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# Sur les holonomies des structures projectives complexes sur les surfaces

Thomas Le Fils

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**École doctorale de sciences mathématiques de Paris centre**

# Thèse de doctorat

Discipline : Mathématiques

présentée par

**Thomas LE FILS**

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# Sur les holonomies des structures projectives complexes sur les surfaces

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dirigée par Maxime WOLFF

Soutenue le 12 juillet 2022 devant le jury composé de :

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# Résumé

Cette thèse a pour objet l'étude des holonomies de certaines structures géométriques sur les surfaces. On s'intéresse en particulier aux structures projectives complexes branchées et aux surfaces de translation. Les espaces de déformations de ces structures géométriques sont naturellement stratifiés, par les sous-espaces où la combinatoire des points de branchement est fixée. Nous étudions les liens entre les holonomies de ces structures et ces stratifications. Les résultats principaux sont démontrés dans trois articles qui forment la plus grande partie de la thèse.

Dans le premier, on définit une classe de représentations d'un groupe de surface de genre 2 dans $\mathrm{PSL}_2(\mathbb{C})$ : les *représentations pentagones*. On montre que ce sont exactement les représentations non-élémentaires d'un groupe de surface dans $\mathrm{PSL}_2(\mathbb{C})$ qui n'admettent pas de décomposition en pantalons telle que la restriction de la représentation à chaque pantalon soit un isomorphisme sur un groupe de Schottky à deux générateurs. Ce faisant, on exhibe une erreur dans la démonstration de Gallo, Kapovich et Marden du fait que toute représentation non-élémentaire d'un groupe de surface dans $\mathrm{PSL}_2(\mathbb{C})$ est l'holonomie d'une structure projective complexe, éventuellement avec un point de branchement d'ordre 1. On montre que les représentations pentagones se réalisent comme de telles holonomies et on répare ainsi cette preuve.

Dans le deuxième article, on s'intéresse aux périodes des surfaces de translation. Soit $\Sigma$ une surface fermée. On caractérise les éléments de $\mathrm{H}^1(\Sigma, \mathbb{C})$ qui se réalisent comme les périodes absolues d'une différentielle abélienne, dont la combinatoire des zéros est fixée. De façon équivalente on caractérise les morphismes $\mathrm{H}_1(\Sigma, \mathbb{Z}) \to \mathbb{C}$ qui sont obtenus comme l'holonomie d'une surface de translation dont la combinatoire des points coniques est prescrite. Cette caractérisation est une généralisation d'un théorème de Haupt et utilise les résultats de Kapovich, qui applique les théorèmes de Ratner à l'étude

de l'action du groupe modulaire sur les espaces de représentations correspondants.

Dans le dernier article, on caractérise les représentations $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ qui se réalisent comme l'holonomie d'une structure projective branchée dont on fixe le nombre de points de branchement, ainsi que les ordres de ceux-ci. Ce résultat étend le précédent, et caractérise en particulier les holonomies des structures affines branchées à singularités prescrites et les holonomies des structures sphériques avec des points coniques d'angles multiples de $2\pi$ fixés.

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# Chapitre 1

# Introduction

## 1.1 Contexte

Le théorème d'uniformisation de Klein, Koebe et Poincaré permet de lier des structures géométriques à des représentations de groupes. En effet ce théorème affirme qu'une surface de Riemann simplement connexe est biholomorphe à l'une des trois surfaces suivantes : la sphère de Riemann $\mathbb{CP}^1$, le plan complexe $\mathbb{C}$ ou le plan hyperbolique $\mathbb{H}^2$. Si $\Sigma$ est une surface de Riemann compacte, son revêtement universel $\widetilde{\Sigma}$ est naturellement une surface de Riemann simplement connexe. Cette dernière s'identifie donc à $X$, où $X$ est $\mathbb{CP}^1$, $\mathbb{C}$ ou $\mathbb{H}^2$. Fixons une identification $\mathrm{dev}\colon \widetilde{\Sigma} \to X$ que l'on appelle développante. Le groupe fondamental de $\Sigma$ agit sur le revêtement $\widetilde{\Sigma} \simeq X$ par biholomorphismes. La surface $\Sigma$ est donc homéomorphe à un quotient de $X$ par un sous-groupe de ses biholomorphismes. Ceci permet de montrer que $X = \mathbb{H}^2$ si et seulement si le genre de $\Sigma$ est au moins deux, ce que l'on suppose désormais. Le groupe des biholomorphismes de $\mathbb{H}^2$ est le groupe $\mathrm{PSL}_2(\mathbb{R})$, qui agit comme suit :

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

L'action par automorphismes de revêtement fournit ainsi une représentation $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ telle que dev soit $\rho$-équivariante :

$$\mathrm{dev}(\gamma \cdot z) = \rho(\gamma) \cdot \mathrm{dev}(z).$$

Cette représentation est appelée l'*holonomie* de la structure complexe. Si l'on avait choisi une autre identification $\widetilde{\Sigma} \simeq \mathbb{H}^2$, ceci aurait changé la

développante dev en $h \circ \mathrm{dev}$, avec $h \in \mathrm{PSL}_2(\mathbb{R})$ et l'holonomie $\rho$ en $h\rho h^{-1}$. Ainsi la classe de conjugaison de $\rho$ est bien définie et l'on a une application d'holonomie de l'espace des structures complexes sur une surface topologique $\Sigma$ dans $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$. Cette application est un homéomorphisme sur son image qui est composée de classes de conjugaison de représentations discrètes et fidèles. Ainsi l'holonomie établit un dictionnaire entre les structures complexes et ces représentations. Ces espaces sont appelés espaces de *Teichmüller*.

L'objet de cette thèse est de généraliser cette interprétation géométrique des représentations discrètes et fidèles des groupes de surfaces dans $\mathrm{PSL}_2(\mathbb{R})$ à d'autres espaces de représentations. Nous donnerons une telle interprétation de façon systématique à toute représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$. Cela s'inscrit dans la continuité des travaux initiés par Goldman qui a étudié comme généralisation des espaces de Teichmüller les espaces de représentations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ pour différents groupes $G$. En particulier Choi et Goldman [CG93] ont montré qu'une des trois composantes connexes de l'espace $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_3(\mathbb{R}))/\mathrm{PSL}_3(\mathbb{R})$ paramétrise les structures projectives convexes sur la surface topologique $\Sigma$. Cette composante connexe est composée uniquement de représentations discrètes et fidèles. C'est l'une des motivations de la théorie de Teichmüller supérieure qui étudie les composantes connexes des espaces $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ composées uniquement de représentations discrètes et fidèles, pour différents groupes de Lie $G$. Cette théorie a connu un très rapide développement ces dernières années, et on renvoie à [Wie18] pour une introduction à ce sujet. Dans cette thèse, nous ne nous intéressons pas uniquement aux représentations discrètes et fidèles mais à tout l'espace de représentations $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$.

Notre interprétation géométrique de ces représentations consiste à les réaliser comme holonomies de structures géométriques sur les surfaces. Une structure géométrique sur la surface $\Sigma$ est entendue au sens des $(G, X)$-structures de Thurston. Il s'agit de la donnée d'un atlas de cartes dans un espace modèle $X$, dont les applications de transition sont dans un groupe $G$ agissant continûment sur $X$. Nous étendons cette définition afin d'y inclure les structure *branchées*, c'est-à-dire dont les cartes ne sont pas nécessairement localement injectives, mais peuvent être ramifiées en certains points. On autorise les cartes à être topologiquement conjuguées à l'application du disque unité dans lui-même $z \mapsto z^{n+1}$ pour un entier $n$. Si $n \geqslant 1$, le point de $\Sigma$ correspondant à 0 est dit de branchement et l'entier $n$ est son ordre. Un procédé général permet d'étendre une carte donnée d'une structure géométrique en

une carte globale appelée *développante* $\mathrm{dev}\colon \widetilde{\Sigma} \to X$ définie sur le revêtement universel de $\Sigma$, et également de globaliser les changements de cartes en un morphisme de groupe $\rho\colon \pi_1(\Sigma) \to G$ qui est l'*holonomie* de la structure géométrique. La fonction dev est alors $\rho$-équivariante. L'application qui à une structure géométrique associe son holonomie n'est pas en général une bijection mais est un outil à la fois pour analyser les structures géométriques à l'aide des représentations, et pour étudier les variétés de caractères avec des méthodes géométriques.

Nous nous intéressons aux structures projectives qui sont les structures géométriques modelées sur $X = \mathbb{CP}^1$, dont les changements de cartes sont dans $G = \mathrm{PSL}_2(\mathbb{C})$. Nous cherchons à déterminer quelles représentations de groupes sont les holonomies de ces structures géométriques. Remarquons que de nombreuses structures géométriques sur les surfaces sont des cas particuliers de structures projectives. En effet les structures sphériques branchées, c'est-à-dire modelées sur $\mathbb{S}^2$ dont les changements de cartes sont dans le groupe $\mathrm{SO}_3(\mathbb{R})$ de ses isométries directes, s'identifient aux structures projectives dont l'application d'holonomie est à valeurs dans $\mathrm{PSU}_2$. De façon équivalente, de telles structures sont données par une métrique de courbure positive constante égale à 1 sur $\Sigma$ avec des singularités coniques dont les angles sont multiples de $2\pi$. Les structures sphériques font l'objet de recherches actives, comme en témoignent le survol [Ere19] et les références qui y sont contenues. On renvoie également aux articles de Mondello et Panov [MP16, MP19] pour des résultats sur les espaces de modules des structures sphériques. Les structures affines, modelées sur $\mathbb{C} \subset \mathbb{CP}^1$ avec changements de cartes dans le sous-groupe $\mathrm{Aff}(\mathbb{C})$ de $\mathrm{PSL}_2(\mathbb{C})$ des transformations de Möbius qui préservent $\mathbb{C}$ représentent un autre cas particulier des structures projectives. En se restreignant aux structures dont les changements de cartes sont dans le sous-groupe $\mathbb{C}$ des translations de $\mathrm{Aff}(\mathbb{C})$ on obtient la notion de surface de translation, très étudiée en raison notamment de ses liens avec la géométrie de l'espace de Teichmüller et avec la dynamique unidimensionnelle, voir par exemple les survols [Zor06, Wri15]. Les surfaces de dilatation modelées sur $\mathbb{C}$, avec des changements de cartes dans $\mathbb{R}^\star \rtimes \mathbb{C} \subset \mathrm{Aff}(\mathbb{C}) = \mathbb{C}^\star \rtimes \mathbb{C}$ constituent une notion intermédiaire entre les surfaces de translation et les surfaces affines. Elles ont été l'objet de travaux récents, voir par exemple [Gha18, DFG19]. Cette notion conserve des liens forts avec la dynamique unidimensionnelle, notamment avec les échanges d'intervalles.

Lorsque l'on restreint le groupe des changements de cartes autorisés, on obtient une nouvelle structure géométrique qui est un cas particulier

de la précédente. Les changements de cartes peuvent alors éventuellement préserver plus de caractéristiques géométriques, comme par exemple les notions de ligne droite, de volume, etc. Ainsi les inclusions des groupes précédents se traduisent par les inclusions des structures géométriques associées.

$$\{\text{surfaces de translation}\} \subset \{\text{surfaces de dilatation}\} \subset \{\text{surfaces affines}\}.$$

Le groupe modulaire $\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Homeo}^+(\Sigma))$ agit sur les espaces de déformations de structures géométriques, ainsi que sur les espaces de représentations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. L'application d'holonomie est équivariante pour ces actions. Ainsi l'ensemble des représentations qui se réalisent comme l'holonomie d'une structure géométrique est invariant par l'action de $\mathrm{Mod}(\Sigma)$ sur les espaces de représentations. C'est pourquoi dans cette thèse nous étudions également la dynamique de l'action de $\mathrm{Mod}(\Sigma) \simeq \mathrm{Out}^+(\pi_1(\Sigma))$ sur des espaces de représentations $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, notamment dans le cas où $G$ est un sous-groupe fermé de $\mathrm{PSL}_2(\mathbb{C})$ comme par exemple $\mathrm{PSU}_2$ ou $\mathrm{Aff}(\mathbb{C})$. La dynamique de l'action du groupe modulaire sur les espaces de représentations est l'objet d'étude d'un domaine entier des mathématiques très actif, qui a été initié par Goldman, voir [Gol97]. Cette étude se concentre souvent sur des propriétés d'ergodicité de l'action. Nous nous intéressons ici à la description de l'adhérence de chaque orbite, complétant ainsi des travaux de Previte et Xia, et de Ghazouani. En effet Previte et Xia [PX00, PX02] ont étudié l'adhérence des orbites de représentations $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}_3(\mathbb{R})$ dont l'image est dense dans $\mathrm{SO}_3(\mathbb{R})$. Il est connu que l'espace $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{SO}_3(\mathbb{R}))/\mathrm{SO}_3(\mathbb{R})$ possède deux composantes connexes. Une des conséquences de leurs travaux est que l'adhérence de l'orbite d'une représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{SO}_3(\mathbb{R})$ d'image dense est toute la composante connexe de $\rho$. Nous donnerons dans cette thèse une description systématique des adhérences de toutes les autres orbites. Nous étudierons aussi les adhérences de chaque représentation dans $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{Aff}(\mathbb{C}))/\mathrm{Aff}(\mathbb{C})$, complétant ainsi des résultats de Ghazouani [Gha16, Gha18].

## 1.2 Quelques contributions sur les holonomies de structures projectives

Les résultats de cette thèse s'inscrivent dans la continuité des travaux de Gallo, Kapovich et Marden sur les holonomies des structures projectives complexes. Dans l'article [GKM00], ces auteurs décrivent les représentations d'un

groupe de surface dans $\mathrm{PSL}_2(\mathbb{C})$ qui se réalisent comme l'holonomie d'une structure projective sans point de branchement. Cette description équivaut à la caractérisation des représentations qui sont les monodromies de certaines équations différentielles linéaires du second ordre sur une surface de Riemann, comme nous le verrons dans la section 2.3.1. La question analogue sur les sphères trouées est abordée dans [Poi84], où Poincaré affirme que « On peut en général trouver une équation du second ordre, sans points à apparence singulière qui admette un groupe donné ». La question pour les surfaces fermées avait fait l'objet de plusieurs tentatives [Gal89, Kap95], avant d'être résolue dans [GKM00].

**Théorème de Gallo, Kapovich et Marden.** Soit $\Sigma$ une surface fermée orientée de genre $g \geqslant 2$. Une représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ est l'holonomie d'une structure projective sur $\Sigma$ si et seulement si $\rho$ est non-élémentaire et $\rho$ se relève à $\mathrm{SL}_2(\mathbb{C})$.

Expliquons les termes utilisés dans ce théorème. Le groupe $\mathrm{PSL}_2(\mathbb{C})$ s'identifie au groupe des isométries directes de l'espace hyperbolique $\mathbb{H}^3$, et agit donc sur ce dernier. Une représentation est dite *élémentaire* si son image est un groupe élémentaire, c'est-à-dire si son action sur $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3 = \mathbb{H}^3 \cup \mathbb{CP}^1$ admet une orbite finie. On dit qu'une représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ se relève à $\mathrm{SL}_2(\mathbb{C})$ s'il existe $\tilde{\rho}\colon \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ telle que $\rho = \pm\tilde{\rho}$. D'après un théorème de Goldman [Gol88b] l'espace $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ possède deux composantes connexes, qui sont distinguées par la propriété de se relever ou non à $\mathrm{SL}_2(\mathbb{C})$.

La démonstration du théorème de Gallo, Kapovich et Marden repose sur une étude de la dynamique de l'action du groupe modulaire sur l'espace de représentations $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$. Les auteurs montrent que toute représentation non-élémentaire dans cet espace possède dans son orbite une représentation suffisamment générique. Plus précisément, la première étape de la démonstration de ce théorème consiste à montrer que pour toute représentation non-élémentaire $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$, il existe une *décomposition de Schottky de la surface* : une décomposition en pantalons de la surface $\Sigma = \cup_i P_i$ telle que la restriction de $\rho$ à chaque pantalon $\rho_{|P_i}\colon \pi_1(P_i) \to \mathrm{PSL}_2(\mathbb{C})$ soit un isomorphisme sur un groupe de Schottky à deux générateurs. La première contribution présentée dans cette thèse consiste à identifier une erreur dans cette première étape : dans le chapitre 3 qui est une reproduction de l'article [LF19], on définit une classe de représentations non-élémentaires d'un groupe de surface de genre 2 dans

$\mathrm{PSL}_2(\mathbb{C})$ qui ne possèdent pas de décomposition de Schottky : *les représentations pentagones*. Le groupe fondamental d'une surface $\Sigma$ de genre 2 est un sous-groupe d'indice 2 du groupe fondamental orbifold de $\Sigma/\iota$, où $\iota$ est un homéomorphisme représentant l'involution hyperelliptique. Ce groupe fondamental orbifold $\pi_1^{\mathrm{orb}}(\Sigma/\iota)$ admet la présentation suivante :

$$\pi_1^{\mathrm{orb}}(\Sigma/\iota) = \langle q_1, \ldots, q_6 \mid q_i^2 = 1, q_1 \ldots q_6 = 1 \rangle.$$

Une représentation pentagone est une représentation non-élémentaire $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ telle qu'il existe une représentation $f\colon \pi_1^{\mathrm{orb}}(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ qui étend $\rho$ (*i.e.* $f_{|\pi_1(\Sigma)} = \rho$) et qui vérifie $f(q_i) = \pm\mathrm{Id}$ pour un $1 \leqslant i \leqslant 6$. On montre dans le chapitre 3 que ces représentations sont exactement les représentations non-élémentaires d'un groupe de surface dans $\mathrm{PSL}_2(\mathbb{C})$ qui n'admettent pas de décomposition de Schottky.

**Théorème 1.2.1.** *Une représentation non-élémentaire $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ admet une décomposition de Schottky si et seulement si ce n'est pas une représentation pentagone.*

Notons que ce théorème peut être vu comme un outil pour étudier les orbites de représentations $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$. En effet il permet par exemple de montrer, comme on le verra dans le chapitre 3, que toute représentation non-élémentaire $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ admet une décomposition en pantalons telle que chaque courbe qui la compose soit envoyée par $\rho$ sur une isométrie hyperbolique de $\mathbb{H}^3$.

La seconde partie de la démonstration de Gallo, Kapovich et Marden consiste à construire, pour une représentation de Schottky $\rho$, une structure projective sur chacun des pantalons $P_i$, qui a pour holonomie $\rho_{|P_i}$. Les auteurs les recollent ensuite à l'aide de cylindres pour obtenir une surface avec une structure projective ayant l'holonomie voulue. Ceci leur permet également de montrer qu'une représentation est l'holonomie d'une structure projective avec un unique point de branchement d'ordre 1 si et seulement si elle est non-élémentaire et ne se relève pas à $\mathrm{SL}_2(\mathbb{C})$. Cette stratégie de preuve ne s'adapte pas pour les représentations pentagones, mais nous construisons des structures projectives explicites avec un unique point de branchement d'ordre 1 pour chaque représentation pentagone dans le chapitre 3, et réparons ainsi la preuve de ce théorème.

La démonstration de Gallo, Kapovich et Marden fait intervenir de façon cruciale la notion de structure branchée pour caractériser les holonomies des

structures projectives sans point de branchement. Ils posent ensuite dans [GKM00, Problem 12.6.1] la question de la caractérisation plus générale des holonomies des structures branchées. Cette caractérisation consiste à déterminer quand une représentation élémentaire se réalise comme l'holonomie d'une structure projective dont les points de branchement sont fixés.

Un premier pas vers cette extension est la caractérisation des holonomies, appelées dans ce contexte périodes, des surfaces de translation dont les angles coniques sont prescrits. Cette question a été posée explicitement par Calsamiglia, Deroin et Francaviglia dans [CDF15], où les auteurs étudient le feuilletage isopériodique de l'espace de modules des surfaces de translation. Les ensembles où l'application d'holonomie a une image fixée forment un feuilletage de l'espace de déformations des surfaces de translation. Ce feuilletage est invariant par l'action du groupe modulaire et donne au quotient un feuilletage du fibré de Hodge, l'espace de modules des surfaces de translation. Ce feuilletage est le feuilletage isopériodique, dont l'étude a été initiée par McMullen [McM14]. Calsamiglia, Deroin et Francaviglia [CDF15], Hamenstädt [Ham18], Ygouf [Ygo21] et Winsor [Win21] ont étudié les propriétés dynamiques de ce feuilletage et en particulier son ergodicité dans les strates.

Le problème que nous considérons revient à déterminer quelles sont les strates qui intersectent une feuille donnée. Nous répondons à cette question dans le chapitre 4, qui est une reproduction de [LF20].

**Théorème 1.2.2.** *Soient $n_1, \dots, n_k$ des entiers tels que $\sum_i n_i = 2g-2$. Une représentation $\chi\colon \pi_1(\Sigma) \to \mathbb{C}$ est l'holonomie d'une surface de translation dont les points de branchements sont d'ordres $n_1, \dots, n_k$ si et seulement si*

1. $\mathrm{Vol}(\chi) > 0$
2. $\mathrm{Vol}(\chi) \geqslant (\max_i n_i + 1)\mathrm{Vol}(\mathbb{C}/\Lambda)$ *si* $\Lambda = \chi(\pi_1(\Sigma))$ *est un réseau de* $\mathbb{C}$.

Ce théorème est un raffinement d'un théorème de Haupt [Hau20], qui caractérise les représentations dans $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C})$ qui sont l'holonomie d'une surface de translation. Dans le théorème ci-dessus, le volume d'une représentation $\chi\colon \pi_1(\Sigma) \to \mathbb{C}$ est la quantité définie comme suit. Comme l'image de $\rho$ préserve la forme volume du plan, son tiré en arrière par n'importe quelle fonction lisse $\widetilde{\Sigma} \to \mathbb{C}$ qui est $\rho$-équivariante induit une forme $\omega$ sur $\Sigma$. Le volume de $\chi$ est sa masse totale :

$$\mathrm{Vol}(\chi) = \int_\Sigma \omega.$$

On verra dans le chapitre 5 qu'une telle fonction lisse existe toujours et que cette quantité ne dépend pas du choix de celle-ci. De façon plus concrète, on peut définir dans notre cas le volume de $\chi$ de la façon suivante :

$$\mathrm{Vol}(\chi) = \sum_{i=1}^{g} \det(\chi(a_i), \chi(b_i)) = \sum_{i=1}^{g} \Im(\overline{\chi(a_i)}\chi(b_i))$$

où $a_1, \ldots, b_g$ est un système de générateur standard de $\pi_1(\Sigma)$, induisant une base symplectique de $\mathrm{H}_1(\Sigma, \mathbb{Z})$. Lorsque $\chi$ est l'holonomie d'une surface de translation, le volume de celle-ci comme surface munie de la métrique plate à singularités coniques induite est égal à $\mathrm{Vol}(\chi)$. La démonstration du théorème 1.2.2 fait intervenir l'étude de la dynamique de l'action du groupe modulaire sur $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C})$. Ceci revient à étudier l'action du groupe symplectique $\mathrm{Sp}(\mathrm{H}_1(\Sigma, \mathbb{Z}))$ sur $\mathrm{H}^1(\Sigma, \mathbb{C})$. Les résultats obtenus par Kapovich dans [Kap20] en utilisant les théorèmes de Ratner, permettent de décrire l'adhérence des orbites. Une autre preuve de ce résultat a été publiée par Bainbridge, Johnson, Judge et Park [BJJP20], se basant sur des résultats de [CDF15].

L'énoncé du théorème 1.2.2 s'inscrit dans une classification plus générale des holonomies des structures projectives branchées dont la combinatoire des points de branchements est prescrite, qui est complétée dans le chapitre 5. Nous allons maintenant expliquer cette classification. Comme précédemment, $\Sigma$ est une surface fermée de genre $g \geqslant 2$, $n_1, \ldots, n_k$ sont des entiers et $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ est une représentation. On cherche à déterminer sous quelles conditions il existe une structure projective avec des points de branchements dont les ordres sont $n_i$ et dont l'holonomie est $\rho$. Il peut exister des obstructions à l'existence d'une telle structure projective branchée sur $\Sigma$, que nous allons lister. Nous ne rentrons pas dans tous les détails des notations utilisées, et nous renvoyons au chapitre 5 pour plus de précisions.

**Obstruction 1.** La somme $\sum_i n_i$ est paire si et seulement si la représentation $\rho$ se relève à $\mathrm{SL}_2(\mathbb{C})$.

Dans le cas où $\rho$ est non-élémentaire, le théorème de Gallo, Kapovich et Marden [GKM00] montre qu'il s'agit de la seule obstruction. La formule de Gauss-Bonnet montre que dans le cas élémentaire, la somme des ordres des points de branchement doit être suffisamment grande.

**Obstruction 2.** Si $\rho$ est élémentaire alors $\sum_i n_i \geqslant 2g - 2$. Si de plus $\rho$ est sphérique, alors $\sum_i n_i \geqslant 2g - 1$.

On dit que $\rho$ est *sphérique* si son image a un conjugué dans $\mathrm{PSU}_2$. La formule de Riemann-Hurwitz donne une autre obstruction.

**Obstruction 3.** Si l'image de $\rho$ est un groupe fini d'ordre $n$ alors

$$n(\chi(\Sigma) + \sum_{i=1}^{k} n_i) \geqslant 2(\max_{1\leqslant i\leqslant k} n_i + 1).$$

Lorsque $\rho$ est *euclidien*, c'est-à-dire que son image a un conjugué dans $\mathrm{Isom}^+(\mathbb{E}^2)$, on peut définir son volume $\mathrm{Vol}(\rho)$ comme précédemment. Si de plus $\sum_i n_i = 2g-2$, alors la structure projective hérite d'une métrique plate à singularités coniques dont le volume est $\mathrm{Vol}(\rho)$.

**Obstruction 4.** Si $\sum_i n_i = 2g-2$ et $\rho$ est Euclidien, alors $\mathrm{Vol}(\rho) > 0$.

L'obstruction du théorème 1.2.2 dans le cas où l'image de $\chi$ est un réseau de $\mathbb{C}$ peut se produire en passant à un revêtement d'indice fini, ce qui donne une autre obstruction.

**Obstruction 5.** Si $\sum_i n_i = 2g-2$ et $\rho$ est euclidien et sa partie linéaire est un groupe fini d'ordre $n$ et le groupe $\Lambda = \{z_0 \in \mathbb{C} \mid z + z_0 \in \rho(\pi_1(\Sigma))\}$ est un réseau de $\mathbb{C}$, alors

$$n\mathrm{Vol}(\rho) \geqslant (\max_{1\leqslant i\leqslant k} n_i + 1)\mathrm{Vol}(\mathbb{C}/\Lambda).$$

Si $\Sigma$ est de genre 2 et que $\rho$ est diédrale (son image fixe deux points dans $\mathbb{CP}^1$), il existe une obstruction qui ne se produit que lorsque la somme $\sum_i n_i$ est minimale : $\sum_i n_i = 2$.

**Obstruction 6.** Si $\rho$ est diédrale et se relève à $\mathrm{SL}_2(\mathbb{C})$ et n'est pas affine et que $\Sigma$ est de genre 2, alors il ne peut pas avoir un unique point de branchement d'ordre 2.

Dans le chapitre 5, on montre qu'il s'agit en fait des seules obstructions à être l'holonomie d'une structure projective branchée dont la combinatoire des points de branchements est fixée.

**Théorème 1.2.3.** *Une représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ est l'holonomie d'une structure projective dont les points de branchements sont d'ordre $n_1, \ldots, n_k$ si et seulement si elle remplit les conditions des 6 obstructions précédentes.*

Remarquons que ce théorème donne en particulier une caractérisation des holonomies des surfaces sphériques branchées et des holonomies des structures affines sur les surfaces, dont les angles coniques sont fixés. De même ce théorème comprend une caractérisation des holonomies des surfaces de demi-translation sans holonomie locale, c'est-à-dire dont tous les angles sont multiples de $2\pi$. Il s'agit donc d'une vaste généralisation du théorème 1.2.2. Dans le cas affine, ce théorème complète un théorème de Ghazouani caractérisant les holonomies des structures affines [Gha18], et répond à une question posée par Chennakkod, Faraco et Gupta [CFG21, Question 1.3].

Comme mentionné précédemment, la démonstration de ce théorème fait intervenir l'étude de l'action de groupe modulaire sur $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, où $G$ est un sous-groupe fermé de $\mathrm{PSL}_2(\mathbb{C})$. En particulier lorsque $G$ est un sous-groupe fini de $\mathrm{PSL}_2(\mathbb{C})$, on obtient une description complète des orbites de cette action.

**Proposition 1.2.4.** *Soit $G$ un sous-groupe fini de $\mathrm{PSL}_2(\mathbb{C})$, alors*

$$|\mathrm{Mod}(\Sigma)\backslash\{\pi_1(\Sigma) \twoheadrightarrow G\}/G| = \begin{cases} 1 & si\ G = \mathbb{Z}/n\mathbb{Z}, \mathrm{D}_{2n+1} \\ 2 & si\ G = \mathrm{D}_{2n}, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5 \end{cases}.$$

L'étude de l'action du groupe modulaire sur ces espaces complète les travaux de Previte-Xia [PX00, PX02] dans le cas où $G$ est dans $\mathrm{PSU}_2$. On obtient en effet dans ce cas une description systématique de l'adhérence de l'orbite de chaque représentation. Dans le cas où $G$ est dans $\mathrm{Aff}(\mathbb{C})$, cette étude est complémentaire de celle de Ghazouani [Gha18, Gha16]. Ces résultats ont d'autres applications que la précédente caractérisation des holonomies des structures projectives branchées. En effet, dans un travail en cours avec Detcherry et Santharoubane, nous planifions de les utiliser pour répondre à une question provenant de la théorie des algèbres d'écheveaux (Skein algebras).

## 1.3 Plan du manuscrit

Expliquons brièvement l'organisation de cette thèse. Dans le chapitre 2, on fait des rappels sur les structures géométriques sur les surfaces. On y explique en particulier les notions de développantes, d'holonomies et on définit les espaces de déformations des structures projectives. On introduit le groupe modulaire et ses actions sur les espaces de structures géométriques et de représentations. On présente en détail les structures projectives et les surfaces

de translation. On y expose ensuite des résultats concernant les holonomies de certaines structures géométriques sur les surfaces.

Les chapitres 3, 4 et 5 constituent l'essentiel de cette thèse et contiennent les preuves des nouveaux résultats. Ils sont une reproduction des articles [LF19, LF20, LF21]. Ils sont repris tels qu'ils ont été écrits pour les revues. C'est la raison pour laquelle ils sont écrits en anglais.

# Chapitre 2

# Structures géométriques branchées sur les surfaces

Soit $\Sigma$ une surface topologique orientée, compacte, connexe et sans bord, de genre $g \geqslant 0$.

## 2.1 Structures géométriques branchées

Nous présentons les structures géométriques branchées sur les surfaces. Cette section s'inspire de Thurston [Thu97] et de Goldman [Gol88a].

### 2.1.1 Structures géométriques

Nous adaptons la définition classique de $(G, X)$-structure au sens de Thurston [Thu97] pour y inclure les structures branchées.

Soit $X$ un espace topologique, et $G$ un groupe agissant continûment sur $X$. Une *carte branchée* est la donnée d'un ouvert $U$ de $\Sigma$ et $V$ de $X$, ainsi que d'une fonction $\varphi : U \to V$ topologiquement conjuguée à la fonction du disque unité dans lui même $z \mapsto z^{n+1}$. Autrement dit il s'agit d'une fonction $\varphi$ telle qu'il existe des homéomorphismes préservant l'orientation $U \simeq \mathbb{D}(0,1)$ et $V \simeq \mathbb{D}(0,1)$ faisant commuter ce diagramme :

$$\begin{tikzcd} U \subset \Sigma \arrow[r, "\varphi"] \arrow[d] & V \subset X \arrow[d] \\ \mathbb{D}(0,1) \arrow[r, "z \mapsto z^n"] & \mathbb{D}(0,1). \end{tikzcd}$$

Deux cartes branchées $\varphi : U \to V$ et $\psi : U' \to V'$ sont dites *compatibles* s'il existe $h \in G$ tel que $\varphi = h \circ \psi$ sur $U \cap U'$. Un *atlas de cartes branchées* est la donnée d'un ensemble de cartes branchées $\varphi_\alpha : U_\alpha \to V_\alpha$ compatibles deux à deux et recouvrant $\Sigma$, c'est-à-dire telles que $\bigcup_\alpha U_\alpha = \Sigma$.

**Definition 2.1.1.** Une *structure géométrique branchée* est la donnée d'un atlas de carte maximal pour l'inclusion.

De façon équivalente, il s'agit d'une classe d'équivalence d'atlas, deux atlas étant considérés comme équivalents si leur union est encore un atlas.

#### Exemples

Avant d'introduire les notions de développante et d'holonomie, donnons quelques exemples de structures géométriques branchées.

- Si l'on prend $X = \mathbb{H}^2$ et $G = \mathrm{PSL}_2(\mathbb{R})$, on obtient la notion de structure hyperbolique branchée.
- Si $X = \mathbb{E}^2$ est le plan Euclidien et $G = \mathbb{R}^2$ est le groupe des translations, alors on obtient la notion de *surface de translation* sur laquelle on reviendra plus en détail.
- Si $X$ est une surface et $G$ est le groupe trivial, on obtient les revêtements ramifiés de $X$.
- Si $X$ est la sphère $\mathbb{S}^2$ et $G = \mathrm{Isom}^+(\mathbb{S}^2) \simeq \mathrm{SO}_3(\mathbb{R})$ est son groupe d'isométries directes, on obtient la notion de surface sphérique à singularités coniques entières. En effet la donnée d'une telle $(G, X)$-structure s'identifie à la donnée d'une métrique de courbure constante égale à 1 avec des singularités coniques d'angles multiples de $2\pi$, en tirant en arrière la métrique ronde à l'aide des cartes.
- Si $X = \mathbb{RP}^2$ et $G = \mathrm{PSL}_3(\mathbb{R})$, on obtient la notion de structure projective réelle branchée.

### 2.1.2 Développante et holonomie

Dans cette section on définit les applications développantes et holonomies associées à des structures géométriques.

#### Développante

On dit que $G$ agit *analytiquement* sur $X$ si pour tout $g_1, g_2 \in G$, il n'existe un ouvert non-vide $U$ de $X$ tel que $g_{1|U} = g_{2|U}$ que lorsque $g_1 = g_2$. On

suppose désormais que $G$ agit analytiquement sur $X$.

Nous expliquons comment construire l'application développante associée à une structure géométrique, en s'inspirant de [Thu97, Chapitre 3]. Considérons une structure géométrique branchée sur $\Sigma$. Notons le revêtement universel de $\Sigma$ par $\widetilde{\Sigma}$, et par $\pi : \widetilde{\Sigma} \to \Sigma$ l'application de revêtement. Choisissons une carte $\varphi : U \subset \Sigma \to X$. Soit $\widetilde{U}$ une composante connexe de $\pi^{-1}(U)$. Il existe une unique fonction appelée *développante* $\mathrm{dev}\colon \widetilde{\Sigma} \to X$ vérifiant les deux propriétés suivantes :

1. $\varphi \circ \pi = \mathrm{dev}$ sur $\widetilde{U}$.
2. La développante est localement le relevé de cartes : pour tout $\tilde{p} \in \widetilde{\Sigma}$, il existe un voisinage ouvert $\widetilde{V}$ de $\tilde{p}$ et une carte $\varphi : \pi(\widetilde{V}) \to X$ telle que $\varphi \circ \pi = \mathrm{dev}$ sur $\widetilde{V}$.

Il s'agit de l'extension de la carte $\varphi$, que l'on peut construire concrètement de proche en proche, en relevant dans le revêtement universel des cartes qui coïncident sur leurs intersections.

L'application développante que nous venons de définir dépend du choix d'une carte $\varphi\colon U \to X$ ainsi que d'une composante connexe $\widetilde{U}$ de $\pi^{-1}(U)$. D'autres choix peuvent résulter en une application différente : par exemple si l'on choisit $h \in G$, et que l'on considère à la place de $\varphi$ la carte $h \circ \varphi$, alors la développante obtenue est $h \circ \mathrm{dev}$. Ces applications forment en fait toutes les développantes possibles. En effet si l'on a deux applications développantes, comme elles sont localement des relevés de cartes, elle diffèrent localement par post-composition par un élément $h \in G$. Mais l'analycité de l'action montre qu'elle diffèrent en fait globalement d'un élément $h \in G$.

### Holonomie

Nous avons vu que l'on peut définir une application développante dev qui étend une carte donnée et la rend globalement définie sur $\widetilde{\Sigma}$. De la même façon, on peut définir l'*holonomie* d'une structure géométrique, qui globalise les changements de cartes, à l'aide de la propriété suivante.

**Proposition 2.1.2.** *Il existe une unique application $\rho\colon \pi_1(\Sigma) \to G$ telle que pour tout $p \in \widetilde{\Sigma}$ et tout $\gamma \in \pi_1(\Sigma)$,*

$$\mathrm{dev}(\gamma \cdot p) = \rho(\gamma) \cdot \mathrm{dev}(p).$$

Cette application $\rho$ est appelée l'*holonomie* de la structure géométrique. L'ouverture de l'application dev, ainsi que l'analycité de l'action de $G$ forcent une telle application $\rho$ à être un homomorphisme de groupe.

Remarquons que la définition de l'holonomie dépend du choix d'une application développante, qui n'est définie qu'à post-composition près par un élément de $G$. Cependant, si on avait choisi une autre développante, qui s'écrirait nécessairement sous la forme $h \circ \mathrm{dev}$ avec $h \in G$, l'application d'holonomie correspondante serait $h\rho h^{-1}$. La classe de conjugaison de l'holonomie est donc bien définie.

#### Structure géométrique comme couple développante holonomie

Nous avons vu que nous pouvons associer à une $(G, X)$-structure sur $\Sigma$ un couple $(\mathrm{dev}, \rho)$ qui est défini à l'action de $G$ près, où $G$ agit par $h \cdot (\mathrm{dev}, \rho) = (h \circ \mathrm{dev}, h\rho h^{-1})$. L'application dev est $\rho$-équivariante, et est localement topologiquement conjuguée à l'une des applications $z \mapsto z^n$ du disque unité dans lui-même.

Réciproquement, si on dispose d'un couple $(f, \rho)$ avec $f \colon \widetilde{\Sigma} \to X$ et $\rho \colon \pi_1(\Sigma) \to G$, tel que $f$ est $\rho$-équivariante et localement topologiquement conjuguée à $z \mapsto z^n$, alors la fonction $f$ permet de définir des cartes dans $X$, qui diffèrent par des éléments de $G$. Ainsi la donnée d'un tel couple équivaut à la donnée d'une structure géométrique modelée sur $X$, avec changements de cartes dans $G$.

Ce point de vue permet de définir une topologie sur l'ensemble $\mathcal{S}_{(G,X)}(\Sigma)$ des $(G, X)$-structures sur $\Sigma$. En effet la projection $(f, \rho) \mapsto f$ injecte ces couples dans l'ensemble des applications continues de $\widetilde{\Sigma}$ dans $X$, et cet ensemble hérite ainsi de la topologie-compacte ouverte de $\mathcal{C}(\widetilde{\Sigma}, X)$. L'ensemble $\mathcal{S}_{(G,X)}(\Sigma)$ des structures géométriques modelées sur $X$ avec transitions dans $G$ est vu comme un quotient de cet ensemble par $G$ et est muni de la topologie quotient. L'application qui à une structure associe son holonomie est continue.

### 2.1.3 Espaces de déformations et stratifications

#### Espace de déformations, de modules

Le groupe $\mathrm{Homeo}^+(\Sigma)$ des homéomorphismes de $\Sigma$ préservant l'orientation agit naturellement sur l'ensemble des structures géométriques $\mathcal{S}_{(G,X)}(\Sigma)$

en tirant en arrière les cartes des atlas.

On définit l'*espace de déformations* $\mathrm{Def}_{(G,X)}(\Sigma)$ des structures $(G, X)$ sur $\Sigma$ comme l'ensemble des structures géométriques modelées sur $X$ avec changements de cartes dans $G$ modulo l'action du groupe $\mathrm{Homeo}_0(\Sigma)$ des homéomorphismes de $\Sigma$ isotopes à l'identité :

$$\mathrm{Def}_{(G,X)}(\Sigma) = \mathcal{S}_{(G,X)}(\Sigma)/\mathrm{Homeo}_0(\Sigma).$$

On définit aussi $\mathcal{M}_{(G,X)}(\Sigma)$ l'*espace de modules* des $(G, X)$-structures sur $\Sigma$. Il s'agit de l'ensemble des structures $(G, X)$ sur $\Sigma$ modulo l'action du groupe des homéomorphismes préservant l'orientation :

$$\mathcal{M}_{(G,X)}(\Sigma) = \mathcal{S}_{(G,X)}(\Sigma)/\mathrm{Homeo}^+(\Sigma).$$

Les espaces de déformations $\mathrm{Def}_{(G,X)}(\Sigma)$ et de modules sont munis de la topologie quotient induite par celle de $\mathcal{S}_{(G,X)}(\Sigma)$. L'application holonomie passe au quotient en une application définie sur l'espace de déformations. On considère donc désormais que cette application est définie sur cet espace :

$$\mathrm{Hol} : \mathrm{Def}_{(G,X)}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), G)/G.$$

### Stratification

Un *point de branchement* est un point de $\Sigma$ qui correspond à zéro dans la conjugaison topologique d'une carte avec $z \mapsto z^{n+1}$, où $n \geqslant 1$. Comme les cartes diffèrent d'un élément de $G$ sur leur intersection, cette notion ne dépend pas du choix d'une carte et l'entier $n$ est caractérisé par la propriété suivante : pour toute carte $\varphi : U \to V$ contenant le point $x$, et $p \in V \setminus \{\varphi(x)\}$, l'ensemble $\varphi^{-1}(p)$ est de cardinal $n + 1$.

L'entier $n$ est appelé *l'ordre* du point de branchement $x$. Remarquons que les points de branchements sont isolés et sont donc en nombre fini car $\Sigma$ est compacte. L'espace de déformations se décompose donc en imposant le nombre et les ordres des points de branchement. En notant $\mathcal{D}(n_1, \ldots, n_k)$ le sous-ensemble de $\mathrm{Def}_{(G,X)}(\Sigma)$ des structures dont les points de branchement sont d'ordre $n_1, \ldots, n_k$, on a :

$$\mathrm{Def}_{(G,X)}(\Sigma) = \bigcup_{k \geqslant 0} \bigcup_{1 \leqslant n_1 \leqslant \ldots \leqslant n_k} \mathcal{D}(n_1, \ldots, n_k).$$

On note simplement $\mathcal{D} = \mathcal{D}()$ l'espace de déformations des structures sans points de branchement.

Lorsque $g \geqslant 2$, et $(G, X) = (\mathrm{PSL}_2(\mathbb{R}), \mathbb{H}^2)$ l'espace $\mathcal{D}$ est appelé l'*espace de Teichmüller* de $\Sigma$ et est noté $\mathcal{T}$.

## 2.2 Action du groupe modulaire

### 2.2.1 Le groupe modulaire

Le groupe modulaire de $\Sigma$ est le groupe $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ des homéomorphismes directs de $\Sigma$, à isotopie près. On peut aussi le définir comme le groupe des composantes connexes des homéomorphismes directs de $\Sigma$ muni de la topologie compacte-ouverte :

$$\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Homeo}^+(\Sigma)).$$

C'est un groupe de type fini, engendré par des classes d'homéomorphismes élémentaires appelés *twists de Dehn*. Un twist de Dehn le long d'une courbe simple $\gamma$ est défini comme suit. Prenons un voisinage tubulaire de $\gamma$ homéomorphe à un cylindre $C = [0,1] \times \mathbb{R}/\mathbb{Z}$. On considère l'homéomorphisme du cylindre défini par $(r,\theta) \mapsto (r, \theta + r)$. Comme il vaut l'identité sur le bord de $C$, il se prolonge, par l'identité, en un homéomorphisme de $\Sigma$ : le twist de Dehn le long de $\gamma$. Un nombre fini de twists de Dehn engendrent le groupe modulaire, voir [FM01, Section 4.4.2].

**Théorème 2.2.1** (Lickorich)**.** *Les twists de Dehn le long des courbes $c_1, \ldots, c_{3g-1}$, voir figure 2.1, engendrent le groupe modulaire.*

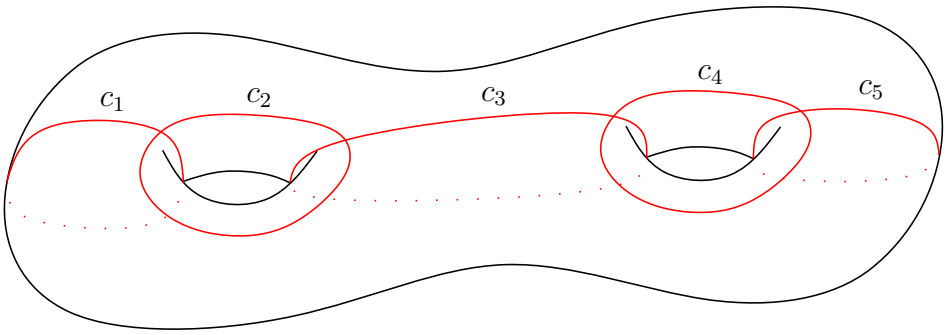


FIGURE 2.1 – Générateurs du groupe modulaire.

Lors des démonstrations des résultats principaux de cette thèse, on fera agir le groupe modulaire sur certains espaces. On fera en particulier agir des twists de Dehn. Nous décrivons maintenant ces actions.

### 2.2.2 Action sur les espaces de déformation

Le groupe modulaire $\mathrm{Mod}(\Sigma)$ agit sur les espaces de déformation $\mathrm{Def}_{(G,X)}(\Sigma)$ de la façon suivante : le groupe des homéomorphismes directs de $\Sigma$ agit sur cet espace en tirant en arrière les cartes, et par construction, cette action factorise en une action de $\mathrm{Mod}(\Sigma)$. Cette action préserve les strates $\mathcal{D}(n_1, \ldots, n_k)$ et induit donc au quotient une stratification de l'espace de modules des structures géométriques $\mathcal{M}_{(G,X)}(\Sigma) = \mathrm{Def}_{(G,X)}(\Sigma)/\mathrm{Mod}(\Sigma)$,

$$\mathcal{M}_{(G,X)}(\Sigma) = \bigcup_k \bigcup_{1 \leqslant n_1 \leqslant \ldots \leqslant n_k} \mathcal{M}(n_1, \ldots, n_k).$$

On note aussi $\mathcal{M} = \mathcal{M}()$ l'espace de modules des structures géométriques sans point de branchement.

#### Exemples

Donnons deux exemples de strates d'espaces de modules de structures géométriques.

— Dans le cas des surfaces de translations, *i.e.* $X = \mathbb{E}^2$ et $G = \mathbb{R}^2$, l'espace $\mathcal{M}(n_1, \ldots, n_k)$ est habituellement noté $\mathcal{H}(n_1, \ldots, n_k)$, et nous allons voir qu'une telle strate est non-vide si et seulement si $\sum_i n_i = 2g - 2$.

— Dans le cas des structures hyperboliques, *i.e.* $X = \mathbb{H}^2$ et $X = \mathrm{PSL}_2(\mathbb{R})$, l'espace $\mathcal{M} = \mathcal{M}_g$ est aussi appelé l'espace de modules des courbes de genre $g$.

### 2.2.3 Action sur les variétés de représentations

Si $\Gamma$ est un groupe, on note $\mathrm{Aut}(\Gamma)$ le groupe de ses automorphismes. On note aussi $\mathrm{Inn}(\Gamma)$ le groupe de ses automorphismes intérieurs : le sous-groupe de $\mathrm{Aut}(\Gamma)$ formé par les automorphismes de la forme $\gamma \mapsto g\gamma g^{-1}$, où $g \in \Gamma$. Enfin, on note $\mathrm{Out}(\Gamma)$ le groupe de ses automorphismes extérieurs : $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. Un élément de $\mathrm{Mod}(\Sigma)$ induit un automorphisme extérieur du groupe fondamental de $\Sigma$. Le sous-groupe $\mathrm{Aut}^+(\pi_1\Sigma)$ de $\mathrm{Aut}(\pi_1\Sigma)$ composé des applications induites par les applications continues $\Sigma \to \Sigma$ qui préservent l'orientation et fixent le point base de $\pi_1(\Sigma)$ est d'indice 2. On définit le groupe des automorphismes extérieurs directs de $\Sigma$ par

$$\mathrm{Out}^+(\pi_1(\Sigma)) = \mathrm{Aut}^+(\pi_1(\Sigma))/\mathrm{Inn}(\pi_1(\Sigma)).$$

Ce groupe est d'indice 2 dans $\mathrm{Out}(\pi_1(\Sigma))$. L'action du groupe modulaire sur le groupe fondamental induit un isomorphisme entre le groupe modulaire et $\mathrm{Out}^+(\pi_1(\Sigma))$, voir [FM01, Chapitre 8].

**Théorème de Dehn, Nielsen et Baer.** L'application naturelle

$$\mathrm{Mod}(\Sigma) \to \mathrm{Out}^+(\pi_1(\Sigma))$$

est un isomorphisme.

Le groupe $G$ agit par conjugaison sur $\mathrm{Hom}(\pi_1(\Sigma), G)$, et on note $R(G)$ la variété des représentations

$$R(G) = \mathrm{Hom}(\pi_1(\Sigma), G)/G.$$

Le groupe des automorphismes directs $\mathrm{Aut}^+(\pi_1(\Sigma))$ agit sur $R(G)$ et cette action factorise en une action de $\mathrm{Out}^+(\pi_1(\Sigma))$. D'après le théorème de Dehn, Nielsen et Baer, le groupe modulaire agit sur $R(G)$ comme $\mathrm{Out}^+(\pi_1(\Sigma))$. On dispose donc de deux actions de $\mathrm{Mod}(\Sigma)$ : une sur $R(G)$ et une sur $\mathrm{Def}_{(G,X)}(\Sigma)$. Ces deux actions sont reliées par l'holonomie :

$$\forall X \in \mathrm{Def}_{(G,X)}(\Sigma), \forall f \in \mathrm{Mod}(\Sigma),\ \mathrm{Hol}(f \cdot X) = f \cdot \mathrm{Hol}(X).$$

En effet si $\rho$ est l'holonomie d'une structure $X$ dont une développante associée est dev et $f \in \mathrm{Mod}(\Sigma)$, alors en relevant un représentant de $f$ en une application $\tilde{f} \colon \widetilde{\Sigma} \to \widetilde{\Sigma}$, l'application $\mathrm{dev} \circ \tilde{f}$ est par définition une développante pour $f \cdot X$ et est $(f \cdot \rho)$-équivariante. En particulier les espaces $\mathrm{Hol}(\mathcal{D}(n_1, \ldots, n_k))$ sont invariants par l'action de $\mathrm{Mod}(\Sigma)$.

## 2.3 Structures projectives et surfaces de translation

Dans cette thèse, nous nous intéressons en particulier aux structures projectives branchées.

### 2.3.1 Structure projectives branchées

Une structure projective branchée est une structure géométrique branchée modelée sur $X = \mathbb{CP}^1$ avec changement de cartes dans $G = \mathrm{PSL}_2(\mathbb{C})$, le

groupe des transformations de Möbius agissant sur $\mathbb{CP}^1$ par

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Les transformations de Möbius sont holomorphes, et une structure projective induit donc une structure complexe sur la surface privée des points de branchement. On peut montrer qu'une telle structure se prolonge sur toute la surface.

**Proposition 2.3.1.** *Il existe une unique structure complexe sur une surface munie d'une structure projective branchée qui rende les cartes holomorphes.*

La développante est alors le prolongement analytique de l'une des cartes pour cette structure complexe.

Rappelons, voir [FM01, Section 11.1.1], que le théorème d'uniformisation permet d'identifier l'espace de Teichmüller à l'espace des structures complexes sur $\Sigma$ modulo isotopie. Il existe donc une application $\Pi : \mathrm{Def}_{(\mathrm{PSL}_2(\mathbb{C}),\mathbb{CP}^1)}(\Sigma) \to \mathcal{T}$ qui oublie la structure projective pour ne garder que la structure complexe sous-jacente.

Remarquons que les structures hyperboliques branchées, les structures de translation (voir section 2.3.2) et les surfaces sphériques à singularités coniques entières induisent naturellement des structures projectives.

**Point de vue analytique** L'espace des structures projectives peut être défini de façon plus analytique comme un sous-ensemble des différentielles quadratiques méromorphes comme suit. Nous reprenons ici les arguments exposés dans le survol de David Dumas [Dum09, Section 3], et les adaptons pour y inclure les structures branchées.

**Definition 2.3.2.** La dérivée Schwarzienne $S(f)$ d'une fonction méromorphe non-constante $f : \Omega \to \mathbb{CP}^1$, où $\Omega$ est un ouvert de $\mathbb{C}$, est la différentielle quadratique définie par :

$$S(f) = \left[ \left( \frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right] dz^2 = \left[ \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right] dz^2.$$

On peut montrer que la dérivée Schwarzienne vérifie les deux propriétés suivantes :

1. $S(f \circ g) = g^* S(f) + S(g)$ pour toutes les fonctions $f$ et $g$ dont la composition est bien définie.
2. $S(A) = 0$ si et seulement si $A$ est localement la restriction d'une transformation de Möbius.

On déduit de ces propriétés que si $\Omega$ est connexe, alors $S(f) = S(g)$ si et seulement si $f = A \circ g$, où $A$ est une transformation de Möbius.

Soit $[(f, \rho)]$ une structure projective sur $\Sigma$. Fixons une identification $\widetilde{\Sigma} \simeq \mathbb{H}^2$ compatible avec la structure complexe induite par la structure projective. Ceci permet de voir $f$ comme une application méromorphe $f \colon \mathbb{H}^2 \to \mathbb{CP}^1$. Sa dérivée Schwarzienne $S(f)$ est une forme différentielle quadratique sur $\widetilde{\Sigma}$ qui est invariante par l'action de $\pi_1(\Sigma)$ :

$$\gamma^* S(f) = S(f \circ \gamma) = S(\rho(\gamma) \circ f) = S(f).$$

Ainsi cette forme passe au quotient en une forme quadratique sur $\Sigma$. En lisant $f$ dans des cartes locales, on observe que cette différentielle quadratique méromorphe a des pôles qui sont d'ordre 2 et qui sont situés précisément aux points de branchement.

Remarquons que l'on peut retrouver la fonction $f$, à post-composition par un élément de $\mathrm{PSL}_2(\mathbb{C})$ près, à partir de sa dérivée Schwarzienne $q = \mathcal{S}(f)$. En effet considérons l'équation différentielle ordinaire qui s'écrit en coordonnées locales

$$u'' + \frac{1}{2} q u = 0.$$

Ses solutions holomorphes forment un espace vectoriel de dimension 2 engendré par $u_2 = \sqrt{\frac{1}{f'}}$ et $u_1 = f u_2$. Prenons deux solutions linéairement indépendantes $v_1, v_2$ de cette équation différentielle. On peut écrire $v_1 = a u_1 + b u_2$ et $v_2 = c u_1 + d u_2$. En faisant le quotient de $v_1$ par $v_2$, on obtient alors :

$$\frac{v_1}{v_2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f.$$

L'holonomie $\rho$ de la structure projective de développante $f$ apparaît alors comme ceci. Soit $\gamma \in \pi_1(\Sigma)$. En prolongeant analytiquement les solutions locales $u_1$ et $u_2$ le long de $\gamma$, on obtient de nouvelles solutions

$$\gamma_* \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = A \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

avec $A = \rho(\gamma) \in \mathrm{PSL}_2(\mathbb{C})$. Ce point de vue permet de montrer que l'holonomie d'une structure projective $X \in \mathcal{P}(n_1, \ldots, n_k)$ se relève en un morphisme $\pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$ si et seulement si la somme des $n_i$ est paire, voir [GKM00, Section 1.4, Lemma 11.2.1]. Il permet également de paramétrer les structures projectives branchées et en particulier de définir d'une autre façon la topologie des espaces de déformations, en les voyant comme un sous-ensemble de l'espace des différentielles quadratiques méromorphes.

### 2.3.2 Les surfaces de translation

Dans cette sous-section nous présentons les *surfaces de translation* : les structures branchées modelées sur le plan Euclidien $X = \mathbb{E}^2$ avec transitions dans $G = \mathbb{R}^2$. On appelle aussi une telle structure une *structure de translation*. Ces structures géométriques interviennent dans de nombreux domaines des mathématiques, en particulier dans la théorie de Teichmüller et la dynamique unidimensionnelle. L'étude des surfaces de translation, de la dynamique de leurs flots géodésiques et de la dynamique de l'action de $\mathrm{GL}_2^+(\mathbb{R})$ sur leurs espaces de modules est un vaste domaine de recherche très actif.

Comme $X = \mathbb{E}^2 \subset \mathbb{CP}^1$ et que les translations sont des transformations de Möbius, une structure de translation induit une structure projective branchée. Les ordres $n_1, \ldots, n_k$ des points de branchement vérifient d'après la formule de Gauss-Bonnet

$$\sum_{i=1}^{k} n_i = 2g - 2.$$

Réciproquement si l'on dispose d'une structure projective dont l'holonomie a un conjugué dans $\mathrm{Hom}(\pi_1(\Sigma), \mathbb{C})$ et dont les ordres des points de branchements vérifient $\sum_i n_i = 2g-2$, alors elle provient d'une surface de translation, comme le montre Kapovich dans [Kap20, Proposition 6.4].

**Proposition 2.3.3.** *Soit $(f, \rho)$ un représentant du couple développante holonomie d'une structure projective avec des points de branchements d'ordre $n_i$ tels que $\sum_i n_i = 2g - 2$ et $\rho\colon \pi_1(\Sigma) \to \mathbb{C}$. La développante $f$ n'atteint pas l'infini : $f(\widetilde{\Sigma}) \subset \mathbb{E}^2$.*

Dans cette thèse nous présentons une généralisation de cet argument aux structures affines, voir la proposition 6.18 dans chapitre 5 qui est une reproduction de [LF21].

Remarquons qu'on ne peut néanmoins pas identifier l'espace de déformations des structures de translation sur $\Sigma$ avec le sous ensemble qu'il induit dans l'espace de déformations des structures projectives. En effet dans l'espace des déformations des structures projectives, on identifie deux structures qui sont obtenues par dilatation et rotation l'une de l'autre, ce qui n'est pas le cas dans l'espace des structures de translation.

Le groupe $\mathrm{Homeo}^+(\Sigma)$ agit sur l'ensemble des couples $(X, \alpha)$ où $X$ est une structure complexe directe sur $\Sigma$, et $\alpha$ une 1-forme holomorphe sur $X$. Notons $\Omega\mathcal{T}$ le quotient de cet ensemble par $\mathrm{Homeo}_0(\Sigma)$ et $\Omega'\mathcal{T}$ le sous-ensemble de $\Omega\mathcal{T}$ formé des formes holomorphes non identiquement nulles. Rappelons que les 1-formes holomorphes sur une surface de Riemann forment un espace vectoriel complexe de dimension $g$. L'espace $\Omega\mathcal{T}$ est donc naturellement un fibré vectoriel au dessus de $\mathcal{T}(\Sigma)$ de dimension complexe $g$. De plus $\Omega'\mathcal{T}$ est égal à $\Omega\mathcal{T}$ privé de l'image de la section nulle.

Les surfaces de translation s'identifient aux éléments de $\Omega'\mathcal{T}$, et l'on dispose d'un dictionnaire entre les *différentielles abéliennes*, les 1-formes holomorphes, et les surfaces de translation.

**Proposition 2.3.4.** *L'espace $\Omega'\mathcal{T}$ s'identifie naturellement à l'espace des déformations* $\mathrm{Def}_{(\mathbb{R}^2,\mathbb{E}^2)}(\Sigma)$.

*Démonstration.* Supposons que l'on dispose d'une structure géométrique modelée sur $X = \mathbb{C}$ avec des changements de cartes qui sont des translations. La forme holomorphe $\alpha = f^*(dz)$ passe au quotient en une 1-forme holomorphe sur $\Sigma$. En effet pour tout $\gamma \in \pi_1(\Sigma)$,

$$\gamma^*\alpha = \gamma^* f^*(dz) = (f \circ \gamma)^*(dz) = (\rho(\gamma) \circ f)^*(dz) = f^*\rho(\gamma)^*(dz) = \alpha.$$

Réciproquement, si l'on dispose d'une structure complexe sur $\Sigma$ ainsi que d'une 1-forme holomorphe non-nulle sur celle-ci, on a un choix privilégié de cartes dans $\mathbb{E}^2$ donnant une structure de surface de translation. En effet, il existe des coordonnées locales holomorphes dans lesquelles $\alpha = dz$. Ces coordonnées diffèrent par des translations. $\square$

Notons $\widehat{\mathcal{H}}(n_1, \ldots, n_k)$ les strates de l'espace de déformations des structures de translation sur $\Sigma$. On note aussi les espaces de modules $\mathcal{M}_{(\mathbb{R}^2,\mathbb{E}^2)}(n_1, \ldots, n_k)$ par $\mathcal{H}(n_1, \ldots, n_k)$. Ainsi l'application d'oubli du marquage des structures de translation se restreint en :

$$\widehat{\mathcal{H}}(n_1, \ldots, n_k) \to \mathcal{H}(n_1, \ldots, n_k).$$

Les espaces $\widehat{\mathcal{H}}(n_1, \ldots, n_k)$ correspondent dans le dictionnaire précédent aux sous-ensembles de $\Omega\mathcal{T}$ formés par les $(X, \alpha)$ où $\alpha$ est une forme holomorphe s'annulant exactement $k$ fois, avec des ordres d'annulation $n_1, \ldots, n_k$. Dans ce langage, c'est la formule de Riemann-Roch qui assure que $\sum_i n_i = 2g - 2$. De plus, cette formule montre que réciproquement la strate $\widehat{\mathcal{H}}(n_1, \ldots, n_k)$ est non-vide si $\sum_i n_i = 2g - 2$.

Dans ce dictionnaire, l'application d'holonomie a une forme particulièrement simple. L'holonomie d'une différentielle abélienne $(X, \alpha)$ est appelée son application de périodes et est définie de la façon suivante :

$$\mathrm{Per}(\alpha) : \gamma \in \pi_1(\Sigma) \mapsto \int_\gamma \alpha \in \mathbb{C}.$$

Rappelons que l'abélianisé du groupe fondamental d'une surface s'identifie naturellement à son premier groupe d'homologie à coefficients entiers. Ainsi, l'application de périodes passe au quotient en une application de $\mathrm{H}_1(\Sigma, \mathbb{Z})$ dans $\mathbb{C}$, et s'identifie à un élément de $\mathrm{H}^1(\Sigma, \mathbb{C})$.

Les périodes des différentielles abéliennes forment une partie de leurs périodes relatives qui paramétrisent les strates de $\Omega\mathcal{T}$. Plus précisément fixons $(X, \omega)$ dans une strate $\widehat{\mathcal{H}}(n_1, \ldots, n_k)$. Notons $Z \subset \Sigma$ l'ensemble des zéros de $\omega$. L'application de périodes relatives $\mathrm{PerRel} : \Omega\mathcal{T} \to \mathrm{Hom}(\mathrm{H}_1(\Sigma, Z, \mathbb{Z}), \mathbb{C})$ est définie de façon analogue par :

$$\mathrm{PerRel}(\alpha) : \gamma \mapsto \int_\gamma \alpha.$$

Cette fonction est un homéomorphisme local au voisinage de $(X, \omega)$ en restriction à $\widehat{\mathcal{H}}(n_1, \ldots, n_k)$, voir par exemple [Vee90, Zor06]. Les choix d'une base symplectique de $\mathrm{H}_1(\Sigma, \mathbb{Z})$ et de $k - 1$ chemins reliant les points de $Z$ permettent d'identifier $\mathrm{Hom}(\mathrm{H}_1(\Sigma, Z, \mathbb{Z}), \mathbb{C})$ à $\mathbb{C}^{2g+k-1}$. Ces identifications définissent une structure de variété complexe sur les strates de $\Omega\mathcal{T}$. Les feuilles du feuilletage isopériodique intersectent donc la strate $\mathcal{H}(n_1, \ldots, n_k)$ en des espaces de dimension complexe $k - 1$.

Pour déterminer les éléments de $\mathrm{H}^1(X, \mathbb{C})$ qui sont les périodes d'une différentielle abélienne dans une strate fixée, on utilisera la notion de *volume* d'une différentielle abélienne. Le volume est défini dans le langage des structures de translation de la façon suivante : si l'on dispose d'un couple $(f, \rho)$ où $f \colon \widetilde{\Sigma} \to \mathbb{C}$ est une développante et $\rho \colon \pi_1(\Sigma) \to \mathbb{C}$ son holonomie, alors on peut tirer en arrière par $f$ la forme $dx \wedge dy$ du plan qui est invariante par

l'action de $\pi_1(\Sigma)$ et induit donc une forme $\omega$ sur $\Sigma$. Le volume de la structure de translation est alors $\int_\Sigma \omega$ : il s'agit du volume Riemannien de la métrique plate à singularités coniques induite par la développante. Si la structure de translation provient de la différentielle abélienne $(X, \alpha)$, alors son volume est donné par

$$\mathrm{Vol}(\alpha) = \frac{i}{2} \int_\Sigma \alpha \wedge \overline{\alpha}.$$

On verra dans le chapitre 4 que ce volume peut se calculer à partir de l'holonomie et permet d'énoncer des conditions nécessaires et suffisantes à l'existence d'une surface de translation d'holonomie fixée.

## 2.4 Résultats connus sur les holonomies de quelques structures géométriques

Cette section a pour but de donner un aperçu de résultats concernant quelques structures géométriques sur les surfaces, leurs espaces de déformations et les liens avec les applications d'holonomies.

### 2.4.1 Les structures hyperboliques branchées

On s'intéresse ici au cas $G = \mathrm{PSL}_2(\mathbb{R})$ et $X = \mathbb{H}^2$.

On rappelle, voir par exemple [Gol80] que l'application d'holonomie permet d'identifier l'espace de Teichmüller à l'une des deux composantes connexes de $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$ qui sont composées de représentations discrètes et fidèles.

**Théorème 2.4.1.** *L'application d'holonomie sur l'espace de Teichmüller* $\mathrm{Hol} : \mathcal{T} \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$ *est un homéomorphisme sur son image qui forme une composante connexe de* $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$ *composée uniquement de classes de conjugaison de représentations discrètes et fidèles.*

La question plus générale de déterminer les représentations $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ qui sont l'holonomie d'une structure hyperbolique branchée, c'est-à-dire qui sont dans $\mathrm{Hol}(\mathrm{Def}_{(G,X)}(\Sigma))$ est ouverte. Rappelons que Goldman [Gol88b] a classifié les composantes connexes de $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$ : elles sont indexées par la classe d'Euler

$$\mathrm{eu}\colon \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R}) \to [2-2g, 2g-2] \cap \mathbb{Z}.$$

La formule de Gauss-Bonnet, voir [Mat12, Proposition 2.1], permet de montrer que les représentations de classe d'Euler négative ou nulle ne peuvent pas se réaliser comme holonomie d'une structure hyperbolique branchée. Dans [Tan94], Tan définit une classe de représentations de classe d'Euler non-nulle qui ne peuvent pas être l'holonomie d'une structure hyperbolique branchée. Il s'agit des surfaces obtenues en précomposant une représentation discrète et fidèle d'un groupe de surface $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$, avec le genre de $S$ strictement inférieur au genre de $\Sigma$, par une application $\pi_1(\Sigma) \to \pi_1(S)$ obtenue en écrasant des anses. Il est conjecturé qu'une représentation $\rho$ telle que $\mathrm{eu}(\rho) > 0$ est l'holonomie d'une structure hyperbolique branchée si et seulement si elle ne provient pas de la construction de Tan. Une démonstration de ce résultat a été annoncée par Deroin et Tholozan pour les représentations de classe d'Euler $2g-3$. Toute ces représentations sont d'après leurs résultats des holonomies de structures hyperboliques branchées. Une conjecture moins forte est que l'ensemble des représentations qui ne se réalisent pas comme une holonomie est de mesure nulle. Cette question est liée à la conjecture de Goldman suivante.

**Conjecture 2.4.2** (Goldman)**.** *L'action de* $\mathrm{Mod}(\Sigma)$ *sur les composantes connexes de* $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$ *qui ne sont pas les deux composantes des représentations discrètes et fidèles est ergodique.*

Cette conjecture a été démontrée par Marché et Wolff en genre 2, [MW16, MW19]. Plus précisément, les auteurs montrent que l'action du groupe modulaire est ergodique sur chacune des deux composantes $\mathrm{eu}^{-1}(\pm 1)$ et qu'elle décompose $\mathrm{eu}^{-1}(0)$ en deux composantes ergodiques de mesure totale. La conjecture de Goldman reste ouverte en genre plus grand et une réponse positive impliquerait que presque toutes les représentations de classe d'Euler strictement positive se réalisent comme l'holonomie d'une structure hyperbolique branchée.

### 2.4.2 Les surfaces de translation

Revenons au cas des surfaces de translation : $X = \mathbb{E}^2$ et $G = \mathbb{R}^2$. Comme nous l'avons mentionné dans l'introduction, un théorème de Haupt de 1920 [Hau20] caractérise les holonomies des surfaces de translation.

**Théorème de Haupt.** Les applications $\chi\colon \pi_1(\Sigma) \to \mathbb{C}$ qui apparaissent comme les périodes absolues d'une différentielle abélienne sont celles qui vérifient les deux conditions suivantes

1. $\mathrm{Vol}(\chi) > 0$
2. $\mathrm{Vol}(\chi) \geqslant 2\mathrm{Vol}(\mathbb{C}/\Lambda)$ si $\chi(\pi_1(\Sigma)) = \Lambda$ est un réseau de $\mathbb{C}$.

L'application Hol définit un feuilletage de l'espace de modules des surfaces de translation comme mentionné dans l'introduction. Rappelons que ce feuilletage a fait l'objet de beaucoup d'attention récemment, notamment de la part de McMullen [McM14], Calsamiglia, Deroin et Francaviglia [CDF14], Hamenstädt [Ham18], Ygouf [Ygo21] et Winsor [Win21]. En particulier les adhérences des feuilles sont décrites dans [CDF14, Théorème 1.5].

Nous précisons le théorème de Haupt en caractérisant les périodes des surfaces de translations dans une strate fixée avec le théorème 1.2.2 que nous démontrons dans le chapitre 4. Nous décrivons ainsi les strates qui intersectent une feuille donnée. Une version de ce résultat pour les structures de translation sur des surfaces épointées a été pré-publiée depuis par Chenakkod, Faraco et Gupta [CFG21].

### 2.4.3 Les structures projectives complexes sans points de branchements

Intéressons-nous maintenant aux structures projectives complexes, *i.e.* $G = \mathrm{PSL}_2(\mathbb{C})$ et $X = \mathbb{CP}^1$, qui sont dans la strate $\mathcal{P} = \mathcal{D}$. Il s'agit des structures projectives sans point de branchement.

Un résultat de Poincaré affirme que les données de l'holonomie et de la structure complexe sous-jacente à une structure projective déterminent cette dernière uniquement [Poi84], voir aussi [LMP09, Theorem 1.4]. Rappelons que l'on note par $\Pi$ l'application qui à une structure projective associe sa structure complexe sous-jacente.

**Théorème 2.4.3.** *L'application*

$$\mathrm{Hol} \times \Pi \colon \mathcal{P} \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C}) \times \mathcal{T}$$

*est injective.*

Nous expliquons maintenant l'opération de greffe (grafting) sur des structures projectives. Pour plus de détails, on renvoie à [Dum09, Section 4.1]. Considérons une structure hyperbolique $X$ sur $\Sigma$. Soient une géodésique simple $\gamma$ sur $X$ et $t > 0$. On peut déformer $X$ dans $\mathcal{P}$ en coupant la surface le

long de $\gamma$ puis en y insérant un cylindre muni d'une structure projective obtenue de la façon suivante. L'holonomie de la structure hyperbolique évaluée en $\gamma$ est conjuguée à l'application $z \mapsto e^l z$ pour un $l \in \mathbb{R}$. Supposons d'abord que $t < 2\pi$. On considère la région bordée par $\mathbb{R}_{>0}$ et $e^{it} \cdot \mathbb{R}_{>0}$, que l'on quotiente par $z \mapsto e^l z$ pour obtenir le cylindre voulu. Si jamais $t > 2\pi$, on recolle $n$ cylindres d'angle $\frac{t}{n}$ avec $2\pi n > t$ à l'aide de rotations $z \mapsto e^{is} z$.

Notons $\mathcal{S}$ l'ensemble des courbes simples sur $\Sigma$. La greffe définit une application

$$\mathrm{Gr}\colon \mathcal{T} \times \mathcal{S} \times \mathbb{R}_{>0} \to \mathcal{P}.$$

On rappelle que les courbes munies d'un poids forment un sous-ensemble $\mathcal{S} \times \mathbb{R}_{>0}$ dense de l'espace $\mathcal{ML}$ des laminations mesurées, voir [Thu02, Chapitres 8 et 9]. Thurston a montré que l'application de greffe s'étend en un homéomorphisme aux laminations mesurées, voir [KT92].

**Théorème 2.4.4** (Thurston)**.** *L'application de greffe*

$$\mathrm{Gr}\colon \mathcal{T} \times \mathcal{ML} \to \mathcal{P}$$

*est un homéomorphisme.*

Notons $\mathcal{F}$ le sous-espace de $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ des représentations *fuchsiennes* : les classes de conjugaison des représentations discrètes et fidèles $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$. On note également $\mathcal{P}_{\mathcal{F}}$ l'espace des structures projectives dont l'holonomie est fuchsienne, c'est-à-dire $\mathcal{P}_{\mathcal{F}} = \mathrm{Hol}^{-1}(\mathcal{F}) \subset \mathcal{P}$.

Goldman montre dans [Gol03] que pour toute structure projective $X$ d'holonomie fuchsienne $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$, il existe une unique multicourbe $\lambda$ avec un poids multiple de $2\pi$ sur chacune de ses composantes, telle que $X$ s'obtient en faisant une greffe le long de $\lambda$ sur la structure hyperbolique d'holonomie $\rho$. On note $\mathcal{ML}(\mathbb{Z}) \subset \mathcal{ML}$ l'espace des laminations mesurées entières : les multicourbes simples avec un poids multiple entier de $2\pi$ sur chaque composante. Une greffe est dite entière si elle est faite sur une lamination entière. Les greffes entières paramétrisent les structure projectives d'holonomie quasi-fuchsienne.

**Théorème 2.4.5** (Goldman)**.** *L'application de greffe* Gr *met en bijection* $\mathcal{T} \times \mathcal{ML}(\mathbb{Z})$ *et* $\mathcal{P}_{\mathcal{F}}$.

L'application $\mathrm{Hol}\colon \mathcal{P} \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ n'est pas injective. En fait les fibres $\mathrm{Hol}^{-1}(\rho)$ sont même infinies, lorsqu'elles ne sont pas

vides. En effet, l'opération de greffe se généralise sur les structures projectives qui ne sont pas nécessairement des structures hyperboliques, le long d'une courbe simple dont l'image est loxodromique, ce qui existe toujours comme montré par Gallo, Kapovich et Marden dans [GKM00, Proposition 3.1.1], voir aussi le corollaire 1.5 dans le chapitre 3.

Il a été montré par Hejhal [Hej75], Earle [Ear81] et Hubbard [Hub81] que l'application d'holonomie sur les structures projectives sans point de branchement est un homéomorphisme local.

**Théorème 2.4.6** (Hejhal, Earle, Hubbard)**.** *L'application*

$$\mathrm{Hol}\colon \mathcal{P} \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$$

*est un homéomorphisme local.*

En revanche, Hejhal [Hej75] montre que cette application n'est pas un revêtement sur son image. Cette image a été caractérisée par Gallo, Kapovich et Marden, voir chapitre 1, comme étant l'ensemble des représentations $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ non-élémentaires qui se relèvent à $\mathrm{SL}_2(\mathbb{C})$.

Comme expliqué dans l'introduction et comme nous allons le voir dans le chapitre 5, nous étendons cette classification à l'ensemble des structures projectives branchées avec une combinatoire de points de branchement fixée.

### 2.4.4 Structures projectives réelles

On suppose désormais que $X = \mathbb{RP}^2$ et $G = \mathrm{PSL}_3(\mathbb{R})$. On s'intéresse ici aux structures non-branchées. On note $\mathcal{RP}(\Sigma) = \mathcal{D}$ l'espace de déformations des structures projectives non-branchées sur $\Sigma$. Goldman [Gol90] a décrit la topologie de cet espace.

**Théorème 2.4.7** (Goldman)**.** *L'espace $\mathcal{RP}(\Sigma)$ est homéomorphe à une union disjointe dénombrable de copies de $\mathbb{R}^{16g-16}$.*

Notons $\mathcal{CRP}(\Sigma)$ le sous-ensemble de $\mathcal{RP}(\Sigma)$ des structures projectives *convexes*, c'est-à-dire dont l'image par une développante est dans un ensemble strictement convexe d'une carte affine de $\mathbb{RP}^2$.

Choi et Goldman ont montré dans [CG93], de façon analogue aux structures hyperboliques, que l'application qui à une structure projective réelle convexe associe son holonomie est un homéomorphisme sur une composante connexe de l'espace de représentations associé.

**Théorème 2.4.8** (Choi-Goldman)**.** *L'application*

$$\mathrm{Hol}\colon \mathcal{CRP}(\Sigma) \to \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_3(\mathbb{R}))/\mathrm{PSL}_3(\mathbb{R})$$

*est un homéomorphisme sur la composante de Hitchin.*

Hitchin a montré dans [Hit92] que l'espace $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_3(\mathbb{R}))/\mathrm{PSL}_3(\mathbb{R})$ possède trois composantes connexes, dont l'une homéomorphe à $\mathbb{R}^{16g-16}$. Labourie [Lab06] a montré qu'elle est composée uniquement de représentations discrètes et fidèles. Il s'agit de celle qui contient des représentations de la forme $\tau \circ \rho$ où $\tau\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_3(\mathbb{R})$ est induit par une représentation irréductible $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})$ et $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{R})$ est discrète et fidèle. Cette composante connexe est appelée la *composante de Hitchin*.

### 2.4.5 Holonomie des structures projectives complexes branchées

Nous donnons ici des résultats sur les holonomies des structures projectives complexes branchées.

#### Isomonodromie

Dans cette thèse on s'intéresse principalement à la question de déterminer l'image de l'application d'holonomie pour différentes structures géométriques sur les surfaces. Ceci permet de décomposer l'espace de déformations de celles-ci en les espaces où l'on fixe l'holonomie $\mathrm{Hol}^{-1}(\rho)$. Étudier les espaces $\mathrm{Hol}^{-1}(\rho)$ permet alors de mieux comprendre les espaces de déformations. Nous donnons ici quelques résultats dans ce sens pour les structures projectives.

Faire une greffe entière sur une structure projective projective ne change pas son holonomie. Dans [Bab17], Baba montre que cette opération permet d'obtenir toutes les structures projectives non branchées d'holonomie donnée suffisamment générique. On dit qu'une représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ est *purement loxodromique* si pour tout $\gamma \in \pi_1(\Sigma) \setminus \{1\}$, $\rho(\gamma)$ est loxodromique, c'est à dire $|\mathrm{Tr}(\rho(\gamma))| > 2$. Cette condition est générique dans $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$, voir [Bab17]. Une dégreffe entière est l'opération inverse d'une greffe entière.

**Théorème 2.4.9** (Baba)**.** *Deux structures projectives non-branchées ayant la même holonomie purement loxodromique sont reliées par une suite de greffes et de dégreffes entières.*

Un autre résultat pour les structures branchées partageant la même holonomie quasi-fuchsienne a été donné par Calsamglia, Deroin et Francaviglia. Rappelons qu'une représentation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ est dite quasi-fuchsienne s'il existe un $z \in \mathbb{H}^3$ tel que l'application $\pi_1(\Sigma) \to \mathbb{H}^3$ définie par $\gamma \mapsto \rho(\gamma) \cdot z$ soit une quasi-isométrie sur son image. Calsamiglia Deroin et Francaviglia considèrent trois types de mouvements permettant de passer d'une structure projective à l'autre sans modifier leur holonomie. Il s'agit de l'opération d'*attacher une sphère*, appelée *bubbling* en anglais. Une autre opération élémentaire consiste en l'opération inverse, d'enlever une sphère attachée de cette façon. Enfin la dernière opération élémentaire est un mouvement de points de branchements, voir [CDF14].

**Théorème 2.4.10** (Calsamiglia-Deroin-Francaviglia)**.** *Soit $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ une représentation quasi-fuchsienne. Soient $X$ et $Y$ deux structures projectives branchées d'holonomie $\rho$. Il existe une séquence d'opérations élémentaires permettant de passer de $X$ à $Y$.*

Dans [Ruf21], Ruffoni montre que l'ensemble des structures projectives que l'on obtient en attachant une sphère à une structure projective non branchée d'holonomie quasi-fuchsienne fixée est dense dans la fibre intersectée avec la strate des structures projectives correspondant aux deux points coniques que l'on crée en collant celle-ci.

**Théorème 2.4.11** (Ruffoni)**.** *Soit $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ une représentation quasi-fuchsienne. L'ensemble des structures projectives obtenues en attachant une sphère à une structure projective $X \in \mathcal{P} \cap \mathrm{Hol}^{-1}(\rho)$ est ouvert et dense dans $\mathcal{P}(1,1) \cap \mathrm{Hol}^{-1}(\rho)$.*

### Autres résultats sur les holonomies des structures projectives

Nous finissons cette section en donnant d'autres résultats sur les holonomies des structures projectives branchées.

**Holonomie des surfaces non-compactes** La question de caractériser les représentations qui se réalisent comme l'holonomie d'une structure projective

peut être posée pour les surfaces épointées. On demande alors à ce que les singularités aux pointes aient une certaine forme. On a par exemple le résultat de Nascimento, obtenu indépendemment par Gupta. Une structure projective de type Fuchsien est une structure projective sur une surface épointés dont les singularités aux pointes sont de la forme $z^\alpha$, $\alpha \in \mathbb{C}$ ou $\log z + \frac{1}{z^n}$ avec $n$ entier. Soit $\Sigma'$ une surface de type fini non-compacte.

**Théorème 2.4.12** (Nascimento, Gupta)**.** *Toute représentation* $\rho\colon \pi_1(\Sigma') \to \mathrm{PSL}_2(\mathbb{C})$ *se réalise comme holonomie d'une structure projective de type Fuchsien.*

Les structures projectives sur des surfaces non-compactes ont reçu beaucoup d'attention récemment et il y a plusieurs résultats sur les holonomies de ces structures [Gup19, GM19, FG21]. Cependant de nombreux problèmes restent ouverts, notamment sur la caractérisation des holonomies des structures projectives dont on contrôle les singularités.

**Holonomie et structure complexe** Rappelons que d'après le théorème de Poincaré, l'application $\Pi : \mathcal{P} \cap \mathrm{Hol}^{-1}(\rho) \to \mathcal{T}$ est injective. En revanche, Baba et Gupta ont démontré que la projection dans l'espace de modules des courbes $\mathcal{P} \cap \mathrm{Hol}^{-1}(\rho) \to \mathcal{M}$ est dense.

**Théorème 2.4.13** (Gupta, Baba)**.** *Pour tout* $X \in \mathcal{M}$ *et tout* $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$ *qui est non-élémentaire et se relevant à* $\mathrm{SL}_2(\mathbb{C})$*, il existe une suite* $Z_n$ *de structures projectives d'holonomie* $\rho$ *telle que la structure complexe sous-jacente de* $Z_n$ *converge vers* $X$ *dans* $\mathcal{M}$*.*

# Chapitre 3

# Représentations pentagones et holonomies des structures projectives complexes sur des surfaces fermées

Cette section est une reproduction de l'article [LF19]. Il contient en particulier la preuve du théorème 1.2.1. On donne un certain nombre d'outils et de résultats sur l'action du groupe modulaire sur des représentations de groupes de surfaces dans $\mathrm{PSL}_2(\mathbb{C})$. On y définit les représentations pentagones et on montre qu'elles se réalisent comme l'holonomie d'une structure projective avec un unique point de branchement d'ordre 1.

On montre également que toute représentation non-élémentaire d'un groupe de surface de $\mathrm{PSL}_2(\mathbb{C})$ admet une décomposition en pantalons dont chaque courbe est envoyé sur un élément loxodromique.

# Pentagon representations and complex projective structures on closed surfaces

Thomas Le Fils

**Abstract**

We define a class of representations of the fundamental group of a closed surface of genus 2 to $\mathrm{PSL}_2(\mathbb{C})$: *the pentagon representations.* We show that they are exactly the non-elementary $\mathrm{PSL}_2(\mathbb{C})$-representations of surface groups that do not admit a Schottky decomposition, *i.e.* a pants decomposition such that the restriction of the representation to each pair of pants is an isomorphism onto a Schottky group. In doing so, we exhibit a gap in the proof of Gallo, Kapovich and Marden that every non-elementary representation of a surface group to $\mathrm{PSL}_2(\mathbb{C})$ is the holonomy of a projective structure, possibly with one branched point of order 2. We show that pentagon representations arise as such holonomies and repair their proof.

## 1 Introduction

Denote by $\Sigma_{g,n}$ an oriented compact surface of genus $g$ with $n$ boundary components, and by $\Gamma_{g,n}$ a fundamental group of $\Sigma_{g,n}$ for all $g, n \geqslant 0$. For simplicity, we denote $\Sigma_{g,0}$ by $\Sigma_g$ and $\Gamma_{g,0}$ by $\Gamma_g$.

A complex projective structure on $\Sigma_g$ is a $(G, X)$ structure with $G = \mathrm{PSL}_2(\mathbb{C})$ and $X = \mathbb{CP}^1$, that is the datum of an atlas of charts with values in $\mathbb{CP}^1$, whose transition maps are restrictions of Möbius transformations. We can also allow branched points in the definition and get the notion of branched projective structure, see [7, Section 1.4] for a concise definition. We denote by $\mathcal{P}(\Sigma_g)$, resp. $\mathcal{P}_b(\Sigma_g)$, the set of unbranched complex projective structures, resp. the set of projective structures with a single branched point, of order 2. The datum of such a structure on $\Sigma_g$ gives rise to a holonomy map, well-defined up to conjugacy (see [19, Chapter 3] for more information on $(G, X)$-structures). Hence we have a map:

$$\mathrm{hol} : \mathcal{P}(\Sigma_g) \sqcup \mathcal{P}_b(\Sigma_g) \to \mathrm{Hom}(\Gamma_g, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C}).$$

This map establishes a relationship between $\mathrm{PSL}_2(\mathbb{C})$-representations of surface groups and projective structures. The study of this relationship has a long history. There is a natural complex structure on $\mathcal{P}(S)$, induced by its identification with the quadratic forms (see [9] for example). Hejhal, Earl and Hubbard

[11, 5, 12] showed that the map $\mathrm{hol}_{|\mathcal{P}(\Sigma_g)}$ is a local biholomorphism. However, it is known that $\mathrm{hol}_{|\mathcal{P}(\Sigma_g)}$ is neither injective nor a covering map. We refer to [4] for more information about projective structures.

The question of finding which representations arise as the holonomy of a complex projective structure has been open for a long time. Poincaré himself asked it in the case where $\Sigma$ is a punctured sphere, see [15, Paragraph 4]. Very recently, Gupta announced an answer for every punctured surface [10]. In [7], Gallo, Kapovich and Marden provided a complete answer for closed surfaces. They showed that the image of hol is the set of non-elementary representations. The main part of the theorem is the proof that every non-elementary representation is in the image of hol.

The strategy of [7] consists in first proving that every non-elementary representation admits a *Schottky decomposition* in the following sense.

**Definition 1.1.** A *Schottky decomposition* for a representation $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ is a pants decomposition of $\Sigma_g = \cup P_i$, such that for all $i$, the restriction $\rho_{|P_i} : \pi_1(P_i) \to \mathrm{PSL}_2(\mathbb{C})$ is an isomorphism onto a Schottky group.

Once such a decomposition is found, the authors put a projective structure on each pair of pants $P_i$, whose holonomy is given by the restriction $\rho_{|P_i}$. Then they glue the pants together with cylinders. It might be required to add a branched point of order 2 in one of the pairs of pants in order to make all the gluings possible.

Perhaps one main contribution of the present article is to exhibit a gap in the proof by Gallo, Kapovich and Marden of the existence of a Schottky decomposition for every non-elementary representation. We will establish later (see Theorem 1.4 below), essentially along their proof, that such a decomposition exists provided $g \geqslant 3$. However, in genus 2, the hyperelliptic involution yields counterexamples that we now introduce.

Let us recall that the mapping class group $\mathrm{Mod}(\Sigma_2)$ of $\Sigma_2$ has its center generated by the hyperelliptic involution. Let $\varphi \in \mathrm{Homeo}^+(\Sigma_2)$ be one of its representatives. The orbifold fundamental group $\Gamma$ of $\Sigma_2/\varphi$ has the following presentation:

$$\Gamma = \langle q_1, q_2, q_3, q_4, q_5, q_6 \mid q_i^2 = 1, q_1 q_2 \dots q_6 = 1 \rangle.$$

The group $\Gamma_2$ is naturally an index two subgroup of $\Gamma$ (see Section 3).

**Definition 1.2.** A *pentagon representation* is a non-elementary representation that is the restriction of a representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho(q_i) = \mathrm{id}$ for exactly one $1 \leqslant i \leqslant 6$.

This definition is reminiscent of the hourglass representations considered in [14]. We will see that the action of $\mathrm{Mod}(\Sigma_2) \simeq \mathrm{Out}^+(\Gamma_2)$ on the conjugacy classes of representations $\Gamma_2 \to \mathrm{PSL}_2(\mathbb{C})$ preserves the pentagon representations. Since there are only two pants decompositions of $\Sigma_2$ up to the action of the mapping class group, it suffices to check that these decompositions cannot be Schottky decompositions of a pentagon representation to prove the following.

**Proposition 1.3.** *A pentagon representation does not admit a Schottky decomposition.*

As a first step into reparing the proof of [7], we prove that there exists a Schottky decomposition for every other non-elementary $\rho$.

**Theorem 1.4.** *A representation $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ admits a Schottky decomposition if and only if $\rho$ is non-elementary and is not a pentagon representation.*

In fact we find a Schottky decomposition with $g$ pairs of pants glued to themselves in the non-pentagon case. This allows us to use the second part of [7], and hence fix the gap of the proof in this case.

As a corollary of this characterization of the pentagon representations, we will see that even if they do not admit a Schottky decomposition, they still have a *loxodromic decomposition.*

**Corollary 1.5.** *If $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary, then there exists a pants decomposition of $\Sigma_g$ whose boundary curves are taken by $\rho$ to loxodromic isometries.*

We hence also repair the proof of this corollary which was used for example in [18, Proposition 1].

We show that the pentagon representations have odd Stiefel-Whitney class, thus they cannot be in $\mathrm{hol}(\mathcal{P}(\Sigma_2))$ (see [7, Corollary 11.2.3]). It remains to understand whether they are in $\mathrm{hol}(\mathcal{P}_b(\Sigma_2))$ or not. We answer positively:

**Theorem 1.6.** *A pentagon representation is the holonomy of a branched projective structure with exactly one branched point, which is of order* 2.

Theorem 1.6 somehow complements a recent theorem of Baba [1], which states that every unbranched complex projective structure is obtained by gluing Schottky pants as in [7]. Indeed the analogous theorem for $\mathcal{P}_b(\Sigma_2)$ cannot hold for the pentagon representations have no Schottky decomposition and yet are in $\mathrm{hol}(\mathcal{P}_b(\Sigma_2))$.

In view of possible generalizations, we provide some alternative (and maybe simpler) proofs of some intermediate results of [6], that we use to prove Theorem 1.4. In particular, we give a new proof of the existence of a special handle in $\Sigma_{g,n}$: a subsurface which is a once punctured torus, whose fundamental group is generated by two elements $a$ and $b$ such that $\rho(a)$ and $\rho(b)$ are loxodromic without a common fixed point, where $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary and $g \geqslant 1$. Our approach begins by finding a handle on which the restriction of $\rho$ is non-elementary. This leads us to study the non-elementary representations of the punctured torus, up to the action of the mapping class group. This study is reminiscent of the work of Goldman in [8]. In particular we prove the following.

**Theorem 1.7.** *If $\rho : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary, then there exists simple loops $a$ and $b$ generating $\Gamma_{1,1}$ such that $\rho(a)$ and $\rho(b)$ are loxodromic.*

It follows that:

**Corollary 1.8.** *If $g \geqslant 1$, and $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary, then there exists a simple curve $\gamma \in \Gamma_{g,n}$ such that $\rho(\gamma)$ is loxodromic.*

Note that this theorem does not hold for $g = 0$. Indeed, Baba observed that some $\mathrm{PSL}_2(\mathbb{R})$-representations of $\Gamma_{0,4}$ studied in [2] are non-elementary and send every simple curve to an elliptic isometry. Later in [3], Deroin and Tholozan exhibited a class of non-elementary representations of $\Gamma_{0,n}$ into $\mathrm{PSL}_2(\mathbb{R})$ that send every simple closed curve to a non-hyperbolic isometry for every $n \geqslant 4$.

Let us now describe the organization of the paper. In Section 2, we recall general facts about curves on surfaces before proving the existence of a special handle given a non-elementary $\rho$. We proceed as described above and prove Theorem 1.7. Then we study pentagon represention in Section 3. Section 4 is devoted to the proof of Theorem 1.4. Finally, we show that the pentagon representations are in $\mathrm{hol}(\mathcal{P}_b(\Sigma_2))$ in Section 5.

**Acknowledgements**

I would like to thank my advisor M. Wolff for his help and guidance. I am also grateful to B. Deroin for his comments. I also thank the anonymous referee for helpful comments and suggestions.

# 2 Special handle

## 2.1 Reminder of curves on surfaces

If $\gamma$ and $\delta$ are elements of $\Gamma_{g,n}$, the loop $\gamma\delta$ is the path following $\delta$ first and then $\gamma$. We also define the commutator of $\gamma$ and $\delta$ to be $[\gamma, \delta] = \delta^{-1}\gamma^{-1}\delta\gamma$.

We fix a system of *standard generators* $a_1, b_1, \dots a_g, b_g, c_1, \dots c_n$ of $\Gamma_{g,n}$, in the same way as Figure 1 for $(g, n) = (2, 1)$.

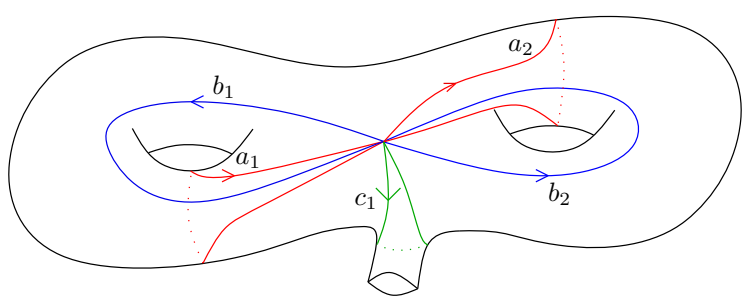


Figure 1: Standard generators for $\Gamma_{2,1}$.

Denote by $\mathcal{S}_{\mathrm{ns}}$ the set of $\gamma \in \Gamma_{g,n}$ such that $\gamma$ is freely homotopic to an essential non-separating simple closed curve. If $f \in \mathrm{Homeo}^+(\Sigma_g, *)$, *i.e.* $f$ fixes the base point $*$ of $\Gamma_g$, we denote by $f_*$ the automorphism of $\Gamma_g$ induced by $f$.

**Lemma 2.1.** *Given $\gamma$ and $\delta$ in $\mathcal{S}_{\mathrm{ns}}$, there exists $f \in \mathrm{Homeo}^+(\Sigma_{g,n}, *)$ such that $f_*(\gamma) \in \{\delta, \delta^{-1}\}$.*

*Proof.* This is a consequence of classification of compact surfaces. See [6, Section 1.3]. Note that if $g \geqslant 1$, and $* \notin \partial\Sigma_{g,n}$, we can even choose $f$ such that $f_*(\gamma) = \delta$. □

Now assume that $g \geqslant 1$, and that $* \notin \partial\Sigma_{g,n}$.

**Lemma 2.2.** *Let $\gamma, \delta \in \mathcal{S}_{\mathrm{ns}}$ be such that they have representatives that cross only at $*$ as in Figure 2.*

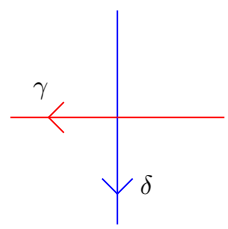


Figure 2: Handle.

*There exists $f \in \mathrm{Homeo}^+(\Sigma_{g,n}, *)$ such that $f_*(\gamma) = a_1$ and $f_*(\delta) = b_1$.*

*Proof.* This is also a consequence of the classification of surfaces, see [6, Section 1.3.3]. □

## 2.2 Existence of a non-elementary handle

Let us recall that $\mathrm{PSL}_2(\mathbb{C})$ is the group of isometries of $\mathbb{H}^3$. Its action on $\mathbb{H}^3$ extends to the Gromov boundary $\partial\mathbb{H}^3 \simeq \mathbb{CP}^1$ where it acts by Möbius transformations. We say that a homomorphism $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ is elementary if there exists $z \in \mathbb{H}^3 \cup \partial\mathbb{H}^3$ such that $\rho(G) \cdot z$ is finite. We have, see for example [17, Chapter 5], the following characterization.

**Proposition 2.3** ([17])**.** *A homomorphism $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ is elementary if and only if there exists a set $S \subset \mathbb{H}^3 \cup \partial\mathbb{H}^3$ containing 1 or 2 points such that $\rho(G)$ stabilizes $S$.*

Suppose $g \geqslant 1$ and $(g, n) \neq (1, 0)$. Fix a non-elementary $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$. We also assume that the base point of $\Gamma_{g,n}$ is not on the boundary $\partial\Sigma_{g,n}$.

**Definition 2.4.** A *non-elementary handle* is a subsurface of $S \subset \Sigma_{g,n}$ which is a once punctured torus, such that the restriction $\rho_{|S} : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ is non-elementary.

Let us recall three lemmas from [7]. We will assume that the reader is familiar with three dimensional hyperbolic geometry, in particular with the notions of loxodromic, elliptic, parabolic isometries and their axes, as introduced for example in [13, Section 3.5].

**Lemma 2.5.** *Suppose $\alpha, \beta \in \mathrm{PSL}_2(\mathbb{C})$.*

- *If $\alpha$ is loxodromic, and $\beta$ does not send its attractive fixed point to its repulsive one (resp. its repulsive one to its attractive one), then there exists $K > 0$ such that $\alpha^k\beta$ is loxodromic for all $k \geqslant K$ (resp. for all $k \leqslant -K$). Moreover, the trace of $\alpha^k\beta$ can be made arbitrarily large.*
- *If $\alpha$ is parabolic and $\beta$ does not fix its fixed point, then $\alpha^k\beta$ is loxodromic for $|k|$ large enough.*

*Proof.* This is a trace computation, see [7, Lemma 2.2.1]. □

**Lemma 2.6.** *If there exists $\gamma \in \mathcal{S}_{\mathrm{ns}}$ such that $\rho(\gamma)$ is loxodromic or parabolic, then there is a handle in $\Sigma_{g,n}$ on which the restriction of $\rho$ is non-elementary.*

*Proof.* We can assume that $\gamma = a_1$. If $\rho(b_1)$ stabilizes the set of fixed points of $\rho(\gamma)$, take $c \in \{a_2^{-1}, b_2, \ldots, b_g, c_1, \ldots, c_n\}$ such that $\rho(c)$ does not, and apply a homeomorphism so that $(a_1, cb_1)$ becomes $(a_1, b_1)$, which exists by Lemma 2.2.

If $\rho(a_1)$ is parabolic, then the group generated by $\rho(a_1)$ and $\rho(b_1)$ is non-elementary. This is also the case if $\rho(a_1)$ is loxodromic and $\rho(b_1)$ does not share a fixed point with $\rho(a_1)$. Otherwise denote by $p$ this common fixed point. Choose $c \in \{a_2^{-1}, b_2, \ldots, b_g, c_1, \ldots, c_n\}$ such that $\rho(c)$ does not fix $p$. Then $(ca_1^{-1}, a_1^k b_1)$ is such a handle for some $k$. Indeed there exists $K > 0$ such that $\rho(a_1^k b_1)$ is loxodromic for $k \geqslant K$ or $k \leqslant -K$. Moreover the fixed point of $\rho(a_1^k b_1)$ which is not $p$ is different for any two $k$ in that range: if $\rho(a_1^k b_1)(q) = \rho(a_1^{k+m} b_1)(q) = q$, then $\rho(a_1)(q) = \rho(b_1)(q) = q$ and $q = p$. Thus we can take a $k$ in that range that does not share a fixed point with $\rho(ca_1^{-1})$. □

**Lemma 2.7.** *If $\alpha$ and $\beta$ are elliptic with different axes and $\alpha\beta$ is elliptic, then:*

1. *The axes of $\alpha$ and of $\beta$ lie in a plane $P$.*
2. *If they are disjoint, they are orthogonal to a plane.*
3. *The axis of $\alpha\beta$ is not contained in $P$.*

*Proof.* Decompose $\alpha$ and $\beta$ as: $\alpha = s_{\ell_2} s_{\ell_1}$ and $\beta = s_{\ell_3} s_{\ell_2}$ where $s_{\ell_i}$ is the elliptic involution with axis $\ell_i$. See [7, Lemma 3.4.1, 3.4.3]. □

**Proposition 2.8.** *Let $g \geqslant 1$ and $n \geqslant 0$ be such that $n \neq 0$ if $g = 1$. Let $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. There exists a non-elementary handle on $\Sigma_{g,n}$.*

*Proof.* We may assume $g \geqslant 2$ or $n \geqslant 2$ because the proposition is obvious otherwise. By contradiction suppose that the restriction of $\rho$ to any handle on $\Sigma_{g,n}$ is elementary. For every $\gamma \in \mathcal{S}_{\mathrm{ns}}$, $\rho(\gamma)$ is elliptic or the identity. As before, we can assume that $\rho(a_1)$ is not the identity, that $\rho(b_1)$ is not the identity and does not have the same axis as $\rho(a_1)$. Their axes cross since $a_1$ and $b_1$ bound a handle on which $\rho$ is elementary. Pick $c' \in \{a_2, b_2^{-1}, \ldots, b_g^{-1}, c_1^{-1}, \ldots, c_n^{-1}\}$

so that $\rho(c')$ does not fix the common fixed point of $\rho(a_1)$ and $\rho(b_1)$. Let $c = c'b_1^{-1}a_1^{-1}$, so that any two out of $a_1, b_1, c$ form a handle.

The axes of $\rho(a_1), \rho(b_1)$ and $\rho(c)$ form a triangle $T$.

Since $\rho(ca_1b_1) = \rho(c')$ is elliptic, the axis of $\rho(c)$ is coplanar with the axis of $\rho(a_1b_1)$. The only plane that contains both the axis of $\rho(c)$ and the common fixed point of $\rho(a_1)$ and $\rho(b_1)$ is the one spanned by $T$. This is a contradition by Lemma 2.7: the axis of $\rho(a_1b_1)$ is not coplanar with both the axis of $\rho(a_1)$ and the axis of $\rho(b_1)$. □

## 2.3 Non-elementary representations of the punctured torus

The aim of this subsection is to prove Theorem 1.7. Let $\rho : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. Our strategy is to precompose $\rho$ by automorphisms of $\Gamma_{1,1}$ induced by Dehn twists along curves homotopic to the standard generators $a$ and $b$ of $\Gamma_{1,1}$. The automorphisms we consider are defined by $(\varphi_k(a), \varphi_k(b)) = (b^k a, b)$ and $(\psi_k(a), \psi_k(b)) = (a, a^k b)$ for $k \in \mathbb{Z}$.

**Lemma 2.9.** *If there exists $\gamma \in \mathcal{S}_{\mathrm{ns}}$ such that $\rho(\gamma)$ is parabolic or loxodromic, then Theorem 1.7 holds.*

*Proof.* We may assume that $a \in \{\gamma, \gamma^{-1}\}$ by Lemma 2.1. If $\rho(a)$ is loxodromic, then so is $\rho(a^k b)$ for some $k$. Apply the Dehn twist that changes $(a, b)$ to $(a, a^k b)$. If $\rho(a)$ is parabolic, then $\rho(a^k b)$ is loxodromic if $|k|$ is large enough. We change $(a, b)$ to $(a, a^k b)$ with a Dehn twist, and return to the previous case. □

### 2.3.1 Existence of a loxodromic

We now show that it is not possible for $\rho$ to send every $\gamma \in \mathcal{S}_{\mathrm{ns}}$ to an elliptic element or to the identity. Assume that is does by contradiction.

**Lemma 2.10.** *The representation $\rho$ has some conjugate into $\mathrm{PSL}_2(\mathbb{R})$: it preserves a plane.*

*Proof.* Since $\rho$ is non-elementary, the isometries $\rho(a)$ and $\rho(b)$ are not the identity, and their axes do not cross. Since $ab \in \mathcal{S}_{\mathrm{ns}}$, it follows from Lemma 2.7 that these axes are orthogonal to a plane. This plane and its boundary circle are preserved by both $\rho(a)$ and $\rho(b)$, hence they are preserved by $\rho(\Gamma_{1,1})$. □

We will thus now assume that $\rho(\Gamma_{1,1}) \subset \mathrm{PSL}_2(\mathbb{R})$ and consider the elements of $\rho(\Gamma_{1,1})$ as isometries of the hyperbolic plane.

**Lemma 2.11.** *There exists $N \geqslant 1$ such that for all $\gamma \in \mathcal{S}_{\mathrm{ns}}$, $\rho(\gamma)^N = \mathrm{id}$.*

Here we denoted by id the element $\pm\mathrm{Id}$ of $\mathrm{PSL}_2(\mathbb{C})$.

*Proof.* Let us first show that $\rho(\gamma)$ has finite order for every $\gamma \in \mathcal{S}_{\mathrm{ns}}$. Suppose for a contradiction that there exists $\gamma \in \mathcal{S}_{\mathrm{ns}}$ such that $\rho(\gamma)$ has infinite order. We may assume without loss of generality that $\gamma = a$. Let us write $\rho(a^n) = s_{\ell_n} s_{\ell'}$ and $\rho(b) = s_{\ell'} s_{\ell''}$ as products of reflections across geodesics, where $\ell'$ is the

geodesic passing through both the fixed point of $\rho(a)$ and the fixed point of $\rho(b)$. Take $n \geqslant 1$ such that $\ell_n$ does not cross $\ell''$. Then $a^n b \in \mathcal{S}_{\mathrm{ns}}$ and $\rho(a^n b)$ is not elliptic, which is a contradiction.

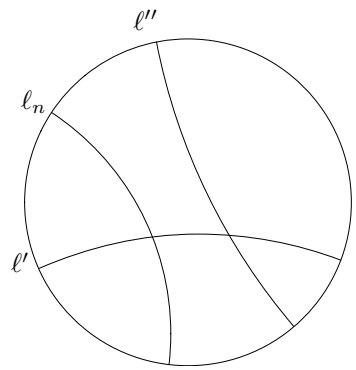


Figure 3: Product of reflections.

This can also be seen with a computation.

By Selberg's lemma, there is a torsion free subgroup $\Lambda$ of $\rho(\Gamma_{1,1})$ of finite index $n$. Let $\gamma \in \mathcal{S}_{\mathrm{ns}}$. Two of the cosets $\Lambda, \rho(\gamma)\Lambda, \dots \rho(\gamma)^n\Lambda$ are equal. Thus $\rho(\gamma)^j \in \Lambda$ for a $j \leqslant n$. This implies that $\rho(\gamma)^j = \mathrm{id}$ and it thus suffices to take $N = n!$. □

We get a contradiction from this lemma.

**Lemma 2.12.** *We can increase the order of $\rho(a)$ or $\rho(b)$ by applying a Dehn twist.*

*Proof.* Write $\rho(a) = s_{\ell_2} s_{\ell_1}$ and $\rho(b) = s_{\ell_3} s_{\ell_2}$ as products of reflections. Since $\rho(ba)$ is elliptic, the lines $\ell_1$, $\ell_2$ and $\ell_3$ form a triangle, see Figure 4. Let us denote by $\theta_i$, for $i \in \{1, 3\}$, the angle of this triangle at the crossing of $\ell_i$ and $\ell_2$.

Suppose $\theta_3 \leqslant \theta_1$. The fixed points of $\rho(b^{-1}a)$ and of $\rho(ba)$ are separated by the line $\ell_2$. Indeed if not, one could construct a triangle with total angle greater than $\pi$. Let $q$ be the intersection of $\ell_1$ with its perpendicular that goes through the fixed point of $\rho(b)$. The function that maps $p \in \ell_1$ to the angle of the triangle formed by $p$ and the two fixed points of $\rho(a)$ and $\rho(b)$ decreases when $p$ moves away from $q$. If $\ell_3$ intersects $\ell_1$ in the part of $\mathbb{H}^2 \setminus \ell_2$ that does not contain $q$, then the order of $\rho(ba)$ is less than the order of $\rho(a)$. In this case change the handle to $(ba, b)$. If not, change the handle to $(b^{-1}a, b)$.

The case where $\theta_1 \leqslant \theta_3$ is symmetric. □

### 2.3.2 Special handle

**Definition 2.13.** A *special handle* is a non-elementary handle with standard generators $a, b$ such that $\rho(a)$ and $\rho(b)$ are loxodromic, and such that $\rho(a)$ does not send a fixed point of $\rho(b)$ to the other.

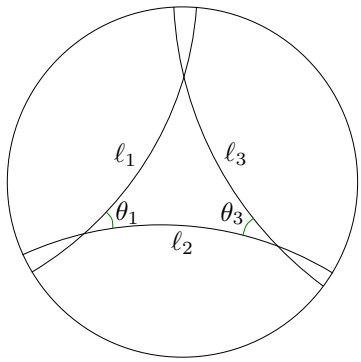


Figure 4: Triangle $(\ell_1, \ell_2, \ell_3)$.

Thanks to Theorem 1.7, we can suppose that the standard generators $a$ and $b$ are sent by $\rho$ to loxodromic isometries. We now modify this handle to get a special one. Our proof relies on the following results of [7, Lemma 2.2.1, 2.2.2, 2.2.3], obtained by trace computations.

**Lemma 2.14** ([7]). *1. If $\xi$ interchanges the fixed points of $\beta^k\alpha$ and of $\beta^{k+1}\alpha$ which are loxodromic, then $\xi\alpha\xi = \alpha^{-1}$ and $\xi\beta\xi = \alpha^{-1}\beta^{-1}\alpha$.*

*2. Suppose $\alpha$ and $\beta$ are loxodromic and $\alpha$ sends a fixed point of $\beta$ to the other. Then $\beta$ sends a fixed point of $\alpha$ to the other if and only if $\mathrm{Tr}^2(\alpha) = \mathrm{Tr}^2(\beta)$.*

*3. If $\alpha$ and $\beta$ are loxodromic, the fixed points of $\alpha^m\beta$ converge to $p^*$ and $\beta(p_*)$) (resp. $p_*$ and $\beta^{-1}(p_*)$) when $m \to \infty$ (resp. $m \to -\infty$) where $p^*$ is the attractive fixed point of $\alpha$ and $p_*$ its repulsive one.*

Later we will need a little bit more than the existence of a special handle, as proved in [7].

**Proposition 2.15.** *Let $\rho : \Gamma_{1,1} \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. There exists a homeomorphism $f$ of $\Sigma_{1,1}$ fixing the boundary pointwise such that $(f(a), f(b))$ is a special handle. Moreover, given a finite set $A \subset \mathbb{CP}^1$ and $\xi \in \mathrm{PSL}_2(\mathbb{C})$ we can assume that no fixed point of $\rho \circ f_*(b)$ lies in $A$ and that $\xi$ does not interchange the fixed points of $\rho \circ f_*(a)$.*

*Proof.* Since $\alpha = \rho(a)$ does not exchange the fixed points of $\beta = \rho(b)$, we have that $\beta^k\alpha$ is loxodromic if $k \geqslant K$ or $k \leqslant -K$ for some $K > 0$. If $\xi$ does not interchange the fixed points of $\beta^k\alpha$ for such a $k$, apply the Dehn twist that changes $(a, b)$ into $(b^k a, b)$.

If however $\xi$ interchanges the fixed points of both $\alpha^k\beta$ and $\alpha^{k+1}\beta$, with $k$ and $k+1$ in that range, then Lemma 2.14 shows that $\xi\alpha\xi = \alpha^{-1}$ and $\xi\beta\xi = \alpha^{-1}\beta^{-1}\alpha$. Note that we cannot have $\xi\beta^{-1}\xi = \beta$, otherwise $\alpha$ and $\beta$ would have the same fixed points.

Thus if it is the case, we apply a sequence of Dehn twists that change the handle like this:

$$(a, b) \to (b^{-1}a, b) \to (b^{-1}, ab) \to (b^{-1}, b^{-1}ab).$$

And we return to the beginning. Therefore we may assume that $\xi$ does not interchange the fixed points of $\alpha$.

Since $\beta$ does not interchange the fixed points of $\alpha$, there exists $K > 0$ such that $\alpha^m\beta$ is loxodromic for $m \geqslant K$ or for $m \leqslant -K$. Now suppose there is an infinite sequence of $m$ in that range such that $\alpha$ sends a fixed point of $\alpha^m\beta$ to the other one. Since the fixed points of $\alpha^m\beta$ tends to $p$ and $\beta^{-1}(q)$ where $p, q$ are the fixed points of $\alpha$, we have $\alpha(p) = \beta^{-1}(q)$ or $\alpha\beta^{-1}(q) = p$. Thus $p = \beta^{-1}(q)$ and $\alpha^m\beta(p) = q$ for all $m$. But if we increase $K$, we can assume $|\mathrm{Tr}(\alpha^m\beta)| \neq |\mathrm{Tr}(\alpha)|$ for $m \geqslant K$ or $m \leqslant -K$. Since $\alpha^m\beta$ sends one fixed point of $\alpha$ to the other and thanks to Lemma 2.14, $\alpha$ does not send a fixed point of $\alpha^m\beta$ to the other, a contradiction.

The fixed points of $\alpha^m\beta$ are disjoint from those of $\alpha^n\beta$ if $n \neq m$. Otherwise, $\alpha$ and $\beta$ would have a common fixed point. Hence we can also suppose that no fixed point of $\alpha^m\beta$ lies in $A$. $\square$

# 3 Pentagon representations

## 3.1 Definition

Let $\Gamma$ be the group with the following presentation:

$$\Gamma = \langle q_1, q_2, q_3, q_4, q_5, q_6 \mid q_i^2 = 1, q_1q_2 \dots q_6 = 1\rangle.$$

Define $\iota$ by $\iota(a_1) = q_2q_1$, $\iota(b_1) = q_2q_3$, $\iota(a_2) = q_5q_4$ $\iota(b_2) = q_5q_6$. We have $\iota([a_2, b_2][a_1, b_1]) = (q_6q_5q_4)^2(q_3q_2q_1)^2 = 1$, so $\iota : \Gamma_2 \to \Gamma$ is a well-defined map. The homomorphism $\iota$ is injective and identifies $\Gamma_2$ with an index two subgroup of $\Gamma$.

Let us recall from the introduction that a pentagon representation is the restriction of a representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ that kills (*i.e.* sends to the identity) exactly one $q_i$, and such that $\rho \circ \iota$ is non-elementary. We leave it as an elementary exercise to show that if two or more $q_i$ are killed by $\rho$, then $\rho$ is elementary. This property is invariant under conjugation, so it is a property of characters.

Before going on to the study of those representations, let us give an example that motivates the terminology. Consider a right-angled pentagon in the hyperbolic plane and denote its vertices by $x_1, \dots, x_5$. Define $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ by $\rho(q_i) = s_{x_i}$ for $i \leqslant 5$ where $s_{x_i}$ is the elliptic involution of the hyperbolic plane fixing $x_i$, and by $\rho(q_6) = \mathrm{id}$. This is well-defined because $s_{x_i}$ is the product of the reflection across the lines $(x_{i-1}x_i)$ and $(x_i, x_{i+1})$ (in cyclic notation) so $\rho(q_1, \dots q_6) = \mathrm{id}$. The representation $\rho \circ \iota$ is non-elementary and is thus a pentagon representation.

We now show that pentagon representations have odd Stiefel-Whitney class.

**Proposition 3.1.** *A pentagon representation does not lift to* $\mathrm{SL}_2(\mathbb{C})$.

*Proof.* Let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be such that $\rho \circ \iota$ is a pentagon representation. Take $\tilde{q}_i \in \mathrm{SL}_2(\mathbb{C})$ such that $\pm\tilde{q}_i = \rho(q_i)$ for $i \leqslant 6$.

Note that $\tilde{q}_i^{-1} = -\tilde{q}_i$ if $\rho(q_i) \neq \mathrm{id}$ because $\rho(q_i)$ is conjugate to $\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Of course if $\rho(q_i) = \mathrm{id}$, then $\tilde{q}_i^{\,2} = \mathrm{Id}$.

Now $\tilde{q}_1\tilde{q}_2\ldots\tilde{q}_6$ is a lift of $\rho(q_1q_2\ldots q_6) = \mathrm{id}$, so $\tilde{q}_1\tilde{q}_2\ldots\tilde{q}_6 = \epsilon\mathrm{Id}$, with $\epsilon = \pm 1$.

Let $\tilde{a_1} = \tilde{q}_2\tilde{q}_1$, $\tilde{b_1} = \tilde{q}_2\tilde{q}_3$, $\tilde{a_2} = \tilde{q}_5\tilde{q}_4$ and $\tilde{b_2} = \tilde{q}_5\tilde{q}_6$.

$$\begin{aligned}
[\tilde{a_2}, \tilde{b_2}][\tilde{a_1}, \tilde{b_1}] &= \tilde{q}_6^{\,-1}\tilde{q}_5^{\,-1}\tilde{q}_4^{\,-1}\tilde{q}_6\tilde{q}_5\tilde{q}_4\tilde{q}_3^{\,-1}\tilde{q}_2^{\,-1}\tilde{q}_1^{\,-1}\tilde{q}_3\tilde{q}_2\tilde{q}_1 \\
&= -(\tilde{q}_6\tilde{q}_5\tilde{q}_4)^2(\tilde{q}_3\tilde{q}_2\tilde{q}_1)^2 \\
&= -(\tilde{q}_6\tilde{q}_5\tilde{q}_4)\epsilon\tilde{q}_3\tilde{q}_2\tilde{q}_1 \\
&= -\epsilon^2\mathrm{Id} = -\mathrm{Id}.
\end{aligned}$$

Therefore $\rho \circ \iota$ does not lift to $\mathrm{SL}_2(\mathbb{C})$. □

## 3.2 Action of the mapping class group

The mapping class group of $\Sigma_2$ acts naturally on $\mathrm{Hom}(\Gamma_2, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ as follows: $[f] \cdot [\rho] = [\rho \circ f_*^{-1}]$ where $f_*$ is the outer automorphism of $\Gamma_2$ induced by $f$.

**Proposition 3.2.** *This action preserves pentagon representations.*

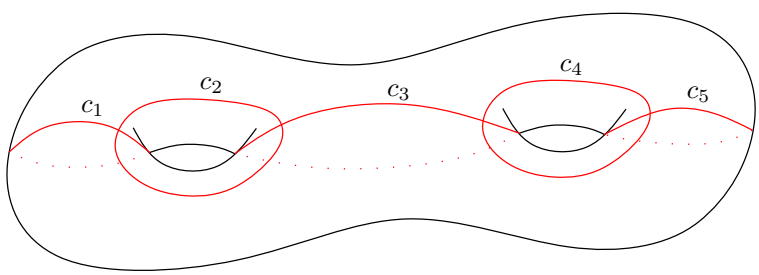


Figure 5: Generators of $\mathrm{Mod}(\Sigma_2)$.

*Proof.* For $1 \leqslant i \leqslant 5$, let $\sigma_i$ be the automorphism of $\Gamma$ defined by $\sigma_i(q_i) = q_{i+1}$, $\sigma_i(q_{i+1}) = q_{i+1}q_iq_{i+1}$ and $\sigma_i(q_j) = q_j$ for $j \neq i, i+1$. The mapping class group is generated by the Lickorish generators: the Dehn twists along the curves $c_i$ drawn in Figure 5; see [6, Chapter 4].

The outer automorphism $[\varphi_i]$ of $\Gamma_2$ induced by a Dehn twist along the curve $c_i$ has a representative $\varphi_i \in \mathrm{Aut}^+(\Gamma_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma_2 & \xrightarrow{\varphi_i} & \Gamma_2 \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota} \\ \Gamma & \xrightarrow{\sigma_i} & \Gamma. \end{array}$$

Hence $[\rho \circ \iota \circ \varphi_i] = [\rho \circ \sigma_i \circ \iota]$ and $\rho \circ \sigma_i$ kills exactly one $q_i$ if $\rho$ does. □

# 4 Schottky decomposition

The aim of this section is to prove Theorem 1.4: the absence of a Schottky decomposition characterizes the pentagon representations among the non-elementary representations.

## 4.1 Pentagons are not Schottky

Let us first show that a pentagon representation does not admit a Schottky decomposition.

*Proof.* Let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be such that $\rho \circ \iota$ is a pentagon representation. Let us consider the two pants decomposition of $\Sigma_2$ as shown in Figure 6. The first one is not a Schottky decomposition for $\rho$. Indeed since there is a $q_i$ killed by $\rho$, its restriction to one of the two handles must be elementary. The second pants decomposition is not a Schottky decomposition for $\rho$ either since one can check that the image of one of the boundary curves has order 2.

For every $g \geqslant 2$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on the set $\mathcal{P}_g$ of pants decompositions of $\Sigma_g$. Let us recall, see for example [16], that the quotient $\mathcal{P}_g/\mathrm{Mod}(\Sigma_g)$ is in bijection with the isomorphism classes of connected trivalent graphs with $2g-2$ vertices. In particular there are only two pants decompositions of a surface of genus 2 up to the action of $\mathrm{Mod}(\Sigma_2)$. Now if there is a Schottky decomposition $P$ for $\rho$, there exists a positive homeomorphism $f$ of $\Sigma$ taking one of these two pants decompositions to $P$. The pentagon representation $[\rho \circ f_*]$ admits a Schottky decomposition with one of those two pants decompositions, which is a contradiction. □

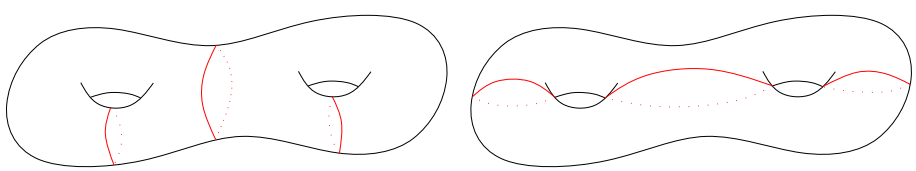

Figure 6: Pants decompositions of $\Sigma_2$.

## 4.2 Non-Schottky are pentagons

### 4.2.1 Tools to form a Schottky decomposition

The following proposition is a rephrasing of the paragraph 4.4 of [7]:

**Proposition 4.1.** *Let $g \geqslant 2$, $n \geqslant 0$ and $\rho : \Gamma_{g,n} \to \mathrm{PSL}_2(\mathbb{C})$. Suppose that $(a_1, b_1)$ is a special handle, and that $\rho(b_2 a_1^{-1})$ does not exchange the fixed points of $\rho(b_1)$. Suppose moreover that $\rho(b_2) \neq \mathrm{id}$ or that $\rho(a_2)$ does not interchange the fixed points of $\rho(a_1)$. A Dehn twist of order $n$, along a curve $d_k$, freely homotopic to $b_2 a_1^{-1} b_1^k$ transforms $(a_1, b_1)$ in a non-elementary handle, and $\rho(a_2)$ (or $\rho(b_2 a_2)$) in a loxodromic isometry for some $k, n$.*

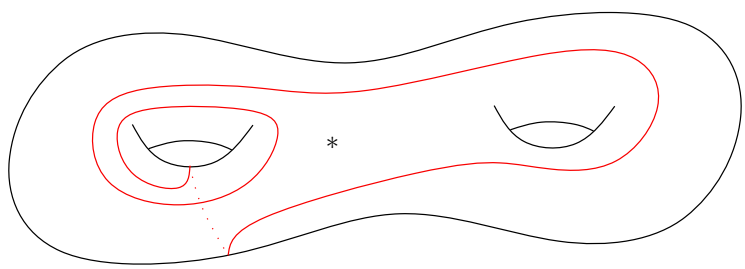


Figure 7: Dehn Twist.

The proof is very similar to the one of [7]; we slightly simplify its beginning and modify its end.

*Proof.* There exists $K > 0$ such that $\rho(a_1^{-1} b_1^k)$ is loxodromic for $|k| \geqslant K$. The isometry $\delta_k = \rho(b_2 a_1^{-1} b_1^k)$ is also loxodromic for $k \geqslant K$ or $k \leqslant -K$, increasing $K$ if necessary.

Fix such a $k$ so that $k+1$ is also in that range, and let $\delta = \delta_k$. There is at most one $n$ such that $\rho(b_1)\delta^n$ shares a given fixed point with $\rho(a_1^{-1} b_1^k)$. For if $\rho(b_1)\delta^n(p) = \rho(a_1^{-1} b_1^k)(p) = p = \rho(b_1)\delta^{n+m}(p)$, then $\delta(p) = p = \rho(b_1)(p) = \rho(a_1)(p)$, which is a contradiction since $(a_1, b_1)$ is a special handle.

Thus there exists $N(k) \geqslant 0$ such that $\rho(b_1)\delta^n$ and $\rho(a_1^{-1} b_1^k)$ do not share a fixed point for $|n| \geqslant N(k)$: the image of the handle is non-elementary.

**Lemma 4.2.** *It is not possible for both $\rho(a_2)$ and $\rho(b_2 a_2)$ to interchange the fixed points of $\delta_k$ and of $\delta_{k+1}$.*

*Proof.* We proceed by contradiction.

- If $\rho(b_2) = \mathrm{id}$, then $\rho(a_2)$ interchanges the fixed points of $\rho(a_1^{-1} b_1^k)$ and of $\rho(a_1^{-1} b_1^{k+1})$. By Lemma 2.14, $\rho(a_2)$ interchanges the fixed points of $\rho(a_1)$.
- If $\rho(b_2) \neq \mathrm{id}$, then $\rho(b_2)$ fixes the fixed points of $\delta_k$ and of $\delta_{k+1}$, and has at most two fixed points in $\partial\mathbb{H}^3$. Hence $\delta_k$ and $\delta_{k+1}$ share a fixed point $p$ with $\rho(b_2)$. This implies that $\rho(b_2 a_1^{-1} b_1^k)(p) = \rho(a_1^{-1} b_1^k)(p) = p$

and $\rho(a_1^{-1}b_1^{k+1})(p) = p$. Hence both $\rho(b_1)$ and $\rho(a_1)$ fix $p$. This is a contradiction since $(a_1, b_1)$ is a special handle. □

It follows that $\rho(a_2)\delta^n$, or $\rho(b_2a_2)\delta^n$ is loxodromic for some $n \geqslant N(k)$ or $n \leqslant -N(k)$, increasing $N(k)$ is necessary. Note that its trace can be made arbitrarily large. □

Let us recall a tool from [7] to construct Schottky groups.

**Lemma 4.3.** *Suppose $\alpha$, $\beta$ and $\delta$ are loxodromic, and that neither $\alpha$ nor $\beta$ shares a fixed point with $\delta$.*

*Then $\delta^n\alpha\delta^{-n}$ and $\beta$ generate a Schottky group for $|n|$ large enough.*

The following proposition explains how we can construct a Schottky decomposition, following [7].

**Proposition 4.4.** *Let $g \geqslant 2$ and $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$. Suppose we can cut the surface along non-separating simple closed curves such that we get a surface of genus 1, with a special handle, and such that the boundary curves are loxodromic with pairwise different images. Then $\rho$ admits a Schottky decomposition.*

*Proof.* Observe that cutting the surface along a non-separating simple closed curve creates two boundary components, hence the number of boundaries is at least 2. Suppose we want to construct a Schottky pair of pants from the boundaries $d_1$ and $d_2$, such that $\rho(d_1) \neq \rho(d_2)$. By Proposition 2.15, we may assume that $\rho(b_1)$ does not fix the fixed points of $\rho(d_1d_2^{-1})$. Then, it is not possible for both $\rho(d_1a_1^{-1})$ and $\rho(d_2a_1^{-1})$ to interchange the fixed points of $\rho(b_1)$.

We can now apply the arguments of 5.2 and 5.3 of [7] to form a Schottky pair of pants from those boundary components, and remove it from the surface. We make sure the trace of the new boundary is larger than the others'. Following [7], we produce a Schottky pants decomposition. □

The mistake of [7] lies in its paragraph 5.5, where a Schottky pair of pants is found, but the non-elementary handle is not always kept. We avoid using this part of their proof.

#### 4.2.2 The genus 2 case

Let $\rho : \Gamma_2 \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation that does not admit a Schottky decomposition.

The following proposition is an adaptation of [7, Paragraph 4.5].

**Proposition 4.5.** *We can change $\rho$ by some $\rho_1$ such that $[\rho_1] = f \cdot [\rho]$ with $f \in \mathrm{Homeo}^+(\Sigma_2)$, so that $(a_1, b_1)$ is a special handle, and $\rho(b_2)$ is loxodromic.*

Let us explain why this result, combined with Theorem 1.4, implies Corollary 1.5. By Theorem 1.4, we just have to consider the pentagon representations. If a pentagon representation $\rho$ is of the form above, it admits an extension $\rho' : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho'(q_4) = \mathrm{id}$. But then $\rho(a_1)$, $\rho([a_2, b_2]) = \rho'((q_5q_6)^2) = \rho(b_2)^2$ and $\rho(b_2)$ are loxodromic. We thus consider a pants decomposition defined by curves freely homotopic to $a_1$, $[a_2, b_2]$ and $b_2$.

*Proof.* We can assume $(a_1, b_1)$ is a special handle. Let us start by changing the handle $(a_2, b_2)$ by $(a_2, a_2b_2)$ in the case where $\rho(a_2) = \rho(b_2)$ and they are of order 2. Return to the notation $(a_2, b_2)$.

If $\rho(b_2) = \mathrm{id}$, apply Proposition 2.15 to make sure that $\rho(a_2)$ does not interchange the fixed points of $\rho(a_1)$. Then we can apply Proposition 4.1 to turn $\rho(a_2)$ into a loxodromic isometry, because $\rho(b_2a_1^{-1})$ is loxodromic and hence cannot interchange two points.

If however $\rho(b_2) \neq \mathrm{id}$, apply Proposition 2.15 to make sure that $\rho(b_2a_2)$ does not interchange the fixed points of $\rho(a_1)$.

If $\rho(b_2a_1^{-1})$ does not interchange the fixed points of $\rho(b_1)$, we can apply Proposition 4.1 to make $\rho(a_2)$ loxodromic.

If it does, then suppose that $\rho(a_2^{-1}a_1^{-1})$ does not interchange the fixed points of $\rho(b_1)$. Then we modify the handle $(a_2, b_2)$ by a homeomorphism to $(b_2a_2, a_2^{-1})$. It is a composition of Dehn twists in the handle that changes it as follows:

$$(a_2, b_2) \to (a_2, a_2^{-1}b_2) \to (b_2a_2, a_2^{-1}).$$

We can apply Proposition 4.1 since we made sure that $\rho(b_2a_2)$ does not interchange the fixed points of $\rho(a_1)$.

Finally if both $\rho(a_2^{-1}a_1^{-1})$ and $\rho(b_2a_1^{-1})$ interchange the fixed points of $\rho(b_1)$, then $\rho(a_2^{\pm 1}b_2a_1^{-1})$ does not, for it would imply that $\rho(a_2)$ fixes them and then $\rho(a_1)$ would interchange them. We can make sure that $\rho(a_2^{\pm 1}b_2) \neq \mathrm{id}$, because we are not in the case where $\rho(b_2) = \rho(a_2)$ is of order 2. We then apply a Dehn twist that does:

$$(a_2, b_2) \to (a_2, a_2^{\pm 1}b_2).$$

We can then use Proposition 4.1.

We have made $\rho(a_2)$ loxodromic. But again we can change the handle $(a_2, b_2)$ as before to make sure $\rho(a_2^{-1})$ is loxodromic. The handle $(a_1, b_1)$ is non-elementary and we can improve it to a special handle with Proposition 2.15. $\square$

We now just have to consider pentagon representations in this special form.

**Proposition 4.6.** *A homomorphism $\rho : \Gamma_2 \to \mathrm{PSL}_2(\mathbb{C})$ that is non-elementary and does not admit a Schottky decomposition is a pentagon representation.*

*Proof.* The axes of $\rho(a_1)$ and of $\rho(b_1)$ do not cross in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$, so there exists a unique line $\ell$ orthogonal to both of them. Let $q_2 = s_\ell$ be the elliptic involution with axis $\ell$. Then $q_1 = q_2\rho(a_1)$ and $q_3 = q_2\rho(b_1)$ are elliptic involutions.

We have $\rho(a_2)^{-1}\rho(b_2)^{-1}\rho(a_2) = \rho(b_2)$. Indeed, otherwise we could cut the surface along a curve freely homotopic to $b_2$, and use Proposition 4.4. The isometry $q_5 = \rho(a_2)$ interchanges the fixed points of $\rho(b_2)$, and is an elliptic involution with an axis orthogonal to the one of $\rho(b_2)$. Hence $q_6 = q_5\rho(b_2)$ has order 2.

We have $\mathrm{id} = \rho([a_2b_2][a_1, b_1]) = (q_6q_5)^2(q_3q_2q_1)^2$. This implies that $q_1q_2q_3 = q_6q_5 \circ r$ where $r$ commutes with $q_6q_5$ and is such that $r^2 = \mathrm{id}$ because of the following lemma.

**Lemma 4.7.** *If $f$ and $g$ are loxodromic isometries such that $f^2 = g^2$, either $f = g$ or $f = g \circ r$ where $r$ is an elliptic involution with the same axis as $f$ and $g$.*

*Proof.* They have the same axis so after conjugating we can write $f(z) = \lambda z$ and $g(z) = \mu z$. We have $\lambda^2 = \mu^2$; hence $\lambda = \mu$ or $\lambda = -\mu$. □

Thanks to Proposition 4.4, $\rho(b_2 a_1^{-1})$ must interchange the fixed points of $\rho(b_1)$. Indeed otherwise we could apply the Proposition 4.1 to improve the situtation and suppose that $\rho(a_2)$ is loxodromic. Then $\rho(a_2)^{-1}\rho(b_2)^{-1}\rho(a_2) = \rho(b_2)$ would be impossible.

We have $\rho(b_2 a_1^{-1}) = q_5 q_6 q_1 q_2 = r q_3$ and $\rho(b_1) = q_2 q_3$. Therefore $r q_3$ is of order 2 and $r q_3 = (r q_3)^{-1} = q_3 r$. Moreover, $(r q_3) q_2 q_3 (r q_3) = q_3 q_2$, thus $r q_3 q_2 r = q_3 q_2$. The centralizer of $r$ contains $q_1 q_2 q_3$, $q_3$ and $q_3 q_2$, hence a non-elementary group. This implies that $r = \mathrm{id}$. □

### 4.2.3 Genus $g \geqslant 3$

In this subsection we consider the case $g \geqslant 3$.

**Proposition 4.8.** *Let $g \geqslant 3$ and $\rho : \Gamma_g \to \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary representation. There exists a Schottky decomposition for $\rho$.*

*Proof.* Thanks to Proposition 4.4, it suffices to show that we can cut the surface along non-separating curves having different loxodromic images.

We can assume that $(a_1, b_1)$ is a special handle. We can also suppose that each $\rho(b_i)$ is loxodromic, by applying successively the arguments of Proposition 4.5. We can moreover suppose that the traces of the $\rho(b_i)$ are pairwise distinct.

Use Proposition 2.15 to make sure that $\rho(b_2)$ does not fix the fixed points of $\rho(b_1)$. It is not possible for both $\rho(b_3 a_1^{-1})$ and $\rho(b_2 b_3 a_1^{-1})$ to interchange the fixed points of $\rho(b_1)$, because that would mean that $\rho(b_2)$ fixes them. We now apply the Proposition 4.1 in the handle $(a_3, b_3)$ or $(a_3, b_2 b_3)$. Note that $\rho(b_3) \neq \mathrm{id}$ and $\rho(b_2 b_3) \neq \mathrm{id}$ for the trace of $\rho(b_2)$ is different from the trace of $\rho(b_3)$. This changes $\rho(a_3)$ into a loxodromic isometry and leaves $\rho(b_3)$ unchanged. Thus both $\rho(a_3)$ and $\rho(b_3)$ are loxodromic; we cut the surface along a curve freely homotopic to $a_3$. We cannot have $\rho(a_3) = \rho(b_3)^{-1}\rho(a_3)^{-1}\rho(b_3)$ for it would imply that $\rho(b_3)$ interchanges the fixed points of $\rho(a_3)$.

We can repeat the argument with the other handles while there are at least two handles to cut. Note that the images of the boundary components are modified by conjugation at each step.

We are left with a special handle, $2(g-2)$ boundary components, and a handle that we want to cut. We may assume that $(a_1, b_1)$ is a special handle, and that the handle we whish to cut is $(a_2, b_2)$, and that $\rho(b_2)$ is loxodromic. Make sure as before that $\rho(b_3)$ does not fix the fixed points of $\rho(b_1)$.

If $\rho(b_2 a_1^{-1})$ does not interchange the fixed points of $\rho(b_1)$, then we apply Proposition 4.1 in the handle $(a_2, b_2)$. Otherwise, we apply the same

proposition in the handle $(a_2, b_2a_3^{-1})$. The boundary components $d_1 = b_3$ and $d_2 = b_3^{-1}a_3^{-1}b_3$ corresponding to the handle $(a_3, b_3)$ are changed as follows: $(d_1, d_2) \to (d_1, \zeta^{-n}d_2\zeta^n)$ where $\zeta = a_1^{-1}b_1^kb_2b_3$, and $k$ and $n$ come from the Proposition 4.1. We cannot have $d_1 = \zeta^{-n}d_2\zeta^n = \zeta^{-n-1}d_2\zeta^{n+1}$ for it would imply $d_1 = d_2$. Thus we can assume that $d_1 \neq d_2$ after this Dehn twist is done.

We can thus cut again, to have a genus 1 surface with a special handle, and $2(g-1)$ boundary components, with any two of them having different images. □

# 5 Projective structure

The goal of this section is to prove Theorem 1.6. As before, we can modify $\rho$ by the action of the mapping class group since the property we are interested in is invariant under this action.

Even if the non-elementary handle is not kept, we still find a Schottky pair of pants following [7, Section 5.5].

**Proposition 5.1.** *There exists $\rho_1$ such that $[\rho_1] = f\cdot[\rho]$ for some $f \in \mathrm{Homeo}^+(\Sigma_2)$, such that $\rho_1(a_2)$ and $\rho_1(b_2^{-1}a_2^{-1}b_2)$ generate a Schottky group, and such that $\rho_1(b_1)$ is loxodromic and $\rho_1(a_1)$ interchanges the fixed points of $\rho_1(b_1)$.*

*Proof.* We may assume that $\rho$ is as in Proposition 4.5. Denote by $\rho'$ an extension of $\rho$ to $\Gamma$. Cut $\Sigma_2$ along a curve freely homotopic to $b_2$, so that we get a genus 1 surface with 2 boundary components having loxodromic images : $d_1 = b_2$ and $d_2 = a_2^{-1}b_2^{-1}a_2$. Their images are equal : $\rho'(q_4) = \mathrm{id}$ and $\rho'(q_5q_6q_5q_5) = \rho'(q_5q_6)$. The isometries $\rho((b_1^{-k}a_1)d_1(a_1^{-1}b_1^k))$ and $\rho(d_2)$ generate a Schottky group for $|k|$ large enough. Indeed $\rho(d_2) = \rho'(q_5q_6)$ cannot fix a fixed point of $\rho(b_1) = \rho'(q_2q_3)$ for $\rho(a_1^{-1}b_2) = \rho'(q_1q_2q_5q_6) = \rho'(q_3)$ interchanges those of $\rho(b_1)$ and thus $\rho(a_1)$ would send a fixed point of $\rho(b_1)$ to the other. Similarly, $\rho(a_1^{-1}a_1d_1a_1^{-1})$ interchanges the fixed points of $\rho(b_1)$, hence $\rho(a_1d_1a_1^{-1})$ does not fix any of them. We thus get a Schottky pair of pants by Lemma 4.3.

We may assume that the Schottky pair of pants comes from cutting the handle $(a_2, b_2)$. One of $q_1$, $q_2$ and $q_3$ is killed by $\rho'$. We can assume that it is $q_1$, applying a homeomorphism of the handle $(a_1, b_1)$ if necessary. Since $\rho(a_1)^{-1}\rho(b_1)\rho(a_1) = \rho(b_1)^{-1}$, the map $\rho(b_1)^2 = \rho([a_1, b_1])^{-1} = \rho([a_2, b_2])$ is loxodromic, and so is $\rho(b_1)$. □

We now put a projective structure on $\Sigma_{1,1}$ whose holonomy is the non-Schottky part of the previous proposition. Namely, this holonomy is given by $\rho$ which maps the standard generators $a_1, b_1$ of $\Gamma_{1,1}$ to $\rho(a_1)$, which is loxodromic, and to $\rho(b_1)$ which is an involution interchanging the fixed points of $\rho(a_1)$.

**Proposition 5.2.** *There exists a projective structure with a single branched point of order 2 on $\Sigma_{1,1}$ such that its holonomy is $\rho$, and such that the developing map embeds the boundary curve in $\mathbb{CP}^1$.*

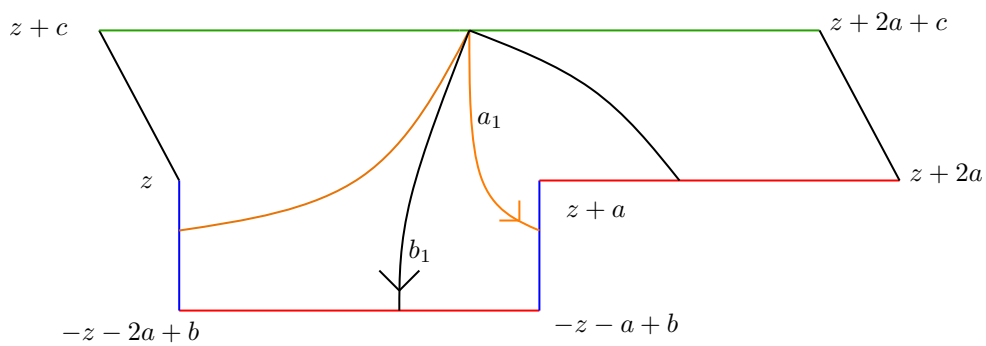


Figure 8: Affine structure on $\Sigma_{1,1}$.

*Proof.* Up to conjugacy we can suppose that $\rho(a_1)(z) = \lambda^{-2}z$ with $|\lambda| > 1$. We can adapt the coordinate $z$ to have $\rho(b_1)(z) = \mu z^{-1}$ with an arbitrary $\mu \in \mathbb{C} \setminus \{0\}$. The choice of $\mu$ will be given in the sequel. Let $a \in \mathbb{C}$ be such that $e^a = \lambda^2$. Choose $z, b, c$ such that the points $-z-2a+b$, $-z-a+b$, $z+a$, $z+2a$, $z+2a+c$, $z+c$ and $z$ form an $L$-shaped polygon $P$ as in Figure 8.

The identifications of the sides by $z \mapsto z+a$ for the blue ones, $z \mapsto -z+b$ for the red ones, and $z \mapsto z+2a$ for the black ones give an affine structure on $S$, with a cone point of angle $4\pi$. Taking the exponential of small enough charts defines a complex projective structure on $\Sigma_{1,1}$ with a single branched point of order 2.

The holonomy of this projective structure maps $a_1$ to $z \mapsto \lambda^{-2}z$. Indeed, if $z = e^\omega$, then $e^{\omega-a} = \lambda^{-2}z$. Similarly, it maps $b_1$ to $z \mapsto e^b z^{-1}$.

Since $\Re(a) > 0$, the developing map embeds the boundary curve in $\mathbb{CP}^1$. $\square$

We are now able to prove Theorem 1.6. We are reduced to the case where $\rho$ is of the form of Proposition 5.1. Put the branched projective structure as above on the handle that is not Schottky. We can put a projective structure on the Schottky handle with the desired holonomy that is compatible (*i.e.* that we can glue to the other one), possibly with a branched point of order 2 (see [7, Paragraph 7,8,9]).

But it is not possible for $\rho$ to be the holonomy of a branched projective structure with two branched points of order 2, since it would imply that it lifts to $\mathrm{SL}_2(\mathbb{C})$ (see [7, Corollary 11.2.3]), contradicting Proposition 3.1.

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# Chapitre 4

# Périodes des différentielles abéliennes à singularités fixées

Cette section contient l'article [LF20]. On y démontre le théorème 1.2.2. En suivant la démarche, et en utilisant des résultats de Kapovich [Kap20], on étudie l'action du groupe symplectique sur $\mathrm{H}^1(\Sigma, \mathbb{C})$, où $\Sigma$ est une surface fermée de genre $g \geqslant 2$. Ceci permet de mettre une représentation $\mathrm{H}_1(\Sigma, \mathbb{Z}) \to \mathbb{C}$ donnée sous une forme qu'il est plus facile de réaliser comme l'holonomie d'une surface de translation que l'on construit explicitement lorsque c'est possible.

# Periods of abelian differentials with prescribed singularities

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**Abstract**

We give a necessary and sufficient condition for a representation of the fundamental group of a closed surface of genus at least 2 to $\mathbb{C}$ to be the holonomy of a translation surface with a prescribed list of conical singularities. Equivalently, we determine the period maps of abelian differentials with prescribed list of multiplicities of zeros. Our main result was also obtained, independently, by Bainbridge, Johnson, Judge and Park.

## 1 Introduction

Let $S$ be a closed oriented surface of genus $g \geqslant 2$. Denote by $\Gamma$ a fundamental group of $S$.

If $X$ is a complex structure on $S$, an *abelian differential* on $X$ is a holomorphic 1-form on $X$. Let us denote by $\Omega$ the space of $(X, \alpha)$ where $X$ is a complex structure on $S$ and $\alpha$ an abelian differential on $X$ and by $\Omega'$ the subset of $\Omega$ where $\alpha$ is not the zero form. The *period* of $\alpha \in \Omega$ is the homomorphism $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ defined by $\chi(\gamma) = \int_\gamma \alpha$. We thus define a map

$$\mathrm{Per} : \Omega \to \mathrm{Hom}(\Gamma, \mathbb{C}).$$

We define the volume of $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ to be $\sum \Im(\overline{\chi(a_i)}\chi(b_i))$, where $(a_1, b_1, \ldots, a_g, b_g)$ is a standard basis of $\Gamma$, that induces a symplectic basis of $H_1(S, \mathbb{Z})$. In [4], Haupt computed the image of $\Omega'$ by Per.

**Haupt's theorem.** *The image of* $\Omega'$ *by* Per *is the set of* $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ *such that*

1. $\mathrm{Vol}(\chi) > 0$
2. *if* $\Lambda = \chi(\Gamma)$ *is a lattice in* $\mathbb{C}$*, then* $\mathrm{Vol}(\chi) \geqslant 2\mathrm{Area}(\mathbb{C}/\Lambda)$.

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Let us recall that the sum of the degrees of zeros of an abelian differential is $2g-2$. Therefore, the space $\Omega'$ is naturally stratified by the subsets $\mathcal{H}(n_1,\ldots,n_k)$ consisting of $\alpha \in \Omega'$ with zeros of multiplicity $n_1,\ldots,n_k$, where $\sum n_i = 2g-2$. As a refinement of Haupt's theorem, we characterise the period maps of abelian differentials in a given stratum, thus answering a question raised by Calsamiglia, Deroin and Francaviglia in [2].

**Theorem 1.1.** *Let $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ be such that $\sum n_i = 2g-2$. The image of $\mathcal{H}(n_1,\ldots,n_k)$ by* Per *is the set of $\chi \in \mathrm{Hom}(\Gamma,\mathbb{C})$ satisfying:*

1. $\mathrm{Vol}(\chi) > 0$
2. *if $\Lambda = \chi(\Gamma)$ is a lattice in $\mathbb{C}$, then* $\mathrm{Vol}(\chi) \geqslant (n_k+1)\mathrm{Area}(\mathbb{C}/\Lambda)$.

Calsamiglia, Deroin and Francaviglia also asked to characterise the image of each connected component of the strata by Per. In [1], Bainbridge, Johnson, Judge and Park completely solved this problem, thus giving a more general counterpart of Theorem 1.1. Their proof relies on the study of the isoperiodic foliation of the Hodge bundle over the moduli space of curves done in [2]. Using explicit constructions, we provide a direct and elementary proof of Theorem 1.1 that does not rely on [2].

The obstructions of Theorem 1.1 on the volume were already observed in [2]. In order to explain their origins, it is convenient to introduce a more geometric point of view on this problem. A *translation surface*, or *flat surface* structure on $S$ is the datum of an atlas of chart in the euclidean plane $\mathbb{E}^2$, with transition maps that are translations. We allow the charts to have conical points, that is to have the local form $z \mapsto z^k$. More precisely, a translation surface structure is a branched $(G,X)$ structure on $S$ where $G=\mathbb{C}$ is the group of translations of $X = \mathbb{E}^2$. The local charts globalize into a *developing map* $f : \tilde{S} \to \mathbb{C}$, where $\tilde{S}$ is the universal cover of $S$, that satisfies $f(\gamma\cdot z) = \chi(\gamma)+f(z)$ for some $\chi \in \mathrm{Hom}(\Gamma,\mathbb{C})$ called the *holonomy* of the translation surface (see [9, Chapter 3] for more information on $(G,X)$ structures). If $\alpha \in \Omega'$, then one can define local charts on $S$ by $\int_{z_0}^{z}\alpha$. In this manner, we get a one-to-one correspondence between translation surfaces and abelian differentials, see [10, Section 3.3]. The period map of an abelian differential corresponds to the holonomy of the associated flat surface, and the stratum $\mathcal{H}(n_1,\ldots,n_k)$ corresponds to the flat surfaces with $k$ conical points of angles $2\pi(n_1+1),\ldots,2\pi(n_k+1)$. Hence the question adressed in this article is to characterise the $\chi \in \mathrm{Hom}(\Gamma,\mathbb{C})$ which are holonomies of flat surfaces with a prescribed list of conical angles. Suppose that $\chi \in \mathrm{Hom}(\Gamma,\mathbb{C})$ is the period map of $\alpha \in \Omega'$. The Riemann bilinear relations (see [6, Section 14]) give:

$$\frac{i}{2}\int_S \alpha\wedge\overline{\alpha} = \frac{i}{2}\sum_{i=1}^{g}\int_{a_i}\alpha\int_{b_i}\overline{\alpha} - \int_{a_i}\overline{\alpha}\int_{b_i}\alpha = \sum_{i=1}^{g}\Im(\overline{\chi(a_i)}\chi(b_i)).$$

Let $z = x+iy$ be a local coordinate on $U \subset S$, and write $\alpha = f(z)dz$. We have $\int_U |f|^2 dxdy = \frac{i}{2}\int_U \alpha\wedge\overline{\alpha} > 0$. A first obstruction to being the holonomy of a flat structure is thus to have positive volume.

Suppose now moreover that $\Lambda = \chi(\Gamma)$ is a lattice in $\mathbb{C}$. Then the developing map of the translation surface induces a holomorphic map $S \to \mathbb{C}/\Lambda$. If $\alpha$ has a zero of degree $n$, then the degree $d$ of this branched cover is at least $n+1$. The volume of $\chi$ is $d \cdot \mathrm{Area}(\mathbb{C}/\Lambda)$. Therefore a second obstruction to being the holonomy of $\alpha \in \mathcal{H}(n_1, \ldots, n_k)$ is to have $\mathrm{Vol}(\chi) \geqslant (n_k+1)\mathrm{Area}(\mathbb{C}/\Lambda)$ when $\Lambda = \chi(\Gamma)$ is a lattice in $\mathbb{C}$.

Theorem 1.1 implies the existence of branched covers of elliptic curves with prescribed induced map on homology.

**Corollary 1.2.** *Let $\Lambda$ be a lattice in $\mathbb{C}$ and $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$. There exists a branched cover $S \to \mathbb{C}/\Lambda$ of degree $d$ with branching points of order $n_1, \ldots, n_k$ if and only if $\sum n_i = 2g-2$ and $d \geqslant n_k + 1$. Moreover, if $\chi \in \mathrm{Hom}(\Gamma, \Lambda)$, then $\chi$ is induced by one of these covering maps if and only if*

1. $\mathrm{Vol}(\chi) = d \cdot \mathrm{Area}(\mathbb{C}/\Lambda)$
2. $\mathrm{Vol}(\chi) \geqslant (n_k+1)\mathrm{Area}(\mathbb{C}/\Lambda')$, *where* $\Lambda' = \chi(\Lambda)$.

*Proof.* Let us consider a covering map $S \to \mathbb{C}/\Lambda$ of degree $d$ with branching points of order $n_1, \ldots, n_k$. Its degree is at least $n_k+1$. The pullback $\alpha$ of $dz$ by this branched cover is in $\mathcal{H}(n_1, \ldots, n_k)$ and thus $\sum n_i = 2g-2$. The period map of $\alpha$ is the map induced on homology by this covering, and has volume $d \cdot \mathrm{Area}(\mathbb{C}/\Lambda)$ by the computation of $\int \alpha \wedge \overline{\alpha}$ above. Observe that $\Lambda' = \chi(\Gamma)$ is a lattice since it is a subgroup of $\Lambda$ and $\mathrm{Vol}(\chi) > 0$. The developing map of the translation surface associated to $\alpha$ gives a branched cover $S \to \mathbb{C}/\Lambda'$, whose degree is at least $n_k+1$ and equals $\mathrm{Vol}(\chi)/\mathrm{Area}(\mathbb{C}/\Lambda')$.

Conversely, given $\chi$ that satisfies those two conditions, then by Theorem 1.1 there exists a translation surface with conical angle $2\pi(n_1+1), \ldots, 2\pi(n_k+1)$ and holonomy $\chi$. Its developing map gives a branched cover $S \to \mathbb{C}/\Lambda$ with induced map on homology $\chi$. The degree of this covering map is $\mathrm{Vol}(\chi)/\mathrm{Area}(\mathbb{C}/\Lambda) = d$, and the orders of its branching points are $n_1, \ldots, n_k$. $\square$

Note that these conditions on the existence of branched covers are not new, see for example [7]. Therefore Corollary 1.2 is actually a characterisation of the induced maps on homology of the branched covers of elliptic curves.

Let us now explain the strategy of the proof. We denote by $\mathrm{Mod}(S)$ the mapping class group of $S$. Our proof relies on the study of the $\mathrm{Mod}(S)$ action on $\mathrm{Hom}(\Gamma, \mathbb{C})$, that was carried out by Kapovich in [5], using Ratner's theory. Starting from an arbitrary $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$, Kapovich finds $\chi' \in \mathrm{Mod}(S) \cdot \chi$ with special properties that allows him to put a translation surface structure on $S$ with holonomy $\chi'$. This strategy is the same as Haupt's original one, except for the use of Ratner's theory. We will adapt the discussion of the classification of $\mathrm{Mod}(S)$ orbits by Kapovich for our purpose in Section 2. Then we will use this classification to construct translation surface structures with given holonomy and given list of conical angles in Section 3. In [5], the classification of the orbits holds only for $g \geqslant 3$, hence we will study the genus 2 case without Ratner's theory in the spirit of Haupt's proof in Section 4.

## 2 Action of the mapping class group

We will often identify $\mathrm{Hom}(\Gamma, \mathbb{C})$ and $\mathbb{C}^{2g}$ with the map $\chi \mapsto (\chi(a_1), \ldots, \chi(b_g))$. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts diagonally on $(\mathbb{R}^2)^{2g} \simeq \mathbb{C}^{2g} \simeq \mathrm{Hom}(\Gamma, \mathbb{C})$. A theorem of Dehn, Nielsen, and Baer, see [3, Chapter 8], states that the natural map from $\mathrm{Mod}(S)$ to the positive outer automorphisms $\mathrm{Out}^+(\Gamma)$ of $\Gamma$ is an isomorphism. This latter group acts naturally on $\mathrm{Hom}(\Gamma, \mathbb{C})$ by precomposition and thus yields a $\mathrm{Mod}(S)$ action on $\mathrm{Hom}(\Gamma, \mathbb{C})$ that commutes with the $\mathrm{GL}_2^+(\mathbb{R})$ action. Recall that $\mathrm{Out}^+(\Gamma)$ acts as $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\Gamma^{\mathrm{ab}} \simeq H_1(S, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ and hence on $\mathbb{C}^{2g}$, see [3, Chapter 6]. Let us state results obtained by Kapovich in [5] using Ratner's theory.

**Theorem 2.1** (Kapovich)**.** *Let $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ be such that $\mathrm{Vol}(\chi) > 0$. After replacing $\chi$ by $A \cdot \chi$ with $A \in \mathrm{GL}_2^+(\mathbb{R})$ if necessary, then*

1. *If $\chi(\Gamma)$ is not a lattice, there exists $\gamma \in \mathrm{Mod}(S)$ such that $\chi' = \gamma \cdot \chi$ satisfies $\Im(\overline{\chi'(a_i)}\chi'(b_i)) > 0$ for all $1 \leqslant i \leqslant g$.*
2. *If $\Lambda = \chi(\Gamma)$ is a lattice and $\mathrm{Vol}(\chi) \geqslant 2\mathrm{Area}(\mathbb{C}/\Lambda)$, then there exists $\gamma \in \mathrm{Mod}(S)$ such that $\chi' = \gamma \cdot \chi$ satisfies:*
   - *(a) $\chi'(\Gamma) = \mathbb{Z} \oplus i\mathbb{Z}$*
   - *(b) $\chi'(a_1)$ is a positive integer and $\chi'(b_1) = i$*
   - *(c) $0 < \chi'(a_i) < \chi'(a_1)$ is an integer and $\chi'(b_i) = 0$ for all $i \geqslant 2$.*

We will need the following improvement of the lattice case.

**Proposition 2.2.** *In the second case of Theorem 2.1, we can moreover assume that $\chi'(a_i) = m_i$ for all $2 \leqslant i \leqslant g$, where $m_2 = 1$ and $m_i \in \mathbb{Z}$ is any fixed integer for $3 \leqslant i \leqslant g$.*

We will give an alternative proof of the second part of Theorem 2.1 as well, based on the following known result.

**Lemma 2.3.** *The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts transitively on the primitive vectors of $\mathbb{Z}^{2g}$.*

We refer to [3, Proposition 6.2] for a proof of Lemma 2.3. We infer the classification of the orbits of the $\mathrm{Sp}_{2g}(\mathbb{Z})$ action on $\mathbb{Z}^{2g}$.

**Corollary 2.4.** *Let $u, v \in \mathbb{Z}^{2g} \setminus \{0\}$. There exists $A \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $Au = v$ if and only if $\gcd(u) = \gcd(v)$.*

We now prove Proposition 2.2. Let $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ be such that $\Lambda = \chi(\Gamma)$ is a lattice and such that $\mathrm{Vol}(\chi) \geqslant 2\mathrm{Area}(\mathbb{C}/\Lambda)$.

*Proof.* We may assume that $\chi(\Gamma) = \mathbb{Z} \oplus i\mathbb{Z}$, changing $\chi$ by $A \cdot \chi$ with $A \in \mathrm{GL}_2^+(\mathbb{R})$ if necessary. Let $u = (\Re(\chi(a_1)), \ldots, \Re(\chi(b_g))$ and $v = (\Im(\chi(a_1)), \ldots, \Im(\chi(b_g))$ and let $(x_1, y_1, \ldots, x_g, y_g) = u + iv$. Since $\gcd(v) = 1$, there exists $A \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that $Av = (0, 1, 0, \ldots, 0)$. Let us write $Au = (p_1, q_1, \ldots, p_g, q_g)$. By Corollary 2.4 we may replace $A$ with $CA$ where $C = \mathrm{Id}_2 \oplus D$ for some $D \in \mathrm{Sp}_{2g-2}(\mathbb{Z})$

so that $Au = (p_1, q_1, 0, l, 0 \dots, 0)$ where $l = \gcd(p_2, q_2, \dots, p_g, q_g)$. Therefore there exists $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that after replacing $\chi$ with $\gamma \cdot \chi$, we have $x_1 = p_1$, $y_1 = q_1 + i$, $x_j = y_j = 0$ for $j \geqslant 2$ unless $j = 2$, where $x_j = 0$ and $y_j = l$. Since $i \in \chi(\Gamma)$, there exists $\lambda, \mu \in \mathbb{Z}$ such that $q_1 + \lambda p_1 + \mu l = 0$. We apply the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \mu \\ -\mu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \oplus \mathrm{Id}_{2g-4}$$

that replaces $(x_1, y_1, x_2, y_2)$ with $(x_1, y_1 + \mu y_2, x_2 - \mu x_1, y_2)$. After this, replace $(x_1, y_1)$ by $(x_1, y_1 + \lambda x_1)$ with the matrix $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \oplus \mathrm{Id}_{2g-2}$. We now have $\chi = (p_1, i, -\mu p_1, l, 0, \dots, 0)$. Changing $(x_2, y_2)$ to $(x_2, y_2 \pm x_2)$ or to $(x_2 \pm y_2, y_2)$, one can make $\max(|x_2|, |y_2|)$ decrease until one of $x_2$ and $y_2$ is 0, since $\chi(\Gamma)$ is discrete. Therefore we may assume that $x_2 = 0$, and that $y_2 = l$ for some new $l \in \mathbb{Z}$, replacing $(x_2, y_2)$ with $(-y_2, x_2)$ if necessary. We replace $(x_1, y_1, x_2, y_2)$ with $(x_1, y_1 - y_2, x_2 + x_1, y_2)$ so that $x_2 = p_1$. Since $\chi(\Gamma) = \mathbb{Z} \oplus i\mathbb{Z}$, we have $\gcd(p_1, l) = 1$. We perform the euclidean algorithm as above in the handle $(a_2, b_2)$ : replace $(x_2, y_2)$ with $(x_2, y_2 \pm x_2)$ or $(x_2 \pm y_2, y_2)$ so that $\max(|x_2|, |y_2|)$ decreases until $x_2 = 0$ or $y_2 = 0$. We can assume that $x_2 = 0$, and $y_2 = 1$, since $\gcd(x_2, y_2)$ is preserved at each step. Indeed, one can change the signs of $x_2$ and $y_2$ if necessary by applying the matrix $\mathrm{Id}_2 \oplus -\mathrm{Id}_2 \oplus \mathrm{Id}_{2g-4}$. Now replace $(x_1, y_1, x_2, y_2)$ by $(x_1, y_1 + ky_2, x_2 - kx_1, y_2)$, where $k$ is such that we get $y_1 = i$. We finally replace $(x_2, y_2)$ with $(y_2, -x_2 - kp_1y_2)$, so that $x_2 = 1$ and $y_2 = 0$. Therefore we have $\chi = (p_1, i, 1, 0, 0, \dots, 0)$. For each $3 \leqslant j \leqslant g$, we perform the symplectic transformations $(x_2, y_2, x_j, y_j) \to (x_2, y_2 - m_jy_j, x_j + m_jx_2, y_j)$, so that $x_j = m_j$ and $y_j = 0$. □

From now on until Section 4, we assume that $g \geqslant 3$. We also adapt the first case of Theorem 2.1.

**Proposition 2.5.** *In the first case of Theorem 2.1 we can also assume the following.*

1. *The parallelogram formed by the vectors $\chi'(a_1)$ and $\chi'(b_1)$ contains in its interior a rectangle $R = I \times J \subset \mathbb{R}^2 \simeq \mathbb{C}$ such that $\lambda(I) \geqslant 2g \cdot \max_{i \geqslant 2} |\Re(\chi'(a_i))|$ and $\lambda(J) \geqslant 2g \cdot \max_{i \geqslant 2} |\Im(\chi'(a_i))|$ where $\lambda$ is the Lebesgue measure.*

2. $\arg(\chi(a_i)) \neq \arg(\chi(a_j))$ *for every* $2 \leqslant i < j \leqslant g$.

*Proof.* Reviewing the arguments of Kapovich in [5], Calsamiglia, Deroin and Francaviglia showed that when we are not in the second case of Theorem 2.1, applying $A \in \mathrm{GL}_2^+(\mathbb{R})$ if necessary, the orbit closure $\overline{\mathrm{Mod}(S) \cdot \chi}$ contains all $\chi' \in \mathbb{C}^{2g}$ such that $\mathrm{Vol}(\chi') = \mathrm{Vol}(\chi)$ and such that $\Im(\chi')$ is in $\mathbb{Z}^{2g}$ and is primitive (see [2, Proposition 8.1]). Take $\chi' \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that $\Im(\chi') \in \mathbb{Z}^{2g}$ is primitive, such that $\Im(\overline{\chi'(a_i)}\chi'(b_i)) > 0$ for all $i$ and such that $\chi'$ satisfies

the two conditions of Proposition 2.5. Rescale $\Re(\chi')$ by a positive constant so that $\mathrm{Vol}(\chi') = \mathrm{Vol}(\chi)$. Then $\chi' \in \overline{\mathrm{Mod}(S)\cdot\chi}$, and $\chi'$ still satisfies the required properties. Since these conditions are open, we may assume that $\chi' \in \mathrm{Mod}(S)\cdot\chi$. $\square$

# 3 Construction of translation surfaces

Let us fix a partition $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ of $2g-2$ and $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that $\mathrm{Vol}(\chi) > 0$ and $\mathrm{Vol}(\chi) \geqslant (n_k+1)\mathrm{Area}(\mathbb{C}/\Lambda)$ if $\chi(\Gamma) = \Lambda$ is a lattice in $\mathbb{C}$. We observe that both the actions of $\mathrm{Mod}(S)$ and $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathrm{Hom}(\Gamma, \mathbb{C})$ preserve the property of being in the image by Per of a given stratum.

**Lemma 3.1.** *Let $\gamma \in \mathrm{Mod}(S)$ and $A \in \mathrm{GL}_2^+(\mathbb{R})$. We have:*

1. $\chi \in \mathrm{Per}(\mathcal{H}(n_1, \ldots, n_k)) \iff A\cdot\chi \in \mathrm{Per}(\mathcal{H}(n_1, \ldots, n_k))$
2. $\chi \in \mathrm{Per}(\mathcal{H}(n_1, \ldots, n_k)) \iff \gamma\cdot\chi \in \mathrm{Per}(\mathcal{H}(n_1, \ldots, n_k))$.

*Proof.* Suppose $\chi$ is the holonomy of a translation surface structure. If $z$ is some local coordinate then define a new chart by $Az$. This gives a new translation surface structure with holonomy $A\cdot\chi$ and the same combinatoric of conical points. If $\gamma$ is represented by a homeomorphism $f$, then the pullback of the translation surface structure by $f$ gives a new translation structure with holonomy $\gamma\cdot\chi$. $\square$

Note that the action of $\mathrm{GL}_2^+(\mathbb{R})$ changes the volume, but does not change its sign, nor the ratio $\mathrm{Vol}(\chi)/\mathrm{Area}(\mathbb{C}/\Lambda)$ in the case where $\chi(\Gamma) = \Lambda$ is a lattice.

Therefore it suffices to consider the $\chi$ which have the form of Proposition 2.5 or of Proposition 2.2 to prove Theorem 1.1. Our goal is to construct a translation surface with holonomy $\chi$ and with conical points of angles $2\pi(n_1+1), \ldots, 2\pi(n_k+1)$.

## 3.1 The generic case

In this section we suppose that $\chi$ has the form of Proposition 2.5.

Consider the parallelogram $P_i \subset \mathbb{C}$ *associated* to $\chi(a_i)$ and $\chi(b_i)$ *i.e.* whose vertices are 0, $\chi(a_i)$, $\chi(b_i)$, $\chi(a_i)+\chi(b_i)$. This gives complex coordinates on the torus $T_1$ obtained by gluing the opposite sides of $P_1$ by translations.

Before going on to the general construction, let us explain how to glue a torus to $T_1$ by adding a single conical point of angle $6\pi$. Make a slit in $P_1$, on a segment of the form $p+\chi(a_2)$. Identify the boundary components of these slits with the edges of $P_2$ corresponding to $\chi(a_2)$, see Figure 1. We also identify the other edges of $P_2$ by the translation $z \mapsto z+\chi(a_2)$.

The two endpoints of the slit on $P_1$ are identified, and yield a conical point of angle $6\pi$ on the resulting genus 2 surface. Observe that its holonomy is given by the vector $(\chi(a_1), \chi(b_1), \chi(a_2), \chi(b_2))$, see Figure 2.

Let us return to the general case. We want to have conical singularities corresponding to $n_1, \ldots, n_k$. For each $1 \leqslant i \leqslant k$, we glue $l_i$ tori as above on

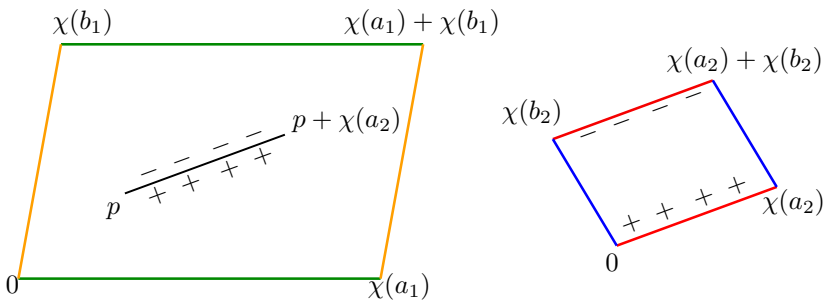


Figure 1: Gluing a handle.

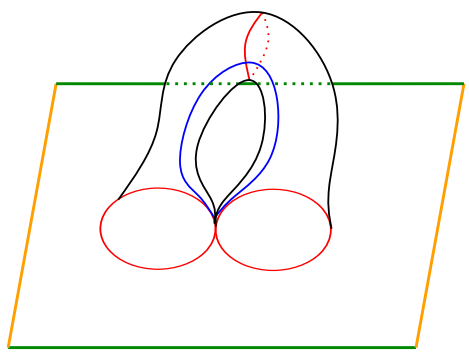

Figure 2: Topological result.

a point $p_i \in P$, where $l_i = \lfloor n_i/2 \rfloor$, see the black starfish formed by the slits meeting at $p_i$ in Figure 3. Each starfish is contained in a rectangle whose sides have length $2\max_{i\geqslant 2}|\Re(\chi(a_i))|$ and $2\max_{i\geqslant 2}|\Im(\chi(a_i))|$ and there are at most $g-1$ of them since $\sum n_i = 2g-2$ and there is exactly one starfish for each $1 \leqslant i \leqslant g$ such that $n_i \geqslant 2$. Since we can choose the $p_i \in P$, and thanks to the assumption that $P$ contains a rectangle of the size given in Proposition 2.5, we can make sure that the starfish do not intersect. Since $\arg\chi(a_i) \neq \arg\chi(a_j)$ for $1 < i < j \leqslant g$, the slits do not overlap outside of the points $p_i$. The conical angle at each $p_i$ is $2\pi(n_i+1)$ if $n_i$ is even and $2\pi n_i$ if $n_i$ is odd.

We now explain how to add $2\pi$ to the total angles of the singularities corresponding to odds $n_i$. Since $\sum n_i$ is even, we can pair the odds $n_i$. We may assume changing the $p_i$ if necessary that if $n_j$ and $n_l$ are paired, and are both at least 3, then the corresponding starfish are close. By close we mean that we can join two of their endpoints with a line segment in $P$ of length less than $\epsilon$, where $\epsilon > 0$ is such that each $P_i$ contains a disk of radius $\epsilon$. We may also assume that this line segment does not intersect any of the starfish save from its endpoints, see the purple line in Figure 3. We make a slit along this line segment, and another along a translate of this segment in a $P_i$ that is not already glued to $P$, and identify the boundary components as indicated in Figure 3. We have room to make this slit in $P_i$ because of the definition of $\epsilon$. We also glue the edges of

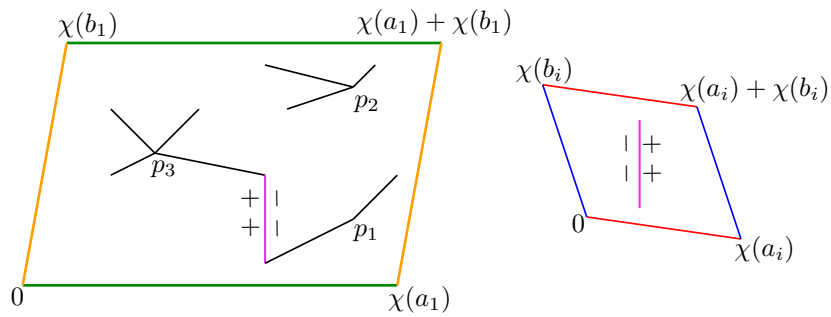


Figure 3: Construction of $\alpha \in \mathcal{H}(5,6,9)$.

$P_i$ with translations. In the case where exactly one of the paired $n_j$ and $n_l$ is one, say $n_j = 1$, we make a slit along a line segment of length less than $\epsilon$, that touches an endpoint of the starfish corresponding to $n_l$ and does not intersect any slit otherwise. In the case where $n_j = n_l = 1$, we choose a purple line in $P$ of length less than $\epsilon$ that does not meet any starfish. This construction adds a $2\pi$ angle at each end of the slit, and allows us to have the desired conical angles. Since $\sum n_i = 2g-2$, the resulting surface has genus $g$. Therefore we have glued each $P_i$ to $P_1$ for $1 < i \leqslant g$.

By construction, the resulting translation surface has holonomy $\chi$, and the corresponding abelian differential is in $\mathcal{H}(n_1, \ldots, n_k)$.

## 3.2 The lattice case

We now suppose that $\chi$ has the form of Proposition 2.2. Note that $\mathrm{Vol}(\chi) = \chi(a_1)$.

### 3.2.1 Multiple singularities

We assume here that $k \geqslant 2$. Let us first explain how to build a translation surface with corresponding abelian differential in $\mathcal{H}(g-1, g-1)$. Consider the parallelogram $P$ in $\mathbb{C}$ associated to $\chi(a_1)$ and $\chi(b_1)$. Consider a segment on $P$ such that its translates by $z \mapsto z + k$, with $1 \leqslant k \leqslant g-1$ are also contained in $P$. Cutting $P$ along those segments yields $2g$ boundaries $\delta_1^+, \delta_1^-, \ldots, \delta_g^+, \delta_g^-$. Identify $\delta_i^+$ with $\delta_{i+1}^-$ for all $1 \leqslant i \leqslant g$, in cyclic notation, as in Figure 4.

Identifying the edges of $P$ by translations, we get a flat surface with two conical points of angle $2g\pi$. The holonomy of this surface is $(\mathrm{Vol}(\chi), i, 1, 0, \ldots, 1, 0)$, see Figure 5, and is in $\mathrm{Mod}(S) \cdot \chi$ by Proposition 2.2. Note that it was possible to put these segments in $P$ because of the assumption that $\mathrm{Vol}(\chi) > g-1$.

Let us return to the general case. We consider the torus obtained by identifying the opposite sides of the parallelogram $P$ associated to $\chi(a_1)$ and $\chi(b_1) = i$. We now explain how to cut and glue it to get the desired conical angles. In the case where $n_1 = n_2$, we make $n_1 + 1$ slits that we glue in a cyclic way as is in Figure 4, which creates two conical points of total angle $2\pi(n_1+1)$ each. If

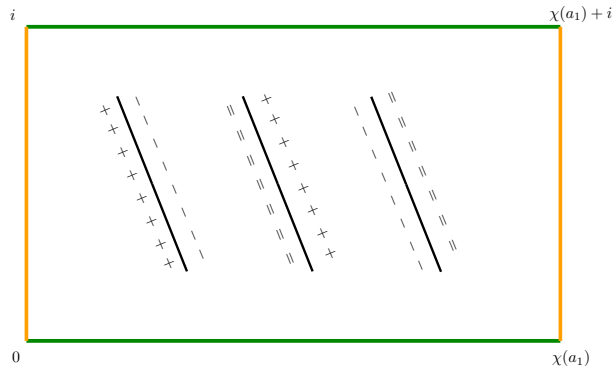


Figure 4: Construction of $\alpha \in \mathcal{H}(2,2)$.

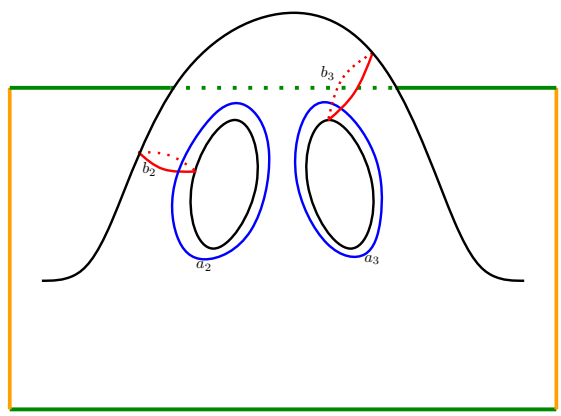


Figure 5: Topological result.

moreover $k \geqslant 4$ and $n_3 = n_4$, we make $n_3 + 1$ slits below, to create two conical angles $2\pi(n_3+1)$. We continue this construction in the same manner as long as we have two equal $n_i$. We may rename the remaining $n_i$, if any, by $n_1, \ldots, n_k$. Either the new $k$ is 1, in which case we make the cuts and gluings explained in Section 3.2.2, or $n_1 < n_2$, in which case we make the cuts and gluings that we now describe.

We make $n_1 + 1$ slits, and glue the boundaries in a cylic way as in Figure 4 thus creating two conical points of angle $2\pi(n_1+1)$. We then make $n_2 - n_1 + 1$ other parallel slits below, such that one of them touches one end of the rightmost preceding slit, that we glue in a cyclic way (see the top of Figure 6). In doing so, we obtain conical angles $2\pi(n_1+1)$ and $2\pi(n_2+1)$ and $2\pi(n_2-n_1+1)$. Iterating this process, we get conical angles $2\pi(n_1+1), \ldots, 2\pi(n_{k-1}+1)$ and $2\pi(l+1)$, with $l \leqslant n_k$. If needed, we make a slit of length one joining the rightmost two ends of the last slits as in the bottom of Figure 6, and some other slits that are translations by multiples of 2 of this one, that we glue in a cyclic way to adjust

the last conical angle.

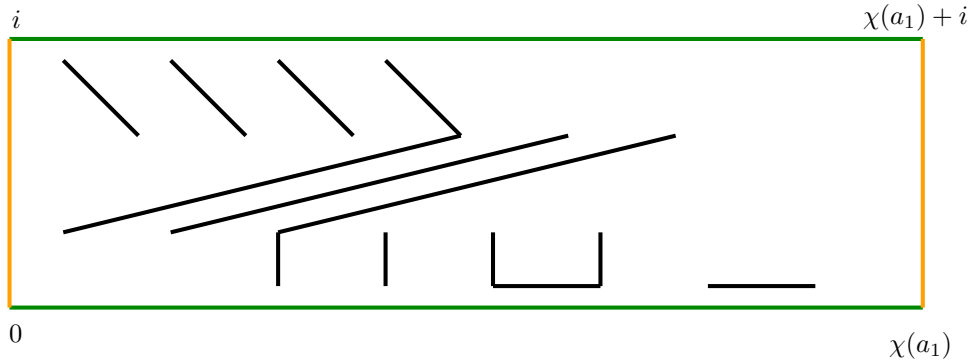


Figure 6: Construction of $\alpha \in \mathcal{H}(3,5,5,5)$.

Each horizontal line has length at most $n_k$, hence we have room to make the slits, since $\mathrm{Vol}(\chi) = \chi(a_1) \geqslant n_k + 1$. We check that the holonomy of the resulting flat surface is given by $(\mathrm{Vol}(\chi), i, 1, 0, \ldots, 1, 0, 2, 0, \ldots, 2, 0)$ and thus is in $\mathrm{Mod}(S) \cdot \chi$ by Proposition 2.2.

### 3.2.2 The minimal stratum

We now suppose that $k = 1$. Let us explain how to cut and glue $P$ to add a single conical point. We make a horizontal slit of length one in $P$. We make other slits along segments which are translates of the first one by $n = 1, 3, 5, 7, \ldots, n_1 - 1$ and we identify their boundaries in a cyclic way as before, see Figure 7.

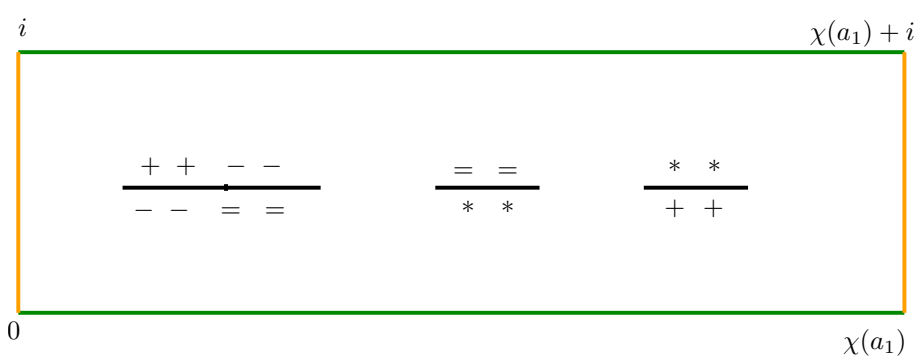


Figure 7: Construction of $\alpha \in \mathcal{H}(6)$.

We obtain a flat surface, with a single conical point of angle $2\pi(n_1+1)$. The total length of this construction is $n_1$. Therefore, we have room to make it in

$P$, since $\mathrm{Vol}(\chi) \geqslant n_1 + 1$. This adds $g-1$ handles to $T_1$ and yields a translation surface with holonomy in $\mathrm{Mod}(S) \cdot \chi$ by Proposition 2.2 and a single conical point.

# 4 The genus 2 case

The aim of this section is to prove Theorem 1.1 when $g = 2$. Let $\chi = (x_1, y_1, x_2, y_2) \in \mathbb{C}^4 \simeq \mathrm{Hom}(\Gamma, \mathbb{C})$ be such that $\mathrm{Vol}(\chi) > 0$. We assume that $\chi(\Gamma)$ is not a lattice in $\mathbb{C}$, since otherwise the above constructions, together with Proposition 2.2, show that Theorem 1.1 holds.

Let us observe that by definition, $\mathrm{Vol}(\chi) = \det(x_1, y_1) + \det(x_2, y_2)$. We may assume that $\det(x_1, y_1) \geqslant \det(x_2, y_2)$, applying $[f] \in \mathrm{Mod}(S)$ that interchanges the two handles if necessary. Since $\mathrm{Vol}(\chi) > 0$, we have $\det(x_1, y_1) > 0$. Let us show that it suffices to have $\det(x_1, y_1)$ and $\det(x_2, y_2)$ positive, with the ratio $\det(x_2, y_2)/\det(x_1, y_1)$ small enough, to prove Theorem 1.1.

**Lemma 4.1.** *If $v_1$ and $v_2$ generate a lattice in $\mathbb{C}$, then there exists $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $(v_1', v_2') = A(v_1, v_2)$ satisfies $\|v_1'\| \leqslant C\sqrt{|\det(v_1, v_2)|}$ where $C = \sqrt{\frac{2}{\sqrt{3}}}$.*

*Proof.* We may assume that $\det(v_1, v_2) < 0$, replacing $v_1$ with $-v_1$ if necessary. It suffices to show that after applying $A \in \mathrm{SL}_2(\mathbb{Z})$, we can assume that the angle between $v_1$ and $v_2$ is $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, since then

$$\frac{\sqrt{3}}{2}(\min(\|v_1\|, \|v_2\|))^2 \leqslant \|v_1\|\|v_2\| \sin\theta = |\det(v_1, v_2)|.$$

Applying $(v_1, v_2) \to (v_2, -v_1)$ if necessary, we have the desired inequality.

Since the action of $\mathbb{C}^*$ on the pairs $(v_1, v_2)$ preserves the angles between the two vectors, we are reduced to the study of the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathbb{H}^2$ by renormalizing $(v_1, v_2)$ so that $v_2 = 1$. It is well known that $D = \{z \in \mathbb{H}^2 \mid |z| > 1, -1/2 \leqslant \Re(z) \leqslant 1/2\}$ is a fundamental domain for this action, see for example [8, Chapter 7]. Therefore, we may assume that the angle between $v_1$ and $v_2$ is in $[\frac{\pi}{3}, \frac{2\pi}{3}]$. □

**Proposition 4.2.** *If $0 < \det(x_2, y_2) < \epsilon \det(x_1, y_1)$, then $\chi \in \mathrm{Per}(\mathcal{H}(2))$ and $\chi \in \mathrm{Per}(\mathcal{H}(1,1))$, where $\epsilon = \frac{1}{C^2}$.*

*Proof.* Applying $A \in \mathrm{GL}_2^+(\mathbb{R})$ does not change the ratio $\det(x_2, y_2)/\det(x_1, y_1)$, hence we may assume that $(x_1, y_1) = (1, i)$. We thus have $\det(x_2, y_2) < \epsilon$, and it follows from Lemma 4.1 that we can assume that $\|x_2\| < C\sqrt{\epsilon} \leqslant 1$. Now we can glue the parallelograms associated to $x_1, y_1$ and $x_2, y_2$ as in Figure 1 since we have room to make the slit. Hence $\chi \in \mathrm{Per}(\mathcal{H}(2))$. We can also glue these parallelograms with a slit as shown in purple in Figure 3, hence $\chi \in \mathrm{Per}(\mathcal{H}(1,1))$. □

It is now enough to prove the following proposition.

**Proposition 4.3.** *There exists $A \in \mathrm{GL}_2^+(\mathbb{R})$ and $\gamma \in \mathrm{Mod}(S)$ such that $\chi' = A \cdot (\gamma \cdot \chi)$ satisfies the condition of Proposition 4.2.*

We may assume that $\det(x_1, y_1) > 0$ and, applying $A \in \mathrm{GL}_2^+(\mathbb{R})$, that $x_1 = 1$ and $y_1 = i$.

**Lemma 4.4.** *If the projection of the group generated by $x_2$ and $y_2$ on the real line is dense, then Proposition 4.3 holds.*

*Proof.* There exists $n, m \in \mathbb{Z}$ such that $x = \Re(nx_2 + my_2)$ satisfies

$$\frac{\mathrm{Vol}(\chi)}{1+\epsilon} < x + 1 < \mathrm{Vol}(\chi).$$

Let $k = \gcd(n, m)$ and $n' = n/k$, $m' = m/k$. We may replace $x_2$ with $n'x_2 + m'y_2$ by applying $A \in \mathrm{Sp}_4(\mathbb{Z})$ of the form $A = \mathrm{Id}_2 \oplus B$ where $B \in \mathrm{SL}_2(\mathbb{Z})$. We now apply the following transformation of $\mathrm{Sp}_4(\mathbb{Z})$ : $(x_1, y_1, x_2, y_2) \to (x_1 + kx_2, y_1, x_2, y_2 - ky_1)$. Thus $\det(x_1, y_1) = x + 1$, and

$$0 < \det(x_2, y_2) = \mathrm{Vol}(\chi) - \det(x_1, y_1) < \epsilon \det(x_1, y_1). \qquad \square$$

We can now suppose that $x_2 = x + iy$ and $y_2 = x' + iy'$ are such that the group generated by $x$ and $x'$ is discrete. We also assume that the group generated by $y$ and $y'$ is dense, otherwise we could return to the previous case by applying the transformation $(1, i, x_2, y_2) \to (-i, 1, x_2, y_2)$.

**Lemma 4.5.** *We can assume that $x' = 0$.*

*Proof.* Apply the transformation $(x_2, y_2) \to (x_2 \pm y_2, y_2)$ or $(x_2, y_2) \to (x_2, y_2 \pm x_2)$ so that $\max(|\Re(x_2)|, |\Re(y_2)|)$ decreases. Since the group generated by $x$ and $x'$ is discrete, iterating this process will end up with $x$ or $x' = 0$. We may assume that it is $x'$, applying $(x_2, y_2) \to (-y_2, x_2)$ if necessary. Note that we have already used this Euclidean algorithm's argument in Proposition 2.2. $\square$

**Lemma 4.6.** *If both $\det(x_1, y_1) \geqslant \det(x_2, y_2)$ are not $0$, there exists $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$ such that $\chi' = \gamma \cdot \chi = (x_1', y_1', x_2', y_2')$ satisfies*

$$0 \leqslant \det(x_1', y_1') \leqslant \frac{\det(x_1, y_1)}{2}.$$

*Moreover, if $\det(x_1, y_1) \neq \det(x_2, y_2)$, we may assume that $0 < \det(x_1', y_1')$.*

*Proof.* Let us assume again that $(x_1, y_1) = (1, i)$. We have $0 < |xy'| = |\det(x_2, y_2)| \leqslant \det(x_1, y_1) = 1$ since $\det(x_1, y_1) + \det(x_2, y_2) = \mathrm{Vol}(\chi)$, thus $0 < |y'| \leqslant 1$ or $0 < |x| \leqslant 1$, and $|x| < 1$ or $|y'| < 1$ if $\det(x_1, y_1) \neq \det(x_2, y_2)$. In the former case, replace $(x_1, y_1, x_2, y_2)$ with $(x_1, y_1 + ky_2, x_2 - ky_1, y_2)$, where $k$ is such that $0 \leqslant (1 + ky') \leqslant \frac{1}{2}$. Such an integer exists: it is equivalent to $\frac{1}{2} \leqslant -ky' \leqslant 1$ which is possible with $k$ an appropriate power of 2. Note that if $|y'| < 1$, we may assume that $0 < 1 + ky'$. In the second case, replace $\chi$ with $(x_1 + kx_2, y_1, x_2, y_2 - ky_1)$ where $0 \leqslant 1 + kx \leqslant \frac{1}{2}$. In both cases the new $\det(x_1, y_1)$ is less than half the old one. $\square$

It might happen that $\det(x_2, y_2) = 0$. Let us explain how to proceed in this case.

**Lemma 4.7.** *If* $\det(x_2, y_2) = 0$*, then there exists* $\gamma \in \mathrm{Sp}_4(\mathbb{Z})$ *such that replacing* $\chi$ *with* $\gamma \cdot \chi$*, we have* $\det(x_1, y_1) > 0$ *and* $\det(x_2, y_2) > 0$ *and* $\det(x_1, y_1) \neq \det(x_2, y_2)$*.*

*Proof.* We can suppose that $\chi = (1, i, z, \lambda z)$ for some $z = x + iy \in \mathbb{C}$, $\lambda \in \mathbb{R}$. Note that $z \neq 0$, otherwise $\chi(\Gamma)$ would be a lattice. Moreover, $\lambda \in \mathbb{Q}$, because otherwise the group generated by $x, \lambda x$ or $y$, $\lambda y$ would be dense. Thus we can assume that $\lambda = 0$ by the argument of Lemma 4.5. We thus have $\chi = (1, i, z, 0)$. Either $x \notin \mathbb{Q}$ or $y \notin \mathbb{Q}$ since $\mathrm{Vol}(\chi) > 0$ and $\chi(\Gamma)$ is not a lattice. If $x \notin \mathbb{Q}$, apply the transformation $(x_1, y_1 - ky_2, x_2 + kx_1, y_2)$ so that $0 < x < 1$. We apply $(x_1 - x_2, y_1, x_2, y_2 + y_1)$, that replaces $\det(x_1, y_1)$ with $0 < \det(x_1, y_1) - x < 1$. Moreover, $\det(x_1, y_1) - x = 1 - x \neq \frac{1}{2}$ since $x \notin \mathbb{Q}$. If $y \notin \mathbb{Q}$, we can proceed in the same way after applying $(x_1, y_1) \to (y_1, -x_1)$. The result follows since $\det(x_1, y_1) + \det(x_2, y_2) = 1$. $\square$

We are now able to prove Proposition 4.3. Let us normalize the volume: $\mathrm{Vol}(\chi) = 1$.

*Proof.* Iterating Lemma 4.6, and using Lemma 4.7 if necessary, we can assume that $x = \det(x_1, y_1) > 0$ and $\det(x_2, y_2) > 0$, *i.e.* that $\frac{1}{2} \leqslant x < 1$. We can moreover assume that $x \neq \frac{1}{2}$. Indeed if $x = \frac{1}{2}$, use one more time Lemma 4.6, and then Lemma 4.7 if necessary.

It suffices to show that we can have $\frac{2}{3} \leqslant x < 1$. Indeed, in this case, $\frac{\det(x_2, y_2)}{\det(x_1, y_1)} \leqslant \frac{1-x}{x} \leqslant \frac{1}{2} < \epsilon$, and we conclude with Proposition 4.2. If $x < \frac{2}{3}$, then applying Lemma 4.6, we replace $x$ with $x'$ that satisfies $\frac{2}{3} < 1 - \frac{x}{2} \leqslant x' < 1$. $\square$


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I would like to thank my advisor Maxime Wolff for his help, and Bertrand Deroin for his advice. While I was writing the present article, an independent proof of Theorem 1.1 was published online by Bainbridge, Johnson, Judge and Park [1]. I wish to thank them for their encouragement to post the present article. I also thank Julien Marché, Léo Bénard and Thibaut Mazuir for helpful conversations. I am grateful to the anonymous referee for helpful comments and suggestions.


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# Chapitre 5

# Holonomie des structures projectives branchées dont la combinatoire des points de branchement est fixée

Cette section est une reproduction de l'article [LF21]. Le résultat principal de cet article est la classification des holonomies des structures projectives complexes branchées, dont on fixe la combinatoire des points de branchements. Il contient également de nouveaux résultats sur l'action du groupe modulaire sur certains espaces de représentations. En particulier on démontre la proposition 1.2.4.

# Holonomy of complex projective structures on surfaces with prescribed branch data

Thomas Le Fils

**Abstract**

We characterize the representations of the fundamental group of a closed surface to $\mathrm{PSL}_2(\mathbb{C})$ that arise as the holonomy of a branched complex projective structure with fixed branch divisor. In particular, we compute the holonomies of the spherical metrics with prescribed integral conical angles and the holonomies of affine structures with fixed conical angles on closed surfaces.

## 1 Introduction

### 1.1 Holonomy of branched projective structures

For each $g \geqslant 0$ and $n \geqslant 0$, we fix $\Sigma_{g,n}$ a connected compact oriented surface of genus $g$ with $n$ boundary components and $\Gamma_{g,n}$ a fundamental group of $\Sigma_{g,n}$. We will be most interested in the case $n = 0$ and we let $\Gamma = \Gamma_{g,0}$ and $\Sigma = \Sigma_{g,0}$.

A *branched projective structure* on $\Sigma$ is the datum of a branched atlas with values in $\mathbb{CP}^1$ and transition maps in $\mathrm{PSL}_2(\mathbb{C})$. By branched we mean that our charts have the local form $z \mapsto z^{n+1}$ for some integer $n \geqslant 0$, called the *order* of the point corresponding to 0 in this chart. A point of order at least 1 is called a *branch point*. The analytic continuation of one of the charts in the universal cover of $\Sigma$ yields a *developing map* that is $\rho$-equivariant for a unique homomorphism $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ called the *holonomy* of the projective structure. The conjugacy class of this holonomy does not depend on the choice of the chart and thus gives a map defined on the set $\mathcal{BP}$ of branched projective structures on $\Sigma$

$$\mathrm{Hol} : \mathcal{BP} \to \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C}).$$

The map Hol and in particular its restriction $\mathrm{Hol}_{|\mathcal{P}}$ to the set $\mathcal{P}$ of projective structures without any branch point has been extensively studied. For example, it is known that $\mathrm{Hol}_{|\mathcal{P}}$ is a local homeomorphism but not a covering map, see [Hej75]. It is also known since Poincaré [Poi84], see also [LMP09], that the restriction of Hol to the subset of $\mathcal{P}$ of projective structures with a fixed underlying complex structure is injective. A celebrated theorem of Gallo, Kapovich and Marden [GKM00] characterizes the image of $\mathrm{Hol}_{|\mathcal{P}}$, see also [LF19].

**Gallo Kapovich Marden.** Suppose $g \geqslant 2$. The image of $\mathcal{P}$ by Hol is the set of nonelementary representations that lift to $\mathrm{SL}_2(\mathbb{C})$. The image of the set of projective structures with a single branch point, of order 1, is the set of nonelementary representations that do not lift to $\mathrm{SL}_2(\mathbb{C})$.

In this article, we adress a problem raised in [GKM00, Problem 12.6.1]: "make precise and optimize the connection between branching divisor and monodromy". The set $\mathcal{BP}$ is naturally stratified by the sets $\mathcal{P}(n_1, \ldots, n_k)$ of branched projective structure with $k$ branch points of orders $n_1, \ldots, n_k$. The aim of this article is thus to characterize the image of the strata $\mathcal{P}(n_1, \ldots, n_k)$ by Hol.

Recall that an elementary representation is a representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ yielding an action with a finite orbit on $\mathbb{H}^3$ or on $\mathbb{CP}^1$. In the first case we say that $\rho$ is *spherical* and we will assume without loss of generality up to conjugating $\rho$, that it takes its values in $\mathrm{PSU}_2 \simeq \mathrm{SO}_3(\mathbb{R})$. In the second case, if $\rho(\Gamma)$ fixes a point in $\mathbb{CP}^1$ we say that $\rho$ is *affine* and if it moreover fixes a horosphere we say that $\rho$ is *Euclidean*. We will assume without loss of generality that this point is $\infty \in \mathbb{CP}^1$ and thus that $\rho$ takes its values in $\mathrm{Aff}(\mathbb{C}) = \{az + b \mid a \in \mathbb{C}^*, b \in \mathbb{C}\}$. If $\rho(\Gamma)$ fixes globally a pair of points in $\mathbb{CP}^1$, we say that $\rho$ is *dihedral* and we will assume that this pair is $\{0, \infty\}$. We refer to [Rat06, Chapter 5] for more information on elementary subgroups of $\mathrm{PSL}_2(\mathbb{C})$.

We wish to *geometrize* a given $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$, that is to find a projective structure $X \in \mathcal{P}(n_1, \ldots, n_k)$ such that $\rho = \mathrm{Hol}(X)$. There are some obstructions to being the holonomy of such a $X$ that we will now explicit. Indeed suppose that $\rho = \mathrm{Hol}(X)$. It is proven in [GKM00] that $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ if and only if $\sum_i n_i$ is even.

**Obstruction 1.** The sum $\sum_i n_i$ must be even if $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ and odd otherwise.

There exist some surgeries on projective structures, such as bubbling (see Proposition 5.7), that allow us to add an even number of branch points to some projective structures. In particular we will see in Section 6.5 that it is a consequence of the theorem of Gallo, Kapovich and Marden cited above that a nonelementary representation of $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if $\sum_i n_i$ has the required parity. Thus we now focus on elementary representations. As a consequence of the Gauss-Bonnet formula, we have another obstruction on $\sum_i n_i$ when $\rho$ is elementary.

**Obstruction 2.** If $\rho$ is elementary, then we must have $\sum_i n_i \geqslant 2g - 2$. If moreover $\rho$ is spherical, then $\sum_i n_i \geqslant 2g - 1$.

The image of $\rho$ might be a finite group. In this case the developing map induces a branched cover $S \to \mathbb{CP}^1$ defined on the cover $S$ of $\Sigma$ associated with the subgroup $\ker \rho$ of $\Gamma$. The Riemann-Hurwitz formula then gives another obstruction.

**Obstruction 3.** If the image of $\rho$ is a finite group of order $n$, then

$$n(\chi(\Sigma) + \sum_{i=1}^{k} n_i) \geqslant 2(\max_{1 \leqslant i \leqslant k} n_i + 1).$$

Let us define an *affine structure* on $\Sigma$ as a branched projective structure on $\Sigma$ with affine holonomy, such that the image of its developing map does not contain $\infty \in \mathbb{CP}^1$. Adapting an argument of Kapovich in [Kap20], we will see in Proposition 6.18 that if $\rho$ is affine, then $\sum_i n_i = 2g-2$ if and only if $X$ is an affine structure. If moreover $\rho$ is Euclidean, then we may define the volume $\mathrm{Vol}(\rho)$ of $\rho$, see Section 4.2. The volume of $\rho$ must be positive, as we shall see in Section 6.3.2.

**Obstruction 4.** If $\sum_i n_i = 2g-2$ and $\rho$ is Euclidean, then $\mathrm{Vol}(\rho) > 0$.

Haupt [Hau20] observed that there is another obstruction to being the holonomy of a *translation surface*, that is an affine structure with holonomy in the group of translations $\mathbb{C}$. Indeed if $g \geqslant 2$ and $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ is the holonomy of a translation surface such that $\Lambda = \chi(\Gamma)$ is a lattice, then $d = \mathrm{Vol}(\chi)/\mathrm{Area}(\Lambda)$ must be at least 2 because the developing map induces a branched cover $\Sigma \to \mathbb{C}/\Lambda$ of degree $d$. This obstruction was generalized by [CDF15] for translation surfaces with prescribed singularities, where the authors observed that $d$ must satisfy $d \geqslant \max_i n_i + 1$ and asked if these were the only obstructions to being the holonomy of a translation surface with prescribed singularities. This latter question was answered in the positive in [LF20, BJJP20]. The *linear part* of an Euclidean representation $\rho$ is the representation $\mathrm{Li} \circ \rho$, where $\mathrm{Li} : \mathrm{Aff}(\mathbb{C}) \to \mathbb{C}^*$ is defined by $az + b \mapsto a$. If the linear part of $\rho$ takes its values in a finite group, then the cover $S$ of $\Sigma$ associated with the subgroup $\ker \mathrm{Li} \circ \rho$ of $\Gamma$ inherits a translation surface structure. Therefore the obstruction of Haupt gives the following.

**Obstruction 5.** If $\sum_i n_i = 2g-2$ and $\rho$ is Euclidean such that the image of its linear part is a finite group of order $n$ and the group $\Lambda = \{z_0 \in \mathbb{C} \mid z + z_0 \in \rho(\Gamma)\}$ is a lattice in $\mathbb{C}$, then we must have

$$n\mathrm{Vol}(\rho) \geqslant (\max_i n_i + 1)\mathrm{Area}(\mathbb{C}/\Lambda).$$

As we will see, $\Lambda$ can be a lattice only if $n \in \{1, 2, 3, 4, 6\}$.

If $\rho$ is dihedral there is an obstruction that only occurs in genus 2, when $\sum_i n_i$ is minimal: that is when $\sum_i n_i = 2$. This new obstruction comes from the observation that every such structure is the exponential of a half-translation structure. We observe thanks to Obstruction 5 that a half-translation structure in genus 2 with a single branch point is in fact a translation surface structure, see Example 6.26. Therefore we show that there is no branched projective structure with holonomy $\rho$ and a single branch point in genus 2.

**Obstruction 6.** Suppose $g = 2$. If $\rho$ is a dihedral representation that is not affine and that lifts to $\mathrm{SL}_2(\mathbb{C})$, then $X$ cannot have a single branch point of order 2.

Our main result is that these are the only obstructions to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

**Theorem 1.1.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if $\rho$ satisfies the conditions of the 6 obstructions above.*

In particular for a given $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ we can compute the number $d(\rho)$ defined as the minimal value of $\sum_i n_i$ such that there exists a branched projective structure in $\mathcal{P}(n_1, \ldots, n_k)$ with holonomy $\rho$.

**Corollary 1.2.** *The image of the function $d : \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C})) \to \mathbb{Z}$ is the set $\{0, 1, 2g-2, 2g-1, 2g, 2g+2\}$ and we explicit $d$, see Figure 1 at the end of this introduction.*

Our proof of Theorem 1.1 consists in the case-by-case study of the possible $n_1, \ldots, n_k$ such that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$, for each form that $\rho(\Gamma)$ can take. In some of these cases we get back to notions that were already studied in the literature. For example if $\rho(\Gamma) \subset \mathrm{PSU}_2$, then we are characterizing the holonomies of conical spherical metrics on $\Sigma$. These results in some special cases that we now explicit have their own interest and form the pieces of our proof of Theorem 1.1.

### 1.1.1 Spherical structures.

Let us first focus on spherical representations. Theorem 1.1 consists in this case in computing the holonomies of spherical metrics with conical singularities of integral angles. For more information on spherical metrics with conical singularities we refer to [Ere19, MP19].

**Proposition 1.3.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$. The representation $\rho$ is the holonomy of a spherical metric with conical singularities of angles $2\pi(n_1+1), \ldots, 2\pi(n_k+1)$ if and only if the following conditions are met.*

1. *$\sum_i n_i$ is even if $\rho$ lifts to $\mathrm{SU}_2$ and odd otherwise.*
2. *$\sum_i n_i \geqslant 2g-1$.*
3. *If the image of $\rho$ is finite of order $n \geqslant 1$, then*

$$n(\chi(\Sigma) + \sum_{i=1}^{k} n_i) \geqslant 2(\max_{1 \leqslant i \leqslant k} n_i + 1).$$

Observe that a branched projective structure with trivial holonomy is just a branched cover of the sphere.

**Proposition 1.4.** *There exists a branched cover $\Sigma \to \mathbb{S}^2$ with branch points of order $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ if and only if $\sum_i n_i = 2(g+d-1)$ for an integer $d \geqslant 1$ such that $1 \leqslant n_i \leqslant d-1$ for every $1 \leqslant i \leqslant k$.*

Note that our convention differs in this setting from the usual one: by a branch point of order $n$ we mean that the branched cover is locally $z \mapsto z^{n+1}$ while this is often called a branch point of order $n+1$. While the present article was being written, an independent proof of Proposition 1.4 appeared in the work of Chenakkod, Faraco and Gupta [CFG21, Corollary D]. Their proof, as well as ours, consists in showing directly that some branched data are realizable. We do

it by following the techniques of [EKS84], while their proof is more geometric. The case $g = 0$ was also proven, with different methods, independently by Kapovich and Tomasini in [Kap17, Tom15].

### 1.1.2 Affine holonomy.

If we consider affine representations and $\sum_i n_i = 2g - 2$ then we compute the holonomies of affine structures with prescribed singularities.

**Proposition 1.5.** *Let $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ be such that $\sum_i n_i = 2g - 2$. Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$.*

1. *If $\rho$ is not Euclidean then $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.*
2. *If $\rho$ is Euclidean, then $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if both $\mathrm{Vol}(\rho) > 0$ and $\rho$ satisfies the condition of Obstruction 5 if the image of its linear part is finite and has order in $\{1, 2, 3, 4, 6\}$.*

This answers a question raised in [CFG21, Question 1.3]. In fact Proposition 1.5 is due to a large extent to Ghazouani [Gha18], where he studies the action of the mapping class group on $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))/\mathrm{Aff}(\mathbb{C})$ and characterizes the representations of $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ that are the holonomy of an affine structure. Ghazouani does not address the problem of characterizing the representations that are holonomies of affine structures with prescribed branch points. Let us observe that Proposition 1.5 gives a new proof of the fact that the strata of quadratic differentials $\mathcal{Q}(2n_1, \ldots, 2n_k)$ are nonempty except if $g = 1$ and $k = 0$, or $g = 2$ and $k = 1$, see Remark 6.28. This was proven by Masur and Smilie in [MS93], where they study the nonemptiness of all the strata of quadratic differentials. See also [Lan08] for another proof and the study of the connectedness of these strata.

In the case where $\sum_i n_i = 2d$ for $d \geqslant g$, we will see that every $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ that is not spherical is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

## 1.2 Action of the mapping class group on some representation spaces

We will prove a Ehresmann-Thurston principle, following an argument of Kapovich [Kap20, Theorem 2.7].

**Proposition 1.6.** *The sets $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ are open.*

The mapping class group $\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Homeo}_+(\Sigma))$ acts on the space of conjugacy classes of representations $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ and this action preserves the sets $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. This group identifies with the positive outer automorphisms of $\Gamma$, $\mathrm{Out}^+(\Gamma) = \mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$ and acts on $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ by precomposition (see Section 2) where $\mathrm{Aut}^+(\Gamma)$ is the index 2 subgroup of $\mathrm{Aut}(\Gamma)$ of automorphisms induced by orientation preserving homeomorphisms. In order to geometrize a representation with prescribed branch data, it suffices

by Proposition 1.6 to geometrize any element of the closure of its orbit under the action of $\mathrm{Mod}(\Sigma)$. Therefore our proof of Theorem 1.1 is based on the description of the closure of the orbits of this action. Indeed this description will leave us with a small number of explicit representations to geometrize.

The study of the mapping class group action on character varieties $\mathrm{Hom}(\Gamma, G)/G$, where $G$ is a group, was pioneered by Goldman and is a wide and active research area. We refer to [Gol03] for an introduction to this subject and [Can15] and references therein for more information on this domain of research. In particular Goldman proved in [Gol97] that the action of the mapping class group on $\mathrm{Hom}(\Gamma, \mathrm{SU}_2)/\mathrm{SU}_2$ is ergodic. While this result provides a description of the closure of almost every orbit, it does not allow us to understand the closure of a given orbit. The description of the closure of the orbit of every representation in $\mathrm{Hom}(\Gamma, \mathrm{SU}_2)/\mathrm{SU}_2$ with dense image was achieved by Previte and Xia in [PX00, PX02]. A consequence of their work is that the orbit of a representation in $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ with image a dense subgroup of $\mathrm{SO}_3(\mathbb{R})$ is dense in its connected component of $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))/\mathrm{SO}_3(\mathbb{R})$, see Lemma 4.4. We study the other orbits of representations in $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ and give a complete description of their closures. This somehow complements the main result of [PX02] in the case of closed surfaces. Let $\epsilon : \mathrm{O}_2(\mathbb{R}) \to \mathrm{O}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) = \mathbb{Z}_2$ be the quotient homomorphism. Goldman showed in [Gol88] that $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ endowed with the pointwise convergence topology has two connected components. Representations in one of these connected components lift to $\mathrm{SL}_2(\mathbb{C})$ while the others do not. Let $\mathrm{sw} : \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C})) \to \{0, 1\}$ be the map that assigns 0 to the representations that lift to $\mathrm{SL}_2(\mathbb{C})$ and 1 to the others. The proper closed subgroups of $\mathrm{SO}_3(\mathbb{R})$ are isomorphic to $\mathrm{SO}_2(\mathbb{R})$, $\mathrm{O}(2)$, the cyclic groups $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, the dihedral groups $\mathrm{D}_n$ of order $2n$, the alternating groups $\mathfrak{A}_4$ and $\mathfrak{A}_5$ and the symmetric group $\mathfrak{S}_4$. Let us start with $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ with infinite image. After conjugating $\rho$, we may assume that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{O}_2(\mathbb{R}))$.

**Proposition 1.7.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{O}_2(\mathbb{R})))$ with infinite image. The closure of $\mathrm{Aut}^+(\Gamma) \cdot \rho$ is the set of representations $\rho_\infty \in \mathrm{Hom}(\Gamma, \mathrm{O}_2(\mathbb{R}))$ such that $\epsilon \circ \rho_\infty$ has the same image as $\epsilon \circ \rho$ and $\mathrm{sw}(\rho_\infty) = \mathrm{sw}(\rho)$.*

We also describe the orbits of representations with finite image in $\mathrm{SO}_3(\mathbb{R})$.

**Proposition 1.8.** *Suppose $g \geqslant 2$. The action $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ is transitive for $G = \mathbb{Z}_n$ or $G = \mathrm{D}_n$ with $n$ odd. There are two orbits for $G = \mathfrak{A}_4, \mathfrak{A}_5, \mathfrak{S}_4, \mathrm{D}_n$ with $n$ even.*

Edmonds showed in [Edm82, Edm83] that if $G$ is abelian or if $G$ is a semidirect product of cyclic groups then two surjective representations $\rho_1$ and $\rho_2$ are in the same orbit under the action of $\mathrm{Aut}^+(\Gamma)$ if and only if $\rho_1([\Sigma]) = \rho_2([\Sigma]) \in \mathrm{H}_2(G, \mathbb{Z})$, see Section 3 for more details on these notations. In particular this proves Proposition 1.8 in the case $G = \mathbb{Z}_n$ and $G = \mathrm{D}_n$. Dunfield and Thurston also showed in [DT06, Section 6] that given a finite group $G$, the action of $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ has $|\mathrm{H}_2(G, \mathbb{Z})|$ orbits for large enough $g$. A consequence of Proposition 1.8 is that in these particular cases, it happens for every $g \geqslant 2$.

Let us now describe the organization of the paper. In Section 2 we recall some well-known facts about fundamental groups of surfaces, their automorphism groups and their representations. In Section 3 we study the action of the mapping class group on the space of conjugacy classes of some representations with finite image. More precisely we prove Proposition 1.8. In Section 4 we provide a description of some orbits of elementary representations with infinite image. In particular we prove Proposition 1.7 and exhibit some particularly simple representations in the closure of the orbits of affine representations. In Section 5 we recall the definition of branched projective structures and prove Proposition 1.6. We also explain some surgeries on these structures that allow us to construct new projectives structures from old ones. In Section 6 we prove Theorem 1.1: we explain the origins of the obstructions and show that a representation satisfying the conditions of these obstructions is *geometric*: it is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. Finally in Section 7 we explicit the function $d$ and prove Corollary 1.2.

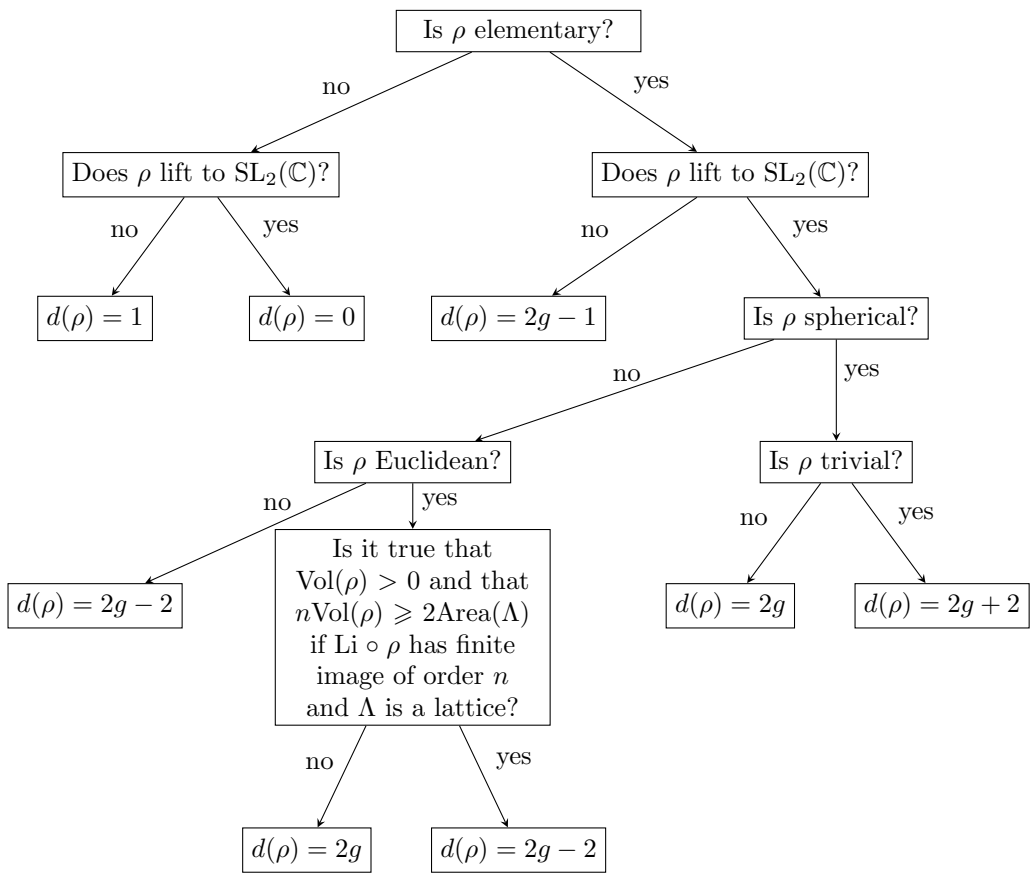


Figure 1: The function $d : \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C})) \to \mathbb{Z}$.

**Ackowledgements**

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# 2 Reminder on the mapping class group

## 2.1 Fundamental groups of surfaces

For each $g, n \geqslant 0$, we fix $\Sigma_{g,n}$ a compact oriented surface of genus $g$ with $n$ boundary components. We also choose a basepoint in each of these surfaces and thus a fundamental group $\Gamma_{g,n}$. Let us fix standard generators $a_1, \ldots, b_g, c_1, \ldots, c_n$ for $\Gamma_{g,n}$, as in Figure 2. These standard generators yield the following presen-

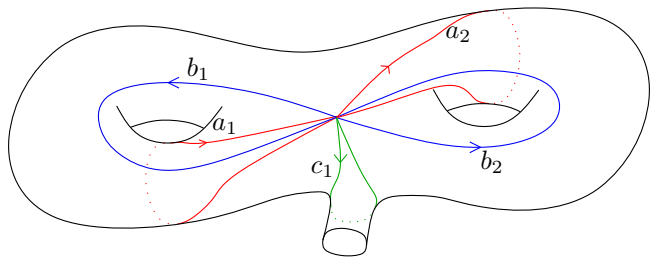


Figure 2: Standard generators of $\Gamma_{2,1}$.

tation of $\Gamma_{g,n}$

$$\Gamma_{g,n} = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \mid \prod_{i=1}^{g} [a_i, b_i] = c_1 \ldots c_n \rangle.$$

In particular for $n = 0$, we have

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = \mathrm{id} \rangle.$$

Recall that every $\sigma \in \mathrm{Aut}(\Gamma)$ is induced by a continuous map $f : \Sigma \to \Sigma$ preserving the basepoint of $\Gamma$. We denote by $\mathrm{Aut}^+(\Gamma)$ the index 2 subgroup of $\mathrm{Aut}(\Gamma)$ induced by the $f$ preserving the orientation of $\Sigma$. Let us denote by $\mathrm{Homeo}_0(\Sigma)$ the connected component of the identity of $\mathrm{Homeo}_+(\Sigma)$, the set of positive homeomorphisms of $\Sigma$, equipped with the compact-open topology. It is the set of homeomorphisms of $\Sigma$ isotopic to the identity, see [FM01, Chapter 2]. The mapping class group is the group of $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ of isotopy class of positive homeomorphisms of $\Sigma$. Every $f \in \mathrm{Mod}(\Sigma)$ induces an outer automorphism of $\Gamma$ by the action on $\Gamma$ of one of its representatives that fixes the basepoint of $\Gamma$. The theorem of Dehn, Nielsen and Baer, see [FM01, Chapter 6], states that this homomorphism $\mathrm{Mod}(\Sigma) \to \mathrm{Out}^+(\Sigma)$ is an isomorphism, where $\mathrm{Out}^+(\Gamma) = \mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$ and $\mathrm{Inn}(\Gamma)$ denotes the set of inner automorphisms of $\Gamma$.

## 2.2 Action on representation spaces

Let $G$ be a group. The group $\mathrm{Aut}^+(\Gamma)$ acts on the set of representations $\mathrm{Hom}(\Gamma, G)$ by precomposition. The group $G$ also acts by conjugation on $\mathrm{Hom}(\Gamma, G)$ and these two actions commute. Therefore $\mathrm{Aut}^+(\Gamma) \times G$ acts on $\mathrm{Hom}(\Gamma, G)$. The group $\mathrm{Inn}(\Gamma)$ of inner automorphisms of $\Gamma$ acts trivially on $\mathrm{Hom}(\Gamma, G)/G$. Hence we have an action of $\mathrm{Out}^+(\Gamma)$ on $\mathrm{Hom}(\Gamma, G)/G$. Therefore the group $\mathrm{Mod}(\Sigma)$ acts on $\mathrm{Hom}(\Gamma, G)/G$. Note that we may restrict both the action of $\mathrm{Aut}^+(\Gamma)$ and the action of $\mathrm{Out}^+(\Gamma)$ on the set of (conjugacy classes) of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$.

*Remark* 2.1. Two surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ are in the same orbit under the action of $\mathrm{Aut}^+(\Gamma) \times G$ if and only if they are in the same orbit under the action of $\mathrm{Aut}^+(\Gamma)$.

In our study of the $\mathrm{Aut}^+(\Gamma)$ action on $\mathrm{Hom}(\Gamma, G)$, we will often consider the restriction of a homomorphism $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho$ kills $c = \prod_{i=1}^{k}[a_i, b_i]$ to $\mathrm{Hom}(\Gamma_{g-k}, G)$. The following lemma ensures that we may then study the action of $\mathrm{Aut}^+(\Gamma_{g-k})$ on this restriction to understand the orbit of $\rho$ under the action of $\mathrm{Aut}^+(\Gamma)$.

**Lemma 2.2.** *Let $\rho \in \mathrm{Hom}(\Gamma, G)$ and $c$ be a simple separating curve in $\Sigma$ such that $\rho(c) = \mathrm{id}$. The curve $c$ induces a splitting $\Sigma = \Sigma_h \sharp \Sigma_l$, where $h + l = g$. For every $\sigma \in \mathrm{Aut}^+(\Gamma_h)$, the representation that we denote by $\sigma \cdot \rho$ satisfying:*

$$\sigma \cdot \rho(\gamma) = \begin{cases} \rho(\sigma \cdot \gamma) & \text{if } \gamma \in \pi_1(\Sigma_h) \\ \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma_l) \end{cases}$$

*lies in* $\mathrm{Aut}^+(\Gamma) \cdot \rho$.

*Proof.* Let us fix such a curve $c$. The set $\mathcal{A}$ of $\sigma \in \mathrm{Aut}^+(\Gamma_h)$ satisfying this property for every $\rho$ such that $\rho(c) = \mathrm{id}$ is stable by multiplication. It contains the automorphisms induced by Dehn twists along nonseparating simple closed curves contained in $\Sigma_h$. The set $\mathcal{A}$ also contains the inner automorphisms of $\Gamma_h$: if $\delta$ is a nonseparating simple closed curve in $\Sigma_h$, then the composition of a Dehn twist along a curve homotopic to $\delta$ and of a Dehn twist along a curve homotopic to $c^{\pm 1}\delta$ conjugates $\rho(\gamma)$ with $\rho(\delta)$ for $\gamma \in \pi_1(\Sigma_h)$ and leaves $\rho(\gamma)$ unchanged for $\gamma \in \pi_1(\Sigma_l)$. Since the Dehn twists along nonseparating simple closed curves generate $\mathrm{Out}^+(\Gamma_h)$, see [FM01, Chapter 4], we have $\mathcal{A} = \mathrm{Aut}^+(\Gamma_h)$. □

Observe that if $G$ is an abelian group, then any homomorphism $\rho \in \mathrm{Hom}(\Gamma, G)$ factors through a homomorphim in $\mathrm{Hom}(\mathrm{H}_1(\Sigma, \mathbb{Z}), G)$ since the abelianization of $\Gamma$ is $\mathrm{H}_1(\Sigma, \mathbb{Z})$. The group $\mathrm{H}_1(\Sigma, \mathbb{Z})$ is the free abelian group on the generators $[a_i], [b_i]$. Therefore, we may identify $\mathrm{Hom}(\Gamma, G)$ with $G^{2g}$, by the map $\rho \mapsto (\rho(a_1), \ldots, \rho(b_g))$. The group $\mathrm{Mod}(\Sigma)$ acts on $\mathrm{H}_1(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}^{2g}$ and it is well known that this action is made by the full symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$, see [FM01, Chapter 6]. The action of $\mathrm{Mod}(\Sigma) \simeq \mathrm{Out}^+(\Gamma)$ on $\mathrm{Hom}(\Gamma, G) \simeq G^{2g}$ is thus by $\mathrm{Sp}_{2g}(\mathbb{Z})$ that acts on $G^{2g}$ seen as a $\mathbb{Z}$-module.

# 3 Mapping class group orbits on representation spaces into some finite groups

In this section we study the orbits of the $\mathrm{Aut}^+(\Gamma)$ action on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ for some finite groups $G$. Observe that the projection $\mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G)/G$ induces a bijection between the $\mathrm{Aut}^+(\Gamma)$ orbits of surjective homomorphisms and the $\mathrm{Mod}(\Sigma)$ orbits of conjugacy class of surjective homomorphisms by Remark 2.1. Let us recall the classification of these orbits when the genus is large enough, from [DT06, Theorem 6.23]. There is a natural map $\Phi : \mathrm{Hom}(\Gamma, G) \to \mathrm{H}_2(G, \mathbb{Z})$ defined as follows. Every $\rho \in \mathrm{Hom}(\Gamma, G)$ induces a map in homology $\rho_\star : \mathrm{H}_2(\Gamma, \mathbb{Z}) \to \mathrm{H}_2(G, \mathbb{Z})$ and we define $\Phi(\rho)$ to be $\rho_\star([\Sigma]) \in \mathrm{H}_2(G, \mathbb{Z})$, where $[\Sigma] \in \mathrm{H}_2(\Sigma, \mathbb{Z}) = \mathrm{H}_2(\Gamma, \mathbb{Z})$ is the fundamental class of $\Sigma$. This map is invariant under the action of $\mathrm{Aut}^+(\Gamma)$, since every positive automorphism of $\Gamma$ maps $[\Sigma]$ to itself.

**Theorem 3.1** (Dunfield-Thurston)**.** *Let $G$ be a finite group. There exists $g_0 \geqslant 0$ such that the natural map from the orbits of surjective homomorphisms of $\mathrm{Hom}(\Gamma_g, G)$ under the action of $\mathrm{Aut}^+(\Gamma)$ to $\mathrm{H}_2(G, \mathbb{Z})$ is a bijection for every $g \geqslant g_0$.*

The proof of Dunfield and Thurston is not constructive and does not give any bound on the minimal $g_0$ required. We will see that $g_0 = 1$ in the case $G = \mathbb{Z}_n$ and that $g_0 = 2$ in the case $G \in \{\mathrm{D}_n, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5\}$. There does not exist any surjective homomorphism in $\mathrm{Hom}(\Gamma_1, G)$ for $G \in \{\mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5\}$ and for $G = \mathrm{D}_n$, $n \geqslant 3$, since these groups are not abelian. Moreover, the image of the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_1, \mathrm{D}_2)$ by $\Phi$ is not the whole group $\mathrm{H}_2(\mathrm{D}_2, \mathbb{Z})$. Hence $g_0 \geqslant 2$ for these groups. However for $g \geqslant 2$ and these groups $G$, the image of the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ by $\Phi$ is surjective. We may check that $|\mathrm{H}_2(G, \mathbb{Z})| \leqslant 2$. We are going to show that the action of $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms with fixed image by $\Phi$ is transitive.

## 3.1 Cyclic groups

Let us start with the case $G = \mathbb{Z}_n$, the cyclic group of order $n \geqslant 2$.

**Proposition 3.2.** *The action of $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$ is transitive.*

This follows from the work of Edmonds in [Edm82], since $\mathrm{H}_2(\mathbb{Z}_n, \mathbb{Z})$ is trivial. Indeed the main result of [Edm82] is the following.

**Theorem 3.3** (Edmonds)**.** *If $G$ is a finite abelian group, then the natural map from the orbits of the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ under the action of $\mathrm{Aut}^+(\Gamma)$ to $\mathrm{H}_2(G, \mathbb{Z})$ is injective.*

This result for the special case $G = \mathbb{Z}_n$ even goes back to Nielsen [Nie37]. Let us nevertheless give a short proof of Proposition 3.2.

*Proof.* Recall from Section 2 that $\mathrm{Aut}^+(\Gamma)$ acts as $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{Hom}(\Gamma, \mathbb{Z}_n) \simeq (\mathbb{Z}_n)^{2g}$. Let us lift $\rho \in \mathrm{Hom}(\Gamma, \mathbb{Z}_n)$ to $\tilde{\rho} \in \mathrm{Hom}(\Gamma, \mathbb{Z}^{2g}) \simeq \mathbb{Z}^{2g}$. It is well known that the orbits of the $\mathrm{Sp}_{2g}(\mathbb{Z})$ action on $\mathbb{Z}^{2g}$ are classified by the greatest common divisor of each of its elements, see for example [FM01, Proposition 6.2]. Hence we may assume that $\tilde{\rho} = (k, 0, \ldots, 0)$ with $k \in \mathbb{Z}$. Note that $\gcd(k, n) = 1$, since $\rho$ is surjective. The homomorphism $\rho' = (k, n, 0, \ldots, 0) \in \mathbb{Z}^{2g}$ is another lift of $\rho$ and with the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$, one can change it to $\rho' = (1, 0, \ldots, 0)$. We now have $\rho(a_1) = 1$, $\rho(b_1) = 0$ and $\rho(a_i) = \rho(b_i) = 0$ for all $i \geqslant 2$. □

## 3.2 Dihedral groups

In this section we consider the case where $G = \mathrm{D}_n$ is the dihedral group of order $2n$, where $n \geqslant 2$, generated by a rotation of the plane of order $n$ and a symmetry $s$ along an axis of the Euclidean plane. Recall that $\epsilon \in \mathrm{Hom}(\mathrm{O}_2(\mathbb{R}), \mathbb{Z}_2)$ denotes the quotient homomorphism $\mathrm{O}_2(\mathbb{R}) \to \mathrm{O}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Observe that there exists a natural isomorphism $\mathrm{D}_n \simeq \mathbb{Z}_2 \rtimes \mathbb{Z}_n$ and that $\epsilon$ restricted to $\mathrm{D}_n$ is the composition of this isomorphism with the projection $\mathbb{Z}_2 \rtimes \mathbb{Z}_n \to \mathbb{Z}_2$. Note that for every $x \in \ker \epsilon$ and every $y \in \mathrm{D}_n \setminus \ker \epsilon$, we have $y^{-1}xy = x^{-1}$ and $y = y^{-1}$.

Let us first deal with the case $g = 1$.

**Proposition 3.4.** *If $g = 1$, then there is no surjective homomorphism $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ unless $n = 2$, in which case the action of $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ is transitive.*

*Proof.* If there exists a surjective homomorphism in $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ then $\mathrm{D}_n$ must be abelian and $n = 2$. The set $\mathrm{Hom}(\Gamma, \mathrm{D}_2)$ has cardinality 16. Among those 16 representations, 6 are surjective. Indeed this set is in bijection with the bases of the $\mathbb{F}_2$ vector space $\mathbb{F}_2 \oplus \mathbb{F}_2$ since $\mathrm{D}_2 \simeq \mathbb{Z}_2^2$. We leave it to the reader to check that the action of $\mathrm{Aut}^+(\Gamma)$ on this set is transitive. □

We now suppose until the end of this section that $g \geqslant 2$.

**Proposition 3.5.** *The action of the group $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ has two orbits if $n$ is even and one if $n$ is odd.*

This follows from this theorem of Edmonds which is the main result of [Edm83].

**Theorem 3.6** (Edmonds)**.** *Let $G$ be a split metacyclic group. The natural map from the $\mathrm{Aut}^+(\Gamma)$ orbits of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ to $\mathrm{H}_2(G, \mathbb{Z})$ is injective.*

The group $\mathrm{D}_n$ is indeed a semi-direct product of cyclic groups: it inserts in the split exact sequence

$$1 \to \mathbb{Z}_n \to \mathrm{D}_n \xrightarrow{\epsilon} \mathbb{Z}_2 \to 1.$$

However for the sake of completeness we will give a direct proof of Proposition 3.5. Let us describe these orbits. We define $\rho \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$ as follows. Let

$\rho(a_1) \in \ker \epsilon$ be an order $n$ rotation and $\rho(b_1) = \mathrm{id}$. We let $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for $1 < i < g$ and $\rho(a_g) = \mathrm{id}$ and $\rho(b_g) = s$. If $n$ is even, we define $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$ by $\rho'(\gamma) = \rho(\gamma)$ for $\gamma \in \{a_1, \ldots, b_g\} \setminus \{a_g\}$ and by $\rho'(a_g) = \rho(a_1)^{n/2}$. Observe that $s$ and $\rho(a_1)^{n/2}$ commute since $s\rho(a_1)^{n/2} = \rho(a_1)^{-n/2}s = \rho(a_1)^{n/2}s$. Therefore $\rho'$ is well-defined. The representations $\rho$ and $\rho'$ are not in the same orbit under the action of $\mathrm{Aut}^+(\Gamma)$ since $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ while $\rho'$ does not because of the following lemma.

**Lemma 3.7.** *Let $\alpha, \beta \in \mathrm{PSL}_2(\mathbb{C})$ be order 2 elements that commute. Either $\alpha = \beta$ or $[\tilde{\alpha}, \tilde{\beta}] = -\mathrm{Id}$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts of $\alpha$ and $\beta$ in $\mathrm{SL}_2(\mathbb{C})$.*

*Proof.* Suppose $\alpha \neq \beta$. We may conjugate $\alpha$ and $\beta$ so that $\alpha = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The map $\beta$ must interchange the fixed points of $\alpha$ in $\mathbb{CP}^1$, thus $\beta = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have $[\tilde{\alpha}, \tilde{\beta}] = -\mathrm{Id}$. □

We turn to the proof of Proposition 3.5.

*Proof.* Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$ with image $\mathrm{D}_n$. The homomorphism $\epsilon \circ \rho \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ is onto, hence by Proposition 3.2, we may assume that $\epsilon \circ \rho(b_g) = 1$, $\epsilon \circ \rho(a_g) = 0$ and $\epsilon \circ \rho(a_i) = \epsilon \circ \rho(b_i) = 0$ for $i < g$. For $i < g$, both $\rho(a_i)$ and $\rho(b_i)$ are in $\mathrm{SO}_2(\mathbb{R})$, hence $\rho([a_i, b_i]) = \mathrm{id}$. Therefore $\rho$ restricts to a homomorphism $\mathrm{Hom}(\Gamma_{g-1}, \mathrm{SO}_2(\mathbb{R}))$ with finite image. By Proposition 3.2 again, we may assume that $\rho(a_1)$ generates the group $\rho(\Gamma_{g-1})$, that $\rho(b_1) = \mathrm{id}$ and that $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for $1 < i < g$. We claim that $\rho(a_1)$ and $\rho(a_g)$ generate $\ker \epsilon$. Indeed every $x \in \ker \epsilon$ is a word in $\rho(a_1)$, $\rho(a_g)$ and $\rho(b_g)$ since $\rho$ is onto. Such a word contains an even number of $\rho(b_g)$ because $x \in \ker \epsilon$. Since $\rho(b_g)^2 = \mathrm{id}$ and $\rho(b_g)\rho(a_i) = \rho(a_i)^{-1}\rho(b_g)$ for all $i$, we can write $x$ without any $\rho(b_g)$. We may apply a Dehn twist along a curve homotopic to $a_1 a_g$ to replace $\rho(b_1) = \mathrm{id}$ with $\rho(a_1)\rho(a_g)$. Applying again an automorphism of $\Gamma_{g-1}$, we may assume that $\rho(a_1)$ is a chosen generator of $\ker \epsilon$, while $\rho(b_1) = \mathrm{id}$ and $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for $1 < i < g$. Let us write $\rho(b_g) = \rho(a_1)^k s$. Let us show that we may assume that $k$ is even. If $k$ is odd and $\rho(a_g) = \rho(a_1)^l$ with $\ell$ odd, then we apply a Dehn twist along a curve homotopic to $a_g$ so that $\rho(b_g)$ is replaced with $\rho(a_g)\rho(b_g) = \rho(a_1)^{k+l}s$. If $\rho(a_g) = \rho(a_1)^l$ with $\ell$ even, then apply a Dehn twist along a curve homotopic to $a_g a_1$. It replaces $\rho(b_g)$ with $\rho(b_g)\rho(a_g)\rho(a_1) = \rho(a_1)^{k+l+1}s$. We may apply an automorphism of $\Gamma_{g-1}$ so that $\rho(b_1) = \mathrm{id}$ again. We now assume that $\rho(b_g) = \rho(a_1)^k s$, with $k = 2k'$. We conjugate $\rho$ with $\rho(a_1)^{k'}$ so that $\rho(b_g) = s$. Since $\rho([a_g, b_g]) = \mathrm{id}$, we have $\rho(b_g)^{-1}\rho(a_g)^{-1}\rho(b_g)\rho(a_g) = \rho(a_g)^2 = \mathrm{id}$. Since $\rho(a_g)$ is in the cyclic group $\ker \epsilon$, either $\rho(a_g) = \mathrm{id}$, or $n$ is even and $\rho(a_g) = \rho(a_1)^{n/2}$. □

## 3.3 The group $\mathfrak{A}_4$

Let us now classify the orbits of the action of $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, G)$ where $G = \mathfrak{A}_4$.

**Proposition 3.8.** *The action of* $\mathrm{Aut}^+(\Gamma)$ *on the set of surjective* $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ *has two orbits.*

Recall that the group $\mathrm{K} = \{\sigma \in \mathfrak{A}_4 \mid \sigma^2 = \mathrm{id}\}$ is an order 4 normal subgroup of $\mathfrak{A}_4$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The quotient $\mathfrak{A}_4/\mathrm{K}$ has order 3; hence there exists a surjective homomorphism $\varphi : \mathfrak{A}_4 \to \mathbb{Z}_3$.

Let us describe a representative of one of these two orbits. Let $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ be such that $\rho(a_1)$ is 3-cycle, $\rho(b_1) = \rho(a_2) = \mathrm{id}$ and $\rho(b_2) \in \mathrm{K} \setminus \{\mathrm{id}\}$ and $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for $3 \leqslant i \leqslant g$. As a first step to prove Proposition 3.8, we show that all the representations of this form lie in the same orbit under the action of $\mathrm{Aut}^+(\Gamma)$.

**Lemma 3.9.** *For every* $\rho_0$ *and* $\rho_1$ *of this form, there exists* $\sigma \in \mathrm{Aut}^+(\Gamma)$ *such that* $\sigma \cdot \rho_0 = \rho_1$.

*Proof.* By Remark 2.1 it suffices to show that they are in the same orbit under the action of $\mathrm{Aut}^+(\Gamma) \times \mathfrak{A}_4$. We may conjugate $\rho_0$ so that $\rho_0(a_1) \in \{\rho_1(a_1), \rho_1(a_1)^{-1}\}$. We can even suppose $\rho_0(a_1) = \rho_1(a_1)$, replacing $\rho_0$ with $\sigma \cdot \rho_0$ by Proposition 3.2, where $\sigma \in \Gamma_1$ following the notation of Lemma 2.2. Conjugating $\rho_0$ with a power of $\rho_0(a_1)$ replaces $\rho_0$ with $\rho_2$. Indeed we have

$$\{\rho_0(a_1)^k \rho_0(b_2) \rho_0(a_1)^{-k} \mid 0 \leqslant k \leqslant 2\} = \mathrm{K} \setminus \{\mathrm{id}\}. \qquad \square$$

Let us now describe the other orbit. Let $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ be such that $\rho(a_1)$ is a 3-cycle, $\rho(b_1) = \mathrm{id}$ and $\rho(a_2)$ and $\rho(b_2)$ are distinct double transpositions and $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for $3 \leqslant i \leqslant g$. Observe that such a representation cannot be in the same orbit as the previous one, since it does not lift to $\mathrm{SL}_2(\mathbb{C})$ by Lemma 3.7.

**Lemma 3.10.** *If* $\rho_0$ *and* $\rho_1$ *have this form, then there exists* $\sigma \in \mathrm{Aut}^+(\Gamma)$ *such that* $\sigma \cdot \rho_0 = \rho_1$.

*Proof.* As before we can assume that $\rho_0(a_1) = \rho_1(a_1)$. The action of $\Gamma_1$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_1, \mathrm{K})$ is transitive, since $\mathrm{K} \simeq \mathrm{D}_2$. Therefore, we may replace $\rho_0$ with $\sigma \cdot \rho_0$ so that $\rho_0 = \rho_1$, where $\sigma \in \Gamma_1$, following the notations of Lemma 2.2. $\square$

We now show that any representation may be taken to the form of Lemma 3.9 or Lemma 3.10. This completes the proof of Proposition 3.8.

*Proof.* Let $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ with image $\mathfrak{A}_4$. The homomorphism $\varphi \circ \rho$ in $\mathrm{Hom}(\Gamma, \mathbb{Z}_3)$ is surjective. We have seen in Proposition 3.2 that the action of $\mathrm{Aut}^+(\Gamma)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, \mathbb{Z}_3)$ is transitive. Hence we may assume that $\varphi \circ \rho(a_i) = \varphi \circ \rho(b_i) = 0$ for all $2 \leqslant i \leqslant g$ and that $\varphi \circ \rho(a_1) = 1$ and $\varphi \circ \rho(b_1) = 0$. For all $2 \leqslant i \leqslant g$, $\rho([a_i, b_i]) = \mathrm{id}$ since $\rho(a_i)$ and $\rho(b_i)$ are in the abelian group K. Since $\varphi \circ \rho(a_1)$ has order 3, $\rho(a_1)$ itself has order 3 and $\rho(a_1)$ is a 3-cycle. Moreover $\rho(b_1) \in \mathrm{K}$ and $\rho(b_1)$ commutes with $\rho(a_1)$. The centralizer of $\rho(b_1)$ is $\mathfrak{A}_4$ thus $\rho(b_1) = \mathrm{id}$ since $\mathfrak{A}_4$ is centerless. We can restrict $\rho$ to the subsurface obtained by collapsing the first

handle and define $\rho' \in \mathrm{Hom}(\Gamma_{g-1}, \mathrm{K})$. The image $\rho'$ cannot be the trivial group, since otherwise $\rho(a_1)$ would generate $\mathfrak{A}_4$. Let us first assume that the image of $\rho'$ is cyclic of order 2. The action of $\mathrm{Aut}^+(\Gamma_{g-1})$ on the set of surjective $\rho' \in \mathrm{Hom}(\Gamma_{g-1}, \mathbb{Z}_2)$ has only one orbit, hence we may assume that $\rho(a_2)$ has order 2, while $\rho(a_i) = \rho(b_i) = \mathrm{id}$ otherwise for $i \geqslant 2$. We now turn to the case where the image of $\rho'$ is K. The action of $\mathrm{Aut}^+(\Gamma_{g-1})$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_{g-1}, \mathrm{K})$ has at most two orbits by Proposition 3.5 since K is isomorphic to $\mathrm{D}_2$. If $\rho'$ does not lift to $\mathrm{SL}_2(\mathbb{C})$, then we may assume that $\rho'(a_2)$ and $\rho'(b_2)$ are different order 2 elements and that $\rho'(a_i) = \rho'(b_i) = \mathrm{id}$ for $3 \leqslant i \leqslant g$. If $\rho'$ lifts to $\mathrm{SL}_2(\mathbb{C})$, then $g \geqslant 3$ and we may assume that $\rho'(a_2)$ and $\rho'(a_3)$ are different elements of K and that $\rho(a_i) = \rho(b_i) = \mathrm{id}$ otherwise for $3 \leqslant i \leqslant g$. After applying Lemma 3.9 in the first two handles, we may assume that $\rho(a_2) = \rho(a_3)$ and get back to the case where the image of $\rho'$ is cyclic. In each case, we can thus assume that $\rho(\gamma) = \mathrm{id}$ for $\gamma \in \{a_3, \ldots, b_g\}$ and that the restriction of $\rho$ on the first two handles has the form of Lemma 3.9 or Lemma 3.10. □

## 3.4 The group $\mathfrak{S}_4$

We turn to the case $G = \mathfrak{S}_4$. As before there are two orbits of class of surjective homomorphisms in $\mathrm{Hom}(\Gamma, G)/G$ under the action of $\mathrm{Mod}(\Sigma)$.

**Proposition 3.11.** *There are two orbits of surjective homomorphisms of* $\mathrm{Hom}(\Gamma, \mathfrak{S}_4)$ *under the action of* $\mathrm{Aut}^+(\Gamma)$.

The group $\mathrm{K} = \{\sigma \in \mathfrak{A}_4 \mid \sigma^2 = 1\}$ is also normal in $\mathfrak{S}_4$ and the quotient group $\mathfrak{S}_4/\mathrm{K}$ is isomorphic to $\mathfrak{S}_3 \simeq \mathrm{D}_3$. Let us denote by $\varphi$ a surjective homomorphism of $\mathrm{Hom}(\mathfrak{S}_4, \mathfrak{S}_3)$. Let us first suppose that $g = 2$.

**Lemma 3.12.** *There exists a nonseparating simple closed curve* $\gamma \in \Gamma$ *such that* $\rho(\gamma) = \mathrm{id}$.

*Proof.* We suppose that $\rho(b_1) \neq \mathrm{id}$ and $\rho(b_2) \neq \mathrm{id}$ since otherwise we are done. Since $\rho(a_1)$ is a 3-cycle, it does not commute with $\rho(b_1) = \rho(b_1)^{-1}$, thus $\rho(b_1)$, $\rho(a_1^{-1}b_1^{-1}a_1)$ and $\rho(a_1b_1^{-1}a_1^{-1})$ are 3 distinct elements of $\mathrm{K} \setminus \{\mathrm{id}\}$. One of them is therefore equal to $\rho(b_2) = \rho(b_2)^{-1}$. The loop $b_2b_1$ in the first case, $b_2a_1^{-1}b_1^{-1}a_1$ in the second case and $b_2a_1b_1^{-1}a_1^{-1}$ in the last one is sent to id and is represented by a simple nonseparating curve. □

We suppose thanks to Lemma 3.12 that $\rho(b_1) = \mathrm{id}$. We also assume that $\rho(a_1) \in \mathfrak{A}_4$. Indeed, if $\rho(a_1) \notin \mathfrak{A}_4$, then suppose $\rho(a_2)$ or $\rho(b_2)$ is not in $\mathfrak{A}_4$. We may assume that $\rho(b_2) \notin \mathfrak{A}_4$ and replace $\rho(a_1)$ with $\rho(b_2)\rho(a_1)$ with a Dehn twist along a curve homotopic to $b_1b_2$. Suppose now that both $\rho(a_2)$ and $\rho(b_2)$ are in $\mathfrak{A}_4$. Since $[\rho(a_2), \rho(b_2)] = \mathrm{id}$, the group generated by $\rho(a_2)$ and $\rho(b_2)$ is either cyclic or K. It cannot be K since one can check that $\mathfrak{S}_4$ is not generated by K together with a single $\sigma \in \mathfrak{S}_4$. Since the group generated by $\rho(a_2)$ and $\rho(b_2)$ is cyclic, we may assume that $\rho(b_2) = \mathrm{id}$ and, interchanging the handle, that $\rho(a_1) \in \mathfrak{A}_4$.

Since $[\rho(a_2), \rho(b_2)] = \mathrm{id}$, and $\rho(a_2)$ and $\rho(b_2)$ are not both in $\mathfrak{A}_4$, we may assume that one of the following holds:

1. $\rho(a_2)$ is a 4-cycle and $\rho(b_2) = \mathrm{id}$
2. $\rho(a_2)$ is a transposition and $\rho(b_2) = \mathrm{id}$
3. $\rho(a_2)$ and $\rho(b_2)$ are two distinct transpositions that commute.

We leave it to the reader to check that such $\rho(a_2)$ and $\rho(b_2)$ together with a double transposition do not generate $\mathfrak{S}_4$. Therefore $\rho(a_1)$ is a 3-cycle. If we are in the second case above, then a Dehn twist replaces $\rho(a_2)$ with $\rho(a_2 a_1)$, which is a 4-cycle. Indeed, it cannot be a transposition: since $\rho(a_1)$ and $\rho(a_2)$ generate $\mathfrak{S}_4$, $\rho(a_2)\rho(a_1)$ does not fix the fixed point of $\rho(a_1)$ nor the fixed points of $\rho(a_2)$. We may then apply a Dehn twist in the first handle so that $\rho(b_1) = \mathrm{id}$ again.

Let us show that two representations $\rho_0$ and $\rho_1$ that are in the first case above are in the same orbit under the action of $\mathrm{Aut}^+(\Gamma)$. By Remark 2.1 we may conjugate $\rho_0$ so that $\rho_0(a_1) = \rho_1(a_1)$. Conjugating $\rho_0$ by some $\rho(a_1)^k$, $0 \leqslant k \leqslant 2$, we may also assume that $\rho_0(a_2) = \rho_1(a_2)^{\pm 1}$, thanks to the following observation whose proof is left to the reader.

**Lemma 3.13.** *If $\sigma$ is a 3-cycle and $\tau$ is a 4-cycle, then $\{\sigma^{-k}\tau\sigma^k\}$ intersect every cyclic subgroup of order 4 of $\mathfrak{S}_4$.*

Applying an automorphism of the second handle, thanks to Proposition 3.2, we may replace $\rho_0(a_2)$ with $\rho_0(a_2)^{-1}$ if necessary.

Let us now suppose that $\rho_0$ and $\rho_1$ both have the form of the third case above: $\rho_0(a_2)$ and $\rho_0(b_2)$ are different transpositions that commute. We may conjugate $\rho_0$ so that $\rho_0(a_1) = \rho_1(a_1)$. There are three possibilities for $\{\rho_0(a_2), \rho_0(b_2)\}$ and we can go from one to the other by conjugating by $\rho_0(a_1)^k$, $0 \leqslant k \leqslant 2$. Therefore we may assume that $\{\rho_0(a_2), \rho_0(b_2)\} = \{\rho_1(a_2), \rho_1(b_2)\}$. Since the action of $\mathrm{Aut}^+(\Gamma_1)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_1, \mathrm{K})$ is surjective, we can assume that $\rho_0 = \rho_1$. Representations in the first case above cannot be in the same orbit as a representation in the third case since the former lifts to $\mathrm{SL}_2(\mathbb{C})$ while the latter does not by Lemma 3.7. As a corollary of this classification in genus 2, we show that if $\rho : \Gamma_2 \to \mathfrak{S}_4$ is surjective, then every $g \in \mathfrak{S}_4$ is the image of a nonseparating simple closed curve by $\rho$.

**Lemma 3.14.** *For every surjective $\rho \in \mathrm{Hom}(\Gamma_2, \mathfrak{S}_4)$ and $g \in \mathfrak{S}_4$, there exists $\sigma \in \mathrm{Aut}^+(\Gamma_2)$ such that $\sigma \cdot \rho(b_1) = g$.*

*Proof.* For every conjugacy class of $g \in \mathfrak{S}_4$, it suffices to exhibit a representation $\rho$ in each of the two $\mathrm{Aut}^+(\Gamma)$-orbits such that $\rho(b_1) = g$. We leave it to the reader to check this for each of the 5 conjugacy classes of $\mathfrak{S}_4$. □

Let us now turn to the case $g \geqslant 3$. The homomorphism $\varphi \circ \rho$ in $\mathrm{Hom}(\Gamma, \mathfrak{S}_3)$ is surjective. Hence we may assume that $\varphi \circ \rho(a_1)$ is a 3-cycle, $\varphi \circ \rho(a_2)$ is a transposition and $\varphi \circ \rho(\gamma) = \mathrm{id}$ for the others $\gamma \in \{a_1, \ldots, b_g\}$. Let us show that

the number of $\gamma \in \{a_3, \dots, b_g\}$ such that $\rho(\gamma) = \mathrm{id}$ can be lowered. Iterating this process, we get back to the genus 2 case.

The group generated by $\rho(\gamma)$ for $\gamma \in \{a_1, b_1, a_2, b_2\}$ has order divisible by 6 and is not included in $\mathfrak{A}_4$. Therefore it is $\mathfrak{S}_4$ or it is isomorphic to $\mathfrak{S}_3$. In the case where it is isomorphic to $\mathfrak{S}_3$, $\rho(a_2)$ must be a transposition and after conjugating we may assume that $\rho(a_2) = (12)$. Let $\gamma \in \{a_3, \dots, b_g\}$ be such that $\rho(\gamma) \neq \mathrm{id}$. We may assume that $\gamma \in \{a_3, b_3\}$, applying an automorphism of $\Gamma_{g-2}$ that interchanges the handles. We can even suppose that $\gamma = a_3$: we may apply the transformation $(x, y) \mapsto (y, -x)$ of $\mathrm{Sp}_2(\mathbb{Z}) \subset \mathrm{Sp}_{2g-2}(\mathbb{Z})$. We can apply a Dehn twist along a curve homotopic to $[a_g, b_g] \dots [a_3, b_3]a_1$ if necessary so that $\rho(\gamma)$ does not preserve the set $\{1, 2\}$. It does not change $\rho(\gamma)$ for $\gamma \in \{a_1, a_2, b_2\}$ and we can turn back $\rho(b_1)$ to its previous value with a Dehn twist in the first handle. A Dehn twist along a curve homotopic to $a_3^{-1}b_2$ replaces $\rho(a_2)$ with a 4-cycle. Hence we may assume that the group generated by $\rho(\gamma)$ for $\gamma \in \{a_1, b_1, a_2, b_2\}$ is $\mathfrak{S}_4$.

Therefore, applying an automorphsim of $\Gamma_2$, we may assume that $\rho(b_1) = \rho(a_3)\rho(b_3)$ by Lemma 3.14. We then apply a Dehn twist along a curve homotopic to $b_3^{-1}b_1$, that replaces $\rho(a_3)$ with $\rho(a_3)\rho(b_1)^{-1}\rho(b_3)^{-1} = \mathrm{id}$. This does not change $\rho(b_3)$ and conjugate the $\rho(a_i)$ and $\rho(b_i)$ for $i \geqslant 4$. Since $\varphi \circ \rho(\gamma) = \mathrm{id}$ for $\gamma \in \{a_3, \dots, b_g\}$, we may apply an automorphism of $\Gamma_2$ to turn back to the case where $\varphi \circ \rho(a_1)$ is a 3-cycle, $\varphi \circ \rho(a_2)$ is a transposition, and $\varphi \circ \rho(b_1) = \varphi \circ \rho(b_2) = \mathrm{id}$.

## 3.5 The group $\mathfrak{A}_5$

Let us classify the orbits of the $\mathrm{Aut}^+(\Gamma)$ action on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma, \mathfrak{A}_5)$.

**Proposition 3.15.** *There are two orbits of surjective homomorphisms of* $\mathrm{Hom}(\Gamma, \mathfrak{A}_5)$ *under the action of* $\mathrm{Aut}^+(\Gamma)$.

We first find a simple closed curve that is sent to a 5-cycle.

**Lemma 3.16.** *Let $g' \geqslant 2$ and $\rho \in \mathrm{Hom}(\Gamma_{g',1}\, \mathfrak{A}_5)$ be such that $\rho(\Gamma_{g',1})$ acts transitively on $\{1, \dots, 5\}$. There exists $\sigma \in \mathrm{Aut}^+(\Gamma_{g',1})$ such that $\sigma \cdot \rho(a_1)$ is a 5-cycle.*

*Proof.* It suffices to find a nonseparating simple closed curve $\gamma \in \Gamma_{g',1}$ such that $\rho(\gamma)$ is a 5-cycle, since for every nonseparating simple closed curve $\gamma \in \Gamma_{g',1}$ there exists $\sigma \in \mathrm{Aut}^+(\Gamma_{g',1})$ such that $\sigma \cdot \gamma \in \{a_1, a_1^{-1}\}$. The product of two double transpositions $\tau$ and $\tau'$ that do not share the same fixed point is not a double transposition. Indeed if $\tau\tau'$ has order 2, then $\tau\tau' = (\tau\tau')^{-1} = \tau'\tau$ thus $\tau\tau'\tau^{-1} = \tau'$ and $\tau$ and $\tau'$ have the same fixed point. Let us first suppose that every $\rho(\gamma)$ for $\gamma \in \{a_1, \dots, b_{g'}\}$ has order 2 or 1. Two of them do not share the same fixed point, hence their product has order 3 or 5. Therefore we may assume that $\rho(a_1)$ has order 3. Let us denote by $p$ and $q$ the fixed points of $\rho(a_1)$.

**Lemma 3.17.** *Let $p \neq q \in \{1, \dots, 5\}$ and $\alpha$, $\beta \in \mathfrak{A}_5$. If $\alpha(p) = p$ and $\alpha(q) \neq q$, and $\beta(q) = q$ and $\beta(p) \neq p$, then $\beta\alpha$ does not interchange $p$ and $q$, nor does it fix $p$ or $q$. If $\tau$ interchanges $p$ and $q$, then $\tau\alpha$ does not fix nor interchanges $p$ and $q$. These two statements also hold for $\alpha\beta$ and $\alpha\tau$.*

*Proof.* For the first part, $\beta(\alpha(p)) = \beta(p) \neq p$ and $\beta(\alpha(q)) \neq q$ because $\alpha(q) \neq q$. Moreover if $\beta\alpha(p) = \beta(p) = q$, then $\beta(q) = q = \beta(p)$ and $q = p$. Let us prove the second part. We have $\tau(\alpha(p)) = \tau(p) = q$ and $\tau(\alpha(q)) \neq p$ since otherwise $\alpha(q) = q$. Moreover $\tau(\alpha(q)) = q$ if and only if $\alpha(q) = p$. But $\alpha(q) = p = \alpha(p)$ implies that $p = q$. □

Let us make some observations on the order of the product of elements of $\mathfrak{A}_5$ whose proof are left to the reader.

**Lemma 3.18.** *If $\sigma$ is a 3-cycle, then the product of $\sigma$ with a double transposition that does not fix its fixed points, nor interchanges them is a 5-cycle. The product of $\sigma$ with a 3-cycle that does not share a fixed point with $\sigma$ is also a 5-cycle. The product of a double transposition with another double transposition that does not share the same fixed point, nor a transposition, is a 5-cycle.* □

If one of $\rho(\gamma)$, $\gamma \in \{b_1, a_2, \dots, b_g\}$ does not fix $p$ nor $q$ nor interchanges them, then the product of $\rho(a_1)$ with this permutation has order 5. If it is not the case, then we can find one of them that does not fix $p$ but fixes $q$ , interchanging $p$ and $q$ if necessary. Indeed since $\rho(\Gamma_{g',1})$ acts transitively on $\{1, \dots, 5\}$, one of them does not preserve the set $\{p, q\}$. Another one of these permutations does not fix $q$ and either interchanges $p$ and $q$, or fixes $p$. Let us say for clarity that these permutations are $\rho(a_2)$ and $\rho(b_2)$. Then $\rho(b_2^{-1}a_2)$ does not fix $p$ nor $q$ nor interchanges them by Lemma 3.17, hence $\rho(b_2^{-1}a_2a_1)$ is a 5-cycle. □

We now show that we may assume that both $\rho(a_i)$ and $\rho(b_i)$ fix 5 for every $i \geqslant 2$.

**Proposition 3.19.** *There exists $\sigma \in \mathrm{Aut}^+(\Gamma)$ such that for all $2 \leqslant i \leqslant g$, both $\sigma \cdot \rho(a_i)$ and $\sigma \cdot \rho(b_i)$ fix 5.*

We may ensure that $\rho(a_i)$ fixes 5 by applying an automorphism of $\mathrm{Aut}^+(\Gamma_{1,1})$, unless both $\rho(a_i)$ and $\rho(b_i)$ are double transpositions that commute.

**Lemma 3.20.** *There exists $\sigma \in \mathrm{Aut}^+(\Gamma_{1,1})$ such that $\sigma \cdot \rho(a_1)$ fixes 5, unless both $\rho(a_1)$ and $\rho(b_1)$ are double transpositions that fix some $p \in \{1, \dots, 4\}$, and $\alpha \neq \beta$.*

*Proof.* Let $\alpha = \rho(a_1)$ and $\beta = \rho(b_2)$. If $\beta$ fixes 5, then one can apply the automorphisms that acts on $\mathrm{Hom}(\Gamma_{1,1}, \mathfrak{A}_5)$ as $(\alpha, \beta) \to (\alpha, \alpha\beta) \to (\beta^{-1}, \alpha\beta)$. If $\alpha(5)$ is fixed by $\beta$, then one can apply the automorphisms $(\alpha, \beta) \to (\alpha, \alpha^{-1}\beta) \to (\alpha^{-1}\beta\alpha, \alpha^{-1}, \beta)$ to replace $\alpha$ with $\alpha^{-1}\beta\alpha$. This new $\alpha$ fixes 5. Similarly, if $\beta(5)$ is fixed by $\alpha$ then one can replace $\beta$ with $\beta^{-1}\alpha\beta$. If $\alpha(5)$ is in the orbit of 5 by the action of the group generated by $\beta$, then one can apply the Dehn twist that replaces $\alpha$ with $\beta^k\alpha$ so that $\beta^k\alpha$ fixes 5. Similarly we can suppose that $\beta(5)$ is

not in the orbit of 5 under the action of the group generated by $\alpha$. Therefore, if $\alpha$ or $\beta$ is a cycle, then there exists such a $\sigma$. Indeed if $\alpha$ is a cycle that does not fix 5, either $\beta(5)$ is fixed by $\alpha$ or $\beta(5)$ is in the orbit of 5 by the action of $\langle\alpha\rangle$. Let us now suppose that both $\alpha$ and $\beta$ are double transpositions. We may replace $\alpha$ with $\beta\alpha$. Either $\beta\alpha$ is not a double transposition, or $\alpha$ and $\beta$ are different double transpositions that share a common fixed point by Lemma 3.18. $\square$

We now prove Proposition 3.19.

*Proof.* Let us assume that for all $k_\rho + 1 \leqslant i \leqslant g$, both $\rho(a_i)$ and $\rho(b_i)$ fix 5, and that $k_\rho$ is minimal for that property. We may also assume that for all $\sigma \in \mathrm{Aut}^+(\Gamma)$ we have $k_\rho \leqslant k_{\sigma\cdot\rho}$. Assume by contradiction that $k_\rho \geqslant 2$. Let us denote by $G$ the group generated by $\rho(\gamma)$ for $\gamma \in \{a_1, \dots, b_{k_\rho}\}$. The group $G$ is included in $\mathfrak{A}_5$, hence it is either $\mathfrak{A}_5$, cyclic or isomorphic to $\mathfrak{A}_4$, $\mathfrak{S}_3$, or $\mathrm{D}_2$.

- If $G$ is cyclic, then by Proposition 3.2, one can assume applying an automorphism of $\Gamma_{k_\rho}$ that $\rho(a_{k_\rho}) = \rho(b_{k_\rho}) = \mathrm{id}$, contradicting the minimality of $k_\rho$.

- If $G$ is isomorphic to $\mathfrak{A}_4$, then it is embedded in $\mathfrak{A}_5$ as the subgroup that fixes a point $p \in \{1, 2, 3, 4\}$. Since

$$\rho([a_1, b_1] \dots [a_{k_\rho}, b_{k_\rho}]) = \rho([a_{k_\rho+1}, b_{k_\rho+1}] \dots [a_g, b_g])^{-1}$$

  the permutation $\rho([a_1, b_1] \dots [a_{k_\rho}, b_{k_\rho}])$ has two fixed points and is in a group isomorphic to $\mathrm{D}_2$, thus it is id. Therefore, we may restrict the representation $\rho$ to $\mathrm{Hom}(\Gamma_{k_\rho}, G)$. By Proposition 3.8, there are two orbits of surjective $\rho' : \Gamma_{k_\rho} \to G$, under the action of $\mathrm{Aut}^+(\Gamma_{k_\rho})$. In each of these orbits, there exists a representation $\rho'$ such that $\rho'(a_{k_\rho})$ is any desired 3-cycle and $\rho'(b_{k_\rho}) = \mathrm{id}$. Hence we may assume that both $\rho(a_{k_\rho})$ and $\rho(b_{k_\rho})$ fix 5, a contradiction.

- Let us suppose that $G$ is isomorphic to $\mathfrak{S}_3$. Since $\rho([a_1, b_1] \dots [a_{k_\rho}, b_{k_\rho}])$ is in the commutator subgroup of $G$, it has order 3 or 1. It is also in the commutator subgroup of $\mathfrak{A}_4$ and thus must be trivial. Hence we may restrict $\rho$ to $\mathrm{Hom}(\Gamma_{k_\rho}, \mathfrak{S}_3)$. The action of $\mathrm{Aut}^+(\Gamma_{k_\rho})$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_{k_\rho}, G)$ is transitive, hence we may assume that $\rho(a_{k_\rho})$ is any fixed element of $G \setminus \{\mathrm{id}\}$ and $\rho(b_{k_\rho}) = \mathrm{id}$ since there exists a surjective representation $\Gamma_{k_\rho} \to G$ with this property. Therefore we can assume that both $\rho(a_{k_\rho})$ and $\rho(b_{k_\rho})$ fix 5, since in each subgroup $G$ of $\mathfrak{A}_5$ isomorphic to $\mathfrak{S}_3$, there exists $g \in G \setminus \{\mathrm{id}\}$ that fixes 5: up to conjugacy we have

$$G = \{\mathrm{id}, (1\ 2\ 3), (1\ 3\ 2), (1\ 2)(4\ 5), (2\ 3)(4\ 5), (1\ 3)(4\ 5)\}.$$

- Let us now assume that $G$ is isomorphic to $\mathrm{D}_2$. We may restrict $\rho$ to $\mathrm{Hom}(\Gamma_{k_\rho}, \mathrm{D}_2)$. If this restriction does not lift to $\mathrm{SL}_2(\mathbb{C})$, then one can assume that $\rho(a_1)$ and $\rho(b_1)$ are different double transpositions and that

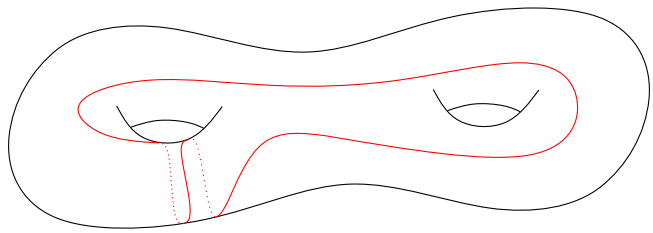

Figure 3: Dehn twist along a simple closed curve.

$\rho(a_{k_\rho}) = \rho(b_{k_\rho}) = \mathrm{id}$, a contradiction. Similarly, if $k_\rho > 2$, one can assume that $\rho$ kills both $a_{k_\rho}$ and $b_{k_\rho}$. Let us thus suppose that $k_\rho = 2$ and that the restriction of $\rho$ to the first two handles lifts to $\mathrm{SL}_2(\mathbb{C})$. We may restrict $\rho$ to the last $g-2$ handles and get a representation $\rho' \in \mathrm{Hom}(\Gamma_{g-2}, \mathfrak{A}_4)$. If $\rho'(\Gamma_{g-2})$ is isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_2$ or $\mathrm{D}_2$ and does not lift to $\mathrm{SL}_2(\mathbb{C})$, then one can assume that $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for all $i > 3$. We may then conjugate $\rho$ so that $\rho(\gamma)$ fix 5 for $\gamma \in \{a_1, b_1, a_2, b_2\}$ and thus get $k_\rho = 1$. If $\rho'(\Gamma_{g-2}) = \mathfrak{A}_4$, we may assume that $\rho(a_3)$ is a 3-cycle that fixes the fixed point of $\rho(\gamma)$ for $\gamma \in \{a_1, b_1, a_2, b_2\}$ and $\rho(b_3) = \mathrm{id}$. We also assume that $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for $i > 4$. Again, conjugating $\rho$ we get $k_\rho = 1$. Let us turn to the case where $\rho'(\Gamma_{g-2})$ is isomorphic to $\mathrm{D}_2$ and lifts to $\mathrm{SL}_2(\mathbb{C})$. After conjugating $\rho$ and applying automorphisms of $\Gamma_{g-2}$ and of $\Gamma_2$, we may assume that $\rho(a_1) = (1\ 3)(2\ 5)$, $\rho(a_2) = (1\ 2)(3\ 5)$, $\rho(a_3) = (1\ 2)(3\ 4)$, $\rho(a_4) = (1\ 3)(2\ 4)$, and $\rho(b_i) = \mathrm{id}$ for all $i$ and $\rho(a_i) = \mathrm{id}$ for all $i > 4$. We may restrict $\rho$ to the second and third handles $\mathrm{Hom}(\Gamma_2, \mathfrak{A}_5)$. The image of $\rho(\Gamma_2)$ is isomorphic to $\mathrm{D}_3 \simeq \mathfrak{S}_3$. Since the action of $\mathrm{Aut}^+(\Gamma_2)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_2, \mathfrak{S}_3)$ is transitive, we may assume that $\rho(a_2) = (3\ 4\ 5)$, $\rho(a_3) = (1\ 2)(3\ 4)$ and $\rho(b_2) = \rho(b_3) = \mathrm{id}$. We now apply a Dehn twist along a curve homotopic to $[a_g, b_g] \dots [a_3, b_3]a_2$. This conjugate the first handle so that $\rho(a_1)$ fixes 5. Hence for all $i \neq 2$, both $\rho(a_i)$ and $\rho(b_i)$ fix 5, thus $k_\rho = 1$.

- Let us suppose that $G = \mathfrak{A}_5$. By Lemma 3.16, we may assume that $\rho(a_1)$ is a 5-cycle. By Lemma 3.20 we may assume that $\rho(b_2)$ fixes 5 or that both $\rho(a_2)$ and $\rho(b_2)$ fix some $p \in \{1, 2, 3, 4\}$. In this latter case we may apply a power of a Dehn twist along a curve homotopic to $[a_2, b_2]a_1$ so that both $\rho(a_i)$ and $\rho(b_i)$ fix $p$ for $k_\rho < i \leqslant g$. We then conjugate $\rho$ to interchange 5 with $p$. We have added a hanlde that fixes 5, which is a contradiction. We thus turn to the case where $\rho(b_2)$ fixes 5. A Dehn twist as in Figure 3 along a curve homotopic to $b_2^{-1}a_1^k b_1^{-1}$ replaces $\rho(a_2)$ with $\sigma_k = \rho(a_2)\rho(b_1^{-1})\rho(a_1)^k\rho(b_2^{-1})$. We may take $k$ so that it fixes 5. $\square$

Let us now observe that if $\rho(\gamma)$ fixes 5 for each $\gamma \in \{a_2, \dots, b_g\}$, then $\rho(a_1)$ commutes with $\rho(b_1)$.

**Lemma 3.21.** *Let $\alpha, \beta \in \mathfrak{A}_5$ be such that $[\alpha, \beta]$ has order 2, and denote by $p \in \{1, \dots, 5\}$ the fixed point of $[\alpha, \beta]$. Both $\alpha$ and $\beta$ fix $p$.*

*Proof.* Let us suppose that $[\alpha, \beta]$ has order 2 and fixes 5. We assume for contradiction that $\alpha$ does not fix 5. If $\alpha$ is a double transposition, then we may assume that $\alpha = (5\ k)(i\ j)$ for $i, j, k \in \{2, 3, 4, 5\}$. The double transposition $\beta\alpha^{-1}\beta$ has the form $(5\ k)(x, y)$, since $[\alpha, \beta]$ must fix $k$. Therefore $[\alpha, \beta]$ fixes both 5 and $k$, this is not possible since it is a double tranposition. If $\alpha = (5\ i\ j)$ is a 3-cycle, then $\beta^{-1}\alpha^{-1}\beta$ is a 3-cycle of the form $(i\ 5\ k)$, since $[\alpha, \beta]$ fixes 5. But $|\{i, j, k, 5\}| \leqslant 4$, hence $[\alpha, \beta]$ fixes some $p \in \{1, 2, 3, 4\}$ and cannot have order 2. If $\alpha$ is a 5-cycle, say $\alpha = (1\ 2\ 3\ 4\ 5)$. We must have $\beta^{-1}\alpha^{-1}\beta = (1\ 5\ i\ j\ k)$ with $\{i, j, k\} = \{2, 3, 4\}$. We cannot have $k = 2$, otherwise $[\alpha, \beta]$ would fix 1. We cannot have $i = 4$, otherwise $[\alpha, \beta]$ would fix 4. If $i = 3$, then $j = 2$ and $[\alpha, \beta]$ fixes 2. If $i = 2$, then $\beta^{-1}\alpha^{-1}\beta = (1\ 5\ 2\ 3\ 4)$ or $\beta^{-1}\alpha^{-1}\beta = (1\ 5\ 2\ 4\ 3)$. In this latter case, $[\alpha, \beta]$ fixes 3. We thus suppose that $\beta^{-1}\alpha^{-1}\beta = (1\ 5\ 2\ 3\ 4)$. Since $\beta' = (2\ 4)$ satisfies $\beta'^{-1}\alpha^{-1}\beta = \beta^{-1}\alpha^{-1}\beta$, the permutation $\beta\beta'^{-1}$ must be in the centralizer of $\alpha^{-1}$, and thus be a power of $\alpha$. Therefore $\beta \in \mathfrak{S}_5 \setminus \mathfrak{A}_5$. $\square$

We now suppose thanks to Proposition 3.19 that $\rho(\gamma)$ fix 5 for $\gamma \in \{a_2, \ldots, b_g\}$. Note that since $\rho(a_1)$ and $\rho(b_1)$ do not both fix 5, the commutator $[\rho(a_1), \rho(b_1)] = \mathrm{id}$ since it is a product of commutators of permutations fixing 5. Therefore the representation restricted on the last $g - 1$ handles gives $\rho : \Gamma_{g-1} \to \mathfrak{A}_4$. The image of this representation is a subgroup of $\mathfrak{A}_4$: it is isomorphic to $\mathfrak{A}_4$, $\mathrm{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_2$ or the trivial group. In each of these cases, we have seen that we may assume that $\rho(a_i) = \rho(b_i) = \mathrm{id}$ for all $i \geqslant 4$, and that $\rho([a_2, b_2]) = \mathrm{id}$. We are thus reduced to the genus $g = 3$ case. In order to deal with this case, we first prove Proposition 3.15 for $g = 2$.

### 3.5.1 The genus 2 case

**Proposition 3.22.** *The action of* $\mathrm{Aut}^+(\Gamma_2)$ *on the set of surjective homomorphisms of* $\mathrm{Hom}(\Gamma_2, \mathfrak{A}_5)$ *has two orbits.*

Let us describe a representation in each of these orbits. Let $\rho_0$ be the representation defined by $\rho_0(a_1) = (1\ 2\ 3\ 4\ 5)$, $\rho_0(b_1) = \rho_0(b_2) = \mathrm{id}$ and $\rho_0(a_2) = (1\ 5\ 4\ 3\ 2)$. Let $\rho_1$ be the representation defined by $\rho_1(a_1) = (1\ 2\ 3\ 4\ 5)$ and $\rho_1(b_1) = \mathrm{id}$, $\rho_1(a_2) = (1\ 2)(3\ 4)$ and $\rho_1(b_2) = (1\ 3)(2\ 4)$. While $\rho_0$ lifts to $\mathrm{SL}_2(\mathbb{C})$, the representation $\rho_1$ does not by Lemma 3.7.

Let us fix a surjective homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_5)$. By Proposition 3.19, we can assume that $\rho(a_2)$ and $\rho(b_2)$ both fix 5, and that $\rho([a_1, b_1]) = \rho([a_2, b_2]) = \mathrm{id}$. Since the centralizer of a cycle in $\mathfrak{A}_5$ is the group it generates, if $\rho(a_1)$ or $\rho(b_1)$ has order 3, we may assume that $\rho(b_1) = \mathrm{id}$. The same holds in the second handle.

Let us first show that if $\rho$ does not lift to $\mathrm{SL}_2(\mathbb{C})$, then there exists $\sigma \in \mathrm{Aut}^+(\Gamma)$ such that $\sigma \cdot \rho = \rho_1$.

**Lemma 3.23.** *The action of* $\mathrm{Aut}^+(\Gamma)$ *on the set of surjective homomorphisms of* $\mathrm{Hom}(\Gamma, \mathfrak{A}_5)$ *that do not lift to* $\mathrm{SL}_2(\mathbb{C})$ *is transitive.*

*Proof.* We suppose that $\rho$ does not lift to $\mathrm{SL}_2(\mathbb{C})$. By Lemma 3.7 and the remark above, $(a_2, b_2)$ must be sent to two different double transpositions. We

may assume that $\rho(b_1) = \mathrm{id}$: if $\rho(a_1)$ is a cycle, since $[\rho(a_1), \rho(b_1)] = \mathrm{id}$, $\rho(b_1)$ is a power of $\rho(a_1)$. If $\rho(a_1)$ is a double transposition, then since $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ and $\rho(b_1)$ commutes with $\rho(a_1)$, $\rho(b_1)$ must again be a power of $\rho(a_1)$. The group generated by $\rho(a_1)$ and $\rho(b_1)$ is thus cyclic and we may assume with an automorphism of $\Gamma_1$ that $\rho(b_1) = \mathrm{id}$. Since $\mathfrak{A}_5$ is not abelian, $\rho(a_1) \neq \mathrm{id}$. We may assume that that $\rho(a_1)$ is a 5-cycle. If it is a double transposition or a 3 cycle, then we may apply an automorphism of $\Gamma_1$ in the handle $(a_2, b_2)$ so that $\rho(a_2)\rho(a_1)$ is a 5-cycle, by Lemma 3.18. A Dehn twist along a curve homotopic to $a_2^{-1}b_1$ replaces $\rho(a_1)$ with $\rho(a_1)\rho(a_2)^{-1}$ which is a 5-cycle. Note that $\rho(a_1)$ and $\rho(b_1)$ are still distinct double transpositions that commute. We can conjugate $\rho$ so that $\rho(a_1) = (1\ 2\ 3\ 4\ 5)$. We can then conjugate $\rho$ by $\rho(a_1)^k$ so that $\rho(a_1)$ and $\rho(b_1)$ fix 5. Since the action of $\mathrm{Aut}^+(\Gamma_1)$ on the set of surjective homomorphisms $\mathrm{Hom}(\Gamma_1, \mathrm{D}_2)$ is transitive, we may assume that $\rho(a_1) = (1\ 2)(3\ 4)$ and $\rho(b_1) = (1\ 3)(2\ 4)$, and $\rho = \rho_1$. □

We now suppose that $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$. We still assume that $\rho([a_1, b_1]) = \rho([a_2, b_2]) = \mathrm{id}$.

**Lemma 3.24.** *There exists a nonseparating simple closed curve $\gamma$ such that $\rho(\gamma) =$ id.*

*Proof.* We have already explained why this holds when one of the handles $(a_1, b_1)$ and $(a_2, b_2)$ does not have its generators mapped by $\rho$ to distinct double transpositions. We now turn to this case. Conjugating $\rho$ if necessary, and then applying a homeomorphism in each handle, we may assume that $\rho(a_1) = (1\ 2)(3\ 4)$, $\rho(b_1) = (1\ 3)(2\ 4)$, $\rho(a_2) = (1\ 3)(2\ 5)$ and $\rho(b_2) = (1\ 5)(2\ 3)$. The loop $\gamma = b_2^{-1}a_2b_1^{-1}a_2b_1^{-1}a_1^{-1}a_2b_1^{-1}$ is simple, nonseparating, and $\rho(\gamma) = \mathrm{id}$. □

We can thus assume that $\rho(b_1) = \mathrm{id}$. Note that we may also assume that $\rho(b_2) = \mathrm{id}$. Indeed if $\rho(a_2)$ and $\rho(b_2)$ are distinct double transpositions, then $\rho$ does not lift do $\mathrm{SL}_2(\mathbb{C})$. Since $[\rho(a_2), \rho(b_2)] = \mathrm{id}$, and that the commutator of a cycle is the group it generates, we may assume that $\rho(b_2) = \mathrm{id}$. One of $\rho(a_1)$ and $\rho(a_2)$ has order at least 3 because the group $\mathfrak{A}_5$ is not generated by two double transpositions. Therefore we suppose $\rho(a_1)$ has order 3 or 5. The following lemma enables us to assume that it has order 3.

**Lemma 3.25.** *If $\rho(a_1)$ is a 5-cycle, then we may replace $\rho(a_1)$ with a double transposition or a 3-cycle without changing $\rho(b_1)$, $\rho(a_2)$ and $\rho(b_2)$.*

*Proof.* Let $p \in \{1, \ldots, 5\}$ be such that $\rho(a_2)$ does not fix $p$. There exists $1 \leqslant k \leqslant 4$ such that $\rho(a_1)^k\rho(a_2)$ fixes $p$. We apply an automorphism of $\Gamma_1$ in the handle $(a_1, b_1)$, that replaces $\rho(a_1)$ with $\rho(a_1)^k$ and leaves $\rho(b_1) = \mathrm{id}$ unchanged; it exists by Proposition 3.2. We then apply a Dehn twist along a curve homotopic to $a_2b_1^{-1}$ to replace it with $\rho(a_1)^k\rho(a_2)$ that fixes $p$. We still have $\rho(b_1) = \mathrm{id}$, and we may apply an automorphism of $\Gamma_1$ in the second handle $(a_2, b_2)$ that does not change $\rho(a_2)$ and turns back $\rho(b_2)$ to id. □

We can now assume that $\rho(a_1)$ is a 3-cycle: we may assume that both $\rho(a_1)$ and $\rho(b_1)$ have order 2 or 3 by Lemma 3.25, and they cannot both have order

2. We may interchange the handles if $\rho(a_1)$ is not a 3-cycle. If $\rho(a_2)$ is a double transposition, then we apply a Dehn twist along a curve homotopic to $a_1 b_2^{-1}$ that replaces it with $\rho(a_2)\rho(a_1)$ which is a 5-cycle by Lemma 3.18. By Lemma 3.25 we may assume that $\rho(a_2)$ is a 3-cycle without modifying $\rho(\gamma)$ for $\gamma \in \{a_1, b_1, b_2\}$.

We now show that all the homomorphisms of this form are in the same orbit under the action of $\mathrm{Aut}^+(\Gamma_2)$, thus proving Proposition 3.22.

*Proof.* We may conjugate $\rho$ so that $\rho(a_1) = (1\ 2\ 3)$. Let us write $\rho(b_2) = (\alpha\ \beta\ \gamma)$, with $\alpha \in \{1, 2, 3\}$ and $\{\beta, \gamma\} = \{4, 5\}$. Conjugating $\rho$ with a power of $\rho(a_1)$, we can suppose that $\alpha = 1$, that is $\rho(a_2) = (1\ 4\ 5)$ or $\rho(a_2) = (1\ 5\ 4)$. By Proposition 3.2, there exists an automorphism of $\Gamma_1$ that replaces $\rho(a_2)$ with $\rho(a_2)^{-1}$, and we can thus assume that $\rho(a_2) = (1\ 4\ 5)$. □

Let us state a corollary of this classification that will be useful to finish our classification of the orbits in genus at least 3.

**Corollary 3.26.** *Let $g \in \mathfrak{A}_5 \setminus \{\mathrm{id}\}$. For every surjective $\rho \in \mathrm{Hom}(\Gamma_2, \mathfrak{A}_5)$, there exists $\sigma \in \mathrm{Aut}^+(\Gamma_2)$ such that $\sigma \cdot \rho(a_1) = g$ and $\sigma \cdot \rho(b_1) = \mathrm{id}$.*

*Proof.* There exists $h \in \mathfrak{A}_5$ such that $g$ and $h$ generate $\mathfrak{A}_5$. The representation $\rho'$ defined by $\rho'(a_1) = g$, $\rho'(b_1) = \rho'(b_2) = \mathrm{id}$ and $\rho'(a_2) = h$ lifts to $\mathrm{SL}_2(\mathbb{C})$. Therefore, if $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$, then there exists $\sigma \in \mathrm{Aut}^+(\Gamma_2)$ such that $\sigma \cdot \rho = \rho'$. There exist two distinct commuting double transpositions $h_1, h_2 \in \mathfrak{A}_5$ such that $h_1$, $h_2$ and $g$ generate $\mathfrak{A}_5$: it suffices to take two distinct transpositions with a fixed point $p \in \{1, \ldots, 5\}$ that is not fixed by $g$, since the proper subgroups of $\mathfrak{A}_5$ of order divisible by 4 have a fixed point in $\{1, \ldots, 5\}$. Let $\rho''$ be the representation defined by $\rho''(a_1) = g$, $\rho''(b_1) = \mathrm{id}$ and $\rho''(a_2) = h_1$ and $\rho''(b_2) = h_2$. The representation $\rho''$ does not lift to $\mathrm{SL}_2(\mathbb{C})$ by Lemma 3.7. Therefore, if $\rho$ does not lift to $\mathrm{SL}_2(\mathbb{C})$, then there exists $\sigma \in \mathrm{Aut}^+(\Gamma_2)$ such that $\sigma \cdot \rho = \rho''$. □

### 3.5.2 Genus $g \geqslant 3$

The proof of Proposition 3.15 is now reduced to the case $g = 3$. Therefore, the following lemma completes its proof.

**Lemma 3.27.** *Let us assume that $\rho \in \mathrm{Hom}(\Gamma_3, \mathfrak{A}_5)$ is surjective. There exists $\sigma \in \mathrm{Aut}^+(\Gamma_3)$ such that $\sigma \cdot \rho(a_3) = \sigma \cdot \rho(b_3) = \mathrm{id}$.*

*Proof.* Recall that we may assume that $\rho([a_1, b_1]) = \rho([a_2, b_2]) = \rho([a_3, b_3]) = \mathrm{id}$, and that the group $H$ generated $\rho(\gamma)$ for $\gamma \in \{a_2, b_2, a_3, b_3\}$ is a subgroup of $\mathfrak{A}_4$, that is embedded as the subgroup whose elements fix 5. Let us act by $\mathrm{Aut}^+(\Gamma_2)$ on the last two handles. The subgroups of $\mathfrak{A}_4$ are cyclic, isomorphic to $\mathrm{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathfrak{A}_4$. We have classified the orbits of the $\mathrm{Aut}^+(\Gamma_2)$ action on $\mathrm{Hom}(\Gamma_2, H)$ in the previous sections. We saw that we may assume that $\rho(a_3) = \rho(b_3) = \mathrm{id}$ unless $H$ is isomorphic to $\mathfrak{A}_4$, or $\mathrm{D}_2$, in which case we can assume that the group $G$ generated by the $\rho(\gamma)$, $\gamma \in \{a_1, b_1, a_2, b_2\}$, is isomorphic to $\mathfrak{A}_5$ and that $\rho(a_3) \neq \mathrm{id}$. Indeed if $H$ is isomorphic to $\mathrm{D}_2$, we

may assume that $\rho(a_2)$ and $\rho(b_2)$ are two distinct double transpositions. Since $\rho(a_1)$ and $\rho(b_1)$ do not both fix 5, the group $G$ is $\mathfrak{A}_5$ because one can check that the proper subgroups of $\mathfrak{A}_5$ of order divisible by 4 have a fixed point $p \in \{1, \dots, 5\}$. If $H$ is isomorphic to $\mathfrak{A}_4$, then we may suppose that $\rho(a_2)$ is a 3-cycle such that $\rho(a_1), \rho(b_1)$ and $\rho(a_2)$ acts transitively on $\{1, \dots 5\}$. We then have $G = \mathfrak{A}_5$ because the only subgroup of order divisible by 3 that acts transitively on $\{1, \dots, 5\}$ is $\mathfrak{A}_5$ itself. We saw that the action of $\mathrm{Aut}^+(\Gamma_2)$ on the set of surjective homomorphism of $\mathrm{Hom}(\Gamma_2, \mathfrak{A}_5)$ has two orbits. Therefore, by Corollary 3.26, we may apply an automorphism of the first two handles so that $\rho(a_2) = \rho(a_3)^{-1}$ and that $\rho(b_2) = \mathrm{id}$. A Dehn twist along a curve homotopic to $a_3 b_2^{-1}$ replaces $\rho(a_2)$ with id, and we still have $\rho(b_2) = \mathrm{id}$. We can permute the second and the third handles so that $\rho(a_3) = \rho(b_3) = \mathrm{id}$. $\square$

# 4 Closure of infinite orbits of the mapping class group action on representation spaces into elementary subgroups of $\mathrm{PSL}_2(\mathbb{C})$

We now describe the closure of the orbits of some representations $\rho \in \mathrm{Hom}(\Gamma, G)$ with infinite image under the action of $\mathrm{Aut}^+(\Gamma)$, where $G$ is an elementary subgroup of $\mathrm{PSL}_2(\mathbb{C})$. We will also show some other results describing these orbits.

## 4.1 Holonomy in $\mathrm{SO}_3(\mathbb{R})$

### 4.1.1 The group $\mathrm{SO}_2(\mathbb{R})$

Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$ with infinite image. The orbit of $\rho$ under the action of $\mathrm{Aut}^+(\Gamma)$ is dense in $\mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$.

**Proposition 4.1.** *The orbit* $\mathrm{Aut}^+(\Gamma) \cdot \rho$ *is dense in* $\mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$.

We identify $\mathrm{SO}_2(\mathbb{R})$ with $\mathbb{R}/\mathbb{Z}$ and $\mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$ with $(\mathbb{R}/\mathbb{Z})^{2g}$ on which $\mathrm{Aut}^+(\Gamma)$ acts as $\mathrm{Sp}_{2g}(\mathbb{Z})$, see Section 2.

*Proof.* Let $\rho_n = (\frac{1}{n}, 0, \dots, 0) \in \mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$. Let us consider a lift $\tilde{\rho}$ of $\rho$ to $\mathrm{Hom}(\Gamma, \mathbb{R}) \simeq \mathbb{R}^{2g}$. Let $x_i = \tilde{\rho}(a_i)$ and $y_i = \tilde{\rho}(b_i)$ for $1 \leqslant i \leqslant g$. One of $x_1, \dots, y_g$ is not rational and we can assume without loss of generality that $x_1$ is as such: we can replace $\rho$ with some $\sigma \cdot \rho$, exchanging the handles so that $x_1$ or $y_1$ is not rational and we may then apply the symplectic transformation $(x_1, y_1) \mapsto (y_1, -x_1)$ if necessary. We can apply the transformation that replaces $(x_1, y_1)$ with $(x_1, y_1 - y_i)$ and $(x_i, y_i)$ with $(x_1 + x_i, y_i)$ for each $i$ if necessary so that $x_i \notin \mathbb{Q}$. For every $i$ such that $x_i\mathbb{Z} + y_i\mathbb{Z}$ is not dense in $\mathbb{R}$, replace $y_i$ with $y_i + 1$. The new $\tilde{\rho}$ is still a lift of $\rho$ and now $x_i\mathbb{Z} + y_i\mathbb{Z}$ is dense in $\mathbb{R}$ because $\frac{y_i}{x_i}$ and $\frac{y_i+1}{x_i}$ cannot both be rationals. We may assume that each $x_i$ and $y_i$ are nonnegative, applying $(x_i, y_i) \mapsto (x_i + ky_i, y_i)$ and $(x_i, y_i) \mapsto (x_i, y_i + lx_i)$, for some $k, l \in \mathbb{Z}$. We now apply the Euclidean algorithm in each handle: if

$x_i > y_i$, replace $x_i$ with $x_i - y_i$ and otherwise replace $y_i$ with $y_i - x_i$. Iterating this process, we can assume that $0 < x_i, y_i < \epsilon$ for every $i$, where $\epsilon > 0$ is fixed. Finally replacing $x_1$ with $x_1 + my_1$, $m \in \mathbb{Z}$, we can assume that $|x_1 - \frac{1}{n}| < \epsilon$ and that the other $x_i$ and $y_i$ are in $[0, \epsilon]$.

We have shown that each $\rho_n$ is in the closure of $\mathrm{Aut}^+(\Gamma)\cdot\rho$. Since this closure is invariant under $\mathrm{Aut}^+(\Gamma)$, it contains every representation in $\mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$ with finite image by Proposition 3.2. Therefore it contains the closure of $(\mathbb{Q}/\mathbb{Z})^{2g}$, hence is equal to $\mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$. □

### 4.1.2 The group $\mathrm{O}_2(\mathbb{R})$

Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{O}_2(\mathbb{R}))$ be such that $\overline{\rho(\Gamma)} = \mathrm{O}_2(\mathbb{R})$. Note that this amounts to requiring that $\rho$ has infinite image in $\mathrm{O}_2$ and is not contained in $\mathrm{SO}_2(\mathbb{R})$. The orbit of such a $\rho$ under the action of $\mathrm{Aut}^+(\Gamma)$ is included in the closed set $\Delta$ of representations $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{O}_2(\mathbb{R}))$ such that $\epsilon \circ \rho'$ is surjective and such that $\mathrm{sw}(\rho') = \mathrm{sw}(\rho)$.

**Proposition 4.2.** *The orbit* $\mathrm{Aut}^+(\Gamma) \cdot \rho$ *is dense in* $\Delta$.

*Proof.* It suffices to show that every representation $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$ with finite image such that $\epsilon \circ \rho'$ is surjective and $\mathrm{sw}(\rho) = \mathrm{sw}(\rho')$ is in $\mathrm{Aut}^+(\Gamma) \cdot \rho$. The representation $\epsilon \circ \rho \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ is surjective, hence we may assume by Proposition 3.2 that $\epsilon \circ \rho(\gamma) = 0$ for $\gamma \in \{b_1, a_2 \ldots, b_g\}$ and that $\epsilon(\rho(a_1)) = 1$. We thus have $[\rho(a_i), \rho(b_i)] = \mathrm{id}$ for every $1 \leqslant i \leqslant g$. Therefore $\rho(b_1)$ is either id or the rotation of order 2 in $\mathrm{SO}_2(\mathbb{R})$. Let us consider a representation $\rho'$ with finite image in $\mathrm{O}_2$, such that $\epsilon \circ \rho$ is surjective and $\mathrm{sw}(\rho) = \mathrm{sw}(\rho')$. By Proposition 3.5, we may assume that $\rho'(\gamma) = \rho(\gamma)$ for $\gamma \in \{a_1, b_1\}$. Since the image of $\rho$ is dense in $\mathrm{O}_2(\mathbb{R})$, one of the $\rho(\gamma)$ has infinite order, where $\gamma \in \{a_2, \ldots, b_g\}$. By Proposition 4.1, there exists a sequence $\sigma_n$ in $\mathrm{Aut}^+(\Gamma_{g-1})$ such that $\sigma_n \cdot \rho \to \rho'$, following the notations of Lemma 2.2. Therefore $\rho' \in \overline{\mathrm{Aut}^+(\Gamma) \cdot \rho}$. □

### 4.1.3 The group $\mathrm{SO}_3(\mathbb{R})$

In this subsection, we consider representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ such that the image of $\rho$ is dense in $\mathrm{SO}_3(\mathbb{R})$. Let us recall the main result of [PX02].

**Theorem 4.3.** *[Previte-Xia] Let* $g' \geqslant 1$ *and* $b \geqslant 0$. *Let* $\rho \in \mathrm{Hom}(\Gamma_{g',b}, \mathrm{SU}_2)$, *and let us denote by* $[\rho]$ *its conjugacy class in* $\mathrm{Hom}(\Gamma_{g',b}, \mathrm{SU}_2)/\mathrm{SU}_2$. *If* $\rho(\Gamma_{g',b})$ *is dense in* $\mathrm{SU}_2$ *then the closure of the orbit* $\mathrm{Mod}(\Sigma) \cdot [\rho]$ *is the set*

$$\{\rho' : \Gamma_{g',b} \to \mathrm{SU}_2 \mid \forall i \leqslant b\ \exists h \in \mathrm{SU}_2, \rho'(c_i) = h\rho(c_i)h^{-1}\}/\mathrm{SU}_2.$$

Let us recall that the representation space $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))/\mathrm{SO}_3(\mathbb{R})$ has two connected components: one of which is the set of conjugacy classes of representations that lift to $\mathrm{SU}_2$, and the other the set of conjugacy classes of representations that do not lift to $\mathrm{SU}_2$. It follows from Theorem 4.3 that the orbit of a representation in $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ with dense image is dense in its connected component of $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))/\mathrm{SO}_3(\mathbb{R})$.

**Lemma 4.4.** *The* $\mathrm{Mod}(\Sigma)$ *orbit of* $[\rho] \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))/\mathrm{SO}_3(\mathbb{R})$ *with dense image is dense in its connected component.*

*Proof.* Let us see $\rho$ in $\mathrm{Hom}(\Gamma_{g,1}, \mathrm{PSU}(2))$ where $\rho(c_1) = \mathrm{id}$. There exists a lift $\tilde{\rho} \in \mathrm{Hom}(\Gamma_{g,1}, \mathrm{SU}(2))$ such that $\tilde{\rho}(\partial\Sigma_{g,1}) = \epsilon\mathrm{Id}$ where $\epsilon = 1$ if $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ and $\epsilon = -1$ otherwise. Theorem 4.3 ensures us that the orbit of $[\tilde{\rho}]$ is dense in its relative character variety, hence its projection in $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))/\mathrm{SO}_3(\mathbb{R})$ is dense in its connected component. $\square$

## 4.2 Affine holonomy

In this subsection we consider representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$, where the group $\mathrm{Aff}(\mathbb{C})$ is the subgroup of $\mathrm{PSL}_2(\mathbb{C})$ that fix $\infty \in \mathbb{CP}^1$. In other words, $\mathrm{Aff}(\mathbb{C}) = \{\pm \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{C}^*, t \in \mathbb{C}\}$. The map $\mathrm{Aff}(\mathbb{C}) \to \mathbb{C}^* \rtimes \mathbb{C}$ defined by $\pm \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \mapsto (\lambda^2, \lambda^{-1}t)$ is an isomorphism. We denote by $az + b$ the element of $\mathrm{Aff}(\mathbb{C})$ corresponding to $(a, b) \in \mathbb{C}^* \rtimes \mathbb{C}$. We identify a representation $\rho$ with the $2g$-tuple $(\rho(\gamma))_{\gamma \in \{a_1, \ldots, b_g\}}$. Let us first observe that a representation in $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ lifts to $\mathrm{SL}_2(\mathbb{C})$.

**Lemma 4.5.** *Every representation* $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ *lifts to* $\mathrm{SL}_2(\mathbb{C})$.

*Proof.* Let us pick lifts $\tilde{\rho}(\gamma) = \begin{pmatrix} \tilde{\lambda}_\gamma & \tilde{t}_\gamma \\ 0 & \tilde{\lambda}_\gamma^{-1} \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{C})$ of $\rho(\gamma)$ for each $\gamma \in \{a_1, \ldots, b_g\}$. The product of the commutators $[\tilde{\rho}(a_i), \tilde{\rho}(b_i)]$ is Id. $\square$

We denote by $\mathrm{Li} : \mathbb{C}^* \rtimes \mathbb{C} \to \mathbb{C}^*$ the projection on the first coordinate and by $\mathrm{Tr} : \mathbb{C}^* \rtimes \mathbb{C} \to \mathbb{C}$ the projection on the second coordinate. Note that Li is a homomorphism while Tr satisfies $\mathrm{Tr}(h_1 h_2) = \mathrm{Tr}(h_1) + \mathrm{Li}(h_1)\mathrm{Tr}(h_2)$ for every $h_1, h_2 \in \mathrm{Aff}(\mathbb{C})$. Recall that an affine representation $\rho$ induces a representation in $\mathbb{C}^*$, *its linear part* $\mathrm{Li} \circ \rho$. We also define the *translation part* of $\rho$ to be the map $\mathrm{Tr} \circ \rho$.

We identify $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ with the set of affine representations $\rho$ such that $\mathrm{Tr}(\rho) = 0$. The linear part of a representation is in the closure of its orbit under the action of $\mathrm{Aff}(\mathbb{C})$ by conjugation.

**Lemma 4.6.** *Let* $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$. *The representation* $\mathrm{Li} \circ \rho$ *is in the closure of* $\mathrm{Aff}(\mathbb{C}) \cdot \rho$.

*Proof.* The sequence $(g_n^{-1} \rho g_n)_n$ tends to $\mathrm{Li} \circ \rho$, where for every $n \geqslant 1$,

$$g_n = \begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix}.$$

$\square$

Let us denote by $\mathbb{S}^1$ the unit circle $\{z \in \mathbb{C} \mid z\overline{z} = 1\}$. The group $\mathrm{Isom}^+(\mathbb{E}^2)$ is the subgroup $\mathbb{S}^1 \rtimes \mathbb{C}$ of $\mathbb{C}^* \rtimes \mathbb{C} = \mathrm{Aff}(\mathbb{C})$. It is the group of positive isometries of the Euclidean plane.

Let us suppose for now that $\rho$ is Euclidean: $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$. The representation $\rho$ acquires a notion of *volume*, see also [Gha16, Section 6].

**Lemma 4.7.** *There exists a smooth $\rho$-equivariant function $f : \tilde{\Sigma} \to \mathbb{C}$. Moreover, the pullback of $dx \wedge dy$ by $f$ induces a form $\omega$ on $\Sigma$ whose volume $\int_\Sigma \omega$ does not depend on $f$.*

*Proof.* Since $\mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ is path-connected, the bundle $\tilde{\Sigma} \times \mathbb{C}/\Gamma$ is topologically trivial, where $\Gamma$ acts by $\gamma \cdot (\tilde{x}, z) = (\gamma \cdot \tilde{x}, \rho(\gamma) \cdot z)$. Hence it admits smooth sections. We can lift such a section to a function $F : \tilde{\Sigma} \to \tilde{\Sigma} \times \mathbb{C}$ of the form $F(\tilde{x}) = (\tilde{x}, f(z))$. The function $f : \tilde{\Sigma} \to \mathbb{C}$ is smooth and $\rho$-equivariant.

Given two smooth $\rho$-equivariant functions $f_0$ and $f_1$, the smooth function $H : (x, t) \in \tilde{\Sigma} \times [0, 1] \mapsto t f_1(x) + (1-t) f_0(x)$ is also $\rho$-equivariant. The pullback $H^*(dx \wedge dy)$ is $\Gamma$-invariant and thus induces a form $\omega$ on $\Sigma \times [0, 1]$. Observe that this form is closed since $dx \wedge dy$ is closed. Therefore Stokes' theorem gives:

$$0 = \int_{\Sigma \times [0,1]} d\omega = \int_{\Sigma \times \{1\}} \omega - \int_{\Sigma \times \{0\}} \omega.$$

The restriction of $\omega$ to $\Sigma \times \{i\}$ is the pullback of $dx \wedge dy$ by $f_i$ for $i = 0, 1$. $\square$

**Definition 4.8.** The volume of $\rho$ is the number $\mathrm{Vol}(\rho) = \int_\Sigma \omega$, where $\omega$ is the induced volume form on $\Sigma$ by the pullback of $dx \wedge dy$ by a smooth $\rho$-equivariant function $f : \tilde{\Sigma} \to \mathbb{C}$.

*Remark* 4.9. The volume is invariant under the action of $\mathrm{Aut}^+(\Gamma)$ and under the action of $\mathrm{Isom}^+(\mathbb{E}^2)$ by conjugation.

### 4.2.1 Finite linear part

Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ be such that $\alpha = \mathrm{Li} \circ \rho$ has finite image of order $n \geqslant 1$. We also suppose that $\rho$ is not spherical.

We exhibit some representations in the closure of the orbit of $\rho$ under the action of $\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C})$. Using our classification of the orbits of $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$ under the action of $\mathrm{Aut}^+(\Gamma)$, we provide a simple form of $\rho$ if $n \geqslant 2$.

**Lemma 4.10.** *If $n \geqslant 2$, then there exists $\rho' \in \mathrm{Aut}^+(\Gamma) \times \mathbb{C} \cdot \rho$ such that*

$$\rho' = (\zeta z, z, z + z_3, z + z_4, \ldots, z + z_{2g}).$$

*Where $\mathbb{C}$ is the subgroup of $\mathrm{Isom}^+(\mathbb{E}^2)$ of translations, and $\zeta = e^{2\pi i/n}$.*

*Proof.* There exists $\sigma \in \mathrm{Aut}^+(\Gamma)$ such that $\sigma \cdot \alpha = (\zeta, 1, \ldots, 1)$ by Proposition 3.2. Therefore, replacing $\rho$ with $\sigma \cdot \rho$ we have

$$\rho = (\zeta z + z_1, z + z_2, z + z_3, \ldots, z + z_{2g}).$$

Now we can conjugate $\rho$ by a translation so that $z_1 = 0$. The product of the commutators $[\rho(a_i), \rho(b_i)]$ is $[\zeta z, z + z_2]$ and must be trivial. Thus $\zeta z$ commutes with $z + z_2$ hence $z_2 = 0$. $\square$

There are even simpler representations in the closure of the orbit of $\rho$ under the action of $\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C})$.

**Lemma 4.11.** *Let $\rho_\infty \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ be the representation defined by*

$$\rho_\infty = \begin{cases} (z+1, z, z, \dots z) & \text{if } n = 1 \\ (\zeta z, z, z+1, z, \dots, z, z) & \text{if } n \geqslant 2. \end{cases}$$

*We have $\rho_\infty \in \overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C}) \cdot \rho}$.*

*Proof.* We first suppose that $n = 1$. Let us write $\rho = (z + z_1, \dots, z + z_{2g})$. We have supposed that $\rho$ is not conjugated in $\mathrm{SO}_3(\mathbb{R})$, hence there exists $1 \leqslant i \leqslant 2g$ such that $z_i \neq 0$. Exchanging the handles if necessary we can assume that $1 \leqslant i \leqslant 2$. Applying a Dehn twist in the first handle if needed, we may even assume that $i = 2$. Conjugating $\rho$ with $z \mapsto z_2 z$, we may assume that $z_2 = 1$. A power of a Dehn twist along a curve homotopic to $b_1$ replaces $\rho$ with

$$\rho_N = (z + N + z_1, z + 1, z + z_3, \dots z + z_{2g}).$$

The representation $h_N^{-1} \rho_N h_N$ converges to $\rho_\infty$, where $h_N = \begin{pmatrix} \sqrt{N} & 0 \\ 0 & 1/\sqrt{N} \end{pmatrix}$. We now suppose that $n \geqslant 2$. We may start with $\rho$ as in Lemma 4.10. Since $\rho$ is not conjugated in $\mathrm{SO}_3(\mathbb{R})$, there exists $3 \leqslant i \leqslant 2g$ such that $z_i \neq 0$. We then proceed as in the case $n = 1$. $\square$

If $n \in \{1, 2\}$ then the group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathrm{Hom}(\Gamma, \mathbb{Z}_n \rtimes \mathbb{C})$. This action is defined as follows: if $A \in \mathrm{GL}_2^+(\mathbb{R})$ and $\rho \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2 \rtimes \mathbb{C})$, then $A \cdot \rho$ is defined to be the representation $\rho' : \gamma \mapsto \alpha(\gamma) z + A \cdot t(\gamma)$, where $\alpha = \mathrm{Li} \circ \rho$ and $t = \mathrm{Tr} \circ \rho$. Let us make an observation that will be useful later in the case $n = 2$.

**Lemma 4.12.** *Let $x, y \in \mathbb{C}$. The representation $(-z, z, z+x, z+y)$ in $\mathrm{Hom}(\Gamma_2, \mathbb{Z}_2 \rtimes \mathbb{C})$ is in the same orbit as the representation $(z + x, -z + y, z - x, -z)$ under the action of $\mathrm{Aut}^+(\Gamma_2)$.*

*Proof.* Let us start with the representation $\rho = (-z, z, z + x, z + y)$. We may apply an automorphism of $\mathrm{Aut}^+(\Gamma_1)$ in the first handle so that $\rho = (z, -z, z + x, z+y)$ by Proposition 3.2. We then apply a Dehn twist along a curve homotopic to $a_2^{-1} b_1$, that replaces $\rho$ with $(-z + x, -z, z + x, -z + x + y)$. A Dehn twist in the second handle replaces $\rho(b_2)$ with $-z + y$. We may now apply a Dehn twist in the first handle and then interchange the two handles so that $\rho = (z + x, -z + y, z - x, -z)$. $\square$

Let us consider the group $\Lambda = \Lambda(\rho) \subset \mathbb{C}$ of $\mathrm{Tr}(\rho(\gamma))$ for $\gamma \in \ker \alpha$. The group $\Lambda$ may be a lattice in $\mathbb{C}$ or not. This property is invariant under the action of $\mathrm{Aut}^+(\Gamma)$. Kapovich gave in [Kap20] a description of the closure of orbits of the $\mathrm{Aut}^+(\Gamma) \times \mathrm{GL}_2^+(\mathbb{R})$ action on $\mathrm{Hom}(\Gamma, \mathbb{C})$, see also [CDF14, Proposition 2.10]. As a corollary of this description we show that if $g \geqslant 3$ and $n = 1$ and $\rho$ of positive volume then there exists $\rho_\infty \in \overline{\mathrm{Aut}^+(\Gamma) \cdot \rho}$ such that $\Lambda(\rho_\infty)$ is a lattice with area a given divisor of $\mathrm{Vol}(\rho)$. Note that if $n = 1$ then $\rho$ may be seen as a representation in $\mathrm{Hom}(\Gamma, \mathbb{C})$ and we have $\Lambda(\rho) = \rho(\Gamma)$.

**Lemma 4.13.** *Let us suppose that $g \geqslant 3$. Let $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ be such that $\mathrm{Vol}(\chi) > 0$ and $\chi(\Gamma)$ is not a lattice in $\mathbb{C}$. For every $m \geqslant 1$ there exists $\chi_\infty \in \overline{\mathrm{Aut}^+(\Gamma) \cdot \chi}$ such that $\Lambda = \chi_\infty(\Gamma)$ is a lattice, and $\mathrm{Vol}(\chi_\infty)/\mathrm{Area}(\Lambda) = m$.*

*Proof.* It follows from [CDF14, Proposition 2.10] that, replacing $\chi$ with $A \cdot \chi$, $A \in \mathrm{GL}_2^+(\mathbb{R})$ if necessary, the closure of $\mathrm{Aut}^+(\Gamma) \cdot \chi$ contains the set of representations $\chi' \in \mathrm{Hom}(\Gamma, \mathbb{C}) \simeq \mathbb{C}^{2g}$ such that $\mathrm{Vol}(\chi') = \mathrm{Vol}(\chi)$ and $\Im(\chi') \in \mathbb{Z}^{2g}$ and $\gcd(\Im(\chi')) = 1$. In particular, $\chi' = (\frac{\mathrm{Vol}(\chi)}{m}, im, i, 0, \ldots, 0)$ is in $\overline{\mathrm{Aut}^+(\Gamma) \cdot \chi}$. $\square$

Let us first classify the orbits of the $\mathrm{Aut}^+(\Gamma)$ action on the set $\Omega(L)$ of representations in $\mathrm{Hom}(\Gamma, \mathbb{C})$ with image a given lattice $L$ in $\mathbb{C}$. If $\rho \in \Omega(L)$ then $\mathrm{Vol}(\rho)$ is a multiple of $\mathrm{Area}(L)$.

**Lemma 4.14.** *If $g \geqslant 2$, then we define a bijection $\Omega(L) \to \mathbb{Z}$ by*

$$[\rho'] \mapsto \mathrm{Vol}(\rho')/\mathrm{Area}(L).$$

*Remark* 4.15. In particular, for each $\rho' \in \mathrm{Hom}(\Gamma, \mathbb{C})$ with image $L$, there exists $\sigma \in \mathrm{Aut}^+(\Gamma)$ such that $\sigma \cdot \rho' = (mu, v, u, 0, \ldots, 0)$, where $m = \mathrm{Vol}(\rho')/\mathrm{Area}(L)$.

We refer to [LF20, Proposition 2.2] for a proof of Lemma 4.14.

We suppose that $\rho = (\zeta z, z, z + x_1, z + y_1, \ldots, z + x_{g-1}, z + y_{g-1})$. Observe that $\Lambda$ is the group generated by the $\xi x_i$ and $\xi y_i$ for $1 \leqslant i < g$ and $\xi$ a $n$-th root of unity.

Let us first suppose that $\Lambda$ is not a lattice in $\mathbb{C}$. We show that we may assume that the group generated by the $x_i$ and $y_i$, $1 \leqslant i < g$, is not a lattice if $g \geqslant 3$.

**Lemma 4.16.** *Suppose $g \geqslant 3$. Replacing $\rho$ with some $\sigma \cdot \rho$ if necessary, we can assume that the group generated by the $x_i$ and $y_i$, $1 \leqslant i < g$, is not a lattice.*

*Proof.* Let us suppose that it is a lattice, and let $u, v$ be generators of this lattice. There exists $\xi$ an $n$-th root of unity such that $\xi u$ or $\xi v$ is not in $\mathbb{Q}u \oplus \mathbb{Q}v$. We can assume that $x_2 = u$, $y_2 = v$, $x_3 = mu$, $y_3 = v$ and $x_j = y_j = 0$ for every $j \geqslant 4$, where $m \in \mathbb{Z}$ by Lemma 4.14 and Lemma 2.2. Again by Lemma 2.2 we may conjugate $\rho(\gamma)$ by $\xi$, $\gamma \in \{a_1, b_1, a_2, b_2\}$, since $\xi$ is a power of $\rho(a_1)$. We thus have replaced $x_2$ and $y_2$ with $\xi x_2$ and $\xi y_2$. These new $x_i$ and $y_i$ do not generate a lattice. $\square$

We now suppose that $\Lambda$ is a lattice. Recall that $\zeta = e^{2\pi i/n}$. We can assume that the map $z \mapsto \zeta z$ is in the image of $\rho$ by Lemma 4.10; conjugating $\rho$ with a translation if necessary. Thus if $a \in \Lambda$, we also have $\zeta a \in \Lambda$ because one can conjugate $z + a$ with $\zeta z$. We also have $\zeta^2 a \in \Lambda$, thus $\zeta^2 \in \mathbb{Q} \oplus \zeta\mathbb{Q}$. Therefore $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leqslant 2$. Since $\varphi(n) = [\mathbb{Q}(\zeta) : \mathbb{Q}]$, we have $\varphi(n) \leqslant 2$, where $\varphi$ denotes the Euler's totient function. This implies $n \in \{1, 2, 3, 4, 6\}$. In fact we may even characterize the lattices that arise in this fashion for $n \in \{3, 4, 6\}$. Let $\omega = \exp(2\pi i/3)$.

**Lemma 4.17.** *There exists $v \in \mathbb{C}^\star$ such that*

$$v\Lambda = \begin{cases} \mathbb{Z}[i] & \text{if } n = 4 \\ \mathbb{Z}[\omega] & \text{if } n \in \{3, 6\}. \end{cases}$$

*Proof.* Choose a vector of minimal length $v$ in $\Lambda \setminus \{0\}$, and replace $\Lambda$ with $\frac{1}{v}\Lambda$. Since $\Lambda$ is stable by multiplication by $e^{2\pi i/n}$, $\mathbb{Z}[e^{2\pi i/n}]$ is a lattice included in $\Lambda$. Let us suppose that $n = 4$. Let $v \in \Lambda$. There exists $w \in \mathbb{Z}[i]$ such that $v - w = a + ib$ satisfies $-\frac{1}{2} \leqslant a, b \leqslant \frac{1}{2}$. Therefore $|v - w| < 1$ and $w = v$. Thus $\Lambda = \mathbb{Z}[i]$. Similarly if $n \in \{3, 6\}$, let $v \in \Lambda$. There exists $w \in \mathbb{Z}[\omega]$ such that $v - w = a + \omega b$, where $-\frac{1}{2} \leqslant a, b \leqslant \frac{1}{2}$. We have $|v - w| \leqslant |a| + |b| \leqslant 1$. Moreover one of these inequalities must be strict, hence $w = v$ thus $\Lambda = \mathbb{Z}[\omega]$. $\square$

We can thus assume that $\Lambda = \mathbb{Z}[i]$ if $n = 4$ and $\Lambda = \mathbb{Z}[\omega]$ if $n \in \{3, 6\}$. We now show that we may also assume that the group $\Lambda$ is generated by the $x_i$ and $y_i$ for $1 \leqslant i < g$.

**Lemma 4.18.** *We may assume that the group generated by $x_i, y_i$ is $\Lambda$, replacing $\rho$ with some $\rho' \in \mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C}) \cdot \rho$.*

*Proof.* Let us suppose that the group $L$ generated by the $x_i, y_i$ is different from $\Lambda$. Observe we cannot have $n \in \{1, 2\}$, thus $n \in \{3, 4, 6\}$. The subgroup $L$ of $\Lambda$ is a lattice in $\mathbb{C}$: the vectors $x_i, y_i$ cannot be colinear since $\mathrm{Vol}(\rho) > 0$. Let $u, v$ be generators of $L$. Let us show that we can assume that the angle between $u$ and $v$ is in $[\pi/3, 2\pi/3]$. We can normalize the lattice $L$ so that $v = 1$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on the generators of $L$ is then the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$. It is well known that the domain $D = \{x \in \mathbb{H}^2 \mid -1/2 \leqslant \Re(x) \leqslant 1/2\} \cap \{x \in \mathbb{H}^2 \mid |x| \geqslant 1\}$ is a fundamental domain for this action, see for example [Ser77, Chapter 7]. Therefore we now assume that the angle $\theta$ between $u$ and $v$ is in $[\pi/3, 2\pi/3]$. We may also assume that $u = e^{i\theta} v$ if $\theta \in \{\pi/3, 2\pi/3\}$. In this case $L = v\mathbb{Z}[\omega]$ and $\Lambda = L$ if $n \in \{3, 6\}$, since $L$ is invariant under multiplication by $e^{i\pi/3}$. By Lemma 4.14 we can suppose that

$$\rho = (\zeta z, z, z + u, z + v, z + u, z + mv, z, \ldots, z)$$

where $m \in \mathbb{Z}$. By Lemma 2.2 we may conjugate $\rho(\gamma)$ with $\rho(a_1)$ for $\gamma \in \{a_1, b_1, a_2, b_2\}$, and thus replace $\rho$ with

$$\rho = (\zeta z, z, z + \zeta u, z + \zeta v, z + u, z + mv, z, \ldots, z).$$

The angle between $u$ and $\zeta v$ is $\theta + \frac{2\pi}{n}$ therefore they generate a lattice unless $(n, \theta) \in \{(4, \frac{\pi}{2}), (3, \frac{\pi}{3}), (6, \frac{2\pi}{3})\}$. Indeed $0 < |\sin(\theta + \frac{2\pi}{n})| < |\sin\theta|$ for others $(n, \theta)$, see Figure 4. Since we are done in the case where $(n, \theta) \notin \{(3, \frac{\pi}{3}), (6, \frac{2\pi}{3})\}$, let us suppose that $(n, \theta) = (4, \frac{\pi}{2})$. Observe that if $|u| = |v|$, then $L$ is invariant by multiplication by $i$, thus $L = \Lambda$. We can thus assume that $|u| < |v|$. There exist $\sigma \in \mathrm{Aut}^+(\Gamma)$ and $m \in \mathbb{Z}$ such that

$$\sigma \cdot \rho = (iz, z, z + u, z, z + u, z + mv, z, \ldots, z).$$

By Lemma 2.2, we may conjugate $\rho(\gamma)$ by $\rho(a_1)$ for $\gamma \in \{a_1, b_1, a_2, b_2\}$ and thus replace $\rho$ as before with

$$\rho = (iz, z, z + iu, z, z + u, z + mv, z \ldots, z).$$

The new lattice $L'$ satisfies $\langle u, iu \rangle \subset L'$. Since $|\det(u, iu)| = |u|^2$ and $|u||v| = |\det(u, v)|$, we have $\mathrm{Area}(L') \leqslant |\det(u, iu)| < |\det(u, v)| \leqslant \mathrm{Area}(L)$. Therefore in each case we may replace the lattice $L$ with another lattice $L'$ such that either $L' = \Lambda$ or $[\Lambda : L'] < [\Lambda : L]$. By iterating this process, we end up after a finite number of steps with $L = \Lambda$. $\square$

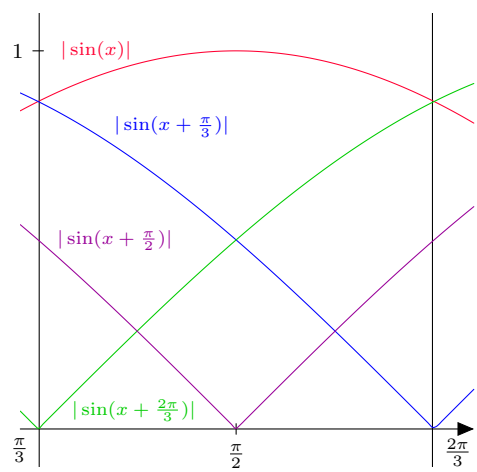


Figure 4: Sinusoidal functions.

#### 4.2.2 Euclidean representation with infinite linear part

In [Gha18], Ghazouani studied the action of the mapping class group on $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))/\mathrm{Aff}(\mathbb{C})$ and in particular he gave a precise description of the closure $H_\alpha \cdot \rho$ in $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))/\mathrm{Aff}(\mathbb{C})$, where $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ and $\alpha = \mathrm{Li} \circ \rho$ and $H_\alpha = \{\sigma \in \mathrm{Aut}^+(\Gamma) \mid \sigma \cdot \alpha = \alpha\}$.
In particular he showed the following, see [Gha18, Lemma 5.3, Proposition 6.4].

**Proposition 4.19** (Ghazouani)**.** *If $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ is such that $\mathrm{Vol}(\rho) > 0$ and $\mathrm{Li} \circ \rho$ has infinite image, then the closure of $H_\alpha \times \mathrm{Aff}(\mathbb{C}) \cdot \rho$ is the set of $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ such that $\mathrm{Vol}(\rho') > 0$ and $\mathrm{Li} \circ \rho' = \mathrm{Li} \circ \rho$.*

It follows that under the same hypotheses, the closure of $H_\alpha \times \mathrm{Isom}^+(\mathbb{E}^2) \cdot \rho$ is the set of $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ such that $\mathrm{Vol}(\rho) = \mathrm{Vol}(\rho')$. Since we have seen that $\mathrm{Aut}^+(\Gamma) \cdot \alpha$ is dense in $\mathrm{Hom}(\Gamma, \mathbb{S}^1)$ when $\alpha \in \mathrm{Hom}(\Gamma, \mathbb{S}^1)$ has infinite image, we get the following.

**Proposition 4.20.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ be such that $\mathrm{Vol}(\rho) > 0$ and $\mathrm{Li} \circ \rho \in \mathrm{Hom}(\Gamma, \mathbb{S}^1)$ has infinite image. The closure of $\mathrm{Aut}^+(\Gamma) \times \mathrm{Isom}^+(\mathbb{E}^2) \cdot \rho$ is the set of $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ such that $\mathrm{Vol}(\rho') = \mathrm{Vol}(\rho)$.*

*Proof.* Let $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ be such that $\mathrm{Vol}(\rho') = \mathrm{Vol}(\rho)$. Let us first show that $\rho' \in \overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C}) \cdot \rho}$. Let $U$ be a neighborhood of $\rho'$ in $\mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$. There exist a neighborhood $U_1$ of $\mathrm{Li} \circ \rho'$ and a neighborhood $U_2$ of $\mathrm{Tr} \circ \rho'$ in the set of maps $\Gamma \to \mathbb{C}$ endowed with the pointwise convergence topology so that $\varphi^{-1}(U_1 \times U_2) \subset U$, where

$$\varphi : p \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C})) \mapsto (\mathrm{Li} \circ p, \mathrm{Tr} \circ p).$$

There exists $\sigma \in \mathrm{Aut}^+(\Gamma)$ such that $\mathrm{Li} \circ \rho'' \in U_1$, where $\rho'' = \sigma \cdot \rho$ by Proposition 4.1. By Proposition 4.19, there exists $f \in \mathrm{Aff}(\mathbb{C})$ and $\sigma' \in H_{\mathrm{Li} \circ \rho''}$ such that $(\sigma', f) \cdot \rho'' \in U$. Therefore $\rho' \in \overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C}) \cdot \rho}$. Let us now show that $\rho' \in \overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{Isom}^+(\mathbb{E}^2) \cdot \rho}$. There exists a sequence $(f_n)_n \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))^{\mathbb{N}}$ and $(\sigma_n)_n \in \mathrm{Aut}^+(\Gamma)^{\mathbb{N}}$ such that $(\sigma_n, f_n) \cdot \rho \to \rho'$. Let us write $f_n = \lambda_n z + \mu_n$, $\lambda_n = \mathrm{Li}(f_n)$ and $\mu_n = \mathrm{Tr}(f_n)$. The sequence $|\lambda_n|$ tends to 1 since $\mathrm{Vol}((\sigma_n, f_n) \cdot \rho) = |\lambda_n|^2 \mathrm{Vol}(\rho)$. The sequence defined by $h_n = \lambda_n / |\lambda_n| z + \mu_n / |\lambda_n|$ is in $\mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ and is such that $(\sigma_n, h_n) \cdot \rho \to \rho'$. $\square$

# 5 Branched projective structures

## 5.1 Definitions

**Definition 5.1.** A branched projective atlas on $\Sigma$ is the datum of an open covering $(U_\alpha)_\alpha$ of $\Sigma$ together with maps, called *charts*, $\varphi_\alpha : U_\alpha \to V_\alpha$, where $V_\alpha$ is an open set of $\mathbb{CP}^1$. We require that the $\varphi_\alpha$ are topologically conjugated to the map from the unit disk to itself $z \mapsto z^n$, for some $n \geqslant 1$. We also require that when $U_\alpha \cap U_\beta \neq \emptyset$, there exists $h \in \mathrm{PSL}_2(\mathbb{C})$ such that $\varphi_\alpha = h \circ \varphi_\beta$ on $U_\alpha \cap U_\beta$.

Two branched atlases are said to be equivalent if their union is still a branched atlas. Let us denote by $\mathbf{BP}(\Sigma)$ the set of equivalence classes of branched projective atlases on $\Sigma$, or equivalently the set of maximal atlases. Given a maximal atlas we may associate to each point $p \in \Sigma$ an integer $n \geqslant 0$ as follows. We can pick a chart $\varphi_\alpha$ around $p$ that is topologically conjuated to $z \mapsto z^{n+1}$ and such that $p$ correspond to 0. This integer $n \geqslant 0$ does not depend on the chart and vanishes everywhere but on a discrete, hence finite, set. We thus define a divisor on $\Sigma$ called the *branch divisor*. A point where this integer $n$ is at least 1 is called a branch point, or a conical point. We say that $n$ is the order of this branch point, or that this conical point has angle $2\pi(n+1)$. The group $\mathrm{Homeo}_+(\Sigma)$ acts on $\mathbf{BP}(\Sigma)$ by precomposing the charts.

**Definition 5.2.** A branched projective structure on $\Sigma$ is an element of $\mathcal{BP}(\Sigma) = \mathbf{BP}(\Sigma)/\mathrm{Homeo}_0(\Sigma)$.

The branch divisor of a branched projective structure is not well-defined but the datum of the orders of the branch points is. Hence $\mathcal{BP}$ decomposes as the union of $\mathcal{P}(n_1, \ldots, n_k)$ of branched projective structures with $k$ branch points of order $n_1, \ldots, n_k$.

Given $X \in \mathbf{BP}(\Sigma)$, we may choose a chart $U_\alpha$ and consider its analytic continuation $f : \widetilde{\Sigma} \to \mathbb{CP}^1$ as usual with geometric structures, see [Thu97, Chapter 3]. There exists a unique $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ such that $f$ is $\rho$-equivariant: $f(\gamma \cdot z) = \rho(\gamma) \cdot f(z)$ for all $z \in \widetilde{\Sigma}$, and $\gamma \in \Gamma$. Choosing another chart would change $f$ to $h \circ f$ and $\rho$ to $h\rho h^{-1}$ for some $h \in \mathrm{PSL}_2(\mathbb{C})$. The map $f$ is called a developing map of $X$ and $\rho$ the holonomy of $X$. Note that one can give an element of $\mathbf{BP}(\Sigma)$ by giving its developing map. Observe that the holonomy $\mathrm{Hol}(X) = [\rho] \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$ satisfies $\mathrm{Hol}(h \cdot X) = \mathrm{Hol}(X)$ for every $h \in \mathrm{Homeo}_0(\Sigma)$. Indeed if $h$ is isotopic to the identity, then it admits a $\Gamma$-equivariant lift $H : \tilde{\Sigma} \to \tilde{\Sigma}$. A developing map for $g \cdot X$ is thus $\mathrm{dev}(X) \circ H$, that is $\rho$-equivariant. Therefore we have a well-defined map

$$\mathrm{Hol} : \mathcal{BP} \to \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C}).$$

In the rest of the paper, we will abuse notation and identify $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ with the subset of $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ of representations $\rho$ whose conjugacy class is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

Let us give some examples of branched projective surfaces.

**Example 5.3.**

1. A (possibly branched) hyperbolic structure on $\Sigma$ gives a developing map $\tilde{\Sigma} \to \mathbb{H}^2$ that is $\rho$-equivariant for some $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$.
2. Suppose $S$ is a surface endowed with a branched projective structure $X$. Let us consider $f : \Sigma \to S$ a branched cover. We define the *pullback* of $X$ by $f$, that we denote $f^*X$, as follows. A developing map for $f^*X$ is given by $\mathrm{dev} \circ \tilde{f}$, where $\mathrm{dev} : \tilde{S} \to \mathbb{CP}^1$ is a developing map for $X$ and $\tilde{f} : \tilde{\Sigma} \to \tilde{S}$ is a lift of $f$.

## 5.2 The mapping class group action on $\mathcal{BP}$

The mapping class group $\mathrm{Mod}(\Sigma) = \mathrm{Homeo}_+(\Sigma)/\mathrm{Homeo}_0(\Sigma)$ acts naturally on $\mathcal{BP}$. This action preserves the sets $\mathcal{P}(n_1, \ldots, n_k)$. The map Hol is equivariant with respect to the actions of $\mathrm{Mod}(\Sigma)$ on $\mathcal{BP}$ and on $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))/\mathrm{PSL}_2(\mathbb{C})$.

**Proposition 5.4.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ and $\sigma \in \mathrm{Aut}^+(\Gamma)$ and $h \in \mathrm{PSL}_2(\mathbb{C})$. For every $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$, we have:*

$$\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k)) \iff h(\sigma \cdot \rho)h^{-1} \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k)).$$

## 5.3 The Ehresmann-Thurston principle

We prove a version of the Ehresman-Thurston principle, Proposition 1.6, adapting an argument of Kapovich [Kap20, Theorem 2.7]. We first show that given a projective structure on $\Sigma$, one can find a triangulation of $\Sigma$ whose edges are sent to broken circular arcs and whose triangles develop injectively. Let us fix $P \subset \tilde{\Sigma}$ a fundamental polygon for the action of $\Gamma$ on $\tilde{\Sigma}$. The edges of $P$ are paired and identified in $\Sigma$ with $\gamma_1, \dots, \gamma_m \in \Gamma$. Let us consider the compact set

$$K = P \cup \bigcup_{i=1}^{m} \gamma_i \cdot P.$$

**Lemma 5.5.** *Let us consider a branched projective structure $X$ on $\Sigma$. There exists a triangulation $T$ of $\Sigma$ such that*

1. *Every conical point of $X$ is a vertex of $T$*
2. *Every edge of $T$ is a broken circular arc: it can be decomposed into segments that are sent to circular arcs by the developing map. We also require that two contiguous circular arcs are not tangent.*
3. *Each lift $\tilde{T}$ of a triangle $T$ that lies in $K$ is send injectively by the developing map onto a set of area less than $\frac{1}{3}\mathrm{Area}(\mathbb{CP}^1)$, where* Area *is the area for the round metric on $\mathbb{CP}^1$.*

*Proof.* Let us start with an arbitrary triangulation of $\Sigma$. We refine this triangulation so that each triangle is included in a chart and such that the image of each lift of these triangles in $K$ has area less than $\frac{1}{3}\mathrm{Area}(\mathbb{CP}^1)$. We can again refine the triangulation so that each branch point is a vertex of this triangulation and such that each triangle develops injectively. We then approximate every edge with a broken circular line. We thus change our triangulation so that it satisfies the hypotheses of Lemma 5.5. □

We now prove Proposition 1.6.

*Proof.* Let $X \in \mathcal{P}(n_1, \dots, n_k)$ and let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ be its holonomy. We fix a sequence $(\rho_n)_n$ in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ such that $\rho_n \to \rho$. Let us choose a triangulation of $X$ as in Lemma 5.5. Let us consider the graph $G$ on $\Sigma$ obtained from the triangulation by subdivising its edges as follows. Let us color the vertices of this triangulation in black. We add green vertices on the points that are sent to endpoints of the circular arcs. We then add a red vertex in the interior of each edge of this new graph. Let us lift $G$ to $\tilde{G}$ on the universal cover $\tilde{\Sigma}$ of $\Sigma$. We fix $p_1, \dots, p_l$ a system of representatives of the vertices of $\tilde{G}$ modulo the action of $\Gamma$. For each vertex $p$ in $\tilde{G}$, we can write in a unique way $p = \gamma p_i$ with $\gamma \in \Gamma$ and $1 \leqslant i \leqslant l$. We let $f_n(p) = \rho_n(\gamma) \cdot f(p_i)$. Let us consider two contiguous edges with vertices $e_1, e_2$ and $e_3$, with $e_2$ the only red one, which is in the middle of $e_1$ and $e_3$. We define $f_n$ on this edge so that it sends it to the circular arc of $\mathbb{CP}^1$ that passes through $f_n(e_2)$ and that is bounded by

$f_n(e_1)$ and $f_n(e_3)$. Let us fix $T_1, \dots, T_l \subset K$ a system of representatives of the triangles. If $n$ is large enough, then the image of the edges of each $T_i$ bound two triangles in $\mathbb{CP}^1$. We extend $f_n$ on $T_i$, by sending $T_i$ homeomorphically onto the one with the smallest area. We then extend $f_n$ by $\rho$-equivariance. Let us now show that $f_n$ is a branched covering map when $n$ is large enough, with singularities of the type $z \mapsto z^{n_1+1}, \dots, z \mapsto z^{n_k+1}$. We must show that $f_n$ is locally injective outside some of the black vertices, where it is a $k$-fold covering map if $f$ is a $k$-fold covering map, for $n$ large enough. By $\rho$-equivariance, it suffices to check it in the interior of $K$. Let $z$ be in the interior of $K$. If $z$ not on the graph $\tilde{G}$, then $f_n$ is locally injective around $z$. Let us suppose that $z$ a black vertex of the graph $\tilde{G}$. The angles formed by the circular arcs around $f_n(z)$ converge to the angles formed by the circular arcs around $f(z)$, and their sum is in $2\pi\mathbb{Z}$. Hence $f_n$ is a $k$-fold cover around $z$ for $n$ large enough, if $f$ is itself a $k$-fold cover around $z$. If $z$ is a point on the interior of an edge of the triangulation, or a vertex that is not black, then $f$ is locally injective around $z$. Indeed, one just need to check that the two triangles around $z$ are not folded on top of each other by $f_n$. Since this is the case with $f$, and that the image of the triangles by $f_n$ are converging to the image of the triangles by $f$, it does not happen when $n$ is large enough. $\square$

For another point of view and proof of Proposition 1.6, we refer to [LMP09, Theorem 1.3, Remark 2.4]. Let us give a useful consequence of Proposition 1.6.

**Corollary 5.6.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$. If there exists $\rho_\infty \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ satisfying $\rho_\infty \in \overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{PSL}_2(\mathbb{C}) \cdot \rho}$, then $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$.*

## 5.4 Surgeries on branched projective structures

We now describe some ways to construct new projective structures from old ones.

### 5.4.1 Cut and paste

Let us consider a non necessarily connected surface $S$ with a branched projective structure. Let $\gamma, \delta : [0,1] \to S$ be injective arcs on $S$ whose image intersect at most in one point, in which case this point is $\gamma(1) = \delta(0)$. Let us suppose that there exists charts $\varphi_\gamma : U_\gamma \to \mathbb{CP}^1$ and $\varphi_\delta : U_\delta \to \mathbb{CP}^1$ such that the image of $\gamma$ (resp. $\delta$) is contained in $U_\gamma$ (resp. $U_\delta$), and such that $\varphi_\gamma \circ \gamma(t) = \varphi_\delta \circ \delta(t)$ for all $0 \leqslant t \leqslant 1$. We cut open $S$ along the image of $\gamma$ and along the image of $\delta$. This yields four boundary components $\gamma^\pm$ and $\delta^\pm$. We glue back the boudary $\gamma^\pm$ with $\delta^\mp$ by identifying $\gamma(t)$ with $\delta(t)$. See also [CDF14, Section 2.2] and in particular [CDF14, Figure 1]. This surgery adds a handle connecting the two arcs and increases the conical angles at the endpoints of the arcs.

### 5.4.2 Bubbling along a closed curve

There exists a surgery on branched projective structures called *bubbling*, see for example [CDF14, Section 2.3]. It amounts to considering an arc $\gamma$ on $\Sigma$ that is contained in a chart and develops injectively and using the cut and paste surgery describe above on the surface $\Sigma \sqcup \mathbb{CP}^1$ with the arcs $\gamma$ and $\delta$, where $\delta$ is the image of $\gamma$ in a chart. We now describe a slightly modified version of this surgery, that increases the branching order of a single point: bubbling along a closed curve.

**Proposition 5.7.** *Let us consider a projective structure $X \in \mathcal{P}(n_1, \dots, n_k)$ with developing map $f$. Let $c\colon [0,1] \to \Sigma$ be a closed curve such that $c(0)$ is a branch point of order $n_1 \geqslant 0$, and let $\tilde{c}\colon [0,1] \to \tilde{\Sigma}$ be a lift of $c$. Suppose that $f \circ \tilde{c}\colon [0,1] \to \mathbb{CP}^1$ is injective. There exists a projective structure $Y \in \mathcal{P}(n_1 + 2n, \dots, n_k)$ with the same holonomy as $X$, for every $n \geqslant 1$.*

In the statment of Proposition 5.7, $n_1 = 0$ means that $c(0)$ is not a branch point.

*Proof.* We may assume that $c$ does not meet any branch point except $c(0)$. Let us cut open $\mathbb{CP}^1$ along the image of $f \circ \tilde{c}$, and denote by $\delta^{\pm}$ the resulting boundary components. Let us cut open $\Sigma$ along the image $c$, and denote by $\gamma^{\pm}$ the resulting boundary components. We glue back $\gamma^{\pm}$ to $\delta^{\mp}$ by identifying $c(t)$ with $f(\tilde{c}(t))$ for each $t \in [0,1]$. This new projective structure has a conical point of order $n_1 + 2$ if $c(0)$ is a branch point of order $n_1$, and has the same holonomy as $X$. Since $c$ still develops injectively, we can repeat this process. □

### 5.4.3 Breaking up a conical point

Eskin, Masur and Zorich described in [EMZ03, 8.1] a surgery on translation surfaces that breaks a conical point into two conical points with smaller angles. Let us describe this surgery, that we adapt to branched projective structures. It breaks a branch point of order $n$ into two branch points of order $a, b$ such that $a + b = n$. This surgery is local: it leaves the charts of the inital projective structures unchanged outside of a disk embedded in $\Sigma$. Therefore it does not modify its holonomy.

**Proposition 5.8.** *Let $X \in \mathcal{P}(n_1, \dots, n_k)$ and let $a, b \geqslant 1$ be such that $a + b = n_1$. There exists $Y \in \mathcal{P}(a, b, n_2, \dots, n_k)$ with the same holonomy as $X$.*

*Proof.* Locally around a branch point of order $n_1$, the projective structure is made of $2(n_1 + 1)$ half-disks glued together as in Figure 5 for $n_1 = 2$. We modify the gluing patern of their boundaries as in Figure 6 for $a = b = 1$. We refer to [EMZ03, Section 8.1] for more details on this construction. See also [KZ03, Section 4.2]. □

*Remark* 5.9. Let $\rho \in \mathrm{Hol}(\mathcal{P}(d))$. It follows from Proposition 5.8 that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ if $\sum_i n_i = d$.

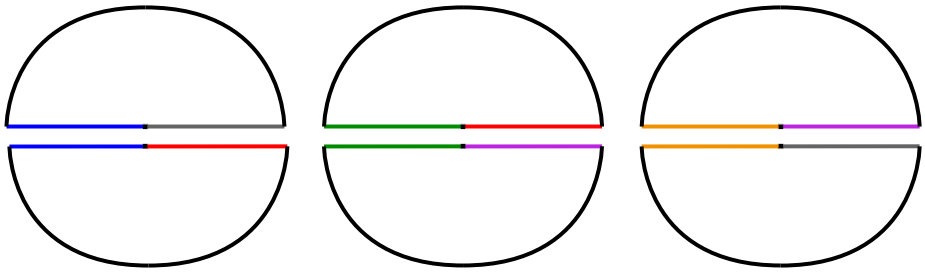

Figure 5: Neighborhood of a branch point of order 2.

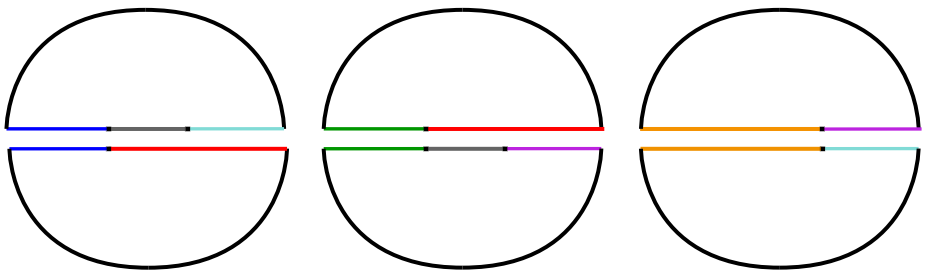

Figure 6: Two branch points of order 1.

# 6 Geometrization

The aim of this section is to prove Theorem 1.1. We will consider representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ that satisfy the conditions of the obstructions and prove that they are *geometric*, that is, that they are the holonomy of a projective structure with the given datum of branch points. Along the way we will also explain the obstructions that we did not explain in the introduction.

## 6.1 Branched cover of the sphere

Let us fix $g \geqslant 0$. We consider the special case where $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is the trivial representation: $\rho(\gamma) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for every $\gamma \in \Gamma$. The aim of this part is to prove the following.

**Proposition 6.1.** *The trivial representation $\rho$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if $\sum_i n_i = 2g + 2l$ for some integer $l \geqslant 1$ such that $\max_i n_i \leqslant l$.*

The developing map $f : \tilde{\Sigma} \to \mathbb{CP}^1$ of a spherical structure with holonomy $\rho$ induces a branched cover $\Sigma \to \mathbb{S}^2$. Conversely, any branched cover $\Sigma \to \mathbb{S}^2$ gives a branched projective structure on $\Sigma$ with trivial holonomy. Therefore we are reduced to the classical problem of the existence of branched covers of the sphere with prescribed branching data. With this point of view it is clear that Proposition 6.1 is equivalent to Proposition 1.4. As observed in [MP19], it is the

$g = 0$ case a necessary and sufficient condition to the existence of a spherical metric with conical angles $2\pi(n_1 + 1), \ldots, 2\pi(n_k + 1)$ since its holonomy must be trivial.

Let us recall the classical statement of the Hurwitz realization problem, following the notations of [EKS84]. A *branch data* is a set of partitions of an integer $d > 0$. A branched cover $\Sigma \to \mathbb{S}^2$ of degree $d$ induces a branch data in the following way: for each point $p \in \mathbb{S}^2$ which is the image of a branch point, the orders $a_i$ of its pre-images give a partition of $d$. A branch data that arises in this way is said to be *realizable*. The Hurwitz realization problem is to characterize the branch data that are realizable. The Riemann-Hurwitz formula gives obstructions to realizability. For a partition $A = [a_1, \ldots, a_l]$, let $v(A) = \sum_i (a_i - 1)$, and for a branch data $\mathcal{D} = \{A_1, \ldots, A_k\}$, let $v(\mathcal{D}) = \sum_i v(A_i)$. If $\mathcal{D}$ is realizable, then the Riemann-Hurwitz formula gives

$$\chi(\Sigma) = 2d - v(\mathcal{D}).$$

In particular, $v(\mathcal{D})$ must be even and at least $2d - 2$. However it is well known that these conditions are not sufficient, see for example [EKS84, PP06]. This problem was considered by Hurwitz in [Hur91] where he showed the following theorem. We identify the conjugacy classes of the symmetric group $\mathfrak{S}_d$ with the partitions of $\{1, \ldots, d\}$ as follows: the lengths of the orbits of the action of the group generated by $\sigma \in \mathfrak{S}_d$ on $\{1, \ldots, d\}$ determine a partition of $d$.

**Hurwitz conditions.** The branch data $\mathcal{D} = \{A_1, \ldots, A_k\}$ is realizable if and only if there exists permutations $\alpha_1 \in A_1, \ldots \alpha_k \in A_k$ such that

1. $\alpha_1 \alpha_2 \ldots \alpha_k = \mathrm{id}$
2. The group generated by the $\alpha_i$ acts transitively on $\{1, \ldots, d\}$.

We show the following statement, that implies Proposition 6.1.

**Proposition 6.2.** *Let $d \geqslant 2$ and $1 \leqslant n_1 \leqslant \ldots \leqslant n_k < d$. The branch data $\mathcal{D} = \{A_1, \ldots, A_k\}$ where $A_i = [n_i + 1, 1, \ldots, 1]$ is realizable if and only $v(\mathcal{D}) = \sum_i n_i$ is even and $v(\mathcal{D}) \geqslant 2d - 2$.*

We follow the reduction methods of [EKS84], and prove this proposition by induction. Let us first check this proposition for $k = 3$.

**Lemma 6.3.** *Proposition 6.2 holds for $k = 3$.*

*Proof.* We proceed by induction on $d$. If $d = 2$ then $n_1 = n_2 = n_3 = 1$ and there is no realizable branch data. Let us suppose that $v(\mathcal{D}) = n_1 + n_2 + n_3$ is even and that $v(\mathcal{D}) \geqslant 2d - 2$. If $n_1 = n_2 = n_3 = d - 1$, then $d$ is odd and we can take $\alpha_1 = \alpha_2$, with $\alpha_1 = (1, \ldots, d)$. Then $\alpha_3 = \alpha_1^{-2}$ is a $d$-cycle, and these $\alpha_i$ satisfy the Hurwitz conditions. Let us now suppose that $\min\{n_1, n_2, n_3\} < d - 1$. We may assume that $n_3 < d - 1$. Let $n_1' = n_1 - 1$, $n_2' = n_2 - 1$ and $n_3' = n_3$ and $\mathcal{D}' = \{[n_1' + 1, 1 \ldots, 1], [n_2' + 1, 1 \ldots, 1], [n_3' + 1, 1 \ldots 1]\}$ . We have $v(\mathcal{D}') = v(\mathcal{D}) - 2$. Hence $v(\mathcal{D}')$ is even and at least $2d' - 1$, where $d' = d - 1$.

Therefore by induction the datum $\mathcal{D}'$ is realizable. Hence there exist $\alpha'_1$, $\alpha'_2$ and $\alpha'_3$ that are cycles in $\mathfrak{S}_{d-1}$ of length $n'_1+1$, $n'_2+1$ and $n'_3+1$ and that generate a group that acts transitively on $\{1,\dots,d-1\}$ and such that $\alpha'_1\alpha'_2\alpha'_3 = \mathrm{id}$. There exists $1 \leqslant i \leqslant d-1$ that is in the support of both $\alpha'_1$ and $\alpha'_2$, because $\alpha'_1\alpha'_2$ is a cycle. We may suppose, conjugating the $\alpha'_i$ if necessary, that $i = d-1$ is in their support. We let $\alpha_1 = \alpha'_1(d-1,d)$, $\alpha_2 = (d-1,d)\alpha'_2$ and $\alpha_3 = \alpha'_3$. The permutations $\alpha_1$, $\alpha_2$ and $\alpha_3$ act transitively on $\{1,\dots,d\}$ and $\alpha_i \in A_i$ for $1 \leqslant i \leqslant 3$. Moreover $\alpha_1\alpha_2\alpha_3 = \mathrm{id}$. □

We now prove Proposition 6.2.

*Proof.* We proceed by induction on $k$. It is obvious for $k = 2$ since in this case $n_1 = n_2 = d-1$. It thus suffices to consider $d$-cycles $\alpha_1$ and $\alpha_2 = \alpha_1^{-1}$ to satisfy the Hurwitz conditions. We now assume that $k > 3$. Let us first suppose that there exist $1 \leqslant i < k$ such that $n_i + n_{i+1} \leqslant d-1$. There exist a $(n_i+1)$-cycle $\alpha$ and a $(n_{i+1}+1)$-cycle $\beta$ such that $\alpha\beta$ is a $(n_i+n_{i+1}+1)$-cycle. Indeed we can give explicit examples, or use Lemma 6.3 in $\mathfrak{S}_{n_i+n_{i+1}+1}$. Let $A$ be the partition associated to $\alpha\beta$. We may assume that $i = 1$. The branch data $\mathcal{D}' = \{A, A_3, \dots, A_k\}$ is realizable by induction, since $v(\mathcal{D}') = v(\mathcal{D})$. Therefore there exist $\sigma \in A$, $\alpha_3 \in A_3, \dots, \alpha_k \in A_4$ that act transitively on $\{1,\dots,d\}$ such that $\sigma\alpha_3\dots\alpha_k = \mathrm{id}$. There exists $\gamma \in \mathfrak{S}_d$ such that $\sigma = \gamma\alpha\beta\gamma^{-1}$. Let $\alpha_1 = \gamma\alpha\gamma^{-1}$ and $\alpha_2 = \gamma\beta\gamma^{-1}$. The permutations $\alpha_1,\dots,\alpha_k$ satisfy the Hurwitz conditions.

We now suppose that for every $1 \leqslant i < k$, we have $n_i + n_{i+1} \geqslant d$. By Lemma 6.3, there exist $\alpha \in A_1$ and $\beta \in A_2$ such that $\alpha\beta$ is a $d$-cycle or a $(d-1)$-cycle. This can also be found in [EKS84, Corollary 4.4, Lemma 4.5]. As before let $A$ be the partition associated to $\alpha\beta$. The branch data $\mathcal{D}' = \{A, A_3, \dots, A_k\}$ satisfies $v(\mathcal{D}') \geqslant d - 2 + n_3 + n_4 \geqslant 2d-2$. Therefore by induction there exist $\sigma \in A$, $\alpha_3 \in A_3, \dots, \alpha_k \in A_4$ that act transitively on $\{1,\dots,d\}$ and such that $\sigma\alpha_3\dots\alpha_k = \mathrm{id}$. There exists $\gamma \in \mathfrak{S}_d$ such that $\sigma = \gamma\alpha\beta\gamma^{-1}$. Let $\alpha_1 = \gamma\alpha\gamma^{-1}$ and $\alpha_2 = \gamma\beta\gamma^{-1}$. The permutations $\alpha_1,\dots,\alpha_k$ satisfy the Hurwitz conditions. □

## 6.2 Spherical structures

In this section we consider representations that fix a point in $\mathbb{H}^3$. We can conjugate $\rho$ so that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$.

### 6.2.1 General obstructions

Suppose we have a branched projective structure $X$ on $\Sigma$ with holonomy $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$. Let us denote by $f : \tilde{\Sigma} \to \mathbb{CP}^1$ its developing map. The pullback $f^*\omega$ of the volume form $\omega$ associated with the round metric is $\Gamma$-invariant. Indeed for every $\gamma \in \Gamma$,

$$\gamma^* f^*\omega = (f \circ \gamma)^*\omega = (\rho(\gamma) \circ f)^*\omega = f^*\rho(\gamma)^*\omega = f^*\omega.$$

Therefore it induces a form $\omega_\Sigma$ on $\Sigma$. The Gauss-Bonnet formula gives, as observed in [MP19, Section 1.2]:

$$\sum_{i=1}^{k} n_i - \frac{1}{2\pi}\int_\Sigma \omega = 2g-2.$$

Therefore, we must have $\sum_i n_i > 2g-2$. This explains Obstruction 2 in the spherical case.

Let us suppose that $\rho$ takes its values in a finite subgroup of $\mathrm{SO}_3(\mathbb{R})$ of order $n \geqslant 1$. We can consider the cover $S$ of $\Sigma$ associated with the subgroup $\ker\rho$ of $\Gamma$. The developing map $f$ induces a branched cover $f : S \to \mathbb{CP}^1$ of degree $d$. The Riemann-Hurwitz formula gives

$$n\chi(\Sigma) = 2d - n\sum_{i=1}^{k} n_i.$$

The degree $d$ must satisfy $d \geqslant \max_i n_i + 1$. This explains Obstruction 3.

### 6.2.2 Cyclic holonomy

We focus here on representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ with finite image included in $\mathrm{SO}_2(\mathbb{R})$. Let us observe that any representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$ lifts to $\mathrm{SL}_2(\mathbb{C})$. Indeed the 2-sheeted cover of $\mathrm{SO}_2(\mathbb{R}) = \{\pm\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} \mid \alpha\overline{\alpha} = 1\}$ is the subgroup $\widetilde{\mathrm{SO}_2(\mathbb{R})} = \{\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} \mid \alpha\overline{\alpha} = 1\}$ of $\mathrm{PSL}_2(\mathbb{C})$ that is abelian, hence every commutator in $\widetilde{\mathrm{SO}_2(\mathbb{R})}$ is trivial. The finite subgroups of $\mathrm{SO}_2(\mathbb{R})$ are cyclic and we fix $\mathbb{Z}_n$ such a cyclic group of order $n \geqslant 2$. Let $\rho$ be a surjective representation in $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$.

**Proposition 6.4.** *Let $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ be such that $\sum_i n_i = 2g+2l$ for some $l \geqslant 0$. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1,\ldots,n_k))$ if and only if $n_k < n(l+1)$.*

Let us denote by $P_n$ a pyramid with basis a regular $n$-gon. We can identify a branched projective structure with holonomy in $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$ with a branched structure modeled on $P_n$ with holonomy in the cyclic of order $n$ isometry group of $P_n$, see Figure 7.

One benefit of this point of view is that a surface equipped with such a structure inherits a conical Euclidean structure as studied for example in [Tro86]: it is obtained by gluing Euclidean triangles along their boundaries. Let us recall the Gauss-Bonnet formula in this setting.

**Lemma 6.5** (Troyanov)**.** *Let us consider a conical Euclidean structure on the surface $\Sigma_{g_S}$ of genus $g_S$. Let us denote by $2\pi(\theta_1+1),\ldots,2\pi(\theta_s+1)$ the total angles of its conical points. We have*

$$\sum_{i=1}^{s}\theta_i = 2g_S - 2.$$

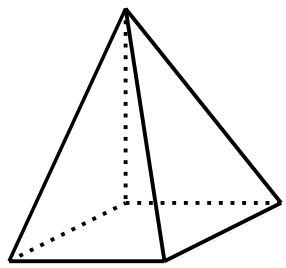

Figure 7: The pyramid model $P_4$.

We refer to [Tro86, Section 3] for a short proof of Lemma 6.5. We now prove Proposition 6.4. It suffices to exhibit a branched projective structure with given branch data and surjective holonomy in $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$ by Proposition 3.2.

*Proof.* We are going to explain how to construct a genus $g$ surface with holonomy a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$ and branch data given by $n_1, \ldots, n_k$. Let us start by considering $l+1$ copies of $\mathbb{CP}^1$. Let us choose an annulus neighborhood of the equator in $\mathbb{CP}^1$. We identify this neighborhood with a rectangle $[0, n] \times [0, 1]$ with its vertical sides identified. A horizontal translation of length 1 in this model corresponds to an order $n$ rotation in $\mathbb{CP}^1$. We are going to cut open the corresponding annuli in these spheres along segments and glue the boundaries back in a different pattern. Let us begin with the case $k = 1$ and $n$ even. We cut open an annulus along a length one horizontal segment and then along another, that is obtained by applying a length 1 horizontal translation to the first one. We then make $g + l - 1$ other slits in the annuli, along segments that are translates of the first one by an integer, see Figure 8. We make sure that there is at least one slit on each annulus. We also make sure that these segments do not intersect. Observe that we can put $\frac{n}{2}$ of these line segments in each annulus. We have room to cut along these $g + l + 1$ segments since we have $n_1 = 2g + 2l$, thus $\frac{n}{2}(l+1) > \frac{n_1}{2} \geqslant g + l$. We glue back the boundary components with translations as indicated in Figure 8.

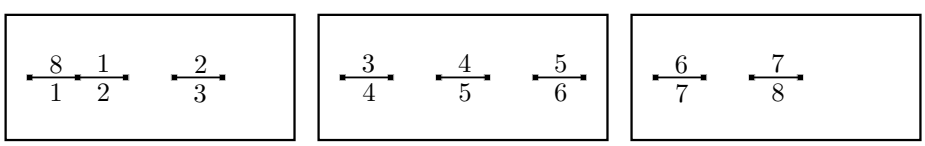


Figure 8: Construction for $(n, k, l, g) = (6, 1, 2, 5)$.

The surface we obtain has genus $g$. Indeed let us denote by $g'$ this genus. Using the pyramidal model, we get a flat structure on this surface. The sum of the angles $\theta_i$ on a pyramid is $-2$ by Lemma 6.5, thus again by Lemma 6.5 we have $2g' - 2 = n_1 - 2(l+1) = 2g - 2$. Moreover the holonomy of the resulting surface is a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$.

Let us now turn to the case $k = 1$ and $n$ odd. Observe that we can now put $\frac{n+1}{2}$ line segments in the first annulus, but only $\frac{n-1}{2}$ ones in the others using the same techniques. If there is enough room to put the $g + l + 1$ line segments as before, then we proceed as in Figure 8. If not, we may increase the total angle contained in two annuli by replacing the cuts of Figure 8 with those of Figure 9.

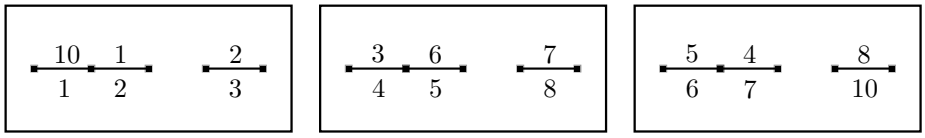


Figure 9: Construction for $(n, k, l, g) = (5, 1, 2, 5)$.

Each of the annuli in which this pattern is used contributes to the total angle by $2\pi n$. Since $n_1 < n(l + 1)$, it is possible the get an angle of $2\pi(n_1 + 1)$. It follows from Lemma 6.5 that we get a genus $g$ surface. It has a projective structure with holonomy a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$.

Let us now turn to the case $k \geqslant 2$. We are again going to make slits in the annuli and glue the boundaries back using a pattern that we now describe. We consider $n_1 + 1$ vertical line segments that are obtained one from the other by horizontal translations. We will identify their boundaries in a cyclic way, see Figure 10. Let us suppose that $n_1 < n_2$. We put $n_2 - n_1$ parallel line segments below, such that the top of exactly one of these segments touches one of the $n_1 + 1$. We will identify these segments in a cylic way, as indicated in Figure 10. Making slits along these segments and identifying the boundaries following this pattern will create three conical points: one of angle $2\pi(n_1 + 1)$, one of angle $2\pi(n_2 + 1)$ and one of angle $2\pi(n_2 - n_1 + 1)$ lines above. We continue similarly: we put $n_3 - (n_2 - n_1)$ line segments below, so that each touches the one above in a single point. We continue this construction until the gluing process gives the desired angles $n_1, \ldots, n_{k-1}$. We then change the last angle with horizontal slits as in Figure 10. If $n_1 = n_2$, we consider $n_1 + 1$ line segments that we identify in a cyclic way, as in the top of Figure 10. We thus make the construction of the pattern for $n_3, \ldots, n_k$ below these segments.

This pattern has $g+l$ connected components, unless $k$ is even and $n_{2i} = n_{2i-1}$ for every $1 \leqslant i \leqslant \frac{k}{2}$, where it has $g + l + 1$ connected components. Indeed one can count directly or argue as follows. If we place each connected component of this pattern on a different sphere, cut open along the segments and glue as indicated in the pattern, we get a genus 1 surface with conical points $n_1, \ldots, n_k$. It follows from Lemma 6.5 that the number $C$ of connected components of this pattern satisfies $\sum_i n_i - 2C = 0$. In the special case $n_{2i} = n_{2i-1}$ for each $i$, then we get a genus 0 surface, thus $\sum_i n_i - 2C = -2$.

We put the connected components of this pattern on the annuli as in Figure 11. We make sure that there is at least one line segment on each of the $l + 1$ spheres. We have room to place the vertical segments in the annuli since $n_i < n(l + 1)$. As before, there may not be enough room to put the horizontal

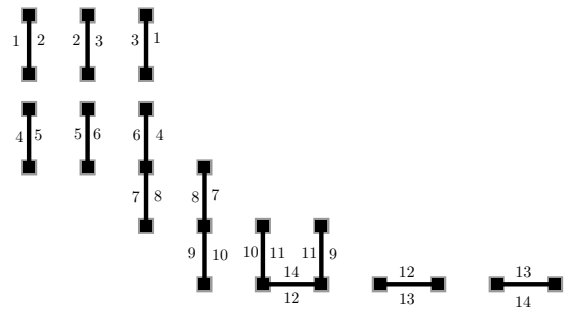


Figure 10: Gluing pattern for $k = 6$ and $(n_i)_{1 \leqslant i \leqslant 6} = (2, 2, 2, 3, 3, 6)$.

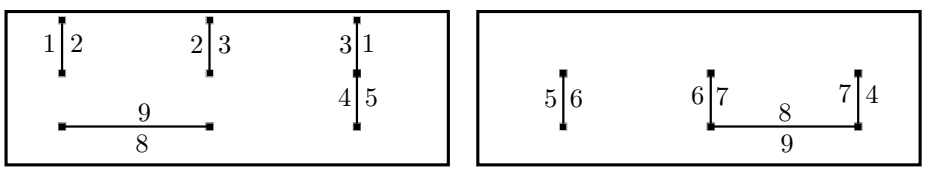


Figure 11: Construction for $(n, k, l, g, n_1, n_2, n_3) = (3, 3, 1, 5, 2, 5, 5)$.

segments when $n$ is odd. We can then use the pattern described in the case $k = 1$ in Figure 9 for the horizontal line segments. The resulting surface is connected, has genus $g$ by Lemma 6.5, and has holonomy a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathbb{Z}_n)$. $\square$

### 6.2.3 Dihedral holonomy

We now consider representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$. We show that Obstruction 1, Obstruction 2 and Obstruction 3 are the only obstructions of being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. Let us consider a bypiramid $B_n$ obtained by gluing two pyramids $P_n$ along their base, see Figure 12. As in the cyclic holonomy case, a branched projective structure with holonomy in $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ may be identified with a structure modeled on $B_n$ with holonomy in its isometry group, which is isomorphic to $\mathrm{D}_n$. It thus inherits an Euclidean structure as before.

**Proposition 6.6.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$ be a surjective homomorphism. The representation $\rho$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots n_k))$ if and only if*

1. *$\sum_i n_i = 2g + 2l$, where $l \geqslant 0$ is such that $\max_i n_i < 2n(l+1)$ if $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$.*
2. *$\sum_i n_i = 2g + 2l - 1$ where $l \geqslant 0$ is such that $\max_i n_i < n(2l+1)$ if $\rho$ does not lift to $\mathrm{SL}_2(\mathbb{C})$.*

**The genus 1 case.** Let us suppose that $g = 1$. Recall that there is no surjective homomorphism in $\mathrm{Hom}(\Gamma_1, \mathrm{D}_n)$ if $n \geqslant 3$ since $\Gamma_1 \simeq \mathbb{Z}^2$ is abelian. Let us

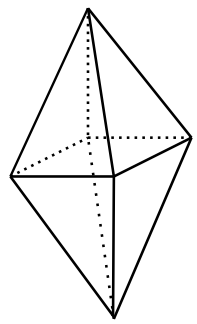

Figure 12: The bipyramid model $B_4$.

also recall that the action of $\mathrm{Aut}^+(\Gamma_1)$ on the set of surjective homomorphisms of $\mathrm{Hom}(\Gamma_1, \mathrm{D}_2)$ is transitive by Proposition 3.4. The surjective homomorphisms of $\mathrm{Hom}(\Gamma_1, \mathrm{D}_2)$ do not lift to $\mathrm{SL}_2(\mathbb{C})$. Obstruction 3 is trivially satisfied since $\sum_i n_i = 2l+1$ thus $\max_i n_i < 2(2l+1)$. Therefore we may rephrase Proposition 6.6 as follows.

**Lemma 6.7.** *Let $\rho \in \mathrm{Hom}(\Gamma_1, \mathrm{D}_2)$ be a surjective homomorphism. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if $\sum_i n_i$ is odd.*

*Proof.* Let $s = \rho(a_1)$ and $t = \rho(b_1)$. Let us consider the round metric on $\mathbb{CP}^1 \simeq \mathbb{S}^2$. Let $C$ be the great circle on $\mathbb{S}^2$ passing through the fixed points of both $s$ and $t$. Let us denote by $S$ and $N$ the fixed points of $t$. Let $d$ be the distance on $\mathbb{CP}^1$ associated with the round metric. We decompose $C$ into 4 arcs: $P_N = \{x \in C \mid d(x, N) \leqslant 1\}$, $P_S = s(P_N) = \{x \in C \mid d(x, S) \leqslant 1\}$, and $P^\pm$ the (closure of the) two connected components of $C \setminus (P_S \cup P_N)$ in $C$. Let us cut $\mathbb{S}^2$ along $C$ to get a half-sphere with boundary $C$. We glue back its boundary in the following way. We glue $P_N$ to $P_S$ with $s$ and $P^-$ to $P^+$ with $t$, see Figure 13. We thus get a torus with holonomy $\rho$ and a single conical point of total angle $2\pi(1+1)$. There exists a curve based at the conical point, namely the image of $P_S$ in the torus, that develops injectively. Hence by Proposition 5.7 we have $\rho \in \mathrm{Hol}(\mathcal{P}(2d+1))$ for every $d \geqslant 0$ and $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ for every $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ such that $\sum_i n_i$ is odd by Remark 5.9. □

**The genus $g \geqslant 2$ case.** We now prove Proposition 6.6 for $g \geqslant 2$.

*Proof.* Let $n_1, \ldots, n_k$ satisfying the hypotheses of Proposition 6.6. By Proposition 3.5, it suffices to exhibit a projective structure with branch data $n_1, \ldots, n_k$ and holonomy a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$. Let us begin with the case where $\sum_i n_i$ is even. Let us consider $l+1$ copies of $\mathbb{S}^2 = \mathbb{CP}^1$. We consider an annular neighborhood of some longitudinal line in the northern hemisphere of $\mathbb{S}^2$, and its image by the flip $s$ that interchanges the north pole with the south pole. We therefore have $2(l+1)$ annuli, see Figure 14. We make the same slits as in the proof of Proposition 6.4. We make sure that there is a slit in each

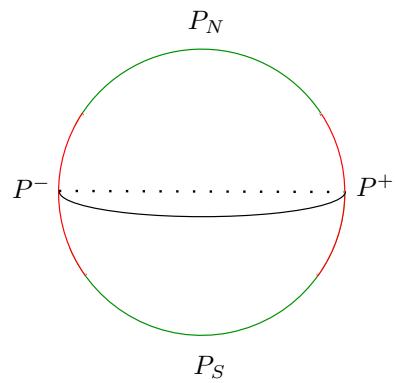


Figure 13: Half-sphere.

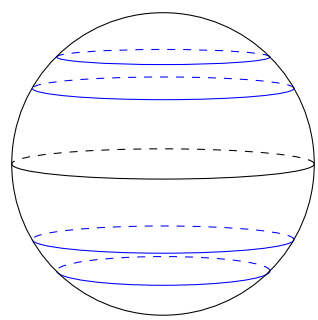

Figure 14: Two annuli embedded in $\mathbb{CP}^1$.

sphere, that there are two slits in an annulus glued with an order $n$ rotation. We also make sure that there are at least two slits, one in the northern annulus and one in the southern annulus of one of the spheres, that are glued together by $s$. This is possible since the pattern of the slits described in Proposition 6.4 has at least $g + l$ connected components and $g \geqslant 2$. Moreover recall that in the case where it has exactly $g + l$ connected components, there are two slits glued by a rotation of order $n$ in the pattern. We get a projective structure on a surface of genus $g$ by the same argument as in Proposition 6.4: it is a consequence of Lemma 6.5. Its holonomy is a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ that lifts to $\mathrm{SL}_2(\mathbb{C})$. Indeed this follows from the fact that $\sum_i n_i$ is even.

Let us now turn to the case where $n$ is even and $\sum_i n_i$ is odd: we have $\sum_i n_i = 2g + 2l - 1$. We consider $l$ copies of $\mathbb{S}^2 = \mathbb{CP}^1$ and a half-sphere: we cut $\mathbb{CP}^1$ along the great circle $C$ that goes through both the fixed points of $s$ and the fixed points $N$ and $S$ of the order 2 element of $\ker \epsilon$. Since $\sum_i n_i$ is odd, at least one of the $n_i$ is odd, say $n_{i_0}$. Let us decompose the boundary of the half-sphere into 4 circular arcs as before. One of these arcs is the set $P_N = \{x \in C \mid d(N, x) < 1\}$, where $d$ denotes the distance associated with

the round metric on $\mathbb{S}^2$, and $P_S = s(P_N) = \{x \in C \mid d(S,x) < 1\}$. The other two parts $P^{\pm}$ are the (closure of the) connected components of $C \setminus (P_N \cup P_S)$, see the circle in green and red of the left bypyramid of Figure 15. We glue $P_N$ with $P_S$, these are the green arcs in Figure 15, with the map $s$. We now make $\frac{n_{i_0}-1}{2}$ slits along circular arcs obtained from $P^-$ by a rotation of order $n$ in the spheres and identify the boundaries components, see the red arcs in Figure 15. We have room to make these slits since we may put $\frac{n}{2} - 1$ of them in the half-sphere, and $n$ in each of the $l$ spheres and $n_{i_0} < 2nl + n$. We now consider $2l$ annular neighborhood of longitudinal circles in the spheres as before, away from these slits, see the blue annuli in the right bypiramid of Figure 15. We also consider the same annular neighborhood of length $n$ in the half-sphere: see the blue annulus on the left side of Figure 15. In these annuli, we make surgeries

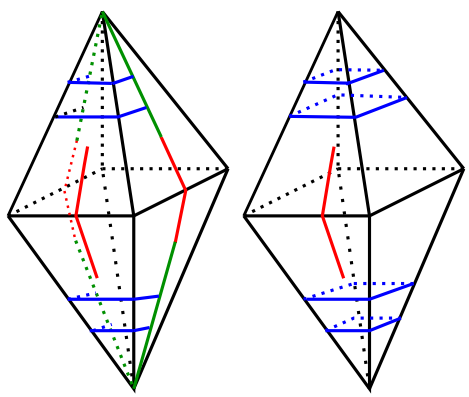

Figure 15: Construction for $(n,k,l,g,n_1) = (4,1,1,2,5)$.

associated with the branch data $\{n_1,\dots,n_k\} \setminus \{n_{i_0}\}$ as explained above in the case where $\rho$ lifts in $\mathrm{SL}_2(\mathbb{C})$. The gluing pattern we have described has at least $\frac{1}{2}(\sum_i n_i - 1) = g + l - 1$ connected components, and since $g \geqslant 2$ it is possible to put a slit on each sphere. Since $\max_i n_i < 2nl + n$, we have room to put these slits in the annuli. We thus get a connected surface, with charts in $\mathbb{CP}^1$, branch datum corresponding to $n_1,\dots,n_k$. This surface has genus $g$ since by Lemma 6.5, its genus $g'$ satisfies $\sum_i n_i = 2g' + 2l - 1$. Its holonomy is a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathrm{D}_n)$ that does not lift so $\mathrm{SL}_2(\mathbb{C})$ since $\sum_i n_i$ is odd. $\square$

### 6.2.4 Tetrahedron

A branched projective structure with holonomy in $\mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ may be identified as before with a branched structure modeled on the regular tetrahedron with holonomy in its isometry group $\mathfrak{A}_4$. We show that the first three obstructions are the only ones to being in $\mathrm{Hol}(\mathcal{P}(n_1,\dots,n_k))$, where $1 \leqslant n_1 \leqslant \dots \leqslant n_k$.

**Proposition 6.8.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ be a surjective homomorphism. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1,\dots,n_k))$ if and only if*

1. $\sum_i n_i = 2g+2l$, *for* $l \geqslant 0$ *such that* $\max_i n_i < 12(l+1)$ *if* $\rho$ *lifts to* $\mathrm{SL}_2(\mathbb{C})$
2. $\sum_i n_i = 2g+2l-1$ *for* $l \geqslant 0$ *such that* $\max_i n_i < 12l+6$ *if* $\rho$ *does not lift to* $\mathrm{SL}_2(\mathbb{C})$.

*Proof.* Let us consider $n_1, \ldots, n_k$ satisfying the hypotheses of Proposition 6.8. It suffices to exhibit a branched projective structure in $\mathcal{P}(n_1, \ldots, n_k)$ with holonomy a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathfrak{A}_4)$ thanks to Proposition 3.8. Let us start with the case where $\sum_i n_i = 2g + 2l$ for some $l \geqslant 0$. Let us consider $l+1$ copies of the regular tetrahedron $T$. We will make slits along line segments in the tetrahedra and glue the boundaries back with isometries of $T$. We use the pattern of Figure 10. Let us describe how to put this pattern in the tetrahedra. Start by considering a line segment in one of the triangles of one of the tetrahedra as in Figure 16. We consider $n_1$ line segments obtained from this one by applying an isometry of $T$. We identify the boundaries of the slits obtained by cutting open the tetrahedra along these segments with the corresponding isometries, following the pattern of Figure 10. We continue to put the pattern of the line segments in this way.

We have room to make these slits since $\max_i n_i < 12(l+1)$. We make sure to have one slit on each tetrahedron and that one of the tetrahedra has slits that are glued together with isometries generating $\mathfrak{A}_4$. This construction gives a projective structure on a closed surface of genus $g$ with holonomy a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathfrak{A}_4)$. Indeed the genus $g'$ of the resulting surface can be computed using Lemma 6.5: it satisfies $2g' - 2 = \sum_i n_i - 2(l+1) = 2g - 2$.

Let us now turn to the case $\sum_i n_i = 2g+2l-1$ for some $l \geqslant 0$. We consider $l$ copies of the regular tetrahedron and a half-tetrahedron: we cut a tetrahedron in half along a plane going through the midpoints of two pairs of opposite edges. We divide the resulting boundary into 2 pairs as in the dihedral case: two red line segments and two green ones. We identify the green lines with an isometry of the tetrahedron. One of the $n_i$, say $n_{i_0}$ must be odd. Let us make $\frac{n_{i_0}-1}{2}$ slits on the tetrahedra and the half-tetrahedron, along line segments obtained by applying isometries to one of the red boundaries. We identify the resulting boundaries in a cyclic way. We then make the slits and gluing associated with the branching data $\{n_1, \ldots, n_k\} \setminus \{n_{i_0}\}$. We have room to make all these slits because of the assumption $\max_i n_i < 12l + 6$. We make sure to obtain a connected surface (each tetrahedron must have at least one slit) and that the resulting holonomy is a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathfrak{A}_4)$. By Lemma 6.5, the genus of the resulting surface is $g$ and its holonomy does not lift to $\mathrm{SL}_2(\mathbb{C})$ since $\sum_i n_i$ is odd. □

### 6.2.5 Cube

We can identify the branched projective strutures with holonomy in the group of the isometries of the cube with a branched structure modeled on the cube with holonomy in its isometry group. Again we may geometrize a surjective

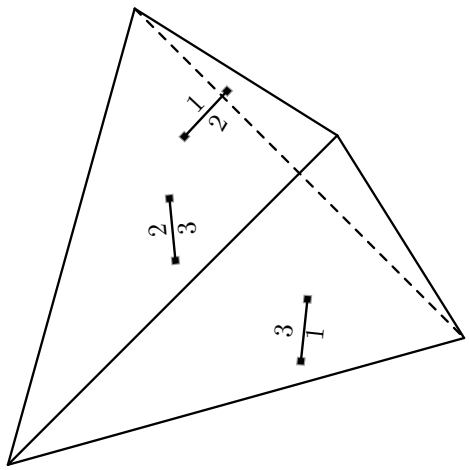


Figure 16: Construction for $(l, g, k, n_1, n_2) = (0, 2, 2, 2, 2)$.

homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{S}_4)$ that satisfies the conditions of the first three obstructions.

**Proposition 6.9.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{S}_4)$ be a surjective homomorphism. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if*

1. $\sum_i n_i = 2g+2l$, *for* $l \geqslant 0$ *such that* $\max_i n_i < 24(l+1)$ *if* $\rho$ *lifts to* $\mathrm{SL}_2(\mathbb{C})$
2. $\sum_i n_i = 2g + 2l - 1$ *for* $l \geqslant 0$ *such that* $\max_i n_i < 24l + 12$ *if* $\rho$ *does not lift to* $\mathrm{SL}_2(\mathbb{C})$.

*Proof.* The construction is analogous to the one of Proposition 6.8, see Figure 17. □

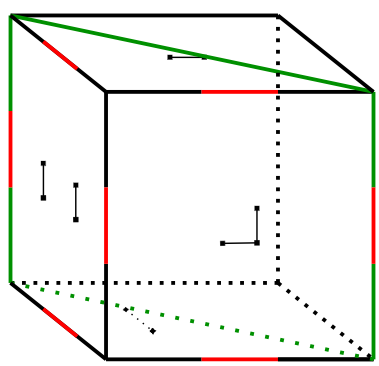

Figure 17: Construction for $(l, g, k, n_1, n_2) = (0, 11, 2, 10, 11)$.

### 6.2.6 Icosahedron

As before, a branched projective structure with holonomy in $\mathrm{Hom}(\Gamma, \mathfrak{A}_5)$ may be identified with a branched structure modeled on the icosahedron. Therefore we will geometrize the surjective representations of $\mathrm{Hom}(\Gamma, \mathfrak{A}_5)$ by gluing together regular icosahedra.

**Proposition 6.10.** *Let $\rho \in \mathrm{Hom}(\Gamma, \mathfrak{A}_5)$ be a surjective homomorphism. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if*

1. $\sum_i n_i = 2g+2l$, *for* $l \geqslant 0$ *such that* $\max_i n_i < 60(l+1)$ *if* $\rho$ *lifts to* $\mathrm{SL}_2(\mathbb{C})$.
2. $\sum_i n_i = 2g + 2l - 1$ *for* $l \geqslant 0$ *such that* $\max_i n_i < 60l + 30$ *if* $\rho$ *does not lift to* $\mathrm{SL}_2(\mathbb{C})$.

Again, the constructions are analogous to those of Proposition 6.8 and Proposition 6.9, see Figure 18.

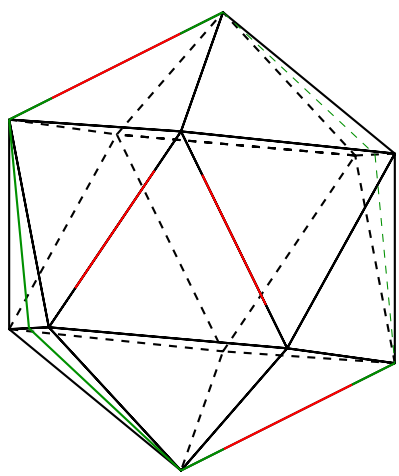

Figure 18: Construction for $(l, g, k, n_1) = (0, 3, 1, 5)$.

### 6.2.7 Holonomy with infinite image in $\mathrm{SO}_3(\mathbb{R})$

We characterize the representations $\mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ with infinite image that are in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. The infinite closed subgroups of $\mathrm{SO}_3(\mathbb{R})$ are isomorphic to $\mathrm{SO}_2(\mathbb{R})$, $\mathrm{O}_2(\mathbb{R})$ or are the whole group $\mathrm{SO}_3(\mathbb{R})$. Let us fix $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$.

**Holonomy with dense image in $\mathrm{SO}_2(\mathbb{R})$** Recall that any representation with image in $\mathrm{SO}_2(\mathbb{R})$ lifts to $\mathrm{SL}_2(\mathbb{C})$. We show that Obstruction 1 and Obstruction 2 are the only obstructions in this case.

**Proposition 6.11.** *A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R})$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ if and only if $\sum_i n_i$ is even and at least $2g$.*

*Proof.* Let us suppose that $\sum_i n_i$ is even and at least $2g$. By Proposition 6.4, there exists a representation in $\mathrm{Hom}(\Gamma, \mathrm{SO}_2(\mathbb{R}))$ with finite image in $\mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$. Therefore since the closure $\mathrm{Aut}^+(\Gamma) \cdot \rho$ contains such a representation byProposition 4.1, it follows from the Ehresmann-Thurston principle (Corollary 5.6) that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$. □

**Holonomy with dense image in $\mathrm{O}_2(\mathbb{R})$** We now suppose that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{O}_2(\mathbb{R}))$ has dense image.

**Proposition 6.12.** *The representation $\rho$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ if and only if $\sum_i n_i > 2g - 2$ and $\sum_i n_i$ is even (resp. odd) if $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ (resp. does not lift to $\mathrm{SL}_2(\mathbb{C})$).*

*Proof.* Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{O}_2)$ with dense image. There exists $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{D}_n)$ in $\mathrm{Hol}(\mathcal{P}(n_1 \dots, n_k))$ for $n$ large enough by Proposition 6.6. By Proposition 4.2, $\rho'$ is in the closure of $\mathrm{Aut}^+(\Gamma) \cdot \rho$ thus $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ by the Ehresmann-Thurston principle Corollary 5.6. □

**Holonomy with dense image in $\mathrm{SO}_3(\mathbb{R})$**

**Proposition 6.13.** *A representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ with dense image is in $\mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ if and only if both $\sum_i n_i > 2g - 2$ and $\sum_i n_i$ is even if $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ and odd otherwise.*

*Proof.* As in Proposition 6.12, there exists $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{SO}_3(\mathbb{R}))$ in $\mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$. For example we can take $\rho'$ with its image in a dihedral group $\mathrm{D}_n$ by Proposition 6.6. This also follows from [MP19, Theorem A] where the existence of spherical metrics with conical points of prescribed angles satisfying the Gauss-Bonnet constraint is shown. By Lemma 4.4, $\rho'$ is in the closure of $\mathrm{Aut}^+(\Gamma) \times \mathrm{SO}_3(\mathbb{R}) \cdot \rho$. Therefore $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ by the Ehresmann-Thurston principle, Corollary 5.6. □

## 6.3 Affine holonomy

In this subsection we geometrize affine representations. An example of a branched projective structure with affine holonomy is an affine structure, as studied for example in [Gha18].

**Definition 6.14.** An *affine structure* is a branched projective structure with affine holonomy whose developing map takes its values in $\mathbb{C} \subset \mathbb{CP}^1$.

This definition generalizes for example the notion of translation surface.

**Definition 6.15.** A translation surface structure $X$ on $\Sigma$ is an affine structure with holonomy in $\mathrm{Hom}(\Gamma, \mathbb{C})$, where $\mathbb{C} \subset \mathrm{Aff}(\mathbb{C})$ is the subgroup of translations. An even half-translation structure is an affine structure with holonomy in $\mathrm{Hom}(\Gamma, \{\pm 1\} \rtimes \mathbb{C})$.

*Remark* 6.16. This amounts to requiring that $\alpha = \mathrm{Li} \circ \rho$ is the trivial homomorphism for the translation surface case, and that $\alpha$ has its image in $\{\pm 1\}$ for the half-translation case.

*Remark* 6.17. Our definition of even half-translation surface is equivalent to the usual definitions of half-translation surfaces, see for example [Zor06, Lan08], with the extra requirement that the total angle of every conical point is even in the following sense: it is of the form $k\pi$, with $k \in 2\mathbb{N}$. In other words, following the notations of [Lan08], we are looking at the half-translation surfaces that are either translation surfaces or correspond to quadratic differentials in

$$\bigcup_{n_i \geqslant 1 | \sum_i n_i = 2g-2} \mathcal{Q}(2n_1, 2n_2, \ldots, 2n_k).$$

This extra requirement is necessary to obtain a holonomy defined on $\Gamma$, and not on the fundamental group of a punctured surface.

### 6.3.1 General obstructions

Let us show that the divisor of a branched projective structure with affine holonomy has degree at least $2g-2$ and that we may read in this degree whether such a structure is an affine one or not.

**Proposition 6.18.** *If $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ is the holonomy of a projective structure in $\mathcal{P}(n_1, \ldots, n_k)$ with developing map $f$, then $\sum_i n_i \geqslant 2g-2$. Moreover $\sum n_i = 2g-2$ if and only if $f(\tilde{\Sigma}) \subset \mathbb{C}$.*

The following proof of Proposition 6.18 is an adaptation of [Kap20, Section 6], where Kapovich shows the same result in the realm of holonomies with values in the subgroup of translations $\mathbb{C} \subset \mathrm{Aff}(\mathbb{C})$.

*Proof.* Let us assume that $\rho$ is the holonomy of a branched projective structure with holomorphic developing map $f : \tilde{\Sigma} \to \mathbb{CP}^1$ for a complex structure on $\Sigma$. Let us choose a lift $\tilde{\rho} \in \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ of $\rho$ and define $\varphi \in \mathrm{Hom}(\Gamma, \mathbb{C}^\star)$ by $\varphi(\gamma) = \lambda_\gamma$ where $\tilde{\rho}(\gamma) = \begin{pmatrix} \lambda_\gamma^{-1} & t_\gamma \\ 0 & \lambda_\gamma \end{pmatrix}$. Let us consider the vector bundles $V = \tilde{\Sigma} \times \mathbb{C}^2/\Gamma$ and $\Lambda = \tilde{\Sigma} \times \mathbb{C}/\Gamma$, where $\Gamma$ acts by $\gamma \cdot (\tilde{x}, v) = (\gamma \cdot \tilde{x}, \tilde{\rho}(\gamma) \cdot v)$ and $\gamma \cdot (\tilde{x}, z) = (\gamma \cdot \tilde{x}, \varphi(\gamma) \cdot z)$. The projection on the second coordinate gives a morphism of holomorphic bundles $\phi : V \to \Lambda$. The developing map gives for each point $\tilde{x} \in \tilde{\Sigma}$ a line $f(\tilde{x}) \subset \mathbb{C}^2$, and by $\rho$-equivariance induces a line subbundle $L \subset V$. We restrict $\phi$ to get a morphism of line bundles $\phi : L \to \Lambda$, which is injective if and only if $f(\tilde{\Sigma})$ does not contain $\infty \in \mathbb{CP}^1$.

**Lemma 6.19.** *We have $\deg L \leqslant \deg \Lambda$ with equality if and only if $\phi$ is injective.*

*Proof.* Let us pick a smooth section $s : \Sigma \to L$ transverse to the zero section whose zeroes are different from those of $\phi : L \to \Lambda$. We also assume that $\phi \circ s$ is transverse to the zero section. The degree of $L$ is the intersection number between $s(\Sigma)$ and the zero section: $\deg L = \sum_{p \in s^{-1}(0)} \mathrm{sign}_p(s)$, where $\mathrm{sign}_p(s)$

is the sign of $\det(ds_p)$ where $s$ is viewed as $s : U \subset \mathbb{C} \to \mathbb{C}$ in a local chart around $p$, and a local trivialization of $L$. The zeroes of the section $\phi \circ s$ of $\Lambda$ are of two types: the zeroes of $s$ and the zeroes of $\phi$. Each zero of $\phi$ has sign 1, because in a local chart centered at 0 and a local trivialization, we have $\phi \circ s : z \mapsto \alpha(z)s(z)$ where $\alpha$ is holomorphic, $\alpha(0) = 0$ and $s(0) \neq 0$. Therefore $d(\phi \circ s) = s(0) \cdot d\alpha_0$ has positive determinant. If $p$ is a zero of $s$, we have $\alpha(0) \neq 0$ and $s(0) = 0$, thus $d(\phi \circ s)_0 = \alpha(0) \cdot ds_0$ and $\mathrm{sign}_p(\phi \circ s) = \mathrm{sign}_p(s)$. We have

$$\deg \Lambda = \sum_{p \in (\phi \circ s)^{-1}(0)} \mathrm{sign}_p(\phi \circ s) = \sum_{p \in s^{-1}(0)} \mathrm{sign}_p(\phi \circ s) + \#\{p \in \Sigma \mid \phi = 0\}.$$

Therefore, $\deg \Lambda = \deg L + \#\{p \in \Sigma \mid \phi = 0\} \geqslant \deg L$ with equality if and only if $\phi$ is injective. □

**Lemma 6.20.** *We have* $\deg \Lambda = 0$.

*Proof.* It suffices to show that this degree does not depend on the homomorphism $\varphi \in \mathrm{Hom}(\Gamma, \mathbb{C}^\star)$, since for $\varphi : \gamma \mapsto 1$, the bundle $\Lambda$ is trivial. Let $(\varphi_t)_{t \in [0,1]}$ be a path in $\mathrm{Hom}(\Gamma, \mathbb{C}^\star)$, and $\Lambda_t$ be the bundle over $\Sigma$ defined as $\Lambda$ with the action of $\varphi_t$. Let $W$ be the vector bundle over $\Sigma \times [0,1]$ defined by $W = \tilde{\Sigma} \times \mathbb{C} \times [0,1]/\Gamma$ where $\Gamma$ acts by $\gamma \cdot (\tilde{x}, z, t) = (\tilde{\gamma} \cdot x, \varphi_t(\gamma) \cdot z, t)$. Let us choose a linear connection on $W$. The parallel transport along curves $x \times [0,1]$, $x \in \Sigma$, defines an isomorphism between the underlying oriented real bundles $\Lambda_0$ and $\Lambda_1$. Therefore $\Lambda_0$ and $\Lambda_1$ have the same Euler class and thus the same degree. Since $\mathrm{Hom}(\Gamma, \mathbb{C}^\star) \simeq (\mathbb{C}^\star)^{2g}$ is path-connected, the degree of $\Lambda$ does not depend on $\varphi$. □

It is shown in [GKM00, Section C] that

$$\deg L = g - 1 - \frac{1}{2} \sum_i n_i.$$

Therefore we have $\sum_i n_i \geqslant 2g - 2$, with equality if and only if $\infty \notin f(\tilde{\Sigma})$. □

We have seen in Section 4.2.1 that the group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathrm{Hom}(\Gamma, \mathbb{Z}_2 \rtimes \mathbb{C})$, the set of homomorphisms with image in the group $\mathrm{Aff}(\mathbb{C})$ and linear part with finite image of order at most 2. As a corollary of Proposition 6.18, we see that this action preserves the property of being in the image of Hol if $\sum_i n_i = 2g - 2$.

**Corollary 6.21.** *Let* $\rho \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2 \rtimes \mathbb{C})$ *and* $A \in \mathrm{GL}_2^+(\mathbb{R})$. *For every* $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ *such that* $\sum_i n_i = 2g - 2$, *we have:*

$$\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k)) \iff A \cdot \rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k)).$$

*Proof.* If $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$, then there exists a developing map $f : \tilde{\Sigma} \to \mathbb{CP}^1$ that is $\rho$-equivariant. By Proposition 6.18, $f$ takes its values in $\mathbb{C}$. Let us see $A$ as a map $A : \mathbb{C} \to \mathbb{C}$. The map $A \circ f$ is a developing map for a projective structure in $\mathcal{P}(n_1, \ldots, n_k)$ with holonomy $A \cdot \rho$. □

### 6.3.2 Euclidean holonomy

We now focus on the case where $\rho$ is Euclidean. Recall that it means that the linear part $\alpha = \mathrm{Li} \circ \rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^\star)$ of $\rho$ has its image in $\mathbb{S}^1 = \{z \in \mathbb{C} \mid z\overline{z} = 1\}$. Let us explain Obstruction 4.

**Proposition 6.22.** *Suppose that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ is the holonomy of an affine structure. We have $\mathrm{Vol}(\rho) > 0$.*

*Proof.* We can identify $\tilde{\Sigma}$ with $Y$, where $Y = \mathbb{H}^2$ if $g \geqslant 2$ and $Y = \mathbb{C}$ if $g = 1$, by the uniformization theorem and consider a holomorphic developing map $f : Y \to \mathbb{C}$. Let us choose $D \subset Y$ a compact fundamental domain for the action of $\Gamma$ on $Y$. We have:

$$\mathrm{Vol}(\rho) = \int_D f^*(dx \wedge dy) = \int_D \det(df) dx \wedge dy.$$

Since $f$ is holomorphic and nonconstant, $\det(df) \geqslant 0$ everywhere and $\det(df)$ is continuous and nonzero. □

**Trivial linear part** Let us first assume that $\alpha$ is the trivial homomorphism. Let $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ be such that $\sum_i n_i = 2g - 2$. By Proposition 6.18 we are asking if $\rho$ is the holonomy of a translation surface with prescribed singularities. This question was addressed by the author in [LF20] and independently by Bainbridge, Johnson, Judge and Park in [BJJP20], where the following is proven.

**Refined Haupt's theorem.** We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if

1. $\mathrm{Vol}(\rho) > 0$
2. $\mathrm{Vol}(\rho) \geqslant (\max_i n_i + 1)\mathrm{Area}(\mathbb{C}/\Lambda)$ if $\Lambda = \{z_0 \in \mathbb{C} \mid z + z_0 \in \rho(\Gamma)\}$ is a lattice in $\mathbb{C}$.

Let us remark that Section 6.3.2 shows that Obstruction 4 and Obstruction 5 are the only obstructions of being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ for a representation $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C})$ if $\sum_i n_i = 2g - 2$.

We now suppose $\sum_i n_i \geqslant 2g$. Since $\sum_i n_i$ is not minimal, the developing maps we are considering have $\infty \in \mathbb{CP}^1$ in their image. Therefore our projective structures may be obtained by performing surgeries on $\mathbb{CP}^1$.

**Proposition 6.23.** *Let $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ be such that $\sum_i n_i$ is even and at least $2g$. There exists a branched projective structure in $\mathcal{P}(n_1, \ldots, n_k)$ with holonomy $\rho$.*

*Proof.* We construct a projective structure on $\Sigma_g$ with branching data $n_1, \ldots, n_k$ and holonomy in $\overline{(\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C})) \cdot \rho}$. It will then follow from Corollary 5.6 that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

Let us consider the Riemann sphere $\mathbb{CP}^1$. We choose $x \in \mathbb{C} \subset \mathbb{CP}^1$ and cut open $\mathbb{CP}^1$ along the segment lines $l_0 = [x, x+1]$ and $l_k = [x + 2k - 1, x + 2k]$

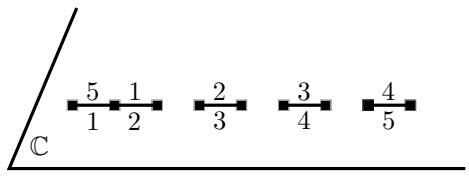


Figure 19: Construction for $g = 4$.

for $1 \leqslant k \leqslant g$. We glue the bottom side of $l_i$ to the top side of $l_{i+1}$, in cyclic notation, as indicated in Figure 19.

We thus get a genus $g$ surface with charts in $\mathbb{CP}^1$, and a unique branch point of order $2g$. Moreover its holonomy is given by $(z+1, z+1, z+2, z, \ldots z+2, z)$, that lies in $\mathrm{Aut}^+(\Gamma) \cdot \rho_\infty$, where $\rho_\infty$ comes from Lemma 4.11. Hence this holonomy is in $\overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C}) \cdot \rho}$ by Lemma 4.11. Observe that the closed curve following $l_0$ develops injectively. Therefore by Proposition 5.7, $\rho_\infty \in \mathrm{Hol}(\mathcal{P}(2k))$ for every $k \geqslant g$. It follows from Remark 5.9 and the Ehresmann-Thurston principle Corollary 5.6 that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. □

**Nontrivial linear part** We now suppose that $\alpha$ has finite image of order $n \geqslant 2$. Let us first suppose that $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ are such that $\sum_i n_i$ is even and at least $2g$. We may perform the same surgeries as in Proposition 6.23.

**Proposition 6.24.** *There exists a branched projective structure in* $\mathrm{Hol}(n_1, \ldots, n_k)$ *with holonomy* $\rho$.

*Proof.* Our construction is very similar to the case $n = 1$, where we cut and glue $\mathbb{CP}^1$ as in Figure 19. We will just flip one of the segment lines, see Figure 20. More precisely, we start as before with the Riemann sphere $\mathbb{CP}^1$. We cut open the sphere $\mathbb{CP}^1$ along the line segments $[x, x+1]$, $[x+1, x+2], [x+3, x+4], \ldots, [x+2g-3, x+2g-2]$ and $[\zeta x, \zeta x + \zeta]$, where $x \in \mathbb{C}$ is chosen so that these line segments do not intersect. We glue back the resulting boundary components as indicated in Figure 20. We thus get a genus $g$ surface with charts

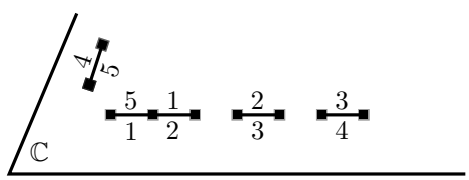


Figure 20: Construction for $g = 4$ and $n = 4$.

in $\mathbb{CP}^1$, a single branch point of order $2g$ and holonomy in $\mathrm{Aut}^+(\Gamma) \cdot \rho_\infty$, thus in $\overline{\mathrm{Aut}^+(\Gamma) \times \mathrm{Aff}(\mathbb{C}) \cdot \rho}$. The curve following the line segment $[x, x+1]$ develops injectively, thus $\rho_\infty \in \mathrm{Hol}(\mathcal{P}(2k))$ for every $k \geqslant g$ by Proposition 5.7. Therefore $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ by Remark 5.9 and Corollary 5.6. □

**Minimal degree** We suppose that $\alpha$ has finite image of order $n \geqslant 2$ and that $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ are such that $\sum_i n_i = 2g-2$. Let us first make an observation on the volume of the pullback.

*Remark* 6.25. Suppose that $f : \Sigma \to S$ is a branched cover and that $S$ is endowed with a branched projective structure $X$ with holonomy $\rho$. The volume of the holonomy of $f^*X$ is $\deg(f) \cdot \mathrm{Vol}(\rho)$.

This observation leads to another obstruction to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ that we now describe. Let $\Lambda$ be the subgroup of $\mathbb{C}$ generated by the complex numbers $a$ such that the Möbius transformation $z+a$ belongs to $\ker \alpha$. Suppose that $\Lambda$ is a lattice in $\mathbb{C}$. If $\rho$ is the holonomy of the developing map $f$, then since $f$ has image in $\mathbb{C}$ by Proposition 6.18, it induces a branched cover $S \to \mathbb{C}/\Lambda$, where $S$ is the cover of $\Sigma$ associated with $\ker \alpha$. The degree $d$ of this branched cover must satisfy $d \geqslant \max_i n_i + 1$, since the branched cover has the local form $z \mapsto z^{n_i+1}$ at the conical points. Moreover $d$ is also given by $d = n\mathrm{vol}(\rho)/\mathrm{Area}(\Lambda)$. This explains Obstruction 5. Note that when $n = 1$ this obstruction is included in the statement of the refined Haupt's theorem.

**Example 6.26.** Every half-translation surface of genus 2 with a single conical point is a translation surface. Indeed let $\rho$ be the holonomy of an even half-translation surface with nontrivial $\alpha = \mathrm{Li} \circ \rho$. Since $\alpha(\Gamma) = \{\pm 1\}$, we can suppose that $\rho = (-z, z, z+z_2, z+z_3)$ by Lemma 4.10. We have $\Lambda = \langle z_2, z_3 \rangle$ and $\mathrm{Area}(\Lambda) = \det(z_2, z_3) = \mathrm{Vol}(\rho)$. Thus the group $\Lambda$ generated by $z_2$ and $z_3$ is a lattice in $\mathbb{C}$. Obstruction 5 gives $2\mathrm{Vol}(\rho) \geqslant (n_i+1)\mathrm{Vol}(\rho)$, thus $n_i = 1$ for all $i$. Since $\sum_i n_i = 2g-2 = 2$, there are 2 conical points. In other words, we have $\mathcal{Q}(4) = \emptyset$, following the notation of [Lan08] ; see also [MS93, Lan08] for alternative proofs of this fact.

Let us show that Obstruction 4 and Obstruction 5 are in this case the only obstructions to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

**Proposition 6.27.** *If* $\mathrm{Vol}(\rho) > 0$ *and* $n\mathrm{Vol}(\rho) \geqslant (\max_i n_i + 1)\mathrm{Area}(\Lambda)$ *if* $\Lambda$ *is a lattice, then* $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

We suppose that $\rho$ satisfies the conditions of Obstruction 4 and Obstruction 5. By Lemma 4.10, we may assume that

$$\rho = (e^{2\pi i/n} z, e^{2\pi i/n} z, z + x_2, z + y_2, \ldots, z + x_g, z + y_g).$$

**Case $g = 2$.** Let us first suppose that $g = 2$ and $n \geqslant 3$. We consider a parallelogram $P \subset \mathbb{C}$ whose vertices are $z_0$, $z_0 + x_2$, $z_0 + y_2$ and $z_0 + x_2 + y_2$, where $z_0 \in \mathbb{C}$ is such that 0 is in the interior of $P$. Note that it is indeed a parallelogram since $\mathrm{Vol}(\rho) = \det(x_2, y_2) > 0$. We are going to add a handle to the torus obtained by gluing its opposite edges. Let $\epsilon > 0$ be such that the disk $D = \{z \in \mathbb{C} \mid |z| < \epsilon\}$ is included in the interior of $P$. Let us cut open $P$ along the circular segment joining $z_0 \in D$ and $e^{2\pi/n} z_0$, and along the circular segment joining $e^{2\pi/n} z_0$ and $e^{4\pi/n} z_0$. We then glue the boundary components as indicated in Figure 21. The topological result of this surgery is to add a

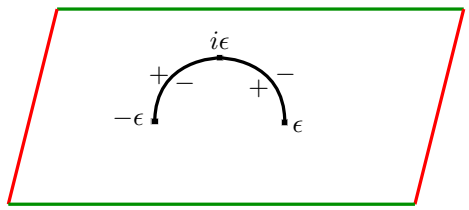


Figure 21: Construction for $g = 2$ and $n = 4$.

handle. Moreover the resulting genus 2 surface comes naturally with charts in $\mathbb{C}$ and has holonomy $\rho$. Thus $\rho \in \mathrm{Hol}(\mathcal{P}(2))$ and also $\rho \in \mathrm{Hol}(\mathcal{P}(1,1))$ by Proposition 5.8. If $n = 2$, then we have seen in Example 6.26 that we cannot have $\rho \in \mathrm{Hol}(\mathcal{P}(2))$. However we have $\rho \in \mathrm{Hol}(\mathcal{P}(1,1))$: it suffices to cut open $P$ along a line segment $l$ and along $-l$. We make sure that these line segments do not intersect and are both in $P$. We then identify the opposites sides of the boundary components as in Figure 22 with the map $z \mapsto -z$. Note that this shows that $\mathcal{Q}(2,2) \neq \emptyset$.

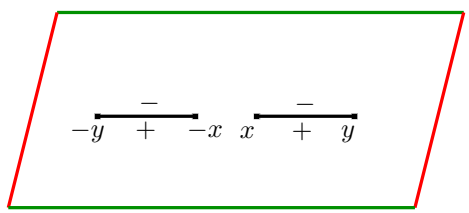


Figure 22: Construction for $g = 2$ and $n = 2$.

*Remark* 6.28. Observe that if $g = 1$, then $\rho$ must be spherical by Lemma 4.10. Indeed, we can suppose that $\rho = (e^{2\pi i/n}z, z)$, therefore we cannot have $\sum_i n_i = 0$. In particular this shows that $\mathcal{Q}(\emptyset) = \emptyset$.

We now suppose that $g \geqslant 3$.

**The generic case.** Let us first deal with the case where the group $\Lambda$ that is generated by the $e^{2\pi ki/n}x_j$ and $e^{2\pi ki/n}y_j$, $k \in \mathbb{Z}$ is not a lattice. We also suppose that $n \geqslant 3$. It follows from Lemma 4.16 that we can assume that the group generated by the $x_j$ and $y_j$ is not a lattice. We may consider a translation surface structure on $\Sigma_{g-1}$ with holonomy $(z+x_2, \ldots, z+y_g)$ and a single conical point $p \in \Sigma_{g-1}$ by the refined Haupt's theorem. We can translate the charts so that there exist $\epsilon > 0$ and a chart $\varphi : U \to V$, where $U \subset S$ and $V \subset \mathbb{C}$ are open, such that $p \in U$, and $\varphi(p) \in D(0,\epsilon) \setminus \{0\}$ and $D(0,\epsilon) \subset V$. We can then add a handle with holonomy $(e^{2\pi i/n}z, z)$ as in the genus $g = 2$ case. We make sure that the new slit starts from the conical point of the surface. The resulting surface has holonomy $\rho$ and a single conical point of order $2g-2$. It follows from Remark 5.9 that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

Let us turn to the case $n = 2$. If $g \geqslant 4$, there exists $\rho_\infty \in \mathrm{Aut}^+(\Gamma) \cdot \rho$ such that $\Lambda_\infty = \{a \in \mathbb{C} \mid z + a \in \rho_\infty(\Gamma)\}$ is a lattice satisfying $\mathrm{Vol}(\rho_\infty) \geqslant (\max_i n_i + 1)\mathrm{Area}(\Lambda_\infty)$ by Lemma 4.13. We will see in the next paragraph that this implies $\rho_\infty \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. Therefore $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ by the Ehresmann-Thurston principle Corollary 5.6. We now turn to the case $g = 3$ and $n = 2$. Let us recall a result from [LF20, Section 4, Proposition 4.2].

**Lemma 6.29.** *Let $\chi \in \mathrm{Hom}(\Gamma_2, \mathbb{C})$ with positive volume and image $\chi(\Gamma_2)$ that is not a lattice. There exists $\chi' \in \mathrm{Aut}^+(\Gamma_2) \times \mathrm{GL}_2^+(\mathbb{R}) \cdot \chi$ such that $\chi'(a_1) = 1$, $\chi'(b_1) = i$ and $|\chi'(a_2)| < 1$.*

We may assume that our representation has the form $\rho = (-z, z, x_1, y_1, x_2, y_2)$, with the representation $\chi'$ defined by $\chi'(a_i) = x_i$ and $\chi'(b_i) = y_i$ for $i = 1, 2$ as in Lemma 6.29, by Lemma 2.2. We consider a parallelogram $P$ in $\mathbb{C}$ whose sides are given by $x_1$ and $y_1$. Let us consider a line segment $\ell = [z_0, z_0 + x_2]$ that is contained in $P$. We may translate both $P$ and $\ell$, so that $\ell$ is so close to 0 that $-\ell$ is also contained in $P$. We make slits along $\ell$ and $-\ell$, and glue back a handle given by identifying two sides of a parallelogram whose sides are given by $x_2$ and $y_2$, as indicated in Figure 23.

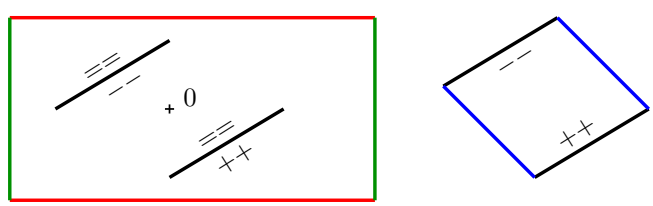


Figure 23: Construction for $g = 3$ and $n = 2$.

We thus get a genus 3 surface with holonomy given by

$$(z + 1, z + i, z + x, -z + y, z - x, -z).$$

It follows from Lemma 4.12 that this representation is in $\mathrm{Aut}^+(\Gamma) \times \mathrm{GL}_2^+(\mathbb{R}) \cdot \rho$. Therefore $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

**The lattice case.** Let us suppose that $n = 4$ and $\Lambda = L = \mathbb{Z}[i]$. We consider a rectangle $R$ whose sides are given by $m = \mathrm{Vol}(\rho)$ and 1. Let us identify the opposite sides of $R$ with translations. Let us draw $m$ annuli in $R$ obtained by translating one of them by integers, see Figure 24. We make the slits and gluing in these annuli explained in Section 6.2.2 for the branched datum $n_1, \ldots, n_k$. Since $\max_i n_i < 4m$, we have room to make the slits. Since $g \geqslant 3$, we can also make sure that the resulting holonomy $\rho_0$ is a surjective homomorphism of $\mathrm{Hom}(\Gamma, \mathbb{Z}_4 \rtimes \mathbb{C})$, with $\Lambda(\rho_0) = \mathbb{Z}[i]$ and such that $\mathrm{Vol}(\rho) = m$. It follows from Lemma 4.14 that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

If $\Lambda = L = \mathbb{Z}[\omega]$, that is $n \in \{3, 6\}$, we consider the parallelogram whose sides are given by $m = \mathrm{Vol}(\rho)$ and $\omega = e^{2\pi i/3}$, and draw $m$ annuli obtained by integer

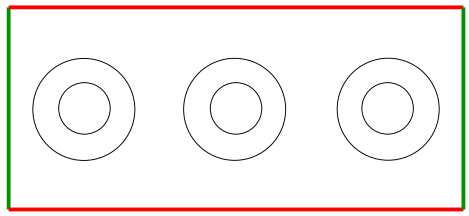

Figure 24: Annuli in $R$ for $(n, m) = (4, 3)$.

translations. We then make slits and gluing in these annuli corresponding to $n_1, \ldots, n_k$.

If $n = 2$, we may assume that $L = \Lambda = \mathbb{Z}[i]$: we may replace $\rho$ with $A \cdot \rho$, $A \in \mathrm{GL}_2^+(\mathbb{R})$ if necessary. The construction is then similar to the case $n = 4$.

**Infinite linear part** Let us suppose that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ is such that $\alpha = \mathrm{Li} \circ \rho$ is infinite. Suppose $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ are such that $\sum_i n_i = 2g - 2$. Let us show that here, Obstruction 4 is the only obstruction to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

**Proposition 6.30.** *The representation $\rho$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if $\mathrm{Vol}(\rho) > 0$.*

*Proof.* Let us suppose that $\mathrm{Vol}(\rho) > 0$. There exists $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{Isom}^+(\mathbb{E}^2))$ such that $\rho' \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ and $\mathrm{Vol}(\rho') = \mathrm{Vol}(\rho)$. Indeed we can for example consider a translation surface with branch data $n_1, \ldots, n_k$ and rescale this translation surface so that it has the required volume. We then take $\rho'$ to be its holonomy. Since $\rho'$ is in the closure of $\mathrm{Aut}^+(\Gamma) \times \mathrm{Isom}^+(\mathbb{E}^2) \cdot \rho$ by Proposition 4.20, we have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ by the Ehresmann-Thurston principle Corollary 5.6. □

### 6.3.3 Strictly affine holonomy

In this section we geometrize *strictly affine* representations. By strictly affine, we mean that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(\mathbb{C}))$ is such that $\alpha = \mathrm{Li} \circ \rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^\star)$ does not have its image contained in $\mathbb{S}^1 \subset \mathbb{C}^\star$. This is equivalent to saying that $\rho$ is not Euclidean. We show that the only obstruction to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ is that $\sum_i n_i$ must be even and at least $2g - 2$.

**Proposition 6.31.** *Let $\rho$ be a strictly affine representation. We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if $\sum_i n_i$ is even and at least $2g - 2$.*

By Lemma 4.6 and the Ehresmann-Thurston principle Corollary 5.6, it suffices to show that $\mathrm{Li} \circ \rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

**Definition 6.32.** The exponential of a representation $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ is the representation defined by $\exp(\chi) : \gamma \in \Gamma \mapsto \exp(\chi(\gamma))z \in \mathrm{Aff}(\mathbb{C})$.

*Remark* 6.33. Suppose $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ is the holonomy of a translation surface structure with developing map $f : \tilde{\Sigma} \to \mathbb{C}$. The map $\exp \circ f$ is $\exp(\chi)$-equivariant.

**Lemma 6.34.** *There exists a representation $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that:*

1. $\exp(\chi) = \mathrm{Li} \circ \rho$
2. *$\chi$ is the holonomy of a translation surface with a single conical point.*

*Proof.* These properties are invariant by the action of $\mathrm{Aut}^+(\Gamma)$. We can thus modify $\rho$ so that both $\alpha(a_1)$ and $\alpha(a_2)$ are not in $\mathbb{S}^1$, where $\alpha = \mathrm{Li} \circ \rho$. Let us first choose $\chi(\gamma) \in \mathbb{C}$ such that $\alpha(\gamma) = \exp(\chi(\gamma))$ for each $\gamma \in \{a_1, \dots, b_g\}$. We thus have $\exp(\chi) = \alpha$. We may now replace $\chi(b_1)$ with $\chi(b_1) + 2\pi i k$ where $k \in \mathbb{Z}$ so that $\det(\chi(a_1), \chi(b_1)) \neq 0$. We can then replace $\chi(b_2)$ with $\chi(b_2) + 2\pi i k$ so that $\mathrm{Vol}(\chi) \geqslant 2g|\det(\chi(a_1), \chi(b_1))|$. Therefore if the group $\Lambda$ generated by the $\chi(\gamma)$ for $\gamma \in \{a_1, \dots, b_g\}$ is a lattice, its area is bounded above by $|\det(\chi(a_1), \chi(b_1))|$, and $\mathrm{Vol}(\chi) \geqslant 2g\mathrm{Area}(\Lambda)$. It follows from the refined Haupt's theorem that $\chi$ is the holonomy of a translation surface with a single conical point of angle $2\pi(2g-1)$. $\square$

The explicit constructions made in [LF20] show that if $\chi$ is the holonomy of a translation surface with a single conical point, then $\chi$ is also the holonomy of another translation surface with a single conical point such that there exists a simple closed curve $c$ that develops injectively on a line that is not parallel to the imaginary axis. The exponential of the associated developing map gives a translation surface with a single conical point and holonomy $\alpha = \exp(\chi)$. The closed curve $c$ based on the conical point develops injectively. Therefore by Proposition 5.7, we have $\alpha \in \mathrm{Hol}(\mathcal{P}(2d))$ for each $d \geqslant g-1$, and thus $\alpha \in \mathrm{Hol}(\mathcal{P}(n_1, \dots, n_k))$ if $\sum_i n_i$ is even and at least $2g-2$ by Remark 5.9.

## 6.4 Dihedral holonomy

In this section, we suppose that $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ is *dihedral* but not affine, that is its image $\rho(\Gamma)$ fixes globally a pair of points in $\mathbb{CP}^1$, but not pointwise. After conjugating $\rho$, we may assume that $\rho(\Gamma)$ is included in the group $\{z \mapsto cz^\epsilon \mid c \in \mathbb{C}, \epsilon = \pm 1\}$. We require that $\rho$ is not conjugated into $\mathrm{SO}_3(\mathbb{R})$ nor in $\mathrm{Aff}(\mathbb{C})$, that is there exist $\gamma$ and $\delta$ in $\Gamma$ such that $\rho(\gamma) = c_\gamma z$ with $|c_\gamma| \neq 1$ and $\rho(\delta) = c_\delta z^{\epsilon(\delta)}$ with $\epsilon(\delta) = -1$.

Observe that $\epsilon : \Gamma \to \{\pm 1\}$ is a group homomorphism. Thus we may assume that it satisfies $\epsilon(a_1) = -1$ and $\epsilon(\gamma) = 1$ for $\gamma \in \{b_1, a_2, \dots b_g\}$ by Proposition 3.2. We may conjugate $\rho$ with an isometry of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ so that $c_\gamma = 1$ for $\gamma = a_1$. Since $\rho([a_i, b_i]) = \mathrm{id}$ for $i \geqslant 2$, we must have $\rho([a_1, b_1]) = \mathrm{id}$. Therefore $c_\gamma \in \{1, -1\}$ for $\gamma = b_1$. It follows from Lemma 3.7 that $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$ if and only if $c_\gamma = 1$, where $\gamma = b_1$. Note that $\rho(\Gamma)$ is not abelian. Therefore we assume that $g \geqslant 2$ in this section.

### 6.4.1 Even Stiefel Whitney class

We first assume that $\rho$ lifts to $\mathrm{SL}_2(\mathbb{C})$: $c_\gamma = 1$ for $\gamma = b_1$. We may extend the definition of the exponential of a representation to the representations of the form $\rho' : \gamma \mapsto \epsilon(\gamma)z + c_\gamma$, where $\epsilon \in \mathrm{Hom}(\Gamma, \{\pm 1\})$.

**Definition 6.35.** 1. The exponential of $\rho'$ is the representation

$$\exp(\rho') : \gamma \mapsto \exp(c_\gamma) z^{\epsilon(\gamma)}.$$

2. The exponential of a half-translation surface $Y$ with developing map $f$ is the projective structure $X = \exp(Y)$ with developing map $\exp \circ f$.

*Remark* 6.36. We have $\mathrm{Hol} \circ \exp = \exp \circ \mathrm{Hol}$.

**Proposition 6.37.** *Let $X \in \mathcal{P}(n_1, \ldots, n_k)$ with $\sum_i n_i = 2g-2$ and holonomy $\rho$. There exists a half-translation surface structure $Y$ on $\Sigma$ such that $X = \exp(Y)$.*

*Proof.* Let us consider a developing map $f : \tilde{\Sigma} \to \mathbb{C}$ associated with $X$. We consider $S$ the double-cover of $\Sigma$ associated with $\ker \epsilon \triangleleft \Gamma$. The developing map induces an affine structure on $S$ with a branching divisor of degree $-2\chi(\Sigma) = -\chi(S)$. Therefore the image of $f$ does not contain $\infty \in \mathbb{CP}^1$ by Proposition 6.18. There exists $\lambda \in \mathbb{C}^\star$ such that $z \mapsto \lambda z^{-1}$ is in $\rho(\Gamma)$, thus $f(\tilde{\Sigma})$ does not contain 0 either. Now let $\tilde{f} : \Sigma \to \mathbb{C}$ be the lift of $f : \tilde{\Sigma} \to \mathbb{C}^\star$ to the universal cover $\mathbb{C}$ of $\mathbb{C}^\star$. Recall that $\exp : \mathbb{C} \to \mathbb{C}^\star$ is the associated covering map. For every $\gamma \in \Gamma$, and for every $z \in \mathbb{C}$, we have $\exp(\tilde{f}(\gamma \cdot z)) = f(\gamma \cdot z) = c_\gamma f(z)^{\epsilon(\gamma)}$, where $\rho(\gamma) : z \mapsto c_\gamma z^{\epsilon(\gamma)}$. Therefore, $\tilde{f}(\gamma \cdot z) = \epsilon(\gamma)\tilde{f}(z) + l_\gamma$ where $\exp(l_\gamma) = c_\gamma$. By continuity, $l_\gamma$ does not depend on $z$. We thus define a map $\rho' : \Gamma \to \mathrm{Isom}^+(\mathbb{E}^2)$ by $\rho'(\gamma) = \epsilon(\gamma)z + l_\gamma$. It follows from the $\rho'$-equivariance of $\tilde{f}$ that $\rho'$ is a group homomorphism. Indeed the relation

$$\tilde{f}(\gamma_1\gamma_2 \cdot z) = \rho'(\gamma_1\gamma_2) \cdot \tilde{f}(z) = \rho'(\gamma_1)\rho'(\gamma_2) \cdot \tilde{f}(z)$$

gives $\rho'(\gamma_1\gamma_2) = \rho'(\gamma_1)\rho'(\gamma_2)$ since $\tilde{f}$ is open and any element of $\mathrm{PSL}_2(\mathbb{C})$ is determined by its action on 3 points in $\mathbb{CP}^1$. The half-translation surface structure $Y$ with developing map $\tilde{f}$ satisfies $\exp(Y) = X$. □

We now explain Obstruction 6. Let us consider a projective structure $X$ on $\Sigma_2$ with branching divisor of degree 2 and with dihedral holonomy that is not affine. There exists a half-translation structure $Y$ on $\Sigma_2$ such that $X = \exp(Y)$. We have seen in Example 6.26 that it is not possible for $Y$ to have a single conical point, thus it must have two conical points of angle $4\pi$ each and so must $X$.

Let us see that this obstruction only occurs when $(g, k) = (2, 1)$.

**Proposition 6.38.** *Suppose $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ are such that $\sum_i n_i = 2g - 2$. The representation $\rho$ is in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if $(g, k) \neq (2, 1)$.*

*Proof.* Let us first suppose that $g \geqslant 3$. We may suppose that $|c_\gamma| \neq 1$ for $\gamma \in \{a_2, a_3\}$, since the map $\gamma \in \Gamma_{g-1} \mapsto \log(|c_\gamma|) \in \mathbb{R}$ is a non trivial homomorphism of $\mathrm{Hom}(\Gamma_{g-1}, \mathbb{R})$. We take a representation $\rho' : \gamma \mapsto \epsilon(\gamma)z + l_\gamma$ with $\exp(\rho') = \rho$. We may also suppose that $\rho'$ is geometric: changing $\rho'(\gamma)$ with $\rho'(\gamma) + 2k\pi i$ for $\gamma \in \{b_2, b_3\}$, we can make sure that $\mathrm{Vol}(\rho') > 0$ and that $2\mathrm{Vol}(\rho') \geqslant (n_k + 1)\mathrm{Area}(\Lambda)$ in the case where $\Lambda = \{z_0 \in \mathbb{C} \mid z + z_0 \in \rho'(\Gamma)\}$ is a lattice in $\mathbb{C}$. Therefore, there exists by Proposition 6.27 a half-translation structure $Y$ with holonomy $\rho'$ and a single conical point. We consider its exponential $X = \exp(Y)$.

Let us now turn to the case $g = 2$ and $k \geqslant 2$. Observe that $k = 2$ and $n_1 = n_2 = 1$ since $\sum_i n_i = 2$. We may consider $\rho'$ as before such that $\rho = \exp(\rho')$, where $\rho' = (z + x, z + y, -z, z)$. We may also assume that $\mathrm{Vol}(\rho) = \det(x_1, y_1)$ is positive; replacing $x$ with $x + 2k\pi i$ for some $k \in \mathbb{Z}$ if necessary (or $y$ with $y + 2k\pi i$). We geometrize $\rho'$ by Proposition 6.27 and take the exponential of the associated developing map. $\square$

We now suppose that $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ are such that $\sum_i n_i$ is even and at least $2g$. We show that there are no more obstructions to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

**Proposition 6.39.** *The representation $\rho$ is in* $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$.

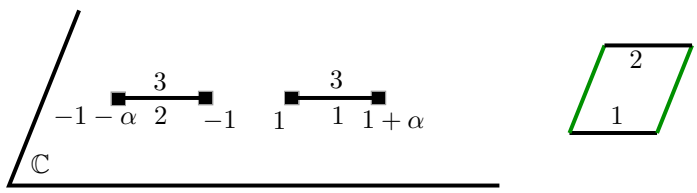


Figure 25: Construction for $(g, k, n_1) = (2, 1, 4)$.

*Proof.* Let us first suppose that $g \geqslant 3$. We have seen that $\rho \in \mathrm{Hol}(\mathcal{P}(2g-2))$. We may check on the constructions we have made that we can suppose that there exists a curve based at the conical point of $Y$ whose exponential develops injectively, where $Y$ comes from the proof of Proposition 6.38. Therefore by Proposition 5.7 we have $\rho \in \mathrm{Hol}(\mathcal{P}(2d))$ for every $d \geqslant g-1$, and $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ by Proposition 5.8.

Let us now suppose that $g = 2$. Let us denote $\rho = (z^{-1}, z, c_2 z, c_3 z)$. We can assume that $|c_2| \neq 1$, since $\rho$ is not spherical. Let us take $\alpha$ and $\beta \in \mathbb{C}$ such that $e^\alpha = c_2$ and $e^\beta = c_3$. We may assume that $\det(\alpha, \beta) > 0$, replacing $\beta$ with $\beta + 2\pi i k$ if necessary. Let us consider a parallelogram in $\mathbb{C}$ whose sides are given by $\alpha$ and $\beta$. We glue the sides corresponding to $\beta$ with the map $z \mapsto z + \alpha$. This gives projective charts on an annulus. We define new charts on this annulus by considering the exponential of the associated developing map. Let us consider the path $\delta_1 = \exp \circ c$, where $c$ is the path along the line segment in $\mathbb{C}$ joining 1 to $1 + \alpha$, and $\delta_2 = \exp \circ d$, where $d = -c$ is the path along the line segments from $-1$ to $-1 - \alpha$. Let us cut open $\mathbb{CP}^1$ along

the image of $\delta_i$ for $i = 1, 2$. We glue the resulting boundaries to those of the annulus, see Figure 25. We obtain a genus 2 surface with holonomy $\rho_1 = (c_2 z, c_3 z^{-1}, c_2^{-1} z, z^{-1}) = \exp(\rho')$, where $\rho' = (z + \alpha, -z + \beta, z - \alpha, -z)$. It follows from Lemma 4.12 that the representation $\rho'' = (-z, z, z + \alpha, z + \beta)$ is in $\mathrm{Aut}^+(\Gamma) \cdot \rho'$. Therefore, $\rho = \exp(\rho'')$ is in $\mathrm{Aut}^+(\Gamma) \cdot \rho_1$. Since the curve $\delta$ develops injectively, we have $\rho \in \mathrm{Hol}(\mathcal{P}(2d))$ for every $d \geqslant 2$ by Proposition 5.7. Hence $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ by Proposition 5.8. □

### 6.4.2 Odd Stiefel Whitney class

In this subsection we suppose that $\rho$ does not lift to $\mathrm{SL}_2(\mathbb{C})$, that is $c_\gamma = -1$ for $\gamma = b_1$.

**Proposition 6.40.** *We have $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if and only if $\sum_i n_i$ is odd and at least $2g - 1$.*

*Proof.* There exists a branched projective structure $X$ on $\Sigma_{g-1}$ with a single branch point of order $2g - 4$ and holonomy $(c_{a_2} z, c_{b_2} z, \ldots, c_{b_g} z)$ by Proposition 6.31, which is obtained by taking the exponential of a translation surface structure $Y$. Observe that we may translate the charts of $Y$, so that the conical point is sent to $z \in \mathbb{R}_{>0}$, such that the line segment $[-z, z]$ is included in the image of this chart. We will now make a surgery in the corresponding chart of $X$ to add a handle to this surface. Let us cut $\Sigma_{g-1}$ open along the segment $[e^{-z}, e^z]$. We will glue the resulting boundaries to the boundary of the hemisphere $\overline{\mathbb{H}^2} = \mathbb{H}^2 \cup \mathbb{RP}^1 \subset \mathbb{CP}^1$. We identify the bottom boundary of the slit with the line segment $[e^{-z}, e^z] \subset \partial\mathbb{H}^2$ with the identity. We glue the top boundary of the slit with the line segment $[-e^z, -e^{-z}]$ of $\partial\mathbb{H}^2$ with the map $z \mapsto -z$. Finally we glue the two line segments $[-e^{-z}, e^{-z}]$ and $(\mathbb{R} \cup \{\infty\}) \setminus [-e^z, e^z]$ of $\partial\mathbb{H}^2$ with the map $z \mapsto \frac{1}{z}$, see Figure 26.

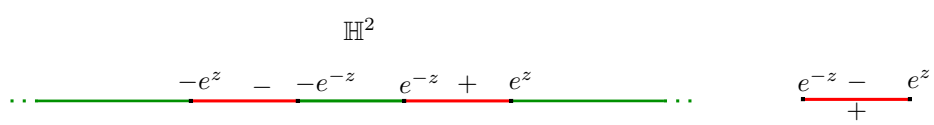


Figure 26: $\overline{\mathbb{H}^2}$ on the left side. Its boundary is glued to those on the right side.

Topologically we have added a handle to $\Sigma_{g-1}$ and thus we get a genus $g$ surface. Moreover it has natural charts in $\mathbb{CP}^1$, that give a projective structure holonomy $\rho$. We have added $3 \times 2\pi$ to the total angle of the conical point thus $\rho \in \mathrm{Hol}(\mathcal{P}(2g - 1))$. The curve on which we have glued the handle develops injectively, hence $\rho \in \mathrm{Hol}(\mathcal{P}(2g - 1 + 2k))$ for every $k \geqslant 0$ by Proposition 5.7. The result follows from Remark 5.9. □

## 6.5 Nonelementary holonomy

The celebrated theorem of Gallo, Kapovich and Marden proven in [GKM00] is in fact enhanced in the same paper, see [GKM00, Theorem 11.2.4].

**Theorem 6.41** (Gallo-Kapovich-Marden)**.** *Let* $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ *be a nonelementary representation. We have* $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ *if and only if* $\sum_i n_i$ *has the same parity as* $\mathrm{sw}(\rho)$*.*

Their proof is based on the existence of Schottky decompositions of nonelementary representations, shown in [GKM00, Part A]. By a Schottky decomposition of $\rho$ we mean a pants decomposition $\Sigma = \cup_i P_i$ such that each restriction $\rho_{|P_i} : \pi_1(P_i) \to \mathrm{PSL}_2(\mathbb{C})$ is an isomorphism onto a Schottky group. Indeed, the techniques of construction of projective structures with given holonomy of Gallo, Kapovich and Marden allow them to prove directly this theorem when $\rho$ has a Schottky decomposition; see [GKM00, Part B, Theorem 11.2.4]. In genus $g = 2$, there exist nonelementary representations $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ that do not admit a Schottky decomposition: the *pentagon representations*, see [LF19]. In the case where $\rho$ is a pentagon representation, there exists a branched projective structure on $\Sigma_2$ with a single conical point of total angle $2\pi(1+1)$ and holonomy $\rho$, see [LF19, Section 5]. Moreover we can assume that there exists a curve based at the conical point that develops injectively, see the red curve in [LF19, Figure 8], whose exponential develops injectively. Thus we have $\rho \in \mathrm{Hol}(\mathcal{P}(2d+1))$ for every $d \geqslant 0$ by Proposition 5.7 and $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ if $\sum_i n_i$ is odd by Remark 5.9. Therefore Obstruction 1 is the only obstruction to being in $\mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$ for nonelementary representations.

# 7 Geometrization in minimal degree

Let us conclude this article with a short section dedicated to the proof of Corollary 1.2. For every $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$, we compute the minimal degree of the branching divisor of a projective structure having $\rho$ as holonomy. In other words, we compute the minimal value of $\sum_i n_i$ such that $\rho \in \mathrm{Hol}(\mathcal{P}(n_1, \ldots, n_k))$. By Remark 5.9, this number $d(\rho)$ is the minimal $m \geqslant 0$ such that $\rho \in \mathrm{Hol}(\mathcal{P}(\underbrace{1, \ldots, 1}_{m}))$. This number was computed by Kapovich for representations $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C})$ with values in the group of translations $\mathbb{C} \subset \mathrm{Aff}(\mathbb{C})$ in [Kap20], and by Gallo Kapovich and Marden for non-elementary representations and the trivial representation in [GKM00]. This computation is a corollary of Theorem 1.1 and is given in Figure 1.

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