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**Applications de la Supergravité et de la
Théorie des Cordes : de la physique des
particules à la cosmologie**

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Résumé

Les sujets de recherche présentés dans cette thèse se situent à l'interface entre certains aspects théoriques et phénoménologiques de la Théorie des Cordes et de la Supergravité, se concentrant sur leurs conséquences à basse énergie. Le premier chapitre traite d'un nouveau type de termes de Fayet-Iliopoulos (FI) en supergravité $\mathcal{N} = 1$, $D = 4$, qui peuvent être écrits sans jauger la symétrie R. Nous introduisons cette construction et développons une nouvelle classe de termes de FI invariants de Kähler, paramétrés par une fonction de la masse du gravitino vue comme fonctionnelle des superchamps chiraux. Nous montrons alors en supergravité de type no-scale que cette fonction peut produire un vide de Sitter et un potentiel inflationnaire compatible avec les données observationnelles. Les deux dernières parties de cette thèse s'intéressent à la possibilité que l'échelle de la corde puisse être beaucoup plus basse que la masse de Planck, en utilisant soit un couplage des cordes très faible, ou des dimensions supplémentaires larges. La première possibilité mène à la théorie appelée *Little String Theory*, dont le dual holographique est étudié dans le second chapitre. Nous revoyons tout d'abord sa structure supersymétrique minimale, qui repose sur un jaugeage abélien d'une supergravité $\mathcal{N} = 2$, $D = 5$ couplée à un multiplet vectoriel, produisant ainsi un background dilaton linéaire. Nous compactifions ce modèle sur S^1/\mathbb{Z}_2 et décrivons la théorie effective $\mathcal{N} = 1$, $D = 4$. Enfin, la troisième partie de cette thèse exploite le cadre des dimensions supplémentaires larges afin d'aborder le problème de l'écart dans le moment magnétique anomal du muon. Nous montrons que la contribution des états de Kaluza-Klein du boson de jauge du nombre leptonique peut combler cet écart, à condition que certains d'entre eux soient plus légers que l'énergie du LEP ~ 200 GeV, et construisons ensuite les configurations minimales de D-branes réalisant cette proposition.

Mots-clés : supergravité, théories effectives supersymétriques à basse énergie, échelle des cordes basse, dimensions supplémentaires (larges), background dilaton linéaire, modèles de D-branes s'entrecoupant.

Abstract

The research topics presented in this thesis lie at the interface between theoretical and phenomenological aspects of String Theory and Supergravity, focusing on their low-energy consequences. The first chapter deals with new kind of Fayet-Iliopoulos (FI) terms in $\mathcal{N} = 1$, $D = 4$ supergravity, which can be written without gauging the R-symmetry. We review this construction and build a new class of Kähler invariant FI terms parametrised by a function of the gravitino mass as functional of the chiral superfields. We then show in no-scale supergravity that this function can provide a de Sitter vacuum and an inflationary potential compatible with the observational data. The last two parts of this thesis deal with the possibility that the string scale might be much lower than the Planck mass, using either an ultra weak string coupling, or large extra dimensions. The first possibility leads to the so-called *Little String Theory*, whose holographic dual is studied in the second chapter. We first revisit the construction of its minimal supersymmetric embedding, which relies on an abelian gauging of a $\mathcal{N} = 2$, $D = 5$ supergravity coupled to one vector multiplet, hence producing a linear dilaton background. We compactify this model on S^1/\mathbb{Z}_2 and describe the effective $\mathcal{N} = 1$, $D = 4$ theory. Finally, the third part of this thesis uses the framework of large extra dimensions to address the issue of the discrepancy in the anomalous magnetic moment of the muon. We show that the contribution of the Kaluza-Klein states of the lepton number gauge boson can bridge the gap in the discrepancy, provided the existence of states lighter than the LEP energy ~ 200 GeV, and then build the minimal D-brane configurations realising this proposal.

Keywords: supergravity, supersymmetric low-energy effective theories, low string scale, (large) extra dimensions, linear dilaton background, intersecting D-brane models.

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Contents

Introduction	1
1 Cosmological models in supergravity through generalized Kähler invariant Fayet-Iliopoulos terms	9
1.1 Introduction	9
1.2 Conventions and notations	10
1.3 Basics of supersymmetry and supergravity	12
1.3.1 Most general chiral models in $\mathcal{N} = 1$ supersymmetry	12
1.3.2 Pure $\mathcal{N} = 1$, $D = 4$ supergravity and Anti-de Sitter supergravity	14
1.4 Spontaneous supersymmetry breaking and Fayet-Iliopoulos terms in supersymmetry and supergravity	16
1.4.1 Spontaneous global supersymmetry breaking	16
1.4.2 Theoretical approach to Fayet-Iliopoulos terms in global supersymmetry	18
1.4.3 Fayet-Iliopoulos terms in supergravity: Freedman model	19
1.5 Superconformal approach to $\mathcal{N} = 1$ supergravity	22
1.5.1 $SU(2, 2 \mathcal{N} = 1)$ superconformal algebra	22
1.5.2 New perspectives from the superconformal formalism	27
1.6 Fayet-Iliopoulos terms in supergravity without gauged R -symmetry	31
1.6.1 The original construction and an improved version	31
1.6.2 A set of Kähler invariant Fayet-Iliopoulos terms	34
1.7 No-scale models and cosmological applications	37
1.7.1 Dilaton case	38
1.7.2 Compact volume case	42
1.7.3 Gauging the axion shift symmetry	44
1.8 Conclusion	46
2 Compactification on a linear dilaton background	47
2.1 Introduction	47
2.2 Conventions and notations	49
2.3 The Linear Dilaton model	51
2.3.1 The $5D$ theory on $\mathbb{R}^{1,3} \times S^1/\mathbb{Z}_2$	51
2.3.2 Spectrum of bosonic fields on a LD background	54
2.4 Kaluza-Klein reduction on a linear dilaton background	58
2.4.1 $4D$ residual gauge symmetry	58
2.4.2 Stückelberg “mechanism”	62
2.4.3 Effective scalar potential	63
2.5 Minimal supersymmetric extension	65

2.5.1	Vector-coupled $\mathcal{N} = 2$, $D = 5$ ungauged supergravity	65
2.5.2	$U(1)_R$ gauging of $\mathcal{N} = 2$, $D = 5$ supergravity	68
2.5.3	Runaway scalar potential from $5D$ gauged supergravity	69
2.5.4	The LD background as a 1/2-BPS solution	73
2.5.5	Preserved supersymmetry and NS5-branes	75
2.6	Compactified $D = 4$ effective theory	78
2.6.1	General considerations on dimensional reduction	78
2.6.2	$\mathcal{N} = 1$, $D = 4$ effective theory	80
2.7	Conclusion	85
3	Intersecting D-brane models and the anomalous magnetic moment of the muon	87
3.1	The anomalous magnetic moment of the muon: a review	87
3.2	Accommodating the $(g - 2)_\mu$ discrepancy with leptophilic Kaluza-Klein states . .	91
3.3	Elements of intersecting D-brane physics	95
3.4	Minimal embedding of the Standard Model into intersecting D-brane configurations with a bulk leptonic $U(1)$	98
3.4.1	The models	100
3.4.2	Anomaly analysis	103
3.4.3	Right-handed neutrino	106
3.4.4	Mass spectrum	107
3.4.5	Dark Matter model	110
3.4.6	Lepton flavour non-universality and the muon $g - 2$	113
3.5	Conclusion	114
A	Kähler geometry	115
A.1	Complex manifolds and Kähler manifolds	115
A.2	Symmetries of Kähler manifolds	117
B	Multiplet calculus	119
C	Fermion masses	121
D	Gravitational action on a bounded manifold	127
D.1	$d + 1$ decomposition and Gibbons-Hawking boundary term: a review	127
D.2	Computation in the framework of the LD background	130
E	Effective theory of the heterotic string	135
	Bibliography	137

List of Illustrations

Figures

1.1	Scalar potential as a function of the canonically normalised field χ , for $\alpha = 1$. . .	41
1.2	Slow-roll parameters ϵ and η as a function of χ , for $\alpha = 1$	41
1.3	Number of e-folds N_+ as a function of α , for $-\frac{2}{3} < \alpha < \frac{\sqrt{3}-2}{3}$	43
1.4	Number of e-folds N_- as a function of α , for $\alpha \geq \frac{\sqrt{3}-2}{3}$	43
3.1	All-order contributions to the anomalous magnetic moment of a lepton l^-	87
3.2	One-loop contribution to the anomalous magnetic moment of a lepton l^-	88
3.3	Scale hierarchy in our model.	99
3.4	Pictorial representation of models A and B.	102
3.5	Pictorial representation of model C.	103
3.6	Dark matter annihilation into SM leptons mediated by the KK excitations $L_\mu^{(n)}$ of the lepton number gauge boson.	110
3.7	Annihilation cross section $10^9 \times \langle \sigma v_r \rangle$ (GeV^{-2}) in terms of the DM mass m_χ (GeV), for a compactification scale $M_L \sim 10$ GeV and a coupling $g_L \sim 10^{-2}$	112

Tables

1	Average size R of large extra dimensions for a given number n of them, in the case of a string scale $M_s = 10$ TeV.	6
1.1	Generators and gauge fields of the superconformal algebra $SU(2, 2 1)$	23
1.2	Gauge fixing conditions reducing the $SU(2, 2 \mathcal{N} = 1)$ superconformal algebra to the $\mathcal{N} = 1$ super-Poincaré algebra	27
2.1	The five different supergravity theories (defined by the C_{IJK} on the left part) coupled to one vector multiplet whose $U(1)_R$ gauging reproduces the potential of LST, and the associated constants (on the right part).	71
2.2	Bosonic content of the $D = 4$, $\mathcal{N} = 2$ multiplets and their \mathbb{Z}_2 -parity.	82
2.3	$\mathcal{N} = 1$, $D = 4$ field content	84
3.1	Model A with $q_Y = \frac{2}{3}q_3 - \frac{1}{2}q_2 + q_1$	102
3.2	Model B with $q_Y = -\frac{1}{3}q_3 - \frac{1}{2}q_2 + q_1$	102
3.3	Model C with $q_Y = -\frac{1}{3}q_3 - \frac{1}{2}q_2 + q_1$	103
3.4	Anomaly-free $U(1)$ in configurations with right-handed neutrinos (model A) . . .	106

Introduction

General Relativity and Quantum Field Theory are two main achievements carried out in theoretical physics during the twentieth century, both of them experimentally tested at an spectacular level of accuracy. General Relativity (GR) describes physics at large scales where gravity dominates. It has predicted new astrophysical objects and phenomena, which have been observed from the very early ages of the theory until our days, like, for the most recent ones, the detection of gravitational waves in 2015, or the first picture of a black hole taken in 2019, a century after their theoretical prediction by Einstein. GR provides the theoretical background upon which has been built modern cosmology, the theory that describes the Universe and its content as a whole. The minimal cosmological model able to describe the main observable properties of our Universe, such as the structure of the Cosmic Microwave Background, the large-scale structure of the galaxies and the accelerating expansion of the Universe, is called the standard model of cosmology or *Lambda-Cold Dark Matter* (Λ CDM) model. On the other hand, Quantum Field Theory (QFT) provides a quantum consistent framework to describe physics at the microscopic scales, combining the laws of quantum mechanics and special relativity in an elegant way. Its theoretical predictions have been experimentally verified with a precision never reached before in the history of science. QFT provides the theoretical background upon which the Standard Model (SM) of particle physics has been built, which describes all known fundamental particles and the way they interact through three of the four known forces: the electromagnetism as well as the strong and weak interactions.

Despite their respective successes, both theories suffer from theoretical and conceptual inconsistencies. General Relativity predicts its own death in region of spacetime with infinite curvature, called singularities, such as the Big Bang or the singularity appearing in the center of black holes. In these regions, quantum mechanical effects cannot be neglected anymore, and a correct description of these extreme regimes would require a consistent theory of Quantum Gravity, which is still unknown so far. On the other hand, many features tend to indicate that the Standard Model of particle physics is not a fundamental theory, but rather an effective theory valid at least up to the electroweak scale and which must break down at higher energies. First, the SM might look in itself inelegant: it does not explain why it is based on the gauge group $SU(3)_c \times SU(2)_w \times U(1)_Y$, why does it contain three generations of fermions neither why there are about 19 free parameters, whose values are not theoretically predicted but have to be measured experimentally. But the biggest conceptual issue of the SM is the so-called *hierarchy problem*, the question of why the electroweak scale and the Planck scale are separated by about 16 orders of magnitude, and how the electroweak scale can be stable under quantum corrections. The SM also does not include gravity, and contains no potential dark matter candidates in its spectrum. Many ideas beyond the SM have been proposed in order to tackle the above mentioned issues, such as Grand Unified Theories, supersymmetry or extra spacetime dimensions, but none of them address the problem of the quantization of gravity.

String Theory is the only theory attempting to unify all the four known fundamental forces in a common framework, and is one of the most promising theory of Quantum Gravity. Its basic assumption is to replace the point-like particles of Quantum Field Theory by 1-dimensional extended objects, called strings. All the elementary matter particles as well as the force carriers that we know then correspond to different excitation modes of such strings. Its theoretical consistency requires in particular two ingredients: supersymmetry and extra spacetime dimensions. The former ensures that the theory contains no tachyon and produces spacetime fermions. On the other hand, while a classical string can live in any spacetime dimensions, its quantization requires the spacetime to be 10-dimensional. In order to accommodate with our observed 4-dimensional world, we must compactify six extra dimensions on a 6-dimensional compact manifold whose characteristic sizes must be sufficiently small to not be accessible by the current experiments.

Supersymmetry and its breaking as well as extra spacetime dimensions lie at the core of this thesis, and we now introduce the basic ideas of these two concepts.

Supersymmetry and supergravity

All modern theoretical physics is based on the concept of symmetries. The Standard Model of Particle Physics is built from the assumption that physical laws are invariant under the action of spacetime rotations and translations. This set of transformations forms the *Poincaré group*, which is the Minkowski spacetime isometry group. One possible fundamental approach to supersymmetry is based on the following question: *is it possible to enlarge this Minkowski spacetime symmetry group?* The Coleman-Mandula theorem [1] brings some parts of the answer. It states that the S -matrix of a quantum field theory in Minkowski space is invariant at most under Poincaré symmetries, and so, that the Poincaré group is a priori the biggest symmetry group of Minkowski spacetime. However, this no-go theorem contains two main loopholes which can be used to circumvent it. The first one is that it assumes the existence of the S -matrix. For a massless theory where the S -matrix does not exist, a new symmetry, the scale invariance, can be added. This leads to the conformal group which contains the Poincaré group¹. The second one arises from the assumption that the spacetime symmetries are described by an algebra, namely that its generators are bosonic. Considering fermionic generators gives a so-called superalgebra, which leads to supersymmetry. In the simplest case when one fermionic generators Q is introduced, called $\mathcal{N} = 1$ supersymmetry, it is defined through the following (anti-)commutation relations:

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{[\mu[\rho} M_{\sigma]\nu]}, & [M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, & [P_\mu, P_\nu] &= 0, \\ \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu, & [Q_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^\beta Q_\beta, & [Q_\alpha, P_\mu] &= 0. \end{aligned} \quad (1)$$

Indices $\{\alpha, \beta, \dot{\alpha}, \dot{\beta}\}$ run from one to two and denote two-component Weyl spinors, while indices $\{\mu, \nu, \rho, \sigma\}$ run from zero to three and denote Lorentz four-vectors. The first line alone defines the Poincaré algebra, spanned by the generators P_μ and $M_{\mu\nu}$ of translations and Lorentz transformations. The four $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}$ introduced in the second line are called supercharges, and transform

1. While the conformal group will not be addressed in this thesis, conformal symmetry seen as a mathematical tool rather than a physical spacetime symmetry will be discussed in Section 1.5.

as spinors under Lorentz transformations. The graded commutation relations (1) define the so-called $\mathcal{N} = 1$ *super-Poincaré algebra*. Representations of this superalgebra are called supermultiplets, or multiplets in the following. Because of the fermionic nature of the supercharges Q 's, different elements of a given multiplet will have different Lorentz spins. Symbolically denoting B and F a bosonic and fermionic field belonging to the same multiplet, a supersymmetry transformation schematically reads:

$$\delta B = \bar{\epsilon}F, \quad \delta F = \not{\partial}B\epsilon, \quad (2)$$

where ϵ has to be a fermionic parameter in order to be consistent with the bosonic and fermionic nature of B and F . In global supersymmetry, this parameter is constant. The commutator of two supersymmetric transformations is given by

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]B \propto \bar{\epsilon}_2 \not{\partial}B\epsilon_1 = \bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu B \equiv \xi^\mu \partial_\mu B, \quad (3)$$

where ξ^μ is a vector field. We deduce that a *globally* invariant supersymmetric theory is also necessarily invariant under *translations*. However, it is known that all fundamental interactions can be described in terms of gauge symmetries, namely symmetries acting differently at each spacetime point. It is therefore legitimate to also try to promote supersymmetry into a local symmetry, that is, replacing the constant parameter ϵ by a spacetime point dependent parameter $\epsilon(x)$. The vector field ξ^μ now also depends on x via $\xi^\mu(x) \equiv \bar{\epsilon}_2(x) \gamma^\mu \epsilon_1(x)$, and the supersymmetry algebra therefore closes into diffeomorphisms:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]B \propto \xi^\mu(x) \partial_\mu B. \quad (4)$$

We conclude that a *locally* invariant supersymmetric theory is also necessarily invariant under *diffeomorphisms*, and therefore *includes gravity*. Such theory of local supersymmetry is therefore called supergravity.

From a top-down perspective, supergravity arises as the low-energy effective realization of superstring theory. Such limit is obtained when the characteristic length scale of the string is much smaller than the curvature of spacetime, so that the effects of the extended nature of the string can be neglected. If String Theory has anything to do with the real world, there must exist a 4-dimensional supergravity theory describing most of its observable low-energy consequences, in particle physics as well as in cosmology.

Cosmological models should be able to describe at least the dark energy content of our Universe, responsible of its currently observed accelerating expansion, as well as the cosmic inflation, which today is very often seen as part of the standard cosmological model as the theory describing its primordial phase. In the minimal Λ CDM model, the dark energy is described by a small and positive cosmological constant Λ , providing a de Sitter (dS) vacuum at the minimum of the scalar potential where the Universe is supposed to sit today. On the other hand, most inflationary models assume a period in early cosmology where the Universe was in a state dominated by an almost flat potential energy, associated to a slowly rolling scalar field: *the inflaton*. The description of these two regimes therefore requires very specific features for the

potential energy. However, supersymmetry severely restricts the form of the scalar potential in supergravity theories, making the construction of such cosmological models challenging. Besides non-linear realization of supersymmetry, another attempt to obtain dS vacua in supergravity relies on the introduction of Fayet-Iliopoulos (FI) terms, in order to uplift the vacuum energy from anti de Sitter (AdS) to dS. However, the standard FI terms in $\mathcal{N} = 1$, $D = 4$ supergravity are highly constrained. They can only be written when the associated $U(1)$ gauge symmetry is a R-symmetry, preventing the presence of a constant superpotential which is required in AdS supergravity. Recently, new FI terms which do not require the gauging of the R-symmetry have been written, first in pure supergravity, and then in the presence of matter, allowing such constant uplift from AdS to dS vacua.

The Chapter 1 of this thesis focuses on the construction of cosmological models based on these new FI terms. We first review the two original constructions, without and with chiral matter multiplets. We then revisit and generalise them in $\mathcal{N} = 1$, $D = 4$ supergravity coupled to one vector multiplet and an arbitrary number of chiral multiplets, building a new class of Kähler invariant FI terms parametrised by a function of the gravitino mass as functional of the chiral matter fields. This function leads to FI terms which can be chosen constant or field dependent, while preserving Kähler invariance in the latter case. As in the previous constructions, a constant term can be used in order to fine-tune the vacuum energy to a positive or null value, while the field dependent terms can now be used in inflationary models. We then apply this construction in two no-scale supergravity models coupled to one vector and one chiral multiplet, showing how our new FI terms can be used to build an inflationary model compatible with the Cosmic Microwave Background observational constraints. The scalar potential obtained in this way satisfies the slow-roll conditions during inflation and possesses a de Sitter vacuum where supersymmetry is spontaneously broken by D and F-terms. The origin of these new FI terms in String Theory is an interesting open problem which might be addressed in future projects.

Supergravity theories arising in the low-energy limit of superstring theories live in ten space-time dimensions, and the associated super-algebras have one or two fermionic generators. The simplest compactification on 6-dimensional torus will preserve all supersymmetries, producing a 4-dimensional theory with $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supercharges. On the other hand, one of the most important property of the SM is the chirality of the gauge interactions in the electroweak sector. However, any 4-dimensional theories with $\mathcal{N} > 1$ supersymmetries are non-chiral, and are thus phenomenologically excluded. One must therefore consider more involved compactification procedures, preserving no more than $\mathcal{N} = 1$ supersymmetry in four spacetime dimensions. Such compactifications are called *supersymmetric compactifications*. They can be obtained through non-perturbative effects, or by applying appropriate projections like orbifold projections, or considering non-trivial backgrounds in the internal manifold.

The Chapter 2 of this thesis deals with this third possibility. We study the linear dilaton background solution of a gauged vector-coupled $\mathcal{N} = 2$, $D = 5$ supergravity theory. The interesting point about this non-trivial background is that it is precisely a 1/2-BPS solution, preserving half of the original supersymmetries. We first describe the breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ induced by the background, then perform the dimensional reduction and find the resulting $\mathcal{N} = 1$, $D = 4$

low-energy effective theory. The phenomenological motivation for studying this model as well as the main results obtained are described in the following section.

Low string scale and (large) extra dimensions

The string scale, the fundamental mass scale of the string excitations, as well as the compactification scale, the inverse of the size of the six extra spacetime dimensions, are the two most fundamental energy scales in String Theory. A natural question which then arises is: what are their characteristic magnitudes? It has long been thought that both of them must be Planck sized. However, one of the main conceptual revolutions in String Theory that arose in the 90's is that this is not necessarily the case: the string scale and the size of the extra dimensions might not be tied to the 4-dimensional Planck mass [2, 3]. This can be easily understood from the 10-dimensional gravitational effective action, common to all superstring theories, given by

$$S_{\text{grav}} = \int d^{10}x \frac{M_s^8}{g_s^2} \mathcal{R}^{(10)}, \quad (5)$$

where g_s is the string coupling, M_s the string scale and $\mathcal{R}^{(10)}$ the 10-dimensional Ricci scalar. Upon compactification to four dimensions on a 6-dimensional compact manifold of volume $V_{(6)}$, the (observed) 4-dimensional Planck Mass M_{Pl} is given in terms of the above quantities by:

$$M_{Pl}^2 = \frac{M_s^8 V_{(6)}}{g_s^2}. \quad (6)$$

Experimental signatures of String Theory at colliders would be available in the case that the fundamental string scale M_s would be much lower than the Planck scale. According to the relation (6), this situation can be obtained in two distinct ways:

- The first possibility arises if the internal volume $V_{(6)}$ is of order the string scale, $V_{(6)} \sim M_s^{-6}$. The observed value for the 4-dimensional Planck mass can then be explained in this case by an ultra weak string coupling, $g_s \ll 1$. Considering gauge fields living on D-branes, the gauge coupling square g_{YM}^2 is proportional to g_s , and therefore infinitesimally small in the limit $g_s \ll 1$. One must then address an important question, whether it is possible to get interacting gauge fields with gauge coupling $g_{YM}^2 \sim \mathcal{O}(1)$. This can be obtained by considering (Neveu-Schwarz) NS5-branes: in this case the gauge coupling of the theory living on the branes is independent of g_s and can thus be of order one even in the limit of an infinitesimally small string coupling. In the strict limit $g_s = 0$, a stack of coincident NS5-branes gives rise to a non-trivial interacting theory, the so-called *Little String Theory*. It is a non-gravitational theory with stringy-like excitations, hence exhibiting an interesting intermediate structure between String Theory and Field Theory. It has been shown that Little String Theory is holographically dual to a gravitational theory in one more spacetime dimension, on a particular background given by the Minkowski metric (in the string frame) times the real line along which the dilaton varies linearly. In order to get a realistic model with gravity, one must then turn on a small non-vanishing string coupling $g_s \ll 1$, hence generating interactions between the theory on the NS5-branes and gravity in the bulk.

- The second possibility arises when g_s is of order 1. The observed value for the 4-dimensional Planck mass can then be explained in this case by an internal volume $V_{(6)}$ large compared to the string scale, $V_{(6)} \gg M_s^{-6}$ [4, 5]. This mechanism does not require String Theory for its low energy realization: it simply relies on the existence of extra dimensions in which gravity spreads. However, String Theory provides a UV consistent framework in which this scenario can be embedded. In String Theory, the Standard Model fields (without gravity) can be localized on $(p+1)$ -dimensional extended objects called *Dp-branes*, and thus cannot probe all spacetime dimensions where only gravity is free to propagate. Among the $(p+1)$ dimensions of the brane, 4 are the non-compact dimensions we observe in experiments, while the remaining $p-3$ longitudinal dimensions must be compactified in order to be inaccessible at current energies, at a scale which can be as small as the TeV. On the other hand, the $9-p$ dimensions transverse to the brane, which are probed only by gravity but not by the Standard Model fields, can be much larger since experimental bounds on gravitational experiments are much weaker: the compactification scale of such extra dimensions can be of order or larger than 10^{-15} TeV. Let us assume that we have $n = 1, \dots, 6$ large extra dimensions (namely with a characteristic length larger than M_s^{-1}), and $6-n$ dimensions at the string scale. The internal volume is then given by $V_{(6)} = V_{(n)} M_s^{n-6}$, so that the relation (6) yields $M_{Pl}^2 = M_s^{2+n} V_{(n)}$. Writing $V_{(n)} = R^n$, one gets for the average size R of the large extra dimensions

$$R = \frac{1}{M_s} \left(\frac{M_{Pl}}{M_s} \right)^{\frac{2}{n}}. \quad (7)$$

For a string scale $M_s = 10$ TeV, we get $R = 10^{\frac{30}{n}-1} \text{ TeV}^{-1}$. The different values of R for each n are given in Table 1, where we have used $\text{TeV}^{-1} \sim 10^{-19}$ m. Obviously, the case $n = 1$ is

n	1	2	3	4	5	6
R	10^{29} TeV^{-1} 10^{10} m	10^{14} TeV^{-1} 10^{-5} m	10^9 TeV^{-1} 10^{-10} m	10^7 TeV^{-1} 10^{-13} m	10^5 TeV^{-1} 10^{-14} m	10^4 TeV^{-1} 10^{-15} m

Table 1 – Average size R of large extra dimensions for a given number n of them, in the case of a string scale $M_s = 10$ TeV.

excluded by observations, while $n = 2$ is in tension with the Cavendish-type experiments probing gravity at short distances. But the cases $n \geq 3$ are compatible with the current experimental bounds.

The Chapter 2 of this thesis deals with the first situation where $g_s \ll 1$. We focus on the holographic dual of Little String Theory in a 5-dimensional spacetime toy model, and in particular on its effective supergravity theory, which in the minimal case is a $\mathcal{N} = 2$, $D = 5$ supergravity theory coupled to one vector multiplet. The linear dilaton (LD) background then arises through a particular gauging along the $U(1)_R$ subgroup of the $5D$ $SU(2)_R$ symmetry group. We first show that such $5D$ vector-coupled supergravity with this background is actually unique, and then perform the dimensional reduction on S^1/\mathbb{Z}_2 , which requires to generalize the standard Kaluza-Klein (KK) compactification in the more general case where the different fields may depend on the compactified coordinate. The main result obtained here is that the

background induces a Higgs mechanism for the KK vector $G_{\mu 5}$ coming from the 5D metric, which becomes massive by absorbing the string frame radion. This mechanism can be mimicked for the 4D vector coming from the 5D Kalb-Ramond two-form, so that the LD background yields two massive vectors which can be packaged into a spin-3/2 massive multiplet. It is then shown that the massless limit yields a 4D $\mathcal{N} = 1$ supersymmetric theory, containing half of the original degrees of freedom.

The Chapter 3 is based on the second scenario of large extra dimensions, $V_{(6)} \gg M_s^{-6}$. We follow a phenomenological approach, motivated by the recent result of the Fermilab Muon $g - 2$ experiment, which has confirmed a deviation of the measured muon anomalous magnetic moment from the Standard Model prediction. Recently, it has been shown that this discrepancy can be explained in the framework of low mass scale strings and large extra dimensions by considering the contribution of KK excitations of the lepton number gauge boson L_μ . Re-investigating this result, we show that the constraints on KK masses and couplings are more stringent than originally thought, our conclusion being that the $g - 2$ muon discrepancy can be explained in this context, with L_μ propagating in one extra dimension transverse to the SM branes, providing the existence of few KK modes with masses lighter than the LEP energy ~ 200 GeV. We then explicitly construct a minimal embedding of the SM into D-brane configurations for such a bulk lepton number gauge boson which does not participate to the hypercharge. We show that the minimal configuration realising this framework contains five stacks of branes: three of them are the SM branes, while two additional $U(1)$ extended in the bulk are required, among which the leptonic $U(1)_L$. In this construction, the total bulk transverse to the SM branes exhibits an interesting inhomogeneous structure. First, we have one large extra dimension in which the L_μ boson propagates, with a size $R_L \sim (10 - 10^2 \text{ GeV})^{-1}$ in order to explain the muon $g - 2$ discrepancy. Then, at most five large extra dimensions, with an average size $R_G \sim (0.1 \text{ GeV})^{-1}$ larger than R_L , needed to lower the string scale in the $\gtrsim \mathcal{O}(10 \text{ TeV})$ region.

This manuscript is based on the following publications:

- [6] I. Antoniadis and F. Rondeau, *New Kähler invariant Fayet–Iliopoulos terms in supergravity and cosmological applications*, *Eur. Phys. J. C* **80** (2020) 346 [1912.08117], whose new results are presented in Sections 1.6.2, 1.7 as well as in Appendix C.
- [7] I. Antoniadis, C. Markou and F. Rondeau, *Aspects of compactification on a linear dilaton background*, *JHEP* **09** (2021) 137 [2106.15184], whose new results are presented in Sections 2.4, 2.5.3, 2.5.5 and 2.6, as well as in Appendix D.2.
- [8] L.A. Anchordoqui, I. Antoniadis, X. Huang, D. Lüst, F. Rondeau and T.R. Taylor, *Leptophilic $U(1)$ Massive Vector Bosons from Large Extra Dimensions: Reexamination of Constraints from LEP Data*, *Phys. Lett. B* **828** (2022) 137014 [2110.01247], whose new results are presented in Section 3.2.
- [9] I. Antoniadis and F. Rondeau, *Minimal embedding of the Standard Model into intersecting D-brane configurations with a bulk leptonic $U(1)$* , *Eur. Phys. J. C* **82** (2022) 701 [2112.07587], whose new results are presented in Section 3.4.

Cosmological models in supergravity through generalized Kähler invariant Fayet-Iliopoulos terms

1.1 Introduction

The simplest extension of pure $\mathcal{N} = 1$ supergravity in flat spacetime is the anti de Sitter (AdS) supergravity, where a negative cosmological constant Λ is included [10]. In order to preserve local supersymmetry, a gravitino effective mass term has to be added, linked to Λ through $\Lambda = -3m_{3/2}^2$, which describes a massless gravitino in AdS spacetime. It is simply obtained by considering a constant superpotential $W = m_{3/2}$. An arbitrary cosmological constant cannot be introduced without breaking explicitly supersymmetry, or considering non-linear realisation [11]. In the presence of an abelian vector multiplet a constant Fayet-Iliopoulos (FI) term can be introduced only if the $U(1)$ gauges the R-symmetry, in which case a constant superpotential is forbidden, leading to a de Sitter (dS) supergravity describing a massive gravitino through curvature effects [12, 13].

Recently, a new type of constant FI-term was introduced which does not require the gauging of the R-symmetry [14]. It assumes that the D-auxiliary component of the $U(1)$ vector multiplet has a non-vanishing vacuum expectation value (VEV) breaking spontaneously supersymmetry, in which case it can be expanded as $D +$ fermion terms of higher dimensions. In the unitary gauge where the gravitino absorbs the $U(1)$ gaugino and becomes massive, the fermion terms vanish and the new FI-term amounts adding a positive contribution to the cosmological constant of the AdS supergravity, since a constant superpotential is now allowed as the $U(1)$ does not gauge the R-symmetry. In the presence of matter, the construction of [14] leads to a scalar potential but breaks Kähler invariance. On the other hand, the new and standard FI-terms can coexist in the case of gauge R-symmetry, providing interesting models of D-term inflation [15]. An alternative construction was made in [16] preserving Kähler invariance and leading to a constant FI-term in the presence of matter, that generates a constant uplift of the vacuum energy. More recently, such FI-terms were written in $\mathcal{N} = 2$ supergravity exhibiting a much richer structure [17].

In this first project, we generalise the above constructions in $\mathcal{N} = 1$ supergravity, preserving the Kähler invariance and keeping the form of the bosonic action to be linear in D up to a field dependent coefficient. We show that the most general FI-term is characterised by an arbitrary function of the gravitino mass, taken as a functional of the chiral superfields. We then study applications to cosmology, building new models of inflation compatible with CMB observations

and possessing a dS vacuum with tuneable (tiny) energy. We specialise to no-scale models [18, 19] of one chiral multiplet containing the inflaton, supplemented by a $U(1)$ gauge symmetry with the new FI-term. Moreover, we choose for the latter a simple characteristic function of the gravitino mass which is a single power and an additive constant, thus depending on three parameters. We show that there is a region in the parameter space where the resulting scalar potential possesses an inflationary plateau describing successfully the cosmological observations with the inflaton rolling down to a minimum with tuneable vacuum energy, where the gravitino mass and the supersymmetry breaking scale are fixed in terms of the parameters of the model.

The Chapter 1 is organised as follows. Important concepts in supersymmetry and supergravity are reminded in Sections 1.3 and 1.4. Section 1.5 introduces the matter-coupled $\mathcal{N} = 1$, $D = 4$ supergravity in the superconformal formalism, which will be the framework used in the rest of this chapter. In Section 1.6.1, we review the recent construction of the new FI-term in $\mathcal{N} = 1$ supergravity without gauging the R -symmetry, and its generalisation to a Kähler invariant FI Lagrangian leading to a positive constant uplift of the scalar potential in the presence of arbitrary matter chiral multiplets. In Section 1.6.2, we propose the most general modification of this construction that preserves Kähler invariance and is characterised by an arbitrary function of the gravitino mass as functional of chiral multiplets. We then study the consequences of such terms on inflation and supersymmetry breaking in a de Sitter vacuum with tuneable energy in Section 1.7, for the case of two no-scale models and for a simple choice of the functional dependence of the new FI D-term. Finally in Section 1.7.3, we discuss the gauging of the shift symmetry that gets rid of the massless particles in the spectrum without altering the inflationary predictions. Moreover, inspired by the low-energy limit of the heterotic string, we identify the inflaton with the string dilaton and gauge the perturbative axionic symmetry by the Green-Schwarz anomaly cancellation mechanism. These models provide new examples of inflation by supersymmetry breaking [20], where the inflaton belongs to the same multiplet with the Goldstino [21, 22], without gauging the R -symmetry. Our conclusions are presented in Section 1.8. Finally, this chapter is related to three appendices, reminding basic elements about Kähler manifolds and their symmetries (Appendix A), containing a summary of the conformal supergravity multiplets calculus (Appendix B) and details of the computation of the fermion masses in our models (Appendix C).

1.2 Conventions and notations

Throughout this thesis, we will use natural units $\hbar = c \equiv 1$. The reduced Planck mass

$$\kappa^{-1} = (8\pi G)^{-1/2} = 2.4 \times 10^{18} \text{ GeV},$$

is set equal to one, and numerical values are given in these units.

In this first chapter, we work exclusively in 4 spacetime dimensions, and adopt the metric convention $(-, +, +, +)$. The indices used throughout the Chapter 1 are defined according to

the following pattern:

$\mu, \nu \dots = 0, \dots, 3$	$4D$ spacetime curved indices
$a, b \dots = 0, \dots, 3$	$4D$ spacetime flat indices
$i, j \dots = 1, \dots, \dim_{\mathbb{R}} \mathcal{M}$	real coordinates of complex manifolds \mathcal{M}
$\alpha, \beta \dots = 1, \dots, \dim_{\mathbb{C}} \mathcal{M}$	complex coordinates of complex manifolds \mathcal{M}
$\alpha, \beta \dots = 1, 2$ or $1, \dots, 4$	spacetime spinor indices for Weyl or Majorana spinors respectively
$A, B \dots = 1, \dots, \dim G$	internal gauge group indices for a given gauge group G

The indices $\alpha, \beta \dots$ denote either complex coordinates of complex manifolds or spacetime spinor indices, depending on the context which will be obvious. Curved and flat indices in $4D$ are related through the vierbein e_{μ}^{α} according to

$$X_{\mu} = e_{\mu}^{\alpha} X_{\alpha}. \quad (1.1)$$

The four-dimensional Dirac matrices γ^a satisfy the Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \mathbb{1}_{4 \times 4}. \quad (1.2)$$

In any spacetime dimension, an irreducible representation of the Clifford algebra induces a spinor representation of the Lorentz algebra, which is in general reducible. This is the case in $D = 4$ spacetime dimension: 4-dimensional Dirac spinors are irreducible representations of the Clifford algebra, but reducible representations of the Lorentz algebra. In $4D$, they contain 4 complex off-shell degrees of freedom. Irreducible spinor representations of the Lorentz algebra can be obtained from a 4-dimensional Dirac spinor either from a chirality projection or a reality projection. The former leads to Weyl spinors, written in a two-component spinor notation, while the latter leads to Majorana spinors, written in a four-component spinor notation. Both of them have 4 real off-shell degrees of freedom. Two-component spinors are commonly used in the literature dealing with 4-dimensional $\mathcal{N} = 1$ global supersymmetry, while four-component spinors are more convenient in 4-dimensional $\mathcal{N} = 1$ supergravity, since they can be easily generalized to higher spacetime dimensions and extended supersymmetry. We will use both formalisms in this chapter, which focuses on $\mathcal{N} = 1$, $D = 4$ global supersymmetry and supergravity.

Two-component Weyl spinors are denoted by θ_{α} and $\bar{\theta}^{\dot{\alpha}}$, and are left-handed and right-handed spinors transforming under the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group respectively. This two-component notation will be used in Sections 1.3.1 and 1.4 following the conventions of [23]. The four-component spinor notation will be used in Sections 1.3.2, 1.5, 1.6, 1.7, as well as in Appendix B and C following the conventions of [10]. These spinors are Majorana spinors, defined by the reality condition

$$\lambda^* = B\lambda, \quad (1.3)$$

where B is a matrix satisfying in four dimensions $\gamma_{\mu}^* = B\gamma_{\mu}B^{-1}$. In this formalism, a barred

spinor $\bar{\lambda}$ is the Majorana conjugate of a four-component spinor λ defined by

$$\bar{\lambda} \equiv \lambda^T C, \quad (1.4)$$

with C the charge conjugation matrix, satisfying in four dimensions $\gamma_\mu^T = -C\gamma_\mu C^{-1}$. Chiral fermions can be obtained from a given four-component fermion by applying the left or right projection operators

$$P_L \equiv \frac{1}{2}(1 + \gamma_*) , \quad P_R \equiv \frac{1}{2}(1 - \gamma_*), \quad (1.5)$$

where $\gamma_* \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$. The chiral projections λ_L and λ_R of a four-component spinor λ are then defined by:

$$\lambda_L \equiv P_L \lambda, \quad \lambda_R \equiv P_R \lambda. \quad (1.6)$$

Let us note that in 4 spacetime dimension, the Weyl and Majorana conditions cannot be imposed simultaneously, so that for a Majorana λ , $P_{L,R}\lambda$ is Weyl but no longer Majorana.

1.3 Basics of supersymmetry and supergravity

1.3.1 Most general chiral models in $\mathcal{N} = 1$ supersymmetry

The aim of this section is to derive the geometric structure of the scalar sector of any supersymmetric theory with chiral matter fields, an important feature which will be ubiquitous in this chapter. In $\mathcal{N} = 1$ supersymmetry, the most general renormalizable supersymmetric Lagrangian involving N chiral and anti-chiral superfields Φ^α , $\Phi^{\dagger\bar{\alpha}}$, α and $\bar{\alpha}$ running from 1 to N , is:

$$\mathcal{L}_\Phi = \int d^2\theta d^2\bar{\theta} \Phi^\dagger_{\bar{\alpha}} \Phi^\alpha + \left[\int d^2\theta \left(a_\alpha \Phi^\alpha + \frac{1}{2} m_{\alpha\beta} \Phi^\alpha \Phi^\beta + \frac{1}{3} y_{\alpha\beta\gamma} \Phi^\alpha \Phi^\beta \Phi^\gamma \right) + h.c. \right], \quad (1.7)$$

where a_α , $m_{\alpha\beta}$, $y_{\alpha\beta\gamma}$ are constant couplings. The θ expansion of the chiral fields are:

$$\Phi^\alpha(y, \theta) = \varphi^\alpha(y) + \sqrt{2}\theta\Omega^\alpha(y) + \theta\theta F^\alpha(y) \quad (1.8)$$

$$\begin{aligned} &= \varphi^\alpha(x) + \sqrt{2}\theta\Omega^\alpha(x) + \theta\theta F^\alpha(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi^\alpha(x) \\ &\quad - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\Omega^\alpha(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\Box\varphi^\alpha(x), \end{aligned} \quad (1.9)$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, and similarly for the anti-chirals $\Phi^{\dagger\bar{\alpha}}$ by taking the hermitian conjugate of the previous expansions. However, we are ultimately interested in supergravity, which is an effective non-renormalizable theory. Thus, renormalizability is in fact not a criterion, and we look at the most general Lagrangian that can be built from chiral superfields. The modification of the Lagrangian (1.7) into its most general form is carried out by introducing two arbitrary real and chiral superfields $K(\Phi, \Phi^\dagger)$ and $W(\Phi)$, respectively real and holomorphic functions of Φ , according to:

$$\mathcal{L}_\Phi = \int d^2\theta d^2\bar{\theta} K(\Phi, \Phi^\dagger) + \left[\int d^2\theta W(\Phi) + h.c. \right]. \quad (1.10)$$

First and foremost, let us notice that this Lagrangian is invariant under a so-called *Kähler transformation*,

$$K(\Phi^\alpha, \Phi^{\dagger\bar{\beta}}) \rightarrow K(\Phi^\alpha, \Phi^{\dagger\bar{\beta}}) + J(\Phi^\alpha) + \bar{J}(\Phi^{\dagger\bar{\beta}}), \quad (1.11)$$

where J and \bar{J} are chiral and anti-chiral superfields, holomorphic and anti-holomorphic functions of Φ^α and $\Phi^{\dagger\bar{\beta}}$ respectively. This symmetry will lie at the core of this first chapter, and in particular will be the cornerstone of the construction carried out in Section 1.6.2.

In order to highlight the geometrical structure hidden behind the superspace Lagrangian (1.7), we would like to write it in terms of component fields. From Eq.(1.8), one can easily expand the superpotential $W(\Phi)$ as:

$$W(\Phi) = W(\varphi) + \sqrt{2}\theta\Omega^\alpha W_\alpha + \theta\theta \left[F^\alpha W_\alpha - \frac{1}{2}\Omega^\alpha\Omega^\beta W_{\alpha\beta} \right], \quad (1.12)$$

with all component fields functions of $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and where we have defined $W_\alpha \equiv \frac{\partial W}{\partial \varphi^\alpha}$. Expanding in terms of x^μ will not bring any new $\theta\theta$ -terms, and we can immediately deduce

$$\left[\int d^2\theta W(\Phi) + h.c. \right] = F^\alpha \bar{W}_\alpha - \frac{1}{2}\Omega^\alpha\Omega^\beta W_{\alpha\beta} + h.c. \quad (1.13)$$

The component expansion of $K(\Phi, \Phi^\dagger)$ is more involved, since now we have to consider the expansion of the different fields in terms of x^μ . We will not present in full details this computation, which is a standard expansion without any conceptual subtleties, and which can be found for instance in [23, 24]. Instead, we focus on the bosonic sector and put the chiral fermions Ω^α to zero. This turns out to be sufficient to highlight the most important geometrical structure we are interested in concerning the supersymmetric chiral model. With $\Omega^\alpha = 0$, the expansion (1.9) reduces to $\Phi^\alpha = \varphi^\alpha(x) + \delta^\alpha(x)$, where we have defined $\delta^\alpha(x) \equiv \theta\theta F^\alpha(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi^\alpha(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box\varphi^\alpha(x)$. Since any cubic or higher order terms in δ vanishes, we deduce the expansion for K :

$$K(\Phi, \Phi^\dagger) = K(\varphi(x), \bar{\varphi}(x)) + \delta^\alpha K_\alpha + \bar{\delta}^{\bar{\beta}} K_{\bar{\beta}} + \frac{1}{2}\delta^\alpha\delta^\beta K_{\alpha\beta} + \frac{1}{2}\bar{\delta}^{\bar{\alpha}}\bar{\delta}^{\bar{\beta}} K_{\bar{\alpha}\bar{\beta}} + \delta^\alpha\bar{\delta}^{\bar{\beta}} K_{\alpha\bar{\beta}}. \quad (1.14)$$

From $\delta^\alpha\delta^\beta = \frac{1}{2}\theta^2\bar{\theta}^2\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta$ and $\delta^\alpha\bar{\delta}^{\bar{\beta}} = \theta^2\bar{\theta}^2\left(-\frac{1}{2}\partial_\mu\varphi^\alpha\partial^\mu\bar{\varphi}^{\bar{\beta}} + F^\alpha\bar{F}^{\bar{\beta}}\right)$, we find the $\theta^2\bar{\theta}^2$ component of K :

$$\begin{aligned} K(\Phi, \Phi^\dagger)\Big|_{\theta^2\bar{\theta}^2} &= \frac{1}{4}\Box\varphi^\alpha K_\alpha + \frac{1}{4}\Box\bar{\varphi}^{\bar{\beta}} K_{\bar{\beta}} + \frac{1}{4}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta K_{\alpha\beta} + \frac{1}{4}\partial_\mu\bar{\varphi}^{\bar{\alpha}}\partial^\mu\bar{\varphi}^{\bar{\beta}} K_{\bar{\alpha}\bar{\beta}} \\ &\quad + \left(-\frac{1}{2}\partial_\mu\varphi^\alpha\partial^\mu\bar{\varphi}^{\bar{\beta}} + F^\alpha\bar{F}^{\bar{\beta}}\right) K_{\alpha\bar{\beta}} \\ &= \frac{1}{4}\partial_\mu\partial^\mu K + \left(-\partial_\mu\varphi^\alpha\partial^\mu\bar{\varphi}^{\bar{\beta}} + F^\alpha\bar{F}^{\bar{\beta}}\right) K_{\alpha\bar{\beta}} \end{aligned} \quad (1.15)$$

The crucial point in this derivation is to notice that the first four terms of $K|_{\theta^2\bar{\theta}^2}$ are equal to $\frac{1}{4}\partial_\mu\partial^\mu K - \frac{1}{2}\partial_\mu\varphi^\alpha\partial^\mu\bar{\varphi}^{\bar{\beta}}$. The total derivative can then be discarded, and as a consequence, the purely holomorphic and anti-holomorphic terms $\propto K_{\alpha\beta}, K_{\bar{\alpha}\bar{\beta}}$ do no longer appear in the Lagrangian, only the mixed terms $\propto K_{\alpha\bar{\beta}}$ remaining. This is a first indication of a geometrical Kähler structure, as we will see soon. Eqs. (1.13) and (1.15) give the bosonic components of the

chiral Lagrangian:

$$\mathcal{L}_\Phi|_{\Omega=0} = \left(-\partial_\mu \varphi^\alpha \partial^\mu \bar{\varphi}^{\bar{\beta}} + F^\alpha \bar{F}^{\bar{\beta}} \right) K_{\alpha\bar{\beta}} + F^\alpha W_\alpha + \bar{F}^{\bar{\beta}} \bar{W}_{\bar{\beta}}. \quad (1.16)$$

Eliminating the auxiliary fields F through their equations of motion, $F^\alpha = -\bar{W}_{\bar{\beta}} K^{\alpha\bar{\beta}}$, $\bar{F}^{\bar{\beta}} = -W_\alpha K^{\alpha\bar{\beta}}$, we end with:

$$\mathcal{L}_\Phi|_{\Omega=0} = -K_{\alpha\bar{\beta}} \partial_\mu \varphi^\alpha \partial^\mu \bar{\varphi}^{\bar{\beta}} - W_\alpha \bar{W}_{\bar{\beta}} K^{\alpha\bar{\beta}}. \quad (1.17)$$

This is the bosonic part of the Lagrangian describing the most general supersymmetric coupling of chiral multiplets. The kinetic term for the complex scalars φ^α takes the form of a non-linear σ -model with target space characterized by the metric $g_{\alpha\bar{\beta}} \equiv \partial_\alpha \partial_{\bar{\beta}} K$. Combined with $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, these three conditions precisely define what is called a *Kähler manifold*. Such manifolds are described in more details in Appendix A.

The analysis of this section has shown that any supersymmetric chiral theory is determined by a superpotential $W(\varphi)$ and a Kähler potential $K(\varphi, \bar{\varphi})$ functions of the complex scalar fields, both of them being the lowest components of the superfields $K(\Phi, \Phi^\dagger)$ and $W(\Phi)$ introduced in Eq. (1.10). The complex scalars can then be seen as the coordinates of a Kähler manifold whose metric is given by the second derivative of this Kähler potential, namely $g_{\alpha\bar{\beta}} \equiv \partial_\alpha \partial_{\bar{\beta}} K$. The Kähler transformations (1.11) on the superfield $K(\Phi, \Phi^\dagger)$ act on the Kähler potential $K(\varphi, \bar{\varphi})$ as $K(\varphi, \bar{\varphi}) \rightarrow K(\varphi, \bar{\varphi}) + f(\varphi) + \bar{f}(\bar{\varphi})$, with f and \bar{f} holomorphic and anti-holomorphic functions respectively. From the expression of $g_{\alpha\bar{\beta}}$, it is therefore obvious that these transformations leave the metric invariant.

1.3.2 Pure $\mathcal{N} = 1$, $D = 4$ supergravity and Anti-de Sitter supergravity

As qualitatively explained in the Introduction of this thesis, a theory of local supersymmetry must contain gravity. Moreover, turning a global symmetry to a local one, that is in our case replacing $\epsilon_\alpha \rightarrow \epsilon_\alpha(x)$, requires the introduction of a gauge field. Since ϵ_α carries a spinor index, the gauge field will be a vector-spinor field $\psi_{\mu\alpha}(x)$. This spin-3/2 field, called the *gravitino*, is the superpartner of the vierbein e_μ^a , the graviton. The Lagrangian of pure $\mathcal{N} = 1$ supergravity will thus contain both the Einstein-Hilbert Lagrangian \mathcal{L}_{EH} of General Relativity, plus the Rarita-Schwinger Lagrangian \mathcal{L}_{RS} for massless free spin-3/2 field:

$$\mathcal{L}_{\text{sugra}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{RS}} = \frac{1}{2} e e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \frac{1}{2} e \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho, \quad (1.18)$$

where we have defined $e \equiv \det e_\mu^a$ and $D_\nu \psi_\rho \equiv \partial_\nu \psi_\rho + \frac{1}{4} \omega_{\nu ab} \gamma^{ab} \psi_\rho$. ψ_μ is a Majorana spinor, and the second and third rank Clifford algebra elements are respectively given by $\gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu]$, $\gamma^{\mu\nu\rho} = \frac{1}{2} \{ \gamma^\mu, \gamma^{\nu\rho} \}$. This Lagrangian is invariant at lowest order in the fermions [10] under the following supersymmetry transformation rules:¹

$$\delta e_\mu^a = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu, \quad \delta \psi_\mu = D_\mu \epsilon(x) \equiv \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu ab} \gamma^{ab} \epsilon. \quad (1.19)$$

1. It is worth noting that the Einstein equations coming from $\delta \mathcal{L}_{\text{RS}}$ arise from the algebra of Dirac γ -matrices. This therefore tells us that there is already some spinorial structure behind General Relativity.

Anti-de Sitter (AdS) supergravity can then be seen as the simplest extension of pure $\mathcal{N} = 1$ supergravity. It naturally arises from the question: can a cosmological constant be compatible with a local supersymmetric action? Adding a cosmological constant to the Lagrangian (1.18), $\mathcal{L}_{\text{cc}} = -e\Lambda$, obviously breaks supersymmetry. However, as we are now going to discuss, supersymmetry can be recovered by adding an effective gravitino mass term to the Lagrangian, $\mathcal{L}_{m_{3/2}} = e\frac{m_{3/2}}{2}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$, and considering the local supersymmetric transformations [25]:

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \quad \delta\psi_\mu = D_\mu\epsilon + g\gamma_\mu\epsilon, \quad (1.20)$$

where a mass-like term $g\gamma_\mu\epsilon$ has been added to the variation of the gravitino. The coupling constant $g \in \mathbb{R}$ will turn out to be the only free parameter of this model. The full AdS supergravity Lagrangian then reads

$$\mathcal{L}_{\text{AdS}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{RS}} + \mathcal{L}_{m_{3/2}} + \mathcal{L}_{\text{cc}}. \quad (1.21)$$

At lowest order in the fermions, its supersymmetric variation can be evaluated via:

$$\begin{aligned} \delta\mathcal{L}_{\text{RS}} &= -e\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\delta\psi_\rho = -\delta\mathcal{L}_{\text{EH}} - e\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(g\gamma_\rho\epsilon) \\ &= -\delta\mathcal{L}_{\text{EH}} - eg\bar{\psi}_\mu \underbrace{\gamma^{\mu\nu\rho}\gamma_\rho}_{(D-2)\gamma^{\mu\nu}} D_\nu\epsilon, \end{aligned} \quad (1.22)$$

$$\delta\mathcal{L}_{m_{3/2}} = em_{3/2}\bar{\psi}_\mu\gamma^{\mu\nu}\delta\psi_\nu = em_{3/2}\bar{\psi}_\mu\gamma^{\mu\nu}D_\nu\epsilon + egm_{3/2}\bar{\psi}_\mu \underbrace{\gamma^{\mu\nu}\gamma_\nu}_{(D-1)\gamma^\mu} \epsilon, \quad (1.23)$$

$$\delta\mathcal{L}_{\text{cc}} = -\Lambda\delta e = -\Lambda\frac{e}{2}\bar{\epsilon}\gamma^\rho\psi_\rho. \quad (1.24)$$

The second term of (1.22) plus the first one of (1.23) cancel if $m_{3/2} = g(D-2)$, while the second term of (1.23) plus (1.24) cancel if $\Lambda = -2gm_{3/2}(D-1)$, which implies in $D = 4$ spacetime dimension that

$$\Lambda = -12g^2 < 0, \quad \Lambda = -3m_{3/2}^2. \quad (1.25)$$

At lowest order in the fermions², we have thus found that only a negative cosmological constant can be added to the pure $\mathcal{N} = 1$ supergravity action in a supersymmetric way. This requires the addition of an effective mass-term³ for the gravitino, which cannot be chosen independently from Λ but has to satisfy the relation $\Lambda = -3m_{3/2}^2$. Any uplift of Λ from this value will then break supersymmetry.

The construction of AdS supergravity in four dimension follows an interesting pattern. Both the shift in the supersymmetry transformation of the gravitino as well as the effective gravitino mass term are linear in the coupling g . Supersymmetry then closes at order g^2 via the introduction of a cosmological constant quadratic in g . This pattern is actually much more general and encodes the gauging procedure of any $\mathcal{N} > 1$ extended supergravity theory. In this case, the cosmological constant is replaced by a scalar potential $\mathcal{O}(g^2)$, while closure of supersymmetry still requires the introduction of a shift $\mathcal{O}(g)$ in the supersymmetry transformations of the fermions as well as $\mathcal{O}(g)$ fermion mass terms. We will encounter such situation in the Chapter 2, whose

2. This analysis turns out to be true to all orders in fermions for $D = 4$, while only true at the quadratic order in fermions in $D = 11$.

3. The correct interpretation of this mass-term is the description of a *massless* gravitino in AdS spacetime [26].

second half-part is based on an abelian gauging of a given $\mathcal{N} = 2$, $D = 5$ supergravity theory, a procedure which will be presented in Section 2.5.2.

1.4 Spontaneous supersymmetry breaking and Fayet-Iliopoulos terms in supersymmetry and supergravity

The commutation relation $[Q_\alpha, P_\mu] = 0$ of the Poincaré superalgebra (1) implies that bosonic and fermionic superpartners have the same mass. This situation being obviously not observed in Nature, supersymmetry must be broken at least at the electroweak energy scale. This can be carried out in two ways:

- An explicit symmetry breaking: the symmetry is broken at the Lagrangian level, for instance by some terms which are negligible in the IR and become only relevant in the UV. In such case, the theory is invariant under the symmetry at low energies, but no longer at high energies.
- A spontaneous symmetry breaking: the Lagrangian and thus the theory are invariant under the symmetry regardless of the considered energy scale, but the vacuum is not invariant below a given energy scale.

In local supersymmetry, the first case is inconsistent as long as there is a spin 3/2 in the spectrum, while it suffers from a lack of predictivity in global supersymmetry. In the same time, the second procedure provides some appealing dynamical processes for symmetry breaking, and is valid in both global and local susy. We will therefore not consider explicit symmetry breaking in the following, and focus only on spontaneous supersymmetry breaking.

1.4.1 Spontaneous global supersymmetry breaking

Let us start this discussion by reminding that the energy $E = P^0$ of any state in the Hilbert space \mathcal{H} of a *global* supersymmetric theory is always positive. To see this, we write $\forall |\Phi\rangle \in \mathcal{H}$, $\forall \alpha, \dot{\alpha} = 1, 2$, $0 \leq \|Q_\alpha |\Phi\rangle\|^2 + \|\bar{Q}_{\dot{\alpha}} |\Phi\rangle\|^2 = \langle \Phi | (\bar{Q}_{\dot{\alpha}} Q_\alpha + Q_\alpha \bar{Q}_{\dot{\alpha}}) | \Phi \rangle = \langle \Phi | \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} | \Phi \rangle = 2\sigma_{\alpha\dot{\alpha}}^\mu \langle \Phi | P_\mu | \Phi \rangle$. Taking the trace of this expression and using $\text{Tr } \sigma^\mu = 2\delta^{\mu 0}$, we get:

$$4 \langle \Phi | P_0 | \Phi \rangle \geq 0. \quad (1.26)$$

Let us now assume a vacuum $|\Omega\rangle$ preserving supersymmetry, *i.e.* being annihilated by all four supercharges, $Q_\alpha |\Omega\rangle = \bar{Q}_{\dot{\alpha}} |\Omega\rangle = 0$. It follows from the discussion above that $\langle \Phi | P_0 | \Phi \rangle = 0$. Conversely, if $\langle \Phi | P_0 | \Phi \rangle > 0$, one needs at least one supercharge such that $Q_\alpha |\Omega\rangle \neq 0$, and supersymmetry is broken in the vacuum. We therefore conclude that in a *globally* supersymmetric theory, ground states of vanishing energy preserve supersymmetry, while those of strictly positive energy break supersymmetry spontaneously.

We then remind that in a globally supersymmetric field theory with chiral multiplets (φ, ψ, F) and vector multiplets (v_μ, λ, D) coupled to each other, the scalar potential reads:

$$\mathcal{V} = \sum_{\text{chiral superfields}} |F|^2 + \frac{1}{2} \sum_{\text{vector superfields}} D^2. \quad (1.27)$$

Therefore, supersymmetry breaking is controlled by the vacuum expectation values of the auxiliary fields, and in order to break global supersymmetry, we must have either $\langle F \rangle \neq 0$, or $\langle D \rangle \neq 0$, or both. The first case is called the F -type or O’Raifeartaigh mechanism [27], while the second one is called the D -type or Fayet-Iliopoulos mechanism [28]. Since this chapter deals with D -term supersymmetry breaking, we are not going to discuss the O’Raifeartaigh mechanism in the following and only focus on the Fayet-Iliopoulos one. In order to introduce it, let us consider the case of two chiral superfields $\Phi_{\pm} = (\varphi_{\pm}, \psi_{\pm}, F_{\pm})$, with the same mass m and $U(1)$ charges ± 1 , coupled to a $U(1)$ vector superfield $V = (v_{\mu}, \lambda, D)$ in the Wess-Zumino gauge. The superspace Lagrangian for this model reads:

$$\mathcal{L}_{\Phi+V} = \int d^4\theta \left(\Phi_+^{\dagger} e^{gV} \Phi_+ + \Phi_-^{\dagger} e^{-gV} \Phi_- \right) + \left[\int d^2\theta \left(m\Phi_+\Phi_- + \frac{1}{4}\mathcal{W}^2 \right) + h.c. \right]. \quad (1.28)$$

The idea of the Fayet-Iliopoulos (FI) procedure is to notice that the integral of V over all the superspace,

$$\mathcal{L}_{FI} = -2\xi \int d^4\theta V = -\xi D, \quad (1.29)$$

is gauge invariant (in the case of an abelian $U(1)$ gauge symmetry), and can therefore be added to the Lagrangian $\mathcal{L}_{\Phi+V}$. The gauge invariance of \mathcal{L}_{FI} can be seen by explicitly writing its gauge transformation $\int d^4\theta V \rightarrow \int d^4\theta (V + \Lambda + \bar{\Lambda}) = \int d^4\theta V - \frac{1}{4} \int d^2\theta \bar{D}\bar{D}\Lambda - \frac{1}{4} \int d^2\bar{\theta} D\bar{D}\bar{\Lambda} = \int d^4\theta V$, where we have used $\int d^2\bar{\theta} = -\frac{1}{4}\bar{D}\bar{D}$, $\int d^2\theta = -\frac{1}{4}D\bar{D}$, and then $\bar{D}_{\dot{\alpha}}\Lambda = 0 = D_{\alpha}\bar{\Lambda}$, for chiral Λ and anti-chiral $\bar{\Lambda}$. The components of the superfields entering in the Lagrangian (1.28) are given by [23]:

$$\mathcal{W}^{\alpha}\mathcal{W}_{\alpha}|_{\theta\theta} = -2i\lambda\sigma^{\mu}\partial_{\mu}\bar{\lambda} - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}F^{\mu\nu}F^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} + D^2, \quad (1.30)$$

$$\Phi_+\Phi_-|_{\theta\theta} = \varphi_+(y)F_-(y) + \varphi_-(y)F_+(y) - \psi_+(y)\psi_-(y), \quad (1.31)$$

$$\begin{aligned} \Phi^{\dagger}e^{gV}\Phi|_{\theta\theta\bar{\theta}\bar{\theta}} &= F\bar{F} + \varphi\Box\bar{\varphi} + i\partial_{\mu}\bar{\psi}\bar{\sigma}^{\mu}\psi + gv_{\mu}\left(\frac{1}{2}\bar{\psi}\bar{\sigma}^{\mu}\psi + \frac{i}{2}\bar{\varphi}\partial^{\mu}\varphi - \frac{i}{2}\partial^{\mu}\bar{\varphi}\varphi\right) \\ &\quad - \frac{ig}{\sqrt{2}}(\varphi\bar{\lambda}\bar{\psi} - \bar{\varphi}\lambda\psi) + \frac{1}{2}\left(gD - \frac{1}{2}g^2v_{\mu}v^{\mu}\right)\bar{\varphi}\varphi, \end{aligned} \quad (1.32)$$

with $y^{\mu} = x^{\mu} - i\theta\sigma^{\mu}\bar{\theta}$, and where the last expansion has been evaluated in the Wess-Zumino gauge, where $V^3 = 0$. Since we are interested in the scalar potential \mathcal{V} of this model, we focus on the scalar sector, putting fermions, vectors and derivative terms to zero. In particular, expanding the fields in terms of x will not bring additional terms to the scalar potential. We deduce:

$$-\mathcal{V} = |F_+|^2 + |F_-|^2 + \frac{gD}{2}(|\varphi_+|^2 - |\varphi_-|^2) + m\left(\varphi_+F_- + \varphi_-F_+ + \bar{\varphi}_+\bar{F}_- + \bar{\varphi}_-\bar{F}_+\right) + \frac{1}{2}D^2 - \xi D. \quad (1.33)$$

Replacing the auxiliary fields D, F_{\pm} using their equations of motion,

$$D = \frac{g}{2}\left(|\varphi_-|^2 - |\varphi_+|^2\right) + \xi, \quad F_+ = -m\bar{\varphi}_-, \quad F_- = -m\bar{\varphi}_+, \quad (1.34)$$

we find the scalar potential

$$\mathcal{V} = \left(m^2 - \frac{g\xi}{2}\right)|\varphi_+|^2 + \left(m^2 + \frac{g\xi}{2}\right)|\varphi_-|^2 + \frac{g^2}{8}\left(|\varphi_+|^2 - |\varphi_-|^2\right)^2 + \frac{\xi^2}{2}. \quad (1.35)$$

If $\xi = 0$, the minimum of \mathcal{V} is obtained for $\varphi_+ = \varphi_- = 0$, and following (1.34), the vacuum is therefore supersymmetric. If $\xi \neq 0$, the system presents two phases:

- $m^2 \pm \frac{\xi g}{2} > 0$: the minimum is reached at $\varphi_+ = \varphi_- = 0$, and $\mathcal{V} \neq 0$ at the minimum: supersymmetry is spontaneously broken while the gauge symmetry remains unbroken. From the auxiliary fields equations of motion (1.34), we see that at the minimum, $\langle F \rangle = 0$ and $\langle D \rangle \neq 0$: supersymmetry breaking is purely of D -type. The Goldstino is the gaugino.
- $m^2 \pm \frac{\xi g}{2} < 0$: the minimum is reached at $\varphi_+ \neq 0$, $\varphi_- \neq 0$, and $\mathcal{V} \neq 0$ at the minimum: both supersymmetry and gauge symmetry are spontaneously broken. We now have $\langle F \rangle \neq 0$ and $\langle D \rangle \neq 0$, and supersymmetry breaking is thus of mixed D and F -types. The Goldstino is a linear combination of the chiral fermions and the gaugino.

Therefore, this model shows how a FI term $\int d^4\theta V$ produces a spontaneous breaking of *global* supersymmetry. The idea to keep in mind is that supersymmetry breaking is controlled by non-vanishing vacuum expectation values (vev) of auxiliary fields, while gauge symmetry breaking is controlled by non-vanishing vev of dynamical fields. In the next section, we discuss the fundamental origin of FI terms in global supersymmetry, before studying in Section 1.4.3 why does the procedure described in the current section fail in supergravity, and how a FI term can be written in this case.

1.4.2 Theoretical approach to Fayet-Iliopoulos terms in global supersymmetry

Before introducing FI terms in supergravity, let us discuss their fundamental origin in global supersymmetry. We explained in Section 1.3.1 how Kähler geometry naturally arises in global supersymmetry. In particular, we found that the scalar sector can be described in terms of a Kähler manifold \mathcal{M} , whose coordinates are the complex scalar fields and the metric is given by the second derivative of a Kähler potential. On the other hand, the FI Lagrangian (1.29) involves the highest component D of a vector multiplet. From the Kähler geometry point of view, vector multiplets are included in a supersymmetric chiral theory by gauging the isometries of \mathcal{M} .

In a very general way, symmetries of a manifold are defined by Killing vectors. In Appendix A.2, we discuss the symmetries of a Kähler manifold and show that such symmetries are determined in a simpler and more restrictive way than those of a given manifold. Indeed, they are defined by holomorphic Killing vectors $k_A^\alpha(\varphi)$, each of them, in turn, being characterized by real functions $\mathcal{P}_A(\varphi, \bar{\varphi})$ called *moment maps* or *Killing potentials*. They are defined up to constants ξ_A , $\mathcal{P}_A \rightarrow \mathcal{P}_A + \xi_A$. Denoting G the Lie group of global symmetries of \mathcal{M} , a subgroup $G_0 \subset G$ is chosen as the gauge group. In general, G_0 will be a non-abelian group, whose generators are labelled by the index A . Each Killing vector k^A will be associated to a vector multiplet $(A_\mu^A, \lambda^A, D^A)$. The k^A generate a Lie algebra whose structure is defined by the Lie brackets $[k_A, k_B] = f_{AB}{}^C k_C$, $[\bar{k}_A, \bar{k}_B] = f_{AB}{}^C \bar{k}_C$, and $[k_A, \bar{k}_B] = 0$. It can be shown [23] that the moment maps \mathcal{P}_A can be chosen to transform in the adjoint representation of G_0 , namely:

$$\left(k_A^\alpha \partial_\alpha + k_A^{\bar{\alpha}} \partial_{\bar{\alpha}}\right) \mathcal{P}_B(\varphi, \bar{\varphi}) = f_{AB}{}^C \mathcal{P}_C(\varphi, \bar{\varphi}). \quad (1.36)$$

This relation, called *equivariance relation*, fixes the constants ξ_A for non-abelian groups. How-

ever, an undetermined ξ_A constant remains for each $U(1)$ factors of G_0 . As we are now going to discuss, these ξ_A constants are actually Fayet-Iliopoulos constants.

In order to get an action both supersymmetric and gauge invariant under the gauge group G_0 , the kinetic term of (1.17) has to be modified into [10]:

$$-K_{\alpha\bar{\beta}}\partial_\mu\varphi^\alpha\partial^\mu\bar{\varphi}^{\bar{\beta}} \rightarrow -K_{\alpha\bar{\beta}}D_\mu\varphi^iD^\mu\bar{\varphi}^{\bar{j}} - iD^Ak_A^\alpha K_\alpha = -K_{\alpha\bar{\beta}}D_\mu\varphi^iD^\mu\bar{\varphi}^{\bar{j}} + D^A\mathcal{P}_A(\varphi, \bar{\varphi}) + iD^Ar_A(\varphi), \quad (1.37)$$

with $D_\mu\varphi^\alpha \equiv \partial_\mu\varphi^\alpha - A_\mu^Ak_A^\alpha$. The last equality follows from the relation (A.2.8) between K and \mathcal{P}_A , derived in Appendix A.2 and which is reproduced here for convenience:

$$\mathcal{P}_A(\varphi, \bar{\varphi}) = i(k_A^\alpha K_\alpha(\varphi, \bar{\varphi}) - r_A(\varphi)). \quad (1.38)$$

In the case when $r_A(\varphi)$ are restricted to be imaginary constants, the moment maps then take the form $\mathcal{P}_A = ik_A^\alpha K_\alpha + \xi_A$, where $\xi_A \equiv -ir_A \in \mathbb{R}$ are called *Fayet-Iliopoulos* constants. The last term of (1.37) then produces a new term in the action,

$$S_{FI} = - \int d^4x \xi_A D^A, \quad (1.39)$$

which is precisely a FI action as already introduced in Eq. (1.29). From the supersymmetric and gauge transformations of D^A , $\delta_s(\epsilon)D^A = \frac{i}{2}\bar{\epsilon}\gamma_*\gamma^\mu(\partial_\mu\lambda^A + \lambda^C A_\mu^B f_{BC}^A)$ and $\delta_g(\theta)D^A = \theta^C D^B f_{BC}^A$, we see that S_{FI} is both supersymmetric and gauge invariant if

$$\xi_A f_{BC}^A = 0, \quad \forall B, C. \quad (1.40)$$

Thus, for a non-abelian index A , ξ_A must vanish, while $\xi_A \neq 0$ is possible for any abelian A .

In summary, for abelian symmetries of \mathcal{M} , constant shifts in the associated moment maps \mathcal{P}_A are mathematically allowed. Physically, these constant shifts correspond to new terms in the action of gauged supersymmetry: the so-called *Fayet-Iliopoulos* terms. Such terms, which have been introduced by hand in the previous section through the Lagrangian (1.29), actually arise from the fact that a $U(1)$ gauge group gives some freedom in the definition of the moment maps. For each $U(1)_A$ factor of the total gauge group G_0 of the isometries of \mathcal{M} , there will be an associated FI parameter ξ_A .

1.4.3 Fayet-Iliopoulos terms in supergravity: Freedman model

In local supersymmetry, the gauge invariance of the Lagrangian (1.29) is no longer guaranteed. This basically comes from the fact that in curved space, the covariant generalizations of the chiral projection operators D^2 and \bar{D}^2 are given by $(\mathcal{D}\mathcal{D} - 8\bar{\mathcal{R}})$ and $(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8\mathcal{R})$, with \mathcal{R} the chiral superspace curvature [23]. While we still have $-\frac{1}{4}\int d^2\Theta 2\mathcal{E} \bar{\mathcal{D}}\bar{\mathcal{D}}\Lambda = 0$, the new curved space contribution $2\int d^2\Theta 2\mathcal{E} \mathcal{R}\Lambda$ does not vanish in general and $\int d^4\theta EV$ is thus not gauge invariant.

The introduction of FI terms in supergravity has first been carried out by Freedman in [12], where it has been shown that such terms require the $U(1)$ symmetry associated to the vector

field V to be a R -symmetry. Before describing the Freedman model, let us first remind the basic ideas of a R -symmetry in supersymmetry. Besides the super-Poincaré generators (1), one can also add internal symmetry generators, which are Lorentz scalars and whose commutation relations define a compact Lie algebra. In supersymmetry, two kinds of internal symmetry are possible:

- A symmetry whose generators commute with the supercharges, called a *gauge symmetry*. As a consequence, such symmetry acts uniformly on all fields in a given multiplet, and has to leave the superpotential invariant.
- A symmetry whose generators do not commute with the supercharges, called a *R -symmetry*. As a consequence, such symmetry acts differently on different components of a given multiplet, and acts as a phase transformation on the gauge field of supersymmetry, that is the gravitino ψ_μ . In addition, the R -symmetry commutes with the Lorentz, translation and gauge generators. In the superspace formalism, rotating the supercharges implies that the R -symmetry rotates the fermionic superspace coordinates $\{\theta, \bar{\theta}\}$. In order for the superpotential Lagrangian $\int d^2\theta W$ to be invariant, this implies that the R -symmetry also rotates the superpotential.

We now show why writing a FI term in supergravity requires the associated $U(1)$ symmetry to be a R -symmetry. Following [13], let us consider the superspace Lagrangian

$$\mathcal{L}_{\text{Freedman FI}} = -3 \int d^4\theta E e^{\frac{2}{3}\xi V}, \quad (1.41)$$

where E is the determinant of the supervierbein, and ξ a constant parameter as in (1.29). This Lagrangian is not invariant under a $U(1)$ gauge transformation

$$V \rightarrow V + \Lambda + \bar{\Lambda}, \quad (1.42)$$

with Λ a chiral superfield, $\bar{D}_{\dot{\alpha}}\Lambda = 0$. However, one can remind that the superspace integral together with E transform under a super-Weyl rescaling according to [23]:

$$\int d^4\theta E \rightarrow \int d^4\theta E e^{2\Sigma + 2\bar{\Sigma}}, \quad (1.43)$$

with Σ a chiral superfield. The Freedman Lagrangian (1.41) is therefore gauge invariant if the gauge transformation (1.42) is followed by a super-Weyl transformation (1.43), provided that Λ and Σ are related through:

$$2\Sigma = -\frac{2}{3}\xi\Lambda. \quad (1.44)$$

The full transformation (1.42)+(1.43) rotates the gravitino, and is thus a R -symmetry. We still have to show that the Lagrangian (1.41) effectively produces a constant FI term. The so-called Freedman model in superspace is given by:

$$\mathcal{L} = \mathcal{L}_{\text{Freedman FI}} + \mathcal{L}_{\text{gauge}} = -3 \int d^4\theta E e^{\frac{2}{3}\xi V} + \frac{1}{4} \left(\int d^2\Theta 2\mathcal{E}\mathcal{W}^2 + h.c. \right). \quad (1.45)$$

The gauge Lagrangian is given in component fields by⁴ $e^{-1}\mathcal{L}_{\text{gauge}}|_{\lambda=0} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2$, where the topological term $F\tilde{F}$ has been discarded since it does not play any role in the following. In the Wess-Zumino gauge, the Freedman Lagrangian can be expanded as

$$\mathcal{L}_{\text{Freedman FI}} = -3 \int d^4\theta E e^{\frac{2}{3}\xi V} = -3 \int d^4\theta E - 2\xi \int d^4\theta EV - \frac{2}{3}\xi^2 \int d^4\theta EV^2. \quad (1.46)$$

To write it in terms of component fields, we need the following components of the superfields V and E , which can be found for instance in [23, 29]:

$$V|_{\lambda=0} = -\theta\sigma^\mu\bar{\theta}v_\mu(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \quad (1.47)$$

$$V^2 = -\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}v^\mu v_\mu, \quad (1.48)$$

$$E|_{\theta=\bar{\theta}=0} = e, \quad (1.49)$$

$$E|_{\theta\bar{\theta}} = -\frac{2}{3}e\theta\sigma^\mu\bar{\theta}b_\mu - \frac{e}{4}(\bar{\theta}\bar{\sigma}^\mu\psi_\mu)(\theta\sigma^\nu\bar{\psi}_\nu) + \frac{e}{4}(\bar{\theta}\bar{\sigma}^\mu\psi_\nu)(\theta\sigma^\nu\bar{\psi}_\mu), \quad (1.50)$$

$$E|_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{6}eR + \frac{1}{12}e\epsilon^{\mu\nu\rho\sigma}(\psi_\mu\sigma_\nu\mathcal{D}_\rho\bar{\psi}_\sigma - \bar{\psi}_\mu\bar{\sigma}_\nu\mathcal{D}_\rho\psi_\sigma) + \frac{e}{9}MM^* - \frac{e}{9}b^\mu b_\mu. \quad (1.51)$$

We want to explicitly show that the $U(1)$ symmetry gauges the R -symmetry (*i.e.* that the gravitino is charged under the $U(1)$), and we thus expect a minimal coupling between the gravitino and the vector field. It turns out that this coupling comes from the two last terms of (1.50). We therefore explicitly detail their integration, while the others, more straightforward, are left implicit. Denoting with ... the remaining terms in the product EV , we have:

$$\begin{aligned} \int d^4\theta EV &= \int d^4\theta \frac{e}{4} \left[v_\rho(\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\rho\bar{\theta}^{\dot{\alpha}})(\bar{\theta}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}\psi_{\mu\beta})(\theta^\gamma\sigma_{\gamma\dot{\gamma}}^\nu\bar{\psi}_{\nu\dot{\gamma}}) \right. \\ &\quad \left. - v_\rho(\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\rho\bar{\theta}^{\dot{\alpha}})(\bar{\theta}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}\psi_{\nu\beta})(\theta^\gamma\sigma_{\gamma\dot{\gamma}}^\nu\bar{\psi}_{\mu\dot{\gamma}}) \right] + \dots \\ &= \int d^4\theta \frac{e}{4} v_\rho \left[\frac{1}{4}\epsilon^{\alpha\gamma}(\sigma_{\alpha\dot{\alpha}}^\rho\delta_{\dot{\beta}}^{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\beta}\beta}\psi_{\mu\beta})(\sigma_{\gamma\dot{\gamma}}^\nu\bar{\psi}_{\nu\dot{\gamma}}) \right. \\ &\quad \left. - \frac{1}{4}\epsilon^{\alpha\gamma}(\sigma_{\alpha\dot{\alpha}}^\rho\delta_{\dot{\beta}}^{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\beta}\beta}\psi_{\nu\beta})(\sigma_{\gamma\dot{\gamma}}^\nu\bar{\psi}_{\mu\dot{\gamma}}) \right] \theta\theta\bar{\theta}\bar{\theta} + \dots \\ &= \frac{e}{16}v_\rho \left[\sigma_{\alpha\dot{\alpha}}^\rho\bar{\sigma}^{\mu\dot{\alpha}\beta}\psi_{\mu\beta}\epsilon^{\alpha\gamma}\sigma_{\gamma\dot{\gamma}}^\nu\epsilon^{\dot{\gamma}\delta}\bar{\psi}_{\nu\dot{\delta}} - \sigma_{\alpha\dot{\alpha}}^\rho\bar{\sigma}^{\mu\dot{\alpha}\beta}\psi_{\nu\beta}\epsilon^{\alpha\gamma}\sigma_{\gamma\dot{\gamma}}^\nu\epsilon^{\dot{\gamma}\delta}\bar{\psi}_{\mu\dot{\delta}} \right] + \dots \\ &= \frac{e}{16}v_\rho \left[\bar{\psi}_{\nu\dot{\delta}}(\bar{\sigma}^{\nu\dot{\delta}\alpha}\sigma_{\alpha\dot{\alpha}}^\rho\bar{\sigma}^{\mu\dot{\alpha}\beta})\psi_{\mu\beta} - \bar{\psi}_{\mu\dot{\delta}}(\bar{\sigma}^{\nu\dot{\delta}\alpha}\sigma_{\alpha\dot{\alpha}}^\rho\bar{\sigma}^{\mu\dot{\alpha}\beta})\psi_{\nu\beta} \right] + \dots \\ &= \frac{e}{16}v_\rho \left[\bar{\psi}_{\nu\dot{\delta}}(\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu - \bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu)^{\dot{\delta}\beta}\psi_{\mu\beta} \right] + \dots, \end{aligned} \quad (1.52)$$

where, in the second equality, we have used $\theta^\alpha\theta^\gamma = -\frac{1}{2}\epsilon^{\alpha\gamma}\theta\theta$, $\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = -\frac{1}{2}\delta_{\dot{\beta}}^{\dot{\alpha}}\bar{\theta}\bar{\theta}$, and in the fourth equality, $\epsilon^{\alpha\gamma}\sigma_{\gamma\dot{\gamma}}^\nu\epsilon^{\dot{\gamma}\delta} = \bar{\sigma}^{\nu\dot{\delta}\alpha}$. From the σ -matrices relation $\bar{\sigma}^\nu\sigma^\rho\bar{\sigma}^\mu - \bar{\sigma}^\mu\sigma^\rho\bar{\sigma}^\nu = -2i\epsilon^{\nu\rho\mu\sigma}\bar{\sigma}_\sigma$, we deduce

$$\int d^4\theta EV = -\frac{ie}{8}\epsilon^{\nu\rho\mu\sigma}\bar{\psi}_{\nu\dot{\delta}}\bar{\sigma}_\sigma\psi_{\mu\beta}v_\rho + \dots \quad (1.53)$$

This term is precisely a minimal coupling between the gravitino ψ_μ and the vector v_ρ . Therefore, the gravitino is charged under the $U(1)$ symmetry, and this $U(1)$ is thus a R -symmetry, as already expected with the superfield point of view from the relation (1.44). Bringing everything together,

4. The gaugino λ is put to zero for compactness in all this section.

the Lagrangian (1.45) in component fields reads:

$$e^{-1} \mathcal{L}|_{\lambda=0} = -\frac{1}{2}R + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\bar{\sigma}_\nu\mathcal{D}_\rho\psi_\sigma - \psi_\mu\sigma_\nu\mathcal{D}_\rho\bar{\psi}_\sigma) - \frac{1}{3}MM^* + \frac{1}{3}b^\mu b_\mu - \frac{i\xi}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\psi_\rho v_\sigma + \frac{2}{3}\xi b^\mu v_\mu + \frac{1}{3}\xi^2 v^\mu v_\mu - \xi D - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^2. \quad (1.54)$$

Eliminating the auxiliary fields through their equations of motion, namely $M = 0$, $b_\mu = -\xi v_\mu$ and $D = \xi$, yields:

$$e^{-1} \mathcal{L}|_{\lambda=0} = -\frac{1}{2}R + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\bar{\sigma}_\nu\mathcal{D}_\rho\psi_\sigma - \psi_\mu\sigma_\nu\mathcal{D}_\rho\bar{\psi}_\sigma) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{i\xi}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\bar{\sigma}_\nu\psi_\rho v_\sigma - \frac{\xi^2}{2}. \quad (1.55)$$

This Lagrangian contains the Einstein-Hilbert, Rarita-Schwinger and Maxwell Lagrangians, together with a gravitino/vector minimal coupling and a constant FI term $\xi^2/2$. We have therefore shown that in supergravity, writing a FI term associated to a $U(1)$ gauge multiplet (v_μ, λ, D) is possible provided that *we promote the vector v_μ to be the gauge field of the R -symmetry*, rotating the gravitino and the other fermions.

1.5 Superconformal approach to $\mathcal{N} = 1$ supergravity

One of the approach to study $\mathcal{N} = 1$, $D = 4$ supergravity coupled to gauge and chiral multiplets is the superconformal formalism. It is based on the idea that $\mathcal{N} = 1$, $D = 4$ supergravity can be seen as a *gauge fixed* superconformal gauge theory, in a similar way that Einstein's General Relativity can itself be seen as a gauge fixed conformal gauge theory. The goal of this section is not to present an exhaustive derivation of the superconformal approach to supergravity, which is too broad to be strictly presented in this thesis and which can be found in details in [10], on which this section is largely based. Rather, it aims to present and summarize the basic ideas of this formalism, to highlight in a different perspective some of the concepts already discussed previously, especially Kähler transformations, FI terms in supergravity and AdS supergravity, as well as to introduce the formalism which will be used in Section 1.6.

1.5.1 $SU(2, 2|\mathcal{N} = 1)$ superconformal algebra

Conformal supergravity can be seen as the gauge theory of the superconformal algebra $SU(2, 2|1)$. This superalgebra contains the conformal algebra $SU(2, 2) \simeq SO(4, 2)$ in its Lie subalgebra, spanned by the generators P_a , M_{ab} , D and K_a of translations, Lorentz transformations, dilatations and special conformal transformations. Its bosonic subalgebra also contains a $U(1)$ factor, which is an R -symmetry and whose generator is denoted T . The fermionic generators are the usual Poincaré supercharge Q_α together with the conformal supercharge S_α . The latter can be understood in the following way: its square is the special conformal generator K_a , in the same way that the square of Q_α is a translation P_a . The superalgebra $SU(2, 2|1)$ is determined by the following set of (anti-)commutation relations. First, the conformal algebra

$SU(2, 2) \simeq SO(4, 2)$ is determined by the non-vanishing commutators:

$$\begin{aligned} [M_{ab}, M_{cd}] &= 4\eta_{[a[c}M_{d]b]}, & [P_a, M_{bc}] &= 2\eta_{a[b}P_{c]}, \\ [K_a, M_{bc}] &= 2\eta_{a[b}K_{c]}, & [P_a, K_b] &= 2(\eta_{ab}D + M_{ab}), \\ [D, P_a] &= P_a, & [D, K_a] &= -K_a. \end{aligned} \quad (1.56)$$

Q_α and S_α being Lorentz spinors, we have:

$$[M_{ab}, Q_\alpha] = -\frac{1}{2}(\gamma_{ab})^\beta{}_\alpha Q_\beta, \quad [M_{ab}, S_\alpha] = -\frac{1}{2}(\gamma_{ab})^\beta{}_\alpha S_\beta. \quad (1.57)$$

Non-vanishing commutators involving D are:

$$[D, P_a] = P_a, \quad [D, Q_\alpha] = \frac{1}{2}Q_\alpha, \quad [D, S_\alpha] = -\frac{1}{2}S_\alpha, \quad [D, K_a] = -K_a. \quad (1.58)$$

The $U(1)$ generator T commutes with all generators of the conformal algebra. T being an R -symmetry, it rotates the supercharges through a chiral transformation:

$$[T, Q_\alpha] = -\frac{3}{2}i(\gamma_*)_\alpha{}^\beta Q_\beta, \quad [T, S_\alpha] = \frac{3}{2}i(\gamma_*)_\alpha{}^\beta S_\beta. \quad (1.59)$$

The last non-vanishing commutators are:

$$[K_a, Q_\alpha] = \gamma_a S_\alpha, \quad [P_a, S_\alpha] = \gamma_a Q_\alpha. \quad (1.60)$$

Finally, the anti-commutators of the fermionic generators are:

$$\begin{aligned} \{Q_\alpha, Q^\beta\} &= -\frac{1}{2}(\gamma^a)_\alpha{}^\beta P_a, & \{S_\alpha, S^\beta\} &= -\frac{1}{2}(\gamma^a)_\alpha{}^\beta K_a, \\ \{Q_\alpha, S^\beta\} &= -\frac{1}{2}\delta_\alpha{}^\beta D - \frac{1}{4}(\gamma^{ab})_\alpha{}^\beta M_{ab} + \frac{1}{2}i(\gamma_*)_\alpha{}^\beta T. \end{aligned} \quad (1.61)$$

In a nutshell, gauging a symmetry group requires to introduce a gauge field B_μ for each generators of the group, and then replace partial derivatives by covariant derivatives of the form $\mathcal{D}_\mu \equiv \partial_\mu - B_\mu^A \Sigma_A$, with Σ_A the generators and A running from 1 to the dimension of the gauge group. For the gauge group $SU(2, 2|1)$, the generators and their associated gauge fields, forming the so-called *Weyl multiplet*⁵, are listed in Table 1.1.

$SU(2, 2 1)$ superconformal algebra										
Bosonic subalgebra: $SU(2, 2) \times U(1)$					Representation formed by fermionic generators					
P_a	M_{ab}	D	K_a	T	Q_α					S_α
e_μ^a	ω_μ^{ab}	b_μ	f_μ^a	A_μ	ψ_μ^α					ϕ_μ^α

Table 1.1 – Generators and gauge fields of the superconformal algebra $SU(2, 2|1)$.

In addition to the usual Lorentz spins (j_1, j_2) , multiplet components are characterized in the superconformal approach by two other parameters (w, c) , respectively called Weyl and chiral weights, specifying the properties under the dilatation D and chiral T transformations. These

5. In the Weyl multiplet $(e_\mu^a, \omega_\mu^{ab}, b_\mu, f_\mu^a, A_\mu, \psi_\mu^\alpha, \phi_\mu^\alpha)$, only e_μ^a , b_μ , A_μ and ψ_μ^α are independent fields. ω_μ^{ab} , f_μ^a and ϕ_μ^α have to be seen as composite fields, functions of the four independent fields.

weights are defined by the transformation property of the lowest component C of the multiplet through:

$$\delta_D(\lambda_D)C = w\lambda_D C, \quad \delta_T(\lambda_T)C = ic\lambda_T C. \quad (1.62)$$

Conformal weights for the Weyl multiplet and gauge multiplet are uniquely fixed, and can be found for instance in [10]. For the following discussion, we need to keep in mind the weights of the vierbein, $w(e_\mu^a) = -1$, $c(e_\mu^a) = 0$. Conformal weights for chiral and real multiplets are arbitrary. Let us consider first a chiral multiplet $(Z, P_L\chi, F)$, with $w(Z) \equiv w$, $c(Z) \equiv c$. Applying the commutator $[D, Q_\alpha] = \frac{1}{2}Q_\alpha$ to Z , and using $DZ = wZ$, $Q_\alpha Z = \frac{1}{\sqrt{2}}P_L\chi_\alpha$, we get $DP_L\chi_\alpha = (w + \frac{1}{2})P_L\chi_\alpha$, that is, $w(P_L\chi) = w + \frac{1}{2}$. Similarly, one can find $w(F) = w + 1$. In a similar way, chiral weights are found from the commutator $[T, Q_\alpha] = -\frac{3}{2}i(\gamma_*)_\alpha^\beta Q_\beta$. If $c(Z) \equiv c$, then $c(P_L\chi) = c - \frac{3}{2}$, and $c(F) = w - 3$. We then consider a real multiplet $(C, P_L\zeta, \mathcal{H}, B_\mu, P_R\lambda, D)$. The lowest component C being real, it cannot undergo a complex transformation, and therefore $c(C) = 0$. With $w(C) \equiv w$, applying the same reasoning as above gives $w(D) = w + 2$ and $c(D) = 0$. This is everything we will need regarding the conformal weights to write invariant actions.

In order to write superconformal invariant actions, we need to generalize the F and D terms of global supersymmetry. Those were the highest components of chiral and real multiplets, and what we need in the superconformal case is thus two applications extracting superconformal invariant quantities from chiral and real multiplets. For this, the operation $[]_F$ is defined as acting on a chiral multiplet $(Z, P_L\chi, F)$ of weights $(3, 3)$ by:

$$[]_F : (Z, P_L\chi, F) \rightarrow [Z]_F \equiv \frac{e}{2} \left[F + \frac{1}{\sqrt{2}}\bar{\psi}_\mu\gamma^\mu P_L\chi + \frac{1}{2}Z\bar{\psi}_\mu\gamma^{\mu\nu}P_R\psi_\nu + h.c. \right]. \quad (1.63)$$

The operation $[]_D$ is defined as acting on a real multiplet $(C, P_L\zeta, \mathcal{H}, B_\mu, P_R\lambda, D)$ of weights $(2, 0)$ by:

$$\begin{aligned} []_D : (C, \zeta, \mathcal{H}, B_\mu, \lambda, D) \rightarrow [C]_D \equiv & \frac{e}{2} \left[D - \frac{1}{2}\bar{\psi}_\mu\gamma^\mu i\gamma_*\lambda - \frac{1}{3}CR(\omega) \right. \\ & + \frac{1}{6} \left(C\bar{\psi}_\mu\gamma^{\mu\rho\sigma} - i\bar{\zeta}\gamma^{\rho\sigma}\gamma_* \right) R'_{\rho\sigma}(Q) \\ & \left. + \frac{1}{4}\epsilon^{abcd}\bar{\psi}_a\gamma_b\psi_c \left(B_d - \frac{1}{2}\bar{\psi}_d\zeta \right) \right], \end{aligned} \quad (1.64)$$

with $R(\omega)$ and $R'(Q)$ the graviton and gravitino curvatures. The weights of the multiplets the operations $[]_F$ and $[]_D$ act on are fixed by the requirement that the total weights of the Lagrangian vanishes, and by the weights of the determinant of the vierbein, $w(e) = -4$, $c(e) = 0$. To compensate them, we need F to have weights $(w, c) = (4, 0)$, which implies from the discussion of the previous paragraph Z to have weights $(w, c) = (3, 3)$. Similarly, we need D to have weights $(4, 0)$, which implies C to have weights $(w, c) = (2, 0)$. As shown in [10], these operations can then be used to build superconformal invariant actions from chiral and real multiplets, respectively according to:

$$S_F = \int d^4x [Z]_F, \quad S_D = \int d^4x [C]_D. \quad (1.65)$$

From now on, we denote by $X = (X^I, I = 0, \dots, n)$ a set of $n + 1$ chiral fields, with Weyl weights chosen as $w(X^I) = 1$, and by $\{\lambda^A, A = 1, \dots, n_V\}$ the gauginos of n_V vector multiplets. Using these notations and the definitions introduced above, the (ungauged) superconformal invariant action of $\mathcal{N} = 1$ supergravity coupled to $n + 1$ chiral and n_V vector multiplets is written as:

$$\mathcal{L} = [N(X, \bar{X})]_D + [\Sigma(X)]_F - \frac{1}{4} [f_{AB}(X) \bar{\lambda}^A P_L \lambda^B]_F. \quad (1.66)$$

The two first terms are respectively the kinetic and potential terms of the chiral multiplets, while the third one is the kinetic term for the gauge multiplets. This action is completely characterized by three functions: a real Kähler potential $N(X, \bar{X})$, a holomorphic superpotential $\Sigma(X)$, and a holomorphic gauge kinetic function $f_{AB}(X)$. The requirement of conformal symmetry imposes the following additional conditions on these three functions⁶:

- The Kähler potential: a conformal invariant action $\int d^4x [N(X, \bar{X})]_D$ can be obtained provided that $w(N(X, \bar{X})) = 2$. Since $N(X, \bar{X})$ must also be real, it is homogeneous of first degree in both X and \bar{X} . We deduce that N must satisfy:

$$N(\lambda X, \bar{\lambda} \bar{X}) = \lambda \bar{\lambda} N(X, \bar{X}), \quad \forall \lambda \in \mathbb{C}. \quad (1.67)$$

- The superpotential: a conformal invariant action $\int d^4x [\Sigma(X)]_F$ can be obtained provided that $w(\Sigma(X)) = 3$. On the one hand, by definition of the Weyl weight w , we have $\delta_D \Sigma(X) = w(\Sigma(X)) \lambda_D \Sigma(X) = 3 \lambda_D \Sigma(X)$. On the other hand, using the chain rule and $w(X^I) = 1$, we have $\delta_D \Sigma(X) = \Sigma_I \delta_D X^I = \Sigma_I w(X^I) \lambda_D X^I = \Sigma_I \lambda_D X^I$. We deduce that Σ must satisfy:

$$X^I \Sigma_I = 3 \Sigma(X). \quad (1.68)$$

- The gauge kinetic function: similarly to the superpotential, the quantity $f_{AB}(X) \bar{\lambda}^A P_L \lambda^B$ must have Weyl weight $w(f_{AB}(X) \bar{\lambda}^A P_L \lambda^B) = 3$. Since $\bar{\lambda}^A P_L \lambda^B$ already has Weyl weight 3, f_{AB} must have vanishing Weyl weight, namely $\delta_D f_{AB}(X) = 0 = f_{AB,I} \delta_D X^I = f_{AB,I} w(X^I) \lambda_D X^I = f_{AB,I} X^I \lambda_D$. We deduce that f_{AB} must satisfy:

$$X^I f_{AB,I} = 0. \quad (1.69)$$

Each terms of (1.66) are independently superconformal invariant, provided that $N(X, \bar{X})$, $\Sigma(X)$ and $f_{AB}(X)$ satisfy the relations (1.67), (1.68), and (1.69).

In Section 1.3.1, we have seen how Kähler manifolds serve as the scalar field target space in a supersymmetric nonlinear σ -model. This is still the case in supergravity⁷. The chiral fields $\{X^I, I = 0, \dots, n\}$ can be seen as the coordinates of a $(n + 1)$ -dimensional Kähler manifold called the *embedding manifold*. So far, $\{X^I, I = 0, \dots, n\}$ was an arbitrary holomorphic set of

6. We use the notations $G_{I\bar{J}} = N_{I\bar{J}} \equiv \frac{\partial^2 N}{\partial X^I \partial \bar{X}^{\bar{J}}}$, $\Sigma_I \equiv \frac{\partial \Sigma(X)}{\partial X^I}$ and $f_{AB,I} \equiv \frac{\partial f_{AB}(X)}{\partial X^I}$.

7. In supergravity, further conditions on the Kähler manifold are required compared to global $\mathcal{N} = 1$ supersymmetry. For instance, in $\mathcal{N} = 1$ supergravity, the target space is not a pure Kähler manifold, but has to be a Kähler-Hodge manifold. We will not discuss them in this thesis, but mathematical definition and their origin in supergravity can be found in [10].

coordinates. Requiring them to be homogeneous, they can be defined by the set $\{s_0, z^\alpha, \alpha = 1, \dots, n\}$ through functions $Z^I(z)$ according to:

$$X^I = s_0 Z^I(z). \quad (1.70)$$

The simplest choice for the Z^I is $Z^0 = 1, Z^\alpha = z^\alpha$. In the variables $\{s_0, z^\alpha\}$, homogeneity requires that N is the product $s_0 \bar{s}_0$ times a function of z and \bar{z} . We can thus write N as:

$$N(X, \bar{X}) = -a s_0 \bar{s}_0 e^{-\frac{K(z, \bar{z})}{a}}, \quad (1.71)$$

where a is a constant which will be specified later. On the other hand, (1.68) means that the superpotential $\Sigma(X)$ on the embedding manifold is a homogeneous holomorphic function of third degree. It is thus given in terms of s_0 and z^α by

$$\Sigma(X) = s_0^3 W(z). \quad (1.72)$$

In the $\{s_0, z^\alpha\}$ coordinates, the superconformal invariant Lagrangian (1.66) therefore reads:

$$\mathcal{L} = -a \left[S_0 \bar{S}_0 e^{-\frac{K(z, \bar{z})}{a}} \right]_D + \left[S_0^3 W(z) \right]_F - \frac{1}{4} \left[f_{AB}(X) \bar{\lambda}^A P_L \lambda^B \right]_F. \quad (1.73)$$

S_0, \bar{S}_0 are chiral and anti-chiral multiplets called *compensator multiplets*, that we write in component form as $S_0 = (s_0, P_L \Omega_0, F_0)$ and $\bar{S}_0 = (\bar{s}_0, P_R \Omega_0, \bar{F}_0)$.

In order to get the physical matter-coupled supergravity Lagrangian from the superconformal Lagrangian (1.73), we need to gauge fix the superconformal symmetries that are not in the super-Poincaré algebra, namely K_a, S, D and T . Special conformal transformations can be fixed by imposing $b_\mu = 0$, while gauge fixing conformal S -supersymmetry is obtained by imposing the chiral (anti-chiral) fermion $P_L \Omega_0$ ($P_R \Omega_0$) superpartner of the compensator scalars s_0 (\bar{s}_0) to vanish, which thus removes one extra fermion field. Since N transforms under the scaling as $\delta N = 2\lambda_D N$, one can fix the dilatation gauge by imposing a constant N ⁸, $N(X, \bar{X}) = -a$. This gauge condition amounts to fix the modulus of s_0 , as can be seen in the following way. Let us write $-a = N(X, \bar{X}) = X^I N_{I\bar{J}} \bar{X}^{\bar{J}} = s_0 \bar{s}_0 Z^I(z) N_{I\bar{J}} \bar{Z}^{\bar{J}}(\bar{z})$, where we have used the homogeneity property of N in the second equality and the decomposition (1.70) in the third one. This implies:

$$s_0 \bar{s}_0 = -a \left(Z^I(z) G_{I\bar{J}} \bar{Z}^{\bar{J}}(\bar{z}) \right)^{-1}, \quad (1.74)$$

so that $|s_0|$ is determined in terms of the physical scalars $z^\alpha, \bar{z}^{\bar{\alpha}}$. Finally, the T superconformal symmetry can be gauge fixed imposing $s_0 = \bar{s}_0$, which amounts to fix the phase of s_0 . These results are summarized in Table 1.2.

8. Restoring the (reduced) Planck mass $\kappa^{-1} = 2.4 \times 10^{18} \text{ GeV}$, we have $N(X, \bar{X}) = -a\kappa^{-2}$. The Planck mass in a (super)gravitational theory can thus be *mathematically* seen as arising from the breaking of the dilatation symmetry in a conformally invariant embedding theory.

Gauge fixing conditions	
K-gauge	$b_\mu = 0$
S-gauge	$\Omega_0 = 0$
D-gauge	$N(X, \bar{X}) = -a$
T-gauge	$s_0 = \bar{s}_0$

Table 1.2 – Gauge fixing conditions reducing the $SU(2, 2|\mathcal{N} = 1)$ superconformal algebra to the $\mathcal{N} = 1$ super-Poincaré algebra

The value of a depends on the theory considered. For $\mathcal{N} = 1$ supergravity, in order to get a canonically normalized Einstein-Hilbert action, we need to choose $a = 3$, which we will keep in the following. Using the D and T-gauge conditions into (1.71), one gets

$$s_0 = \bar{s}_0 = e^{\frac{K(z, \bar{z})}{6}}, \quad (1.75)$$

which will be the gauge fixing condition for the compensating scalars that we will use in the following. In pure $\mathcal{N} = 1$ supergravity where $K(z, \bar{z}) = 0$, it reduces to $s_0 = \bar{s}_0 = 1$.

In conclusion, the S-gauge condition has eliminated the fermion Ω_0 , while the D and T-gauge conditions have fixed the s_0, \bar{s}_0 scalars in terms of the z^α and $\bar{z}^{\bar{\alpha}}$, so that s_0 and \bar{s}_0 are no longer independent physical scalars. The gauge-fixing procedure has therefore removed the chiral and anti-chiral compensating multiplets S_0 and \bar{S}_0 , so that the gauge fixed theory (*i.e.* $\mathcal{N} = 1$ supergravity) contains n chiral multiplets while the superconformal theory contained $n + 1$ ones. Only the n z^α will remain as the physical chiral matter fields. Their complex scalar lowest components form the coordinates of a n -dimensional Kähler manifold, called the *projective manifold*. The real function $K(z, \bar{z})$ is the Kähler potential of this physical scalar field target space.

1.5.2 New perspectives from the superconformal formalism

Kähler transformations

In the superconformal formalism, Kähler transformations arise from the relation (1.70) between the arbitrary holomorphic basis $\{X^I\}$ and the physical basis $\{s_0, z^\alpha\}$ of the embedding Kähler manifold. Indeed, we see that if we redefine the second set according to

$$s_0 \rightarrow s'_0 = s_0 e^{\frac{f(z)}{a}}, \quad Z^I(z) \rightarrow Z'^I(z) = Z^I(z) e^{-\frac{f(z)}{a}}, \quad (1.76)$$

then the original ones $\{X^I\}$ are unchanged. Combining Eqs. (1.74) and (1.75), we get $K(z, \bar{z}) = -a \ln(-a^{-1} Z^I G_{I\bar{J}} \bar{Z}^{\bar{J}})$. Using Z'^I in this latter equation, we see that under a transformation (1.76), K transforms according to

$$K(z, \bar{z}) \rightarrow K'(z, \bar{z}) = K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}), \quad (1.77)$$

which is precisely a Kähler transformation as already defined in Section 1.3.1. While the homogeneity property (1.67) of the Kähler potential $N(X, \bar{X})$ of the embedding Kähler manifold does

not permit Kähler transformations on N , Kähler transformations of the Kähler potential $K(z, \bar{z})$ of the projective Kähler manifold are allowed. From the superconformal point of view, Kähler transformations can therefore be seen as transformations in the projective space, arising from the ambiguity (1.76) to define coordinates in the embedding space. Let us consider a general function $V(z, \bar{z})$ transforming under Kähler transformations as

$$V(z, \bar{z}) \rightarrow V'(z, \bar{z}) = V(z, \bar{z})e^{-a^{-1}(w_+f(z)+w_-\bar{f}(\bar{z}))}. \quad (1.78)$$

Such functions are said to have Kähler weights (w_+, w_-) . Assuming V to be scalar under complex coordinate transformations on the projective manifold⁹, it is therefore natural to define its covariant derivatives as:

$$\nabla_\alpha V(z, \bar{z}) \equiv \partial_\alpha V(z, \bar{z}) + w_+ a^{-1} (\partial_\alpha K) V(z, \bar{z}), \quad \bar{\nabla}_{\bar{\alpha}} V(z, \bar{z}) \equiv \partial_{\bar{\alpha}} V(z, \bar{z}) + w_- a^{-1} (\partial_{\bar{\alpha}} K) V(z, \bar{z}). \quad (1.79)$$

One can check that these quantities are indeed covariant under Kähler transformations as required for a covariant derivative, namely $\nabla_\alpha V(z, \bar{z}) \rightarrow \nabla_\alpha V(z, \bar{z})e^{-a^{-1}(w_+f(z)+w_-\bar{f}(\bar{z}))}$. In this terminology, $Z^I(z), \bar{Z}^{\bar{I}}(\bar{z})$ have Kähler weights $(1, 0)$ and $(0, 1)$, while s_0, \bar{s}_0 have weights $(-1, 0)$ and $(0, -1)$. On the other hand, we have already seen that the superpotentials of the embedding and the projective manifold Σ and W are related by $\Sigma = s_0^3 W(z)$. For Σ to be Kähler invariant, $W(z)$ must then carry Kähler weights $(3, 0)$. With the choice $a = 3$, the Kähler transformations of W and s_0 are thus given by:

$$W(z) \rightarrow W'(z) = W(z)e^{-f(z)}, \quad (1.80)$$

$$s_0 \rightarrow s'_0 = s_0 e^{\frac{f(z)}{3}}, \quad (1.81)$$

while the covariant derivatives of Z^I and W read:

$$\nabla_\alpha Z^I \equiv \partial_\alpha Z^I + \frac{1}{3} (\partial_\alpha K) Z^I, \quad (1.82)$$

$$\nabla_\alpha W \equiv \partial_\alpha W + (\partial_\alpha K) W. \quad (1.83)$$

These transformations and Kähler covariant quantities will be at the core of the constructions presented in Section 1.6.

Fayet-Iliopoulos terms and R -symmetry

The supergravity action (1.73) discussed so far is invariant under the $SU(2, 2|1)$ superconformal algebra, as well as under Yang-Mills gauge symmetries which commute with local superconformal symmetries. Yang-Mills symmetries are local internal symmetries of the embedding Kähler manifold. Symmetries of Kähler metrics are discussed in Appendix A.2. The main result obtained here is that such symmetries are determined by real *moment maps*:

$$\mathcal{P}_A(z, \bar{z}) = i(k_A^\alpha \partial_\alpha K(z, \bar{z}) - r_A(z)) = -i(k_A^{\bar{\alpha}} \partial_{\bar{\alpha}} K(z, \bar{z}) - \bar{r}_A(\bar{z})), \quad (1.84)$$

9. If V does transform under complex coordinate transformations, that is if V carries α or $\bar{\alpha}$ indices, then the Christoffel connection has to be added to its covariant derivatives.

where $r_A(z)$ are arbitrary holomorphic functions, and k_A^α are the holomorphic Killing vectors characterizing each symmetries labelled by the index A . They are holomorphic functions of the scalars $\{X^I, I = 0, \dots, n\}$, and can be expanded on this basis according to $k_A(X) = k_A^I \frac{\partial}{\partial X^I}$. Denoting θ^A the gauge parameters of the gauge transformations, infinitesimal gauge transformations on the scalar fields $\{X^I, I = 0, \dots, n\}$ are given by:

$$\delta X^I = \theta^A k_A^J \frac{\partial X^I}{\partial X^J} = \theta^A k_A^I, \quad (1.85)$$

while those on the coordinates $\{s_0, z^\alpha\}$ are [10]:

$$\delta z^\alpha = \theta^A k_A^\alpha(z), \quad \delta s_0 = \frac{1}{3} \theta^A s_0 r_A(z). \quad (1.86)$$

We would like to study the relation between FI terms ir_A and the R -symmetry. To this purpose, let us first compute the relation between the Killing vectors k_A^I of the embedding manifold and those k_A^α of the projective manifold. Using the relation (1.70) $X^I = s_0 Z^I(z^\alpha)$ and the chain rule, infinitesimal gauge transformations on X^I can be written as

$$\begin{aligned} \delta X^I &= \delta s_0 Z^I(z^\alpha) + s_0 \delta Z^I(z^\alpha) \\ &= \delta s_0 Z^I(z^\alpha) + s_0 \delta z^\alpha \partial_\alpha Z^I(z^\alpha) \\ &= \frac{1}{3} \theta^A s_0 r_A Z^I + s_0 \theta^A k_A^\alpha \partial_\alpha Z^I, \end{aligned} \quad (1.87)$$

where the last equality follows from the gauge transformations (1.86) of the physical coordinates $\{s_0, z^\alpha\}$. Expressing r_A in terms of the moment maps \mathcal{P}_A using (1.84), we get:

$$\begin{aligned} \delta X^I &= \theta^A s_0 \left[\frac{1}{3} (i\mathcal{P}_A + k_A^\alpha \partial_\alpha K) Z^I + k_A^\alpha \partial_\alpha Z^I \right] \\ &= \theta^A s_0 \left[\frac{i}{3} \mathcal{P}_A Z^I + k_A^\alpha \nabla_\alpha Z^I \right], \end{aligned} \quad (1.88)$$

where in the second equality we have brought together the second and third term of the first equality into the covariant derivative $\nabla_\alpha Z^I$ (1.82). Identification with $\delta X^I = \theta^A k_A^I$ gives the following relation between k_A^I and k_A^α :

$$k_A^I = s_0 \left[\frac{i}{3} \mathcal{P}_A Z^I + k_A^\alpha \nabla_\alpha Z^I \right], \quad \forall A. \quad (1.89)$$

We now use the invariance of Σ under gauge symmetries, namely $\delta \Sigma = \theta^A k_A^I \Sigma_I = 0$. Eq.(1.89) therefore implies that

$$\begin{aligned} \forall A, \quad 0 &= s_0 \left[\frac{i}{3} \mathcal{P}_A Z^I + k_A^\alpha \nabla_\alpha Z^I \right] \Sigma_I \\ &= i\mathcal{P}_A W + k_A^\alpha \nabla_\alpha W, \end{aligned} \quad (1.90)$$

where the last equality follows from the relations $s_0 \Sigma_I Z^I = \Sigma_I X^I = 3\Sigma = 3s_0^3 W$ and $s_0 \Sigma_I \nabla_\alpha Z^I = \nabla_\alpha \Sigma = s_0^3 \nabla_\alpha W$. Finally, using again the expression of the moment map \mathcal{P}_A (1.84) as well as the

covariant derivative $\nabla_\alpha W$ (1.83), we deduce that:

$$k_A^\alpha \partial_\alpha W(z) = -r_A(z)W(z), \quad \forall A. \quad (1.91)$$

The relation (1.91) is of utmost importance for the physics of Fayet-Iliopoulos terms in supergravity. It tells us that the gauge properties of the physical superpotential $W(z)$ are related to the holomorphic functions $r_A(z)$. In particular, we find that if a Fayet-Iliopoulos constant $\xi_A \equiv -ir_A(z) \neq 0$ (that is, if the compensator field s_0 is charged under the gauge transformations according to (1.86)), then the superpotential $W(z)$ cannot be gauge invariant under the associated $U(1)_A$. This gauge symmetry is therefore a R -symmetry. We thus find back through the superconformal formalism the result already derived in Section 1.4.3.

Anti-de Sitter supergravity

In the superconformal formalism, the Lagrangian of pure $\mathcal{N} = 1$ supergravity is simply the D-term of (1.73) with a vanishing Kähler potential and $a = 3$, namely

$$\mathcal{L}_{\text{sugra}} = -3 \left[S_0 \bar{S}_0 \right]_D, \quad (1.92)$$

with the chiral and anti-chiral compensator multiplets given by $S_0 = (s_0, P_L \Omega_0, F_0)$ and $\bar{S}_0 = (\bar{s}_0, P_R \Omega_0, \bar{F}_0)$. After gauge fixing the conformal symmetries, it yields the action [10]:

$$S_{\text{sugra}} = \int d^4x \frac{e}{2} \left[R(\omega(e, \psi)) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + 6A_\mu A^\mu - 6F_0 \bar{F}_0 \right], \quad (1.93)$$

with $D_\nu \psi_\rho \equiv \left(\partial_\nu + \frac{1}{4} \omega_{\nu ab}(e, \psi) \gamma^{ab} \right) \psi_\rho$. The complex scalar F_0 and the real vector A_μ are the auxiliary fields of the gravitational multiplet, corresponding to the auxiliary complex scalar M and real vector b_μ introduced in the superspace formalism in Section 1.4.3. They can be eliminated from the action using their trivial classical equations of motion $F_0 = 0$, $A_\mu = 0$. We then find back the pure $\mathcal{N} = 1$ supergravity action (1.18).

As explained in Section 1.3.2, the simplest extension of pure $\mathcal{N} = 1$ supergravity is the so-called anti-de Sitter supergravity. It is obtained from the superconformal formalism by supplementing the pure supergravity action with a constant superpotential of the projective space, $W(z) = W_0$. The Lagrangian thus reads:

$$\mathcal{L}_{\text{AdS}} = -3 \left[S_0 \bar{S}_0 \right]_D + \left[S_0^3 W_0 \right]_F. \quad (1.94)$$

Using the result (B.0.9) of Appendix B, the composite chiral multiplet with lowest component $s_0^3 W_0$ is given by

$$\begin{aligned} S_0^3 W_0 &= W_0 (s_0^3, 3s_0^2 P_L \Omega_0, 3s_0^2 F_0 - 3s_0 \bar{\Omega}_0 P_L \Omega_0) \\ &= (W_0, 0, 3W_0 F_0), \end{aligned} \quad (1.95)$$

where in the second line we have used the S-gauge fixing condition $\Omega_0 = 0$ as well as the D and T-gauge fixing condition (1.75) $s_0 = \bar{s}_0 = 1$. Using (1.63), the new term in the action is given

by:

$$S_{W_0} = \int d^4x \frac{e}{2} \left[3W_0(F_0 + \bar{F}_0) + \frac{1}{2}W_0\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \right]. \quad (1.96)$$

Considering the full action $S_{\text{AdS}} \equiv S_{\text{sugra}} + S_{W_0}$, the field equations for the auxiliary fields give $A_\mu = 0$ as previously, but now a non-vanishing $F_0 = \frac{W_0}{2}$. Using this expression for F_0 in the total action, we find:

$$S_{\text{AdS}} = \int d^4x \frac{e}{2} \left[R(\omega(e, \psi)) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + m_{3/2}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu - 2\Lambda \right], \quad (1.97)$$

with $m_{3/2} = \frac{W_0}{2}$, $\Lambda = -\frac{3}{4}W_0^2 = -3m_{3/2}^2$.

Again, we find back with the superconformal formalism a result already derived in Section 1.3.2: the value of the gravitino mass and the AdS cosmological constant are not independent parameters, both of them being set by the constant superpotential W_0 . In order to be able to tune the cosmological constant independently from the gravitino mass, one might want to add a constant FI term to the action (1.97). However, as explained in Sections 1.4.3 and 1.5.2, this necessarily implies the gauging of the R -symmetry and so forbids the constant superpotential written there. This issue motivates the construction of a new kind of FI term which does not require the gauging of the R -symmetry, which is the topic of the next section.

1.6 Fayet-Iliopoulos terms in supergravity without gauged R -symmetry

1.6.1 The original construction and an improved version

So far, we have discussed through the Freedman model in Section 1.4.3 and the superconformal approach in Section 1.5.2 why, in $\mathcal{N} = 1$ supergravity, does a Fayet-Iliopoulos term associated to a $U(1)$ gauge multiplet require this $U(1)$ to be a R -symmetry. However, in [14], a new type of FI term which does not imply the gauging of the R -symmetry has been developed. The idea is to replace the Freedman Lagrangian (1.41) by:

$$\mathcal{L}_{\text{Freedman FI}} = -3 \int d^4\theta E e^{\frac{2}{3}\xi V} \quad \rightarrow \quad -3 \int d^4\theta E \quad + \quad \mathcal{L}_{\text{new FI}}, \quad (1.98)$$

with

$$\mathcal{L}_{\text{new FI}} = 8\xi \int d^4\theta E \frac{\mathcal{W}^2\bar{\mathcal{W}}^2}{\mathcal{D}^2\mathcal{W}^2\bar{\mathcal{D}}^2\bar{\mathcal{W}}^2} \mathcal{D}^\alpha\mathcal{W}_\alpha. \quad (1.99)$$

Let us first look how such Lagrangian produces a constant FI term in the case of global supersymmetry. Putting the gaugino λ to zero, the component expansion of the square of the supersymmetric field strength is¹⁰

$$\mathcal{W}^2 \Big|_{\lambda=0} = (D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu})\theta^2. \quad (1.100)$$

10. Up to the Chern-Simons term $\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ which does not play a role in this discussion and is thus discarded.

$\mathcal{W}^2\bar{\mathcal{W}}^2$ already being a $\theta^2\bar{\theta}^2$ term, only the lowest components of the remaining superfields entering in the Lagrangian (1.99) will play a role in the integration. They are given by

$$\mathcal{D}^\alpha \mathcal{W}_\alpha| = -2D, \quad (1.101)$$

$$\mathcal{D}^2 \mathcal{W}^2| = 2F^{\mu\nu} F_{\mu\nu} - 4D^2, \quad (1.102)$$

where $X|$ denotes the lowest component of the superfield X . In global supersymmetry, the bosonic part of the new FI Lagrangian is thus given by

$$8\xi \int d^4\theta \frac{\mathcal{W}^2\bar{\mathcal{W}}^2}{D^2\mathcal{W}^2\bar{D}^2\bar{\mathcal{W}}^2} \mathcal{D}^\alpha \mathcal{W}_\alpha = 8\xi \frac{(D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu})^2}{(2F^{\mu\nu}F_{\mu\nu} - 4D^2)^2} (-2D) + \xi \mathcal{O}(\lambda, \bar{\lambda}) = -\xi D + \xi \mathcal{O}(\lambda, \bar{\lambda}), \quad (1.103)$$

which is a constant FI term. One can henceforth notice that this term makes sense only when $\langle D \rangle \neq 0$: supersymmetry is thus spontaneously broken by a D -term, and the goldstino is identified with the gaugino λ .

In order to discuss the supergravity generalisation of this new FI Lagrangian, we now adopt the superconformal formalism presented in Section 1.5. In this formalism, denoting $S_0 = (s_0, P_L\Omega_0, F_0)$ and $\bar{S}_0 = (\bar{s}_0, P_R\Omega_0, \bar{F}_0)$ the chiral and anti-chiral compensator fields, with conformal weights $(1, 1)$ and $(1, -1)$ respectively, the new FI Lagrangian (1.99) reads:

$$\mathcal{L}_{FI} = -\xi \left[S_0 \bar{S}_0 \frac{\mathcal{W}^2 \bar{\mathcal{W}}^2}{T(\bar{\mathcal{W}}^2) \bar{T}(\mathcal{W}^2)} (V)_D \right]_D. \quad (1.104)$$

ξ is a constant parameter, $(V)_D$ is a real linear multiplet defined by

$$(V)_D = (D, \not{D}\lambda, 0, \mathcal{D}^b \hat{F}_{ab}, -\not{D}\not{D}\lambda, -\square^C D), \quad (1.105)$$

whose lowest component D is the real auxiliary field of the vector superfield V , the latter having (anti)-chiral field strength $(\bar{\mathcal{W}}) \mathcal{W}$ given by

$$\mathcal{W}^2 = \frac{\bar{\lambda} P_L \lambda}{S_0^2}, \quad \bar{\mathcal{W}}^2 = \frac{\bar{\lambda} P_R \lambda}{\bar{S}_0^2}, \quad (1.106)$$

so that $(V)_D$ is given by the super-covariant derivative of \mathcal{W} . The chiral projection operator T acts on an anti-chiral multiplet \bar{X} of weights $(1, -1)$ to produce a chiral multiplet $T(\bar{X})$ of weights $(2, 2)$ according to:

$$T : \bar{X} = (\bar{X}, P_R\Omega, \bar{F}) \quad \mapsto \quad T(\bar{X}) = (\bar{F}, \not{D}P_R\Omega, \square^C \bar{X}). \quad (1.107)$$

$\bar{\lambda} P_L \lambda$ has weights $(3, 3)$ and reads, in components form:

$$\bar{\lambda} P_L \lambda = \left(\bar{\lambda} P_L \lambda; \sqrt{2} P_L \left(-\frac{1}{2} \gamma \cdot \hat{F} + iD \right) \lambda; 2\bar{\lambda} P_L \not{D}\lambda + \hat{F}^- \cdot \hat{F}^- - D^2 \right), \quad (1.108)$$

with the covariant field strength \hat{F}_{ab} and the self-dual and anti self-dual tensors \hat{F}_{ab}^\pm given by

$$\hat{F}_{ab} = e_a^\mu e_b^\nu (2\partial_{[\mu} A_{\nu]} + \bar{\psi}_{[\mu} \gamma_{\nu]} \lambda), \quad \hat{F}_{ab}^\pm = \frac{1}{2}(\hat{F}_{ab} \pm \tilde{\hat{F}}_{ab}). \quad (1.109)$$

The dual field strength is $\tilde{\hat{F}}_{ab} = -\frac{1}{2}i\epsilon_{abcd}\hat{F}^{cd}$, while the covariant derivative $\mathcal{D}_\mu\lambda$ is defined by

$$\mathcal{D}_\mu\lambda = \left(\partial_\mu - \frac{3}{2}b_\mu + \frac{1}{4}w_\mu^{ab}\gamma_{ab} - \frac{3}{2}i\gamma_*\mathcal{A}_\mu \right) \lambda - \left(\frac{1}{4}\gamma^{ab}\hat{F}_{ab} + \frac{1}{2}i\gamma_*D \right) \psi_\mu. \quad (1.110)$$

As discussed in Section 1.5.1, the fields b_μ , w_μ^{ab} and \mathcal{A}_μ are the gauge fields corresponding to dilatations, Lorentz transformations and T symmetry of the conformal algebra respectively. In this subsection, we summarize the results obtained in [14, 16]. The technical details are postponed to Section 1.6.2, which presents a generalization of the scalar potential obtained in these two papers that has been carried out during this first project.

Let us first consider the Lagrangian of Anti-de Sitter supergravity, described in Sections 1.3.2 and 1.5.2, coupled to an abelian $U(1)$ gauge multiplet plus the FI term (1.104). The full Lagrangian reads:

$$\mathcal{L} = -3 \left[S_0 \bar{S}_0 \right]_D + \left[S_0^3 W_0 \right]_F - \frac{1}{4g^2} \left[\bar{\lambda} P_L \lambda \right]_F + \mathcal{L}_{FI}. \quad (1.111)$$

Supersymmetry is broken via a non-vanishing VEV of the D-auxiliary component of the vector multiplet driven by the linear term in D, with the goldstino being the $U(1)$ gaugino. In component form, after having gauge fixed the scalar compensator through $s_0 = 1$, integrated the auxiliary fields, and in the unitary gauge where the goldstino vanishes, one gets [14]:

$$e^{-1}\mathcal{L} = \frac{1}{2} \left(R - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + m_{3/2} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \right) - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \left(-3m_{3/2}^2 + \frac{1}{2}\xi^2 \right). \quad (1.112)$$

with $m_{3/2} = \frac{W_0}{2}$, a constant superpotential. Therefore, in the absence of chiral matter superfields, any $\xi \neq 0$ uplifts the vacuum energy by a constant term $\mathcal{V}_{FI} = \xi^2/2$ and breaks supersymmetry. One can then tune ξ to get a de Sitter vacuum configuration, matching with the observational data. For instance, $\xi = \sqrt{6}m_{3/2}$ gives a massive gravitino in flat Minkowski spacetime with spontaneously broken supersymmetry.

Introducing chiral matter multiplets X^i in the previous model, the Lagrangian is now given by:

$$\mathcal{L} = -3 \left[S_0 \bar{S}_0 e^{-K(X^i, \bar{X}^i)/3} \right]_D + \left[S_0^3 W(X^i) \right]_F - \frac{1}{4g^2} \left[\bar{\lambda} P_L \lambda \right]_F + \mathcal{L}_{FI}. \quad (1.113)$$

In component form, after having gauge fixed the scalar compensator through $s_0 = e^{K/6}$ and integrated the auxiliary fields, the bosonic part of the previous Lagrangian reads [14]:

$$e^{-1}\mathcal{L}|_{\text{bos}} = \frac{1}{2}R - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - G_{i\bar{j}} \partial X^i \cdot \partial \bar{X}^{\bar{j}} - \left(e^K (|\nabla_i W|^2 - 3|W|^2) + \frac{\xi^2 g^2}{2} e^{2/3K} \right). \quad (1.114)$$

Therefore, when matter fields are coupled, the scalar potential contribution from (1.104) becomes field dependent, $\mathcal{V}_{FI} = \frac{\xi^2 g^2}{2} e^{2K/3}$, and no longer Kähler invariant, which basically comes from the fact that the FI Lagrangian (1.104) is not itself Kähler invariant. To remedy this, a generalized

Kähler invariant FI term has been built in [16]. From the generic Kähler transformations for a Kähler potential $K(X, \bar{X})$, a superpotential $W(X)$ and the compensator S_0 ,

$$\begin{aligned} K(X, \bar{X}) &\rightarrow K(X, \bar{X}) + J(X) + \bar{J}(\bar{X}), \\ W(X) &\rightarrow W(X)e^{-J(X)}, \\ S_0 &\rightarrow S_0e^{J(X)/3}, \end{aligned} \tag{1.115}$$

this new construction is based on the modification of the FI term (1.104) by introducing in it the Kähler potential according to

$$\mathcal{L}_{FI} = -\xi \left[(S_0 \bar{S}_0 e^{-K/3})^{-3} \frac{(\bar{\lambda} P_L \lambda)(\bar{\lambda} P_R \lambda)}{T(\mathcal{W}^2) \bar{T}(\bar{\mathcal{W}}^2)} (V)_D \right]_D. \tag{1.116}$$

The modified and henceforth Kähler invariant gauge field strengths are given by¹¹

$$\mathcal{W}^2 = \frac{\bar{\lambda} P_L \lambda}{(S_0 \bar{S}_0 e^{-K/3})^2}, \quad \bar{\mathcal{W}}^2 = \frac{\bar{\lambda} P_R \lambda}{(S_0 \bar{S}_0 e^{-K/3})^2}. \tag{1.117}$$

The new bosonic contribution to the scalar potential arising from this new term reads $\mathcal{V}_{FI} = \frac{\xi^2 g^2}{2}$, which is constant whether matter fields are included or not. The first aim of the project presented in this chapter has been to generalise the work carried out in [14, 16] by building the most general extended FI terms whose bosonic component is linear in the auxiliary field D , up to a general field dependent coefficient, while preserving Kähler invariance at the same time. This work is presented in the next subsection.

1.6.2 A set of Kähler invariant Fayet-Iliopoulos terms

The starting point of our new construction is to modify the field strengths (1.106) by introducing the superpotential W in order to make them Kähler invariant. This can be done in the following way¹²:

$$\mathcal{W}^2 = \frac{\bar{\lambda} P_L \lambda}{S_0^2 W(X)^{\frac{2}{3}}}, \quad \bar{\mathcal{W}}^2 = \frac{\bar{\lambda} P_R \lambda}{\bar{S}_0^2 \bar{W}(\bar{X})^{\frac{2}{3}}}, \tag{1.118}$$

where the $\frac{2}{3}$ exponent of $W(X)$ is uniquely fixed by the Kähler transformations (1.115) to get Kähler invariant \mathcal{W}^2 and $\bar{\mathcal{W}}^2$. The superpotential W has vanishing Weyl and Chiral weights and is assumed to have a non-vanishing VEV. Therefore, \mathcal{W}^2 and $\bar{\mathcal{W}}^2$ have the same (Weyl, Chiral) weights (1, 1) and (1, -1) as those of (1.106), and one can thus still apply the (anti-) chiral projection operators (\bar{T}) T . The resulting multiplets $T(\mathcal{W}^2)$ and $\bar{T}(\bar{\mathcal{W}}^2)$ then carry weights (2, 2) and (2, -2). The operation $[\]_D$ has to act on a multiplet of weights (2, 0). $(V)_D$ having already weights (2, 0), we need to multiply it with a multiplet with vanishing weights, which can be chosen $(S_0 \bar{S}_0)^{-1} \frac{(\bar{\lambda} P_L \lambda)(\bar{\lambda} P_R \lambda)}{T(\mathcal{W}^2) \bar{T}(\bar{\mathcal{W}}^2)}$ as it can be easily checked knowing the weights (1, 1) and (3, 3) of S_0 and $\bar{\lambda} P_L \lambda$, respectively. The Kähler potential K and the superpotential W having vanishing weights, we can include them for free in the previous combination in the following

11. Note that these superfields are no longer chiral, contrary to the previous ones of Eq. (1.106).

12. Besides Kähler invariance, the new gauge field strengths (1.118) are again chiral like those of Eq. (1.106).

form:

$$\mathcal{R}(V)_D \equiv (S_0 \bar{S}_0)^{-1} e^{nK} W^\alpha \bar{W}^\alpha \frac{(\bar{\lambda} P_L \lambda)(\bar{\lambda} P_R \lambda)}{T(\bar{\mathcal{W}}^2) \bar{T}(\mathcal{W}^2)} (V)_D, \quad (1.119)$$

where at this point the parameters n and α are arbitrary. As it can be seen from (1.115), the quantity $\mathcal{R}(V)_D$ is Kähler invariant provided that n and α are related by

$$n = \frac{1}{3} + \alpha. \quad (1.120)$$

Therefore, the most general Kähler invariant FI term involving both the Kähler potential and the superpotential is in fact a set of Lagrangians labelled by one free parameter α according to:

$$\mathcal{L}_{FI}^{(\alpha)} = -\xi_\alpha \left[(S_0 \bar{S}_0)^{-1} e^{(\frac{1}{3} + \alpha)K} W^\alpha \bar{W}^\alpha \frac{(\bar{\lambda} P_L \lambda)(\bar{\lambda} P_R \lambda)}{T(\bar{\mathcal{W}}^2) \bar{T}(\mathcal{W}^2)} (V)_D \right]_D. \quad (1.121)$$

We now add a series of terms (1.121) in $\mathcal{N} = 1$ supergravity coupled to the $U(1)$ gauge multiplet (whose gauge kinetic function is chosen to be one for simplicity), plus a set of matter chiral multiplets denoted generically $\{X\}$. Before gauge fixing the superconformal generators, the lagrangian for this model reads:

$$\mathcal{L} = -3 \left[S_0 \bar{S}_0 e^{-\frac{K(X, \bar{X})}{3}} \right]_D + \left[S_0^3 W(X) \right]_F - \frac{1}{4g^2} \left[\bar{\lambda} P_L \lambda \right]_F + \sum_i \mathcal{L}_{FI}^{(\alpha_i)}, \quad (1.122)$$

where the sum is running for now over an arbitrary set of parameters α_i .

We are interested in the contribution of Eq. (1.121) to the scalar potential, and in particular we would like to check that Kähler invariance is preserved. For simplicity and in order to highlight the cosmological applications, we focus on the bosonic sector. The contribution to the fermion masses arising from these new FI terms is studied in Appendix C. Putting all fermions to zero for now, the remaining components of the chiral multiplet $\bar{\lambda} P_L \lambda$ given in Eq. (1.108) are:

$$\bar{\lambda} P_L \lambda = (0, 0, \hat{F}^- \cdot \hat{F}^- - D^2). \quad (1.123)$$

With the chiral and anti-chiral compensators $S_0 = (s_0, P_L \Omega_0, F_0)$ and $\bar{S}_0 = (\bar{s}_0, P_R \Omega_0, \bar{F}_0)$, one can use the composition laws (B.0.9) to find the bosonic components of the composite chiral and anti-chiral multiplets $S_0^{-1} \bar{\lambda} P_L \lambda$ and $\bar{S}_0^{-1} \bar{\lambda} P_R \lambda$:

$$\begin{aligned} S_0^{-1} \bar{\lambda} P_L \lambda &= (0, 0, s_0^{-1}(\hat{F}^- \cdot \hat{F}^- - D^2)) \\ &= (0, 0, -2s_0^{-1}(\hat{F}^- \cdot \hat{F}^- - D^2), 0, 0, 0, 0), \end{aligned} \quad (1.124)$$

$$\begin{aligned} \bar{S}_0^{-1} \bar{\lambda} P_R \lambda &= (0, 0, \bar{s}_0^{-1}(\hat{F}^+ \cdot \hat{F}^+ - D^2)) \\ &= (0, 0, 0, -2\bar{s}_0^{-1}(\hat{F}^+ \cdot \hat{F}^+ - D^2), 0, 0, 0). \end{aligned} \quad (1.125)$$

Similarly, the bosonic components of the chiral superfield \mathcal{W}^2 given in Eq. (1.118) are:

$$\mathcal{W}^2 = (0, 0, s_0^{-2} W^{-\frac{2}{3}} (\hat{F}^- \cdot \hat{F}^- - D^2)), \quad (1.126)$$

from which we deduce the components of the anti-chiral superfield $\bar{T}(\mathcal{W}^2)$:

$$\bar{T}(\mathcal{W}^2) = \left(s_0^{-2} W^{-\frac{2}{3}} (\hat{F}^- \cdot \hat{F}^- - D^2), 0, 0 \right). \quad (1.127)$$

The product of the multiplets (1.124) and (1.125) already being a $\theta^2 \bar{\theta}^2$ term, only the lowest components of the remaining quantities entering in the Lagrangian (1.121) will contribute to the bosonic sector. We can therefore rewrite it as:

$$\mathcal{L}_{FI}^{(\alpha_i)} = -\xi_i \frac{e^{(\frac{1}{3} + \alpha_i)K} W^{\alpha_i} \bar{W}^{\alpha_i} D}{(s_0 \bar{s}_0)^{-2} (W \bar{W})^{-\frac{2}{3}} (F^- \cdot F^- - D^2) (F^+ \cdot F^+ - D^2)} [\mathcal{R}]_D, \quad (1.128)$$

with the real multiplet \mathcal{R} defined as:

$$\mathcal{R} \equiv (S_0^{-1} \bar{\lambda} P_L \lambda) (\bar{S}_0^{-1} \bar{\lambda} P_R \lambda). \quad (1.129)$$

Looking at the seven-components notation (1.124) and (1.125) for the multiplets $S_0^{-1} \bar{\lambda} P_L \lambda$ and $\bar{S}_0^{-1} \bar{\lambda} P_R \lambda$ and the multiplication law (B.0.8), we see that the only non-vanishing *bosonic* term of \mathcal{R} arises from $\frac{1}{2} f_{ij} K^i \mathcal{H}^j$ in its D-component. More precisely, it reads

$$(\mathcal{R})_D = \frac{1}{2} f_{ij} K^i \mathcal{H}^j = \frac{1}{2} f_{21} K^2 \mathcal{H}^1 = 2(s_0 \bar{s}_0)^{-1} (\hat{F}^- \cdot \hat{F}^- - D^2) (\hat{F}^+ \cdot \hat{F}^+ - D^2). \quad (1.130)$$

The operation $[\]_D$ defined in (B.0.7) immediately leads to

$$[\mathcal{R}]_D = \frac{e}{2} (\mathcal{R})_D = e(s_0 \bar{s}_0)^{-1} (\hat{F}^- \cdot \hat{F}^- - D^2) (\hat{F}^+ \cdot \hat{F}^+ - D^2). \quad (1.131)$$

The FI Lagrangian (1.128) is therefore given by:

$$e^{-1} \mathcal{L}_{FI}^{(\alpha_i)} = -\xi_i s_0 \bar{s}_0 e^{(\alpha_i + \frac{1}{3})K} (W \bar{W})^{\alpha_i + \frac{2}{3}} D. \quad (1.132)$$

Since we are interested in matter coupled $\mathcal{N} = 1$ supergravity, we use the Einstein frame where the conformal symmetry is gauge fixed through $s_0 = \bar{s}_0 = e^{\frac{K}{6}}$. This leads to a set of Kähler invariant terms parametrised by some constants $\{\alpha_i, \xi_i\}$ according to:

$$e^{-1} \mathcal{L}_{FI}^{(\alpha_i)} = -\xi_i e^{(\alpha_i + \frac{2}{3})K} (W \bar{W})^{\alpha_i + \frac{2}{3}} D = -\xi_i e^{(\alpha_i + \frac{2}{3})\mathcal{G}} D, \quad (1.133)$$

where $\mathcal{G} \equiv K + \ln |W|^2$. Therefore, after gauge fixing the conformal symmetry and integrating out the auxiliary fields, the pure bosonic sector arising from the Lagrangian (1.122) is given by

$$e^{-1} \mathcal{L}^{(B)} = \frac{1}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - G_{I\bar{J}} \partial X^I \cdot \partial \bar{X}^{\bar{J}} - \mathcal{V} \quad (1.134)$$

with the scalar potential

$$\mathcal{V} = e^{\mathcal{G}} \left(\partial_I \mathcal{G} G^{I\bar{J}} \partial_{\bar{J}} \mathcal{G} - 3 \right) + \mathcal{V}_{FI}. \quad (1.135)$$

The new FI contribution to the scalar potential, \mathcal{V}_{FI} , arising from Eq. (1.133) reads

$$\mathcal{V}_{FI} = \frac{1}{2} \left(\sum_i \xi_i e^{(\alpha_i + \frac{2}{3})\mathcal{G}} \right)^2, \quad (1.136)$$

which is obviously Kähler invariant while field dependent at the same time.

The above construction therefore provides a way to obtain an arbitrary set of (Kähler invariant) FI terms from a single $U(1)$ gauge field, in the presence of a superpotential W with non-vanishing expectation value. Each term of the sum is parametrised by two real constants ξ_i and α_i . A constant FI term is obviously recovered by choosing one $\alpha_{i_0} = -2/3$. For this value, the bosonic part of the Lagrangian (1.121) is equal to the one of the Lagrangian (1.116). Being independent of W , it is in particular valid even for vanishing superpotential, like the new FI term (1.116). However, it is not clear that the fermionic parts of the Lagrangians (1.121) and (1.116) are equal, as well, for $\alpha_{i_0} = -2/3$.

A general sum appearing in the Lagrangian (1.122), involving terms of the form (1.133), using that $e^{\mathcal{G}/2} = m_{3/2}[X]$, amounts to adding a general function of the gravitino mass $m_{3/2}[X]$ considered as a functional of the scalar fields $\{X\}$:

$$e^{-1}\mathcal{L}_{FI}^{(B)} = -f(m_{3/2}[X])D \quad \longrightarrow \quad \mathcal{V}_{FI} = \frac{g^2}{2} |f(m_{3/2}[X])|^2 \quad (1.137)$$

This construction allows us to refine the scalar potential by adding new field dependent and Kähler invariant terms. In the following, we will restrict ourselves as an illustration to the study of the simple case of one term of the type (1.133) up to an additive constant, corresponding to the choice $i = 1, 2$ with α_1 an arbitrary parameter and $\alpha_2 = -2/3$. Considering the Kähler potential of no-scale type and a constant superpotential, we will show that this choice is sufficient to produce inflationary models compatible with the slow-roll conditions and consistent with the CMB observations, with the inflaton rolling towards a de Sitter vacuum with tuneable energy and spontaneously broken supersymmetry.

1.7 No-scale models and cosmological applications

In this section, we study the cosmological consequences of the previous modified FI-term construction in the case of simple no-scale models. Considering one chiral superfield X associated to the inflaton, we successively choose the Kähler potentials

$$K(X, \bar{X}) = -\ln(X + \bar{X}) \quad \text{and} \quad K(X, \bar{X}) = -3\ln(X + \bar{X}), \quad (1.138)$$

together with a constant superpotential $W = W_0$ and an exponential one $W(X) = e^{\beta X}$, respectively. In the context of string theory, these forms of Kähler potentials arise in all toroidal/orbifold compactifications as well as in the large volume limit of Calabi-Yau compactifications, both in heterotic string and in type II orientifolds. In this context, the first Kähler potential could describe for instance the kinetic term of the dilaton, associated to the string coupling, while the second may describe the internal volume of the 3-complex dimensional compact space. We will therefore refer to the “dilaton case” and “compact volume case” to describe these two models. From now on, we also restrict the sum (1.136) to only two terms parametrised by three constants ξ_1 , $\alpha_1 \equiv \alpha$ and ξ_2 , while $\alpha_2 = -\frac{2}{3}$.

1.7.1 Dilaton case

We first consider the Kähler potential $K = -\ln(X + \bar{X})$. In terms of the gravitino mass $m_{3/2}^2 = e^{\mathcal{G}}$, this yields the scalar potential:

$$\mathcal{V} = -2m_{3/2}^2 + \frac{1}{2} \left(\xi_1 (m_{3/2}^2)^{\alpha+2/3} + \xi_2 \right)^2, \quad (1.139)$$

where we have redefined the parameters ξ_i to absorb the gauge coupling constant g . As we will show below, there is a region of the parameter space ξ_1 , ξ_2 and α , such that the above potential has an inflational plateau allowing slow-roll inflation compatible with the cosmological observations, and a minimum, where supersymmetry is spontaneously broken, with a tuneable vacuum energy by a fine tuning of the parameters (for instance to obtain a vanishing cosmological constant in the vacuum).

In order to compute the slow-roll parameters, one needs to work with the canonically normalised field χ , defined by its kinetic term through

$$\frac{\partial_\mu X \partial^\mu \bar{X}}{(X + \bar{X})^2} = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi + \dots \quad (1.140)$$

where the dots denote terms containing the imaginary part of X , which has no influence on the discussion of this section. We will come back to it in Section 1.7.3, where the shift symmetry associated to this imaginary part will be gauged by the $U(1)$. Focusing on the real part for now, we deduce from (1.140)

$$\text{Re}X = e^{\sqrt{2}\chi}, \quad (1.141)$$

and thus

$$m_{3/2}^2 = e^{\mathcal{G}} = \frac{|W_0|^2}{2} e^{-\sqrt{2}\chi}. \quad (1.142)$$

In the following, the ‘dilaton’ χ will be identified with the inflaton, dynamically driving inflation starting from a large value, slightly rolling down along the potential, attaining the horizon exit denoted by χ_* and ending at a value χ_{end} when slow-roll stops. The field then continues to fall down towards the minimum, when reheating takes place. From now on, quantities observed at the horizon exit are specified with a star *, and the approximation of large inflaton field $\chi \gg 1$ is assumed in this region¹³. The gravitino mass (1.142) vanishes exponentially and the potential (1.139) for $\alpha \geq -2/3$ is therefore dominated by a constant, as required by slow-roll inflation. In the following, we will thus restrict to the region $\alpha > -2/3$, with $\mathcal{V}_* \simeq \frac{\xi_2^2}{2}$. In terms of the canonical field χ , the slow-roll parameters are given as usual by

$$\epsilon \equiv \frac{1}{2} \left(\frac{\partial V / \partial \chi}{V} \right)^2 \quad ; \quad \eta \equiv \frac{\partial^2 V / \partial \chi^2}{V}. \quad (1.143)$$

At large field χ , by further assuming $\alpha > 1/3$, they can be expanded into

$$\epsilon \underset{\chi \gg 1}{\approx} \left(\frac{4m_{3/2}^2}{\xi_2^2} \right)^2 \underset{\chi \gg 1}{\approx} \frac{1}{4} \eta^2 \quad ; \quad \eta \underset{\chi \gg 1}{\approx} -\frac{8m_{3/2}^2}{\xi_2^2}. \quad (1.144)$$

13. $\chi \gg 1$ corresponds to weak coupling, which is necessary for the validity of an effective supergravity theory. However, the large field approximation is not really needed; instead, the required condition is that $m_{3/2} \rightarrow 0$.

Actually, the large field condition is not really necessary. The required approximation is that the gravitino mass (1.142) should be small during inflation, so that the potential (1.139) is approximately constant. This is an important point, implying that the models we study are consistent with small field inflation, avoiding trans-planckian initial conditions for the normalised inflaton field.

A central quantity to be taken into account in inflation is the number N of e-folds between the horizon exit and the end of inflation, a period observable through the CMB. This quantity, which must be set within the range $[40, 60]$ to satisfy CMB observations, is given by:

$$N = \int_{\chi_*}^{\chi_{\text{end}}} \frac{d\chi}{\sqrt{2\epsilon(\chi)}} \in [40, 60]. \quad (1.145)$$

Two other observable quantities at the horizon exit are the amplitude of primordial density fluctuations A_S and the spectral index, or tilt n_S , respectively given by

$$A_S = \frac{\mathcal{V}_*}{24\pi^2\epsilon_*} = 2.2 \times 10^{-9}, \quad (1.146)$$

$$n_S = 1 + 2\eta_* - 6\epsilon_* = 0.96, \quad (1.147)$$

where the numerical equalities also follow from the CMB data.

To be consistent with observations, the inflaton potential during inflation should respect the three conditions (1.145), (1.146) and (1.147), which we now use in order to constrain the three parameters ξ_1 , ξ_2 and α . In the large field limit at the horizon exit, the tilt (1.147) gives $n_S = 1 + 2\eta_* - 6\epsilon_* \simeq 1 + 2\eta_* - \frac{3}{2}\eta_*^2 \approx 1 + 2\eta_* = 1 - 16m_{3/2}^{*2}/\xi_2^2 \simeq 0.96$, from which we deduce that $\xi_2^2 \simeq 400m_{3/2}^{*2}$. Moreover, the amplitude (1.146) leads to $\frac{1}{768\pi^2} \frac{\xi_2^6}{m_{3/2}^{*4}} = 2.2 \times 10^{-9}$. Therefore, in the large field limit, the ξ_1 and α dependence drops, and one can immediately find from these two relations the numerical values for the parameter ξ_2 and the gravitino mass at the horizon exit $m_{3/2}^{*2}$, namely:

$$\xi_2^2 = 1.04 \times 10^{-10}, \quad (1.148)$$

$$m_{3/2}^{*2} = 2.6 \times 10^{-13}. \quad (1.149)$$

From the value of ϵ at the horizon, given by the first equation of (1.144), and the two relations (1.148) and (1.149), we find the predicted value for the tensor-to-scalar ratio of primordial perturbations to be:

$$r = 16\epsilon_* \simeq \left(\frac{16m_{3/2}^{*2}}{\xi_2^2} \right)^2 = 1.6 \times 10^{-3}, \quad (1.150)$$

which is fixed and independent of any parameters of the model, as long as $\alpha > 1/3$ is considered.

On the other hand, the condition to have a (almost) vanishing potential at its minimum¹⁴, for a value of the gravitino mass denoted $\tilde{m}_{3/2}^2$ in what follows, can then be used in order to determine the parameter ξ_1 in terms of α . This is obtained by numerically solving $\mathcal{V}(\tilde{m}_{3/2}^2) = 0$, with the result denoted $\xi_1(\alpha)$ in the following. In order to constrain the last remaining parameter

14. This is obviously an approximation since the cosmological constant is extremely small but nonzero. But the point to keep in mind is that in this model, the cosmological constant can be tuned to *any* small positive value.

α via the number of e-folds equation (1.145), we first need to determine the value of the inflaton field at the end of inflation, depending on α . Inflation stops when χ reaches a value χ_{end} such that $\epsilon(\chi_{\text{end}}) = 1$ or $|\eta(\chi_{\text{end}})| = 1$. In this model, the condition first fulfilled turns out to be $\eta(\chi_{\text{end}}) = -1$, which leads to the equation:

$$6m_{3/2}^{2\text{end}} - \xi_1^2(\alpha)(m_{3/2}^{2\text{end}})^{2\alpha+4/3} \left(\frac{1}{2} + 4 \left(\alpha + \frac{2}{3} \right)^2 \right) - \xi_1(\alpha)\xi_2(m_{3/2}^{2\text{end}})^{\alpha+2/3} \left(1 + 2 \left(\alpha + \frac{2}{3} \right)^2 \right) - \frac{\xi_2^2}{2} = 0. \quad (1.151)$$

This equation is solved numerically to get $m_{3/2}^{2\text{end}}$ in terms of α . The number of e-folds is then used in order to determine the parameter α . Indeed, equation (1.145) becomes:

$$N(\alpha) = -\frac{1}{4} \int_{m_{3/2}^{*2}}^{m_{3/2}^{2\text{end}}(\alpha)} \frac{-4m_{3/2}^2 + \left[\xi_1(\alpha)(m_{3/2}^2)^{\alpha+2/3} + \xi_2 \right]^2}{2m_{3/2}^2 - \left(\alpha + \frac{2}{3} \right) \xi_1^2(\alpha)(m_{3/2}^2)^{2\alpha+4/3} - \left(\alpha + \frac{2}{3} \right) \xi_1(\alpha)\xi_2(m_{3/2}^2)^{\alpha+2/3}} \frac{dm_{3/2}^2}{m_{3/2}^2}. \quad (1.152)$$

Using $m_{3/2}^{2\text{end}}$ given by the largest solution of Eq. (1.151), the value for ξ_2^2 (1.148), and the expression for $\xi_1(\alpha)$ given from the solution of $\mathcal{V}(\tilde{m}_{3/2}^2) = 0$, the above integral can be numerically evaluated in terms of α . It turns out that any α larger or approximately equal to 1 leads to an acceptable e-fold number $N \in [40, 60]$. Thus, the only fine tuning of the model, besides fixing the overall scale of the potential by its asymptotic value determined by ξ_2 , is related to the vacuum energy. The gravitino mass at the minimum of the potential $\tilde{m}_{3/2}$ can be between 10^{10} TeV and the Planck scale by choosing α between 1 and 10.5 respectively.

As an illustration, we now choose $\alpha \simeq 1$, which gives $\xi_1(\alpha = 1) \simeq 10^{11}$.¹⁵ With these values, the scalar potential and the slow-roll parameters are plotted in terms of χ in Figures 1.1 and 1.2 respectively, where we have also set $W_0 = \sqrt{2}$. The vertical grey lines indicate the horizon exit and the end of inflation (from the right to the left). The corresponding numerical values for the gravitino mass are $m_{3/2}^{*2} = 2.64 \times 10^{-13}$ and $m_{3/2}^{2\text{end}} = 1.56 \times 10^{-11}$ in Planck units. The minimum is reached at $\tilde{m}_{3/2}^2 = 5.29 \times 10^{-11}$. The associated values for the inflaton field are $\chi_* = 20.48 + \frac{1}{\sqrt{2}} \ln \frac{|W_0|^2}{2}$, $\chi_{\text{end}} = 17.59 + \frac{1}{\sqrt{2}} \ln \frac{|W_0|^2}{2}$ and $\tilde{\chi} = 16.73 + \frac{1}{\sqrt{2}} \ln \frac{|W_0|^2}{2}$. Notice that the values of the inflaton can be made less than one for an appropriate choice of W_0 , as we already mentioned in the beginning of the section. Finally, note that because of the space-time curvature during inflation, the value of $m_{3/2}$ entering in the Lagrangian is not the physical gravitino mass, which should be computed taking into account the curvature contribution in an approximate de Sitter spacetime [30, 31].

The spectrum at the minimum contains the imaginary part of X and the $U(1)$ gauge boson, which remain massless in this model, as it can be seen from the expression of the scalar potential,

15. Despite the large value of ξ_1 , one can check that the approximation $\mathcal{V}_* = \xi_2^2/2$ at the horizon, assumed in the computation of the tilt and of the amplitude, is valid. Indeed, with the numerical values $\alpha \sim 1$, $\xi_1 \sim 10^{11}$, $\xi_2 \sim 10^{-5}$ and $m_{3/2}^{*2} = 2.64 \times 10^{-13}$, we get $\frac{\xi_1(m_{3/2}^{*2})^{\alpha+2/3}}{\xi_2} \sim 6 \times 10^{-5}$.

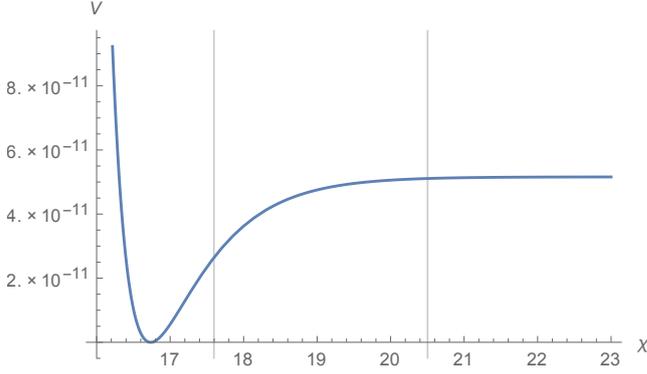


Figure 1.1 – Scalar potential as a function of the canonically normalised field χ , for $\alpha = 1$

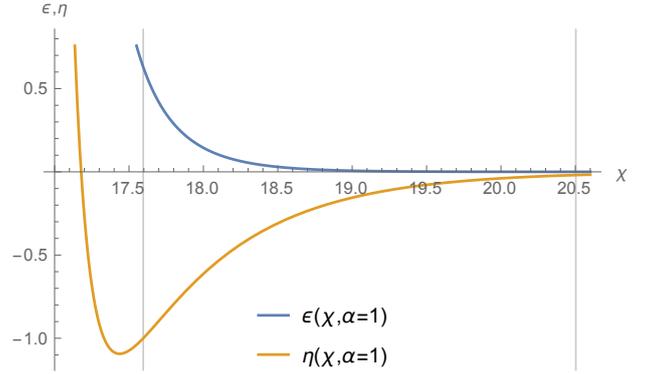


Figure 1.2 – Slow-roll parameters ϵ and η as a function of χ , for $\alpha = 1$

as well as the massive gravitino and inflaton whose masses are given by:

$$\tilde{m}_{3/2}^2 = 5.29 \times 10^{-11}, \quad \tilde{m}_\chi^2 = \left. \frac{\partial^2 V}{\partial \chi^2} \right|_{\chi=\chi_{\min}} = 2.46 \times 10^{-10}. \quad (1.153)$$

There is also a massive spin-1/2 fermion corresponding to a linear combination of the $U(1)$ gaugino and the fermionic component of the inflaton superfield, orthogonal to the Goldstino direction. Indeed at the minimum, supersymmetry is spontaneously broken by a non-vanishing expectation value of both a D and F-term. The Goldstino $P_L \nu$ is thus a linear combination of the gaugino λ and of the chiral fermion Ω : $P_L \nu = \frac{1}{\sqrt{2}} \Omega^X g_{X\bar{X}} \bar{F}^{\bar{X}} - \frac{i}{2} D P_L \lambda$, with $\bar{F}^{\bar{X}} \equiv -e^{K/2} g^{X\bar{X}} \nabla_X W$, evaluated at the minimum. In order to compute the direction of supersymmetry breaking, we consider:

$$\|F\| \equiv \sqrt{F^X g_{X\bar{X}} \bar{F}^{\bar{X}}} = \sqrt{e^{\mathcal{G}} \partial_X \mathcal{G} g^{X\bar{X}} \partial_{\bar{X}} \mathcal{G}} = m_{3/2}, \quad (1.154)$$

$$\|D\| \equiv \xi_1 e^{5/3 \mathcal{G}} + \xi_2 = \xi_1 (m_{3/2}^2)^{5/3} + \xi_2. \quad (1.155)$$

At the minimum, we have:

$$\left\| \frac{D}{F} \right\|_{\tilde{m}_{3/2}} = \left[\xi_1 (\tilde{m}_{3/2}^2)^{7/6} + \xi_2 (\tilde{m}_{3/2}^2)^{-1/2} \right] \simeq 1.5, \quad (1.156)$$

where we have used the values $\xi_1 = 10^{11}$, $\xi_2 = 10^{-5}$ and $\tilde{m}_{3/2}^2 = 5.29 \times 10^{-11}$ obtained previously. At the minimum, the Goldstino is thus an approximately equal mixing of the chiral fermion Ω and the gaugino λ .

The computation of the fermion masses is detailed in Appendix C. The mass squared m_f^2 of the physical fermion which remains after elimination of the Goldstino is given in Eq. (C.0.36). For $p = 1$, its numerical value at the minimum where $\tilde{m}_{3/2}^2 = 5.29 \times 10^{-11}$ is (in Planck units):

$$m_f^2 = 5.9 \times 10^{-12}. \quad (1.157)$$

1.7.2 Compact volume case

In this subsection, we consider the no-scale model with Kähler potential $K(X, \bar{X}) = -3 \ln(X + \bar{X})$. If one takes a constant superpotential as in the previous subsection, the F-term of the scalar potential will vanish, and the new Fayet-Iliopoulos term will be ill-defined at the minimum, where D now vanishes¹⁶. Instead, we consider a superpotential of the form $W(X) = e^{\beta X}$, with β a real constant. Note that the imaginary shift of X becomes now a (global) R-symmetry [20]. The full scalar potential is then given by:

$$\mathcal{V} = m_{3/2}^2 \left[-3 + \frac{1}{3} \left(\beta(X + \bar{X}) - 3 \right)^2 \right] + \frac{1}{2} \left(\xi_1 (m_{3/2}^2)^{\alpha+2/3} + \xi_2 \right)^2. \quad (1.158)$$

Choosing $\beta \ll (X + \bar{X})^{-1}|_*$, the first term of (1.158) can be neglected at the horizon exit as well as during the inflationary period. However, outside of the inflationary plateau, the D-term starts decreasing significantly and the F-term cannot be neglected anymore. Supersymmetry at the minimum of the potential is then spontaneously broken by non-vanishing expectation values of both D and F-terms, and a tuning of the parameters would be required in order to get a vanishing potential at its minimum, as in the previous case studied above. We will not study this region in the following, focusing on the inflationary period where the F contribution to \mathcal{V} can be neglected and the scalar potential is only given by its D-term:

$$\mathcal{V}|_{\text{infla.}} = \frac{1}{2} \left(\xi_1 (m_{3/2}^2)^{\alpha+2/3} + \xi_2 \right)^2. \quad (1.159)$$

Now the normalised field χ and the gravitino mass are given by:

$$\text{Re}X = e^{\sqrt{\frac{2}{3}}\chi} \quad ; \quad m_{3/2}^2 = \frac{|W|^2}{(X + \bar{X})^3} = \frac{|W|^2}{8} e^{-\sqrt{6}\chi}. \quad (1.160)$$

Like in the previous subsection, the potential at the horizon exit, where $\chi \gg 1$ is assumed¹⁷, is given by $\mathcal{V}_* = \xi_2^2/2$. The slow-roll parameters expanded in this limit read:

$$\epsilon \underset{\chi \gg 1}{\approx} \frac{12\xi_1^2 \left(\alpha + \frac{2}{3}\right)^2 (m_{3/2}^2)^{2\alpha+4/3}}{\xi_2^2} \quad ; \quad \eta \underset{\chi \gg 1}{\approx} \frac{12\xi_1 \left(\alpha + \frac{2}{3}\right)^2 (m_{3/2}^2)^{\alpha+2/3}}{\xi_2}, \quad (1.161)$$

and thus $\eta^2 \underset{\chi \gg 1}{\approx} 12 \left(\alpha + \frac{2}{3}\right)^2 \epsilon$. With these two quantities, the tilt and amplitude analysis yields:

$$\xi_2(\alpha) = \frac{6 \times 10^{-6}}{\alpha + 2/3} \quad , \quad (m_{3/2}^{*2})^{\alpha+2/3}(\xi_1, \alpha) = -\frac{1}{\xi_1} \frac{1.02 \times 10^{-8}}{(\alpha + 2/3)^3}. \quad (1.162)$$

The gravitino mass at the end of inflation, $m_{3/2}^{2 \text{ end}}$, is still given by the condition $\eta(m_{3/2}^{2 \text{ end}}) =$

16. Of course, the vanishing of the F-part of the scalar potential is a tree-level result and can be circumvented by considering quantum corrections in the Kähler potential.

17. $\chi \gg 1$ now corresponds to a large volume of the compact space, which is compatible with the effective theory where higher derivatives are neglected.

± 1 , which is now solution of the equation:

$$\xi_1^2 (m_{3/2}^{\text{end}})^{2\alpha+4/3} \left(\frac{1}{2} \mp 12 \left(\alpha + \frac{2}{3} \right)^2 \right) + \xi_1 \xi_2(\alpha) (m_{3/2}^{\text{end}})^{\alpha+2/3} \left(1 \mp 6 \left(\alpha + \frac{2}{3} \right)^2 \right) + \frac{\xi_2(\alpha)^2}{2} = 0. \quad (1.163)$$

This can be solved analytically at fixed α , yielding:

$$(m_{3/2}^{\text{end}})^{\alpha+2/3}_{\pm}(\xi_1, \alpha) = -\frac{\xi_2(\alpha)}{\xi_1} \times \frac{1 \mp 6(\alpha + 2/3)^2 - 2\sqrt{3}(\alpha + 2/3)\sqrt{3(\alpha + 2/3)^2 \pm 1}}{1 \mp 24(\alpha + 2/3)^2}. \quad (1.164)$$

On the other hand, the number of e-folds is given by:

$$N_{\pm}(\alpha) = \frac{-1}{12(\alpha + \frac{2}{3})} \int_{m_{3/2}^{*\pm}(\xi_1, \alpha)}^{m_{3/2}^{\text{end}}_{\pm}(\xi_1, \alpha)} \frac{[\xi_1 (m_{3/2}^2)^{\alpha+2/3} + \xi_2(\alpha)]^2}{\xi_1^2 (m_{3/2}^2)^{2\alpha+4/3} + \xi_1 \xi_2(\alpha) (m_{3/2}^2)^{\alpha+2/3}} \frac{dm_{3/2}^2}{m_{3/2}^2}, \quad (1.165)$$

which is independent of ξ_1 , as can be seen from the change of variable $m_{3/2}^2 \rightarrow m_{3/2}^2 \xi_1^{\frac{1}{\alpha+2/3}}$ and by using the second equation of (1.162) and Eq. (1.164).

Two regions for α have to be considered: (i) $-\frac{2}{3} < \alpha < \frac{\sqrt{3}-2}{3}$, where $\eta(m_{3/2}^{\text{end}}) = 1$ is first fulfilled, and where the gravitino mass at the end of inflation and the number of e-folds are respectively given by $(m_{3/2}^{\text{end}})_+$ and N_+ ; (ii) $\alpha \geq \frac{\sqrt{3}-2}{3}$, where $\eta(m_{3/2}^{\text{end}}) = -1$ is first fulfilled, and where the gravitino mass at the end of inflation and the number of e-folds are respectively given by $(m_{3/2}^{\text{end}})_-$ and N_- . Both e-fold numbers are plotted in terms of α in Figures 1.3 and 1.4. Any $\alpha \gtrsim -0.46$ leads to an acceptable $N \in [40, 60]$, while the parameter ξ_1 remains undetermined. From the value of ϵ at the horizon given by the first equation of (1.161) and

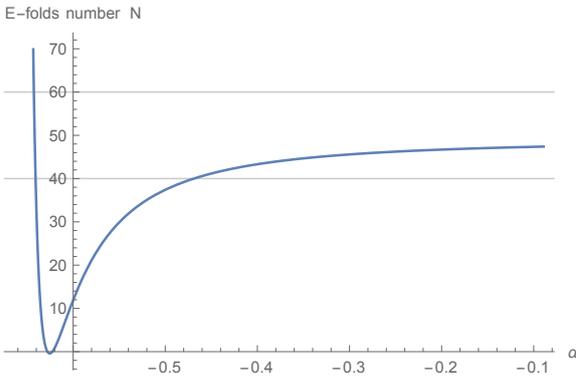


Figure 1.3 – Number of e-folds N_+ as a function of α , for $-\frac{2}{3} < \alpha < \frac{\sqrt{3}-2}{3}$

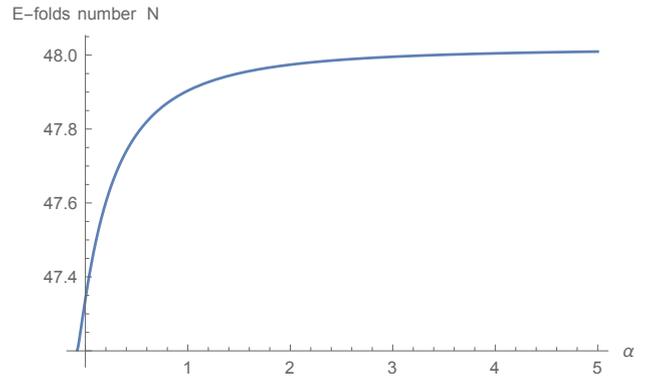


Figure 1.4 – Number of e-folds N_- as a function of α , for $\alpha \geq \frac{\sqrt{3}-2}{3}$

the two relations (1.162), one sees that the predicted value for the tensor-to-scalar ratio r of primordial perturbations remains independent of ξ_1 , and depends only on α :

$$r(\alpha) = 16 \epsilon_*(\alpha) = 16 \frac{12(\alpha + 2/3)^2 \xi_1^2 (m_{3/2}^*)^{2\alpha+4/3}(\xi_1, \alpha)}{\xi_2^2(\alpha)} \simeq \frac{5.4 \times 10^{-4}}{(\alpha + 2/3)^2}. \quad (1.166)$$

Thus, α can be chosen such that the tensor-to-scalar ratio is large and close to the experimental bound, for instance $r(\alpha = -0.45) \simeq 10^{-2}$ with $N_+(\alpha = -0.45) \simeq 41$.

1.7.3 Gauging the axion shift symmetry

In the two previous models, the spectrum contained two massless particles: the imaginary part of the complex inflaton field X , and the $U(1)$ gauge boson A_μ , which is an unwanted phenomenological property. This can be avoided by gauging the imaginary shift symmetry by the $U(1)$. Under a gauge transformation $A_\mu \rightarrow A_\mu - 2\partial_\mu\lambda$, one then has for the complex scalar $X \rightarrow X + ic\lambda$, with λ the gauge parameter and c a constant related to the charge e^c of the field e^X . In terms of superfields, this transformation reads $X \rightarrow X + c\Lambda$, with Λ a chiral superfield gauge parameter. The gauge transformation of the vector superfield V is $V \rightarrow V - \Lambda - \bar{\Lambda}$. In order to keep a gauge invariant Kähler potential with shift symmetry, $K(X + \bar{X})$ must be modified as:

$$K(X + \bar{X}) \rightarrow K(X + \bar{X} + cV). \quad (1.167)$$

Note that this modification does not change the pure bosonic part of the FI Lagrangian (1.121). Indeed, when fermions are put to zero, the only non-vanishing components of the chiral multiplets $\bar{\lambda}P_L\lambda$ and $\bar{\lambda}P_R\lambda$ are their $\theta\theta$ and $\bar{\theta}\bar{\theta}$ components. Therefore, only the lowest components of the other superfields involved in (1.121) contribute to the bosonic sector, and the lowest component of $e^{(\alpha+\frac{1}{3})K}$ does not receive additional contributions from cV in the Wess-Zumino gauge.

In order to see how a massive gauge boson arises from this gauging, we work in global supersymmetry and compute the (bosonic) new terms appearing from this modification. Putting fermions to zero and expanding in components, we have

$$X + \bar{X} + cV \Big|_{\text{bos}} = 2\text{Re}X - \theta\sigma^\mu\bar{\theta}(cA_\mu + 2\partial_\mu\text{Im}X) + \frac{1}{2}\theta^2\bar{\theta}^2(cD - \partial^2\text{Re}X) + \theta^2F + \bar{\theta}^2\bar{F}, \quad (1.168)$$

from which we deduce:

$$K(X + \bar{X} + cV) \Big|_{\theta^2\bar{\theta}^2} = \frac{K'}{2}(cD - \partial^2\text{Re}X) - \frac{K''}{4}(cA_\mu + 2\partial_\mu\text{Im}X)^2 + K''F\bar{F}. \quad (1.169)$$

It follows that

$$\int d^4\theta K(X + \bar{X} + cV) = \int d^4\theta K(X + \bar{X}) - \frac{c^2}{4}K''A_\mu A^\mu - cK''A_\mu\partial^\mu\text{Im}X + \frac{c}{2}K'D + \text{fermions}. \quad (1.170)$$

As a result, there is a mass term for the gauge boson A_μ , as well as a new field dependent FI term $-\xi(X)D$, with $\xi(\chi) = -cK'/2$. It modifies the D-term of the scalar potential (1.136) according to $D = g^2 \left[-cK'/2 + \sum_i \xi_i e^{(\alpha_i+2/3)\mathcal{G}} \right]$, which leads to the following D-term contribution to the scalar potential:

$$\mathcal{V}_{FI} = \frac{g^2}{2} \left(\sum_i \xi_i e^{(\alpha_i+\frac{2}{3})\mathcal{G}} - \frac{c}{2}K' \right)^2. \quad (1.171)$$

It is easy to show that the extra contribution proportional to c , due to the gauging of the shift symmetry, does not alter the inflationary predictions discussed in the previous section, when restricting the D-auxiliary field to only two non-vanishing terms, as in the previous section. Consider for example the compact volume case with a Kähler potential $K = -3\ln(X + \bar{X})$. The second term in (1.171) then becomes proportional to $m_{3/2}^{2/3}$ which may be identified as a particular case of the potential (1.159) studied before for $\alpha = -1/3$ and $\xi_1 = (3cg)/(2|W_0|^{2/3})$.

ξ_2 can then be obtained from the first sum in (1.171) by choosing one non-vanishing term, say ξ_2 with $\alpha_2 = -2/3$. The analysis is then reduced to the one of the last section, in the compact volume case with a fixed value of the parameter $\alpha = -1/3$, which is within the allowed region of the parameter space compatible with observational data, as seen in Fig. 1.3.

Let us finally consider another example inspired by the heterotic string with X identified with the string dilaton, as in the first model considered in the previous section, where its axionic imaginary part is dual to the Neveu-Schwarz antisymmetric tensor in four dimensions. In this case, the constant c is related to a $U(1)$ anomaly which is cancelled by a Green-Schwarz term. The gauging of the shift symmetry is a consequence of the anomaly cancellation and the axion is absorbed by the $U(1)$ becoming massive and no massless particle remains in the spectrum [32]. The gauge coupling is not anymore constant but is fixed by X : $g^2 = 1/\text{Re}X$, corresponding to a gauge kinetic function linear in X . The scalar potential can be easily obtained from Eq. (1.171) using $K = -\ln(X + \bar{X})$ and the expressions (1.141) and (1.142):

$$\mathcal{V} = -2m_{3/2}^2 + \frac{e^{-\sqrt{2}\chi}}{2} \left(\sum_i \xi_i e^{(\alpha_i + \frac{2}{3})G} + \frac{c}{4} e^{-\sqrt{2}\chi} \right)^2. \quad (1.172)$$

Again we restrict the D-term to only two non-vanishing contributions. In order to get an asymptotically constant potential at infinity, we choose $\alpha_2 = -\frac{7}{6}$, while $\alpha_1 = \frac{1}{3}$ is chosen to be able to absorb the constant c in ξ_1 . We obtain in this way a potential with the same form as in Eq. (1.139):

$$\mathcal{V} = -2m_{3/2}^2 + \frac{1}{2} \left(\xi'_1 (m_{3/2}^2)^{3/2} + \xi'_2 \right)^2, \quad (1.173)$$

where we have defined $\xi'_1 \equiv \frac{\xi_1 \sqrt{2}}{W_0} + \frac{c}{\sqrt{2}W_0^3}$ and $\xi'_2 \equiv \frac{\xi_2 \sqrt{2}}{W_0}$.

The potential is thus the same as the one of Eq. (1.139), with $\alpha = \frac{5}{6}$. This is an acceptable value since it leads to a number of e-folds $N(5/6) \simeq 51$. The numerical predictions obtained in Section 1.7.1 are not modified by the gauging of the shift symmetry. The main improvement is that now the imaginary part of the inflaton has been absorbed by the $U(1)$ gauge boson which acquires a mass. In order to compute this mass, one needs to rescale $A_\mu \rightarrow gA_\mu$ so that the gauge field kinetic term becomes canonical. After this rescaling, the gauge boson mass square reads:

$$m_A^2(\chi) = \frac{e^{-3\sqrt{2}\chi}}{8} c^2 = \frac{g^6}{8} c^2. \quad (1.174)$$

With the values of χ at the horizon crossing and at the minimum found above, we get m_A^2 in terms of the parameters c and W_0 :

$$m_A^{*2} = 1.84 \times 10^{-38} \frac{c^2}{W_0^6}, \quad (1.175)$$

$$m_A^{2\text{end}} = 3.05 \times 10^{-34} \frac{c^2}{W_0^6}, \quad (1.176)$$

$$\tilde{m}_A^2 = 1.93 \times 10^{-31} \frac{c^2}{W_0^6}, \quad (1.177)$$

which can therefore vary in a large range of values consistent with all experimental bounds. The rest of the masses do not present any significant change from the previous analysis in the dilaton

case without the gauging.

1.8 Conclusion

In this first project, we generalised the construction of new FI D-terms in $\mathcal{N} = 1$ supergravity that do not require the gauging of R-symmetry and preserve invariance under ordinary Kähler transformations. Their bosonic part is just linear in the D-auxiliary field with a multiplicative factor which is an arbitrary function of the gravitino mass, expressed as a functional of the chiral multiplets. We then used these terms to construct new models of D-term inflation. Considering just a $U(1)$ and the inflaton multiplet with a no-scale Kähler potential and constant superpotential, we restricted to a simple form of the function associated to the new FI D-term which is a single positive power of the gravitino mass up to an additive constant. This later dominates the inflationary period by an asymptotically de Sitter regime, because the gravitino mass vanishes asymptotically in this region. The resulting models are consistent with observational CMB data and share common properties with the Starobinsky R^2 model [33] on one hand and with the models of inflation by supersymmetry breaking on the other hand, where the inflaton is identified with the superpartner of the goldstino [15, 20]. Moreover, they predict a variable range of primordial gravitational waves that can be within experimental reach. After the end of inflation, the inflaton rolls down to the minimum of the potential which has a tuneable (tiny) vacuum energy and supersymmetry is broken by a combination of F and D-term VEVs. An interesting open problem is whether there exists a microscopic origin of these new FI D-terms, for instance within string theory.

Compactification on a linear dilaton background

2.1 Introduction

Little String Theory (LST) is a 6-dimensional non-gravitational theory, obtained for instance in type IIB or heterotic string theory by considering a stack of k coincident (Neveu-Schwarz) NS5-branes, in the limit of vanishing string coupling constant g_S [34, 35, 36, 37]. With closed string amplitudes being proportional to g_S , in this limit the branes and bulk dynamics decouple. However, and in contrast with the D-brane case, the gauge coupling of the effective $U(k)$ gauge theory on the NS5-branes is independent of g_S and therefore remains finite in the limit $g_S \rightarrow 0$. We are left with an interacting non-gravitational theory living on the NS5-branes.

Being a non-local and strongly coupled theory without any Lagrangian description, LST is easier studied through its 7-dimensional holographic dual, obtained in the near horizon limit of the NS5-branes. It is a weakly coupled string theory on the background $\mathbb{R}^{5,1} \times \mathbb{R}_y$, with the dilaton varying linearly in the coordinate y of the real line \mathbb{R}_y [38]. The main features of LST phenomenology can be captured in a simpler model where two dimensions of the bulk are compactified on T^2 , leading to a gravitational holographic dual of LST on $\mathbb{R}^{3,1} \times \mathbb{R}_y$ whose action is a simple graviton-dilaton model with a runaway scalar potential [39, 40].

The vanishing limit of the string coupling constant leads to an interesting phenomenological application of LST in the context of the hierarchy problem [40, 41]. The string scale can be indeed separated from the Planck mass to much lower energies, such as in the (multi)TeV region using an ultra-weak string coupling. The hierarchy problem then amounts to explain the smallness of the coupling [42]. This question has also been addressed more recently in the so-called clockwork mechanism [43, 44], which was shown to have as continuum limit the five-dimensional ($5D$) linear dilaton theory [45, 46, 47].

In order to obtain a finite string coupling and thus a four-dimensional Planck mass, the extra dimension \mathbb{R}_y is compactified on a segment S^1/\mathbb{Z}_2 . The dimensional reduction on a linear dilaton (LD) background reveals very distinguishable features. In particular the generic spectrum, such as that of the graviton, is a massless $4D$ zero mode with a flat wave-function along the extra dimension, followed by an infinite tower of Kaluza-Klein (KK) states starting from a mass gap fixed by the slope of the linear dilaton. In this thesis, we first extend this analysis to the KK gauge sector of the metric which has not been studied so far. It turns out that the situation is different from the scalar and spin-2 excitations: due to the non-trivial background, we show that the zero mode of the KK vector acquires a mass by absorbing the scalar radion, while its wave-function is no longer flat along the extra dimension but rather localised around the origin and

away exponentially suppressed. This result follows from a theoretical gauge symmetry analysis, which generalises the standard KK reduction in the more general case where the fields may depend on the extra coordinate. Within the gravitational sector, the reduction is performed in the ADM formalism [48], crucially taking into account the Gibbons-Hawking boundary terms at the end-points of the interval.

Since the holographic dual of LST is actually a string theory, one can study its effective supergravity theory, a construction which has been first carried out in [49, 50]. It relies on a $U(1)_R$ gauging of the $SU(2)$ R-symmetry of $\mathcal{N} = 2$, $D = 5$ supergravity coupled to one vector multiplet [51, 52, 53]. Besides minimal couplings between the gauge field and the fermions, the gauging generates fermion mass terms as well as a scalar potential, both highly constrained by supersymmetry. The family of the scalar potentials obtained in this way is parametrised by two independent parameters. The observation made in [49, 50] is that the vanishing of one parameter precisely reproduces the runaway potential leading to the LD background solution. Here we revisit this construction in order to show that the five-dimensional supergravity theory with this property is actually unique.

In addition to possible phenomenological implications mentioned above, the gravity dual of LST can also be used as a framework to study supersymmetry breaking, which will be the viewpoint adopted in the second part of this chapter. The crucial point is that the LD background preserves 1/2 of the original supersymmetries [50]. We shall show that the NS5-branes, already introduced at the bosonic level by the junction conditions, remain consistent in a supersymmetric context, namely they preserve the remaining supersymmetries, and this choice of branes is unique. Motivated by the massive vector field obtained in the KK reduction of the metric, we perform the full dimensional reduction of the bosonic sector of the supergravity action in order to find a similar mechanism in the Maxwell sector. Obtaining a second massive vector, one can arrange the massive (non KK) spectrum into a massive $4D$ $\mathcal{N} = 1$ spin-3/2 multiplet which contains half of the total degrees of freedom. With the two free parameters of the theory, the slope α of the LD background and the compactification radius L , one can then consider two different limits.

- The low energy limit $|\alpha| \rightarrow \infty$, $L \rightarrow 0$, where all masses are sent to infinity and only the massless spectrum remains, which we show to be described by an effective $D = 4$, $\mathcal{N} = 1$ supergravity. Moreover, such a truncation is consistent with a $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ orbifold projection, under which half of the degrees of freedom are assigned a \mathbb{Z}_2 -odd parity.
- The intermediate limit where α is kept finite and $L \rightarrow 0$, keeping all massive and massless zero modes and projecting out only the KK excitations. The possibility to describe a consistent $4D$ supergravity theory in this limit is not clear and remains a non-trivial open question. This would potentially lead to a very interesting example of partial supersymmetry breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ using only vector multiplets.

The Chapter 2 is organised as follows. In Section 2.2 we summarise our conventions and notations used throughout this chapter. In Section 2.3, we review the main characteristics of the five dimensional graviton-dilaton system, in the presence of a runaway scalar potential corresponding to a non-critical string, stressing some important points often left implicit in the litterature. The compactification of this model on a LD background is then performed in Section

2.4, shedding light in particular on the KK gauge sector. The minimal supersymmetrisation of this model, seen as the holographic dual of LST, is then introduced in Section 2.5. We first remind the formalism of $\mathcal{N} = 2$, $D = 5$ supergravity, together with its $U(1)_R$ gauging, before listing the different supergravity theories coupled to one vector multiplet whose gauging reproduces the scalar potential of LST and we show that they are all equivalent. We then study the supersymmetries preserved both by the background and by the NS5-branes sitting at the boundaries. The complete compactification down to $D = 4$ is presented in Section 2.6, where we highlight a similar Higgs mechanism for a second vector, and show that a $\mathcal{N} = 1$ supersymmetric theory can be obtained in the massless limit. Our conclusions are presented in Section 2.7. Finally, this chapter is related to two appendices, reviewing some important aspects of General Relativity on a manifold with boundaries (Appendix D) and then identifying the $5D$ supergravity studied here with the effective action of a (non-critical) heterotic string theory (Appendix E).

Our new results are presented in Sections 2.4, 2.5.3, 2.5.5 and 2.6, as well as in Appendix D.2.

2.2 Conventions and notations

In this second chapter, we start working in 5 spacetime dimensions before compactifying down to $4D$. We adopt the metric convention $(-, +, +, +, +)$, and write the five-dimensional Minkowski space coordinates as $x^M = (x^\mu, y)$. The indices used throughout the Chapter 2 are defined according to the following pattern:

$M, N \dots = 0, \dots, 3, 5$	$5D$ spacetime curved indices
$m, n \dots = 0, \dots, 3, 5$	$5D$ spacetime flat indices
$\mu, \nu \dots = 0, \dots, 3$	$4D$ spacetime curved indices
$a, b \dots = 0, \dots, 3$	$4D$ spacetime flat indices
$I, J \dots = 0, \dots, n_V$	vector field labels
$x, y \dots = 1, \dots, n_V$	scalar manifold curved indices
$\tilde{a}, \tilde{b} \dots = 1, \dots, n_V$	scalar manifold flat indices
$i, j \dots = 1, 2$	fundamental representation of $SU(2)_R$

Curved and flat indices in $5D$ ($4D$) are related through the fünfbein e_M^m (vierbein e_μ^a) according to

$$X_M = e_M^m X_m \quad (X_\mu = e_\mu^a X_a). \quad (2.1)$$

Similarly, curved and flat indices of the scalar manifold are related through the n_V -bein $f_x^{\tilde{a}}$ according to

$$\varphi^x = (f_x^{\tilde{a}})^{-1} \varphi^{\tilde{a}} = f_{\tilde{a}}^x \varphi^{\tilde{a}}. \quad (2.2)$$

The five-dimensional Dirac matrices γ^m satisfy the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \mathbb{1}_{4 \times 4}. \quad (2.3)$$

In particular,

$$(\gamma^5)^2 = (\gamma_5)^2 = 1. \quad (2.4)$$

Antisymmetrized products of γ matrices are defined with weight one, $\gamma^{m_1 \dots m_r} = \gamma^{[m_1 \dots m_r]}$. In particular,

$$\gamma^{mn} \equiv \frac{1}{2}(\gamma^m \gamma^n - \gamma^n \gamma^m). \quad (2.5)$$

Spinors in $D = 5$ dimensions can be equivalently described either in terms of unconstrained Dirac spinors, which have 4 complex components, or either in terms of pairs of spinors χ^i , $i = 1, \dots, \mathcal{N} = 2k$ satisfying a reality condition. The pairs are called symplectic since the position of the indices is raised and lowered according to

$$\chi^i = \Omega^{ij} \chi_j, \quad \chi_i = \chi^j \Omega_{ji}, \quad (2.6)$$

where Ω_{ij} is a $2k \times 2k$ matrix satisfying

$$\Omega^{ik} \Omega_{kj} = -\delta_j^i. \quad (2.7)$$

The Dirac and Majorana conjugates of a spinor χ^i are respectively defined by

$$\bar{\chi}^i \equiv (\chi_i)^\dagger \gamma^0, \quad (\chi^i)^C \equiv (\chi^i)^T C, \quad (2.8)$$

with C the charge conjugation matrix, satisfying in five dimensions $\gamma_M^T = C \gamma_M C^{-1}$. The symplectic-Majorana condition then imposes the Dirac conjugate of χ^i to be equal to its Majorana conjugate, namely:

$$\chi_i^\dagger \gamma^0 = (\chi^i)^T C. \quad (2.9)$$

Since (2.9) relates the components of χ^i to those of its complex conjugate, this is a reality condition which projects out half of the degrees of freedom of the k pairs of spinors, ending with $2 \times 4k$ real components. Both descriptions in terms of one Dirac spinor or one pair of symplectic-Majorana spinors are thus equivalent, both of them describing 8 real off-shell degrees of freedom. In practice however, only the symplectic formulation is used, since it makes explicit the action of the R-symmetry group $USp(\mathcal{N})$ in $D = 5$ dimensions.

For the case $\mathcal{N} = 2$ we are considering in this chapter, $\Omega_{ij} = \epsilon_{ij}$. Since ϵ_{ij} is an $SU(2)$ -invariant tensor, the indices $i, j = 1, 2$ are referred to the $SU(2)_R$ indices, therefore raised and lowered according to the NorthWest-SouthEast convention

$$\chi^i = \epsilon^{ij} \chi_j, \quad \chi_i = \chi^j \epsilon_{ji}, \quad (2.10)$$

and where we choose $\epsilon_{12} = \epsilon^{12} = 1 = -\epsilon_{21} = -\epsilon^{21}$. In particular, bilinears of fermions satisfy

$$\bar{\lambda}^i \chi_i = \bar{\chi}_i \lambda^i = -\bar{\chi}^i \lambda_i, \quad (2.11)$$

where the first equality is a standard Majorana flip in $D = 5$ dimension and the second one follows from (2.10). Let us finally highlight that in five dimensions, the bilinear quantities $\bar{\lambda}^i \chi_i$ and $\bar{\lambda}^i \gamma_\mu \chi_i$ built from symplectic-Majorana spinors are pure imaginary and real respectively.

2.3 The Linear Dilaton model

2.3.1 The 5D theory on $\mathbb{R}^{1,3} \times S^1/\mathbb{Z}_2$

The work carried out in this chapter is based on the five-dimensional dilaton-gravity theory whose action is given by

$$S = \int d^5x \sqrt{-g} e^{-\sqrt{3}\phi} \left[\frac{1}{2} \mathcal{R}^{(5)}[g_{MN}] + \frac{3}{2} (\partial\phi)^2 - \Lambda \right], \quad (2.12)$$

where g_{MN} is the five-dimensional metric in the string frame (not to be confused with the Einstein frame metric G_{MN} which will be introduced below), ϕ the dilaton field, and Λ a constant introducing a runaway dilaton potential, characteristic of non-critical string theory. Varying S with respect to ϕ and g^{MN} yields the equation of motion of the dilaton and the five-dimensional Einstein equations, respectively given by:

$$\sqrt{-g} e^{-\sqrt{3}\phi} \left\{ \frac{\sqrt{3}}{2} \mathcal{R}^{(5)} - \frac{3\sqrt{3}}{2} (\partial\phi)^2 + 3\Box_5\phi - \sqrt{3}\Lambda \right\} = 0, \quad (2.13)$$

$$\sqrt{-g} e^{-\sqrt{3}\phi} \left\{ R_{MN} - \frac{1}{2} g_{MN} \mathcal{R}^{(5)} + \sqrt{3} \nabla_M \partial_N \phi - \sqrt{3} g_{MN} g^{PQ} \nabla_P \partial_Q \phi + \frac{3}{2} g_{MN} (\partial\phi)^2 + g_{MN} \Lambda \right\} = 0. \quad (2.14)$$

One can easily check that these equations are solved by the five-dimensional Minkowski metric, in addition of a linearly varying dilaton along the fifth direction y , breaking the 5D Poincaré invariance into a 4D one,

$$g_{MN} = \eta_{MN}, \quad (2.15)$$

$$\phi = \alpha y, \quad (2.16)$$

provided that the bulk “cosmological constant” Λ and the parameter α are related by

$$\Lambda = -\frac{3}{2} \alpha^2. \quad (2.17)$$

In order to have a finite four-dimensional Planck mass, the fifth direction \mathbb{R}_y must be compactified, on a space chosen to be an interval S^1/\mathbb{Z}_2 of length L , in complete analogy with the Randall-Sundrum model [54]. The \mathbb{Z}_2 symmetry which must have the background therefore imposes to replace the solution (2.16) by

$$\phi = \alpha |y|. \quad (2.18)$$

Denoting here and all along this chapter differentiation with respect to the fifth coordinate y with a prime, we have $\phi' = \alpha \operatorname{sgn} y$ ¹ and $\phi'' = 2\alpha [\delta(y) - \delta(y - L)]$, so that boundary terms will arise from the terms $\Box_5\phi$ and $g^{PQ} \nabla_P \partial_Q \phi$ of the equations of motion. Consistency of the

1. The sign function is defined as yielding ± 1 depending of the sign of its argument. An ambiguity remains at 0, and it is worth noticing that as long as $\operatorname{sgn} 0 \neq 0$, no inconsistency appears and one can arbitrarily choose $\operatorname{sgn} 0 = \pm 1$.

equations of motion then requires to add boundary terms to the action (2.12), which is modified into:

$$S = \int d^5x \left\{ \sqrt{-g} e^{-\sqrt{3}\phi} \left[\frac{1}{2} \mathcal{R}^{(5)} [g_{MN}] + \frac{3}{2} (\partial\phi)^2 - \Lambda \right] - \sqrt{-g_1} e^{-\alpha_1\phi} V_1 \delta(y) - \sqrt{-g_2} e^{-\alpha_2\phi} V_2 \delta(y-L) \right\}, \quad (2.19)$$

with g_1 and g_2 the determinant of the induced metrics at the two four-dimensional boundaries. The additional terms describe branes located at the fixed points $y = 0$ and $y = L$ of the orbifold S^1/\mathbb{Z}_2 , and contain four parameters: α_1 and α_2 characterizing the nature of the branes, and their tensions V_1 and V_2 . As we are now going to show, these parameters can be fully determined by the classical equations of motion following from the action (2.19).

The dilaton equation of motion (2.13) is modified into

$$\sqrt{-g} e^{-\sqrt{3}\phi} \left\{ \frac{\sqrt{3}}{2} \mathcal{R}^{(5)} - \frac{3\sqrt{3}}{2} (\partial\phi)^2 + 3\Box_5\phi - \sqrt{3}\Lambda \right\} - \alpha_1 \sqrt{-g_1} e^{-\alpha_1\phi} V_1 \delta(y) - \alpha_2 \sqrt{-g_2} e^{-\alpha_2\phi} V_2 \delta(y-L) = 0, \quad (2.20)$$

while the five-dimensional Einstein equations split into equations for the 4-dimensional scalar g_{55} , the 4-dimensional metric $g_{\mu\nu}$ and the 4-dimensional vector $g_{\mu 5}$ as:

$$R_{55} - \frac{1}{2} g_{55} \mathcal{R}^{(5)} + \sqrt{3} \nabla_5 \partial_5 \phi - \sqrt{3} g_{55} g^{PQ} \nabla_P \partial_Q \phi + \frac{3}{2} g_{55} (\partial\phi)^2 + g_{55} \Lambda = 0, \quad (2.21)$$

$$\sqrt{-g} e^{-\sqrt{3}\phi} \left\{ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}^{(5)} + \sqrt{3} \nabla_\mu \partial_\nu \phi - \sqrt{3} g_{\mu\nu} g^{PQ} \nabla_P \partial_Q \phi + \frac{3}{2} g_{\mu\nu} (\partial\phi)^2 + g_{\mu\nu} \Lambda \right\} + g_{\mu\nu} \left\{ \sqrt{-g_1} e^{-\alpha_1\phi} V_1 \delta(y) + \sqrt{-g_2} e^{-\alpha_2\phi} V_2 \delta(y-L) \right\} = 0, \quad (2.22)$$

$$R_{\mu 5} - \frac{1}{2} g_{\mu 5} \mathcal{R}^{(5)} + \sqrt{3} \nabla_\mu \partial_5 \phi - \sqrt{3} g_{\mu 5} g^{PQ} \nabla_P \partial_Q \phi + \frac{3}{2} g_{\mu 5} (\partial\phi)^2 + g_{\mu 5} \Lambda = 0, \quad (2.23)$$

where greek indices $\mu, \nu \dots$ denote only 4D spacetime. The dilaton and 4-dimensional graviton equations of motion (2.20) and (2.22) can be respectively rewritten as:

$$\frac{\sqrt{3}}{2} \mathcal{R}^{(5)} - \frac{3\sqrt{3}}{2} (\partial\phi)^2 + 3\Box_5\phi - \sqrt{3}\Lambda - \alpha_1 g_{55}^{-\frac{1}{2}} e^{(\sqrt{3}-\alpha_1)\phi} V_1 \delta(y) - \alpha_2 g_{55}^{-\frac{1}{2}} e^{(\sqrt{3}-\alpha_2)\phi} V_2 \delta(y-L) = 0, \quad (2.24)$$

and

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}^{(5)} + \sqrt{3} \nabla_\mu \partial_\nu \phi - \sqrt{3} g_{\mu\nu} g^{PQ} \nabla_P \partial_Q \phi + \frac{3}{2} g_{\mu\nu} (\partial\phi)^2 + g_{\mu\nu} \Lambda + g_{\mu\nu} g_{55}^{-\frac{1}{2}} \left\{ e^{(\sqrt{3}-\alpha_1)\phi} V_1 \delta(y) + e^{(\sqrt{3}-\alpha_2)\phi} V_2 \delta(y-L) \right\} = 0. \quad (2.25)$$

It is then straightforward to check that they are still solved by $g_{MN} = \eta_{MN}$ and $\phi = \alpha|y|$,

provided that the bulk “cosmological constant” Λ and the parameter α are related by²

$$\Lambda = -\frac{3}{2}\alpha^2 \operatorname{sgn}^2 y. \quad (2.26)$$

In addition, the boundary terms at $y = 0$ and $y = L$ lead to four equations relating α , α_1 , α_2 , V_1 and V_2 ,

$$6\alpha = \alpha_1 V_1 \qquad 6\alpha = -\alpha_2 e^{(\sqrt{3}-\alpha_2)\alpha L} V_2, \quad (2.27)$$

$$2\sqrt{3}\alpha = V_1 \qquad 2\sqrt{3}\alpha = -e^{(\sqrt{3}-\alpha_2)\alpha L} V_2, \quad (2.28)$$

which are obviously solved by

$$V_1 = 2\sqrt{3}\alpha = -V_2, \quad (2.29)$$

$$\alpha_1 = \alpha_2 = \sqrt{3}. \quad (2.30)$$

Eq. (2.29) thus shows that consistency of the equations of motion requires a system of branes of opposite tensions to set at the fixed points $y = 0$ and $y = L$ of the S^1/\mathbb{Z}_2 orbifold, similarly to the Randall-Sundrum model.³ Eq. (2.30) tells us about the nature of these branes: from the action (2.19), one sees that $\alpha_1 = \alpha_2 = \sqrt{3}$ correspond to a brane tension $\propto \frac{1}{g_s^2}$, with $g_s^2 = e^{\sqrt{3}\phi}$ the string coupling constant. Therefore, the solution at the boundaries arising from the classical equations of motion consists of NS5-branes, as expected from the LST string theory approach.

Having found the nature of the branes sitting at the two boundaries, one can then move from the string frame to the Einstein frame metric by means of the Weyl transformation

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\phi} g_{MN}, \quad (2.31)$$

where G_{MN} is the Einstein frame metric and g_{MN} the string frame one. The Einstein frame bulk action $S_{\mathcal{M}}$ then reads

$$S_{\mathcal{M}} = S_{EH} + S_{dil} + S_{\Lambda}, \quad (2.32)$$

with the Einstein-Hilbert action S_{EH} , the dilaton action S_{dil} and the 5D cosmological constant action S_{Λ} given by:

$$S_{EH} = \frac{1}{2} \int d^5x \sqrt{-G} \mathcal{R}^{(5)}[G_{MN}], \quad (2.33)$$

$$S_{dil} = -\frac{1}{2} \int d^5x \sqrt{-G} G^{MN} \partial_M \phi \partial_N \phi, \quad (2.34)$$

$$S_{\Lambda} = - \int d^5x \sqrt{-G} e^{\frac{2}{\sqrt{3}}\phi} \Lambda. \quad (2.35)$$

On a bounded manifold, it is well-known that the Einstein-Hilbert action must be supplemented by the so-called Gibbons-Hawking (GH) boundary action S_{GH} , given in the Einstein frame by the integral over the boundary of the square root of the determinant of the induced metric on it

2. One sees here a first inconsistency the choice $\operatorname{sgn} 0 = 0$ would imply: a cosmological constant Λ constant everywhere except at $y = 0$ where it would vanish, while $\operatorname{sgn} 0 = \pm 1$ remain consistent.

3. In the string context, a negative contribution to the tension can be provided by orientifolds.

times the trace of its extrinsic curvature tensor⁴. Although the GH term is seldom considered in the literature dealing with this linear dilaton model, it is an important piece which will turn out to be crucial in the Kaluza-Klein reduction performed in Section 2.4. Thus, in addition to the bulk action (2.32), the five dimensional LD model presented here contains also a boundary action $S_{\partial\mathcal{M}}$ given by

$$S_{\partial\mathcal{M}} = \sum_{i=1}^2 (S_{NS5_i} + S_{GH_i}), \quad (2.36)$$

with, in the Einstein frame:

$$S_{NS5_i} = -V_i \int d^5x \sqrt{-g} e^{\frac{1}{\sqrt{3}}\phi} \delta(y - y_i), \quad (2.37)$$

$$S_{GH_i} = \int d^4x \sqrt{-g} K_i. \quad (2.38)$$

The indice i labels the two four-dimensional boundaries of \mathcal{M} , located at $y = y_i$. K_i is the trace of their extrinsic curvature tensor, g the determinant of the 4D induced metric on them, and the constants Λ , V_1 and V_2 have been computed in (2.17) and (2.29), and are given by $\Lambda = -\frac{3}{2}\alpha^2$, $V_1 = -V_2 = 2\sqrt{3}\alpha$.

The total 5D action we will work with in Section 2.4 is then

$$S = S_{\mathcal{M}} + S_{\partial\mathcal{M}}, \quad (2.39)$$

with $S_{\mathcal{M}}$ and $S_{\partial\mathcal{M}}$ respectively given by (2.32) and (2.36).

2.3.2 Spectrum of bosonic fields on a LD background

The aim of this subsection is to remind some general results regarding the spectrum of bosonic fields on a linear dilaton background, and then motivate the first part of the work carried out in this chapter, which will be described in Section 2.4. We start with the case of a bulk scalar and vector fields in five dimensions, an analysis which can be found for instance in [55, 56], before moving to the spin-0 and spin-1 spectrum of the 5D dilaton-gravity theory described in the previous subsection.

Scalar field on a LD background

Let us first consider a given 5D massless scalar field χ on a LD background, distinct from the dilaton field ϕ of the previous section. The string frame Lagrangian of such a scalar χ would be obtained by merely adding the standard kinetic term $-\frac{1}{2}(\partial\chi)^2$ in the bracket of the Lagrangian (2.12), which yields at quadratic order

$$\mathcal{L} = -\frac{1}{2}e^{-Qy} \partial_M \chi \partial^M \chi, \quad (2.40)$$

4. In the string frame, the background metric is trivial, so that the GH action does not modify the analysis on the nature of the branes carried out above and can thus be neglected. However, in the Einstein frame, second derivatives of the background metric now contain delta functions, and we need the GH term in order to cancel them and clearly identify the spectrum of the KK vector $G_{\mu 5}$, as done below.

where Q is a constant related to α in (2.16), but kept as an arbitrary parameter in this subsection. The equation of motion for χ following from (2.40) is easily found to be

$$\square_4 \chi + \chi'' - Q\chi' = 0. \quad (2.41)$$

Considering the mode expansion

$$\chi(x, y) = \sum_{n=0}^{\infty} f_{(n)}(y) e^{ip_n \cdot x}, \quad (2.42)$$

we get for the internal wave functions $f_{(n)}(y)$ of the Kaluza-Klein modes:

$$f_{(n)}''(y) - Qf_{(n)}'(y) - p_n^2 f_{(n)}(y) = 0, \quad \forall n \geq 0, \quad (2.43)$$

with $p_n^2 = -m_n^2$. The most general solution reads

$$f_{(n)}(y) = Ae^{x+y} + Be^{x-y}, \quad (2.44)$$

where A and B are two constants, and x_{\pm} are given by $x_{\pm} = \frac{Q \pm \sqrt{Q^2 - 4m_n^2}}{2}$. Imposing Neumann boundary conditions at $y = 0$ and $y = L$, $\partial_5 f_{(n)}(y)|_{y=0,L} = 0$, it is then easy to see that the lowest mode compatible with the boundary conditions is massless, $m_0 = 0$, with a constant internal wave function, $f_0(y) = \text{constant}$. We then have excited modes with masses $m_n^2 = \left(\frac{n\pi}{L}\right)^2 + \frac{Q^2}{4}$, $\forall n > 0$, with wavefunctions given by

$$f_{(n)}^N(y) = e^{\frac{Q}{2}y} \left(\sin \frac{n\pi y}{L} - \frac{2n\pi}{QL} \cos \frac{n\pi y}{L} \right), \quad \forall n > 0, \quad (2.45)$$

up to an overall normalisation constant, irrelevant for the discussion here. In the case of Neumann boundary conditions, a 5D massless bulk scalar therefore gives rise, from the 4D point of view, to a single massless zero mode with constant wavefunction, followed by discrete KK excited states on top of a mass gap fixed by Q :

$$\begin{cases} m_0 &= 0, \\ m_n^2 &= \left(\frac{n\pi}{L}\right)^2 + \frac{Q^2}{4}, \quad \forall n > 0, \end{cases} \quad (2.46)$$

which is a distinctive feature of the linear dilaton background.

Vector field on a LD background

A very similar analysis can be carried out for a given 5D massless bulk vector field A_M in the LD background. The equations of motion for A_M , following from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} e^{-Qy} F_{MN} F^{MN} = -\frac{1}{2} e^{-Qy} \left(\partial_M A_N \partial^M A^N - \partial_M A_N \partial^N A^M \right), \quad (2.47)$$

read

$$\partial_M F^{MN} - QF^{5N} = 0, \quad (2.48)$$

which split into:

$$\square_4 A_5 - \partial_5 \partial_\mu A^\mu = 0, \quad (2.49)$$

$$\square_4 A_\nu + A_\nu'' - Q A_\nu' - \partial_\nu \left(\partial_\mu A^\mu + \partial_5 A^5 - Q A^5 \right) = 0. \quad (2.50)$$

Similarly to the scalar case described above, we expand A_μ in modes

$$A_\mu(x, y) = \sum_{n=0}^{\infty} f_{(n)}(y) A_\mu^{(n)}(x), \quad (2.51)$$

and impose in addition the gauge condition

$$\partial_5 A^5 = Q A_5, \quad (2.52)$$

which in the flat case $Q = 0$ reduces to the standard gauge of toroidal KK compactification where A_5 is a function of x only. For the zero mode with a flat internal wave function $f_0(y) = \text{constant}$, one gets the equation of motion

$$\square_4 A_\mu^{(0)}(x) - \partial_\nu \partial_\mu A^{\mu(0)}(x) = 0, \quad (2.53)$$

which describes a massless vector with its remaining 4-dimensional gauge symmetry. Applying ∂^ν to (2.50), we see that the KK modes $A_\mu^{(n)}$, $n > 0$, satisfy $\partial_\mu A^{\mu(n)} = 0$, so that the differential equation satisfied by the internal profiles $f_{(n)}(y)$ reads:

$$f_{(n)}''(y) - Q f_{(n)}'(y) - p_n^2 f_{(n)}(y) = 0, \quad \forall n \geq 0, \quad (2.54)$$

with $p_n^2 = -\square_4 A_\mu^{(n)}$. This is the same equation as obtained previously in the scalar case, and we thus conclude that the spin-1 KK masses are given by:

$$m_n^2 = \left(\frac{n\pi}{L} \right)^2 + \frac{Q^2}{4}, \quad \forall n > 0. \quad (2.55)$$

Regarding the scalar A_5 , the gauge condition (2.52) imposes the zero mode of A_5 to be of the form $A_5^{(0)}(x, y) = e^{Qy} \tilde{A}_5(x)$, describing a massless 4-dimensional scalar as follows from the equation of motion (2.49).

The dimensional reduction of a given 5-dimensional vector field on a LD background therefore leads to a 4-dimensional massless scalar, together with a 4-dimensional massless vector followed by massive KK vector excitations, as in the toroidal case. The LD background prints distinguishable features both on the scalar sector, through the exponential profile of its zero mode, as well as in the vector sector, through the mass gap above which the KK tower of massive states starts.

Dilaton-gravity sector on a LD background

In the five-dimensional LD model introduced in Section 2.3.1, the spectrum for the four-dimensional graviton has been computed in [40]. Starting from the 5D Einstein-frame metric

parametrisation

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|} \begin{pmatrix} \eta_{\mu\nu} + h_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.56)$$

the spectrum of the gravitational excitations $h_{\mu\nu}$, for Neumann boundary conditions, has been found to be the same as the ones described in the above subsections: a massless zero mode with wavefunction independent of y , followed by discrete KK modes with masses $m_n^2 = (\frac{n\pi}{L})^2 + \frac{3}{4}\alpha^2$, $\forall n > 0$.

The spectrum analysis in the scalar sector has been carried out in [42]. Putting the 4D vector and tensor fluctuations to zero, the metric parametrisation considered here, at linear order in the scalars, is of the form

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|} \begin{pmatrix} (1 + 2\sigma)\eta_{\mu\nu} & 0 \\ 0 & 1 + 2r \end{pmatrix}, \quad (2.57)$$

where σ and r are respectively the trace of the 4D metric excitations $h_{\mu\nu}$ and the radion G_{55} . A crucial point to notice is that the three scalar fluctuations $\delta\phi$, σ and r are not independent, but instead related by the following two constraints [57]:

$$r + 2\sigma = 0, \quad (2.58)$$

$$\sqrt{3}\alpha r + 3 \operatorname{sgn} y \sigma' + \alpha \delta\phi = 0. \quad (2.59)$$

The first one is the usual constraint on the trace of the metric tensor, corresponding to the Weyl transformation one has to perform on the four dimensional metric in order to bring its kinetic term into a canonical form. The second one is only relevant in the case of a LD background⁵: in the general case $\alpha \neq 0$, it implies that only one linear combination of the scalars r and $\delta\phi$ is dynamical [58], while the orthogonal combination can be eliminated by a gauge choice on part of the 5-dimensional diffeomorphisms, and is thus unphysical. The spectrum for the remaining physical scalar is given in [42]. Although the analysis has been done in the more general case where the radion is stabilised, the unstabilised case we are considering here can be easily recovered: one finds exactly the same spectrum as for the 4D metric, namely a massless zero mode with wavefunction independent of y , followed by massive KK excitations with masses $m_n^2 = (\frac{n\pi}{L})^2 + \frac{3}{4}\alpha^2$, $\forall n > 0$.

So far two points were left out in the literature: what is happening to the vanishing 0-mode scalar degree of freedom, which must be recovered as a physical degree of freedom in the limit $\alpha \rightarrow 0$, as well as the analysis for the Kaluza-Klein vector $G_{\mu 5}$? The aim of the next section is to clarify these two points. We will show that the zero mode of the unphysical scalar is actually absorbed by the zero mode of the KK vector, providing it with a mass via a gauge-fixing that is reminiscent of the Stückelberg term. To this purpose, we first have to find the most general parametrisation for the metric tensor G_{MN} including the KK vector $G_{\mu 5}$, which reduces to (2.56) or (2.57) in the relevant limits, and whose components transform appropriately under four-dimensional diffeomorphisms. We also want this parametrisation to be valid not only at

5. In the flat case $\alpha = 0$, it simply imposes $r' = 0$, which is the usual gauge choice for the radion in standard Kaluza-Klein reduction on a circle.

the linear level in the scalars, but to all orders in order to be able to find not only the scalar spectrum but also the full scalar potential of the dimensionally reduced 4D theory.

2.4 Kaluza-Klein reduction on a linear dilaton background

2.4.1 4D residual gauge symmetry

The five-dimensional Einstein frame metric G_{MN} is written, in the most general case, as function of x and y according to

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & G_{\mu 5} \\ G_{\nu 5} & G_{55} \end{pmatrix} (x, y). \quad (2.60)$$

In order to parametrise the form of the metric as well as its y dependence compatible with the dimensional reduction $D = 5 \rightarrow D = 4$, we start from the 5D diffeomorphism transformations, with parameter $\xi^P = (\xi^\mu(x, y), \xi^5(x, y))$. Under ξ^P , the transformation of G_{MN} ,

$$\delta G_{MN} = \xi^P \partial_P G_{MN} + G_{MP} \partial_N \xi^P + G_{NP} \partial_M \xi^P, \quad (2.61)$$

splits for $G_{\mu\nu}$, $G_{\mu 5}$ and G_{55} into:

$$\delta G_{\mu\nu} = \xi^\rho \partial_\rho G_{\mu\nu} + 2G_{\rho(\mu} \partial_{\nu)} \xi^\rho + \xi^5 \partial_5 G_{\mu\nu} + 2G_{5(\mu} \partial_{\nu)} \xi^5, \quad (2.62)$$

$$\delta G_{\mu 5} = \xi^\nu \partial_\nu G_{\mu 5} + G_{\mu\nu} \partial_5 \xi^\nu + G_{5\nu} \partial_\mu \xi^\nu + \xi^5 \partial_5 G_{\mu 5} + G_{\mu 5} \partial_5 \xi^5 + G_{55} \partial_\mu \xi^5, \quad (2.63)$$

$$\delta G_{55} = \xi^\mu \partial_\mu G_{55} + 2G_{\mu 5} \partial_5 \xi^\mu + \xi^5 \partial_5 G_{55} + 2G_{55} \partial_5 \xi^5. \quad (2.64)$$

Let us first consider the 4D diffeomorphisms, parametrised by the 4-vector $\xi^\mu(x, y)$. From (2.62), one sees that $G_{\mu\nu}$ already transforms as a rank-2 tensor under ξ^μ , namely

$$\delta_{\xi^\rho} G_{\mu\nu} = \xi^\rho \partial_\rho G_{\mu\nu} + 2G_{\rho(\mu} \partial_{\nu)} \xi^\rho. \quad (2.65)$$

The transformation of $G_{\mu 5}$ reads:

$$\delta_{\xi^\nu} G_{\mu 5} = \xi^\nu \partial_\nu G_{\mu 5} + G_{5\nu} \partial_\mu \xi^\nu + G_{\mu\nu} \partial_5 \xi^\nu. \quad (2.66)$$

The last term, being independent of $G_{\mu 5}$, is a shift in the transformation which can be used in order to gauge fix to zero the KK excitations of $G_{\mu 5}$, as long as $\partial_5 \xi^\nu \neq 0$. Since all functions $\xi^\nu(x, y)$ which are not constant in y can be used for this purpose, we end up with a residual 4D diffeomorphisms invariance parametrised by all functions ξ^μ constant in y . Under such $\xi^\mu(x)$, one sees that $G_{\mu 5}$ and G_{55} transform indeed correctly as a 4D vector and a Lorentz scalar respectively.

We next turn to the $U(1)_{KK}$ transformations, parametrised by the 4D Lorentz scalar function $\xi^5(x, y)$. Let us first recall that the equations of motion are solved, in the Einstein frame, by the background metric

$$\bar{G}_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|} \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.67)$$

We thus define the radion r , which is the fluctuation of G_{55} around the background solution through:

$$G_{55}(x, y) = e^{-\frac{2}{\sqrt{3}}\alpha|y|} e^{2r(x, y)}. \quad (2.68)$$

The ξ^5 part of the transformation (2.64) on r then reads:

$$\delta_{\xi^5} r = \partial_5 \xi^5 - \frac{\alpha}{\sqrt{3}} \operatorname{sgn} y \xi^5 + \xi^5 \partial_5 r. \quad (2.69)$$

Defining

$$K_\mu \equiv \frac{G_{\mu 5}}{G_{55}}, \quad (2.70)$$

and using the transformations (2.63) for $G_{\mu 5}$ and (2.64) for G_{55} , one finds:

$$\delta_{\xi^5} K_\mu = \partial_\mu \xi^5 - K_\mu \partial_5 \xi^5 + \xi^5 \partial_5 K_\mu. \quad (2.71)$$

In order for K_μ to transform as a gauge field under $U(1)_{KK}$, and assuming that the x and y dependences of ξ^5 and K_μ factorize, one sees that they must satisfy

$$\xi^5 = f(y) \tilde{\xi}^5(x), \quad K_\mu = f(y) \tilde{K}_\mu(x), \quad (2.72)$$

where $f(y)$ is an arbitrary function of y , and $\tilde{\xi}^5(x)$ and $\tilde{K}_\mu(x)$ two arbitrary functions of x . Similarly, defining

$$g_{\mu\nu} \equiv \frac{G_{\mu\nu}}{G_{55}} - K_\mu K_\nu, \quad (2.73)$$

we have

$$\delta_{\xi^5} g_{\mu\nu} = \xi^5 \partial_5 g_{\mu\nu} - 2g_{\mu\nu} \partial_5 \xi^5. \quad (2.74)$$

In a similar way that we have used a subset of the functions $\xi^\mu(x, y)$ to gauge fix the KK modes of the vector $G_{\mu 5}$, one can now use a subset of the functions $\xi^5(x, y)$ in order to gauge fix a series of KK modes of the scalars. From the $5D$ point of view, two scalar fields are present: the radion r and the dilaton $\delta\phi$, whose transformations with respect to ξ^5 are respectively given by:

$$\delta_{\xi^5} r = \partial_5 \xi^5 - \frac{\alpha}{\sqrt{3}} \operatorname{sgn} y \xi^5 + \xi^5 \partial_5 r, \quad (2.75)$$

$$\delta_{\xi^5}(\delta\phi) = \alpha \operatorname{sgn} y \xi^5 + \xi^5 \partial_5 \delta\phi. \quad (2.76)$$

One can then choose to gauge fix the excitations of r , of $\delta\phi$, or of a combination of both. All possibilities are obviously physically equivalent, and our choice will be motivated by the requirement that the $4D$ metric must remain inert under the residual gauge freedom which has not been fixed. The canonically normalised $4D$ metric $\tilde{g}_{\mu\nu}$ is related to $g_{\mu\nu}$ by

$$g_{\mu\nu} = e^{-3r} \tilde{g}_{\mu\nu}. \quad (2.77)$$

From the transformations (2.74) and (2.75), one finds the transformation of $\tilde{g}_{\mu\nu}$ under ξ^5 ,

$$\delta_{\xi^5} \tilde{g}_{\mu\nu} = \partial_5 \xi^5 \tilde{g}_{\mu\nu} - \sqrt{3} \alpha \operatorname{sgn} y \xi^5 \tilde{g}_{\mu\nu} + \xi^5 \partial_5 \tilde{g}_{\mu\nu}. \quad (2.78)$$

This transformation motivates us to consider the linear combination $r - \frac{2}{\sqrt{3}}\delta\phi$, which transforms under ξ^5 according to:

$$\delta_{\xi^5} \left(r - \frac{2}{\sqrt{3}}\delta\phi \right) = \partial_5 \xi^5 - \sqrt{3}\alpha \operatorname{sgn} y \xi^5 + \xi^5 \partial_5 \left(r - \frac{2}{\sqrt{3}}\delta\phi \right). \quad (2.79)$$

Again $\forall \xi^5(x, y) \neq e^{\sqrt{3}\alpha|y|}\tilde{\xi}^5(x)$, the two first terms of the transformation are non zero constant shifts, which can thus be used to gauge fix the KK modes of the scalar $r - \frac{2}{\sqrt{3}}\delta\phi$. Therefore, we end up with a residual gauge freedom which cannot be fixed for now, associated with the parameter

$$\xi^5 \equiv e^{\sqrt{3}\alpha|y|}\tilde{\xi}^5(x). \quad (2.80)$$

Under such ξ^5 , one immediately verifies that the constant terms in the transformation (2.78) of $\tilde{g}_{\mu\nu}$ vanish. Then, the zero mode of $\tilde{g}_{\mu\nu}$, independent of y , does not transform under ξ^5 as required. Having found the form of ξ^5 , we deduce from (2.72) that

$$K_\mu = e^{\sqrt{3}\alpha|y|}\tilde{K}_\mu(x). \quad (2.81)$$

The final parametrisation of the 5D metric G_{MN} therefore reads:

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|} e^{2r} \begin{pmatrix} e^{-3r}\tilde{g}_{\mu\nu}(x) + e^{2\sqrt{3}\alpha|y|}\tilde{K}_\mu\tilde{K}_\nu(x) & e^{\sqrt{3}\alpha|y|}\tilde{K}_\mu(x) \\ e^{\sqrt{3}\alpha|y|}\tilde{K}_\nu(x) & 1 \end{pmatrix}, \quad (2.82)$$

whose inverse is given by:

$$G^{MN} = e^{\frac{2}{\sqrt{3}}\alpha|y|} e^{-2r} \begin{pmatrix} e^{3r}\tilde{g}^{\mu\nu}(x) & -e^{\sqrt{3}\alpha|y|}e^{3r}\tilde{K}^\mu(x) \\ -e^{\sqrt{3}\alpha|y|}e^{3r}\tilde{K}^\nu(x) & 1 + e^{2\sqrt{3}\alpha|y|}e^{3r}\tilde{K}^2(x) \end{pmatrix}, \quad (2.83)$$

with $\tilde{K}^\mu = \tilde{g}^{\mu\nu}\tilde{K}_\nu$, $\tilde{K}^2 = \tilde{g}^{\lambda\sigma}\tilde{K}_\lambda\tilde{K}_\sigma$. The metric G_{MN} thus reduces in the appropriate limits to the two forms (2.56) or (2.57) previously studied in the literature, namely by turning off the spin-1 excitations, as well as the scalar or the graviton fluctuations.

At that point, let us summarise the approach followed here. Using the gauge freedom associated to the $\xi^\mu(x, y)$, we first set to zero the KK modes of the vector $G_{\mu 5}$, ending with a residual 4D diffeomorphism invariance associated with the vectors $\xi^\mu(x)$. Under such $\xi^\mu(x)$, all the fields $G_{\mu\nu}$, $G_{\mu 5}$ and G_{55} respectively transform as a 4D rank-2 tensor, Lorentz vector and Lorentz scalar. Next, we have used the gauge freedom associated to the $\xi^5(x, y)$ in order to get rid of the KK modes of the scalar $r - \frac{2}{\sqrt{3}}\delta\phi$. The remaining $U(1)_{KK}$ gauge transformation is associated with the scalar function $\xi^5 \equiv e^{\sqrt{3}\alpha|y|}\tilde{\xi}^5(x)$, under which the different fields transform according to:

$$\delta_{\xi^5}\tilde{g}_{\mu\nu} = \xi^5\partial_5\tilde{g}_{\mu\nu}, \quad (2.84)$$

$$\delta_{\xi^5}\tilde{K}_\mu = \partial_\mu\tilde{\xi}^5(x), \quad (2.85)$$

$$\delta_{\xi^5}r = \frac{2\alpha}{\sqrt{3}}\operatorname{sgn} y \xi^5 + \xi^5\partial_5 r, \quad (2.86)$$

$$\delta_{\xi^5}(\delta\phi) = \alpha \operatorname{sgn} y \xi^5 + \xi^5\partial_5\delta\phi. \quad (2.87)$$

Denoting by φ_- and φ_+ the orthogonal combinations

$$\varphi_- \equiv r - \frac{2}{\sqrt{3}}\delta\phi, \quad \varphi_+ \equiv r + \frac{1}{\sqrt{3}}\delta\phi, \quad (2.88)$$

their transformations under ξ^5 are given by:

$$\delta_{\xi^5}\varphi_- = \xi^5\partial_5\varphi_-, \quad (2.89)$$

$$\delta_{\xi^5}\varphi_+ = \sqrt{3}\alpha \operatorname{sgn} y \xi^5 + \xi^5\partial_5\varphi_+. \quad (2.90)$$

The shift in the transformation of φ_+ resembles that of a Goldstone mode, and can be used in order to eliminate it through a suitable $U(1)_{KK}$ gauge transformation. The breaking of the remaining $U(1)_{KK}$ invariance will be reflected in the emergence of a mass term for the KK vector K_μ in a manner that is reminiscent of the Stückelberg term, which is why we will be referring to φ_+ as the Stückelberg field in the following. The aim of the next two subsections is to compute the dimensionally reduced action in order to clarify the origin of the mass term for the KK vector, and then find the scalar potential associated to the zero mode of the remaining physical scalar φ_- .

At this point, it is instructive to move from the Einstein to the string frame in order to get more physical intuition about the fields φ_+ and φ_- . The Einstein frame metric (2.82), given in the basis (φ_-, φ_+) by

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|} e^{\frac{2}{3}(\varphi_- + 2\varphi_+)} \begin{pmatrix} e^{-(\varphi_- + 2\varphi_+)} \tilde{g}_{\mu\nu}(x) + e^{2\sqrt{3}\alpha|y|} \tilde{K}_\mu \tilde{K}_\nu(x) & e^{\sqrt{3}\alpha|y|} \tilde{K}_\mu(x) \\ e^{\sqrt{3}\alpha|y|} \tilde{K}_\nu(x) & 1 \end{pmatrix}, \quad (2.91)$$

can be brought, by means of the 5D Weyl transformation (2.31), to the string frame metric

$$G_{MN}^S = \begin{pmatrix} g_{\mu\nu}^S(x) + e^{2\varphi_+} e^{2\sqrt{3}\alpha|y|} \tilde{K}_\mu \tilde{K}_\nu(x) & e^{2\varphi_+} e^{\sqrt{3}\alpha|y|} \tilde{K}_\mu(x) \\ e^{2\varphi_+} e^{\sqrt{3}\alpha|y|} \tilde{K}_\nu(x) & e^{2\varphi_+} \end{pmatrix}, \quad (2.92)$$

from which we deduce that φ_+ is actually the string frame radion. The 4D string frame metric $g_{\mu\nu}^S$ is related to the 4D Einstein frame metric $\tilde{g}_{\mu\nu}$ by the 4D Weyl transformation

$$g_{\mu\nu}^S = e^{-\varphi_-} \tilde{g}_{\mu\nu}. \quad (2.93)$$

Regarding φ_- , we remind that in our normalisation in which the 5D action reads

$$S^{(5)} = \int d^5x \sqrt{-G^S} e^{-\sqrt{3}\delta\phi} \frac{1}{2} \left[\mathcal{R}^{(5)}[G^S] + 3(\partial\delta\phi)^2 + \dots \right], \quad (2.94)$$

the 4D dilaton $\delta\phi_4$ is defined such that the 4D action is

$$S^{(4)} = \int d^4x \sqrt{-g^S} e^{-\sqrt{3}\delta\phi_4} \frac{1}{2} \left[\mathcal{R}^{(4)}[g^S] + 3(\partial\delta\phi_4)^2 + \dots \right]. \quad (2.95)$$

From the above equation then it follows that φ_- is simply proportionnal to the 4D dilaton according to

$$\varphi_- = -\sqrt{3}\delta\phi_4. \quad (2.96)$$

2.4.2 Stückelberg “mechanism”

Let us now compute the four-dimensional effective bosonic action from the five-dimensional one introduced in Section 2.3.1, whose structure is reminded here for convenience:

$$S = S_{EH} + S_{dil} + S_{\Lambda} + \sum_{i=1}^2 (S_{NS5_i} + S_{GH_i}). \quad (2.97)$$

The three first terms correspond to the bulk action given in (2.33)-(2.35), while the last two are the boundary terms given in (2.37)-(2.38). The computation of the dilaton kinetic term S_{dil} , cosmological constant S_{Λ} and NS5-brane contributions S_{NS5_i} is straightforward. Using the general metric decomposition found in (2.82), its inverse (2.83) and $\phi = \alpha|y| + \delta\phi$, we get:

$$S_{dil} = -\frac{1}{2} \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ e^{-3r} \alpha^2 + (\alpha \operatorname{sgn} y K_{\mu} - \partial_{\mu} \delta\phi)^2 \right. \\ \left. + (e^{-3r} + \tilde{K}^2) \left((\delta\phi')^2 + 2\alpha \operatorname{sgn} y \delta\phi' \right) - 2K^{\mu} \partial_{\mu} \delta\phi \delta\phi' \right\}, \quad (2.98)$$

$$S_{\Lambda} = \frac{3}{2} \alpha^2 \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} e^{-r + \frac{2}{\sqrt{3}} \delta\phi}, \quad (2.99)$$

$$S_{NS5_i} = \pm 2\sqrt{3}\alpha \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} e^{-2r + \frac{1}{\sqrt{3}} \delta\phi} \delta(y - y_i). \quad (2.100)$$

The computation of the gravitational action $S_G \equiv S_{EH} + S_{GH}$ is more involved. We perform the analysis in analogy to the ADM framework, splitting the five dimensional spacetime \mathcal{M} into timelike slices of constant y . The general framework of a $d + 1$ spacetime decomposition with boundaries is reviewed in Appendix D.1, and the particular computation for the metric (2.82) is detailed in Appendix D.2. The total gravitational action (D.2.20), up to quadratic order in K_{μ} , reads:

$$S_G = \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ e^{-3r} \left[2\alpha^2 + 2\sqrt{3}\alpha \operatorname{sgn} y r' + \frac{3}{2} (r')^2 \right] \right. \\ \left. - \frac{3}{2} \tilde{K}^{\mu} \partial_{\mu} r' - \frac{1}{4} e^{3r} \tilde{g}^{\mu\nu} \tilde{g}^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \left(\alpha \operatorname{sgn} y K_{\mu} - \frac{\sqrt{3}}{2} \partial_{\mu} r \right)^2 \right\}, \quad (2.101)$$

where we have ignored the term $\tilde{K}^{\mu} \partial_{\mu} (r' r)$ which is irrelevant for the discussion here, used (2.81) to write $(K^2)' = 2\sqrt{3}\alpha \operatorname{sgn} y K^2$, and finally arranged the radion kinetic term, the mass term for K_{μ} and their mixing into a perfect square. From (2.98) and (2.101), one sees that the dilaton and gravitational actions contain the terms

$$S_{dil} + S_G \supset \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ -\frac{3}{2} \left[\alpha \operatorname{sgn} y K_{\mu} - \frac{1}{\sqrt{3}} \partial_{\mu} \varphi_+ \right]^2 - \frac{1}{4} (\partial_{\mu} \varphi_-)^2 \right\} \quad (2.102)$$

where we have introduced back the combinations φ_- and φ_+ defined in (2.88). Using the gauge transformation (2.90) to fix the Stückelberg field φ_+ to zero, one immediately reads the mass term for the $U(1)_{KK}$ boson K_{μ} :

$$m_{U(1)_{KK}}^2 = \frac{3}{2} \alpha^2. \quad (2.103)$$

The dimensional reduction from five to four dimensions of the metric tensor on a linear dilaton

background therefore provides a very peculiar spectrum, parametrised by two parameters: the slope of the linear dilaton α , and the radius of compactification L . In the KK excitation modes, the situation is similar to the flat case ($\alpha = 0$): a scalar combination φ_- of the radion and the dilaton as well as the $U(1)_{KK}$ vector K_μ have infinite towers of states which can be gauge fixed to zero, providing three additional polarisations to the excitation modes of the 4D graviton, which acquire masses $\propto 1/L$. The difference from the flat case comes from the mass gap above the zero mode, on top of which the tower of KK states starts. Proportional to α , this gap is a characteristic feature of the LD background, and vanishes in the flat limit $\alpha \rightarrow 0$, thus recovering the usual case of toroidal compactification. In the zero mode sector, the 4D graviton as well as φ_- have a massless zero mode. The zero mode of the scalar orthogonal combination φ_+ is absorbed by the zero mode of K_μ , which acquires a mass $\propto \alpha$, in sharp contrast with the flat case where all the zero modes of the different components of the metric remain massless. The main result here is thus that the residual $U(1)_{KK}$ symmetry, which remains in standard KK compactification on a flat background $\alpha = 0$, is here broken at a scale $m_{U(1)_{KK}}^2 = \frac{3}{2}\alpha^2$ fixed by the slope α of the LD background.

One may wonder if the low-energy limit that corresponds to decouple all massive states, not only the KK modes, by sending formally the LD slope $|\alpha| \rightarrow \infty$, corresponds to an orbifold reduction. Indeed, the y -parity projects out the KK vector boson K_μ but keeps the dilaton and radion which should also be projected, as it plays the role of the longitudinal polarisation of K_μ (in the string frame). A possible answer is to extend the orbifold by the T-duality which in the string frame inverts the radion and thus makes φ_+ odd. Consistency of the theory then requires to add also the NS antisymmetric tensor B_{MN} since the vector $B_{5\mu}$ is exchanged with $G_{5\mu}$ under T-duality (to linear order). This will be done indeed in the supersymmetric case that we study in the following.

2.4.3 Effective scalar potential

Having found the mass term for the KK vector K_μ as well as its associated Stückelberg field, we can now focus on the scalar sector, putting $K_\mu = 0$. Combining the results found above, the total scalar action, before gauge fixing φ_+ to zero, is given by:

$$\begin{aligned}
S_{\text{scalar}} = & \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ -\frac{1}{2}(\partial_\mu \delta\phi)^2 - \frac{3}{4}(\partial_\mu r)^2 \right. \\
& + e^{-3r} \left[\frac{3}{2}\alpha^2 \left(1 + e^{2(r+\frac{1}{\sqrt{3}}\delta\phi)} \right) - 2\sqrt{3}\alpha e^{r+\frac{1}{\sqrt{3}}\delta\phi} (\delta(y) - \delta(y-L)) \right. \\
& \left. \left. + \frac{3}{2}(r')^2 - \frac{1}{2}(\delta\phi')^2 + 2\sqrt{3}\alpha \operatorname{sgn} yr' - \alpha \operatorname{sgn} y \delta\phi' \right] \right\}. \quad (2.104)
\end{aligned}$$

Varying it with respect to $\delta\phi$ and r , we respectively get the equations of motion:

$$\begin{aligned} \square_4 \quad & \delta\phi + e^{-3r} \delta\phi'' - 3\alpha \operatorname{sgn} y e^{-3r} \left(r + \frac{1}{\sqrt{3}} \delta\phi \right)' - 3e^{-3r} r' \delta\phi' \\ & - \sqrt{3}\alpha^2 e^{-3r} \left(1 - e^{2(r+\frac{1}{\sqrt{3}}\delta\phi)} \right) + 2\alpha e^{-3r} \left(1 - e^{r+\frac{1}{\sqrt{3}}\delta\phi} \right) (\delta(y) - \delta(y-L)) = 0, \end{aligned} \quad (2.105)$$

$$\begin{aligned} \square_4 \quad & r - 2e^{-3r} r'' + 2\sqrt{3}\alpha \operatorname{sgn} y e^{-3r} \left(r + \frac{1}{\sqrt{3}} \delta\phi \right)' + e^{-3r} ((\delta\phi')^2 + 3(r')^2) \\ & + \alpha^2 e^{-3r} \left(1 - e^{2(r+\frac{1}{\sqrt{3}}\delta\phi)} \right) - \frac{8}{\sqrt{3}}\alpha e^{-3r} \left(1 - e^{r+\frac{1}{\sqrt{3}}\delta\phi} \right) (\delta(y) - \delta(y-L)) = 0. \end{aligned} \quad (2.106)$$

Integrating these two equations on a infinitesimal interval $[y_i - \epsilon, y_i + \epsilon]$, where y_i denotes the location of the two branes, we find the jump conditions for the derivatives of the scalar fluctuations $\delta\phi$ and r :

$$\Delta\delta\phi' = -2\alpha \left(1 - e^{r+\frac{1}{\sqrt{3}}\delta\phi} \right), \quad \Delta r' = -\frac{4\alpha}{\sqrt{3}} \left(1 - e^{r+\frac{1}{\sqrt{3}}\delta\phi} \right), \quad (2.107)$$

where $\Delta X \equiv X(y_i + \epsilon) - X(y_i - \epsilon)$. In terms of the fields φ_- and φ_+ introduced in (2.88), the equations of motion (2.105) and (2.106) yields for φ_- :

$$\begin{aligned} \square_4 \quad & \varphi_- - 2e^{-3r} \varphi_+'' + 4\sqrt{3}\alpha \operatorname{sgn} y e^{-3r} \varphi_+' + 3e^{-3r} \varphi_+'^2 + 3\alpha^2 e^{-3r} (1 - e^{2\varphi_+}) \\ & - 4\sqrt{3}\alpha e^{-3r} (1 - e^{\varphi_+}) (\delta(y) - \delta(y-L)) = 0, \end{aligned} \quad (2.108)$$

with the jump condition

$$\Delta\varphi_-' = 0. \quad (2.109)$$

Therefore, imposing for the zero modes the gauge condition

$$\varphi_+ = 0, \quad (2.110)$$

the equation of motion of the zero mode of φ_- simplifies into the one of a massless $4D$ scalar field,

$$\square_4 \varphi_- = 0. \quad (2.111)$$

It follows that the zero mode of the only physical scalar field φ_- is not only massless, but has also vanishing scalar potential. This is expected from the fact that the interbrane distance L has not been stabilised, and φ_- plays the role of the modulus for this parameter. The vanishing of the full scalar potential can of course be checked at the level of the action (2.104), first noticing that $\frac{3}{2}(r')^2 - \frac{1}{2}(\delta\phi')^2 = 0$ in the gauge $\varphi_+ = r + \frac{1}{\sqrt{3}}\delta\phi = 0$. Then, regarding the last two terms in S_{scalar} , they read $2\sqrt{3}\alpha \operatorname{sgn} y r' - \alpha \operatorname{sgn} y \delta\phi' = \sqrt{3}\alpha \operatorname{sgn} y \varphi_+' + \sqrt{3}\alpha \operatorname{sgn} y \varphi_+'_-$. The first term vanishes due to the gauge condition, while the second is zero on the zero mode of φ_- which is independent of y . Hence, for the zero mode of the physical scalar φ_- , the last line of (2.104) vanishes and we end up with

$$S_{\text{scalar}} = \int d^5 x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ -\frac{1}{4} (\partial_\mu \varphi_-)^2 + e^{-3r} \left[3\alpha^2 - 2\sqrt{3}\alpha (\delta(y) - \delta(y-L)) \right] \right\}. \quad (2.112)$$

Denoting \mathcal{V} the volume factor

$$\mathcal{V} \equiv \int_{-L}^L dy e^{-\sqrt{3}\alpha|y|} = 2 \int_0^L dy e^{-\sqrt{3}\alpha y} = 2 \frac{1 - e^{-\sqrt{3}\alpha L}}{\sqrt{3}\alpha}, \quad (2.113)$$

the integration of the second term yields

$$\int_{-L}^L dy e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} e^{-3r} 3\alpha^2 = 3\alpha^2 \sqrt{-\tilde{g}} e^{-3r} \mathcal{V} = 2\sqrt{3}\alpha \sqrt{-\tilde{g}} e^{-3r} (1 - e^{-\sqrt{3}\alpha L}), \quad (2.114)$$

which is cancelled by the delta function terms. Hence, the full scalar potential of the zero mode of φ_- vanishes, and the effective 4D scalar action is simply given by the kinetic term of φ_- , in agreement with the analysis of the equations of motion carried out above:

$$S_{\text{scalar}} = -\frac{\mathcal{V}}{4} \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\mu \varphi_- \partial_\nu \varphi_-. \quad (2.115)$$

2.5 Minimal supersymmetric extension

The minimal supersymmetric extension of the bosonic linear dilaton model introduced above has been constructed in [49, 50]. It is based on a gauging of $\mathcal{N} = 2$ supergravity coupled to one vector multiplet along the $U(1)_R$ subgroup of the $SU(2)_R$ symmetry group, a construction holographically dual to Little String Theory.

The formalism of ungauged $\mathcal{N} = 2$, $D = 5$ supergravity coupled to an arbitrary number n_V of vector multiplets as well as its abelian gauging are reminded in the following Subsections 2.5.1 and 2.5.2 respectively. Following [50], we will restrict ourselves to the case $n_V = 1$, and thus only one physical real scalar φ^1 is present, associated to the dilaton degree of freedom. The multiplet content of the theory is then

$$\left(e_M^m, \psi_M^i, A_M^0 \right), \quad \left(A_M^1, \lambda^i, \varphi^1 = e^{\frac{\phi}{\sqrt{3}}} \right), \quad (2.116)$$

with ψ_M^i the two gravitini, A_M^0 the graviphoton, A_M^1 the $U(1)$ vector field, λ^i the two dilatini and ϕ the canonically normalized dilaton. The coordinates of the $n_V + 1 = 2$ -dimensional embedding scalar manifold are called $\xi^1 \equiv \varphi^1$ and ξ^0 , the latter being an unphysical scalar field which will be fixed in terms of the physical one φ^1 after imposing the constraint

$$\mathcal{F} \equiv \beta^3 C_{IJK} \xi^I \xi^J \xi^K = 1, \quad (2.117)$$

where C_{IJK} are completely symmetric real constants which define the theory, and with $\beta \equiv \sqrt{\frac{2}{3}}$. The functions $\xi^I(\phi)$ are related to the $h^I(\phi)$ introduced in the next Subsection 2.5.1 by $\xi^I(\phi) = \beta^{-1} h^I(\phi)$.

2.5.1 Vector-coupled $\mathcal{N} = 2$, $D = 5$ ungauged supergravity

Pure $\mathcal{N} = 2$, $D = 5$ supergravity is built out of the supergravity multiplet

$$\left(e_M^m, \psi_M^i, A_M^0 \right), \quad (2.118)$$

which contains, in addition of the graviton e_M^m , two symplectic-Majorana gravitini ψ_M^i , with $i = 1, 2$ the $SU(2)_R$ index, and a vector field A_M^0 , called the graviphoton. The most general $\mathcal{N} = 2$, $D = 5$ matter-coupled supergravity can then be obtained by coupling the supergravity multiplet with a given numbers of vector, tensor and hypermultiplets. In this thesis, we will not consider the case of tensor and hypermultiplets and refer to [59, 60] for a general description of $\mathcal{N} = 2$, $D = 5$ supergravity coupled to these matter multiplets. Instead, we focus on $\mathcal{N} = 2$ Maxwell-Einstein supergravity, obtained by coupling a given number n_V of vector multiplets to the supergravity multiplet (2.118) [52]. A $\mathcal{N} = 2$ vector multiplet,

$$(A_M, \lambda^i, \varphi), \quad (2.119)$$

contains a vector field A_M , a $SU(2)_R$ doublet of symplectic-Majorana spin-1/2 fermions λ^i , called dilatini, and a real scalar φ . The total field content of the theory is thus

$$e_M^m, \psi_M^i, A_M^I, \lambda_i^{\tilde{a}}, \varphi^x, \quad (2.120)$$

with $I = 0, 1, \dots, n_V$. The real scalars φ^x describe a real n_V -dimensional manifold \mathcal{M} , whose structure has been called very special real geometry, and whose coordinate and local frame indices are respectively written $x = 1, \dots, n_V$ and $\tilde{a} = 1, \dots, n_V$. \mathcal{M} is equipped with a metric g_{xy} and a n_V -bein $f_x^{\tilde{a}}$, related through:

$$g_{xy} = f_x^{\tilde{a}} \delta_{\tilde{a}\tilde{b}} f_y^{\tilde{b}}. \quad (2.121)$$

It turns out that \mathcal{M} is better described as a submanifold of a $(n_V + 1)$ -dimensional Riemannian space, with coordinates $h^I(\varphi^x)$, with an embedding defined through the constraint

$$\mathcal{F} \equiv C_{IJK} h^I h^J h^K = 1. \quad (2.122)$$

C_{IJK} are completely symmetric real constants, which will turn out to uniquely determine the whole theory. From C_{IJK} and $h^I(\varphi^x)$, we define another set of variables $h_I(\varphi^x)$ through

$$h_I \equiv \frac{1}{3} \frac{\partial}{\partial h^I} C_{JKL} h^J h^K h^L = C_{IJK} h^J h^K, \quad (2.123)$$

so that $h_I h^I = 1$, as well as a symmetric tensor $G_{IJ}(\varphi^x)$ which can be seen as the metric tensor of the embedding $(n_V + 1)$ -dimensional space, raising and lowering the indices I, J, \dots according to

$$h_I \equiv G_{IJ} h^J, \quad h^I \equiv G^{IJ} h_J. \quad (2.124)$$

$G_{IJ}(\varphi^x)$ will appear to be the kinetic matrix of the vector fields. From the additional condition that $G_{IK} G^{KJ} = \delta_I^J$, it is easy to check that it can be written as

$$G_{IJ} = -2C_{IJK} h^K + 3h_I h_J. \quad (2.125)$$

Introducing the quantities

$$h_x^I \equiv -\sqrt{\frac{3}{2}} \partial_x h^I, \quad h_{Ix} \equiv G_{IJ} h_x^J = \sqrt{\frac{3}{2}} \partial_x h_I, \quad (2.126)$$

the metric $g_{xy}(\varphi^x)$ of \mathcal{M} is then defined as being the pullback of $G_{IJ}(\varphi^x)$ to \mathcal{M} :

$$g_{xy} \equiv G_{IJ} h_x^I h_y^J = -2C_{IJK} h_x^I h_y^J h_z^K. \quad (2.127)$$

Finally, we define the symmetric $T_{xyz}(\varphi^x)$ tensor by

$$T_{xyz} \equiv \sqrt{\frac{3}{2}} h_{Ix;y} h_z^I = -\sqrt{\frac{3}{2}} h_{Ix} h_{y;z}^I = C_{IJK} h_x^I h_y^J h_z^K, \quad (2.128)$$

where a semicolon ";" denotes the covariant derivative associated with the Levi-Civita connection on \mathcal{M} , such that $g_{xy;z} = 0$, as well as the quantity $\Phi_{Ixy}(\varphi^x)$ symmetric in its last two indices:

$$\Phi_{Ixy} \equiv \sqrt{\frac{2}{3}} \left(\frac{1}{4} g_{xy} h_I + T_{xyz} h_I^z \right). \quad (2.129)$$

With the formalism and notations introduced above, the Lagrangian of $\mathcal{N} = 2$, $D = 5$ supergravity coupled to n_V vector multiplets can then be written as [52]:

$$\begin{aligned} e^{-1} \mathcal{L}^{(0)} &= \frac{1}{2} \mathcal{R}^{(5)} - \frac{1}{2} g_{xy} \partial_M \phi^x \partial^M \phi^y - \frac{1}{4} G_{IJ} F_{MN}^I F^{MNJ} + \frac{e^{-1}}{6\sqrt{6}} C_{IJK} \epsilon^{MNPQR} F_{MN}^I F_{PQ}^J A_R^K \\ &- \frac{1}{2} \bar{\psi}_M^i \gamma^{MNP} D_N \psi_{Pi} - \frac{1}{2} \bar{\lambda}^{i\bar{a}} \gamma^M \left(D_M \delta^{\bar{a}\bar{b}} + \Omega_x^{\bar{a}\bar{b}} \partial_M \phi^x \right) \lambda_i^{\bar{b}} - \frac{i}{2} \bar{\lambda}^{i\bar{a}} \gamma^M \gamma^N \psi_{Mi} f_x^a \partial_N \phi^x \\ &+ \frac{1}{4} h_I^{\bar{a}} \bar{\lambda}^{i\bar{a}} \gamma^M \gamma^{NP} \psi_{Mi} F_{NP}^I + \frac{i}{4} \Phi_{I\bar{a}\bar{b}} \bar{\lambda}^{i\bar{a}} \gamma^{MN} \lambda_i^{\bar{b}} F_{MN}^I \\ &- \frac{3i}{8\sqrt{6}} h_I \left(\bar{\psi}_M^i \gamma^{MNPQ} \psi_{Ni} F_{PQ}^I + 2\bar{\psi}^{Mi} \psi_i^N F_{MN}^I \right) + \mathcal{L}_{4\text{-fermions}}. \end{aligned} \quad (2.130)$$

The action $S = \int d^5x \mathcal{L}^{(0)}$ is invariant under the following $\mathcal{N} = 2$ supersymmetry transformations:

$$\delta e_M^m = \frac{1}{2} \bar{\epsilon}^i \gamma^m \psi_{Mi}, \quad (2.131)$$

$$\begin{aligned} \delta \psi_{Mi} &= D_M(\hat{\omega}) \epsilon_i + \frac{i}{4\sqrt{6}} h_I e_M^m (\gamma_{mnl} \epsilon_i - 4\eta_{mn} \gamma_l \epsilon_i) \hat{F}^{nlI} - \frac{1}{12} e_M^m \gamma_{mn} \epsilon^j \bar{\lambda}_i^{\bar{a}} \gamma^n \lambda_j^{\bar{a}} \\ &+ \frac{1}{48} e_M^m \gamma_{mnl} \epsilon^j \bar{\lambda}_i^{\bar{a}} \gamma^{nl} \lambda_j^{\bar{a}} + \frac{1}{6} e_M^m \epsilon^j \bar{\lambda}_i^{\bar{a}} \gamma_m \lambda_j^{\bar{a}} - \frac{1}{12} e_M^m \gamma^n \epsilon^j \bar{\lambda}_i^{\bar{a}} \gamma_{mn} \lambda_j^{\bar{a}}, \end{aligned} \quad (2.132)$$

$$\delta \varphi^x = \frac{i}{2} \bar{\epsilon}^i \lambda_i^{\bar{a}} f_{\bar{a}}^x, \quad (2.133)$$

$$\delta A_M^I = -\frac{1}{2} e_M^m \bar{\epsilon}^i \gamma_m \lambda_i^{\bar{a}} h_{\bar{a}}^I + \frac{i\sqrt{6}}{4} \bar{\psi}_M^i \epsilon_i h^I, \quad (2.134)$$

$$\begin{aligned} \delta \lambda_i^{\bar{a}} &= -\frac{i}{2} f_x^{\bar{a}} \gamma^M (\hat{\partial}_M \varphi)^x \epsilon_i - \delta \varphi^x \Omega_x^{\bar{a}\bar{b}} \lambda_i^{\bar{b}} + \frac{1}{4} h_I^{\bar{a}} \gamma^{mn} \epsilon_i \hat{F}_{mn}^I \\ &- \frac{i}{4\sqrt{6}} T_{\bar{b}\bar{c}}^{\bar{a}} \left(-3\epsilon^j \bar{\lambda}_i^{\bar{b}} \lambda_j^{\bar{c}} + \gamma_m \epsilon^j \bar{\lambda}_i^{\bar{b}} \gamma^m \lambda_j^{\bar{c}} + \frac{1}{2} \gamma_{mn} \epsilon^j \bar{\lambda}_i^{\bar{b}} \gamma^{mn} \lambda_j^{\bar{c}} \right). \end{aligned} \quad (2.135)$$

The hatted quantities \hat{X} are the supercovariantization of the unhatted ones X , namely:

$$\hat{\omega}_{Mmn}(e) \equiv \omega_{Mmn}(e) - \frac{1}{4} \left(\bar{\psi}_n^i \gamma_M \psi_{mi} + 2 \bar{\psi}_M^i \gamma_{[n} \psi_{m]i} \right), \quad (2.136)$$

$$D_M(\hat{\omega})\epsilon_i \equiv \partial_M \epsilon_i + \frac{1}{4} \hat{\omega}_M^{mn}(e) \gamma_{mn} \epsilon_i, \quad (2.137)$$

$$\hat{F}_{MN}^I \equiv F_{MN}^I + \frac{i\sqrt{6}}{4} h^I \bar{\psi}_{[M}^i \psi_{N]i} + h_{\tilde{a}}^I \bar{\psi}_{[M}^j \gamma_{N]} \lambda_{\tilde{a}}^j, \quad (2.138)$$

$$(\hat{\partial}_M \varphi)^x \equiv \partial_M \varphi^x - \frac{i}{2} f_{\tilde{a}}^x \bar{\psi}_M^j \lambda_{\tilde{a}}^j. \quad (2.139)$$

Knowing the symmetric constants C_{IJK} , one can find the functions $h^I(\varphi^x)$ by solving the constraint (2.122), then deduce the functions $h_I(\varphi^x)$, $G_{IJ}(\varphi^x)$, $g_{xy}(\varphi^x)$, $T_{xyz}(\varphi^x)$ and $\Phi_{Ixy}(\varphi^x)$ using Eqs. (2.123), (2.125), (2.127), (2.128) and (2.129) respectively, and thus completely determine the above Lagrangian and susy transformations. Therefore, even for a fixed number n_V of vector multiplets, several matter-coupled $\mathcal{N} = 2$, $D = 5$ supergravity theories are possible, depending on the geometry of the scalar manifold \mathcal{M} , in turn determined by the constants C_{IJK} . In Section 2.5.3, we list the different $\mathcal{N} = 2$, $D = 5$ supergravity theories coupled to $n_V = 1$ vector multiplet whose $U(1)_R$ gauging produces the runaway scalar potential of the non-critical string, and we now recall the main ideas of the $U(1)_R$ gauging of $\mathcal{N} = 2$, $D = 5$ supergravity.

2.5.2 $U(1)_R$ gauging of $\mathcal{N} = 2$, $D = 5$ supergravity

The global symmetry group G of the Lagrangian (2.130) can be written as $G = H \times SU(2)_R$, where H is the group of linear transformations acting on h^I and leaving C_{IJK} invariant⁶, and $SU(2)_R$ the R -symmetry group acting on the fermions ψ_M^i and $\lambda_{\tilde{a}}^i$. One can then arbitrarily choose to gauge the $U(1)_R$ subgroup of $SU(2)_R$ ⁷, a subgroup of H or a combination of both. These general gaugings have been described in [61], and we will only consider in the following the simplest case of the $U(1)_R$ gauging alone, following [51, 53].

The gauging along the $U(1)_R$ subgroup of the $SU(2)_R$ R -symmetry group is achieved by defining the $U(1)_R$ gauge field as a linear combination of the $n_V + 1$ vector fields A_M^I ,

$$A_M \equiv v_I A_M^I, \quad (2.140)$$

with v_I a set of $n_V + 1$ real constants. In the same time, we promote the Lorentz covariant derivatives of the fermionic fields to Lorentz- $U(1)_R$ covariant derivatives⁸

$$D_M \lambda^{\tilde{a}i} \rightarrow (\mathcal{D}_M \lambda^{\tilde{a}i}) \equiv D_M \lambda^{\tilde{a}i} + g A_M \delta^{ij} \lambda_{\tilde{a}}^j, \quad (2.141a)$$

$$D_M \psi_N^i \rightarrow (\mathcal{D}_M \psi_N^i) \equiv D_M \psi_N^i + g A_M \delta^{ij} \psi_{Nj}, \quad (2.141b)$$

where g is the $U(1)_R$ coupling constant. These replacements in the original Lagrangian $\mathcal{L}^{(0)}$ (2.130) and in the susy transformations (2.131)-(2.135) will break the supersymmetry. It can be

6. Note that in the most general case, H is only a subgroup of the isometry group of \mathcal{M} .

7. Since the vector fields are invariant under $SU(2)_R$, they cannot be used as non-Abelian gauge fields for $SU(2)_R$, and the full $SU(2)_R$ group cannot be gauged.

8. The scalars φ^x are kept uncharged under the $U(1)_R$, and their partial derivatives are thus not replaced by covariant derivatives in the gauging procedure.

recovered via the addition to $\mathcal{L}^{(0)}$ of a scalar potential P and fermion mass terms, given by

$$e^{-1}\mathcal{L}' = -g^2P - \frac{i\sqrt{6}}{8}g\bar{\psi}_M^i\gamma^{MN}\psi_N^j\delta_{ij}P_0 - \frac{g}{\sqrt{2}}\bar{\lambda}^{i\tilde{a}}\gamma^M\psi_M^j\delta_{ij}P_{\tilde{a}} + \frac{ig}{2\sqrt{6}}\bar{\lambda}^{i\tilde{a}}\lambda^{j\tilde{b}}\delta_{ij}P_{\tilde{a}\tilde{b}}, \quad (2.142)$$

as well as adding new g -dependent parts to the original susy transformations (2.132) and (2.135) of the gravitini and dilatini, of the form:

$$\delta'\psi_{Mi} = -\frac{ig}{2\sqrt{6}}P_0\gamma_M\epsilon_{ij}\delta^{jk}\epsilon_k, \quad (2.143)$$

$$\delta'\lambda_i^{\tilde{a}} = -\frac{g}{\sqrt{2}}P_{\tilde{a}}\epsilon_{ij}\delta^{jk}\epsilon_k. \quad (2.144)$$

Supersymmetry then requires the new functions $P(\varphi^x)$, $P_0(\varphi^x)$, $P_{\tilde{a}}(\varphi^x)$ and $P_{\tilde{a}\tilde{b}}$ to satisfy

$$P = -P_0^2 + P_{\tilde{a}}P^{\tilde{a}}, \quad (2.145)$$

$$P_0 = 2h^I v_I, \quad (2.146)$$

$$P^{\tilde{a}} = \sqrt{2}h^{I\tilde{a}}v_I, \quad (2.147)$$

$$P_{\tilde{a}\tilde{b}} = \frac{1}{2}\delta_{\tilde{a}\tilde{b}}P_0 + 2\sqrt{2}T_{\tilde{a}\tilde{b}\tilde{c}}P^{\tilde{c}}. \quad (2.148)$$

2.5.3 Runaway scalar potential from 5D gauged supergravity

In this subsection we want to come back on the work carried out in [49, 50] in order to classify all possible $D = 5$, $\mathcal{N} = 2$ supergravity theories, coupled to $n_V = 1$ vector multiplet, whose $U(1)_R$ gauging exactly produces the runaway scalar potential of the non-critical string. The approach followed here is to start from the wanted scalar potential and trace back the construction carried out in [51] towards the different allowed 5D prepotentials, which completely define the theory. In the $n_V = 1$ case, the two functions P_0 and $P_{\tilde{a}}$ introduced by the gauging and defined in Appendix 2.5.2 are expressed in terms of two arbitrary constants A and B by:

$$P_0 = Ae^{-\frac{2}{\sqrt{3}}\phi} + Be^{\frac{\phi}{\sqrt{3}}}, \quad P_{\tilde{a}} = Ae^{-\frac{2}{\sqrt{3}}\phi} - \frac{B}{2}e^{\frac{\phi}{\sqrt{3}}}, \quad (2.149)$$

so that the full scalar potential P reads:

$$P = -P_0^2 + P_{\tilde{a}}P^{\tilde{a}} = -3B\left(Ae^{-\frac{\phi}{\sqrt{3}}} + \frac{B}{4}e^{\frac{2}{\sqrt{3}}\phi}\right), \quad (2.150)$$

where \tilde{a} denotes the flat indices of the scalar manifold, which take just one value if $n_V = 1$; ϕ is the canonically normalised dilaton whose kinetic term is $-\frac{1}{2}(\partial\phi)^2$. The scalar metric $g_{xy} = g_{xx}$ in the $n_V = 1$ case is therefore simply given in this basis by $g_{xx} = 1$, and similarly for the scalar manifold einbein $f_x^{\tilde{a}}$ defined in (2.121). In the following we will thus not distinguish flat indices \tilde{a} and curved indices x of the scalar manifold, and simply write P_x . The P_0 and P_x can be seen as the coordinates of the constants v_I defined in (2.140) in the basis (h_I, h'_I) according to

$$v_I = \frac{1}{2}P_0h_I(\phi) + \frac{\sqrt{3}}{2}P_xh'_I(\phi), \quad \forall I = 0, 1, \quad (2.151)$$

where the prime denotes differentiation with respect to ϕ . Knowing from (2.149) the P_0 and P_x , (2.151) is a first-order differential equation for the h_I , whose general solution is given by

$$h_I(\phi) = \left(\frac{4v_I}{3B} + 2AC_I \right) e^{-\frac{\phi}{\sqrt{3}}} - BC_I e^{\frac{2}{\sqrt{3}}\phi}, \quad \forall I = 0, 1, \quad (2.152)$$

with C_I the constants of integration. The completely symmetric constants C_{IJK} , which enter the 5D prepotential \mathcal{F} through $\mathcal{F} = C_{IJK} h^I h^J h^K$, are then related to the h_I and their derivatives according to:

$$C_{IJK} = h_I h_J h_K - \frac{3}{2\beta^2} h'_{(I} h'_J h'_K) + \frac{1}{\sqrt{2}\beta^3} h'_I h'_J h'_K. \quad (2.153)$$

Plugging the result (2.152) and its derivative into the relation (2.153), we then find the four independent constants C_{IJK} expressed in terms of the six constants A , B , v_I and C_I by:

$$C_{IJJ} = -9B \left(AC_J + \frac{2v_J}{3B} \right) \left[2 \left(AC_I + \frac{2v_I}{3B} \right) C_J + \left(AC_J + \frac{2v_J}{3B} \right) C_I \right], \quad \forall I, J = \{0, 1\}. \quad (2.154)$$

After these general considerations, we now focus on the particular case $A = 0$, $B \neq 0$, which reproduces the runaway potential of the non-critical string⁹

$$P = -P_0^2 + P_x P^x = -\frac{3}{4} B^2 e^{\frac{2}{\sqrt{3}}\phi}, \quad (2.155)$$

with $P_0 = B e^{\frac{\phi}{\sqrt{3}}}$, $P_x = -\frac{B}{2} e^{\frac{\phi}{\sqrt{3}}}$, and gives for the constants C_{IJK} :

$$C_{IJJ} = -\frac{4v_J}{B} (2v_I C_J + v_J C_I), \quad \forall I, J = \{0, 1\}. \quad (2.156)$$

Since we are interested in $n_V = 1$ gauged supergravity theories leading to the runaway scalar potential (2.155), we need to investigate which choices of the different constants are compatible with the system of the four equations (2.156).

Without loss of generality, one can impose as a starting point C_{011} to be non-vanishing, which from (2.156) implies $v_1 \neq 0$. Two cases can then be considered, depending if C_1 vanishes or not.

- If $C_1 = 0$, it implies $C_{111} = 0$, and necessarily $C_0 \neq 0$ since $C_{011} \neq 0$. But there remains a freedom on the choice of v_0 which can vanish or not, leading to $C_{100} = C_{000} = 0$ or $C_{100} \neq 0, C_{000} \neq 0$.
- In the second case $C_1 \neq 0$, we have $C_{111} \neq 0$. There are then three subcases, depending on the choice of v_0 and C_0 . If $v_0 = 0$, since $C_{011} \neq 0$, we must have $C_0 \neq 0$. But if v_0 is non vanishing, the two possibilities for C_0 to vanish or not are allowed, respectively leading to $(C_{000} = 0, C_{100} \neq 0)$ and $(C_{000} \neq 0, C_{100} \neq 0)$.

These results are summarised in Table 2.1. We end up with five different possibilities for the choice of the constants C_{IJK} , and therefore five different theories leading to the scalar potential (2.155). The associated values for the constants v_I and C_I are given in the second vertical part of the table, from which one can find all relevant quantities, among which the functions h_I and the kinetic matrix G_{IJ} .

9. The A term cannot be considered as a string loop correction either.

	C_{011}	C_{111}	C_{100}	C_{000}	v_1	C_1	v_0	C_0
(a)	$\neq 0$	0	0	0	$\neq 0$	0	0	$\neq 0$
(b)	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$
(c)	$\neq 0$	$\neq 0$	0	0	$\neq 0$	$\neq 0$	0	$\neq 0$
(d)	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	0
(e)	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$

Table 2.1 – The five different supergravity theories (defined by the C_{IJK} on the left part) coupled to one vector multiplet whose $U(1)_R$ gauging reproduces the potential of LST, and the associated constants (on the right part).

However, while being a priori distinct, these five cases have actually the same physical content. The scalar potential as well as the scalar metric having been fixed, it is sufficient to check that the actions of the vectors are equivalent for these different cases. According to (2.130), the gauge field part of the action contains the kinetic term $-\frac{1}{4}G_{IJ}F^IF^J$ and the topological term $C_{IJK}A^IF^JF^K$, where spacetime indices, respectively contracted with the 5D metric and 5D Levi-Civita tensor, have been left implicit in both cases. In the simplest case (a), the vector kinetic matrix $G_{IJ}^{(a)}$ and topological term $\mathcal{L}_{CS}^{(a)}$ reads:

$$G_{IJ}^{(a)} = \begin{pmatrix} 3B^2C_0^2e^{\frac{4}{\sqrt{3}}\phi} & 0 \\ 0 & \frac{8v_1^2}{3B^2}e^{-\frac{2}{\sqrt{3}}\phi} \end{pmatrix}, \quad (2.157)$$

$$\mathcal{L}_{CS}^{(a)} = C_{011}A^0F^1F^1 + 2C_{011}A^1F^1F^0. \quad (2.158)$$

Starting with the case (e), the kinetic metric and the topological terms are respectively given by:

$$G_{IJ}^{(e)} = \begin{pmatrix} \frac{8v_0^2}{3B^2}e^{-\frac{2}{\sqrt{3}}\phi} + 3B^2C_0^2e^{\frac{4}{\sqrt{3}}\phi} & \frac{8v_0v_1}{3B^2}e^{-\frac{2}{\sqrt{3}}\phi} + 3B^2C_0C_1e^{\frac{4}{\sqrt{3}}\phi} \\ \frac{8v_0v_1}{3B^2}e^{-\frac{2}{\sqrt{3}}\phi} + 3B^2C_0C_1e^{\frac{4}{\sqrt{3}}\phi} & \frac{8v_1^2}{3B^2}e^{-\frac{2}{\sqrt{3}}\phi} + 3B^2C_1^2e^{\frac{4}{\sqrt{3}}\phi} \end{pmatrix}, \quad (2.159)$$

$$\begin{aligned} \mathcal{L}_{CS}^{(e)} = & C_{011}A^0F^1F^1 + 2C_{011}A^1F^1F^0 + C_{100}A^1F^0F^0 + 2C_{100}A^0F^1F^0 \\ & + C_{111}A^1F^1F^1 + C_{000}A^0F^0F^0. \end{aligned} \quad (2.160)$$

Since $C_0 \neq 0$ and $v_1 \neq 0$, we are free to consider the field redefinitions

$$A_M^0 \rightarrow A_M^0 - \frac{C_1}{C_0}A_M^1, \quad A_M^1 \rightarrow A_M^1 - \frac{v_0}{v_1}A_M^0, \quad (2.161)$$

which bring $G_{IJ}^{(e)}$ into the diagonal form

$$G_{IJ}^{(e)} \rightarrow G_{IJ}^{(e)} = \begin{pmatrix} 3B^2 \left(C_0 - C_1 \frac{v_0}{v_1}\right)^2 e^{\frac{4}{\sqrt{3}}\phi} & 0 \\ 0 & \frac{8}{3B^2} \left(v_1 - v_0 \frac{C_1}{C_0}\right)^2 e^{-\frac{2}{\sqrt{3}}\phi} \end{pmatrix}, \quad (2.162)$$

while the Chern-Simons terms are sent to:

$$\mathcal{L}_{CS}^{(e)} \rightarrow \mathcal{L}_{CS}^{(e)} = \left(C_{011} + \frac{20v_0v_1C_1}{B} - \frac{12v_0^2C_1^2}{BC_0} + \frac{4v_0^3C_1^3}{Bv_1C_0^2} \right) (A^0F^1F^1 + 2A^1F^0F^1). \quad (2.163)$$

These are indeed equivalent to (2.157)-(2.158) of the case (a). The theories (b) and (c) are immediately obtained from (e) by simply turning off C_1 or v_0 , respectively. Finally, for the case (d) where $C_0 = 0$, one has to consider instead the field redefinition

$$A_M^0 \rightarrow A_M^0 - \frac{v_1}{v_0} A_M^1, \quad (2.164)$$

which brings $G_{IJ}^{(d)}$ and $\mathcal{L}_{CS}^{(d)}$ into

$$G_{IJ}^{(d)} \rightarrow G_{IJ}^{(d)} = \begin{pmatrix} \frac{8v_0^2}{3B^2} e^{-\frac{2}{\sqrt{3}}\phi} & 0 \\ 0 & 3B^2 C_1^2 e^{\frac{4}{\sqrt{3}}\phi} \end{pmatrix}, \quad (2.165)$$

$$\mathcal{L}_{CS}^{(d)} \rightarrow \mathcal{L}_{CS}^{(d)} = C_{100} A^1 F^0 F^0 + 2C_{100} A^0 F^0 F^1. \quad (2.166)$$

This simply corresponds to an exchange between the 0 and 1 vectors compared to the case (a), as can be immediately checked from Eqs. (2.157)-(2.158).

Hence, after appropriate field redefinitions, the five theories of Table 2.1 turn out to be physically equivalent, and we can then restrict without loss of generality to the simplest one (a), where $C_{111} = C_{100} = C_{000} = 0$. Its prepotential

$$\mathcal{F} = \xi^0 (\varphi^1)^2 \quad (2.167)$$

has been considered in the five-dimensional heterotic string theory compactified on $K_3 \times S^1$ in [62], where $1/\xi^0$ has been identified with the heterotic string coupling. It corresponds to the choice of the only non-vanishing constant C_{IJK}

$$C_{011} = \frac{1}{3\beta^3}. \quad (2.168)$$

In particular, one can easily find from (2.168) the gauge kinetic matrix

$$G_{IJ} = \begin{pmatrix} \frac{1}{2} e^{\frac{4}{\sqrt{3}}\phi} & 0 \\ 0 & e^{-\frac{2}{\sqrt{3}}\phi} \end{pmatrix}, \quad (2.169)$$

as well as the various functions h^I , h_I , h_a^I and $h_{\bar{I}}^{\tilde{a}}$ defined in Appendix 2.5.1:

$$h^0 = \sqrt{\frac{2}{3}} e^{-\frac{2}{\sqrt{3}}\phi}, \quad h^1 = \sqrt{\frac{2}{3}} e^{\frac{1}{\sqrt{3}}\phi}, \quad (2.170)$$

$$h_0 = \frac{1}{\sqrt{6}} e^{\frac{2}{\sqrt{3}}\phi}, \quad h_1 = \sqrt{\frac{2}{3}} e^{-\frac{1}{\sqrt{3}}\phi}, \quad (2.171)$$

$$h_{\bar{1}}^0 = \frac{2}{\sqrt{3}} e^{-\frac{2}{\sqrt{3}}\phi}, \quad h_{\bar{1}}^1 = -\frac{1}{\sqrt{3}} e^{\frac{\phi}{\sqrt{3}}}, \quad (2.172)$$

$$h_{\bar{0}}^1 = \frac{1}{\sqrt{3}} e^{\frac{2}{\sqrt{3}}\phi}, \quad h_{\bar{1}}^{\tilde{1}} = -\frac{1}{\sqrt{3}} e^{-\frac{\phi}{\sqrt{3}}}. \quad (2.173)$$

The minimal supersymmetric extension of the dilaton-gravity theory we are working with is therefore described in the Einstein frame by the Lagrangian $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}'$, where $\mathcal{L}^{(0)}$ is the Lagrangian of the ungauged theory given in (2.130) and \mathcal{L}' the part added by the gauging

(2.142). From the results for G_{IJ} and P found above, it is easy to see that its bosonic part is given by:

$$\begin{aligned}
e^{-1}\mathcal{L}^{\text{bos}} &= \frac{1}{2}\mathcal{R}^{(5)}[G_{MN}] - \frac{1}{2}\partial_M\phi\partial^M\phi - e^{\frac{2}{\sqrt{3}}\phi}\Lambda \\
&\quad - \frac{1}{8}e^{\frac{4}{\sqrt{3}}\phi}F_{MN}^0F^{MN0} - \frac{1}{4}e^{-\frac{2}{\sqrt{3}}\phi}F_{MN}^1F^{MN1} \\
&\quad + \frac{e^{-1}}{6\sqrt{6}}C_{011}\epsilon^{MNPQR}\left(A_M^0F_{NP}^1F_{QR}^1 + 2A_M^1F_{NP}^1F_{QR}^0\right).
\end{aligned} \tag{2.174}$$

In addition of the terms on the first line, already present in the analysis of the Sections 2.3 and 2.4, the Lagrangian also contains the kinetic terms for the two gauge bosons as well as a five-dimensional Chern-Simons term, with F_{MN}^0 and F_{MN}^1 the abelian field strengths of the graviphoton and the $U(1)$ vector respectively. From the 5D point of view, the only effect of the gauging in the bosonic sector appears in the scalar potential.

This theory is invariant under $\mathcal{N} = 2$ supersymmetries in four dimensions, namely has 8 real conserved supercharges. In the following subsection 2.5.4, we remind that the background solution $\phi = \alpha|y|$ preserves 1/2 of the supersymmetries, and compute the direction of the unbroken one. A second source of supersymmetry breaking can then arise from the compactification of the theory on S^1/\mathbb{Z}_2 , and from the introduction of branes on the boundaries. We will then check in subsection 2.5.5 that the supersymmetry preserved by the LD background is also preserved by the branes located at the singular points of the orbifold.

2.5.4 The LD background as a 1/2-BPS solution

In the vacuum of the theory where all fermions and vectors vanish, the relevant parts of the supersymmetry transformations for the gravitini and dilatini read:

$$\delta(\epsilon_1, \epsilon_2)\psi_{Mi} = \partial_M\epsilon_i + \frac{1}{4}\omega_M^{mn}(e)\gamma_{mn}\epsilon_i - \frac{i\alpha}{2\sqrt{3}}e^{\frac{\phi}{\sqrt{3}}}e_M^m\gamma_m\epsilon_{ij}\delta^{jk}\epsilon_k, \tag{2.175}$$

$$\delta(\epsilon_1, \epsilon_2)\lambda_i = -\frac{i}{2}e_m^M\gamma^m\partial_M\phi\epsilon_i + \frac{\alpha}{2}e^{\frac{\phi}{\sqrt{3}}}\epsilon_{ij}\delta^{jk}\epsilon_k, \tag{2.176}$$

where the last terms proportional to α arise from the gauging. In order to compute their explicit forms in the background, we recall the background metric and frame field in the Einstein frame:

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|}\eta_{MN}, \quad e_M^m = e^{-\frac{1}{\sqrt{3}}\alpha|y|}\delta_M^m. \tag{2.177}$$

The relation between the spin-connection $\omega_M^{mn}(e)$ and the frame field e_M^m ,

$$\omega_M^{mn}(e) = 2e^{N[m}\partial_{[M}e_{N]}^n] - e^{N[m}e^{n]Q}e_{Mq}\partial_N e_Q^q, \tag{2.178}$$

leads to the following components for the spin-connection:

$$\omega_\mu^{ab}(e) = \omega_5^{ab}(e) = \omega_5^{a5}(e) = 0, \tag{2.179a}$$

$$\omega_\mu^{a5}(e) = -\frac{\alpha}{\sqrt{3}}|y|'\delta_\mu^a = -\frac{\alpha}{\sqrt{3}}\text{sgn}(y)\delta_\mu^a. \tag{2.179b}$$

The transformations (2.175) and (2.176) in the background $\phi = \alpha|y|$ therefore read:

$$\delta(\epsilon_1, \epsilon_2)\psi_{Mi} = \partial_M \epsilon_i - \frac{\alpha}{2\sqrt{3}} \operatorname{sgn} y \delta_M^a \gamma_a \gamma_5 \epsilon_i - \frac{i\alpha}{2\sqrt{3}} \delta_M^m \gamma_m \epsilon_{ij} \delta^{jk} \epsilon_k, \quad (2.180)$$

$$\delta(\epsilon_1, \epsilon_2)\lambda_i = \frac{\alpha}{2} e^{\frac{\phi}{\sqrt{3}}} \left[-i \operatorname{sgn} y \gamma_5 \epsilon_i + \epsilon_{ij} \delta^{jk} \epsilon_k \right]. \quad (2.181)$$

Setting the transformations (2.180) and (2.181) to zero, we obtain the Killing equations which need to be solved in order to study the existence or not of preserved supersymmetries in the vacuum. We start with the Killing equations for the fifth components of the two gravitini, which form the system of coupled partial differential equations

$$\delta(\epsilon_1, \epsilon_2)\psi_{51} = \partial_5 \epsilon_1 - \frac{i\alpha}{2\sqrt{3}} \gamma_5 \epsilon_2 = 0, \quad (2.182a)$$

$$\delta(\epsilon_1, \epsilon_2)\psi_{52} = \partial_5 \epsilon_2 + \frac{i\alpha}{2\sqrt{3}} \gamma_5 \epsilon_1 = 0, \quad (2.182b)$$

whose solutions are given by:

$$\epsilon_1(x, y) = e^{-\frac{\alpha}{2\sqrt{3}}|y|} \epsilon(x), \quad \epsilon_2(x, y) = i \operatorname{sgn}(y) e^{-\frac{\alpha}{2\sqrt{3}}|y|} \gamma_5 \epsilon(x). \quad (2.183)$$

Plugging these solutions into the Killing equations for the 4-dimensional gravitini,

$$\delta(\epsilon_1, \epsilon_2)\psi_{\mu i} = \partial_\mu \epsilon_i - \frac{\alpha}{2\sqrt{3}} \operatorname{sgn}(y) \delta_\mu^a \gamma_a \gamma_5 \epsilon_i - \frac{i\alpha}{2\sqrt{3}} \delta_\mu^a \gamma_a \epsilon_{ij} \delta^{jk} \epsilon_k = 0 \quad (2.184)$$

we find that they are satisfied provided that $\epsilon(x)$ is a constant spinor, $\epsilon(x) = \epsilon$. The solutions to the Killing equations are thus given by

$$\epsilon_1(y) = e^{-\frac{\alpha}{2\sqrt{3}}|y|} \epsilon, \quad (2.185a)$$

$$\epsilon_2(y) = i \operatorname{sgn}(y) e^{-\frac{\alpha}{2\sqrt{3}}|y|} \gamma_5 \epsilon, \quad (2.185b)$$

and the direction of unbroken supersymmetry is obviously:

$$\epsilon_2(y) = i \operatorname{sgn}(y) \gamma_5 \epsilon_1(y). \quad (2.186)$$

It is then straightforward to check that the dilatini Killing equations are also satisfied, namely $\delta(\epsilon_1, \epsilon_2 = i \operatorname{sgn} y \gamma_5 \epsilon_1)\lambda_1 = \delta(\epsilon_1, \epsilon_2 = i \operatorname{sgn} y \gamma_5 \epsilon_1)\lambda_2 = 0$.

We thus conclude that the linear dilaton background $\phi = \alpha|y|$ breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$, with the 4 remaining real supercharges associated with the 4 components of the Killing spinor ϵ .

In the following we will define the supersymmetric transformation in the direction of the preserved supersymmetry by:

$$\delta_L \equiv \delta(\epsilon_1, \epsilon_2 = i \operatorname{sgn}(y) \gamma_5 \epsilon_1), \quad (2.187)$$

while the transformation in the direction of the broken supersymmetry would be given by $\delta_{NL} \equiv \delta(\epsilon_1, \epsilon_2 = -i \operatorname{sgn}(y) \gamma_5 \epsilon_1)$.

2.5.5 Preserved supersymmetry and NS5-branes

As described in Section 2.3.1, introducing a \mathbb{Z}_2 symmetry on the background solution produces discontinuous terms in the equations of motion, whose cancellation requires adding brane contributions to the original Lagrangian. In the Einstein frame, these brane Lagrangians are given by:

$$\mathcal{L}_1 = -2\sqrt{3}\alpha e^{(4)} e^{\frac{\phi}{\sqrt{3}}} \delta(y), \quad \mathcal{L}_2 = 2\sqrt{3}\alpha e^{(4)} e^{\frac{\phi}{\sqrt{3}}} \delta(y-L), \quad (2.188)$$

with $e^{(4)}$ the Einstein frame four-dimensional vierbein induced on the branes. The aim of this subsection is to show how the boundary terms coming from the supersymmetric variation of the bulk Lagrangian are cancelled by the supersymmetric variations of the brane Lagrangians (2.188), up to linear order in the dilatini λ_i and gravitini ψ_{Mi} . The part of the bulk Lagrangian whose supersymmetric variation brings terms linear in λ reads:

$$\mathcal{L} = \mathcal{L}_{kin}^\phi + \mathcal{L}_{kin}^\lambda + \mathcal{L}_{\phi\lambda\psi} + \mathcal{L}_\Lambda^{(\alpha)} + \mathcal{L}_{\lambda\lambda}^{(\alpha)} + \mathcal{L}_{\lambda\psi}^{(\alpha)}, \quad (2.189)$$

with, in the Einstein frame:

$$e^{-1} \mathcal{L}_{kin}^\phi = -\frac{1}{2} \partial_M \phi \partial^M \phi, \quad (2.190)$$

$$e^{-1} \mathcal{L}_{kin}^\lambda = -\frac{1}{2} \bar{\lambda}^i \gamma^M D_M(\omega) \lambda_i = -\frac{1}{2} \bar{\lambda}^i \gamma^M (\partial_M + \frac{1}{4} \omega_M^{mn} \gamma_{mn}) \lambda_i, \quad (2.191)$$

$$e^{-1} \mathcal{L}_{\phi\lambda\psi} = -\frac{i}{2} \partial_N \phi \bar{\lambda}^i \gamma^M \gamma^N \psi_{Mi}, \quad (2.192)$$

$$e^{-1} \mathcal{L}_\Lambda^{(\alpha)} = \frac{3}{2} \alpha^2 e^{\frac{2}{\sqrt{3}} \phi}, \quad (2.193)$$

$$e^{-1} \mathcal{L}_{\lambda\lambda}^{(\alpha)} = -\frac{i\alpha}{4\sqrt{3}} e^{\frac{\phi}{\sqrt{3}}} \bar{\lambda}^i \lambda^j \delta_{ij}, \quad (2.194)$$

$$e^{-1} \mathcal{L}_{\lambda\psi}^{(\alpha)} = \frac{\alpha}{2} e^{\frac{\phi}{\sqrt{3}}} \bar{\lambda}^i \gamma^M \psi_M^j \delta_{ij}, \quad (2.195)$$

and where the superscript (α) means that the corresponding terms arise from the gauging. The relevant parts of the supersymmetric transformations of the dilaton, dilatini and gravitini are respectively given by:

$$\delta\phi = \frac{i}{2} \bar{\epsilon}^i \lambda_i, \quad (2.196)$$

$$\delta\lambda^i = -\frac{i}{2} \not{\partial} \phi \epsilon^i - \frac{\alpha}{2} e^{\frac{\phi}{\sqrt{3}}} \delta^{ij} \epsilon_j, \quad (2.197)$$

$$\delta\psi_M^i = D_M(\omega) \epsilon^i + \frac{i\alpha}{2\sqrt{3}} \delta_M^m \gamma_m \delta^{ij} \epsilon_j. \quad (2.198)$$

In the bulk, the variation of the dilaton and dilatini kinetic terms, as well as $\mathcal{L}_{\phi\lambda\psi}$, yields:

$$e^{-1} \delta \mathcal{L}_{kin}^\phi = -\frac{i}{2} \partial_M \phi \bar{\epsilon}^i \partial^M \lambda_i = \frac{i}{2} \partial^M \partial_M \phi \bar{\epsilon}^i \lambda_i, \quad (2.199)$$

$$\begin{aligned} e^{-1} \delta \mathcal{L}_{kin}^\lambda &= \frac{i}{2} \partial^M \partial_M \phi \bar{\lambda}^i \epsilon_i + \frac{i}{2} \partial_N \phi \bar{\lambda}^i \gamma^M \gamma^N D_M(\omega) \epsilon_i \\ &\quad - \frac{\alpha}{2\sqrt{3}} e^{\frac{\phi}{\sqrt{3}}} \bar{\lambda}^i \not{\partial} \phi \epsilon_k \epsilon_{ij} \delta^{jk} - \frac{\alpha}{2} e^{\frac{\phi}{\sqrt{3}}} \bar{\lambda}^i \not{D}(\omega) \epsilon_k \epsilon_{ij} \delta^{jk}, \end{aligned} \quad (2.200)$$

$$e^{-1} \delta \mathcal{L}_{\phi\lambda\psi} = -\frac{i}{2} \partial_N \phi \bar{\lambda}^i \gamma^M \gamma^N D_M(\omega) \epsilon_i + \frac{3\alpha}{4\sqrt{3}} e^{\frac{\phi}{\sqrt{3}}} \bar{\lambda}^i \not{\partial} \phi \epsilon_k \epsilon_{ij} \delta^{jk}. \quad (2.201)$$

One can already check that without the gauging, i.e. taking $\alpha = 0$, and using $\bar{\epsilon}^i \lambda_i = -\bar{\lambda}^i \epsilon_i$, the three above variations cancel, $\delta(\mathcal{L}_{kin}^\phi + \mathcal{L}_{kin}^\lambda + \mathcal{L}_{\phi\lambda\psi}) = 0$ as expected. The gauging produces extra terms in $\delta\mathcal{L}_{kin}^\lambda$ and $\delta\mathcal{L}_{\phi\lambda\psi}$, proportional to α , which are cancelled by the variation of $\mathcal{L}_\Lambda^{(\alpha)} + \mathcal{L}_{\lambda\lambda}^{(\alpha)} + \mathcal{L}_{\lambda\psi}^{(\alpha)}$, as it can be checked by computing:

$$e^{-1}\delta\mathcal{L}_\Lambda^{(\alpha)} = \frac{i\sqrt{3}}{2}\alpha^2 e^{\frac{2}{\sqrt{3}}\phi} \bar{\epsilon}^i \lambda_i, \quad (2.202)$$

$$e^{-1}\delta\mathcal{L}_{\lambda\lambda}^{(\alpha)} = -\frac{\alpha}{4\sqrt{3}}e^{\frac{\phi}{\sqrt{3}}}\bar{\lambda}^i \not{\partial}\phi \epsilon^j \delta_{ij} + \frac{i\alpha^2}{4\sqrt{3}}e^{\frac{2}{\sqrt{3}}\phi} \bar{\lambda}^i \epsilon_k \delta_i^k \quad (2.203)$$

$$e^{-1}\delta\mathcal{L}_{\lambda\psi}^{(\alpha)} = \frac{\alpha}{2}e^{\frac{2}{\sqrt{3}}\phi} \bar{\lambda}^i \gamma^M \delta\psi_M^j \delta_{ij} \quad (2.204)$$

$$= \frac{\alpha}{2}e^{\frac{\phi}{\sqrt{3}}}\bar{\lambda}^i \not{D}(\omega) \epsilon^j \delta_{ij} + \frac{5i\alpha^2}{4\sqrt{3}}e^{\frac{2}{\sqrt{3}}\phi} \bar{\lambda}^i \epsilon_k \delta_i^k. \quad (2.205)$$

From (2.199), (2.200), (2.201), (2.202), (2.203) and (2.205), we see that $\delta(\mathcal{L}_{kin}^\phi + \mathcal{L}_{kin}^\lambda + \mathcal{L}_{\phi\lambda\psi} + \mathcal{L}_\Lambda^{(\alpha)} + \mathcal{L}_{\lambda\lambda}^{(\alpha)} + \mathcal{L}_{\lambda\psi}^{(\alpha)}) = 0$ at linear order in λ .

We now consider the boundary terms on S^1/\mathbb{Z}_2 . They arise from integrations by part done in the bulk variations, as well as from the brane Lagrangians (2.188). In the bulk analysis carried out above, two integrations by parts have been done, in the variations of the dilatini and dilaton kinetic terms:

(i) In $\delta\mathcal{L}_{kin}^\lambda$, the integration by parts brings a total derivative of the form $\partial_5(ee^{\frac{\phi}{\sqrt{3}}}\bar{\lambda}^i \gamma^5 \delta\lambda_i)$. At the linear level in the fluctuations $\bar{\lambda}_i$, we evaluate $\delta\lambda_i$ at the background level, which vanishes in the direction of the preserved supersymmetry

$$\delta_L \lambda_i \equiv \delta(\epsilon_1, \epsilon_2 = i \operatorname{sgn} y \gamma_5 \epsilon_1) \lambda_i = 0. \quad (2.206)$$

We therefore conclude that the dilatini kinetic term does not bring additional boundary contributions in the variation of the Lagrangian.

(ii) In $\delta\mathcal{L}_{kin}^\phi$, the integration by parts yields

$$\delta\mathcal{L}_{kin}^\phi = -\frac{i}{2}\partial_5(e\partial^5\phi\bar{\epsilon}^i\lambda_i) + \frac{i}{2}e\partial^M\partial_M\phi\bar{\epsilon}^i\lambda_i. \quad (2.207)$$

The first term is a total derivative integrated on the interval $[-L, L]$. The integrand $\partial_5(e\partial^5\phi\bar{\epsilon}^i\lambda_i)$ being an even function of y , its integral on $[-L, L]$ is 2 times its integral on $[0, L]$, giving, at linear order in λ_i , $-i\alpha\left[e^{-\sqrt{3}\alpha|y|}\bar{\epsilon}^i\lambda_i\right]_{y=0}^{y=L}$. The second term of (2.207) cancels with the first one of (2.200), like in the bulk analysis, and we thus conclude that in the direction of the preserved supersymmetry $\epsilon_2 = i \operatorname{sgn} y \gamma_5 \epsilon_1$, $\delta(\mathcal{L}_{kin}^\phi + \mathcal{L}_{kin}^\lambda)$ brings the boundary contribution

$$\delta(\mathcal{L}_{kin}^\phi + \mathcal{L}_{kin}^\lambda) = -i\alpha\left[e^{-\sqrt{3}\alpha|y|}\bar{\epsilon}^i\lambda_i\right]_{y=0}^{y=L}. \quad (2.208)$$

It is then straightforward to check that this contribution is indeed cancelled by the supersymmetric variation of the brane Lagrangians (2.188). Using (2.196), the variation of the dilaton ϕ yields:

$$\delta_\phi\mathcal{L}_1 = -i\alpha e^{(4)}e^{\frac{\phi}{\sqrt{3}}}\bar{\epsilon}^i\lambda_i\delta(y), \quad \delta_\phi\mathcal{L}_2 = i\alpha e^{(4)}e^{\frac{\phi}{\sqrt{3}}}\bar{\epsilon}^i\lambda_i\delta(y-L). \quad (2.209)$$

Since we are interested in the variations linear in the fluctuations λ_i , we replace ϕ and $e^{(4)}$ by their background values $\phi = \alpha|y|$ and $e^{(4)} = e^{-\frac{4}{\sqrt{3}}\alpha|y|}$, leading to

$$\delta_\phi \mathcal{L}_1 = -i\alpha e^{-\sqrt{3}\alpha|y|} \bar{\epsilon}^i \lambda_i \delta(y), \quad \delta_\phi \mathcal{L}_2 = i\alpha e^{-\sqrt{3}\alpha|y|} \bar{\epsilon}^i \lambda_i \delta(y-L), \quad (2.210)$$

so that:

$$\delta_\phi (\mathcal{L}_1 + \mathcal{L}_2) = i\alpha \left[e^{-\sqrt{3}\alpha|y|} \bar{\epsilon}^i \lambda_i \right]_{y=0}^{y=L}, \quad (2.211)$$

which exactly cancels the boundary terms (2.208) coming from the bulk variations.

Finally, we consider the supersymmetric variations linear in the gravitini ψ_{Mi} . The part of the bulk Lagrangian whose supersymmetric variation brings terms linear in ψ_{Mi} is simply the Einstein-Hilbert action,

$$\mathcal{L}_{EH} = \frac{1}{2} e \mathcal{R}^{(5)}(\omega) = \frac{1}{2} e e_m^M e_n^N R_{MN}{}^{mn}(\omega), \quad (2.212)$$

whose supersymmetric variation contains two terms:

$$\delta \mathcal{L}_{EH} = -\frac{1}{2} e \left(R_{MN} - \frac{1}{2} G_{MN} \mathcal{R}^{(5)} \right) \bar{\epsilon}^i \gamma^M \psi_i^N + \frac{1}{2} e e_m^M e_n^N \delta R_{MN}{}^{mn}(\omega). \quad (2.213)$$

From $\delta R_{MN}{}^{mn}(\omega) = D_M \delta \omega_N{}^{mn} - D_N \delta \omega_M{}^{mn}$, we see that the second term is a total derivative whose integral would vanish in the absence of boundaries. Taking into account the boundaries in the fifth direction, it remains the total derivative $\partial_5 (e e_n^5 e_p^N \delta \omega_N{}^{np})$. Contracting the general variation of the spin connection ω_{Mnp}

$$e_N^n e_P^p \delta \omega_{Mnp} = (D_{[M} \delta e_{N]}^n) e_{Pn} - (D_{[N} \delta e_{P]}^n) e_{Mn} + (D_{[P} \delta e_{M]}^n) e_{Nn} \quad (2.214)$$

with G^{PM} , we see that $e_n^5 e_p^N \delta \omega_N{}^{np} = 0$ and thus the second term of (2.213) vanishes. In order to evaluate the first one at linear order in the perturbation ψ_i^N , we plug the background values in the Einstein tensor, which yields

$$R_{MN} - \frac{1}{2} G_{MN} \mathcal{R}^{(5)} = -2\sqrt{3}\alpha \eta_{MN} [\delta(y) - \delta(y-L)] + 2\sqrt{3}\alpha \delta_M^5 \delta_N^5 [\delta(y) - \delta(y-L)] + \dots, \quad (2.215)$$

where the dots denote bulk terms. In the contraction with $\bar{\epsilon}^i \gamma^M \psi_i^N$, the terms with $M = N = 5$ cancel, and after considering the background value $e = e^{-\frac{5}{\sqrt{3}}\alpha|y|}$ and writing $\gamma^\mu = e_a^\mu \gamma^a = e^{\frac{\alpha}{\sqrt{3}}|y|} \delta_a^\mu \gamma^a$, $\psi_i^\nu = G^{\nu\rho} \psi_{\rho i} = e^{\frac{2}{\sqrt{3}}\alpha|y|} \eta^{\nu\rho} \psi_{\rho i}$, we end with

$$\delta \mathcal{L}_{EH} = \sqrt{3}\alpha e^{-\frac{2}{\sqrt{3}}\alpha|y|} \bar{\epsilon}^i \delta_a^\mu \gamma^a \psi_{\mu i} [\delta(y) - \delta(y-L)]. \quad (2.216)$$

Again we want to check that this contribution is cancelled by the supersymmetric variation of the brane Lagrangians (2.188). From the variation of the determinant of the four-dimensional vierbein

$$\delta e^{(4)} = \frac{1}{2} e^{(4)} \bar{\epsilon}^i \gamma^\mu \psi_{\mu i}, \quad (2.217)$$

we deduce that

$$\delta_e \mathcal{L}_1 = -\sqrt{3}\alpha e^{(4)} e^{\frac{\phi}{\sqrt{3}}} \bar{\epsilon}^i \gamma^\mu \psi_{\mu i} \delta(y), \quad \delta_e \mathcal{L}_2 = \sqrt{3}\alpha e^{(4)} e^{\frac{\phi}{\sqrt{3}}} \bar{\epsilon}^i \gamma^\mu \psi_{\mu i} \delta(y-L). \quad (2.218)$$

Since we are interested in the variations linear in the fluctuations $\psi_{\mu i}$, we replace the other fields by their background values $\phi = \alpha|y|$, $e^{(4)} = e^{-\frac{4}{\sqrt{3}}\alpha|y|}$, and write again $\gamma^\mu = e_a^\mu \gamma^a = e^{\frac{\alpha}{\sqrt{3}}|y|} \delta_a^\mu \gamma^a$, which leads to:

$$\delta_e \mathcal{L}_1 = -\sqrt{3}\alpha e^{-\frac{2}{\sqrt{3}}\alpha|y|} \bar{\epsilon}^i \delta_a^\mu \gamma^a \psi_{\mu i} \delta(y), \quad \delta_e \mathcal{L}_2 = \sqrt{3}\alpha e^{-\frac{2}{\sqrt{3}}\alpha|y|} \bar{\epsilon}^i \delta_a^\mu \gamma^a \psi_{\mu i} \delta(y-L). \quad (2.219)$$

Again, these brane variations exactly cancel the boundary terms (2.216) coming from the bulk variation of \mathcal{L}_{EH} . We thus conclude that the original $\mathcal{N} = 1$ supersymmetry preserved by the linear dilaton background on $\mathbb{R}^{1,4}$ remains preserved after the compactification of the fifth direction on S^1/\mathbb{Z}_2 , provided the branes added at the two boundaries of the interval are NS5-branes.

2.6 Compactified $D = 4$ effective theory

The total bosonic Lagrangian of the $\mathcal{N} = 2$, $D = 5$ supergravity theory introduced in Section 2.5.3 has been written in (2.174). The compactification of the graviton-dilaton system performed in Section 2.4 has revealed two important features: on the LD background, the KK vector K_μ coming from the $5D$ metric becomes massive by absorbing a scalar combination φ_+ , identified with the string frame radion, while only the orthogonal combination φ_- , identified as the $4D$ dilaton, remains massless, with a vanishing effective scalar potential. Since we know that the LD background breaks 1/2 of the original supersymmetries, we should be able to write the effective $\mathcal{N} = 1$, $D = 4$ supergravity after identifying all massive states and decoupling them from the massless spectrum. To this purpose, in the following two subsections we will first dimensionally reduce the remaining part of the bosonic action, taking into account that the zero modes of the fields may depend on the compactified coordinate y , in contrast with standard KK reduction. Then, in the scalar sector we will first need to identify the massless $4D$ scalars and complexify them in a consistent way to form a chiral matter multiplet. In the vector sector, since we already know that the KK vector K_μ becomes massive, we will need to identify a second massive vector, so that both of them could form the bosonic content of a massive spin-3/2 multiplet, decoupled from the spectrum in the low energy limit.

2.6.1 General considerations on dimensional reduction

The five-dimensional action of the two vectors A_M^0 and A_M^1 contains, in addition to the kinetic terms S_{kin} , a Chern-Simons term S_{CS} ,

$$S = S_{kin} + S_{CS}, \quad (2.220)$$

with

$$S_{kin} = \int d^5x \sqrt{-G} \left\{ -\frac{1}{8} e^{\frac{4}{\sqrt{3}}\phi} \hat{F}_{MN}^0 \hat{F}^{MN0} - \frac{1}{4} e^{-\frac{2}{\sqrt{3}}\phi} \hat{F}_{MN}^1 \hat{F}^{MN1} \right\}, \quad (2.221)$$

$$S_{CS} = \int d^5x \frac{1}{6\sqrt{6}} C_{011} \hat{\epsilon}^{MNPQR} \left(\hat{A}_M^0 \hat{F}_{NP}^1 \hat{F}_{QR}^1 + 2\hat{A}_M^1 \hat{F}_{NP}^1 \hat{F}_{QR}^0 \right), \quad (2.222)$$

where we have denoted all the five dimensional quantities with a hat in order to distinguish them from their four dimensional counterparts. The standard dimensional reduction of a given tensor field is usually performed in the vielbein formalism rather than the metric one. Using local Lorentz transformations, one can write the 5D Einstein frame fünfbein \hat{e}_M^m and its inverse \hat{e}_m^M as:

$$\hat{e}_M^m = e^{-\frac{\alpha}{\sqrt{3}}|y|} \begin{pmatrix} e^{-\frac{1}{2}r} \tilde{e}_\mu^a(x) & e^r K_\mu(x, y) \\ 0 & e^r \end{pmatrix}, \quad \hat{e}_m^M = e^{\frac{\alpha}{\sqrt{3}}|y|} \begin{pmatrix} e^{\frac{1}{2}r} \tilde{e}_a^\mu(x) & -e^{\frac{1}{2}r} K_a(x, y) \\ 0 & e^{-r} \end{pmatrix}, \quad (2.223)$$

whose squares obviously reproduce the metric (2.82) and its inverse (2.83). The dimensional reduction of the vector kinetic terms is then carried out in the following way: we first identify the five and four dimensional vectors \hat{A}_a and A_a on flat indices, and then we use the fünfbein parametrisation (2.223) to relate five and four-dimensional vectors with curved indices:

$$A_a \equiv \hat{A}_a = \hat{e}_a^M \hat{A}_M = \hat{e}_a^\mu \hat{A}_\mu + \hat{e}_a^5 \hat{A}_5 = e^{\frac{\alpha}{\sqrt{3}}|y|} e^{\frac{1}{2}r} \tilde{e}_a^\mu (\hat{A}_\mu - A_5 K_\mu), \quad (2.224)$$

where we have defined $\hat{A}_5 = A_5$. This construction automatically implies invariance of the 4D vector $A_\mu = \hat{A}_\mu - A_5 K_\mu$ under the $U(1)_{KK}$ ¹⁰. A similar analysis for the field strength yields:

$$F_{ab} \equiv \hat{F}_{ab} = e^{\frac{2}{\sqrt{3}}\alpha|y|} e^r \tilde{e}_a^\mu \tilde{e}_b^\nu \left\{ 2\partial_{[\mu} A_{\nu]} + 2A_5 \partial_{[\mu} K_{\nu]} - 2K_{[\mu} \partial_5 (A_{\nu]} + A_5 K_{\nu]} \right\}, \quad (2.225)$$

$$\hat{F}_{a5} = e^{\frac{2}{\sqrt{3}}\alpha|y|} e^{-\frac{1}{2}r} \tilde{e}_a^\mu \left\{ \partial_\mu A_5 - \partial_5 (A_\mu + A_5 K_\mu) \right\}. \quad (2.226)$$

Putting everything together, using $\sqrt{-G} = e^{-\frac{5\alpha}{\sqrt{3}}|y|} e^{-r} \sqrt{-\tilde{g}}$ and moving from the scalar basis $(r, \delta\phi)$ to (φ_-, φ_+) , we get the vector kinetic action in terms of the four-dimensional quantities:

$$\begin{aligned} S_{kin} = & - \frac{1}{8} \int d^5x e^{\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ e^{2\varphi_+ - \varphi_-} \left[F_{\mu\nu}(A^0) + A_5^0 F_{\mu\nu}(K) - 2K_{[\mu} \partial_5 (A_{\nu]} + A_5^0 K_{\nu]} \right]^2 \right. \\ & \left. + 2e^{-2\varphi_-} \left[\partial_\mu A_5^0 - \partial_5 (A_\mu^0 + A_5^0 K_\mu) \right]^2 \right\} \\ & - \frac{1}{4} \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ e^{\varphi_-} \left[F_{\mu\nu}(A^1) + A_5^1 F_{\mu\nu}(K) - 2K_{[\mu} \partial_5 (A_{\nu]} + A_5^1 K_{\nu]} \right]^2 \right. \\ & \left. + 2e^{-2\varphi_+} \left[\partial_\mu A_5^1 - \partial_5 (A_\mu^1 + A_5^1 K_\mu) \right]^2 \right\}, \quad (2.227) \end{aligned}$$

where in the right-hand-side (RHS), contractions are made with the 4D metric $\tilde{g}_{\mu\nu}(x)$. The dimensional reduction of the Chern-Simons term works in a similar way. We first write its expression in terms of flat indices and then identify the four and five-dimensional Levi-Civita tensors to be equal on flat indices, $\hat{\epsilon}^{abcd5} = \epsilon^{abcd}$. This leads to:

$$\begin{aligned} C_{IJK} \hat{\epsilon}^{MNPQR} \hat{A}_M^I \hat{F}_{NP}^J \hat{F}_{QR}^K &= C_{IJK} \sqrt{-G} \hat{\epsilon}^{mnpqr} \hat{A}_m^I \hat{F}_{np}^J \hat{F}_{qr}^K \\ &= C_{IJK} \sqrt{-G} \epsilon^{abcd} \left(\hat{F}_{ab}^I \hat{F}_{cd}^J \hat{A}_5^K - 4\hat{F}_{ab}^I \hat{F}_{c5}^J \hat{A}_d^K \right). \quad (2.228) \end{aligned}$$

10. $\delta_{\xi^5} A_\mu = 0$ holds only in the standard case when A_μ is independent of the compactified coordinate. If A_μ has a y -dependence, it transforms under the $U(1)_{KK}$ according to $\delta_{\xi^5} A_\mu = \xi^5 \partial_5 A_\mu$.

Using the expressions for \hat{F}_{ab} , \hat{F}_{a5} and \hat{A}_a obtained above, converting back flat into curved indices and integrating by parts, we obtain:

$$\begin{aligned}
C_{IJK} \hat{\epsilon}^{MNPQR} \hat{A}_M^I \hat{F}_{NP}^J \hat{F}_{QR}^K &= C_{IJK} \epsilon^{\mu\nu\rho\sigma} \left\{ 3A_5^I F_{\mu\nu}^J(A) F_{\rho\sigma}^K(A) + 3A_5^I A_5^J F_{\mu\nu}^K(A) F_{\rho\sigma}(K) \right. \\
&+ A_5^I A_5^J A_5^K F_{\mu\nu}(K) F_{\rho\sigma}(K) - 4(F_{\mu\nu}^I(A) + A_5^I F_{\mu\nu}(K)) K_{[\rho} \partial_5 \hat{A}_{\sigma]}^J A_5^K \\
&+ \left. 4(F_{\mu\nu}^I(A) + A_5^I F_{\mu\nu}(K)) \partial_5 \hat{A}_\rho^J A_\sigma^K + 8K_{[\mu} \partial_5 \hat{A}_{\nu]}^I (\partial_\rho A_5^J - \partial_5 \hat{A}_\rho^J) A_\sigma^K \right\}. \quad (2.229)
\end{aligned}$$

The difference compared to standard KK compactification, where the zero modes of the fields are assumed to be independent of the compactified coordinates, lies in the terms proportional to $\partial_5(\dots)$: they must be kept in the framework of the LD background, since the latter may introduce an explicit y -dependence even on the zero modes of the fields. These considerations are general in the sense that they do not depend on the background, and can be used in other frameworks where the fields may have given dependences on the compactified coordinates.

2.6.2 $\mathcal{N} = 1$, $D = 4$ effective theory

Having performed the dimensional reduction of the spin-1 action, one can now find the $4D$ spectrum of the zero modes of the different fields, and in particular check that the massless spectrum arranges into a $\mathcal{N} = 1$ supersymmetric effective theory. Since we are ultimately interested in the massless limit, we can set the massive Kaluza-Klein vector $K_\mu = 0$ in the dimensionally reduced action obtained above. Up to quadratic order in A_μ^I , A_5^I , the kinetic action (2.227) then reads:

$$\begin{aligned}
S_{kin} &= \int d^5x \sqrt{-\tilde{g}} \left\{ -\frac{1}{8} e^{\sqrt{3}\alpha|y|} e^{-\varphi_- + 2\varphi_+} F_{\mu\nu}^0 F^{0\mu\nu} - \frac{1}{4} e^{-\sqrt{3}\alpha|y|} e^{\varphi_-} F_{\mu\nu}^1 F^{1\mu\nu} \right. \\
&\quad \left. - \frac{1}{4} e^{\sqrt{3}\alpha|y|} e^{-2\varphi_-} (\partial_\mu A_5^0 - \partial_5 A_\mu^0)^2 - \frac{1}{2} e^{-\sqrt{3}\alpha|y|} e^{-2\varphi_+} (\partial_\mu A_5^1 - \partial_5 A_\mu^1)^2 \right\}, \quad (2.230)
\end{aligned}$$

while the Chern-Simons action (2.229), for $C_{011} \neq 0$ only, yields:

$$S_{CS} = \int d^5x \frac{1}{6\sqrt{6}} C_{011} \epsilon^{\mu\nu\rho\sigma} \left(3A_5^0 F_{\mu\nu}^1 F_{\rho\sigma}^1 + 6A_5^1 F_{\mu\nu}^1 F_{\rho\sigma}^0 \right). \quad (2.231)$$

In order to ease the comparison between the Einstein and Maxwell sectors, both of which should contribute to the $\mathcal{N} = 1$ effective supergravity, we also recall the (quadratic) dimensionally reduced action of the Kaluza-Klein vector K_μ and the scalars φ_- and φ_+ coming from the $5D$ dilaton and gravitational actions, obtained previously in Section 2.4.2:

$$\begin{aligned}
S_{dil+G} &= \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ -\frac{1}{4} e^{\varphi_- + 2\varphi_+} F^2(K) \right. \\
&\quad \left. - \frac{3}{2} \left[\alpha \operatorname{sgn} y K_\mu - \frac{1}{\sqrt{3}} \partial_\mu \varphi_+ \right]^2 - \frac{1}{4} (\partial_\mu \varphi_-)^2 \right\}. \quad (2.232)
\end{aligned}$$

The point is that we would like to determine which are the massless $\mathcal{N} = 1$ multiplets and the corresponding truncation of the dimensionally reduced action.

In the scalar sector, the exponentials in front of the kinetic terms of A_5^0 and A_5^1 in (2.230) tell us that φ_- and φ_+ must be complexified with A_5^0 and A_5^1 respectively. From the action for

φ_- found in section 2.4.3, $S[\varphi_-] = -\frac{1}{4} \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} (\partial_\mu \varphi_-)^2$, one sees that the exponential of y factorizes out of the kinetic terms of A_5^0 and φ_- if and only if A_5^0 is of the form

$$A_5^0 = e^{-\sqrt{3}\alpha|y|} \tilde{A}_5^0(x), \quad (2.233)$$

which gives for the scalar action:

$$S_{\text{scalar}} = -\frac{1}{4} \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ (\partial_\mu \varphi_-)^2 + e^{-2\varphi_-} (\partial_\mu \tilde{A}_5^0(x))^2 \right\}. \quad (2.234)$$

Defining the complex scalar field

$$S = e^{\varphi_-} + i \tilde{A}_5^0(x), \quad (2.235)$$

the above scalar action can then be rewritten as

$$S_{\text{scalar}} = - \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \frac{\partial_\mu S \partial^\mu \bar{S}}{(S + \bar{S})^2}, \quad (2.236)$$

from which we deduce the Kahler potential to be $K(S, \bar{S}) = -\ln(S + \bar{S})$.

In the vector sector, the $\mathcal{N} = 1$ spectrum can be found by considering the 4D Chern-Simons term (2.231). Denoting $f(S)$ the holomorphic gauge kinetic function, the resulting $\mathcal{N} = 1$ theory must have a topological term of the form $\text{Im} f F \wedge F$. Such term can only come from the first one of (2.231) $A_5^0 F^1 F^1$, which is consistent with the fact that A_5^1 , being complexified with φ_+ , must ultimately be projected out of the spectrum. We therefore deduce that the A_μ^1 vector remains massless and sit in a $\mathcal{N} = 1$ vector multiplet, while A_μ^0 must acquire a mass, by absorbing A_5^1 . From the kinetic terms (2.230), one sees that such a mechanism can be obtained by assuming a mixing of the zero modes of the vectors of the form

$$A_\mu^1 = \tilde{A}_\mu^1(x) + e^{\sqrt{3}\alpha|y|} A_\mu^0(x). \quad (2.237)$$

The factor in the exponential, a priori arbitrary, is found by requiring the two massive vectors K_μ and A_μ^0 to have the same mass, since they should form together with the massive gravitino a massive $\mathcal{N} = 1$ spin-3/2 multiplet, namely

$$m_{U(1)_{KK}}^2 = m_{U(1)^0}^2 = \frac{3}{2} \alpha^2. \quad (2.238)$$

In the low energy limit, obtained by truncating the massive spectrum, the remaining bosonic action reads (we denote $F^1 \equiv F(\tilde{A}^1(x))$):

$$S_{\text{bos.}} = \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ \partial_S \partial_{\bar{S}} K \partial_\mu S \partial^\mu \bar{S} - \frac{1}{4} \text{Re} f F_{\mu\nu}^1 F^{1\mu\nu} + \frac{1}{8} \frac{\text{Im} f}{\sqrt{-\tilde{g}}} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^1 F_{\rho\sigma}^1 \right\}, \quad (2.239)$$

which, after integration on y , gives the bosonic Lagrangian of a $D = 4$, $\mathcal{N} = 1$ supersymmetric theory, defined by the Kahler potential K , superpotential W and gauge kinetic function f given by:

$$K(S, \bar{S}) = -\ln(S + \bar{S}), \quad W(S) = 0, \quad f(S) = S. \quad (2.240)$$

It is easy to check that these results are consistent with a $\mathcal{N} = 1$ supersymmetric spectrum

$\mathcal{N} = 2$ supergravity multiplet		$\mathcal{N} = 2$ vector multiplet		$\mathcal{N} = 2$ vector multiplet	
$g_{\mu\nu}$	$g_{\mu 5}$	A_μ^0	$e^{\varphi^-} + i\tilde{A}_5^0$	A_μ^1	$e^{\varphi^+} + iA_5^1$
+	–	–	+	+	–

Table 2.2 – Bosonic content of the $D = 4$, $\mathcal{N} = 2$ multiplets and their \mathbb{Z}_2 -parity.

obtained from a standard orbifold compactification. The four-dimensional theory contains two $\mathcal{N} = 2$ vector multiplets $(A_\mu^0, e^{\varphi^-} + i\tilde{A}_5^0)$ and $(A_\mu^1, e^{\varphi^+} + iA_5^1)$. Two truncations $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ can then be considered, putting A_μ^1 and $e^{\varphi^-} + i\tilde{A}_5^0$ to zero, or A_μ^0 and $e^{\varphi^+} + iA_5^1$ to zero. The first case is obviously excluded, since no topological term would remain in (2.231), which is consistent with the gauge $\varphi_+ = 0$ set in Section 2.4.2, where we also identified φ_- with the massless scalar field. We are thus forced to truncate the A_μ^0 and A_5^1 , by assigning them a \mathbb{Z}_2 -odd parity, while A_μ^1 and A_5^0 are kept even. In Appendix E we show that the $5D$ vector A_M^0 is dual to the $5D$ Kalb-Ramond two form B_{MN} . Truncating A_μ^0 therefore amounts to assigning an orbifold \mathbb{Z}_2 -odd parity to $B_{\mu 5}$, and thus an even parity to $B_{\mu\nu}$, the $4D$ dual of A_5^0 . Regarding the string frame radion φ_+ , it is obviously even under the \mathbb{Z}_2 of the orbifold, and odd under the discrete \mathbb{Z}_2 subgroup of the T-duality group, which inverts the radius of compactification and interchanges at the same time $B_{\mu 5}$ with the KK vector K_μ . The full truncation must therefore combine both \mathbb{Z}_2 transformations; it projects out half of the degrees of freedom of the original theory, namely 6 bosonic and 6 fermionic on-shell degrees of freedom. These results are summarised in Table 2.2.

The $\mathcal{N} = 1$, $D = 4$ multiplet content is then easily obtained after dimensional reduction of the supersymmetry (susy) transformations introduced in Section 2.5.1, using in particular the $5D$ frame field parametrisation (2.223). In the following, we will also work with the linear combinations of the fermions λ_+ and λ_- defined by

$$\lambda_+ \equiv \lambda_1 + i \operatorname{sgn} y \gamma_5 \lambda_2, \quad \lambda_- \equiv \lambda_1 - i \operatorname{sgn} y \gamma_5 \lambda_2. \quad (2.241)$$

From the susy transformation of the dilaton (2.133), we deduce the transformation of the physical scalar φ_- in the direction of the preserved supersymmetry:

$$\delta_L e^{\varphi^-} = -\frac{i}{\sqrt{3}} e^{\varphi^-} \bar{\epsilon}^1 \lambda_+. \quad (2.242)$$

Since we know the y -dependence of the spinor $\bar{\epsilon}^1$,

$$\bar{\epsilon}^1 = e^{-\frac{\alpha}{2\sqrt{3}}|y|} \bar{\epsilon}, \quad (2.243)$$

with ϵ the constant Killing spinor associated with the $\mathcal{N} = 1$ preserved supersymmetry, we deduce that the zero mode of the spinor λ_+ must satisfy

$$\lambda_+ = e^{\frac{\alpha}{2\sqrt{3}}|y|} \tilde{\lambda}_+(x), \quad (2.244)$$

where $\tilde{\lambda}_+(x)$ is a spinor independent of y . Similarly, the transformations of the $5D$ vectors (2.134)

yields for the 4D scalars A_5^0 :

$$\delta_L A_5^0 = -\frac{1}{\sqrt{3}} e^{-\sqrt{3}\alpha|y|} e^{\varphi_-} \bar{\epsilon}^1 \gamma_5 \lambda_+. \quad (2.245)$$

The combination $\bar{\epsilon}^1 \gamma_5 \lambda_+$ being independent of y , we find again that A_5^0 must be of the form $A_5^0 = e^{-\sqrt{3}\alpha|y|} \tilde{A}_5^0(x)$, as already obtained above. The transformation of the complex scalar $S = e^{\varphi_-} + i\tilde{A}_5^0$ is thus given by:

$$\delta_L S = -\frac{i}{\sqrt{3}} e^{\varphi_-} \bar{\epsilon}^1 (1 + \gamma_5) \lambda_+. \quad (2.246)$$

In the vector sector, the susy transformations of A_μ^1 in the direction of the preserved supersymmetry is given by:

$$\delta_L A_\mu^1 = \frac{1}{2\sqrt{3}} e^{-\frac{1}{2}\varphi_-} \bar{\epsilon}^1 \tilde{e}_\mu^a(x) \gamma_a \lambda_-. \quad (2.247)$$

Similarly as for λ_+ , using $\bar{\epsilon}^1 = e^{-\frac{\alpha}{2\sqrt{3}}|y|} \bar{\epsilon}$ and the fact that A_μ^1 is a function of x only tell us that λ_- must have a y -dependence of the form

$$\lambda_- = e^{\frac{\alpha}{2\sqrt{3}}|y|} \tilde{\lambda}_-(x). \quad (2.248)$$

The transformations of the fermions λ_+ and λ_- respectively read:

$$\begin{aligned} \delta_L \lambda_+ &= \frac{i}{\sqrt{3}} e^{\frac{\alpha}{\sqrt{3}}|y|} e^{\frac{1}{6}(\varphi_- + 2\varphi_+)} \tilde{e}_a^\mu(x) \gamma^a \partial_\mu (\varphi_- + 2\varphi_+) \epsilon_1 \\ &+ \frac{1}{\sqrt{3}} \tilde{e}_a^\mu(x) \gamma^a \gamma^5 \left[e^{\frac{4\alpha}{\sqrt{3}}|y|} e^{-\frac{5}{6}\varphi_- + \frac{1}{3}\varphi_+} (\partial_\mu A_5^0 - \partial_5 A_\mu^0) - e^{\frac{\alpha}{\sqrt{3}}|y|} e^{\frac{1}{6}\varphi_- - \frac{2}{3}\varphi_+} (\partial_\mu A_5^1 - \partial_5 A_\mu^1) \right] \epsilon_1, \end{aligned} \quad (2.249)$$

$$\delta_L \lambda_- = \frac{1}{2\sqrt{3}} \tilde{e}_a^\mu(x) \tilde{e}_b^\nu(x) \gamma^a \gamma^b \left[e^{\frac{4\alpha}{\sqrt{3}}|y|} e^{-\frac{1}{3}\varphi_- + \frac{4}{3}\varphi_+} F_{\mu\nu}^0 - e^{\frac{\alpha}{\sqrt{3}}|y|} e^{\frac{2}{3}\varphi_- + \frac{1}{3}\varphi_+} F_{\mu\nu}^1 \right] \epsilon_1. \quad (2.250)$$

After gauge fixing the scalars φ_+ and A_5^1 to zero, truncating A_μ^0 and using $A_5^0 = e^{-\sqrt{3}\alpha|y|} \tilde{A}_5^0(x)$ and that $A_\mu^1 = \tilde{A}_\mu^1(x)$ when $A_\mu^0 = 0$, they lead for the four-dimensional spinors $\tilde{\lambda}_+(x)$ and $\tilde{\lambda}_-(x)$:

$$\delta_L \tilde{\lambda}_+(x) = \frac{i}{\sqrt{3}} e^{\frac{1}{6}\varphi_-} \tilde{e}_a^\mu(x) \gamma^a \partial_\mu \varphi_-(x) + \frac{1}{\sqrt{3}} \tilde{e}_a^\mu(x) \gamma^a \gamma^5 e^{-\frac{5}{6}\varphi_-} \partial_\mu \tilde{A}_5^0(x) \epsilon(x), \quad (2.251)$$

$$\delta_L \tilde{\lambda}_-(x) = -\frac{1}{2\sqrt{3}} e^{\frac{2}{3}\varphi_-} \tilde{e}_a^\mu(x) \tilde{e}_b^\nu(x) \gamma^a \gamma^b F_{\mu\nu}^1 \epsilon(x). \quad (2.252)$$

The gravitational multiplet is then easily obtained from

$$\delta_L \tilde{e}_\mu^a(x) = \frac{1}{2} e^{\frac{\alpha}{\sqrt{3}}|y|} e^{\frac{1}{6}\varphi_- + \frac{1}{3}\varphi_+} \bar{\epsilon}^1 \gamma^a \psi_{\mu-}, \quad (2.253)$$

where we have defined $\psi_{\mu-} \equiv \psi_{\mu 1} - i \operatorname{sgn} y \gamma_5 \psi_{\mu 2}$ and used $e_\mu^a = e^{-\frac{\alpha}{\sqrt{3}}|y|} e^{-\frac{1}{2}r} \tilde{e}_\mu^a(x)$. The same argument as for λ_\pm imposes the 4D gravitino $\psi_{\mu-}$ to be of the form

$$\psi_{\mu-} = e^{-\frac{\alpha}{2\sqrt{3}}|y|} \tilde{\psi}_{\mu-}(x), \quad (2.254)$$

so that the 4D transformation, after gauge fixing $\varphi_+ = 0$, reads:

$$\delta_L \tilde{e}_\mu^a(x) = \frac{1}{2} e^{\frac{1}{6}\varphi_-} \bar{\epsilon}(x) \gamma^a \tilde{\psi}_{\mu-}(x). \quad (2.255)$$

Finally, one has to rescale the different fermions in order to get both canonical kinetic terms and standard susy transformations. Since we want to keep standard gravitino transformation of the form $\delta_L \psi_\mu = \partial_\mu \epsilon + \dots$, the parameter ϵ and the gravitino ψ_μ have to be rescaled by the same powers of e^{φ^-} , and we thus define the normalised fermions $\tilde{\epsilon}(x)$ and $\tilde{\psi}_{\mu-}(x)$ by:

$$\tilde{\epsilon}(x) = e^{\frac{1}{12}\varphi^-} \tilde{\epsilon}(x), \quad \tilde{\psi}_{\mu-}(x) = e^{\frac{1}{12}\varphi^-} \tilde{\psi}_{\mu-}(x). \quad (2.256)$$

The correct normalisations for the chiral fermion $\tilde{\lambda}_+$ and the gaugino $\tilde{\lambda}_-$ can be obtained from their kinetic terms, which after dimensional reduction are found to be:

$$\mathcal{L}_{kin}(\tilde{\lambda}_\pm) = -\frac{1}{4} e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} e^{-\frac{1}{6}\varphi^- - \frac{1}{3}\varphi_+} \tilde{\lambda}_\pm(x) \tilde{e}_\mu^a(x) \gamma^a \partial_\mu \tilde{\lambda}_\pm(x). \quad (2.257)$$

Defining the normalised fermions $\tilde{\lambda}_+(x)$ and $\tilde{\lambda}_-(x)$ by,

$$\tilde{\lambda}_+(x) = e^{\frac{11}{12}\varphi^-} \tilde{\lambda}_+(x), \quad \tilde{\lambda}_-(x) = e^{-\frac{7}{12}\varphi^-} \tilde{\lambda}_-(x), \quad (2.258)$$

their kinetic terms have now the correct powers of e^{φ^-} matching with the ones of their bosonic partners, namely (in the gauge $\varphi_+ = 0$):

$$\mathcal{L}_{kin}(\tilde{\lambda}_+) = -\frac{1}{4} e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} e^{-2\varphi^-} \tilde{\lambda}_+(x) \tilde{e}_\mu^a(x) \gamma^a \partial_\mu \tilde{\lambda}_+(x), \quad (2.259)$$

$$\mathcal{L}_{kin}(\tilde{\lambda}_-) = -\frac{1}{4} e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} e^{\varphi^-} \tilde{\lambda}_-(x) \tilde{e}_\mu^a(x) \gamma^a \partial_\mu \tilde{\lambda}_-(x). \quad (2.260)$$

With these normalisations, the different factors of e^{φ^-} disappear in the susy transformations, and we end up with the standard $\mathcal{N} = 1$ supersymmetric transformations involving only four-dimensional x -dependent fields:

$$\delta_L S = -\frac{i}{\sqrt{3}} \bar{\epsilon} (1 + \gamma_5) \tilde{\lambda}_+, \quad \delta_L A_\mu^1 = \frac{1}{2\sqrt{3}} \bar{\epsilon} \gamma_\mu \tilde{\lambda}_-, \quad (2.261)$$

$$\delta_L [(1 + \gamma_5) \tilde{\lambda}_+] = \frac{i}{\sqrt{3}} (1 + \gamma_5) \not{\partial} S \epsilon, \quad \delta_L \tilde{\lambda}_- = -\frac{1}{2\sqrt{3}} \gamma^\mu \gamma^\nu F_{\mu\nu}^1 \epsilon, \quad (2.262)$$

where all x -dependences have been now left implicit for compactness. These results are summarised in Table 2.3.

$\mathcal{N} = 1$ supergravity multiplet		$\mathcal{N} = 1$ vector multiplet		$\mathcal{N} = 1$ chiral multiplet	
e_μ^a	$\tilde{\psi}_{\mu-}$	A_μ^1	$\tilde{\lambda}_-$	$S = e^{\varphi^-} + i\tilde{A}_5^0$	$(1 + \gamma_5) \tilde{\lambda}_+$
$-\frac{1}{3}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	(0 ; -1)	$\frac{1}{6}$

Table 2.3 – $\mathcal{N} = 1$, $D = 4$ field content

In the second row we have listed the constants n which appear in the exponent of the internal wave functions of the zero modes of the five-dimensional fields $\Phi(x, y)$ through $\Phi^{(0)}(x, y) = e^{n\sqrt{3}\alpha|y|} \tilde{\Phi}(x)$.

2.7 Conclusion

The work carried out in this chapter analysed different aspects of the linear dilaton background arising from a runaway scalar potential in five dimensions, in relation to two different perspectives: compactification and supersymmetry breaking.

On the one hand, we performed the KK compactification down to four dimensions of the dilaton-gravity action of a non-critical string, emphasising the new features emerging from the non-trivial background compared to the standard (toroidal) flat case. Besides the known mass gap between the $4D$ zero mode of the $5D$ fields and the lowest state of their KK tower, it has been shown how the LD background may induce an exponential profile of the wavefunction of the zero modes of some fields. In the case of spin-1, this behaviour brings a vector mass term which breaks the residual gauge symmetry, inherited from the original higher dimensional one, at a scale proportional to the slope of the LD background. This mechanism has been explicitly described in the case of the KK vector $G_{\mu 5}$ arising from the $5D$ metric G_{MN} , through a gauge symmetry analysis which generalises the well-known results of the toroidal case.

On the other hand in a supersymmetric theory, the LD background behaves as a 1/2-BPS solution and can thus be used to study supersymmetry breaking. This aspect has been investigated in the framework of the $\mathcal{N} = 2$, $D = 5$ gauged supergravity coupled to one vector multiplet, built as a holographic dual of Little String Theory.

Both aspects are then combined together by carrying out the compactification down to four dimensions of the full bosonic sector of the above $5D$ supergravity theory. We have shown how the lowest massive spectrum associated to the mass gap, induced by an exponential profile of the vector wavefunctions, can be decoupled from the massless sector in the low energy limit, so that the remaining degrees of freedom arrange into $\mathcal{N} = 1$ multiplets under the supersymmetry preserved by the background. The corresponding supersymmetric truncation is consistent with a standard orbifold projection.

The possibility to describe a consistent $\mathcal{N} = 2$, $D = 4$ supergravity theory by including the above lowest massive modes is an interesting open problem. A starting point would be to examine if the LD background induces a gauging in the 4-dimensional $\mathcal{N} = 2$ theory that can lead to a (spontaneous) partial supersymmetry breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$. This could bring new insights regarding the partial supersymmetry breaking in supergravity, which is highly restrictive and requires so far the use of hypermultiplets [63, 64, 65] (see also [10]).

Intersecting D-brane models and the anomalous magnetic moment of the muon

3.1 The anomalous magnetic moment of the muon: a review

The *Landé g-factor* of a particle is a proportionality constant between its spin \vec{s} and magnetic moment $\vec{\mu}$ defined by

$$\vec{\mu} = g \frac{e}{2m} \vec{s}, \quad (3.1)$$

where e and m are the electric charge and mass of the particle, and $\vec{s} = 1/2\vec{\sigma}$ for a spin-1/2 particle, with $\vec{\sigma}$ the Pauli matrices. The Dirac equation for an elementary spin-1/2 particle implies that $g = 2$. However, quantum corrections will deviate the value of g from 2, leading to an *anomalous magnetic moment* defined by $a \equiv (g - 2)/2$.

In Quantum Electrodynamics (QED), the value of the Landé g-factor of a lepton and its loop corrections can be extracted from the general lepton scattering process depicted in Figure 3.1, as discussed in standard textbooks like [66, 67]. Using Lorentz invariance, the Ward identity $p_\mu \mathcal{M}^\mu = 0$ and the fact that QED is parity-conserving, we can parametrize this amplitude as:

$$i\mathcal{M}_{\text{all-orders}}^\mu = -ie \bar{u}(q_2) \left[\gamma^\mu f_1 \left(\frac{p^2}{m^2} \right) + (q_1 + q_2)^\mu f_2 \left(\frac{p^2}{m^2} \right) \right] u(q_1). \quad (3.2)$$

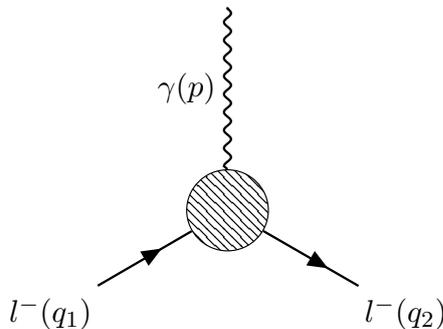


Figure 3.1 – Amplitude $i\mathcal{M}_{\text{all-orders}}^\mu$ containing all-order contributions to the anomalous magnetic moment of a lepton l^- .

The Gordon identity for on-shell spinors,

$$2m \bar{u}(q_2)\gamma^\mu u(q_1) = \bar{u}(q_2)(q_1^\mu + q_2^\mu + i\gamma^{\mu\nu}p_\nu)u(q_1), \quad (3.3)$$

allows us to rewrite the second term of (3.2) in terms of $\gamma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, yielding:

$$i\mathcal{M}_{\text{all-orders}}^\mu = -ie \bar{u}(q_2) \left[\gamma^\mu F_1 \left(\frac{p^2}{m^2} \right) + \frac{i\gamma^{\mu\nu}p_\nu}{2m} F_2 \left(\frac{p^2}{m^2} \right) \right] u(q_1). \quad (3.4)$$

F_1 and F_2 are functions of $\frac{p^2}{m^2}$ called *form factors*. At tree level, it is easy to see that $F_1 = 1$ and $F_2 = 0$. These functions contain all the properties of the interaction between a lepton with an electromagnetic field: $F_1(0)$ gives the electric charge of the lepton, while $F_2(0)$ gives its Landé g-factor according to:

$$F_2(0) = \frac{g - 2}{2}. \quad (3.5)$$

Therefore, computing the QED one-loop correction to the anomalous magnetic moment amounts to extract from the process represented in Figure 3.1 the one-loop correction to F_2 . This is a standard computation which can be found for instance in [66, 67]. In this section, we summarise the main steps of this derivation, in the more general case where the boson propagating in the loop is massive. This will be the starting point of the analysis carried out in the next section, where we consider the correction to g coming from an infinite tower of (leptophilic) massive vectors.

There is a single one-loop Feynman diagram contributing to F_2 , depicted in Figure 3.2. Straight lines represent the lepton of mass m , the external wavy line a photon, while the internal wavy line depicts in the general case a massive gauge boson of mass M . The particular QED result, involving a photon running in the loop, will be recovered by taking $M = 0$.

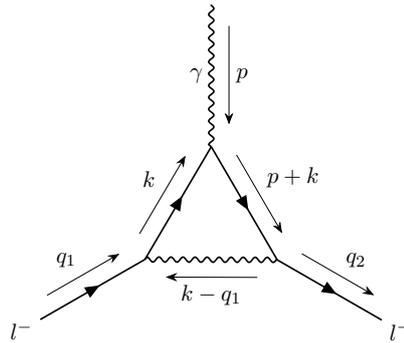


Figure 3.2 – Amplitude $i\mathcal{M}_{1\text{-loop}}^\mu$ containing the one-loop contribution to the anomalous magnetic moment of a lepton l^- .

The corresponding amplitude is given by:

$$\begin{aligned} i\mathcal{M}_{1\text{-loop}}^\mu &= \int \frac{d^4k}{(2\pi)^4} D^{\nu\rho}(k - q_1) \bar{u}(q_2)(-ig\gamma_\nu) \frac{i(\not{p} + \not{k} + m)}{(p + k)^2 - m^2 + i\epsilon} \\ &\quad \times (-ie\gamma^\mu) \frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon} (-ig\gamma_\rho) u(q_1), \end{aligned} \quad (3.6)$$

where $D^{\nu\rho}(k - q_1)$ is the propagator of the massive gauge boson, which reads in the unitary gauge:

$$D^{\nu\rho}(k - q_1) = \frac{-i}{(k - q_1)^2 - M^2 + i\epsilon} \left(g^{\nu\rho} - \frac{(k - q_1)^\nu (k - q_1)^\rho}{M^2} \right). \quad (3.7)$$

The second term in (3.7) will not contribute to the lepton magnetic moment, so that the relevant part of the amplitude (3.6) we have to consider reads:

$$i\mathcal{M}^\mu = -g^2 e \int \frac{d^4 k}{(2\pi)^4} \frac{N^\mu}{ABC}, \quad (3.8)$$

where we have defined

$$A \equiv k^2 - m^2 + i\epsilon, \quad B \equiv (p + k)^2 - m^2 + i\epsilon, \quad (3.9)$$

$$C \equiv (k - q_1)^2 - M^2 + i\epsilon, \quad N^\mu \equiv \bar{u}(q_2) \gamma_\nu (\not{p} + \not{k} + m) \gamma^\mu (\not{k} + m) \gamma^\nu u(q_1). \quad (3.10)$$

In order to evaluate the integral (3.8), we use the Feynman parametrization

$$\frac{1}{ABC} = 2 \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{1}{(xA + yB + zC)^3} \quad (3.11)$$

$$= 2 \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{1}{(\ell^2 - \Delta + i\epsilon)^3}, \quad (3.12)$$

where we moved from the first to the second line using $x + y + z = 1$ and momentum conservation $p^\mu + q_1^\mu = q_2^\mu$, introducing the quantities

$$\ell^\mu \equiv k^\mu + yp^\mu - zq_1^\mu, \quad \Delta \equiv -xyp^2 + (1 - z)^2 m^2 + zM^2. \quad (3.13)$$

Let us now focus on the numerator N^μ in (3.10). Using the relation $\gamma_\nu \gamma^\mu \gamma^\nu = -2\gamma^\mu$, we can first rewrite it as $N^\mu = -2\bar{u}(q_2) \left(\not{k} \gamma^\mu \not{p} + \not{k} \gamma^\mu \not{k} + m^2 \gamma^\mu - 2m(2k^\mu + p^\mu) \right) u(q_1)$. Writing it in terms of ℓ^μ , using the relations

$$\int \frac{d^4 \ell}{(2\pi)^4} \ell^\mu f(\ell^2) = 0, \quad \int \frac{d^4 \ell}{(2\pi)^4} \ell^\mu \ell^\nu f(\ell^2) = \int \frac{d^4 \ell}{(2\pi)^4} \frac{g^{\mu\nu} \ell^2}{4} f(\ell^2), \quad (3.14)$$

for any function f of the magnitude of ℓ , and after some Dirac algebra, one obtains:

$$N^\mu = -2\bar{u}(q_2) \left[\gamma^\mu \left(-\frac{1}{2} \ell^2 + (1 - x)(1 - y)p^2 + (1 - 2z - z^2)m^2 \right) + (q_1^\mu + q_2^\mu) m z (z - 1) + p^\mu m (z - 2)(x - y) \right] u(q_1). \quad (3.15)$$

The third term proportional to p^μ is odd under $x \leftrightarrow y$, while the denominator of (3.8) is even under $x \leftrightarrow y$, as can be seen from (3.12). This term therefore vanishes after integration. The second term of (3.15) proportional to $q_1^\mu + q_2^\mu$ can be rewritten using the Gordon identity (3.3), leading to:

$$N^\mu = -2\bar{u}(q_2) \left[\gamma^\mu \left(-\frac{\ell^2}{2} + (1 - x)(1 - y)p^2 + (1 - 4z + z^2)m^2 \right) - i\gamma^{\mu\nu} p_\nu m z (z - 1) \right] u(q_1). \quad (3.16)$$

As explained above, only the second term proportional to $\gamma^{\mu\nu}$ will contribute to the magnetic

moment. Using the identity

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(l^2 - \Delta + i\epsilon)^3} = -\frac{i}{32\pi^2\Delta}, \quad (3.17)$$

the part of the amplitude (3.8) proportional to $\gamma^{\mu\nu}$ then reads:

$$i\mathcal{M}^\mu \supset \frac{g^2 em}{16\pi^2} \bar{u}(q_2)\gamma^{\mu\nu}u(q_1)p_\nu \int_0^1 dx dy dz \delta(x+y+z-1) \frac{z(z-1)}{\Delta}. \quad (3.18)$$

Comparing with (3.4), we find that the one-loop correction to the form factor F_2 is given by:

$$F_2\left(\frac{p^2}{m^2}\right) = \frac{g^2 m^2}{4\pi^2} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{z(1-z)}{-xy p^2 + (1-z)^2 m^2 + zM^2}, \quad (3.19)$$

so that the one-loop correction to the anomalous magnetic moment $a \equiv \frac{g-2}{2} = F_2(0)$ reads:

$$a = \frac{g^2 m^2}{4\pi^2} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{z(1-z)}{(1-z)^2 m^2 + zM^2}. \quad (3.20)$$

In QED, the one-loop correction to the magnetic moment comes from the diagram represented in Figure 3.2 with a photon running in the loop. It is simply obtained from (3.20) by taking $M = 0$, leading to

$$a^{\text{QED}} = \frac{\alpha}{2\pi}, \quad (3.21)$$

with $\alpha = \frac{e^2}{4\pi}$ the fine structure constant. This is a historically famous result obtained by Schwinger in 1948 [68], whose agreement with experimental data for the magnetic moment of the electron has been one of the biggest success of Quantum Electrodynamics.

The case of the anomalous magnetic moment of the muon ($(g-2)_\mu$) is more involved. The theoretical calculation, involving QED, electroweak and hadronic contributions, yields a result $a_\mu^{\text{SM}} \equiv (g-2)_\mu^{\text{SM}}/2 = 116\,591\,810(43) \times 10^{-11}$ [69], smaller than the experimental measurement from the Brookhaven National Laboratory (BNL) experiment E821 by 3.7σ [70]. The recent Muon $g-2$ experiment at Fermilab has confirmed the BNL results [71], pushing the discrepancy with the SM theoretical prediction to 4.7σ , with a difference ¹

$$\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (2.51 \pm 0.59) \times 10^{-9}. \quad (3.22)$$

The anomalous magnetic moment of the muon might therefore be one of the most promising signals of new physics beyond the Standard Model, and it is worth studying new ingredients able to explain the above mentioned discrepancy, which is the aim of the third chapter of this thesis.

1. This result must be tempered by the theoretical uncertainties coming from strong interaction effects. The analysis for the hadronic vacuum polarisation contribution has been carried out in [72, 73, 74]. A recent result from lattice QCD reduce the discrepancy to 1.6σ [75], while producing tensions with other quantities at the same time [76, 77], so that the hadronic contributions to the muon $(g-2)_\mu$ remains an open issue to be addressed by future lattice simulations.

The Chapter 3 is organised as follows. In Section 3.2, we revisit a new solution to the muon $g - 2$ discrepancy recently proposed in the framework of low mass scale strings and large extra dimensions. It relies on the contribution to the $(g - 2)_\mu$ of the Kaluza-Klein modes of the lepton number gauge boson, which could bridge the gap in the discrepancy if some of them are sufficiently light. The Section 3.3 reminds some basic concepts of the physics of intersecting D-branes, which are then used in Section 3.4 in order to build a minimal embedding of the SM into such intersecting D-brane configurations realising this proposal. In Section 3.4.1, we show that the gauge group of this model is $SU(3)_c \times SU(2)_w \times U(1)_c \times U(1)_w \times U(1) \times U(1)' \times U(1)_L$ and construct various D-brane configurations realising it, listing the different possibilities for the quantum numbers of the SM spectrum and the allowed Yukawa couplings for the quarks and leptons. The anomaly analysis of these models is performed in Section 3.4.2 without the right-handed neutrino, in the non-supersymmetric case as well as in the minimal supersymmetric extension, and then in Section 3.4.3 with the inclusion of the right-handed neutrino. Section 3.4.4 discusses the gauge bosons mass spectrum arising from the four-dimensional generalisation of the Green-Schwarz mechanism. The inclusion of Dark Matter candidates in this framework is described in Section 3.4.5. Finally, in Section 3.4.6, we investigate the possibility of introducing lepton flavour non-universality by gauging only the muonic lepton number that can explain the $(g - 2)_\mu$ discrepancy due to the exchange of KK excitations that couple only to muons and are thus not constrained by the LEP and LHC bounds. Our conclusions are presented in Section 3.5.

3.2 Accommodating the $(g - 2)_\mu$ discrepancy with leptophilic Kaluza-Klein states

The possibility to explain the muon $g - 2$ discrepancy in the framework of low mass scale strings and large extra dimensions has been recently studied in [78], where three contributions to the muon anomalous magnetic moment have been examined: from Regge excitations of the string, from anomalous $U(1)$ gauge bosons, as well as from Kaluza-Klein (KK) modes of a bulk vector field. While the first contribution is strongly suppressed, it has been shown how the second one can reduce, but not fully bridge, the discrepancy.

However, the most interesting contribution comes from the KK modes of the lepton number gauge boson, denoted L_μ in the following, living on a lepton brane $U(1)_L$ extended along some extra dimensions of the bulk. The zero mode of L_μ is anomalous and acquires a mass through a four-dimensional generalisation of the Green-Schwarz mechanism. If its mass is of order the string scale, it is too heavy to accommodate the $(g - 2)_\mu$ discrepancy. If it is lighter due to volume suppression, it can only partially explain the discrepancy since the zero mode is a linear combination of the various $U(1)$ factors, coupled to both quarks and leptons and thus subject to stringent LHC bounds [78, 79]. On the other hand, the KK excitations couple only to leptons to lowest order. Therefore, such modes evade the LHC bounds, and their masses can be made sufficiently light to provide a significant contribution to the $(g - 2)_\mu$ [79]. The aim of this section is to reexamine the consistency of this scenario with model constraints from LEP data. While the constraints on KK masses and couplings are more stringent than earlier thought in [79], we

show that the discrepancy can still be explained assuming the existence of a few leptophilic KK modes lighter than the LEP energy, $\sqrt{s}|_{\text{LEP}} \sim 200$ GeV.

At the leading order in the $U(1)_L$ coupling constant g_L , the contribution of massive vector bosons to the $(g-2)_\mu$ has been derived in Section 3.1. We have obtained the result (3.20) which is reproduced here for convenience:

$$\Delta a_\mu = \frac{\alpha_L m_\mu^2}{\pi} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{z(1-z)}{(1-z)^2 m_\mu^2 + zM^2}, \quad (3.23)$$

where M is the mass of the boson, m_μ the muon mass and $\alpha_L = g_L^2/(4\pi)$. One can then consider three different cases, depending whether $M \gg m_\mu$, $M \sim m_\mu$ or $M \ll m_\mu$.

Case 1: $M \gg m_\mu$

When all KK states have masses much bigger than the muon mass, the sum of the integral (3.23) over all the KK states can be approximated by

$$\Delta a_\mu^{(1)} = \sum_n \frac{1}{3} \frac{\alpha_L(n)}{\pi} \frac{m_\mu^2}{M_n^2}, \quad (3.24)$$

where M_n is the mass of the n th KK excitation [79].

The bound from LEP data on the so-called compositeness scale associated to 4-fermion operators is given by [80]:

$$\left| \sum_n \frac{\alpha_L(n)}{s - M_n^2} \right| < B \sim (10 \text{ TeV})^{-2}, \quad (3.25)$$

where s is the square of the center-of-mass energy². For $M_n \gg \sqrt{s}$, (3.25) reduces to $\sum_n \alpha_L(n)/M_n^2 < B$. Thus, the sum of the KK exchange given in (3.24) is constrained by the compositeness bound, yielding $\Delta a_\mu^{(1)} \sim \mathcal{O}(10^{-11})$; a result which is independent on the number of extra dimensions. Hence, one needs at least few KK modes lighter than LEP energy in order to provide a significant contribution able to bridge the gap in the muon anomalous magnetic moment.

A crucial point to take into account is that the gauge coupling is suppressed by the volume of the compact space $V_\perp \sim (RM_s)^d$,

$$g_L^2 = g_s/V_\perp, \quad (3.26)$$

where g_s is the string coupling, R is the compactification scale, M_s is the string scale, and d stands for the number of extra dimensions in which L propagates. For $d = 1$, we have $M_n = n/R$

2. For fine-tuned values of M_n close to \sqrt{s} , the vector boson propagator appearing in the left-hand side of (3.25) is regulated by replacing $\frac{1}{s-M_n^2}$ by $\frac{1}{s-M_n^2+i\Gamma_n M_n}$, with Γ_n the decay rate of the n -th KK mode. Since the number of possible decay channels of the KK excitations increase for higher modes, Γ_n increases with n and its explicit computation would require a model dependent analysis.

and after substituting these figures into (3.24), $\Delta a_\mu^{(1)}$ becomes³

$$\Delta a_\mu^{(1)} = \frac{g_s m_\mu^2}{72 M_1 M_s}. \quad (3.27)$$

The observed value of Δa_μ then implies

$$M_1 M_s \sim g_s \times 5 \times 10^4 \text{ GeV}^2, \quad (3.28)$$

where $g_s \lesssim 4\pi$ to remain in the perturbative regime.

As an illustration, if we take $M_s = 10 \text{ TeV}$ ⁴ then we have $M_1 \sim g_s \times 5 \text{ GeV}$, so that the highest possible value for the compactification scale M_1 , obtained for $g_s = 4\pi$, is of order $M_1 \sim 60 \text{ GeV}$, which is consistent with the condition $m_\mu \ll M_1 \ll \sqrt{s}$ for all the approximations. The associated gauge coupling is then of order $g_L \sim 10^{-1}$. Taking $\sqrt{s}|_{\text{LEP}} = 209 \text{ GeV}$, the total KK contribution to the LEP bound is given by

$$\left| \sum_n \frac{g_L^2}{4\pi(s - n^2 M_1^2)} \right| \sim 10^{-2} \text{ TeV}^{-2}, \quad (3.29)$$

and hence the bound (3.25) is satisfied.

Case 2: $M \sim m_\mu$

In the case of a massive boson with a mass of order of the muon mass m_μ , its contribution (3.23) to $(g-2)_\mu$ is given by

$$\Delta a_\mu^{(2)} = \frac{\alpha_L}{\pi} \frac{-9 + 2\sqrt{3}\pi}{18}. \quad (3.30)$$

If the lightest KK state have a mass $M_1 \sim m_\mu$, the total contribution to the muon anomalous magnetic moment is therefore the sum of $\Delta a_\mu^{(1)}$ (Eq. (3.24) for $n > 1$) and $\Delta a_\mu^{(2)}$ (Eq. (3.30)), which in the case of one extra dimension yields:

$$\Delta a_\mu = \frac{g_s}{4\pi^2} \frac{m_\mu}{M_s} \left(\frac{-9 + 2\sqrt{3}\pi}{18} + \frac{1}{3} \sum_{n>1} \frac{1}{n^2} \right). \quad (3.31)$$

The $(g-2)_\mu$ discrepancy can then be accommodated for a string scale at $M_s \sim g_s \times 3 \times 10^2 \text{ TeV}$, yielding a coupling $g_L \sim 5 \times 10^{-4}$, now independent of g_s . With $M_1 = m_\mu = 105 \text{ MeV}$, we now get

$$\left| \sum_n \frac{g_L^2}{4\pi(s - n^2 M_1^2)} \right| \sim 10^{-4} \text{ TeV}^{-2}, \quad (3.32)$$

so that the bound (3.25) is also satisfied.

3. We have neglected here the n -dependence of the gauge coupling of the n -th KK excitation, given in the case of one extra dimension by $g_L(n) = g_L \exp\left\{-cn^2 \frac{M_1^2}{M_s^2}\right\}$, with c a positive (model dependent) numerical constant. When $M_1 \ll M_s$, as it is the case in the large extra dimension scenario considered here, the exponential is of order 1 for all $n \lesssim \frac{M_s}{M_1}$, and the gauge coupling can indeed be taken constant. The exponential suppression of g_L becomes significant only for higher KK modes with $n \gg \frac{M_s}{M_1}$, which give a negligible contribution to $\Delta a_\mu^{(1)}$.

4. In order to lower the string scale in this region, one assumes in general additional large extra dimensions transverse to both SM and L stacks of branes. They do not play any role in the current analysis and will be briefly discussed in Section 3.4.

Case 3: $M \ll m_\mu$

We can also consider the situation where some of the lightest KK states have masses much lower than the muon mass, in which case the integral (3.23) gives a constant contribution $\frac{\alpha_L}{2\pi}$. Multiplying by $\frac{m_\mu}{M_1}$, the number of states with masses below m_μ , and assuming again one extra dimension, we get the contribution

$$\Delta a_\mu^{(3)} = \frac{g_s}{8\pi^2} \frac{m_\mu}{M_s}. \quad (3.33)$$

The total contribution to the muon anomalous magnetic moment is then the sum of $\Delta a_\mu^{(1)}$ (Eq. (3.24) for $n > \frac{m_\mu}{M_1} + 1$), $\Delta a_\mu^{(2)}$ (Eq. (3.30)) and $\Delta a_\mu^{(3)}$ (Eq. (3.33)), that is, in the case of one extra dimension:

$$\Delta a_\mu = \frac{g_s}{8\pi^2} \frac{m_\mu}{M_s} \left(1 + 2 \cdot \frac{-9 + 2\sqrt{3}\pi}{18} + \frac{2}{3} \frac{m_\mu}{M_1} \sum_{n=\frac{m_\mu}{M_1}+2} \frac{1}{n^2} \right). \quad (3.34)$$

As an example, let us take $\frac{m_\mu}{M_1} = 10$, in which case $\Delta a_\mu \sim \frac{g_s}{8\pi^2} \frac{m_\mu}{M_s}$, accommodating the discrepancy for a string scale $M_s \sim g_s \times 5 \times 10^2$ TeV. With $M_1 = \frac{m_\mu}{10} = 10.5$ MeV, one gets a coupling $g_L \sim 10^{-4}$, again independent of g_s , from which we can evaluate

$$\left| \sum_n \frac{g_L^2}{4\pi(s - n^2 M_1^2)} \right| \sim 6 \times 10^{-4} \text{ TeV}^{-2}, \quad (3.35)$$

again satisfying the bound (3.25).

Let us note that unlike the discrepancy between the experimental value and the SM prediction of the muon anomalous magnetic moment which is positive, $\Delta a_\mu^{\text{exp}} \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (251 \pm 59) \times 10^{-11}$, the discrepancy of the electron anomalous magnetic moment is negative, $\Delta a_e^{\text{exp}} \equiv a_e^{\text{exp}} - a_e^{\text{SM}} = -88(36) \times 10^{-14}$ [81]. The contributions coming from the KK excitations being positive, they will increase the discrepancy of $(g - 2)_e$, and we thus have to check that this contribution is lower than or of order of the experimental error on $(g - 2)_e$, that is $\lesssim 10^{-13}$. Assuming $M_1 \gg m_e$ where m_e is the electron mass, this contribution is simply obtained by replacing the muon mass m_μ by the electron mass m_e in (3.24), namely

$$\Delta a_e = \frac{m_e^2}{m_\mu^2} \Delta a_\mu^{(1)} = \frac{m_e^2 g_s}{72 M_1 M_s}. \quad (3.36)$$

For the different values obtained above for M_1 and M_s , we get in the case 1 $\Delta a_e \sim 10^{-14}$, and in the cases 2 and 3 $\Delta a_e \sim 10^{-13}$, indeed smaller than or of order of the error on $(g - 2)_e$.

This scenario can be easily described in terms of intersecting D-brane models, where extra dimensions and extra $U(1)$ bosons are ubiquitous. In the next section, we first remind the basic ideas of such D-brane configurations, and then use this framework in Section 3.4 in order to embed the Standard Model in this scenario with a leptophilic gauge boson propagating along one extra dimension.

3.3 Elements of intersecting D-brane physics

A D p -brane is an object extended along p spatial dimensions. It spans a $(p+1)$ -dimensional worldvolume, which can be seen at weak coupling as a subspace of spacetime on which open strings end. Let us denote d the number of spatial dimensions, $D = d + 1$ the total number of spacetime dimensions, $\{x^0, \dots, x^p\}$ the spacetime coordinates tangential to the D p -brane and $\{x^{p+1}, \dots, x^d\}$ the spacetime coordinates normal to the brane. Denoting $\tau \in \mathbb{R}$ and $\sigma \in [0, \pi]$ the string worldsheet coordinates, its embedding spacetime coordinates $X^M(\tau, \sigma)$, $M = 0, \dots, D-1$, are split in a similar pattern according to $\{X^+, X^-, X^\mu, \mu = 2, \dots, p\}$, and $\{X^i, i = p+1, \dots, d\}$, for the coordinates tangential and normal to the brane respectively. We now consider an open string whose both endpoints lie on the D p -brane. Its coordinates normal to the brane must therefore satisfy Dirichlet-Dirichlet (DD) boundary conditions, while the ones tangential to the brane must satisfy Neumann-Neumann (NN) boundary conditions:

$$\begin{aligned} X^i(\tau, \sigma) \Big|_{\sigma=0, \pi} &= 0, \quad i = p+1, \dots, d, \quad (\text{DD}) \\ \partial_\sigma X^\mu(\tau, \sigma) \Big|_{\sigma=0, \pi} &= 0, \quad \mu = 2, \dots, p, \quad (\text{NN}). \end{aligned} \quad (3.37)$$

The mode expansions compatible with these boundary conditions read:

$$X^i(\tau, \sigma) = \sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^i}{n} \sin(n\sigma) e^{-in\tau}, \quad (\text{DD}) \quad (3.38)$$

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} \cos(n\sigma) e^{-in\tau}, \quad (\text{NN}) \quad (3.39)$$

where x^μ and p^μ are the center of mass coordinate and momentum of the string and α_n the oscillation modes. The 2-dimensional worldsheet fermions $\psi^M(\tau, \sigma)$, superpartners of the worldsheet bosons $X^M(\tau, \sigma)$, are written $\psi^M = \begin{pmatrix} \psi_+^M \\ \psi_-^M \end{pmatrix}$, where ψ_+^M and ψ_-^M are $2M$ 2-dimensional Majorana-Weyl spinors. Splitting again the spacetime index M into $\mu = 2, \dots, p$ for the directions tangential to the brane (in the light-cone gauge), and $i = p+1, \dots, d$ for the directions normal to the brane, the ψ_\pm^i must satisfy DD boundary conditions, while the ψ_\pm^μ must satisfy NN boundary conditions, which are respectively given by:

$$\begin{aligned} \psi_+^i(\tau, \sigma=0) &= -\psi_-^i(\tau, \sigma=0), \quad \psi_+^i(\tau, \sigma=\pi) = -\epsilon \psi_-^i(\tau, \sigma=\pi) \quad i = p+1, \dots, d, \quad (\text{DD}) \\ \psi_+^\mu(\tau, \sigma=0) &= \psi_-^\mu(\tau, \sigma=0), \quad \psi_+^\mu(\tau, \sigma=\pi) = \epsilon \psi_-^\mu(\tau, \sigma=\pi), \quad \mu = 2, \dots, p, \quad (\text{NN}), \end{aligned} \quad (3.40)$$

with $\epsilon = -1$ for the worldsheet fermions in the Neveu-Schwarz (NS) sector, and $\epsilon = +1$ for the worldsheet fermions in the Ramond (R) sector. The mode expansions compatible with these boundary conditions read:

$$\psi_\pm^i(\tau, \sigma) = \pm i \sqrt{\frac{\alpha'}{2}} \sum_r \psi_r^i e^{-ir(\tau \pm \sigma)}, \quad (\text{DD}) \quad (3.41)$$

$$\psi_\pm^\mu(\tau, \sigma) = i \sqrt{\frac{\alpha'}{2}} \sum_r \psi_r^\mu e^{-ir(\tau \pm \sigma)}, \quad (\text{NN}) \quad (3.42)$$

with $r \in \mathbb{Z} + \frac{1}{2}$ in the NS sector, and $r \in \mathbb{Z}$ in the R sector.

Since we are interested in the spacetime bosons, we focus on the NS sector. The NS ground-state $|0\rangle_{\text{NS}}$ is defined as being annihilated by positive modding operators, namely

$$\alpha_n^P |0\rangle_{\text{NS}} = 0, \quad \psi_{n-1/2}^P |0\rangle_{\text{NS}} = 0, \quad \forall n > 0, \quad \forall P = 2, \dots, d. \quad (3.43)$$

The bosonic massless spectrum of an open superstring with both ends satisfying DD boundary conditions in $d - p$ directions and NN boundary conditions in $p + 1$ directions then arises from $\psi_{-1/2}^P |0\rangle_{\text{NS}}$. Since a Dp -brane breaks the Lorentz group into $SO(1, d) \rightarrow SO(1, p) \times SO(d - p)$, we must therefore arrange the states into representations of the residual symmetry group according to:

$$\psi_{-1/2}^i |0\rangle_{\text{NS}} \rightarrow \phi^i, \quad (3.44)$$

$$\psi_{-1/2}^\mu |0\rangle_{\text{NS}} \rightarrow A_\mu, \quad (3.45)$$

where ϕ^i is a $SO(d - p)$ vector and A_μ a $SO(1, p)$ vector. From the point of view of the Dp -brane worldvolume with the Lorentz group $SO(1, p)$, we thus get a vector field A_μ and $d - p$ real scalars ϕ^i . The ϕ^i can be seen as Goldstone bosons arising from the spontaneous breaking of the translational symmetry induced by the brane, and describe the fluctuations of the brane in the $d - p$ transverse directions.

Let us then consider a stack of N parallel and coincident Dp -branes. An open string stretching from the brane $a = 1, \dots, N$ to the brane $b = 1, \dots, N$ is said to sit in the ab sector. Since strings are oriented, the ab and ba sectors are different, and we end with N^2 different open string sectors. The open string spectrum of a stack of N parallel and coincident Dp -branes is thus given by N^2 copies of the spectrum of a single Dp -brane that we found above. The N^2 gauge bosons can then interact in the following way. Let us consider two open strings in the ab and bc sectors. Since the end of the first string and the beginning of the second string lie on the same brane, they can join together yielding an open string in the ac sector. We thus get N^2 interacting massless gauge fields, so that the gauge group is enhanced to $U(N)$. The two independent labels $a, b = 1, \dots, N$ at both ends of the open strings are discrete degrees of freedom called *Chan-Paton* indices. The addition of parallel and coincident Dp -branes therefore provides an easy way to generate non-abelian gauge groups in String Theory, which lie at the core of any models of particle physics. However, another cornerstone of the Standard Model is the chirality of its fermionic spectrum, which cannot be obtained yet in such coincident Dp -brane framework.

Chirality in four dimensions is a violation of 4-dimensional parity. However, 4-dimensional parity is related to the parity in the six extra dimensions. In order to violate 4-dimensional chirality, one must therefore violate 6-dimensional parity. This cannot be obtained with a single stack of N coincident branes, since this configuration does not produce a preferred orientation in the 6-dimensional internal space. To this purpose, one must consider at least two stacks of N_1 and N_2 branes which intersect over a 4-dimensional subspace of their worldvolumes. The rotation from the first stack to the second one provides an orientation in the transverse 6-dimensional space, breaking $6D$ parity and hence leading to 4-dimensional chirality. In the R sector, an open string stretching between two intersecting stacks of N_1 and N_2 branes leads to

a massless 4-dimensional chiral fermion localized at the intersection of their worldvolumes. If a string stretching from the stack 1 to the stack 2 gives a left-handed fermion, then its right-handed counterpart will correspond to an oppositely oriented string stretching from the stack 2 to the stack 1. Such states transform in the bi-fundamental representation $(\mathbf{N}_1, \overline{\mathbf{N}}_2)_{1,-1}$ of the gauge group $U(N_1) \times U(N_2)$, with the subscripts denoting the $U(1)_1 \times U(1)_2$ charges. This charge assignment is required to get consistent string interactions. Let us consider an open string stretching from the stack 1 to the stack 2, in the $a_1 b_2$ sector, and another string stretching from the stack 2 to the stack 1, in the $b_2 c_1$ sector, with $a_1, c_1 = 1, \dots, N_1$ and $b_2 = 1, \dots, N_2$. Interactions of such open strings happen on the b_2 brane in the stack 2, when the end of the first one joins the beginning of the second one. This requires their $U(1)_2$ charges to be opposite, hence justifying the charge assignment given above. Finally, in the NS sector, an open string stretching between two intersecting stacks of branes gives a light spectrum of complex scalars, in which can be embedded the Higgs sector of the SM.

To summarise, a single D-brane gives rise to a $U(1)$ gauge theory living on its worldvolume, with the associated gauge boson corresponding to an open string with both ends attached to this brane. Non-abelian gauge symmetries arise from a stack of N coincident D-branes, giving rise to a $U(N)$ gauge symmetry. The (chiral) matter fields then correspond to open strings stretching between intersecting D-branes, living in their common worldvolume. These so-called *intersecting D-brane models* thus provide an elegant framework to obtain phenomenologically consistent particle physics models from String Theory [82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92].

The minimal intersecting D-brane models which can reproduce the SM gauge group $G_{SM} = SU(3)_c \times SU(2)_w \times U(1)_Y$ and its matter spectrum charged under G_{SM} contain three stacks of branes giving rise to a gauge symmetry $U(3)_c \times U(2)_w \times U(1)$ [93, 94]⁵. The ‘‘color’’ stack $U(3)_c$ and ‘‘weak’’ stack $U(2)_w$ are obtained by considering three and two coincident D-branes respectively. For phenomenological reasons, a third $U(1)$ factor arising from a single D-brane is necessary to accommodate the SM.

An open string with one end on the color branes transforms in the $\mathbf{3}$ (or $\overline{\mathbf{3}}$) of $SU(3)_c$; similarly, an open string with one end on the weak branes transforms as a doublet of $SU(2)_w$. The non-abelian structure partially fixes the assignments of the SM particles. The quark doublet Q corresponds to an open string with one end on the color stack and the other on the weak stack of D-branes, while the anti-quark singlets u^C and d^C have one of their ends on the color stack. The lepton doublet L as well as Higgs doublet(s) H must have one of their ends attached to the weak stack of branes. However, the abelian structure is not uniquely determined since the hypercharge can be a linear combination of the different abelian factors.

The standard normalisation for the $U(N) \simeq SU(N) \times U(1)_N$ generators that we will consider in the next section is $\text{Tr} T^a T^b = \delta^{ab}/2$, while the corresponding $U(1)_N$ charges are measured with respect to the coupling $g_N/\sqrt{2N}$, with g_N the $SU(N)$ coupling constant, so that the fundamental representation of $SU(N)$ has $U(1)_N$ charge unity.

5. The $U(2)_w$ may be reduced to $Sp(1) \simeq SU(2)$, reducing the number of $U(1)$ factors to two.

3.4 Minimal embedding of the Standard Model into intersecting D-brane configurations with a bulk leptonic $U(1)$

The aim of this section is to realise the proposal of Section 3.2 in the framework of intersecting D-brane models presented in Section 3.3, with matter and gauge fields corresponding to open strings ending on D-branes. We build a minimal embedding of the Standard Model (SM) into such intersecting D-brane configurations with a gauged lepton number associated to an abelian $U(1)_L$ D-brane. The crucial point to take into account is that such leptonic brane should extend into some extra dimensions transverse to the SM stacks of branes, in order to give rise to the leptophilic KK excitations required to bridge the gap in the muon anomalous magnetic moment. As a consequence, the corresponding $U(1)_L$ cannot contribute to the hypercharge linear combination, since this would lead to an unrealistic small gauge coupling suppressed by the volume of the extra dimensions. Assuming that the lepton number gauge boson propagates into one “large” extra dimension, the associated compactification scale M_L must then satisfy $M_L \sim \mathcal{O}(10 - 10^2 \text{ GeV})$ for a string scale $M_s \gtrsim 10 \text{ TeV}$, as described in Section 3.2. The total bulk transverse to the SM branes is then made of two inhomogeneous parts. First, the one with one large extra dimension described above along which the $U(1)_L$ brane extends, called “ L -bulk” in the following, whose size $R_L = M_L^{-1} \sim (10 - 10^2 \text{ GeV})^{-1}$ must be sufficiently low to explain the $(g - 2)_\mu$ discrepancy. Then, a second part with the remaining (at most five) additional extra dimensions transverse to both SM and L stacks of branes, called “gravitational bulk”, with an average size larger than the L -bulk in order to lower the string scale in the $\gtrsim \mathcal{O}(10 \text{ TeV})$ region [4, 5]. From the string theoretic relation $M_{Pl}^2 = g_s^{-2} M_s^8 V_{(6)}$, with M_{Pl} the four-dimensional Planck mass, g_s the string coupling (assumed to be of order 1), M_s the string scale and $V_{(6)} = R_L R_G^5$ the volume of the six-dimensional internal space, one gets an average size R_G of the gravitational bulk of the order $R_G \sim (0.1 \text{ GeV})^{-1}$.⁶ This special hierarchy of scales is summarized and depicted in Figure 3.3. In the following, using the term “bulk” alone will refer to some extra dimensions transverse to the SM branes when there is no need to specify whether such dimension(s) are in the L -bulk or in (part of) the gravitational bulk.

Besides the Standard Model color $U(3)_c$, weak $U(2)_w$ and abelian $U(1)$ stacks of branes localised effectively in four dimensions and the L -bulk $U(1)_L$ brane, we show that the minimal embedding of the SM in such configuration requires a fifth $U(1)'$ brane extended in the bulk, so that the total gauge group is $SU(3)_c \times SU(2)_w \times U(1)_c \times U(1)_w \times U(1) \times U(1)' \times U(1)_L$. Identifying $U(1)_c$ with the baryon number (B), we find two possible models, depending on whether the anti-quark u^C or d^C couples to the $U(1)'$. If $U(1)_c$ is not identified with the baryon number, a third model is possible, described here for completeness although not phenomenologically relevant since it forbids the presence of a mass term for the up-type quarks and may lead to dangerous baryon number violating processes. We will thus focus in the models where $U(1)_c$ is identified with the baryon number. In this case, it is shown that there is one anomaly-free $U(1)$ combination besides the hypercharge, which does not couple to the SM spectrum and which can thus remain massless or acquire a mass. The minimal supersymmetric extension of our model is also briefly discussed in the context of the anomaly analysis, and we show that the inclusion of the Higgsinos

6. In case of less than six large extra dimensions, $V_{(6)} = R_L R_G^n$ with $n < 5$, R_G becomes larger.

in the spectrum does not modify the result for the non-anomalous vector bosons obtained in the non-supersymmetric case. The presence of two branes extended in the bulk allows to introducing the right-handed neutrino ν_R as a state of an open string stretched between these two branes. When ν_R is included in the spectrum, the SM particles are now charged under a second anomaly-free $U(1)$ combination given by $B - L$. Depending on the charges of the right-handed neutrino, a third non-anomalous vector can arise, which remains invisible from the SM spectrum as in the situation without ν_R . These two bulk branes can also be used to introduce in a similar way Dark Matter (DM) particle candidates, as Dirac fermions corresponding to open strings stretched between them. By computing the cross-section of the annihilation process of the DM fermions into SM leptons, mediated by the KK excitations of L_μ , we find the masses of such DM particles which yield the correct DM relic density, in terms of the compactification scale M_L .

As already mentioned above, the anomalous gauge bosons must get a mass through a four-dimensional generalisation of the Green-Schwarz (GS) mechanism [95, 96, 97]. Such bosons form in general linear combinations of the various abelian factors associated to each stack of branes, some of them being (effectively) localised in four dimensions while some others propagating into (large) extra dimensions. We will show that such combinations are dominated in the large volume limit by the (zero mode of the) vectors propagating in the bulk which enter in the linear combination defining the anomalous bosons and can become massive with a string scale mass $\mathcal{O}(M_s)$ independent of the compactification scale.

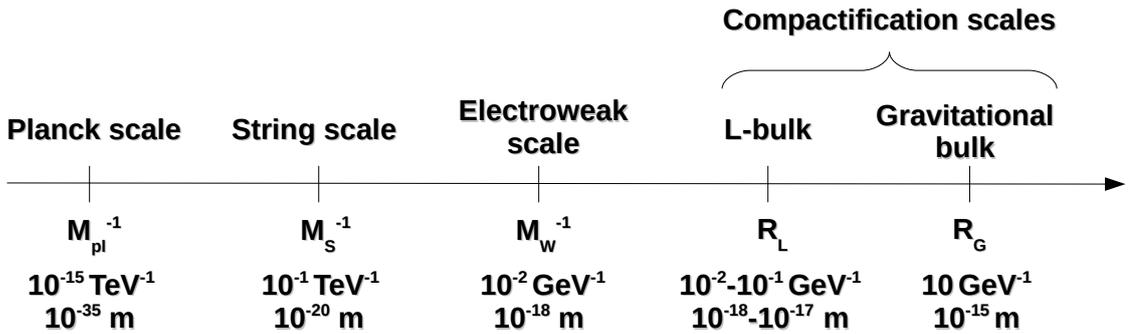


Figure 3.3 – Scale hierarchy in our model.

3.4.1 The models

In our construction, since we want to obtain leptophilic KK excitations, we require the lepton number gauge boson to be in the bulk and therefore one must consider a bulk leptonic $U(1)_L$ brane. As a starting point, let us consider it to be added to the minimal three stacks model $U(3)_c \times U(2)_w \times U(1)$ mentioned in Section 3.3, namely we consider the total gauge group:

$$G = SU(3)_c \times SU(2)_w \times U(1)_c \times U(1)_w \times U(1) \times U(1)_L. \quad (3.46)$$

Only the lepton doublet L and singlet e^C must have an end attached to the $U(1)_L$. In order to have a lepton number, L and e^C must have opposite q_L charges, chosen to be $+1$ and -1 respectively. The other ends of L and e^C must be attached to the $U(2)_w$ and $U(1)$ branes respectively. The q_1 charge of e^C and the $q_2 \equiv q_w$ charge of L can be defined to be $+1$. The q_2 charge v of the quark doublet Q can be either $+1$ or -1 if Q belongs to the fundamental $\mathbf{2}$ or anti-fundamental $\bar{\mathbf{2}}$ of $SU(2)_w$. The $q_3 \equiv q_c$ charge of Q is fixed to 1, while the ones of u^C and d^C are chosen to be -1 in order to get a baryon number⁷. As in the three stacks configuration described above, u^C and d^C must have one of their ends attached on the $U(3)_c$ branes, while the other end can be attached to the $U(1)$ brane ($\{x, y\} = \pm 1$) or be in the bulk ($x = 0$ and/or $y = 0$). The total matter content and their quantum numbers therefore reads:⁸

$$Q \quad (\mathbf{3}, \mathbf{2}; 1, v, 0, 0)_{1/6} \quad (3.47)$$

$$u^C \quad (\bar{\mathbf{3}}, \mathbf{1}; -1, 0, x, 0)_{-2/3} \quad (3.48)$$

$$d^C \quad (\bar{\mathbf{3}}, \mathbf{1}; -1, 0, y, 0)_{1/3} \quad (3.49)$$

$$L \quad (\mathbf{1}, \mathbf{2}; 0, 1, 0, 1)_{-1/2} \quad (3.50)$$

$$e^C \quad (\mathbf{1}, \mathbf{1}; 0, 0, 1, -1)_1, \quad (3.51)$$

with the hypercharge of each species indicated as a subscript for completeness. Since $U(1)_L$ is in the bulk, it should not contribute to the hypercharge, thus given by the combination

$$q_Y = c_3 q_3 + c_2 q_2 + c_1 q_1, \quad (3.52)$$

where c_3 , c_2 and c_1 are constants. The quantum numbers v , x and y as well as the constants c_i are now constrained by requiring the different states to have the correct hypercharge.

The charges of the leptons e^C and L fixes $c_1 = 1$ and $c_2 = -\frac{1}{2}$ respectively. The charges of the quark doublet Q then imposes $c_3 = \frac{1}{6} + \frac{v}{2}$. Finally, the anti-quarks u^C and d^C respectively leads to $x = -\frac{1}{2} + \frac{v}{2}$ and $y = \frac{1}{2} + \frac{v}{2}$. Since $v = \pm 1$, we have either $x = 0$ or $y = 0$, so that there is at least one end of the u^C or d^C strings elsewhere, which requires the existence of an additional brane $U(1)'$ besides the SM and L branes. To leave open the possibility of having this brane extended in the bulk, in order to avoid again an extremely small gauge coupling, we assume that $U(1)'$ does not participate to the hypercharge, therefore still given by (3.52). The

7. The other possible choice $+2$ for the q_3 charge of the anti-quarks u^C or d^C , which breaks the baryon number symmetry, is discussed below.

8. Here we are considering identical embedding for each of the three generations of quarks and leptons.

total gauge group now reads

$$G = SU(3)_c \times SU(2)_w \times U(1)_c \times U(1)_w \times U(1) \times U(1)' \times U(1)_L, \quad (3.53)$$

under which the matter content has the following quantum numbers:

$$Q \quad (\mathbf{3}, \mathbf{2}; 1, v, 0, 0, 0)_{1/6} \quad (3.54)$$

$$u^C \quad (\bar{\mathbf{3}}, \mathbf{1}; -1, 0, x, z, 0)_{-2/3} \quad (3.55)$$

$$d^C \quad (\bar{\mathbf{3}}, \mathbf{1}; -1, 0, y, w, 0)_{1/3} \quad (3.56)$$

$$L \quad (\mathbf{1}, \mathbf{2}; 0, 1, 0, 0, 1)_{-1/2} \quad (3.57)$$

$$e^C \quad (\mathbf{1}, \mathbf{1}; 0, 0, 1, 0, -1)_1. \quad (3.58)$$

Again, the constant $v = \pm 1$ specifies whether the quark doublet Q belongs to the fundamental $\mathbf{2}$ or anti-fundamental $\bar{\mathbf{2}}$ representation of $SU(2)_w$. The only ambiguities lie in the quantum numbers of the anti-quarks u^C, d^C : they must have one of their ends attached to the $U(3)_c$ branes, while the other ends can be tied to the $U(1)$ or $U(1)'$ branes. This choice, which will be fixed by assigning the correct hypercharges to the states, is encoded in the constants $x, y, z, w = \{0, \pm 1\}$.

We have as previously $c_1 = 1$, $c_2 = -\frac{1}{2}$ and $c_3 = \frac{1}{6} + \frac{v}{2}$, while the charges x and y are given by $x = -\frac{1}{2} + \frac{v}{2}$ and $y = \frac{1}{2} + \frac{v}{2}$. Two different models can then be considered, depending on whether $v = +1$ or $v = -1$.

- In the first case $v = +1$, we get $c_3 = \frac{2}{3}$, $x = 0$ and $y = 1$, which enforces z to be non-vanishing and $w = 0$. The u^C string has thus one end on the bulk brane $U(1)'$ while d^C is stretched between two branes participating to the hypercharge.
- The second case $v = -1$ amounts to exchange the u^C and d^C anti-quarks. Here we get $c_3 = -\frac{1}{3}$, $x = -1$ and $y = 0$, which implies necessarily that $z = 0$ and w is non-vanishing, so that u^C is now stretched between two branes participating to the hypercharge while d^C has one end on the bulk brane $U(1)'$.

These two models, defined by the choice $v = \pm 1$, will be respectively denoted A and B in the following. Since $U(1)'$ does not contribute to the hypercharge, the non-vanishing constants z in model A and w in model B can be independently chosen to ± 1 . Without loss of generality, we fix them to be $+1$.

One can then implement the Higgs doublets. It is easy to check that for each of the models A and B, two Higgs doublets (together with their complex conjugates) with vanishing charges Q_3 and Q_L and hypercharge $\pm 1/2$ are possible, given by:

$$\text{Model A} \quad : \quad H_d(\mathbf{1}, \mathbf{2}; 0, -1, -1, 0, 0)_{-1/2}, \quad H_u(\mathbf{1}, \mathbf{2}; 0, -1, 0, -1, 0)_{1/2}, \quad (3.59)$$

$$\text{Model B} \quad : \quad H_u(\mathbf{1}, \mathbf{2}; 0, 1, 1, 0, 0)_{1/2}, \quad H_d(\mathbf{1}, \mathbf{2}; 0, 1, 0, -1, 0)_{-1/2}. \quad (3.60)$$

The allowed Yukawa couplings then read:

$$\text{Model A} \quad : \quad Qd^C H_d, \quad Qu^C H_u, \quad Le^C H_d, \quad (3.61)$$

$$\text{Model B} \quad : \quad Qd^C H_d, \quad Qu^C H_u, \quad Le^C H_u^\dagger. \quad (3.62)$$

These results are summarized in Tables 3.1 and 3.2, and the two models are represented pictorially in Figure 3.4.

	q_3	q_2	q_1	q'_1	q_L	q_Y
Q	1	1	0	0	0	$\frac{1}{6}$
u^C	-1	0	0	1	0	$-\frac{2}{3}$
d^C	-1	0	1	0	0	$\frac{1}{3}$
L	0	1	0	0	1	$-\frac{1}{2}$
e^C	0	0	1	0	-1	1
H_d	0	-1	-1	0	0	$-\frac{1}{2}$
H_u	0	-1	0	-1	0	$\frac{1}{2}$

Table 3.1 – Model A with $q_Y = \frac{2}{3}q_3 - \frac{1}{2}q_2 + q_1$

	q_3	q_2	q_1	q'_1	q_L	q_Y
Q	1	-1	0	0	0	$\frac{1}{6}$
u^C	-1	0	-1	0	0	$-\frac{2}{3}$
d^C	-1	0	0	1	0	$\frac{1}{3}$
L	0	1	0	0	1	$-\frac{1}{2}$
e^C	0	0	1	0	-1	1
H_u	0	1	1	0	0	$\frac{1}{2}$
H_d	0	1	0	-1	0	$-\frac{1}{2}$

Table 3.2 – Model B with $q_Y = -\frac{1}{3}q_3 - \frac{1}{2}q_2 + q_1$

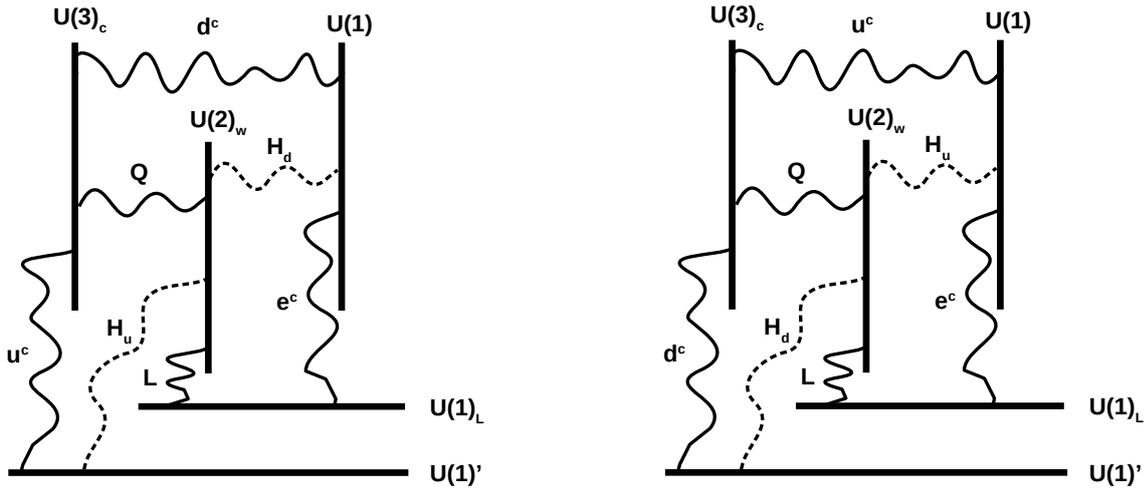


Figure 3.4 – Pictorial representation of models A and B.

One can also consider a model where the anti-quark u^C has both ends attached to the $U(3)$ stack of branes and its orientifold image, corresponding to the $\bar{\mathbf{3}}$ of $SU(3)$ obtained as the antisymmetric product of two $\mathbf{3}$'s. Repeating the analysis described above, we get the matter content summarized in Table 3.3.

	q_3	q_2	q_1	q'_1	q_L	q_Y
Q	1	-1	0	0	0	$\frac{1}{6}$
u^C	2	0	0	0	0	$-\frac{2}{3}$
d^C	-1	0	0	1	0	$\frac{1}{3}$
L	0	1	0	0	1	$-\frac{1}{2}$
e^C	0	0	1	0	-1	1
H_e	0	1	1	0	0	$\frac{1}{2}$
H_d	0	-1	0	1	0	$\frac{1}{2}$

Table 3.3 – Model C with $q_Y = -\frac{1}{3}q_3 - \frac{1}{2}q_2 + q_1$

In this case, an up quark mass term is no longer allowed since Qu^C has a non-vanishing q_3 charge. The only possible Yukawa couplings are

$$Qd^C H_d^\dagger, \quad Le^C H_e^\dagger, \quad (3.63)$$

where H_d and H_e have been defined in Table 3.3. Let us note that in this model, the role of u^C and d^C is not symmetric, and there is no consistent model with d^C in the $\bar{\mathbf{3}}$ of $SU(3)$. The model C is represented pictorially in Figure 3.5.

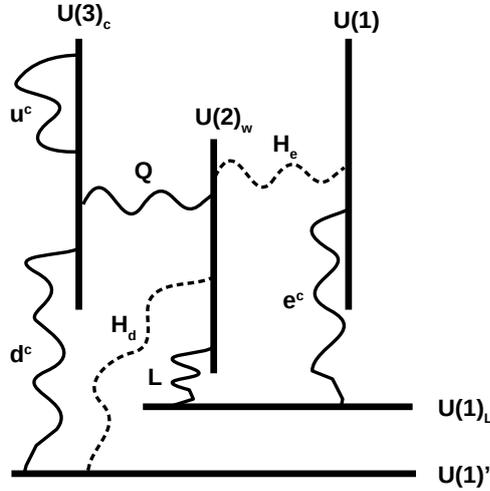


Figure 3.5 – Pictorial representation of model C.

3.4.2 Anomaly analysis

In order to find the anomalous and anomaly free combinations of the $U(1)$'s in the different models constructed above, we compute the anomaly matrix $K_{IJ} = \text{Tr} T_I^2 Q_J$, built from the mixed gauge and gravitational anomalies. The column of indices $J = \{c, w, 1, 1', L\}$ labels the abelian generators Q_J , while the line indices are $I = \{SU(3), SU(2), Y, \text{Grav}\}$, with $T_{\text{Grav}} = \mathbb{1}$

for the gravitational anomalies. One gets, for the two models A and B:

$$\text{Model A : } K^{(A)} = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & 2 & 0 & 0 & \frac{1}{2} \\ -\frac{3}{2} & \frac{2}{3} & \frac{4}{3} & \frac{4}{3} & -\frac{1}{2} \\ 0 & 8 & 4 & 3 & 1 \end{pmatrix}, \quad (3.64)$$

$$\text{Model B : } K^{(B)} = \begin{pmatrix} 0 & -1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -1 & 0 & 0 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \\ 0 & -4 & -2 & 3 & 1 \end{pmatrix}. \quad (3.65)$$

Diagonalizing the matrices $K^\top K$, one finds the anomaly free $U(1)$'s as the eigenvectors associated to the vanishing eigenvalues, the other eigenvectors being anomalous. In addition to the hypercharge q_Y , one finds in both models a second anomaly free $U(1)$ given by:

$$\text{Model A : } q \equiv \frac{1}{3}q_3 - \frac{1}{2}q_2 + q'_1 + q_L, \quad (3.66)$$

$$\text{Model B : } q \equiv -\frac{2}{3}q_3 - \frac{1}{2}q_2 - q'_1 + q_L. \quad (3.67)$$

It is easy to see that the q charges of the SM particles are minus their hypercharge q_Y , namely the SM sees only one anomaly free $U(1)$ as expected. The second anomaly free combination reads $q_3 - q_2 + q_1 + q'_1 + q_L$, and is invisible from the SM as one can easily check.

The analysis in the model C is carried out in a similar way. In that case, the anomaly matrix reads

$$K^{(C)} = \begin{pmatrix} \frac{3}{2} & -1 & 0 & \frac{1}{2} & 0 \\ \frac{3}{2} & -1 & 0 & 0 & \frac{1}{2} \\ \frac{5}{2} & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{1}{2} \\ 9 & -4 & 1 & 3 & 1 \end{pmatrix}, \quad (3.68)$$

and it turns out that the hypercharge q_Y is the only anomaly free $U(1)$.

One can wonder whether the supersymmetrisation of our models modify the results of the anomaly analysis. In the Minimal Supersymmetric extension of the Standard Model (MSSM), each SM particle gets a superpartner: the quark and lepton fermions are paired with the sleptons and squarks, the SM gauge bosons are paired with the gauginos, while the spin-1/2 fermionic partners of the Higgs scalars are the Higgsinos. Being chiral, the Higgsinos \tilde{H} can modify the anomalous and anomaly-free $U(1)$ combinations obtained in the non-supersymmetric models. In our case, their introduction in the spectrum gives new contributions to the entries K_{IJ} for $I, J = \{2, 3, 4\}$ of the anomaly matrix K . For model A, where the quantum numbers of the Higgsinos are

$$\tilde{H}_d(\mathbf{1}, \mathbf{2}; 0, -1, -1, 0, 0)_{-1/2}, \quad \tilde{H}_u(\mathbf{1}, \mathbf{2}; 0, -1, 0, -1, 0)_{1/2}, \quad (3.69)$$

one gets the anomaly matrix

$$K_{\text{MSSM}}^{(A)} = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{3} & \frac{5}{6} & \frac{5}{6} & -\frac{1}{2} \\ 0 & 4 & 2 & 1 & 1 \end{pmatrix}. \quad (3.70)$$

It is then easy to check that the vectors associated to the zero eigenvalues of the matrix $\left(K_{\text{MSSM}}^{(A)}\right)^\top K_{\text{MSSM}}^{(A)}$ are q_y and q given by (3.66), so that the minimal supersymmetric extension of model A does not modify the result obtained in the non-supersymmetric case.

On the other hand, the minimal supersymmetric extension of model B would require the introduction of a third Higgs doublet, since with the two Higgs doublets (3.60) the Yukawa couplings (3.62) would violate holomorphy of the superpotential.

Let us note that an anomaly-free U(1) is not necessarily massless because of underlying 6-dimensional anomalies [98, 99, 100]. One therefore needs to impose additional model-dependent constraints to ensure that the hypercharge remains massless in four dimensions. As an example, let us assume that our framework arises from a given type IIA orientifold compactification with D6-branes and orientifold O6-planes: we denote by i a stack of N_i D6-branes giving rise to a factor $U(N_i)$ in the gauge group, and \tilde{i} its orientifold image. The D6 $_i$ -branes span 4-dimensional Minkowski space and are wrapped on 3-cycles Π_i in the internal space X_6 . In general, 3-cycles in 6-dimensional compact space intersect several times. Introducing the 3-homology class $[\Pi_i]$ of the 3-cycle Π_i , the homological intersection number of the stacks i and j is given by $I_{ij} = [\Pi_i] \cdot [\Pi_j]$. The data N_i and I_{ij} are then sufficient to determine the massless chiral spectrum of the 4-dimensional theory⁹:

- ij sector: open strings stretching between the stacks i et j correspond to I_{ij} 4D chiral fermions in the bi-fundamental representation $(\mathbf{N}_i, \overline{\mathbf{N}}_j)$ of $U(N_i) \times U(N_j)$.
- $i\tilde{j}$ sector: open strings stretching between the stacks i and the orientifold image \tilde{j} of the stack j correspond to $I_{i\tilde{j}}$ 4D chiral fermions in the bi-fundamental representation $(\mathbf{N}_i, \mathbf{N}_j)$ of $U(N_i) \times U(N_j)$.

In this framework, an anomaly free U(1) linear combination

$$q_Y = \sum_i c_i q_i \quad (3.71)$$

remains massless if the following condition holds [101]:

$$\sum_{i \neq j} c_i N_i (I_{ji} - I_{j\tilde{i}}) = 0, \quad (3.72)$$

for every j , where the sum runs over i . Let us take as an example the minimal supersymmetric extension of model B built in Section 3.4.1, where the constants c_i defining the hypercharge linear combination are $c_3 = -\frac{1}{3}$, $c_2 = -\frac{1}{2}$, $c_1 = 1$ and $c_{1'} = c_{1_L} = 0$. Using the intersection

9. We do not discuss the sector $i\tilde{i}$ which does not play a role in our analysis.

numbers corresponding to the fermionic spectrum of Table 3.2,

$$I_{32} = 3, \quad I_{3\bar{1}} = -3, \quad I_{31'} = -3, \quad I_{2\bar{1}L} = 3, \quad I_{11L} = 3, \quad (3.73)$$

one can easily check that the constraints (3.72) for $j = \{3, 1_L\}$ are indeed satisfied. With the intersection numbers $I_{2\bar{1}} = 1$ and $I_{21'} = 1$, corresponding to the two Higgsinos doublet superpartners of the Higgs scalars H_u and H_d defined in Table 3.2, the constraints (3.72) for $j = \{2, 1, 1'\}$ are not satisfied. It would be the case if we could have for instance the intersection number $I_{2\bar{1}} = 3$ and $I_{21'} = 3$. This amounts of introducing two additional Higgs doublet pairs in the $(\mathbf{2}, 1)$ and $(\mathbf{2}, 1')$, which do not modify our phenomenological analysis presented here.

Note finally that the local D-brane configurations built in this section may also be obtained from other string constructions which do not admit such interpretation in terms of intersecting D-branes (such as ordinary type I orbifolds, or non-commuting magnetized D-branes), for which the above conditions do not apply as such.

3.4.3 Right-handed neutrino

Finally, we can implement the right-handed neutrino ν_R . The existence in the total gauge group of two abelian factors which do not participate to the hypercharge easily allows to introduce such SM singlet state ν_R , corresponding either to an open string with one end on the $U(1)'$ brane and the other on the $U(1)_L$ brane, or with both ends on one of the two branes.¹⁰ The right-handed neutrino enters the anomaly analysis only through the gravitational anomalies, modifying the entries K_{44} and K_{45} of the anomaly matrices computed above. In the following we will focus on model A, the analysis in the two other models can be carried out in a completely similar way and does not bring any new relevant physical results. In the case when the open string associated to ν_R stretches between the $U(1)_L$ and $U(1)'$ branes, the four different possibilities for the charge assignments of ν_R together with the associated anomaly-free $U(1)$ combinations is listed in Table 3.4¹¹.

Configurations	ν_R quantum numbers	Anomaly-free $U(1)$ (besides the hypercharge)
1	$\nu_R(\mathbf{1}, \mathbf{1}; 0, 0, 0, -1, 1)_0$	$q \equiv \frac{1}{3}q_3 - \frac{1}{2}q_2 + q'_1 + q_L$
2	$\nu_R(\mathbf{1}, \mathbf{1}; 0, 0, 0, 1, 1)_0$	$\tilde{q} \equiv \frac{2}{3}q_3 - \frac{1}{2}q_2 + q'_1$
3	$\nu_R(\mathbf{1}, \mathbf{1}; 0, 0, 0, -1, -1)_0$	$-\frac{1}{3}q_3 + q_L$
4	$\nu_R(\mathbf{1}, \mathbf{1}; 0, 0, 0, 1, -1)_0$	$\tilde{q} \equiv \frac{2}{3}q_3 - \frac{1}{2}q_2 + q'_1 \quad ; \quad -\frac{1}{3}q_3 + q_L$

Table 3.4 – Anomaly-free $U(1)$ in configurations with right-handed neutrinos (model A)

The first configuration does not modify the result (3.66) obtained in the absence of ν_R : the SM particles see one anomaly-free $U(1)$, the hypercharge q_Y , while there is a second anomaly free combination $q_3 - q_2 + q_1 + q'_1 + q_L$ invisible from the SM. The second configuration contains also an extra anomaly-free $U(1)$, $\tilde{q} \equiv \frac{2}{3}q_3 - \frac{1}{2}q_2 + q'_1$. In that case, one observes that the \tilde{q} charges of the SM particles are given by $B - L - q_Y$ ¹²: the SM spectrum is thus charged under

10. For notational simplicity, we call a charged open string with ends on the same brane when it stretches between the brane and its orientifold image.

11. It is easy to check that these results remain unchanged in the minimal supersymmetric extension of the model, introducing the Higgsinos (3.69) in the spectrum.

12. The $B - L$ charges are defined as usual: $1/3$ for Q , $-1/3$ for u^C and d^C , -1 for L and 1 for e^C and ν_R .

two anomaly-free $U(1)$'s, q_Y and $B - L$. The situation is similar in the third configuration, with $B - L$ still anomaly free and now given by $\frac{1}{3}q_3 - q_L$. Finally, the fourth configuration combines the features of the three previous ones: there are now two anomaly-free $U(1)$'s besides the hypercharge, $\tilde{q} \equiv \frac{2}{3}q_3 - \frac{1}{2}q_2 + q_1'$ and $\frac{1}{3}q_3 - q_L$. The \tilde{q} charges of the SM particles are given, as in the second configuration, by $B - L - q_Y$, so that the SM sees again two anomaly-free $U(1)$'s, q_Y and $B - L$. The third anomaly-free combination, invisible from the SM, is given as in the first configuration by $q_3 - q_2 + q_1 + q_1' + q_L$.

In configurations 2, 3 and 4, the anomaly-free $B - L$ boson may then acquire a mass as a consequence of 6-dimensional anomalies as mentioned at the end of Section 3.4.2, or through a standard Higgs mechanism.

Regarding the ν_R mass, only the fourth configuration allows for a Yukawa coupling $L\nu^R H_u$. One way to obtain a small Dirac neutrino mass compatible with the experimental bounds is to allow ν^R to propagate in the bulk, in which case the Dirac mass m_{ν_R} coming from such Yukawa coupling is suppressed by the volume V_\perp of the extra transverse dimensions, namely $m_{\nu_R} \sim \frac{v}{\sqrt{V_\perp}}$, with v the vev of the Higgs field [102]. This can be obtained in our framework if the $U(1)'$ brane extends along the extra dimension of the $U(1)_L$, so that ν_R propagates in the L -bulk. In that case however, the L -bulk having one extra dimension, it is easy to see that such Dirac neutrino mass is much above the upper limit $\sum m_{\nu_i} \lesssim 0.1$ eV, so that the fourth configuration is phenomenologically excluded. A Yukawa coupling $L\nu^R H_u$ is forbidden in the three first cases, since such a term would not be neutral under $U(1)'$ or $U(1)_L$. However, a Dirac mass term can still arise through non-perturbative effects, taking the form $L\nu^R H_u e^{-\frac{\alpha}{g_s}}$, with g_s the string coupling and α a model-dependent numerical factor [103].

On the other hand, neutrino Majorana masses are perturbatively forbidden since such terms break the (global) lepton number symmetry, but can also arise from non-perturbative instanton effects [104].

3.4.4 Mass spectrum

The $U(1)$ combinations orthogonal to the anomaly free $U(1)$'s, Y , (3.66), (3.67) or listed in Table 3.4, are anomalous and acquire a mass through a 4-dimensional generalisation of the Green-Schwarz (GS) mechanism. In our model, such anomalous bosons form linear combination of some $U(1)$ localised in 4-dimensions and some others extended in the bulk. The aim of this section is to clarify how this situation impacts the anomaly analysis, and in particular if the compactification scale enters the mass of the anomalous bosons.

To simplify the analysis, let us consider a toy model with three $U(1)$ bosons: $B_\mu(x)$ and $C_\mu(x)$ are localised in four dimensions, while $X_\mu(x, y)$ is a bulk vector. Their $U(1)$ charges are respectively denoted Q_3 , Q_2 and Q_1 , while their kinetic action is given by

$$S_{kin} = \int d^4x \left[-\frac{1}{4g_3^2} F^2(B) - \frac{1}{4g_2^2} F^2(C) \right] - \frac{1}{4g_{1(5)}^2} \int d^5x F^2(X). \quad (3.74)$$

The standard KK reduction is carried out by expanding X_μ and X_5 according to $X_\mu(x, y) = \sum_{n \in \mathbb{Z}} X_\mu^{(n)}(x) e^{\frac{iny}{R}}$ and $X_5(x, y) = \sum_{n \in \mathbb{Z}} X_5^{(n)}(x) e^{\frac{iny}{R}}$. Integrating then the second term of (3.74)

over y , we get:

$$\begin{aligned}
S_{kin} = & \int d^4x \left[-\frac{1}{4g_3^2} F^2(B) - \frac{1}{4g_2^2} F^2(C) \right] \\
& - \frac{1}{4g_1^2} \int d^4x \left[F_{(0)}^2(X) + \sum_{n \neq 0} F_{\mu\nu}^{(n)} F^{\mu\nu(-n)} + 2 \left(\partial_\mu X_5^{(0)} \right)^2 \right] \\
& - \frac{1}{2g_1^2} \int d^4x \sum_{n \neq 0} \left(\partial_\mu X_5^{(n)}(x) - \frac{in}{R} X_\mu^{(n)}(x) \right) \left(\partial_\mu X_5^{(-n)}(x) + \frac{in}{R} X_\mu^{(-n)}(x) \right),
\end{aligned} \tag{3.75}$$

where we have defined the four-dimensional effective gauge coupling g_1 from the five-dimensional one $g_{1(5)}$ by

$$\frac{1}{g_1^2} = \frac{V_\perp}{g_{1(5)}^2}, \tag{3.76}$$

with V_\perp the volume of the extra dimensions. In the gauge in which $\forall n \neq 0, X_5^{(n)} = 0$, the KK excitations $X_\mu^{(n)}, n \neq 0$, become massive. The gauge symmetries associated to these states having been fixed, the bosons $X_\mu^{(n)}, n \neq 0$, do not contribute to anomalies, so that only combinations of $B_\mu(x), C_\mu(x)$ and the zero mode $X_\mu^{(0)}(x)$ of $X_\mu(x, y)$ can be anomalous. In the following, we will denote the basis $(B_\mu, C_\mu, X_\mu^{(0)})$ as the ‘‘D-brane basis’’.

We next consider the basis formed by the hypercharge Y_μ and two anomalous vectors A_μ and A'_μ , all of three orthogonal to each other, denoted ‘‘hypercharge basis’’ in the following. In the most general case, the hypercharge Y_μ is a linear combination of all the bosons of the D-brane basis localised in four dimensions, namely in this model $B_\mu(x)$ and $C_\mu(x)$, while A_μ and A'_μ can be combinations of all the D-brane basis bosons, including $X_\mu^{(0)}(x)$. We parametrise these combinations as:

$$Q_Y = c_3 Q_3 + c_2 Q_2, \tag{3.77a}$$

$$Q_A = c_2 Q_3 - c_3 Q_2 + c_1 Q_1, \tag{3.77b}$$

$$Q_{A'} = c_2 Q_3 - c_3 Q_2 - \frac{c_2^2 + c_3^2}{c_1} Q_1. \tag{3.77c}$$

In order to relate the original D-brane basis $(B_\mu, C_\mu, X_\mu^{(0)})$ to the hypercharge basis (Y_μ, A_μ, A'_μ) , we write the covariant derivatives of the bosons in both bases (assuming a canonical normalisation of their kinetic terms):

$$\mathcal{D}_\mu = \partial_\mu - i \frac{g_3}{\sqrt{6}} Q_3 B_\mu(x) - i \frac{g_2}{2} Q_2 C_\mu(x) - i g_1 Q_1 X_\mu^{(0)}(x) \tag{3.78}$$

$$= \partial_\mu - i g_Y Q_Y Y_\mu(x) - i g_A Q_A A_\mu(x) - i g_{A'} Q_{A'} A'_\mu(x). \tag{3.79}$$

Using the relations (3.77), one can identify the different terms and get the resulting 3×3 rotation matrix \mathcal{R} relating $(B_\mu, C_\mu, X_\mu^{(0)})$ to (Y_μ, A_μ, A'_μ) :

$$\begin{pmatrix} Y_\mu \\ A_\mu \\ A'_\mu \end{pmatrix} = \begin{pmatrix} \sqrt{6} c_3 \frac{g_Y}{g_3} & 2c_2 \frac{g_Y}{g_2} & 0 \\ \sqrt{6} c_2 \frac{g_A}{g_3} & -2c_3 \frac{g_A}{g_2} & c_1 \frac{g_A}{g_1} \\ \sqrt{6} c_2 \frac{g_{A'}}{g_3} & -2c_3 \frac{g_{A'}}{g_2} & -\frac{c_2^2 + c_3^2}{c_1} \frac{g_{A'}}{g_1} \end{pmatrix} \begin{pmatrix} B_\mu \\ C_\mu \\ X_\mu^{(0)} \end{pmatrix}. \tag{3.80}$$

Imposing the orthogonality condition for \mathcal{R} , $\sum_{j=1}^3 \mathcal{R}_{ij}^2 = 1, \forall i = 1, 2, 3$, we get the well-known relations between the coupling constants of the vector bosons in the two bases:

$$\frac{1}{g_Y^2} = \frac{6c_3^2}{g_3^2} + \frac{4c_2^2}{g_2^2}, \quad (3.81a)$$

$$\frac{1}{g_A^2} = \frac{6c_2^2}{g_3^2} + \frac{4c_3^2}{g_2^2} + \frac{c_1^2}{g_1^2}, \quad (3.81b)$$

$$\frac{1}{g_{A'}^2} = \frac{6c_2^2}{g_3^2} + \frac{4c_3^2}{g_2^2} + \left(\frac{c_2^2 + c_3^2}{c_1} \right)^2 \frac{1}{g_1^2}. \quad (3.81c)$$

Using then (3.76) relating the five dimensional coupling constant $g_{1(5)}$ and the four dimensional one g_1 , one sees that the relations for g_A and $g_{A'}$ are dominated in the large volume limit by the g_1 coupling, namely

$$\frac{1}{g_Y^2} = \frac{6c_3^2}{g_3^2} + \frac{4c_2^2}{g_2^2}, \quad (3.82a)$$

$$\frac{1}{g_A^2} \sim \frac{c_1^2}{g_1^2} = V_\perp \frac{c_1^2}{g_{1(5)}^2}, \quad (3.82b)$$

$$\frac{1}{g_{A'}^2} \sim \left(\frac{c_2^2 + c_3^2}{c_1} \right)^2 \frac{1}{g_1^2} = V_\perp \left(\frac{c_2^2 + c_3^2}{c_1} \right)^2 \frac{1}{g_{1(5)}^2}. \quad (3.82c)$$

The above rotation matrix in Eq. (3.80) has thus the following structure:

$$\begin{pmatrix} Y_\mu \\ A_\mu \\ A'_\mu \end{pmatrix} \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & 0 \\ \mathcal{O}\left(\frac{1}{\sqrt{V_\perp}}\right) & \mathcal{O}\left(\frac{1}{\sqrt{V_\perp}}\right) & \mathcal{O}(1) \\ \mathcal{O}\left(\frac{1}{\sqrt{V_\perp}}\right) & \mathcal{O}\left(\frac{1}{\sqrt{V_\perp}}\right) & \mathcal{O}(1) \end{pmatrix} \begin{pmatrix} B_\mu \\ C_\mu \\ X_\mu^{(0)} \end{pmatrix}, \quad (3.83)$$

so that, in the large volume limit, the bosons A_μ and A'_μ are simply given by the zero mode of the bulk vector X_μ .

The anomalies being localised in four dimensions, the effective action involved in the GS anomaly cancellation is given by

$$\begin{aligned} S = & \int d^4x \left[-\frac{1}{4g_A^2} F_A^2 - \frac{1}{2} (\partial_\mu a + M_s A_\mu)^2 + \frac{a}{M_s} \sum_I k_I \text{Tr} F_I \wedge F_I \right] \\ & + \int d^4x \left[-\frac{1}{4g_{A'}^2} F_{A'}^2 - \frac{1}{2} (\partial_\mu a' + M_s A'_\mu)^2 + \frac{a'}{M_s} \sum_I k'_I \text{Tr} F_I \wedge F_I \right], \quad (3.84) \end{aligned}$$

where F_A ($F_{A'}$) is the field strength of the anomalous $U(1)_A$ ($U(1)_{A'}$), g_A ($g_{A'}$) the associated gauge coupling, and a (a') the pseudo-scalar axion responsible for the anomaly cancellation. The indice I in the sum over Pontryagin densities denotes $SU(3)$, $SU(2)$ and Y for the mixed gauge anomalies, F_I are the associated field strengths and the constants k_I (k'_I) are given by $k_I = \text{Tr} T_I^2 Q_A$ ($k'_I = \text{Tr} T_I^2 Q_{A'}$)¹³.

13. The gravitational anomalies are also canceled by a similar term of the form $\frac{a}{M_s} k_G R \wedge R$, where $k_G = \text{Tr} Q_A$, and similarly for A' , a' .

In the most general case where both the axion and the anomalous vector propagate into some extra dimensions of the bulk, the mass M_A of the anomalous gauge boson is of order $M_A \propto \sqrt{\frac{V_a}{V_A}} M_s$, where V_a and V_A denote the volume of the space where the axion a and the vector A_μ propagate, respectively [98]. If a is localised in four dimensions, then $M_A \propto \frac{M_s}{\sqrt{V_A}}$, and thus, for one extra dimension at a scale $M_L \sim 10$ GeV and a string scale $M_s \sim 10$ TeV, we get $M_A \sim 10^2$ GeV. This mass being too low and subject to stringent phenomenological constraints, one needs to have $V_a = V_A$ in order to get a mass of the anomalous boson of the order of the string scale. We conclude that the axions which cancel the anomalies of anomalous $U(1)$ combinations which have a component along L_μ must also propagate in the L -bulk.

3.4.5 Dark Matter model

In this section, we briefly describe how the models built in this chapter can easily provide Dark Matter (DM) candidates, as Dirac fermions (called χ in the following) corresponding to open strings stretched between bulk branes, similar to the right-handed neutrinos. The simplest situation arises when χ has one end on the $U(1)_L$ brane and the other on the $U(1)'$ brane. If the extra dimension(s) along which the $U(1)'$ extends is (are) orthogonal to the L -bulk, then χ is localised in four dimensions and has no KK modes. The KK excitations $L_\mu^{(n)}$ of the lepton number gauge boson, which couple to both χ and the SM leptons l , then provide a mediator between the dark and the (leptonic) visible sector, allowing DM annihilation process $\chi\bar{\chi} \rightarrow l\bar{l}$ through the s-channel diagram represented in Figure 3.6. Its amplitude is given by:

$$\mathcal{M}_n = \frac{-i}{s - n^2 M_L^2} [\bar{v}(p_2)(-ig_L\gamma^\mu)u(p_1)] [\bar{u}(p_3)(-ig_L\gamma_\mu)v(p_4)], \quad (3.85)$$

with M_L the compactification scale of the extra dimension of the L -bulk where L_μ propagates, u and v the on-shell external spinors satisfying $\not{p}_1 u(p_1) = m u(p_1)$, $\bar{v}(p_2)\not{p}_2 = -m\bar{v}(p_2)$, and g_L the $U(1)_L$ coupling assumed to be independent of n ¹⁴.

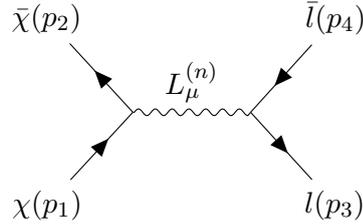


Figure 3.6 – Dark matter annihilation into SM leptons mediated by the KK excitations $L_\mu^{(n)}$ of the lepton number gauge boson.

The total tree-level amplitude \mathcal{M} for the process $\chi\bar{\chi} \rightarrow l\bar{l}$ is given by the sum over the

14. The gauge coupling of the n -th KK excitation is given by $g_L(n) = g_L e^{-cn^2 \frac{M_L^2}{M_s^2}}$, with c a positive numerical constant. When $M_L \ll M_s$, as it is the case in the large extra dimension scenario considered in this chapter, the exponential is of order 1 for all $n \lesssim \frac{M_s}{M_L}$, and the gauge coupling can indeed be taken constant. For higher KK modes with $n \gg \frac{M_s}{M_L}$, one has to consider the exponential suppression of g_L and the analysis would be modified. For the values $M_s \sim 10$ TeV and $M_L \sim 10$ GeV that will be considered below, the result presented in this section is thus valid for the first $\mathcal{O}(10^3)$ KK excitations.

KK modes of the vector boson mediator, $\mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}_n$. Averaging $|\mathcal{M}|^2$ over the incoming helicities and summing over the outgoing helicities of the fermions, one gets

$$\overline{|\mathcal{M}|^2} = 2g_L^4 \left| \sum_{n=1}^{\infty} \frac{1}{s - n^2 M_L^2} \right|^2 (t^2 + u^2 + 4s(m_\chi^2 + m_l^2) - 2(m_\chi^2 + m_l^2)^2), \quad (3.86)$$

where we have defined $\overline{|\mathcal{M}|^2} \equiv \frac{1}{4} \sum_{\text{Spins}} |\mathcal{M}|^2$, m_χ and m_l are the DM and lepton masses respectively, and s , t and u are the Mandelstam variables defined by

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \quad (3.87)$$

In the center of mass frame, writing the 4-momenta $p_1 = (E, \vec{p}_i)$, $p_2 = (E, -\vec{p}_i)$, $p_3 = (E, \vec{p}_f)$, $p_4 = (E, -\vec{p}_f)$ with $E = \sqrt{s}/2$, θ the angle between the incoming \vec{p}_i and outgoing \vec{p}_f , and neglecting the lepton mass m_l , we get:

$$\overline{|\mathcal{M}|^2} = g_L^4 \left| \sum_{n=1}^{\infty} \frac{1}{1 - n^2 \frac{M_L^2}{s}} \right|^2 \left[1 + \cos^2 \theta + \frac{4m_\chi^2}{s} (1 - \cos^2 \theta) \right]. \quad (3.88)$$

We can then compute the differential cross-section of the DM annihilation process, given in the center of mass frame for $2 \rightarrow 2$ particle scattering by:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} \overline{|\mathcal{M}|^2}. \quad (3.89)$$

Explicitly writing the sum in Eq. (3.88) in terms of the cotangent function and using $|\vec{p}_i| = \sqrt{E^2 - m_\chi^2}$, $|\vec{p}_f| = E$, one gets:

$$\frac{d\sigma}{d\Omega} = \frac{g_L^4}{64\pi^2 s} \frac{1}{\sqrt{1 - \frac{4m_\chi^2}{s}}} \left(-\frac{1}{2} + \frac{\pi\sqrt{s}}{2M_L} \cot\left(\frac{\pi\sqrt{s}}{M_L}\right) \right)^2 \left[1 + \cos^2 \theta + \frac{4m_\chi^2}{s} (1 - \cos^2 \theta) \right]. \quad (3.90)$$

In the non-relativistic limit, s can be expressed in terms of the relative velocity v_r of the annihilating particles as

$$s = 4m_\chi^2 + m_\chi^2 v_r^2 + \mathcal{O}(v_r^4). \quad (3.91)$$

Expanding the expression (3.90) in terms of v_r yields:

$$\frac{d\sigma}{d\Omega} = \frac{g_L^4}{256\pi^2 m_\chi^2} \left(-1 + \frac{2m_\chi\pi}{M_L} \cot\left(\frac{2m_\chi\pi}{M_L}\right) \right)^2 \frac{1}{v_r} + \mathcal{O}(v_r). \quad (3.92)$$

At lowest order in v_r , the differential cross section is thus independent of θ so that the total annihilation cross section reads:

$$\sigma v_r = \frac{g_L^4}{64\pi m_\chi^2} \left(-1 + \frac{2m_\chi\pi}{M_L} \cot\left(\frac{2m_\chi\pi}{M_L}\right) \right)^2. \quad (3.93)$$

For a string scale $M_s \sim 10$ TeV, M_L and g_L must be of order $M_L \sim 10$ GeV and $g_L \sim 10^{-2}$ in order to accommodate the muon anomalous magnetic moment discrepancy, as described in

Section 3.2. For such values, the annihilation cross-section (3.93) is plotted in terms of the DM mass m_χ in Figure 3.7. The horizontal grey line indicates the annihilation cross section which yields the observed value for the DM relic density, given by¹⁵

$$\Omega h^2 \sim \frac{10^{-26} \text{ cm}^3 \cdot \text{s}^{-1}}{\langle \sigma v_r \rangle} \sim 0.1 \times \frac{10^{-9} \text{ GeV}^{-2}}{\langle \sigma v_r \rangle}. \quad (3.94)$$

The cross-section diverges for all $m_\chi = n \frac{M_L}{2}$, and is regulated taking into account the width

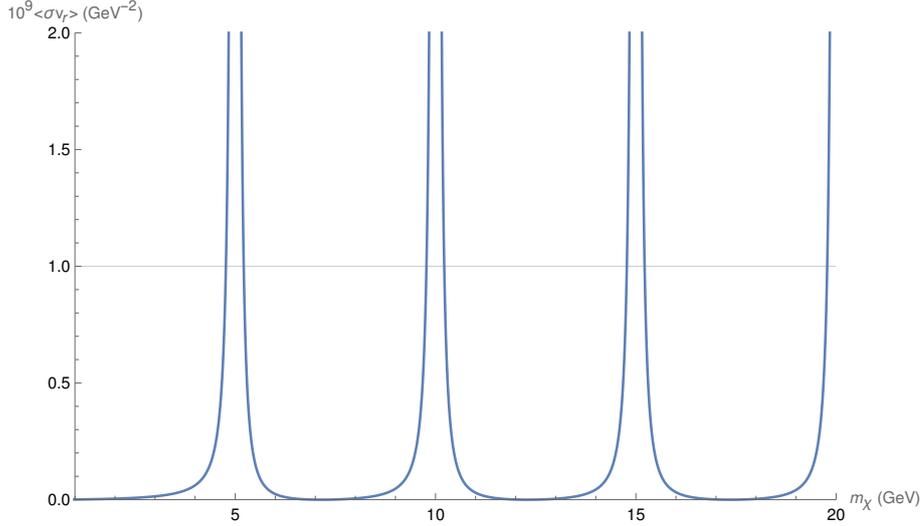


Figure 3.7 – Annihilation cross section $10^9 \times \langle \sigma v_r \rangle$ (GeV^{-2}) in terms of the DM mass m_χ (GeV), for a compactification scale $M_L \sim 10$ GeV and a coupling $g_L \sim 10^{-2}$.

of the KK resonances, by replacing the vector boson propagator $\frac{1}{s-M_n^2}$ by $\frac{1}{s-M_n^2+i\Gamma_n M_n}$, with Γ_n the decay rate of $L_\mu^{(n)}$. We thus have to check how this regularisation modifies the function $\langle \sigma v_r \rangle$ and in particular if it brings the local maxima of the cross-section below the critical value $10^9 \times \langle \sigma v_r \rangle = 1 \text{ GeV}^{-2}$. For the lowest KK modes n , the width of $L_\mu^{(n)}$ is dominated by decays into SM leptons, with the corresponding decay rate given by

$$\Gamma_n(L_\mu^{(n)} \rightarrow l\bar{l}) = \frac{g_L^2}{4\pi} M_n \left(1 + 2 \frac{m_l^2}{M_n^2} \right) \sqrt{1 - 4 \frac{m_l^2}{M_n^2}} \Theta(M_n - 2m_l), \quad (3.95a)$$

$$= \frac{g_L^2}{4\pi} M_n \Theta(M_n - 2m_l) + \mathcal{O}(m_l^4). \quad (3.95b)$$

For $g_L \sim 10^{-2}$, we get $\Gamma_n(L_\mu^{(n)} \rightarrow l\bar{l}) \sim 10^{-5} M_n$, so that Γ_n can indeed be neglected for the lowest n . For higher KK modes, other decay channels contribute to the width, such as the decay of $L_\mu^{(n)}$ into lighter KK states, and the analysis would be modified. Focusing on the lightest modes from now on, one thus gets from the plot of Figure 3.7 that the correct DM relic density is obtained for several values of the χ -fermion mass m_χ centred around integer multiple of $\frac{M_L}{2}$. For $M_L \sim 10$ GeV, the two first lightest possible DM masses are in tension with the phenomenological constraints coming from dwarf galaxies γ -ray and CMB observations, which yield a lower bound on the DM mass around 10 – 15 GeV [105, 106]. In our example, these

15. The annihilation cross section σv_r given in Eq. (3.93) being independent of v_r , the thermal average $\langle \sigma v_r \rangle$ entering in the expression of the DM relic density is trivial in our case: $\langle \sigma v_r \rangle = \sigma v_r$.

constraints are thus satisfied for DM masses centred around $m_\chi = n \frac{M_L}{2}$, $n \gtrsim 3$. Obviously, the constraints are automatically satisfied for $M_L \gtrsim 30$ GeV.

Since direct couplings between the DM χ and the SM particles are forbidden, no Yukawa couplings between χ and the SM leptons are allowed. In model A introduced in Section 3.4.1, this means that among the four possible configurations for χ , the one $\chi(\mathbf{1}, \mathbf{1}; 0, 0, 0, 1, -1)_0$ is not allowed. For the three other quantum number assignments, namely $\chi(\mathbf{1}, \mathbf{1}; 0, 0, 0, -1, 1)_0$, $\chi(\mathbf{1}, \mathbf{1}; 0, 0, 0, 1, 1)_0$ and $\chi(\mathbf{1}, \mathbf{1}; 0, 0, 0, -1, -1)_0$, the Yukawa coupling $L\chi H_u$ is forbidden, and the mass of χ can arise for instance from brane separation, when the $U(1)_L$ and $U(1)'$ branes are localised at two different points in the extra (large) dimensions of the gravitational bulk.

Obviously, a global (top-down) construction of a fully consistent string model may require the presence of additional branes. Such (hidden) branes or/and corresponding ‘‘messenger’’ states may provide alternative DM candidates besides the minimal possibility discussed above.

3.4.6 Lepton flavour non-universality and the muon $g - 2$

Implicit in the construction carried out above was the assumption of lepton flavour universality, namely that the three families of charged leptons are identical copies of each other (apart from the mass) and that the lepton number gauge boson L_μ couples with the same strength to each of them. Another possibility to address the discrepancy of the anomalous magnetic moment of the muon would be to gauge only the muonic lepton number $U(1)_{L^{(\mu)}}$ replacing the L -brane in Figure 3.4 and identifying the leptons of the first and third generation by open strings that do not end on the $L^{(\mu)}$ brane. They could for instance end on a 6-th brane that gauges $L^{(e)} + L^{(\tau)}$, or end on the $U(1)'$ brane in the minimal case, breaking the total lepton number. As a result, this construction leads to lepton flavour non-universality (LFNU) but its main achievement is to avoid LEP and LHC bounds while still use light KK-excitations of the $U(1)_{L^{(\mu)}}$ gauge boson $L_\mu^{(\mu)}$ in order to account for the $(g - 2)_\mu$ discrepancy. Their contribution to the muon vertex correction is given by:

$$\Delta a_\mu = \sum_n \frac{g_{L^{(\mu)}}^2 m_\mu^2}{12\pi^2 M_n^2}, \quad (3.96)$$

where m_μ is the muon mass, $g_{L^{(\mu)}}$ the gauge coupling of the $U(1)_{L^{(\mu)}}$ and M_n the mass of the n th KK excitation of $L_\mu^{(\mu)}$. Since $L_\mu^{(\mu)}$ does not couple to electrons, its coupling and KK masses evade the LEP bounds and are thus now completely unconstrained.

As mentioned above, a LFNU model can easily be obtained in the framework built in this chapter, by replacing the $U(1)_L$ brane by a muonic $U(1)_{L^{(\mu)}}$ associated to a gauge boson $L_\mu^{(\mu)}$ with corresponding gauge couplings $g_{L^{(\mu)}}$. Assuming again that $U(1)_{L^{(\mu)}}$ extends along one large extra dimension with a compactification scale $M_{L^{(\mu)}}$, we have $g_{L^{(\mu)}}^2 = g_s \frac{M_{L^{(\mu)}}}{M_s}$, with g_s the string coupling, and $M_n = nM_{L^{(\mu)}}$, so that the contribution (3.96) reads:

$$\Delta a_\mu = \frac{g_s m_\mu^2}{72 M_{L^{(\mu)}} M_s}. \quad (3.97)$$

Such contribution can therefore accommodate the discrepancy (3.22) for a compactification scale

and a string scale satisfying

$$M_{L^{(\mu)}} M_s \sim g_s \times 5 \times 10^4 \text{ GeV}^2. \quad (3.98)$$

For a string scale $M_s = 10 \text{ TeV}$, we thus get a compactification scale $M_{L^{(\mu)}} \sim g_s \times 5 \text{ GeV}$. In the perturbative regime where $\frac{g_s}{4\pi} \lesssim 1$, $M_{L^{(\mu)}}$ can therefore vary from the $\mathcal{O}(\text{GeV})$ to $\mathcal{O}(10^2 \text{ GeV})$. Let us note that if $L_{\mu}^{(\mu)}$ also propagates along some extra smaller dimensions with a size larger but near the string length, g_s is suppressed by the volume of these dimensions, further increasing the range of possible values for the compactification scale $M_{L^{(\mu)}}$.

3.5 Conclusion

The Brookhaven National Laboratory experiment E821 together with the recent Muon $g - 2$ experiment at Fermilab have pushed the discrepancy between the measured value of the muon anomalous magnetic moment and its Standard Model prediction to 4.7σ , providing a strong hint of new physics beyond the SM. This discrepancy can be explained in the framework of low mass scale strings and large extra dimensions, assuming that the SM lepton number global symmetry (or even the muonic lepton number) is gauged and that the associated gauge boson propagates along (at least) one large extra dimension, so that the main contribution to $(g - 2)_{\mu}$ is due to the exchange of its lightest Kaluza-Klein excitations. The work carried out here realised this proposal, by building the minimal embedding of the Standard Model into intersecting D-brane configurations with a gauged lepton number associated to a $U(1)_L$ brane extended along one large extra dimension and which does not participate to the hypercharge combination, as required for phenomenological reasons. Consistency of the models requires the introduction of a fifth brane in a way that all SM mixed anomalies cancel.

The presence of the two extra branes, beyond the SM ones, allows to introduce in a simple way the right handed neutrino as well as a Dark Matter candidate. For a string scale $M_s \gtrsim 10 \text{ TeV}$, the bulk of these models exhibits an interesting non-homogeneous structure, with one large extra dimension with a compactification scale in the range of $\mathcal{O}(10 - 10^2 \text{ GeV})$ required to explain the $(g - 2)_{\mu}$ discrepancy, and several larger extra dimensions with an average compactification scale $\lesssim \mathcal{O}(0.1 \text{ GeV})$ in order to lower the string scale in the $\mathcal{O}(10 \text{ TeV})$ region. Within this framework, the anomalous magnetic moment of the muon may provide a hint for the low mass scale strings proposal accessible in future high energy particle colliders.

Kähler geometry

A.1 Complex manifolds and Kähler manifolds

Let us consider a smooth even dimensional manifold \mathcal{M} , with $\dim_{\mathbb{R}}\mathcal{M} = 2m$, and two coordinate patches \mathcal{U}_1 and \mathcal{U}_2 on \mathcal{M} which intersect each others. Given a point $p \in \mathcal{U}_1 \cap \mathcal{U}_2$, its local coordinates in the patches \mathcal{U}_1 and \mathcal{U}_2 are respectively denoted by (x^1, \dots, x^{2m}) and (y^1, \dots, y^{2m}) . The coordinate transformation $(x^1, \dots, x^{2m}) \rightarrow (y^1, \dots, y^{2m})$ is then required to be smooth, usually C^∞ . It is then natural to try to pair up these coordinates into $z^i = x^{2i-1} + ix^{2i}$ and $w^i = y^{2i-1} + iy^{2i}$, for all $i = 1, \dots, m$. However, the coordinate transformation on the $2m$ real coordinates $x^i \rightarrow y^i(x)$, $i = 1, \dots, 2m$ will, in general, lead to the coordinate transformations on the m complex coordinates $z^i \rightarrow w^i(z^1, \dots, z^m, \bar{z}^1, \dots, \bar{z}^m)$, which is usually not holomorphic. A complex manifold is not only a smooth even dimensional manifold, but also a manifold where such change of coordinates can be done holomorphically, namely such that $\forall i, j = 1, \dots, m$, $\frac{\partial}{\partial \bar{z}^i} w^j = 0$. To this end, an additional structure needs to be added on \mathcal{M} .

In a similar way that a Riemannian manifold is built from a given smooth manifold by adding on it a Riemannian structure, namely the metric tensor $g_{ij}(x)$, a complex manifold can be built from a given even dimensional smooth manifold by adding the so-called *almost complex structure* on it. This is a real-valued tensor $J_i^j(x)$ on the tangent space of the manifold, satisfying the property

$$J_i^k J_k^j = -\delta_i^j, \quad (\text{A.1.1})$$

which obviously mimics $i^2 = -1$. One can show [107] that the coordinate transformations of the complex coordinates can then be done holomorphically if and only if the almost complex structure satisfies the additional condition:

$$J_i^l (\partial_l J_j^k - \partial_j J_l^k) - J_j^l (\partial_l J_i^k - \partial_i J_l^k) = 0. \quad (\text{A.1.2})$$

\mathcal{M} equipped with an almost complex structure J_i^j satisfying Eq.(A.1.2) is then called a *complex manifold*. One can then consider the coordinate basis $\{\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \bar{z}^\alpha}\}$, in which the complex structure takes the form:

$$J = \begin{pmatrix} J_\alpha^\beta & J_\alpha^{\bar{\beta}} \\ J_\alpha^{\bar{\beta}} & J_\alpha^\beta \end{pmatrix} = \begin{pmatrix} i\delta_\alpha^\beta & 0 \\ 0 & -i\delta_\alpha^{\bar{\beta}} \end{pmatrix}. \quad (\text{A.1.3})$$

In physical applications, a manifold is generally already endowed by a Riemannian structure g_{ij} , and it is therefore natural to try to combine it with the complex structure introduced above. On the one hand, in analogy with the Levi-Civita connection, we impose the complex structure to be covariantly constant. On the other hand, we require the metric to be invariant under

the action of the complex structure. These two requirements lead to the so-called *compatibility conditions*:

$$\nabla_i J_k^j = 0, \quad g_{kl} J_i^k J_j^l = g_{ij}. \quad (\text{A.1.4})$$

A manifold equipped with a Riemannian and a complex structure satisfying the compatibility conditions (A.1.4) is then called a *Kähler manifold*. It is then convenient to introduce the 2-form

$$\Omega = -g_{ij} J_k^i dx^k \wedge dx^j = -2ig_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}, \quad (\text{A.1.5})$$

written in real and complex coordinates, called the *Kähler form*. The second compatibility condition is equivalent to $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, as can be seen from (A.1.3). Such a metric is said to be hermitian. One can show that the first of the compatibility conditions (A.1.4) is satisfied if and only if the Kähler form is closed, namely $d\Omega = 0$. The definition (A.1.4) of a Kähler manifold can thus be rephrased into:

A Kähler manifold is a complex manifold with hermitian metric whose fundamental 2-form Ω is closed.

The closeness of Ω leads to

$$\begin{aligned} 0 = d\Omega &= -2i\partial_\gamma g_{\alpha\bar{\beta}} dz^\gamma \wedge dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + c.c. \\ &= -i(\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}}) dz^\gamma \wedge dz^\alpha \wedge d\bar{z}^{\bar{\beta}} + c.c., \end{aligned} \quad (\text{A.1.6})$$

or, in components, $\partial_\gamma g_{\alpha\bar{\beta}} - \partial_\alpha g_{\gamma\bar{\beta}} = 0 = \partial_{\bar{\beta}} g_{\alpha\bar{\gamma}} - \partial_{\bar{\gamma}} g_{\alpha\bar{\beta}}$. This implies that locally, the metric can be expressed as

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z}). \quad (\text{A.1.7})$$

The real locally defined function $K(z, \bar{z})$ is called the *Kähler potential*. The so-called *Kähler transformations*

$$K(z, \bar{z}) \rightarrow K'(z, \bar{z}) = K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}) \quad (\text{A.1.8})$$

obviously leaves $g_{\alpha\bar{\beta}}$ invariant. The compatibility conditions (A.1.4) defining a Kähler manifold can therefore be rephrased into

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \quad g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z}), \quad (\text{A.1.9})$$

the last equation being valid in each coordinate patches, but not globally over the manifold¹.

1. If K were a globally defined function on the manifold, then the Kähler form $\Omega = -2i\partial_\alpha \partial_{\bar{\beta}} K dz^\alpha \wedge d\bar{z}^{\bar{\beta}}$ would be exact, $\Omega = d\Lambda$, with Λ globally defined. Denoting $m = \dim_{\mathbb{C}} \mathcal{M}$, Ω^m is a top degree form and can therefore be chosen as a volume form. However, on a compact manifold without boundary, this would lead to $\int_{\mathcal{M}} (d\Lambda)^m = 0$ for a globally defined Λ . Therefore, for compact manifolds without boundary, the Kähler potential cannot be globally defined.

A.2 Symmetries of Kähler manifolds

A symmetry of a Kähler manifold is defined as a coordinate change preserving both the hermitian and the complex structures. As in the real case, the generators of such coordinate transformations are vector fields k called *Killing vectors*, defined in the Kähler case such that the Lie derivatives of the metric tensor and the complex structure both vanish:

$$\mathcal{L}_k g_{ij} = \nabla_i k_j + \nabla_j k_i = 0, \quad (\text{A.2.1})$$

$$\mathcal{L}_k J_i^j = \underbrace{k^l \nabla_l J_i^j}_{=0} + \nabla_i k^l J_l^j - \nabla_l k^j J_i^l = 0, \quad (\text{A.2.2})$$

where the first term of the second equation vanishes by definition of a Kähler manifold.

Let us first look at the consequences of the second equality. Taking $i \rightarrow \alpha$ and $j \rightarrow \beta$, we get $\nabla_\alpha k^l J_l^\beta - \nabla_l k^\beta J_\alpha^l = i(\nabla_\alpha k^\beta - \nabla_\alpha k^\beta) = 0$, which vanishes trivially and thus do not bring any new informations. However, considering $i \rightarrow \bar{\alpha}$ and $j \rightarrow \beta$, we get $\nabla_{\bar{\alpha}} k^l J_l^\beta - \nabla_l k^\beta J_{\bar{\alpha}}^l = 2i \nabla_{\bar{\alpha}} k^\beta = 2i(\partial_{\bar{\alpha}} k^\beta + \Gamma_{\bar{\alpha}i}^\beta k^i) = 2i \partial_{\bar{\alpha}} k^\beta$, where the last equality follows from $\Gamma_{\bar{\alpha}i}^\beta = 0$ for Kähler metrics, for any i holomorphic or anti-holomorphic indices. We deduce that k^β are functions of z^α only and not $\bar{z}^{\bar{\alpha}}$ ², and Killing vectors can thus be written as $k^i = \{k^\alpha(z), k^{\bar{\alpha}}(\bar{z})\}$.

The conditions (A.2.1) and (A.2.2) imply that $\mathcal{L}_k \Omega = \mathcal{L}_k (-g_{ij} J_k^i dx^k \wedge dx^j) = 0$. From the definition $\mathcal{L}_k = i_k d + di_k$, where i_k and d are the interior and exterior derivatives respectively, and using the fact that Ω is closed, namely $d\Omega = 0$, we deduce that $di_k \Omega = 0$. The Poincaré lemma then states that locally, there exists a 0-form \mathcal{P} such that

$$i_k \Omega = -2d\mathcal{P}(z, \bar{z}), \quad (\text{A.2.3})$$

where the -2 factor is introduced for convenience. From the definition of the interior derivative, we have $i_k \Omega = i_k (-J_{ij} dx^i \wedge dx^j) = -2k^l J_{li} dx^i = -2k^l J_l^k g_{ki} dx^i = -2ik^\alpha g_{\alpha\bar{\beta}} d\bar{z}^{\bar{\beta}} + 2ik^{\bar{\beta}} g_{\alpha\bar{\beta}} dz^\alpha$, where in the last equality we moved from real to complex coordinates and have used (A.1.3). Eq. (A.2.3) can then be written as:

$$g_{\alpha\bar{\beta}} k^\alpha = -i \partial_{\bar{\beta}} \mathcal{P}(z, \bar{z}), \quad g_{\alpha\bar{\beta}} k^{\bar{\beta}} = i \partial_\alpha \mathcal{P}(z, \bar{z}). \quad (\text{A.2.4})$$

Let us now consider the consequences of Eq. (A.2.1), which splits in complex coordinates into $\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = 0$ and $\nabla_\alpha k_{\bar{\beta}} + \bar{\nabla}_{\bar{\beta}} k_\alpha = 0$. The first one is trivial, since $\nabla_\alpha k_\beta = g_{\beta\bar{\gamma}} \nabla_\alpha k^{\bar{\gamma}} = g_{\beta\bar{\gamma}} \partial_\alpha k^{\bar{\gamma}} = 0$, where in the penultimate equality we have used again $\Gamma_{\alpha i}^{\bar{\gamma}} = 0$ for Kähler metrics. The second one is satisfied provided that there is a moment map \mathcal{P} satisfying the equations (A.2.4). Indeed, if (A.2.4) are satisfied, we have $\nabla_\alpha k_{\bar{\beta}} + \bar{\nabla}_{\bar{\beta}} k_\alpha = \nabla_\alpha (g_{\gamma\bar{\beta}} k^{\bar{\gamma}}) + \bar{\nabla}_{\bar{\beta}} (g_{\alpha\bar{\gamma}} k^{\bar{\gamma}}) = -i \nabla_\alpha (\partial_{\bar{\beta}} \mathcal{P}) + i \bar{\nabla}_{\bar{\beta}} \partial_\alpha \mathcal{P} = -i \partial_\alpha \partial_{\bar{\beta}} \mathcal{P} + i \partial_{\bar{\beta}} \partial_\alpha \mathcal{P} = 0$. Therefore, a symmetry of a Kähler manifold defined by the holomorphic Killing vector k is characterized by the real moment map $\mathcal{P}(z, \bar{z})$ satisfying:

$$k^\alpha(z) = -i g^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \mathcal{P}(z, \bar{z}), \quad k^{\bar{\alpha}}(\bar{z}) = i g^{\beta\bar{\alpha}} \partial_\beta \mathcal{P}(z, \bar{z}). \quad (\text{A.2.5})$$

2. Let us notice that this is only true for the upper vector components $k^\alpha, k^{\bar{\alpha}}$, since for the lower vector components we have $\partial_{\bar{\alpha}} k_\beta = \partial_{\bar{\alpha}} [g_{\beta\bar{\gamma}}(z, \bar{z}) k^{\bar{\gamma}}(\bar{z})] \neq 0$ in general.

Applying ∇_α to the second equation above, using $\nabla_\alpha k^{\bar{\alpha}}(\bar{z}) = 0$ and the fact that $g^{\beta\bar{\alpha}}$ is invertible, we deduce that $\mathcal{P}(z, \bar{z})$ satisfies the equation:

$$\nabla_\alpha \partial_\beta \mathcal{P}(z, \bar{z}) = 0. \quad (\text{A.2.6})$$

We can now understand why Kähler manifolds are so attractive and in fact simpler than an arbitrary manifold. In a similar way that a Kähler metric is determined by a single real function, the Kähler potential $K(z, \bar{z})$, a Killing vector of a Kähler manifold is also determined by a single real function, the *moment map* or *Killing potential* $\mathcal{P}(z, \bar{z})$. With the ingredients presented above, we can now find a relation between these two functions. By definition, an isometry of a manifold leaves the metric invariant, which is the meaning of Eq. (A.2.1). However, in the case of a Kähler manifold, the Kähler potential does not need to be invariant, and can transform by a Kähler transformation (A.1.8) under the symmetry:

$$\delta K = \theta \left(k^\alpha \partial_\alpha + k^{\bar{\alpha}} \partial_{\bar{\alpha}} \right) K(z, \bar{z}) = \theta(r(z) + \bar{r}(\bar{z})), \quad (\text{A.2.7})$$

with $r(z)$ and $\bar{r}(\bar{z})$ arbitrary holomorphic and anti-holomorphic functions. On the one hand, we have $k_{\bar{\alpha}} = g_{\beta\bar{\alpha}} k^\beta = k^\beta \partial_\beta \partial_{\bar{\alpha}} K = \partial_{\bar{\alpha}}(k^\beta \partial_\beta K)$. On the other hand, Eq. (A.2.4) gives $k_{\bar{\alpha}} = -i \partial_{\bar{\alpha}} \mathcal{P}$. Integrating these equations yields $\mathcal{P} = i(k^\alpha \partial_\alpha K + \zeta(z))$, with $\zeta(z)$ an arbitrary holomorphic function. To determine it, we use the reality of \mathcal{P} , $\mathcal{P} - \bar{\mathcal{P}} = 0$, which gives $(k^\alpha \partial_\alpha + k^{\bar{\alpha}} \partial_{\bar{\alpha}}) K(z, \bar{z}) = -\zeta(z) - \bar{\zeta}(\bar{z})$. Comparing this with Eq. (A.2.7) gives $\zeta(z) = -r(z)$, $\bar{\zeta}(\bar{z}) = -\bar{r}(\bar{z})$. Therefore, the Killing potential \mathcal{P} and Kähler potential K are related by:

$$\mathcal{P}(z, \bar{z}) = i(k^\alpha \partial_\alpha K(z, \bar{z}) - r(z)) = -i \left(k^{\bar{\alpha}} \partial_{\bar{\alpha}} K(z, \bar{z}) - \bar{r}(\bar{z}) \right). \quad (\text{A.2.8})$$

Multiplet calculus

This appendix, based on [108], summarises the conformal supergravity multiplets calculus used in the Chapter 1. A general complex scalar multiplet is given by

$$\mathcal{C} = (\mathcal{C}, \mathcal{Z}, \mathcal{H}, \mathcal{K}, \mathcal{B}_\mu, \Lambda, \mathcal{D}), \quad (\text{B.0.1})$$

where \mathcal{C} , \mathcal{H} , \mathcal{K} and \mathcal{D} are complex scalars, \mathcal{Z} and Λ are Dirac fermions, and \mathcal{B}_μ is a Lorentz vector. A chiral multiplet is obtained from a complex multiplet by imposing $P_R \mathcal{Z} = 0$, $\mathcal{K} = 0$, $\mathcal{B}_\mu = i\mathcal{D}_\mu \mathcal{C}$, $\Lambda = 0$ and $\mathcal{D} = 0$. Renaming $\mathcal{C} = Z$, it is written, in a seven-components notation, as

$$(Z, -i\sqrt{2}P_L \chi, -2F, 0, i\mathcal{D}_\mu Z, 0, 0), \quad (\text{B.0.2})$$

and similarly for its anti-chiral counterpart:

$$(\bar{Z}, i\sqrt{2}P_R \chi, 0, -2\bar{F}, -i\mathcal{D}_\mu \bar{Z}, 0, 0). \quad (\text{B.0.3})$$

The chiral and anti-chiral multiplets are also usually written in a three-components notation according to:

$$(Z, P_L \chi, F), \quad (\bar{Z}, P_R \chi, \bar{F}). \quad (\text{B.0.4})$$

A real multiplet is obtained from a complex multiplet by imposing its lowest component $\mathcal{C} = C$ to be real. This implies $\mathcal{Z} = \zeta$ and $\Lambda = \lambda$ to be Majorana spinors, $\mathcal{B}_\mu = B_\mu$ and $\mathcal{D} = D$ to be real, while $\mathcal{K} = \bar{\mathcal{H}}$ is still complex. A real multiplet is thus written in a six-components notation according to

$$(C, \zeta, \mathcal{H}, B_\mu, \lambda, D). \quad (\text{B.0.5})$$

Throughout the Chapter 1, the operation $[]_F$ is defined as acting on a chiral multiplet $(Z, P_L \chi, F)$ of weights $(3, 3)$ by:

$$[]_F : (Z, P_L \chi, F) \rightarrow [Z]_F \equiv \frac{e}{2} \left[F + \frac{1}{\sqrt{2}} \bar{\psi}_\mu \gamma^\mu P_L \chi + \frac{1}{2} Z \bar{\psi}_\mu \gamma^{\mu\nu} P_R \psi_\nu + h.c. \right]. \quad (\text{B.0.6})$$

The operation $[]_D$ is defined as acting on a real multiplet $(C, \zeta, \mathcal{H}, B_\mu, \lambda, D)$ of weights $(2, 0)$ by:

$$\begin{aligned} []_D : (C, \zeta, \mathcal{H}, B_\mu, \lambda, D) \rightarrow [C]_D \equiv & \frac{e}{2} \left[D - \frac{1}{2} \bar{\psi}_\mu \gamma^\mu i\gamma_* \lambda - \frac{1}{3} C R(\omega) \right. \\ & + \frac{1}{6} \left(C \bar{\psi}_\mu \gamma^{\mu\rho\sigma} - i\bar{\zeta} \gamma^{\rho\sigma} \gamma_* \right) R'_{\rho\sigma}(Q) \\ & \left. + \frac{1}{4} \epsilon^{abcd} \bar{\psi}_a \gamma_b \psi_c \left(B_d - \frac{1}{2} \bar{\psi}_d \zeta \right) \right], \end{aligned} \quad (\text{B.0.7})$$

where $R(\omega)$ and $R'_{\rho\sigma}(Q)$ are the graviton and gravitino curvatures. Both operations are used to build superconformal invariant actions from chiral and real multiplets, respectively, according to $S_F = \int d^4x [Z]_F$ and $S_D = \int d^4x [C]_D$.

Given a set of complex multiplets $\mathcal{C}^i = (\mathcal{C}^i, \mathcal{Z}^i, \mathcal{H}^i, \mathcal{K}^i, \mathcal{B}_\mu^i, \Lambda^i, \mathcal{D}^i)$, $i = 1 \dots n$, one can build another complex multiplet $\mathcal{C} = (\mathcal{C}, \mathcal{Z}, \mathcal{H}, \mathcal{K}, \mathcal{B}_\mu, \Lambda, \mathcal{D})$ whose lowest component is given by an arbitrary function f of the first components of \mathcal{C}^i 's: $\mathcal{C} = f(\mathcal{C}^i)$. The other components of \mathcal{C} are then given by:

$$\begin{aligned}
\mathcal{Z} &= f_i \mathcal{Z}^i, \\
\mathcal{H} &= f_i \mathcal{H}^i - \frac{1}{2} f_{ij} \bar{\mathcal{Z}}^i P_L \mathcal{Z}^j, \\
\mathcal{K} &= f_i \mathcal{K}^i - \frac{1}{2} f_{ij} \bar{\mathcal{Z}}^i P_R \mathcal{Z}^j, \\
\mathcal{B}_\mu &= f_i \mathcal{B}_\mu^i + \frac{1}{2} i f_{ij} \bar{\mathcal{Z}}^i P_L \gamma_\mu \mathcal{Z}^j, \\
\Lambda &= f_i \Lambda^i + \frac{1}{2} f_{ij} \left(i \gamma_* \mathcal{B}^i + P_L \mathcal{K}^i + P_R \mathcal{H}^i - \mathcal{D} \mathcal{C}^i \right) \mathcal{Z}^j - \frac{1}{4} f_{ijk} \mathcal{Z}^i \bar{\mathcal{Z}}^j \mathcal{Z}^k, \\
\mathcal{D} &= f_i \mathcal{D}^i + \frac{1}{2} f_{ij} \left(\mathcal{K}^i \mathcal{H}^j - \mathcal{B}^i \cdot \mathcal{B}^j - \mathcal{D} \mathcal{C}^i \cdot \mathcal{D} \mathcal{C}^j - 2 \bar{\Lambda}^i \mathcal{Z}^j - \bar{\mathcal{Z}}^i \mathcal{D} \mathcal{Z}^j \right), \\
&\quad - \frac{1}{4} f_{ijk} \bar{\mathcal{Z}}^i \left(i \gamma_* \mathcal{B}^j + P_L \mathcal{K}^j + P_R \mathcal{H}^j \right) \mathcal{Z}^k + \frac{1}{8} f_{ijkl} \bar{\mathcal{Z}}^i P_L \mathcal{Z}^j \bar{\mathcal{Z}}^k P_R \mathcal{Z}^l,
\end{aligned} \tag{B.0.8}$$

with $f_i \equiv \frac{\partial f}{\partial \mathcal{C}^i}$ and so on for higher order derivatives. The bar on spinors are the Majorana conjugate defined by $\bar{\psi} = \psi^T C$, with C the charge conjugation matrix satisfying $\gamma_\mu^T = -C \gamma_\mu C^{-1}$.

In the first chapter of this thesis, we often deal with chiral multiplets, and it is therefore convenient to write the form taken by the composition laws (B.0.8) in that particular case. Considering one chiral multiplet, written in the three-components notation as $(X_1, P_L \Omega_1, F_1)$, one can consider a second chiral multiplet $(X_2, P_L \Omega_2, F_2)$ whose lowest component is built from a given function $f(X_1)$. One can also consider the product of two chiral multiplets X_3 and X_4 , yielding another chiral multiplet X_5 . The composition laws (B.0.8) then simplify and give the following chiral multiplets:

$$\begin{aligned}
X_2 &= f(X_1), & X_5 &= X_3 X_4, \\
P_L \Omega_2 &= f_1 P_L \Omega_1, & P_L \Omega_5 &= X_3 P_L \Omega_4 + X_4 P_L \Omega_3, \\
F_2 &= f_1 F_1 - \frac{1}{2} f_{11} \bar{\Omega}_1 P_L \Omega_1, & F_5 &= F_3 X_4 + F_4 X_3 - \bar{\Omega}_3 P_L \Omega_4.
\end{aligned} \tag{B.0.9}$$

Fermion masses

In this appendix, we compute the fermion masses in the no-scale models presented in Section 1.7. We recall the full Lagrangian considered here:

$$\mathcal{L} = -3 \left[S_0 \bar{S}_0 e^{-\frac{K(X, \bar{X})}{3}} \right]_D + \left[S_0^3 W(X) \right]_F - \frac{1}{4} \left[\bar{\lambda} P_L \lambda \right]_F + \mathcal{L}_{FI}^{(\alpha)} + \mathcal{L}_{FI}^{(-2/3)}, \quad (\text{C.0.1})$$

with the new FI-terms given by:

$$\mathcal{L}_{FI}^{(\alpha)} = -\xi \left[(S_0 \bar{S}_0)^{-1} e^{(\frac{1}{3} + \alpha)K} W^\alpha \bar{W}^\alpha \frac{(\bar{\lambda} P_L \lambda)(\bar{\lambda} P_R \lambda)}{T(\mathcal{W}^2) \bar{T}(\mathcal{W}^2)} (V)_D \right]_D. \quad (\text{C.0.2})$$

We write the fermion mass terms as $\mathcal{L}_m = \mathcal{L}_m^{(0)} + \mathcal{L}_m^{FI}$, with $\mathcal{L}_m^{(0)}$ arising from the usual matter-coupled $\mathcal{N} = 1$ supergravity Lagrangian, namely the three first terms of (C.0.1), and \mathcal{L}_m^{FI} arising from the FI Lagrangians $\mathcal{L}_{FI}^{(\alpha)} + \mathcal{L}_{FI}^{(-2/3)}$. $\mathcal{L}_m^{(0)}$ reads:

$$\begin{aligned} e^{-1} \mathcal{L}_m^{(0)} &= \frac{1}{2} m_{3/2} \bar{\psi}_\mu P_R \gamma^{\mu\nu} \psi_\nu + \bar{\psi}_\mu \gamma^\mu \left(\frac{1}{\sqrt{2}} \Omega^\alpha e^{K/2} \nabla_\alpha W + \frac{i}{2} \mathcal{P}_A P_L \lambda^A \right) \\ &\quad - \frac{1}{2} m_{\alpha\beta}^{(0)} \bar{\Omega}^\alpha \Omega^\beta - m_{\alpha A}^{(0)} \bar{\Omega}^\alpha \lambda^A - \frac{1}{2} m_{AB}^{(0)} \bar{\lambda}^A P_L \lambda^B + h.c., \end{aligned} \quad (\text{C.0.3})$$

where ψ_μ denotes the gravitino, Ω^α the chiral fermions, and λ^A the gauginos. The various masses are given by [10]:

$$m_{3/2} = e^{K/2} W, \quad (\text{C.0.4})$$

$$m_{\alpha\beta}^{(0)} = e^{K/2} \nabla_\alpha \nabla_\beta W \equiv e^{K/2} (\partial_\alpha + \partial_\alpha K) \nabla_\beta W - e^{K/2} \Gamma_{\alpha\beta}^\gamma \nabla_\gamma W, \quad (\text{C.0.5})$$

$$m_{\alpha A}^{(0)} = i\sqrt{2} \left(\partial_\alpha \mathcal{P}_A - \frac{1}{4} f_{AB, \alpha} (\text{Re} f)^{-1 BC} \mathcal{P}_C \right) = m_{A\alpha}^{(0)}, \quad (\text{C.0.6})$$

$$m_{AB}^{(0)} = -\frac{1}{2} e^{K/2} f_{AB, \alpha} g^{\alpha\bar{\beta}} \bar{\nabla}_{\bar{\beta}} \bar{W}. \quad (\text{C.0.7})$$

In the no-scale models with Kähler potential $K = -p \ln(X + \bar{X})$ studied in Section 1.7, we consider only one chiral matter multiplet and one gauge multiplet, therefore the index α and A take only one value. The Christoffel symbols are given by $\Gamma_{\alpha\beta}^\gamma = g^{\gamma\bar{\delta}} \partial_\alpha g_{\beta\bar{\delta}}$, which reduce to only one non-vanishing component $\Gamma_{XX}^X = -\frac{2}{X + \bar{X}}$. The moment map \mathcal{P} is defined by $\mathcal{P} = i(k^\alpha \partial_\alpha K - r)$, where k^α is the Killing vector associated to the gauged symmetry and r is the corresponding FI constant. In Section 1.7 we considered $r = 0$. When the chiral multiplet becomes charged under the gauged shift symmetry, the associated constant Killing vector is $k^X = ic$. Focussing on the ungauged case considered in Section 1.7, we thus have $\mathcal{P} = 0$. Finally,

the gauge kinetic function f_{AB} being constant, we end up with:

$$m_{3/2} = e^{K/2}W, \quad m_{\Omega\Omega}^{(0)} = m_{3/2} \frac{p(p-1)}{(X+\bar{X})^2}, \quad m_{\Omega\lambda}^{(0)} = 0, \quad m_{\lambda\lambda}^{(0)} = 0. \quad (\text{C.0.8})$$

We now turn to the fermion mass contributions arising from the FI Lagrangians. Keeping only quadratic terms in fermions containing no derivatives, the FI Lagrangian (C.0.2) can be written as

$$\mathcal{L}_{FI}^{(\alpha)} = -\xi \frac{D(W\bar{W})^\alpha}{T(\mathcal{W}^2)\bar{T}(\bar{\mathcal{W}}^2)} \left[\hat{\mathcal{R}}^{(\alpha)} \right]_D, \quad (\text{C.0.9})$$

with $\hat{\mathcal{R}}^{(\alpha)}$ the real multiplet defined by

$$\hat{\mathcal{R}}^{(\alpha)} \equiv e^{(\frac{1}{3}+\alpha)K(X,\bar{X})} (S_0^{-1}\bar{\lambda}P_L\lambda) (\bar{S}_0^{-1}\bar{\lambda}P_R\lambda), \quad (\text{C.0.10})$$

which is a function of the chiral multiplets $S_0^{-1}\bar{\lambda}P_L\lambda$, X , and their anti-chiral counterparts. In the seven-components notation, they are given by:

$$S_0^{-1}\bar{\lambda}P_L\lambda = (s_0^{-1}\bar{\lambda}P_L\lambda, 2s_0^{-1}DP_L\lambda, 2s_0^{-2}F_0\bar{\lambda}P_L\lambda + 2s_0^{-1}D^2, 0, 0, 0, 0), \quad (\text{C.0.11})$$

$$X = (X, -i\sqrt{2}P_L\Omega, -2F, 0, 0, 0, 0), \quad (\text{C.0.12})$$

$$\bar{S}_0^{-1}\bar{\lambda}P_R\lambda = (\bar{s}_0^{-1}\bar{\lambda}P_R\lambda, 2\bar{s}_0^{-1}DP_R\lambda, 0, 2\bar{s}_0^{-2}\bar{F}_0\bar{\lambda}P_R\lambda + 2\bar{s}_0^{-1}D^2, 0, 0, 0), \quad (\text{C.0.13})$$

$$\bar{X} = (\bar{X}, i\sqrt{2}P_R\Omega, 0, -2\bar{F}, 0, 0, 0), \quad (\text{C.0.14})$$

Writing $\hat{\mathcal{R}}^{(\alpha)} \equiv (0, 0, 0, (\hat{\mathcal{R}}^{(\alpha)})_{\mathcal{B}_\mu}, (\hat{\mathcal{R}}^{(\alpha)})_\lambda, (\hat{\mathcal{R}}^{(\alpha)})_D)$, its contribution to the fermion masses arises from:

$$\left[\hat{\mathcal{R}}^{(\alpha)} \right]_D = \frac{e}{2} \left((\hat{\mathcal{R}}^{(\alpha)})_D - \frac{i}{2} \bar{\psi} \cdot \gamma \gamma_* (\hat{\mathcal{R}}^{(\alpha)})_\lambda \right). \quad (\text{C.0.15})$$

The tensor calculus (B.0.8) gives:

$$\begin{aligned} (\hat{\mathcal{R}}^{(\alpha)})_D &= \left[i2\sqrt{2}\partial_X e^{(\alpha+1/3)K} s_0^{-1}\bar{s}_0^{-1}D^3\bar{\lambda}P_L\Omega - 2\partial_X e^{(\alpha+1/3)K} s_0^{-1}\bar{s}_0^{-1}D^2F\bar{\lambda}P_L\lambda \right. \\ &\quad \left. + 2e^{(\alpha+1/3)K} s_0^{-1}s_0^{-2}D^2F_0\bar{\lambda}P_L\lambda \right] + h.c. + 2e^{(\alpha+1/3)K} s_0^{-1}\bar{s}_0^{-1}D^4, \end{aligned} \quad (\text{C.0.16})$$

$$(\hat{\mathcal{R}}^{(\alpha)})_\lambda = 2e^{(\alpha+1/3)K} D^3 s_0^{-1}\bar{s}_0^{-1}\lambda. \quad (\text{C.0.17})$$

Combining this with

$$\frac{D(W\bar{W})^\alpha}{T(\mathcal{W}^2)\bar{T}(\bar{\mathcal{W}}^2)} = (W\bar{W})^{\alpha+2/3} D^{-3} s_0^2 \bar{s}_0^2 - 2(W\bar{W})^{\alpha+2/3} \left(s_0 \bar{s}_0^2 \frac{F_0}{D^5} \bar{\lambda}P_L\lambda + h.c. \right), \quad (\text{C.0.18})$$

one obtains:

(i) The gravitino-gaugino mixing:

$$e^{-1} \mathcal{L}_{FI}^{(\alpha)} \supset \frac{i}{2} \bar{\psi}_\mu \gamma^\mu \xi e^{(\alpha+2/3)\mathcal{G}} \gamma_* \lambda. \quad (\text{C.0.19})$$

Considering $\mathcal{L}^{(0)} + \mathcal{L}_{FI}^{(\alpha)} + \mathcal{L}_{FI}^{(-2/3)}$, we get the following gravitino/spin-1/2 mixing Lagrangian:

$$e^{-1} \mathcal{L}_{\text{mix}} = \bar{\psi}_\mu \gamma^\mu \left(\frac{1}{\sqrt{2}} \Omega e^{K/2} \nabla_X W + \frac{i}{2} (\xi_1 e^{(\alpha+2/3)\mathcal{G}} + \xi_2) P_L \lambda \right) + h.c., \quad (\text{C.0.20})$$

from which we identify the Goldstino as the linear combination:

$$P_L\nu = -\frac{1}{\sqrt{2}}\Omega e^{K/2}\nabla_X W - \frac{i}{2}(\xi_1 e^{(\alpha+2/3)\mathcal{G}} + \xi_2)P_L\lambda. \quad (\text{C.0.21})$$

(ii) The fermion mass terms:

$$\begin{aligned} e^{-1}\mathcal{L}_{FI}^{(\alpha)} \supset & -i\sqrt{2}\xi\left(\alpha + \frac{1}{3}\right)\partial_X K e^{(\alpha+1/3)\mathcal{G}}(W\bar{W})^{1/3}s_0\bar{s}_0\bar{\lambda}P_L\Omega \\ & +\xi\left(\alpha + \frac{1}{3}\right)\partial_X K e^{(\alpha+1/3)\mathcal{G}}(W\bar{W})^{1/3}s_0\bar{s}_0\frac{F}{D}\bar{\lambda}P_L\lambda \\ & +\xi e^{(\alpha+1/3)\mathcal{G}}(W\bar{W})^{1/3}s_0\bar{s}_0\frac{s_0^{-1}F_0}{D}\bar{\lambda}P_L\lambda + h.c. \end{aligned} \quad (\text{C.0.22})$$

Considering from now on $\mathcal{L}_{FI}^{(\alpha)} + \mathcal{L}_{FI}^{(-2/3)}$, we define for simplicity of the expressions the following quantities:

$$\begin{aligned} D_{\text{bos}} & \equiv \left(\xi_1 e^{(\alpha_1+1/3)\mathcal{G}} + \xi_2 e^{-1/3\mathcal{G}}\right)(W\bar{W})^{1/3}s_0\bar{s}_0 \\ & = \xi_1 e^{(\alpha_1+2/3)\mathcal{G}} + \xi_2, \end{aligned} \quad (\text{C.0.23})$$

$$\begin{aligned} \gamma & \equiv \partial_X K \left(\xi_1\left(\alpha_1 + \frac{1}{3}\right)e^{(\alpha_1+1/3)\mathcal{G}} - \frac{\xi_2}{3}e^{-1/3\mathcal{G}}\right)(W\bar{W})^{1/3}s_0\bar{s}_0 \\ & = \partial_X K \left(\xi_1\left(\alpha_1 + \frac{1}{3}\right)e^{(\alpha_1+2/3)\mathcal{G}} - \frac{\xi_2}{3}\right). \end{aligned} \quad (\text{C.0.24})$$

One can already read from (C.0.22) the gaugino/chiral fermion mixing mass term:

$$m_{\Omega\lambda}^{FI} = i\sqrt{2}\gamma. \quad (\text{C.0.25})$$

In order to find the gaugino mass term $m_{\lambda\lambda}^{FI}$, we have to eliminate the auxiliary fields D , F and F_0 using their equations of motion. The part of the total Lagrangian containing the auxiliary field D , up to quadratic order in fermions, is:

$$e^{-1}\mathcal{L} \supset \frac{1}{2}D^2 - D_{\text{bos}}D + \left(D_{\text{bos}}\frac{s_0^{-1}F_0}{D}\bar{\lambda}P_L\lambda + \gamma\frac{F}{D}\bar{\lambda}P_L\lambda + h.c.\right), \quad (\text{C.0.26})$$

so that the equation of motion for D reads $D^3 - D_{\text{bos}}D^2 - \left[(D_{\text{bos}}s_0^{-1}F_0 + \gamma F)\bar{\lambda}P_L\lambda + h.c.\right] = 0$. Solving it analytically and expanding the solution up to quadratic order in fermions, we find

$$D = D_{\text{bos}} + \left(\frac{D_{\text{bos}}s_0^{-1}F_0 + \gamma F}{D_{\text{bos}}^2}\bar{\lambda}P_L\lambda + h.c.\right) + \text{higher order in fermions}. \quad (\text{C.0.27})$$

Replacing (C.0.27) in (C.0.26), we find the following quadratic contribution in fermions:

$$e^{-1}\mathcal{L} \supset \frac{D_{\text{bos}}s_0^{-1}F_0 + \gamma F}{D_{\text{bos}}}\bar{\lambda}P_L\lambda + h.c. \quad (\text{C.0.28})$$

We now eliminate the auxiliary fields F_0 and F , associated to the compensator and X chiral multiplet, respectively. The part of the total Lagrangian containing the auxiliary fields F_0 , F ,

up to the quadratic order in fermions, reads:

$$\begin{aligned}
e^{-1}\mathcal{L} \supset & -3e^{-K/3}F_0\bar{F}_0 + 3e^{K/3}WF_0 + 3e^{K/3}\bar{W}\bar{F}_0 + \frac{1}{9}g_{X\bar{X}}F\bar{F} + \frac{1}{3}e^{K/2}\nabla_X WF \\
& + \frac{1}{3}e^{K/2}\bar{\nabla}_{\bar{X}}\bar{W}\bar{F} + \frac{1}{D_{\text{bos}}}(D_{\text{bos}}e^{-K/6}F_0 + \gamma F)\bar{\lambda}P_L\lambda \\
& + \frac{1}{D_{\text{bos}}}(D_{\text{bos}}e^{-K/6}\bar{F}_0 + \gamma\bar{F})\bar{\lambda}P_R\lambda,
\end{aligned} \tag{C.0.29}$$

which yields, after elimination of F_0 and F :

$$e^{-1}\mathcal{L} \supset e^K \left(3W\bar{W} - \nabla_X W g^{X\bar{X}} \bar{\nabla}_{\bar{X}} \bar{W} \right) + \frac{e^{K/2}}{D_{\text{bos}}} \left(D_{\text{bos}} \bar{W} \bar{\lambda} P_L \lambda - 3g^{X\bar{X}} \bar{\nabla}_{\bar{X}} \bar{W} \gamma \bar{\lambda} P_L \lambda + h.c. \right). \tag{C.0.30}$$

The first two terms correspond to the usual F -contribution to the scalar potential, while the last two terms give the contribution to the gaugino mass $m_{\lambda\lambda}^{FI}$ from the FI-terms. For a constant superpotential, it reads:

$$m_{\lambda\lambda}^{FI} = -2 \frac{m_{3/2}}{D_{\text{bos}}} \left[D_{\text{bos}} + 3(X + \bar{X})\gamma \right]. \tag{C.0.31}$$

At the minimum of the potential where $\partial_X \mathcal{V} = 0$ and $\mathcal{V} = 0$, $m_{\Omega\Lambda}^{FI}$ and $m_{\lambda\lambda}^{FI}$ given in Eqs. (C.0.25) and (C.0.31) simplify, and the entries of the fermion mass matrix can be written as:

$$\begin{aligned}
m_{\Omega\Omega} &= m_{\Omega\Omega}^{(0)} = m_{3/2} \frac{p(p-1)}{(X + \bar{X})^2}, \\
m_{\Omega\lambda} &= m_{\Omega\lambda}^{FI} = -\frac{i\sqrt{2}}{6} \frac{pD_{\text{bos}}}{X + \bar{X}}, \\
m_{\lambda\lambda} &= m_{\lambda\lambda}^{FI} = -2m_{3/2} \left(1 - \frac{p}{2} \right).
\end{aligned} \tag{C.0.32}$$

In order to study the spin-1/2 fermions mass matrix, we have to get rid of the gravitino-Goldstino mixing (C.0.20). This can be done by carrying out a supersymmetry transformation, bringing the gravitino ψ_μ into the physical, massive, one Ψ_μ through [10]:

$$P_L \psi_\mu \rightarrow P_L \Psi_\mu = P_L \psi_\mu - \frac{2}{3m_{3/2}^2} \partial_\mu P_L \nu - \frac{1}{3m_{3/2}} \gamma_\mu P_R \nu. \tag{C.0.33}$$

The mixing term between the gravitino and the Goldstino then vanishes, and Ψ_μ is the massive gravitino in Minkowski space with physical mass $m_{3/2}$. In addition, the transformation (C.0.33) brings new contributions to the spin-1/2 fermion mass terms. Writing the Goldstino $P_L \nu$ as a linear combination of the gaugino λ and the chiral fermion Ω , namely $P_L \nu = A\Omega + B P_L \lambda$ where A and B are given in this model by Eq. (C.0.21), these new contributions read:

$$\begin{aligned}
m_{\Omega\Omega}^{(\nu)} &= -\frac{4}{3m_{3/2}^2} A^2, \\
m_{\Omega\lambda}^{(\nu)} &= -\frac{4}{3m_{3/2}^2} AB, \\
m_{\lambda\lambda}^{(\nu)} &= -\frac{4}{3m_{3/2}^2} B^2.
\end{aligned} \tag{C.0.34}$$

The most general structure of a spin-1/2 mass term $m^{(g)}$ is then given by $m^{(g)} \equiv m^{(0)} + m^{FI} + m^{(\nu)}$, with $m^{(0)}$ the contribution from the original Lagrangian $-3 \left[S_0 \bar{S}_0 e^{-\frac{K(X, \bar{X})}{3}} \right]_D + [S_0^3 W(X)]_F - \frac{1}{4} [\bar{\lambda} P_L \lambda]_F$, m^{FI} the contribution from the new FI terms $\mathcal{L}_{FI}^{(\alpha)} + \mathcal{L}_{FI}^{(-2/3)}$, and $m^{(\nu)}$ the shifts (C.0.34) upon elimination of the gravitino-Goldstino mixing. From Eqs. (C.0.8), (C.0.32) and (C.0.34), we deduce the fermion mass matrix at the minimum of the potential:

$$M = \begin{pmatrix} m_{\Omega\Omega}^{(g)} & m_{\Omega\lambda}^{(g)} \\ m_{\Omega\lambda}^{(g)} & m_{\lambda\lambda}^{(g)} \end{pmatrix} = \begin{pmatrix} \frac{p-3}{3} m_{3/2} & \frac{i\sqrt{2}}{6} \sqrt{p} D_{\text{bos}} \\ \frac{i\sqrt{2}}{6} \sqrt{p} D_{\text{bos}} & \frac{p}{3} m_{3/2} \end{pmatrix} \quad (\text{C.0.35})$$

A normalisation factor for $m_{\Omega\Omega}^{(g)}$ and $m_{\Omega\lambda}^{(g)}$ has been introduced due to the non-canonical kinetic term of the chiral fermion, while the gaugino already has canonical kinetic term since the gauge kinetic function f has been set to one. Using $D_{\text{bos}}^2 = -2(p-3)m_{3/2}^2$ at the minimum of the potential, one immediately sees that the determinant of M vanishes, while its non-zero eigenvalue m_f , corresponding to the mass of the physical fermion, is given by:

$$m_f^2 = m_{3/2}^2 \left(\frac{4}{9} p^2 - \frac{4}{3} p + 1 \right), \forall p \neq 3, \quad (\text{C.0.36})$$

where we excluded the value $p = 3$ for which, in the case of a constant superpotential, the D-term vanishes in the minimum, making the new FI-term singular, and a different superpotential is used in Section 1.7.2.

Gravitational action on a bounded manifold

D.1 $d + 1$ decomposition and Gibbons-Hawking boundary term: a review

Let us consider a $d + 1$ -dimensional space-time manifold \mathcal{M} , equipped with a metric \hat{G}_{MN} and foliated with a set of co-dimension 1 hypersurfaces Σ_y . Such hypersurfaces can be defined by an arbitrary scalar field $y(x^M)$ such that y is constant on each of these hypersurfaces. The unit normal vector n_M to Σ_y is then proportional to $\partial_M y$, and normalized such that $n^M n_M = \hat{G}^{MN} n_M n_N = \epsilon$, with $\epsilon = +1$ if Σ_y is timelike, and $\epsilon = -1$ if Σ_y is spacelike.

In order to relate the coordinate systems on each hypersurfaces, we consider a congruence of curves which intersect each Σ_y once and only once. The coordinates z^μ on each hypersurfaces are now chosen such that points on the same curves have the same coordinates z^μ on the different slices Σ_y . Therefore, considering two infinitesimally closed hypersurfaces Σ_y and Σ_{y+dy} , the vector y^M tangent to the curve points from a point with coordinates z^μ on Σ_y to a point with the same coordinates on Σ_{y+dy} . This construction therefore allows us to move from the original coordinate system x^M to a new one (y, z^μ) . The vectors y^M tangent to the curves and E_μ^M tangent to Σ_y are respectively given by:

$$y^M = \left. \frac{\partial x^M}{\partial y} \right|_{z^\mu}, \quad E_\mu^M = \left. \frac{\partial x^M}{\partial z^\mu} \right|_y. \quad (\text{D.1.1})$$

The E_μ^M can be seen as a map from $\otimes^q T_P^*(\mathcal{M})$, an arbitrary tensor product of the cotangent spaces of \mathcal{M} , to $\otimes^q T_P^*(\Sigma_y)$, an arbitrary tensor product of the cotangent spaces of Σ_y , projecting any $X_{M_1 \dots M_q} \in \otimes^q T_P^*(\mathcal{M})$ down to a $X_{\mu_1 \dots \mu_q} \in \otimes^q T_P^*(\Sigma_y)$ through

$$E_\mu^M : X_{M_1 \dots M_q} \mapsto X_{\mu_1 \dots \mu_q} = X_{M_1 \dots M_q} E_{\mu_1}^{M_1} \dots E_{\mu_q}^{M_q}. \quad (\text{D.1.2})$$

With this projection map, one can define the induced metric $\hat{g}_{\mu\nu}$ and the extrinsic curvature $\hat{K}_{\mu\nu}$ respectively by

$$\hat{g}_{\mu\nu} = \hat{G}_{MN} E_\mu^M E_\nu^N, \quad (\text{D.1.3})$$

$$\hat{K}_{\mu\nu} = E_\mu^M E_\nu^N \hat{\nabla}_M n_N, \quad (\text{D.1.4})$$

with $\hat{\nabla}_M$ the covariant derivative compatible with the metric \hat{G}_{MN} . While $\hat{g}_{\mu\nu}$ characterises

the local intrinsic geometry of Σ_y , $\hat{K}_{\mu\nu}$ describes how the hypersurface is embedded in the ambient space \mathcal{M} , and the data of both is sufficient to completely characterise the geometry of Σ_y embedded into \mathcal{M} .

The vector y^M tangent to the curves can be decomposed in the basis provided by normal n^M and tangent vectors E_μ^M according to

$$y^M = \hat{N}n^M + \beta^\mu E_\mu^M. \quad (\text{D.1.5})$$

The components \hat{N} and β^μ are respectively called the lapse function and the shift vector. Their physical interpretation follows from the construction described above: the lapse describes the orthogonal distance between the two slices, while the shift describes how the coordinate systems of the two infinitesimally closed hypersurfaces are transversally shifted one with respect to the other. Using

$$dx^M = \frac{\partial x^M}{\partial y} dy + \frac{\partial x^M}{\partial z^\mu} dz^\mu = (\hat{N}n^M + \beta^\mu E_\mu^M) dy + E_\mu^M dz^\mu, \quad (\text{D.1.6})$$

it is easy to find the following decomposition of the line element in terms of the lapse \hat{N} , the shift β^μ and the induced metric $\hat{g}_{\mu\nu}$:

$$ds^2 = \epsilon \hat{N}^2 dy^2 + \hat{g}_{\mu\nu} (\beta^\mu dy + dz^\mu) (\beta^\nu dy + dz^\nu). \quad (\text{D.1.7})$$

This is the famous ADM decomposition, first introduced in [48], which corresponds to the ADM metric tensor

$$\hat{G}_{MN} = \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu\rho} \beta^\rho \\ \hat{g}_{\nu\rho} \beta^\rho & \epsilon \hat{N}^2 + \hat{g}_{\rho\sigma} \beta^\rho \beta^\sigma \end{pmatrix}, \quad (\text{D.1.8})$$

whose inverse reads

$$\hat{G}^{MN} = \begin{pmatrix} \hat{g}^{\mu\nu} + \epsilon \hat{N}^{-2} \beta^\mu \beta^\nu & -\epsilon \hat{N}^{-2} \beta^\mu \\ -\epsilon \hat{N}^{-2} \beta^\nu & \epsilon \hat{N}^{-2} \end{pmatrix}. \quad (\text{D.1.9})$$

We now suppose that \mathcal{M} is bounded in the y direction by two hypersurfaces Σ_{y_1} and Σ_{y_2} located at y_1 and y_2 . The total gravitational action S_G is now the sum of the Einstein-Hilbert bulk term S_{EH} and the Gibbons-Hawking boundary term S_{GH} :

$$S_G = S_{EH} + S_{GH}, \quad (\text{D.1.10})$$

with

$$S_{EH} = \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \sqrt{-\hat{G}} R^{(d+1)}[\hat{G}_{MN}], \quad (\text{D.1.11})$$

$$S_{GH} = -\epsilon \int_{\Sigma_{y_1}} d^d x \sqrt{|\hat{g}|} \hat{K} + \epsilon \int_{\Sigma_{y_2}} d^d x \sqrt{|\hat{g}|} \hat{K}. \quad (\text{D.1.12})$$

Here \hat{K} is the trace of the extrinsic curvature, $\hat{K} \equiv \hat{g}^{\mu\nu} \hat{K}_{\mu\nu}$, and the relative sign between the two terms in the GH action comes from the fact that both normals to Σ_{y_1} and Σ_{y_2} point along the increasing y , and are thus directed respectively inward and outward of \mathcal{M} .

Starting from the Gauss equation

$$R_{MNPQ}E_\mu^ME_\nu^NE_\rho^PE_\sigma^Q = R_{\mu\nu\rho\sigma} + \epsilon(\hat{K}_{\mu\sigma}\hat{K}_{\nu\rho} - \hat{K}_{\mu\rho}\hat{K}_{\nu\sigma}), \quad (\text{D.1.13})$$

which relates the tangential components of the curvature tensor of \mathcal{M} on the LHS with the intrinsic and extrinsic curvature tensors of Σ on the RHS, one can find the relation between the $(d + 1)$ -dimensional Ricci scalar $R^{(d+1)}$ evaluated on Σ_y and the intrinsic curvature scalar $R^{(d)}$ of Σ_y :

$$R^{(d+1)}[\hat{G}_{MN}] = R^{(d)}[\hat{g}_{\mu\nu}] + \epsilon(\hat{K}^2 - \hat{K}^{\mu\nu}\hat{K}_{\mu\nu}) + 2\epsilon\hat{\nabla}_M(n^N\hat{\nabla}_N n^M - n^M\hat{\nabla}_N n^N). \quad (\text{D.1.14})$$

The first two terms $R^{(d)}[\hat{g}_{\mu\nu}]$ and $(\hat{K}^2 - \hat{K}^{\mu\nu}\hat{K}_{\mu\nu})$ depend respectively on the intrinsic and extrinsic geometry of Σ_y . The third term contains second normal derivatives, which will cancel with the GH boundary term as we are now going to show. In the foliated spacetime $\mathcal{M} = \Sigma_y \times S^1/\mathbb{Z}_2$, the EH action (D.1.11) reads:

$$\begin{aligned} \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \sqrt{-\hat{G}} R^{(d+1)}[\hat{G}_{MN}] &= \frac{1}{2} \int_{y_1}^{y_2} dy \int_{\Sigma_y} d^d x \hat{N} \sqrt{|\hat{g}|} \left[R^{(d)}[\hat{g}_{\mu\nu}] + \epsilon(\hat{K}^2 - \hat{K}^{\mu\nu}\hat{K}_{\mu\nu}) \right] \\ &\quad + \epsilon \int_{\mathcal{M}} d^{d+1}x \sqrt{-\hat{G}} \hat{\nabla}_M \left[n^N \hat{\nabla}_N n^M - n^M \hat{\nabla}_N n^N \right]. \end{aligned} \quad (\text{D.1.15})$$

Using Gauss's theorem, the second term can be written as:

$$\begin{aligned} I &\equiv \epsilon \int_{\mathcal{M}} d^{d+1}x \sqrt{-\hat{G}} \hat{\nabla}_M \left[n^N \hat{\nabla}_N n^M - n^M \hat{\nabla}_N n^N \right], \\ &= \epsilon \int_{\partial\mathcal{M}} \left[n^N \hat{\nabla}_N n^M - n^M \hat{\nabla}_N n^N \right] d\Sigma_M. \end{aligned} \quad (\text{D.1.16})$$

I is the sum of two contributions at each boundaries Σ_{y_1} and Σ_{y_2} of \mathcal{M} . On Σ_{y_1} (Σ_{y_2}), the surface element is given by $d\Sigma_M = -\epsilon n_M \sqrt{|\hat{g}|} d^d x$ ($d\Sigma_M = +\epsilon n_M \sqrt{|\hat{g}|} d^d x$), where the minus (plus) sign takes into account the inward (outward) direction of the normal of Σ_{y_1} (Σ_{y_2}). Using $n_M n^M = \epsilon$ and $n_M \hat{\nabla}_N n^M = 0$, we find:

$$I = \epsilon \int_{\Sigma_{y_1}} d^d x \sqrt{|\hat{g}|} \hat{K} - \epsilon \int_{\Sigma_{y_2}} d^d x \sqrt{|\hat{g}|} \hat{K}, \quad (\text{D.1.17})$$

which is exactly cancelled by the GH boundary term (D.1.12). Hence, the total gravitational action $S_G = S_{EH} + S_{GH}$ reads:

$$S_G = \frac{1}{2} \int_{y_1}^{y_2} dy \int_{\Sigma_y} d^d x \hat{N} \sqrt{|\hat{g}|} \left[R^{(d)}[\hat{g}_{\mu\nu}] + \epsilon(\hat{K}^2 - \hat{K}^{\mu\nu}\hat{K}_{\mu\nu}) \right]. \quad (\text{D.1.18})$$

The d -dimensional Ricci scalar is given by:

$$\begin{aligned} R^{(d)}[\hat{g}_{\mu\nu}] &= \frac{1}{2} \partial_\mu \left(\sqrt{|\hat{g}|} \hat{g}^{\mu\nu} \hat{N} \right) \partial_\nu \ln \hat{g} + \partial_\mu \hat{N} \partial_\nu \left(\sqrt{|\hat{g}|} \hat{g}^{\mu\nu} \right) - \frac{1}{2} \hat{N} \sqrt{|\hat{g}|} \hat{\Gamma}_{\mu\nu}^\rho \partial_\rho \hat{g}^{\mu\nu} \\ &\quad - \partial_\mu \left(\hat{N} \hat{g}^{\mu\nu} \partial_\nu \sqrt{|\hat{g}|} + \hat{N} \partial_\nu (\sqrt{|\hat{g}|} \hat{g}^{\mu\nu}) \right), \end{aligned} \quad (\text{D.1.19})$$

with $\hat{\Gamma}_{\mu\nu}^\rho$ the d -dimensional Christoffel symbols computed from the metric $\hat{g}_{\mu\nu}$. The last term

in the above expression is a total derivative in the unbounded x^μ directions, and can thus be discarded. In order to express the extrinsic curvature tensor $\hat{K}_{\mu\nu} = E_\mu^M E_\nu^N \hat{\nabla}_M n_N$ in terms of the induced metric $\hat{g}_{\mu\nu}$ and the ADM variables \hat{N} and β_μ , we start by writing the derivative of $\hat{g}_{\mu\nu}$ with respect to y as:

$$\hat{g}'_{\mu\nu} \equiv \mathcal{L}_y \hat{g}_{\mu\nu} = \mathcal{L}_y (\hat{G}_{MN} E_\mu^M E_\nu^N) = (\mathcal{L}_y \hat{G}_{MN}) E_\mu^M E_\nu^N, \quad (\text{D.1.20})$$

where, in the last equality, we have used

$$\mathcal{L}_y E_\mu^M = [y, E_\mu]^M = [\partial_y, \partial_{x^\mu}]^M = 0. \quad (\text{D.1.21})$$

The Lie derivative of the metric \hat{G}_{MN} is given by:

$$\begin{aligned} \mathcal{L}_y \hat{G}_{MN} &= \hat{\nabla}_M y_N + \hat{\nabla}_N y_M \\ &= \hat{\nabla}_M (\hat{N} n_N + \beta_N) + \hat{\nabla}_N (\hat{N} n_M + \beta_M) \\ &= n_N \partial_M \hat{N} + n_M \partial_N \hat{N} + \hat{N} (\hat{\nabla}_M n_N + \hat{\nabla}_N n_M) + \hat{\nabla}_M \beta_N + \hat{\nabla}_N \beta_M. \end{aligned} \quad (\text{D.1.22})$$

Contracting with $E_\mu^M E_\nu^N$, the two first terms vanish since E_μ^M and n_M are orthogonal. The third term yields the extrinsic curvature, while the last ones give the intrinsic covariant derivative of the shift vector. We deduce the extrinsic curvature tensor $\hat{K}_{\mu\nu}$ expressed in terms of the induced metric $\hat{g}_{\mu\nu}$, the lapse \hat{N} and shift β^ρ :

$$\hat{K}_{\mu\nu} = \frac{1}{2\hat{N}} \left(\hat{g}'_{\mu\nu} - {}^{(d)}\hat{\nabla}_\mu (\hat{g}_{\nu\rho} \beta^\rho) - {}^{(d)}\hat{\nabla}_\nu (\hat{g}_{\mu\rho} \beta^\rho) \right), \quad (\text{D.1.23})$$

an expression which is directly used in the computation of the gravitational action in Section 2.4.2, whose technical details are developed in the following subsection.

D.2 Computation in the framework of the LD background

We now want to compute the gravitational action in the framework of the Section 2.4.2, on a $5D$ manifold with two boundaries along the fifth direction, with the metric (2.82) and its inverse (2.83) which we reproduce here¹:

$$G_{MN} = e^{-\frac{2}{\sqrt{3}}\alpha|y|} e^{2r(x,y)} \begin{pmatrix} e^{-3r(x,y)} \tilde{g}_{\mu\nu}(x) + B_\mu B_\nu(x,y) & B_\mu(x,y) \\ B_\nu(x,y) & 1 \end{pmatrix}, \quad (\text{D.2.1})$$

$$G^{MN} = e^{\frac{2}{\sqrt{3}}\alpha|y|} e^{-2r} \begin{pmatrix} e^{3r} \tilde{g}^{\mu\nu}(x,y) & -g^{\mu\rho} B_\rho(x,y) \\ -g^{\nu\rho} B_\rho(x,y) & 1 + g^{\lambda\sigma} B_\lambda B_\sigma(x,y) \end{pmatrix}. \quad (\text{D.2.2})$$

Rescaling the ADM $4D$ induced metric $\hat{g}_{\mu\nu}$ according to $\hat{g}_{\mu\nu} \rightarrow \hat{N}^2 \hat{g}_{\mu\nu}$, the ADM metric \hat{G}_{MN} (D.1.8) and its inverse \hat{G}^{MN} (D.1.9), in the case of timelike hypersurfaces ($\epsilon = 1$) we are

1. In this subsection we denote the KK vector B_μ instead of K_μ to avoid confusion with the extrinsic curvature, also called K .

interested in, read:

$$\hat{G}_{MN} = \hat{N}^2 \begin{pmatrix} \hat{g}_{\mu\nu} & \hat{g}_{\mu\rho}\beta^\rho \\ \hat{g}_{\nu\rho}\beta^\rho & 1 + \hat{g}_{\rho\sigma}\beta^\rho\beta^\sigma \end{pmatrix}, \quad \hat{G}^{MN} = \hat{N}^{-2} \begin{pmatrix} \hat{g}^{\mu\nu} + \beta^\mu\beta^\nu & -\beta^\mu \\ -\beta^\nu & 1 \end{pmatrix}. \quad (\text{D.2.3})$$

They turn out to be very similar to the parametrisations (D.2.1) and (D.2.2), and are in fact equivalent, noticing the identification:

$$\begin{aligned} \hat{N}^2(1 + \hat{g}_{\rho\sigma}\beta^\rho\beta^\sigma) &= N^{-2}, & \hat{N}^{-2} &= N^2(1 + g^{\rho\sigma}B_\rho B_\sigma), \\ \hat{N}^2\hat{g}_{\mu\rho}\beta^\rho &= N^{-2}B_\mu, & \hat{N}^{-2}\beta^\mu &= N^2g^{\mu\rho}B_\rho, \\ \hat{N}^2\hat{g}_{\mu\nu} &= N^{-2}(g_{\mu\nu} + B_\mu B_\nu), & \hat{N}^{-2}(\hat{g}^{\mu\nu} + \beta^\mu\beta^\nu) &= N^2g^{\mu\nu}, \end{aligned} \quad (\text{D.2.4})$$

with the overall factor N given by

$$N = e^{\frac{\alpha}{\sqrt{3}}|y|-r}. \quad (\text{D.2.5})$$

One easily finds for the inverse metric of $\hat{N}^2\hat{g}_{\mu\nu}$, $\hat{N}^{-2}\hat{g}^{\mu\nu} = N^2(g^{\mu\nu} - xB^\mu B^\nu)$, $x \equiv \frac{1}{1+B^2}$.

The main result (D.1.18) of the previous subsection giving the total gravitational action $S_G = S_{EH} + S_{GH}$, in the case of timelike ($\epsilon = 1$) hypersurfaces of dimension $d = 4$, and for the rescaled 4D induced metric $\hat{N}^2\hat{g}_{\mu\nu}$ reads:

$$S_G = \frac{1}{2} \int_{y_1}^{y_2} dy \int_{\Sigma_y} d^4x \hat{N} \sqrt{|\hat{N}^8 \hat{g}|} \left[R^{(4)}[\hat{N}^2\hat{g}_{\mu\nu}] + (\hat{K}^2 - \hat{K}^{\mu\nu}\hat{K}_{\mu\nu}) \right], \quad (\text{D.2.6})$$

where $\hat{g} \equiv \det \hat{g}_{\mu\nu}$, $\hat{K}_{\mu\nu}$ is the extrinsic curvature associated to the metric $\hat{N}^2\hat{g}_{\mu\nu}$, $\hat{K}^{\mu\nu} = \hat{N}^{-4}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}\hat{K}_{\rho\sigma}$, and $\hat{K} = \hat{N}^{-2}\hat{g}^{\mu\nu}\hat{K}_{\mu\nu}$. According to the result (D.1.23) derived earlier, the extrinsic curvature tensor is given by

$$\hat{K}_{\mu\nu} = \frac{1}{2\hat{N}} \left((\hat{N}^2\hat{g}_{\mu\nu})' - {}^{(4)}\hat{\nabla}_\mu(\hat{N}^2\hat{g}_{\nu\rho}\beta^\rho) - {}^{(4)}\hat{\nabla}_\nu(\hat{N}^2\hat{g}_{\mu\rho}\beta^\rho) \right), \quad (\text{D.2.7})$$

where ${}^{(4)}\hat{\nabla}_\mu$ denotes the covariant derivative compatible with the metric $\hat{N}^2\hat{g}_{\mu\nu}$. Using the identification (D.2.4), one can express the extrinsic curvature tensor in terms of the variables N , $g_{\mu\nu}$ and B_ρ . The computation yields:

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}N\sqrt{1+B^2} \left\{ \left[N^{-2}(g_{\mu\nu} + B_\mu B_\nu) \right]' - N^{-2}(\partial_\mu B_\nu + \partial_\nu B_\mu) + 2xN^{-2}\Gamma_{\mu\nu}^\rho B_\rho \right. \\ &\quad \left. + 2xN^{-2}B^\rho \left[B_{(\nu}F_{\mu)\sigma} + B_\sigma\partial_{(\mu}B_{\nu)} \right] + 2xN^{-3}(g_{\mu\nu} + B_\mu B_\nu)B^\rho\partial_\rho N \right\}, \end{aligned} \quad (\text{D.2.8})$$

where $\Gamma_{\mu\nu}^\rho$ are the Christoffel symbols computed from the metric $g_{\mu\nu}$, and $F_{\mu\nu}$ is the field strength of B_μ , $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. Since we are interested in the spectrum of the theory and especially in the mass term for the KK vector B_μ , we work up to quadratic order in B . Ignoring interaction terms, we get for the extrinsic curvature

$$\begin{aligned} K_{\mu\nu} &= \frac{1}{2}N\sqrt{1+B^2} \left\{ \left[N^{-2}(g_{\mu\nu} + B_\mu B_\nu) \right]' - N^{-2}(\partial_\mu B_\nu + \partial_\nu B_\mu) \right. \\ &\quad \left. + 2N^{-2}\Gamma_{\mu\nu}^\rho B_\rho + 2N^{-3}g_{\mu\nu}B^\rho\partial_\rho N \right\} + \text{h.o.t.}, \end{aligned} \quad (\text{D.2.9})$$

where here and in the following, h.o.t. will denote higher order terms in B , but also terms of the form $B^2 \times r$. From $K = N^2(g^{\mu\nu} - xB^\mu B^\nu)K_{\mu\nu} = N^2(g^{\mu\nu} - B^\mu B^\nu)K_{\mu\nu} + \text{h.o.t.}$, we get

$$\begin{aligned} K^2 &= N^2(1 + B^2) \left\{ \frac{16}{3}\alpha^2 + \frac{16\alpha}{\sqrt{3}} \operatorname{sgn} yr' + 4(r')^2 \right. \\ &\quad + \left(\frac{8\alpha}{\sqrt{3}} \operatorname{sgn} y + 4r' \right) \left(g^{\mu\nu} \partial_\mu B_\nu - B_\rho \Gamma_{\mu\nu}^\rho g^{\mu\nu} + 4B^\rho \partial_\rho r \right) \\ &\quad \left. + g^{\mu\nu} g^{\rho\sigma} \partial_\mu B_\nu \partial_\rho B_\sigma - \frac{4\alpha}{\sqrt{3}} \operatorname{sgn} y (B^2)' \right\} + \text{h.o.t.} \end{aligned} \quad (\text{D.2.10})$$

Similarly, using $K^{\mu\nu} = N^4(g^{\mu\rho} - xB^\mu B^\rho)(g^{\nu\sigma} - xB^\nu B^\sigma)K_{\rho\sigma} = N^4(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\rho} B^\nu B^\sigma - g^{\nu\sigma} B^\mu B^\rho)K_{\rho\sigma} + \text{h.o.t.}$, one finds

$$\begin{aligned} K^{\mu\nu} K_{\mu\nu} &= \frac{1}{4} N^2(1 + B^2) \left\{ \frac{16}{3}\alpha^2 + \frac{16\alpha}{\sqrt{3}} \operatorname{sgn} yr' + 4(r')^2 \right. \\ &\quad + \left(\frac{8\alpha}{\sqrt{3}} \operatorname{sgn} y + 4r' \right) \left(g^{\mu\nu} \partial_\mu B_\nu - B_\rho \Gamma_{\mu\nu}^\rho g^{\mu\nu} + 4B^\rho \partial_\rho r \right) \\ &\quad \left. + 4g^{\mu\rho} g^{\nu\sigma} \partial_\mu B_\nu \partial_\rho B_\sigma - \frac{4\alpha}{\sqrt{3}} \operatorname{sgn} y (B^2)' \right\} + \text{h.o.t.}, \end{aligned} \quad (\text{D.2.11})$$

and therefore

$$\begin{aligned} K^2 - K^{\mu\nu} K_{\mu\nu} &= \frac{3}{4} N^2(1 + B^2) \left\{ \frac{16}{3}\alpha^2 + \frac{16\alpha}{\sqrt{3}} \operatorname{sgn} yr' + 4(r')^2 \right. \\ &\quad + \left(\frac{8\alpha}{\sqrt{3}} \operatorname{sgn} y + 4r' \right) \left(g^{\mu\nu} \partial_\mu B_\nu - B_\rho \Gamma_{\mu\nu}^\rho g^{\mu\nu} + 4B^\rho \partial_\rho r \right) \\ &\quad \left. + \frac{4}{3} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \partial_\mu B_\nu \partial_\rho B_\sigma - \frac{4\alpha}{\sqrt{3}} \operatorname{sgn} y (B^2)' \right\} + \text{h.o.t.} \end{aligned} \quad (\text{D.2.12})$$

Regarding the determinant of the metric $\det(\hat{N}^2 \hat{g}_{\mu\nu}) = \det(N^{-2}(g_{\mu\nu} + B_\mu B_\nu))$, we use the relation $\det(g_{\mu\nu} + B_\mu B_\nu) = (\det g_{\mu\rho}) \det(\delta_\nu^\rho + B^\rho B_\nu) = (\det g_{\mu\rho})(1 + B^\sigma B_\sigma) + \text{h.o.t.}$ Using $g_{\mu\rho} = e^{-3r} \tilde{g}_{\mu\rho}$, we deduce:

$$\sqrt{-\det(\hat{N}^2 \hat{g}_{\mu\nu})} = N^{-4} e^{-6r} \sqrt{-\tilde{g}} \left(1 + \frac{B^2}{2} \right) + \text{h.o.t.} \quad (\text{D.2.13})$$

From the identification (D.2.4), we have $\hat{N} = N^{-1}(1 + B^2)^{-1/2} = N^{-1}(1 - \frac{B^2}{2}) + \text{h.o.t.}$, so that

$$\hat{N} \sqrt{-\det(\hat{N}^2 \hat{g}_{\mu\nu})} = N^{-5} e^{-6r} \sqrt{-\tilde{g}} + \text{h.o.t.} \quad (\text{D.2.14})$$

Combining the results (D.2.12) and (D.2.14), we get:

$$\begin{aligned} \frac{1}{2} \hat{N} \sqrt{|\hat{N}^8 \hat{g}|} (K^2 - K_{\mu\nu} K^{\mu\nu}) &= e^{-\sqrt{3}\alpha|y|} e^{-3r} \sqrt{-\tilde{g}} \left\{ 2\alpha^2 + 2\sqrt{3}\alpha \operatorname{sgn} yr' + \frac{3}{2}(r')^2 \right. \\ &\quad + \left(\sqrt{3}\alpha \operatorname{sgn} y + \frac{3}{2}r' \right) \left(g^{\mu\nu} \partial_\mu B_\nu - B_\rho \Gamma_{\mu\nu}^\rho g^{\mu\nu} + 4B^\rho \partial_\rho r \right) \\ &\quad + \frac{1}{2} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \partial_\mu B_\nu \partial_\rho B_\sigma \\ &\quad \left. - \frac{\sqrt{3}\alpha}{2} \operatorname{sgn} y (B^2)' + 2\alpha^2 B^2 \right\} + \text{h.o.t.} \end{aligned} \quad (\text{D.2.15})$$

The third line is the kinetic term for B_μ , $-\frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}$, up to higher order terms of the form $B^2 \times r$, as can be seen after integrations by part on the unbounded directions x^μ , whose total derivatives vanish. The linear terms in B on the second line can be arranged noticing that $g^{\mu\nu}\partial_\mu B_\nu - B_\rho\Gamma_{\mu\nu}^\rho g^{\mu\nu} = \partial_\mu B^\mu + B^\sigma\Gamma_{\mu\sigma}^\sigma = \partial_\mu B^\mu + B^\sigma(\tilde{\Gamma}_{\mu\sigma}^\sigma - 6\partial_\sigma r)$, where $\tilde{\Gamma}_{\mu\nu}^\rho$ are the Christoffel symbols computed from the metric $\tilde{g}_{\mu\nu}$. Using $\tilde{\Gamma}_{\mu\sigma}^\mu = \frac{1}{\sqrt{-\tilde{g}}}\partial_\sigma\sqrt{-\tilde{g}}$, integrating by part and again discarding a total derivative ∂_σ , we get the final result:

$$\begin{aligned} \frac{1}{2}\hat{N}\sqrt{|\hat{N}^8\hat{g}|}(K^2 - K_{\mu\nu}K^{\mu\nu}) &= e^{-\sqrt{3}\alpha|y|}e^{-3r}\sqrt{-\tilde{g}}\left\{2\alpha^2 + 2\sqrt{3}\alpha\operatorname{sgn}yr' + \frac{3}{2}(r')^2\right. \\ &+ B^\mu\partial_\mu\left[-\frac{3}{2}r' + \sqrt{3}\alpha\operatorname{sgn}yr + \frac{3}{2}r'r\right] \\ &\left.- \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma} - \frac{\sqrt{3}\alpha}{2}\operatorname{sgn}y(B^2)' + 2\alpha^2B^2\right\} + \text{h.o.t.} \end{aligned} \quad (\text{D.2.16})$$

Finally, the d -dimensional Ricci scalar (D.1.19),

$$\begin{aligned} R^{(d)}[\hat{N}^2\hat{g}_{\mu\nu}] &= \frac{1}{2}\partial_\mu\left(\sqrt{-\det(\hat{N}^2\hat{g}_{\mu\nu})}\hat{N}^{-2}\hat{g}^{\mu\nu}\hat{N}\right)\partial_\nu\ln\left(-\det(\hat{N}^2\hat{g}_{\mu\nu})\right) \\ &+ \partial_\mu\hat{N}\partial_\nu\left(\sqrt{-\det(\hat{N}^2\hat{g}_{\mu\nu})}\hat{N}^{-2}\hat{g}^{\mu\nu}\right) \\ &- \frac{1}{2}\hat{N}\sqrt{-\det(\hat{N}^2\hat{g}_{\mu\nu})}\hat{\Gamma}_{\mu\nu}^\rho\partial_\rho\left(\hat{N}^{-2}\hat{g}^{\mu\nu}\right), \end{aligned} \quad (\text{D.2.17})$$

expressed in terms of the variables N , $g_{\mu\nu}$ and B_ρ , reads:

$$\begin{aligned} R^{(d)}[N^{-2}(g_{\mu\nu} + B_\mu B_\nu)] &= \frac{1}{2}\partial_\nu\ln\left(-\det(N^{-2}(g_{\mu\nu} + B_\mu B_\nu))\right) \\ &\times \partial_\mu\left(\sqrt{-\det(N^{-2}(g_{\mu\nu} + B_\mu B_\nu))}N^2(g^{\mu\nu} - xB^\mu B^\nu)N^{-1}(1 + B^2)^{-1/2}\right) \\ &+ \partial_\mu\left(N^{-1}(1 + B^2)^{-1/2}\right)\partial_\nu\left(\sqrt{-\det(N^{-2}(g_{\mu\nu} + B_\mu B_\nu))}(N^2(g^{\mu\nu} - xB^\mu B^\nu))\right) \\ &- \frac{1}{2}N^{-1}(1 + B^2)^{-1/2}\sqrt{-\det(N^{-2}(g_{\mu\nu} + B_\mu B_\nu))}\Gamma_{\mu\nu}^\rho\partial_\rho\left(N^2(g^{\mu\nu} - xB^\mu B^\nu)\right), \end{aligned} \quad (\text{D.2.18})$$

with $\Gamma_{\mu\nu}^\rho$ the Christoffel symbols computed from the metric $N^{-2}(g_{\mu\nu} + B_\mu B_\nu)$. Although rather involved, it is easy to see that this expression does not bring additional B_μ terms up to quadratic order in B : in the first two terms of (D.2.18), quadratic terms in B always appear in the form $\sim B^2\partial_\mu N \sim B^2\partial_\mu r$, which is an interaction term, while in the third term, quadratic terms in B are multiplied by Christoffel symbols, and are thus again of the form $B^2 \times$ fluctuations. One can hence set B_μ to zero in the previous expression, and the computation then gives:

$$\frac{1}{2}R^{(d)}[N^{-2}g_{\mu\nu}] = -\frac{3}{4}e^{-\sqrt{3}\alpha|y|}\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\mu r\partial_\nu r + \text{h.o.t.} \quad (\text{D.2.19})$$

Combining the results (D.2.16) and (D.2.19), we deduce the total gravitational action, up to

quadratic order in the KK vector B_μ :

$$\begin{aligned}
S_G = \int d^5x e^{-\sqrt{3}\alpha|y|} \sqrt{-\tilde{g}} \left\{ -\frac{3}{4}(\partial_\mu r)^2 + e^{-3r} \left[2\alpha^2 + 2\sqrt{3}\alpha \operatorname{sgn} y r' + \frac{3}{2}(r')^2 \right] \right. \\
+ \tilde{B}^\mu \partial_\mu \left(-\frac{3}{2}r' + \sqrt{3}\alpha \operatorname{sgn} y r + \frac{3}{2}r' r \right) \\
\left. - \frac{1}{4} e^{3r} \tilde{g}^{\mu\nu} \tilde{g}^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} - \frac{\sqrt{3}\alpha}{2} \operatorname{sgn} y (\tilde{B}^2)' + 2\alpha^2 \tilde{B}^2 \right\} + \text{h.o.t.}, \quad (\text{D.2.20})
\end{aligned}$$

where we have replaced $B^\mu = g^{\mu\nu} B_\nu = e^{3r} \tilde{g}^{\mu\nu} B_\nu \equiv e^{3r} \tilde{B}^\mu$, $B^2 = g^{\mu\nu} B_\mu B_\nu = e^{3r} \tilde{g}^{\mu\nu} B_\mu B_\nu \equiv e^{3r} \tilde{B}^2$.

Effective theory of the heterotic string

In this appendix, we check that the supergravity theory introduced in Section 2.5 is an effective theory of an heterotic string theory, in agreement with [62]. We recall the Lagrangian (2.174) of its bosonic sector, in the Einstein frame:

$$\begin{aligned}
e_E^{-1} \mathcal{L}^{\text{bos}} = & \frac{1}{2} \mathcal{R}^{(5)}[G_{MN}^E] - \frac{1}{2} \partial_M \phi \partial^M \phi - e^{\frac{2}{\sqrt{3}} \phi} \Lambda \\
& - \frac{1}{8} e^{\frac{4}{\sqrt{3}} \phi} F_{MN}^0 F^{MN0} - \frac{1}{4} e^{-\frac{2}{\sqrt{3}} \phi} F_{MN}^1 F^{MN1} \\
& + \frac{e_E^{-1}}{6\sqrt{6}} C_{011} \epsilon^{MNPQR} \left(A_M^0 F_{NP}^1 F_{QR}^1 + 2A_M^1 F_{NP}^1 F_{QR}^0 \right),
\end{aligned} \tag{E.0.1}$$

where e_E stands for the 5D Einstein frame fünfbein, while e_S will denote later its string frame counterpart. In order to show the heterotic nature of this action, one must dualize the graviphoton A_M^0 into the Kalb-Ramond (KR) two-form B_{MN} , whose completely antisymmetric three-form field strength will be written $H_{MNP} = \partial_{[M} B_{NP]}$. To this purpose, we consider the action as being a functional of F_{MN}^0 rather than A_M^0 , and add the Lagrange multiplier term

$$\mathcal{L}_{LM} = \frac{1}{4} \epsilon^{MNPQR} F_{MN}^0 H_{PQR}, \tag{E.0.2}$$

so that the equation of motion of B_{MN} enforces F^0 to be closed. On the other hand, the equation of motion for F_{MN}^0 ,

$$F^{MN0} = e_E^{-1} e^{-\frac{4}{\sqrt{3}} \phi} \epsilon^{MNPQR} \left(H_{PQR} + \frac{2C_{011}}{\sqrt{6}} A_P^1 F_{QR}^1 \right), \tag{E.0.3}$$

can be used in order to completely eliminate it in $\mathcal{L}^{\text{bos}} + \mathcal{L}_{LM}$, leading to:

$$\begin{aligned}
e_E^{-1} \mathcal{L}^{\text{bos}} + \mathcal{L}_{LM} = & \frac{1}{2} \mathcal{R}^{(5)}[G_{MN}^E] - \frac{1}{2} \partial_M \phi \partial^M \phi - \frac{3}{2} e^{-\frac{4}{\sqrt{3}} \phi} \left(H_{PQR} + \frac{2C_{011}}{\sqrt{6}} A_{[P}^1 F_{QR]}^1 \right)^2 \\
& - \frac{1}{4} e^{-\frac{2}{\sqrt{3}} \phi} F_{MN}^1 F^{MN1} - e^{\frac{2}{\sqrt{3}} \phi} \Lambda.
\end{aligned} \tag{E.0.4}$$

In order to move from the Einstein to the string frame, we perform the Weyl transformation

$$G_{MN}^E = e^{-2\sigma} G_{MN}^S, \tag{E.0.5}$$

which leads, in D space-time dimensions, to the well-known relations

$$e_E = e^{-D\sigma} e_S, \quad (\text{E.0.6})$$

$$\mathcal{R}^{(D)}[G_{MN}^E] = e^{2\sigma} \left[\mathcal{R}^{(D)}[G_{MN}^S] + 2(D-1)\square\sigma - (D-1)(D-2)G_S^{MN}\partial_M\sigma\partial_N\sigma \right], (\text{E.0.7})$$

For $\sigma = \frac{\phi}{\sqrt{3}}$, $D = 5$ and after discarding a total derivative, we get the string frame Lagrangian

$$\begin{aligned} \mathcal{L}_S^{\text{bos}} = e_S e^{-\sqrt{3}\phi} & \left\{ \frac{1}{2} \mathcal{R}^{(5)}[G_{MN}^S] + \frac{3}{2} \partial_M \phi \partial^M \phi - \frac{3}{2} \left(H_{PQR} + \frac{2C_{011}}{\sqrt{6}} A_{[P}^1 F_{QR]}^1 \right)^2 \right. \\ & \left. - \frac{1}{4} F_{MN}^1 F^{MN1} - \Lambda \right\}, \quad (\text{E.0.8}) \end{aligned}$$

which is indeed the effective Lagrangian density of an heterotic superstring theory. The topological term present in (E.0.1) in the vector formulation is translated in the KR formulation into a gauge Chern-Simons term $\omega_3 = A^1 \wedge dA^1$, which combines with H_{PQR} to form the generalized field strength $\mathcal{H}_3 \sim H_3 + \omega_3$ satisfying the modified Bianchi identity $d\mathcal{H}_3 \sim dA^1 \wedge dA^1$.

This analysis therefore shows that the $4D$ vector A_μ^0 , which becomes massive after compactification on a LD background as shown in Section 2.6, is the dual of the vector $B_{\mu 5}$ coming from the dimensional reduction of the $5D$ KR two-form B_{MN} .

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