$ est une suite de solutions approchées et que $\psi^\tau \to \psi$ quand $\tau \to 0$, alors ψ est une solution faible du problème sur un intervalle $[0, T[$. Il reste alors à montrer que cette suite de solutions approchées converge bien vers une fonction ψ solution de (1.3.6). Cependant, la preuve de ce résultat est basée sur l'utilisation des estimations de Strichartz, qui ne semblent pas fournir d'informations utiles dans le cas des non-linéarités logarithmiques, et donc dans le cadre de loi de pression isotherme. En effet, plutôt que d'estimer $\nabla \left(\psi |\psi|^{\gamma-1}\right)$, nous devons regarder la quantité $\nabla \left(\psi \log(|\psi|^2)\right)$, qui n'est pas bornée quand la fonction d'onde $|\psi|^2$ s'approche de 0. De même, on serait tenté d'utiliser les résultats de la seconde partie de [34] où les auteurs montrent l'existence de solutions faibles au système d'Euler-Korteweg isotherme sans terme de dissipation en μ, mais encore une fois leur preuve est basée sur le lien entre ce système et l'équation de Schrödinger logarithmique (1.2.12) et les propriétés d'existence de cette dernière, lien qui ne peut être utilisé dans le cas de l'équation (1.2.22).

Une seconde approche a alors été de se tourner vers les résultats plus classiques de la mécanique des fluides compressibles décrits dans la Section 1.2.2. L'idée est alors de construire une solution faible du système de *Navier-Stokes-Langevin-Korteweg* sur $\mathbb{T}^d$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.3.7a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \lambda \nabla \rho + \mu \rho u = \dfrac{\hbar^2}{2} \rho \nabla \left(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \nu \operatorname{div}(\rho \mathbb{D} u), & (1.3.7b) \end{cases}$$

via les constructions classiques de solution faible effectuées dans [34], où le terme de dissipation de Langevin n'introduit pas de difficulté supplémentaire. Cette solution faible $(\rho_\nu, (\rho u)_\nu)_{\nu>0}$ est également solution faible du système de *Navier-Stokes-Langevin-Korteweg augmenté*

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (1.3.8a) \\ \partial_t(\rho w) + \operatorname{div}(\rho w \otimes u) + \lambda' \nabla \rho + \mu \rho w = \dfrac{\hbar_\nu}{2} \operatorname{div}(\rho \nabla v) + \dfrac{\nu}{2} \operatorname{div}(\rho \nabla w), & (1.3.8b) \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes u) + \operatorname{div}\left(\rho \nabla u^\top \right) = 0, & (1.3.8c) \end{cases}$$

où $\hbar_\nu^2 = \hbar^2 - \nu^2 > 0$ et $\lambda' = \lambda - \frac{\mu\nu}{2} > 0$, solution que l'on note $(\rho_\nu, w_\nu, v_\nu) := (\rho_\nu, u_\nu + \frac{\nu}{2} \nabla \log \rho_\nu, \log \overline{\rho})$. Le point clef ici est que le terme de pression isotherme $\lambda \nabla \rho$ peut "absorber" la contribution issue du terme d'amortissement en μ par l'identité

$$\lambda \nabla \rho + \mu \rho u = \lambda \nabla \rho + \mu \rho (w - \frac{\nu}{2} \nabla \log \rho) = (\lambda - \frac{\mu\nu}{2}) \nabla \rho + \mu \rho w.$$

En supposant $\lambda > 0$ (on se place donc dans le cas d'une non-linéarité défocalisante côté Schrödinger), et comme ν a pour vocation de tendre vers 0, on peut prendre ν suffisamment petit de sorte que $\lambda' = \lambda - \mu\nu/2 > 0$, point essentiel pour construire une entropie relative positive. On construit alors une entropie relative du système précédent, en notant $\overline{v} = \hbar_\nu v/2$,

$$\begin{aligned} \mathcal{E}_{NSLK}(\rho, w, \overline{v} | R, W, \overline{V})(t) &= \frac{1}{2} \int_{\mathbb{T}^d} \rho (|\overline{v} - \overline{V}|^2 + |w - W|^2) + \lambda' \int_{\mathbb{T}^d} H(\rho|R) \\ &+ \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} \rho \left(\left| \frac{\mathbb{T}_N(\overline{v})}{\sqrt{\rho}} - \nabla \overline{V} \right|^2 + \left| \frac{\mathbb{T}_N(w)}{\sqrt{\rho}} - \nabla W \right|^2 \right) + \mu \int_0^t \int_{\mathbb{T}^d} \rho |w - W|^2, \end{aligned}$$

où

$$H(\rho|R) := H(\rho) - H(R) - H'(R)(\rho - R)$$

et $H(\rho) = \rho \log \rho - \rho$, et où les tenseurs $\mathbb{T}_N(\overline{v})$ et $\mathbb{T}_N(w)$ sont définis via les conditions de compatibilité suivantes

$$\sqrt{\rho}\mathbb{T}_N(\overline{v}) = \nabla(\sqrt{\rho}\sqrt{\rho}\overline{v}) - 2\sqrt{\rho}\overline{v} \otimes \nabla\sqrt{\rho} \quad \text{et} \quad \sqrt{\rho}\mathbb{T}_N(w) = \nabla(\sqrt{\rho}\sqrt{\rho}w) - 2\sqrt{\rho}w \otimes \nabla\sqrt{\rho}.$$

De même on définit une entropie relative

$$\mathcal{E}_{ELK}(\rho, u, \overline{v}|R, U, \overline{V})(t) = \frac{1}{2}\int_{\mathbb{T}^d} \rho(|\overline{v} - \overline{V}|^2 + |u - U|^2) + \lambda \int_{\mathbb{T}^d} H(\rho|R) + \mu \int_0^t \int_{\mathbb{T}^d} \rho|u - U|^2,$$

associée au système d'*Euler-Langevin-Korteweg augmenté*

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & (1.3.9a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \lambda\nabla\rho + \mu\rho u = \dfrac{\hbar}{2}\operatorname{div}(\rho\nabla\overline{v}), & (1.3.9b) \\ \partial_t(\rho\overline{v}) + \operatorname{div}(\rho\overline{v} \otimes u) + \dfrac{\hbar}{2}\operatorname{div}\left(\rho\nabla u^\top\right) = 0. & (1.3.9c) \end{cases}$$

Il est important de noter que les deux entropies relatives ci-dessus sont bien positives grâce au lemme de Csiszár–Kullback, qui implique bien la positivité du terme $\lambda\int_{\mathbb{T}^d} H(\rho|R)$ dans le cas où $\lambda > 0$. Le reste de la preuve consiste alors à montrer que l'entropie relative $\mathcal{E}_{NSLK}$ satisfait une inégalité de type Gronwall, stable par passage à la limite $\nu \to 0$, et on obtient alors le théorème suivant :

Théorème 6. *Voir [42, Theorem 1.4].*
Soit $T > 0$ et $(\rho_0, u_0, \overline{v}_0 = \frac{\hbar}{2}\nabla\log\rho_0)$ tel que $\mathcal{E}_{ELK}(\rho_0, u_0, \overline{v}_0) < \infty$, alors il existe une solution dissipative au système d'Euler-Langevin-Korteweg sur $[0, T[\, \times \mathbb{T}^d$ avec condition initiale $(\rho_0, u_0, \overline{v}_0)$, au sens où :

$$\mathcal{E}_{ELK}(\rho, u, \overline{v}|R, U, \overline{V})(t) \leq \mathcal{E}_{ELK}(\rho, u, \overline{v}|R, U, \overline{V})(0)e^{Ct} + b_{ELK}(t) + C\int_0^t b_{ELK}(s)e^{C(t-s)}ds$$

pour presque tout $t \in [0, T[$, avec

$$b_{ELK}(t) = \int_0^t \int_{\mathbb{T}^d} \frac{\rho}{R}|\mathscr{E}(R, U) \cdot (U - u)|$$

pour toute fonction suffisamment régulière U, et où $(R, \overline{V}, \mathscr{E})$ est défini via la condition initiale

$$\int_{\mathbb{T}^d} R_0 = \int_{\mathbb{T}^d} \rho_0,$$

et le système

$$\partial_t R + \operatorname{div}(RU) = 0, \tag{1.3.10a}$$

$$\overline{V} = \frac{\hbar}{2}\nabla \log R, \tag{1.3.10b}$$

$$\mathscr{E}(R,U) = R(\partial_t U + U\cdot\nabla U) + \lambda\nabla R + \mu RU - \frac{\hbar}{2}\operatorname{div}(R\nabla\overline{V}), \tag{1.3.10c}$$

et $C = C(\hbar, R, U, \overline{V})$ est une constante uniformément bornée sur $\mathbb{R}_+ \times \mathbb{T}^d$.

On peut faire plusieurs remarques à partir de ce théorème :

- contrairement au cas d'une pression barotropique traité dans [27], l'hypothèse supplémentaire $\int R_0 = \int \rho_0$ est nécessaire pour pouvoir appliquer le lemme de Csiszár–Kullback au terme $\lambda \int_{\mathbb{T}^d} H(\rho|R)$ de l'entropie relative $\mathcal{E}_{ELK}$.
- le système (1.3.10) représente le système de "solutions fortes" d'Euler-Langevin-Korteweg augmenté : comme nous n'avons pas de véritable notion de solution forte associée à ce système, le terme $\mathscr{E}(R,U)$ correspond à l'erreur introduite par les fonctions R et U pour avoir une solution forte (en d'autres termes, si $\mathscr{E}(R,U) = 0$, alors (R,U) constitue bien une solution forte de (1.3.9)). Cette erreur se transmet ensuite à l'inégalité de type Gronwall sur $\mathcal{E}_{ELK}$ via le terme d'intégration b_{ELK}.

1.3.4 Limite isotherme pour le système d'Euler compressible avec amortissement

Article de référence : **The isothermal limit for the compressible Euler equations with damping**, à paraître à *Discrete and Continuous Dynamical Systems Series B* [43].

Une remarque intéressante est que la propriété de propagation des solutions gaussiennes pour l'équation d'Euler-Langevin-Korteweg sur $\mathbb{R}^d$ ne nécessite pas la présence du potentiel Bohmien $\hbar^2\rho\nabla(\Delta\sqrt{\rho}/\sqrt{\rho})$. De plus, comme on a vu que le terme en $1/r^3$ dans l'équation (1.3.1) n'influençait pas la dynamique de l'équation, on peut s'attendre à obtenir le même type de résultat de convergence vers une gaussienne limite pour des solutions faibles du système d'Euler isotherme en dimension $d = 1$

$$\begin{cases} \partial_t\rho + \partial_x m = 0, & (1.3.11a)\\ \partial_t m + \partial_x\left(\dfrac{m^2}{\rho}\right) + \partial_x\rho + m = 0, & (1.3.11b)\end{cases}$$

(on adopte ici la convention de notation $m = \rho u$, plus commune en mécanique des fluides compressibles en dimension 1). En notant τ l'unique solution $\mathcal{C}^\infty([0,\infty))$ de l'équation différentielle

$$\ddot{\tau} = \frac{2}{\tau} - \dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0,$$

on sait par [41] que pour $t \to +\infty$,

$$\tau(t) \sim 2\sqrt{t} \quad \text{et} \quad \dot{\tau}(t) \sim \frac{1}{\sqrt{t}}.$$

En introduisant la notation de la gaussienne fondamentale $\Gamma := e^{-y^2}$, et par le changement de variable $y = x/\tau(t)$ que nous avons déjà rencontré pour l'équation de Schrödinger-Langevin, mais cette fois-ci dans le cadre de la mécanique des fluides,

$$\rho(t,x) = \frac{1}{\tau(t)} R\left(t, \frac{x}{\tau(t)}\right) \frac{\|\rho_0\|_{L^1}}{\|\Gamma\|_{L^1}} \quad \text{et} \quad m(t,x) = \frac{1}{\tau(t)^2} M\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot{\tau}(t)}{\tau(t)} yR\left(t, \frac{x}{\tau(t)}\right),$$

on obtient un système d'Euler isotherme amorti renormalisé sur le couple (R, M) qui s'écrit

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2}\partial_y M = 0, & (1.3.12a) \\ \partial_t M + \dfrac{1}{\tau^2}\partial_y \left(\dfrac{M^2}{R}\right) + \partial_y R + 2yR + M = 0. & (1.3.12b) \end{cases}$$

En supposant l'existence de solutions faibles d'énergie finie pour ce système, et en supposant également la condition de régularité supplémentaire sur le moment d'ordre 2 de la densité

$$(t,x) \mapsto x^2 \rho(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R})),$$

on montre d'une façon analogue à [41] le théorème suivant :

Théorème 7. *Voir [43, Proposition 2.6].*

$$R(t) \underset{t\to\infty}{\rightharpoonup} \Gamma \textit{ faiblement dans } L^1(\mathbb{R}).$$

Via l'étude d'équations différentielles non-linéaires (rendue plus accessibles grâce à l'absence de terme quantique en $\hbar$ et au cas scalaire de la dimension $d = 1$), on peut également montrer la convergence du moment d'ordre 1 de R vers celui de Γ, et sous une hypothèse de convergence de l'énergie cinétique

$$\mathcal{E}_{\mathrm{kin}}(t) := \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t,y)}{R(t,y)} dy \longrightarrow 0,$$

on montre de plus la convergence du moment d'ordre 2 de R, ce qui donne lieu au théorème suivant :

Théorème 8. *Voir [43, Proposition 2.5].*
Sous l'hypothèse précédente sur $\mathcal{E}_{\text{kin}}$*, on a*

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} R(t,y)dy \longrightarrow \int_{\mathbb{R}} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} \Gamma(y)dy \quad \textit{quand } t \to \infty.$$

L'hypothèse précédente sur la convergence de l'énergie cinétique du système renormalisé nécessite quelques commentaires. Le système (1.3.12) vérifie une inégalité de dissipation d'énergie qui s'écrit

$$\frac{1}{2\tau(t)^2}\int_{\mathbb{R}} \frac{M^2}{R}dy + \int_{\mathbb{R}} R\log\left(\frac{R}{\Gamma}\right)dy + \int_0^t \frac{\dot{\tau}(t)}{\tau^3(t)}\int_{\mathbb{R}} \frac{M^2}{R}dyds \leq C, \tag{1.3.13}$$

de laquelle on peut déduire grâce aux asymptotiques de τ et $\dot{\tau}$ les propriétés suivantes sur $\mathcal{E}_{\text{kin}}$:

$$\mathcal{E}_{\text{kin}} \in L^\infty(\mathbb{R}_+) \quad \text{et} \quad \int_0^\infty \frac{\mathcal{E}_{\text{kin}}(t)}{1+t}dt < \infty,$$

de sorte qu'il existe une suite $(t_n)_n$ telle que $t_n \to \infty$ et

$$\mathcal{E}_{\text{kin}}(t_n) \to 0 \quad \text{quand } n \to \infty.$$

Nous n'avons cependant aucune preuve que ce résultat soit vérifié uniformément en temps, ce qui impose l'hypothèse énoncée précédemment. À noter que dans le cas Gaussien, on observe bien une décroissance de l'énergie cinétique en $t^{-1/2}$ pour toute solution.

Ces résultats font écho à une vaste littérature reliée à l'étude du système d'Euler isentropique en dimension 1 d'espace :

$$\begin{cases} \partial_t \rho + \partial_x m = 0, & \text{(1.3.14a)} \\ \partial_t m + \partial_x \left(\dfrac{m^2}{\rho}\right) + \partial_x \rho^\gamma + m = 0, & \text{(1.3.14b)} \end{cases}$$

avec $\gamma > 1$. Il est maintenant bien connu que ce système admet des solutions faibles entropiques L^∞ (on renvoie à [106], [78] ou au Chapitre 5 pour une définition précise de ce type de solutions) dont le comportement asymptotique est gouverné par le profil diffusif suivant :

$$\begin{cases} \partial_t \overline{\rho} = \partial_x^2 \overline{\rho}^\gamma, & \text{(1.3.15a)} \\ \overline{m} = -\partial_x \overline{\rho}^\gamma, & \text{(1.3.15b)} \end{cases}$$

où l'équation (1.3.15a) désigne l'équation des milieux poreux, dont les solutions fondamentales sont appelées solutions de Barenblatt [16], et l'équation (1.3.15b) désigne la loi de Darcy. Plusieurs articles de F. Huang & al. ont cherché à trouver des taux de convergence de plus en plus précis en $t^{-\alpha}$ avec $\alpha > 0$ de ρ vers ces profils de Barenblatt limites $\overline{\rho}$, tout en réussissant à accroître la validité de cette convergence pour des intervalles de $\gamma > 1$ de plus en plus grands : on citera notamment l'article [79] pour l'obtention d'une convergence en norme $L^1(\mathbb{R})$ (qui constitue la bonne notion de norme puisque l'on a conservation de la masse pour la fonction ρ) pour $1 < \gamma < 3$, convergence améliorée récemment pour tout $\gamma > 1$ dans [67].

Notre travail consiste donc en un prolongement logique de ce cadre d'étude pour le cas d'une loi de pression isotherme $\gamma = 1$, d'autant plus que les fonctions de Barenblatt convergent uniformément vers des fonctions gaussiennes quand $\gamma \to 1$ (cf Proposition 5.10 et Figure 5.1 au Chapitre 5). Cependant, une question encore ouverte est celle de l'obtention d'un taux de convergence explicite de ces solutions faibles vers leur gaussienne limite : en effet, la méthode de F. Huang & al., basée sur le développement de Taylor par rapport à la variable ρ de la loi de pression $P(\rho) = \rho^\gamma$, est difficilement applicable dans le cas d'une pression isotherme. Une approche pertinente semble néanmoins être l'utilisation de l'inégalité de Csiszár-Kullback, qui peut s'énoncer ainsi dans le cadre de notre problème :

$$\|R - \Gamma\|_{L^1}^2 \leq 2\|\rho_0\|_{L^1} \int_{\mathbb{R}} R \log\left(\frac{R}{\Gamma}\right).$$

On constate donc que l'obtention d'un taux de décroissance explicite de l'entropie $\int R \log(R/\Gamma)$ fournirait un taux de convergence de R vers son profil gaussien limite. L'existence d'un tel taux de décroissance aurait également des conséquences fortes dans le cadre de l'étude de l'équation de Schrödinger logarithmique (1.2.12).

Chapitre 2

Dynamics of the Schrödinger-Langevin equation

Abstract: We consider the nonlinear Schrödinger-Langevin equation for both signs of the logarithmic nonlinearity. We explicitly compute the dynamics of Gaussian solutions for large times, which is obtained through the study of a particular nonlinear differential equation of order 2. We then give the asymptotic behavior of general energy weak solutions under some regularity assumptions. Some numerical simulations are performed in order to corroborate the theoretical results.

2.1 Introduction

We consider the following nonlinear Schrödinger equation:

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{1}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right), \tag{2.1.1}$$

with $\psi(0,x) = \psi_0(x)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, $\mu > 0$ and $\lambda \in \mathbb{R}^*$. This equation first appears in Nassar's paper [104] as a possible way to give a stochastic interpretation of quantum mechanics in the context of Bohmian mechanics. It had a recent renewed interest in the physics community, in particular in quantum mechanics in order to describe the continuous measurement of the position of a quantum particle (see for example [103], [118] or [102]) and in cosmology and statistical mechanics (see [45], [46] or [47]). Note that in its physical interpretation, $\lambda = 2k_BT/\hbar$ corresponds to a quantum friction coefficient, so both positive and negative signs could be of interest (k_B and $\hbar$ denotes respectively the Boltzmann and the normalized Planck constant, and T is an effective temperature), unlike the real friction coefficient μ which is taken positive (see [45]).

The nonlinear potential $\arg(\psi) = \frac{1}{2i}\log(\psi/\psi^*)$, which could be seen as the argument of the wave function ψ, is not defined when ψ is equal to zero. In order to circumvent this difficulty, we will rather consider the fluid formulation of equation (2.1.1). Plugging the Madelung transform $\psi = \sqrt{\rho}e^{iS}$, or in a more rigorous way the change of unknown functions $\rho = |\psi|^2$ and $J = \operatorname{Im}(\psi^*\nabla\psi)$, we get the following quantum hydrodynamical system

$$\begin{cases} \partial_t \rho + \mathrm{div} J = 0 & (2.1.2a) \\ \partial_t J + \mathrm{div}\left(\dfrac{J \otimes J}{\rho}\right) + \lambda \nabla \rho + \mu J = \dfrac{1}{2}\rho \nabla \left(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}}\right), & (2.1.2b) \end{cases}$$

where

$$\mathrm{div}\left(\frac{J \otimes J}{\rho}\right) = \left(\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\frac{J_i J_j}{\rho}\right)\right)_{1 \le i \le d}$$

denoting $J = (J_i)_{1 \le i \le d}$. We note that the last nonlinear term of equation (2.1.2b), usually called the Bohm potential, could also be expressed as

$$\frac{1}{2}\rho \nabla \left(\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\Delta \nabla \rho - \mathrm{div}(\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) \tag{2.1.3}$$

under some regularity assumptions, and its study is at the heart of the Bohmian dynamics theory (see [56], [64] and [115]).

Equations (2.1.2a)-(2.1.2b) stand as an isothermal fluid system with an additional dissipation term. In [33], the study of the isothermal Euler-Korteweg in the case $\lambda > 0$ and $\mu = 0$ reveals an asymptotic vanishing Gaussian profile for every rescaled global weak solution with well-prepared initial data. This property, rather unusual in the hydrodynamic setting, is a direct consequence of the link with the defocusing logarithmic Schrödinger equation [36], namely

$$i\partial_t \psi + \frac{1}{2}\Delta \psi = \lambda \psi \log(|\psi|^2). \tag{2.1.4}$$

On the contrary, in the focusing case (which is way more studied in the mathematical literature, see [38] or [5]), the existence and stability of periodic non-dispersive solutions, called solitons, is proved and well-understood. The existence of shape-moving periodic solutions, called *breathers*, is also known [61], but no stability results has yet been proved. Let us finally note that in the formal energy estimate of equation (2.1.1)

$$\int_{\mathbb{R}^d} |\nabla \psi|^2 + \lambda \int_{\mathbb{R}^d} |\psi|^2 \log |\psi|^2 + \mu \int_0^t \int_{\mathbb{R}^d} |\nabla \arg(\psi)|^2 |\psi|^2 \le C,$$

the potential energy of this equation

$$\lambda \int_{\mathbb{R}^d} |\psi|^2 \log |\psi|^2,$$

has no definite sign regardless of the sign of λ, so the focusing or defocusing behavior of its solutions may not be clear at first glance.

The question of the existence of solutions to this kind of quantum system is already dealt with in the case of barotropic pressure of the form $P(\rho) = \lambda\rho^\gamma$ in [4], where $\gamma > 1$. However, the proof is based on the link with the power-like Schrödinger equation

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi|\psi|^{\gamma-1} \tag{2.1.5}$$

and the use of Strichartz estimates which do not seem to be helpful for the logarithmic non-linearity. The global existence of weak solutions of equations (2.1.2a)-(2.1.2b) will be the aim of a forthcoming paper. We will concentrate here on the study of the large-time behavior of solutions.

This paper is organized as follows. In Section 2, we provide energy estimates, assumptions about existence and regularity of solutions of equations (2.1.2a)-(2.1.2b), and we state the main results of this paper. In Section 3, we explicitly compute the behavior of Gaussian solutions of (2.1.1) for both the focusing ($\lambda < 0$) and the defocusing ($\lambda > 0$) case, which will be crucial to the study of the universal dynamics of our solutions. In Section 4 we prove the theorems stated in Section 2. Section 5 is devoted to the simulation of numerical trajectories of solutions of (2.1.1), in order to illustrate our results and comfort our initial assumptions.

2.2 First properties and main results

Equation (2.1.2a) formally induces some mass conservation for all $t \geq 0$:

$$\|\rho(t)\|_{L^1(\mathbb{R}^d)} = \|\rho_0\|_{L^1(\mathbb{R}^d)} \tag{2.2.1}$$

Note that system (2.1.2a)-(2.1.2b) can also be written in terms of unknown function $u = J/\rho$, which is more convenient in order to express the following energy estimate, which holds for all $t \geq 0$:

$$E(t) + \mu\int_0^t D(s)ds \leq E_0, \tag{2.2.2}$$

where

$$E(t) = E(\rho, u) = E_c(t) + E_p(t), \tag{2.2.3}$$

$$E_c(t) = \frac{1}{2}\int_{\mathbb{R}^d} \rho|u|^2 dx, \tag{2.2.4}$$

$$E_p(t) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla\sqrt{\rho}|^2 + \lambda\rho\log\rho\right) dx, \tag{2.2.5}$$

$$D(s) = D(\rho, u) = 2E_c(s), \tag{2.2.6}$$

and $E_0 := E(0)$. We note that equation (2.1.1) has a conserved L^2-norm but a dissipated energy, which is a typical feature in the context of fluid dynamics but a quite unusual one in the framework of nonlinear Schrödinger equations. In particular, for the focusing case $\lambda < 0$, the dissipation term in estimate (2.2.2) implies that $E_c \in L^1(\mathbb{R})$. In fact, this is a direct consequence of the mass conservation (2.2.1) and the logarithmic Sobolev inequality

$$\int \rho \log \rho \leq \frac{\alpha^2}{\pi} \|\nabla \sqrt{\rho}\|_{L^2}^2 + \left(\log \|\rho\|_{L^1} - d\left(1 + \log \alpha\right)\right) \|\rho\|_{L^1} \tag{2.2.7}$$

which holds for any $\alpha > 0$ and for any function $\rho \in W^{1,1}(\mathbb{R}^d) \backslash \{0\}$ (for a proof of this inequality we refer to [94]). Note that $E_c \in L^1(\mathbb{R})$ still holds in the defocusing case $\lambda > 0$ under some regularity assumptions (see Remark 2.9 below).

A first consequence of this important property is that the center of mass of every solution converges, which could be seen through the formal integration by part of equation (2.1.2b) (using the alternative formulation of the Bohm potential (2.1.3)),

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \rho(t,x)u(t,x)dx\right) = \int_{\mathbb{R}^d} \partial_t(\rho(t,x)u(t,x))dx = -\mu \int_{\mathbb{R}^d} \rho(t,x)u(t,x)dx,$$

and of equation (2.1.2a),

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} x\rho(t,x)dx\right) = \int_{\mathbb{R}^d} x\partial_t\rho(t,x)dx = \int_{\mathbb{R}^d} \rho(t,x)u(t,x)dx = e^{-\mu t} \int_{\mathbb{R}^d} \rho_0(x)u_0(x)dx,$$

so finally

$$\int_{\mathbb{R}^d} x\rho(t,x)dx \underset{t\to+\infty}{\to} \int_{\mathbb{R}^d} x\rho_0(x)dx + \frac{1}{\mu}\int_{\mathbb{R}^d} \rho_0(x)u_0(x)dx. \tag{2.2.8}$$

As for the logarithmic Schrödinger equation, an important feature of (2.1.1) is that the evolution of initial Gaussian data remains Gaussian (see [20] and [36]). In order to shorten the calculations, we may consider centered Gaussian initial data (cf Remark 2.17 for the behavior of moving-center Gaussian). The following asymptotic result for Gaussian functions will be a crucial guide for the general case, and its proof will be the aim of Section 3.

Theorem 2.1. ***(Gaussian behavior).***
Let $\lambda \in \mathbb{R}^*$, and consider the initial data

$$\psi_0(t,x) = b_0 \exp\left(-\frac{1}{2}\sum_{j=1}^d a_{0j}x_j^2\right)$$

with b_0, $a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. Then the solution ψ to (2.1.1) *is given by*

$$\psi(t,x) = b_0 \prod_{j=1}^d \frac{1}{\sqrt{r_j(t)}} \exp\left(i\phi_j(t) - \alpha_{0j}\frac{x_j^2}{2r_j^2(t)} + i\frac{\dot{r}_j(t)}{r_j(t)}\frac{x_j^2}{2}\right)$$

for some real-valued functions ϕ_j, r_j depending only on time, such that, as $t \to \infty$,

- *if* $\lambda < 0$,

$$r_j(t) \to \sqrt{\frac{\alpha_{0j}}{-2\lambda}} \quad \textit{and} \quad \dot{r}_j(t) \to 0,$$

so the Gaussian solution ψ *tends to a universal Gaussian profile, namely*

$$|\psi(t,x)| \underset{t\to+\infty}{\longrightarrow} Ce^{\lambda|x|^2} \quad \textit{for all } x \in \mathbb{R}^d,$$

where $C = C(\lambda, \alpha_0, b_0, d) > 0$ *denotes a generic constant depending only on* λ, $\|\psi_0\|_{L^2(\mathbb{R}^d)}$ *and the dimension* d,

- *and if* $\lambda > 0$, *for large* t,

$$r(t) \sim 2\sqrt{\frac{\lambda\alpha_0}{\mu}t} \quad \textit{and} \quad \dot{r}(t) \sim \sqrt{\frac{\lambda\alpha_0}{\mu t}},$$

so that

$$\|\psi(t)\|_{L^\infty} = \hat{C}t^{-d/4} \quad \textit{and} \quad \|\nabla\psi(t)\|_{L^2} \sim \tilde{C}\frac{1}{\sqrt{t}}.$$

where $\hat{C} = \hat{C}(\alpha_0, b_0, \mu, \lambda, d)$ *and* $\tilde{C} = \tilde{C}(\alpha_0, b_0, \mu, \lambda, d) > 0$ *denote two generic constants depending only on* λ, μ, *the initial conditions and the dimension* d.

We now give the notion of weak solution we will use throughout this paper. Following [4], we express:

Definition 2.2. *We say that* (ρ, J) *is an* ***energy weak solution*** *of system* (2.1.2a)-(2.1.2b) *in* $[0,T[\times\mathbb{R}^d$ *with initial data* $(\rho_0, J_0) \in L^1(\mathbb{R}^d)\times L^1(\mathbb{R}^d)$, *if there exists locally integrable functions* $\sqrt{\rho}$, Λ *such that, by defining* $\rho := \sqrt{\rho}^2$ *and* $J = \sqrt{\rho}\Lambda$, *the following holds:*

- *The global regularity:*

$$\sqrt{\rho} \in L^\infty([0,T[\,;H^1(\mathbb{R}^d)), \quad \Lambda \in L^2([0,T[\,;L^2(\mathbb{R}^d)),$$

with the compatibility condition

$$\sqrt{\rho} \geq 0 \ \textit{a.e. on} \ (0,\infty)\times\mathbb{R}^d, \quad \Lambda = 0 \ \textit{a.e. on} \ \{\rho = 0\}\,.$$

- *For any test function* $\eta \in \mathcal{C}_0^\infty([0,T[\times\mathbb{R}^d)$,

$$\int_0^T\int_{\mathbb{R}^d}(\rho\partial_t\eta + J\cdot\nabla\eta)dxdt + \int_{\mathbb{R}^d}\rho_0\eta(0)dx = 0,$$

and for any test function $\zeta \in \mathcal{C}_0^\infty([0,T[\times\mathbb{R}^d;\mathbb{R}^d)$,

$$\int_0^T\int_{\mathbb{R}^d}\left(J\cdot\partial_t\zeta+\Lambda\otimes\Lambda:\nabla\zeta+\lambda\rho\mathrm{div}(\zeta)-\mu J\cdot\zeta+\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}:\nabla\zeta-\frac{1}{4}\rho\Delta\mathrm{div}\zeta\right)dxdt$$
$$+\int_{\mathbb{R}^d}J_0\zeta(0)dx=0.$$

- *(Generalized irrotationality condition) For almost every* $t\geq 0$,

$$\nabla\wedge J=2\nabla\sqrt{\rho}\wedge\Lambda$$

holds in the sense of distributions.

- *For almost every* $t\in[0,T)$, *equation* (2.2.2) *holds.*

In the focusing case, under some regularity and tightness assumption, we will show that every energy weak solution tends weakly in $L^1(\mathbb{R}^d)$ to a universal Gaussian profile of same mass. We denote

$$W(\mathbb{R}^d):=\left\{\psi\in H^1(\mathbb{R}^d)\;\middle|\;|\psi|^2\log|\psi|^2\in L^1(\mathbb{R}^d)\right\}.$$

Assumption 2.3. ***(Regularity in the focusing case).***
Let $\lambda<0$ *and* $(\rho_0,J_0)\in L^1(\mathbb{R}^d)\times L^1(\mathbb{R}^d)$. *We assume there exists an energy weak solution* (ρ,J) *of system* (2.1.2a)-(2.1.2b) *with initial data* (ρ_0,J_0) *in the sense of Definition 2.2 with the additional regularity:*

$$\sqrt{\rho}\in L^\infty([0,T[\,;H^2(\mathbb{R}^d)\cap W(\mathbb{R}^d)).$$

Remark 2.4. The fourth condition in Definition 2.2 insures the energy (2.2.3) to be bounded by above, however the term $\int\rho\log\rho$ has no definite sign, hence the condition $\sqrt{\rho}\in L^\infty([0,T[\,;W(\mathbb{R}^d))$ ensures that the energy is bounded from below. The regularity assumption on H^2 will be used in the following in order to perform the limit in equation (2.1.2b).

Assumption 2.5. ***(Tightness in the focusing case).***
If $\lambda<0$, *we assume that every energy weak solution* (ρ,J) *of system* (2.1.2a)-(2.1.2b) *in the sense of Definition 2.2 has a bounded moment of order 2, namely*

$$\sup_{t\geq 0}\int_{\mathbb{R}^d}|x|^2\rho(t,x)dx<\infty.$$

Theorem 2.6. ***(Focusing case).***
If $\lambda < 0$, under Assumption 2.3 and Assumption 2.5, and assuming $\rho_0 \neq 0$, we have

$$\rho(t,.) \underset{t\to+\infty}{\rightharpoonup} c_\lambda e^{\lambda|x-x_\infty|^2} \quad \textit{weakly in } L^1(\mathbb{R}^d),$$

where $c_\lambda := \|\rho_0\|_{L^1(\mathbb{R}^d)}(-\lambda/\pi)^{d/2}$ and x_∞ is determined by

$$x_\infty \|\rho_0\|_{L^1(\mathbb{R}^d)} = \lim_{t\to\infty} \int_{\mathbb{R}^d} x\rho(t,x)dx = \int_{\mathbb{R}^d} x\rho_0(x)dx + \frac{1}{\mu}\int_{\mathbb{R}^d} \rho_0(x)u_0(x)dx \tag{2.2.9}$$

using (2.2.8)*.*

Remark 2.7. Assumption 2.5 deserves a few remarks. First of all, this hypothesis seems to be true for every energy weak solution which satisfies it at $t = 0$, as it holds for every Gaussian solutions (see Remark 2.17), and seems to be verified through numerical simulations.

Despite probably not being optimal, this hypothesis is crucial in order to show the convergence of ρ to a universal Gaussian profile. In fact, energy estimate (2.2.2) prevents oscillations and finite-time blow-up of the density ρ, however the boundedness of the center of mass $\int x\rho$ still allows two symmetric Gaussian functions to symmetrically diverge with respect to the origin at infinity (although numerics seems to invalidate this kind of scenario).

Let us explain why the method we use in order to bound $\int x\rho$ fails when it comes to estimate $\int |x|^2\rho$. As we would like to exploit the crucial property that $E_c \in L^1(\mathbb{R})$, we try to calculate

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d} |x|^2\rho(t,x)dx\right) = \int_{\mathbb{R}^d} x\rho(t,x)u(t,x)dx := f(t).$$

Using equation (2.1.2b), we get

$$f' + \mu f = \|\nabla\sqrt{\rho}\|^2_{L^2(\mathbb{R}^d)} + \|\sqrt{\rho}u\|^2_{L^2(\mathbb{R}^d)} + \lambda d\|\rho_0\|_{L^1(\mathbb{R}^d)} = 2E(t) - 2\lambda\int_{\mathbb{R}^d}\rho\log\rho + \lambda d\|\rho_0\|_{L^1(\mathbb{R}^d)},$$

which implies that f is bounded using the logarithmic Sobolev inequality (2.2.7) and the energy estimate (2.2.2). Unfortunately, this is insufficient to show that f is integrable.

For the focusing case ($\lambda > 0$), we are going to see that we have an analogous result of what holds for the logarithmic Schrödinger equation. We denote

$$\mathcal{F}(H^1)(\mathbb{R}^d) := \left\{\psi \in L^2(\mathbb{R}^d) \;\middle|\; \langle x\rangle\psi(x) \in L^2(\mathbb{R}^d)\right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$.

Assumption 2.8. ***(Regularity in the defocusing case).***
Let $\lambda > 0$ and $(\rho_0, J_0) \in L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$. We assume there exists an energy weak solution

(ρ, J) *of system* (2.1.2a)-(2.1.2b) *with initial data* (ρ_0, J_0) *in the sense of Definition 2.2 with the additional regularity:*

$$\sqrt{\rho} \in L^\infty(\mathbb{R}; H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d)).$$

Remark 2.9. Note that the above condition ensures that the term $\int \rho \log \rho$ in the energy (2.2.3) is bounded in the defocusing case (cf [36] or below in the proof of Lemma 2.26).

Theorem 2.10. ***(Defocusing case).***
If $\lambda > 0$*, under Assumption 2.8, we denote by* $\tilde{\rho}$ *the scaled function defined by*

$$\rho(t,x) = \frac{1}{\tau(t)^{d/2}} \tilde{\rho}\left(t, \frac{x}{\tau(t)}\right),$$

where τ *denotes the unique* $\mathcal{C}^\infty([0,\infty))$ *solution of the differential equation*

$$\ddot{\tau} = \frac{2\lambda}{\tau} - \mu\dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0,$$

which satisfies, as $t \to +\infty$*,*

$$\tau(t) \sim 2\sqrt{\frac{\lambda t}{\mu}}.$$

Then

$$\tilde{\rho}(t,.) \underset{t \to +\infty}{\rightharpoonup} c_0 e^{-|x|^2} \quad \textit{weakly in } L^1(\mathbb{R}^d),$$

where $c_0 := \|\rho_0\|_{L^1(\mathbb{R}^d)} \pi^{-d/2}$.

Remark 2.11. *(Effect of scaling factors)*
We conclude this section with a last property that equation (2.1.1) shares with the logarithmic Schrödinger equation (2.1.4). Unlike the typical power-like nonlinear Schrödinger equation (2.1.5), replacing ψ with $\kappa\psi$ ($\kappa > 0$) in (2.1.1) has only little effect. In fact, if ψ is solution of (2.1.1) with initial datum $\psi(0,x) = \psi_0$, then the purely time-dependent gauge transform

$$\kappa\psi \exp\left(-i\frac{\lambda}{\mu}\log|\kappa|^2\left(1 - e^{-\mu t}\right)\right)$$

is also a solution of (2.1.1) with initial datum $\psi(0,x) = \kappa\psi_0$. In particular, the L^2-norm of the initial datum has no influence on the dynamics of the solution.

2.3 Gaussian solutions

2.3.1 Propagation of Gaussian data

The following calculations will be made in dimension $d = 1$ for reader's convenience, but they can all be adapted in any dimension d (cf Remark 2.24 at the end of this section).
We plug into (2.1.1) Gaussian functions of the form:

$$\psi(t,x) = b(t)\exp\left(-\frac{1}{2}a(t)x^2\right), \tag{2.3.1}$$

so we have the equation

$$i\dot{b} - \frac{1}{2}i\dot{a}bx^2 - \frac{1}{2}ab + \frac{1}{2}a^2bx^2 = \lambda b\log(|b|^2) - \lambda b\,\mathrm{Re}(a)x^2 - \frac{1}{2}i\mu b\log\left(\frac{b}{b^*}\right) - \frac{1}{2}\mu b\,\mathrm{Im}(a)x^2.$$

Equating the constant in x and the factors of x^2, we get the following non-linear differential equations system:

$$i\dot{b} - \frac{1}{2}ab = \lambda b\log(|b|^2) - \frac{1}{2}i\mu b\log\left(\frac{b}{b^*}\right), \tag{2.3.2}$$

$$i\dot{a} - a^2 = 2\lambda\,\mathrm{Re}(a) + \mu\,\mathrm{Im}(a). \tag{2.3.3}$$

We first look at equation (2.3.2) in order to express $|b|$ as a function of a. Multiplying (2.3.2) by b^*, we get

$$i\dot{b}b^* - \frac{1}{2}a|b|^2 = \lambda|b|^2\log(|b|^2) - \frac{1}{2}i\mu|b|^2\log\left(\frac{b}{b^*}\right).$$

As $i\log(b/b*) \in \mathbb{R}$, by taking the imaginary part, we obtain

$$\mathrm{Im}\left(i\dot{b}b^* - a\frac{|b|^2}{2}\right) = 0.$$

We recall that $\mathrm{Re}(\dot{b}b^*) = \frac{1}{2}\frac{d}{dt}(|b|^2)$, so we have the differential equation

$$\frac{d}{dt}(|b|^2) = |b|^2\,\mathrm{Im}(a),$$

so

$$|b(t)| = |b(0)|\exp\left(\frac{1}{2}\,\mathrm{Im}\,A(t)\right) \tag{2.3.4}$$

with $A(t) = \int_0^t a(s)ds$.

Remark 2.12. We can also express b as a function of a. We then look at b under the form

$$b(t) = r(t)e^{i\phi(t)},$$

where $r(t) = |b(t)|$. By plugging this expression into (2.3.2), we obtain the equation

$$i\dot{r} - \dot{\phi}r - \frac{1}{2}ar = \lambda r \log(r^2) + \mu\phi r.$$

As $\dot{r}/r = \frac{1}{2}\operatorname{Im}(a(t))$, we have

$$\dot{\phi} + \mu\phi = \frac{1}{2}i\operatorname{Im}(a) - \frac{1}{2}a - \lambda\log(r^2) = \frac{1}{2}i\operatorname{Im}(a) - \frac{1}{2}a - \lambda\log(|b_0|^2) - \lambda\operatorname{Im}(A).$$

Thanks to the variation of constant formula, we get ϕ as the solution of this ordinary differential equation:

$$\phi(t) = \phi_0 e^{-\mu t} + \int_0^t e^{-\mu(t-s)}\left(\frac{1}{2}i\operatorname{Im}(a(s)) - \frac{1}{2}a(s) - \lambda\log(|b_0|^2) - \lambda\operatorname{Im}(A(s))\right)ds,$$

and so we get b as a function of a.

We now want to solve equation (2.3.3) with initial conditions $a(0) = \alpha_0 + i\beta_0$, with $\alpha_0 > 0$. As suggested in [36], we denote

$$a = -i\frac{\dot{\omega}}{\omega},$$

so we have

$$i\dot{a} - a^2 = \frac{\ddot{\omega}\omega - \dot{\omega}^2}{\omega^2} + \left(\frac{\dot{\omega}}{\omega}\right)^2 = 2\lambda\operatorname{Re}\left(-i\frac{\dot{\omega}}{\omega}\right) + \mu\operatorname{Im}\left(-i\frac{\dot{\omega}}{\omega}\right),$$

that gives the equation

$$\ddot{\omega} = 2\lambda\omega\operatorname{Im}\left(\frac{\dot{\omega}}{\omega}\right) - \mu\omega\operatorname{Re}\left(\frac{\dot{\omega}}{\omega}\right).$$

Denoting $\omega = re^{i\theta}$, we calculate

$$\begin{aligned}
\dot{\omega} &= \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta},\\
\ddot{\omega} &= \ddot{r}e^{i\theta} + 2i\dot{r}\dot{\theta}e^{i\theta} - r\dot{\theta}^2e^{i\theta} + ir\ddot{\theta}e^{i\theta},\\
2\lambda\omega\operatorname{Im}\left(\frac{\dot{\omega}}{\omega}\right) &= 2\lambda re^{i\theta}\operatorname{Im}\left(\frac{\cancel{\dot{r}}e^{i\theta} + ir\dot{\theta}e^{i\theta}}{re^{i\theta}}\right) = 2\lambda r\dot{\theta}e^{i\theta},\\
\mu\omega\operatorname{Re}\left(\frac{\dot{\omega}}{\omega}\right) &= \mu re^{i\theta}\operatorname{Re}\left(\frac{\dot{r}e^{i\theta} + \cancel{ir\dot{\theta}}e^{i\theta}}{re^{i\theta}}\right) = \mu\dot{r}e^{i\theta}.
\end{aligned}$$

Then we get the equation

$$\ddot{r} + 2i\dot{r}\dot{\theta} - r\dot{\theta}^2 + ir\ddot{\theta} = 2\lambda r\dot{\theta} - \mu\dot{r}.$$

By taking the real and imaginary part, we get the real system of equations:

$$\ddot{r} - r\dot{\theta}^2 = 2\lambda r\dot{\theta} - \mu\dot{r} \tag{2.3.5}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{2.3.6}$$

We know that

$$\dot{\theta}(0) = \alpha_0 \quad \text{and} \quad \left(\frac{\dot{r}}{r}\right)(0) = -\beta_0.$$

We can fix $r(0) = 1$ so that $\dot{r}(0) = -\beta_0$. We note that (2.3.6) gives us the conservation of the quantity

$$\frac{d}{dt}(r^2\dot{\theta}) = r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0,$$

so $\dot{\theta} = \alpha_0/r^2$. By plugging $\dot{\theta}$ into (2.3.5), we finally obtain the following equation on r:

$$\boxed{\ddot{r} = \frac{\alpha_0^2}{r^3} + \frac{2\lambda\alpha_0}{r} - \mu\dot{r},} \tag{2.3.7}$$

with $r(0) = 1$ and $\dot{r}(0) = -\beta_0$. By multiplying (2.3.7) by $\dot{r}$ and integrating between 0 and t, we get:

$$\dot{r}(t)^2 = \dot{r}(0)^2 + \alpha_0^2\left(1 - \frac{1}{r(t)^2}\right) + 4\lambda\alpha_0\log(r(t)) - 2\mu\int_0^t \dot{r}(s)^2 ds. \tag{2.3.8}$$

Recalling the expression of a, we get

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot{r}(t)}{r(t)}, \tag{2.3.9}$$

in particular $\mathrm{Re}(a) \geq 0$. We note that we have an explicit expression of our Gaussian ψ as function of r. As we are facing a non-linear equation of order 2, we will now look at the asymptotic behavior of its solutions for both the defocusing and the focusing case. This will give the asymptotic dynamics of our Gaussian solutions.

2.3.2 Focusing case

We first look at the focusing case which corresponds to $\lambda < 0$ in (2.3.7). The Cauchy theory ensures the existence of a $\mathcal{C}^2$-solution (denoted by r in the following) of (2.3.7) as long as $r \neq 0$. We will show in this section that every Gaussian tends to a multiple of the Gaussian limit $e^{\lambda x^2}$.

Lemma 2.13. *There exist m, $M > 0$ such that for all $t \geq 0$,*

$$0 < m \leq r(t) \leq M.$$

Proof. Using equation (2.3.8), as

$$\dot{r}(t)^2 + \frac{\alpha_0^2}{r(t)^2} + 2\mu \int_0^t \dot{r}(s)^2 ds \geq 0,$$

we get that

$$\dot{r}(0)^2 + \alpha_0^2 + 4\lambda\alpha_0 \log(r(t)) \geq 0.$$

Then

$$\log(r(t)) \leq \frac{\dot{r}(0)^2 + \alpha_0^2}{-4\lambda\alpha_0}$$

as $\lambda < 0$, so $r(t) \leq M := \exp(-(\dot{r}(0)^2 + \alpha_0^2)/4\lambda\alpha_0)$. In order to get a lower bound, we remark that equation (2.3.8) also gives by triangle inequality

$$\frac{\alpha_0^2}{r(t)^2} \leq \dot{r}(0)^2 + \alpha_0^2 + 4\lambda\alpha_0 |\log(r(t))|,$$

so $r(t) \geq m := \alpha_0 / \sqrt{\dot{r}(0)^2 + \alpha_0^2 + 4\lambda\alpha_0 |\log(M)|}$.

□

Before going for the proof of Proposition 2.15, we recall a lemma from real analysis:

Lemma 2.14. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative uniformly continuous function, and assume that $\int_0^\infty f(x)dx < \infty$. Then $f(x) \to 0$ as $x \to +\infty$.*

Proposition 2.15. *We denote by r the $\mathcal{C}^2$-solution on $[0, +\infty[$ of equation (2.3.7) with $\lambda < 0$. Then, as $t \to +\infty$, we have*

$$r(t) \to \sqrt{\frac{\alpha_0}{-2\lambda}} \quad \textit{and} \quad \dot{r}(t) \to 0.$$

Proof. Still using equation (2.3.8), as we can remark that

$$\frac{\alpha_0^2}{r(t)^2} + 2\mu \int_0^t \dot{r}(s)^2 ds \geq 0,$$

we get that

$$\dot{r}(t)^2 \leq \dot{r}(0)^2 + \alpha_0^2 + 4\lambda\alpha_0 \log(r(t)),$$

and as we know that $m \leq r(t) \leq M$ for all $t \geq 0$, we also get that $\dot{r}$ is bounded, as well as $\ddot{r}$ using equation (2.3.7). So $\dot{r}\ddot{r}$ is also bounded on $\mathbb{R}_+$, hence $\dot{r}^2$ is Lipschitz continuous so uniformly continuous. We assume by contradiction that $\dot{r}(t) \not\to 0$ at $+\infty$, so according to Lemma 2.14 we have

$$\int_0^t \dot{r}(s)^2 ds \underset{t \to +\infty}{\longrightarrow} +\infty.$$

By the integral expression (2.3.8) and as r is bounded and $\mu > 0$ we get that $\dot{r}(t)^2 \to -\infty$ as $t \to \infty$, which is obviously impossible. So we get that

$$\dot{r}(t) = \dot{r}(0) + \int_0^t \ddot{r}(s)ds \underset{t\to+\infty}{\longrightarrow} 0,$$

so $\int_0^\infty \ddot{r}(s)ds$ has a finite limit (in particular, it is uniformly bounded). Multiplying equation (2.3.7) by $\ddot{r}$, integrating over $[0,t]$ and using the lower bound of r, we get that

$$\int_0^t \ddot{r}(s)^2 ds \leq \left(\frac{\alpha_0^2}{m^3} + \frac{2\lambda\alpha_0}{m}\right)\int_0^t \ddot{r}(s)ds - \frac{\mu}{2}(\dot{r}(t)^2 - \dot{r}(0)^2),$$

so $\int_0^\infty \ddot{r}(s)^2 ds < \infty$. Differentiating equation (2.3.7), we have

$$\dddot{r} = -\frac{3\alpha_0^2\dot{r}}{r^4} - \frac{2\lambda\alpha_0\dot{r}}{r^2} - \mu\ddot{r},$$

so $\dddot{r}$ is bounded on $\mathbb{R}_+$, and $\ddot{r}^2$ is Lipschitz so uniformly continuous. Then by applying Lemma 2.14 we have $\ddot{r}(t)^2 \to 0$ as $t \to \infty$, so $\ddot{r} \to 0$.
We take $(t_n)_n$ a sequence such that $t_n \to +\infty$ as $n \to \infty$. As r is bounded, there is a subsequence (not relabeled here for convenience) such that $r(t_n) \to L$. We already know that $\dot{r}(t_n) \to 0$ and $\ddot{r}(t_n) \to 0$, so by passing in the limit in equation (2.3.7), we get that

$$\frac{\alpha_0^2}{L^3} + \frac{2\lambda\alpha_0}{L} = 0,$$

and we deduce that $L = \sqrt{-\alpha_0/2\lambda}$. As the sequence $(r(t_n))_n$ has a unique point of accumulation, by sequential characterisation of continuity we get that

$$r(t) \to \sqrt{\frac{\alpha_0}{-2\lambda}}.$$

□

Hence

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot{r}(t)}{r(t)} \underset{t\to+\infty}{\longrightarrow} -2\lambda,$$

so for all $x \in \mathbb{R}$ we get that

$$|\psi(t,x)| \underset{t\to+\infty}{\longrightarrow} Ce^{\lambda x^2},$$

where $C = C(\lambda, \alpha_0, \beta_0) > 0$ is a constant depending only on the initial conditions and λ.

Remark 2.16. As we have $\|\psi(t)\|_{L^2(\mathbb{R})} = \|\psi_0\|_{L^2(\mathbb{R})}$ for all $t \in \mathbb{R}_+$, we can make explicit the constant $C(\lambda, \alpha_0, \beta_0) = \sqrt[4]{-2\lambda/\pi}\|\psi_0\|_{L^2(\mathbb{R})}$.

Remark 2.17. In order to check the tightness hypothesis in the Gaussian case, we can make the same calculations for Gaussian functions with moving centers as performed in [33], using the hydrodynamical equations of motion (2.1.2a)-(2.1.2b) this time, where ρ denotes the density and u the velocity of our Gaussian functions. Taking different initial centers leads to considering

$$\rho(0,x) = b_0 e^{-\alpha_0 x^2}, \quad u(0,x) = \beta_0 x + c_0,$$

for all $x \in \mathbb{R}$, with b_0, $\alpha_0 > 0$ and β_0, $c_0 \in \mathbb{R}$. We then look at solutions of the form

$$\rho(t,x) = b(t) e^{-\alpha(t)(x-\overline{x}(t))^2}, \quad u(t,x) = \beta(t)x + c(t), \tag{2.3.10}$$

which corresponds to

$$\psi(t,x) = \sqrt{b(t)} e^{-\frac{\alpha(t)}{2}(x-\overline{x}(t))^2} e^{i(\beta(t)x^2 + c(t)x + \gamma)}$$

with $\gamma \in \mathbb{R}$ in the Schrödinger formulation, and we plug this ansatz in equations (2.1.2a)-(2.1.2b), leading to the system of ordinary differential equations

$$\dot{\alpha} + 2\alpha\beta = 0, \quad \dot{\beta} + \beta^2 + \mu\beta = 2\lambda\alpha + \alpha^2, \tag{2.3.11}$$

$$\dot{\overline{x}} = \beta\overline{x} + c, \quad \dot{c} + \beta c + \mu c = -2\lambda\alpha\overline{x} - \alpha^2\overline{x}, \tag{2.3.12}$$

$$\dot{b} = b(\dot{\alpha}\overline{x}^2 + 2\alpha\overline{x}(\dot{\overline{x}} - c) - \beta). \tag{2.3.13}$$

In order to solve this system, mimicking [93] and [33], we can check that the two equations of (2.3.11) are satisfied if and only if α and β are of the form

$$\alpha(t) = \frac{\alpha_0}{r(t)^2}, \quad \beta(t) = \frac{\dot{r}(t)}{r(t)},$$

where we recall that r is the solution of equation (2.3.7) with $r(0) = 1$ and $\dot{r}(0) = \beta_0$. Plugging these expressions into (2.3.13), we also get that

$$b(t) = \frac{b_0}{r(t)}.$$

We know that ρ is a Gaussian function centered in $\overline{x}$, so we get that for all $t \geq 0$,

$$\int_{\mathbb{R}} (x - \overline{x}(t))\rho(t,x)dx = 0,$$

hence $\overline{x}(t) = \int_{\mathbb{R}} x\rho(t,x)dx/\|\rho_0\|_{L^1(\mathbb{R})}$, and we get from (2.2.8) that $\overline{x}(t)$ has a finite limit as $t \to +\infty$. In particular, we note that the dissipation implies that the center of our Gaussian solutions is bounded, contrary to the logarithmic case ($\mu = 0$). This stands as an evidence that the Galilean invariance principle is no longer verified due to the dissipation term ($\mu > 0$). In order to get the behavior of the second moment of ρ, we calculate:

$$\begin{aligned}
\int_{\mathbb{R}} x^2\rho(t,x)dx &= \int_{\mathbb{R}} (x-\overline{x}(t)+\overline{x}(t))^2\rho(t,x)dx \\
&= \int_{\mathbb{R}} (x-\overline{x}(t))^2\rho(t,x)dx + 2\overline{x}(t)\int_{\mathbb{R}} (x-\overline{x}(t))\rho(t,x)dx + \int_{\mathbb{R}} \overline{x}(t)^2\rho(t,x)dx \\
&= \frac{b(t)}{2\alpha(t)} + \overline{x}(t)^2\int_{\mathbb{R}} \rho_0(x)dx = \frac{b_0}{2\alpha_0}r(t) + \overline{x}(t)^2\|\rho_0\|_{L^1(\mathbb{R})}.
\end{aligned}$$

As we know that r and $\overline{x}$ converges thanks to the previous computations, we get that $\int_{\mathbb{R}} x^2\rho$ is uniformly bounded in time, which corroborates Assumption 2.5.

2.3.3 Defocusing case

We now look at the defocusing case by taking $\lambda > 0$ in (2.3.7). By calculating an equivalent of $r(t)$ as $t \to \infty$, we will see that every Gaussian vanishes to 0 in L^∞-norm in $t^{-\frac{1}{4}}$ (we recall that $d = 1$ for the present computations). We will also make explicit the decay of $\|\nabla\psi(t)\|_{L^2}$ thanks to an equivalent of $\dot{r}$.

Lemma 2.18. *There exists a constant $m > 0$ such that $\forall t \geq 0$, $r(t) \geq m$.*

Proof. Let $t \geq 0$, and we rewrite the expression (2.3.8) under the form:

$$\dot{r}(t)^2 + \frac{\alpha_0^2}{r(t)^2} + 2\mu\int_0^t \dot{r}(s)^2 ds = \dot{r}(0)^2 + \alpha_0^2 + 4\lambda\alpha_0\log(r(t)) \geq 0,$$

so

$$\log(r(t)) \geq -\frac{\dot{r}(0)^2 + \alpha_0^2}{4\lambda\alpha_0}.$$

Denoting $m = \exp\left(-\frac{\dot{r}(0)^2+\alpha_0^2}{4\lambda\alpha_0}\right)$, we immediately get that $\forall t \geq 0$, $r(t) \geq m$. □

Lemma 2.19. *There exists $T_0 \geq 0$ such that for all $t > T_0$, $\dot{r}(t) > 0$.*

Proof. We first show the existence of a time T_0 such that $\dot{r}(T_0) \geq 0$. If $\dot{r}(0) \geq 0$, the result is trivial. Otherwise, $\dot{r}(0) < 0$, and we denote

$$T_0 = \inf\{t \geq 0 \mid \dot{r}(t) \geq 0\} \leq +\infty.$$

So $\forall t \in [0, T_0]$, we have

$$\ddot{r}(t) = \frac{\alpha_0^2}{r^3(t)} + \frac{2\lambda\alpha_0}{r(t)} - \mu\dot{r}(t) \geq \frac{\alpha_0^2}{r^3(t)} + \frac{2\lambda\alpha_0}{r(t)}.$$

As $\dot{r} < 0$ on $[0, T_0[$, r is decreasing on this interval, so $\forall t \in [0, T_0[$,

$$\ddot{r}(t) \geq \frac{\alpha_0^2}{r^3(0)} + \frac{2\lambda\alpha_0}{r(0)} > 0.$$

By integration,

$$\dot{r}(t) \geq \dot{r}(0) + \left(\frac{\alpha_0^2}{r^3(0)} + \frac{2\lambda\alpha_0}{r(0)}\right) t.$$

By linear growth, we deduce that T_0 is finite, and that $\dot{r}(T_0) = 0$ by continuity. We remark that we have the bound

$$T_0 \leq -\frac{\dot{r}(0)}{\alpha_0^2 + 2\lambda\alpha_0},$$

still in the case where $\dot{r}(0) < 0$ (otherwise $T_0 = 0$).
We are going to show now that $\forall t > T_0$, $\dot{r}(t) > 0$. We denote

$$T = \inf\left\{t \geq T_0 \mid \dot{r}(t) \leq 0\right\} \leq +\infty.$$

If $T = +\infty$, we immediately get the result. Otherwise, by a continuity argument, $\dot{r}(T) = 0$ and

$$\ddot{r}(T) = \frac{\alpha_0^2}{r(T)^3} + \frac{2\lambda\alpha_0}{r(T)} > 0,$$

so for η small enough, $\ddot{r} > 0$ on $]T - \eta, T + \eta[$, so $\dot{r}$ is non-decreasing on this interval. As $\dot{r}(T) = 0$, there exists $\varepsilon > 0$ such that $\dot{r}(T - \varepsilon) < 0$, which contradicts the definition of T, hence we see that $\forall t > T_0$, $\dot{r}(t) > 0$. □

Lemma 2.20. *$r(t) \to +\infty$ as $t \to +\infty$.*

Proof. According to the previous lemma, we know that r is non-decreasing from a time T_0 on, so r is either diverging to $+\infty$, or converging. We assume by contradiction that there exists $\ell > 0$ (as $r \geq m > 0$) such that $r(t) \to \ell$ when $t \to +\infty$.
We assume, still by contradiction, that $\dot{r}(t) \nrightarrow 0$ at $+\infty$. As $\dot{r}^2$ is positive and uniformly continuous, we have

$$\int_0^t \dot{r}(s)^2 ds \to +\infty,$$

and as r converges by hypothesis, we get from (2.3.8) that $r(t)^2 \to -\infty$ when $t \to +\infty$, which is obviously impossible, so $\dot{r}(t) \to 0$.

Then we deduce that
$$\ddot{r}(t) \to \frac{\alpha_0^2}{\ell^3} + \frac{2\lambda\alpha_0}{\ell} > 0,$$
so $\exists T \geq 0$ such that $\forall t \geq T$, $\ddot{r}(t) > 0$, *ie* $\dot{r}$ is increasing on $[T, +\infty[$. However we saw that $\dot{r}(t) \to 0$, so $\forall t \geq T$, $\dot{r}(t) \leq 0$, which is in contradiction with the previous lemma. We finally conclude that $r(t) \to +\infty$. □

Lemma 2.21. *There exists $T_1 \geq 0$ such that for all $t \geq T_1$, $\ddot{r}(t) \leq 0$.*

Proof. Case 1: $\ddot{r}(T_0) \leq 0$.
Then we can take $T_1 = T_0$. We assume that there exists $T \geq T_1$ such that $\ddot{r}(T) = 0$, then

$$\dddot{r}(T) = -\frac{3\alpha_0^2\dot{r}(T)}{r(T)^4} - \frac{2\lambda\alpha_0\dot{r}(T)}{r(t)^2} - \mu\ddot{r}(T) = -\left(\frac{3\alpha_0^2}{r(T)^4} + \frac{2\lambda\alpha_0}{r(T)^2}\right)\dot{r}(T),$$

but $T_1 \geq T_0$ so $\dot{r}(T) > 0$, and so $\dddot{r}(T) < 0$. We deduce that there exists $\varepsilon > 0$ small enough such that $\forall t \in]T, T+\varepsilon[$, $\ddot{r}(t) < 0$. We have just shown that the set $\{t \geq T_1 \mid \ddot{r}(t) \leq 0\}$ is open, and it is clearly closed as $\ddot{r}$ is continuous, and non-empty because T_0 belongs to it, so we see that $\forall t \geq T_1$, $\ddot{r}(t) \leq 0$ as $[T_1, +\infty[$ is a connected space.

Case 2: $\ddot{r}(T_0) > 0$.
Rewriting this condition with equation (2.3.7), we have:

$$\mu\dot{r}(T_0) < \frac{\alpha_0^2}{r(T_0)^3} + \frac{2\lambda\alpha_0}{r(T_0)}.$$

As taking $T_0 = T_0 + \eta$ with η small enough, we can assume that $\dot{r}(T_0) > 0$ and that $\forall t \geq T_0$, $\dot{r}(t) > 0$. First of all, we show that $\dot{r}$ is bounded. We denote

$$t_{\min} = \inf\left\{ t \geq T_0 \;\middle|\; \mu\dot{r}(t) = \frac{\alpha_0^2}{r(T_0)^3} + \frac{2\lambda\alpha_0}{r(T_0)} \right\}.$$

We assume (by contradiction) that $t_{\min} < \infty$. We know that $\forall t \geq T_0$, $\dot{r}(t) > 0$, so r is increasing on $[T_0, +\infty[$. We deduce that

$$\mu\dot{r}(t_{\min}) = \frac{\alpha_0^2}{r(T_0)^3} + \frac{2\lambda\alpha_0}{r(T_0)} > \frac{\alpha_0^2}{r(t_{\min})^3} + \frac{2\lambda\alpha_0}{r(t_{\min})},$$

so that $\ddot{r}(t_{\min}) < 0$. However $t_{\min}$ is the smallest time t such that $\mu\dot{r}(t) = \alpha_0^2/r(T_0)^3 + 2\lambda\alpha_0/r(T_0)$, then for all $t < t_{\min}$, $\dot{r}(t) < \dot{r}(t_{\min})$, and

$$\frac{\dot{r}(t_{\min}) - \dot{r}(t)}{t_{\min} - t} > 0,$$

so we have $\ddot{r}(t_{\min}) \geq 0$ when $t \to t_{\min}$, hence the contradiction. We can conclude that $t_{\min} = +\infty$, so $\forall t \geq T_0$,

$$\dot{r}(t) \leq \frac{1}{\mu}\left(\frac{\alpha_0^2}{r(T_0)^3} + \frac{2\lambda\alpha_0}{r(T_0)}\right),$$

which means that $\dot{r}$ is bounded.
We now assume, still by contradiction, that $\forall t \geq T_0$, $\ddot{r}(t) > 0$, so $\dot{r}$ is increasing, and since $\dot{r}$ is bounded, it converges to a real $\ell > 0$ (because $\dot{r}(T_0) > 0$). As $r(t) \to +\infty$, by (2.3.7), we have

$$\ddot{r}(t) \underset{t\to+\infty}{\longrightarrow} -\mu\ell < 0,$$

leading to an absurdity. So there exists a time $T_1 \geq T_0$ such that $\ddot{r}(T_1) \leq 0$. Then we conclude like the Case 1.

□

Proposition 2.22. *There exists $C_{T_1,\lambda,\mu,\alpha_0} > 0$ such that for all $t \geq T_1$, As $t \to +\infty$, we have*

$$2\sqrt{\frac{\lambda\alpha_0}{\mu}t} \leq r(t) \leq C_{T_1,\lambda,\mu,\alpha_0} + 2\sqrt{\frac{\lambda\alpha_0}{\mu}t}.$$

Proof. We know from the previous lemma that $\forall t \geq T_1$, $\ddot{r}(t) \leq 0$, so

$$\frac{\alpha_0^2}{r^3(t)} + \frac{2\lambda\alpha_0}{r(t)} \leq \mu\dot{r}(t),$$

and $\alpha_0^2/r^3(t) \geq 0$, so $\mu\dot{r}(t) \geq \frac{2\lambda\alpha_0}{r(t)}$. We denote by g the solution of the differential equation

$$\mu\dot{g} = \frac{2\lambda\alpha_0}{g}, \tag{2.3.14}$$

with $g(0) = r(0)$, so that $\forall t \geq T_1$, $r(t) \geq g(t)$. Solving (2.3.14), we get the lower bound:

$$r(t) \geq g(t) = \sqrt{\frac{4\lambda\alpha_0}{\mu}t + g(0)^2} \geq 2\sqrt{\frac{\lambda\alpha_0}{\mu}t}. \tag{2.3.15}$$

We now need an upper bound for r. We rewrite the equation (2.3.7) under the form $\mu\dot{r} = \alpha_0^2/r^3 + 2\lambda\alpha_0/r - \ddot{r}$, then by integrating between the times T_1 and t, we have

$$\mu r(t) = \mu r(T_1) + \int_{T_1}^t \left(\frac{\alpha_0^2}{r^3(s)} + \frac{2\lambda\alpha_0}{r(s)}\right) ds - \dot{r}(t) + \dot{r}(T_1) \leq \mu r(T_1) + \dot{r}(T_1) + \int_{T_1}^t \left(\frac{\alpha_0^2}{r^3(s)} + \frac{2\lambda\alpha_0}{r(s)}\right) ds,$$

because $\dot{r}(t) \geq 0$. Using the previous lower bound r in order to bound $1/r$ by above, we get

$$r(t) \leq \frac{r(T_1) + \dot{r}(T_1)}{\mu} + \int_{T_1}^{t} \frac{1}{8\lambda}\sqrt{\frac{\alpha_0 \mu}{\lambda s^3}} ds + \int_{T_1}^{t} 2\sqrt{\frac{\lambda \alpha_0}{\mu s}} ds.$$

Recalling that the function $s \mapsto -\frac{3}{2\sqrt{s}}$ is an anti-derivative of $s \mapsto \frac{1}{\sqrt{s}^3}$, and that the function $s \mapsto 2\sqrt{s}$ is an anti-derivative of $s \mapsto 1/\sqrt{s}$, we can write that

$$r(t) \leq C_{T_1,\lambda,\mu,\alpha_0} + 2\sqrt{\frac{\lambda \alpha_0}{\mu}t}, \tag{2.3.16}$$

so by using (2.3.15) and (2.3.16), we get the result. □

From expressions (2.3.1), (2.3.9) and (2.3.4), we can calculate the L^∞ norm of our Gaussian solutions:

$$\|\psi(t)\|_{L^\infty} = |b(0)| \exp\left(\frac{1}{2}\int_0^t \operatorname{Im}(a(s))ds\right) = \frac{|b(0)|}{\sqrt{r(t)}} \sim \sqrt{2}|b(0)| \left(\frac{\mu}{\lambda \alpha_0 t}\right)^{\frac{1}{4}}.$$

In order to express the H^1 norm of ψ, by the following computations,

$$\|\nabla \psi(t)\|_{L^2}^2 = \frac{1}{2}\sqrt{\pi}\frac{|b(t)|^2}{\sqrt{\operatorname{Re}(a(t))}}\frac{|a(t)|^2}{\operatorname{Re}(a(t))} = \frac{1}{2}\sqrt{\pi}\frac{|b(0)|^2}{\alpha_0\sqrt{\alpha_0}}\left(\frac{\alpha_0^2}{r(t)^2} + \dot{r}(t)^2\right),$$

we see that we also need to have an equivalent of $\dot{r}$, which is the aim of the following proposition:

Proposition 2.23. *As $t \to +\infty$, we have*

$$\dot{r}(t) \sim \sqrt{\frac{\lambda \alpha_0}{\mu t}}.$$

Proof. We already know from the previous proposition that there exists a generic constant $C > 0$ such that

$$r(t) \leq C + 2\sqrt{\frac{\lambda \alpha_0}{\mu}t},$$

and also that $\ddot{r}(t) \leq 0$ for all $t \geq T_1$, so using equation (2.3.7) we get that

$$\mu \dot{r}(t) \geq \frac{\alpha_0^2}{\left(C + 2\sqrt{\frac{\lambda \alpha_0 t}{\mu}}\right)^3} + \frac{2\lambda \alpha_0}{C + 2\sqrt{\frac{\lambda \alpha_0 t}{\mu}}} \geq \frac{\lambda \alpha_0}{\sqrt{\frac{\lambda \alpha_0 t}{\mu}}},$$

hence

$$\dot{r}(t) \geq \sqrt{\frac{\lambda \alpha_0}{\mu t}}.$$

In order to get an upper bound, we recall that

$$r(t) \geq 2\sqrt{\frac{\lambda\alpha_0 t}{\mu}}.$$

Then, still using equation (2.3.7), we get that

$$\ddot{r} + \mu\dot{r} = \frac{\alpha_0^2}{r^3} + \frac{2\lambda\alpha_0}{r} \leq \frac{\sqrt{\alpha_0\mu}}{8\sqrt{\lambda t}^3} + \sqrt{\frac{\lambda\alpha_0\mu}{t}} =: g(t).$$

We can rewrite the previous inequality as

$$\frac{d}{dt}\left(\dot{r}(t)e^{\mu t}\right) \leq g(t)e^{\mu t},$$

so we get

$$\dot{r}(t) \leq \dot{r}(0)e^{-\mu t} + e^{-\mu t}\int_0^t g(s)e^{\mu s}ds.$$

Finally, thanks to the asymptotic expansion

$$\int_0^t \frac{1}{\sqrt{s}}e^{\mu s}ds = \frac{1}{\mu}e^{\mu t}\left(\frac{1}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)\right),$$

we get the estimate

$$\dot{r}(t) \leq \sqrt{\frac{\lambda\alpha_0}{\mu t}} + o\left(\frac{1}{\sqrt{t}}\right).$$

□

Thanks to the previous proposition, we finally get

$$\|\nabla\psi(t)\|_{L^2} \sim C_{\alpha_0,\mu,\lambda}|b(0)|\frac{1}{\sqrt{t}}.$$

Remark 2.24. All the results above can easily be generalized to any dimension d. In fact, if we now consider Gaussian functions of the form

$$\psi(t,x) = b(t)\exp\left(-\frac{1}{2}\sum_{j=1}^{d} a_j(t)x_j^2\right), \tag{2.3.17}$$

then plugging it into (2.1.1) gives the system

$$i\dot{b} - \frac{1}{2}b\sum_{j=1}^{d} a_j = \lambda b\log(|b|^2) - \frac{1}{2}i\mu b\log\left(\frac{b}{b^*}\right), \tag{2.3.18}$$

$$i\dot{a}_j - a_j^2 = 2\lambda\,\mathrm{Re}(a_j) + \mu\,\mathrm{Im}(a_j), \quad j = 1, ..., d \tag{2.3.19}$$

which is the d-dimensional analogue of system (2.3.2) − (2.3.3). Using equation (2.3.18) we get the explicit formulation of b with respect to $(a_j)_{1\leq j\leq d}$:

$$|b(t)| = |b(0)| \exp\left(\frac{1}{2}\sum_{j=1}^{d} \operatorname{Im} A_j(t)\right) \tag{2.3.20}$$

where

$$A_j(t) := \int_0^t a_j(s)ds.$$

As the d equations (2.3.19) are decoupled as j varies, we are now back to the study of equation (2.3.7) on each independent dimension component.

2.4 Long-time behavior

2.4.1 Focusing case

Before going for the proof of Theorem 2.6, a first look at the equation, as suggested in [45], would be the study of oscillating solutions of equation (2.1.1) of the form

$$\psi(t,x) = f(x)e^{iS(t)}. \tag{2.4.1}$$

It is already well known since 1976 [20] (see also [37]) that standing waves of the form $f(x)e^{i\omega t}$ are the stationary solutions for the logarithmic Schrödinger equation, where $\omega \in \mathbb{R}$ and $f \in W(\mathbb{R}^d)$ stands as a solution of the semilinear elliptic equation

$$-\frac{1}{2}\Delta f + \omega f + \lambda f \log|f|^2 = 0. \tag{2.4.2}$$

In our case, dealing with the dissipation term $\log(\psi/\psi^*)$, standing waves are no longer relevant, and stationary solutions are rather of the form (2.4.1) with

$$S_\omega(t) = \frac{\omega}{\mu} + e^{-\mu t} \tag{2.4.3}$$

and f still a solution of (2.4.2) belonging to $W(\mathbb{R}^d)$. Moreover, it is already known (see [20]) in the case $\lambda < 0$ that the Gausson

$$\phi_\omega(x) := e^{\frac{\omega}{2\lambda}+d}e^{\lambda|x|^2} \tag{2.4.4}$$

solves equation (2.4.2) for any dimension d, and up to translations, it is the unique strictly positive $\mathcal{C}^2$-solution for (2.4.2) such that $f(x) \to 0$ as $|x| \to \infty$ (see [5]).
Ardila recently showed in [5] the orbital stability of the Gausson by minimizing some energy

functionals. As suggested by the study of the Gaussian case, we will show here that every smooth solution of our system (2.1.2a)-(2.1.2b) tends to the Gausson at large times, using both energy minimization and dynamical system theory, as well as the additional regularity and tightness assumptions made in Section 2.

Proof of Theorem 2.6. We recall that $(\rho, \rho u)$ is an energy weak solution of system (2.1.2a)-(2.1.2b), and we also assume that $\sqrt{\rho} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}^d))$. We remark that the dissipation term in (2.2.2) being a multiple of the kinetic energy, we have the following differential equation:

$$\dot{E}_c(t) + \dot{E}_p(t) = -2\mu E_c(t) \leq 0. \tag{2.4.5}$$

We are now going to invoke the LaSalle invariance principle [90], that we briefly recalls as follows: for an autonomous dynamical system

$$\frac{dX}{dt}(t) = F(X(t)) \tag{2.4.6}$$

with $X \in \mathbb{R}$ and $F \in \mathcal{C}^1(\mathbb{R})$, we assume there exists a function $V \in \mathcal{C}^1(\mathbb{R})$ such that

$$\frac{d}{dt}V(X) \leq 0$$

for all $X \in \mathbb{R}$. Then, defining

$$\Omega := \left\{ X \in \mathbb{R} \;\middle|\; \frac{d}{dt}V(X) = 0 \right\},$$

we get that every bounded solution of (2.4.6) converges as $t \to \infty$ to an invariant subspace of Ω under the dynamic of Ω, called ω-limit set. In our case, by LaSalle's invariance principle, the ω-limit set of any solution of (2.2.2) is thus reduced to the set

$$\Omega = \left\{ (\rho, u) \;\middle|\; \dot{E}_c(\rho, u) + \dot{E}_p(\rho, u) = 0 \right\} = \{ (\rho, u) \mid E_c(\rho, u) = 0 \}$$

by equation (2.4.5).
Denote (ρ, u) a solution to (2.1.2a)-(2.1.2b). Taking a sequence $t_n \to +\infty$, we denote

$$\rho_n(t,x) = \rho(t + t_n, x) \quad \text{and} \quad u_n(t,x) = u(t + t_n, x)$$

for all $t \in (0,1)$ and $x \in \mathbb{R}^d$. From (2.2.1) we get that $(\rho_n)_n$ is bounded in $L^1(\mathbb{R}^d)$, and from (2.2.2) we get that for all $t \geq 0$,

$$\sup_n \int_{\mathbb{R}^d} \rho_n |\log(\rho_n)| < \infty.$$

In fact, writing

$$\int_{\mathbb{R}^d} \rho_n |\log(\rho_n)| = \int_{\rho_n>1} \rho_n |\log(\rho_n)| - \int_{\rho_n<1} \rho_n |\log(\rho_n)|,$$

we get that

$$\int_{\rho_n>1} \rho_n |\log(\rho_n)| \leq \int_{\mathbb{R}^d} \sqrt{\rho_n}^{2+\varepsilon} \lesssim \|\sqrt{\rho_n}\|_{L^2(\mathbb{R}^d)}^{(2+\varepsilon)(1-\alpha)} \|\nabla\sqrt{\rho_n}\|_{L^2(\mathbb{R}^d)}^{(2+\varepsilon)\alpha} \lesssim \|\nabla\sqrt{\rho_n}\|_{L^2(\mathbb{R}^d)}^{\frac{\varepsilon}{2}}$$

by Gagliardo-Nirenberg inequality with $\alpha = 1/2 - 1/(2+\varepsilon) > 0$, for small $\epsilon > 0$, so $\int \rho_n |\log(\rho_n)|$ is uniformly bounded using the energy estimate (2.2.2). Then, by de la Vallée-Poussin theorem (see Lemma 5 or [53]), $(\rho_n)_n$ is equi-integrable, and so it converges weakly (up to a subsequence) by Dunford-Pettis theorem (see e.g.[54]):

$$\rho_n(t,\cdot) \underset{n\to+\infty}{\rightharpoonup} \rho_\infty(t,\cdot) \quad \text{weakly in } L^1(\mathbb{R}^d),$$

for all $t \in (0,1)$. Furthermore, using Assumption 2.5 we get that the sequence $(\rho_n)_n$ is tight in $L^1(\mathbb{R}^d)$, so

$$\|\rho_\infty\|_{L^1(\mathbb{R}^d)} = \|\rho_0\|_{L^1(\mathbb{R}^d)}. \tag{2.4.7}$$

From LaSalle's invariance principle we get that $E_c(\rho_n, u_n) \to 0$, *ie*

$$\|\sqrt{\rho_n} u_n\|_{L^2(\mathbb{R}^d)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Multiplying equation (2.1.2a) by a test function $\varphi \in \mathcal{D}((0,1)\times\mathbb{R}^d)$ and integrating over space and time, we have

$$\int_0^1 \int_{\mathbb{R}^d} (\rho_n \partial_t \varphi + \rho_n u_n \cdot \nabla\varphi)\, dxdt = 0.$$

By dominated convergence using equation (2.2.1) and from weak convergence of $(\rho_n)_n$ we get that

$$\int_0^1 \int_{\mathbb{R}^d} \rho_n \partial_t \varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_0^1 \int_{\mathbb{R}^d} \rho_\infty \partial_t \varphi.$$

Furthermore, writing $\rho_n u_n = \sqrt{\rho_n}\sqrt{\rho_n} u_n$, we get, still by dominated convergence using equation (2.2.3), that

$$\left| \int_0^1 \int_{\mathbb{R}^d} \sqrt{\rho_n}\sqrt{\rho_n} u_n \cdot \nabla\varphi dxdt \right| \leq C_\varphi \|\sqrt{\rho_0}\|_{L^2(\mathbb{R}^d)} \int_0^1 \|\sqrt{\rho_n} u_n\|_{L^2(\mathbb{R}^d)} dt \underset{n\to+\infty}{\longrightarrow} 0.$$

So we have that in the distribution sense,

$$\partial_t \rho_\infty = 0. \tag{2.4.8}$$

Now multiplying equation (2.1.2b) by a test function $\varphi \in \mathcal{D}((0,1)\times\mathbb{R}^d)$ and integrating over space and time, we get by integrating by parts:

$$\int_0^1 \int_{\mathbb{R}^d} (\rho_n u_n \partial_t \varphi + \rho_n u_n \otimes u_n \nabla\varphi + \lambda\rho_n \mathrm{div}\varphi - \mu\rho_n u_n \varphi)\, dxdt$$
$$= \int_0^1 \int_{\mathbb{R}^d} \left(\frac{1}{4}\rho_n \mathrm{div}\Delta\varphi - \nabla\sqrt{\rho_n} \otimes \nabla\sqrt{\rho_n}\nabla\varphi\right) dxdt.$$

Denoting $\rho_n u_n \otimes u_n = \sqrt{\rho_n} u_n \otimes \sqrt{\rho_n} u_n$, by the same arguments as above, we get that

$$\int_0^1 \int_{\mathbb{R}^d} (\rho_n u_n \partial_t \varphi + \sqrt{\rho_n} u_n \otimes \sqrt{\rho_n} u_n \nabla\varphi + \lambda\rho_n \mathrm{div}\varphi - \mu\rho_n u_n \varphi)\, dxdt \underset{n\to+\infty}{\longrightarrow} \int_0^1 \int_{\mathbb{R}^d} \lambda\rho_\infty \mathrm{div}\varphi.$$

From weak convergence of $(\rho_n)_n$ we get that

$$\int_0^1 \int_{\mathbb{R}^d} \rho_n \mathrm{div}\Delta\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_0^1 \int_{\mathbb{R}^d} \rho_\infty \mathrm{div}\Delta\varphi dxdt,$$

so the only remaining term is

$$\int_0^1 \int_{\mathbb{R}^d} \nabla\sqrt{\rho_n} \otimes \nabla\sqrt{\rho_n}\nabla\varphi dxdt \underset{n\to+\infty}{\longrightarrow} \int_0^1 \int_{\mathbb{R}^d} \nabla\sqrt{\rho_\infty} \otimes \nabla\sqrt{\rho_\infty}\nabla\varphi dxdt.$$

In fact, we recall that from Assumption 2.3 we get that $\nabla\sqrt{\rho_n}$ is uniformly bounded in L^2, and so

$$\nabla\sqrt{\rho_n} \underset{n\to+\infty}{\rightharpoonup} \nabla\sqrt{\rho_\infty} \quad \text{weakly in } L^2(\mathbb{R}^d).$$

As $f \mapsto \int_0^1 \int_{\mathbb{R}^d} f \otimes \nabla\sqrt{\rho_\infty}\nabla\phi$ is a linear form on $L^2(\mathbb{R}^d)$, we have

$$\left|\int_0^1 \int_{\mathbb{R}^d} (\nabla\sqrt{\rho_n} - \nabla\sqrt{\rho_\infty}) \otimes \nabla\sqrt{\rho_\infty}\nabla\varphi dxdt\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

Then, using (2.2.3) and the additional assumption that $\sqrt{\rho_n} \in L^\infty(\mathbb{R}_+, H^2(\mathbb{R}^d))$, we get that

$$\left|\int_0^1 \int_{\mathbb{R}^d} (\nabla\sqrt{\rho_n} - \nabla\sqrt{\rho_\infty}) \otimes \nabla\sqrt{\rho_\infty}\nabla\varphi dxdt\right|$$
$$\leq \|\nabla\sqrt{\rho_\infty}\|_{L^2(\mathbb{R}^d)} \int_0^1 \int_{\mathbb{R}^d} |\nabla\sqrt{\rho_n} - \nabla\sqrt{\rho_\infty}|^2 \, |\nabla\varphi|^2 dxdt \underset{n\to+\infty}{\longrightarrow} 0,$$

since $H^2(\mathbb{R}^d)$ is compactly embedded into $H^1_{\mathrm{loc}}(\mathbb{R}^d)$, and recalling that $|\nabla\varphi|^2$ has a compact support, so finally

$$\left|\int_0^1 \int_{\mathbb{R}^d} \nabla\sqrt{\rho_n} \otimes \nabla\sqrt{\rho_n}\nabla\varphi dxdt - \int_0^1 \int_{\mathbb{R}^d} \nabla\sqrt{\rho_\infty} \otimes \nabla\sqrt{\rho_\infty}\nabla\varphi dxdt\right| \underset{n\to+\infty}{\longrightarrow} 0,$$

so we have the result. Under Assumption 2.5, we now get that ρ_n converges weakly in L^1 to ρ_∞

which is solution of equation (2.4.8) and

$$\lambda \nabla \rho_\infty = \frac{1}{2}\rho_\infty \nabla \left(\frac{\Delta\sqrt{\rho_\infty}}{\sqrt{\rho_\infty}} \right). \tag{2.4.9}$$

As mentioned in [82], and as $\|\rho_\infty\|_{L^1(\mathbb{R}^d)} > 0$ in view of (2.4.7), there exists a domain $\Omega \subset \mathbb{R}^d$ on which $\rho_\infty > 0$, so we can divide equation (2.4.9) by $\rho_\infty > 0$. Then, integrating over Ω and multiplying by $f := \sqrt{\rho_\infty}$, equation (2.4.9) is linked to the already mentioned second-order nonlinear elliptic equation:

$$-\frac{1}{2}\Delta f + \omega f + \lambda f \log |f|^2 = 0,$$

where ω denotes the integration constant. From [49] we know that every solution f of (2.4.2) such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\Delta f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ in the sense of distribution is either trivial or strictly positive.

In our case, as $f = \sqrt{\rho_\infty} \in L^2(\mathbb{R}^d) \subset L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\Delta\sqrt{\rho_\infty} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ from the previous calculations, we get that $\Omega = \mathbb{R}^d$ which means that $\rho_\infty > 0$ on $\mathbb{R}^d$, so by standard elliptic regularity arguments, f is $\mathcal{C}^2(\mathbb{R}^d)$, and from [49] we infer that, up to translations, f is equal to the Gausson (2.4.4). Hence for all $(t_n)_n$, the sequence $(\rho_n)_n$ has the same limit, so we get that

$$\rho(t,.) \underset{t\to+\infty}{\rightharpoonup} \rho_\infty \quad \text{weakly in } L^1(\mathbb{R}^d), \tag{2.4.10}$$

with $\rho_\infty = c_\lambda e^{\lambda|x|^2}$ up to translations. Furthermore, from the conservation of mass (2.4.7) we get that $c_\lambda = \|\rho_0\|_{L^1(\mathbb{R}^d)}(-\lambda/\pi)^{d/2}$. We now denote by x_∞ the (unique) center of the Gaussian function ρ_∞, and for all $R > 0$, we call $\xi_R \in \mathcal{C}^\infty(\mathbb{R}^d)$ the following smooth function such that $\xi_R = 1$ on $\{|x| \leq R\}$ and $\xi_R = 0$ on $\{|x| \geq 2R\}$. In order to determine x_∞, we use the weak convergence (2.4.10) to infer that, $\forall R > 0$,

$$\int_{\mathbb{R}^d} x\xi_R(x)\rho(t,x)dx \underset{t\to+\infty}{\longrightarrow} \int_{\mathbb{R}^d} x\xi_R(x)\rho_\infty(x)dx.$$

As ρ_∞ is a Gaussian function, we get that

$$\int_{\mathbb{R}^d} x\xi_R(x)\rho_\infty(x)dx = c_\lambda \int_{\mathbb{R}^d} x\xi_R(x)e^{\lambda|x-x_\infty|^2}dx \underset{R\to+\infty}{\longrightarrow} x_\infty\|\rho_0\|_{L^1(\mathbb{R}^d)}.$$

What's more, as we know that $\int_{\mathbb{R}^d}|x|^2\rho(t,x)dx < \infty$ for all $t \geq 0$ by Assumption 2.5, we get by dominated convergence theorem that

$$\int_{\mathbb{R}^d} x\rho(t,x)dx \underset{t\to+\infty}{\longrightarrow} x_\infty\|\rho_0\|_{L^1(\mathbb{R}^d)},$$

which concludes the proof.

2.4.2 Defocusing case

We are now going to prove Theorem 2.10. Following [36], we define ϕ according to the formal solution ψ of (2.1.1):

$$\psi(t,x) = \frac{1}{\tau(t)^{d/2}}\phi\left(t, \frac{x}{\tau(t)}\right)\frac{\|\psi_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\exp\left(i\frac{\dot{\tau}(t)}{\tau(t)}\frac{|x|^2}{2}\right), \tag{2.4.11}$$

where $\gamma := e^{-|x|^2/2}$ denotes the usual centered Gaussian function and τ is now the unique solution of the differential equation

$$\ddot{\tau} = \frac{2\lambda}{\tau} - \mu\dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.$$

Mimicking the proof of Proposition 2.22 and Proposition 2.23 (with $\alpha_0 = 1$), we also get that

$$\tau(t) \sim 2\sqrt{\frac{\lambda}{\mu}t} \quad \text{and} \quad \dot{\tau}(t) \sim \sqrt{\frac{\lambda}{\mu t}}.$$

In fact, the previous study of equation (2.3.7) in the defocusing case shows that the α_0^2/r^3 has no influence in the long-time behavior of the solution r, so we can get rid of it in the forthcoming calculations. Note that this last remark is no longer true in the focusing case, where the α_0^2/r^3 term plays a crucial role in the asymptotic behavior of the solution.

Plugging (2.4.11) into (2.1.1), we get that ϕ satisfy the equation

$$i\partial_t\phi + \frac{1}{2\tau^2}\Delta\phi = \lambda\phi\log\left(\left|\frac{\phi}{\gamma}\right|^2\right) + \frac{1}{2i}\mu\phi\log\left(\frac{\phi}{\phi^*}\right) - d\lambda\phi\log\tau + 2\lambda\phi\log\left(\frac{\|\psi_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\right),$$

where the initial datum for ϕ is

$$\phi(0) = \phi_0 := \frac{\|\gamma\|_{L^2(\mathbb{R}^d)}}{\|\psi_0\|_{L^2(\mathbb{R}^d)}}\psi_0. \tag{2.4.12}$$

Using the gauge transform consisting in replacing ϕ with $\phi e^{-i\theta(t)}$, where θ is the unique solution of the linear differential equation

$$\dot{\theta} + \mu\theta = -d\lambda\log\tau - 2\lambda\log\phi_0,$$

we may assume that the last two terms are absent, and we look into the following scaled equation:

$$i\partial_t\phi + \frac{1}{2\tau^2}\Delta\phi = \lambda\phi\log\left(\left|\frac{\phi}{\gamma}\right|^2\right) + \frac{1}{2i}\mu\phi\log\left(\frac{\phi}{\phi^*}\right). \tag{2.4.13}$$

Of course all the above formal calculations rigorously stand in the fluid case, so the hydrodynamical formulation of (2.4.13) is the following:

$$\begin{cases} \partial_t \varrho + \dfrac{1}{\tau^2}\mathrm{div}\mathcal{J} = 0, & (2.4.14a) \\ \partial_t \mathcal{J} + \dfrac{1}{\tau^2}\mathrm{div}\left(\dfrac{\mathcal{J}\otimes\mathcal{J}}{\varrho}\right) + \lambda\nabla\varrho + 2\lambda y\varrho + \mu\mathcal{J} = \dfrac{1}{2\tau^2}\varrho\nabla\left(\dfrac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right), & (2.4.14b) \end{cases}$$

where $y = x/\tau$. The couple $(\varrho, \mathcal{J})$ induced by Assumption 2.8 and the above scaling will be called energy weak solution of (2.4.14a)-(2.4.14b), according to Definition 2.2.

Let us now explain the heuristic behind the calculations of the forthcoming proof about the long time behavior of our solutions. Letting $\tau(t) \to \infty$, equation (2.4.14b) simplifies into

$$\partial_t \mathcal{J} + \lambda\nabla\varrho + 2\lambda y\varrho + \mu\mathcal{J} = 0, \tag{2.4.15}$$

then differentiating equation (2.4.14a) in time,

$$\mathrm{div}\partial_t\mathcal{J} = -\partial_t(\tau^2\partial_t\varrho), \tag{2.4.16}$$

and plugging it into (2.4.15), we get

$$-\partial_t(\tau^2\partial_t\varrho) - \mu\tau^2\partial_t\varrho + \lambda\Delta\varrho + 2\lambda\mathrm{div}(y\varrho) = 0.$$

As

$$\partial_t(\tau^2\partial_t\varrho) = \tau^2\partial_t^2\varrho + 2\tau\dot{\tau}\partial_t\varrho,$$

and recalling that $\tau \sim 2\sqrt{\lambda t/\mu}$ and $\dot{\tau} \sim \sqrt{\lambda/(\mu t)}$, we get that $\tau\dot{\tau} \to 2\lambda/\mu$ and $\tau^2 \ll (\tau^2)^2$ as $t \to +\infty$, so we can neglect the $\partial_t^2\varrho$ term. Introducing another scaling in time s defined by

$$\partial_s = \frac{\mu}{\lambda}\tau^2\partial_t, \tag{2.4.17}$$

and discarding the lower-order terms, we finally find that

$$\partial_s\varrho = L\varrho,$$

where $L := \lambda\Delta + 2\lambda\mathrm{div}(y.)$ denotes a Fokker-Planck operator, for which it is well known (under some hypothesis that we will recall in the following) that in large times the solution converges strongly to a Gaussian. Note that unlike the elliptic equation (2.4.2) of the focusing case, the Fokker-Planck operator L is not invariant by translations, hence the limit will be unique (and not only unique up to translations).

Remark 2.25. Note that the definition (2.4.17) of s can be rewritten

$$s = \frac{\mu}{\lambda} \int \frac{1}{\tau^2},$$

that gives $s \sim \log(t)/4$. This scaling is different from the one for the logarithmic Schrödinger equation described in [36](which was of order $s \sim \log\log(t)/4$). However, we will see that the same proof stands in order to get the long-time dynamics of our solutions.

In order to perform the limit in a rigorous way, we will first look at the energy quantities of our system (2.4.14a)-(2.4.14b). The scaled form of the energy inequality (2.2.2) referring to system (2.4.14a)-(2.4.14b) is the following:

$$\mathcal{E}(t) + \mu \int_0^t \mathcal{D}(s) ds \leq \mathcal{E}_0, \tag{2.4.18}$$

where

$$\mathcal{E}(t) = \frac{1}{2\tau(t)^2} \mathcal{E}_{\text{kin}}(t) + \lambda \mathcal{E}_{\text{ent}}(t), \tag{2.4.19}$$

$$\mathcal{E}_{\text{kin}}(t) = \frac{1}{2} \int_{\mathbb{R}^d} \left(\varrho |u|^2 + |\nabla \sqrt{\varrho}|^2 \right) dy, \tag{2.4.20}$$

$$\mathcal{E}_{\text{ent}}(t) = \int_{\mathbb{R}^d} \varrho \log \left(\frac{\varrho}{\Gamma} \right) dx \tag{2.4.21}$$

$$\mathcal{D}(s) = \int_{\mathbb{R}^d} \varrho |u|^2 dx, \tag{2.4.22}$$

$\mathcal{E}_0 := E_0$ and $\Gamma := e^{-|x|^2} = \gamma^2$. Now we recall that from (2.4.12) and (2.4.14a), for all $t \geq 0$,

$$\|\varrho(t)\|_{L^1(\mathbb{R}^d)} = \|\varrho_0\|_{L^1(\mathbb{R}^d)} = \|\Gamma\|_{L^1(\mathbb{R}^d)}.$$

As $\varrho, \Gamma \geq 0$, we can apply the Csiszár-Kullback inequality (see e.g. [3]) that gives

$$\mathcal{E}_{\text{ent}}(t) \geq \frac{1}{2\|\Gamma\|_{L^1}} \|\varrho(t) - \Gamma\|_{L^1}^2 \geq 0.$$

We now state the first lemma of this section, which will be useful in the following to get some convergence estimates:

Lemma 2.26. *There holds*

$$\sup_{t \geq 0} \left(\int_{\mathbb{R}^d} \varrho(t,y)(1 + |y|^2 + |\log \varrho(t,y)|) dy + \frac{1}{\tau(t)^2} \mathcal{E}_{\text{kin}}(t) \right) < \infty \tag{2.4.23}$$

and

$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau^3(t)} \mathcal{E}_{\text{kin}}(t) dt < \infty. \tag{2.4.24}$$

Proof. Denoting $\mathcal{E}_+$ the positive part of the scaled energy $\mathcal{E}$, we have

$$\mathcal{E}_+ = \frac{1}{2\tau^2}\mathcal{E}_{\text{kin}} + \lambda \int_{\varrho>1} \varrho \log \varrho + \lambda \int_{\mathbb{R}^d} |y|^2 \varrho,$$

so equation (2.4.18) gives

$$\mathcal{E}_+(t) + \mu \int_0^t \mathcal{D}(s)ds \leq \mathcal{E}_0 + \lambda \int_{\varrho<1} \varrho \log \left(\frac{1}{\varrho}\right).$$

The last term is controlled by

$$\int_{\varrho<1} \varrho \log \left(\frac{1}{\varrho}\right) \lesssim \int_{\mathbb{R}^d} \varrho^{1-\varepsilon}$$

for all $\varepsilon > 0$ arbitrary small. By an interpolation formula, we get that

$$\int_{\mathbb{R}^d} \varrho^{1-\varepsilon} \lesssim \|\varrho\|_{L^1}^{1-\varepsilon\frac{d+2}{2}} \||y|^2\varrho\|_{L^1}^{\frac{d\varepsilon}{2}}$$

for all $0 < \varepsilon < 2/(d+2)$. This implies that for all $t \geq 0$,

$$\mathcal{E}_+(t) + \mu \int_0^t \mathcal{D}(s)ds \lesssim 1 + \mathcal{E}_+^{\frac{d\varepsilon}{2}}(t),$$

thus $\mathcal{E}_+(t) \in L^\infty(\mathbb{R}_+)$, and equation (2.4.23) follows. Differentiating $\mathcal{E}$, we get

$$\dot{\mathcal{E}} = -2\frac{\dot{\tau}}{\tau}\mathcal{E}_{\text{kin}},$$

and since $\mathcal{E}(t) \geq 0$, (2.4.24) follows from (2.4.23). $\square$

Proposition 2.27. *Let $(\varrho, \mathcal{J})$ be a smooth solution of the scaled hydrodynamical system* (2.4.14a)-(2.4.14b)*. Then*

$$\varrho(t) \underset{t\to\infty}{\rightharpoonup} \Gamma \textit{ weakly in } L^1(\mathbb{R}^d).$$

Proof. By the elimination of $\mathcal{J}$ described above, using equation (2.4.16), we get

$$-\partial_t(\tau^2\partial_t\varrho) - \mu\tau^2\partial_t\varrho + \lambda L\varrho = \frac{1}{2\tau^2}\varrho\nabla\left(\frac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\right) - \frac{1}{\tau^2}\text{div}\left(\frac{\mathcal{J}\otimes\mathcal{J}}{\varrho}\right).$$

Using the change of variables (2.4.17) and introducing the notation

$$\tilde{\varrho}(s(t), y) := \varrho(t, y),$$

we calculate the quantities

$$\partial_t(\tau^2\partial_t\varrho) = \frac{\lambda^2}{\mu^2}\frac{1}{\tau^2}\partial_s^2\tilde{\varrho} \text{ and } \mu\tau^2\partial_t\varrho = \lambda\partial_s\tilde{\varrho},$$

hence we obtain the following equation:

$$-\frac{\lambda^2}{\mu^2}\frac{1}{\tau^2}\partial_s^2\tilde{\varrho}-\lambda\partial_s\tilde{\varrho}+\lambda L\tilde{\varrho}=\frac{1}{2\tau^2}\tilde{\varrho}\nabla\left(\frac{\Delta\sqrt{\tilde{\varrho}}}{\sqrt{\tilde{\varrho}}}\right)-\frac{1}{\tau^2}\mathrm{div}\left(\frac{\mathcal{J}\otimes\mathcal{J}}{\tilde{\varrho}}\right). \tag{2.4.25}$$

Now we remark that equation (2.4.23) induces

$$\sup_{s\geq 0}\int_{\mathbb{R}^d}\tilde{\varrho}(s,y)(1+|y|^2+|\log\tilde{\varrho}(s,y)|)dy<\infty, \tag{2.4.26}$$

(2.4.24) gives

$$\int_0^\infty\frac{d\tau}{ds}(s)\mathcal{E}_{\mathrm{kin}}(s)ds<\infty, \tag{2.4.27}$$

and that

$$\tau(s)\sim 2\sqrt{\frac{\lambda}{\mu}}e^{2s}\text{ and }\frac{d\tau}{ds}(s)\sim\sqrt{\frac{\lambda}{\mu}}e^{-2s},$$

so we can conclude like in [36]. Let a sequence $s_n\to\infty$, take $s\in[-1,2]$, and denote

$$\tilde{\varrho}_n(s,y):=\tilde{\varrho}(s+s_n,y).$$

From (2.4.26) along with the de la Vallée-Poussin and Dunford-Pettis theorems, we get the following weak convergence (up to a subsequence, not relabeled for reader's convenience), for all $p\in[1,\infty)$,

$$\tilde{\varrho}_n\underset{t\to\infty}{\rightharpoonup}\tilde{\varrho}_\infty\quad\text{in }L^p(-1,2;L^1(\mathbb{R}^d)).$$

We also get the weak convergence of the initial datum, up to another subsequence:

$$\tilde{\varrho}_n(0)\underset{t\to\infty}{\rightharpoonup}\tilde{\varrho}_{0,\infty}\quad\text{in }L^1(\mathbb{R}^d).$$

Thanks to (2.4.26), we also get that the family $(\tilde{\varrho}(s_n,.))_n$ is tight, so

$$\int_{\mathbb{R}^d}\tilde{\varrho}_{0,\infty}(y)dy=\int_{\mathbb{R}^d}\Gamma(y)dy$$

and

$$\int_{\mathbb{R}^d}\tilde{\varrho}_{0,\infty}(y)(1+|y|^2+|\log\tilde{\varrho}_{0,\infty}(y)|)dy<\infty.$$

Then, denoting $\tau_n(s)=\tau(s+s_n)$, equation (2.4.27) implies that

$$\frac{1}{2\tau_n^2}\tilde{\varrho}_n\nabla\left(\frac{\Delta\sqrt{\tilde{\varrho}_n}}{\sqrt{\tilde{\varrho}_n}}\right)-\frac{1}{\tau_n^2}\mathrm{div}\left(\frac{\mathcal{J}_n\otimes\mathcal{J}_n}{\tilde{\varrho}_n}\right)\underset{t\to\infty}{\rightharpoonup}0\quad\text{in }L^1(-1,2;W^{-2,1}(\mathbb{R}^d)).$$

In addition, in (2.4.25), all the other terms but two obviously go weakly to zero, which yields

$$\partial_s \tilde{\varrho}_\infty = L\tilde{\varrho}_\infty \tag{2.4.28}$$

in $\mathcal{D}'((-1,2)\times\mathbb{R}^d)$, with $\tilde{\varrho}_\infty(0,;) = \tilde{\varrho}_{0,\infty} \in L^1(\mathbb{R}^d)$. Thanks to the above bounds on $\tilde{\varrho}_{0,\infty}$, it is known (see [6]) that the solution $\tilde{\varrho}_\infty$ to (2.4.28) is actually defined for all $s \geq 0$ and satisfies

$$\|\tilde{\varrho}_\infty(s,\cdot) - \Gamma\|_{L^1(\mathbb{R}^d)} \underset{s\to\infty}{\longrightarrow} 0. \tag{2.4.29}$$

Going back to system (2.4.14a)-(2.4.14b), we need to show that $\tilde{\varrho}_\infty$ is independent of s. In the s variable, system (2.4.14a) becomes

$$\partial_s \tilde{\varrho} + \frac{\mu}{\lambda}\mathrm{div}\tilde{\mathcal{J}} = 0, \tag{2.4.30}$$

and (2.4.27) implies that $\tilde{\mathcal{J}} \in L^2(-1,2;L^1(\mathbb{R}^d))$. With $\tilde{\mathcal{J}}_n(s) := \tilde{\mathcal{J}}(s = s_n)$, we have

$$\mathrm{div}\tilde{\mathcal{J}}_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1}(\mathbb{R}^d)),$$

so

$$\partial_s \tilde{\varrho}_\infty = 0.$$

Combining this last equality with equation (2.4.29), we infer that $\tilde{\varrho}_\infty = \Gamma$. The limit being unique, no extraction of a subsequence is needed, and we conclude that

$$\tilde{\varrho}(s) \underset{s\to\infty}{\rightharpoonup} \Gamma \text{ weakly in } L^1(\mathbb{R}^d).$$

$\square$

2.5 Numerical simulations

In this section we will plot some numerical trajectories of equation (2.1.1). As performed in [13] and [14] in the logarithmic case $\mu = 0$, in order to avoid numerical blow-up and round-off error due to the first logarithmic component of equation (2.1.1), we are going to discretize the following regularized equation, with a small parameter $\varepsilon > 0$:

$$\begin{cases} i\partial_t \psi^\varepsilon + \frac{1}{2}\Delta\psi^\varepsilon = \lambda\psi^\varepsilon \log(|\psi^\varepsilon|^2 + \varepsilon) + \frac{1}{2i}\mu\psi^\varepsilon \log\left(\frac{\psi^\varepsilon}{(\psi^\varepsilon)^*}\right), \quad x \in \Omega, \quad t > 0, \\ \psi^\varepsilon(0,x) = \psi_0(x), \end{cases}$$

where $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet boundary condition or periodic boundary condition posted on the boundary. We remark that mass conservation still holds, in the sense that

$$\forall t \geq 0, \quad \|\psi^\varepsilon(t,.)\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)},$$

and the new regularized energy is defined as follows:

$$\forall t \geq 0, \quad E^\varepsilon(t) = \int_\Omega \Big(|\nabla \psi^\varepsilon(t,x)|^2 + 2\lambda\varepsilon|\psi^\varepsilon(t,x)| + \lambda|\psi^\varepsilon(t,x)|^2 \log(|\psi^\varepsilon(t,x)|^2 + \varepsilon) \\ -\lambda\varepsilon^2 \log(1 + |\psi^\varepsilon(t,x)|/\varepsilon)^2 \Big)\, dx.$$

We will perform a semi-discretization in time with a Lie-Trotter splitting method. The operator splitting methods for the time integration of (2.5) are based on the following splitting

$$\partial_t \psi^\varepsilon = A(\psi^\varepsilon) + B(\psi^\varepsilon) + C(\psi^\varepsilon),$$

where

$$A(\psi^\varepsilon) = \frac{1}{2} i\Delta\psi^\varepsilon, \quad B(\psi^\varepsilon) = -i\psi^\varepsilon \log(|\psi^\varepsilon|^2 + \varepsilon), \quad C(\psi^\varepsilon) = -\frac{1}{2}\mu\psi^\varepsilon \log\left(\frac{\psi^\varepsilon}{(\psi^\varepsilon)^*}\right),$$

and the solutions of the subproblems

$$i\partial_t u(t,x) = -\frac{1}{2}\Delta u(t,x), \quad u(0,x) = u_0(x), \quad x \in \Omega, \quad t > 0,$$

$$i\partial_t v(t,x) = \lambda v(t,x) \log(v(t,x) + \varepsilon), \quad v(0,x) = v_0(x), \quad x \in \Omega, \quad t > 0,$$

$$i\partial_t w(t,x) = \frac{\mu}{2i} w(t,x) \log\left(\frac{w(t,x)}{w^*(t,x)}\right), \quad w(0,x) = w_0(x), \quad x \in \Omega, \quad t > 0.$$

The associated operators are explicitly given, for $t \geq 0$, by

$$u(t,.) = \Phi_A^t(u_0) = e^{it\Delta} u_0,$$

$$v(t,.) = \Phi_B^t(v_0) = v_0 e^{-it\log(|v_0|^2+\varepsilon)},$$

$$w(t,.) = \Phi_C^t(w_0) = a_0 e^{i\theta_0 e^{-\mu t}}, \ w_0 = a_0 e^{i\theta_0}.$$

All the numerical simulations will be made in dimension $d = 1$ on the interval $\Omega = [a,b]$. The computation of Φ_A^t is made by a Fast Fourier Transformation. Let N be a positive even integer and denote $\Delta x = (b-a)/N$ and the grid points $x_j = a + k\Delta x$ for $0 \leq k \leq N-1$. Denote by $\psi^{N,j}$ the discretized solution vector over the grid $(x_k)_{0\leq k\leq N-1}$ at time $t = t_j = j\Delta t$. Let $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$ denote the discrete Fourier transform and its inverse, respectively. With this notation, $\Phi_A^t(\psi^{N,j})$ can be obtained by

$$\Phi_A^t(\psi^{N,j}) = \mathcal{F}_N^{-1}\left(e^{-i\Delta t(\mu^N)^2}\mathcal{F}_N(\psi^{N,j})\right),$$

where

$$\mu^N = \frac{2\pi}{b-a}\left[0, 1, \ldots, \left(\frac{N}{2}-1\right), -\frac{N}{2}, \ldots, -1\right],$$

and the multiplication of two vectors is taken as point-wise.

We perform our simulations with a space step $\Delta x = 0.2$ on the interval $\Omega = [-100, 100]$. We take the saturation constant $\varepsilon = 10^{-3}$, and the dissipation constant $\mu = 1$, using a time step $\Delta t = 10^{-3}$ on the interval $[0, T_{\max}]$ with $T_{\max} = 1000$. We perform two simulations with λ either equal to 0.1 or -0.1, with the initial function

$$\psi_0(x) = |\sin x| e^{-0.1(x-3)^2} + |\cos x| e^{-0.2(x+4)^2}.$$

In the focusing case, we clearly observe the convergence to a Gaussian function of the same mass (Figure 2.1). In the defocusing case, our solution vanishes with a Gaussian profile (Figure 2.2).

In order to corroborate the tightness hypothesis Assumption 2.5 made in Section 2 in the focusing case $\lambda < 0$, we are also going to test the worst case scenario of two symmetric Gaussian function diverging at both infinities. Here we take the initial function

$$\varphi_0(x) = e^{-(x-10)^2+100ix} + e^{-(x+10)^2-100ix},$$

so our two Gaussian functions have huge initial velocities trying to make them go to infinity. This is indeed the case for the linear Schrödinger equation (see Figure 2.3). We observe that the dissipation (taking $\lambda = -0.1$ and $\mu = 10$ in order to enhance the nonlinear effect) seems to prevent this behavior from happening by quickly freezing their diverging dynamics (cf Figure 2.4). Here we do not go for large times, as we take $T_{\max} = 10$ with $\Delta t = 10^{-4}$, in order to avoid edge effect from our moving Gaussian functions in the free case. Other constants are taken as above.

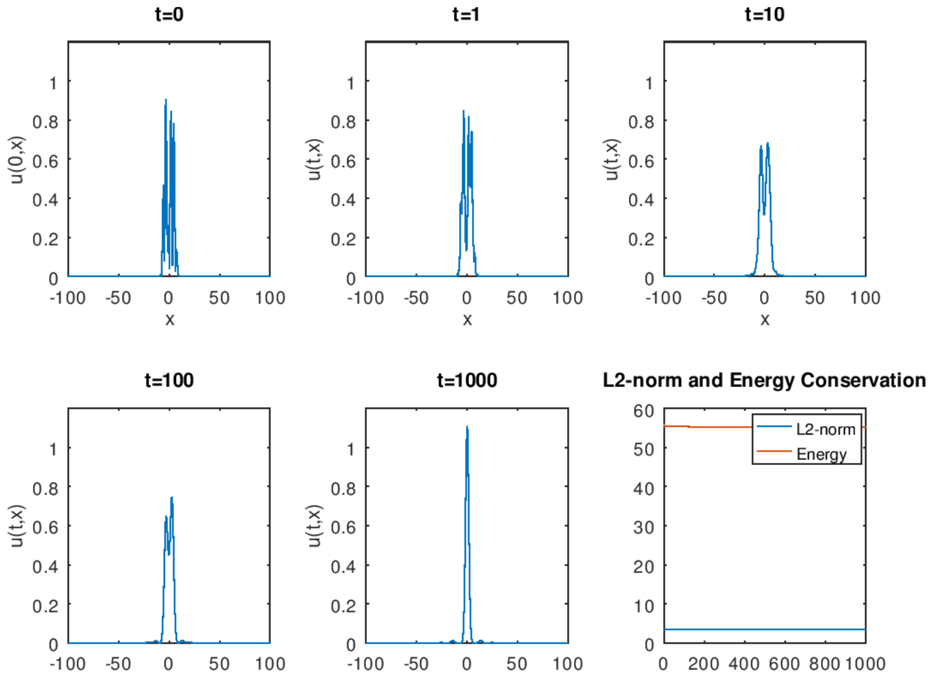

Figure 2.1 – Solution of equation (2.1.1) with initial datum ψ_0 in the focusing case ($\lambda = -0.1$, $\mu = 1$).

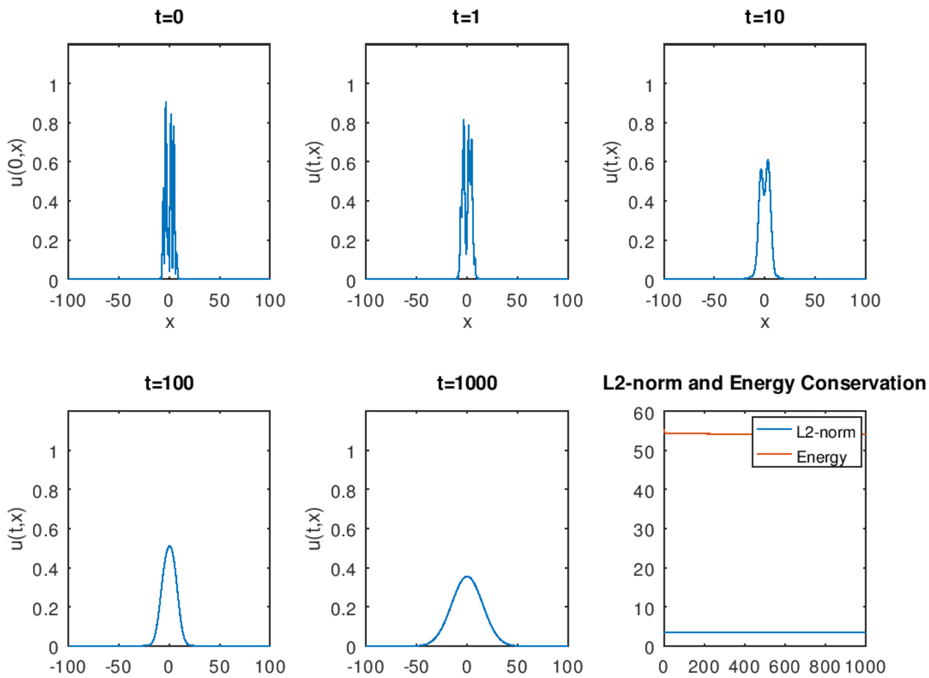

Figure 2.2 – Solution of equation (2.1.1) with initial datum ψ_0 in the defocusing case ($\lambda = 0.1$, $\mu = 1$).

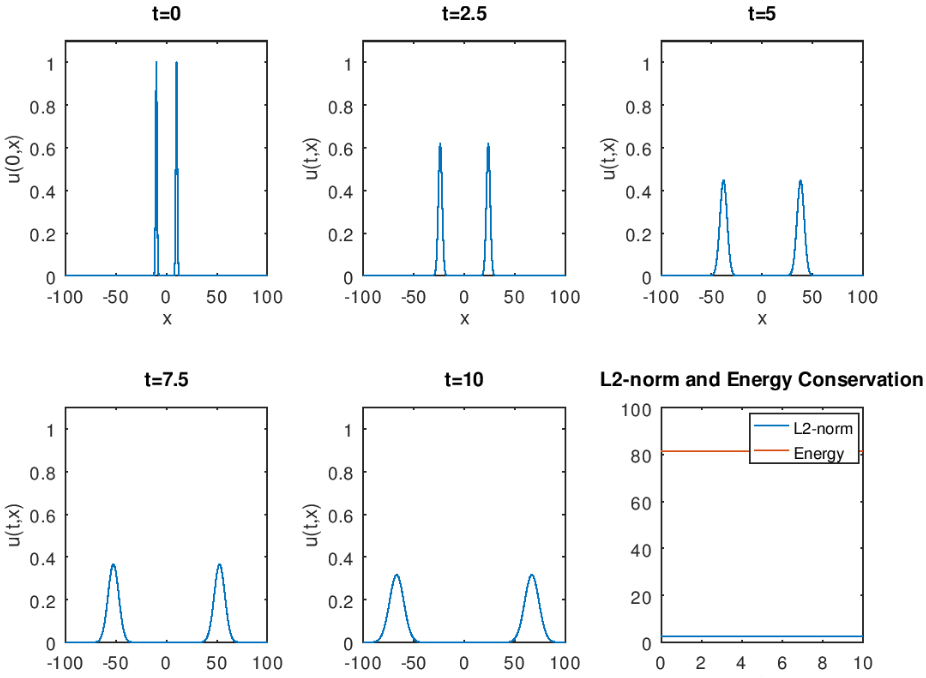

Figure 2.3 – Solution of equation (2.1.1) with initial datum φ_0 in the free case ($\lambda = 0$, $\mu = 0$).

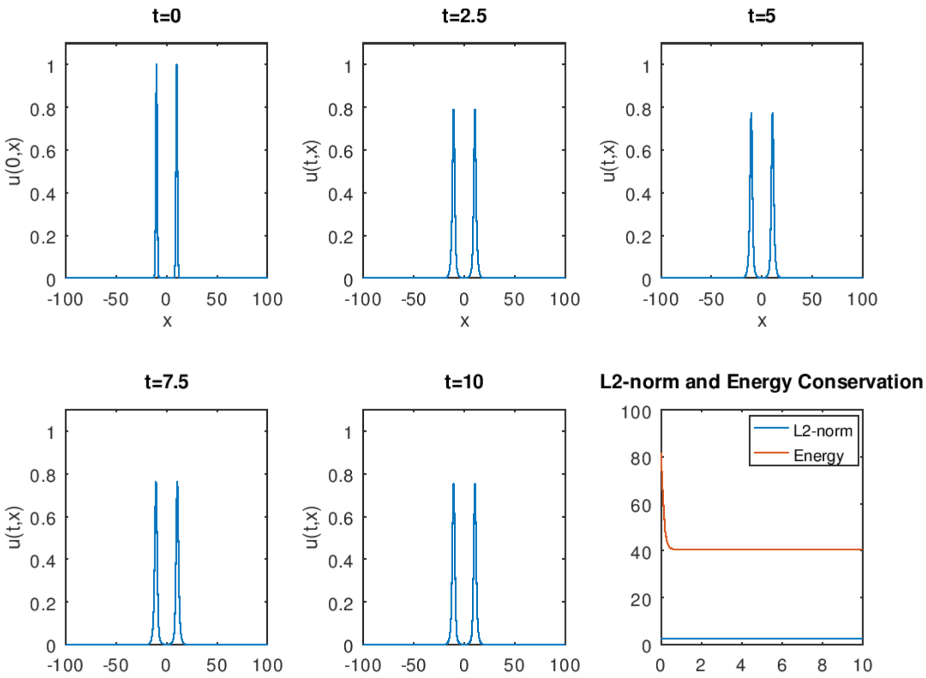

Figure 2.4 – Solution of equation (2.1.1) with initial datum φ_0 in the focusing case ($\lambda = -0.1$, $\mu = 10$).

Chapitre 3

Around plane waves solutions of the Schrödinger-Langevin equation

Abstract: We consider the logarithmic Schrödinger equations with damping, also called Schrödinger-Langevin equation. On a periodic domain, this equation possesses plane wave solutions that are explicit. We prove that these solutions are asymptotically stable in Sobolev regularity. In the case without damping, we prove that for almost all value of the nonlinear parameter, these solutions are stable in high Sobolev regularity for arbitrary long times when the solution is close to a plane wave. We also show and discuss numerical experiments illustrating our results.

3.1 Introduction

Let $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ denote the d-dimensional torus ($d \in \mathbb{N}^*$). We consider the logarithmic Schrödinger equation with damping, also called Schrödinger-Langevin equation,

$$i\partial_t\psi + \Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{\mu}{2i}\psi\log\left(\frac{\psi}{\psi^*}\right), \tag{3.1.1}$$

with $t \geq 0$, $x \in \mathbb{T}^d$, $\psi(0,x) = \psi^0(x)$, $\lambda < 0$ (focusing case) or $\lambda > 0$ (defocusing case), and $\mu \geq 0$. We also denote ψ^* (or also $\overline{\psi}$) the complex conjugate of a complex function ψ. Note that when $\mu = 0$, this equation is the logarithmic Schrödinger equation

$$i\partial_t\psi + \Delta\psi = \lambda\psi\log|\psi|^2. \tag{3.1.2}$$

This latter equation was introduced in [20] as a model of nonlinear wave mechanics, in particular for its tensorization property, a feature shared with the Schrödinger-Langevin equation (3.1.1). Indeed, if the initial datum ψ^0 is a tensor product $\psi^0(x) = \prod_{j=1}^d \psi_j^0(x_j)$, then the solution to the d-dimensional equation (3.1.1) is given by

$$\psi(t,x) = \prod_{j=1}^d \psi_j(t,x_j)$$

where each ψ_j solves the same equation (3.1.1) but in dimension one. It can also easily be shown that the logarithmic nonlinearities above are the only ones satisfying this tensorization property. It has then been proposed to model various phenomena such as quantum optics [32], [87], nuclear physics [73] or transport and diffusion phenomena [50, 91], [72]. On the other hand, the Schrödinger-Langevin equation (3.1.1) first appears in [104] as a possible way to give a stochastic interpretation of quantum mechanics in the context of Bohmian mechanics. It had a recent renewed interest in the physics community, in particular in quantum mechanics in order to describe the continuous measurement of the position of a quantum particle (see for example [103], [118] or [102]) and in cosmology and statistical mechanics (see [45], [46] or [47]).

A lot of properties of these two equations are already known on the whole space $\mathbb{R}^d$. The mathematical study of equation (3.1.2) for the focusing case $\lambda < 0$ goes back to [37], and the global existence of solutions is now well understood (see [38] and [49]), as well as their qualitative behaviors (see for instance [5], [61] or [63]). On the other hand, the defocusing case $\lambda > 0$ for (3.1.2) has received some recent attention, in particular from the work [36] which established the global existence and the uniqueness of solutions as well as their asymptotic behavior. For the Schrödinger-Langevin equation (3.1.1), the long-time behavior of solutions is given in [41] under some global existence assumptions.

However, few results are known about these equations on the d-dimensional torus geometry. The existence and uniqueness of global weak solutions to the logarithmic Schrödinger equation (3.1.2) are given in [13], and the existence of global dissipative solutions to the Euler-Langevin-Korteweg equations, which is the fluid counterpart of the Schrödinger-Langevin equation through the Madelung transform $\psi = \sqrt{\rho}e^{iS}$, is established in [42]. As no behaviour properties or asymptotic features are currently known up to the authors knowledge, this paper is a first step in order to give some qualitative description of the solutions of these equations on $\mathbb{T}^d$.

The most striking property of (3.1.1) is the preservation of the L^2 norm even in the damped case $\mu > 0$, hence this equation is always conservative in this sense. On $\mathbb{T}^d$, a first natural question is to analyze the existence and stability of plane waves solutions. They are of the form

$$\nu_m(t,x) = \rho e^{i(m\cdot x - \omega t)} \tag{3.1.3}$$

for $\rho > 0$, $m \in \mathbb{Z}^d$ and $\omega \in \mathbb{R}$, they belong to any Sobolev space on the torus, and they usually play the role of ground states for nonlinear equations on $\mathbb{T}^d$. For instance, in the case of the classical cubic nonlinear Schrödinger

$$i\partial\psi + \Delta\psi = \lambda|\psi|^2\psi, \tag{3.1.4}$$

plane-wave solutions (3.1.3) correspond to $\omega = |m|^2 + \lambda\rho^2$, and it has been shown in [58] that these solutions are stable in large Sobolev spaces H^s: to be more precise, small perturbations of (3.1.3) in H^s remain essentially localized in the m-th Fourier mode over very long times in H^s for sufficiently large Sobolev exponent s, and *almost all* ρ which corresponds here to the L^2 norm of the plane wave [1].

In the case of the logarithmic Schrödinger equation (3.1.2), for initial data made of a single Fourier mode $u(0,x) = \rho e^{im\cdot x}$ (with $\rho > 0$ and $m \in \mathbb{Z}^d$), the equation has the unique plane-wave solution $\nu_m(t,x) = \rho e^{i(m\cdot x - \omega t)}$ localized at the m-th Fourier mode, with

$$\omega = |m|^2 + 2\lambda \log \rho.$$

As performed in [58], a natural question is then to study the stability of these solutions in Sobolev spaces over very long time. Indeed, we state such a result in Theorem 3.2, and give a proof based on normal transformations as in [71, 11]. The main difference with the result for (3.1.4) is that the frequencies of the linearized operator do not depend on ρ, by a subtle mechanism of scaling invariance of (3.1.2). This result is thus obtained for *almost all* λ, to avoid resonances in the equation.

In contrast, the damping effect in the Schrödinger-Langevin induces that the only stationary plane wave solutions of (3.1.1) are the constant plane wave functions of the form

$$\nu = \rho e^{-2i\lambda \log \rho/\mu}, \tag{3.1.5}$$

where $\rho > 0$. We study the dynamics of perturbations of such a solution in Theorem 3.1. We will show that every small perturbations of (complex) constants (3.1.5) converges exponentially fast in $H^s(\mathbb{T}^d)$ with $s > \frac{d}{2}$ towards the constant solution (3.1.5) where ρ denotes the L^2 norm of the solution, which is preserved by the dynamics. The rate of convergence depends on arithmetic relations between λ and μ which can also generate Jordan block dynamics. Surprisingly, we also have situations where for given λ, the relaxation is slower for large μ than for small μ, and where the damping rate depend on the modes. This is exemplified by numerical experiments and proved in detail in Section 3.

This paper is organized as follows. In Section 2, we will recall some notations and properties of functional analysis and nonlinear PDE analysis in order to state our results Theorem 3.1 and Theorem 3.2. Then, Section 3 is dedicated to the proof of Theorem 3.1, and Section 4 to the proof of Theorem 3.2. Finally, in Section 5, we show some numerical simulations in order to illustrate our results and explore situations not covered by our analysis.

1. For the orbital stability in the energy space H^1, results can also be found in Zhidkov [120, Sect. 3.3] and Gallay & Haragus [66, 65].

3.2 Algebraic context and main results

With a periodic function $u \in L^2(\mathbb{T}^d)$ we can associate the Fourier coefficients u_n for $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ defined by

$$u_n = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-in\cdot x} \mathrm{d}x,$$

with $n \cdot x = n_1 x_1 + \cdots n_d x_d$, $x = (x_1, \dots, x_d)$. For the average, we will use the specific notation

$$\langle u \rangle = u_0 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) \mathrm{d}x.$$

We define the Sobolev space $H^s(\mathbb{T}^d)$ associated with the norm

$$\|u\|_{H^s} = \left(\sum_{n \in \mathbb{Z}^d} \left(1 + |n|^2\right)^s |u_n|^2 \right)^{\frac{1}{2}},$$

where $|n|^2 = n_1^2 + \cdots + n_d^2$ for $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$. We recall that $H^s(\mathbb{T}^d)$ is an algebra when $s > d/2$, namely there exists a constant C_s such that for all u, $v \in H^s$, we have

$$\|uv\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s}. \tag{3.2.1}$$

Note also that we have

$$\|u\|_{L^2}^2 = \sum_{n \in \mathbb{Z}^d} |u_n|^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |u(x)|^2 \mathrm{d}x = (u, u)_{L^2},$$

where

$$(u, v)_{L^2} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \overline{u(x)} v(x) \mathrm{d}x.$$

We now define the notion of solution to the Schrödinger-Langevin equation. For $T > 0$, and an application $t \mapsto u(t, x) \in \mathcal{C}([0, T], H^s)$ such that $\langle u \rangle \neq 0$, by classical lifting theorem, we can define $a(t) > 0$ and $\theta(t) \in \mathbb{R}$ such that $\langle u(t) \rangle = a(t) e^{i\theta(t)}$ and $\theta(0) \in [-\pi, \pi[$, and such that the application $t \mapsto (a(t), \theta(t))$ is continuous on $[0, T]$. We define the function

$$w = -a + e^{-i\theta} u = |\langle u \rangle| \left(-1 + \frac{u}{\langle u \rangle} \right),$$

so that we have the following parametrization, valid for all u such that $\langle u \rangle \neq 0$:

$$u(t, x) = e^{i\theta(t)} (a(t) + w(t, x)), \tag{3.2.2}$$

where $\theta(t)$, $a(t)$ and $w(t,x)$ are continuous in time, as long as $\langle u\rangle \neq 0$. In this case, we can define the logarithm

$$\log(u(t,x)) := i\theta(t) + \log a(t) + \log\left(1 + \frac{w(t,x)}{a(t)}\right).$$

This application is well defined and smooth for curves on the domain

$$\mathcal{U}_s \quad = \quad \left\{u = e^{i\theta}(a+w) \;\middle|\; (a,\theta,w) \in \mathbb{R}_+ \times \mathbb{T} \times H^s,\ a>0,\ \langle w\rangle = 0, \quad \left\|\frac{w}{a}\right\|_{H^s} < \frac{1}{C_s}\right\}$$

where C_s is the constant appearing in (3.2.1). Note that this set contains arbitrary large functions as both a and w can become large.

On $\mathcal{U}_s$, and owing to the analytic series

$$\log\left(1+\frac{w}{a}\right) = -\sum_{n\geq 1}\frac{1}{n}\left(-\frac{w}{a}\right)^n, \tag{3.2.3}$$

we see that the application $u \mapsto \log u$ defined above is analytic. With this definition of the logarithm, it is clear that for a curve $t \mapsto u(t) \in \mathcal{U}_s$ we have $u^*(t) \in \mathcal{U}_s$, and $\log|u(t)|^2 = \log u(t) + \log u^*(t)$. Moreover, we have $\log(u^*(t)) = (\log u(t))^*$. Hence we deduce that $u(t) \mapsto \lambda u(t)\log|u(t)|^2 = \lambda u(t)(\log u(t) + \log u^*(t))$ is well defined on $\mathcal{C}([0,T],\mathcal{U}_s)$. Similarly, the function $u(t) \mapsto \log(u(t)) - \log(u^*(t))$ is well defined on $\mathcal{C}([0,T],\mathcal{U}_s)$, and hence so is the function

$$u \in \mathcal{C}([0,T],\mathcal{U}_s) \mapsto \frac{\mu}{2i}u\log\left(\frac{u}{u^*}\right) \in \mathcal{C}([0,T],H^s).$$

This thus allows to define mild and strong solutions to the Schrödinger-Langevin equation (3.1.1) than can be written

$$i\partial_t\psi + \Delta\psi = z\psi\log(\psi) + z^*\psi\log(\psi^*), \quad \text{with} \quad z = \lambda + \frac{\mu}{2i}. \tag{3.2.4}$$

Note that the L^2 norm is preserved along the dynamics of this equation, as we can check that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{L^2}^2 = 2\mathrm{Re}(\psi,\partial_t\psi)_{L^2} = -2\,\mathrm{Im}(\psi,\Delta\psi)_{L^2} + 4\,\mathrm{Im}(\psi,\psi\,\mathrm{Re}(z\log\psi))_{L^2} = 0.$$

We now state our first theorem concerning the Schrödinger-Langevin equation:

Theorem 3.1. *Let $s > \frac{d}{2}$, $\lambda > -\frac{1}{2}$, $\mu > 0$. Then there exists $\varepsilon_0 > 0$ such that for all $\rho > 0$, if the initial datum $\psi^0 \in \mathcal{U}_s$ satisfies*

$$\|\psi^0 - \langle\psi^0\rangle\|_{H^s} \leq \varepsilon_0\rho \quad \textit{and} \quad \|\psi_0\|_{L^2} = \rho$$

then the solution of (3.1.1) *with* $\psi(0,\cdot) = \psi^0 \in \mathcal{U}_s$ *satisfies for* $t \geq 0$,

$$\|\psi(t,\cdot) - \rho e^{-2i\lambda \log \rho/\mu}\|_{H^s} \leq Ce^{-\alpha t}(1+\beta t), \tag{3.2.5}$$

where the constant $C > 0$ *depends on* s, d, ρ *and* ε_0, *and where* $\alpha > 0$ *and* $\beta \in \{0,1\}$ *depend on* μ *and* λ *as follows:*

(i) If $\mu < 2\sqrt{1+2\lambda}$ *then* $\alpha = \frac{\mu}{2}$ *and* $\beta = 0$.

(ii) If $\mu = 2\sqrt{1+2\lambda}$ *then* $\alpha = \frac{\mu}{2}$ *and* $\beta = 1$.

(iii) If $\mu > 2\sqrt{1+2\lambda}$ *then*

$$\alpha = \frac{\mu}{2} - \left(\frac{\mu^2}{4} - 1 - 2\lambda\right)^{\frac{1}{2}} \in \left(0, \frac{\mu}{2}\right),$$

and if there exists $n \geq 2$ *and* $j \in \mathbb{Z}^d$ *such that* $\mu^2 = 4n^2 + 8\lambda n$ *and* $n = |j|^2$, *we have* $\beta = 1$, *and if this is not the case,* $\beta = 0$.

Let us make the following remarks:

- The presence of $\beta = 1$ reflects the possibility of Jordan block in the reduced dynamics.
- There is no restriction on ρ, the plane wave solution can be arbitrarily large, a phenomenon which can seem rather unusual in the context of nonlinear Schrödinger equations. This is due to a scaling invariance of equation (3.1.1).
- When λ is fixed, $\mu \to \infty$, the damping rate

$$\alpha = \frac{2\lambda + 1}{\mu} + \mathcal{O}\left(\frac{1}{\mu^3}\right)$$

  goes to 0. Thus a larger damping coefficient implies a slowlier global relaxation to the equilibrium. This echoes some known behaviors in other context, see for instance [74] and the reference therein. To our knowledge, such behaviour was however never identified in conservative models coming from quantum mechanics.
- Note that the exponential damping rate α is always given by the first modes damping rates α_j with $|j| = 1$, but higher modes may decay at faster speed depending on $|j|$. We refer the reader to Section 3.5 for a formal discussion about damping rates of higher modes of the solution ψ.
- The hypothesis $\lambda > -\frac{1}{2}$ seems to be crucial to obtain a uniform global convergence result towards a constant. Indeed, numerical experiments as performed in Figure 3.6 for $\lambda = -1$ tend to show that the solution converges to a non trivial quasi-periodic solution (which would be a limit torus in the dynamics). A complete analysis of such case would be very interesting and will be the subject of further studies.

We now state our second result, which analyzes the case $\mu = 0$, that is the logarithmic Schrödinger equation (3.1.2). Recall that for $m \in \mathbb{Z}^d$, plane wave solutions of (3.1.2) are explicitly given by

$$\nu_m(t,x) = \rho e^{i\theta_0} e^{im\cdot x} e^{-i\left(|m|^2+2\lambda\log\rho\right)t} \tag{3.2.6}$$

for all $t \geq 0$ and $x \in \mathbb{T}^d$, and $\rho = \|\nu_m(t,x)\|_{L^2(\mathbb{T}^d)} > 0$. We prove the stability of such functions by H^s perturbations. In this case we use a *normal form* strategy already employed in the cubic case (see [58]) which ensures the H^s-stability of plane waves solutions of (3.1.2) for sufficiently large Sobolev exponent s, and in the following sense:

Theorem 3.2. *Let $m \in \mathbb{Z}^d$ be fixed. Let $-\frac{1}{2} < \lambda_- < \lambda_+$, $\theta_0 \in \mathbb{R}$ and $N > 1$ be fixed arbitrarily. Then there exist $s_0 > 0$, $C \geq 1$ and a set of full measure Λ in the interval $[\lambda_-, \lambda_+]$ such that for every $s \geq s_0$ and every $\lambda \in \Lambda$, there exists $\varepsilon_0 > 0$ such that the following holds: for all $\rho > 0$, if the initial data $\psi^0 \in \mathcal{U}_s$ is such that*

$$\|e^{-im.x}\psi^0 - \langle e^{-im.x}\psi^0\rangle\|_{H^s} = \varepsilon\rho \leq \varepsilon_0\rho, \quad \textit{and} \quad \|\psi^0\|_{L^2} = \rho,$$

then the solution $\psi(t)$ of (3.1.2) with $\psi(0,x) = \psi^0(x)$ satisfies

$$\|e^{-im.x}\psi(t,\cdot) - \rho e^{i\theta(t)}\|_{H^s} \leq C\rho\varepsilon \quad \textit{for} \quad t \leq (\rho\varepsilon)^{-N},$$

where $\theta(t)$ is such that

$$|\dot\theta + |m|^2 - 2\lambda\log\rho| \leq C\varepsilon^2\rho^2.$$

The proof relies on a Birkhoff normal form Theorem (see [71] and [11]), which will be recalled in Section 3.6. with Theorem 3.6.

Let us remark that in the energy space H^1, the previous Theorem holds for all times $t \in \mathbb{R}$ and for all values of $\lambda > -\frac{1}{2}$ by using energy estimates as in [120, 66, 65]. It is a straightforward consequence of the Hamiltonian formulation (3.4.5) in the w variable after elimination of the zero mode by L^2 preservation and gauge invariance. As for the cubic case, the case of Sobolev space with large index H^s cannot be performed for all times and all values of the parameters as resonnance phenomena can appear.

3.3 Asymptotic stability for the Schrödinger-Langevin equation

3.3.1 Elimination of the zero mode

Note that if ψ is solution of (3.1.1) with initial datum $\psi(0,x) = \psi^0$, then the purely time-dependent gauge transform

$$\kappa\psi \exp\left(-2i\frac{\lambda}{\mu}\log\kappa\left(1-e^{-\mu t}\right)\right) \tag{3.3.1}$$

is also a solution of (3.1.1) with initial datum $\psi(0,x) = \kappa\psi^0$. Hence by taking $\kappa = \rho^{-1}$ and modifying ε_0 and C accordingly, we see that it is sufficient to prove the result for $\rho = 1$.

We search a solution under the form

$$\psi(t,x) = e^{i\theta(t)}(a(t) + w(t))$$

with $a(t) > 0$, $\theta(t) \in \mathbb{R}$, and

$$w(t,x) = \sum_{j\neq 0} w_j(t)e^{ij\cdot x}$$

on the Fourier basis. For $r_0 > 0$, we denote

$$\mathcal{B}_s(r_0) = \{w \in H^s \mid \langle w\rangle = 0, \quad \text{and} \quad \|w\|_{H^s} \leq r_0\}.$$

Note that we have $\psi - \langle\psi\rangle = e^{i\theta}w$ and hence we can assume $w(0,\cdot) \in \mathcal{B}_s(r_0)$ at time $t = 0$, for r_0 small enough. We also recall that the L^2 norm of the solution is $\|\psi\|_{L^2}^2 = a^2 + \|w\|_{L^2}^2 = \rho^2$, and it is preserved along the dynamics, hence we have

$$a = \sqrt{1 - \|w\|_{L^2}^2}. \tag{3.3.2}$$

Now we calculate with $\psi = e^{i\theta}(a + w)$, using the representation (3.2.4) with $z = \lambda + \frac{\mu}{2i}$:

$$\begin{aligned}
&i\partial_t\psi = -e^{i\theta}(a+w)\dot{\theta} + e^{i\theta}i\dot{a} + e^{i\theta}i\partial_t w,\\
&\Delta\psi = e^{i\theta}\Delta w,\\
&z\psi\log(\psi) = ze^{i\theta}(a+w)\left(i\theta + \log a + \log\left(1+\frac{w}{a}\right)\right),\\
&z^*\psi\log(\psi^*) = z^*e^{i\theta}(a+w)\left(-i\theta + \log a + \log\left(1+\frac{w^*}{a}\right)\right).
\end{aligned}$$

We then see that (3.2.4) is equivalent to

$$\begin{aligned}&-(a+w)\dot{\theta}+i\dot{a}+i\partial_t w+\Delta w=\\&\qquad z(a+w)\left(i\theta+\log a+\log\left(1+\frac{w}{a}\right)\right)+z^*(a+w)\left(-i\theta+\log a+\log\left(1+\frac{w^*}{a}\right)\right),\end{aligned}$$

or equivalently

$$-(a+w)(\dot{\theta}+\mu\theta+2\lambda\log(a))+i\dot{a}+i\partial_t w+\Delta w=zw+z^*w^*+X(w,w^*),\tag{3.3.3}$$

where

$$\begin{aligned}X(w,w^*)&:=z(a+w)\log\left(1+\frac{w}{a}\right)-zw+z^*(a+w)\log\left(1+\frac{w^*}{a}\right)-z^*w^*\\&=-z(a+w)\sum_{n\geq 1}\frac{1}{n}\left(-\frac{w}{a}\right)^n-zw-z^*(a+w)\sum_{n\geq 1}\frac{1}{n}\left(-\frac{w^*}{a}\right)^n-z^*w^*.\end{aligned}$$

Hence, as a is given by (3.3.2), we have, for r_0 small enough,

$$w\in\mathcal{B}_s(r_0)\quad\Longrightarrow\quad\|X(w,w^*)\|_{H^s}\leq C_{r_0}\|w\|_{H^s}^2.$$

Taking the average $\langle\,\cdot\,\rangle$ of (3.3.3), we obtain

$$-a(\dot{\theta}+\mu\theta+2\lambda\log(a))+i\dot{a}=\langle X(w,w^*)\rangle,$$

which yields an equation for θ:

$$\dot{\theta}+\mu\theta+2\lambda\log(a)=\frac{1}{a}\mathrm{Re}\langle X(w,w^*)\rangle.\tag{3.3.4}$$

We also note that as long as $w\in\mathcal{B}_s(r_0)$, we have

$$\left|\frac{1}{a}\mathrm{Re}\langle X(w,w^*)\rangle\right|\leq C\|w\|_{H^s}^2.$$

Projecting again (3.3.3) on the non zero modes, we obtain

$$i\partial_t w+\Delta w=zw+z^*w^*+X(w,w^*)-\langle X(w,w^*)\rangle+w(\dot{\theta}+\mu\theta+2\lambda\log(a)).$$

This shows that the equation for w can be written

$$i\partial_t w=-\Delta w+zw+z^*w^*+F(w,w^*),\tag{3.3.5}$$

where

$$F(w,w^*) = X(w,w^*) - \langle X(w,w^*)\rangle + \frac{w}{a}\mathrm{Re}\langle X(w,w^*)\rangle$$

satisfies the estimate

$$w \in \mathcal{B}_s(r_0) \quad \Longrightarrow \quad \|F(w,w^*)\|_{H^s} \leq C\|w\|^2_{H^s}\,.$$

3.3.2 Diagonalization and Jordan blocks

To analyze the long time behavior of (3.3.5), we embed it into the complex system

$$\left|\begin{array}{rcl} i\partial_t\xi &=& -\Delta\xi + z\xi + z^*\eta + F(\xi,\eta) \\ i\partial_t\eta &=& \Delta\eta - z^*\eta - z\xi - F(\xi,\eta)^* \end{array}\right. \tag{3.3.6}$$

and we consider this system in the ball $(\xi,\eta) \in \mathcal{B}_s(r_0)\times\mathcal{B}_s(r_0)$ equipped with the norm product

$$\|(\xi,\eta)\|^2_{H^s} = \|\xi\|^2_{H^s} + \|\eta\|^2_{H^s}\,.$$

We can write this system as

$$i\partial_t\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}-\Delta+z & z^*\\ -z & \Delta - z^*\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} + Q(\xi,\eta), \tag{3.3.7}$$

where $Q : \mathcal{B}_s(r_0)\times\mathcal{B}_s(r_0) \to H^s\times H^s$ satisfy the estimate

$$\|Q(\xi,\eta)\|_{H^s} \leq C\|(\xi,\eta)\|^2_s\,.$$

Now, if we consider the extended system (3.3.6) with an initial condition satisfying

$$\xi(0) = \eta(0)^*$$

then for all times $\xi(t) = \eta(t)^*$ and $\xi(t) = w(t)$ coincides with the solution of (3.3.5). Note that in Fourier, this condition yields $\xi_j = (\eta^*)_j = (\eta_{-j})^*$.

Taking the Fourier transform of the equation (3.3.7), we obtain the collection of equations

$$\forall\, j \in \mathbb{Z}^d\backslash\,\{0\}\,,\quad i\partial_t\begin{pmatrix}\xi_j\\ \eta_j\end{pmatrix} = A_{|j|^2}\begin{pmatrix}\xi_j\\ \eta_j\end{pmatrix} + Q_j(\xi,\eta). \tag{3.3.8}$$

where for $n = |j|^2$,

$$A_n = \left(\begin{array}{cc} n+z & z^*\\ -z & -n-z^*\end{array}\right) = \left(\begin{array}{cc} n+\lambda+\frac{\mu}{2i} & \lambda-\frac{\mu}{2i}\\ -\lambda-\frac{\mu}{2i} & -n-\lambda+\frac{\mu}{2i}\end{array}\right).$$

In particular, the eigenvalues of the matrix A_n are

$$\frac{\mu}{2i} \pm \sqrt{n^2 + 2\lambda n - \frac{\mu^2}{4}}.$$

We can distinguish three cases:
(i) $\underline{4n^2 + 8\lambda n - \mu^2 > 0}$. In this case the eigenvalues are of the form $\frac{\mu}{2i} \pm \delta_n$, with

$$\delta_n = \sqrt{n^2 + 2\lambda n - \frac{\mu^2}{4}} > 0.$$

We can explicit the diagonalization $A_n = P_n D_n P_n^{-1}$, with the diagonal matrix

$$D_n = \begin{pmatrix} \frac{\mu}{2i} - \delta_n & 0 \\ 0 & \frac{\mu}{2i} + \delta_n \end{pmatrix},$$

and the matrix of change of coordinates

$$P_n^{-1} = \frac{1}{\sqrt{2\delta_n(\delta_n + n + \lambda)}} \begin{pmatrix} n + \lambda + \delta_n & \lambda - \frac{\mu}{2i} \\ \lambda + \frac{\mu}{2i} & n + \lambda + \delta_n \end{pmatrix} \tag{3.3.9}$$

and

$$P_n = \frac{1}{\sqrt{2\delta_n(\delta_n + n + \lambda)}} \begin{pmatrix} n + \lambda + \delta_n & -\left(\lambda - \frac{\mu}{2i}\right) \\ -\left(\lambda + \frac{\mu}{2i}\right) & n + \lambda + \delta_n \end{pmatrix}. \tag{3.3.10}$$

Note that P_n is hermitian, has condition number smaller than 2 and that

$$\det(P_n) = \det\left(P_n^{-1}\right) = 1.$$

(ii) $\underline{4n^2 + 8\lambda n - \mu^2 < 0}$. Note that for given λ and μ, this situation occurs only a finite number of times, as when n becomes large, $n^2 + 2\lambda n$ goes to $+\infty$. In this case the two eigenvalues are under the form $-i\alpha_n$ and $-i\beta_n$, with

$$\alpha_n = \frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} - n^2 - 2\lambda n}, \qquad \beta_n = \frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} - n^2 - 2\lambda n}.$$

Now as $n \geq 1$ and $\lambda > -\frac{1}{2}$, we have that $n^2 + 2\lambda n > 0$ and hence $\beta_n > \alpha_n > 0$. Let us finally note that the sequence $n^2 + 2\lambda n$ is increasing, so the minimum of all the α_n and β_n is α_1, and this case happen if $4 + 8\lambda - \mu^2 < 0$. The diagonalization matrices are the same as in (3.3.9) and (3.3.10), and we have

$$D_n := P_n^{-1} A_n P_n = \begin{pmatrix} -i\beta_n & 0 \\ 0 & -i\alpha_n \end{pmatrix},$$

but note that as $\delta_n = i(\frac{\mu^2}{4} - n^2 - 2\lambda n)^{\frac{1}{2}}$ is now imaginary, these matrices are not hermitian anymore and have no special geometric structure.
(iii) $\underline{4n^2 + 8\lambda n - \mu^2 = 0}$. In this case $\frac{\mu}{2i}$ is a double eigenvalue. As

$$(n+\lambda)^2 = (\lambda - \frac{\mu}{2i})(\lambda + \frac{\mu}{2i}),$$

we have that $(n+\lambda, -\lambda - \frac{\mu}{2i})$ is an eigenvector and the matrix A_n can be put under the Jordan form

$$T_n := P_n^{-1} A_n P_n = \begin{pmatrix} \frac{\mu}{2i} & \lambda - \frac{\mu}{2i} \\ 0 & \frac{\mu}{2i} \end{pmatrix},$$

with

$$P_n^{-1} = \frac{1}{n+\lambda} \begin{pmatrix} 2(n+\lambda) & \lambda - \frac{\mu}{2i} \\ \lambda + \frac{\mu}{2i} & (n+\lambda) \end{pmatrix}$$

and

$$P_n = \frac{1}{n+\lambda} \begin{pmatrix} n+\lambda & -\lambda + \frac{\mu}{2i} \\ -\lambda - \frac{\mu}{2i} & 2(n+\lambda) \end{pmatrix}.$$

Again, the matrices P_n and P_n^{-1} have condition number smaller than 2, uniformly in n.

We now go back to (3.3.7), and we define $V = \begin{pmatrix} f \\ g \end{pmatrix} = P^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ in Fourier by the formula

$$\forall\, j \in \mathbb{Z}^d \backslash \{0\}, \quad V_j = \begin{pmatrix} f_j \\ g_j \end{pmatrix} = P_{|j|^2}^{-1} \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}.$$

From the properties of the matrices P_n exhibited in the previous three cases, the application $(\xi, \eta) \mapsto V$ is bounded and invertible in $H^s \times H^s$, the inverse being given by the multiplication with the matrices $P_{|j|^2}$. Hence the system becomes

$$i\partial_t V_j = D_{|j|^2} V_j + R_j(V),$$

where $R(V) = P^{-1} Q(PV)$ satisfies

$$\|R(V)\|_{H^s} \leq C \|V\|_{H^s}^2$$

for $V \in \mathcal{B}_s(r_0) \times \mathcal{B}_s(r_0)$ and for some r_0 small enough. Here, the two by two matrices $D_{|j|^2}$ are given explicitly in term of λ, μ and $|j|^2$.

Let $\alpha > 0$. We now define the operator Λ as

$$(\Lambda V)_j = \alpha V_j \quad \text{if} \quad 4n^2 + 8\lambda n - \mu^2 \neq 0$$

and

$$(\Lambda V)_j = \begin{pmatrix} \alpha & -i\lambda + \frac{\mu}{2} \\ 0 & \alpha \end{pmatrix} V_j \quad \text{if} \quad 4n^2 + 8\lambda n - \mu^2 = 0.$$

We calculate that in this latter case, we have for $t \in \mathbb{R}$,

$$e^{t\Lambda_j} \begin{pmatrix} f_j \\ g_j \end{pmatrix} = \begin{pmatrix} 1 & t(-i\lambda + \frac{\mu}{2}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_j \\ g_j \end{pmatrix}.$$

We can define the change of variable $U = e^{t\Lambda} V$ so we have

$$\begin{aligned} i\partial_t U &= e^{t\Lambda}(i\Lambda + D)e^{-t\Lambda}U + e^{t\Lambda}R(e^{-\Lambda t}U) \\ &= (i\Lambda + D)U + e^{t\Lambda}R(e^{-\Lambda t}U), \end{aligned}$$

where the last equality comes from the fact that if $4n^2 + 8\lambda n - \mu^2 \neq 0$ the block operator $e^{t\Lambda_j}$ is the scalar multiplication by $e^{t\alpha}$, and in the Jordan block case,

$$i\Lambda_j + D_{|j|^2} = \begin{pmatrix} i(\alpha - \frac{\mu}{2}) & 0 \\ 0 & i(\alpha - \frac{\mu}{2}) \end{pmatrix}. \tag{3.3.11}$$

This leads to the following estimate: when $U \in \mathcal{B}_s(r_0)$, as R is at least quadratic, we have that for $t \geq 0$,

$$\|e^{t\Lambda}R(e^{-\Lambda t}U)\|_{H^s} \leq Ce^{-\alpha t}(1 + \beta t^2)\|U\|_{H^s}^2$$

where $\beta = 0$ in the case where $4n^2 + 8\lambda n - \mu^2 \neq 0$ for all $n = |j|^2$ with $j \in \mathbb{Z}^d$, *i.e.* when no Jordan block is present.

Let us examine the operator $i\Lambda + D$. In the Jordan block case *(iii)*, we have seen that the operator is given by (3.3.11). In the case (i), it is

$$i\Lambda_j + D_{|j|^2} = \begin{pmatrix} i(\alpha - \frac{\mu}{2}) - \delta_n & 0 \\ 0 & i(\alpha - \frac{\mu}{2}) + \delta_n. \end{pmatrix}$$

and in the case *(ii)*,

$$i\Lambda_j + D_{|j|^2} = \begin{pmatrix} i(\alpha - \beta_n) & 0 \\ 0 & i(\alpha - \alpha_n) \end{pmatrix}.$$

Hence, if $\alpha = \min(\alpha_n, \frac{\mu}{2}) = \min(\alpha_1, \frac{\mu}{2})$, we have that

$$i\Lambda + D = -iA + B,$$

where A and B are diagonal with real coefficients, the coefficients of A being nonnegative.

Let L be the operator defined on $H^s \times H^s$ by the formula

$$(LU)_j = |j|U_j.$$

As $i\Lambda + D$ is diagonal, it commutes with L. Then we have for $U = \begin{pmatrix} f \\ g \end{pmatrix}$:

$$\begin{aligned}\|U\|_{H^s}^2 &= \sum_{j\neq 0} |f_j|^2 |n|^{2s} + \sum_{j\neq 0} |g_j|^2 |n|^{2s} \\ &= (L^s U, L^s U)_{L^2} = (U, L^{2s} U)_{L^2}\end{aligned}$$

where for $U = \begin{pmatrix} f \\ g \end{pmatrix}$ and $V = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$,

$$(U, V)_{L^2} = \frac{1}{2\pi} \int_{\mathbb{T}^d} \overline{f(x)} \xi(x) \mathrm{d}x + \frac{1}{2\pi} \int_{\mathbb{T}^d} \overline{g(x)} \eta(x) \mathrm{d}x.$$

Let us calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \|U\|_{H^s}^2 &= 2\mathrm{Re}(L^s U, L^s \partial_t U)_{L^2} \\ &= 2\mathrm{Re}(L^s U, L^s(-A - iB)U)_{L^2} + K(t, U) \\ &= -2\mathrm{Re}(L^s U, A L^s U)_{L^2} + K(t, U),\end{aligned}$$

where we have the estimate

$$|K(t, U)| \leq C e^{-\alpha t}(1 + \beta t^3) \|U\|_{H^s}^3.$$

Moreover, with the choice of α, we have

$$-2\mathrm{Re}(L^s U, A L^s U)_{L^2} \leq 0,$$

so we end up with the estimate

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \|U\|_{H^s}^2 \right| \leq C e^{-\alpha t}(1 + \beta t^3) \|U\|_{H^s}^3 \leq M e^{-\frac{\alpha}{2} t} \|U\|_{H^s}^3$$

for some positive constant M, which is valid as long as $U(t) \in \mathcal{B}_s(r_0) \times \mathcal{B}_s(r_0)$ with r_0 small enough. If we denote $y = \|U\|_{H^s(\mathbb{T}^d)}^2$, we have to study the differential inequation

$$|\dot{y}| \leq M e^{-\frac{\alpha}{2} t} y^{3/2}.$$

We look at the following differential equation, for all $t \geq 0$,

$$\dot{f}(t) = Me^{-\frac{\alpha}{2}t} f(t)^{3/2}.$$

Note that as the function $(t, f(t)) \mapsto e^{-\frac{\alpha}{2}t} f(t)^{3/2}$ is locally lipschitz with respect to its second variable f, the classical theory of Cauchy-Lipschitz theorem applies, and as $f(0) = y(0) = \|U(0)\|_{H^s}^2 = \|V(0)\|_{H^s}^2 > 0$, we know that $f(t) > 0$ as long as the solution f exists. In particular, we can write

$$\frac{\dot{f}(t)}{f(t)^{3/2}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{-2}{\sqrt{f(t)}}\right) = Me^{-\frac{\alpha}{2}t},$$

so

$$\sqrt{f(t)} = \frac{1}{\frac{1}{\sqrt{f(0)}} + \frac{M}{\alpha}\left(e^{-\frac{\alpha}{2}t} - 1\right)}$$

by integrating, so f remains uniformly bounded under the condition

$$f(0) < \left(\frac{\alpha}{M}\right)^2,$$

and more precisely

$$f(0) \leq f(t) \leq \frac{f(0)}{\left(1 - \frac{M}{\alpha}\sqrt{f(0)}\right)^2} < \infty,$$

in particular the solution f do not blow-up in finite time and exists for all time $t \geq 0$. As $f(0) = \|U(0)\|_{H^s}^2$, we have by comparison that $y(t) \leq f(t)$ for all $t \geq 0$, so

$$\sup_{t \geq 0} \|U(t)\|_{H^s}^2 < \infty$$

under the condition

$$\|U(0)\|_{H^s} < \frac{\alpha}{M}.$$

Recalling that $U(t) = e^{\Lambda t} V(t)$, we get that

$$\|V(t)\|_{H^s} \leq C(\mu, s, d, \|V(0)\|_{H^s}) e^{-\alpha t}(1 + \beta t),$$

and this shows Theorem 3.1, $\beta = 1$ or $\beta = 0$ reflecting the presence of Jordan block in the dynamics.

Now if the initial conditions $\xi^* = \eta$ are satisfied, it gives a solution of the initial system. It remains to control a and θ, which is read in a non trivial relation between the components of U

and V, but without affecting the fact that $V \in \mathcal{B}_s$. But from (3.3.2) we obtain directly

$$|a-1| \leq C\|V\|_{H^s}^2 \leq Ce^{-2\alpha t}(1+\beta t)^2,$$

and from (3.3.4)

$$|\dot{\theta}+\mu\theta| \leq Ce^{-2\alpha t}(1+\beta t)^2 \leq Ce^{-\frac{3}{2}\alpha t}.$$

We deduce that

$$|\theta(t)| \leq Ce^{-\mu t}\int_0^t e^{(\mu-\alpha)s}\mathrm{d}s = \frac{1}{\mu-\frac{3}{2}\alpha}(e^{-\frac{3}{2}\alpha t}-e^{-\mu t})$$

and this shows the result, as $\alpha \leq \frac{\mu}{2}$.

3.4 Stability for the logarithmic Schrödinger equation

We now prove Theorem 3.2. The strategy follows the lines of [58] and uses a Birkhoff normal form reduction as in [9, 11]. As in the previous section, we can eliminate the zero mode and perform block diagonalization of the linear operator, but the preservation of the Hamiltonian structure is crucial. Indeed, the logarithmic Schrödinger equation is associated with the energy

$$H(\psi,\psi^*) := \|\nabla\psi\|_{L^2}^2 + \frac{\lambda}{(2\pi)^d}\int_{\mathbb{T}^d}|\psi(t,x)|^2\left(\log|\psi(t,x)|^2-1\right)dx, \tag{3.4.1}$$

which is preserved for all time $t \geq 0$, as equation (3.1.2) can be written

$$i\partial_t\psi = \frac{\partial H}{\partial\psi^*}(\psi,\psi^*) \quad \text{and hence} \quad \frac{\mathrm{d}}{\mathrm{d}t}H(\psi(t),\psi(t)^*) = 0.$$

3.4.1 Reduction to the case $m = 0$

Equation (3.1.2) satisfies the galilean invariance principle, which means that if ψ is a solution, then

$$\varphi(t,x) = \psi(t,x-vt)e^{-i\left(|v|^2t/2-v\cdot x\right)}$$

is also a solution for every $v \in \mathbb{Z}^d$. Using this property on the plane wave ν_m with $v=-m$, we get that

$$\nu_m(t,x+mt)e^{-i\left(|m|^2t+m\cdot x\right)} = \rho e^{-2i\lambda t\log\rho} = \nu_0(t,x),$$

which means that we can restrict our attention to the case $m=0$. Note that this transformation preserves the L^2-norm.

Another important feature of (3.1.2) is the effect of a scaling factor (cf [36]) : unlike what happens in the case of the usual power-like nonlinearity $\lambda\psi|\psi|^\gamma$ for $\gamma > 1$, if ψ is a solution to

(3.1.2), then

$$\kappa\psi(t,x)e^{-2it\lambda\log\kappa}$$

also solves (3.1.2) with initial datum $\kappa\psi^0$ (compare (3.3.1)). Keeping this property in mind, and using the conservation of the L^2 norm of any solution of (3.1.2), we will take in the following, for all $t \geq 0$,

$$\|\psi(t)\|_{L^2} = \|\psi^0\|_{L^2} = 1,$$

which means that we can consider only the case $\rho = 1$.

3.4.2 Elimination of the zero mode

We perform the same decomposition as in the previous section: we have

$$\psi = e^{i\theta}(a+w) \tag{3.4.2}$$

with $\langle w\rangle = 0$ and $a > 0$. The preservation of the L^2 norm guarantees that $a = \sqrt{1-\|w\|_{L^2}^2}$ as in (3.3.2). We obtain exactly the same equations for θ - (3.3.4) and w - (3.3.5), but with $\mu = 0$. The main point is to check that F is Hamiltonian, and that we can apply a Birkhoff normal procedure, *i.e.* that the frequencies are generically non resonant.

We decompose the solution on the usual Fourier basis $\psi(t,x) = \sum_{j\in\mathbb{Z}^d}\psi_j(t)e^{ij\cdot x}$. The Hamiltonian $H(\psi,\psi^*)$ (see (3.4.1)) can be viewed as a function of the coefficients ψ_j and ψ_j^*, and we have

$$i\dot{\psi}_j = \frac{\partial H}{\partial\psi_j^*}(\psi,\psi^*).$$

Now (3.4.2) is viewed as a change of variable for the Fourier modes: it defines a function $\psi \mapsto (a,\theta,w)$ with $w = (w_j)_{0\neq j\in\mathbb{Z}^d}$, $0 \leq a \in \mathbb{R}$ and $\theta \in \mathbb{R}$ defined by

$$\psi_0 = ae^{i\theta} \quad \text{and} \quad \psi_j = w_je^{i\theta} \quad \text{for} \quad j \in \mathcal{Z} := \mathbb{Z}^d\setminus\{0\}.$$

In these new variables, the Hamiltonian function can be written

$$H(\psi,\psi^*) = \mathcal{H}(a,w,w^*),$$

as the Hamiltonian is gauge invariant (*i.e.* invariant by the transformation $\psi \mapsto e^{i\theta}\psi$), however this transformation is not symplectic. As $\psi_0 = e^{i\theta}a$, we obtain the following collection of equations:

$$\left|\begin{aligned} &i\dot{a} - \dot{\theta}a = \frac{\partial\mathcal{H}}{\partial a}(a,w,w^*),\\ &i\dot{w}_j - \dot{\theta}w_j = \frac{\partial\mathcal{H}}{\partial w_j^*}(a,w,w^*), \quad j\neq 0.\end{aligned}\right.$$

As $a \in \mathbb{R}_+$, taking the real part of the first equation in the previous system shows that

$$\dot{\theta} = -\frac{1}{a}\frac{\partial \mathcal{H}}{\partial a}(a, w, w^*), \tag{3.4.3}$$

so θ is well controlled by a and w. Inserting this equation in the equation for w_j, we then get the equation on the j-th mode:

$$i\dot{w}_j = \frac{\partial \mathcal{H}}{\partial w_j^*}(a, w, w^*) - \frac{w_j}{a}\frac{\partial \mathcal{H}}{\partial a}(a, w, w^*). \tag{3.4.4}$$

Now as $a = \sqrt{1 - \|w\|_{L^2}^2}$ we also have

$$\frac{\partial a}{\partial w_j^*} = \frac{-w_j}{2a},$$

and therefore the equations of motion (3.4.4) for $w = (w_j)_{j\in\mathcal{Z}}$ are Hamiltonian, namely

$$i\dot{w}_j = \frac{\partial \hat{H}}{\partial w_j^*}(w, w^*), \quad j \in \mathcal{Z}, \quad \hat{H}(w, w^*) = \mathcal{H}(a, w, w^*), \tag{3.4.5}$$

with the real-valued Hamiltonian

$$\begin{aligned}\hat{H}(w, w^*) &= \mathcal{H}(a, w, w^*) \\ &= \|\nabla w\|_{L^2}^2 + \frac{\lambda}{(2\pi)^d}\int_{\mathbb{T}^d} |a + w|^2(\log(a + w) + \log(a + w^*) - 1)dx.\end{aligned}$$

As in [58], we can use the expansion (3.2.3) for the logarithm as well as the expansion

$$a = 1 + \sum_{m\geq 1} \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\ldots\left(\frac{1}{2} - m + 1\right)}{m!}\left(\sum_{j\neq 0}|w_j|^2\right)^m$$

and

$$\frac{1}{a} = 1 + \sum_{m\geq 1} \frac{-\frac{1}{2}\left(-\frac{1}{2} - 1\right)\ldots\left(-\frac{1}{2} - m + 1\right)}{m!}\left(\sum_{j\neq 0}|w_j|^2\right)^m,$$

to obtain a representation of the form

$$\hat{H}(w, \overline{w}) = \|\nabla w\|_{L^2}^2 + \lambda\|\psi\|_{L^2}^2 + \frac{\lambda}{2}\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}(w^2 + \overline{w}^2)\mathrm{d}x + \mathcal{P}(w, \bar{w}),$$

$$\text{with } \mathcal{P}(w, \overline{w}) = \sum_{r\geq 3}\mathcal{P}_r(w, \overline{w}),$$

where $\mathcal{P}_r(w,\overline{w})$ denotes a polynomial of degree r of the form

$$\mathcal{P}_r(w,\overline{w}) = \sum_{p+q=r} \sum_{\substack{(j,\ell)\in\mathcal{Z}^p\times\mathcal{Z}^q \\ \mathcal{M}(j,\ell)=0}} \mathcal{P}_{k,\ell} w_{j_1}\dots w_{j_p} \overline{w}_{\ell_1}\dots\overline{w}_{\ell_q},$$

and

$$\mathcal{M}(j,\ell) = j_1 + \ldots + j_p - \ell_1 - \ldots - \ell_q \tag{3.4.6}$$

denotes the momentum of the multi-index (k,ℓ), and that the Taylor expansion of the Hamiltonian $\mathcal{P}$ contains only terms with zero momentum. Moreover, this Hamiltonian function is smooth on $\mathcal{B}_s(r_0) := \{w \in H^s \mid \|w\|_{H^s} \le r_0\}$ for r_0 small enough owing to estimates of the form

$$|\mathcal{P}_{k,\ell}| \le M L^{p+q},$$

which holds for some positive constants M and L independent of k and l. This bound comes from the analyticity of the logarithm expansion (3.2.3) and of the function $w \mapsto \sqrt{1-\|w\|_{L^2}^2}$. The equation for w can thus be written (compare (3.3.5))

$$i\partial_t w = \frac{\partial \hat{H}}{\partial w^*}(w,w^*) = -\Delta w + \lambda w + \lambda w^* + \frac{\partial \mathcal{P}}{\partial w^*}(w,w^*),$$

or equivalently, for $j \in \mathcal{Z}$,

$$i\partial_t w_j = (|j|^2+\lambda)w_j + \lambda \overline{w}_{-j} + \frac{\partial \mathcal{P}}{\partial \overline{w}_j}(w,\overline{w}),$$

owing to the fact that $(w^*)_j = \overline{w}_{-j}$.

3.4.3 Diagonalization and non-resonant frequencies

The linear part in the differential equation (3.4.5) for w_j is $(|j|^2+\lambda)w_j + \lambda\overline{w}_{-j}$. By using the same strategy as in the previous section, taking the equation for w_j together with that for $\overline{w}_{-j}$, we are thus led to consider the matrix (for $n = |j|^2 \ge 1$)

$$A_n = \begin{pmatrix} n+\lambda & \lambda \\ -\lambda & -n-\lambda \end{pmatrix}.$$

Hence we have the eigenvalues $\pm\sqrt{n^2+2\lambda n}$ which are all real if we assume $\lambda > -\frac{1}{2}$, which will be the case from now on.

Lemma 3.3. *Let $\lambda > -1/2$. Then, for all $n \ge 1$, the matrix A_n is diagonalized by a 2×2*

matrix S_n that is real symplectic and hermitian and has condition number smaller than 2:

$$S_n^{-1} A_n S_n = \begin{pmatrix} \Omega_n & 0 \\ 0 & -\Omega_n \end{pmatrix} \quad \textit{with} \quad \Omega_n = \sqrt{n^2 + 2\lambda n}.$$

Proof. By direct diagonalization we have

$$S_n = \frac{1}{\sqrt{(n+\Omega_n)(n+2\lambda+\Omega_n)}} \begin{pmatrix} n+\lambda+\Omega_n & -\lambda \\ -\lambda & n+\lambda+\Omega_n \end{pmatrix}$$

and

$$S_n^{-1} = \frac{1}{\sqrt{(n+\Omega_n)(n+2\lambda+\Omega_n)}} \begin{pmatrix} n+\lambda+\Omega_n & \lambda \\ \lambda & n+\lambda+\Omega_n \end{pmatrix},$$

which are indeed real and self-adjoint. Denoting

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we can also easily check that S_n and S_n^{-1} are in fact symplectic, namely

$$S_n^T J S_n = J \quad \text{and} \quad \left(S_n^{-1}\right)^T J S_n^{-1} = J.$$

□

Using the real symplecticity of the matrix S_n^{-1} of Lemma 3.3, we make the symplectic change of variables

$$\begin{pmatrix} \xi_j \\ \overline{\xi}_{-j} \end{pmatrix} = S_n^{-1} \begin{pmatrix} w_j \\ \overline{w}_{-j} \end{pmatrix}$$

for $j \in \mathbb{Z}^d \setminus \{0\}$ and $n = |j|^2 \geq 1$. This linear transformation applied to the Hamiltonian system (3.4.5) gives the new Hamiltonian system

$$i \frac{\mathrm{d}}{\mathrm{d}t} \left(\xi_j(t)\right) = \frac{\partial \tilde{H}}{\partial \overline{\xi}_j} \left(\xi(t), \overline{\xi}(t)\right) \tag{3.4.7}$$

and define a new Hamiltonian $\tilde{H}(\xi, \overline{\xi}) = H(w, \overline{w})$ which can be written under the form

$$\tilde{H}(\xi, \overline{\xi}) = H_0 + P = \sum_{j \neq 0} \omega_j |\xi_j|^2 + P(\xi, \overline{\xi}),$$

where we denote the frequencies $\omega_j = \Omega_n = \sqrt{n^2 + 2\lambda n}$ for $|j|^2 = n \geq 1$, and the non-quadratic term P is of the form

$$P(\xi, \bar{\xi}) = \sum_{p+q\geq 3} \sum_{\substack{(j,\ell)\in \mathcal{Z}^p\times\mathcal{Z}^q \\ \mathcal{M}(j,\ell)=0}} P_{k,\ell}\xi_{k_1}\ldots\xi_{k_p}\bar{\xi}_{\ell_1}\ldots\bar{\xi}_{\ell_q}, \tag{3.4.8}$$

where the sum is still only over multi-indices with zero momentum (3.4.6), since the transformation mixes only terms that give the same contribution to the momentum. From the properties of the original Hamiltonian $\mathcal{P}$ and the bound on the transformations S_n, we can state the following bound: for $k \in \mathcal{Z}^p$, $\ell \in \mathcal{Z}^q$, the coefficients in (3.4.8) are bounded by

$$|P_{k,\ell}| \leq ML^{p+q},$$

for some constants M and L independent of p and q.

Remark 3.4. The H^s-norm of the sequence ξ is equivalent to the Sobolev norm of the perturbation of the m-th plane-wave of Theorem 3.2 in the case $\rho = 1$ namely

$$c\|\xi\|_{H^s(\mathbb{T}^d)} \leq \|\psi(t,\cdot) - e^{i\theta}\|_{H^s(\mathbb{T}^d)} \leq C\|\xi\|_{H^s(\mathbb{T}^d)},$$

where c and C denote two positive constants depending on λ. In particular, under the assumptions of Theorem 3.2, the system (3.4.7) has small initial values whose H^s-norm is of order ε.

3.4.4 Non resonance condition and normal form

We are now going to prove that the frequencies $(\Omega_n)_n$ of our linearized system satisfy Bambusi's non-resonance inequality [9] for almost all values of λ:

Lemma 3.5. *Let $\lambda_- > -1/2$, $\lambda_+ > \max(\lambda_-, 1/2)$ and $r > 1$. There exists $\alpha = \alpha(r) > 0$ and a set of full Lebesgue measure $\Lambda \subset [\lambda_-, \lambda_+]$ such that for every $\lambda \in \Lambda$ there is a $\gamma > 0$ such that, for all integers p, q with $p+q \leq r$ and for all $m = (m_1, \ldots, m_p) \in \mathbb{N}^p$ and $n = (n_1, \ldots, n_q) \in \mathbb{N}^q$,*

$$|\Omega_{m_1} + \ldots + \Omega_{m_p} - \Omega_{n_1} - \ldots - \Omega_{n_q}| \geq \frac{\gamma}{\mu_3(m,n)^\alpha}, \tag{3.4.9}$$

except if the frequencies cancel pairwise. Here, $\mu_3(m,n)$ denotes the third-largest among the integers $m_1, \ldots, m_p$, $n_1, \ldots, n_q$.

Proof. The proof is exactly the same as in the cubic nonlinear Schrödinger case, and can be found in [58, Lemma 2.2]. Indeed, in this latter reference, the frequencies where under the form $\sqrt{n^4 + 2n^2\lambda_0\rho^2}$ where λ_0 was fixed and ρ the varying parameter. The condition (3.4.9) is thus a consequence of this result by replacing $\lambda_0\rho^2$ by the varying parameter λ in the proof of [58, Lemma 2.2]. Note that the arguments are similar to the one used in [9, 11, 116]. □

We are now in position to apply the normal form result, Theorem 7.2 of [71] (see also [11]). We denote the ball of radius $r > 0$:

$$\mathcal{O}_s(r) := \left\{(\xi, \bar{\xi}) \in \mathbb{C}^{\mathcal{Z}} \times \mathbb{C}^{\mathcal{Z}} \;\middle|\; \|\xi\|_{H^s} \leq r\right\},$$

where the Sobolev norm $\|\cdot\|_{H^s}$ can be written

$$\|\xi\|_{H^s} = \sum_{n\geq 1} n^s J_n(\xi, \bar{\xi}) \quad \text{with the super-actions} \quad J_n(\xi, \bar{\xi}) = \sum_{|j|^2=n} |\xi_j|^2.$$

For a given Hamiltonian $K \in \mathcal{C}^\infty(\mathcal{O}_s(r), \mathbb{C})$ satisfying $K(\xi, \bar{\xi}) \in \mathbb{R}$, we denote by $X_K(\xi, \bar{\xi})$ the Hamiltonian vector field

$$X_k(\xi, \bar{\xi})_j = \left(i\frac{\partial K}{\partial \xi_j}, -i\frac{\partial K}{\partial \bar{\xi}_j}\right), \quad j \in \mathcal{Z},$$

associated with the Poisson bracket

$$\{K, G\} = i \sum_{j\in\mathcal{Z}} \frac{\partial K}{\partial \xi_j}\frac{\partial G}{\partial \bar{\xi}_j} - \frac{\partial K}{\partial \bar{\xi}_j}\frac{\partial G}{\partial \xi_j}.$$

We can now apply Theorem 7.2 of [71], which can be expressed in our setting as follows:

Theorem 3.6. *Let λ be in the set Λ of full measure as given in Lemma 3.5 for some $N \geq 3$. There exists s_0 such that for any $s \geq s_0$ there exist two neighborhoods $\mathcal{U} = \mathcal{O}_s(r_0)$ and $\mathcal{V} = \mathcal{O}_s(r_1)$ of the origin and a canonical transformation $\tau : \mathcal{V} \to \mathcal{U}$ which puts $\tilde{H} = H_0 + P$ in normal form up to order N, i.e.,*

$$\tilde{H} \circ \tau = H_0 + Z + R,$$

where

- $H_0 = \sum_{j\in\mathcal{Z}} \omega_j |\xi_j|^2$,
- *Z is a polynomial of degree N which commutes with all the super-actions, namely $\{Z, J_n\} = 0$ for all $n \geq 1$,*
- *$R \in \mathcal{C}^\infty(\mathcal{V}, \mathbb{R})$ and $\|X_R(\xi, \bar{\xi})\|_{H^s} \leq C_s \|\xi\|^N_{H^s}$ for $\xi \in \mathcal{V}$,*
- *τ is close to the identity: $\|\tau(\xi, \bar{\xi}) - (\xi, \bar{\xi})\|_{H^s} \leq C_s \|\xi\|^2_{H^s}$ for all $\xi \in \mathcal{V}$.*

The verification of the hypotheses for our Hamiltonian $\tilde{H}$ is pretty standard and can be performed as in [58]. The consequence of this Theorem are well known from [71, 11]: there exists ε_0 such that if $\|\xi(0)\|_{H^s} \leq \varepsilon < \varepsilon_0$, then we have $\|\xi(t)\|_{H^s} \leq 2\varepsilon$ for a time $t \leq \varepsilon^{-N}$. As a consequence, the same result holds true for w solution of the system (3.4.5) up to changes of the constants. This implies that $|a - 1| = \mathcal{O}\left(\|w\|^2_{L^2}\right) \leq C\varepsilon^2$ and $|\dot{\theta}| = \mathcal{O}\left(\|w\|^2_{L^2}\right) \leq C\varepsilon^2$ for

$t \leq \varepsilon^{-N}$, and we can then conclude the proof owing to Remark 3.4. Note that for a fixed N, we need only a finite number of iterations in the normal form construction, and thus the use of the estimates in Lemma 3.5 for a finite number of r. Typically, the r needed is linear in N, see [11] for details.

3.5 Numerical simulations

In this section we will perform a semi-discretization in time with a Lie-Trotter splitting method of the nonlinear Schrödinger-Langevin equation (3.1.1). The operator splitting methods for the time integration of (3.1.1) are based on the following splitting

$$\partial_t \psi = A(\psi) + B(\psi) + C(\psi),$$

where

$$A(\psi) = i\Delta\psi, \quad B(\psi) = -i\lambda\psi \log(|\psi|^2), \quad C(\psi) = -\frac{1}{2}\mu\psi \log\left(\frac{\psi}{\psi^*}\right),$$

and the solutions of the subproblems

$$i\partial_t u(t,x) = -\Delta u(t,x), \quad u(0,x) = u_0(x), \quad x \in \mathbb{T}^d, \quad t > 0,$$

$$i\partial_t v(t,x) = \lambda v(t,x) \log\left(|v(t,x)|^2\right), \quad v(0,x) = v_0(x), \quad x \in \mathbb{T}^d, \quad t > 0,$$

$$i\partial_t w(t,x) = \frac{\mu}{2i} w(t,x) \log\left(\frac{w(t,x)}{w^*(t,x)}\right), \quad w(0,x) = w_0(x), \quad x \in \mathbb{T}^d, \quad t > 0.$$

The associated operators are explicitly given, for $t \geq 0$, by

$$u(t,\cdot) = \Phi_A^t(u_0) = e^{it\Delta} u_0,$$

$$v(t,\cdot) = \Phi_B^t(v_0) = v_0 e^{-it\log(|v_0|^2)},$$

$$w(t,\cdot) = \Phi_C^t(w_0) = a_0 e^{i\theta_0 e^{-\mu t}}, \ w_0 = a_0 e^{i\theta_0}.$$

Note that in our simulations the initial functions will be some small perturbations of plane waves and should stay away from zero over large times as induced by Theorem 3.2 and Theorem 3.1, so we do not need to saturate the logarithm nonlinearity by an ε-approximation $\psi \mapsto \psi^\varepsilon \log\left(|\psi^\varepsilon|^2 + \varepsilon\right)$ as performed in [13] or [14].

All the numerical simulations will be made in dimension $d = 1$ on the torus $\mathbb{T} = [-\pi, \pi]$. The computation of Φ_A^t is made by a Fast Fourier Transformation. Let K be a positive even integer and denote $\Delta x = 2\pi/K$ and the grid points $x_j = -\pi + k\Delta x$ for $0 \leq k \leq K-1$. Denote by $\psi^{K,j}$

the discretized solution vector over the grid $(x_k)_{0\leq k\leq K-1}$ at time $t = t_j = j\Delta t$, which can be written with the discrete Fourier ansatz

$$\psi^{K,j}(n) = \sum_{k=0}^{K-1} \psi_k^{K,j} e^{in\cdot x_k}, \quad 0 \leq n \leq K-1.$$

Let $\mathcal{F}_K$ and $\mathcal{F}_K^{-1}$ denote the discrete Fourier transform and its inverse, respectively. With this notation, $\Phi_A^t(\psi^{K,j})$ can be obtained by

$$\Phi_A^t(\psi^{K,j}) = \mathcal{F}_K^{-1}\left(e^{-i\Delta t(\sigma^K)^2}\mathcal{F}_K(\psi^{K,j})\right),$$

where

$$\sigma^K = \left[0, 1, \ldots, \left(\frac{K}{2}-1\right), -\frac{K}{2}, \ldots, -1\right],$$

and the multiplication of two vectors is taken as point-wise. In the following we will both plot the dynamics of the absolute value of the solution $\left(|\psi^{K,j}|\right)_j$ and the evolution of the actions $\left(|\psi_k^{K,j}|\right)_j$ for $0 \leq k \leq K-1$ over the discrete time $\{0, \ldots, T_{\max}\}$ with $T_{\max} = J\Delta t$ and $J \in \mathbb{N}^*$, which corresponds to the number of discretization points of time.

In the following we perform our simulations with $K = 2^7$ space discretization points on the interval $\mathbb{T} = [-\pi, \pi]$, using a time step $\Delta t = 10^{-2}$ on the interval $[0, T_{\max}]$ with $T_{\max} = 100$. We also take the initial function

$$\psi_0(x) = \frac{1}{1 + 0.2\cos(x)}.$$

3.5.1 Evolution of the solution

Here we plot the absolute value of the solution $\left(|\psi^{K,j}|\right)_{0\leq j\leq J}$. We take the constant $\lambda = 0.5$ and we perform two simulations of $u(t,x) = |\psi(t,x)|^2$ with μ either equal to 0 or 2. In the case without dissipation ($\mu = 0$), we clearly observe the stability of the solution (Figure 3.1), whereas in the case with dissipation ($\mu = 2$) we see that the solution converges quickly to its limit (Figure 3.2). We also get in the second case a numerical exponential decreasing rate in H^s norm of $\alpha_{num} = 1.0002$ for the function

$$\psi - \rho e^{-2i\lambda \log\rho/\mu}$$

with $\rho = \|\psi^0\|_{L^2}$, which well corroborates the theoretical damping rate $\alpha_{theo} = \mu/2 = 1$ of Theorem 3.1.

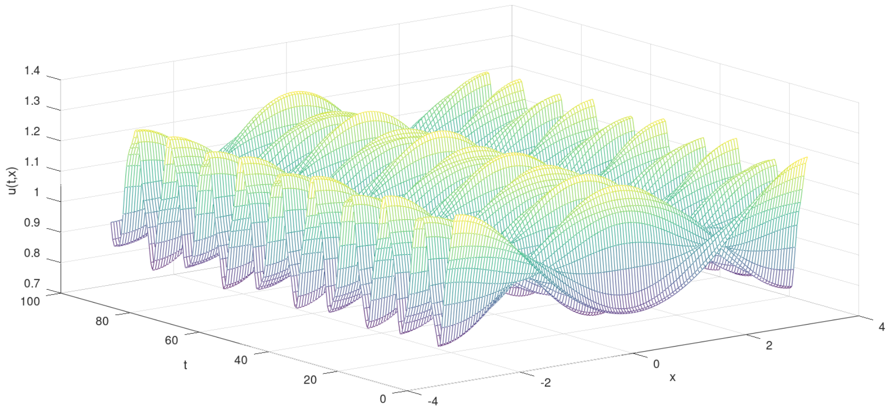

Figure 3.1 – Solution of equation (3.1.1) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 0$).

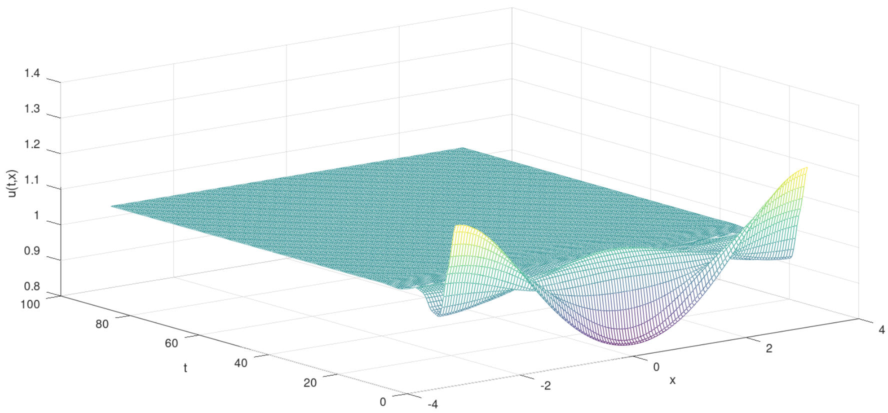

Figure 3.2 – Solution of equation (3.1.1) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 2$).

3.5.2 Evolution of the actions

We now plot the evolution of the actions $\left(|\psi_k^{K,j}|\right)_j$ with $0 \leq k \leq K-1$, in logarithmic scale, with $T_{\max} = 20$. We will first take the constant $\lambda = 0.5$ and μ equal to 0 (Figure 3.3), 2 (Figure 3.4) and 4 (Figure 3.5) in order to corroborate the state of Theorem 3.2 and Theorem 3.1. We then take a focusing constant $\lambda = -0.5$ with $\mu = 2$ (Figure 3.6).

In Figure 3.3 we well observe the near-conservation of the actions of our stable solution. In Figure 3.4, all the actions drop to zero at the same exponential rate, except the mode 0 which stays constant, as described in the proof of Theorem 3.1 (in fact, in this case $\mu = 2 < 2\sqrt{2} = 2\sqrt{1+2\lambda}$). The actions still decay to zero in Figure 3.5, and we observe that the first action well numerically decreases at the rate $-0.5858 \approx 2 - \sqrt{2} = \mu/2 - \sqrt{\mu^2/4 - 1 - 2 * \lambda} =: \alpha$, which fall under the scope of point (iii) of Theorem 3.1. However, other modes decay faster: the second mode has a numerical exponential rate of $-1.1743 \approx 2\alpha$, and all the higher modes decay at the same rate of $\mu/2$. For a general damping $\mu > 0$ we can conjecture that the j-th mode exponentially decreases at the rate

$$\min\left(\alpha_j, \sum_{\substack{k_1 \pm \ldots \pm k_j = j \\ 0 \leq k_1, \ldots, k_j < j}} \alpha_{k_j}\right).$$

This phenomenon reflects the possible nonlinear effects affecting the evolution of each mode: for instance, in the case of $\mu = 4$, the quadratic remainder Q_2 in the equation of second mode

$$i\partial_t \begin{pmatrix} w_2 \\ \overline{w}_{-2} \end{pmatrix} = P_2 \begin{pmatrix} \frac{\mu}{2i} - \delta_2 & 0 \\ 0 & \frac{\mu}{2i} + \delta_2 \end{pmatrix} P_2^{-1} \begin{pmatrix} w_2 \\ \overline{w}_{-2} \end{pmatrix} + Q_2(w, \overline{w}),$$

possibly contains the product of two first-order modes $w_1 \overline{w}_{-1}$, which contributes to a smaller decreasing rate $2\alpha_1 = 4 - 2\sqrt{2}$ than the linear one $\alpha_2 = \mu/2 = 2$. However, a rigorous proof of this statement would require some precise understanding of the dynamics of each individual mode through their nonlinear interactions which goes way beyond the analysis covered in this paper.

Finally, the case of Figure 3.6 which is not covered by our analysis ($\lambda = -0.5$) is way more exotic: the zero and first mode are conserved, and it also becomes true for the other modes after a transitory state where the actions decrease exponentially fast.

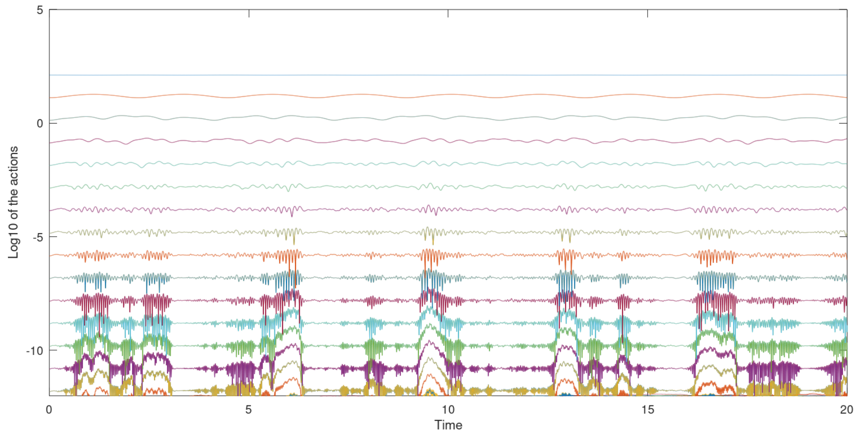

Figure 3.3 – Evolution of the actions of solution of equation (3.1.1) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 0$).

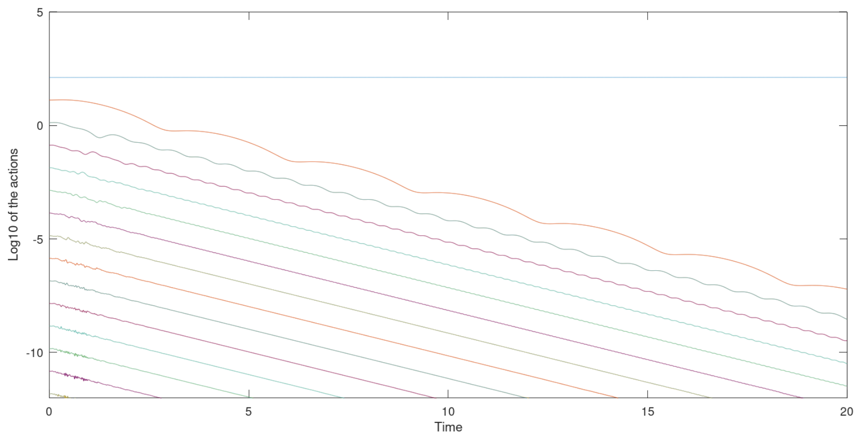

Figure 3.4 – Evolution of the actions of solution of equation (3.1.1) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 2$).

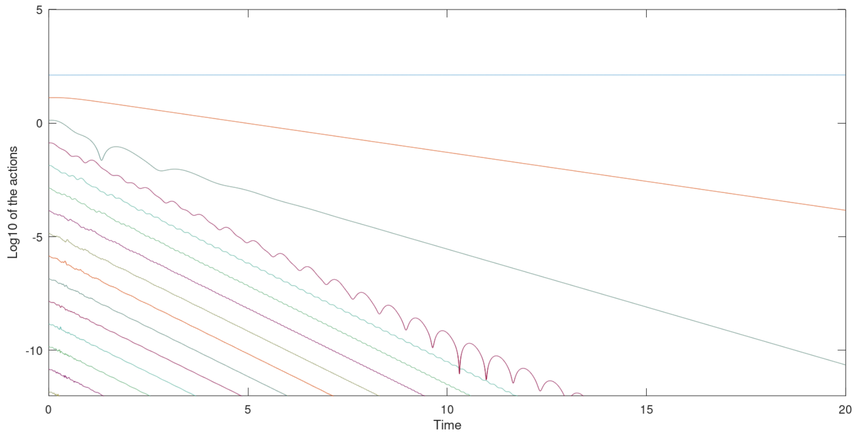

Figure 3.5 – Evolution of the actions of solution of equation (3.1.1) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 4$).

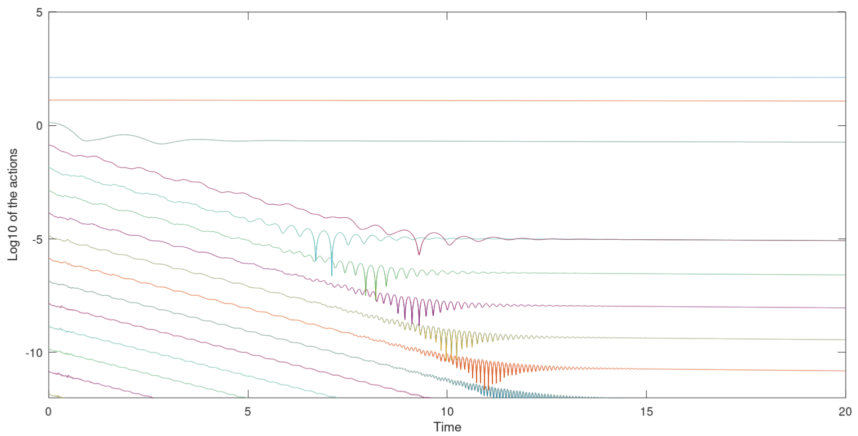

Figure 3.6 – Evolution of the actions of solution of equation (3.1.1) with initial datum ψ_0 ($\lambda = -0.5$, $\mu = 2$).

Chapitre 4

Global dissipative solutions of the defocusing isothermal Euler-Langevin-Korteweg equations

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Abstract: We construct global dissipative solutions on the torus of dimension at most three of the defocusing isothermal Euler-Langevin-Korteweg system, which corresponds to the Euler-Korteweg system of compressible quantum fluids with an isothermal pressure law and a linear drag term with respect to the velocity. In particular, the isothermal feature prevents the energy and the BD-entropy from being positive. Adapting standard approximation arguments we first show the existence of global weak solutions to the defocusing isothermal Navier-Stokes-Langevin-Korteweg system. Introducing a relative entropy function satisfying a Gronwall-type inequality we then perform the inviscid limit to obtain the existence of dissipative solutions of the Euler-Langevin-Korteweg system.

4.1 Introduction

The aim of this paper is to give a proper notion of solution and to prove the global existence of such solutions of the isothermal Euler-Langevin-Korteweg system of equations (denoted ELK in the following for reader convenience):

$$
\begin{cases}
\partial_t\rho + \operatorname{div}(\rho u) = 0, & \text{(4.1.1a)}\\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \lambda\nabla\rho + \mu\rho u = \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), & \text{(4.1.1b)}
\end{cases}
$$

where $t \in [0,T[$ for a fixed $T > 0$. The unknown functions are the density $\rho : [0,T[\times\mathbb{T}^d \to \mathbb{R}_+$ and the velocity field $u : [0,T[\times\mathbb{T}^d \to \mathbb{R}^d$ of the fluid, with the initial condition $\rho(0,x) = \rho_0(x)$ and $u(0,x) = u_0(x)$ for all $x \in \mathbb{T}^d$, where $\mathbb{T}^d$ denotes the d-dimensional torus with $1 \leq d \leq 3$. We also denote $\lambda > 0$ the pressure constant, $\mu > 0$ the dissipation constant and $\hbar > 0$ the renormalized Planck constant. Using the Madelung transform $\psi = \sqrt{\rho}e^{iS/\varepsilon}$, or in a more rigorous way the change of unknown $\rho = |\psi|^2$ and $\rho u = \hbar\,\mathrm{Im}(\psi^*\nabla\psi)$, this system is directly linked with the

Schrödinger-Langevin equation:

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{\hbar}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right). \tag{4.1.2}$$

This equation first appears in Nassar's paper [104] as a possible way to give a stochastic interpretation of quantum mechanics in the context of Bohmian mechanics. It had a recent renewed interest in the physics community, in particular in quantum mechanics in order to describe the continuous measurement of the position of a quantum particle (see for example [103], [118] or [102]) and in cosmology and statistical mechanics (see [45], [46] or [47]). Note that in its physical interpretation, $\lambda = 2k_B\tau/\hbar$ corresponds to a quantum friction coefficient, so both positive and negative signs could be of interest (k_B and $\hbar$ denotes respectively the Boltzmann and the normalized Planck constant, and τ is an effective temperature), unlike the real friction coefficient μ which is taken positive (see [45]). In the mathematics community, this equation has not seen much interest yet, despite the presence of unusual nonlinear effects. In [41], the author has shown that in the focusing case ($\lambda < 0$) every well-prepared density function of the solution of equation (4.1.2) on $\mathbb{R}^d$ converges to a Gaussian function weakly in $L^1(\mathbb{R}^d)$, whose mass and center are uniquely determined by its initial data ρ_0 and u_0, whereas in the defocusing case ($\lambda > 0$) every solution disperses to 0, with a slower dispersion rate than usual directly affected by the nonlinear Langevin potential $\frac{1}{2i}\log(\psi/\psi^*)$. Still in the defocusing case, up to a space-time rescaling incorporating dispersive effects, the density ρ of every rescaled solution of equation (4.1.2) on $\mathbb{R}^d$ also converges to a Gaussian function weakly in $L^1(\mathbb{R}^d)$, a phenomenon which is reminiscent of the defocusing logarithmic Schrödinger equation [36] (corresponding to the case $\mu = 0$ in (4.1.2)).

The question of the existence of solutions to this kind of quantum system is already dealt with in the case of barotropic pressure of the form $P(\rho) = \lambda\rho^\gamma$ in [4], where $\gamma > 1$ (and $\mu = 0$). However, the proof is based on the link with the power-like Schrödinger equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = \lambda\psi|\psi|^{\gamma-1} \tag{4.1.3}$$

and the use of Strichartz estimates which do not seem to be helpful for the logarithmic nonlinearity. In fact, rather than estimate $\nabla(\psi|\psi^{\gamma-1}|)$, we have to deal with the quantity $\nabla(\psi\log(|\psi|^2))$ which becomes unbounded when the wave function ψ vanishes to 0, preventing us from applying the fractional steps method of [4]. In the second part of [34], the authors have shown the existence of solutions to the isothermal Euler-Korteweg system (which corresponds to the case $\mu = 0$ in the system (4.1.1)), but again their proof is strongly based on the link with the logarithmic Schrödinger equation, which does not seem to be useful in our case due to the ill-posed nonlinear Langevin potential $\frac{1}{2i}\log(\psi/\psi^*)$. In a more regular framework, note that in the recent work [62], the author shows the local existence in time of solutions to the Euler-Korteweg system satisfying some analytic regularity, which may be extended to the Euler-Langevin-Korteweg equations.

Unlike the Navier-Stokes-Korteweg system of equations (which corresponds to system (4.1.1) adding the viscous term $\nu \mathrm{div}\,(\rho \mathbb{D}u)$ on the right hand side of (4.1.1b), see [84] or the system (4.2.1) below), the lack of viscous term in the Euler-Korteweg system prevents us from adding some smoothing terms in the equations and then make these terms tend to 0. However, in the recent paper [27], the authors manage to pass to the viscous limit $\nu \to 0$ for a specific notion of weak solution called dissipative solution. We will base our proof of existence on this approach, with the specificity of taking an isothermal pressure law $\lambda\rho$ (which leads to the use of energy with no definite sign, see below) and adding a dissipation term $\mu\rho u$. Note that in the recent paper [28] the authors have shown the exponential decay to equilibrium of global weak solutions of the Navier-Stokes-Korteweg system with such a dissipation term and for both barotropic and isothermal pressure laws.

Dissipative solutions were introduced by Lions [96], and were recently used in fluid models in order to give a rigorous justification of some viscous singular limits (see [60]). They are based on the use of some particular functionals called *relative entropies*, namely

$$\mathcal{E}(U|V) : X \times Y \mapsto \mathbb{R}_+,$$

where X denotes a Banach space of *weak solutions* U, and $Y \subset X$ denotes a Banach space of *strong solutions* V embedded into our space of weak solutions of the fluid system

$$\frac{d}{dt}U(t) = \mathcal{A}(t, U(t)),\ t > 0,\ U(0) = U_0,$$

where $\mathcal{A}$ is a (nonlinear) generator. A relative entropy has to enjoy the following three properties:

- **Distance property.** For every $(U, V) \in X \times Y$,

$$\mathcal{E}(U|V) \geq 0 \quad \text{and} \quad \mathcal{E}(U|V) = 0 \text{ only if } U = V.$$

- **Lyapounov functional.** If V is an *equilibrium solution*, namely $\mathcal{A}(t, V(t)) = 0$ for all $t \geq 0$, then $V \in Y$ and

$$\frac{d}{dt}\mathcal{E}(U|V) \leq 0.$$

- **Gronwall inequality.** For every $(U, V) \in X \times Y$, for a.a. $t \geq 0$,

$$\mathcal{E}(U|V)(t) - \mathcal{E}(U|V)(0) \leq \int_0^t \mathcal{E}(U|V)(s)ds.$$

The presence of the nonlinear quantum viscous term $\hbar^2\rho\nabla\left(\Delta\sqrt{\rho}/\sqrt{\rho}\right)$, usually called Bohm potential, makes difficult to define a relative entropy for the ELK system (4.1.1). In order to circumvent this difficulty, we denote $\overline{v} = \frac{\hbar}{2}\nabla\log\rho$ and introduce the augmented ELK system:

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho u) = 0, & (4.1.4a)\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \lambda\nabla\rho + \mu\rho u = \dfrac{\hbar}{2}\operatorname{div}(\rho\nabla\overline{v}), & (4.1.4b)\\ \partial_t(\rho\overline{v}) + \operatorname{div}(\rho\overline{v}\otimes u) + \dfrac{\hbar}{2}\operatorname{div}\left(\rho\nabla u^\top\right) = 0. & (4.1.4c)\end{cases}$$

Note that this augmented formulation wasz introduced in [21] as a numerical tool in order to compute quantum fluid systems, and is based on the identity

$$\frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{\hbar^2}{4}\operatorname{div}(\rho\nabla^2\log\rho).$$

Formally differentiating in space equation (4.1.1a), we see that a solution of the ELK system (4.1.1) also stands as a solution of (4.1.4) (see Remark 4.2 below). We also define the functional $H(\rho) = \rho\log\rho - \rho$, so we can introduce the following relative entropy functional for our system:

$$\mathcal{E}_{ELK}(\rho,u,\overline{v}|R,U,\overline{V})(t) = \frac{1}{2}\int_{\mathbb{T}^d}\rho(|\overline{v}-\overline{V}|^2 + |u-U|^2) + \lambda\int_{\mathbb{T}^d}H(\rho|R) + \mu\int_0^t\int_{\mathbb{T}^d}\rho|u-U|^2,$$

where

$$H(\rho|R) := H(\rho) - H(R) - H'(R)(\rho - R).$$

Note that here $(\rho,u,\overline{v})$ stands as a weak solution of (4.1.4), where $(R,U,\overline{V})$ has to be seen as a strong solution of this system. We will also use the convention

$$\mathcal{E}_{ELK}(\rho,u,\overline{v}) := \mathcal{E}_{ELK}(\rho,u,\overline{v}|1,0,0).$$

We now introduce the concept of dissipative solution, which is induced by the Gronwall inequality of our relative entropy functional as follows:

Definition 4.1. *Let $T>0$ and $(\rho_0,u_0,\overline{v}_0 = \frac{\hbar}{2}\nabla\log\rho_0)$ such that $\mathcal{E}_{ELK}(\rho_0,u_0,\overline{v}_0) < \infty$. We say that $(\rho,u,\overline{v})$ is a **dissipative solution** of the augmented ELK system (4.1.4) in $[0,T[\times\mathbb{T}^d$ with initial data $(\rho_0,u_0,\overline{v}_0)$ if there exists locally integrable functions $\sqrt{\rho}$, $\sqrt{\rho}u$ and $\sqrt{\rho}v$ such that, by defining $\rho = \sqrt{\rho}^2$, $\rho u = \sqrt{\rho}\sqrt{\rho}u$ and $\rho v = \sqrt{\rho}\sqrt{\rho}v$, the following holds:*

- *The global regularity:*

$$\rho\in L^\infty([0,T[\,;L^1(\mathbb{T}^d)),\ \ \rho\geq 0\ \ a.a.\ \ in\,[0,T[\,\times\mathbb{T}^d,$$

$$\sqrt{\rho}u \text{ and } \sqrt{\rho}v \in L^\infty([0,T[\,; L^2(\mathbb{T}^d)).$$

- *The mass conservation: for a.e.* $t \in [0,T[$,

$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx. \tag{4.1.5}$$

- *The relative entropy:*

$$\mathcal{E}_{ELK}(\rho,u,\overline{v}|R,U,\overline{V})(t) \leq \mathcal{E}_{ELK}(\rho,u,\overline{v}|R,U,\overline{V})(0)e^{Ct} + b_{ELK}(t) + C\int_0^t b_{ELK}(s)e^{C(t-s)}ds,$$

for a.e. $t \in [0,T[$, *where*

$$b_{ELK}(t) = \int_0^t \int_{\mathbb{T}^d} \frac{\rho}{R}|\mathscr{E}(R,U)\cdot(U-u)|$$

for all smooth function U *and* $(R,\overline{V},\mathscr{E})$ *defined respectively through the initial condition*

$$\int_{\mathbb{T}^d} R_0 = \int_{\mathbb{T}^d} \rho_0,$$

and the system

$$\partial_t R + \operatorname{div}(RU) = 0, \tag{4.1.6a}$$

$$\overline{V} = \frac{\hbar}{2}\nabla \log R, \tag{4.1.6b}$$

$$\mathscr{E}(R,U) = R(\partial_t U + U\cdot\nabla U) + \lambda\nabla R + \mu RU - \frac{\hbar}{2}\operatorname{div}(R\nabla\overline{V}), \tag{4.1.6c}$$

and $C = C(\hbar,R,U,\overline{V})$ *is a constant uniformly bounded on* $\mathbb{R}_+\times\mathbb{T}^d$.

Remark 4.2. As we are unable to have the existence of strong solutions to the system (4.1.4), we have to introduce the function $\mathscr{E}(R,U)$ which has to be understood as an error function. In fact, if $\mathscr{E}(R,U)=0$ then we recover a strong solution of (4.1.4) through the system (4.1.6) by differentiating in space the mass equation (4.1.6a), which gives

$$\partial_t\nabla R + \nabla\operatorname{div}(RU) = \partial_t\nabla R + \operatorname{div}(\nabla(RU)^\top) = 0,$$

that could be written

$$\partial_t\nabla R + \operatorname{div}(R\nabla\log R\otimes U) + \operatorname{div}(\nabla(RU)^\top - R\nabla\log R\otimes U) = 0$$

in order to show that $\overline{V}$ fulfills equation (4.1.4c). Of course, we have to take into account this error term in the Gronwall inequality of Definition 4.1 through the function b_{ELK}, whose expression will make sense with the calculations of the proof of Proposition 4.30.

Remark 4.3. We have to state at this stage that the positivity of our entropy functional $\mathcal{E}_{ELK}$ is unclear due to the presence of the logarithmic function in the isothermal contribution $\lambda \int_{\mathbb{T}^d} H(\rho|R)$. In order to get positivity, we have to invoke the Csiszár-Kullback inequality (see e.g. [3]) which tells that for $f, g \geq 0$ such that $\int f = \int g$, we have

$$\|f-g\|_{L^1(\mathbb{T}^d)}^2 \leq 2\|f\|_{L^1(\mathbb{T}^d)} \int_{\mathbb{T}^d} f(x) \log\left(\frac{f(x)}{g(x)}\right) dx.$$

Recalling that $\int r_0 = \int \rho_0$ and using equation (4.1.5) that ensures that for a.a. $t \in [0, T[$, $\|\rho(t)\|_{L^1(\mathbb{T}^d)} = \|R(t)\|_{L^1(\mathbb{T}^d)}$, we get that

$$\int_{\mathbb{T}^d} H(\rho|R) = \int_{\mathbb{T}^d} (\rho \log \rho - \rho - (R \log R - R) - \log R(\rho - R)) = \int_{\mathbb{T}^d} \rho \log\left(\frac{\rho}{R}\right) \geq 0.$$

Under the assumption $\lambda > 0$, we then recover the positivity of $\mathcal{E}_{ELK}$, which will be crucial in the proof of Theorem 4.31. Note that in the focusing case $\lambda < 0$, Csiszár-Kullback inequality does not ensure positivity on our relative entropy anymore, hence we cannot define a proper notion of dissipative solution in the focusing case and apply the results from Section 3. However, all the results from Section 2 are still valid in the case $\lambda < 0$ (see Remark 4.12 below).

Remark 4.4. As pointed out by Feireisl [59], the mass conservation property of a function $\rho \in L^\infty([0, T[\,; L^1(\mathbb{R}^d))$ satisfying the continuity equation (4.1.4a) in the distributional sense is a direct consequence of the Lebesgue's dominated convergence theorem, taking mollifiers as test functions. However, this condition is not fulfilled in the case of dissipative solutions, which justify the second point of Definition 4.1.

We can now state the main result of this paper:

Theorem 4.5. ***(Global existence for the augmented isothermal ELK).***
Let $T > 0$ and $(\rho_0, u_0, \overline{v}_0 = \frac{\hbar}{2}\nabla \log \rho_0)$ smooth enough such that $\rho_0 \in L^1(\mathbb{T}^d)$, $\sqrt{\rho_0}u_0$, $\sqrt{\rho_0}\overline{v}_0 \in L^2(\mathbb{T}^d)$ and $\mathcal{E}_{ELK}(\rho_0, u_0, \overline{v}_0) < \infty$, then there exists a dissipative solution of the augmented ELK system (4.1.4) *in* $[0, T[\, \times \mathbb{T}^d$ *with initial data* $(\rho_0, u_0, \overline{v}_0)$.

The rest of this paper is devoted to the proof of Theorem 4.5, and is organized as follows. In Section 2, we introduce several regularizing viscous terms in the ELK system which enable us to construct a solution of our regularized system. Letting these several viscosity terms one by one go to zero, we recover the existence of a weak solution to the Navier-Stokes-Langevin-Korteweg equation, which corresponds to the ELK system with an additional term $\nu \mathrm{div}\,(\rho \mathbb{D}u)$ in the momentum equation. Section 3 is then dedicated to the inviscid limit $\nu \to 0$, where we have to introduce augmented formulation for both ELK and NSLK systems.

4.2 The isothermal Navier-Stokes-Langevin-Korteweg system

We look at the Navier-Stokes-Langevin-Korteweg system (denoted NSLK in the following for reader convenience), namely:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & (4.2.1a) \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \lambda \nabla \rho + \mu \rho u = \dfrac{\hbar^2}{2} \rho \nabla \left(\dfrac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \nu \operatorname{div}(\rho \mathbb{D} u), & (4.2.1b) \end{cases}$$

where $\mathbb{D}u = (\nabla u + \nabla u^\top)/2$. This system formally enjoys the mass conservation

$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx$$

and the following energy estimate:

$$E_{NSLK}(\rho,u)(t) + \int_0^t D_{NSLK}(\rho,u)(s)ds \leq E_0, \tag{4.2.2}$$

with

$$E_{NSLK}(\rho,u) = \frac{1}{2} \int_{\mathbb{T}^d} \left(\rho |u|^2 + \hbar^2 |\nabla \sqrt{\rho}|^2 \right) + \lambda \int_{\mathbb{T}^d} H(\rho), \tag{4.2.3}$$

$$D_{NSLK}(\rho,u) = \mu \int_{\mathbb{T}^d} \rho |u|^2 + \nu \int_{\mathbb{T}^d} \rho |\mathbb{D}u|^2. \tag{4.2.4}$$

Remark 4.6. Similarly to Remark 4.3, in order to use (4.2.2) to get some regular bounds on our solutions, we need to show that E_{NSLK} is also bounded from below, which is not obvious due to the presence of the logarithmic term $\lambda \int_{\mathbb{T}^d} H(\rho)$. Denoting $E(t) + \lambda \|\rho_0\|_{L^1} = E^+(t) - E^-(t)$, with

$$E^+(t) = \int_{\mathbb{T}^d} \left(\frac{1}{2} \rho |u|^2 + \hbar^2 |\nabla \sqrt{\rho}|^2 \right) dx + \lambda \int_{\rho > 1} \rho \log \rho \geq 0,$$

$$E^-(t) = -\lambda \int_{\rho<1} \rho \log \rho = \lambda \int_{\rho<1} \rho \log \left(\frac{1}{\rho} \right) \geq 0,$$

we remark that $E^-(t)$ is controlled by

$$\int_{\rho<1} \rho \log \left(\frac{1}{\rho} \right) \lesssim \int_{\mathbb{T}^d} \rho^{1-\varepsilon} \leq C \|\rho_0\|_{L^1}^{1-\varepsilon}$$

by the Hölder inequality on the compact set $\mathbb{T}^d$ with $\varepsilon > 0$ small enough, hence $E^-(t)$ and $E^+(t) \in L^\infty$, so the energy E_{NSLK} is indeed bounded from below.

The BD-entropy of a quantum fluid system, firstly introduced in [25] and [22], is now a classical tool in order to get further regularity bounds on our solution. This system also enjoys a BD-entropy estimate:

$$\mathcal{E}_{NSLK}(\rho, u)(t) + \int_0^t \mathcal{D}_{NSLK}(\rho, u)(s)ds \leq \mathcal{E}_0, \tag{4.2.5}$$

with

$$\mathcal{E}_{NSLK}(\rho, u) = \frac{1}{2}\int_{\mathbb{T}^d} \left(\rho|u + \frac{\nu}{2}\nabla \log \rho|^2 + \hbar^2|\nabla\sqrt{\rho}|^2\right) + \lambda' \int_{\mathbb{T}^d} H(\rho), \tag{4.2.6}$$

$$\mathcal{D}_{NSLK}(\rho, u) = \mu \int_{\mathbb{T}^d} \rho|u|^2 + \frac{\nu}{2}\int_{\mathbb{T}^d} \rho|\mathbb{A}u|^2 + \frac{\nu\hbar^2}{2}\int_{\mathbb{T}^d} \rho|\nabla^2 \log \rho|^2 + 2\nu \int_{\mathbb{T}^d} |\nabla\sqrt{\rho}|^2, \tag{4.2.7}$$

where $\lambda' := \lambda - \mu\nu$ and $\mathbb{A}u = (\nabla u - \nabla u^\top)/2$.

Remark 4.7. Note that, as performed in [29] and [28], we could use a generalization of the BD-entropy called κ-entropy (originally introduced in [26]) in order to consider more general density dependent viscosities of the type $\mathrm{div}(\nu(\rho)\mathbb{D}u)$ with $\nu(\rho)$ not necessarily equal to $\nu\rho$. However, as this is not the main focus of our study, we prefer to stay in the more succinct Navier-Stokes framework.

We now give the notion of weak solution induced by these quantities:

Definition 4.8. *Let $T > 0$ and (ρ_0, u_0) such that $E_{NSLK}(\rho_0, u_0) < \infty$. We say that (ρ, u) is a* ***weak solution*** *of the NSLK system* (4.2.1) *in $[0, T[\, \times \mathbb{T}^d$ with initial data (ρ_0, u_0), if there exists locally integrable functions $\sqrt{\rho}$, $\sqrt{\rho}u$ such that, by defining $\rho := \sqrt{\rho}^2$ and $\rho u := \sqrt{\rho}\sqrt{\rho}u$, the following holds:*

- *The global regularity:*

$$\sqrt{\rho} \in L^\infty([0, T[\,; H^1(\mathbb{T}^d)), \quad \sqrt{\rho}u \in L^\infty([0, T[\,; L^2(\mathbb{T}^d)),$$

  with the compatibility condition

$$\sqrt{\rho} \geq 0 \text{ a.e. on } (0, \infty) \times \mathbb{T}^d, \quad \sqrt{\rho}u = 0 \text{ a.e. on } \{\rho = 0\}.$$

- *For any test function $\eta \in \mathcal{C}_0^\infty([0, T[\, \times \mathbb{T}^d)$,*

$$\int_0^T \int_{\mathbb{T}^d} (\rho\partial_t\eta + \rho u \cdot \nabla\eta)dxdt + \int_{\mathbb{T}^d} \rho_0\eta(0)dx = 0, \tag{4.2.8}$$

and for any test function $\zeta \in \mathcal{C}_0^\infty([0,T[\times \mathbb{T}^d;\mathbb{T}^d)$,

$$\int_0^T \int_{\mathbb{T}^d} \Big(\rho u \cdot \partial_t \zeta + \sqrt{\rho} u \otimes \sqrt{\rho} u : \nabla \zeta + \lambda \rho \mathrm{div}(\zeta) - \mu \rho u \cdot \zeta + \hbar^2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho} : \nabla \zeta \\ - \frac{\hbar^2}{4} \rho \Delta \mathrm{div} \zeta - \nu \sqrt{\rho} \mathbb{S}_N(u) \cdot \nabla \zeta \Big) dxdt + \int_{\mathbb{T}^d} \rho_0 u_0 \zeta(0) dx = 0, \quad (4.2.9)$$

with $\mathbb{S}_N(u) = (\mathbb{T}_N(u) + \mathbb{T}_N(u)^\top)/2$, where $\mathbb{T}_N(u)$ is defined through the compatibility condition

$$\sqrt{\rho}\ \mathbb{T}_N(u) = \nabla(\sqrt{\rho}\sqrt{\rho} u) - 2\sqrt{\rho} u \otimes \nabla \sqrt{\rho}.$$

- *For any test function $\xi \in \mathcal{C}_0^\infty(\mathbb{T}^d)$,*

$$\lim_{t\to 0} \int_{\mathbb{T}^d} \rho(t,x)\xi(x)dx = \int_{\mathbb{T}^d} \rho_0(x)\xi(x)dx,$$

$$\lim_{t\to 0} \int_{\mathbb{T}^d} \rho(t,x)u(t,x)\xi(x)dx = \int_{\mathbb{T}^d} \rho_0(x)u_0(x)\xi(x)dx.$$

Remark 4.9. The tensor valued function $\mathbb{T}_N(u)$ has to be understood as $\sqrt{\rho}\nabla u$. Of course, we can not define this term properly, so we use the algebraic identity given in the previous definition, which makes sense in the distribution sense in view of the regularity assumption.

Remark 4.10. Note that we have used in the momentum equation (4.2.9) the identity

$$\frac{\hbar^2}{2}\rho\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{\hbar^2}{4}\Delta\nabla\rho - \hbar^2 \mathrm{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

The aim of this section is now to prove the following theorem:

Theorem 4.11. ***(Global existence for the isothermal NSLK).***
Let $T > 0$, $\nu > 0$ and (ρ_0, u_0) such that $E_{NSLK}(\rho_0, u_0) < \infty$, then there exists a weak solution of system (4.2.1a)-(4.2.1b) *in $[0,T[\times\mathbb{T}^d$ with initial data (ρ_0, u_0). Furthermore, for almost every $t \in [0,T[$, equations* (4.2.2) *and* (4.2.5) *holds.*

Remark 4.12. Note that Theorem 4.11 still holds in the focusing case $\lambda < 0$. In fact, to adapt the proof, we just have to show that E_{NSLK} is bounded from below similarly to Remark 4.6. Writing $E_{NSLK}(t) = E^+(t) - E^-(t)$, where

$$E^+(t) = \int_{\mathbb{T}^d}\left(\frac{1}{2}\rho|u|^2 + \hbar^2|\nabla\sqrt{\rho}|^2 - \lambda\rho\right)dx + \lambda\int_{\rho<1}\rho\log\rho \geq 0,$$

$$E^-(t) = -\lambda\int_{\rho>1}\rho\log\rho \geq 0,$$

and using the conservation of mass $\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$, we get that

$$\int_{\rho>1} \rho \log \rho \lesssim \int_{\mathbb{T}^d} \sqrt{\rho}^{2+\varepsilon} \leq \|\sqrt{\rho}\|_{L^2}^{(2+\varepsilon)(1-\alpha)} \|\nabla\sqrt{\rho}\|_{L^2}^{(2+\varepsilon)\alpha} = C\|\nabla\sqrt{\rho}\|_{L^2}^{(2+\varepsilon)\alpha}$$

by the Gagliardo-Nirenberg inequality with $\alpha = 1/2 - 1/(2+\varepsilon)$, for $\varepsilon > 0$ small enough (see Lemma 2). Finally,

$$E^+(t) \leq E_0 + E^-(t) \leq E_0 + CE^+(t)^\delta$$

with $\delta < 1$, inducing that $E^+(t) \in L^\infty$ and so $E^-(t)$. Unfortunately, we are unable to define a positive relative entropy functional in the focusing case, so we can't make the viscous limit $\nu \to 0$ in this case.

4.2.1 Regularized NSLK system

We first give the following lemma, which is reminiscent of [23, Lemma 2.1], with a slightly different statement, see [34]:

Lemma 4.13. *For $n \in \mathbb{N}^*$, there holds*

$$\|\nabla^n(f^{-1})\|_{L^2(\mathbb{T}^d)} \lesssim (1 + \|f^{-1}\|_{L^4(\mathbb{T}^d)} + \|f^{-1}\|_{L^{2(n+1)}(\mathbb{T}^d)})^{n+1}(1 + \|f\|_{H^\sigma(\mathbb{T}^d)})^n$$

with $\sigma > n + d/2$.

Following [111], we first aim at proving the existence of a solution to the following regularized NSLK system:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = \delta_1\Delta\rho, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \lambda\nabla\rho + \mu\rho u + r_0 u + r_1\rho|u|^2 u = \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + \nu\operatorname{div}(\rho\mathbb{D}u) \\ + \delta_2\Delta^2 u - \delta_1(\nabla\rho\cdot\nabla)u + \eta_1\nabla\rho^{-\alpha} + \eta_2\rho\nabla\Delta^{2s+1}\rho, \end{cases}$$

(4.2.10a)

(4.2.10b)

where the regularization parameters verify $0 < \delta_1, \delta_2, \eta_1, \eta_2, r_0, r_1 < 1$ and $\alpha > 0$, $s \in \mathbb{N}^*$ are chosen sufficiently large (to be fixed later on). Integrating equation (4.2.10a) in space, we get the conservation of mass, namely for all $t \geq 0$,

$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx. \tag{4.2.11}$$

Then, multiplying formally equation (4.2.10b) with u and combining it with equation (4.2.10a) we get the energy estimate, for almost all $t \geq 0$:

$$E_{\text{reg}}(\rho, u)(t) + \int_0^t D_{\text{reg}}(\rho, u)(s)ds \leq E_0, \tag{4.2.12}$$

where

$$E_{\text{reg}}(\rho, u) = \frac{1}{2}\int_{\mathbb{T}^d} \left(\rho|u|^2 + \hbar^2|\nabla\sqrt{\rho}|^2\right) + \lambda\int_{\mathbb{T}^d} H(\rho) + \frac{\eta_1}{\alpha+1}\int_{\mathbb{T}^d} \rho^{-\alpha} + \frac{\eta_2}{2}\int_{\mathbb{T}^d} |\nabla\Delta^s\rho|^2, \tag{4.2.13}$$

$$\begin{aligned} D_{\text{reg}}(\rho, u) = \mu\int_{\mathbb{T}^d} \rho|u|^2 + \nu\int_{\mathbb{T}^d} \rho|\mathbb{D}u|^2 + \delta_2\int_{\mathbb{T}^d} |\Delta u|^2 + \delta_1\eta_2\int_{\mathbb{T}^d} |\Delta^{s+1}\rho|^2 \\ + 4\delta_1\int_{\mathbb{T}^d} |\nabla\sqrt{\rho}|^2 + \frac{4\delta_1\eta_1}{\alpha}\int_{\mathbb{T}^d} |\nabla\rho^{-\alpha/2}|^2 + r_0\int_{\mathbb{T}^d} |u|^2 + r_1\int_{\mathbb{T}^d} \rho|u|^4 + \frac{\delta_1\hbar^2}{2}\int_{\mathbb{T}^d} \rho|\nabla^2\log\rho|^2. \end{aligned} \tag{4.2.14}$$

Definition 4.14. *Let $T > 0$. We say that (ρ, u) is a **weak solution** of the regularized NSLK system (4.2.10) in $[0, T[\times \mathbb{T}^d$ with initial data $(\rho_0, u_0) \in L^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$, if the following holds:*

- *The global regularity:*

$$\rho \in H^1([0, T[\,; H^1(\mathbb{T}^d)) \cap \mathcal{C}^0([0, T[\,; H^{2s}(\mathbb{T}^d)) \cap L^2([0, t[\,; H^{2s+2}(\mathbb{T}^d)),$$

$$1/\rho \in \mathcal{C}^0([0, T[\times \mathbb{T}^d),$$

$$u \in L^\infty([0, T[\,; L^2(\mathbb{T}^d)) \cap L^2([0, T[\,; H^2(\mathbb{T}^d)).$$

- *For any test function $\eta \in \mathcal{C}_0^\infty([0, T[\times \mathbb{T}^d)$,*

$$\int_0^T \int_{\mathbb{T}^d} (\rho\partial_t\eta + \rho u \cdot \nabla\eta + \delta_1\rho\Delta\eta)dxdt + \int_{\mathbb{T}^d} \rho_0\eta(0)dx = 0, \tag{4.2.15}$$

  and for any test function $\zeta \in \mathcal{C}_0^\infty([0, T[\times \mathbb{T}^d; \mathbb{T}^d)$,

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \Big(\rho u \cdot \partial_t\zeta + \rho u \otimes u : \nabla\zeta + \lambda\rho\text{div}(\zeta) - \mu\rho u \cdot \zeta + \hbar^2\nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} : \nabla\zeta - \frac{\hbar^2}{4}\rho\Delta\text{div}\zeta \\ -\nu\rho\mathbb{D}u : \nabla\zeta - r_0 u \cdot \zeta - r_1\rho|u|^2 u \cdot \zeta - \delta_1\nabla u : \nabla\rho \otimes \zeta - \delta_2\Delta u \cdot \delta\zeta - \eta_1\rho^{-\alpha}\text{div}\zeta \\ -\eta_2\Delta^{s+1}\rho\Delta^s\left[\nabla\rho \cdot \zeta + \rho\text{div}\zeta\right]\Big)\,dxdt + \int_{\mathbb{T}^d} \rho_0 u_0\zeta(0)dx = 0. \end{aligned} \tag{4.2.16}$$

Proposition 4.15. *Let $T > 0$ and*

$$\rho_0 \in \mathcal{C}_0^\infty(\mathbb{T}^d), \quad u_0 \in L^2(\mathbb{T}^d), \quad \inf_{x \in \mathbb{T}^d} \rho_0(x) \geq \theta > 0, \tag{4.2.17}$$

then there exists a weak solution (ρ, u) *of system 4.2.10 in* $[0,T[\times\mathbb{T}^d$ *with initial data* (ρ_0, u_0) *which satisfies moreover the conservation of mass* (4.2.11) *and the energy estimate* (4.2.12).

Remark 4.16. We note that the conservation mass (4.2.11) and the energy estimate (4.2.13) induce the following uniform bounds with respect to $E(\rho_0, u_0)$:

$$\begin{aligned}
\rho(1+|\log\rho|) \in L^\infty([0,T[\,;L^1(\mathbb{T}^d)), \quad \sqrt{\rho}u \in L^\infty([0,T[\,;L^2(\mathbb{T}^d)),\\
\nu\sqrt{\rho}\nabla u \in L^2([0,T[\,;L^2(\mathbb{T}^d)), \quad \hbar\nabla\sqrt{\rho} \in L^\infty([0,T[\,;L^2(\mathbb{T}^d)),\\
\sqrt{r_0}u \in L^2([0,T[\,;L^2(\mathbb{T}^d)), \quad \sqrt{r_1}\rho^{\frac{1}{4}}u \in L^4([0,T[\,;L^4(\mathbb{T}^d)),\\
\eta_1^{\frac{1}{\alpha}}\rho^{-1} \in L^\infty([0,T[\,;L^\alpha(\mathbb{T}^d)), \quad \sqrt{\eta_2}\rho \in L^\infty([0,T[\,;H^{s+1}(\mathbb{T}^d)),\\
\sqrt{\delta_1\eta_1}\nabla\rho^{-\frac{\alpha}{2}} \in L^2([0,T[\,;L^2(\mathbb{T}^d)), \quad \sqrt{\delta_1\eta_2}\Delta^{s+1}\rho \in L^2([0,T[\,;L^2(\mathbb{T}^d)),\\
\sqrt{\delta_2}\Delta u \in L^2([0,T[\,;L^2(\mathbb{T}^d)).
\end{aligned}$$

Also, the energy estimate (4.2.12) ensures that

$$\sqrt{\nu\hbar^2}\nabla^2\sqrt{\rho} \in L^2([0,T[\,;L^2(\mathbb{T}^d)), \quad (\nu\hbar^2)^{\frac{1}{4}}\nabla\rho^{\frac{1}{4}} \in L^4([0,T[\,;L^4(\mathbb{T}^d)).$$

Finally, combining these bounds with Lemma 4.13 we obtain that there exists a positive constant $C(E_{reg}(\rho_0, u_0), \eta_1, \eta_2, \theta)$ such that

$$\|1/\rho\|_{L^\infty([0,T[\times\mathbb{T}^d)} \leq C(E_{reg}(\rho_0, u_0), \eta_1, \eta_2, \theta).$$

Proof. In all the convergences mentioned in the following proof, we have to extract subsequences that we do not relabel for conciseness.

Step 1: Faedo-Galerkin approximation. Following [84] and [111], we denote $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{T}^d)$ (which is also an orthogonal basis of $H^2(\mathbb{T}^d)$) and we introduce the finite-dimensional space $X_N = \operatorname{span}\{e_1, \dots, e_N\}$ with $N \in \mathbb{N}^*$ fixed. Note that $u_N \in \mathcal{C}([0,T]\,;X_N)$ is given by

$$u_N(t,x) = \sum_{i=1}^{N} \lambda_i(t)e_i(x)$$

for all $(t,x) \in [0,T]\times\mathbb{T}^d$ and for some functions $t \mapsto \lambda_i(t)$, and the norm of u can be written

$$\|u_N\|_{\mathcal{C}([0,T];X_N)} = \sup_{t\in[0,T]} \sum_{i=1}^{N} |\lambda_i(t)|.$$

In particular, u_N is bounded in $\mathcal{C}([0,T]\,;\mathcal{C}^k(\mathbb{T}^d))$ for any $k \geq 0$, namely there exists $C_k > 0$ such that

$$\|u_N\|_{\mathcal{C}([0,T];\mathcal{C}^k(\mathbb{T}^d))} \leq C_k \|u_N\|_{\mathcal{C}([0,T];L^2(\mathbb{T}^d))}.$$

By the classical theory of parabolic equations, for any $u_N \in \mathcal{C}([0,T]\,;X_N)$ there exists an unique solution

$$\rho_N \in \mathcal{C}^0\left([0,T[\,;H^{2s+1}(\mathbb{T}^d)\right)$$

satisfying equation (4.2.15) on $[0,T[\times\mathbb{T}^d$ with the initial condition $\rho_0 \in \mathcal{C}^\infty(\mathbb{T}^d)$ and $\rho_0 \geq \theta_0 > 0$. The maximum principle provides the inequality

$$\begin{aligned}\inf_{x\in\mathbb{T}^d} \rho_0(x) \exp\left(-\int_0^T \|\mathrm{div}(u_N(s,\cdot))\|_{L^\infty(\mathbb{T}^d)} ds\right) &\leq \rho_N(t,x) \\ &\leq \sup_{x\in\mathbb{T}^d} \rho_0(x) \exp\left(\int_0^T \|\mathrm{div}(u_N(s,\cdot))\|_{L^\infty(\mathbb{T}^d)} ds\right),\end{aligned}$$

so there exists a constant $\theta > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{T}^d$,

$$0 < \theta \leq \rho_N(t,x) \leq \frac{1}{\theta}.$$

We then introduce, for any $k \geq 1$, the linear continuous operator $\mathcal{S} : \mathcal{C}([0,T]\,;X_N) \to \mathcal{C}([0,T]\,;\mathcal{C}^k(\mathbb{T}^d))$ such that $\mathcal{S}(u_N) = \rho_N$, which is Lipschitz continuous as there exists a constant $C_{N,k} > 0$ such that

$$\|\mathcal{S}(u_N) - \mathcal{S}(\tilde{u}_N)\|_{\mathcal{C}([0,T];\mathcal{C}^k(\mathbb{T}^d))} \leq C_{N,k} \|u_N - \tilde{u}_N\|_{\mathcal{C}([0,T];L^2(\mathbb{T}^d))}.$$

The Faedo-Galerkin approximation for the weak formulation of the momentum balance can then be written as follows, for any $t \in [O,T]$ and any test function $\zeta \in X_N$,

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}^d} \Big(\rho_N u_N \cdot \partial_t \zeta + \rho_N u_N \otimes u_N : \nabla\zeta + \lambda \rho_N \mathrm{div}(\zeta) - \mu \rho_N u_N \cdot \zeta + \hbar^2 \nabla\sqrt{\rho_N} \otimes \nabla\sqrt{\rho_N} : \nabla\zeta \\ &-\frac{\hbar^2}{4}\rho_N \Delta \mathrm{div}\zeta - \nu\rho_N \mathbb{D}u_N : \nabla\zeta - r_0 u_N \cdot \zeta - r_1 \rho_N |u_N|^2 u_N \cdot \zeta - \delta_1 \nabla u_N : \nabla\rho_N \otimes \zeta - \delta_2 \Delta u_N \cdot \delta\zeta \\ &\qquad -\eta_1 \rho_N^{-\alpha} \mathrm{div}\zeta - \eta_2 \Delta^{s+1} \rho_N \Delta^s \left[\nabla\rho_N \cdot \zeta + \rho_N \mathrm{div}\zeta\right]\Big) \, dxdt + \int_{\mathbb{T}^d} \rho_0 u_0 \zeta(0) dx = 0.\end{aligned}$$

In order to solve this equation, we introduce the following family of operators, given a density function $\rho_N \in L^1(\mathbb{T}^d)$ with $\rho \geq \theta > 0$,

$$\mathcal{M}[\rho_N] : X_N \to X_N^*, \quad \langle \mathcal{M}[\rho_N]u, w\rangle = \int_{\mathbb{T}^d} \rho_N u \cdot w,$$

for $u, w \in X_N$. These operators are symmetric positive definite with smallest eigenvalue

$$\inf_{\|w\|_{L^2(\mathbb{T}^d)}=1} \langle \mathcal{M}[\rho_N]v, v\rangle = \inf_{\|w\|_{L^2(\mathbb{T}^d)}=1} \int_{\mathbb{T}^d} \rho_N |w|^2 \geq \inf_{x\in\mathbb{T}^d} \rho_N(x) \geq \theta,$$

hence as X_N is finite-dimensional, they are invertible and

$$\|\mathcal{M}^{-1}[\rho_N]\|_{\mathcal{L}(X_N^*, X_N)} \leq \frac{1}{\theta},$$

where we have denoted $\mathcal{L}(X_N^*, X_N)$ the set of all bounded linear mappings from X_N^* to X_N. Moreover, $\mathcal{M}^{-1}$ is Lipschitz continuous in the sense that there exists a constant $C_{N,\theta} > 0$ such that

$$\|\mathcal{M}^{-1}[\rho_N] - \mathcal{M}^{-1}[\tilde{\rho}_N]\|_{\mathcal{L}(X_N^*, X_N)} \leq C_{N,\theta} \|\rho_N - \tilde{\rho}_N\|_{L^1(\mathbb{T}^d)},$$

for all ρ_N, $\tilde{\rho}_N$ in the set

$$\left\{ \rho \in L^1(\mathbb{T}^d) \;\middle|\; \inf_{x\in\mathbb{T}^d} \rho(x) \geq \theta > 0 \right\}.$$

For more details, we refer to [59]. We are then looking for $u_N \in \mathcal{C}([0,T]\,; X_N)$ solution of the following integral equation

$$u_N = \mathcal{M}^{-1}[\mathcal{S}(u_N)](t) \left(\mathcal{M}[\rho_0] + \int_0^T \mathcal{N}\left[\mathcal{S}(u_N), u_N\right](s) \right),$$

where

$$\begin{aligned}\mathcal{N}\left[\mathcal{S}(u_N), u_N\right] = &-\mathrm{div}(\rho_N u_N \otimes u_N) - \lambda \nabla \rho_N - \mu \rho_N u_N - r_0 u_N - r_1 \rho_N |u_N|^2 + \frac{\hbar^2}{2} \rho_N \nabla \left(\frac{\Delta\sqrt{\rho_N}}{\sqrt{\rho_N}} \right) \\ &+ \nu \mathrm{div}(\rho_N \mathbb{D} u_N) + \delta_2 \Delta^2 u_N - \delta_1 (\nabla \rho_N \cdot \nabla) u_N + \eta_1 \nabla \rho_N^{-\alpha} + \eta_2 \rho_N \nabla \Delta^{2s+1} \rho_N\end{aligned}$$

and $\rho_N = \mathcal{S}(u_N)$. Thanks to the Lipschitz properties of $\mathcal{S}$ and $\mathcal{M}^{-1}$, we can then apply a fixed point theorem of Banach spaces on a short time $0 < T' \leq T$ in the space $\mathcal{C}([0,T']\,; X_N)$, so there exists a local solution (ρ_N, u_N) of (4.2.15)-(4.2.16). Using the conservation of mass of ρ_N (4.2.11), and multiplying equation (4.2.16) with u_N and combining with equation (4.2.10a) we get the energy estimate (4.2.12) for (ρ_N, u_N), which then allows to extend T' to T by classical arguments, as performed in [110].

Step 2: Uniform estimates on the approximate solutions. We note that $\rho_N(0,.) = \rho_0$ and $\rho_N u_N(0,.) = \mathbb{P}_N[\rho_0 u_0]$, where $\mathbb{P}_N$ denotes the $L^2(\mathbb{T}^d)$-projection onto X_N. In particular, since by assumption $\rho_0 u_0 \in L^2(\mathbb{T}^d)$, we have that

$$E_{\mathrm{reg}}(\rho_N, u_N)(0) \leq E_{\mathrm{reg}}(\rho_0, u_0),$$

and so with the energy inequality (4.2.12) satisfied by (ρ_N, u_N) we get that, for all $N \in \mathbb{N}^*$,

$$\sup_{0\leq t\leq T} E_{\mathrm{reg}}(\rho_N, u_N)(t) + \int_0^T D_{\mathrm{reg}}(\rho_N, u_N)(s)ds \leq C(E_{\mathrm{reg}}(\rho_0, u_0)). \tag{4.2.18}$$

From this inequality we get several uniform bounds on (ρ_N, u_N) with respect to N, in particular we get the existence of some functions ρ and V such that

$$\begin{aligned}\rho_N &\rightharpoonup \rho \text{ weakly-* in } L^\infty([0,T[\,;H^{2s+1}(\mathbb{T}^d)),\\ \sqrt{\rho_N}u_N &\rightharpoonup V \text{ weakly-* in } L^\infty([0,T[\,;L^2(\mathbb{T}^d)).\end{aligned}$$

Using the bounds

$$\rho_N \in L^\infty([0,T[\,;H^{2s+1}(\mathbb{T}^d)) \quad \text{and} \quad \frac{1}{\rho_N} \in L^\infty([0,T[\,;L^\alpha(\mathbb{T}^d))$$

together with Lemma 4.13, we get that there exists a constant $C(E_{\mathrm{reg}}(\rho_0,u_0),\eta_1,\eta_2,\theta,T)$ such that

$$\rho \geq C(E_{\mathrm{reg}}(\rho_0,u_0),\eta_1,\eta_2,\theta,T) > 0, \tag{4.2.19}$$

and we may set $u = V/\sqrt{\rho}$.

Step 3: The limit $N \to \infty$. In order to pass to the limit $N \to \infty$ into equations (4.2.15) and (4.2.16), we need the convergence of our functions ρ_N and u_N in a stronger sense. As (ρ_N, u_N) satisfies the energy inequality (4.2.12), it also gets the uniform bounds of Remark 4.16, in particular:

$$\begin{aligned}\rho_N \in L^\infty([0,T[\,;H^{2s+1}(\mathbb{T}^d)) \cap L^2([0,T[\,;H^{2s+2}(\mathbb{T}^d)),\\ 1/\rho_N \in L^\infty([0,T[\,\times\mathbb{T}^d), \quad u_N \in L^2([0,T[\,;H^2(\mathbb{T}^d)),\end{aligned}$$

and from the continuity equation (4.2.10a) satisfied by ρ_N we know that $\partial_t\rho_N$ is bounded in $L^2([0,T[\,;H^1(\mathbb{T}^d))$. By classical weak-convergence results and Ascoli-Arzelà type argument we get that

$$\begin{aligned}\rho_N &\to \rho \text{ strongly in } \mathcal{C}^0([0,T[\,;H^{2s}(\mathbb{T}^d)),\\ \rho_N &\rightharpoonup \rho \text{ weakly in } L^2([0,T[\,;H^{2s+2}(\mathbb{T}^d)),\\ \rho_N &\rightharpoonup \rho \text{ weakly in } H^1([0,T[\,;H^1(\mathbb{T}^d)),\end{aligned}$$

and from the bound from below on ρ_N (4.2.19) we also get that

$$1/\rho_N \to 1/\rho \text{ strongly in } \mathcal{C}^0([0,T[\,\times\mathbb{T}^d).$$

Furthermore, given the uniform bounds for ρ_N and u_N, and since $(e_k)_k$ is orthogonal for the H^2-scalar product, we can get that $(\mathbb{P}_N[\rho_N u_N])_N$ and $(\rho_N u_N)_N$ both converge in $L^2([0,T[\,;H^1(\mathbb{T}^d))$ (see [34] for the details). Together with the fact that $(1/\rho_N)_N$ is uniformly bounded, we get that

$$u_N \to u \text{ strongly in } L^2([0,T[\,;H^1(\mathbb{T}^d)).$$

Note that the uniform estimates satisfied by (ρ_N, u_N) also entail that u_N (and so u) is bounded in $L^\infty([0,T[\,;L^2(\mathbb{T}^d)) \cap L^2([0,T[\,;H^2(\mathbb{T}^d))$. The only remaining problematic term in (4.2.16) is $r_1\rho_N|u_N|^2u_N$. However as u_N is bounded in $L^\infty([0,T[\,;L^2(\mathbb{T}^d))$ and in $L^2([0,T[\,;H^1(\mathbb{T}^d))$, we get by interpolation that u_N is also bounded in $L^4([0,T[\,;L^3(\mathbb{T}^d))$.
As each term in (4.2.15) and (4.2.16) is now handle by the previous regularities of ρ_N and u_N, we can pass to the limit $N \to \infty$ into these equations, so finally (ρ, u) is a weak solution of the regularized NSLK system (4.2.10) in $[0,T[\,\times \mathbb{T}^d$ with initial data (ρ_0, u_0). □

In order to pass into the limits $\delta_1, \delta_2, \eta_1, \eta_2 \to 0$, we will need further estimates on our solution (ρ, u). A common way to get other estimates is to introduce the following BD-entropy:

$$\begin{aligned}\mathcal{E}_{\text{reg}}(\rho,u) = \frac{1}{2}\int_{\mathbb{T}^d}\left(\rho|u+\nu\nabla\log\rho|^2 + \hbar^2|\nabla\sqrt{\rho}|^2\right) + \lambda'\int_{\mathbb{T}^d}H(\rho)\\ + \frac{\eta_1}{\alpha+1}\int_{\mathbb{T}^d}\rho^{-\alpha} + \frac{\eta_2}{2}\int_{\mathbb{T}^d}|\nabla\Delta^s\rho|^2 - r_0\int_{\mathbb{T}^d}\log\rho,\end{aligned} \tag{4.2.20}$$

$$\begin{aligned}\mathcal{D}_{\text{reg}}(\rho,u) = \mu\int_{\mathbb{T}^d}\rho|u|^2 + \nu\int_{\mathbb{T}^d}\rho|\mathbb{A}u|^2 + \left(\nu\hbar^2 + \delta_1\nu^2 + \frac{\delta_1\hbar^2}{2}\right)\int_{\mathbb{T}^d}\rho|\nabla^2\log\rho|^2\\ + 4(\nu+\delta_1)\int_{\mathbb{T}^d}|\nabla\sqrt{\rho}|^2 + \left(\frac{\eta_1\nu\alpha}{4} + \frac{4\delta_1\eta_1}{10}\right)\int_{\mathbb{T}^d}|\nabla\rho^{-\frac{\alpha}{2}}|^2\\ + \eta_2(\nu+\delta_1)\int_{\mathbb{T}^d}|\Delta^{s+1}\rho|^2 + \delta_2\int_{\mathbb{T}^d}|\Delta u|^2 + r_0\int_{\mathbb{T}^d}|u|^2 + r_1\int_{\mathbb{T}^d}\rho|u|^4.\end{aligned} \tag{4.2.21}$$

Proposition 4.17. *Let $T > 0$ and assume (ρ_0, u_0) satisfies the regularity (4.2.17). We denote (ρ, u) the weak solution of (4.2.10) constructed in Proposition 4.15. Then there exists constants C_1 and C_2 with dependencies mentioned in parentheses, such that:*

$$\sup_{t\in[0,T[}\mathcal{E}_{\text{reg}}(\rho,u)(t) + \int_0^T\mathcal{D}_{\text{reg}}(\rho,u)(s)ds \le C_1(E_{\text{reg}}(0)) + (\delta_1+\delta_2)C_2(r_0,r_1,\eta_1,\eta_2,E_{\text{reg}}(0)). \tag{4.2.22}$$

Proof. The proof of this identity is mostly technical, and we refer to [34] for the details (as the new dissipation term $\mu\rho u$ is essentially harmless in the calculation). The idea is to differentiate and multiply by ν^2 the continuity equation (4.2.10a), which gives

$$\nu^2(\partial_t(\rho\nabla\log\rho) + \operatorname{div}(\rho\nabla\log\rho\otimes u) + \operatorname{div}(\rho\nabla^\top u) - \delta_1\Delta\nabla\rho) = 0,$$

and we also take $\zeta = (\nu\nabla\log\rho)\xi$, where $\xi \in \mathcal{C}_0^\infty([0,T[\times\mathbb{T}^d;\mathbb{T}^d)$, as a test function in the weak formulation of the momentum equation (4.2.16). We then combine these two equations and integrate in time. The technical part is then to prove that each term appearing in this equation is well defined, and using some classical inequality from functional analysis (namely Hölder or Young inequalities and Sobolev embedding) we finally get (4.2.22). □

Remark 4.18. Note that the presence of the term $-r_0\int_{\mathbb{T}^d}\log\rho$ in $\mathcal{E}_{\mathrm{reg}}(\rho,u)$ prevents this quantity from being positive, however as for the logarithmic energy term we can get a lower bound for this term by controlling his negative part

$$-r_0\int_{\rho>1}\log\rho \geq -r_0\int_{\rho>1}\rho \geq -r_0\int_{\mathbb{T}^d}\rho = -r_0\|\rho_0\|_{L^1(\mathbb{T}^d)}$$

as ρ satisfies the mass conservation (4.2.11). Hence we can properly get some new uniform bounds on (ρ,u) using equation (4.2.22).

4.2.2 NSLK with drag forces

We are now going to prove the existence of a weak solution to the following system, which will be called NSLK system with drag forces:

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \lambda\nabla\rho + \mu\rho u + r_0 u + r_1\rho|u|^2 u = \dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) + \nu\operatorname{div}(\rho\mathbb{D}u).\end{cases}$$

(4.2.23a)

(4.2.23b)

We define the energy of this system and its corresponding dissipation from the ones of the previous regularized system,

$$E_{\mathrm{drag}}(\rho,u) := E_{\mathrm{reg}}(\rho,u)|_{\delta_1,\delta_2,\eta_1,\eta_2=0} \quad\text{and}\quad D_{\mathrm{drag}}(\rho,u) := D_{\mathrm{reg}}(\rho,u)|_{\delta_1,\delta_2,\eta_1,\eta_2=0},$$

as well as the BD-entropy and its corresponding flux,

$$\mathcal{E}_{\mathrm{drag}}(\rho,u) := \mathcal{E}_{\mathrm{reg}}(\rho,u)|_{\delta_1,\delta_2,\eta_1,\eta_2=0} \quad\text{and}\quad \mathcal{D}_{\mathrm{drag}}(\rho,u) := \mathcal{D}_{\mathrm{reg}}(\rho,u)|_{\delta_1,\delta_2,\eta_1,\eta_2=0}.$$

Definition 4.19. *Let $T > 0$. We say that (ρ, u) is a* ***weak solution*** *of the NSLK system with drag forces* (4.2.23) *in* $[0, T[\times\mathbb{T}^d$ *with initial data* $(\rho_0, u_0) \in L^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$*, if the following holds:*

- *The global regularity:*

$$\sqrt{\rho} \in \mathcal{C}^0([0, T[\,; H^1(\mathbb{T}^d)), \quad \nabla^2\sqrt{\rho} \in L^2([0, T[\,; L^2(\mathbb{T}^d)),$$

$$u \in L^2([0, T[\,; L^2(\mathbb{T}^d)), \quad \sqrt{\rho}u \in \mathcal{C}^0([0, T[\,; L^2(\mathbb{T}^d)).$$

- *For any test function* $\eta \in \mathcal{C}_0^\infty([0, T[\times\mathbb{T}^d)$*, equation* (4.2.15) *holds, and for any test function* $\zeta \in \mathcal{C}_0^\infty([0, T[\times\mathbb{T}^d; \mathbb{T}^d)$*, equation* (4.2.16) *holds, taking* $\delta_1, \delta_2, \eta_1, \eta_2 = 0$ *for both equations.*

Remark 4.20. As in [34], remark that in presence of drag forces $r_0, r_1 > 0$, u is well defined as function, ∇u as a distribution and $\sqrt{\rho}\mathbb{D}u$ is also well defined, unlike in the original system (4.2.1) without drag forces where the regularity induced by the energy estimate (4.2.2) is insufficient to define u and so $\sqrt{\rho}\mathbb{D}u$ has to be understood as $\mathbb{S}_N(u)$.

Proposition 4.21. *Let $T > 0$ and assume (ρ_0, u_0) satisfies the regularity* (4.2.17) *such that*

$$E_{\text{drag}}(\rho, u)(0) = E_{\text{drag}}(\rho_0, u_0) < \infty \quad \textit{and} \quad \mathcal{E}_{\text{drag}}(\rho, u)(0) = \mathcal{E}_{\text{drag}}(\rho_0, u_0) < \infty.$$

Then there exists a weak solution of the NSLK system with drag forces (4.2.10) *in* $[0, T[\times\mathbb{T}^d$ *with initial data* (ρ_0, u_0)*. Furthermore, there exists constants C_1 and C_2 with dependencies mentioned in parentheses, such that*

$$\sup_{t\in[0,T[} E_{\text{drag}}(\rho, u)(t) + \int_0^T D_{\text{drag}}(\rho, u)(s)ds \le C_1(E_{\text{drag}}(0)), \tag{4.2.24}$$

and

$$\sup_{t\in[0,T[} \mathcal{E}_{\text{drag}}(\rho, u)(t) + \int_0^T \mathcal{D}_{\text{drag}}(\rho, u)(s)ds \le C_2(E_{\text{drag}}(0), \mathcal{E}_{\text{drag}}(0)). \tag{4.2.25}$$

Proof. The proof is the exact same as the one appearing in [34], where the authors first let $\delta_1, \delta_2 \to 0$ and then perform the limit $\eta_1, \eta_2 \to 0$, so we refer to it for the details. □

4.2.3 The limit $r_0, r_1 \to 0$

In [88], in order to pass to the limit $r_0, r_1 \to 0$, the authors have to introduce a new type of solutions to the Navier-Stokes-Korteweg system called renormalised solutions. In our framework these solutions are defined as follows:

Definition 4.22. *Let $T > 0$. We say that (ρ, u) is a* ***renormalised weak solution*** *of the NSLK system with drag forces* (4.2.10) *in* $[0, T[\times\mathbb{T}^d$ *with initial data* $(\sqrt{\rho_0}, (\sqrt{\rho}u)_0) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$, *if there exists locally integrable functions $\sqrt{\rho}$, $\sqrt{\rho}u$ such that, by defining $\rho := \sqrt{\rho}^2$ and $u := \sqrt{\rho}u/\sqrt{\rho}$, the following holds:*

- *The global regularity:*

$$\begin{aligned}
\sqrt{\rho} \in L^\infty([0,T[\,;H^1(\mathbb{T}^d)), \quad \sqrt{\rho}u \in L^\infty([0,T[\,;L^2(\mathbb{T}^d)),\\
\hbar\nabla^2\sqrt{\rho} \in L^2([0,T[\,;L^2(\mathbb{T}^d)), \quad \mathbb{T}_N(u) \in L^2([0,T[\,;L^2(\mathbb{T}^d)),\\
\sqrt{\hbar}\nabla\rho^{1/4} \in L^4([0,T[\,;L^4(\mathbb{T}^d)), \quad r_1^{1/4}\rho^{1/4}u \in L^4([0,T[\,;L^4(\mathbb{T}^d)),\\
r_0^{1/2}u \in L^2([0,T[\,;L^2(\mathbb{T}^d)), \quad r_0\log\rho \in L^\infty([0,T[\,;H^1(\mathbb{T}^d)),
\end{aligned}$$

  where $\mathbb{T}_N(u)$ is defined as in Definition 4.8, and with the compatibility condition

$$\sqrt{\rho} \geq 0 \ \textit{a.e. on}\ [0,T[\times\mathbb{T}^d, \quad \sqrt{\rho}u = 0 \ \textit{a.e. on}\ \{\rho = 0\}\,.$$

- *For any function $\varphi \in W^{2,\infty}(\mathbb{R}^d)$, there exists two measures f_φ, $g_\varphi \in \mathcal{M}(]0,T[\times\mathbb{T}^d)$ with*

$$\|f_\varphi\|_{\mathcal{M}(]0,T[\times\mathbb{T}^d)} + \|g_\varphi\|_{\mathcal{M}(]0,T[\times\mathbb{T}^d)} \leq C\|\nabla^2\varphi\|_{L^\infty(\mathbb{R}^d)},$$

  where the constant C depends only on the solution (ρ,u) such that for any test function $\eta \in \mathcal{C}_0^\infty([0,T[\times\mathbb{T}^d)$,

$$\int_0^T\int_{\mathbb{T}^d}(\rho\partial_t\eta + \rho u\cdot\nabla\eta)dxdt + \int_{\mathbb{T}^d}\rho_0\eta(0)dx = 0, \tag{4.2.26}$$

  and for any test function $\zeta \in \mathcal{C}_0^\infty([0,T[\times\mathbb{T}^d)$,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\Big(\rho\varphi(u)\cdot\partial_t\zeta + \rho\varphi(u)u:\nabla\zeta + \Big[\lambda\rho\mathrm{div}(\zeta) - \mu\rho u\cdot\zeta - r_0u - r_1\rho|u|^2u\\
&+\hbar^2\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}:\nabla\zeta - \frac{\hbar^2}{4}\rho\Delta\mathrm{div}\zeta - \nu\sqrt{\rho}\mathbb{S}_N(u)\cdot\nabla\zeta\Big]\varphi'(u)\Big)\,dxdt + \int_{\mathbb{T}^d}\rho_0u_0\zeta(0)dx = \langle f_\varphi,\zeta\rangle,
\end{aligned} \tag{4.2.27}$$

  with $\mathbb{S}_N(u) = (\mathbb{T}_N(u) + \mathbb{T}_N(u)^\top)/2$ and where $\mathbb{T}_N(u)$ is defined through the compatibility condition

$$\sqrt{\rho}\ \varphi_i'(u)(\mathbb{T}_N(u))_{j,k} = \partial_j(\rho\varphi_i'(u)u_k) - 2\sqrt{\rho}u_k\partial_j\sqrt{\rho} + g_\varphi, \quad \forall i,j,k \in \{1,\dots,d\}\,.$$

- *For any test function $\xi \in \mathcal{C}_0^\infty(\mathbb{T}^d)$,*

$$\lim_{t\to0}\int_{\mathbb{T}^d}\rho(t,x)\xi(x)dx = \int_{\mathbb{T}^d}\rho_0(t,x)\xi(x)dx,$$

$$\lim_{t\to 0}\int_{\mathbb{T}^d}\rho(t,x)u(t,x)\xi(x)dx = \int_{\mathbb{T}^d}\rho_0(t,x)u_0(t,x)\xi(x)dx.$$

Note that renormalized solutions allow us to obtain some stability on our weak solutions, since the notion avoid the problem of concentration. In fact, in [88], the authors have proved the following lemma:

Lemma 4.23. *Let $T > 0$, then:*

- *For r_0, $r_1 \geq 0$, any renormalised weak solution of the NSLK system with drag forces* (4.2.23) *is also a weak solution (in the sense of Definition 4.19 if r_0, $r_1 > 0$ or Definition 4.8 is r_0, $r_1 = 0$).*
- *For r_0, $r_1 > 0$, the two notions are equivalent: any weak solution of the NSLK system with drag forces* (4.2.23) *is also a renormalized solution of the same system.*

This lemma, also used in [34] in the case of an isothermal pressure, is proven by the use of Lebesgue's dominated convergence Theorem in the limit $n \to \infty$ (together with the extra regularity on the density provided by the quantum term and the BD entropy inequality) for some well suited sequences of test functions φ_n, and the presence of the damping term $\mu\rho u$ is essentially harmless.

Thanks to Lemma 4.23 and Proposition 4.21, we have constructed a renormalized solution of the NSLK system with drag forces (4.2.23). In order to prove Theorem 4.11, the only remaining step is to prove the compactness of this renormalized solution in terms of the parameters r_0 and r_1.

Proof of Theorem 4.11. In this proof, we will denote $r = (r_0, r_1)$ and $(\rho_r, u_r)_r$ the sequence of weak solutions to the NSLK system with drag forces (4.2.23) constructed in Proposition 4.21. Again, all the convergences below are made up to extraction that we do not relabel for conciseness.

We are first going to pass into the limit in the continuity equation (4.2.26) only using a priori estimates (i) from Definition 4.22 which do not depend on r. From the continuous embedding $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$ as $d \leq 3$, we get that $\sqrt{\rho_r} \in L^\infty([0,T[\,;L^6(\mathbb{T}^d))$, so combined with the fact that $\sqrt{\rho_r}u_r \in L^\infty([0,T[\,;L^2(\mathbb{T}^d))$ we get that $\rho_r u_r$ is uniformly bounded in $L^\infty([0,T[\,;L^{3/2}(\mathbb{T}^d))$. Moreover, using the continuity equation (4.2.26) we get that $\partial_t\rho_r$ is uniformly bounded in $L^\infty([0,T[\,;W^{-1,3/2}(\mathbb{T}^d))$. Writing $\nabla\rho_r = 2\sqrt{\rho_r}\nabla\sqrt{\rho_r}$, and recalling that $\nabla\sqrt{\rho_r} \in L^\infty([0,T[\,;L^2(\mathbb{T}^d))$, we get that $\nabla\rho_r$ is also uniformly bounded in $L^\infty([0,T[\,;L^{3/2}(\mathbb{T}^d))$, hence by Aubin-Lions lemma we get that

$$\rho_r \to \rho \text{ strongly in } \mathcal{C}^0([0,T[\,;L^p(\mathbb{T}^d)), \quad \text{for } 1 \le p < 3.$$

From (4.2.27) we get that $\partial_t(\rho_r u_r)$ is uniformly bounded in $L^2([0,T[\,;H^{-N}(\mathbb{T}^d))$ for a N large enough, and from the identity

$$\nabla(\rho_r u_r) = \sqrt{\rho_r}\sqrt{\rho_r}\nabla u_r + \nabla\rho_r \cdot u_r = \sqrt{\rho_r}\mathbb{T}_N(u) + 2\nabla\sqrt{\rho_r}\cdot\sqrt{\rho_r}u_r,$$

and the a priori estimates (i) from Definition 4.22 we get that $\nabla(\rho_r u_r)$ is uniformly bounded in $L^2([0,T[\,;L^{3/2}(\mathbb{T}^d))$. As we already know that $\rho_r u_r$ is uniformly bounded in $L^\infty([0,T[\,;L^{3/2}(\mathbb{T}^d))$, we get from Aubin-Lions lemma that

$$\begin{aligned} &\rho_r u_r \rightharpoonup \rho u \text{ weakly in } \mathcal{C}^0([0,T[\,;L^{3/2}(\mathbb{T}^d)), \quad \text{for } 1 \le p < 3. \\ &\rho_r u_r \to \rho u \text{ strongly in } L^p([0,T[\,;L^q(\mathbb{T}^d)), \quad \text{for } 1 \le p < \infty \text{ and } 1 \le q < \frac{3}{2}. \end{aligned}$$

In particular, $(\rho_r)_r$ and $(\rho_r u_r)_r$ are uniformly bounded in $\mathcal{C}^0([0,T[\,;L^1(\mathbb{T}^d))$, so we can pass to the limit $r \to 0$ in the continuity equation (4.2.26), and we also get part (iii) of Definition 4.22.

We are now going to pass to the limit in the momentum equation (4.2.27). Using the previous convergences and the estimates independent of r on $\nabla^2\sqrt{\rho_r}$ and $\mathbb{T}_{N,r}$ that ensures some weak convergence of these quantities in $L^2([0,T[\,;L^2(\mathbb{T}^d))$, we can pass to the limit $r \to 0$ in the left hand side of equation (4.2.27). Indeed, as in [88] and [34], introducing $u = \rho u/\rho \mathbf{1}_{\{\rho>0\}}$ we can show with the previous convergences that $\rho_r \to \rho$ and $u_r \to u$ a.e., and consequently that $\rho_r^\alpha \phi(u_r) \to \rho^\alpha\phi(u)$ in $L^p([0,T[\,\times \mathbb{T}^d)$ for any bounded $\phi : \mathbb{R}^d \to \mathbb{R}^d$, with $\alpha < 6$ and $p < 6/\alpha$. For the right hand side of equation (4.2.27), we remark that the sequence $(f_{\phi,r})_r$ is uniformly bounded in measures, so it converges to a measure f_ϕ with the same bound. Note that we can similarly pass to the limit in the renormalized compatibility condition for $\mathbb{T}_{N,r}$ and obtain the renormalized condition for $\mathbb{T}_N(u)$.

Hence (ρ, u) defines a renormalized weak solution of the NSLK system (4.2.1) in the sense of Definition 4.22 taking $r_0 = r_1 = 0$, which also stands as a weak solution of the NSLK system (4.2.1). By Fatou's lemma, we then get that (ρ, u) satisfies (4.2.2), which completes the proof of Theorem 4.11. □

Remark 4.24. Defining

$$\mathcal{E}(\rho,u) := \mathcal{E}_{\text{drag}}(\rho,u)|_{r_0,r_1=0} \quad \text{and} \quad \mathcal{D}(\rho,u) := \mathcal{D}_{\text{drag}}(\rho,u)|_{r_0,r_1=0},$$

and as

$$-r_0\int_{\rho>1}\log\rho \geq -r_0\|\rho_0\|_{L^1(\mathbb{T}^d)} \quad \text{and} \quad 0\leq -r_0\int_{\rho<1}\log\rho \leq C_2(E_{\text{drag}}(0),\mathcal{E}_{\text{drag}}(0))+r_0\|\rho_0\|_{L^1(\mathbb{T}^d)},$$

from respectively Remark 4.18 and equation (4.2.25), which implies that

$$-r_0\int_{\mathbb{T}^d}\log\rho \underset{r_0\to 0}{\longrightarrow} 0,$$

we also gets from the previous proof that (ρ,u) satisfies the following BD-entropy estimate (which will be useful in the following section):

$$\sup_{t\in[0,T[}\mathcal{E}(\rho,u)(t)+\int_0^T\mathcal{D}(\rho,u)(s)ds\leq C(E(0),\mathcal{E}(0)). \tag{4.2.28}$$

4.3 The isothermal Euler-Langevin-Korteweg system

Following [27], by denoting $w=u+\frac{\nu}{2}\nabla\log\rho$ and $v=\nabla\log\rho$, we consider the augmented Navier-Stokes-Langevin-Korteweg system of equations:

$$\begin{cases}\partial_t\rho+\operatorname{div}(\rho u)=0, & (4.3.1\text{a})\\ \partial_t(\rho w)+\operatorname{div}(\rho w\otimes u)+\lambda'\nabla\rho+\mu\rho w=\dfrac{\hbar_\nu}{2}\operatorname{div}(\rho\nabla v)+\dfrac{\nu}{2}\operatorname{div}(\rho\nabla w), & (4.3.1\text{b})\\ \partial_t(\rho v)+\operatorname{div}(\rho v\otimes u)+\operatorname{div}\left(\rho\nabla u^\top\right)=0. & (4.3.1\text{c})\end{cases}$$

where we denote $\hbar_\nu^2=\hbar^2-\nu^2>0$ and $\lambda'=\lambda-\frac{\mu\nu}{2}>0$ (as we are letting $\nu\to 0$, cf Remark 4.25 below).

Remark 4.25. Note that here the isothermal pressure term $\lambda\nabla\rho$ can absorb the contribution of the dissipative Langevin term, through the identity

$$\lambda\nabla\rho+\mu\rho u=\lambda\nabla\rho+\mu\rho(w-\frac{\nu}{2}\nabla\log\rho)=(\lambda-\frac{\mu\nu}{2})\nabla\rho+\mu\rho w.$$

As we assume that $\lambda>0$ and as we are going to let $\nu\to 0$ in this section, we can take ν small enough such that $\lambda'=\lambda-\frac{\mu\nu}{2}>0$, which will be crucial in the following. In fact the compatibility of these two terms is a special feature of the isothermal pressure and the Langevin potential, which may not work for other pressure laws (for example the classical barotropic pressure $\lambda\rho^\gamma$, $\gamma>1$), and reinforce the link between these two quantities.

By denoting $\overline{v}=\hbar_\nu v/2$, we also rewrite the associated BD entropy estimate (4.2.5) in terms of ρ, w and $\overline{v}$, which stands as the energy estimate of the augmented system (4.3.1):

$$\mathcal{E}_{NSLK}(t) + \int_0^t \mathcal{D}_{NSLK}(s)ds \leq E_0,$$

where

$$\mathcal{E}_{NSLK}(\rho, w, \overline{v}) = \int_{\mathbb{T}^d} \left(\frac{1}{2}\rho|w|^2 + \frac{1}{2}\rho|\overline{v}|^2 + \lambda' H(\rho)\right),$$
$$\mathcal{D}_{NSLK}(\rho, w, \overline{v}) = \frac{\nu}{2}\int_{\mathbb{T}^d} \left(\rho|\nabla w|^2 + \rho|\nabla \overline{v}|^2 + \frac{4\lambda'}{\hbar_\nu^2}\rho|\overline{v}|^2\right) + \mu \int_{\mathbb{T}^d} \rho|w|^2.$$

We also introduce the relative entropy functional of the augmented NSLK system:

$$\begin{aligned}\mathcal{E}_{NSLK}(\rho, w, \overline{v}|R, W, \overline{V})(t) = &\frac{1}{2}\int_{\mathbb{T}^d} \rho(|\overline{v} - \overline{V}|^2 + |w - W|^2) + \lambda' \int_{\mathbb{T}^d} H(\rho|R) \\ &+ \frac{\nu}{2}\int_0^t \int_{\mathbb{T}^d} \rho \left(\left|\frac{\mathbb{T}_N(\overline{v})}{\sqrt{\rho}} - \nabla \overline{V}\right|^2 + \left|\frac{\mathbb{T}_N(w)}{\sqrt{\rho}} - \nabla W\right|^2\right) + \mu\int_0^t \int_{\mathbb{T}^d} \rho|w - W|^2,\end{aligned}$$

where $\mathbb{T}_N(\overline{v})$ and $\mathbb{T}_N(w)$ are defined through the compatibility conditions

$$\sqrt{\rho}\mathbb{T}_N(\overline{v}) = \nabla(\sqrt{\rho}\sqrt{\rho}\overline{v}) - 2\sqrt{\rho}\overline{v} \otimes \nabla\sqrt{\rho} \quad \text{and} \quad \sqrt{\rho}\mathbb{T}_N(w) = \nabla(\sqrt{\rho}\sqrt{\rho}w) - 2\sqrt{\rho}w \otimes \nabla\sqrt{\rho}.$$

We now introduce the definitions of weak and strong solutions to the augmented NSLK system:

Definition 4.26. *Let $T > 0$ and $(\rho_0, w_0, \overline{v}_0 = \frac{\hbar}{2}\nabla \log \rho_0)$ such that $\mathcal{E}_{NSLK}(\rho_0, u_0, \overline{v}_0) < \infty$. We say that $(\rho, w, \overline{v})$ is a **weak solution** of the augmented system (4.3.1) in $[0, T[\times \mathbb{T}^d$ with initial data $(\rho_0, w_0, \overline{v}_0)$, if there exists locally integrable functions $\sqrt{\rho}$, $\sqrt{\rho}u$ such that, by defining $\rho := \sqrt{\rho}^2$, $\rho u := \sqrt{\rho}\sqrt{\rho}u$, $v = \nabla \log \rho$ and $w = u + \frac{\nu}{2}v$, the following holds:*

- *The global regularity (i) of Definition 4.8 on $\sqrt{\rho}$ and $\sqrt{\rho}u$ is verified.*
- *For any test function $\eta \in \mathcal{C}_0^\infty([0, T[\times \mathbb{T}^d)$, ρ and ρu verify equation (4.2.8), for any test function $\zeta \in \mathcal{C}_0^\infty([0, T[\times \mathbb{T}^d; \mathbb{T}^d)$,*

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} \Big(\rho w \cdot \partial_t \zeta + \sqrt{\rho}w \otimes \sqrt{\rho}u : \nabla\zeta + \lambda'\rho\mathrm{div}(\zeta) - \mu\rho w \cdot \zeta + \sqrt{\rho}\mathbb{T}_N(\overline{v}) : \nabla\zeta \\ + \frac{\nu}{2}\sqrt{\rho}\mathbb{T}_N(w) : \nabla\zeta\Big)\, dxdt + \int_{\mathbb{T}^d} \rho_0 u_0 \zeta(0)dx = 0,\end{aligned}$$

and for any test function $\xi \in \mathcal{C}_0^\infty([0, T[\times \mathbb{T}^d; \mathbb{T}^d)$,

$$\int_0^T \int_{\mathbb{T}^d} \left(\rho v \cdot \partial_t \xi \sqrt{\rho}v \otimes \sqrt{\rho}u : \nabla\xi + \sqrt{\rho}\mathbb{T}_N(u)^\top : \nabla\xi\right) dxdt + \int_{\mathbb{T}^d} \rho_0 v_0 \xi(0)dx = 0.$$

- *The compatibility condition (iii) of Definition 4.8 on ρ and ρu is satisfied.*

Remark 4.27. Let us remark here that weak solutions of the NSLK system (4.2.1) are also weak solutions of the augmented NSLK system (4.3.1). In fact, taking the gradient of equation (4.2.1a) (which is satisfied in the distribution sense), we get that

$$\partial_t \nabla\rho + \operatorname{div}(\nabla(\rho u)^\top) = 0,$$

so by definition of $\sqrt{\rho}\mathbb{T}_N$ and expression of v we can write

$$\partial(\rho v) + \operatorname{div}(\rho v \otimes u) + \operatorname{div}(\sqrt{\rho}\mathbb{T}_N(u)^\top) = 0.$$

Definition 4.28. *Let $T > 0$ and $(R_0, W_0, \overline{V}_0 = \frac{\hbar_\nu}{2}\nabla \log R_0)$ such that $\mathcal{E}_{NSLK}(R_0, W_0, \overline{V}_0) < \infty$. We say that $(R, W, \overline{V})$ is a **strong solution** of the augmented system (4.3.1) in $[0, T[\times \mathbb{T}^d$ with initial data $(R_0, W_0, \overline{V}_0)$, if the following holds:*

- *The global regularity:*

$$\begin{gathered}
0 < \inf_{]0,T[\times\mathbb{T}^d} R \le R \le \sup_{]0,T[\times\mathbb{T}^d} R < \infty, \\
\nabla R \in L^2([0,T[\,; L^\infty(\mathbb{T}^d)) \cap L^1([0,T[\,; W^{1,\infty}(\mathbb{T}^d)) \\
W, \overline{V} \in L^\infty([0,T[\,; W^{2,\infty}(\mathbb{T}^d)) \cap W^{1,\infty}([0,T[\,; L^\infty(\mathbb{T}^d)) \\
\partial_t H'(R),\ \nabla H'(R) \in L^1([0,T[\,; L^\infty(\mathbb{T}^d)).
\end{gathered}$$

- *The function $R, \overline{V}$ and W satisfies, for a.a. $(t,x) \in [0,T] \times \mathbb{T}^d$,*

$$\partial_t R + \operatorname{div}(RU) = 0, \tag{4.3.2a}$$

$$R(\partial_t W + \nabla W U) + \lambda' \nabla R + \mu R W = \frac{\hbar_\nu}{2}\operatorname{div}(R\nabla\overline{V}) + \frac{\nu}{2}\operatorname{div}(R\nabla W), \tag{4.3.2b}$$

$$R(\partial_t \overline{V} + \nabla\overline{V} U) + \operatorname{div}\left(R\nabla U^\top\right) = 0, \tag{4.3.2c}$$

where $U = W - \frac{\nu}{2}V$, and $\overline{V} = \frac{\hbar_\nu}{2}V$.

4.3.1 A Gronwall inequality

We are going to mimic the steps of the proof given in [27], giving a special attention to the isothermal pressure law $\lambda'\rho$ and to the new dissipation term:

$$\mu \int_0^t \int_{\mathbb{T}^d} \rho |w - W|^2.$$

Proposition 4.29. *Let $(\rho, w, \overline{v})$ be a weak solution of the augmented system* (4.3.1)*, and let*

$$R \in \mathcal{C}^1([0,T] \times \mathbb{T}^d), \quad R > 0, \quad W, \overline{V} \in \mathcal{C}^2([0,T] \times \mathbb{T}^d).$$

Then we have the following inequality:

$$\begin{aligned}
\mathcal{E}_{NSLK}(\rho, w, \overline{v}|R, W, \overline{V})(t) \leq\ & \mathcal{E}_{NSLK}(\rho, w, \overline{v}|R, W, \overline{V})(0) \\
& + \int_0^t \int_{\mathbb{T}^d} \rho(\partial_t \overline{V} \cdot (\overline{V} - \overline{v}) + (\nabla \overline{V} u) \cdot (\overline{V} - \overline{v})) \\
& + \int_0^t \int_{\mathbb{T}^d} \rho(\partial_t W \cdot (W - w) + (\nabla W u) \cdot (W - w)) \\
& - \lambda' \int_0^t \int_{\mathbb{T}^d} (\partial_t (H'(R))(\rho - R) + \rho \nabla (H'(R)) \cdot u + \rho \mathrm{div} W) \\
& - 2\nu\lambda' \int_0^t \int_{\mathbb{T}^d} |\nabla \sqrt{\rho}|^2 + \mu \int_0^t \int_{\mathbb{T}^d} \Big(\rho |W|^2 - \rho w \cdot W\Big) + I_1,
\end{aligned}$$

where

$$\begin{aligned}
I_1 = \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} & \rho(|\nabla \overline{V}| + |\nabla W|^2) - \sqrt{\rho}(\mathbb{T}(\overline{v}) : \nabla \overline{V} + \mathbb{T}(w) : \nabla W) \\
& + \frac{\hbar_\nu}{2} \int_0^t \int_{\mathbb{T}^d} \sqrt{\rho}(\mathbb{T}(\overline{v}) : \nabla W - (\mathbb{T}(w))^\top : \nabla \overline{V}).
\end{aligned}$$

Proof. As we know that $(\rho, w, \overline{v})$ is a solution of (4.3.1), we can use (4.2.2), and using the fact that for any function $f \in W^{1,1}([0,T[\,; L^1(\mathbb{T}^d))$ we have

$$\int_{\mathbb{T}^d} f(t,x) dx - \int_{\mathbb{T}^d} f(0,x) dx = \int_0^t \int_{\mathbb{T}^d} \partial_t f(s,x) dx ds,$$

we get that

$$\begin{aligned}
\mathcal{E}_{NSLK}(t) - \mathcal{E}_{NSLK}(0) \leq\ & \int_0^t \int_{\mathbb{T}^d} \partial_t \left(\frac{1}{2} \rho |\overline{V}|^2 - \rho \overline{v} \cdot \overline{V} \frac{1}{2} \rho |W|^2 - \rho w \cdot W \right) \\
& + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} \rho(|\nabla \overline{V}| + |\nabla W|^2) - 2\sqrt{\rho}(\mathbb{T}(\overline{v}) : \nabla \overline{V} + \mathbb{T}(w) : \nabla W) \\
& - \lambda' \int_0^t \int_{\mathbb{T}^d} \partial_t \left(H(R) + H'(R)(\rho - R) \right) - 2\nu\lambda' \int_0^t \int_{\mathbb{T}^d} |\nabla \sqrt{\rho}|^2 \\
& + \mu \int_0^t \int_{\mathbb{T}^d} \Big(\rho |W|^2 - 2\rho w \cdot W \Big).
\end{aligned}$$

We are now going to use equations from the augmented system (4.3.1) in order to develop the terms in the previous inequality. Using equation (4.3.1a), we get that

$$\begin{aligned}\int_0^t \int_{\mathbb{T}^d} \partial_t \left(H(R) + H'(R)(\rho - R)\right) &= \int_0^t \int_{\mathbb{T}^d} (H'(R)\partial_t R + \partial_t(H'(R))(\rho - R) + H'(R)\partial_t(\rho - R)) \\ &= \int_0^t \int_{\mathbb{T}^d} (\partial_t(H'(R))(\rho - R) - H'(R)\mathrm{div}(\rho u)) \\ &= \int_0^t \int_{\mathbb{T}^d} (\partial_t(H'(R))(\rho - R) + \rho\nabla(H'(R)) \cdot u).\end{aligned}$$

Then using respectively equations (4.3.1b) and (4.3.1c), we have

$$\begin{aligned}&\partial_t(\rho w \cdot W) = \partial_t(\rho w) \cdot W + \rho w \cdot \partial_t W \\ &= \langle -\mathrm{div}\,(\rho w \otimes u) - \lambda' \nabla\rho - \mu\rho w + \frac{\hbar_\nu}{2}\mathrm{div}(\sqrt{\rho}\mathbb{T}(\overline{v})) + \frac{\nu}{2}\mathrm{div}(\sqrt{\rho}\mathbb{T}(w)) | W\rangle_{W^{-2,1}(\mathbb{T}^d)\times W^{2,\infty}(\mathbb{T}^d)} + \rho w \cdot \partial_t W,\end{aligned}$$

and

$$\begin{aligned}\partial_t(\rho\overline{v} \cdot \overline{V}) &= \partial_t(\rho\overline{v}) \cdot \overline{V} + \rho\overline{v} \cdot \partial_t\overline{V} \\ &= \langle -\mathrm{div}\,(\rho\overline{v} \otimes u) - \frac{\hbar_\nu}{2}\mathrm{div}(\sqrt{\rho}\mathbb{T}(w)^\top) + \frac{\nu}{2}\mathrm{div}(\sqrt{\rho}\mathbb{T}(\overline{v}) | \overline{V}\rangle_{W^{-2,1}(\mathbb{T}^d)\times W^{2,\infty}(\mathbb{T}^d)} + \rho\overline{v} \cdot \partial_t\overline{V}.\end{aligned}$$

Finally, we develop the quantities

$$\partial_t(\frac{1}{2}\rho|\overline{V}|^2) = \frac{1}{2}\partial_t\rho|\overline{V}|^2 + \rho\overline{V} \cdot \partial_t\overline{V}, \quad \partial_t(\frac{1}{2}\rho|W|^2) = \frac{1}{2}\partial_t\rho|W|^2 + \rho W \cdot \partial_t W,$$

and since $\nabla\overline{v}$, $\nabla\overline{V}$ are symmetric matrices (recall that v and V are gradient of functions), we get from (4.3.1a) and integration by parts:

$$
\begin{aligned}
\mathcal{E}_{NSLK}(t) - \mathcal{E}_{NSLK}(0) \leq & \int_0^t \int_{\mathbb{T}^d} \rho \partial_t \overline{V} \cdot (\overline{V} - \overline{v}) + \int_0^t \int_{\mathbb{T}^d} \rho (\nabla \overline{V} u) \cdot (\overline{V} - \overline{v}) \\
& + \int_0^t \int_{\mathbb{T}^d} \rho \partial_t W \cdot (W - w) + \int_0^t \int_{\mathbb{T}^d} \rho (\nabla W u) \cdot (W - w) \\
& + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} \rho (|\nabla \overline{V}| + |\nabla W|^2) - 2\sqrt{\rho} (\mathbb{T}(\overline{v}) : \nabla \overline{V} + \mathbb{T}(w) : \nabla W) \\
& - \lambda' \int_0^t \int_{\mathbb{T}^d} \partial_t (H'(R))(\rho - R) - \lambda' \int_0^t \int_{\mathbb{T}^d} \rho \nabla (H'(R)) \cdot u \\
& - 2\nu\lambda' \int_0^t \int_{\mathbb{T}^d} |\nabla \sqrt{\rho}|^2 + \mu \int_0^t \int_{\mathbb{T}^d} \left(\rho |W|^2 - 2\rho w \cdot W \right) \\
& - \lambda' \int_0^t \int_{\mathbb{T}^d} \rho \mathrm{div} W + \mu \int_0^t \int_{\mathbb{T}^d} \rho w \cdot W \\
& + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} \sqrt{\rho} (\mathbb{T}(\overline{v}) : \nabla \overline{V} + \mathbb{T}(w) : \nabla W) \\
& + \frac{\hbar_\nu}{2} \int_0^t \int_{\mathbb{T}^d} \sqrt{\rho} (\mathbb{T}(\overline{v}) : \nabla W - (\mathbb{T}(w))^\top : \nabla \overline{V}).
\end{aligned}
$$

The result follows from simplifying the previous expression. $\square$

Proposition 4.30. *Let $(R, W, \overline{V})$ be a strong solution of* (4.3.1)*, then any weak solution $(\rho, w, \overline{v})$ of the augmented system* (4.3.1) *satisfies:*

$$
\begin{aligned}
\mathcal{E}_{NSLK}(t) - \mathcal{E}_{NSLK}(0) \leq & \int_0^t \int_{\mathbb{T}^d} \rho (\nabla \overline{V} (u - U)) \cdot (\overline{V} - \overline{v}) + \int_0^t \int_{\mathbb{T}^d} \rho (\nabla W (u - U)) \cdot (W - w) \\
& + \lambda' \frac{\nu}{\hbar_\nu} \int_0^t \int_{\mathbb{T}^d} (\rho \nabla (H'(R)) \cdot (\overline{v} - \overline{V}) - \rho \mathrm{div} \overline{V}) - 2\nu\lambda' \int_0^t \int_{\mathbb{T}^d} |\nabla \sqrt{\rho}|^2 \\
& + \frac{\nu}{2} \int_0^t \int_{\mathbb{T}^d} \frac{\rho}{R} (\mathrm{div}(R \nabla \overline{V}) \cdot (\overline{V} - \overline{v}) + \mathrm{div}(R \nabla W) \cdot (W - w)) \\
& + \frac{\hbar_\nu}{2} \int_0^t \int_{\mathbb{T}^d} \frac{\rho}{R} (\mathrm{div}(R \nabla \overline{V}) \cdot (W - w) - \mathrm{div}(R \nabla W^\top) \cdot (\overline{V} - \overline{v})) + I_1.
\end{aligned}
$$

Proof. Multiplying equation (4.3.2a) by $\frac{\rho}{R}(W - w)$ and equation (4.3.2c) by $\frac{\rho}{R}(\overline{V} - \overline{v})$, and integrating with respect to time and space, we can replace the terms $\rho(\partial_t W \cdot (W - w))$ and

$\rho(\partial_t\overline{V}\cdot(\overline{V}-\overline{v}))$ in the inequality of Proposition 4.29, which gives:

$$\begin{aligned}\mathcal{E}_{NSLK}(t)-\mathcal{E}_{NSLK}(0)\leq{}&\frac{\nu}{2}\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}\mathrm{div}(R\nabla\overline{V})\cdot(\overline{V}-\overline{v})-\frac{\hbar_\nu}{2}\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}\mathrm{div}(R\nabla W^\top)\cdot(\overline{V}-\overline{v})\\&+\int_0^t\int_{\mathbb{T}^d}\rho(\nabla\overline{V}(u-U))\cdot(\overline{V}-\overline{v})+\int_0^t\int_{\mathbb{T}^d}\rho(\nabla W(u-U))\cdot(W-w)\\&+\frac{\nu}{2}\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}\mathrm{div}(R\nabla W)\cdot(W-w)+\frac{\hbar_\nu}{2}\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}\mathrm{div}(R\nabla\overline{V})\cdot(W-w)\\&-2\nu\lambda'\int_0^t\int_{\mathbb{T}^d}|\nabla\sqrt{\rho}|^2+I_1+\lambda' I_2,\end{aligned}$$

where

$$\begin{aligned}I_2=-\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}\nabla R\cdot(W-w)-\int_0^t\int_{\mathbb{T}^d}(\partial_t(H'(R))(\rho-R)+\rho\nabla(H'(R))\cdot u)\\+\int_0^t\int_{\mathbb{T}^d}H'(R)\partial_t R+\int_0^t\int_{\mathbb{T}^d}(R\mathrm{div}U-\rho\mathrm{div}W).\end{aligned}$$

Note that the dissipation term in μ has disappeared from the expression. In fact, by multiplying by $\frac{\rho}{R}(W-w)$, the contribution of (4.3.2a) in the previous inequality is

$$-\mu\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}(W-w)\cdot RW=-\mu\int_0^t\int_{\mathbb{T}^d}\left(\rho|W|^2-\rho w\cdot W\right),$$

which is exactly the opposite of the one in the expression of Proposition 4.29. Using equation (4.3.2a) $H'(R)\partial_t R+H'(R)\mathrm{div}(rU)=0$, and as $R\nabla(H'(R))=\nabla R$, we get by integration by parts that

$$0=\int_0^t\int_{\mathbb{T}^d}(H'(R)\partial_t R-R\nabla(H'(R))\cdot U)=\int_0^t\int_{\mathbb{T}^d}(H'(R)\partial_t R+R\mathrm{div}U).$$

Observing that

$$\partial_t(H'(R))=-\mathrm{div}U-H''(R)\nabla R\cdot U=-\mathrm{div}U-\nabla(H'(R))\cdot U,$$

we get by integration by parts that

$$\begin{aligned}I_2=\int_0^t\int_{\mathbb{T}^d}\rho\nabla(H'(R))\cdot(-W+w+U-u)+\int_0^t\int_{\mathbb{T}^d}\left(\mathrm{div}U(\rho-R)-R\nabla(H'(R))\cdot U-\rho\mathrm{div}W\right)\\=\lambda'\frac{\nu}{\hbar_\nu}\int_0^t\int_{\mathbb{T}^d}(\rho\nabla(H'(R))\cdot(\overline{v}-\overline{V})-\rho\mathrm{div}\overline{V}),\end{aligned}$$

recalling that $U=W-\frac{\nu}{2}V$ and $\overline{V}=\frac{\hbar_\nu}{2}V$, which gives the result. □

We can now state the main theorem of this section:

Theorem 4.31. *Let $(R, W, \overline{V})$ be a strong solution of (4.3.1) and assume $\lambda' > 0$, then any weak solution $(\rho, w, \overline{v})$ of the augmented system (4.3.1) satisfies:*

$$\mathcal{E}_{NSLK}(t) - \mathcal{E}_{NSLK}(0) \leq C\left(1 + \frac{\nu}{\hbar_\nu^2}\right)\int_0^t \mathcal{E}_{NSLK}(s)ds, \tag{4.3.3}$$

where $C = C(R, W, \overline{V})$ denotes a constant independent of λ, μ, ν and $\hbar$.

Proof. The idea is to bound every term on the right hand side of the inequality of Proposition 4.30 by a multiple of $\int_0^t \mathcal{E}_{NSLK}$. Recalling that $W, \overline{V} \in L^\infty([0,T[\,; W^{2,\infty}(\mathbb{T}^d)) \cap W^{1,\infty}([0,T[\,; L^\infty(\mathbb{T}^d))$, we immediately get by Cauchy-Schwarz inequality that

$$\begin{aligned}\int_0^t \int_{\mathbb{T}^d} \rho(\nabla\overline{V}(u-U)) \cdot (\overline{V} - \overline{v}) + \int_0^t \int_{\mathbb{T}^d} \rho(\nabla W(u-U)) \cdot (W - w)\\ \leq C\int_0^t \int_{\mathbb{T}^d} \rho(|u-U|^2 + |\overline{v} - \overline{V}|^2 + |w - W|^2).\end{aligned}$$

For the second line of the inequality of Proposition 4.30, as $\nabla(H'(R)) = \nabla \log R = V$, we get by integration by parts that

$$\begin{aligned}&\lambda' \frac{\nu}{\hbar_\nu}\int_0^t \int_{\mathbb{T}^d} (\rho\nabla(H'(R)) \cdot (\overline{v} - \overline{V}) - \rho\mathrm{div}\overline{V}) - \frac{\nu}{2}\lambda' \int_0^t \int_{\mathbb{T}^d} \frac{1}{\rho}|\nabla\rho|^2\\ &= -2\lambda' \frac{\nu}{\hbar_\nu^2}\int_0^t \int_{\mathbb{T}^d} \left(\rho\overline{V} \cdot (\overline{V} - \overline{v}) - \lambda'\rho\overline{V} \cdot \overline{v}\right) - 2\lambda' \frac{\nu}{\hbar_\nu^2}\int_0^t \int_{\mathbb{T}^d} \rho|\overline{v}|^2 = -2\lambda' \frac{\nu}{\hbar_\nu^2}\int_0^t \int_{\mathbb{T}^d} \rho|\overline{V} - \overline{v}|^2.\end{aligned}$$

For the third and fourth lines of the inequality of Proposition 4.30, we can also show in the same way that

$$\begin{aligned}&\frac{\nu}{2}\int_0^t \int_{\mathbb{T}^d} \frac{\rho}{R}(\mathrm{div}(R\nabla\overline{V}) \cdot (\overline{V} - \overline{v}) + \mathrm{div}(R\nabla W) \cdot (W - w))\\ &\quad + \frac{\hbar_\nu}{2}\int_0^t \int_{\mathbb{T}^d} \frac{\rho}{R}(\mathrm{div}(R\nabla\overline{V}) \cdot (W - w) - \mathrm{div}(R\nabla W^\top) \cdot (\overline{V} - \overline{v})) + I_1\\ &\qquad = \frac{\nu}{2}\int_0^t \int_{\mathbb{T}^d} \rho(V - v) \cdot \left[\nabla\overline{V}(\overline{V} - \overline{v}) + \nabla W^\top (W - w)\right]\\ &\qquad\quad + \frac{\hbar_\nu}{2}\int_0^t \int_{\mathbb{T}^d} \rho(V - v) \cdot \left[\nabla\overline{V}(W - w) - \nabla W(\overline{V} - \overline{v})\right],\end{aligned}$$

hence we get the result by bounding these integrals like the first term. $\square$

4.3.2 The viscous limit $\nu \to 0$

Theorem 4.32. *Let $T > 0$, $\lambda' > 0$ and $(\rho_0, u_0, \overline{v}_0 = \frac{\hbar}{2}\nabla \log \rho_0)$ such that $\mathcal{E}_{ELK}(\rho_0, u_0, \overline{v}_0) < \infty$. Let $(\rho^\nu, u^\nu, \overline{v}^\nu = \frac{\hbar_\nu}{2}\nabla \log \rho^\nu)$ be a weak solution to the augmented NSLK system (4.3.1) in $[0,T[\, \times \mathbb{T}^d$ with initial data $(\rho_0, u_0, \overline{v}_0^\nu)$. Let $(\rho, u, \overline{v})$ be the weak limit of $(\rho^\nu, u^\nu, \overline{v}^\nu)$ when ν tends* *to 0 in the sense*

$$\rho^\nu \rightharpoonup \rho \text{ weakly in } L^\infty([0,T[\,;L^1(\mathbb{T}^d)),$$

$$\sqrt{\rho^\nu}w^\nu \rightharpoonup \sqrt{\rho}u \text{ weakly in } L^\infty([0,T[\,;L^2(\mathbb{T}^d)),$$

$$\sqrt{\rho^\nu}\overline{v}^\nu \rightharpoonup \sqrt{\rho}\overline{v} \text{ weakly in } L^\infty([0,T[\,;L^2(\mathbb{T}^d)),$$

with $\rho\overline{v} = \frac{\hbar}{2}\nabla\rho$. *Then* $(\rho, u, \overline{v})$ *is a dissipative solution of the augmented ELK system* (4.1.4) *in* $[0,T[\,\times\mathbb{T}^d$ *with initial data* $(\rho_0, u_0, \overline{v}_0)$.

Proof. Let U be a smooth enough function such that, defining $(R, \overline{V}, \mathscr{E})$ through (4.1.6) and

$$\overline{V}^\nu = \frac{\hbar}{2}\nabla\log R \quad \text{and} \quad W^\nu = U + \frac{\nu}{2}V,$$

we have the global regularity (i) of Definition 4.28 on $(R, W^\nu, \overline{V}^\nu)$ and the compatibility condition $\int R_0 = \int \rho_0$. We also define

$$\mathscr{E}^\nu(R,U) = R(\partial_t U + U\cdot\nabla U) + \lambda\nabla R + \mu\rho U - 2\frac{\nu}{2}\mathrm{div}(R\mathbb{D}U) - \frac{\hbar}{2}(\mathrm{div}(R\nabla\overline{V})),$$

such that

$$\mathscr{E}^\nu(R,U) = \mathscr{E}(R,U) - \nu\mathrm{div}(R\mathbb{D}U).$$

Using equation (4.1.6a), we can easily check that $(R, W^\nu, \overline{V}^\nu)$ satisfies (4.3.2c) and that

$$\mathscr{E}^\nu(R,U) = R(\partial_t W + U\cdot\nabla W) + \lambda'\nabla R + \mu RW - \frac{\nu}{2}\mathrm{div}(R\nabla W) - \frac{\hbar_\nu}{2}\mathrm{div}(R\nabla\overline{V}).$$

As $\mathscr{E}^\nu(R,U)$ is not necessarily equal to 0, $(R, W^\nu, \overline{V}^\nu)$ fails to be a strong solution of the augmented system (4.3.1) in the sense of Definition 4.28, however we can show in the exact same way as for the proof of Theorem 4.31 that

$$\mathcal{E}_{NSLK}(t) - \mathcal{E}_{NSLK}(0) \le C^\nu\int_0^t \mathcal{E}_{NSLK}(s)ds + b^\nu_{NSLK}(t),$$

where

$$b^\nu_{NSLK}(t) = \int_0^t\int_{\mathbb{T}^d}\frac{\rho^\nu}{R}(\mathscr{E}^\nu(R,U)\cdot(W^\nu - w^\nu)) \quad \text{and} \quad C^\nu = C\left(1 + \frac{\nu}{\hbar_\nu^2}\right).$$

By Gronwall lemma, we get that

$$\mathcal{E}_{NSLK}(t) \le \mathcal{E}_{NSLK}(0)e^{C^\nu t} + C^\nu\int_0^t b^\nu_{NSLK}(s)e^{C^\nu(t-s)}ds + b^\nu_{NSLK}(t). \tag{4.3.4}$$

By definition, we have

$$\frac{1}{2}\int_{\mathbb{T}^d}\rho(|\overline{v}-\overline{V}|^2+|w-W|^2)+\lambda'\int_{\mathbb{T}^d}H(\rho|R)+\mu\int_0^t\int_{\mathbb{T}^d}\rho|w-W|^2\leq\mathcal{E}_{NSLK}(\rho^\nu,w^\nu,\overline{v}^\nu|R,W^\nu,\overline{V}^\nu)(t),$$

so (4.3.4) gives

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{T}^d}\rho^\nu(|\overline{v}^\nu-\overline{V}^\nu|^2+|w^\nu-W^\nu|^2)+\lambda'\int_{\mathbb{T}^d}H(\rho^\nu|R)+\mu\int_0^t\int_{\mathbb{T}^d}\rho|w-W|^2\\ \leq\mathcal{E}_{NSLK}(\rho^\nu,w^\nu,\overline{v}^\nu|R,W^\nu,\overline{V}^\nu)(0)e^{C^\nu t}+C^\nu\int_0^t b^\nu_{NSLK}(s)e^{C^\nu(t-s)}ds+b^\nu_{NSLK}(t).\end{aligned}$$

We are now going to pass to the limit $\nu\to 0$ in the previous inequality. By the lower semi-continuity of the term $\mathcal{E}_{NSLK}(\rho^\nu,w^\nu,\overline{v}^\nu|R,W^\nu,\overline{V}^\nu)$, the left hand-side is not smaller than

$$\frac{1}{2}\int_{\mathbb{T}^d}\rho(|\overline{v}-\overline{V}|^2+|u-U|^2)+\lambda\int_{\mathbb{T}^d}H(\rho|R)+\mu\int_0^t\int_{\mathbb{T}^d}\rho|u-U|^2,$$

which corresponds to $\mathcal{E}_{ELK}(\rho,u,\overline{v}|R,U,\overline{V})(t)$. On the right-hand side, as $b^\nu_{NSLK}(t)$ tends to

$$b_{ELK}(t)(t)=\int_0^t\int_{\mathbb{T}^d}\frac{\rho}{R}\mathscr{E}\cdot(U-u),$$

for all $t\geq 0$, and by the direct limit of $\mathcal{E}_{NSLK}(\rho^\nu,w^\nu,\overline{v}^\nu|R,W^\nu,\overline{V}^\nu)(0)$, we conclude that

$$\mathcal{E}_{ELK}(t)\leq\mathcal{E}_{ELK}(0)e^{Ct}+b_{ELK}(t)+C\int_0^t b_{ELK}(s)e^{C(t-s)}ds,$$

which shows that $(\rho,u,\overline{v})$ is indeed a dissipative solution in the sense of Definition 4.1, as mass conservation property (4.1.5) directly follows from dominated convergence theorem as $\|\rho^\nu(t,\cdot)\|_{L^1(\mathbb{T}^d)}=\|\rho_0\|_{L^1(\mathbb{T}^d)}$ for a.a. $t\in[0,T[$. $\square$

Chapitre 5

The isothermal limit for the compressible Euler equations with damping

Abstract: We consider the isentropic Euler system with damping. We rigorously show the convergence of Barenblatt solutions towards a limit Gaussian profile in the isothermal limit $\gamma \to 1$, and we explicitly compute the propagation and the behavior of Gaussian initial data. We then show the weak L^1 convergence of the density as well as the asymptotic behavior of its first and second moments.

5.1 Introduction

We consider the isentropic compressible Euler equations with frictional damping:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, & (5.1.1a) \\ \partial_t m + \partial_x \left(\dfrac{m^2}{\rho} \right) + \partial_x \rho^\gamma + m = 0, & (5.1.1b) \end{cases}$$

with $\rho(0,x) = \rho_0(x) \geq 0$ and $m(0,x) = m_0(x)$ for $x \in \mathbb{R}$, $t \geq 0$, and the adiabatic gas exponent $\gamma > 1$. Such a system appears in the mathematical modeling of compressible flow through a porous medium, and its study has drawn a lot of attention over the last decades [100]. The global existence of L^∞ weak entropy solutions to the Cauchy problem of (5.1.1) is now well established (see for instance [106] and [78]), and it is known that their long-time asymptotic behavior [76] is governed by the limit diffusive profile:

$$\begin{cases} \partial_t \overline{\rho} = \partial_x^2 \overline{\rho}^\gamma, & (5.1.2a) \\ \overline{m} = -\partial_x \overline{\rho}^\gamma, & (5.1.2b) \end{cases}$$

where equation (5.1.2a) is the porous media equation, whose fundamental solutions are called Barenblatt solutions [16], and equation (5.1.2b) is the famous Darcy law. In [97], the author constructs a class of particular solutions for (5.1.1) which tend to the Barenblatt solutions $\overline{\rho}$

asymptotically in time in $L^\infty(\mathbb{R})$, with an explicit rate $\log(t)/t$. In [77], the authors find the explicit rates

$$\|\rho-\overline{\rho}\|^2_{L^2(\mathbb{R})} \leq C(1+t)^{-k_1+\varepsilon} \quad \text{if } 1<\gamma\leq 2,$$

and

$$\|\rho-\overline{\rho}\|^\gamma_{L^\gamma(\mathbb{R})} \leq C(1+t)^{-k_2+\varepsilon} \quad \text{if } \gamma> 2,$$

for any $\varepsilon>0$, with $k_1=\min\left(\frac{\gamma^2}{(\gamma+1)^2},\frac{\gamma-1}{\gamma}\right)$ and $k_2=\min\left(\frac{\gamma^2}{(\gamma+1)^2},\frac{1}{\gamma}\right)$, for every L^∞ weak entropy solution (ρ,m) of (5.1.1). Unfortunately, the decay rates are not in L^1-norm, which is the natural norm as (5.1.1a) and (5.1.2a) both satisfies the conservation of norm

$$\|\rho(t,.)\|_{L^1(\mathbb{R})}=\|\rho_0\|_{L^1(\mathbb{R})} \quad \text{and} \quad \|\overline{\rho}(t,.)\|_{L^1(\mathbb{R})}=\|\overline{\rho}_0\|_{L^1(\mathbb{R})} \quad \text{for all } t\geq 0.$$

Decay rates in L^1-norm were achieved for a particular range of γ in [79], where the authors show that

$$\|\rho-\overline{\rho}\|_{L^1(\mathbb{R})} \leq C(1+t)^{-\frac{1}{4(\gamma+1)}+\varepsilon} \quad \text{if } 1<\gamma<3,$$

which was recently improved and extended in [67] with the estimate

$$\|\rho-\overline{\rho}\|_{L^1(\mathbb{R})} \leq C(1+t)^{-\frac{1}{(\gamma+1)^2}+\varepsilon} \quad \text{for all } \gamma>1.$$

Throughout the years, a lot of effort has been put to extend the range of γ to finally achieve the full range $\gamma>1$ of barotropic pressure laws $P(\rho)=\rho^\gamma$. The next natural step is the study of the case $\gamma=1$, which leads to the isothermal pressure law $P(\rho)=\rho$. However, in the compressible fluid literature, much fewer results are known for the isothermal Euler system with damping

$$\begin{cases} \partial_t\rho+\partial_x m=0, & (5.1.3\text{a}) \\ \partial_t m+\partial_x\left(\dfrac{m^2}{\rho}\right)+\partial_x\rho+m=0, & (5.1.3\text{b}) \end{cases}$$

which stands as the limit $\gamma\to 1$ of the isentropic system (5.1.1). In [78], the authors show the existence of L^∞ entropy weak solutions to the Cauchy problem of (5.1.3). They also prove, up to a scaling in space $z=x/\sqrt{1+t}$, the L^p_{loc} convergence of the density ρ towards a diffusive profile $\overline{\rho}$ which satisfies the heat equation

$$\partial_t\overline{\rho}=\partial_x^2\overline{\rho}, \tag{5.1.4}$$

in the case $\max(\overline{\rho}_+,\overline{\rho}_-)>0$, where $\overline{\rho}(\pm\infty)=\overline{\rho}_\pm$. However, time dependent Gaussian functions, which stand as particular solutions of the heat equation on the whole space, do not satisfy this condition. In a bounded domain $\Omega\subset\mathbb{R}^d$, the author shows in [119] the exponential convergence of any global solution (ρ, m) with small initial data (ρ_0, m_0) towards the limit profile $(\|\rho_0\|_{L^1}/|\Omega|, 0)$. In fact, at least formally, equation (5.1.3) satisfies the energy inequality

$$\frac{1}{2}\int_{\mathbb{R}} \frac{m^2}{\rho} + \int_{\mathbb{R}} \rho \log \rho + \int_0^t \int_{\mathbb{R}} \frac{m^2}{\rho} \leq E_0 \quad \text{for all } t \geq 0, \tag{5.1.5}$$

but the left-hand side has no definite sign due to the logarithmic contribution in the potential energy, a property that causes many technical difficulties such as a lack of compactness on the whole space $\mathbb{R}$ when the density $t \mapsto \rho(t, .) \in L^1(\mathbb{R})$ vanishes at infinity.
In [33], in the case of the compressible Euler equations without frictional damping, the authors show that Gaussian functions stand as particular solutions of their system. They also prove that every global weak solution disperses and converges to a universal asymptotic Gaussian profile, up to a rescaling by their dispersion rate, in the weak L^1 topology, under some integrability assumptions on the moment of order 2 of the initial data ρ_0. We will show in this paper that this kind of property still holds in the case with damping (5.1.3), and we will study the asymptotic behavior of Gaussian solutions and moments of order 1 and 2 of the density ρ as they stand as a key ingredient for the proof of this kind of feature.

The formal limit $\gamma \to 1$ is singular regarding many features of the isentropic case, especially for the energy inequality of system (5.1.1):

$$\frac{1}{2}\int_{\mathbb{R}} \frac{m^2}{\rho} + \frac{1}{\gamma - 1}\int_{\mathbb{R}} \rho^\gamma + \int_0^t \int_{\mathbb{R}} \frac{m^2}{\rho} \leq E_0. \tag{5.1.6}$$

In this paper we propose to give sense to the formal isothermal limit $\gamma \to 1$, in particular we will rigorously show and illustrate the convergence of the Barenblatt solutions of (5.1.2a) to the Gaussian solutions of (5.1.4), a feature which does not seem to be so much known in the community to the best of the author knowledge.

From now on, (ρ_γ, m_γ) with $\gamma > 1$ will denote a solution of the isentropic Euler system (5.1.1), whereas (ρ, m) will denote a solution of the isothermal Euler system. The overline bar will be use for solutions of the different limit diffusive profiles. The notation C will denote a generic constant $C > 0$.

This paper is organized as follows. In Section 2, we provide energy estimates, assumptions about existence and regularity of solutions of equations (5.1.3), and we state the main results of this paper. In Section 3, we show that Barenblatt solutions converge to a Gaussian profile as $\gamma \to 1$. In Section 4, we explicitly compute the behavior of Gaussian solutions of (5.1.3). In Section 5, we study the evolution of the first and second moments of every weak solution of

(5.1.3). In Section 6, we prove the weak L^1 convergence of every solution towards a universal Gaussian profile. Finally, in Section 7, we give some perspectives about the open question of explicit convergence rate in the isothermal case.

5.2 Assumptions and main results

We first give a notion of global weak solution for the 1-dimensional isothermal Euler equations with damping, and we will assume the global existence of this kind of solutions through the rest of the paper:

Definition 5.1. *We say that (ρ, m) is a* ***weak solution*** *of system* (5.1.3) *in* $[0,T[$ *with initial data $(\rho_0, m_0) \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$, if there exists locally integrable functions $\sqrt{\rho}$, Λ such that, by defining $\rho := \sqrt{\rho}^2$ and $m = \sqrt{\rho}\Lambda$, the following holds:*

- *The global regularity:*

$$\sqrt{\rho} \in L^\infty([0,T[\,; L^2(\mathbb{R})), \quad \Lambda \in L^2([0,T[\,; L^2(\mathbb{R})),$$

with the compatibility condition

$$\sqrt{\rho} \geq 0 \text{ a.e. on } (0,\infty) \times \mathbb{R}, \quad \Lambda = 0 \text{ a.e. on } \{\rho = 0\}\,.$$

- *For any test function $\eta \in \mathcal{C}_0^\infty([0,T[\, \times \mathbb{R})$,*

$$\int_0^T \int_{\mathbb{R}} (\rho \partial_t \eta + m \partial_x \eta) dx dt + \int_{\mathbb{R}} \rho_0 \eta(0) dx = 0,$$

and for any test function $\zeta \in \mathcal{C}_0^\infty([0,T[\, \times \mathbb{R}; \mathbb{R})$,

$$\int_0^T \int_{\mathbb{R}} \left(m \partial_t \zeta + \Lambda^2 \partial_x \zeta + \rho \partial_x \zeta - m \zeta \right) dx dt + \int_{\mathbb{R}} m_0 \zeta(0) dx = 0.$$

Assumption 5.2. *Let $(\rho_0, m_0) \in L^1(\mathbb{R}) \times L^1(\mathbb{R})$. We assume that there exists a weak solution (ρ, m) of system* (5.1.3) *with initial data (ρ_0, m_0) in the sense of Definition 5.1 which satisfies, for all $t \geq 0$, the energy estimate*

$$\frac{1}{2} \int_{\mathbb{R}} \frac{m^2(t,x)}{\rho(t,x)} dx + \int_{\mathbb{R}} \rho(t,x) \log(\rho(t,x)) dx + \int_0^t \int_{\mathbb{R}} \frac{m^2(s,x)}{\rho(s,x)} dx ds \leq C,$$

with the additional regularity

$$(t,x) \mapsto x^2 \rho(t,x) \in L^\infty([0,T[\,; L^1(\mathbb{R})).$$

Remark 5.3. The L^∞ entropy weak solutions of system (5.1.3) described in [78] are in fact weak solutions of (5.1.3) when the initial data $(\rho_0, m_0) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$ satisfies the estimates

$$0 \leq \rho_0 \leq C \quad \text{and} \quad |m_0| \leq C\rho_0 |\log \rho_0|$$

(see [78, Definition 1] for a precise definition of L^∞ entropy weak solutions of system (5.1.3)). Note that the existence of L^∞ entropy weak solutions of the isothermal Euler system without damping is proved in [92] under the same initial data conditions.

Remark 5.4. This assumption is also satisfied for instance in the case of Gaussian initial data, where ρ_0 is a Gaussian function and m_0 is of the form $m_0 = (\beta x + c)\rho_0$, with β, $c \in \mathbb{R}$ (see Section 5.4).

We now have to introduce several quantities in order to state our results. Let τ be the unique $\mathcal{C}^\infty([0,\infty))$ solution of the differential equation

$$\ddot{\tau} = \frac{2}{\tau} - \dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = 0.$$

From [41], we know that this function satisfies, as $t \to +\infty$,

$$\tau(t) \sim 2\sqrt{t} \quad \text{and} \quad \dot{\tau}(t) \sim \frac{1}{\sqrt{t}}.$$

We also introduce the Gaussian function

$$\Gamma := e^{-y^2}.$$

We make the change of variable $y = x/\tau(t)$,

$$\rho(t,x) = \frac{1}{\tau(t)} R\left(t, \frac{x}{\tau(t)}\right) \frac{\|\rho_0\|_{L^1}}{\|\Gamma\|_{L^1}} \quad \text{and} \quad m(t,x) = \frac{1}{\tau(t)^2} M\left(t, \frac{x}{\tau(t)}\right) + \frac{\dot{\tau}(t)}{\tau(t)} y R\left(t, \frac{x}{\tau(t)}\right).$$

We first remark that this change of variable implies that for all $t \geq 0$,

$$\int_{\mathbb{R}} R(t,y)dy = \int_{\mathbb{R}} R_0(y)dy = \int_{\mathbb{R}} \Gamma(y)dy.$$

System (5.1.3) becomes, in the terms of the new unknown (R, M),

$$\begin{cases} \partial_t R + \dfrac{1}{\tau^2}\partial_y M = 0, & (5.2.1a) \\ \partial_t M + \dfrac{1}{\tau^2}\partial_y \left(\dfrac{M^2}{R}\right) + \partial_y R + 2yR + M = 0, & (5.2.1b) \end{cases}$$

which have the following energy inequality:

$$\mathcal{E}(t) + \int_0^t \frac{\dot{\tau}(t)}{\tau^3(t)} \int_{\mathbb{R}} \frac{M^2}{R} dyds \leq C, \tag{5.2.2}$$

where

$$\mathcal{E}(t) = \frac{1}{2\tau(t)^2} \int_{\mathbb{R}} \frac{M^2}{R} dy + \int_{\mathbb{R}} R \log\left(\frac{R}{\Gamma}\right) dy. \tag{5.2.3}$$

In fact, differentiating with respect to time the left part of (5.2.2), using equations (5.2.1a) and (5.2.1b) and by integration by parts, we get that

$$\frac{d}{dt}\left(\frac{1}{2\tau(t)^2}\int_{\mathbb{R}} \frac{M^2}{R} + \int_{\mathbb{R}} R\log R + \int_{\mathbb{R}} y^2 R + \int_0^t \frac{\dot{\tau}(s)}{\tau(t)^3}\int_{\mathbb{R}} \frac{M^2}{R} ds\right) = -\frac{1}{\tau^2}\int_{\mathbb{R}} \frac{M^2}{R} \leq 0.$$

Note that from the Csiszár-Kullback inequality (see e.g. Lemma 4 or [3])

$$\int_{\mathbb{R}} R \log\left(\frac{R}{\Gamma}\right) \geq \frac{1}{2\|R_0\|_{L^1}} \|R - \Gamma\|_{L^1}^2 \geq 0,$$

we get that for all $t \geq 0$, $\mathcal{E}(t) \geq 0$.

Denoting by $\mathcal{E}_{\text{kin}}$ the kinetic energy

$$\mathcal{E}_{\text{kin}}(t) := \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t,y)}{R(t,y)} dy$$

for all $t \geq 0$, the asymptotics of τ and $\dot{\tau}$ and the following Lemma 5.14 give that

$$\mathcal{E}_{\text{kin}} \in L^\infty(\mathbb{R}_+) \quad \text{and} \quad \int_0^\infty \frac{\mathcal{E}_{\text{kin}}(t)}{1+t} dt < \infty,$$

so we know there exists a sequence $(t_n)_n$, such that $t_n \to \infty$ and

$$\mathcal{E}_{\text{kin}}(t_n) \to 0 \quad \text{as } n \to \infty.$$

Unfortunately, we have no proof that this property is actually satisfied uniformly in time, even if it seems to be a reasonable assumption that we state in the following:

Assumption 5.5. *We assume that*

$$\mathcal{E}_{\text{kin}}(t) = \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t,y)}{R(t,y)} dy \to 0 \quad \textit{as } t \to \infty.$$

This assumption is for instance satisfied for Gaussians solutions of system (5.1.3), with an explicit convergence rate in $t^{-1/2}$ (see Remark 5.16 below). We now state the two main results of this paper:

Proposition 5.6. *Under Assumption 5.2 and Assumption 5.5, we have*

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} R(t,y)dy \longrightarrow \int_{\mathbb{R}} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} \Gamma(y)dy \quad \text{as } t \to \infty.$$

Proposition 5.7. *Under Assumption 5.2, we have*

$$R(t) \underset{t\to\infty}{\rightharpoonup} \Gamma \text{ weakly in } L^1(\mathbb{R}).$$

Remark 5.8. In Proposition 5.6, Assumption 5.5 is only required to show the convergence of the moment of order 2. Without this assumption, we are only able to show that the moment of order 2 is uniformly bounded (see Lemma 5.14).

Remark 5.9. Note that Propositions 5.6 and 5.7 are also satisfied in higher space dimensions $\mathbb{R}^d$ with $d \geq 1$, in the sense that

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} R(t,y)dy \longrightarrow \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \Gamma(y)dy \quad \text{as } t \to \infty$$

with $\Gamma = e^{-|y|^2}$, and

$$R(t) \underset{t\to\infty}{\rightharpoonup} \Gamma \text{ weakly in } L^1(\mathbb{R}^d).$$

Of course, our notion of global weak solution has to be adapted to the d-dimensional case, as well as system (5.1.3), and we refer to [33] and [41] for some d-dimensional analogue of Definition 5.1.

In the same vein, the following discussion about the particular Gaussian solutions of (5.1.3) can easily be generalized to any dimension d by a tensorization property. We also refer the reader to [33] and [41] for some d-dimensional analogue of system (5.4.1)-(5.4.2)-(5.4.3).

5.3 The limit $\gamma \to 1$ of Barenblatt's solutions

We consider the unique fundamental solution of the porous media equation (with the Dirac delta function as initial data)

$$\begin{cases} \partial_t \overline{\rho}_\gamma = \partial_x^2 \left[\left(\overline{\rho}_\gamma \right)^\gamma \right], \\ \overline{\rho}_\gamma(-1,x) = \lambda\delta(x), \quad \lambda > 0, \end{cases} \tag{5.3.1}$$

which is called the Barenblatt solution [7]. Note that we take the initial data at $t = -1$ to avoid the singularity at $t = 0$. We recall that this solution can be written

$$\overline{\rho}_\gamma(t,x) = (1+t)^{-\frac{1}{\gamma+1}} \left[A - B\xi^2\right]_+^{\frac{1}{\gamma-1}} \tag{5.3.2}$$

with $\xi = x(1+t)^{-\frac{1}{\gamma+1}}$, $[f]_+ = \max\{f, 0\}$, $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$ and A determined by

$$2A^{\frac{\gamma+1}{2(\gamma-1)}} B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\cos\theta)^{\frac{\gamma+1}{\gamma-1}} d\theta = \lambda. \tag{5.3.3}$$

Moreover, $\overline{\rho}_\gamma$ is continuous on $\mathbb{R}$. It satisfies the conservation of mass

$$\int_{\mathbb{R}} \overline{\rho}_\gamma(t,x)dx = \lambda$$

for all $t \geq -1$, and has compact support for any finite time $T > 0$, namely

$$\overline{\rho}_\gamma = 0 \quad \text{if} \quad |\xi| \geq \sqrt{A/B}.$$

The power $\frac{1}{\gamma-1}$ in (5.3.2) could make the limit $\gamma \to 1$ unclear, however we can show:

Proposition 5.10. *We make the change of variable $\xi = x(1+t)^{-\frac{1}{\gamma+1}}$ and*

$$\overline{\rho}_\gamma(t,x) = (1+t)^{-\frac{1}{\gamma+1}} \mathcal{B}_\gamma\left(x(1+t)^{-\frac{1}{\gamma+1}}\right)$$

which preserves the L^1-norm. Then,

$$\mathcal{B}_\gamma \to \frac{\lambda}{2\sqrt{\pi}} e^{-\frac{\xi^2}{4}} \quad \text{in } L^\infty(\mathbb{R}) \text{ as } \gamma \to 1.$$

In particular, for all $x \in \mathbb{R}$, the pointwise estimate

$$|\overline{\rho}_\gamma(t,x)| \leq A^{\frac{1}{\gamma-1}}(1+t)^{-\frac{1}{\gamma+1}}$$

has the following continuous limit when $\gamma \to 1$:

$$|\overline{\rho}(t,x)| \leq \frac{\lambda}{2\sqrt{\pi}}(1+t)^{-\frac{1}{2}},$$

where $\overline{\rho}$ denotes the Gaussian limit of the Barenblatt profile $\overline{\rho}_\gamma$ as $\gamma \to 1$ (namely $\overline{\rho} := \overline{\rho}_1$, we omit the index γ when $\gamma = 1$).

Proof. We have, for $|\xi| < \sqrt{A/B}$,

$$\mathcal{B}_\gamma(\xi) = \left[A - B\xi^2\right]_+^{\frac{1}{\gamma-1}} = \exp\left(\frac{1}{\gamma-1}\log A + \frac{1}{\gamma-1}\log\left(1 - \frac{B}{A}\xi^2\right)\right),$$

as $A > 0$ from expression (5.3.3) and $\frac{B}{A}\xi^2 < 1$. We have now two terms to handle when $\gamma \to 1$. Taking the logarithm in equation (5.3.3), and recalling that $B = \frac{\gamma-1}{2\gamma(\gamma+1)}$, we have

$$\frac{1}{\gamma-1}\log A = \frac{2}{\gamma+1}\left(\log\left(\frac{\lambda}{2\sqrt{2\gamma(\gamma+1)}}\right) - \log\left(\frac{1}{\sqrt{\gamma-1}}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}d\theta\right)\right).$$

We denote $\varepsilon = \gamma - 1$, so we look at the limit $\varepsilon \to 0$ of the following Wallis integral

$$W_{1+\frac{2}{\varepsilon}} = \int_0^{\frac{\pi}{2}}(\cos\theta)^{1+\frac{2}{\varepsilon}}d\theta = \int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}d\theta.$$

We recall the well-known equivalent of Wallis integral when $\varepsilon \to 0$,

$$W_{1+\frac{2}{\varepsilon}} = \sqrt{\frac{\pi}{2\left(1+\frac{2}{\varepsilon}\right)}} + O\left(\frac{1}{1+\frac{2}{\varepsilon}}\right) = \sqrt{\frac{\pi\varepsilon}{4+2\varepsilon}} + O(\sqrt{\varepsilon}),$$

so it is easy to check that

$$\frac{1}{\sqrt{\gamma-1}}\int_0^{\frac{\pi}{2}}(\cos\theta)^{\frac{\gamma+1}{\gamma-1}}d\theta = \sqrt{\frac{\pi}{4}}\left(1 + O(\sqrt{\varepsilon})\right)$$

as $\gamma \to 1$, and then

$$\exp\left(\frac{1}{\gamma-1}\log A\right) = \frac{\lambda}{2\sqrt{\pi}}\frac{1}{1+O(\sqrt{\varepsilon})}.$$

For the second term, as we have

$$\frac{B}{A} = B\left(\frac{2W_{1+\frac{2}{\varepsilon}}}{\lambda B^{\frac{1}{2}}}\right)^{\frac{2\varepsilon}{2+\varepsilon}} = \frac{\varepsilon}{4} + O(\varepsilon^2)$$

from the previous equivalent, we write

$$\frac{1}{\varepsilon}\log\left(1 - \frac{B}{A}\xi^2\right) = -\frac{1}{\varepsilon}\sum_{n\geq 1}\frac{1}{n}\left(\frac{B}{A}\xi^2\right)^n \sim -\frac{1}{4}\xi^2 + \varepsilon\sum_{n\geq 2}\frac{1}{n}\left(\frac{\xi^2}{4}\right)^n\varepsilon^{n-2},$$

so finally

$$\exp\left(\frac{1}{\gamma-1}\log\left(1 - \frac{B}{A}\xi^2\right)\right) \longrightarrow e^{-\frac{\xi^2}{4}},$$

and
$$\mathcal{B}_\gamma(\xi) - \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}} = \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}}\left(\frac{1}{1+O(\varepsilon)}e^{O(\varepsilon)} - 1\right) = O(\varepsilon).$$
On $|\xi| \geq \sqrt{A/B}$, we know that $\mathcal{B}_\gamma(\xi) = 0$, and as $\xi \to e^{-\xi^2/4}$ is a Gaussian,
$$\sup_{|\xi|\geq\sqrt{A/B}}\left|\mathcal{B}_\gamma(\xi) - \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}}\right| = \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{A}{B}} \leq Ce^{-\frac{1}{\varepsilon}},$$
so finally
$$\sup_{\xi\in\mathbb{R}}\left|\mathcal{B}_\gamma(\xi) - \frac{\lambda}{2\sqrt{\pi}}e^{-\frac{\xi^2}{4}}\right| \leq Ce^{-\frac{1}{\gamma-1}} + O(\gamma-1) \longrightarrow 0 \text{ as } \gamma \to 1,$$
which ends the proof. □

In order to illustrate this result, we make the following plot of the function $\mathcal{B}_\gamma$ in Figure 5.1 for several values of γ (namely $\gamma = 2$, 1.5 and 1.1), and the initial condition constant $\lambda = 1$. We also plot the limit Gaussian profile $e^{-\xi^2/4}/2\sqrt{\pi}$, which corresponds to the case $\gamma = 1$.

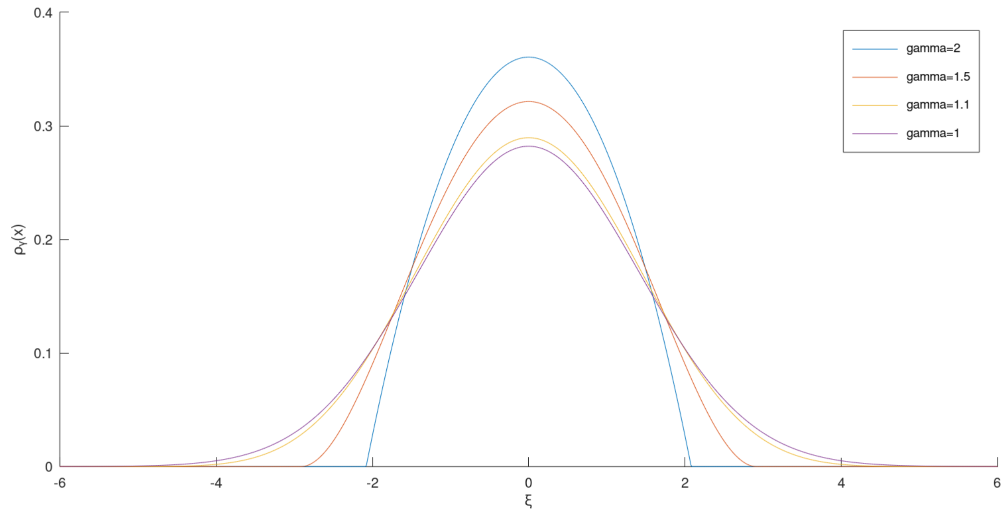

Figure 5.1 – Convergence of $\mathcal{B}_\gamma$ towards its limit Gaussian profile.

5.4 Gaussian solutions

In this section, following [33], we seek for particular Gaussian solutions of (5.1.3) of the form
$$\overline{\rho}(t,x) = b(t)e^{-\alpha(t)(x-\overline{x}(t))^2}$$

and

$$\overline{m}(t,x) = (\beta(t)x + c(t))\overline{\rho}(t,x),$$

with the initial conditions $b(0) = b_0 > 0$, $\alpha(0) = \alpha_0 > 0$, $\beta(0) = \beta_0 \in \mathbb{R}$, $c(0) = c_0 \in \mathbb{R}$. As system (5.1.3) is invariant by translation, we assume that $\overline{x}(0) = 0$. We also denote $\rho_0 = \overline{\rho}(0,.)$ and $m_0 = \overline{m}(0,.)$. Plugging these expressions into (5.1.3), we obtain the following set of differential equations:

$$\dot{\alpha} + 2\alpha b = 0, \quad \dot{\beta} + \beta^2 + \beta = 2\alpha, \tag{5.4.1}$$

$$\dot{\overline{x}} = \beta\overline{x} + c, \quad \dot{b} = b(\dot{\alpha}\overline{x}^2 + 2\alpha c\overline{x}(\dot{\overline{x}} - c) - \beta), \tag{5.4.2}$$

$$\dot{c} + \beta c + c = -2\alpha\overline{x}. \tag{5.4.3}$$

In order to solve this system, mimicking [93], we can check that the two equations of (5.4.1) are satisfied if and only if α and β are of the form

$$\alpha(t) = \frac{\alpha_0}{\tau(t)^2}, \quad \beta(t) = \frac{\dot{\tau}(t)}{\tau(t)},$$

where τ is the global solution of the differential equation

$$\ddot{\tau} = \frac{2\alpha_0}{\tau} - \dot{\tau}, \quad \tau(0) = 1, \quad \dot{\tau}(0) = \beta_0. \tag{5.4.4}$$

We recall (see [41]) that there exists a unique global solution $\tau \in \mathcal{C}^\infty([0,\infty))$ to this nonlinear ODE, and that this solution remains uniformly bounded from below by a strictly positive constant.

Plugging these expressions into the second equation of (5.4.2), we also get that

$$b(t) = \frac{b_0}{\tau(t)}.$$

We also get an expression of c in terms of the center $\overline{x}$ of our Gaussian:

$$c(t) = c_0 - \left(1 + \frac{\dot{\tau}}{\tau}\right)\overline{x}.$$

Let us first show that the center of the Gaussian has an explicit expression, which does not depend on the function τ:

Proposition 5.11. *We have*

$$\overline{x}(t) = \frac{1}{\|\rho_0\|_{L^1(\mathbb{R})}} \left(\int_{\mathbb{R}} x\rho_0 - (1 - e^{-t}) \int_{\mathbb{R}} m_0 \right).$$

In particular, there exists $\overline{x}_\infty \in \mathbb{R}$ such that

$$\overline{x}(t) \to \overline{x}_\infty \quad \text{as} \quad t \to \infty.$$

Proof. We know that $\overline{\rho}$ is a Gaussian function centered in $\overline{x}$, so we get that for all $t \geq 0$,

$$\int_{\mathbb{R}} (x - \overline{x}(t))\overline{\rho}(t, x)dx = 0,$$

hence we only have to study the first moment of $\overline{\rho}$ keeping in mind that

$$\overline{x}(t) = \frac{1}{\|\rho_0\|_{L^1(\mathbb{R})}} \int_{\mathbb{R}} x\overline{\rho}(t, x)dx.$$

Integrating equation (5.1.3b) over $\mathbb{R}$ we get that

$$\frac{d}{dt} \left(\int_{\mathbb{R}} \overline{m}(t, x)dx \right) = \int_{\mathbb{R}} \partial_t \overline{m}(t, x)dx = - \int_{\mathbb{R}} \overline{m}(t, x)dx,$$

so we get by integration by parts that

$$\frac{d}{dt} \left(\int_{\mathbb{R}} x\overline{\rho}(t, x)dx \right) = \int_{\mathbb{R}} x\partial_t \overline{\rho}(t, x)dx = - \int_{\mathbb{R}} \overline{m}(t, x)dx = -e^{-t} \int_{\mathbb{R}} m_0(x)dx,$$

so finally integrating this expression over $[0, t]$ we get the result. □

In order to know the behavior of our Gaussian functions when $t \to \infty$, we only need to get some equivalents of the real function τ, which is the goal of the following proposition. We refer to [41] for a complete study of the differential equation (5.4.4).

Proposition 5.12. *Let τ be the unique $\mathcal{C}^\infty([0, \infty))$ solution of the differential equation* (5.4.4)*, then we have*

$$\tau(t) \sim 2\sqrt{\alpha_0 t} \quad \text{and} \quad \dot{\tau}(t) \sim \sqrt{\frac{\alpha_0}{t}}.$$

In particular,

$$\alpha(t) \sim \frac{1}{4t}, \quad \beta(t) \sim \frac{1}{2t}, \quad b(t) \sim \frac{b_0}{2\sqrt{\alpha_0 t}} \quad \text{and} \quad c(t) \to c_0 - \overline{x}_\infty.$$

5.5 Evolution of certain quantities

In this section, we are going to give two propositions that describe the behavior of respectively the first and second moment of the renormalized density R.

Proposition 5.13. *We denote*

$$I_1 = \int_{\mathbb{R}} M dy \quad \textit{and} \quad I_2 = \int_{\mathbb{R}} yR dy.$$

Then,

- *if* $I_1(0) \neq I_2(0)$,

$$I_1(t) \sim \frac{I_1(0) - I_2(0)}{\sqrt{t}} \quad \textit{and} \quad I_2(t) \sim \frac{I_2(0) - I_1(0)}{2\sqrt{t}} \quad \textit{as } t \to \infty,$$

- *if* $I_1(0) = I_2(0)$,

$$I_1(t) \sim 4te^{-t} I_1(0) \quad \textit{and} \quad I_2(t) \sim \frac{e^{-t}}{\sqrt{t}} I_1(0) \quad \textit{as } t \to \infty.$$

Proof. First off, mimicking the calculus of the proof of Proposition 5.11, we have by integrating equation (5.1.3b) over $\mathbb{R}$ and integration by parts that for all $t \geq 0$,

$$\frac{d}{dt}\left(\int_{\mathbb{R}} m(t,y)dy\right) = -\int_{\mathbb{R}} m(t,y)dy,$$

so we easily get that

$$\int_{\mathbb{R}} m(t,y)dy = e^{-t}\int_{\mathbb{R}} m_0(y)dy.$$

Now, by integration by parts, and using (5.2.1), we get the system of coupled differential equations:

$$\dot{I}_1 = \int_{\mathbb{R}} \partial_t M = -\int_{\mathbb{R}} \left[\frac{1}{\tau^2}\partial_x\left(\frac{M^2}{R}\right) + \partial_y R + 2yR + M\right] = -I_1 - 2I_2,$$

and

$$\dot{I}_2 = \int_{\mathbb{R}} y\partial_t R = -\int_{\mathbb{R}} y\frac{\partial_x M}{\tau^2} = \frac{1}{\tau^2} I_1.$$

We denote $\tilde{I}_2 = \tau I_2$, so that

$$\dot{\tilde{I}}_2 = \dot{\tau} I_2 + \tau \dot{I}_2 = \dot{\tau} I_2 + \frac{1}{\tau} I_1,$$

and

$$\ddot{\tilde{I}}_2 = \ddot{\tau} I_2 + \dot{\tau}\dot{I}_2 - \frac{\dot{\tau}}{\tau^2} I_1 + \frac{1}{\tau}\dot{I}_1 = -(\dot{\tau} I_2 + \frac{1}{\tau} I_1) = -\dot{\tilde{I}}_2,$$

hence

$$\dot{\tilde{I}}_2(t) = \dot{\tilde{I}}_2(0)e^{-t}$$

and

$$\tilde{I}_2(t) = \tilde{I}_2(0) - \dot{\tilde{I}}_2(0)(e^{-t} - 1).$$

We easily compute the initial conditions

$$\dot{\tilde{I}}_2(0) = I_1(0) \quad \text{and} \quad \tilde{I}_2(0) = I_2(0),$$

so finally

$$I_2(t) = \frac{1}{\tau(t)}\left[I_2(0) - I_1(0)(1 - e^{-t})\right] \sim \frac{1}{\tau(t)}\left[I_2(0) - I_1(0)\right]$$

when the initial condition are not well prepared in the sense that $I_2(0) \neq I_1(0)$. Note that if $I_2(0) = I_1(0)$, we have

$$I_2(t) = \frac{e^{-t}}{\tau(t)} I_1(0).$$

We know recall that

$$\int_{\mathbb{R}} m = \frac{1}{\tau^2}\int_{\mathbb{R}} M + \frac{\dot{\tau}}{\tau}\int yR = e^{-t} I_1(0),$$

from the previous calculations, so

$$I_1(t) = \tau^2(t) e^{-t} I_1(0) - \dot{\tau}(t)\left[I_2(0) - I_1(0)(1 - e^{-t})\right],$$

which concludes the proof. □

We now give a useful lemma, which induces the boundedness of the second moment of the density R uniformly with respect to time:

Lemma 5.14. *There holds*

$$\sup_{t \geq 0} \int_{\mathbb{R}} R(t,y)(1 + y^2 + |\log R(t,y)|)dy < \infty \tag{5.5.1}$$

and

$$\int_0^\infty \frac{\dot{\tau}(t)}{\tau^3(t)} \int_{\mathbb{R}} \frac{M^2}{R} dydt < \infty. \tag{5.5.2}$$

Proof. Since $\mathcal{E}(t) \geq 0$, (5.5.2) follows from (5.2.2). We define

$$\mathcal{E}_+ := \frac{1}{2\tau^2}\int_{\mathbb{R}} \frac{M^2}{R} dy + \int_{R>1} R\log R + \int_{\mathbb{R}} y^2 R,$$

such that Equation (5.2.2) gives

$$\mathcal{E}_+(t) + \int_0^t \frac{\dot{\tau}(t)}{\tau^3(t)} \int_{\mathbb{R}} \frac{M^2}{R} dyds \leq C + \int_{R<1} R\log\left(\frac{1}{R}\right).$$

The last term is controlled by

$$\int_{R<1} R\log\left(\frac{1}{R}\right) \leq C_\varepsilon \int_{\mathbb{R}} R^{1-\varepsilon}$$

for all $\varepsilon > 0$ arbitrary small. By an interpolation formula, we get that

$$\int_{\mathbb{R}} R^{1-\varepsilon} \lesssim \|R\|_{L^1}^{1-\frac{3\varepsilon}{2}} \|y^2 R\|_{L^1}^{\varepsilon/2}$$

for all $0 < \varepsilon < \frac{2}{3}$. This implies that for all $t \geq 0$,

$$\mathcal{E}_+(t) + \int_0^t \frac{\dot\tau(t)}{\tau^3(t)} \int_{\mathbb{R}} \frac{M^2}{R} dyds \lesssim 1 + \mathcal{E}_+^{\varepsilon/2}(t),$$

thus $\mathcal{E}_+(t) \in L^\infty(\mathbb{R}_+)$, and estimate (5.5.1) follows. □

With Assumption 5.5, we can then show a much better result on the second moment of the density R than its boundedness, which is the convergence of its second moment towards the second moment of the Gaussian Γ:

Proposition 5.15. *We denote*

$$J_1 = \int_{\mathbb{R}} y^2 (R - \Gamma) dy.$$

Then, under Assumption 5.5,

$$|J_1(t)| \to 0 \quad \textit{as } t \to +\infty.$$

Proof. Denoting

$$J_2 = \int_{\mathbb{R}} yM dy,$$

we first write the system of differential equations satisfied by J_1 and J_2. By integration by parts and using equation (5.2.1a), we compute

$$\dot J_1 = \frac{d}{dt}\int_{\mathbb{R}} y^2 \partial_t (R-\Gamma) = -\int_{\mathbb{R}} y^2 \frac{\partial_y M}{\tau^2} = \frac{2}{\tau^2}\int_{\mathbb{R}} yM = \frac{2}{\tau^2} J_2.$$

Still by integration by parts, and using equation (5.2.1b), we get that

$$\dot J_2 = \int_{\mathbb{R}} y\partial_t M = -\int_{\mathbb{R}} y\left[\frac{1}{\tau^2}\partial_y\left(\frac{M^2}{R}\right) + \partial_y R + 2yR + M\right] = \frac{1}{\tau^2}\int_{\mathbb{R}} \frac{M^2}{R} + \int_{\mathbb{R}} R - \int_{\mathbb{R}} 2y^2 R - \int_{\mathbb{R}} yM.$$

Here we introduce the function Γ, writing

$$\int_{\mathbb{R}} R - 2\int_{\mathbb{R}} y^2 R = \int_{\mathbb{R}} \Gamma - 2\int_{\mathbb{R}} y^2\Gamma - \int_{\mathbb{R}} 2y^2(R-\Gamma),$$

and we can remark by an easy calculation that

$$\int_{\mathbb{R}} y^2\Gamma = \frac{1}{2}\int_{\mathbb{R}}\Gamma,$$

so that finally

$$\dot{J}_2 + J_2 + 2J_1 = \frac{1}{\tau^2}\int_{\mathbb{R}}\frac{M^2}{R}.$$

We denote $\tilde{J}_1 = \tau J_1$, so that

$$\dot{\tilde{J}}_1 = \dot{\tau}J_1 + \tau\dot{J}_1 = \dot{\tau}J_1 + \frac{2}{\tau}J_2,$$

and

$$\ddot{\tilde{J}}_1 = \ddot{\tau}J_1 + \dot{\tau}\dot{J}_1 - \frac{2\dot{\tau}}{\tau}J_2 + \frac{2}{\tau}\dot{J}_2 = -\dot{\tilde{J}}_1 - \frac{2}{\tau}J_1 + \frac{2}{\tau^3}\int_{\mathbb{R}}\frac{M^2}{R},$$

so we write

$$\ddot{\tilde{J}}_1 + \dot{\tilde{J}}_1 + \frac{1}{\tau^2}\tilde{J}_1 = \frac{2}{\tau^3}\int_{\mathbb{R}}\frac{M^2}{R}.$$

Rather than trying to solve directly this non-autonomous differential equation of order 2, we will work on the following approximate equation, for $t \geq 1$,

$$\ddot{f} + \dot{f} + \frac{1}{4t}f = \frac{2}{\tau}\mathcal{E}_{\text{kin}}. \tag{5.5.3}$$

In fact, denoting $w = \tilde{J}_1 - f$, we see that w satisfies the differential equation

$$\ddot{w} + \dot{w} + \frac{1}{4t}w + \left(\frac{1}{\tau^2} - \frac{1}{4t}\right)\tilde{J}_1 = 0,$$

with

$$\left(\frac{1}{\tau^2} - \frac{1}{4t}\right) = -\frac{(\tau - 2\sqrt{t})(\tau + 2\sqrt{t})}{4t\tau^2} = O\left(\frac{1}{t^{\frac{3}{2}}}\right)$$

as we actually know from [41, Proposition 3.11] that there exists a time $T > 0$ such that, for all $t \geq T$,

$$2\sqrt{t} \leq \tau(t) \leq C_T + 2\sqrt{t}$$

with $C_T \geq 0$, so the function $t \mapsto \tau(t) - 2\sqrt{t}$ is uniformly bounded, and from the equivalent $\tau(t) \sim 2\sqrt{t}$ we get that

$$\frac{\tau + 2\sqrt{t}}{4t\tau^2} \sim \frac{\sqrt{t}}{t^2} = \frac{1}{t^{\frac{3}{2}}}.$$

The homogeneous part of equation (5.5.3) can be written as a Kummer's type equation

$$t\ddot{f} + t\dot{f} + \frac{1}{4}f = 0$$

which has two fundamental independent solutions (see [1, Chapter 13])

$$f_1(t) = t\mathcal{M}\left(\frac{5}{4}, 2, -t\right) = te^{-t}\mathcal{M}\left(\frac{3}{4}, 2, t\right)$$

and

$$f_2(t) = e^{-t}\mathcal{U}\left(-\frac{1}{4}, 0, t\right) = te^{-t}\mathcal{U}\left(\frac{3}{4}, 2, t\right),$$

where $\mathcal{M}(a, b, z)$ and $\mathcal{U}(a, b, z)$ respectively denotes the Krummer's and the Tricomi's function, which both stand as confluent hypergeometric functions and are independent solutions of the Kummer's equation

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0.$$

In particular, from the asymptotic properties [1]

$$\mathcal{M}(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}\left(1 + O\left(|z|^{-1}\right)\right), \quad z > 0,$$

and

$$\mathcal{U}(a, b, z) = z^{-a}\left(1 + O\left(|z|^{-1}\right)\right),$$

we get the following asymptotic for our fundamental solutions

$$f_1(t) \sim \frac{C}{t^{\frac{1}{4}}}\left(1 + O\left(\frac{1}{t}\right)\right) \tag{5.5.4}$$

and

$$f_2(t) \sim e^{-t}t^{\frac{1}{4}}\left(1 + O\left(\frac{1}{t}\right)\right). \tag{5.5.5}$$

From the classical theory of linear differential equations, we know that every solution of equation (5.5.3) can be written under the form

$$f(t) = c_1 f_1(t) + c_2 f_2(t) + f_0(t), \tag{5.5.6}$$

where c_1 and c_2 denote two real numbers, and f_0 is a particular solution of (5.5.3). We easily compute the Wronskian function

$$W(t) = f_1(t)\dot{f}_2(t) - \dot{f}_1(t)f_2(t)$$

by solving the differential equation

$$\dot{W} = \dot{f}_1\dot{f}_2 + f_1\ddot{f}_2 - \ddot{f}_1 f_2 - \dot{f}_1\dot{f}_2 = f_1(-\dot{f}_2 - \frac{1}{\tau^2}f_2) + (\dot{f}_1 + \frac{1}{\tau^2}f_1)f_2 = -W,$$

which leads to

$$W(t) = W_0 e^{-t}.$$

Then, by a variation of constant formula, we can find a particular solution of (5.5.3) under the form

$$f_0(t) = f_2(t) \int_0^t \frac{f_1(s)}{W(s)} \frac{\mathcal{E}_{\mathrm{kin}}(s)}{\tau(s)} ds - f_1(t) \int_0^t \frac{f_2(s)}{W(s)} \frac{\mathcal{E}_{\mathrm{kin}}(s)}{\tau(s)} ds,$$

so that using Assumption 5.5 and the equivalents (5.5.4) and (5.5.5), we get that every solution of (5.5.3) has the asymptotic:

$$|f(t)| = o(\sqrt{t}).$$

In fact, in the expression of (5.5.6),

$$\left| f_1(t) \int_0^t e^s f_2(s) \frac{\mathcal{E}_{\mathrm{kin}}(s)}{\tau(s)} ds \right| \lesssim \frac{1}{t^{\frac{1}{4}}} \int_1^t s^{-\frac{1}{4}} \mathcal{E}_{\mathrm{kin}}(s) ds = o(\sqrt{t}),$$

and every other terms is bounded as $t \to \infty$. The same result applies for the function $w = \tilde{J}_1 - f$, so we can write that for all $t \geq t_0 \geq 0$,

$$\begin{aligned} \tilde{J}_1(t) = f(t) + c_1 f_1(t) + c_2 f_2(t) + f_2(t) \int_{t_0}^t \frac{f_1(s)}{W(s)} \left(\frac{1}{4s} - \frac{1}{\tau(s)^2} \right) \tilde{J}_1(s) ds \\ - f_1(t) \int_{t_0}^t \frac{f_2(s)}{W(s)} \left(\frac{1}{4s} - \frac{1}{\tau(s)^2} \right) \tilde{J}_1(s) ds. \quad (5.5.7) \end{aligned}$$

We already know from Lemma 5.14 that $\tilde{J}_1$ is $O(\sqrt{t})$, and as we have just shown that f is $o(\sqrt{t})$, for $\varepsilon > 0$ fixed, there exists $t_0 \geq 0$ such that, for all $t \geq t_0$,

$$|\tilde{J}_1(t)| \leq C\sqrt{t}, \quad \left| \frac{1}{4s} - \frac{1}{\tau(s)^2} \right| \leq C t^{-\frac{3}{2}} \quad \text{and} \quad \frac{1}{\sqrt{t}} \left| f(t) + c_1 f_1(t) + c_2 f_2(t) \right| \leq \frac{\varepsilon}{2}.$$

Injecting these inequalities in the right-hand side of (5.5.7), we have

$$\frac{1}{\sqrt{t}} |\tilde{J}_1(t)| \leq \frac{\varepsilon}{2} + \frac{C}{t^{\frac{1}{4}}} + \frac{C}{t^{\frac{3}{4}}} \leq \varepsilon$$

for t large enough, so we finally get by a bootstrap argument that

$$|\tilde{J}_1(t)| = o(\sqrt{t}).$$

Finally, recalling that $\tilde{J}_1 = \tau J_1$ and that $\tau \sim 2\sqrt{t}$, we can conclude that

$$|J_1| = \left| \int_{\mathbb{R}} y^2 (R - \Gamma) dy \right| = o(1),$$

which ends the proof. □

Remark 5.16. Note that in the Gaussian case, we can explicitly compute the kinetic energy

$$\mathcal{E}_{\text{kin}}(t) = \frac{1}{\tau^2(t)} \int_{\mathbb{R}} \frac{M^2(t,y)}{R(t,y)} dy = \frac{c(t)}{\tau(t)} \|\rho_0\|_{L^1(\mathbb{R})} = O\left(\frac{1}{\sqrt{t}}\right),$$

unless the initial data are well prepared in the sense that $c_0 = \overline{x}_\infty$ (which would actually improve this rate). This feature gives an explicit rate of convergence of $\mathcal{E}_{\text{kin}}$ towards 0 that can be propagated to the convergence of the second momentum of R by adapting the proof of Proposition 5.15, namely

$$|J_1| = \left| \int_{\mathbb{R}} y^2 (R - \Gamma) dy \right| = O\left(\frac{1}{\sqrt{t}}\right).$$

5.6 Convergence

Proof of Proposition 5.7. We are first going to try to eliminate the momentum M of our target equation. Differentiating equation (5.2.1b) with respect to y, and using equation (5.2.1a) in order to express

$$\partial_t(\partial_y M) = -\partial_t(\tau^2 \partial_t R),$$

we get that

$$-\partial_t(\tau^2 \partial_t R) - \tau^2 \partial_t R + LR = -\frac{1}{\tau^2} \partial_y^2 \left(\frac{M^2}{R}\right),$$

where we have defined the Fokker-Planck operator

$$L := \partial_y^2 + 2\partial_y(y\cdot).$$

Following [41], we introduce another scaling in time s defined by

$$\partial_s = \tau^2 \partial_t, \tag{5.6.1}$$

and the notation

$$\tilde{R}(s(t), y) := R(t, y).$$

We calculate the quantities

$$\partial_t(\tau^2 \partial_t R) = \frac{1}{\tau^2} \partial_s^2 \tilde{R} \text{ and } \tau^2 \partial_t R = \partial_s \tilde{R},$$

hence we obtain the following equation:

$$-\frac{1}{\tau^2}\partial_s^2\tilde{R}-\partial_s\tilde{R}+L\tilde{R}=-\frac{1}{\tau^2}\partial_y^2\left(\frac{M^2}{\tilde{R}}\right). \tag{5.6.2}$$

Now we remark that equation (5.5.1) induces

$$\sup_{s\geq 0}\int_{\mathbb{R}}\tilde{R}(s,y)(1+|y|^2+|\log\tilde{R}(s,y)|)dy<\infty, \tag{5.6.3}$$

that equation (5.5.2) gives

$$\int_0^\infty \dot{\tau}(t)\int_{\mathbb{R}}\frac{\tilde{M}^2}{\tilde{R}}dydt<\infty, \tag{5.6.4}$$

and that

$$\tau(s)\sim 2e^{2s} \text{ and } \dot{\tau}(s)\sim e^{-2s},$$

so we can conclude like in [36]. Let a sequence $s_n\to\infty$, take $s\in[-1,2]$, and denote

$$\tilde{R}_n(s,y):=\tilde{R}(s+s_n,y).$$

From (5.6.3) along with the de la Vallée-Poussin and Dunford-Pettis theorems (cf Lemma 5 and 6), we get the following weak convergence (up to a subsequence, not relabeled for reader's convenience), for all $p\in[1,\infty)$,

$$\tilde{R}_n \underset{n\to\infty}{\rightharpoonup} \tilde{R}_\infty \quad \text{in } L^p(-1,2;L^1(\mathbb{R})).$$

We also get the weak convergence of the initial datum, up to another subsequence:

$$\tilde{R}_n(0) \underset{n\to\infty}{\rightharpoonup} \tilde{R}_{0,\infty} \quad \text{in } L^1(\mathbb{R}).$$

Thanks to (5.6.3), we also get that the family $(\tilde{R}(s_n,.))_n$ is compact, so

$$\int_{\mathbb{R}}\tilde{R}_{0,\infty}(y)dy=\int_{\mathbb{R}}\Gamma(y)dy$$

and

$$\int_{\mathbb{R}}\tilde{R}_{0,\infty}(y)(1+|y|^2+|\log\tilde{R}_{0,\infty}(y)|)dy<\infty.$$

Then, denoting $\tau_n(s)=\tau(s+s_n)$, equation (5.6.4) implies that

$$-\frac{1}{\tau_n^2}\partial_y^2\left(\frac{M_n^2}{\tilde{R}_n}\right) \underset{n\to\infty}{\rightharpoonup} 0 \quad \text{in } L^1(-1,2;W^{-2,1}(\mathbb{R})).$$

In addition, in (5.6.2), all the other terms but two obviously go weakly to zero, which yields

$$\partial_s \tilde{R}_\infty = L\tilde{R}_\infty \tag{5.6.5}$$

in $\mathcal{D}'((-1,2)\times\mathbb{R})$, with $\tilde{R}_\infty(0,;) = \tilde{R}_{0,\infty} \in L^1(\mathbb{R})$. Thanks to the above bounds on $\tilde{R}_{0,\infty}$, it is known (see [6]) that the solution $\tilde{R}_\infty$ to (5.6.5) is actually defined for all $s \geq 0$ and satisfies

$$\|\tilde{R}_\infty(s,\cdot) - \Gamma\|_{L^1(\mathbb{R})} \underset{s\to\infty}{\longrightarrow} 0. \tag{5.6.6}$$

Going back to system (5.2.1), we need to show that $\tilde{R}_\infty$ is independent of s. In the s variable, equation (5.2.1a) becomes

$$\partial_s \tilde{R} + \partial_y \tilde{M} = 0,$$

and (5.6.4) implies that $\tilde{M} \in L^2(-1,2;L^1(\mathbb{R}))$. With $\tilde{M}_n(s) := \tilde{M}(s+s_n)$, we have

$$\partial_y \tilde{M}_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1}(\mathbb{R})),$$

so

$$\partial_s \tilde{R}_\infty = 0.$$

Combining this last equality with equation (5.6.6), we infer that $\tilde{R}_\infty = \Gamma$. The limit being unique, no extraction of a subsequence is needed, and we conclude that

$$\tilde{R}(s,\cdot) \underset{s\to\infty}{\rightharpoonup} \Gamma \text{ weakly in } L^1(\mathbb{R}).$$

5.7 Conclusion

We have shown that Gaussian functions play an important role in the study of the isothermal compressible Euler equation, standing both as particular solutions of the system (5.1.3) and time-asymptotic limit of solutions of this system. They also make the link between the isentropic and the isothermal system, as limit of the Barenblatt solutions of (5.1.2a).

The next step of the analysis of the long-time behavior of this system would be to find an explicit convergence rate of any L^∞ weak solution of (5.1.3) towards the limit Gaussian profile, adapting the work from [77], [79] and [67]. A good approach seems to be the use of the Csiszár-Kullback inequality that gives a lower bound of the entropy by the L^1-norm of $\rho - \overline{\rho}$:

$$\|\rho - \overline{\rho}\|_{L^1}^2 \leq 2\|\rho_0\|_{L^1} \int_{\mathbb{R}} \rho \log\left(\frac{\rho}{\overline{\rho}}\right) = 2\|\rho_0\|_{L^1} \int_{\mathbb{R}} R \log\left(\frac{R}{\Gamma}\right).$$

Unfortunately, no decreasing rate of the entropy function is currently known. This is still an open question that appears in other fields of the analysis of PDEs, for instance the study of the logarithmic Schrödinger equation [36]. Note that in [119], the author indeed has an upper bound for the entropy, assuming $\overline{\rho} \geq c > 0$, which is true on the compact set $\Omega \subset \mathbb{R}$ but obviously false on the whole space $\mathbb{R}$.

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DOCTORAT BRETAGNE LOIRE MATHSTIC

UNIVERSITÉ DE RENNES 1

Titre : Autour de l'équation de Schrödinger-Langevin et du système d'Euler isotherme amorti

Mot clés : équations de Schrödinger logarithmique, mécanique des fluides compressibles, stabilité d'ondes planes, solutions gaussiennes

Résumé : On s'intéresse dans cette thèse à la dynamique de l'équation de Schrödinger-Langevin et son lien avec le système d'Euler-Korteweg isotherme amortie via la transformation de Madelung. L'étude des solutions particulières gaussiennes sur l'espace permet d'expliciter le comportement en temps long des solutions de cette équation. Sur le tore, on montre la stabilité asymptotique des solutions de type onde plane. L'existence de solutions dissipatives au système d'Euler est obtenue par limite visqueuse du système de Navier-Stokes-Korteweg avec amortissement et la construction d'une entropie relative adéquate. On étudie également la dynamique du système d'Euler isotherme amorti.

Title: Around Schrödinger-Langevin equation and isothermal Euler system with damping

Keywords: logarithmic Schrödinger equations, compressible fluids, plane wave stability, Gaussian solutions

Abstract: This manuscript deals with the dynamics of the Schrödinger-Langevin equation and its relation with the isothermal Euler-Korteweg system through the Madelung transform. The study of Gaussian solutions on the whole space highlights the long-time behaviour of the solutions of this equation. On the torus, we show the asymptotic stability of plane wave solutions. The global existence of dissipative solutions of the Euler system is obtained through the viscous limit of the damped Navier-Stokes-Korteweg system and the use of a particular relative entropy. We also look at the dynamics of the Euler system with damping.