The structure of string theory compactification

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À mon grand-père, Léon.

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Résumé court en français

Notre compréhension de l'univers repose sur d'une part le modèle standard pour les interactions de la matière à des échelles infiniment petites et la relativité général qui nous permet de comprendre notre univers à des échelles infiniment grandes. Néanmoins, en étudiant des objets tels que les trous noirs où ni les effets de la gravité ni les effets quantique ne sont négligeable nous arrivons à la conclusion que le modèle standard et la relativité général sont incompatible. Nous avons donc besoin d'une théorie qui puisse inclure à la fois le modèle standard et la relativité générale. Cette théorie est appelé gravité quantique.

La théorie des cordes est le candidat le plus prometteur pour la gravité quantique reproduisant dans certaines limites de basses énergies la relativité général et le modèle standard. Néanmoins la théories des cordes possèdes des particularités intrigantes voir problématiques. Une de ces particularités est que la théorie des cordes requiert des dimensions d'espaces temps supplémentaires: dix dimensions d'espace temps pour les versions supersymétrique de la théorie des cordes et onze dimensions pour la théorie M qui unifie les cinq théories des cordes. Dans notre vie quotidienne nous n'observons que quatre dimensions d'espace temps. Un problème central de la théorie des cordes est donc de comprendre où serait cachées ces dimensions supplémentaires.

Une façon simple de comprendre ces dimensions supplémentaire est de prendre l'exemple d'un funambule sur une corde. Le funambule ne peut se déplacer que dans une seule direction, il ne peut aller que soit en avant soit en arrière. Maintenant prenons une fourmi se déplaçant elle aussi sur la corde du funambule. La taille de la fourmi étant de l'ordre du rayon de la corde la fourmi pourra aussi se déplacer le long de la dimension circulaire de la corde.

Nous utiliserons un point de vue exactement analogue pour les dimensions supplémentaires de la théorie des cordes. Nous considérerons que ces dimensions supplémentaires forment une variété compact que nous appellerons variété interne en opposition à la variété externe sur laquelle nous vivons. Exactement comme dans l'exemple du funambule cette variété interne est si petite que nous ne pouvons pas l'observer.

Tant que notre but est de donner des prédictions de la théorie des cordes et de la théorie M pour l'univers de dimension quatre que nous observons, nous pouvons nous concentrer sur la théorie effective qui ne garde que les excitations de la théorie qui correspondent à notre échelle d'énergie.

Tout d'abord au lieu de considérer la théorie des cordes nous nous concentrerons sur sa limite de basse énergie, la supergravité. La théorie des cordes contient un ensemble infini d'états dont la masse est inversement proportionnelle au carré de la longueur de la corde, l, et un ensemble finis de mode non massif. Étant donné que l est petit pour des

processus à notre échelle d'énergies nous pouvons négliger les modes massifs et garder seulement les modes non massifs qui sont décris par la supergravité dix dimensionnelle (onze dimensionnelle pour la théorie M).

Nous devons maintenant prendre en compte les effets de la compactification sur cette supergravité. Pour étudier comment la réduction dimensionnelle fonctionne et comprendre idées que nous utiliserons dans le reste de la thèse nous allons présenter un exemple plus simple, la réduction de Kaluza-Klein de cinq à quatre dimensions.

Considérons la gravité d'Einstein à cinq dimensions. En plus d'un espace Minkowski à cinq dimensions il existe d'autre solutions aux équations d'Einstein notamment une solution correspondant à une compactification spontanée

$$\mathcal{M}_5 = \mathcal{M}_4 \times S^1 \tag{.1}$$

où l'espace-temps est le produit d'un espace Minkowski de quatre dimensions $\mathcal{M}_4$ avec un espace compact très simple, le cercle S^1. On notera $x^M = \{x^\mu, y\}$ les coordonnées cinq dimensionnelles, x^μ les coordonnées sur $\mathcal{M}_4$ et $0 \leq y < 2\pi R$ la coordonnée du cercle.

La théorie effective est obtenue en étudie les fluctuations autour du vide (.1). Étant donné que la solution (.1) brise l'invariance de Poincaré à $SO(3,1) \times U(1)$, on peut organiser les degrés de liberté en fonction de leur nombre quantique quatre dimensionnel.

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} - \phi A_\mu A_\nu & -\phi A_\mu \\ -\phi A_\nu & -\phi \end{pmatrix} , \tag{.2}$$

où $g_{\mu\nu}$ est la métrique à quatre dimensions, A_μ est un vecteur 4-d et ϕ est un scalaire 4-d. Leur valeur moyenne du vide donne la métrique du vide

$$\langle g_{\mu\nu} \rangle = \eta_{\mu\nu} \, , \, \langle A_\mu \rangle = 0 \, , \, \langle \phi \rangle = 1 \, .$$

Comme S^1 est compact on peut développer en série de Fourier sur S^1 tout les champs. Par exemple le développement du scalaire ϕ est

$$\phi(x,y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \phi_n(x) \mathrm{e}^{\frac{i\, n\, y}{R}}$$

où x représente les coordonnées quatre dimensionnelles. Les coefficient dans les développement représentent les champs de quatre dimensions dont la masse est donnée par la valeur propre du d'Alembertien de S^1, ces champs sont les états de Kaluza Klein. Pour les fluctuations du scalaire on obtient:

$$\partial^\mu \partial_\mu \phi_n(x^\mu) - m^2 \phi_n(x^\mu) = 0 \quad \Leftrightarrow \quad m^2 = \frac{n^2}{R^2} \, .$$

Des développements similaires peuvent être obtenu pour les autres champs de (.2). Et donc compactifier la gravité à cinq dimensions sur un cercle donne une gravité à quatre dimensions couplé à des tours infinis d'états de Kaluza-Klein dont les masses sont proportionnelles à l'inverse du rayon du cercle.

Le rayon R du cercle précise l'échelle d'énergie à laquelle les modes massifs peuvent être exclus: en prenant R très petit les modes massifs deviennent extrêmement massifs

et peuvent être tronqués et donc on obtient une théorie effective avec un nombre finis de champs correspondant aux champs non massifs.

L'utilisation d'un tel argument est possible seulement lorsque le rayon de l'espace interne peut être arbitrairement petit. Lorsque cela est le cas nous avons ce que l'on appelle un séparation d'échelle entre l'espace interne et externe. C'est le cas par exemple pour les compactification sur des espace de Calabi-Yau avec comme espace externes un espace Minkowski. Néanmoins en supregravité nous rencontrons beaucoup d'exemple où il n'y a pas de tel séparation d'échelle. C'est le cas des compactifications ou l'espace externe est un espace Anti de Sitter. Nous pourrions aussi être intéressé par garder des états massifs et non massifs dans la théorie effective.

Dans ce cas nous avons recourt à ce que l'on appel une troncature cohérente. Une troncature cohérente est une procédure de sélection d'un sous ensembles de champs de la théorie de sorte que les champs gardés ne soient couplé à aucun des champs non sélectionnés. Donnons nous un exemple pour illustrer cette définition. Nous voulons tronquer le modèle suivant

$$\mathcal{L} = \tfrac{1}{2}(\partial\lambda)^2 + \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}g\lambda\varphi^2 - \tfrac{1}{2}m^2\lambda^2\,,$$

à un modèle avec un champ. En étudiant les équations du mouvement

$$\partial^2\varphi = g\lambda\varphi \quad \text{and} \quad \partial^2\lambda = m^2\lambda + \tfrac{1}{2}g\varphi^2\,,$$

nous observons qu'à cause du terme φ^2, le scalaire λ ne peut pas être tronqué de façon cohérente, tandis que cela est possible pour φ. Le résultat est le Lagrangien d'un champ massif λ. Nous pouvons comprendre ce résultat en terme de symétries. Le Lagrangien de départ possède une symétrie $\mathbb{Z}_2$ selon laquelle le champ φ est chargé alors que le champ λ est invariant.

$$\begin{matrix}\varphi \to -\varphi \\ \lambda \to \lambda\end{matrix}\,.$$

La troncature cohérente est obtenue en ne gardant que les modes invariant selon ce $\mathbb{Z}_2$ de symétrie. L'idée d'utiliser un groupe de symétrie pour sélectionner un nombre finis d'états qui définissent une théorie effective est au coeur des troncatures cohérentes et de cette thèse.

Dans cette thèse nous appliquerons cette idée de troncature cohérente à la supergravité à 10/11-dimensions. Nous utiliserons le formalisme de la géométrie généralisée exceptionnelle, une extension de la géométrie différentielle qui permet d'unifier transformation de coordonnés de l'espace-temps et transformations de jauge des potentiels de la théorie de cordes dans des difféomorphismes généralisés. Nous verrons que nous pourrons utiliser des groupes de structures sur le tangent exceptionnel de l'espace interne pour obtenir de façon systématique des troncature cohérente. Cela nous permettra dans le cas de troncature à des théories effective de cinq dimensions de donner une classifications des théories qui peuvent être obtenues comme un troncature cohérente de la théorie des cordes. Ensuite nous donnerons pour certaines troncations le calculs explicite de la théorie effective.

Chapter I
Introduction

Our understanding of the universe is very well described at small scales by the Standard Model and at very large scales by General Relativity. Unfortunately the two theories show signs of incompatibility when combined to describe systems, black hole for instance, where neither gravity nor quantum effects can be neglected. The theory that would embed the quantum field theory of Standard Model and General Relativity is usually referred to as quantum gravity.

String theory is one of the most promising theory of quantum gravity and is believed to reproduce the Standard Model and general relativity as low energy limits. But string theory have some very particular and intriguing features. One of them is that it requires extra dimensions of space-time: in superstring theory the space-time must be ten-dimensional while for M-theory, the theory that unifies all five versions of string theory, the dimensions are eleven. In our everyday life we only experience four dimensions of space-time. Hence a central problem of string theory is what to do with the extra dimensions.

A simple way of thinking of the extra dimensions is to take the example of a tightrope walker. On the tightrope the walker is able to walk only in one direction, either he goes forward or backwards. Now take an ant on that tightrope. Since the ant is much smaller than the walker and its scale is about the scale of the radius of the tightrope the ant has access to the circular direction of the rope.

In string compactifications we will take exactly this point of view. We will consider that the extra dimensions of string theory form a small compact manifold, which we will call internal as opposed to the non compact external manifold that we experience. In this way the space-time of string theory will be a product of the external and internal manifolds. As in the example of the tightrope walker, the internal manifold is so small compared to our scale that we cannot have access to it.

As long as we are interested in making predictions from string theory or M-theory about observables in our four dimensional, non-compact universe, we can rely on effective theories that only capture the excitations that are relevant at our energy scales.

First, rather than considering the full string theory, we will consider its low energy limit, supergravity. String theory contains an infinite set of modes with masses inversely proportional to the square of the string length, l, and a finite set of massless modes. Because of the smallness of l for processes at our energy scales we can neglect the mas-

sive modes and only keep the massless ones, which are described by a ten-dimensional supergravity.

Once we have supergravity, we must take into account the effect of the compactification. To see how it works and illustrate the ideas we will use in the rest of the thesis, we can start with the simple example of the Kaluza-Klein reduction from five to four dimensions.

Consider pure Einstein gravity in five dimensions. Beside 5-dimensional Minkowski space, there exists another solution of Einstein equation corresponding to a spontaneous compactification

$$\mathcal{M}_5 = \mathcal{M}_4 \times S^1 \tag{I.1}$$

where the space-time is the product of four-dimensional Minkowksi $\mathcal{M}_4$ times a very simple compact space, the circle S^1. We denote $x^M = \{x^\mu, y\}$ the 5-dimensional coordinates with x^μ the coordinates on $\mathcal{M}_4$ and $0 \leq y < 2\pi R$ the coordinate on the circle. The effective theory is constructed by looking at the fluctuations around the vacuum (I.1). Since the solution (I.1) breaks the five-dimensional Poincaré invariance to $SO(3,1) \times U(1)$, we can arrange the degrees of freedom according to their four-dimensional quantum numbers

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} - \phi A_\mu A_\nu & -\phi A_\mu \\ -\phi A_\nu & -\phi \end{pmatrix} , \tag{I.2}$$

where $g_{\mu\nu}$ is the 4-d metric, A_μ is a 4-d vector, and ϕ is a 4-d scalar. Their vacuum expectation values give the background metric

$$\langle g_{\mu\nu} \rangle = \eta_{\mu\nu} \, , \, \langle A_\mu \rangle = 0 \, , \, \langle \phi \rangle = 1 \, .$$

Since S^1 is compact, we can Fourier expand all the fields on S^1. For example, for the scalar ϕ this gives

$$\phi(x,y) = \frac{1}{\sqrt{2\pi R}} \sum_{n \in \mathbb{Z}} \phi_n(x) \mathrm{e}^{\frac{i\,n\,y}{R}}$$

where x denote the four-dimensional coordinates. The coefficients in the expansion represent 4d fields with mass given by the eigenvalues of the wave operator on S^1: these are the Kaluza Klein states. For instance, for the scalar fluctuation one has

$$\partial^\mu \partial_\mu \phi_n(x^\mu) - m^2 \phi_n(x^\mu) = 0 \quad \Leftrightarrow \quad m^2 = \frac{n^2}{R^2}$$

Similar expansions hold for the other fields in (I.2). Thus compactifying pure five-dimensional gravity on a circle gives a four-dimensional gravity theory coupled to infinite towers of fields, the Kaluza-Klein states, with masses proportional to the inverse of the circle radius. The radius R of the circle set the energy scale at which the massive modes can be discarded: by taking R very small the massive modes become heavy and, for energies lower than R^{-2}, can be truncated away to get an effective theory with only a finite set of fields, the massless ones.

A key element here is that the scale R of the internal circle is completely uncorrelated from the scale of the external manifold. This implies that we can take R arbitrarily small and induce a separation of scales between the massless modes of the truncated theory and the rest of the KK towers. In string theory, if the compactification admits a separation

of scales, we can obtain a lower-dimensional low-energy effective supergravity theory by truncating out modes above the cut-off scale. This is what happens in compactifications on special holonomy manifolds to Minkowski space-time, where the effective theory is obtained by keeping only the massless modes, namely the zero-modes of appropriate differential operators on the internal space.

However in supergravity/string theory we encounter many examples where we cannot rely on scale separation. This is the case of compactifications to Anti de Sitter space-times, where the scales of the external and internal manifolds are related. Or we might be interested in constructions where we keep both some light and massive modes within the truncation.

In this case we must instead resort to what is called a *consistent truncation* [4]. A consistent truncation is a procedure to truncate a theory to a subset of fields in such a way that there is no coupling between the modes that are kept and those that are discarded. Let us consider again a simple a toy model. We would like to truncate the following model

$$\mathcal{L} = \tfrac{1}{2}(\partial\lambda)^2 + \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}g\lambda\varphi^2 - \tfrac{1}{2}m^2\lambda^2\,,$$

to a single field. From the equations of motion

$$\partial^2\varphi = g\lambda\varphi \quad \text{and} \quad \partial^2\lambda = m^2\lambda + \tfrac{1}{2}g\varphi^2\,,$$

we see that, because of the term φ^2, the scalar λ cannot be truncated away in a consistent way, while this is possible for φ. The results is a Lagrangian for one massive field λ. We could interpret it in terms of symmetries. The original Lagrangian has a $\mathbb{Z}_2$ symmetry under which the field φ is charged whereas the field λ is invariant

$$\begin{matrix}\varphi \to -\varphi \\ \lambda \to \lambda\end{matrix}\,.$$

The consistent truncation is obtained by keeping only the singlet under the $\mathbb{Z}_2$ symmetry group. The idea of using a symmetry group to select the finite set of fields of the effective theory is at the heart of consistent truncations and of the rest of this thesis.

When compactifying string-theory/supergravity on an internal manifold M, the Kaluza-Klein towers are obtained by expanding on a basis of tensors on the internal space, a generalization of the Fourier modes, which are associated to symmetries of the internal manifolds and contain the dependence on the coordinates of the internal manifold. A consistent truncation is a choice of such a basis that allows to truncate to a finite set of singlet fields in such a way that the dependence of the higher-dimensional fields on the internal manifold factorises out once the truncation ansatz is plugged in the equations of motion. This condition is what makes consistent truncations relatively rare and hard to prove (see for instance [4, 5].

When looking for supergravity solutions, such as domain wall or black-holes, it is often convenient to do that in lower-dimensional gauged supergravity models corresponding to truncations of the 10/11-dimensional supergravity, since the equations of motion or the supersymmetry variations are simpler. If the truncation is consistent, all solutions of the lower-dimensional theory also satisfy the equations of motion of 10-/11-dimensional supergravity. This is very important as, for example, a vacuum that appears stable within

a lower-dimensional supergravity might suffer from instabilities triggered by modes not kept in the truncation [6], or vacua which appear different within the lower-dimensional model may actually be identified in the full 10/11-dimensional theory [7]. These examples highlight how important it is to know which lower-dimensional theories can arise as consistent truncations of 10-/11-dimensional supergravity.

The best known examples of consistent truncations are provided by Scherk-Schwarz reductions on group manifolds. These are parallelisable manifolds admitting a basis of left(right) invariant forms. A consistent truncation is obtained by expanding all the fields of the higher-dimensional theory on such basis and only keeping the singlet components. The truncation procedure preserves all supersymmetries of the higher-dimensional theory and gives a maximally supersymmetric truncated theory.

In recent years a lot of progress has been done in the study of consistent truncations by exploiting the geometric structures of the compactification manifolds.

For instance it has been shown [1] that G-structures allow to construct consistent truncations on manifolds that are not necessarily homogeneous. Suppose a manifold M has a reduced structure group G_S with singlet intrinsic torsion. Then a consistent truncation is obtained by expanding all the field in G_S representations and keeping only the G_S-singlets. This can preserve different fractions of supersymmetry, depending on how many G_S-invariant spinors exist on M.

Moreover the data of the G_S structure, namely the G_S-invariant no-where vanishing tensors, determine the full field content and gauge interactions of the truncated theory. Examples of this type in M-theory are the truncations based on Sasaki–Einstein and weak-G_2 holonomy manifolds of [8], and the tri-Sasakian reduction of [9].

However there are consistent truncations that cannot be interpreted in terms of conventional G-structures. Classic examples are the maximally supersymmetric consistent truncations on spheres, such as eleven-dimensional supergravity on S^7 [10] and S^4 [11].

These examples have a natural interpretation in Exceptional Generalised Geometry and Exceptional Field Theory. In this thesis I will focus on the Exceptional Generalised Geometry approach of consistent truncations. Exceptional Generalised Geometry is a reformulation of 11/10-dimensional supergravity that allows to treat diffeomorphisms and gauge transformations of the higher-rank gauge fields of supergravity in a geometric way. It replaces the tangent bundle TM with an extended tangent bundle E, whose fibres transform in a representation of the exceptional group $E_{d(d)}$ rather than $GL(d,\mathbb{R})$. Then diffeomorphism and gauge transformations are unified as generalised diffeomorphisms on E. As for ordinary G_S structures, if the generalised structure group of E can be reduced to a subgroup $G_S \subset E_{6(6)}$ we say the manifold admits a generalised G_S structure.

The notion of generalised G_S structure plays a central role in this thesis since it allows for a systematic approach to consistent truncations with different amounts of supersymmetry: there is a consistent truncation any time a supergravity theory is reduced on a manifold M admitting a generalised G_S structure with constant singlet intrinsic torsion [1].

In this framework all maximally supersymmetric truncations, both conventional Scherk–Schwarz reductions and sphere truncations, can be seen as generalised Scherk–Schwarz reductions on generalised parallelisable manifolds [12, 13, 14, 15, 16, 17, 18, 19, 20]. In particular, the notion of a generalised parallelisation allowed to prove the long-standing conjecture of the consistency of type IIB supergravity on S^5 [12, 21, 16]. Considering

larger generalised G_S-structure group allows to treat half-maximal truncations [22, 23, 24, 25, 26, 1] and $\mathcal{N} = 2$ truncations.

As for generalised Scherk–Schwarz reductions one can show that it is possible to determine the lower-dimensional supergravity directly from the data of the generalised G_S structure, a priori of any explicit substitution into the equations of motion.

This approach also provides a proof of the conjecture of [27] that for any supersymmetric $AdS_d \times M$ vacuum there is consistent truncation on M to a gauged supergravity with the same amount of supersymmetry [24, 1], based on the fact that the conditions for a supersymmetric $\mathrm{AdS}_d \times_{\mathrm{w}} M$ vacuum can be rephrased as the requirement that M admits a generalised G_S structure with vanishing non-singlet intrinsic torsion [28, 29, 30].

This thesis collect the work I have done during my PhD on the derivation of consistent truncations in the framework of Exceptional Generalised Geometry.

In the first chapter I will describe the general framework for constructing consistent truncations with different amounts of supersymmetry (including non-supersymmetric truncations), based on the generalized G-structures. I will first recall how the construction works for ordinary G_S-structures and then show how this naturally extends to generalised ones. The key requirement is that G-structure has only singlet "intrinsic torsion" [28]. Then I will discuss how this formalism allows one to easily determine all the features of the lower-dimensional gauged supergravity, such as the amount of supersymmetry, the coset manifold of the scalars, the number of gauge and tensor fields, and the gauging, all directly from the geometry.

In order to make the formalism more explicit I will focus on truncations of eleven dimensional supergravity to five dimensional gauged supergravities with $\mathcal{N} = 2$ supersymmetry. The analysis for half-maximal truncations can be found treated in [1].

I will then apply this formalism to two different kind of problems. I will apply it to derive a classification of which $\mathcal{N} = 2$ gauged supergravities in five dimensions can be obtained as consistent truncations of of eleven dimensional or type IIB supergravity. In this case the relevant exceptional group is $E_{6(6)}$. The classification is purely algebraic: we identify the possible $G_S \subset E_{6(6)}$ structures that lead to $\mathcal{N} = 2$ supergravities, we work under the hypothesis that they only admit constant singlet intrinsic torsion, and then we analyse the field content and gaugings. This means that the list of 5d supergravities we find might be further reduced once the differential condition of having constant intrinsic torsion are verified in explicit examples.

Finally I will show how this formalism can be used concretely to derive explicit examples of five dimensional $\mathcal{N} = 2$ consistent truncations: the largest truncation containing $\mathcal{N} = 2$ Maldacena–Nuñez [31] and also the one containing the "BBBW" [32] backgrounds.

The main text of the thesis only covers part of the results I obtained during my PhD. These are contained in three papers [1, 2, 3] which are presented in the Appendix.

Chapter II

Consistent truncation and generalised G_S structures

We are interested in consistent truncations of eleven/ten-dimensional supegravity on backgrounds of the type

$$\mathcal{M} = X \times M$$

where M_d is a compact d-dimensional manifold and X is a non compact external space of dimension $11-d$ or $10-d$, respectively.

Typically a consistent truncation relies on some geometrical properties of the compactification manifold M_d. The best known examples are Scherk–Schwarz reductions on group manifolds. A d-dimensional group manifold $M = \mathscr{G}$ admits a basis of globally defined left-invariant vectors and a basis of dual of left-invariant one-forms $\{e^a\}$, $a = 1, \ldots, d$. By expanding all fields of the higher-dimensional theory on these basis and keeping only modes invariant under the group action provides a consistent truncation.

Scherk–Schwarz reductions are particularly simple examples of a more general construction, based on the notion of G-structure, which allows to derive consistent truncations for more general manifolds than group manifolds or homogeneous spaces [8, 33, 34, 35, 36, 9, 37].

Consider a d-dimensional manifold M. At each point p of M we can define a local frame $e_m^{(\alpha)}$, namely a local basis for the tangent bundle TM, so that any vector v can be expanded as $v = v^m_{(\alpha)} e_m^{(\alpha)}$. On two overlapping patches U_α and U_β the components of v are related by a $GL(d, \mathbb{R})$ transformation

$$v^m_{(\alpha)} = (M_{\alpha\beta})^m{}_n v^n_{(\beta)} \,.$$

Since one can repeat this construction at every point on M, the the matrices $M_{\alpha\beta}$ can be seen as functions from $U_\alpha \cap U_\beta$ to $GL(d, \mathbb{R})$. These are called transition functions and contain all the information about the non-trivial topology of the bundle TM. They must satisfy the consistency conditions

$$\begin{aligned} &M_{\alpha\beta} M_{\beta\alpha} && U_\alpha \cap U_\beta \\ &M_{\alpha\beta} M_{\beta\gamma} = M_{\alpha\gamma} && U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

so that they form a group, the structure group.

In the general case the structure group is of a d-dimensional manifold is $GL(d,\mathbb{R})$. If the structure group of M is a subgroup $G_S \subset GL(d,\mathbb{R})$ we say that the manifold has a G_S-structure. The G_S-structure is defined by a set of G_S-invariant, nowhere vanishing tensors $\{\Xi_i\}$.[1] For example, a Riemannian manifold admits a no-where vanishing metric g or, equivalently, a subset of orthonormal frames on M, which defines a $G_S = O(d)$ structure. This also implies that for Riemannian manifolds the possible G_S-structure are all subgroups of $O(d)$.

A G_S-structure is characterised by its intrinsic torsion. For Riemannian manifolds the intrinsic torsion can be defined via the action of the Levi-Civita connection on the invariant tensors Ξ_i

$$\begin{aligned}\nabla_m \Xi_i{}^{n_1\ldots n_r}{}_{p_1\ldots p_s} = {} & K_m{}^{n_1}{}_q \Xi_i{}^{q\ldots n_r}{}_{p_1\ldots p_s} + \cdots + K_m{}^{n_r}{}_q \Xi_i{}^{n_1\ldots q}{}_{p_1\ldots p_s} \\ & - K_m{}^q{}_{p_1} \Xi_i{}^{n_1\ldots n_r}{}_{q\ldots p_s} + \cdots - K_m{}^q{}_{p_s} \Xi_i{}^{n_1\ldots n_r}{}_{p_1\ldots q}\,,\end{aligned} \tag{II.1}$$

with the indices m and n, p spanning T^*M and $\Lambda^2 T^*M$, respectively. From its index structure it follows that $K_m{}^n{}_p$ must be a section of $T^*M \otimes \Lambda^2 T^*M$. However decomposing $\Lambda^2 T^*M \simeq SO(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}$ is the Lie algebra of G_S and using the fact that Ξ_i are G_S-invariant, we see that K is actually a section of $T^*M \otimes \mathfrak{g}^\perp$.

The intrinsic torsion is the part of the torsion that does not depend on the choice of connection and is defined in terms of K as

$$(T_{\text{int}})_{mn}{}^p = K_n{}^p{}_m - K_m{}^p{}_n\,.$$

T_{int} can be decomposed into G_S representations, known as the "torsion classes" of the structure. For consistent truncations we are interested in G_S-structures whose non-zero torsion components are singlets under G_S.

Let us now see how G-structures are related to consistent truncations. Suppose a manifold M admits a G_S structure, with G_S invariant tensors Ξ_i and constant singlet intrinsic torsion. The 11/10-dimensional supergravity can be consistently truncated on M by expanding all bosonic fields on the basis of tensors Ξ_i, which encode the dependence on the internal space, and only keeping the fields that are G_S singlets. Since the intrinsic torsion has only singlet components, (II.1) implies that the derivatives of the singlet fields can only contain singlets. Thus the truncation is necessarily consistent, since products of singlet representations can never source the non-singlet representations that were truncated away. For the spinors the truncations works in the same way. The G_S-structure lifts to a $\tilde{G}_S \subset Spin(d)$ structure and we simply have to expand the spinor fields in terms of the spinors invariant under $\tilde{G}_S$.

The G_S-structure completely determines the field content and gauge interactions of the truncated theory. For example it is easy to find the scalar and vector fields that come from the reduction of the higher-dimensional metric. The scalars are the G_S singlet components of the metric. Since the metric parameterises the coset $GL(d,\mathbb{R})/O(d)$, these are given by the $GL(d,\mathbb{R})$ deformations of a reference metric that commute with G_S modulo the $O(d)$ deformations that commute with G_S

$$\text{metric scalars} \quad \Leftrightarrow \quad H \in \frac{\mathrm{C}_{GL(d,\mathbb{R})}(G_S)}{\mathrm{C}_{O(d,\mathbb{R})}(G_S)}\,,$$

[1] Formally a G_S-structure defines a G_S-principal sub-bundle P of the $GL(d,\mathbb{R})$ frame bundle. In most cases the two definitions are equivalent.

where $\mathrm{C}_B(A)$ denotes the commutant of the subgroup A of B inside B.

The vectors coming from the metric are given by the G_S-invariant one-forms $\eta^a \in \{\Xi_i\}$. If we call $\hat{\eta}_a$ the singlet vectors dual to η^a we have

$$\text{metric gauge fields} \quad \Leftrightarrow \quad A^a_\mu \, \hat{\eta}_a \, .$$

The components of the singlet intrinsic torsion are completely determined by the Lie derivatives of the invariant tensors

$$\mathcal{L}_{\hat{\eta}_a} \Xi_i = f_{ai}{}^j \, \Xi_j \, , \tag{II.2}$$

where $f_{ai}{}^j$ are constants. They also give the gauge algebra of the metric gauge fields via the Lie bracket

$$[\hat{\eta}_a, \hat{\eta}_b] = f_{ab}{}^c \, \hat{\eta}_c \, . \tag{II.3}$$

Let us consider again Scherk–Schwarz reductions on a group manifold $M = \mathcal{G}$. The basis of globally defined (left-invariant) one-forms, $\{e^a\} \in T^*M$, reduce the structure group to $G_S = \mathbb{1}$ (i.e. M is parallelisable). Furthermore, the group action implies that

$$\mathrm{d}e^a = \frac{1}{2} f_{bc}{}^a \, e^b \wedge e^c \, ,$$

where $f_{bc}{}^a$ are the structure constants of the Lie algebra $\mathrm{Lie}\mathscr{G}$. This means that the identity structure has singlet, constant intrinsic torsion (singlet because $\mathrm{d}e^a$ is expressed in terms of the invariant $\{e^a\}$ basis, and constant because the coefficients of the expansion are constant). Then the truncated theory is obtained by expanding all higher-dimensional fields on the basis of left-invariant forms and plugging them in the equations of motion. Since only singlet tensors are generated we conclude that the truncation is consistent. Examples of such consistent truncations in the context of M-theory can be found in [38, 39, 40, 37].

The scalar fields of the truncated theory parameterise the coset

$$\frac{\mathrm{C}_{GL(d)}(\mathbb{1})}{\mathrm{C}_{SO(d)}(\mathbb{1})} = \frac{GL(d,\mathbb{R})}{SO(d)} \, .$$

The one-forms define d gauge fields with a Lie algebra given by the Lie bracket (II.3). The consistent truncation ansatz for the metric is

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + h_{ab} (e^a + A^a)(e^b + A^b) \, ,$$

where $h_{ab}(x)$ is matrix of scalar fields and $A^a_\mu(x)$ are gauge fields in the adjoint of G_S. Since the spin bundle is also trivialised, Scherk–Schwarz reductions preserve the full supersymmetry of the higher-dimensional theory.

Another interesting example is the reduction of M-theory and type IIB on a Sasaki–Einstein manifold M of dimension $d = 2n + 1$ [8, 35, 36]. The manifold admits an $G_S = SU(n) \subset GL(d,\mathbb{R})$ structure defined by a real one-form η, a real two-form ω and a complex n-form Ω satisfying

$$\mathrm{d}\eta = 2\omega, \qquad \mathrm{d}\Omega = i \, (n+1) \, \eta \wedge \Omega \, . \tag{II.4}$$

Since only invariant tensors appear on the right-hand side of the differential conditions (II.4), the intrinsic torsion has only constant singlet components. In this case the metric scalar manifold is

$$\frac{\mathrm{C}_{GL(2n+1,\mathbb{R})}(SU(n))}{\mathrm{C}_{SO(2n+1,\mathbb{R})}(SU(n))} = \frac{\mathbb{R}^+ \times \mathbb{C}}{U(1)} = \mathbb{R}^+ \times \mathbb{R}^+.$$

As there is a single invariant one-form η the truncated theory will contain only one gauge field $A_\mu(x)$ coming from the metric. The ansatz for the metric is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2U}\mathrm{d}s^2_{2n} + e^{2V}(\eta + A)^2,$$

where $\mathrm{d}s^2_{2n}$ is the (local) $2n$-dimensional Kähler–Einstein metric defined by (ω, Ω). The scalars fields $U(x)$ and $V(x)$ parametrise the scalar manifold.

The construction of consistent truncations based on G_S-structures is very powerful, but does not cover very well-known examples such as the reductions of eleven-dimensional supergravity on S^7 [10] and on S^4 [11], where the consistency is not a consequence of any manifest symmetry.

The main result of this thesis is that extending the notion of the G_S-structure to exceptional generalised geometry [41, 42] allows the previous examples of compactification on S^7 and S^7 to be treated on the same footing as the conventional Scherk–Schwarz reductions. More generally, it provides a new systematic way to study consistent truncations with a generic amount of supersymmetry: reducing a supergravity theory on any manifold M admitting a generalised G_S-structure with constant singlet intrinsic torsion gives a consistent truncation [1]. In the rest of the chapter, we will give the main ideas without entering into the details of a specific theory or compactification. We will then specify to truncations of M-theory to five dimensions.

Exceptional generalised geometry replaces the tangent bundle TM with a larger bundle E on M, whose fibres transform in a representation of the exceptional group $E_{d(d)}$. In this way, the diffeomorphisms and gauge symmetries of higher-dimensional supergravity are unified as generalised diffeomorphisms on E. Then, one can generalise all conventional notions of differential geometry such as tensors, connections, and Lie derivatives.

The bundle E is called the generalised tangent bundle, and its sections are generalised vectors. The dual generalised vectors are sections of the bundle E^*, and generalised tensors are obtained by tensoring E and/or E^*. For example, we will need the dual weighted vectors $Z_\flat$, which are sections of the bundle[2] $N \sim \det T^*M \otimes E^*$, and the generalised metric, which is a section of the symmetric product $S^2(E^*)$. In analogy with an ordinary metric on M, a generalised metric G parameterises, at each point on M, the coset

$$G \in \frac{E_{d(d)}}{H_d},$$

where H_d is the maximally compact subgroup of $E_{d(d)}$. Spinors can also be introduced as sections of the spinor bundle $\mathcal{S}$, transforming in the spinorial representation of $\tilde{H}_d$, the double cover of the group H_d.

The action of an infinitesimal generalised diffeomorphism is generated by the generalised Lie derivative along a generalised vector. We denote by $\mathrm{ad}F$ the adjoint bundle,

[2] We consider only orientable manifolds. Then, $\det T^*M$ is trivial and we can define arbitrary powers $(\det T^*M)^p$ for any real p.

namely the bundle whose fibres transform in the adjoint of $E_{d(d)}$. Then, in analogy with the conventional Lie derivative, we define the generalised one as [43],

$$(L_V V')^M = V^N \partial_N V'^M - (\partial \times_{\text{adj}} V)^M{}_N V'^N \,, \tag{II.5}$$

where V^M are the components of the generalised vector V in a standard coordinate basis, $\partial_M = \partial_m$ are viewed as sections of the dual tangent bundle, and the projection onto the adjoint bundle is $\times_{\text{adj}} : E^* \otimes E \to \text{ad}F$.

The definition of a generalised G_S-structure is a natural extension of the conventional one. A generalised G_S-structure on M is the reduction of the generalised structure group $E_{d(d)}$ to a subgroup G_S, and it is defined by a set of nowhere vanishing G_S-invariant generalised tensors $\{Q_i\}$. For instance, the generalised metric defines a $G_S = H_d$ structure on M [43, 44]. In what follows, we will always assume that M admits an H_d structure, and we will always consider generalised structures $G_S \subset H_d$.

Given a generalised G_S-structure, with $G_S \subseteq H_d$, defined by a set of G_S-invariant generalised tensors $\{Q_i\}$, we can define its intrinsic torsion from the Lie derivative of a generalised tensor α along a generalised vector V [28]:

$$(L_V^{\tilde{D}} - L_V)\,\alpha = T(V)\cdot\alpha$$

Here L_V is the generalised Lie derivative defined in (II.5), and $L_V^{\tilde{D}}$ is the generalised Lie derivative calculated using a G_S-compatible connection [3] $\tilde{D}$. The torsion can be seen as a map from the generalised tangent bundle into the adjoint one, $T : \Gamma(E) \to \Gamma(\text{ad}F)$, so that $T(V)$ acts on α via the adjoint action. The intrinsic torsion T_{int} is then the component of T that is independent of the choice of compatible connection $\tilde{D}$ and can be decomposed into representations of G_S.

Consider now eleven-dimensional or type II supergravity on a product space $X \times M$, where M is a d-dimensional manifold and X is a D-dimensional Lorentzian space with $D = 11 - d$ in M-theory and $D - 10$ in type II supergravity. We assume $d \leq 7$.

As we discussed above, the $GL(D,\mathbb{R})$ structure group of conventional geometry on M is extended to $E_{d(d)}$. The idea is then to rearrange the supergravity fields into generalised tensors transforming as representations of $GL(D,\mathbb{R}) \times E_{d(d)}$ and to interpret the theory as a D-dimensional theory on X with an infinite number of fields. The fields in X will be scalar, vectors, and two-forms according to their $GL(D,\mathbb{R})$ representation [4].

The scalar degrees of freedom on X are given by the components of all supergravity fields (metric and higher-rank potentials) with all internal indices and are repackaged into a generalised metric. The $GL(D,\mathbb{R})$ one-forms and vectors are sections of the generalised

[3] A generalised connection $\tilde{D}$ is compatible with the G_S-structure if $\tilde{D}Q_i = 0$ for all Q_i. The definition of a generalised connection is the same as in conventional differential geometry. However, in generalised geometry, the conditions of being torsion free and metric compatible do not uniquely determine the connection. However, only certain projections of the action of the connection appear in the supergravity, and these are unique [43].

[4] We do not consider higher form-field degrees of freedom, as in the tensor hierarchy [45, 46], since they are dual to the scalar, vector, and two-forms and therefore do not introduce new degrees of freedom. In particular, this means that for $D = 4$, $\mathcal{A}_\mu{}^M$ contain both the vectors and their duals, and in $D = 6$, $\mathcal{B}_{\mu\nu}{}^{MN}$ contain both the two-forms and their duals.

tangent space E, while the two-forms are sections of the bundle N. In summary, we have

$$\begin{aligned} \text{scalars:} \qquad & G_{MN}(x,y) \in \Gamma(S^2 E^*)\,, \\ \text{vectors:} \qquad & \mathcal{A}_\mu{}^M(x,y) \in \Gamma(T^*X \otimes E)\,, \\ \text{two-forms:} \qquad & \mathcal{B}_{\mu\nu}{}^{MN}(x,y) \in \Gamma(\Lambda^2 T^*X \otimes N)\,, \end{aligned}$$

where x and y are coordinates on X and M, respectively, and the capital index M denotes components of vectors in E or E^*.

The equations of motion and the supersymmetry variations are also organised according to the representations above, and the dynamics of the supergravity is completely determined by the Levi–Civita connection on the external space X and a generalised connection on M.

If the manifold M has a generalised G_S-structure, $G_S \subset H_d$, with only constant, singlet intrinsic torsion, we can construct a consistent truncation in the following way. Expand all bosonic fields in terms of the G_S invariant tensors $\{Q_i\}$ defining the structure, and keep only those transforming as singlets under the structure group. The coefficient of the expansion will depend on the external coordinates x, while the dependence on the internal space is only in the tensors $\{Q_i\}$.

Since there are only singlet representations in the intrinsic torsion, the generalised Levi–Civita connection acts on any invariant generalised tensor Q_i as

$$D_M Q_i = \Sigma_M \cdot Q_i\,,$$

where Σ_M is a section of $E^* \otimes \operatorname{ad} H_d$ that is completely determined in terms of the constant singlet torsion. Here, $\operatorname{ad} H_d$ denotes the bundle of tensors transforming in the adjoint representation of H_d. This means the derivatives of all the truncated fields are also expanded in terms of singlets only. Since products of singlet representations cannot source non-singlet representations, keeping only all possible singlets gives a consistent truncation.

To extend the truncation to the fermionic sector of the supergravity theory, it is enough to lift the structure group G_S to $\tilde{G}_S \subset \tilde{H}_d$ and to expand all the fermionic fields in terms of $\tilde{G}_S$ singlets.

From the data of the G_S-structure, we can determine the number of scalars, vectors, one-forms, and two-forms of the truncated theory, as well as the possible gaugings.

All scalars of the truncated theory are given by the G_S singlets in the generalised metric G_{MN}. These are singlet deformations of the structure modulo, those singlet deformations that do not deform the metric

$$\text{scalars:}\, \mathcal{M} = \frac{\mathrm{C}_{E_{d(d)}}(G_S)}{\mathrm{C}_{H_d}(G_S)} = \frac{\mathcal{G}}{\mathcal{H}}\,. \tag{II.6}$$

Consider now the vectors of the truncated theory. Being sections of $T^*X \otimes E$, they are determined by the number of G_S invariant generalised vectors $\{K_I\}$:

$$\text{vectors:} \qquad \mathcal{A}^M_\mu(x,y) = A_\mu{}^I(x)\, K^M_I \;\in\; \Gamma(T^*M \otimes \mathcal{V})\,, \tag{II.7}$$

where $\mathcal{V} \subset \Gamma(E)$ is the vector space spanned by the $\{\mathcal{K}_I\}$.

Similarly the two-forms are determined by the G_S singlets in the bundle N:

$$\text{two-forms:} \qquad \mathcal{B}_{\mu\nu}{}^{MN}(x,y) = \mathcal{B}_{\mu\nu\, I}(x) K_{\flat}^{IMN} \in \Gamma(\Lambda^2 T^* X \otimes \mathcal{B})\,,$$

where $\{K_{\flat}^I\}$ is a basis generating the G_S-invariant vector space $\mathcal{B} \subset \Gamma(N)$.

The representations above determine the full content of the theory, namely the fields coming from the reduction of the metric and the higher-rank potentials of the supergravity theory. In particular, this means that the vectors K_I generate all symmetries of the reduced theories, coming both from the metric and the higher-rank potentials. This is an important difference with respect to the reductions based on the conventional G_S-structure.

The G_S-structure also determines the embedding tensor (see [47, 48] for a review of this formalism) and hence the gaugings of the reduced theory in terms of the singlet intrinsic torsion. Since the G_S-structure has only singlet intrinsic torsion, in analogy with (II.2), the generalised Lie derivative of the G_S-invariant generalised tensors along any invariant generalised vector K_I can be written as

$$L_{K_I} Q_i = -T_{\text{int}}(K_I) \cdot Q_i\,, \tag{II.8}$$

where T_{int} is a map from the space $\mathcal{V}$ of the G_S invariant vector to the G_S singlets in the adjoint bundle. This is the embedding tensor. Notice that $T_{\text{int}}(K_I)$ must correspond to the elements in the adjoint that commute with G_S, namely the Lie algebra of the commutant group $\mathcal{G} = \mathrm{C}_{E_{d(d)}}(G_S)$. $\mathcal{G}$ is the subgroup of the isometry group of the scalar manifold that can a priori be gauged in the truncated theory.

Chapter III

M-theory truncation to five dimensions

In this chapter we will make the previous discussion more concrete by focusing on consistent truncations of the eleven dimensional supergravity on

$$\mathcal{M}_{11} = X_5 \times M_6$$

to five dimensional supergravity with $\mathcal{N} = 2$ supersymmetry.

Our conventions for eleven-dimensional supergravity are the same as in [43]. The eleven-dimensional bosonic action is (we denote by a hat the 11d quantities)

$$\hat{S} = \frac{1}{2}\int \left(\hat{R}\,\hat{*}\,1 - \tfrac{1}{2}\hat{F}\wedge *\hat{F} - \tfrac{1}{6}\hat{A}\wedge\hat{F}\wedge\hat{F}\right),$$

where $\hat{F} = \mathrm{d}\hat{A}$ and $\hat{A}$ is the three-form potential. The equations of motion are

$$\hat{R}_{\hat{\mu}\hat{\nu}} - \tfrac{1}{12}\left(\hat{F}_{\hat{\mu}\hat{\rho}_1\hat{\rho}_2\hat{\rho}_3}\hat{F}_{\hat{\nu}}{}^{\hat{\rho}_1\hat{\rho}_2\hat{\rho}_3} - \tfrac{1}{12}\,\hat{g}_{\hat{\mu}\hat{\nu}}\hat{F}^2\right) = 0\,,$$
$$\mathrm{d}\,\hat{*}\,\hat{F} + \tfrac{1}{2}\hat{F}\wedge\hat{F} = 0\,.$$

The six-form potential $\hat{\tilde{A}}$ dual to the three-form $\hat{A}$ may be introduced via the first-order relation

$$\hat{*}\,\mathrm{d}\hat{A} + \tfrac{1}{2}\,\hat{A}\wedge\mathrm{d}\hat{A} \;=\; \mathrm{d}\hat{\tilde{A}}\,, \tag{III.1}$$

whose exterior derivative gives the Maxwell equation.

III.1 $E_{6(6)}$ generalised geometry

Eleven-dimensional supergravity compactified on a six-dimensional manifold can be reformulated in terms of $E_{6(6)} \times \mathbb{R}^+$ generalised geometry, which we will review below.

To the internal manifold M we associate a generalised tangent bundle E, whose sections transform in the real $\mathbf{27}^*$ representation[1] of $E_{6(6)}$, the generalised structure group, with weight one under $\mathbb{R}^+$. The ordinary structure group $GL(6)$ embeds in $E_{6(6)} \times \mathbb{R}^+$ and can be used to decompose the generalised tangent bundle as

$$E \;\simeq\; TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M\,. \tag{III.2}$$

[1] Given a representation $\mathbf{n}$ we will use $\mathbf{n}^*$ and $\overline{\mathbf{n}}$ for the dual and conjugate representations, respectively. For non-compact groups these may not be equivalent.

The sections of E are called generalised vectors and, using (III.2), can be seen as (local) sums of a vector, a two-form and a five-form on M,

$$V = v + \omega + \sigma \,.$$

The dual bundle E^* is defined as

$$E^* \simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM \,,$$

with sections

$$Z = \hat{v} + \hat{\omega} + \hat{\sigma} \,,$$

where $\hat{v}$ is one-form, $\hat{\omega}$ is a two-vector and $\hat{\sigma}$ is a five-vector. Generalised vectors and dual generalised vectors have a natural pairing given by

$$\langle Z, V \rangle = \hat{v}_m v^m + \tfrac{1}{2}\, \hat{\omega}^{mn} \omega_{mn} + \tfrac{1}{5!}\, \hat{\sigma}^{mnpqr} \sigma_{mnpqr} \,.$$

The $E_{6(6)}$ cubic invariant is defined on E and E^* as[2]

$$\begin{aligned} c(V,V,V) &= -\,6\, \iota_v\, \omega \wedge \sigma - \omega \wedge \omega \wedge \omega \,, \\ c^*(Z,Z,Z) &= -\,6\, \iota_{\hat{v}}\, \hat{\omega} \wedge \hat{\sigma} - \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \,. \end{aligned} \tag{III.3}$$

The frame bundle F for E defines an $E_{6(6)} \times \mathbb{R}^+$ principal bundle. By considering bundles whose fibres transform in different representations of $E_{6(6)} \times \mathbb{R}^+$, we can then define other generalised tensors. To describe the bosonic sector of the supergravity theories we will need, besides the generalised vectors, weighted dual vectors, adjoint tensors and the generalised metric. Adjoint tensors R are sections of the adjoint bundle $\mathrm{ad}\, F$ of the form

$$\begin{aligned} \mathrm{ad}F &\simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM \,, \\ R &= l + r + a + \tilde{a} + \alpha + \tilde{\alpha} \,, \end{aligned}$$

and hence transform in the $\mathbf{1} + \mathbf{78}$ of $E_{6(6)}$ with weight zero under the $\mathbb{R}^+$ action. Locally l is a function, r a section of $\mathrm{End}(TM)$, a is a three-form and so on. One notes that in the exceptional geometric reformulation, the internal components of the gauge potentials of type II or M-theory, are embedded in the adjoint bundle.

The action of an adjoint element R on a generalised vector $V \in \Gamma(E)$ and on a dual generalised vector Z is also denoted by $\cdot$ and is defined as

$$V' = R \cdot V \qquad\qquad Z' = R \cdot Z \,,$$

where the components of V' are

$$\begin{aligned} v' &= lv + r \cdot v + \alpha \lrcorner\, \omega - \tilde{\alpha} \lrcorner\, \sigma \,, \\ \omega' &= l\omega + r \cdot \omega + v \lrcorner\, a + \alpha \lrcorner\, \sigma \,, \\ \sigma' &= l\sigma + r \cdot \sigma + v \lrcorner\, \tilde{a} + a \wedge \omega \,, \end{aligned} \tag{III.4}$$

[2] This is 6 times the cubic invariant given in [49]. Because of this, we introduced a compensating factor of 6 in the formulae (III.16) and (III.17).

and those of Z' are

$$\begin{aligned}\hat{v}' &= -l\hat{v} + r\cdot\hat{v} - \hat{\omega}\lrcorner a + \hat{\sigma}\lrcorner\tilde{a}\,,\\ \hat{\omega}' &= -l\hat{\omega} + r\cdot\hat{\omega} - \alpha\lrcorner\hat{v} - \hat{\sigma}\lrcorner a\,,\\ \hat{\sigma}' &= -l\hat{\sigma} + r\cdot\hat{\sigma} - \tilde{\alpha}\lrcorner\hat{v} - \alpha\wedge\hat{\omega}\,.\end{aligned} \tag{III.5}$$

The action of an adjoint element R on another adjoint element R' is given by the commutator, $R'' = [R, R']$. In components, R'' reads

$$\begin{aligned}l'' &= \tfrac{1}{3}(\alpha\lrcorner a' - \alpha'\lrcorner a) + \tfrac{2}{3}(\tilde{\alpha}'\lrcorner\tilde{a} - \tilde{\alpha}\lrcorner\tilde{a}')\,,\\ r'' &= [r, r'] + j\alpha\lrcorner ja' - j\alpha'\lrcorner ja - \tfrac{1}{3}(\alpha\lrcorner a' - \alpha'\lrcorner a)\,\mathbb{1}\,,\\ &\quad + j\tilde{\alpha}'\lrcorner j\tilde{a} - j\tilde{\alpha}\lrcorner j\tilde{a}' - \tfrac{2}{3}(\tilde{\alpha}'\lrcorner\tilde{a} - \tilde{\alpha}\lrcorner\tilde{a}')\,\mathbb{1}\,,\\ a'' &= r\cdot a' - r'\cdot a + \alpha'\lrcorner\tilde{a} - \alpha\lrcorner\tilde{a}'\,,\\ \tilde{a}'' &= r\cdot\tilde{a}' - r'\cdot\tilde{a} - a\wedge a'\,,\\ \alpha'' &= r\cdot\alpha' - r'\cdot\alpha + \tilde{\alpha}'\lrcorner a - \tilde{\alpha}\lrcorner a'\,,\\ \tilde{\alpha}'' &= r\cdot\tilde{\alpha}' - r'\cdot\tilde{\alpha} - \alpha\wedge\alpha'\,,\end{aligned} \tag{III.6}$$

where $\cdot$ denotes the $\mathfrak{gl}(6)$ action.

It will be useful to also define weighted dual vectors $Z_\flat$ as sections of the bundle $N \simeq \det T^*M \otimes E^*$ which has $\mathbb{R}^+$ weight two[3]. Concretely one finds

$$\begin{aligned}N &\simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M)\,,\\ Z_\flat &= \lambda + \rho + \tau\,.\end{aligned}$$

An important object in our construction is generalised metric, which encodes the internal components of all bosonic fields. The generalised metric G is a positive-definite, symmetric rank-2 tensor

$$G \in \Gamma(\det T^*M \otimes S^2 E^*)\,,$$

so that, given two generalised vectors $V, W \in \Gamma(E)$, the inner product $G(V, W)$ is a top form. Just as an ordinary metric g, at each point on M, parameterises the coset $\mathrm{GL}(6)/\mathrm{O}(6)$, a generalised metric at a point $p \in M$ corresponds to an element of the coset

$$G|_p \in \frac{E_{6(6)} \times \mathbb{R}^+}{USp(8)/\mathbb{Z}_2}\,.$$

In order to include fermionic fields of M-theory we arrange them into representations of $USp(8)$, the double cover of the maximal compact subgroup $USp(8)/\mathbb{Z}_2$ of $E_{6(6)}$. For instance, supersymmetry parameters are section of the generalsied spinor bundle $\mathcal{S}$, transforming in the $\mathbf{8}$ of $USp(8)$. The R-symmetry of the reduced five-dimensional theory is in general then some subgroup $G_R \subseteq USp(8)$.

Now that we have introduce all the relevant objects of the exceptional generalised geometry. We now arrange the eleven-dimensional bosonic fields into generalised tensors transforming in representations of $GL(5, \mathbb{R}) \times E_{6(6)}$, where $GL(5, \mathbb{R})$ gives the tensorial

[3]Note that $\det T^*M$ is just a different notation for the top-form bundle $\Lambda^6 T^*M$ that stresses that it is a real line bundle. In the following we will assume that the manifold is orientable and hence $\det T^*M$ is trivial. Thus, we can define arbitrary powers $(\det T^*M)^p$ for any real p.

structure of the fields in the five-dimensional theory obtained after reduction. We separate the eleven-dimensional coordinates in coordinates x^μ, $\mu = 0, \ldots, 4$, on the external spacetime X, and z^m, $m = 1, \ldots, 6$, on the internal manifold M.

The bosonic fields of eleven-dimensional supergravity are decomposed as

$$\begin{aligned}
\hat{g} &= \mathrm{e}^{2\Delta}\, g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dz^m Dz^n \ , \\
\hat{A} &= \tfrac{1}{3!} A_{mnp} Dz^{mnp} + \tfrac{1}{2} A_{\mu mn} \mathrm{d}x^\mu \wedge Dz^{mn} + \tfrac{1}{2}\, \bar{A}_{\mu\nu m} \mathrm{d}x^{\mu\nu} \wedge Dz^m + \tfrac{1}{3!}\, \bar{A}_{\mu\nu\rho}\, \mathrm{d}x^{\mu\nu\rho} \,, \\
\hat{\tilde{A}} &= \tfrac{1}{6!} \tilde{A}_{m_1 \ldots m_6} Dz^{m_1 \ldots m_6} + \tfrac{1}{5!} \tilde{A}_{\mu m_1 \ldots m_5} \mathrm{d}x^\mu \wedge Dz^{m_1 \ldots m_5} + \tfrac{1}{2 \cdot 4!} \bar{\tilde{A}}_{\mu\nu m_1 \ldots m_4} \mathrm{d}x^{\mu\nu} \wedge Dz^{m_1 \ldots m_4} \\
&\quad + \ldots ,
\end{aligned} \tag{III.7}$$

where $Dz^m = \mathrm{d}z^m - h_\mu{}^m \mathrm{d}x^\mu$, and all tensor field components may depend both on x^μ and z^m, except for the external metric, for which we assume a dependence on the external coordinates only, $g_{\mu\nu} = g_{\mu\nu}(x)$.

The barred fields need to be redefined. In Appendix D of [2] we provide a justification for these redefinitions by studying the gauge transformations of the metric and three-form potential. For the three-form components we introduce the new fields $A_{\mu\nu m}$, $A_{\mu\nu\rho}$ via

$$\bar{A}_{\mu\nu m} = A_{\mu\nu m} - \, h_{[\mu}{}^n A_{\nu] n m} \,, \qquad \bar{A}_{\mu\nu\rho} = A_{\mu\nu\rho} + h_{[\mu}{}^n h_\nu{}^p A_{\rho] n p} \,. \tag{III.8}$$

Similar redefinitions apply to the six-form components with at least two external indices, however we will not discuss them in detail here.

The supergravity fields having all components on the internal manifold M arrange into the inverse generalised metric

$$G^{MN} \ \leftrightarrow \ \{\Delta,\, g_{mn},\, A_{mnp},\, \tilde{A}_{m_1 \ldots m_6}\} \ ,$$

in the following way[4]

$$\begin{aligned}
(G^{-1})^{mn} &= \mathrm{e}^{2\Delta} g^{mn} \\
(G^{-1})^m{}_{n_1 n_2} &= \mathrm{e}^{2\Delta} g^{mp} A_{p n_1 n_2} \\
(G^{-1})^m{}_{n_1 \ldots n_5} &= \mathrm{e}^{2\Delta} g^{mp} (A_{p[n_1 n_2} A_{n_3 n_4 n_5]} + \tilde{A}_{p n_1 \ldots n_5}) \\
(G^{-1})_{m_1 m_2\, n_1 n_2} &= \mathrm{e}^{2\Delta} (g_{m_1 m_2, n_1 n_2} + g^{pq} A_{p m_1 m_2} A_{q n_1 n_2]}) \\
(G^{-1})_{m_1 m_2\, n_1 \ldots n_5} &= \mathrm{e}^{2\Delta} [g_{m_1 m_2, [n_1 n_2} A_{n_3 n_4 n_5]} \\
&\qquad + g^{pq} (A_{p m_1 m_2} (A_{q[n_1 n_2} A_{n_3 n_4 n_5]} + \tilde{A}_{q n_1 \ldots n_5})] \\
(G^{-1})_{m_1 \ldots m_5\, n_1 \ldots n_5} &= \mathrm{e}^{2\Delta} [g_{m_1 \ldots m_5,\, n_1 \ldots n_5} \\
&\qquad + g^{pq} (A_{p[m_1 m_2} A_{m_3 m_4 m_5]} + \tilde{A}_{p m_1 \ldots m_5}) (A_{q[n_1 n_2} A_{n_3 n_4 n_5]} + \tilde{A}_{q n_1 \ldots n_5})] \,,
\end{aligned} \tag{III.9}$$

where $g_{m_1 m_2, n_1 n_2} = g_{m_1 [n_1} g_{|m_2| n_2]}$, and similarly for $g_{m_1 \ldots m_5, n_1 \ldots n_5}$. Since the generalised metric is a scalar on the external spacetime, after imposing our truncation ansatz it will provide the scalar fields of the reduced five-dimensional theory.

[4] This expression follows straightforwardly from the elements of the conformal split frame given in [43].

The density $\boldsymbol{\kappa}$ introduced in the next section when defining the HV structure is related to the determinant of the generalised metric and is an $E_{6(6)}$ invariant. For eleven-dimensional metrics of the form (III.7), this is given by [43, 49]

$$\boldsymbol{\kappa}^2 = \mathrm{e}^{3\Delta}\sqrt{\det g_{mn}}\,. \tag{III.10}$$

The tensors with one external leg arrange into a generalised vector $\mathcal{A}_\mu$ on M, with components

$$\mathcal{A}_\mu{}^M = \{h_\mu{}^m,\, A_{\mu mn}, \tilde{A}_{\mu m_1\ldots m_5}\}\,, \tag{III.11}$$

and will provide the gauge potentials of the reduced theory. The tensors with two anti-symmetrised external indices define a weighted dual vector $\mathcal{B}_{\mu\nu}$ on M, which is a section of $\det T^*M\otimes E^*$, with components

$$\mathcal{B}_{\mu\nu\,M} = \{A_{\mu\nu m},\, \tilde{A}_{\mu\nu m_1\ldots m_4},\, \tilde{g}_{\mu\nu m_1\ldots m_6,n}\}\,, \tag{III.12}$$

and will give the two-form fields of the reduced theory. The last term in (III.12) is related to the dual graviton and we will not discuss it further here.

The tensors with three antisymmetrised external indices arrange into the generalised tensor

$$\mathcal{C}_{\mu\nu\rho}{}^{\hat{\alpha}} = \{A_{\mu\nu\rho},\, \tilde{A}_{\mu\nu\rho m_1m_2m_3},\, \tilde{g}_{\mu\nu\rho m_1\ldots m_5,n}\}\,,$$

which is a section of (a sub-bundle of) the weighted adjoint bundle $\det T^*M\otimes \mathrm{ad}\, F$, whose components are labeled by $\hat{\alpha}=1,\ldots,57$. See e.g. [50, 46] for more details on this tensor hierarchy.

So far the formalism is fully generic for truncations preserving any amount of supersymmetry. In the rest of this chapter we will focus on $\mathcal{N}=2$ five dimensional truncations. The study of the half maximal case is in Section 3 of [1].

III.2 Five dimensional $\mathcal{N}=2$ truncations

In this section we focus on consistent truncations with $\mathcal{N}=2$ supersymmetry. For $\mathcal{N}=2$ supersymmetry we need two invariant supercharges in the spinor bundle $\mathcal{S}$ implying that we need subgroups $G_S\subset USp(8)$ that give only two singlets when decomposing the $\mathbf{8}$ of $USp(8)$.

The largest structure group giving $\mathcal{N}=2$ supersymmetry is $G_S=USp(6)$: under the breaking

$$USp(8)\supset USp(6)\times SU(2)_R\,, \tag{III.13}$$

the spinorial representation decomposes as

$$\mathbf{8} = (\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})\,. \tag{III.14}$$

The $SU(2)_R$ factor in (III.13) is the R-symmetry of the reduced theory under which the two spinors singlets form a doublet, as expected for $\mathcal{N}=2$ supersymmetry parameters. Under (III.13) the fundamental and adjoint of $E_{6(6)}$ decompose as

$$\begin{aligned}\mathbf{27}^* &= (\mathbf{1},\mathbf{1})\oplus(\mathbf{14},\mathbf{1})\oplus(\mathbf{6},\mathbf{2})\,,\\ \mathbf{78} &= (\mathbf{1},\mathbf{3})\oplus(\mathbf{6},\mathbf{2})\oplus(\mathbf{21},\mathbf{1})\oplus(\mathbf{14},\mathbf{1})\oplus(\mathbf{14}',\mathbf{2})\,.\end{aligned} \tag{III.15}$$

The $G_S = USp(6)$ structure is often called an *HV structure* [51, 49, 30] and can also be defined in terms of a triplet of non-vanishing invariant adjoint tensors and a generalised vector, corresponding to the singlets under $G_S = USp(6)$ in (III.15). As they will be useful in the rest of the thesis, let us first introduce the vector and hypermultiplet structures that these tensors separately define.

A vector-multiplet structure, or *V structure*, is given by a globally defined generalised vector $K \in \Gamma(E)$ of positive norm with respect to the $E_{6(6)}$ cubic invariant,

$$c(K,K,K) := 6\,\boldsymbol{\kappa}^2 > 0\,, \tag{III.16}$$

where $\boldsymbol{\kappa}$, the density introduced earlier, is a section of $(\det T^*M)^{1/2}$. The vector K is the $(\mathbf{1},\mathbf{1})$ singlet in the decomposition of the $\mathbf{27}^*$ in (III.15) and is stabilised by $\mathrm{F}_{4(4)} \subset E_{6(6)}$. A hypermultiplet structure, or *H structure*, is determined by a pair $(J_\alpha, \boldsymbol{\kappa}^2)$ where $J_\alpha \in \Gamma(\mathrm{ad}F)$ $(\alpha = 1,2,3)$ is a triplet that define a basis for a highest root $\mathfrak{su}_2$ subalgebra of $\mathfrak{e}_{6(6)}$ and hence satisfy

$$[J_\alpha, J_\beta] \;=\; 2\epsilon_{\alpha\beta\gamma} J_\gamma\,, \qquad \mathrm{tr}(J_\alpha J_\beta) \;=\; -\delta_{\alpha\beta}\,,$$

while $\boldsymbol{\kappa}$ is a section of $(\det T^*M)^{1/2}$ as above. The J_α correspond to the $(\mathbf{1},\mathbf{3})$ triplet in the decomposition of the $\mathbf{78}$ in (III.15) and are stabilised by $\mathrm{SU}^*(6) \subset E_{6(6)}$.

The HV structure corresponds to a V and an H structure, such that the two $\boldsymbol{\kappa}$ densities are the same and in addition compatibility constraint

$$J_\alpha \cdot K \;=\; 0\,, \tag{III.17}$$

is satisfied, where $\cdot$ denotes the adjoint action. The common stabiliser of compatible K and J_α is

$$\mathrm{SU}^*(6) \cap \mathrm{F}_{4(4)} \simeq USp(6)\,.$$

From the general dictionary (II.7) we see immediately that an HV structure will give truncations to minimal five dimensional $\mathcal{N} = 2$ supergravity since there is only one singlet vector, which gives the graviphoton. In order to have matter multiplets we need to look for further reduced structure groups $G_S \subset USp(6)$ such that in the decomposition

$$USp(8) \supset USp(6) \times SU(2)_R \supset G_S \times SU(2)_R\,,$$

additional G_S singlets beyond those defined by the $USp(6)$ structure appear in $\mathbf{27}^*$ and the $\mathbf{78}$, but none in the $\mathbf{8}$. This means the $\mathbf{6}$ in the decomposition (III.14) cannot contain any singlets, and hence that all the singlets in the $\mathbf{27}^*$ must transform trivially under $SU(2)_R$.

Each $G_S \subset USp(6)$ singlet will give a G_S-invariant generalised tensor in the corresponding bundle. In particular, the singlets in $\mathbf{27}^*$ will span a sub-bundle E_{singlet}

$$E \supset E_{\mathrm{singlet}} \simeq M \times \mathcal{V}\,. \tag{III.18}$$

The bundle is by definition trivial and hence can be written as a product where $\mathcal{V}$ is the fibre. The vector space $\mathcal{V}$ transforms as a representation of the commutant $\mathrm{C}_{E_{6(6)}}(G_S)$ of G_S in $E_{6(6)}$. In particular, from the discussion above, there must be an R-symmetry subgroup $SU(2)_R \subset \mathrm{C}_{E_{6(6)}}(G_S)$ that acts trivially on $\mathcal{V}$ (and hence E_{singlet}). Furthermore,

the corresponding Lie algebra $\mathfrak{su}(2)$ must correspond to a highest root in $\mathfrak{e}_{6(6)}$. Let us define $G_{\rm H}$ as the simple subgroup of $\mathrm{C}_{E_{6(6)}}(G_S)$ that contains such a highest root $SU(2)$. We can then also identify the corresponding trivial sub-bundle of the adjoint bundle[5]

$$\operatorname{ad} F \supset \operatorname{ad} F_{G_{\rm H}} \simeq M \times \mathfrak{g}_{\rm H}\,, \tag{III.19}$$

where $\mathfrak{g}_{\rm H}$ is the Lie algebra of $G_{\rm H}$. Note that by definition $R\cdot v = 0$ for all $v \in \Gamma(E_{\rm singlet})$ and $R \in \Gamma(\operatorname{ad} F_{G_{\rm H}})$.

Given any trivial G_S-invariant vector bundle $P \simeq M \times \mathbb{R}^n$ and G_S-compatible generalised connection $\tilde{D}$, one can define a constant section $s \in \Gamma(P)$ by $\tilde{D}s = 0$. Furthermore, the definition is independent of the choice of $\tilde{D}$ since the bundle transforms trivially under G_S. For the sub-bundles $E_{\rm singlet}$ and $\operatorname{ad} F_{G_{\rm H}}$ we can identify $\mathcal{V}$ and $\mathcal{U} \simeq \mathfrak{g}_{\rm H}$ with the spaces of constant sections

$$\begin{aligned}\mathcal{V} &= \left\{ v \in \Gamma(E_{\rm singlet}) : \tilde{D}v = 0 \right\}\,,\\ \mathfrak{g}_{\rm H} \simeq \mathcal{U} &= \left\{ R \in \Gamma(\operatorname{ad} F_{G_{\rm H}}) : \tilde{D}R = 0 \right\}\,,\end{aligned}$$

giving a natural realisation of the isomorphisms (III.18) and (III.19). Note that the elements of $\mathcal{U}$ generate a global $G_{\rm H}$ symmetry. The G_S-structure also defines a constant invariant section $\boldsymbol{\kappa}^2 \in \Gamma(\det T^*M)$. Hence for each $v \in \mathcal{V}$ the expression

$$C(v,v,v) = \boldsymbol{\kappa}^{-2} c(v,v,v)\,, \tag{III.20}$$

where c is the $E_{6(6)}$ cubic invariant, defines a map into $\mathbb{R}$ (or more precisely to constant functions on M). We can always choose a basis of normalised nowhere-vanishing linearly independent vectors and adjoint elements for $\mathcal{V}$ and $\mathcal{U}$

$$\{K_{\tilde{I}}\,, J_A\}\,, \qquad \tilde{I} = 0, \ldots, \dim \mathcal{V} - 1, \quad A = 1, \ldots, \dim G_{\rm H}\,,$$

where by definition we have

$$J_A \cdot K_{\tilde{I}} = 0\,, \qquad\qquad \forall\, \tilde{I},\, A\,.$$

In this basis, the components $C_{\tilde{I}\tilde{J}\tilde{K}}$ of the map (III.20) are given by

$$c(K_{\tilde{I}}, K_{\tilde{J}}, K_{\tilde{K}}) = 6\,\boldsymbol{\kappa}^2 C_{\tilde{I}\tilde{J}\tilde{K}}\,, \tag{III.21}$$

and define a symmetric, constant tensor, while the adjoint tensor basis J_A satisfy

$$[J_A, J_B] = f_{AB}{}^C J_C\,,$$

where $f_{AB}{}^C$ are the structure constants of $\mathfrak{g}_{\rm H}$. Finally, we can normalise

$$\operatorname{tr}(J_A J_B) = \eta_{AB}\,,$$

[5]Note that there are singlets in the adjoint bundle that are not in $\operatorname{ad} F_{G_{\rm H}}$. In addition to elements generating the other possible factors in $\mathrm{C}_{E_{6(6)}}(G_S)$ there are also elements of the form $V \otimes_{\rm ad} W$, where V is a section of $E_{\rm singlet}$, W is a section of the dual bundle $E^*_{\rm singlet}$ and $\otimes_{\rm ad}$ is the projection onto the adjoint bundle. However these will not play a relevant role in our construction.

where η_{AB} is a diagonal matrix with -1 and $+1$ entries in correspondence with compact and non-compact generators of $G_{\rm H}$, respectively. Note that in the "minimal" case of $G_S = USp(6)$ with the HV structure (K, J_α) the spaces $\mathcal{V}$ and $\mathcal{U}$ are one- and three-dimensional, with basis vectors K and J_α, respectively.

When $G_S \subset USp(6)$ we naturally obtain a moduli space of G_S-invariant HV structures. Note that the moduli do not necessarily consist of massless scalar fields from the point of view of the reduced $\mathcal{N} = 2$ five-dimensional theory, but rather will lead to a consistent truncation. The idea is that we can use the G_S singlet vectors and adjoint tensors to define reference $USp(6)$ structure. Then we can use it to build a reference generalised metric using the definition (IV.26). The physical moduli are then generated by acting on the structure with elements of $E_{6(6)}$ that commute with G_S, modulo elements of $USp(8)/\mathbb{Z}_2$, that leave the generalised metric invariant. The moduli obtained this way hence parameterise the coset

$$\mathcal{M} \;=\; \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)}\,. \tag{III.22}$$

By definition we are only considering G_S that only admits $\mathcal{N} = 2$ supersymmetry, in other words we are not interested in theories that are subsectors of more supersymmetric ones. This means there are no elements of $\mathrm{C}_{E_{6(6)}}(G_S)$ that lead to two different $USp(6)$ structures with the same generalised metric. Hence $\mathrm{C}_{E_{6(6)}}(G_S)$ must factorise into groups that act separately on $\mathcal{V}$ and $\mathcal{U}$, that is

$$\mathrm{C}_{E_{6(6)}}(G_S) = \mathrm{C}_{G_\mathcal{U}}(G_S) \times \mathrm{C}_{G_\mathcal{V}}(G_S)\,, \tag{III.23}$$

where $G_\mathcal{U}$ and $G_\mathcal{V}$ are the subgroups of $E_{6(6)}$ that leave fixed all elements of $\mathcal{U}$ and $\mathcal{V}$, respectively. Consequently, the moduli space $\mathcal{M}$ factorises into V structure and H structure moduli spaces, as expected from $\mathcal{N} = 2$ supergravity,

$$\mathcal{M} \;=\; \mathcal{M}_{\rm VT} \times \mathcal{M}_{\rm H} = \frac{\mathrm{C}_{G_\mathcal{U}}(G_S)}{\mathrm{C}_{H_\mathcal{U}}(G_S)} \times \frac{\mathrm{C}_{G_\mathcal{V}}(G_S)}{\mathrm{C}_{H_\mathcal{V}}(G_S)} = \frac{G_{\rm VT}}{H_{\rm VT}} \times \frac{G_{\rm H}}{H_{\rm H}}\,, \tag{III.24}$$

where, similarly, $H_\mathcal{U}$ and $H_\mathcal{V}$ are the subgroups of $USp(8)/\mathbb{Z}_2$ that leave $\mathcal{U}$ and $\mathcal{V}$ fixed, respectively. In general there are common factors that cancel between the numerators and denominators in the commutator group expression for the cosets; for example the centre $C(G_S)$ is always a subgroup common to both. Thus it is useful to introduce the notation $G_{\rm VT}$, $G_{\rm H}$, $H_{\rm VT}$ and $H_{\rm H}$ for the numerators and denominators that remain in the quotients in (III.24) once all the common factors have been cancelled (except when there are no hypermultiplets in which case we take $G_{\rm H} = H_{\rm H} = SU(2)$). For $\mathcal{M}_{\rm H}$, one finds $G_{\rm H}$ is the simple subgroup of $\mathrm{C}_{E_{6(6)}}(G_S)$ that contains a highest root $SU(2)$, consistent with our definition of $G_{\rm H}$ above.

The V structure moduli space corresponds to deformations of K that leave J_α invariant, while the H structure moduli space describes deformations of J_α that leave K invariant. When given a dependence on the external spacetime coordinates, these deformations provide the scalar fields in the truncated theory, with $\mathcal{M}_{\rm VT}$ and $\mathcal{M}_{\rm H}$ being identified with the vector multiplet and the hypermultiplet scalar manifolds, respectively.

We can identify the moduli explicitly as follows. Consider first $\mathcal{M}_{\rm VT}$. Using the basis $K^{\tilde{I}}$, a general vector $K \in \mathcal{V}$ can be written as a linear combination

$$K = h^{\tilde{I}} K_{\tilde{I}}\,,$$

where $h^{\tilde{I}}$, $\tilde{I}=0,\ldots,n_{\rm VT}$, are real parameters. Fixing $\boldsymbol{\kappa}^2$ in (III.16), and using (III.21), gives

$$C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}}=1\,,$$

showing that the $n_{\rm VT}+1$ parameters $h^{\tilde{I}}$ are constrained by one real relation and thus define an $n_{\rm VT}$-dimensional hypersurface, just as in (II.6),

$$\mathcal{M}_{\rm VT}=\{\,h^{\tilde{I}}:\ C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}}=1\,\}\,.$$

The space $\mathcal{M}_{\rm VT}$ is the moduli space of the V structure and, in the truncation, will determine the vector multiplet scalar manifold of the five-dimensional theory. The metric on $\mathcal{M}_{\rm VT}$ is obtained by evaluating the generalised metric on the invariant generalised vectors,

$$a_{\tilde{I}\tilde{J}}=\tfrac{1}{3}\,G(K_{\tilde{I}},K_{\tilde{J}})\,.$$

It is straightforward to verify that, using (IV.26), the expression above reproduces the five-dimensional expression (IV.35).

Consider now $\mathcal{M}_{\rm H}$. The family of H structures is obtained by parameterising the possible choices of $\mathfrak{su}_2$ algebra. Recall that by definition $\mathcal{U}\simeq\mathfrak{g}_{\rm H}$, so we are interested in the space of highest root $\mathfrak{su}(2)\subset\mathfrak{g}_{\rm H}$ subalgebras. Fixing $\boldsymbol{\kappa}^2$ and modding out by the $SU\,2$ symmetry that relates equivalent triples J_α we have the moduli space

$$\mathcal{M}_{\rm H}\;=\;\frac{G_{\rm H}}{SU(2)_R\cdot{\rm C}_{G_{\rm H}}(SU(2)_R)}\,,\tag{III.25}$$

that is, comparing with (III.24), we have $H_{\rm H}=SU(2)_R\cdot{\rm C}_{G_{\rm H}}(SU(2)_R)$. Points in $\mathcal{M}_{\rm H}$ can be parameterised by starting from a reference subalgebra $\mathfrak{j}\simeq\mathfrak{su}_2\subset\mathfrak{g}_{\rm H}$ and then acting on a basis $\{j_1,j_2,j_3\}$ of $\mathfrak{j}$ by the adjoint action of group elements $h\in G_{\rm H}$, defined as

$$J_\alpha={\rm ad}_{G_{\rm H}}\,j_\alpha=h\,j_\alpha\,h^{-1}\,.$$

One has to mod out by the elements of $G_{\rm H}$ that have a trivial action, namely $h\in SU(2)_R\simeq\exp(\mathfrak{j})$ and $h\in{\rm C}_{G_{\rm H}}(SU(2)_R)$. The resulting symmetric spaces (III.25) and are all quaternionic–Kähler, in agreement with the identification of $\mathcal{M}_{\rm H}$ with the hyperscalar manifold in five-dimensional supergravity.

III.2.1 The data of the truncation

Any generalised G_S structure on a manifold M with only constant, singlet intrinsic torsion gives rise to a consistent truncation of eleven-dimensional or type II supergravity with spacetime $X\times M$ to a gravitational theory on X [1, 2]. In this section we focus on truncations to five-dimensional $\mathcal{N}=2$ supergravity and recall how the the generalised $G_S\subseteq USp(6)$ structure encodes the data of the truncated theory, as summarised in Section 2 of [3].

The field content of the truncated theory is completely determined by the G_S-invariant spaces $\mathcal{U}$ and $\mathcal{V}$ and the moduli space of HV structures, while the gauging is determined by the singlet torsion.

The scalars of the truncated theory are given by the moduli space (III.22) of generalised metrics on M that factors (III.24) into

$$\begin{aligned} \text{VM scalars:} \quad & \phi(x)^i \leftrightarrow \mathcal{M}_{\rm VT} = \frac{G_{\rm VT}}{H_{\rm VT}}\,, \\ \text{HM scalars:} \quad & q(x)^X \leftrightarrow \mathcal{M}_{\rm H} = \frac{G_{\rm H}}{H_{\rm H}} = \frac{G_{\rm H}}{SU(2)_R \times \mathrm{C}_{G_{\rm H}}(SU(2)_R)}\,, \end{aligned}$$

where x^μ are the coordinates on X.

By construction, both spaces are homogeneous and so correspond to one of the cases listed in Section 2 of [3]. As discussed in Section III.2, the metrics can be explicitly constructed in terms of the basis vectors $K_{\tilde I}$ and J_A. In particular, the cubic invariant on $\mathcal{V}$, which fixes the metric on $\mathcal{M}_{\rm VT}$, is given by (III.20).

The other bosonic fields are the vectors and two-forms. As we will see in a moment, the singlet intrinsic torsion allows one to decompose the space of constant vectors as $\mathcal{V} = \mathcal{R} \oplus \mathcal{T}$ so that the basis vectors split

$$\{K_{\tilde I}\} = \{K_I\} \cup \{K_M\}\,,$$

where $\{K_I\}$ with $I = 0, \ldots, n_{\rm V}$ are a basis for $\mathcal{R}$ and $\{K_M\}$ with $M = n_{\rm V}+1, \ldots, n_{\rm VT}$ are a basis for $\mathcal{T}$. The vector fields and two-forms are in one-to-one correspondence with a basis in $\mathcal{R}$ and $\mathcal{T}$ respectively[6]

$$\begin{aligned} \text{vectors:} \quad & A_\mu{}^I(x) \leftrightarrow K_I\,, \\ \text{two-forms:} \quad & B_{\mu\nu}{}^M(x) \leftrightarrow K_M\,. \end{aligned}$$

The gauge interactions of the truncated theory are determined by the intrinsic torsion of the G_S-structure, which in turn is captured by the constants appearing in (II.8). The first relation defines a bracket $[\![\cdot,\cdot]\!] : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ on $\mathcal{V}$ given by

$$[\![v, w]\!]^{\tilde I} := (L_v w)^{\tilde I} = t_{\tilde J \tilde K}{}^{\tilde I} v^{\tilde J} w^{\tilde K}\,, \qquad \forall\, v, w \in \mathcal{V}\,.$$

Since the generalised Lie derivative satisfies $L_u(L_v w) = L_{L_u v} w + L_v(L_u w)$ the bracket defines a Leibniz algebra. As in Section 2 of [3], one can then choose a splitting $\mathcal{V} = \mathcal{R} \oplus \mathcal{T}$, where $\mathcal{T}$ is the image of the symmetrised bracket, such that $\mathcal{R}$ is the space of vector multiplets and $\mathcal{T}$ the space of tensors.

For a consistent gauging we need to check the conditions given in Section 2 of [3]. They follow from the properties of the generalised Lie derivative as we now show. Recall first that $(t_v)^{\tilde J}{}_{\tilde I} = v^{\tilde K} t_{\tilde K \tilde I}{}^{\tilde J}$ is an element of the Lie algebra $\mathrm{Lie}\, G_{\rm VT} \subset \mathfrak{e}_{6(6)}$. Since c and $\boldsymbol{\kappa}^2$ are $E_{6(6)}$ invariants, the action of t_v must preserve the cubic tensor C given by (III.20) and hence we satisfy one of the conditions mentionned before. Furthermore, by definition

$$L_v w + L_w v = \mathrm{d}(v \otimes_N w)\,, \tag{III.26}$$

[6] In the general formalism given in [1, 2] the two-forms were valued in constant sections of the singlet sub-bundle of $N \simeq \det T^*M \otimes E^*$, written using dual basis vectors $K^{\flat\tilde I}$, and isomorphic to elements of $\mathcal{V}^*$. The relation to the fields here is that the $\tilde I$ index is raised using the symplectic form Ω^{-1} defined by the singlet torsion. Note also that one can consider $A_\mu^{\tilde I}$ and $B_{\mu\nu}^{\tilde I}$ defined for all values of $\tilde I$. However, once the non-propagating fields are eliminated only A_μ^I and $B_{\mu\nu}^M$ are dynamical and the Lagrangian takes the form given in Section 2 of [3]

where d is the exterior derivative and $\otimes_N$ is the projection onto $N \simeq \det T^*M \otimes E^*$ given by $v \otimes_N w = c(v,w,\cdot)$. If $v, w \in \mathcal{V}$ then the left-hand side of (III.26) is by definition an element of $\mathcal{T}$. Using (III.20), the right-hand side is just the sequence of maps described in Section of [3], where the symplectic form on $\mathcal{T}$ is defined by the composition $\Omega^{-1} = \mathrm{d} \circ \boldsymbol{\kappa}^2$. Hence (III.26) implies we satisfy the second condition required for a consistent gauging.

To complete the description of the gauging we identify the embedding tensor and the Killing vector fields on $\mathcal{M}_{\rm VT}$ and $\mathcal{M}_{\rm H}$. Since both manifolds are coset spaces, from (III.23), the group of isometries is $G_{\rm iso} = G_{\rm VT} \times G_{\rm H}$ and the embedding tensor is a map

$$\Theta : \mathcal{V} \to \mathfrak{g}_{\rm iso} = \mathrm{Lie}\, G_{\rm VT} \oplus \mathrm{Lie}\, G_{\rm H}\,.$$

The corresponding gauged Killing vectors $k^i_{\tilde{I}}(\phi)$ and $\tilde{k}^X_{\tilde{I}}(q)$ on $\mathcal{M}_{\rm VT}$ and $\mathcal{M}_{\rm H}$ are given in Section 2 of [3]. If we view $K = h^{\tilde{I}}(\phi) K_{\tilde{I}}$ as giving the embedding of $\mathcal{M}_{\rm VT}$ in $\mathcal{V}$ and $J_\alpha = m^A_\alpha(q) J_A$ as giving the embedding of $\mathcal{M}_{\rm H}$ in $\mathcal{U}$ then, we can identify the Killing vectors explicitly from the relations

$$\begin{aligned} k^i_{\tilde{I}} \partial_i h^{\tilde{J}} &= \Theta_{\tilde{I}}{}^a k^i_a \partial_i h^{\tilde{J}} = t_{\tilde{I}\tilde{K}}{}^{\tilde{J}} h^{\tilde{K}}\,, \\ \tilde{k}^X_{\tilde{I}} \partial_X m^A_\alpha &= \Theta_{\tilde{I}}{}^m \tilde{k}^X_m \partial_X m^A_\alpha = p_{\tilde{I}B}{}^A m^B_\alpha\,. \end{aligned}$$

Thus we can identify the embedding tensor as an element of $\mathrm{Lie}\, G_{\rm VT} \oplus \mathrm{Lie}\, G_{\rm H}$

$$\Theta_{\tilde{I}} = \begin{pmatrix} (t_{\tilde{I}})_{\tilde{J}}{}^{\tilde{K}} & 0 \\ 0 & (p_{\tilde{I}})_A{}^B \end{pmatrix}\,.$$

Using the Leibniz property that $L_{K_{\tilde{I}}}(L_{K_{\tilde{J}}}\alpha) = L_{(L_{K_{\tilde{I}}}K_{\tilde{J}})}\alpha + L_{K_{\tilde{J}}}(L_{K_{\tilde{I}}}\alpha)$ for any generalised tensor α, it follows that each set of vectors forms a representation of $\mathfrak{g}_{\rm gauge}$. In other words, we have

$$[t_{\tilde{I}}, t_{\tilde{J}}] = t_{\tilde{I}\tilde{J}}{}^{\tilde{K}} t_{\tilde{K}}\,, \qquad [p_{\tilde{I}}, p_{\tilde{J}}] = t_{\tilde{I}\tilde{J}}{}^{\tilde{K}} p_{\tilde{K}}\,.$$

Finally, it is worth noting that the Killing prepotentials descend directly from the moment maps for generalised diffeomorphisms that appear in integrability conditions for an HV structure [49] and are given by

$$g\, P^\alpha_{\tilde{I}} = \tfrac{1}{8}\, \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}\left(J_\beta (L_{K_{\tilde{I}}} J_\gamma)\right)\,,$$

where as above $J_\alpha = m^A_\alpha(q) J_A$ is the dressed triplet.

It is important to note that generic $\mathcal{N}=2$ supergravity allows gaugings defined by an embedding tensor Θ that is a general element of $\mathcal{V}^* \otimes \mathfrak{g}_{\rm iso}$. However, the fact that our theory comes from a consistent truncations will typically restrict the form of Θ to only lie in certain $G_{\rm VT} \times G_{\rm H}$ representations in the decomposition of $\mathcal{V}^* \otimes \mathfrak{g}_{\rm iso}$. For this reason, in the following we will use T to denote the embedding tensor that appears in the consistent truncations to distinguish it from the more general Θ. As a consequence, we will see that not all the allowed $\mathcal{N}=2$ gaugings can arise from consistent truncations.

III.3 Classification of 5d $\mathcal{N} = 2$ truncations

We can use the formalism presented in the previous section to provide a classification of the five-dimensional gauged supergravities that can be obtained as consistent truncations of M-theory or type IIB.

As discussed above there are two conditions in the derivation of a consistent truncation using a generalised G_S structure: one topological, the existence of G_S, and one differential, the constant singlet intrinsic torsion.

For our classification we will focus on the topological part and perform a scan of the possible subgroups G_S of $USp(6)$ that gives inequivalent $\mathcal{N} = 2$ truncations. We will assume that the differential condition of having constant singlet intrinsic torsion is satisfied. Under this assumption a purely algebraic analysis allow to classify the field content and gaugings of the truncated theory [3]. Already look at the algebraic constaints consideably reduces the number of possible theory and their gaugings. From the example of maximally supersymmetric gauged supergravity we know that the differential condition puts important restrictions on the allowed gauged supergravities [20, 52, 53]. Thus We expect that not all cases we find in our analyis are actually realised on explicit constructions. can be solved.

The algebraic problem then consists of the following steps. We first scan for all possible inequivalent ways of breaking $USp(8)$ to $G_S \subset USp(6)$ that admit only two singlets in the fundamental representation of $USp(8)$. Given a G_S with these features, it will embed in $E_{6(6)}$ as

$$E_{6(6)} \supset G_S \cdot \mathrm{C}_{E_{6(6)}}(G_S)\,,$$

where we are using the "central product"[7]. We then check whether under this breaking the $\mathbf{27}^*$ and $\mathbf{78}$ of $E_{6(6)}$ contain G_S singlets, which will determine the vector and hyper-multiplets of the truncated theory. In each case the singlets will transform under $\mathrm{C}_{E_{6(6)}}(G_S)$ which also determines the form of the scalar manifold $\mathcal{M}$ of the truncated theory

$$\mathcal{M} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)}\,. \tag{III.27}$$

Note that by construction the scalar manifolds are always necessarily symmetric spaces and furthermore are always a product $\mathcal{M} = \mathcal{M}_{\mathrm{VT}} \times \mathcal{M}_{\mathrm{H}}$ of vector-tensor multiplet and hypermultiplet scalar manifolds as in (III.24).

By scanning all continuous Lie subgroups $G_S \subset USp(6)$, we find that there are only a small number of inequivalent G_S structures with the properties above. We list them here according to the type of breaking of $USp(6)$ that they correspond to. All other cases either give rise to extra singlets in the $\mathbf{6}$ of $USp(6)$ or can be obtained as subgroups of the G_S-structures listed below without giving rise to any new fields in the consistent truncation.

Br.1 $G_S = SU(2) \times \mathrm{Spin}(p)$, $2 \leq p \leq 5$.

[7] By definition, for any group G and subgroup H, the commutant $\mathrm{C}_G(H)$ of H in G includes the centre $Z(H)$ of H. The central product is defined to be $H \cdot \mathrm{C}_G(H) = (H \times \mathrm{C}_G(H))/Z(H)$ where one modes out by the diagonal $Z(H)$ subgroup. In this case $Z(USp(6)) = \mathbb{Z}_2$ and the central product reflects the fact that the maximal compact subgroup of $E_{6(6)}$ is $USp(8)/\mathbb{Z}_2$ and not $USp(8)$.

These are obtained from the embedding

$$USp(6) \supset USp(4) \times SU(2) \simeq \mathrm{Spin}(5) \times SU(2)\,,$$

which gives

$$\mathbf{6} = (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,,$$

and by further breaking the $USp(4)$ factor

$$\begin{aligned} &USp(4) \supset SU(2) \times SU(2) \simeq \mathrm{Spin}(4)\,,\\ &USp(4) \supset SU(2) \times SU(2) \supset SU(2)_D \simeq \mathrm{Spin}(3)\,,\\ &USp(4) \subset SU(2) \times SU(2) \supset SU(2)_D \subset U(1)_D \simeq \mathrm{Spin}(2)\,. \end{aligned}$$

The corresponding branching of the **6** of $USp(6)$ are

$$\begin{aligned} \mathbf{6} &= (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})\,,\\ \mathbf{6} &= 2\cdot(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,,\\ \mathbf{6} &= 2\cdot\mathbf{1}_1 \oplus 2\cdot\mathbf{1}_{-1} \oplus \mathbf{2}_0\,, \end{aligned}$$

for the breaking to $\mathrm{Spin}(4)\times SU(2)$, $\mathrm{Spin}(3)\times SU(2)$ and $\mathrm{Spin}(2)\times SU(2)$, respectively.

Br.2 $G_S = SO(3)$ and $G_S = SU(2)$.

The relevant breaking is

$$USp(6) \supset SO(3) \times SU(2)\,,$$

with the **6** of $USp(6)$ branching as

$$\mathbf{6} = (\mathbf{3},\mathbf{2})\,. \tag{III.28}$$

Taking $G_S = SO(3)$ or $G_S = SU(2)$ leads to two different consistent truncations.

Br.3 $G_S = SU(3)$.

This comes from the breaking

$$USp(6) \supset SU(3) \times U(1)$$

which gives

$$\mathbf{6} = \mathbf{3}_1 \oplus \overline{\mathbf{3}}_{-1}\,.$$

Br.4 $G_S = SU(2) \times U(1)$

This truncation is obtained by further breaking the $SU(3)$ group of the previous case. Under $SU(3) \supset SU(2) \times U(1)$, we get

$$\mathbf{6} = \mathbf{2}_{1,1} \oplus \mathbf{1}_{-2,1} \oplus \mathbf{2}_{-1,-1} \oplus \mathbf{1}_{2,-1}\,.$$

Br.5 $G_S = U(1)$.

This comes from the same breaking $SU(3) \supset SU(2) \times U(1)$ as Br.4 but taking only the $U(1)$ factor as the structure group.

Br.6 $G_S = U(1)$.

This comes from the same breaking as Br.3 and taking the $U(1)$ factor as structure group.

The breaking listed above can give rise to gauged supergravities with different matter content and we summarise our results in Table III.1: we list the G_S structure group, the number of vector/tensor multiplets $n_{\rm VT}$ and hypermultiplets $n_{\rm H}$, and the associated scalar manifolds. We see that the possible consistent truncations are limited. In particular, we find the largest possible truncation consists of only 14 vector/tensor multiplets.

$n_{\rm VT}$ \ $n_{\rm H}$	0	1	2
0	$G_S = USp(6)$ $\mathcal{M} = 1$	$G_S = SU(3)$ $\mathcal{M} = \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}$	$G_S = SO(3)$ $\mathcal{M} = \frac{\mathrm{G}_{2(2)}}{SO(4)}$
1	$G_S = SU(2)\times \mathrm{Spin}(5)$ $\mathcal{M} = \mathbb{R}^+$	$G_S = SU(2)\times U(1)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}$	-
2	$G_S = SU(2)\times \mathrm{Spin}(4)$ $\mathcal{M} = \mathbb{R}^+ \times SO(1,1)$	$G_S = U(1)$ $\mathcal{M} = \mathbb{R}^+ \times SO(1,1) \times \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}$	-
3	$G_S = SU(2)\times \mathrm{Spin}(3)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SO(2,1)}{SO(2)}$	$G_S = U(1)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SO(2,1)}{SO(2)} \times \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}$	-
4	$G_S = SU(2)\times \mathrm{Spin}(2)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)}$	$G_S = U(1)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)} \times \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}$	-
5	$G_S = SU(2)$ $\mathcal{M} = \frac{\mathrm{SL}(3,\mathbb{R})}{SO(3)}$ $G_S = SU(2)\times \mathbb{Z}_2$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SO(4,1)}{SO(4)}$	-	-
6	$G_S = SU(2)\times \mathbb{Z}_2$ $\mathcal{M} = \mathbb{R}^+ \times \frac{SO(5,1)}{SO(5)}$	-	-
8	$G_S = U(1)$ $\mathcal{M} = \frac{\mathrm{SL}(3,\mathbb{C})}{SU(3)}$	-	-
14	$G_S = \mathbb{Z}_2$ $\mathcal{M} = \frac{\mathrm{SU}^*(6)}{USp(6)}$	-	-

Table III.1: List of all possible consistent truncation with $n_{\rm VT}$ vector/tensor multiplets, n_H hypermultiplets, and the required $G_S \subset E_{6(6)}$ structure group, as well as the associated scalar manifold $\mathcal{M}$.

Let us again reiterate that the consistent truncations that can be actually realised will be a subset of those presented in the group-theoretic analysis here. This is because the requirement that a given G_S structure has singlet intrinsic torsion will introduce non-trivial differential constraints that a given manifold M must satisfy and which we do not analyse here.

For every consistent truncations, by looking at the intrinsic torsion it is possible to determine the allowed gauging. We refer to Section 4 of [3] for the detailed analysis, Here we will simply summarise our results and stress that only a few gaugings are possible.

As already mentioned we did not Verify that the differential constraint of having constant intrinsic torsion are actually satisfied. Instead we can show that in some cases it is possible to find a geometry that realises the G_S structure with constant singlet intrinsic torsion. This will be the subject of our next chapter.

III.3.1 Truncations to only vector and tensor multiplets

We analyse first the possible consistent truncations that give rise to a theory with only vector/tensor multiplets. Since a consistent truncation necessarily gives rise to a symmetric scalar manifold (see Section 2 of [3]), the vector/tensor scalar manifolds that one can obtain must be symmetric "very special real" manifolds, as classified in [54, 55, 56].

This classification consists of a generic case, possible for arbitrary number of vector/tensor multiplets, where the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ factorises, with the only non-zero components given by

$$C_{0ij} = \eta_{ij}\,, \qquad i,j = 1,\ldots,n_{\rm VT}\,. \tag{III.29}$$

Here η_{ij} has signature $(1, n_{\rm VT}-1)$ and the scalar manifold is given by

$$\mathcal{M}_{\rm VT} = \mathbb{R}^+ \times \frac{SO(n_{\rm VT}-1,1)}{SO(n_{\rm VT}-1)}\,. \tag{III.30}$$

Additionally, there are a number of "special" cases that only exist for specific values of $n_{\rm VT}$ and for which $C_{\tilde{I}\tilde{J}\tilde{K}}$ does not factorise. These are given by

$$\begin{aligned}
\mathcal{M}_{\rm VT} &= \frac{\mathrm{SL}(3,\mathbb{R})}{SO(3)}\,, && n_{\rm VT} = 5\,,\\
\mathcal{M}_{\rm VT} &= \frac{\mathrm{SL}(3,\mathbb{C})}{SU(3)}\,, && n_{\rm VT} = 8\,,\\
\mathcal{M}_{\rm VT} &= \frac{\mathrm{SU}^*(6)}{USp(6)}\,, && n_{\rm VT} = 14\,,\\
\mathcal{M}_{\rm VT} &= \frac{\mathrm{E}_{(6,-26)}}{\mathrm{F}_4}\,, && n_{\rm VT} = 26\,.
\end{aligned} \tag{III.31}$$

Finally, there is a second "generic case", which exists for arbitrary $n_{\rm VT} > 1$, but where the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ does not factorise [56]. The associated scalar manifolds are given by

$$\mathcal{M}_{\rm VT} = \frac{SO(n_{\rm VT},1)}{SO(n_{\rm VT})}\,. \tag{III.32}$$

We want to determine which of these gauged supergravities can arise from a consistent truncation and how can they be classified in terms of the structure groups G_S listed in the previous section.

In order to have a consistent truncation with only vector/tensor multiplets, the generalised tensors defining the G_S structure must consist of the triplet of adjoint tensor J_α, $\alpha = 1, 2, 3$ corresponding to an H-structure (see Section III.2) and of $n_{\rm VT} + 1$ generalised vectors $K_{\tilde{I}}$, $\tilde{I} = 0, 1, \ldots n_{\rm VT}$ satisfying

$$\begin{aligned} J_\alpha \cdot K_{\tilde{I}} &= 0 \,, \\ \boldsymbol{\kappa}^{-2} c(K_{\tilde{I}}, K_{\tilde{J}}, K_{\tilde{K}}) &= C_{\tilde{I}\tilde{J}\tilde{K}} \,, \end{aligned} \tag{III.33}$$

with constant $C_{\tilde{I}\tilde{J}\tilde{K}}$.

Since the J_α are stabilised by $\mathrm{SU}^*(6) \subset E_{6(6)}$, the structure group must be a subgroup of $\mathrm{SU}^*(6)$. Under the breaking $E_{6(6)} \supset SU^*(6) \cdot SU(2)_R$, we have

$$\begin{aligned} \mathbf{27}^* &= (\mathbf{15}^*, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{2}) \,, \\ \mathbf{78} &= (\mathbf{35}, \mathbf{1}) \oplus (\mathbf{20}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3}) \,, \end{aligned}$$

where the triplet of J_α belong to $(\mathbf{1}, \mathbf{3})$ and generate the $SU(2)_R$ symmetry. Then, the first condition in (III.33), implies that the vectors $K_{\tilde{I}}$ must be invariant under $SU(2)_R$ and therefore must lie in the real vector space

$$\mathcal{V} \subseteq (\mathbf{15}^*, \mathbf{1}) \,.$$

Thus, we can have at most $n_{\rm VT} = 14$ vector/tensor multiplets and we can immediately rule out the case $n_{\rm VT} = 26$ in (III.31), as well as the case $n_{\rm VT} > 14$ in (III.30).

The family (III.32) is also ruled out, because the isometries of the corresponding scalar manifolds are not linearly realised. As we discussed in Section III.2, the isometry group of the scalar manifold is the commutant in $E_{6(6)}$ of the structure group and by construction it acts linearly on the set of singlet generalised vectors. As a result, the gauged supergravities with vector/tensor scalar manifolds (III.32) do not arise from consistent truncations.

All other cases can in principle arise in consistent truncations and in the next subsection we will discuss from which generalised structure G_S they can be obtained and then use G_S to study the intrinsic torsion and hence find the admissible gaugings.

a) Generic case

The generic case with scalar manifold (III.30) corresponds to the structure groups

$$G_S = \mathrm{Spin}(6 - n_{\rm VT}) \times SU(2) \,, \tag{III.34}$$

of item (Br.1) of the list in the previous section, where for notational convenience we let $\mathrm{Spin}(1) = \mathrm{Spin}(0) = \mathbb{Z}_2$. Note that (III.34) implies that we can have at most $n_{\rm VT} = 6$ vector/tensor multiplets in the truncation. Moreover, the case $n_{\rm VT} = 5$ and $n_{\rm VT} = 6$ have identical structure groups. This means that any background admitting a truncation with $n_{\rm VT} = 5$ actually admits a truncation with $n_{\rm VT} = 6$, with the former truncation being a subtruncation of the latter.

To see how these structure groups arise, note that the structure (III.29) of the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ implies that the vectors $K_{\tilde{I}}$ can be split into a vector K_0 and $n_{\rm VT}$ vectors K_i such that for any $i, j, k = 1, \ldots, n_{\rm VT}$,

$$c(K_0, K_0, \cdot) = 0 \,, \qquad c(K_i, K_j, K_k) = 0 \,, \qquad c(K_0, K_i, K_j) = \eta_{ij} \,, \tag{III.35}$$

where η_{ij} has signature $(5,1)$. The vector K_0 corresponds to the graviphoton of the truncated theory.

By studying the form of (III.35), we can deduce the stabiliser group of the generalised vector fields $K_{\tilde{I}}$ as follows. Being in the $\mathbf{15}^*$ of SU*(6), the vectors $K_{\tilde{I}}$ can be seen as six-dimensional two-forms. Then the first condition in (III.35) is equivalent to

$$K_0 \wedge K_0 = 0\,,$$

with $\wedge$ the standard wedge product of p-forms. Thus, K_0 must be decomposable and we can choose a basis of independent six-dimensional one-forms such that

$$K_0 = e_5 \wedge e_6\,.$$

The stabiliser of K_0 is SU*(4) $\times\, SU(2)$, embedded in SU*(6) as

$$\begin{aligned}\text{SU}^*(6) &\supset \text{SU}^*(4) \times SU(2) \times U(1)\,,\\ \mathbf{15}^* &= (\mathbf{4}^*,\mathbf{2})_1 \oplus (\mathbf{6},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{1})_4\,,\end{aligned}$$

with $K_0 \in (\mathbf{1},\mathbf{1})_4$. This forces the G_S structure to be a subgroup of SU$^*(4) \times SU(2)$. The other conditions in (III.35) become

$$K_0 \wedge K_i \wedge K_j = \eta_{ij}\,, \qquad K_i \wedge K_j \wedge K_k = 0\,, \tag{III.36}$$

where the metric η_{ij} is invariant under SU$^*(4) \simeq \text{Spin}(5,1)$. From (III.36) it follows that

$$K_i \in (\mathbf{6},\mathbf{1})_{-2}\,.$$

Thus, there can be at most six vector multiplets of this type.

The structure group G_S can now be easily determined. Since the n_{VT} singlets K_i satisfy the inner product (III.36) of signature $(1, n_{\text{VT}}-1)$ they break SU*(4) to

$$\text{SU}^*(4) \simeq \text{Spin}(5,1) \supset \text{Spin}(6-n_{\text{VT}}) \times \text{Spin}(n_{\text{VT}}-1,1)\,,$$

where the factor $\text{Spin}(6-n_{\text{VT}})$ is the stabiliser of the K_i while the factor $\text{Spin}(n_{\text{VT}}-1,1)$ rotates the K_i into each other. Thus, the structure group is given by

$$G_S = \text{Spin}(6-n_{\text{VT}}) \times SU(2)\,.$$

Although the structure groups and the isometry groups are Spin subgroups of $E_{6(6)}$, the generalised vectors K_i never appear in spinorial representations of G_S and hence only see the orthogonal groups and not their double covers. This is the reason why the case with $n_{\text{VT}} = 5$ vectors/tensors can always be enhanced to $n_{\text{VT}} = 6$: on the two-forms K_i the $\mathbb{Z}_2$ structure group acts trivially. Moreover, this is why the coset spaces can be reduced to take the form (III.30):

$$\mathcal{M} = \mathcal{M}_{\text{VT}} = \frac{\text{C}_{E_{6(6)}}(G_S)}{\text{C}_{USp(8)/\mathbb{Z}_2}(G_S)} = \mathbb{R}^+ \times \frac{SO(n_{\text{VT}}-1,1)}{SO(n_{\text{VT}}-1)}\,.$$

The corresponding isometry group is

$$G_{\text{iso}} = \mathbb{R}^+ \times SO(n_{\text{VT}}-1,1) \times SU(2)_R\,,$$

where as discussed above we take $G_{\rm H} = SU(2)_R$, even though there are no hypermultiplets, in order to include the R-symmetry. Under $G_{\rm iso}$ the space of vectors transforms as

$$\mathcal{V} = (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{n})_{-1} \ni (v^0, v^i)\,,$$

where the first entries are the $SU(2)_R$ representations, $\mathbf{n}$ is the vector representation of $SO(n_{\rm VT}-1,1)$, the subscripts are the $\mathbb{R}^+$ charges, and $i = 1,\ldots,n_{\rm VT}$ denotes $SO(n_{\rm VT}-1,1)$ indices.

We can also determine the embedding tensor of the truncated theory and the possible gaugings. These are encoded in the intrinsic torsion of the G_S structure, which must only contain G_S singlets for the truncation to be consistent. The detailed analysis is given in [3]. In Table III.2 we summarise the allowed gaugings for truncations with only vectors/tensor multiplet of generic type. Whenever we list a product group, the individual factors can also be gauged separately even though they are not listed as such. Whenever there are abelian factors in $G_{\rm gauge}$, the $U(1)_R$ can also be gauged diagonally with some combination of these factors.

$n_{\rm VT}$	$G_{\rm iso}$	$G_{\rm gauge}$	$n_{\rm T}$
1	$SU(2)_R \times \mathbb{R}^+$	$U(1)_R$	–
2	$SU(2)_R \times SO(1,1) \times \mathbb{R}^+$	$U(1)_R$	–
		$SO(1,1)$	2
3	$SU(2)_R \times SO(2,1) \times \mathbb{R}^+$	$SO(2,1)\times U(1)_R$	–
		$SO(2),\ \ SO(1,1)$	2
4	$SU(2)_R \times SO(3,1) \times \mathbb{R}^+$	$SO(2,1)\times U(1)_R,\ \ SO(3)\times U(1)_R,$ $ISO(2)\times U(1)_R,\ \ SU(2)_R$	–
		$SO(2)\times U(1)_R,\ \ SO(1,1)\times U(1)_R,$	2
		$SO(1,1)$	4
6	$SU(2)_R \times SO(5,1) \times \mathbb{R}^+$	$SO(3)\times SO(2,1)\times U(1)_R,\ \ SO(2,1)\times SU(2)_R,$ $ISO(2)\times U(1)_R$	–
		$SO(2,1)\times U(1)\times U(1)_R,\ \ SO(3)\times SO(2)\times U(1)_R,$ $SO(3)\times SO(1,1)\times U(1)_R,\ \ ISO(2)\times U(1)\times U(1)_R,$ $SO(2)\times SU(2)_R,\ \ SO(1,1)\times SU(2)_R$	2
		$U(1)\times U(1)_R,\ \ SO(1,1)\times U(1)_R$	4
		$SO(1,1)$	6

Table III.2: Allowed gaugings $G_{\rm gauge}$ of the global isometry groups $G_{\rm iso}$ in the generic cases with $n_{\rm VT}$ vector/tensor multiplets. The first column gives the total number of vectors and tensor multiplets, the second the global isometry group, the third the allowed gaugings and the last one the number of vectors that are dualised to tensors in each case.

b) Special cases

The special cases (III.31) are also associated to some of the generalised G_S-structures we listed at the beginning of this section. We now discuss case by case what the associated

structure groups are, we determine the corresponding embedding tensor and hence the possible gaugings of the truncated theory.

Differently from the generic case it is quite cumbersome to analyse in full generality the constraints imposed on the gaugings by the Leibniz condition and hence the allowed gaugings. Thus in this section we will limit ourselves to study what are the largest reductive groups and largest compact groups that can be gauged.

b).1 $\mathbf{n_{VT}=5}$: This truncation is associated to a $G_S = SU(2)$ generalised structure. The structure group is taken to be the $SU(2)$ factor in the breaking (Br.2) of $USp(6)$ and it embeds in $\mathrm{SU}^*(6)$ as $\mathrm{SU}^*(6) \supset \mathrm{SL}(3,\mathbb{R}) \times SU(2)$. Under this embedding we have

$$\mathbf{15}^* = (\mathbf{6}^*, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{3}) \,,$$

so that $\mathcal{V} = (\mathbf{6}^*, \mathbf{1})$ and there are six independent singlet vectors giving rise to $n_{\mathrm{VT}} = 5$ vector multiplets. It is easy to check that we also get the expected scalar manifold

$$\mathcal{M} = \mathcal{M}_{\mathrm{VT}} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)} = \frac{\mathrm{SL}(3,\mathbb{R})}{SO(3)} \,,$$

with isometry group

$$G_{\mathrm{iso}} = SU(2)_R \times \mathrm{SL}(3,\mathbb{R}) \,.$$

b).2 $\mathbf{n_{VT}=8}$: This truncation arises for the case (Br.6) and corresponds to a $G_S = U(1)$ structure group. Under the branching $\mathrm{SU}^*(6) \supset SL(3,\mathbb{C}) \times U(1)$ the vectors decompose as[8]

$$\begin{aligned} \mathbf{15}^* &= (\mathbf{3} \otimes \overline{\mathbf{3}})_0 \oplus \mathbf{3}^*_2 \oplus \overline{\mathbf{3}}^*_{-2} \,, \\ &\ni (v^{\alpha\dot{\alpha}}, v_\alpha, \bar{v}_{\dot{\alpha}}) \,, \end{aligned}$$

where raised α and $\dot{\alpha}$ indices denote the fundamental representation $\mathbf{3}$ and conjugate-fundamental representation $\overline{\mathbf{3}}$ of $\mathrm{SL}(3,\mathbb{C})$ respectively. Thus for example, since $\mathbf{15}^*$ is real, the two components v^α and $\bar{v}^{\dot{\alpha}}$ are related by complex conjugation $(v^\alpha)^* = \bar{v}^\alpha$ and $(v^{\alpha\dot{\beta}})^* = v^{\beta\dot{\alpha}}$. We see that the U 1-singlet space $\mathcal{V} = (\mathbf{3} \otimes \overline{\mathbf{3}})_0$ is nine-dimensional giving rise to $n_{\mathrm{VT}} = 8$ vector multiplets.

It is easy to check that (III.27) gives the expected scalar manifold

$$\mathcal{M} = \mathcal{M}_{\mathrm{VT}} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)} = \frac{\mathrm{SL}(3,\mathbb{C})}{SU(3)} \,,$$

with isometry group

$$G = \mathrm{SL}(3,\mathbb{C}) \times SU(2)_R \,.$$

[8] Recall that for $\mathrm{SL}(3,\mathbb{C})$ the dual and conjugate representations are not equivalent. Here we denote them by $\mathbf{n}^*$ and $\overline{\mathbf{n}}$, respectively.

b).3 $\mathbf{n_{VT} = 14}$: This is the maximal case, where the invariant vectors span the whole $\mathcal{V} = \mathbf{15}^*$ of SU*(6). It does not correspond to any of the generalised structures listed at the beginning of this section and therefore must correspond to a discrete structure group. Indeed, since all the $K_{\tilde{I}}$ are stabilised and from (III.27) we have

$$\mathcal{M} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(6)}(G_S)} = \frac{SU^*(6)}{USp(6)}\,,$$

it is easy to identify the generalised structure as

$$G_S = \mathbb{Z}_2 \subset E_{6(6)}\,.$$

The $\mathbb{Z}_2$ acts diagonally as $-\mathbb{1}$ in $USp(6)$, leading to the global isometry group

$$G_{\rm iso} = C_{E_{6(6)}}(\mathbb{Z}_2) = SU(2)_R \cdot SU^*(6)\,.$$

As before, by studying the intrinsic torsion we can determine the possible gaugings. In Table III.3 we summarise the maximal reductive and compact gauge groups for the special cases of purely vector/tensor multiplet truncations of this section. As in the previous table, whenever we list a product of groups, the individual factors can also be gauged separately even though they are not listed as such. Whenever there are abelian factors in $G_{\rm gauge}$, the $U(1)_R$ can also be gauged diagonally with some combination of these factors.

$n_{\rm VT}$	$G_{\rm iso}$	$G_{\rm gauge}$	$n_{\rm T}$
5	$SU(2)_R \times SL(3,\mathbb{R})$	$SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ $SL(2,\mathbb{R}) \times U(1)_R$	– 2
8	$SU(2)_R \times SL(3,\mathbb{C})$	$SU(3) \times U(1)_R$, $SU(2,1) \times U(1)_R$	–
14	$SU(2)_R \times SU^*(6)$	$SU(3,1)$ $SU(3) \times U(1)_R$, $SU(3) \times U(1)$	– 6

Table III.3: Maximal reductive and compact gauge groups in the special cases of purely vector/tensor multiplet truncations. The first column gives the total number of vectors and tensor multiplets, the second the global isometry group, the third the allowed gaugings and the last one the number of vectors that are dualised to tensors in each case.

III.3.2 Truncations with only hypermultiplets

Let us now analyse which consistent truncations are possible with only hypermultiplets and no vector multiplets.

Truncations of this kind are associated to a generalised structures G_S that is defined by a single generalised vector K in the $\mathbf{27}^*$ of $E_{6(6)}$, defining a V-structure, and a set of adjoint tensors J_A, $A = 1, \ldots, \dim(G_{\rm H})$, satisfying

$$J_A \cdot K = 0\,.$$

Since the stabiliser of the V-structure is $\mathrm{F}_{4(4)} \subset E_{6(6)}$, we must have $G_S \subset \mathrm{F}_{4(4)}$. Finally, by construction, the scalar manifold must be symmetric

$$\mathcal{M}_{\mathrm{H}} = \frac{G_{\mathrm{H}}}{SU(2)_R \cdot \mathrm{C}_{USp(6)}(G_S)}\,, \tag{III.37}$$

where $G_{\mathrm{H}} = \mathrm{C}_{E_{6(6)}}(G_S)$ is the group generated by the singlets J_A.

The above considerations already restrict the possible scalar manifolds for the hypermultiplets to the following list [57, 58]

$$\begin{aligned}
\mathcal{M}_{\mathrm{H}} &= \frac{\mathrm{F}_{4(4)}}{SU(2)\cdot USp(6)}\,, && n_{\mathrm{H}} = 7\,,\\
\mathcal{M}_{\mathrm{H}} &= \frac{\mathrm{SO}_0(4,p)}{SO(4)\times SO(p)}\,, && n_{\mathrm{H}} = p\,, \quad p \leq 5\,,\\
\mathcal{M}_{\mathrm{H}} &= \frac{\mathrm{G}_{2(2)}}{SO(4)}\,, && n_{\mathrm{H}} = 2\,,\\
\mathcal{M}_{\mathrm{H}} &= \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}\,, && n_{\mathrm{H}} = 1\,,
\end{aligned} \tag{III.38}$$

where $\mathrm{SO}_0(4,p)$ denotes the connected component of the $SO(4,p)$.

However the first two manifolds do not arise from truly $\mathcal{N}=2$ truncations. This is because they correspond to generalised structure groups that lead to extra singlets in the decomposition of the **6** of $USp(6)$. For $\mathcal{M}_{\mathrm{H}} = \frac{\mathrm{F}_{4(4)}}{SU(2)\cdot USp(6)}$, the structure group is trivial, $G_S = \mathbb{1}$, since it is given by the commutant in $\mathrm{F}_{4(4)}$ of the isometry group. Thus this truncation always comes from a sub-truncation of five-dimensional maximal supergravity. Similarly, for the $\mathcal{M}_{\mathrm{H}} = \frac{\mathrm{SO}_0(4,p)}{SO(4)\times SO(p)}$, with $p \leq 5$, the structure group has to be

$$G_S = \mathrm{Spin}(5-p)\,,$$

with $\mathrm{Spin}(0) = \mathrm{Spin}(1) = \mathbb{Z}_2$. The decomposition of the **6** of $USp(6)$ under G_S always contains two extra singlets, so that these cases are sub-truncations of half-maximal gauged supergravity. Indeed, from the commutant of G_S in the full $E_{6(6)}$ and $USp(8)$ groups,

$$\mathrm{C}_{E_{6(6)}}(\mathrm{Spin}(5-p)) = \mathrm{Spin}(5,p)\times\mathbb{R}^+\,, \qquad \mathrm{C}_{USp(8)}(\mathrm{Spin}(5-p)) = USp(4)\times\mathrm{Spin}(p)\,,$$

one can easily check that $G_S = \mathrm{Spin}(5-p)$ actually allows for a half-maximal truncation with p vector multiplets and scalar manifold

$$\mathcal{M} = \frac{\mathrm{Spin}(5,p)}{USp(4)\times\mathrm{Spin}(p)}\times\mathbb{R}^+\,.$$

This leaves only the two last manifolds in (III.38) as truly $\mathcal{N}=2$ truncations.

- The case with $\mathbf{n}_{\mathrm{H}} = \mathbf{2}$ hypermultiplets corresponds to a $G_S = SO\,3$ that is obtained from (Br.2). The structure group embeds as

$$\begin{aligned}
\mathrm{F}_{4(4)} &\supset SU(2)\times\mathrm{G}_{2(2)}\,,\\
USp(6) &\supset SU(2)\times SU(2)\,.
\end{aligned} \tag{III.39}$$

Decomposing the **78** of $E_{6(6)}$ in representations of $G_S = SU(2)$ gives 6 compact and 8 non-compact singlets. Altogether they correspond to the generators of $\mathrm{G}_{2(2)}$, while the compact ones give its $SO(4)$ maximal compact subgroup. Then (III.37) gives the expected scalar manifold

$$\mathcal{M} = \mathcal{M}_{\mathrm{H}} = \frac{\mathrm{G}_{2(2)}}{SO(4)} \,.$$

It is also easy to check that there are no vector/tensor multiplets in the truncation, since there are no singlets in the **26** of $\mathrm{F}_{4(4)}$ under the branching (III.39).

- The case with $\mathbf{n}_{\mathrm{H}} = \mathbf{1}$ tensor multiplet corresponds to the generalised structure $G_S = SU(3)$ (Br.3). This is embedded as

$$\begin{aligned} \mathrm{F}_{4(4)} &\supset SU(3) \times SU(2,1) \,, \\ USp(6) &\supset SU(3) \times U(1) \,. \end{aligned}$$

In the decomposition of the **78** of $SU(3)$ one finds 4 compact and 4 non-compact singlets, which generate $SU(2,1)$. The compact ones give the compact subgroup $SU(2) \times U(1)$ so that we recover the hyperscalar manifold

$$\mathcal{M} = \mathcal{M}_{\mathrm{H}} = \frac{SU(2,1)}{\mathrm{S}(U(2) \times U(1))} \,.$$

As, again, there are no singlets in the **26** of $\mathrm{F}_{4(4)}$ under the branching to $G_S = SU(3)$, there are no vector multiplets.

The study of the intrinsic torsions and the gauging for the truncations with only hypermultiplets is very simple. As the only vector in the theory is the graviphoton in the universal multiplet, only abelian gaugings are possible. Moreover, in all cases, the intrinsic torsion only contains the adjoint representation of the isometry group

$$W_{\mathrm{int}} = \mathrm{ad}\, G_{\mathrm{H}} \ni \tau_0{}^A{}_B \,,$$

with $A, B = 1, \ldots \dim G_{\mathrm{H}}$ so that the map $T : \mathcal{V} \to \mathfrak{g}_{\mathrm{iso}}$ is

$$T(v^0) = v^0 \tau_0{}^A{}_B \,, \qquad A = 1, \ldots, \dim G_{\mathrm{H}} \,.$$

The generalised Lie derivative on the adjoint singlets is

$$L_{K_0} J_A = [J_{K_0}, J_A] = -T(K_0) \cdot J_A = p_{0A}{}^B J_B \,,$$

with the component of the embedding tensor

$$p_{0A}{}^B = \tau_0{}^A{}_B \,,$$

and the graviphoton can gauge any one-dimensional subgroup of G_{H}.

III.3.3 Truncations with vector/tensor and hypermultiplets

The last class of truncations that can arise consists of truncations with both vector/tensor and hypermultiplets. One way to study this class is to start from the truncations with only hypermultiplets discussed in the previous section and look for a subgroup of the structure group G_S that allows for extra singlet vectors but no extra singlets in the branching of the **6** under $USp(6) \supset G_S$. This last condition is necessary to have a truly $\mathcal{N}=2$ truncation and leaves only two possible cases: $n_{\rm H}=2$ with $G_S = SU(2)$ (Br.2) or $n_{\rm H}=1$ and $G_S = SU(3)$ (Br.3).

The case with $n_{\rm H}=2$ hypermultiplets and hyperscalar manifold

$$\mathcal{M}_{\rm H} = \frac{\mathrm{G}_{2(2)}}{SO(4)}\,, \tag{III.40}$$

is immediately ruled out since any further reduction of the $G_S = SU(2)$ structure group necessarily gives rise to a singlet in the **6** of $USp(6)$. This can be easily see from (III.28) by breaking the second $SU(2)$ factor. Therefore consistent truncations with hypermultiplets forming the scalar manifold (III.40) and vector/tensor multiplets necessarily arise from subtruncations of $\mathcal{N}>2$ gauged supergravity.

We are left with the case with $n_{\rm H}=1$ hypermultiplet and hyperscalar manifold

$$\mathcal{M}_{\rm H} = \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}\,.$$

The structure group is $SU(3)$ and we can consider two non-trivial subgroups $G_S = SU(2)\times U(1)$ (Br.4) and $G_S = U(1)$ (Br.5). As we will discuss below, they allow for $n_{\rm VT}=1$ and $n_{\rm VT}=4$ vector multiplets, respectively. Cases with $n_{\rm VT}=2,3$ can only be obtained as sub-truncations of the $n_{\rm VT}=4$ case and therefore we will not discuss them here.

Recall that the scalar manifold of the vector/tensor multiplets in the truncation can now be computed from the commutant of G_S within the stabiliser groups $G_{\mathcal{U}}$ and $H_{\mathcal{U}}$, in $E_{6(6)}$ and $USp\,8/\mathbb{Z}_2$ respectively, of the space $\mathcal{U}$ of J_A that define the hypermultiplet moduli. One finds

$$G_{\mathcal{U}} = SL(3,\mathbb{C}) \subset E_{6(6)}\,,$$

with compact subgroup

$$H_{\mathcal{U}} = SU(3) \subset USp(8)/\mathbb{Z}_2\,.$$

The scalar manifold of the vector/tensor multiplets is then

$$\mathcal{M}_{\rm VT} = \frac{\mathrm{C}_{G_{\mathcal{U}}}(G_S)}{\mathrm{C}_{H_{\mathcal{U}}}(G_S)} = \frac{\mathrm{C}_{\mathrm{SL}(3,\mathbb{C})}(G_S)}{\mathrm{C}_{SU(3)}(G_S)}\,. \tag{III.41}$$

We thus find the two following possible truncations.

- $\mathbf{n_{VT}=1, n_H=1}$: Consider first the structure group $G_S = SU(2)\times U(1)$.

  The $\mathbf{27}^*$ of $E_{6(6)}$ contains two G_S singlets so that $\mathcal{V}$ is two-dimensional and $n_{\rm VT}=1$. Thus, the scalar manifold is

$$\mathcal{M}_{\rm VT} = \mathbb{R}^+\,, \qquad \mathcal{M}_{\rm H} = \frac{SU(2,1)}{\mathrm{S}(U(2)\times U(1))}\,.$$

The decomposition of the adjoint of $E_{6(6)}$ gives four compact and five non-compact G_S singlets that are the generators of the isometry group

$$G_{\rm iso} = \mathbb{R}^+ \times SU(2,1)\,.$$

- $\mathbf{n}_{\rm VT} = \mathbf{4}, \mathbf{n}_{\rm H} = \mathbf{1}$: Keeping only $G_S = U(1) \subset SU(2) \times U(1)$ as structure group the $\mathbf{27}^*$ contains five G_S singlets so that $\mathcal{V}$ is five-dimensional and $n_{\rm VT} = 4$. The commutators of $G_S = U(1)$ in $\mathrm{SL}(3,\mathbb{C})$ and $SU(3)$

$$\begin{aligned} \mathrm{C}_{\mathrm{SL}(3,\mathbb{C})}(U(1)) &= \mathrm{SL}(2,\mathbb{C}) \times U(1) \times \mathbb{R}^+\,, \\ \mathrm{C}_{SU(3)}(U(1)) &= SU(2) \times U(1)\,. \end{aligned}$$

and hence, from (III.41), the scalar manifold is

$$\mathcal{M}_{\rm VT} = \frac{SO(3,1)}{SO(3)} \times \mathbb{R}^+\,, \qquad \mathcal{M}_{\rm H} = \frac{SU(2,1)}{\mathrm{S}(U(2) \times U(1))}\,.$$

The adjoint of $E_{6(6)}$ contains seven compact and seven non-compact G_S singlet elements corresponding to the isometry group

$$G_{\rm iso} = SO(3,1) \times \mathbb{R}^+ \times SU(2,1)\,.$$

The analysis of the gauging of the vector/tensor multiplet isometries is the same as for the $n_{\rm VT} = 4$ generic case without hypermultiplets, so that the possible gauge groups are $SO(2,1)$, $SO(3)$, $ISO(2)$, when there are no tensor multiplets, and $SO(2)$ or $SO(1,1)$ with tensor multiplets. In Table III.4 we give the list of possible gauging for truncations with vector/tensor and hypermultiplets. For simplicity we give a list of product groups, but the individual factors can also be gauged separately. $G_{\rm gauge}$, the $U(1)_R$ can also be gauged diagonally with some combination of these factors.

$n_{\rm VT}$	$n_{\rm H}$	$G_{\rm iso}$	$G_{\rm gauge}$	$n_{\rm T}$
1	1	$SU(2,1) \times SO(1,1) \times \mathbb{R}^+$	$U(1)_R \times \mathbb{R}^+$	–
4	1	$SU(2,1) \times SO(3,1) \times \mathbb{R}^+$	$SO(2,1) \times \mathbb{R}^+ \times U(1)_R$, $SO(3) \times \mathbb{R}^+ \times U(1)_R$, $\mathrm{ISO}(2) \times \mathbb{R}^+ \times U(1)_R$, $SU(2)_R \times \mathbb{R}^+$	–
			$SO(2) \times U(1)_R \times \mathbb{R}^+$, $SO(1,1) \times U(1)_R \times \mathbb{R}^+$	2
			$SO(1,1)$	4

Table III.4: Summary of the gauge groups in the mixed cases. The first column gives the total number of vectors and tensor multiplets, the second the global isometry group, the third the allowed gaugings and the last one the number of vectors that are dualised to tensors in each case.

Chapter IV

Examples

In this chapter we will show that some of the truncations we classified in the previous section are actually geometrically realised in the sense that we can find a geometry that admit a certain structure group with a constant singlet intrinsic torsion.

We will derive two consistent truncations corresponding to two different $U(1)$ structure group. The first one gives four vector multiplets and one hypermultipet and contains the $\mathcal{N} = 2$ Maldacena-Nunez solution, while the second one will give two vector multiplets and one hypermultipet and contains the Bah, Beem, Bobev, Wecht (BBBW) solution.

The second example is a consistent truncation that contains a more general class of warpped M5-branes the Bah, Beem, Bobev, Wecht solutions.

IV.1 Truncation containing Maldacena Nunez solution

The Maldacena-Nuñez [31] correspond to warped AdS_5 backgrounds of eleven-dimensional supergravity that describe the near-horizon region of M5-branes wrapping a a Riemann surface $\mathbb{\Sigma}$ (of negative constant curvature) in a Calabi–Yau geometry.

The world volume theory is topologically twisted in order to preserve supersymmetry. This amounts in requiring that the spin connection on $\mathbb{\Sigma}$ is identified with a $U(1)$ connection the $SO(5)$ R-symmetry group of the M5-brane theory. The theory preserves $\mathcal{N} = 2$ or $\mathcal{N} = 1$ superconformal symmetry in four dimensions, depending on how the $U(1)$ is chosen inside $SO(5)$.

The corresponding supergravity solutions are warped products of AdS_5 times a six-dimensional manifold, M, which is the fibration of a deformed S^4 over $\mathbb{\Sigma}$. The $SO(5)$ is realised via the action of the isometry group of the round S^4. The structure of the fibration reflects the twist of the world-volume theory and determines the amount of supersymmetry of the solutions, which in five-dimensional language is either $\mathcal{N} = 4$ or $\mathcal{N} = 2$, respectively. In this chapter we will focus on the $\mathcal{N} = 2$ Maldacena Nunez solution, which we will call "MN1". The derivation of the largest consistent truncation containing the $\mathcal{N} = 4$ MN solution is worked out in Section 5 of [1].

The eleven-dimensional metric of MN1 is [1]

$$\hat{g} \;=\; \mathrm{e}^{2\Delta}\, g_{\mathrm{AdS}_5} + g_6\,,$$

where g_{AdS_5} is the Anti de Sitter metric with radius $\ell = \frac{3}{2}R$, R being the length scale of the internal space M, the warp factor is

$$\mathrm{e}^{2\Delta} \;=\; \left(\frac{2}{3}\right)^{2/3} (3+\cos^2\zeta)^{1/3}\,.$$

The metric on M takes the form

$$g_6 \;=\; R^2\, \frac{3^{1/3}}{2^{4/3}} \left(3+\cos^2\zeta\right)^{1/3} \left[g_{\mathbb{\Sigma}} + \mathrm{d}\zeta^2 + \frac{\sin^2\zeta}{3+\cos^2\zeta} \left(\sigma_1^2+\sigma_2^2+(\sigma_3+\upsilon)^2\right)\right]\,.$$

Here, $g_{\mathbb{\Sigma}}$ is the uniform metric on (a quotient of) the hyperbolic plane $\mathbb{\Sigma} = H^2$, with Ricci scalar curvature $\mathcal{R}_{\mathbb{\Sigma}} = -2$, while υ is the spin connection on $\mathbb{\Sigma}$ satisfying

$$\mathrm{d}\upsilon = -\,\mathrm{vol}_{\mathbb{\Sigma}}\,,$$

with $\mathrm{vol}_{\mathbb{\Sigma}}$ the volume form on $\mathbb{\Sigma}$.[2] The deformed S^4 is described as a foliation of a round S^3 over an interval, with the interval coordinate being $\zeta \in [0,\pi]$, while σ_α, $\alpha = 1,2,3$, are the standard $SU(2)_{\mathrm{left}}$-invariant forms on S^3, expressed in terms of Euler angles $\{\theta,\phi,\psi\}$. Their explicit expression can be found in Appendix D of [2], together with more details on the parameterisation of S^4.

The four-form reads

$$\begin{aligned}\hat{F} \;=\; \frac{R^3}{4}\Bigg[& \frac{15+\cos^2\zeta}{(3+\cos^2\zeta)^2}\sin^3\zeta\,\mathrm{d}\zeta\wedge\sigma_1\wedge\sigma_2\wedge(\sigma_3+\upsilon) \\ & + \sin\zeta\left(-\mathrm{d}\zeta\wedge\sigma_3 + \frac{\sin(2\zeta)}{3+\cos^2\zeta}\,\sigma_1\wedge\sigma_2\right)\wedge\mathrm{vol}_{\mathbb{\Sigma}}\Bigg]\,.\end{aligned} \tag{IV.1}$$

Note that the invariant volume form (III.10) is given by

$$\boldsymbol{\kappa}^2 = R^2\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{vol}_4\,, \tag{IV.2}$$

where vol_4 is the volume form of the round S^4 of radius R.

The solution has $SU(2)_{\mathrm{left}}\times U(1)_{\mathrm{right}}$ symmetry, which embeds in the $SO(5)$ isometry group of a round S^4 as

$$SO(5) \;\supset\; SO(4) \;\simeq\; SU(2)_{\mathrm{left}}\times SU(2)_{\mathrm{right}} \;\supset\; SU(2)_{\mathrm{left}}\times U(1)_{\mathrm{right}}\,. \tag{IV.3}$$

[1] We present the solution in a form similar to the one given in [59, Sect. 5]. The precise dictionary with this reference is: $\alpha=\zeta$, $\nu=-\phi$, $\psi_{\mathrm{GMSW}}=\psi$, $\mathrm{e}^{2\lambda}=\mathrm{e}^{2\Delta}$, $m^{-1}=\ell_{\mathrm{AdS}_5}=\frac{3}{2}R$, where the variables on the left-hand side are those of [59] while the variables on the right-hand side are those used here. The length scale R that appears in our expressions is equal to the radius of S^4 in the related $\mathrm{AdS}_7\times S^4$ Freund-Rubin solution of eleven-dimensional supergravity. The four-form $\hat{F}$ in (IV.1) has an overall opposite sign with respect to the one of [59], $\hat{F}=-F_{\mathrm{GMSW}}$; this sign does not affect the equations of motion, it just modifies the projection condition satisfied by the supersymmetry spinor parameter.

[2] Choosing local coordinates x,y on the hyperbolic plane, one can write $g_{\mathbb{\Sigma}}=\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$, $\mathrm{vol}_{\mathbb{\Sigma}}=\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}$, and $\upsilon=-\frac{\mathrm{d}x}{y}$.

This symmetry is manifest as the solution is given in terms of the σ_α. The globally-defined combination $(\sigma_3 + \upsilon)$ describes a fibration of S^4 over $\mathbb{\Sigma}$, such that the $U(1)_{\text{right}}$ action on S^4 is used to cancel the $U(1)$ holonomy of $\mathbb{\Sigma}$.

The $U(1)_{\text{right}}$ factor provides the R-symmetry of the holographically dual $\mathcal{N} = 1$ SCFT, while $SU(2)_{\text{left}}$ corresponds to a flavour symmetry. The dual $\mathcal{N} = 1$ SCFT has been described in [60].

IV.1.1 Generalised $U(1)$ structure of the MN1 solution

The MN1 solution admits a generalised $U(1)_S$ structure, which will be the basis for constructing the consistent truncation. In order to characterise it we proceed in two steps. The first is purely group theoretical: it consists in embedding the relevant $U(1)_S$ in $E_{6(6)}$, computing its commutant and the corresponding decompositions of the generalised tangent and adjoint bundles. To this end, it is convenient to decompose $E_{6(6)}$ according to its maximal compact subgroup $USp(8)/\mathbb{Z}_2$. Since the $\mathfrak{usp}(8)$ algebra can be given in terms of Cliff(6) gamma matrices (see Appendix E of [2]), this reduces the problem to gamma matrix algebra. The details of the derivation can be found in Appendix E of [2]; here we will just give the results. Once the relevant $U(1)_S$ singlets are identified, the second step is to express them in terms of the geometry of the six-dimensional manifold M.

The generalised $U(1)_S$ structure of the MN1 solution is the diagonal of the ordinary geometrical $U(1) \simeq SO(2) \subset GL(2,\mathbb{R})$ structure on the Riemann surface and a $U(1)$ factor in the $SO(5) \subset \mathrm{SL}(5,\mathbb{R}) \simeq E_{4(4)}$ generalised structure for the generalised tangent space of the four-sphere. In terms of the isometry group decomposition (IV.3) this can be identified with $U(1)_{\text{right}}$. If we denote by 1 to 4 the directions in M along S^4 and by 5,6 those along $\mathbb{\Sigma}$, the generator of $U(1)_S$ can be written as a $\mathfrak{usp}(8)$ element as

$$\mathfrak{u}(1)_S = \mathrm{i}\,\hat{\Gamma}_{56} - \frac{\mathrm{i}}{2}(\hat{\Gamma}_{12} - \hat{\Gamma}_{34})\,, \tag{IV.4}$$

where $\hat{\Gamma}_m$ are six-dimensional gamma matrices. The first term corresponds to the $U(1)$ holonomy of $\mathbb{\Sigma}$ while the second one is the $U(1)_{\text{right}}$ in $SO(5)$. By computing the commutators of (IV.4) in $USp(8)$ we find that the $U(1)_S$ structure embeds in $USp(8)$ as[3]

$$USp(8) \supset SU(2) \times SU(2)_H \times U(1) \times U(1)_S\,,$$

where as above we distinguish the factor $SU(2)_H$ that gives the R-symmetry of the five-dimensional supergravity theory. Under this splitting, the spinorial representation of $USp(8)$ decomposes as

$$\mathbf{8} = (\mathbf{1},\mathbf{2})_0 \oplus (\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}\,,$$

where the elements in the brackets denote the $SU(2) \times SU(2)_H$ representations and the subscript gives the $U(1)_S$ charge. We then see that there are only two spinors that are singlets under $U(1)_S$ and that transform as a doublet of $SU(2)_H$ as required by $\mathcal{N} = 2$ supersymmetry.

[3]Here and below we give expressions ignoring subtleties involving the centres of each group; thus for instance we will not distinguish between embeddings in $USp(8)$ and $USp(8)/\mathbb{Z}_2$.

The embedding of the $U(1)_S$ structure in the full $E_{6(6)}$ is obtained in a similar way (see Appendix E of [2] for details)

$$E_{6(6)} \supset \mathrm{C}_{E_{6(6)}}(U(1)_S) = \mathbb{R}^+ \times \mathrm{Spin}(3,1) \times SU(2,1) \times U(1)_S\,, \tag{IV.5}$$

where $\mathrm{C}_{E_{6(6)}}(U(1)_S)$ is the commutant of $U(1)_S$ in $E_{6(6)}$. We can now determine how many generalised vectors and adjoint elements are $U(1)_S$ singlets. Under (IV.5) the **27** decomposes as

$$\begin{aligned}\mathbf{27} \;=\; & (\mathbf{1},\mathbf{1})_{(0,8)} \oplus (\mathbf{4},\mathbf{1})_{(0,-4)} \oplus (\mathbf{2},\mathbf{1})_{(3,-2)} \oplus (\bar{\mathbf{2}},\mathbf{1})_{(-3,-2)} \\ & \oplus (\mathbf{1},\mathbf{3})_{(2,-4)} \oplus (\mathbf{1},\bar{\mathbf{3}})_{(-2,-4)} \oplus (\bar{\mathbf{2}},\mathbf{3})_{(1,2)} \oplus (\mathbf{2},\bar{\mathbf{3}})_{(-1,2)}\,,\end{aligned} \tag{IV.6}$$

where the first subscript denotes the $U(1)_S$ charge and the second one the $\mathbb{R}^+$ charge. We see that there are five singlets K_I, $I = 0, 1, \ldots, 4$, where

$$K_0 \in (\mathbf{1},\mathbf{1})_{(0,8)}$$

is only charged under the $\mathbb{R}^+$, while

$$\{K_1, K_2, K_3, K_4\} \in (\mathbf{4},\mathbf{1})_{(0,-4)}$$

form a vector of $SO(3,1)$.

The singlets in the **78** adjoint representation are the generators of the commutant $C_{E_{6(6)}}(U(1)_S)$. However only the generators of the $SU(2,1)$ subgroup are relevant for the structure. Indeed, (IV.6) shows that the generators of $\mathbb{R}^+ \times SO(3,1)$ do not leave the singlet vectors invariant, and therefore do not contribute to the truncation. We denote by J_A, $A = 1, \ldots, 8$, the elements of the adjoint bundle generating $\mathfrak{su}_{2,1}$. Four of them are in the **36** of $USp(8)$ and generate the compact subalgebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1$, and four more are in the **42** of $USp(8)$ and generate the rest of $\mathfrak{su}_{2,1}$.

The $U(1)_S$ structure is then defined by

$$\{K_I, J_A\}\,, \qquad I = 0, \ldots, 4\,, \quad A = 1, \ldots, 8\,.$$

The derivation of the explicit expressions for these generalised tensors relies on the way the solution of [31] is constructed by deforming the $\mathrm{AdS}_7 \times S^4$ background dual to flat M5-branes so as to describe their backreaction when wrapping a Riemann surface $\mathbb{\Sigma}$. The world-volume theory on the wrapped M5-branes is made supersymmetric by a topological twist, where the spin connection on the Riemann surface is cancelled by switching on a background gauge field for a $U(1)$ subgroup of the $SO(5)$ R-symmetry. On the dual background the topological twist implies that M is an S^4 fibration over $\mathbb{\Sigma}$

$$\begin{array}{ccc} S^4 & \xrightarrow{\;i\;} & M \\ & & \big\downarrow{\scriptstyle \pi} \\ & & \mathbb{\Sigma} \end{array} \tag{IV.7}$$

The generalised tangent bundle for S^4 is given by

$$E_4 \;\simeq\; TS^4 \oplus \Lambda^2 T^* S^4\,, \tag{IV.8}$$

and transforms under $\mathrm{SL}(5,\mathbb{R}) \simeq E_{4(4)}$. It is generalised parallelisable, meaning it admits a globally defined frame [12]. The idea is then to consider first the direct product $\mathbb{\Sigma} \times S^4$, express the $E_{6(6)}$ generalised tensors on this manifold in terms of the frame on $\mathbb{\Sigma}$ and the parallelisation on S^4, and then implement the twist of S^4 over $\mathbb{\Sigma}$ so as to make globally well-defined objects. In the decomposition

$$E_{6(6)} \supset GL(2,\mathbb{R}) \times \mathrm{SL}(5,\mathbb{R})\,,$$

where $GL(2,\mathbb{R})$ is the structure group of the conventional tangent bundle on $\mathbb{\Sigma}$ and $\mathrm{SL}(5,\mathbb{R}) \simeq E_{4(4)}$ is the structure group of the generalised tangent bundle on S^4, the $E_{6(6)}$ generalised tangent bundle on $\mathbb{\Sigma} \times S^4$ decomposes as

$$E \simeq T\mathbb{\Sigma} \oplus (T^*\mathbb{\Sigma} \otimes N_4) \oplus (\Lambda^2 T^*\mathbb{\Sigma} \otimes N_4') \oplus E_4\,, \tag{IV.9}$$

and the adjoint bundle as

$$\begin{aligned} \mathrm{ad}F \simeq\ & \mathrm{ad}F_4 \oplus (T\Sigma \otimes T^*\Sigma) \oplus (T^*\Sigma \otimes E_4) \\ & \oplus (\Lambda^2 T^*\Sigma \otimes N_4) \oplus (T\Sigma \otimes E_4^*) \oplus (\Lambda^2 T\Sigma \otimes N_4^*)\,. \end{aligned} \tag{IV.10}$$

In the expressions above E_4 is the generalised tangent bundle on S^4 introduced in (IV.8), $\mathrm{ad}\,F_4$ is the adjoint bundle on S^4,

$$\mathrm{ad}\,F_4 \simeq \mathbb{R} \oplus (TS^4 \otimes T^*S^4) \oplus \Lambda^3 T^*S^4 \oplus \Lambda^3 TS^4\,,$$

and N_4 and N_4' are the following bundles on S^4,

$$\begin{aligned} N_4 &\simeq T^*S^4 \oplus \Lambda^4 T^*S^4\,, \\ N_4' &\simeq \mathbb{R} \oplus \Lambda^3 T^*S^4\,. \end{aligned}$$

The bundles E_4, N_4 and N_4' admit the globally defined generalised frames

$$E_{ij} \in \Gamma(E_4)\,, \qquad E_i \in \Gamma(N_4)\,, \qquad E_i' \in \Gamma(N_4')\,, \qquad i,j = 1,\ldots,5\,,$$

see Appendix D of [2] for their expression in a coordinate basis and note that they include a contribution from the three-form gauge potential A_{S^4} of the flux on the S^4. Geometrically this defines a generalised identity structure on S^4. Given the way $U(1)_S$ is embedded in $USp(8)$, we will find it useful to also introduce the following linear combinations of the generalised frame elements E_{ij} on S^4,

$$\begin{aligned} &\Xi_1 = E_{13} + E_{24}\,, \quad \Xi_2 = E_{14} - E_{23}\,, \quad \Xi_3 = E_{12} - E_{34}\,, \\ &\widetilde{\Xi}_1 = E_{13} - E_{24}\,, \quad \widetilde{\Xi}_2 = E_{14} + E_{23}\,, \quad \widetilde{\Xi}_3 = E_{12} + E_{34}\,. \end{aligned}$$

Since their restriction to TM corresponds to the Killing vectors generating the $SU(2)_{\text{left}} \times SU(2)_{\text{right}} \simeq SO(4) \subset SO(5)$ isometries of S^4 (see Appendix D of [2] for their explicit expression), Ξ_α and $\widetilde{\Xi}_\alpha$, $\alpha = 1,2,3$, may be seen as generalised Killing vectors generating the corresponding generalised isometries.

As for the Riemann surface $\mathbb{\Sigma}$, it can be (a quotient of) the hyperbolic plane H^2 as in the MN1 solution reviewed earlier, but we can also take a torus T^2, or a sphere S^2. We

introduce orthonormal co-frame one-forms e_1, e_2 on $\mathbb{\Sigma}$, such that the constant curvature metric and the compatible volume form on $\mathbb{\Sigma}$ are given by

$$g_{\mathbb{\Sigma}} = (e_1)^2 + (e_2)^2\,, \qquad \text{vol}_{\mathbb{\Sigma}} = e_1 \wedge e_2\,. \tag{IV.11}$$

The metric is normalised so that the Ricci scalar curvature is $\mathcal{R}_{\mathbb{\Sigma}} = 2\kappa$, where $\kappa = +1$ for S^2, $\kappa = 0$ for T^2 and $\kappa = -1$ for H^2 (and quotients thereof). We also define the $U(1)$ spin connection, υ, on $\mathbb{\Sigma}$ as

$$\mathrm{d}(e_1 + \mathrm{i}\, e_2) \,=\, \mathrm{i}\,\upsilon \wedge (e_1 + \mathrm{i}\, e_2)\,, \qquad \mathrm{d}\upsilon \,=\, \kappa\ \text{vol}_{\mathbb{\Sigma}}\ . \tag{IV.12}$$

The decompositions (IV.9) and (IV.10) allow us to express the $U(1)_S$ invariant generalised tensors in terms of tensors on $\mathbb{\Sigma}$ and the S^4 generalised frames introduced above. We provide the derivation in Appendix D of [2] and here just present the resulting expressions. Let us first focus on the singlet generalised vectors K_I. These can be written as

$$K_0 \,=\, \mathrm{e}^{\Upsilon} \cdot (R^2\,\text{vol}_{\mathbb{\Sigma}} \wedge\, E_5')\,, \qquad K_{1,2,3} \,=\, \mathrm{e}^{\Upsilon} \cdot \widetilde{\Xi}_{1,2,3}\,, \qquad K_4 \,=\, \mathrm{e}^{\Upsilon} \cdot \Xi_3\,, \tag{IV.13}$$

where Υ is a section of the adjoint bundle implementing the twist of S^4 over $\mathbb{\Sigma}$ as in (IV.7), ensuring that these are globally defined objects on the six-dimensional manifold. Recall that in the MN1 solution, the $U(1)$ that is used to twist the four-sphere and compensate the spin connection υ on $\mathbb{\Sigma}$ is the Cartan of $SU(2)_{\text{right}} \subset SO(5)$. The $E_{6(6)}$ twist element Υ is constructed in a way similar to the one used in [1], albeit with a different choice of $U(1)$ in $SO(5)$. We embed the connection one-form υ in a generalised dual vector, the Killing vector generating the Cartan of $SU(2)_{\text{right}}$ in the generalised vector Ξ_3 introduced above, and we project their product onto the adjoint of $E_{6(6)}$. That is,

$$\Upsilon \,=\, -\frac{R}{2}\,\upsilon \times_{\text{ad}} \Xi_3\,,$$

where $\times_{\text{ad}}$ denotes the projection onto the adjoint and again R is the radius of S^4. Evaluating its action in (IV.13), we find that this is trivial for all the K_I's except for K_4, and we obtain our final expressions

$$K_0 \,=\, R^2\,\text{vol}_{\mathbb{\Sigma}} \wedge\, E_5'\,, \qquad K_{1,2,3} \,=\, \widetilde{\Xi}_{1,2,3}\,, \qquad K_4 \,=\, \Xi_3 - R\,\upsilon \wedge E_5\,. \tag{IV.14}$$

A similar procedure applies to the singlets J_A, $A = 1, \ldots, 8$, in the adjoint bundle. In

this way we obtain (see Appendix E of [2] for the derivation)

$$
\begin{aligned}
J_1 &= \tfrac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(-R\,e_1\times_{\rm ad}\Xi_1 - R\,e_2\times_{\rm ad}\Xi_2 + R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_1 + R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_2\big),\\
J_2 &= \tfrac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(R\,e_1\times_{\rm ad}\Xi_2 - R\,e_2\times_{\rm ad}\Xi_1 - R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_1 + R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_3 &= \tfrac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(\hat{e}_1\otimes e_2 - \hat{e}_2\otimes e_1 - R\,e_2\times_{\rm ad}\Psi_{15} + R^{-1}\,\Psi_{15}^*\times_{\rm ad}\hat{e}_2\\
&\qquad - E^*_{5[1}\times_{\rm ad}E_{2]5} + E^*_{5[3}\times_{\rm ad}E_{4]5}\big)\,,\\
J_4 &= \tfrac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(R\,e_1\times_{\rm ad}\Xi_2 - R\,e_2\times_{\rm ad}\Xi_1 + R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_1 - R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_5 &= \tfrac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(R\,e_1\times_{\rm ad}\Xi_1 + R\,e_2\times_{\rm ad}\Xi_2 + R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_1 + R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_6 &= -\tfrac{1}{3}\,\mathrm{e}^{\Upsilon}\cdot\big(\hat{e}_1\otimes e_1 + \hat{e}_2\otimes e_2 + \sum_{i=1}^{4}E_{i5}^*\times_{\rm ad}E_{i5} + 2\big)\,,\\
J_7 &= \mathrm{e}^{\Upsilon}\cdot\big(R\,e_2\times_{\rm ad}\Psi_{15} + R^{-1}\,\Psi_{15}^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_8 &= \tfrac{1}{2\sqrt{3}}\,\mathrm{e}^{\Upsilon}\cdot\big(\hat{e}_1\otimes e_2 - \hat{e}_2\otimes e_1 - 3R\,e_2\times_{\rm ad}\Psi_{15} + 3R^{-1}\,\Psi_{15}^*\times_{\rm ad}\hat{e}_2\\
&\qquad - E^*_{5[1}\times_{\rm ad}E_{2]5} + E^*_{5[3}\times_{\rm ad}E_{4]5}\big)\,,
\end{aligned}
\tag{IV.15}
$$

where the superscript $*$ denotes dual generalised vectors, transforming in the $\overline{\mathbf{27}}$, and we introduced $\Psi_{1i} = R\,e_1\wedge E_i$ and $\Psi_{2i} = R\,e_2\wedge E_i$. The adjoint action of e^{Υ} is evaluated using the formula (III.4); we do not show the resulting expressions as they are rather lengthy. Evaluating the commutators $[J_A, J_B]$ using again (III.6), we checked that the J_A satisfy precisely the $SU(2,1)$ commutation relations (the choice of $SU(2,1)$ structure constants is given in the Appendix F of [2]).

IV.1.2 The V and H structure moduli spaces

We now construct the V structure and H structure moduli spaces. Applying the general discussion of Section III.2 we have

$$
\mathcal{M}_{\rm V}\times\mathcal{M}_{\rm H} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)} = \mathbb{R}^+\times\frac{\mathrm{Spin}(3,1)}{SU(2)}\times\frac{SU(2,1)}{SU(2)_H\times U(1)}\,,
\tag{IV.16}
$$

As we now show the first two factors give the V structure moduli space and the last factor the H structure moduli space.

The V structure

Evaluating (III.21) for the K_I constructed above we obtain the constant, symmetric tensor C_{IJK}. Using the invariant volume (IV.2), we find that the non-vanishing components of C_{IJK} are given by

$$
C_{0IJ} = C_{I0J} = C_{IJ0} = \tfrac{1}{3}\,\eta_{IJ}\,,\qquad \text{for } I,J = 1,\ldots,4\,,
$$

where

$$
\eta = \mathrm{diag}(-1,-1,-1,1)\,.
$$

A family of V structures is then obtained by defining K as the linear combination

$$K = h^{\tilde{I}} K_{\tilde{I}}\,,$$

where h^I, $I = 0, \ldots, n_{\rm V}$, are real parameters spanning $\mathcal{M}_{\rm V}$. Fixing $\boldsymbol{\kappa}$ and imposing (III.21) it follows that our V structure moduli space is the hypersurface

$$C_{IJK} h^I h^J h^K \;=\; h^0 \left(-(h^1)^2 - (h^2)^2 - (h^3)^2 + (h^4)^2\right) \;=\; 1\,. \tag{IV.17}$$

It will be convenient to redefine the h^I in terms of the parameters

$$\{\Sigma, H^1, H^2, H^3, H^4\}$$

as

$$\begin{aligned} h^0 &= \Sigma^{-2}\,, \\ h^I &= -\Sigma\, H^I\,, \quad I = 1, \ldots, 4\,, \end{aligned} \tag{IV.18}$$

so that

$$K \;=\; \Sigma^{-2} K_0 - \Sigma \left(H^1 K_1 + H^2 K_2 + H^3 K_3 + H^4 K_4\right)\,.$$

From (IV.17) we see that H^I are coordinates on the unit hyperboloid $\frac{SO(3,1)}{SO(3)}$,

$$-(H^1)^2 - (H^2)^2 - (H^3)^2 + (H^4)^2 = 1\,,$$

while Σ (that we assume strictly positive) is a coordinate on $\mathbb{R}^+$, whose powers in (IV.18) are dictated by the weight of the K_I's under the action of the $\mathbb{R}^+$ that commutes with the generalised structure.

The resulting V structure moduli space thus is

$$\mathcal{M}_{\rm V} \;=\; \mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)}\,,$$

and will determine $n_{\rm V} = 4$ vector multiplets in five-dimensional $\mathcal{N} = 2$ supergravity. Note that by identifying $SU(2) \simeq \mathrm{Spin}(3)$ this matches the first two factors in (IV.16). The isometry group is $SO(3,1)$ because the h^I form a vector rather than a spinor representation of $\mathrm{Spin}(3,1)$.

The H structure

We next turn to the H structure moduli space, again following the general discussion given in Section III.2. Since the commutant of $SU(2)_H$ in $SU(2,1)$ is $U(1)$, from (III.22) we obtain that the H structure moduli space is[4]

$$\mathcal{M}_{\rm H} = \frac{SU(2,1)}{SU(2)_H \times U(1)}\,. \tag{IV.19}$$

[4] More precisely one has $\mathcal{M}_{\rm H} = SU(2,1)/S(U(2) \times U(1))$.

This is a simple quaternionic-Kähler manifold of quaternionic dimension $n_{\rm H} = 1$. We will denote by

$$\{\varphi, \xi, \theta_1, \theta_2\}$$

the coordinates on this space. In Appendix F of [2] we give the explicit parameterisation chosen for the coset space as well as the explicit form of the "dressed" $\mathfrak{su}(2)$ elements J_α, depending on $\{\varphi, \xi, \theta_1, \theta_2\}$, in terms of $\mathfrak{su}(2,1)$ elements. Below we will use this dressed triplet to construct the generalised metric. In Appendix F of [2] we also give the $SU(2,1)$ invariant metric on $\mathcal{M}_H$, which will provide the hyperscalar kinetic term in the five-dimensional theory.

IV.1.3 Intrinsic torsion and gauging

For the $U(1)_S$ structure constructed in the previous section to lead to a consistent truncation, it must be checked that its intrinsic torsion only contains $U(1)_S$ singlets, and that these are constant. In particular we need to show that the generalised Lie derivatives closes on the singlets of the truncation (II.8), on the V structure and on the H structure. For the two conditions we evaluate the generalised Lie derivatives of the tensors K_I and J_A in (IV.14) and (IV.15), using the action of generalised Lie derivative on a generalised vector and on sections of the adjoint bundle given in Appendix A of [2].

Consider first the algebra of the generalised vectors (IV.14). Using the fact that, under the generalised Lie derivative, the S^4 frames E_{ij} generate an $\mathfrak{so}(5)$ algebra

$$L_{E_{ij}} E_{kl} = -R^{-1} \left(\delta_{ik} E_{jl} - \delta_{il} E_{jk} + \delta_{jl} E_{ik} - \delta_{jk} E_{il}\right) ,$$

one can show that the only non-zero Lie derivatives are

$$L_{K_\alpha} K_\beta \;=\; -\tfrac{2}{R}\, \epsilon_{\alpha\beta\gamma} K_\gamma \,, \qquad\qquad \alpha, \beta, \gamma = 1, 2, 3 \,, \tag{IV.20}$$

so that the vectors K_α, $\alpha = 1, 2, 3$, lead to an $SO(3)$ factor in the gauge group in the five-dimensional supergravity.[5] This embeds in the $SO(3,1)$ factor of the global symmetry group of the ungauged theory in the obvious way. Hence (IV.20) determines the components of the embedding tensor acting on the vector multiplet sector of the five-dimensional supergravity theory.

We thus have that the non-vanishing structure constants are $f_{\alpha\beta\gamma} = -2\,\epsilon_{\alpha\beta\gamma}$ and the gauge coupling constant is $g = \frac{1}{R}$. The non-trivial vector multiplet scalar covariant derivatives are

$$\mathcal{D} H^\alpha = \mathrm{d} H^\alpha - \tfrac{2}{R}\, \epsilon^\alpha{}_{\beta\gamma} \mathcal{A}^\beta\, H^\gamma \,,$$

while the gauge field strengths read

$$\mathcal{F}^0 = \mathrm{d}\mathcal{A}^0 \,, \qquad \mathcal{F}^\alpha = \mathrm{d}\mathcal{A}^\alpha - \tfrac{1}{R}\, \epsilon^\alpha{}_{\beta\gamma} \mathcal{A}^\beta \wedge \mathcal{A}^\gamma \,, \qquad \mathcal{F}^4 = \mathrm{d}\mathcal{A}^4 \,. \tag{IV.21}$$

In order to determine the gauging in the hypersector we also need to compute the Lie derivative of the J_A along the generalised vectors K_I. We find that the J_A are neutral under the action of the $SO(3)$ generators K_α,

$$L_{K_\alpha} J_A = 0 \,, \qquad A = 1, \ldots 8 \,,$$

[5] For simplicity, we use the indices $\alpha, \beta = 1, 2, 3$ both to label the generators of the $SU(2)_H$ entering in the definition of the H structure and the generators of the $SU(2)$ in the V structure, although these are different subgroups of $E_{6(6)}$.

consistently with the fact that the gauging in the vector multiplet sector does not affect the hypersector. On the other hand, the remaining generalised vectors K_0 and K_4 act non-trivially on the J_A, and determine an abelian gauging of the $SU(2,1)$ generators. In detail, the generalised Lie derivative of the J_A along K_0 gives

$$\begin{aligned} &L_{K_0}(J_1 - J_5) = 0\,, && L_{K_0} J_3 = -\tfrac{1}{2R} J_6\,, \\ &L_{K_0}(J_1 + J_5) = \tfrac{1}{R}(J_2 + J_4)\,, && L_{K_0} J_6 = -\tfrac{1}{2R}(J_3 + 2J_7 - \sqrt{3}J_8)\,, \\ &L_{K_0}(J_2 + J_4) = 0\,, && L_{K_0} J_7 = \tfrac{1}{R} J_6\,, \\ &L_{K_0}(J_2 - J_4) = -\tfrac{1}{R}(J_1 - J_5)\,, && L_{K_0} J_8 = \tfrac{\sqrt{3}}{2R} J_6\,, \end{aligned} \tag{IV.22}$$

while the one along K_4 yields

$$\begin{aligned} &L_{K_4}(J_1 - J_5) = -\tfrac{2}{R}(J_2 + J_4)\,, && L_{K_4} J_3 = \tfrac{\kappa}{2R} J_6\,, \\ &L_{K_4}(J_1 + J_5) = -\tfrac{2}{R}(J_2 - J_4) - \tfrac{\kappa}{R}(J_2 + J_4)\,, && L_{K_4} J_6 = \tfrac{\kappa}{2R}(J_3 + 2J_7 - \sqrt{3}J_8)\,, \\ &L_{K_4}(J_2 + J_4) = \tfrac{2}{R}(J_1 - J_5)\,, && L_{K_4} J_7 = -\tfrac{\kappa}{R} J_6\,, \\ &L_{K_4}(J_2 - J_4) = \tfrac{2}{R}(J_1 + J_5) + \tfrac{\kappa}{R}(J_1 - J_5)\,, && L_{K_4} J_8 = -\tfrac{\sqrt{3}\kappa}{2R} J_6\,. \end{aligned} \tag{IV.23}$$

The actions (IV.22) and (IV.23) can equivalently be expressed in terms of an adjoint action as

$$L_{K_0} J_A = [J_{(K_0)}, J_A]\,, \qquad L_{K_4} J_A = [J_{(K_4)}, J_A]\,, \qquad A = 1, \ldots, 8\,,$$

where the sections of the adjoint bundle

$$\begin{aligned} J_{(K_0)} &= \tfrac{1}{4R}\,(J_3 + 2J_7 - \sqrt{3}J_8)\,, \\ J_{(K_4)} &= -\tfrac{\kappa}{4R}\,(J_3 + 2J_7 - \sqrt{3}J_8) - \tfrac{1}{R}\,(J_3 + \tfrac{1}{\sqrt{3}}J_8) \end{aligned}$$

correspond to $SU(2,1)$ generators acting on the H-structure moduli space (IV.19) as isometries. We denote by k_0 and k_4 the corresponding Killing vectors on $\mathcal{M}_{\rm H}$. We refer to the Appendix B of [2] for this computation.

$$\begin{aligned} k_0 &= \partial_\xi\,, \\ k_4 &= -\kappa\,\partial_\xi + 2\,(\theta_2 \partial_{\theta_1} - \theta_1 \partial_{\theta_2})\,. \end{aligned} \tag{IV.24}$$

These Killing vectors specify the isometries of $\mathcal{M}_{\rm H}$ that are gauged in the five-dimensional supergravity. The hyperscalar covariant derivatives are determined as

$$\begin{aligned} \mathcal{D}(\theta_1 + \mathrm{i}\,\theta_2) &= \mathrm{d}(\theta_1 + \mathrm{i}\,\theta_2) - \tfrac{2}{R}\,\mathrm{i}\,\mathcal{A}^4\,(\theta_1 + \mathrm{i}\,\theta_2)\,, \\ \mathcal{D}\xi &= \mathrm{d}\xi + \tfrac{1}{R}\mathcal{A}^0 - \tfrac{\kappa}{R}\mathcal{A}^4\,. \end{aligned} \tag{IV.25}$$

The triholomorphic Killing prepotentials P_I^α obtained by evaluating the moment maps read

$$\begin{aligned} P_0^\alpha &= \{0\,,\, 0\,,\, \tfrac{1}{4}\,\mathrm{e}^{2\varphi}\}\,, \\ P_4^\alpha &= \{\sqrt{2}\,\mathrm{e}^{\varphi}\theta_1\,,\, \sqrt{2}\,\mathrm{e}^{\varphi}\theta_2\,,\, -1 + \tfrac{1}{4}\,\mathrm{e}^{2\varphi}(2\theta_1^2 + 2\theta_2^2 - \kappa)\}\,, \end{aligned}$$

with $P_1^\alpha = P_2^\alpha = P_3^\alpha = 0$.

The information above completely characterises the five-dimensional $\mathcal{N} = 2$ supergravity obtained upon reduction on M. This will be discussed in Section IV.1.5. Before coming to that, we provide the explicit bosonic truncation ansatz.

IV.1.4 The truncation ansatz

The bosonic part of the truncation ansatz is obtained by imposing that the generalised tensors are expanded in singlets of the G_S structure. The generalised metric is obtained by constructing the K and J_α parameterising a family of HV structures as detailed in Section III.2, and plugging these generalised tensors in the formula (IV.26). The resulting generalised metric depends on the H and V structure moduli; when given a dependence on the external coordinates x^μ, these are then identified with the hyperscalar and vector multiplet scalar fields of the truncated $\mathcal{N} = 2$ theory, respectively. Thus we have

$$\left.\begin{aligned} K &= h^I(x) K_I \\ J_\alpha &= L(x) j_\alpha L(x)^{-1} \end{aligned}\right\} \qquad \text{giving } G^{MN}(x) \text{ from (IV.27)}\,,$$

where L is the representative of the coset $\mathcal{M}_{\rm H}$. Given this HV structure, one can construct the generalised metric as

$$G(V,V) \;=\; 3\left(3\,\frac{c(K,K,V)^2}{c(K,K,K)^2} - 2\frac{c(K,V,V)}{c(K,K,K)} + 4\,\frac{c(K,J_3\cdot V, J_3\cdot V)}{c(K,K,K)}\right)\,, \tag{IV.26}$$

where c is the $E_{6(6)}$ cubic invariant and V is a generalised vector.

For the purpose of constructing the truncation ansatz by comparing with (III.11), we will also need the inverse generalised metric. We can exploit the isomorphism between the generalised tangent bundle E and its dual E^* provided by the generalised metric to construct a $USp(6)$ singlet $K^* \in \Gamma(E^*)$ as $K^*(V) = G(K,V)$, where V is any generalised vector. Then, denoting by $Z \in \Gamma(E^*)$ a generic dual vector, the inverse generalised metric is given by

$$G^{-1}(Z,Z) \;=\; 3\left(3\,\frac{c^*(K^*,K^*,Z)^2}{c^*(K^*,K^*,K^*)^2} - 2\frac{c^*(K^*,Z,Z)}{c^*(K^*,K^*,K^*)} + 4\,\frac{c^*(K^*,J_3\cdot Z, J_3\cdot Z)}{c^*(K^*,K^*,K^*)}\right)\,, \tag{IV.27}$$

the cubic invariant c^* and of the adjoint elements J_α on the dual generalised vectors have been defined in (III.3) and (III.5). Comparing the expression for the inverse generalised metric with its general form (III.9), we obtain the truncation ansatz for Δ, g_{mn}, A_{mnp} (as well as $\tilde{A}_{m_1\ldots m_6}$, whenever it is needed). Note that $\boldsymbol{\kappa}^2$ given in (III.10) is independent of the scalar fields $h^I(x)$ and $L(x)$, so it can be evaluated using any chosen representative of the family of HV structures defined by the G_S structure.

The gauge potentials $\mathcal{A}_\mu{}^I(x)$ on the external space-time are defined by taking

$$\mathcal{A}_\mu \;=\; \mathcal{A}_\mu{}^I(x)\, K_I \quad \in \Gamma(T^*X) \otimes \mathrm{span}(\{K_I\}) \tag{IV.28}$$

where $\mathrm{span}(\{K_I\}) \subset \Gamma(E)$ is the vector space spanned by the set of G_S singlets K_I, $I = 0, 1, \ldots, n_{\rm V}$. Similarly the two-form fields are given by

$$\mathcal{B}_{\mu\nu} \;=\; \mathcal{B}_{\mu\nu\,I}(x)\, K_\flat^I \quad \in \Gamma(\Lambda^2 T^*X) \otimes \mathrm{span}(\{K_\flat^I\})\,, \tag{IV.29}$$

where $\text{span}(\{K_\flat^I\}) \subset \Gamma(\det T^*M \otimes E^*)$ is the vector space spanned by the weighted dual basis vectors $K_\flat^I$, the latter being defined by $K_\flat^I(K_J) = 3\boldsymbol{\kappa}^2\,\delta^I{}_J$. We also have

$$\mathcal{C}_{\mu\nu\rho} \;=\; \mathcal{C}_{\mu\nu\rho}{}^A(x)\, J_A^\flat \quad \in \Gamma(\Lambda^3 T^*X) \otimes \text{span}(\{J_A^\flat\})\,, \tag{IV.30}$$

where $\text{span}(\{J_A^\flat\}) \subset \Gamma(\det T^*M \otimes \text{ad}(F))$ is spanned by the G_S singlets in the weighted adjoint bundle, here denoted by $J_A^\flat$ and given by $J_A^\flat = \boldsymbol{\kappa}^2 J_A$. In Appendix C of [2] we show that these expressions, together with the field redefinitions (III.8), lead to the correct five-dimensional covariant objects, consistent with the expected gauge transformations.

We compute the inverse generalised metric (IV.27) out of the $U(1)_S$ invariant generalised tensors. This depends on the V structure moduli $\{\Sigma, H^1, H^2, H^3, H^4\}$ and on the H structure moduli $\{\varphi, \xi, \theta_1, \theta_2\}$, which are now promoted to scalar fields in the external, five-dimensional spacetime. Then we evaluate the generalised tensors $\mathcal{A}_\mu, \mathcal{B}_{\mu\nu}, \mathcal{C}_{\mu\nu\rho}$ using (IV.28)–(IV.30). Separating the components of these tensors as described in Appendix C of [3], we obtain the ansatz for the eleven-dimensional metric $\hat{g}$ and three-form potential $\hat{A}$.

We start from the covariantised differentials $Dz^m \;=\; \mathrm{d}z^m - h_\mu{}^m \mathrm{d}x^\mu$ of the coordinates on M, that appear in (III.7). From (III.11) and (IV.28) we see that $h_\mu = h_\mu{}^m \partial_m$ is given by

$$h_\mu \;=\; \mathcal{A}_\mu^I\, K_I|_{TM}\,,$$

where $K_I|_{TM}$ is the restriction of K_I to the tangent bundle of M. Evaluating the right hand side using the explicit form (IV.14) of the generalised vectors K_I, we obtain

$$h_\mu \;=\; \tfrac{2}{R}\,(\mathcal{A}_\mu^\alpha\, \tilde{\xi}_\alpha + \mathcal{A}_\mu^4\, \xi_3)\,, \tag{IV.31}$$

where we recall that ξ_α, $\tilde{\xi}_\alpha$, $\alpha = 1,2,3$, are the pull-back to TM of the $SU(2)_{\text{left}}$- and $SU(2)_{\text{right}}$-invariant vectors on S^3, respectively, whose coordinate expression is given in the corresponding paper in (VI). It follows that Dz^m, and thus both the eleven-dimensional metric and three-form, contain the five-dimensional gauge potentials $\mathcal{A}^\alpha$, $\mathcal{A}^4$, gauging the $SU(2)_{\text{left}} \times U(1)_{\text{right}}$ isometries of S^3 in M. Notice that $\mathcal{A}^0$ does not appear in (IV.31) as K_0 does not have a component in TM, hence it will not enter in the eleven-dimensional metric. However K_0 will appear in the ansatz for the three-form, as it does have a component in $\Lambda^2 T^*M$.

In order to express our ansatz in a more compact way, it will be convenient to introduce new one-forms Ω_α and $\widetilde{\Omega}_\alpha$, $\alpha = 1,2,3$, adapted to the symmetries of the problem, that incorporate the covariantised differentials above but also include some more terms. Recall that we describe S^4 as a foliation of S^3, parameterised by Euler angles $\{\theta, \phi, \psi\}$, over an interval, parameterised by ζ. We define

$$\begin{aligned}
\Omega_1 &= \cos\psi\, \mathsf{D}\theta + \sin\psi \sin\theta\, \mathsf{D}\phi\,, & \widetilde{\Omega}_1 &= \cos\phi\, \mathsf{D}\theta + \sin\phi \sin\theta\, \mathsf{D}\psi\,,\\
\Omega_2 &= -\sin\psi\, \mathsf{D}\theta + \cos\psi \sin\theta\, \mathsf{D}\phi\,, & \widetilde{\Omega}_2 &= \sin\phi\, \mathsf{D}\theta - \cos\phi \sin\theta\, \mathsf{D}\psi\,,\\
\Omega_3 &= \mathsf{D}\psi + \cos\theta\, \mathsf{D}\phi\,, & \widetilde{\Omega}_3 &= \mathsf{D}\phi + \cos\theta\, \mathsf{D}\psi\,,
\end{aligned}$$

which are analogous to the left- and right-invariant forms $\sigma_\alpha, \tilde{\sigma}_\alpha$ given in (VI), but with the ordinary differential of the coordinates being replaced by the new covariantised differential D. This extends the differential D given above and is defined as

$$\mathsf{D}z^m = \mathrm{d}z^m - \tfrac{2}{R}(\mathsf{A}^\alpha \xi_\alpha^m + \tilde{\mathsf{A}}^\alpha \tilde{\xi}_\alpha^m)\,,$$

with

$$\begin{aligned}&\mathsf{A}^1 = \tfrac{R}{\sqrt{2}}\,(\theta_2 e_1 - \theta_1 e_2)\,, \qquad \mathsf{A}^2 = \tfrac{R}{\sqrt{2}}\,(\theta_1 e_1 + \theta_2 e_2)\,, \qquad \mathsf{A}^3 = -\tfrac{R}{2}\,\upsilon + \mathcal{A}^4\,,\\ &\tilde{\mathsf{A}}^\alpha = \mathcal{A}^\alpha\,, \qquad \alpha = 1,2,3\,,\end{aligned} \tag{IV.32}$$

where the five-dimensional scalars θ_1, θ_2 are two of the H structure moduli, and we recall that e_1, e_2 are the vielbeine on the Riemann surface $\mathbb{\Sigma}$ while υ is the connection on $\mathbb{\Sigma}$. The local one-forms $\tilde{\mathsf{A}}^\alpha, \mathsf{A}^\alpha$ gauge all the left- and right- isometries of S^3, respectively, and would correspond to $SO(4) \simeq SU(2)_{\text{left}} \times SU(2)_{\text{right}}$ gauge potentials in the reduction of eleven-dimensional supergravity on S^4 down to seven-dimensional supergravity. However, in the further reduction on $\mathbb{\Sigma}$ of interest here only $\mathcal{A}^\alpha, \mathcal{A}^4$ become five-dimensional gauge fields, while the rest of (IV.32) implements the twist on the Riemann surface and introduces the five-dimensional scalars θ_1, θ_2.

We are now in the position to give the truncation ansatz for the eleven-dimensional metric

$$\hat{g} \;=\; \mathrm{e}^{2\Delta} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dz^m Dz^n\,.$$

The warp factor is

$$\mathrm{e}^{2\Delta} \;=\; \bar{\Delta}^{1/3}\,(\mathrm{e}^{\varphi}\Sigma)^{4/5}\;,$$

while the part with at least one internal leg reads

$$\begin{aligned}g_{mn}Dz^m Dz^n = R^2\bar{\Delta}^{1/3}(\mathrm{e}^{\varphi}\Sigma)^{-6/5}\, g_{\mathbb{\Sigma}} + R^2\bar{\Delta}^{-2/3}\mathrm{e}^{2\varphi/5}\Sigma^{-3/5}\Big[\Big(\mathrm{e}^{-2\varphi}\Sigma^3\sin^2\zeta + H_-\cos^2\zeta\Big)\mathrm{d}\zeta^2\\ + \tfrac{1}{4}H_+\sin^2\zeta\,\delta^{\alpha\beta}\Omega_\alpha\otimes\Omega_\beta - \tfrac{1}{2}\sin^2\zeta\,H^\alpha\,\widetilde{\Omega}_\alpha\otimes_s\Omega_3 - \cos\zeta\sin\zeta\,\mathrm{d}\zeta\otimes_s \mathrm{d}_6 H_+\Big]\,,\end{aligned}$$

where $\otimes_s$ is the symmetrised tensor product, defined as $\Omega\otimes_s\widetilde{\Omega} = \frac{1}{2}(\Omega\otimes\widetilde{\Omega} + \widetilde{\Omega}\otimes\Omega)$. In these expressions we introduced the function

$$\bar{\Delta} \;=\; \left(\mathrm{e}^{-2\varphi}\Sigma^3\right)^{-4/5}\cos^2\zeta + \left(\mathrm{e}^{-2\varphi}\Sigma^3\right)^{1/5} H_+\sin^2\zeta\,,$$

as well as

$$\begin{aligned}H_\pm \;&=\; H^4 \pm \Big(H^1\sin\theta\sin\phi - H^2\sin\theta\cos\phi + H^3\cos\theta\Big)\;,\\ \mathrm{d}_6 H_+ \;&=\; H^1\,\mathrm{d}(\sin\theta\sin\phi) - H^2\,\mathrm{d}(\sin\theta\cos\phi) + H^3\,\mathrm{d}\cos\theta\,.\end{aligned}$$

Note that in the last expression the exterior derivative acts on the internal coordinates and not on the scalars H^I, which only depend on the external coordinates.

We next come to the eleven-dimensional three-form potential $\hat{A}$. We first give our result and then make some comments. The ansatz for $\hat{A}$ reads

$$\begin{aligned}\hat{A} = &-\tfrac{1}{8}R^3\cos\zeta\,[2+\sin^2\zeta\,\bar{\Delta}^{-1}(\mathrm{e}^{-2\varphi}\Sigma^3)^{-4/5}\,]\,\Omega_1\wedge\Omega_2\wedge\Omega_3\\ &+\tfrac{1}{4}R^3\sin^3\zeta\,\bar{\Delta}^{-1}(\mathrm{e}^{-2\varphi}\Sigma^3)^{1/5}\,\mathrm{d}\zeta\wedge H^\alpha\,\widetilde{\Omega}_\alpha\wedge\Omega_3\\ &+R^3\,\cos\zeta\,(\mathcal{D}\xi - \theta_1\mathcal{D}\theta_2 + \theta_2\mathcal{D}\theta_1)\wedge\mathrm{vol}_{\mathbb{\Sigma}} + \tfrac{1}{4}R^3\cos\zeta\,\Big(2\theta_1^2 + 2\theta_2^2 - \kappa\Big)\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\Omega_3\\ &+\tfrac{1}{2}R^2\cos\zeta\,(\mathcal{F}^4\wedge\Omega_3 - \mathcal{F}^\alpha\wedge\widetilde{\Omega}_\alpha) + R\,\cos\zeta\,\Sigma^4\,*_5\,\mathcal{F}^0\\ &+\tfrac{1}{2\sqrt{2}}\,R^3\cos\zeta\,\Big[\Big(\mathcal{D}\theta_2\wedge e^1 - \mathcal{D}\theta_1\wedge e^2\Big)\wedge\Omega_1 + \Big(\mathcal{D}\theta_1\wedge e^1 + \mathcal{D}\theta_2\wedge e^2\Big)\wedge\Omega_2\Big]\end{aligned} \tag{IV.33}$$

where the five-dimensional gauge field strengths, $\mathcal{F}$, and the covariant derivatives, $\mathcal{D}$, of the five-dimensional scalars were given in (IV.21) and (IV.25), respectively.

Equation (IV.33) has been obtained by first computing $\hat{A}$ through the general procedure of Section III.1, then implementing a gauge shift by an exact three-form so as to obtain a nicer expression (this is why derivatives of the external fields appear), and finally dualising away the five-dimensional two- and three-forms, so that the only five-dimensional degrees of freedom contained in the ansatz are scalar and vector fields, in addition to the metric $g_{\mu\nu}$. Let us outline how this dualisation is performed. Evaluating (IV.29) and (IV.30), we find that only one external two-form $\mathcal{B}$ and one external three-form $\mathcal{C}$ appear in the ansatz for $\hat{A}$. These are paired up with the generalised tensors E_5 and E_5' on S^4, which, as generalised tensors on M, are sections of $\det T^*M \otimes E^*$ and $\det T^*M \otimes \mathrm{ad}\, F$, respectively. The combination entering in $\hat{A}$ is

$$\left[\mathcal{B}E_5 + \mathcal{C}E_5'\right]_3 \;=\; R\,\mathcal{B}\wedge \mathrm{d}\cos\zeta + R\,\mathcal{C}\cos\zeta \;=\; (\mathcal{C}-\mathrm{d}\mathcal{B})\,R\cos\zeta + \mathrm{d}\left(\mathcal{B}\,R\cos\zeta\right)\,,$$

where the subscript on the left-hand side indicates the restriction to the three-form part, and the last term in the expression is removable via a gauge transformation of $\hat{A}$. Hence $\mathcal{B}$ and $\mathcal{C}$ only appear in the combination $(\mathcal{C}-\mathrm{d}\mathcal{B})$. This means that the two-form gets eaten by the three-form via the Stuckelberg mechanism, giving it a mass. While a massless three-form in five-dimensions is dual to a scalar field, here we dualise the two-form at the same time and also obtain a vector field. The duality relation is obtained considering the duality between the eleven-dimensional three-form $\hat{A}$ and six-form $\hat{\tilde{A}}$ given in (III.1), and looking at the relevant terms with three external indices. In this way we find that

$$\mathcal{C}-\mathrm{d}\mathcal{B} \;=\; \Sigma^4 *_5 \mathrm{d}\mathcal{A}^0 - \mathcal{A}^4\wedge\mathrm{d}\mathcal{A}^4 + \mathcal{A}^\alpha\wedge\mathrm{d}\mathcal{A}^\alpha - \tfrac{1}{3R}\,\epsilon_{\alpha\beta\gamma}\mathcal{A}^\alpha\wedge\mathcal{A}^\beta\wedge\mathcal{A}^\gamma\,.$$

We have used this expression to eliminate $(\mathcal{C}-\mathrm{d}\mathcal{B})$ completely from the truncation ansatz. This explains the $*_5\mathcal{F}^0$ term appearing in (IV.33).

Our truncation ansatz reproduces the Maldacena–Nuñez solution upon taking $\kappa=-1$ and setting the scalars to

$$H^1=H^2=H^3=\theta_1=\theta_2=\xi=0\,,\quad H^4=\Sigma=1\,,\quad \varphi=\tfrac{1}{2}\log\tfrac{4}{3}\,. \tag{IV.34}$$

The consistent truncation of [61] is recovered as a subtruncation that projects out the fields transforming under $SU(2)_{\text{left}}$, that is setting $\tilde{\mathcal{A}}^\alpha = H^\alpha = 0$, $\alpha=1,2,3$, which also implies $H^4=1$.[6] The further truncation to minimal gauged supergravity is obtained by setting the scalars to their AdS value (IV.34) and taking $\mathcal{A}^0=-\mathcal{A}^4$.

One can obtain a slightly larger subtruncation by projecting out only the modes charged under $U(1)_{\text{left}}$, rather than $SU(2)_{\text{left}}$, namely setting $\tilde{\mathcal{A}}^1=\tilde{\mathcal{A}}^2=H^1=H^2=0$. This leaves us with two vector multiplets, one hypermultiplet and just the abelian gauging generated by the Killing vectors (IV.24), which is the same as the one in the truncation of [61]. A generalisation of this subtruncation will be discussed in Section IV.2.

[6] Then the one-forms Ω_α essentially reduce to those in [61], up to slightly different conventions, while $\widetilde{\Omega}_\alpha$ drop out of the ansatz. When comparing our truncation ansatz with the one given in Section 4.1 of [61], one should take into account that $\hat{A}^{\text{here}} = -\hat{A}^{\text{FNR}}$ (this is seen from comparing our 11d Maxwell equation with the one in [62], which provides the 7d to 11d uplift formulae used in [61]). Moreover $\zeta^{\text{here}} = \zeta^{\text{FNR}} + \pi/2$, $\mathcal{A}^4 \propto \mathcal{A}^{\text{FNR}}$, $\mathcal{A}^0 \propto \chi_1^{\text{FNR}}$, $\Sigma = 2^{1/3}\Sigma^{\text{FNR}}$, $\mathrm{e}^{2\varphi} = 2(\mathrm{e}^{2\varphi})^{\text{FNR}}$, $|\theta_{1,2}| = \frac{1}{\sqrt{2}}|\theta_{1,2}|^{\text{FNR}}$, $\xi = \frac{1}{2}\xi^{\text{FNR}}$.

The truncation of [61] was obtained via a reduction of gauged seven-dimensional supergravity on the Riemann surface $\mathbb{\Sigma}$. Similarly, we can obtain our truncation ansatz by combining the well-known truncation of eleven-dimensional supergravity on S^4 [11], leading to seven-dimensional maximal $SO(5)$ supergravity, with a further truncation reducing the seven-dimensional theory on $\mathbb{\Sigma}$. Starting from the convenient form of the bosonic truncation ansatz on S^4 given in [63], we have explicitly checked that this procedure works out as expected and reproduces the ansatz above.

IV.1.5 The five-dimensional theory

We now put together the ingredients defining the truncated five-dimensional theory and discuss it in more detail. This is an $\mathcal{N}=2$ gauged supergravity coupled to four vector multiplets and one hypermultiplet. The vector multiplet scalar manifold is

$$\mathcal{M}_{\rm V} \;=\; \mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)}\,,$$

while the hypermultiplet scalar manifold is

$$\mathcal{M}_{\rm H} \;=\; \frac{SU(2,1)}{SU(2)_H \times U(1)}\,.$$

As discussed before, these have a geometric origin as the V and H structure moduli spaces of the internal manifold. At the bosonic level, the vector multiplets are made of gauge fields $\mathcal{A}^I$ and constrained scalar fields h^I, $I=0,1,\ldots,4$, which we have parameterised in terms of Σ and H^I, $I=1,\ldots,4$, in (IV.18). The latter scalars satisfy the constraint $\eta_{IJ}H^IH^J=1$, with $\eta=\mathrm{diag}(-1,-1,-1,1)$. We have also found that the non-vanishing components of the symmetric tensor C_{IJK} are given by

$$C_{0IJ}=C_{I0J}=C_{IJ0}=\tfrac{1}{3}\,\eta_{IJ}\,,\qquad\qquad I,J=1,\ldots,4\,.$$

The kinetic terms in the vector multiplet sector are controlled by the matrix a_{IJ}, given by the general formula

$$a_{IJ}=3h_Ih_J-2C_{IJK}h^K\,, \tag{IV.35}$$

which in our case reads

$$\begin{aligned}
&a_{00}=\tfrac{1}{3}\,\Sigma^4\,,\\
&a_{0J}=0\,,\\
&a_{IJ}=\tfrac{2}{3}\,\Sigma^{-2}\left(2\eta_{IK}H^K\eta_{JL}H^L-\eta_{IJ}\right)\,,\qquad\qquad I,J=1,\ldots,4\,.
\end{aligned} \tag{IV.36}$$

The hypermultiplet comprises the scalars $q^X=\{\varphi,\xi,\theta_1,\theta_2\}$, and the kinetic term is given by the quaternionic-Kähler metric on $\mathcal{M}_{\rm H}$ that we derived in Appendix B of [2],

$$g_{XY}\mathrm{d}q^X\mathrm{d}q^Y \;=\; 2\,\mathrm{d}\varphi^2+\mathrm{e}^{2\varphi}\left(\mathrm{d}\theta_1^2+\mathrm{d}\theta_2^2\right)+\tfrac{1}{2}\,\mathrm{e}^{4\varphi}\left(\mathrm{d}\xi-\theta_1\mathrm{d}\theta_2+\theta_2\mathrm{d}\theta_1\right)^2\,. \tag{IV.37}$$

The gauge group is $SO(3)\times U(1)\times\mathbb{R}$. The symmetries being gauged are the $SO(3)\subset SO(3,1)$ isometries of $\mathcal{M}_{\rm V}$ and two abelian isometries in $\mathcal{M}_{\rm H}$, generated by the Killing

vectors (IV.24). Note that the ∂_ξ term generates the non-compact $\mathbb{R}$ factor and the $\theta_2\partial_{\theta_1} - \theta_1\partial_{\theta_2}$ term generates the compact $U(1)$.

We recall for convenience the gauge field strengths

$$\mathcal{F}^0 = \mathrm{d}\mathcal{A}^0\,, \qquad \mathcal{F}^\alpha = \mathrm{d}\mathcal{A}^\alpha - g\,\epsilon^\alpha{}_{\beta\gamma}\mathcal{A}^\beta\wedge\mathcal{A}^\gamma\,, \qquad \mathcal{F}^4 = \mathrm{d}\mathcal{A}^4\,, \quad \alpha = 1,2,3\,,$$

and the covariant derivatives of the charged scalars,

$$\begin{aligned}\mathcal{D}H^\alpha &= \mathrm{d}H^\alpha - \tfrac{2}{R}\,\epsilon^\alpha{}_{\beta\gamma}\mathcal{A}^\beta\,H^\gamma\,,\\ \mathcal{D}(\theta_1+\mathrm{i}\,\theta_2) &= \mathrm{d}(\theta_1+\mathrm{i}\,\theta_2) - \tfrac{2}{R}\,\mathrm{i}\,\mathcal{A}^4\,(\theta_1+\mathrm{i}\,\theta_2)\,,\\ \mathcal{D}\xi &= \mathrm{d}\xi + \tfrac{1}{R}\mathcal{A}^0 - \tfrac{\kappa}{R}\,\mathcal{A}^4\,,\end{aligned}$$

where the gauge coupling constant is given by the inverse S^4 radius, $g = \frac{1}{R}$. The scalars Σ, H^4 and φ remain uncharged. The gauging in the hypersector is the same as in [61], while the gauging in the vector multiplet sector is a novel feature of our truncation.

Plugging these data in the general form of the $\mathcal{N}=2$ supergravity action given in Appendix B of [2], we obtain the bosonic action for our model,

$$\begin{aligned}S = \frac{1}{16\pi G_5}\int\Big[&(\mathcal{R}-2\mathcal{V})*1 - \tfrac{1}{2}\,\Sigma^4\mathcal{F}^0\wedge*\mathcal{F}^0 - \tfrac{3}{2}\sum_{I,J=1}^{4}a_{IJ}\mathcal{F}^I\wedge*\mathcal{F}^J - 2\Sigma^{-2}\mathrm{d}\Sigma\wedge*\mathrm{d}\Sigma\\ &- \tfrac{3}{2}\sum_{I,J=1}^{4}a_{IJ}\mathcal{D}(\Sigma H^I)\wedge*\mathcal{D}(\Sigma H^J) - g_{XY}\mathcal{D}q^X\wedge*\mathcal{D}q^Y + \sum_{I,J=1}^{4}\eta_{IJ}\mathcal{A}^0\wedge\mathcal{F}^I\wedge\mathcal{F}^J\Big]\,,\end{aligned}$$

where G_5 is the five-dimensional Newton constant.[7] The scalar potential $\mathcal{V}$ is obtained from the Killing prepotentials of the gauged isometries as in Appendix B of [2].

$$\begin{aligned}\mathcal{V} = \frac{1}{R^2}\Big\{&\frac{\mathrm{e}^{4\varphi}}{4\Sigma^4} - \frac{2\,H^4\,\mathrm{e}^{2\varphi}}{\Sigma} + \Sigma^2\Big[-2+\mathrm{e}^{2\varphi}\Big(2\big((H^4)^2-1\big)\big(\theta_1^2+\theta_2^2\big)-\kappa\Big)\\ &+\frac{1}{8}\,\mathrm{e}^{4\varphi}\big(2(H^4)^2-1\big)\big(2\theta_1^2+2\theta_2^2-\kappa\big)^2\Big]\Big\}\,.\end{aligned} \tag{IV.38}$$

The supersymmetric AdS vacuum conditions summarised in are easily solved and give the scalar field values

$$H^1 = H^2 = H^3 = \theta_1 = \theta_2 = 0\,, \quad H^4 = \Sigma = 1\,, \quad \varphi = \tfrac{1}{2}\log\tfrac{4}{3}\,, \tag{IV.39}$$

that is precisely the values (IV.34) that reproduce the MN1 solution reviewed in earlier in the section. The negative curvature $\kappa = -1$ for the Riemann surface arises as a positivity condition for the scalars Σ and $\mathrm{e}^{2\varphi}$. The critical value of the scalar potential yields the cosmological constant $\Lambda \equiv \mathcal{V} = -\frac{8}{3R^2}$, corresponding to an AdS_5 radius $\ell = \frac{3}{2}R$, again in harmony with the MN1 solution.

[7] As discussed in [30], the five-dimensional Newton constant is given by $(G_5)^{-1} \propto \int_M \mathrm{e}^{3\Delta}\,\mathrm{vol}_6 = \int_M \boldsymbol{\kappa}^2$. In the present case, $\int_M \boldsymbol{\kappa}^2 = R^2\,\mathrm{Vol}_\Sigma\,\mathrm{Vol}_4$, where $\mathrm{Vol}_\Sigma = \frac{4\pi(1-g)}{\kappa}$ is the standard volume of a Riemann surface of genus g and $\mathrm{Vol}_4 = \frac{8\pi^3}{3}R^4$ is the volume of a round S^4 with radius R.

By extremising the scalar potential (IV.38) we can search for further AdS_5 vacua within our truncation. Then, by analysing the mass matrix of the scalar field fluctuations around the extrema we can test their perturbative stability. In the following we discuss the outcome of this analysis for the three extrema that we have found.

- We recover the supersymmetric vacuum (IV.39). Being supersymmetric, this is stable. The supergravity field fluctuations source $SU(2,2|1)$ superconformal multiplets in the dual $\mathcal{N}=1$ SCFT [60], with the supergravity mass eigenvalues providing the conformal dimension Δ of the operators in the multiplets. The field fluctuations that were also considered in [61] correspond to the energy-momentum multiplet (containing the energy-momentum tensor with $\Delta=4$ and the R-current with $\Delta=3$) and to a long vector multiplet of conformal dimension $\Delta=1+\sqrt{7}$ (see [61] for more details). The additional $SO(3)$ vector multiplet included in this paper sources a conserved $SO(3)$ flavour current multiplet in the dual SCFT. The three scalar operators in this multiplet have conformal dimension $\Delta=2$ (once) and $\Delta=4$ (twice), while the $SO(3)$ flavour current has conformal dimension $\Delta=3$, as required for a conserved current. Another piece of information about the dual SCFT is given by the Weyl anomaly coefficients; these are obtained from the five-dimensional Newton constant G_5 and the AdS_5 radius ℓ through the formula $a=c=\frac{\pi\ell^3}{8G_5}$.

- When $\kappa=-1$ we also recover the non-supersymmetric vacuum discussed in [61], that was originally found in [64]. The analysis of the scalar mass matrix shows that the fluctuation of H^4 has a mass squared $m^2\ell^2\simeq -4.46$, which is below the Breitenlohner–Freedman bound $\ell^2 m^2_{\mathrm{BF}}=-4$. We thus establish that this vacuum is perturbatively unstable. Note that the unstable mode lies outside the truncation of [61].

- For $\kappa=+1$, we find a non-supersymmetric vacuum with non zero value of the H-scalars, given by

$$\Sigma=\frac{2^{1/3}}{5^{1/6}}\,,\qquad \mathrm{e}^{2\varphi}=\frac{8}{3}\,,\qquad H^4=\frac{3\sqrt{5}}{4}\,,\qquad \theta_1=\theta_2=0\,,\qquad \ell=3\,\frac{2^{1/6}}{5^{5/6}}R\,,$$

  where ℓ is the AdS radius. This appears to be a new solution. It represents an $SO(3)$ worth of vacua really, since the scalars H^α, $\alpha=1,2,3$, can take any value such that $\sqrt{(H^1)^2+(H^2)^2+(H^3)^2}=\sqrt{(H^4)^2-1}=\frac{\sqrt{29}}{4}$. We find that a linear combination of the fluctuations of Σ,φ and H^4 has mass squared $m^2\ell^2\simeq -5.86<m^2_{\mathrm{BF}}\ell^2$, hence this vacuum is perturbatively unstable. Nevertheless, it allowed us to perform a non-trivial check of our truncation ansatz for non-vanishing H-fields, as we have verified that its uplift does satisfy the equations of motion of eleven-dimensional supergravity.

IV.2 Truncations for more general wrapped M5-branes

The $\mathcal{N}=2$ and $\mathcal{N}=4$ Maldacena–Nuñez solutions are special cases of an infinite family of $\mathcal{N}=2$ solutions [65, 32],[8] describing M5-branes wrapping a Riemann surface in a

[8] See also [66], where a subset of the solutions was previously found.

Calabi–Yau geometry. These solutions, which we will denote as BBBW solutions, have the same general features of the MN1 solution. In particular, they all admit a generalised $U(1)_S$ structure, which we use to derive the most general consistent truncation to $\mathcal{N}=2$ gauged supergravity in five dimensions associated with such backgrounds. As we will see, the truncated theory has two vector multiplets, one hypermultiplet and gauge group $U(1)\times\mathbb{R}$. It generalises the $U(1)_{\text{right}}$ invariant subtruncation of the truncation presented in the previous section: the matter content is the same and the gauging is deformed by one (discrete) parameter. Our systematic approach allows us to complete the consistent truncation derived from seven-dimensional maximal $SO(5)$ supergravity on $\mathbb{\Sigma}$ previously presented in [67] by including all scalar fields in the hypermultiplet and directly deriving the gauging.

IV.2.1 The BBBW solutions

The BBBW solutions describe M5-branes wrapped on a Riemann surface $\mathbb{\Sigma}$, such that the $(2,0)$ theory on the branes has a twisting over $\mathbb{\Sigma}$ depending on two integer parameters p and q. The way the Riemann surface is embedded in the ambient space determines the local structure of the latter. The authors of [65, 32] showed that there is an infinite family of allowed geometries, corresponding to the fibration $\mathcal{L}_1\oplus\mathcal{L}_2\hookrightarrow\mathbb{\Sigma}$ of two complex line bundles over the Riemann surface, so that the total space is Calabi–Yau. The degrees of these line bundles are identified with the integers that parameterise the twist of the M5 world-volume theory, $p=\deg\mathcal{L}_1$ and $q=\deg\mathcal{L}_2$. By the Calabi–Yau condition p and q must satisfy $p+q=2g-2$, with g the genus of $\mathbb{\Sigma}$. In this setup, the $\mathcal{N}=1$ and $\mathcal{N}=2$ twistings considered in [31] arise from setting $p=q$ and $q=0$ (or $p=0$), respectively.

The corresponding $\text{AdS}_5\times_{\text{w}}M$ supergravity solutions are generalisation of the MN1 solution reviewed in Section IV.1. The eleven-dimensional metric is a warped product

$$\hat{g}=\mathrm{e}^{2\Delta}g_{\text{AdS}_5}+g_6\,,$$

with warp factor

$$\mathrm{e}^{2\Delta}\,\ell^2\;=\;\mathrm{e}^{2f_0}\bar{\Delta}^{1/3}\,.$$

where ℓ is the AdS radius. The six-dimensional manifold M is still a fibration of a squashed four-sphere over the Riemann surface, with metric

$$g_6=\bar{\Delta}^{1/3}\mathrm{e}^{2g_0}g_{\mathbb{\Sigma}}+\tfrac{1}{4}\,\bar{\Delta}^{-2/3}g_4\,,$$

where the Riemann surface metric $g_{\mathbb{\Sigma}}$ satisfies (IV.11), (IV.12), and the metric on the squashed and fibered S^4 is

$$g_4=X_0^{-1}\mathrm{d}\mu_0^2+\sum_{i=1,2}X_i^{-1}\left(\mathrm{d}\mu_i^2+\mu_i^2(\mathrm{d}\varphi_i+A^{(i)})^2\right). \tag{IV.40}$$

The angles φ_1,φ_2 vary in $[0,2\pi]$,[9] and

$$\mu_0=\cos\zeta\,,\qquad \mu_1=\sin\zeta\cos\tfrac{\theta}{2}\,,\qquad \mu_2=\sin\zeta\sin\tfrac{\theta}{2}\,,$$

[9] They are related to the angles of Section IV.1 by $\varphi_1=-(\phi+\psi)/2$ and $\varphi_2=(\phi-\psi)/2$.

with $\zeta, \theta \in [0, \pi]$. The two circles φ_1 and φ_2 are independently fibered over the Riemann surface, with connections

$$A^{(1)} = -\frac{1+z}{2}\,\upsilon \qquad A^{(2)} = -\frac{1-z}{2}\,\upsilon\,,$$

where υ is again the connection on $\mathbb{\Sigma}$ and the discrete parameter z is related to the integers p and q as

$$z \;=\; \frac{p-q}{p+q}\,.$$

The warping function $\bar{\Delta}$ and the constants f_0, g_0 depend on z and on the curvature κ of the Riemann surface as

$$\bar{\Delta} = \sum_{I=0}^{2} X_I \mu_I^2\,, \qquad \mathrm{e}^{f_0} = X_0^{-1}\,, \qquad \mathrm{e}^{2g_0} = -\tfrac{1}{8}\,\kappa\, X_1 X_2\,[(1-z)X_1 + (1+z)X_2]\,,$$

with

$$\begin{aligned} X_0 &= (X_1 X_2)^{-2}\,,\\ X_1 X_2^{-1} &= \frac{1+z}{2z - \kappa\sqrt{1+3z^2}}\,,\\ X_1^5 &= \frac{1+7z+7z^2+33z^3+\kappa(1+4z+19z^2)\sqrt{1+3z^2}}{4z(1-z)^2}\,. \end{aligned}$$

The four-form flux is given by

$$\begin{aligned} \hat{F} = &-\frac{1}{4}\,\bar{\Delta}^{-5/2}\Big[\sum_{I=0}^{2}(X_I^2\mu_I^2 - \bar{\Delta}X_I) + 2\bar{\Delta}X_0\Big]\,\mathrm{vol}_4\\ &+\frac{1}{16}\bar{\Delta}^{-1/2}\sum_{i=1}^{2} X_i^{-2}\,*_4\,[\mathrm{d}(\mu_i^2)\wedge(\mathrm{d}\varphi_i + A^{(i)})]\wedge \mathrm{d}A^{(i)}\,, \end{aligned}$$

where the Hodge star $*_4$ is computed using the metric (IV.40).

The solution has two $U(1)$ isometries corresponding to shifts of the angles φ_1, φ_2 that parameterise the two diagonal combinations of the $U(1)_{\text{right}}$ and $U(1)_{\text{left}}$ subgroups of $SO(5)$. It turns out that neither of them corresponds to the superconformal R-symmetry of the dual $\mathcal{N}=1$ SCFT, which is given by a linear combination involving X_1, X_2 [65, 32].

IV.2.2 Generalised $U(1)_S$ structure

The construction of the generalised structure associated to the BBBW solutions follows the same lines as for the MN1 solution. We first embed the ordinary $U(1)$ structure in $E_{6(6)}$ and then look for the invariant generalised tensors. The generalised $U(1)_S$ structure of the solutions is determined by the topological twist of the M5 world-volume theory, as a linear combination of the $U(1)$ holonomy of $\mathbb{\Sigma}$ and the $U(1)_{\text{right}}$ and $U(1)_{\text{left}}$ subgroups of the $SO(5)$ R-symmetry group

$$U(1)_S \;\sim\; U(1)_{\mathbb{\Sigma}} - U(1)_{\text{right}} - z\,U(1)_{\text{left}}\,.$$

This embeds in $E_{6(6)}$ as an element of its compact subgroup $USp(8)$ with generator

$$\mathfrak{u}(1)_S \;=\; \mathrm{i}\,\hat{\Gamma}_{56} - \frac{\mathrm{i}}{p+q}(p\,\hat{\Gamma}_{12} - q\,\hat{\Gamma}_{34})\,, \tag{IV.41}$$

where $\hat{\Gamma}_{56}$ is the $\mathfrak{usp}_8$ element generating $U(1)_\Sigma$ and $\frac{1}{2}(\hat{\Gamma}_{12} \pm \hat{\Gamma}_{34})$ generate $U(1)_{\text{left/right}}$. When $p = q$ we recover the $U(1)_S$ structure group of the MN1 solution, whereas $q = 0$ (or $p = 0$) gives the MN2 structure considered in [1]. Below we assume that p, q are generic, and do not fulfill these special conditions which as we have seen lead to a larger truncation.

By looking at the singlets under $\mathfrak{u}(1)_S$ in the **27** and **78** representations of $E_{6(6)}$, we find that the $U(1)_S$ structure is defined by eight J_A, $A = 1, \ldots, 8$, in the adjoint bundle and three generalised vectors K_I, $I = 0, 1, 2$. The singlets in the adjoint bundle have the same form (IV.15) as for the MN1 solution, while the three singlet generalised vectors take the same form as a subset of the MN1 generalised vectors,[10]

$$\begin{aligned} K_0 \;&=\; \mathrm{e}^{\Upsilon}\cdot(R^2\,\mathrm{vol}_\Sigma \wedge E_5')\,,\\ K_1 \;&=\; \mathrm{e}^{\Upsilon}\cdot\widetilde{\Xi}_3\,,\\ K_2 \;&=\; \mathrm{e}^{\Upsilon}\cdot\Xi_3\,. \end{aligned}$$

However now the twisting element Υ has a more general form dictated by the embedding (IV.41), that is

$$\Upsilon \;=\; -\frac{R}{p+q}\,\upsilon \times_{\text{ad}} (p\,E_{12} - q\,E_{34})\,. \tag{IV.42}$$

This makes our generalised tensors globally well-defined. We emphasise that these depend on the integers p, q only through (IV.42).

IV.2.3 Features of the truncation

The number of $U(1)_S$ singlets in the **27** and **78** implies that the truncated supergravity theory contains two vector multiplets and one hypermultiplet. The H structure moduli space is the same as for the MN1 case,

$$\mathcal{M}_{\text{H}} \;=\; \frac{SU(2,1)}{SU(2)_H \times U(1)}\,.$$

As before, this is parameterised by real coordinates $q^X = \{\varphi, \xi, \theta_1, \theta_2\}$ and the metric is given by Eq. (IV.37). The V structure moduli space is determined again following our discussion in Section III.2, and is a subspace of the one for the MN1 truncation.

[10] Before acting with Υ, the singlets for the BBBW solutions are related to those used for the MN1 solutions as

$$K_0 = K_0^{\text{MN1}}\,, \quad K_1 = K_3^{\text{MN1}}\,, \quad K_2 = K_4^{\text{MN1}}\,,$$

and to the structure of the MN2 solution in [1] as

$$K_0 = \tfrac{1}{2}(K_5^{\text{MN2}} - K_8^{\text{MN2}})\,, \qquad \begin{aligned} K_1 &= K_0^{\text{MN2}} + \tfrac{1}{2}(K_5^{\text{MN2}} + K_8^{\text{MN2}})\,,\\ K_2 &= K_0^{\text{MN2}} - \tfrac{1}{2}(K_5^{\text{MN2}} + K_8^{\text{MN2}})\,. \end{aligned}$$

Evaluating the cubic invariant on the singlets K_I as in (III.21), we obtain that the non-zero components of the C_{IJK} tensor are

$$C_{0IJ} = C_{I0J} = C_{IJ0} = \tfrac{1}{3}\,\eta_{IJ}\,, \qquad \text{for } I,J = 1,2\,,$$

with $\eta = \mathrm{diag}(-1,1)$. Parameterising the V structure moduli as in (IV.18), with $I = 1,2$, the constraint (III.21) gives the equation of the unit hyperboloid $SO(1,1)$,

$$-(H^1)^2 + (H^2)^2 = 1\,,$$

while again Σ parameterises $\mathbb{R}^+$. Thus the V structure moduli space is

$$\mathcal{M}_{\rm V} \;=\; \mathbb{R}^+ \times SO(1,1)\,.$$

The kinetic matrix a_{IJ} then takes the same form (IV.36), that is

$$\begin{aligned} a_{00} &= \tfrac{1}{3}\,\Sigma^4\,,\\ a_{01} &= a_{02} = 0\,,\\ a_{IJ} &= \tfrac{2}{3}\,\Sigma^{-2}\begin{pmatrix} 2(H^1)^2+1 & -2H^1H^2 \\ -2H^1H^2 & 2(H^2)^2-1 \end{pmatrix}\,, \qquad I,J=1,2\,. \end{aligned}$$

The gauging of the reduced theory is obtained from the generalised Lie derivative L_{K_I} acting on the K_J and the J_A. The Lie derivatives among vectors are now trivial,

$$L_{K_I} K_J = 0\,, \qquad I,J = 0,1,2\,. \tag{IV.43}$$

The Lie derivatives $L_{K_I} J_A$ are conveniently expressed as the adjoint action of $SU(2,1)$ generators,

$$L_{K_0} J_A = [J_{(K_0)}, J_A]\,, \qquad L_{K_1} J_A = [J_{(K_1)}, J_A]\,, \qquad L_{K_2} J_A = [J_{(K_2)}, J_A]\,.$$

Evaluating the generalised Lie derivatives we find

$$\begin{aligned} J_{(K_0)} &= \tfrac{1}{4R}\,(J_3 + 2J_7 - \sqrt{3}J_8)\,,\\ J_{(K_1)} &= \tfrac{1}{4R}\,\kappa\, z\,(J_3 + 2J_7 - \sqrt{3}J_8)\,,\\ J_{(K_2)} &= -\tfrac{1}{4R}\,\kappa\,(J_3 + 2J_7 - \sqrt{3}J_8) - \tfrac{1}{R}\,(J_3 + \tfrac{1}{\sqrt{3}}J_8)\,. \end{aligned} \tag{IV.44}$$

Eq. (IV.43) implies that the vector multiplet sector is not gauged, so the field strengths are all abelian,

$$\mathcal{F}^I = \mathrm{d}\mathcal{A}^I\,,$$

while (IV.44) specifies the gauging in the hypermultiplet sector in terms of κ and z. The $SU(2,1)$ generators act as isometries on $\mathcal{M}_{\rm H}$; the corresponding Killing vectors can again be computed as before and read

$$\begin{aligned} k_0 &= \partial_\xi\,,\\ k_1 &= \kappa\, z\,\partial_\xi\,,\\ k_2 &= -\kappa\,\partial_\xi + 2\,(\theta_2\partial_{\theta_1} - \theta_1\partial_{\theta_2})\,. \end{aligned}$$

It follows that the covariant derivatives of the charged scalars are

$$\begin{aligned}\mathcal{D}(\theta_1 + \mathrm{i}\,\theta_2) &= \mathrm{d}(\theta_1 + \mathrm{i}\,\theta_2) - \tfrac{2}{R}\,\mathrm{i}\mathcal{A}^2\,(\theta_1 + \mathrm{i}\,\theta_2)\,,\\ \mathcal{D}\xi &= \mathrm{d}\xi + \tfrac{1}{R}\,\mathcal{A}^0 + \tfrac{1}{R}\,\kappa\left(z\mathcal{A}^1 - \mathcal{A}^2\right)\,,\end{aligned}$$

where again the inverse S^4 radius $\frac{1}{R}$ plays the role of the gauge coupling constant. The Killing prepotentials can be computed using formulas in the Appendix B of [2], and read

$$\begin{aligned}P_0^\alpha &= \{0\,,\,0\,,\,\tfrac{1}{4}\,\mathrm{e}^{2\varphi}\}\,,\\ P_1^\alpha &= \{0\,,\,0\,,\,\tfrac{1}{4}\,\kappa\,z\,\mathrm{e}^{2\varphi}\}\,,\\ P_2^\alpha &= \{\sqrt{2}\,\mathrm{e}^{\varphi}\theta_1\,,\,\sqrt{2}\,\mathrm{e}^{\varphi}\theta_2\,,-1+\tfrac{1}{4}\,\mathrm{e}^{2\varphi}\,(2\theta_1^2+2\theta_2^2-\kappa)\,\}\,.\end{aligned}\qquad\text{(IV.45)}$$

Notice that for $z = 0$ (that is $p = q$), the quantities above reduce to those obtained for the MN1 structure in Section IV.1.3.

The five-dimensional bosonic action is then determined to be

$$\begin{aligned}S = \frac{1}{16\pi G_5}\int\Big[&(\mathcal{R}-2\mathcal{V})*1 - \tfrac{1}{2}\,\Sigma^4\mathcal{F}^0\wedge*\mathcal{F}^0 - \tfrac{3}{2}\sum_{I,J=1}^{2}a_{IJ}\mathcal{F}^I\wedge*\mathcal{F}^J - 2\Sigma^{-2}\mathrm{d}\Sigma\wedge*\mathrm{d}\Sigma\\ &- \tfrac{3}{2}\sum_{I,J=1}^{2}a_{IJ}\,\mathrm{d}(\Sigma H^I)\wedge*\mathrm{d}(\Sigma H^J) - g_{XY}\mathcal{D}q^X\wedge*\mathcal{D}q^Y - \mathcal{A}^0\wedge\left(\mathcal{F}^1\wedge\mathcal{F}^1 - \mathcal{F}^2\wedge\mathcal{F}^2\right)\Big]\,,\end{aligned}$$

where the scalar potential reads

$$\begin{aligned}\mathcal{V} = \frac{1}{R^2}\Big\{&\frac{\mathrm{e}^{4\varphi}}{4\Sigma^4} - \frac{2\,\mathrm{e}^{2\varphi}H^2}{\Sigma} + \Sigma^2\Big[-2+\mathrm{e}^{2\varphi}\left(2(H^1)^2(\theta_1^2+\theta_2^2)-\kappa\right)\\ &+\frac{1}{8}\,\mathrm{e}^{4\varphi}\big((H^1)^2+(H^2)^2\big)\big(2\theta_1^2+2\theta_2^2-\kappa\big)^2\\ &+z\,\kappa\left(z\kappa\,(H^1)^2+z\kappa\,(H^2)^2+4H^1H^2\big(2\theta_1^2+\theta_2^2-\kappa\big)\right)\Big]\Big\}\,.\end{aligned}$$

It is straightforward to analyse the supersymmetric AdS_5 vacuum conditions. The hyperino equation gives

$$\begin{aligned}&\theta_1 = \theta_2 = 0\,,\\ &\Sigma^{-3} = \kappa\left(zH^1 - H^2\right)\,,\end{aligned}\qquad\text{(IV.46)}$$

where we assume $\kappa = \pm 1$ (hence leaving aside the case $\kappa = 0$). The gaugino equation gives

$$\begin{aligned}&2\Sigma^{-3}\,P_0^\alpha + H^1P_1^\alpha + H^2P_2^\alpha = 0\,,\\ &H^2P_1^\alpha + H^1P_2^\alpha = 0\,.\end{aligned}$$

Plugging the Killing prepotentials (IV.45) and using (IV.46) we obtain

$$\begin{aligned}3\kappa\,\mathrm{e}^{2\varphi}\left(zH^1 - H^2\right) - 4H^2 &= 0\,,\\ \kappa\,\mathrm{e}^{2\varphi}(z\,H^2 - H^1) - 4H^1 &= 0\,.\end{aligned}$$

Taking into account the allowed range of the scalar fields, the solution to these equations is

$$\frac{H^1}{H^2} = \frac{1+\kappa\sqrt{1+3z^2}}{3z}\,, \qquad \mathrm{e}^{2\varphi} = \frac{4}{\sqrt{1+3z^2}-2\kappa}\,.$$

For $\kappa = 1$, well-definiteness of the fields requires $|z| > 1$, as in [32], while z can be generic for $\kappa = -1$. The MN1 case $z = 0$ is recovered as a limiting case after fixing $\kappa = -1$. The critical value of the scalar potential determines the AdS radius ℓ as

$$\ell = \left(\frac{\kappa - 9\kappa z^2 + (1+3z^2)^{3/2}}{4z^2}\right)^{1/3} R\,.$$

Although we do not present the uplift formulae for this truncation, we have checked that the supersymmetric vacuum identified above matches the BBBW solution summarised in Section IV.2.1. To do so, we have computed the inverse generalised metric G^{-1} associated with the $U(1)_S$ structure under consideration; this depends on the V structure and H structure parameters. From the generalised metric we have reconstructed the ordinary metric g_6 and the three-form potential on M, as well as the warp factor $\mathrm{e}^{2\Delta}$. Substituting the values for the scalars found above, we find agreement with the solution in Section IV.2.1 upon fixing the S^4 radius as $R = \frac{1}{2}$ and implementing the following dictionary:

$$\begin{aligned}
\mathrm{e}^{2\varphi} &= 4\,\mathrm{e}^{-2g_0-\frac{1}{2}f_0}\,,\\
\Sigma^3 &= 4\,\mathrm{e}^{-2g_0+\frac{3}{4}f_0}\,,\\
H^1 &= \tfrac{1}{2}X_0^{\frac{1}{4}}(X_1 - X_2)\,,\\
H^2 &= \tfrac{1}{2}X_0^{\frac{1}{4}}(X_1 + X_2)\,,
\end{aligned}$$

with our AdS radius being given in terms of the quantities appearing there as

$$\ell = 2^{2/3}\,\mathrm{e}^{f_0+\frac{2}{3}g_0}R\,.$$

By extremising the scalar potential[11] we recover the supersymmetric vacuum and also find new non-supersymmetric vacua, where the scalar field values are rather complicated functions of the parameter z. As an example, we give the numerical values for one chosen value of z, that we take $z = \frac{1}{2}$. When $\kappa = -1$ we find a new extremum of the potential at

$$\Sigma \simeq 0.9388\,, \quad \varphi \simeq 0.1109\,, \quad H^2 \simeq 1.0217\,, \quad \theta_1 = \theta_2 = 0\,, \qquad \ell = 1.5276\,R\,,$$

while when $\kappa = 1$ we find an extremum at

$$\Sigma \simeq 0.8631\,, \quad \varphi \simeq 0.2812\,, \quad H^2 \simeq 1.5506\,, \quad \theta_1 = \theta_2 = 0\,, \qquad \ell = 1.0644\,R\,,$$

and another one at

$$\Sigma \simeq 1.1580\,, \quad \varphi \simeq 0.8455\,, \quad H^2 \simeq 1.9847\,, \quad \theta_1 = \theta_2 = 0\,, \qquad \ell = 0.6198\,R\,,$$

where for each solution we have also indicated the corresponding AdS radius ℓ.

[11] To do so, it is convenient to parameterise $H^1 = \sinh\alpha$, $H^2 = \cosh\alpha$, and extremise with respect to α.

Chapter V
Conclusions

In this thesis we discussed how generalised geometry provides a systematic approach to consistent truncations. Generalised geometry is a reformulation of eleven or ten dimensional supergravity on a d-dimensional manifold M that treats diffeomorphism and gauge transformations of higher rank potentials as a generalised diffeomorphism on an extension of the tangent space to M whose generalised structure group is $E_{d(d)}$. This allows to generalised the ordinary notion of G_S structure to a generalised one. A generalised G_S structure is associated to a set of nowhere-vanishing tensors that are invariant under G_S. As for ordinary G_S structure, a generalised G_S structure is characterised by its intrinsic torsion.

The main result of this thesis is the prove that any manifold M that admits an exceptional G_S-structure with singlet intrinsic generalised torsion can give rise to a consistent truncation of eleven or ten dimensional supergravity on it. The consistent truncation is given by expanding the eleven/ten dimensional fields in terms of the G_S invariant generalised tensors. Plugging the fields in the equations of motion, the singlet intrinsic torsion guarantees that only singlet fields can appear and hence the truncation is consistent. Moreover the G_S singlet invariant tensors and the intrinsic torsion completely fix the field content and the gauging of the truncated theory. In particular, the number of G_S singlets in the fundamental and adjoint representations of $E_{6(6)}$ determine the vector and hyermultiplets, respectively.

This construction encompasses all the truncations obtained in the past using ordinary G_S structures. However, since a generalised G_S structures need not to correspond to an ordinary one, our construction enlarges the class of consistent truncations one can construct and moreover allows for a systematic way to study and classify truncations in different dimensions and with different supersymmetry.

As an example, after the discussion of the general results, we focused on truncations of the eleven dimensional or type IIB supergravity to five dimensional $\mathcal{N} = 2$ gauged supergravity. In this case the relevant generalised geometry is $\mathbb{R}^+ \times E_{6(6)}$, whose maximal compact subgroup $USp(8)$ determines the R-symmetry of the reduced theory. In order to have $\mathcal{N} = 2$ supersymmetry in five dimensions, the R-symmetry must be $SU(2) \subset USp(8)$ and the spinorial representation must contain only two singlets under the generalised structure group transforming as a doublet of the $SU(2)$. This implies that the relevant generalised structures are $G_S \subseteq USp(6)$. In particular an $USp(6)$ structure corresponds to the minimal five dimensional $\mathcal{N} = 2$ supergravity, while smaller structure group allows

for extra matter. Scanning all the possible continuous subgroups of $USp(6)$ and the corresponding singlet invariant tensors allows to give a classification of all five dimensional $\mathcal{N} = 2$ theories that can be obtained as consistent truncations (corresponding to continuous structure groups). The classification is given in Table III.1. For each case we give the number of vector/tensor multiplets and hypermultiplets and their scalar manifolds. We also assumed that the generalised G_S structure has singlet intrinsic and then analyzed the possible gaugings. We refer to the Section 4 of [3] for this analysis. The result is that a very limited number of theories can be found. For the half maximal case a similar classification, although simpler, has been done in Section 3 of [1].

Our classification is of particular interest for theories with AdS vacua. It is conjectured that no AdS vacua of string theory admit scale separation [68]. Hence it is not possible to write an effective $\mathcal{N} = 2$ theory in this case. Thus we are led to conjecture that those gauged supergravities that cannot come from consistent truncations and which have AdS vacua must belong to the "swampland". Put differently, these gauged supergravities are lower-dimensional artefacts that are not related to string theory.

We should stress that our classification only determines what are the possible 5d theories that have an eleven or ten dimensional origin but do not guarantees that they are actually realised. An important issue that we do not address here is whether we can actually solve the differential conditions imposed by the intrinsic torsion, that are required for the consistent truncation to exist.

This would require determining the explicit form of the manifold M and checking the differential constraints associated to having singlet intrinsic torsion. The analogous condition is known to limit the possible gaugings in the maximally supersymmetric case [12, 69, 19, 20, 52, 53]. So it is to be expected that the number of actual truncations is even more restricted than what we present here.

We leave this issue for future work. In this thesis we provided particular examples where we found an internal geometry on which the constraint is solved.

One example is the most general truncation around the Maldacena–Nuñez [31] with $\mathcal{N} = 2$ supersymmetries. The solution describes the near horizon geometry of M5-branes wrapping a Riemann surface of negative curvature. The geometry is a warped product of AdS_5 times a 6-dimensional manifold that is a fibration of S^4 over the Riemann surface. The truncation gives a five-dimensional $\mathcal{N} = 2$ supergravity with four vector multiplets, one hypermultiplet and a non-abelian gauging. This is one of the cases in our list. We worked out the full bosonic truncation ansatz and checked that the truncated theory admits an $\mathcal{N} = 2$ AdS_5 vacuum corresponding to the MN solution. This extends the truncation of [61] by $SO(3)$ vector multiplets.

The Maldacena–Nuñez solution is a particular case of a family of solutions founds in [65, 32], which corresponding to different ways in which the M5 wrap the Riemann surfaces. It is easy to extend our construction to truncations on these geometries. we obtained a truncation featuring two vector multiplets, one hypermultiplet and an abelian gauging, completing the truncation in [67]. Although in this case we did not give all details of the truncation ansatz, it should be clear that it can be obtained by following precisely the same steps presented for the case of Maldacena–Nuñez geometry.

Note that our consistent truncations can equivalently be obtained as truncations on a Riemann surface of maximal $SO(5)$ supergravity, that comes from the reduction of eleven-dimensional supergravity on S^4 [70].

What is discussed in this thesis does not cover all the results I obtained during the PhD. The rest can be found in the three papers in the appendix.

There are many directions one can explore in the future.

One direction is to continue the classification programme. Combined with the results obtained for the maximal five dimensional truncations, a classification for any amount of supersymmetry for five dimensional truncation is almost complete, up to discrete structure group and solving the differential constraints in $\mathcal{N} = 2$. The next challenge is to do the same kind of classification in different dimensions. We have developed an algorithmic way of deriving all the consistent truncation for a given dimension with any amount of supersymmetry, with a continuous structure group. The algorithm relies on roots systems associated to the different exceptional groups and their maximal compact subgroup associated to the particular dimension. We are currently considering the case of reductions to four dimensions and we hope to be able to provide soon an equivalent classification as for the five dimensional truncations.

As already mentioned above, it is important to have a better control on the constraint of having a singlet constant generalised intrinsic torsion. It would also be interesting to see whether the approach of [20] can be extended to non-maximally super- symmetric truncations and to use the the five-dimensional embedding tensor to determine what the uplifted geometry should be. We leave this to future work.

It would also be interesting to study whether or not in the truncations we have found there exist an AdS vacua. The existence conditions of such a vacua are given in [71] and can be easily translated into the exceptional geometry language.

Another direction is to construct new explicit examples of truncations. In M-theory, it would be nice to apply the construction of Section IV.1 to the general ansatz for half-maximal AdS_5 solutions of [72]. In particular, this would provide new consistent truncations containing the AdS_5 solutions of [73], describing M5-branes wrapped on Riemann surfaces with punctures. A first step in that direction might be given by extending the work of [74] to non maximal cases. A similar construction is conceivable for the supergravity description of D3-branes wrapped on Riemann surfaces [31], however in this case one would need to use the $E_{8(8)}$ generalised geometry, which is not fully developed yet (though see [75, 76]).

Finally, the work of [77] showed that the generalised formalism can be used to compute the Kaluza-Klein spectrum for deformations of maximally supersymmetric compactifications. It would be very interesting to extend the spectroscopy for solutions with reduced supersymmetry. It is likely that the generalisation relies on an accurate study of the generalised intrinsic torsion.

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Chapter VI
Publications

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Systematics of consistent truncations from generalised geometry

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Abstract: We present a generalised geometry framework for systematically constructing consistent truncations of ten- and eleven-dimensional supergravity preserving varying fractions of supersymmetry. Truncations arise when there is a reduced structure group G_S of the exceptional generalised geometry, such that the intrinsic torsion is a G_S-singlet. The matter content of the truncated theory follows from group-theoretical arguments, while the gauging is determined by the sub-algebra of generalised diffeomorphisms generated by the G_S-singlet vectors. After discussing the general ideas across different spacetime dimensions and amounts of supersymmetry, we provide detailed formulae for truncations to gauged half-maximal supergravity in five dimensions. In particular, we establish an expression for the generalised metric on the exceptional tangent bundle, which determines the scalar truncation ansatz. As applications, we show that this formalism gives a simple derivation of a new consistent truncation of type IIB supergravity on β-deformed Lunin-Maldacena geometries, yielding half-maximal supergravity coupled to two vector multiplets, and of the truncation of eleven-dimensional supergravity on Maldacena-Núñez geometries, given by S^4 twisted over a Riemann surface, which leads to half-maximal supergravity coupled to three vector multiplets.

Keywords: Flux compactifications, Supergravity Models, AdS-CFT Correspondence

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Contents

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1 Introduction

A common problem in string theory and supergravity is how to derive lower-dimensional effective theories. Given a Kaluza-Klein reduction on a compact manifold, a consistent truncation is a procedure to truncate the infinite tower of Kaluza-Klein states to a finite set in a consistent way, such that solutions of equations of motion of the truncated system are also always solutions of the original theory. In other words, the dependence of the higher-dimensional fields on the internal manifold factorises out once the truncation ansatz is plugged in the equations of motion. The classic example, known as a Scherk-Schwarz reduction, is when the internal space is a group manifold $\mathcal{G}$ (or a quotient $\mathcal{G}/\Gamma$ thereof by a freely-acting discrete group Γ) [1]. Consistency is a consequence of keeping only modes invariant under the group action. Aside from these cases, consistent truncations are relatively rare and hard to construct, see for instance [2, 3]. Classic examples of consistent truncations on spaces that are not group manifolds are the truncations of eleven-dimensional supergravity on S^7 [4] and on S^4 [5] both leading to a maximally supersymmetric truncated theory.

Recently, the reformulation of supergravity using Generalised Geometry and Exceptional Field Theory has provided a new framework for giving a systematic geometrical description of maximally supersymmetric consistent truncations, both of conventional Scherk-Schwarz type and the exotic sphere truncations [6–11]. In particular, the notion of a generalised parallelisation allows one to show that all known such truncations are a form of generalised Scherk-Schwarz reductions and to prove the long-standing conjecture of the consistency of type IIB supergravity on S^5 [6, 10, 12]. Extensions of these ideas have also recently been considered in the case of half-maximal truncations in [13–17], mostly focused on reductions to seven- and six-dimensional supergravities, although [15] also discusses more general cases. An appealing feature of the maximal generalised Scherk-Schwarz reductions is that one can determine the lower-dimensional supergravity directly from the generalised geometry, a priori of any explicit substitution into the equations of motion. It is therefore natural to ask whether generalised geometry can give a similar characterisation of generic consistent truncations with any amount of supersymmetry.

In this paper, we derive such a unified framework for constructing consistent truncations with different amounts of supersymmetry (including non-supersymmetric truncations), based on the G-structure of the generalised geometry. The key requirement is that the so-called "intrinsic torsion" [18] of the G-structure contains only singlets. This formalism allows one to easily determine all the features of the lower-dimensional gauged supergravity, such as the amount of supersymmetry, the coset manifold of the scalars, the number of gauge and tensor fields, and the gauging, all directly from the geometry. It also provides a general proof of the conjecture of [19], stating that to any supersymmetric solution to ten- or eleven-dimensional supergravity of the warped product form $\mathrm{AdS}_D \times_{\mathrm{w}} M$, there is a consistent truncation to pure gauged supergravity in D dimensions containing that solution and having the same supersymmetry. As we will see, this statement follows from observing that supersymmetric $\mathrm{AdS}_D \times_{\mathrm{w}} M$ solutions always define a "maximal" supersymmetric generalised G-structure, and the G-invariant tensors then can be used to define a consistent truncation. When the actual generalised G-structure is a subgroup of

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the maximal one, we show that one may go further and obtain a consistent truncation which includes matter multiplets and in some cases preserves more supersymmetry than the vacuum.

The structure of the paper and the main results are as follows. In section 2 we describe the general ideas and apply them to a number of simple cases, notably identifying the maximal G-structure for a given amount of supersymmetry and thus proving the conjecture of [19], and also deriving the field content of the supersymmetric truncations that arise from reductions to $D = 4$ and $D = 5$ on conventional G-structure manifolds, reproducing a number of known results in the literature. In section 3 we focus on truncations leading to half-maximal supergravity in five dimensions, which are based on $\mathrm{E}_{6(6)}$ generalised geometry. This case was first considered in the general analysis of [15], but here we give a number of new results. In particular, we show that the relevant $\mathrm{SO}(5-n) \subset \mathrm{SO}(5,5) \subset \mathrm{E}_{6(6)}$ generalised structure is fully specified by a set of $6+n$ generalised vectors on the internal manifold. We argue that if the algebra of generalised diffeomorphisms (that is, diffeomorphisms together with form-field gauge transformations) generated by these vectors closes with constant coefficients, then the generalised structure has singlet intrinsic torsion and the consistent truncation exists. The resulting five-dimensional half-maximal supergravity is coupled to n vector multiplets, and its gauge algebra is the one generated by the $6+n$ generalised vectors. We give detailed formulae based on these vectors specifying the full bosonic truncation ansatz. In particular, we provide an expression for the generalised metric on the internal manifold, which gives the complete scalar truncation ansatz. This is one of the main results of our work.

In section 4 we apply our formalism to consistent truncations of type IIB supergravity on five-dimensional manifolds preserving half-maximal supersymmetry (that is, 16 out of 32 supercharges). We first illustrate how the formalism works by reproducing the truncation of type IIB supergravity on squashed Sasaki-Einstein manifolds derived in [20, 21]. This is half-maximal supergravity coupled to two vector multiplets and with a $\mathrm{U}(1)\times\mathrm{Heis}_3$ gauging, where Heis_3 denotes the Heisenberg group. Then we argue that when the Sasaki-Einstein manifold is toric, the exact same truncated theory is also obtained by deforming the internal geometry via the TsT transformation of [22] with parameter β. Another way to say this is that we TsT-transform the full truncation ansatz, rather than just the AdS solution. We thus obtain a continuous family of uplifts of the $\mathrm{U}(1) \times \mathrm{Heis}_3$ gauged five-dimensional supergravity, parameterised by β. At the technical level, this is shown by exploiting the fact that the TsT transformation has a simple action in generalised geometry via a bi-vector field. It was recently shown in the S^5 case that such backgrounds admitted a truncation to minimal gauged supergravity (8 supercharges) [23]. Our result shows that they in fact admit a much larger truncation to half-maximal supergravity with two vector multiplets.

In section 5 we derive a consistent truncation of eleven-dimensional supergravity on Maldacena-Núñez geometries where S^4 fibers over a Riemann surface [24], leading to half-maximal supergravity coupled to three vector multiplets and with a $\mathrm{U}(1) \times \mathrm{ISO}(3)$ gauge algebra. We note that the existence of such a consistent truncation, as well as an analysis of its sub-truncations and vacua, was very recently proven using a different approach, considering the explicit truncation directly from seven-dimensional maximal gauged su-

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pergravity [25]. We conclude in section 6 outlining some directions of future research including some more consistent truncations that it would be interesting to explore using our approach.

2 Consistent truncations from G-structures

2.1 Conventional G-structure constructions

Before turning to the generalised geometry picture, let us review the role of conventional G-structures in consistent truncations. Through the study of several cases such as [20, 21, 26–30], it is now understood that any G-structure with constant, singlet intrinsic torsion leads to a consistent truncation.

The idea is as follows. In conventional Scherk-Schwarz reductions on a group manifold $M = \mathcal{G}$ all the higher-dimensional fields in the theory can be decomposed into representations of $\mathcal{G}$. By keeping all the singlet representations and nothing else, one ensures that the truncation is consistent, since products of singlet representations can never source the non-singlet representations that were truncated away. However, this argument extends: the key point is not that the manifold has isometries but that the structure group G_S is reduced, since this allows one to decompose all tensor fields into G_S representations and then keep only those fields transforming as singlets. In the case of a group manifold the structure group is trivial since the manifold is parallelisable, but more generally one can consider cases with larger structure groups.[1] Explicitly, one has

Theorem 1. *Let M be a d-dimensional manifold with a G_S-structure defining a set of invariant tensors $\{\Xi_i\}$ with $G_S \subset \mathrm{O}(n)$ and only constant, singlet intrinsic torsion. Any field theory has a consistent truncation on M defined by expanding all fields in terms of the invariant tensors.*

If the theory includes spinors, then the G_S-structure lifts to a $\tilde{G}_S \subset \mathrm{Spin}(d)$ structure and we can include fermions in the truncation by expanding any spinor fields in terms of spinors invariant under $\tilde{G}_S$.

To explain this in a little more detail, first recall that a choice of G_S-structure on a d-dimensional manifold M is a reduction of the structure group. Formally, a G_S-structure defines a G_S-principal sub-bundle P of the $\mathrm{GL}(d, \mathbb{R})$ frame bundle. In most cases, the structure can equivalently be defined by a set of G_S-invariant, nowhere vanishing tensors $\{\Xi_i\}$. The existence of a G_S-structure means that all tensor fields can be decomposed into irreducible representations of G_S. For example, a choice of $G_S = \mathrm{O}(d)$ structure defines a subset of orthonormal frames, or equivalently is defined by an invariant metric tensor g. A given G_S-structure P is characterised by its intrinsic torsion. If $G_S \subset \mathrm{O}(d)$, this is defined

[1] The same symmetry argument used for Scherk-Schwarz reductions implies that dimensional reductions on coset manifolds $M = \mathcal{G}/\mathcal{H}$ keeping all $\mathcal{G}$-invariant Kaluza-Klein modes and nothing else are consistent. In this case, there is a nice connection with the other argument given above, based on the G-structure of M. Indeed one can show that if $\mathcal{H}$ contains no nontrivial invariant subgroup of $\mathcal{G}$, then $\mathcal{G}/\mathcal{H}$ admits a $\mathcal{G}$-invariant $\mathcal{H}$-structure (see e.g. [31, appendix A]). The $\mathcal{G}$-invariant truncation and the truncation based on singlets of the $\mathcal{H}$-structure then coincide.

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in the following simple way (see for example [32]). Since $G_S \subset \mathrm{O}(d)$ the structure defines a metric g and hence a corresponding Levi-Civita connection ∇. Acting on each invariant tensor Ξ_i we have

$$\begin{aligned}\nabla_m \Xi_i{}^{n_1\ldots n_r}{}_{p_1\ldots p_s} = K_m{}^{n_1}{}_q \Xi_i{}^{q\ldots n_r}{}_{p_1\ldots p_s} + \cdots + K_m{}^{n_r}{}_q \Xi_i{}^{n_1\ldots q}{}_{p_1\ldots p_s} \\ - K_m{}^q{}_{p_1} \Xi_i{}^{n_1\ldots n_r}{}_{q\ldots p_s} + \cdots - K_m{}^q{}_{p_s} \Xi_i{}^{n_1\ldots n_r}{}_{p_1\ldots q},\end{aligned} \tag{2.1}$$

which uniquely defines $K_m{}^n{}_p$ as a section of $T^*M \otimes \mathfrak{g}^\perp$ with m and n, p denoting the T^*M and $\mathfrak{g}^\perp$ indices respectively. Here we have decomposed $\Lambda^2 T^*M \simeq \mathfrak{so}(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ with $\mathfrak{g}$ the Lie algebra of G_S. Note that the $T^*M \otimes \mathfrak{g}$ part is missing in K because, by definition Ξ_i is G_S-invariant. The tensor K defines the intrinsic torsion $(T_{\rm int})_{mn}{}^p = K_n{}^p{}_m - K_m{}^p{}_n$. Note that equivalently one can define a new torsionful connection $\tilde{\nabla} = \nabla - K$ that is compatible with the structure, that is $\tilde{\nabla}\Xi_i = 0$ for all Ξ_i. The intrinsic torsion $T_{\rm int}$ is then the torsion of $\tilde{\nabla}$. In general $T_{\rm int}$ will decompose into G_S representations, known as the "torsion classes" of the G_S-structure. Note that in many examples, the invariant tensors Ξ_i are all differential forms and the intrinsic torsion is completely determined by the exterior derivatives $\mathrm{d}\Xi_i$.

As for reduction on group manifolds, the proof of theorem 1 is very straightforward. By expanding in terms of invariant tensors, all the fields one keeps transform as singlets under the structure group, with the only dependence on the internal space coming from the $\{\Xi_i\}$. Furthermore since the intrinsic torsion has only singlet components (and is independent of the internal space) any derivative of a field is given by the right-hand side of (2.1) and is itself an expansion in terms of singlets. So long as we keep all possible singlets and nothing else, given the equations of motion can be written as generalised tensors, the truncation is then necessarily consistent, since products of singlet representations can never source the non-singlet representations that were truncated away.

Focusing on the gravity sector, the scalars and vector fields in the consistent truncation appear in the following way. Recall that the choice of metric parameterises a $\mathrm{GL}(d,\mathbb{R})/\,\mathrm{O}(d)$ coset. To count the number of G_S singlets in the metric we can use the commutant of G_S in $\mathrm{GL}(d,\mathbb{R})$ and $\mathrm{O}(d)$. The scalars in the consistent truncation coming from the metric thus parameterise

$$\text{metric scalars} \quad \Leftrightarrow \quad H \in \frac{\mathrm{C}_{\mathrm{GL}(d,\mathbb{R})}(G_S)}{\mathrm{C}_{\mathrm{O}(d)}(G_S)}\,, \tag{2.2}$$

where $\mathrm{C}_K(A)$ denotes the commutant of $A \subset K$ inside K. We can also count the number of vectors coming from the metric, by counting the number of invariant one-forms $\eta^a \in \{\Xi_i\}$, giving

$$\text{metric gauge fields} \quad \Leftrightarrow \quad \mathcal{A}^a\,\hat{\eta}_a, \tag{2.3}$$

where $\hat{\eta}_a$ are the dual singlet vectors. For singlet torsion, the torsion is completely determined by the Lie derivatives of the invariant tensors

$$\mathcal{L}_{\hat{\eta}_a}\Xi_i = f_{ai}{}^j\,\Xi_j\,, \tag{2.4}$$

where $f_{ai}{}^j$ are constants, fixed by the intrinsic torsion. For example, the gauging of the truncated theory depends on the Lie bracket

$$\left[\hat{\eta}_a, \hat{\eta}_b\right] = f_{ab}{}^c\,\hat{\eta}_c\,, \tag{2.5}$$

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and we see that the singlet intrinsic torsion determines the gauge algebra of the metric gauge fields.

To see how the construction works in practice consider the reduction on a Sasaki-Einstein manifold M of dimension $d = 2n+1$, which appeared in the context of reductions of M-theory and type IIB in [26] and [20, 21] respectively. The invariant tensors (η, ω, Ω), where η is a real one-form, ω a real two-form and Ω a complex n-form on M, define an $G_S = \mathrm{SU}(n) \subset \mathrm{GL}(d, \mathbb{R})$ structure and satisfy

$$\mathrm{d}\eta = 2\omega, \qquad \mathrm{d}\Omega = \mathrm{i}\,(n+1)\,\eta \wedge \Omega\,, \tag{2.6}$$

implying we indeed have constant singlet torsion, since only invariant tensors appear on the right-hand sides of these equations. In this case the metric scalar manifold is

$$\frac{\mathrm{C}_{\mathrm{GL}(2n+1,\mathbb{R})}(\mathrm{SU}(n))}{\mathrm{C}_{\mathrm{SO}(2n+1)}(\mathrm{SU}(n))} = \frac{\mathbb{R}^+ \times \mathbb{C}}{\mathrm{U}(1)} = \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.7}$$

where the first $\mathbb{R}^+$ comes from $\mathrm{C}_{\mathrm{GL}(2n+1)}(\mathrm{GL}(2n))$ and $\mathbb{C}$ from $\mathrm{C}_{\mathrm{GL}(2n)}(\mathrm{SU}(n))$. There is a single invariant one-form η and so there will be a single gauge field $\mathcal{A}_\mu(x)$ coming from the metric. Concretely the consistent truncation on M is defined by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2U}\mathrm{d}s^2_{2n} + \mathrm{e}^{2V}(\eta + \mathcal{A})^2, \tag{2.8}$$

where $\mathrm{d}s^2_{2n}$ is the (local) $2n$-dimensional Kähler-Einstein metric defined by (ω, Ω). The scalars fields $U(x)$ and $V(x)$ parametrise the scalar manifold H.

The Scherk-Schwarz reduction $M = \mathcal{G}$ is of course itself also an example. The group structure picks out a preferred co-frame $\{e^a\} \in T^*M$ of (say) left-invariant one-forms. Geometrically the one-forms define an "identity structure" $G_S = \mathbb{1} \supset \mathrm{GL}(d)$ (or parallelisation). Since $\mathrm{C}_K(\mathbb{1}) = K$, the scalar fields are in the coset

$$\frac{\mathrm{C}_{\mathrm{GL}(d)}(\mathbb{1})}{\mathrm{C}_{\mathrm{SO}(d)}(\mathbb{1})} = \frac{\mathrm{GL}(d,\mathbb{R})}{\mathrm{SO}(d)}\,. \tag{2.9}$$

The one-forms define d gauge fields with a Lie algebra given by the Lie bracket (2.5). The consistent truncation ansatz for the metric is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + h_{ab}\big(e^a + \mathcal{A}^a\big)\big(e^b + \mathcal{A}^b\big)\,, \tag{2.10}$$

where $h_{ab}(x)$ is matrix of scalar fields and the $\mathcal{A}^a_\mu(x)$ are gauge fields in the adjoint of G_S.

Any number of other examples can be constructed. We note that the standard consistent truncation keeping a volume modulus on an orientable manifold can be thought of arising from the corresponding $\mathrm{SL}(n, \mathbb{C})$-structure. Similarly the universal sector of type II Calabi-Yau compactifications arises from keeping $\mathrm{SU}(n)$-singlet fields in the metric and form-field degrees of freedom.

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2.2 Generalised G-structure constructions

We can now extend this picture to generalised geometry to describe the consistent truncations of eleven-dimensional and type II supergravities on d- and $(d-1)$-dimensional manifolds M respectively. The generalisation is straightforward: we replace the conventional G_S-structures with generalised G-structure on the generalised tangent space E associated to M. The generic structure group on E is the exceptional group $\mathrm{E}_{d(d)}$ which has a maximal compact R-symmetry subgroup H_d (see table 1). If a $G_S \subset H_d$ structure is defined by a set of generalised invariant tensors the idea is then to expand the supergravity fields in terms of the tensors used to define the consistent truncation. This is a generalisation of the construction given in [6], where it was shown that maximally supersymmetric consistent truncations corresponded to "Leibniz parallelisations", that is, identity structures $G_S = \mathbb{1}$ with constant intrinsic torsion.

2.2.1 Main theorem

Let us start by stating the result and then discuss more details of the generalised geometry and the proof of the statement. We claim

Theorem 2. *Let M be a d-dimensional (respectively $(d-1)$-dimensional) manifold with a generalised G_S-structure defining a set of invariant tensors $\{Q_i\}$ with $G_S \subset H_d$ and only constant, singlet intrinsic torsion. Then there is a consistent truncation of eleven-dimensional (respectively type II) supergravity on M defined by expanding all bosonic fields in terms of the invariant tensors. If $\tilde{H}_d$ is the double cover of H_d, acting on fermions the structure group lifts to $\tilde{G}_S \subset \tilde{H}_d$ and the truncation extends to the fermionic sector, provided again one expands the spinor fermion fields in terms of $\tilde{G}_S$ singlets.*

To see how this works, we start by summarising the generalised geometry reformulation of eleven-dimensional or type II supergravity on a product space $X \times M$ where X is a D-dimensional Lorentzian space, and the internal manifold M is d-dimensional, or, in the case of type II supergravity, $(d-1)$-dimensional. In generalised geometry, the $\mathrm{GL}(d,\mathbb{R})$ or $\mathrm{GL}(d-1,\mathbb{R})$ structure group of conventional geometry on M is extended to $\mathrm{E}_{d(d)}$ for $d \leq 7$ [33, 34]. This allows one to reformulate supergravity, so that the bosonic supergravity fields and their equations of motion are rearranged into generalised tensors transforming as representations of $\mathrm{GL}(D,\mathbb{R}) \times \mathrm{E}_{d(d)}$. The $\mathrm{GL}(D,\mathbb{R})$ scalar degrees of freedom are repackaged into a generalised metric, that is a symmetric generalised tensor $G \in \Gamma(S^2 E^*)$ which is invariant under the R-symmetry subgroup $H_d \subset \mathrm{E}_{d(d)}$. Thus geometrically the generalised metric defines an H_d-structure [35, 36]. The $\mathrm{GL}(D,\mathbb{R})$ one-form, vector degrees of freedom are sections of the generalised tangent space E, while the two-form tensor degrees of freedom are sections of a generalised tensor bundle here denoted N [37–41]. In summary we have

$$\begin{aligned} \text{scalars:} \quad & G_{MN}(x,y) \in \Gamma(S^2 E^*)\,, \\ \text{vectors:} \quad & \mathcal{A}_\mu{}^M(x,y) \in \Gamma(T^*X \otimes E)\,, \\ \text{two-forms:} \quad & \mathcal{B}_{\mu\nu}{}^{MN}(x,y) \in \Gamma(\Lambda^2 T^*X \otimes N)\,, \end{aligned} \tag{2.11}$$

where x and y are coordinates on X and M respectively, the index M denotes components of vectors in E (or E^* if lowered) and we are using the fact that $N \subset S^2 E$. One

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$\mathrm{E}_{d(d)}$	E	N	W	$\tilde{H}_d$	$\mathcal{S}$
$\mathrm{E}_{7(7)}$	**56**	**133**	$\mathbf{912} \oplus \mathbf{56}$	SU(8)	$\mathbf{8} \oplus \bar{\mathbf{8}}$
$\mathrm{E}_{6(6)}$	**27**	$\mathbf{27}'$	$\mathbf{351} \oplus \mathbf{27}'$	USp(8)	**8**
Spin(5, 5)	$\mathbf{16}^s$	**10**	$\mathbf{144}^s \oplus \mathbf{16}^c$	USp(4) × USp(4)	$(\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4})$
$\mathrm{SL}(5, \mathbb{R})$	**10**	$\mathbf{5}'$	$\mathbf{40} \oplus \mathbf{15}' \oplus \mathbf{10}'$	USp(4)	**4**

Table 1. Generalised geometry groups, bundles and representations.

can also further introduce higher form-field degrees of freedom following the tensor hierarchy [42, 43]. However, these do not introduce new degrees of freedom but are dual to the scalar, vector and two-forms.[2] The relevant groups and $\mathrm{E}_{d(d)}$ representations are all listed in table 1. Note that $\tilde{H}_d$ is actually the double cover of H_d. The dynamics of the supergravity is completely determined by the Levi-Civita connection on the external space and a generalised connection D on the internal space. The latter is the generalised analogue of the Levi-Civita connection: it has vanishing generalised torsion and is compatible with the generalised metric. We also include in table 1 the $\mathrm{E}_{d(d)}$ representation of the generalised tensor bundle W in which the generalised torsion lies and the $\tilde{H}_d$ representation of the spinor bundle $\mathcal{S}$ in which the supersymmetry parameter lies [36].

Now suppose we have a reduced structure group $G_S \subset H_d$ defined by a set of G_S-invariant generalised tensors $\{Q_i\}$. As described in [18], one can again define an intrinsic torsion T_{int} for the generalised G_S-structure, and decompose it into representations of G_S. The definition is as follows. Let $\tilde{D}$ be a generalised connection compatible with the G_S-structure, that is, sastisfying $\tilde{D}Q_i = 0$ for all Q_i. Formally, the generalised torsion T of $\tilde{D}$ is defined by, acting on any generalised tensor α,

$$\left(L_V^{\tilde{D}} - L_V\right)\alpha = T(V) \cdot \alpha \tag{2.12}$$

where L is the generalised Lie derivative, $L^{\tilde{D}}$ is the generalised Lie derivative calculated using $\tilde{D}$ and we view the torsion as a map $T : \Gamma(E) \to \Gamma(\mathrm{ad}\,\tilde{F})$ where $\mathrm{ad}\,\tilde{F}$ is the $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ adjoint bundle, so that $T(V)$ acts via the adjoint action on α. The intrinsic torsion is then the component of T that is independent of the choice of compatible connection $\tilde{D}$. We are interested in the case where only singlet representations appear in the intrinsic torsion. This means we can define a generalised Levi-Civita connection such that, in analogy with (2.1), acting on any invariant generalised tensor Q_i,

$$D_M Q_i = \Sigma_M \cdot Q_i \tag{2.13}$$

where Σ_M is a section of $E^* \otimes \mathrm{ad}\,P_{H_d}$ that is completely determined in terms of the singlet torsion.[3] Here we are using a notion where $\mathrm{ad}\,P_{H_d}$ is the bundle of tensors transforming on the adjoint representation of H_d.

[2] Note that for $D = 4$ this means the $\mathcal{A}_\mu{}^M$ contain both the vectors and their duals, and in $D = 6$ the $\mathcal{B}_{\mu\nu}{}^{MN}$ contain both the two-forms and their duals.

[3] Note there is a subtlety that the connection D is not uniquely determined by the conditions of compatibility with the generalised metric and being torsion-free. However only certain projections of the action of D appear in the supergravity and these are unique [35]. In equation (2.13), we are choosing a particular torsion-free compatible D. Equivalently, one can show that the unique projected operators, acting on Q_i, are completely determined by the singlet intrinsic torsion.

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The proof of consistency is just as before. By expanding in terms of invariant tensors, all the fields one keeps transform as singlets under the structure group, with the only dependence on the internal space coming from the $\{Q_i\}$. Furthermore from (2.13) the derivatives of all the truncated fields also have expansions in terms of singlets. So long as we keep all possible singlets and nothing else, the truncation is then necessarily consistent, since products of singlet representations can never source the non-singlet representations that were truncated away.

2.2.2 Structure of the truncated theory

So far we have made a general argument that a G_S-structure with singlet intrinsic torsion will lead to a consistent truncation of eleven-dimensional or type II supergravity. However, one can go further and deduce the structure of the truncated theory from the G_S-structure and the torsion. We will find that in all cases, even when there is no preserved supersymmetry, it is described by a version of the embedding tensor formalism (see e.g. [44, 45] for a review of this formalism).

We start by identifying the G_S-singlet truncated degrees of freedom. Since $G_S \subset H_d$ the structure encodes the generalised metric G_{MN}. In the truncation we want to keep singlet deformations of the structure, modulo those singlet deformations that do not deform the metric. At each point in M the metric is an element of the coset $\mathrm{E}_{d(d)}/H_d$, thus we can generate the singlet deformations of the metric by acting on the structure by elements of $\mathrm{E}_{d(d)}$ that commute with G_S modulo elements of H_d that commute with G_S, since the latter will not change the metric. Thus we find the scalars parametrise the coset

$$\text{scalars:} \qquad h^I(x) \in \mathcal{M}_{\text{scal}} = \frac{\mathrm{C}_{\mathrm{E}_{d(d)}}(G_S)}{\mathrm{C}_{H_d}(G_S)} := \frac{\mathcal{G}}{\mathcal{H}}\,. \tag{2.14}$$

Recall that the vector fields are sections of $T^*X \otimes E$. If $\{K_{\mathcal{A}}\}$ is a basis for the G_S-invariant generalised vectors, spanning a vector space $\mathcal{V} \subset \Gamma(E)$, then we have

$$\text{vectors:} \qquad \mathcal{A}_\mu{}^{\mathcal{A}}(x)\, K_{\mathcal{A}} \in \Gamma(T^*M) \otimes \mathcal{V}. \tag{2.15}$$

If $\{J_\Sigma\}$ is a basis generating the G_S-invariant vector space $\mathcal{B} \subset \Gamma(N)$, we similarly have the two-form degrees of freedom

$$\text{two-forms:} \qquad \mathcal{B}_{\mu\nu}{}^{\Sigma}(x) J_\Sigma \in \Gamma(\Lambda^2 T^*X) \otimes \mathcal{B}. \tag{2.16}$$

Note that by definition $\mathcal{V}$ and $\mathcal{B}$ are both representation spaces for the action of the commutant group $\mathcal{G}$. Note we also have $N \subset S^2 E$ and so we can use the projection map $\times_N$ and embedding to define the constants $d_{\mathcal{A}\mathcal{B}}{}^{\Sigma}$ and $\tilde{d}_{\Sigma}{}^{\mathcal{A}\mathcal{B}}$

$$K_{\mathcal{A}} \times_N K_{\mathcal{B}} = d_{\mathcal{A}\mathcal{B}}{}^{\Sigma} J_\Sigma, \qquad J_\Sigma = \tilde{d}_{\Sigma}{}^{\mathcal{A}\mathcal{B}} K_{\mathcal{A}} \otimes K_{\mathcal{B}}\,, \tag{2.17}$$

intertwining the representation spaces.

Turning to the singlet intrinsic torsion, we note that, since $\tilde{D} K_{\mathcal{A}} = 0$, in analogy with (2.4), we have

$$L_{K_{\mathcal{A}}} Q_i = -T_{\text{int}}(K_{\mathcal{A}}) \cdot Q_i\,, \tag{2.18}$$

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where we recall that L is the generalised Lie derivative. Since $T_{\rm int}$ is a singlet, then $T_{\rm int}(K_{\mathcal{A}})$ must be a singlet of $\mathrm{ad}\,\tilde{F}$, but such singlets are precisely the Lie algebra of the commutant group $\mathcal{G} = \mathrm{C}_{\mathrm{E}_{d(d)}}(G_S)$. Thus $-T_{\rm int}$ defines an "embedding tensor" [44, 45], that is a linear map

$$\Theta : \mathcal{V} \to \mathrm{Lie}\,\mathcal{G}\,. \tag{2.19}$$

Acting on the $K_{\mathcal{A}}$, we get

$$L_{K_{\mathcal{A}}} K_{\mathcal{B}} = \Theta_{\mathcal{A}} \cdot K_{\mathcal{B}} = \Theta_{\mathcal{A}}{}^{\hat{\alpha}} (t_{\hat{\alpha}})_{\mathcal{B}}{}^{\mathcal{C}} K_{\mathcal{C}} := X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} K_{\mathcal{C}}\,, \tag{2.20}$$

where $(t_{\hat{\alpha}})_{\mathcal{B}}{}^{\mathcal{C}}$ are the representations of the generators of $\mathrm{Lie}\,\mathcal{G}$ acting on $\mathcal{V}$. The Leibniz property of the generalised Lie derivative then implies [6, 35] the standard quadratic condition on the embedding tensor

$$[X_{\mathcal{A}}, X_{\mathcal{B}}] = -X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} X_{\mathcal{C}}\,, \tag{2.21}$$

where we are viewing $(X_{\mathcal{A}})_{\mathcal{B}}{}^{\mathcal{C}} = X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}$ as a matrix. Thus we can view the $K_{\mathcal{A}}$ as generating a Lie algebra with structure constants $X_{[\mathcal{A}\mathcal{B}]}{}^{\mathcal{C}}$. Since the image of Θ may not be the whole of $\mathrm{Lie}\,\mathcal{G}$, we see that the vector fields describe a gauge group

$$\text{gauge group:} \qquad G_{\rm gauge} \subseteq \mathcal{G}\,, \tag{2.22}$$

where $\mathrm{Lie}\,G_{\rm gauge} = \mathrm{Im}\,\mathcal{V} \subseteq \mathrm{Lie}\,\mathcal{G}$ under the embedding tensor map Θ. The $X_{\mathcal{A}}$ then define the adjoint representation and Θ defines how the gauge action embeds as an action in $\mathcal{G}$.

By reducing the generalised geometry/EFT reformulation of supergravity of [35–41], we can then summarise the structure and gauging of the truncated theory, which match the standard formulae for gauging of a tensor hierarchy via an embedding tensor [44, 45]:

- *The fields in the truncated theory are as follows*

$$\begin{aligned} &\textit{scalars:} && h^I(x) \in \mathcal{M}_{\rm scal} = \frac{\mathrm{C}_{\mathrm{E}_{d(d)}}(G_S)}{\mathrm{C}_{H_d}(G_S)} := \frac{\mathcal{G}}{\mathcal{H}}\,, \\ &\textit{vectors:} && \mathcal{A}^{\mathcal{A}}_{\mu}(x)\, K_{\mathcal{A}} \quad \in \Gamma(T^*X) \otimes \mathcal{V}\,, \\ &\textit{two-forms:} && \mathcal{B}^{\Sigma}_{\mu\nu}(x)\, J_{\Sigma} \quad \in \Gamma(\Lambda^2 T^*X) \otimes \mathcal{B}\,. \end{aligned} \tag{2.23}$$

- *The theory is gauged by $G_{\rm gauge} \subseteq \mathcal{G}$ with the scalar covariant derivatives*

$$\hat{D}_{\mu} h^I = \partial_{\mu} h^I - \mathcal{A}^{\mathcal{A}}_{\mu}\, \Theta_{\mathcal{A}}{}^{\hat{\alpha}} k_{\hat{\alpha}}{}^I\,, \tag{2.24}$$

  where $k_{\hat{\alpha}}$ are the Killing vectors on $\mathcal{M}_{\rm scal}$ generating the action of the $\mathrm{Lie}\,\mathcal{G}$*.*

- *The gauge transformations of the vectors and two-forms are*

$$\begin{aligned} \delta \mathcal{A}^{\mathcal{A}}_{\mu} &= \partial_{\mu} \Lambda^{\mathcal{A}} + X_{\mathcal{B}\mathcal{C}}{}^{\mathcal{A}} \big(\mathcal{A}^{\mathcal{B}}_{\mu}\, \Lambda^{\mathcal{C}} - \Xi^{\mathcal{B}\mathcal{C}}_{\mu} \big)\,, \\ \delta \mathcal{B}^{\Sigma}_{\mu\nu} &= 2 d_{\mathcal{A}\mathcal{B}}{}^{\Sigma} \Big(\partial_{[\mu} \Xi^{\mathcal{A}\mathcal{B}}_{\nu]} + 2 X_{\mathcal{C}\mathcal{D}}{}^{\mathcal{A}} \mathcal{A}^{\mathcal{C}}_{[\mu}\, \Xi^{\mathcal{D}\mathcal{B}}_{\nu]} - \Lambda^{\mathcal{A}} \mathcal{H}^{\mathcal{B}}_{\mu\nu} - \mathcal{A}^{\mathcal{A}}_{[\mu}\, \delta \mathcal{A}^{\mathcal{B}}_{\nu]} \Big)\,, \end{aligned} \tag{2.25}$$

  where $\Xi^{\mathcal{A}\mathcal{B}}_{\mu} = \Xi_{\mu}{}^{\Sigma} \tilde{d}_{\Sigma}{}^{\mathcal{A}\mathcal{B}}$ *and* $\mathcal{H}^{\mathcal{A}} = \mathrm{d}\mathcal{A}^{\mathcal{A}} + X_{\mathcal{B}\mathcal{C}}{}^{\mathcal{A}} (\mathcal{A}^{\mathcal{B}} \wedge \mathcal{A}^{\mathcal{C}} + \mathcal{B}^{\Sigma} \tilde{d}_{\Sigma}{}^{\mathcal{B}\mathcal{C}})$*.*

- *Given a lift $\tilde{G}_S \subseteq \tilde{H}_d$, the number of supersymmetries preserved by the truncated theory is given by the number of $\tilde{G}_S$-singlets in the generalised spinor bundle $\mathcal{S}$.*

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$\tilde{H}_d$	$G_{\mathcal{N}}$
SU(8)	SU$(8-\mathcal{N})$
USp(8)	USp$(8-2\mathcal{N})$
USp(4) × USp(4)	USp$(4-2\mathcal{N}_+)\times$ USp$(4-2\mathcal{N}_-)$
USp(4)	USp$(4-2\mathcal{N})$

Table 2. Maximal generalised structure subgroups $G_{\mathcal{N}} \subset \tilde{H}_d$ preserving $\mathcal{N}$ supersymmetries in the truncated theory. Note that for $d = 5$ we have six-dimensional supergravity with $(\mathcal{N}_+, \mathcal{N}_-)$ supersymmetry.

The key point here is that the geometrical data of the G_S-structure and its singlet intrinsic torsion completely determine the truncated theory. The precise relationship between these expressions and the uplifted supergravity fields depends on the normalisations of the basis vectors $K_{\mathcal{A}}$ and J_Σ and the explicit expression for the generalised metric G_{MN} in terms of the relevant normalised invariant tensors. We will turn to the details of these relationships in the explicit example of half-maximal truncations in the following sections.

2.3 Maximal structure groups and pure supergravities

To see how the truncated theories arise for some specific structure groups and match known consistent truncations, in this and the next sub-section let us focus on truncations preserving a given amount of supersymmetry in $D = 11 - d$ dimensions. For $\mathcal{N}$ supersymmetries the generalised spinor bundle $\mathcal{S}$ must have $\mathcal{N}$ singlets when decomposed under the structure group $G_S \subset \tilde{H}_d$.[4] Let $G_{\mathcal{N}}$ be the *maximal* subgroup of $\tilde{H}_d$ for which this is true, that is the largest possible generalised structure group that preserves $\mathcal{N}$ supersymmetries. These groups are listed in table 2.

We can then use our formalism to determine the corresponding consistent truncations. In each case we need to find the commutant groups $\mathcal{G}$ and $\mathcal{H}$ and the spaces of vector and tensor multiplets. Both are fixed once one knows the embedding $G_{\mathcal{N}} \subset \mathrm{E}_{d(d)}$. The results are summarised in table 3. For the vector and two-form degrees of freedom we include only the minimum dynamical set. In particular, in $D = 4$ and $D = 5$, the two-forms are dual to scalars and vectors respectively, and so are not listed. For vectors in $D = 4$ and two-forms in $D = 6$ we include both the fields and their duals. In $D = 6$ the self- and anti-self-dual two-forms are distinguished by their transformation under the two R-symmetry groups. Comparing with the standard literature (see for example the review in [46]) we see that these theories are in one-to-one correspondence with the possible pure supergravity theories. This includes, in particular, the maximally supersymmetric cases of the sphere reductions. In each case, the gauging of the theory will depend on the singlet torsion, as described for the sphere cases in [6].

From one perspective, this is not surprising — the representation theory is the same as that giving each pure supergravity theory as truncation of the maximally supersymmetric

[4]Note that here and in the following subsection we will ignore discrete factors in the structure group and hence ignore the possible distinction between G_S and $\tilde{G}_S$.

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$\mathrm{E}_{d(d)}$	$\mathcal{N}$	$\mathcal{G} = \mathrm{C}_{\mathrm{E}_{d(d)}}(G_S)$	$\mathcal{H} = \mathrm{C}_{\tilde{H}_d}(G_S)$	$\mathcal{V}$	$\mathcal{B}$
$\mathrm{E}_{7(7)}$	1	$\mathrm{U}(1)$	$\mathrm{U}(1)$	–	
	2	$\mathrm{SU}(2)\times\mathrm{U}(1)$	$\mathrm{SU}(2)\times\mathrm{U}(1)$	$\mathbf{1}\oplus\mathbf{1}$	
	3	$\mathrm{SU}(3)\times\mathrm{U}(1)$	$\mathrm{SU}(3)\times\mathrm{U}(1)$	$\mathbf{3}\oplus\bar{\mathbf{3}}$	
	4	$\mathrm{SU}(4)\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SU}(4)\times\mathrm{U}(1)$	$(\mathbf{6},\mathbf{2})$	
	5	$\mathrm{SU}(5,1)$	$\mathrm{SU}(5)\times\mathrm{U}(1)$	$\mathbf{20}$	
	6	$\mathrm{SO}^*(12)$	$\mathrm{SU}(6)\times\mathrm{U}(1)$	$\mathbf{32}$	
	8	$\mathrm{E}_{7(7)}$	$\mathrm{SU}(8)$	$\mathbf{56}$	
$\mathrm{E}_{6(6)}$	1	$\mathrm{USp}(2)$	$\mathrm{USp}(2)$	$\mathbf{1}$	
	2	$\mathrm{USp}(4)\times\mathbb{R}^+$	$\mathrm{USp}(4)$	$\mathbf{5}+\mathbf{1}$	
	3	$\mathrm{SU}^*(6)$	$\mathrm{USp}(6)$	$\mathbf{15}$	
	4	$\mathrm{E}_{6(6)}$	$\mathrm{USp}(8)$	$\mathbf{27}$	
$\mathrm{Spin}(5,5)$	$(1,0)$	$\mathrm{USp}(2)$	$\mathrm{USp}(2)$	–	$\mathbf{1}$
	$(1,1)$	$\mathrm{USp}(2)\times\mathrm{USp}(2)\times\mathbb{R}^+$	$\mathrm{USp}(2)\times\mathrm{USp}(2)$	$(\mathbf{2},\mathbf{2})$	$2\cdot(\mathbf{1},\mathbf{1})$
	$(2,0)$	$\mathrm{USp}(4)$	$\mathrm{USp}(4)$	–	$\mathbf{5}$
	$(2,1)$	$\mathrm{SU}^*(4)\times\mathrm{USp}(2)$	$\mathrm{USp}(4)\times\mathrm{USp}(2)$	$(\mathbf{4},\mathbf{2})$	$(\mathbf{6},\mathbf{1})$
	$(2,2)$	$\mathrm{Spin}(5,5)$	$\mathrm{USp}(4)\times\mathrm{USp}(4)$	$\mathbf{16}$	$\mathbf{10}$
$\mathrm{SL}(5,\mathbb{R})$	1	$\mathrm{USp}(2)\times\mathbb{R}^+$	$\mathrm{USp}(2)$	$\mathbf{3}$	$\mathbf{1}$
	2	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{USp}(4)$	$\mathbf{10}$	$\mathbf{5}$

Table 3. Commutant groups and $\mathcal{G}$-representations of vectors and two-forms for $G_{\mathcal{N}}$-structure consistent truncations.

one in that dimensions. However, this analysis does allow us to give a proof of the conjecture in [19] (see also [47, 48]):

Corollary. *Any supergravity solution with a D-dimensional AdS (or Minkowski) factor preserving $\mathcal{N}$ supersymmetries, defines a consistent truncation to the corresponding pure supergravity theory.*

The proof follows from the analysis of supersymmetric background in [18, 49, 50]. There it was showed that solutions with AdS (or Minkowski) factors with $\mathcal{N}$ supersymmetries correspond to $G_{\mathcal{N}}$ generalised structures with singlet torsion. The corollary then follows as a direct application of Theorem 2. For the Minkowski space case, the intrinsic torsion vanishes and the truncated theory is ungauged.

2.4 Supersymmetric truncations from conventional G-structures

The more interesting case is when the structure group G_S is a subgroup of $G_{\mathcal{N}}$ but one still has the same number of supersymmetries, that is, the same number of G_S-singlets in the generalised spin bundle $\mathcal{S}$, since this can allow for truncated theories with non-trivial matter

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$\mathrm{E}_{d(d)}$	$\mathcal{N}$	G_S	$\mathcal{G} = \mathrm{C}_{\tilde{H}_d}(G_S)$	$\mathcal{H} = \mathrm{C}_{\mathrm{E}_{d(d)}}(G_S)$	$\mathcal{V}$
$\mathrm{E}_{7(7)}$	1	G_2	$\mathrm{SL}(2,\mathbb{R})$	$\mathrm{U}(1)$	–
	2	$\mathrm{SU}(3)$	$\mathrm{SU}(2,1) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{U}(1)^2$	$2 \cdot (\mathbf{1},\mathbf{2})$
	4	$\mathrm{SU}(2)$	$\mathrm{SO}(6,3) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(6) \times \mathrm{SO}(3) \times \mathrm{U}(1)$	$(\mathbf{9},\mathbf{2})$
$\mathrm{E}_{6(6)}$	1	$\mathrm{SU}(3)$	$\mathrm{SU}(2,1)$	$\mathrm{SU}(2) \times \mathrm{U}(1)$	$\mathbf{1}$
	2	$\mathrm{SU}(2)$	$\mathrm{SO}(5,2) \times \mathbb{R}^+$	$\mathrm{SO}(5) \times \mathrm{SO}(2)$	$\mathbf{7} \oplus \mathbf{1}$

Table 4. Commutant groups and $\mathcal{G}$-representations of the vector fields for consistent truncations using conventional G-structures.

content. A simple way to achieve this situation is to consider the case of a conventional G-structure that corresponds to the appropriate number of supersymmetries. This analysis will allow us to connect to a number of known consistent truncations, including cases that require considerable calculation to derive the structure of the truncated theory.

For definiteness we consider the cases of truncations of eleven-dimensional and type IIB supergravity to $D = 4$ or $D = 5$ on manifolds with G_2, SU(3) or SU(2) conventional G-structures. Calculating the commutant groups and the representation of the space of vector fields $\mathcal{V}$ we find the structure of the truncated theory is the same, independent of whether it came from eleven-dimensional or type IIB supergravity. We list the relevant groups and representations in table 4. Note that for $D = 4$ we give both the vectors and their duals, forming doublets of the $\mathrm{SL}(2,\mathbb{R})$ subgroup of $\mathcal{G}$.

In each case we can identify the multiplet structure of the truncated theory and match to known examples of truncations, as follows:

$G_2 \subset \mathrm{E}_{7(7)}$ structure. This case only refers to eleven-dimensional supergravity. Singlet intrinsic torsion implies a *weak G_2 manifold*. The $D = 4$ truncated theory is $\mathcal{N} = 1$ supergravity coupled to a single chiral multiplet

$$\mathcal{M}_{\mathrm{scal}} = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}, \tag{2.26}$$

and there are no vector multiplets, matching the truncation first derived in [26].

$\mathrm{SU}(3) \subset \mathrm{E}_{7(7)}$ structure. The $D = 4$ truncated theory is $\mathcal{N} = 2$ supergravity coupled to a single hypermultiplet and a single vector multiplet, with the scalar manifolds

$$\mathcal{M}_{\mathrm{scal}} = \mathcal{M}_{\mathrm{hyper}} \times \mathcal{M}_{\mathrm{vector}} = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)}. \tag{2.27}$$

For eleven-dimensional supergravity this includes the case of consistent truncation on a *Sasaki-Einstein seven-manifold* first derived in [26]. For type IIB, it includes the case of the universal sector of *nearly Kähler reductions*, the analogue of the IIA case considered in [27, 28].

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SU(2) $\subset$ $\mathbf{E_{7(7)}}$ structure. The $D = 4$ truncated theory is $\mathcal{N} = 4$ (half-maximal) supergravity coupled to three vector multiplets, with scalar manifold

$$\mathcal{M}_{\rm scal} = \frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,3)}{\mathrm{SO}(6)\times\mathrm{SO}(3)} \,. \tag{2.28}$$

For eleven-dimensional supergravity this includes the case consistent truncation on a *tri-Sasaki seven-manifold* first derived in [29].

SU(3) $\subset$ $\mathbf{E_{6(6)}}$ structure. This case only refers to eleven-dimensional supergravity. The $D = 5$ truncated theory is minimal supergravity coupled to a single hypermultiplet

$$\mathcal{M}_{\rm scal} = \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)} \,, \tag{2.29}$$

and has only the graviphoton with no extra gauge fields. For the case of vanishing intrinsic torsion the theory is just the universal sector of eleven-dimensional supergravity compactified on a Calabi-Yau manifold.

SU(2) $\subset$ $\mathbf{E_{6(6)}}$ structure. The $D = 5$ truncated theory is half-maximal supergravity coupled to two vector multiplets, with the scalar manifolds

$$\mathcal{M}_{\rm scal} = \mathbb{R}^+ \times \frac{\mathrm{SO}(5,2)}{\mathrm{SO}(5)\times\mathrm{SO}(2)} \,. \tag{2.30}$$

For type IIB supergravity this includes the case of consistent truncation on a *Sasaki-Einstein five-manifold* derived in [20, 21]. We will analyse this case in considerable detail in section 4.2.

In each of these cases the gauging of the theory will depend on the particular intrinsic torsion, via the embedding tensor Θ defined by (2.18). Rather than work through the details in each case here we will focus in the following sections on the particular class of half-maximal $D = 5$ truncations. This will in particular include the details of the Sasaki-Einstein five-manifold example. We will also go further and discuss more involved examples. Finally, we note that we could also have considered cases above where G_S is a subgroup of the conventional SU(3) or SU(2) structure groups such that we still have the same amount of supersymmetry. These would be relevant for example, to the consistent truncation of type IIB on the $T^{1,1}$ coset space [51, 52] (which admits a left-invariant $\mathrm{U}(1) \subset \mathrm{SU}(2)$ structure) and of eleven-dimensional supergravity on the various coset spaces considered in [30].

3 Half-maximal truncations to five dimensions

In order to make the general formalism more explicit, in the following sections we will focus on the case of consistent truncations of type IIB and eleven-dimensional supergravity to five dimensions, preserving half-maximal supersymmetry. In this section we will give the details of the generic formalism, identifying the possible structure groups G_S, the invariant generalised tensors and, in particular, how they determine the generalised metric. Concrete examples will be discussed in the following sections. We note that the case of half-maximal

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truncations to five (and other) dimensions using exceptional field theory was first considered in the general analysis of [15]. Here we give a number of new results, both for how the generalised structure is defined and how the truncations are constructed. For the general structure of half-maximal supergravity in five dimensions we refer to [53] (see also [54]).

3.1 SO$(5-n)$ generalised structures

Dimensional reductions of eleven-dimensional supergravity on a six-dimensional manifold or of type IIB supergravity on a five-dimensional manifold are described by $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ generalised geometry. The R-symmetry group of five-dimensional supergravity is contained in USp(8), the maximal compact subgroup of $\mathrm{E}_{6(6)}$. For half-maximal supergravity, USp(8) must be broken to

$$\mathrm{USp}(8) \supset \mathrm{USp}(4)_R \times \mathrm{USp}(4)_S \supseteq \mathrm{USp}(4)_R \times \tilde{G}_S\,, \tag{3.1}$$

where the factor $\mathrm{USp}(4)_R$ is identified with the R-symmetry of half-maximal supergravity, while the other $\mathrm{USp}(4)_S$ factor contains the (double cover of) the reduced structure group, $\tilde{G}_S \subseteq \mathrm{USp}(4)$. Under the first embedding in (3.1), the spinorial representation of USp(8) decomposes as $\mathbf{8} = (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4})$, and we can identify the four spinor parameters of half-maximal supergravity as those that transform in the $(\mathbf{4},\mathbf{1})$ representation, in the $\mathbf{4}$ of $\mathrm{USp}(4)_R$ and singlets of $\tilde{G}_S$. Since we are focussing on dimensional reductions that do not have more than half-maximal supersymmetry, we also require that there are no further $\tilde{G}_S$-singlets in the $(\mathbf{1},\mathbf{4})$ representation. This (essentially) restricts the possible structure groups[5] to be $G_S = \mathrm{SO}(5-n)$, $n = 0,\ldots,3$. (Here we are ignoring the possibility of finite structure groups, hence exclude $n = 4$). Thus half-maximal truncations correspond to dimensional reductions on (the double cover of) $G_S = \mathrm{SO}(5-n)$ generalised structures. This structure group is embedded in $\mathrm{E}_{6(6)}$ as:

$$G_S = \mathrm{SO}(5-n) \subseteq \mathrm{SO}(5)_S \subset \mathrm{SO}(5,5) \subset \mathrm{E}_{6(6)}\,. \tag{3.2}$$

There are two extra cases of $G_S \subset \mathrm{SO}(5)_S$ not included in this sequence. These come from the embeddings

$$\mathrm{SO}(5)_S \supset \mathrm{SO}(4) = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2} \supset \mathrm{SU}(2)\times\mathrm{U}(1) \supset \mathrm{U}(1)\times\mathrm{U}(1). \tag{3.3}$$

Choosing either $G_S = \mathrm{SU}(2)\times\mathrm{U}(1)$ or $G_S = \mathrm{U}(1)^2$ still gives a half-maximal truncation. However, it is easy to show that the commutant subgroups and G_S-singlets are the same as the case of $G_S = \mathrm{SO}(4)$. Thus although the structure is different the resulting truncated theory is the same, meaning we can restrict to the sequence (3.2).

As discussed in the previous section, the vector fields in the truncation are in one-to-one correspondence with the G_S-singlets in the fundamental representation of $\mathrm{E}_{6(6)}$, while the scalar fields parameterise the coset

$$\mathcal{M}_{\mathrm{scal}} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(8)}(G_S)} = \mathrm{O}(1,1)\times\frac{\mathrm{SO}(5,n)}{\mathrm{SO}(5)\times\mathrm{SO}(n)} := \frac{\mathcal{G}}{\mathcal{H}}\,, \tag{3.4}$$

[5]For spinorial representations we of course need the double cover $\tilde{G}_S$. Thus, for instance USp(4) is the double cover of SO(5), but when discussing bosonic representations we can use SO(5) at the place of USp(4).

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where as before $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ and $\mathrm{C}_{\mathrm{USp}(8)}(G_S)$ are the commutants of G_S in $\mathrm{E}_{6(6)}$ and USp(8), respectively. This matches the standard structure of the scalar manifold for half-maximal supergravity coupled to n vector multiplets [53]. The single scalar in the gravity multiplet parameterises the $\mathrm{O}(1,1)$ factor[6] while the scalars in the vector multiplets parameterise the $\frac{\mathrm{SO}(5,n)}{\mathrm{SO}(5)\times\mathrm{SO}(n)}$ coset space.

We can also identify the number of singlets in the generalised tangent space, which determines the number of vector fields in the truncation. They also form a representation of $\mathcal{G}$. Recall that the generalised tangent space E transforms in the **27** of $\mathrm{E}_{6(6)}$. Under $\mathrm{SO}(1,1)\times\mathrm{SO}(5,5)\subset\mathrm{E}_{6(6)}$ we have the decomposition

$$\begin{aligned} E &= E_0 \oplus E_{10} \oplus E_{16}\,, \\ \mathbf{27} &= \mathbf{1}_{-4} \oplus \mathbf{10}_2 \oplus \mathbf{16}_{-1}\,, \end{aligned} \tag{3.5}$$

where the subscripts denote the $\mathrm{SO}(1,1)$ weights. Under $\mathrm{SO}(5)\times\mathrm{SO}(5)$ we have $\mathbf{16}_{-1}=(\mathbf{4},\mathbf{4})$. By construction the **4** representation has no singlets under G_S and hence there are no singlets in the $\mathbf{16}_{-1}$ component. On the other hand, the $\mathbf{10}_2$ representation decomposes as

$$\mathbf{10}_2 = (\mathbf{5+n},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{5-n})_2, \tag{3.6}$$

under $\mathrm{O}(1,1)\times\mathrm{SO}(5,n)\times\mathrm{SO}(5-n)\subset\mathrm{O}(1,1)\times\mathrm{SO}(5,5)$. Thus we see that we get $6+n$ singlets, one from the $\mathbf{1}_{-4}$ representation and $5+n$ from $\mathbf{10}_2$. In summary, as a $\mathcal{G}=\mathrm{O}(1,1)\times\mathrm{SO}(5,n)$ representation, we have the space of vector fields

$$\begin{aligned} \mathcal{V} &= \mathbf{1}_{-4} \oplus (\mathbf{5+n})_2, \\ \{K_{\mathcal{A}}\} &= \{K_0, K_A : A = 1,\ldots,5+n\}, \end{aligned} \tag{3.7}$$

where we are using the index $\mathcal{A}=0,1,\ldots,5+n$. In terms of the half-maximal supergravity six of these vectors come from the gravity multiplet and n of them from the additional vector multiplets.

In generalised geometry, the $\mathrm{E}_{6(6)}$ cubic invariant, acting on the generalised tangent space E, gives a map $c : S^3E \to \det T^*M$, which can be used to choose a natural parametrisation of the invariant generalised vectors. From the decompositions (3.5) and (3.6) we have

$$\begin{aligned} c(K_0,K_0,V) &= 0\,, \qquad \forall\, V\in\Gamma(E)\,, \\ c(K_A,K_B,K_C) &= 0\,, \qquad \forall\, A,B,C\,, \end{aligned} \tag{3.8}$$

and hence, independent of the choice of K_A, an $\mathrm{SO}(5,5)$ metric η on E_{10} given by

$$c(K_0,V,W) = \eta(V,W)\,\mathrm{vol}\,, \tag{3.9}$$

where vol is a volume form on $\det T^*M$. Since the K_A are fixed up to $\mathrm{SO}(5,5)$ rotations, we can use this to fix an orthonormal basis, and hence also the volume form vol, by

$$\eta(K_A,K_B) = \eta_{AB}\,, \tag{3.10}$$

[6]Previously we denoted such factor by $\mathbb{R}^+$, while here we use $\mathrm{O}(1,1)$ to match the standard supergravity literature.

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where

$$\eta_{AB} = \mathrm{diag}(-1,-1,-1,-1,-1,+1,\ldots,+1) \tag{3.11}$$

is the flat $\mathrm{SO}(5,n)$ metric.[7] Note that the freedom in the normalisation of η in (3.9) and hence of the $K_{\mathcal{A}}$ vectors via rescaling $K_0 \mapsto \lambda^2 K_0$ with $K_A \mapsto \lambda^{-1} K_A$ is just the action of the $\mathrm{O}(1,1)$ subgroup of $\mathcal{G}$. Note that specifying a set of vectors $\{K_{\mathcal{A}}\}$ satisfying (3.8) and (3.10) fixes an $\mathrm{SO}(5-n) \subset \mathrm{E}_{6(6)}$ structure. That is, the structure is completely determined by the vectors and no other generalised tensors are needed.

Turning to the two-form fields, for $\mathrm{E}_{6(6)}$ generalised geometry we have

$$\begin{aligned} N \simeq \det T^*M \otimes E^* &= N_0 \oplus N_{10} \oplus N_{16}\,, \\ \mathbf{27'} &= \mathbf{1}_4 \oplus \mathbf{10}_{-2} \oplus \mathbf{16'}_1, \end{aligned} \tag{3.12}$$

where again we decompose under $\mathrm{SO}(1,1) \times \mathrm{SO}(5,5) \subset \mathrm{E}_{6(6)}$. The same argument as for E then gives the space of singlet two-forms J_Σ

$$\begin{aligned} \mathcal{B} &= \mathbf{1}_4 \oplus (\mathbf{5}+\boldsymbol{n})_{-2}\,, \\ \{J^{\mathcal{A}}\} &= \{J^0, J^A : A = 1,\ldots,5+n\}\,, \end{aligned} \tag{3.13}$$

where the isomorphism $N \simeq \det T^*M \otimes E^*$ allows us to identify the usual Σ index on the basis with the dual of the index on $K_{\mathcal{A}}$. It is natural to normalise

$$\langle J^{\mathcal{A}}, K_{\mathcal{B}} \rangle = \delta^{\mathcal{A}}{}_{\mathcal{B}}\, \mathrm{vol}\,, \tag{3.14}$$

where $\langle W, V \rangle$ denotes the natural pairing between a vector and the (weighted) dual vector. The cubic invariant provides the intertwining maps (2.17) via

$$\begin{aligned} J^0 &= \frac{1}{5+n}\, \eta^{AB}\, c(K_A, K_B, \cdot)\,, \\ J^A &= \eta^{AB} c(K_0, K_B, \cdot)\,. \end{aligned} \tag{3.15}$$

It will be helpful in what follows to also define

$$J^0 = \mathrm{vol}\cdot K_0^*\,, \qquad J^A = \eta^{AB}\, \mathrm{vol}\cdot K_B^*\,, \tag{3.16}$$

so that $\{K^*_{\mathcal{A}}\}$ are a set on E^*, dual to $\{K_{\mathcal{A}}\}$, satisfying

$$\langle K_0^*, K_0 \rangle = 1\,, \qquad \langle K_A^*, K_B \rangle = \eta_{AB}\,, \qquad \langle K_0^*, K_A \rangle = \langle K_A^*, K_0 \rangle = 0\,. \tag{3.17}$$

Having identified the matter content of the truncated theory, we now turn to its gauging. From the general discussion, this is determined by the intrinsic torsion of the structure, which encodes an embedding tensor. Since in this case, the generalised vectors determine the G_S-structure, all the information of the intrinsic torsion should be encoded in (2.20), namely

$$L_{K_{\mathcal{A}}} K_{\mathcal{B}} = X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} K_{\mathcal{C}}\,. \tag{3.18}$$

[7] The overall sign in η is chosen so as to allow a straightforward identification with the $\mathrm{SO}(5,n)$ metric normally used in half-maximal supergravity [53].

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The analysis of gaugings of half-maximal supergravity in five dimensions can be found in [53]. The embedding tensor has components $f_{ABC} = f_{[ABC]}$, $\xi_{AB} = \xi_{[AB]}$ and ξ_A. For simplicity we will only discuss the case $\xi_A = 0$, although it would be straightforward to include the general case $\xi_A \neq 0$. The remaining components have to satisfy the conditions

$$f_{[AB}{}^{E} f_{CD]E} = 0\,, \qquad \xi_A{}^{D} f_{DBC} = 0\,, \tag{3.19}$$

where the indices are raised/lowered using the $\mathrm{SO}(5,n)$ metric η_{AB}. Using the composite index $\mathcal{A} = \{0, A\}$, the components can be assembled into the gauge group generators $(X_{\mathcal{A}})_{\mathcal{B}}{}^{\mathcal{C}} = X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}$ as:

$$X_{AB}{}^{C} = -f_{AB}{}^{C}\,, \qquad X_{0A}{}^{B} = -\xi_A{}^{B}\,, \tag{3.20}$$

with the other components vanishing. Then the $(X_{\mathcal{A}})_{\mathcal{B}}{}^{\mathcal{C}}$ generators satisfy the commutation relations:

$$[X_{\mathcal{A}}, X_{\mathcal{B}}] = -X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}} X_{\mathcal{C}}\,. \tag{3.21}$$

Thus, in general, we expect that any consistent truncation (leading to a gauging with $\xi_A = 0$) should have a generalised Lie derivative algebra (3.18) with the components of $X_{\mathcal{A}\mathcal{B}}{}^{\mathcal{C}}$ given by (3.20). Note that, in the generalised geometry, the algebraic conditions $f_{ABC} = f_{[ABC]}$, $\xi_{AB} = \xi_{[AB]}$ follow from consistency of the generalised algebra (3.18) with the conditions (3.8) and (3.10).

Having determined the number n of vector multiplets and the embedding tensor from the generalised $\mathrm{SO}(5-n)$ structure, we have fully characterised the five-dimensional half-maximal supergravity theory that is obtained after truncation. However we still need to provide the truncation ansatz, namely the embedding of the lower-dimensional fields into the higher-dimensional ones. This is necessary to uplift any solution of the lower-dimensional theory. In order to be able to do this we need a further geometrical ingredient, that is the construction of the generalised metric on the exceptional tangent bundle starting from the generalised vectors defining the $\mathrm{SO}(5-n)$ structure. This will be instrumental to specifying the scalar truncation ansatz.

3.2 The generalised metric

Recall that, in the generalised geometry reformulation, the generalised metric G_{MN} can be viewed as an element of the coset $\mathrm{E}_{6(6)} \times \mathbb{R}^+/(\mathrm{USp}(8)/\mathbb{Z}_2)$. Here we have a $G_S = \mathrm{SO}(5-n)$ structure. Given the embedding (3.1), since $\tilde{G}_S \subset \mathrm{USp}(8)$, the structure determines the metric. Since the structure is completely determined by the vectors $\{K_{\mathcal{A}}\}$ this means we should be able to use them to construct G explicitly.

The easiest way to see how this construction works is to use the embedding (3.2). The choice of K_0 and K_0^* fixes the $\mathrm{SO}(1,1) \times \mathrm{SO}(5,5) \subset \mathrm{E}_{6(6)}$ subgroup and gives a decomposition of the generalised tangent space (3.5). This in turn gives a decomposition of the metric into orthogonal metrics on E_0, E_{10} and E_{16} subspaces,

$$G = G_0 + G_{10} + G_{16}\,. \tag{3.22}$$

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We can then use our knowledge of $\mathrm{SO}(5)\times\mathrm{SO}(5)\subset\mathrm{SO}(5,5)$ generalised structures to construct the three pieces of the metric as:

$$G_0(V,V) = \big\langle K_0^*, V\big\rangle\big\langle K_0^*, V\big\rangle\,, \tag{3.23}$$
$$G_{10}(V,V) = 2\,\delta^{ab}\langle K_a^*, V\rangle\langle K_b^*, V\rangle + \eta(V,V)\,, \tag{3.24}$$
$$G_{16}(V,V) = -4\sqrt{2}\,\langle K_1\cdots K_5\cdot V, V\rangle\,, \tag{3.25}$$

where we have denoted the first five generalised vectors $\{K_a\}$ by an index $a=1,\ldots,5$. Recall from (3.10) that these satisfy $\eta(K_a,K_b)=-\delta_{ab}$.

Let us explain these formulae. The metric G_0 is simply obtained by projecting onto the singlet. For G_{10}, we use the fact that E_{10} is the generalised tangent bundle for the $\mathrm{SO}(5,5)$ geometry and that the structure $\mathrm{SO}(5)\times\mathrm{SO}(5)\subset\mathrm{SO}(5,5)$ induces a split of E_{10} into positive- and negative-definite eigenspaces

$$E_{10} = C_+\oplus C_-\,. \tag{3.26}$$

Then the $\mathrm{SO}(5,5)$ invariant metric η given in (3.9) and the generalised metric G_{10} can be written as

$$\begin{aligned}\eta(V,V) &= G_+ - G_-\,,\\ G_{10} &= G_+ + G_-\,,\end{aligned} \tag{3.27}$$

where $G_\pm$ are metrics on $C_\pm$. Since the K_a form a basis for C_-, we have

$$G_-(V,V) = \delta^{ab}\big\langle K_a^*, V\big\rangle\big\langle K_b^*, V\big\rangle\,. \tag{3.28}$$

Hence $G_{10}=G_++G_-=2G_-+\eta$, and we recover (3.24).

For G_{16} we recall that, given the $\mathrm{SO}(5)\times\mathrm{SO}(5)$ structure, the positive definite inner product on $\mathrm{SO}(5,5)$ spinors is

$$\big\langle\Psi,\Gamma^{(+)}\Psi\big\rangle\,, \tag{3.29}$$

where $\langle\cdot,\cdot\rangle$ is the Mukai pairing and $\Gamma^{(+)}$ is the chirality operator on C_+, that is

$$\Gamma^+ = \Gamma_1^+\cdots\Gamma_5^+\,, \tag{3.30}$$

where we decompose the $\mathrm{SO}(5,5)$ gamma matrices into $\{\Gamma_a^+\}\cup\{\Gamma_{\hat a}^-\}$ spanning C_+ and C_-. In this case, the Mukai pairing is just the natural pairing between $\Psi\in\Gamma(E)$ and $\Psi^*\in\Gamma(E^*)$. Thus we can write G_{16} as

$$G_{16}(V,V) = -4\sqrt{2}\,\big\langle K_1\cdots K_5\cdot V, V\big\rangle\,, \tag{3.31}$$

where the Clifford actions of K_a map between E and E^* and are given by

$$\begin{aligned}W\cdot\Psi &:= W^M\Gamma_M\Psi = \frac{c(W,\Psi,\cdot)}{\mathrm{vol}} &&\in\Gamma(E^*)\,,\\ W\cdot\Psi^* &:= W^M\Gamma_M\Psi^* = \mathrm{vol}\cdot c^*(W^*,\Psi^*,\cdot) &&\in\Gamma(E)\,.\end{aligned} \tag{3.32}$$

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Here we define $V^* = \eta(V,\cdot) \in \Gamma(E^*)$, and $c^*(\cdot,\cdot,\cdot)$ is the $\mathrm{E}_{6(6)}$ cubic invariant on E^*. Note that one does not need to project V onto E_{16} since as defined G_{16} will vanish identically when acting on sections of E_0 or E_{10}.

We will also need the inverse generalised metric G^{-1}, which acts on dual generalised vectors $Z \in \Gamma(E^*)$. Its expression is closely related to the one for G and reads

$$\begin{aligned} G_0^{-1}(Z,Z) &= \langle Z, K_0\rangle\langle Z, K_0\rangle\,, \\ G_{10}^{-1}(Z,Z) &= 2\delta^{ab}\langle Z, K_a\rangle\langle Z, K_b\rangle + \eta^{-1}(Z,Z)\,, \\ G_{16}^{-1}(Z,Z) &= -4\sqrt{2}\,\langle Z, K_1\cdots K_5\cdot Z\rangle\,, \end{aligned} \tag{3.33}$$

where $\eta^{-1}(Z,Z) = \mathrm{vol}\cdot c^*(K_0^*, Z, Z)$ is the inverse of the $\mathrm{SO}(5,5)$ metric η.

3.3 The truncation ansatz

We provide here the main steps of the construction of the truncation ansatz, which is entirely based on the generalised vectors $K_{\mathcal{A}}$ defining the $\mathrm{SO}(5-n)$ structure. More explicit formulae will be provided in the next sections, where we will specialise the formalism to both type IIB supergravity or M-theory, and discuss some concrete examples.

We start from the ansatz for the vector fields. By taking the higher-dimensional supergravity fields with one external index we make a generalised vector $\mathcal{A}_\mu^M$, where we recall that μ is an external spacetime index while M labels the components of a generalised vector, which in $\mathrm{E}_{6(6)}$ generalised geometry transform in the **27**. We expand this generalised vector as in (2.15)

$$\mathcal{A}_\mu^M(x,y) = \sum_{\mathcal{A}=0}^{5+n} \mathcal{A}_\mu^{\mathcal{A}}(x) K_{\mathcal{A}}^M(y)\,, \tag{3.34}$$

where $\mathcal{A}_\mu^{\mathcal{A}}$ are the five-dimensional supergravity vector fields. Similarly, the supergravity fields with two antisymmetrised external indices can be arranged in a generalised tensor, as a section of the bundle N. Exploiting the isomorphism $N \simeq \det T^*M \otimes E^*$, we can write this as a weighted dual vector $\mathcal{B}_{\mu\nu\,M}$, and express the truncation ansatz (2.16) as

$$\mathcal{B}_{\mu\nu\,M}(x,y) = \sum_{\mathcal{A}=0}^{5+n} \mathcal{B}_{\mu\nu\,\mathcal{A}}(x) J^{\mathcal{A}}{}_M(y)\,. \tag{3.35}$$

The ansatz for the scalar fields is more elaborated as it requires the generalised metric. This is specified by choosing a metric on the coset space (3.4), which is also the scalar manifold of half-maximal supergravity in five dimensions. We parameterise the $\mathrm{O}(1,1)$ factor by a non-vanishing scalar Σ. The $\frac{\mathrm{SO}(5,n)}{\mathrm{SO}(5)\times\mathrm{SO}(n)}$ factor is described by a coset representative $(\mathcal{V}_A{}^a, \mathcal{V}_A{}^{\underline{a}}) \in \mathrm{SO}(5,n)$ and its inverse $(\mathcal{V}_a{}^A, \mathcal{V}_{\underline{a}}{}^A)^T$, where $a = 1,\ldots,5$ and $\underline{a} = 1,\ldots,n$ are local $\mathrm{SO}(5)$ and $\mathrm{SO}(n)$ indices, respectively. The coset representative satisfies

$$\begin{aligned} \eta_{AB} &= -\delta_{ab}\,\mathcal{V}_A{}^a\mathcal{V}_B{}^b + \delta_{\underline{ab}}\,\mathcal{V}_A{}^{\underline{a}}\mathcal{V}_B{}^{\underline{b}}\,, \\ M_{AB} &= \delta_{ab}\,\mathcal{V}_A{}^a\mathcal{V}_B{}^b + \delta_{\underline{ab}}\,\mathcal{V}_A{}^{\underline{a}}\mathcal{V}_B{}^{\underline{b}}\,. \end{aligned} \tag{3.36}$$

Note that the matrix M_{AB} is a metric on the coset, with inverse $M^{AB} = \eta^{AC} M_{CD}\,\eta^{DB}$.

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The construction of the generalised metric now goes as follows. We introduce the "dressed" generalised vectors

$$\tilde{K}_0 = \Sigma^2 K_0\,, \qquad \tilde{K}_a = \Sigma^{-1}\,\mathcal{V}_a{}^A K_A\,, \qquad \tilde{K}_{\underline{a}} = \Sigma^{-1}\,\mathcal{V}_{\underline{a}}{}^A K_A\,, \tag{3.37}$$

and their duals

$$\tilde{K}^*_0 = \Sigma^{-2} K^*_0\,, \qquad \tilde{K}^*_a = \Sigma\,\mathcal{V}_a{}^A K^*_A\,, \qquad \tilde{K}^*_{\underline{a}} = \Sigma\,\mathcal{V}_{\underline{a}}{}^A K^*_A\,. \tag{3.38}$$

The generalised metric and its inverse are defined as in (3.23)–(3.25), this time using the dressed generalised vectors $\tilde{K}_0$ and $\tilde{K}_a$, $a = 1, \ldots, 5$. The generalised metric is then $G = G_0 + G_{10} + G_{16}$, with

$$\begin{aligned}
G_0(V,V) &= \Sigma^{-4}\,\langle K^*_0, V\rangle\langle K^*_0, V\rangle\,,\\
G_{10}(V,V) &= \Sigma^2\left(2\,\delta^{ab}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\,\langle K^*_A, V\rangle\langle K^*_B, V\rangle + \eta(V,V)\right)\,,\\
G_{16}(V,V) &= -\frac{4\sqrt{2}}{5!}\,\Sigma^{-1}\,\epsilon^{abcde}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\mathcal{V}_c{}^C\mathcal{V}_d{}^D\mathcal{V}_e{}^E\,\langle K_A\cdots K_E\cdot V, V\rangle\,.
\end{aligned} \tag{3.39}$$

Similarly, the inverse generalised metric $G^{-1} = G_0^{-1} + G_{10}^{-1} + G_{16}^{-1}$ is given by

$$\begin{aligned}
G_0^{-1}(Z,Z) &= \Sigma^4\,\langle Z, K_0\rangle\langle Z, K_0\rangle\,,\\
G_{10}^{-1}(Z,Z) &= \Sigma^{-2}\left(2\,\delta^{ab}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\langle Z, K_A\rangle\langle Z, K_B\rangle + \eta^{-1}(Z,Z)\right)\,,\\
G_{16}^{-1}(Z,Z) &= -\frac{4\sqrt{2}}{5!}\,\Sigma\,\epsilon^{abcde}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\mathcal{V}_c{}^C\mathcal{V}_d{}^D\mathcal{V}_e{}^E\,\langle Z, K_A\cdots K_E\cdot Z\rangle\,.
\end{aligned} \tag{3.40}$$

Notice that the $\mathrm{SO}(5)\times\mathrm{SO}(n)$ invariant matrices $2\,\delta^{ab}\mathcal{V}_a{}^A\mathcal{V}_b{}^B = M^{AB} - \eta^{AB}$, $M^{ABCDE} = \epsilon^{abcde}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\mathcal{V}_c{}^C\mathcal{V}_d{}^D\mathcal{V}_e{}^E$ are familiar from the construction of half-maximal supergravity in five dimensions [53]. Also note that to get the correct power of Σ in the G_{16} and G_{16}^{-1} expressions it is important to keep track of how many of the Clifford actions are with $\tilde{K}_a$ and how many with $\tilde{K}^*_a$.

The scalar ansatz is obtained by equating the inverse generalised metric with the one obtained from the split frame [6, 35], which encodes all supergravity fields with purely internal indices (including the warp factor of the external metric). By separating the different tensorial structures on the internal manifold M, we obtain the scalar ansatz for the individual higher-dimensional supergravity fields.

4 Type IIB truncations

In this section we specialise our formalism to dimensional reductions of type IIB supergravity on five-dimensional manifolds. To this end, we first recall the details of type IIB $\mathrm{E}_{6(6)}$ geometry and present the truncation anzatz adapted to the type IIB fields. Then we discuss concrete examples of consistent truncations. The first is the truncation on squashed Sasaki-Einstein manifolds of [20, 21], leading to half-maximal supergravity coupled to two vector multiplets. Although this truncation is not new and can be understood

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based on ordinary SU(2) structure, it will serve to illustrate the validity of our approach in a relatively simple case. This will also make clear how generalised goemetry fully characterises the lower-dimensional theory even before the lower-dimensional Lagrangian is constructed from the truncation of the higher-dimensional equations of motion. We will then consider a β-deformed Sasaki-Einstein manifold and will show that there is a consistent truncation on such manifolds leading to the same half-maximal supergravity obtaind from the Sasaki-Einstein truncation. This truncation includes the supersymmetric, β-deformed AdS_5 solution.

4.1 $E_{6(6)}$ geometry for type IIB

We recall here some basic definitions of the $E_{6(6)}$ generalised geometry for type IIB supergravity on a five-dimensional manifold M. A more detailed account is given in appendix A following the conventions of [55, appendix E].

It is convenient to decompose the generalised tangent bundle E, whose fibers transform in the **27** of $E_{6(6)}$, according to the $GL(5) \times SL(2)$ subgroup of $E_{6(6)}$

$$E \simeq TM \oplus (T^*M \oplus T^*M) \oplus \Lambda^3 T^*M \oplus (\Lambda^5 T^*M \oplus \Lambda^5 T^*M)\,, \tag{4.1}$$

where the two copies of T^*M and the two copies of $\Lambda^5 T^*M$ transform as SL(2) doublets. A generalised vector can be written as

$$V = v + \lambda^\alpha + \rho + \sigma^\alpha\,, \tag{4.2}$$

where v is a vector, λ^α is an SL(2) doublet of one-forms, ρ is a three-form and σ^α is an SL(2) doublet of five-forms, $\alpha = \{+,-\}$ being the SL(2) index. The dual bundle decomposes accordingly as

$$E^* \simeq T^*M \oplus (TM \oplus TM) \oplus \Lambda^3 TM \oplus (\Lambda^5 TM \oplus \Lambda^5 TM)\,, \tag{4.3}$$

with sections

$$Z = \hat{v} + \hat{\lambda}_\alpha + \hat{\rho} + \hat{\sigma}_\alpha\,, \tag{4.4}$$

where $\hat{v}$ is a one-form, $\hat{\lambda}_\alpha$ is an SL(2) doublet of vectors, $\hat{\rho}$ is a three-vector, and $\hat{\sigma}_\alpha$ is an SL(2) doublet of five-vectors. The natural pairing between a generalised vector and a dual one is

$$\langle Z, V \rangle = \hat{v}_m v^m + \hat{\lambda}^m_\alpha \lambda^\alpha_m + \frac{1}{3!}\,\hat{\rho}^{mnp}\rho_{mnp} + \frac{1}{5!}\,\hat{\sigma}^{mnpqr}_\alpha \sigma^\alpha_{mnpqr}\,. \tag{4.5}$$

The cubic invariant is defined on E and E^*, respectively, as

$$c(V,V,V) = -3\left(\iota_v\rho \wedge \rho + \epsilon_{\alpha\beta}\,\rho \wedge \lambda^\alpha \wedge \lambda^\beta - 2\,\epsilon_{\alpha\beta}\,\iota_v\,\lambda^\alpha\sigma^\beta\right), \tag{4.6}$$

$$c^*(Z,Z,Z) = -3\left(\hat{v}\lrcorner\hat{\rho} \wedge \hat{\rho} + \epsilon^{\alpha\beta}\hat{\rho} \wedge \hat{\lambda}_\alpha \wedge \hat{\lambda}_\beta - 2\,\epsilon^{\alpha\beta}\,\hat{v}\lrcorner\,\hat{\lambda}_\alpha\hat{\sigma}_\beta\right). \tag{4.7}$$

The bosonic fields of type IIB supergravity are the metric, the dilaton ϕ, the axion C_0, an SL(2) doublet of two-form potentials $\hat{B}^\alpha$ ($\hat{B}^+$ being the NSNS two-form and $\hat{B}^-$ being the RR one), a self-dual four-form $\hat{C}$, and a doublet of six-form potentials $\hat{\tilde{B}}^\alpha$ that are on-shell dual to the two-forms.[8] When dimensionally reducing on a five-dimensional manifold,

[8] In this subsection the symbol hat denotes ten-dimensional fields.

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the ten-dimensional fields are decomposed according to the $\mathrm{SO}(1,9) \supset \mathrm{SO}(1,4) \times \mathrm{SO}(5)$ splitting of the Lorentz group. We will use coordinates x^μ, $\mu = 0, \ldots, 4$ for the external spacetime and y^m, $m = 1, \ldots, 5$ for the internal manifold M. Then the type IIB metric takes the form

$$g_{10} = \mathrm{e}^{2\Delta}\, g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dy^m Dy^n \,, \tag{4.8}$$

where $Dy^m = \mathrm{d}y^m - h_\mu{}^m \mathrm{d}x^\mu$ and $\Delta(x,y)$ is the warp factor of the external metric $g_{\mu\nu}(x)$. The form fields decompose as

$$\begin{aligned}
\hat{B}^\alpha &= \frac{1}{2} B^\alpha_{m_1 m_2} Dy^{m_1 m_2} + \overline{B}^\alpha_{\mu m} \mathrm{d}x^\mu \wedge Dy^m + \frac{1}{2} \overline{B}^\alpha_{\mu\nu} \mathrm{d}x^{\mu\nu} \,, \\
\hat{C} &= \frac{1}{4!} C_{m_1 \ldots m_4} Dy^{m_1 \ldots m_4} + \frac{1}{3!} \overline{C}_{\mu m_1 m_2 m_3} \mathrm{d}x^\mu \wedge Dy^{m_1 m_2 m_3} + \frac{1}{4} \overline{C}_{\mu\nu m_1 m_2} \mathrm{d}x^{\mu\nu} \wedge Dy^{m_1 m_2} + \ldots \,, \\
\hat{\tilde{B}}^\alpha &= \frac{1}{5!} \overline{\tilde{B}}^\alpha_{\mu m_1 \ldots m_5} \mathrm{d}x^\mu \wedge Dy^{m_1 \ldots m_5} + \frac{1}{2 \cdot 4!} \overline{\tilde{B}}^\alpha_{\mu\nu m_1 \ldots m_4} \mathrm{d}x^{\mu\nu} \wedge Dy^{m_1 \ldots m_4} + \ldots \,,
\end{aligned} \tag{4.9}$$

where $\mathrm{d}x^{\mu\nu} = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and $Dy^{m_1 \ldots m_p} = Dy^{m_1} \wedge \cdots \wedge Dy^{m_p}$. The ellipsis denote forms with more than two external indices which we will not need. The expansion in Dy instead of $\mathrm{d}y$ ensures covariance of the components under internal diffeomorphisms.

As discussed in e.g. [56, 57], covariance under generalised diffeomorphisms also requires a redefinition of the barred fields in the expansion above. We adopt a notation such that $B_{\mu,p}$ indicates the components of a one-form in the external spacetime which are p-forms in the internal manifold. Similarly, $B_{\mu\nu,p}$ are the components of a two-form in the external spacetime that are p-forms in the internal manifold. We perform the following field redefinitions of the one-forms in the external spacetime:

$$\begin{aligned}
\overline{B}^\alpha_{\mu,1} &= B^\alpha_{\mu,1} \,, \\
C_{\mu,3} &= \overline{C}_{\mu,3} + \frac{1}{2} \epsilon_{\alpha\beta} B^\alpha_{\mu,1} \wedge B^\beta \,, \\
\tilde{B}^\alpha_{\mu,5} &= \overline{\tilde{B}}^\alpha_{\mu,5} - \frac{1}{2} \overline{B}^\alpha_{\mu,1} \wedge C - \frac{1}{2} \overline{C}_{\mu,3} \wedge B^\alpha \,,
\end{aligned} \tag{4.10}$$

where B^α, C are just internal. The external two-forms are redefined as

$$\begin{aligned}
B^\alpha_{\mu\nu} &= \overline{B}^\alpha_{\mu\nu} + h_{[\mu} \lrcorner \overline{B}^\alpha_{\nu]} \,, \\
C_{\mu\nu,2} &= \overline{C}_{\mu\nu,2} + \frac{1}{2} \epsilon_{\alpha\beta} B^\alpha_{\mu\nu}\, B^\beta \,, \\
\tilde{B}^\alpha_{\mu\nu,4} &= \overline{\tilde{B}}^\alpha_{\mu\nu,4} + \frac{1}{2} \overline{B}^\alpha_{\mu\nu}\, C + \frac{1}{2} \overline{C}_{\mu\nu,2} \wedge B^\alpha - \overline{B}^\alpha_{[\mu,1} \wedge \overline{C}_{\nu],3} \,.
\end{aligned} \tag{4.11}$$

The new (unbarred) fields transform covariantly both under internal diffeomorphisms and form gauge transformations, that is under generalised diffeomorphisms.

The next step is to arrange the redefined fields into the inverse generalised metric G^{MN}, the generalised vectors $\mathcal{A}^M_\mu$ and the tensors $\mathcal{B}_{\mu\nu M}$. The generalised metric is made by all the type IIB supergravity fields with only internal indices, including the warp factor Δ,

$$G^{MN} \;\leftrightarrow\; \{\Delta,\, g_{mn},\, \phi,\, C_0,\, B^\alpha_{m_1 m_2},\, C_{m_1 \ldots m_4}\} \,. \tag{4.12}$$

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Its precise expression is given in (A.23). The fields with one external index can be arranged into the generalised vector $\mathcal{A}_\mu^M \in \Gamma(E)$,

$$\mathcal{A}_\mu{}^M = \{h_\mu{}^m,\, B^\alpha_{\mu m},\, C_{\mu m_1 m_2 m_3},\, \tilde{B}^\alpha_{\mu m_1 \dots m_5}\}\,. \tag{4.13}$$

Similarly the fields with two external indices form the generalised tensor $\mathcal{B}_{\mu\nu\,M}$, that is a section of $N \simeq \det T^*M \otimes E^*$ (see (A.11), (A.12) for its $\mathrm{GL}(5) \times \mathrm{SL}(2)$ decomposition),

$$\mathcal{B}_{\mu\nu\,M} \;=\; \{B_{\mu\nu\,\alpha},\, C_{\mu\nu m_1 m_2},\, \tilde{B}_{\mu\nu m_1 \dots m_4\,\alpha},\, \tilde{g}_{\mu\nu m_1 \dots m_5, n}\}\,. \tag{4.14}$$

Here, the SL(2) index α on the type IIB fields has been lowered with $\epsilon_{\alpha\beta}$, and $\tilde{g} \in \Gamma(\Lambda^7 T^*M_{10} \otimes T^*M_{10})$ is a tensor related to the dual graviton in ten dimensions. The latter is not part of type IIB supergravity in its standard form and will not play a role in the specific truncations we will discuss below.

We have thus decomposed the ten-dimensional tensors according to their external or internal legs and repackaged the components into generalised geometry objects. We can then specify the dependence of these fields on the internal coordinates by making the consistent truncation ansatz described in section 3.3.

4.2 Truncation from generalised SU(2) structure on Sasaki-Einstein manifolds

We discuss type IIB supergravity on a five-dimensional a Sasaki-Einstein manifold M, which admits a consistent truncation to half-maximal gauged supergravity with two vector multiplets [20], see also [21, 58, 59].

4.2.1 Generalised SU(2) structure

Five-dimensional Sasaki-Einstein (SE_5) structures are examples of ordinary SU(2) structures, whose torsion is also an SU(2)-singlet. The SU(2) structure is defined by a vector ξ, a one-form η and a triplet of real two-forms j_i, $i = 1, 2, 3$, satisfying the compatibility conditions[9]

$$\begin{aligned} \xi \lrcorner j_i &= 0 & \xi \lrcorner \eta &= 1\,, \\ j_i \wedge j_j &= 0 \quad \text{for } i \neq j\,, & \frac{1}{2}\, j_i \wedge j_j \wedge \eta &= \delta_{ij}\, \mathrm{vol}\,, \end{aligned} \tag{4.15}$$

where vol is the volume form compatible with the SE_5 metric

$$g_{\mathrm{SE}_5} = g_{\mathrm{KE}} + \eta^2\,, \tag{4.16}$$

which is taken with canonical normalization $R_{mn} = 4 g_{mn}$. Locally this metric describes a fibration over a four-dimensional Kähler-Einstein base with metric g_{KE}. In a neighbourhood, the fibre direction corresponds to the orbit of the vector ξ, which is also an isometry and is called the Reeb vector. In addition the SU(2) invariant forms satisfy the differential conditions

$$\mathrm{d}\eta = 2 j_3\,, \qquad \mathrm{d} j_3 = 0\,, \qquad \mathrm{d}(j_1 + \mathrm{i}\, j_2) = 3\,\mathrm{i}\,\eta \wedge (j_1 + \mathrm{i}\, j_2)\,. \tag{4.17}$$

[9] The j_i are identified with the forms used in eq. (2.6) as $j_3 = \omega$ and $j_1 + \mathrm{i}\, j_2 = \Omega$.

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The $\mathrm{AdS}_5 \times \mathrm{SE}_5$ supersymmetric solution of type IIB supergravity has string frame metric

$$g_{10} = \ell^2 \left(g_{\mathrm{AdS}_5} + g_{\mathrm{SE}_5} \right) , \tag{4.18}$$

where g_{AdS_5} is the unit AdS_5 metric and ℓ sets the overall scale. The solution also contains a non-trivial self-dual five-form flux whose internal part is proportional to the SE_5 volume,

$$F^{\mathrm{fl}} = \mathrm{d}C = \kappa \, \mathrm{vol} \, , \qquad\qquad \xi \lrcorner C = 0 \, , \tag{4.19}$$

where κ is a constant related to the overall scale as $\ell^4 = \frac{\kappa}{4} \, \mathrm{e}^{\phi_0}$.[10] The second expression in (4.19) is just a convenient gauge choice for the four-form potential.

The consistent truncation was originally constructed by expanding the type IIB supergravity fields in the most general way possible in the basis of SU(2)-singlets given above [20, 21]. We now show how this truncation is easily derived from $\mathrm{E}_{6(6)}$ generalised geometry; this will also give the opportunity to illustrate the general statements made in section 3 in a concrete example. We thus lift the Sasaki-Einstein SU(2) structure to the generalised tangent bundle, and take $G_S = \mathrm{SU}(2) \subset \mathrm{USp}(4)$. Under

$$\mathrm{USp}(8) \supset \mathrm{USp}(4)_R \times \mathrm{USp}(4) \supset \mathrm{USp}(4)_R \times \mathrm{U}(1) \times \mathrm{SU}(2)_S \, , \tag{4.20}$$

the spinorial representation decomposes as

$$\mathbf{8} \to (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}) \to (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2_1}) \oplus (\mathbf{1}, \mathbf{2_{-1}}) \, , \tag{4.21}$$

so we have precisely four G_S-singlets and the truncation preserves half-maximal supersymmetry. In order to count the vector fields in the truncation, we embed $\mathrm{SU}(2) \sim \mathrm{SO}(3)$ in $\mathrm{E}_{6(6)}$

$$\mathrm{E}_{6(6)} \supset \mathrm{SO}(5,5) \times \mathrm{SO}(1,1) \supset \mathrm{SO}(1,1) \times \mathrm{SO}(5,2) \times \mathrm{SO}(3)_S \, , \tag{4.22}$$

and decompose the fundamental representation of $\mathrm{E}_{6(6)}$,

$$\mathbf{27} \to \mathbf{10}_2 \oplus \mathbf{16}_{-1} \oplus \mathbf{1}_{-4} \to (\mathbf{7}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{3})_2 \oplus (\mathbf{8}, \mathbf{2})_{-1} \oplus (\mathbf{1}, \mathbf{1})_{-4} \, . \tag{4.23}$$

We find 8 singlets of SO(3), 7 transforming in the fundamental of $\mathrm{SO}(5,2)$ and one neutral. This matches the vector field content of half-maximal supergravity coupled to two vector multiplets.

From (4.20), (4.22), we see that the scalar manifold of the truncated theory is

$$\mathcal{M}_{\mathrm{scal}} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(\mathrm{SU}(2)_S)}{\mathrm{C}_{\mathrm{USp}(8)}(\mathrm{SU}(2)_S)} = \mathrm{SO}(1,1) \times \frac{\mathrm{SO}(5,2)}{\mathrm{SO}(5) \times \mathrm{SO}(2)} \, , \tag{4.24}$$

that is the scalar manifold of half-maximal supergravity coupled to two vector multiplets.[11]

The eight generalised vectors $K_{\mathcal{A}}$, with $\mathcal{A} = \{0, A\} = 0, 1, \ldots, 7$, defining the generalised SU(2) structure are constructed from the tensors defining the ordinary SU(2)

[10] The parameter κ is related to the N units of five-form flux as $\kappa = 27\pi N$.

[11] Precisely the same group-theoretical arguments described here were used in [60, 61] to identify a consistent truncation of maximal SO(6) supergravity to half-maximal supergravity with two vector multiplets. Although the matter content of the five-dimensional theory is the same, the gauging is different.

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structure on the Sasaki-Einstein manifold. For the generalised vectors to contain all the information about the background, we should also include a twist by the four-form C satisfying (4.19),

$$K_{\mathcal{A}} = \mathrm{e}^{C} \check{K}_{\mathcal{A}} \,, \tag{4.25}$$

where $\check{K}_{\mathcal{A}}$ denotes the untwisted vectors and the adjoint action of C on a generalised vector is given in (A.10). We find that the generalised SU(2) structure is defined by

$$\begin{aligned}
K_0 &= \xi \,, \\
K_i &= \frac{1}{\sqrt{2}} \, \eta \wedge j_i \qquad\qquad i = 1,2,3 \,, \\
K_4 &= \frac{1}{\sqrt{2}} \left(n\,\eta - r\,\mathrm{vol} - n\,\eta \wedge C \right) \,, \\
K_5 &= \frac{1}{\sqrt{2}} \left(-r\,\eta - n\,\mathrm{vol} + r\,\eta \wedge C \right) \,, \\
K_6 &= \frac{1}{\sqrt{2}} \left(n\,\eta + r\,\mathrm{vol} - n\,\eta \wedge C \right) \,, \\
K_7 &= \frac{1}{\sqrt{2}} \left(-r\,\eta + n\,\mathrm{vol} + r\,\eta \wedge C \right) \,,
\end{aligned} \tag{4.26}$$

where

$$n^{\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\alpha} \,, \qquad r^{\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\alpha} \tag{4.27}$$

are a basis for the SL(2) doublets. Using (4.6) for the cubic invariant, it is straightforward to verify that the compatibility relations (3.8), (3.10) are satisfied, with $n = 2$.

We will also need the dual vectors $K^*_{\mathcal{A}}$. Evaluating (3.15), (3.16), we find that these are

$$\begin{aligned}
K^*_0 &= \eta \,, \\
K^*_i &= -\frac{1}{\sqrt{2}} \, \hat{j}_i \wedge \xi \,, \qquad\qquad i = 1,2,3 \,, \\
K^*_4 &= \frac{1}{\sqrt{2}} \left(-\hat{r}\,\xi + \hat{n}\, C \lrcorner \hat{\mathrm{vol}} + \hat{n}\,\hat{\mathrm{vol}} \right) \,, \\
K^*_5 &= \frac{1}{\sqrt{2}} \left(\hat{n}\,\xi + \hat{r}\, C \lrcorner \hat{\mathrm{vol}} + \hat{r}\,\hat{\mathrm{vol}} \right) \,, \\
K^*_6 &= \frac{1}{\sqrt{2}} \left(\hat{r}\,\xi + \hat{n}\, C \lrcorner \hat{\mathrm{vol}} + \hat{n}\,\hat{\mathrm{vol}} \right) \,, \\
K^*_7 &= \frac{1}{\sqrt{2}} \left(-\hat{n}\,\xi + \hat{r}\, C \lrcorner \hat{\mathrm{vol}} + \hat{r}\,\hat{\mathrm{vol}} \right) \,,
\end{aligned} \tag{4.28}$$

where $\hat{j}_i$, are the two-vectors dual to the two forms j_i, $\hat{\mathrm{vol}}$ is the five-vector dual to the volume form, and

$$\hat{r}_{\alpha} = \epsilon_{\alpha\beta}\, r^{\beta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\alpha} \,, \qquad \hat{n}_{\alpha} = -\epsilon_{\alpha\beta}\, n^{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\alpha} \,. \tag{4.29}$$

The gauging of the five-dimensional theory is obtained by computing the generalised Lie derivative between the set of generalised vectors, as in (3.18). The definition of the

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type IIB generalised Lie derivative can be found in (A.16). We find that the algebra closes into the non-vanishing structure constants

$$\begin{aligned}
&X_{01}{}^{2} = -X_{02}{}^{1} = 3\,,\\
&X_{04}{}^{5} = -X_{05}{}^{4} = -X_{04}{}^{7} = -X_{07}{}^{4} = X_{05}{}^{6} = X_{06}{}^{5} = -X_{06}{}^{7} = X_{07}{}^{6} = \frac{\kappa}{2},\\
&X_{34}{}^{5} = -X_{34}{}^{7} = -X_{35}{}^{4} = X_{35}{}^{6} = X_{36}{}^{5} = -X_{36}{}^{7} = -X_{37}{}^{4} = X_{37}{}^{6} = \sqrt{2}\,,\\
&X_{45}{}^{3} = X_{47}{}^{3} = -X_{56}{}^{3} = X_{67}{}^{3} = \sqrt{2}\,,
\end{aligned} \tag{4.30}$$

where the terms in the last two lines are antisymmetric in the lower indices. From (3.20) we conclude that the embedding tensor components are

$$\begin{aligned}
&\xi_{12} = 3\,, \qquad \xi_{45} = \xi_{47} = -\xi_{56} = \xi_{67} = \frac{\kappa}{2}\,,\\
&f_{345} = f_{347} = -f_{356} = f_{367} = \sqrt{2}\,.
\end{aligned} \tag{4.31}$$

This is fully consistent with the embedding tensor found in [20].[12] As discussed there, the corresponding gauge algebra is $\mathrm{Heis}_3 \times \mathrm{U}(1)$, where Heis_3 is the three-dimensional Heisenberg algebra. The remaining four generators, that transform in a non-adjoint representation of the gauge algebra, determine the vector fields that are eaten-up by two-form fields via a Stückelberg mechanism.

4.2.2 Generalised metric

In order to recover the scalar truncation ansatz we need to construct the generalised metric evaluating the formulae (3.40). We first derive the generalised metric for the background solution $\mathrm{AdS}_5 \times \mathrm{SE}_5$ using (3.33), since this is simpler and it allows one to see how the construction works. Then in the next subsection we will discuss the generalised metric for the dressed generalised vectors, allowing for general Σ, $\mathcal{V}$, and extract the scalar ansatz. For simplicity, we also momentarily set the four-form C to zero, that is we work with the untwisted vectors, and reintroduce it in a second step.

Recalling the decomposition (4.4) of the arbitrary dual generalised vector Z, we find that G_0^{-1} in (3.40) is

$$G_0^{-1}(Z,Z) = (\xi^m \hat{v}_m)^2\,, \tag{4.32}$$

while the two terms defining G_{10}^{-1} evaluate to

$$2\delta^{ab}\langle Z, K_a\rangle\langle Z, K_b\rangle = \frac{1}{4}\sum_{i=1,2,3}\Big(\eta_m j_{i\,np}\,\hat{\rho}^{mnp}\Big)^2 + \sum_{\alpha=1,2}(\eta_m \hat{\lambda}^m_\alpha)^2 + \frac{1}{5!}\sum_{\alpha=1,2}(\hat{\sigma}^{mnpqr}_\alpha)^2\,, \tag{4.33}$$

and

$$\frac{c^*(K_0^*, Z, Z)}{\hat{\mathrm{vol}}} = -\frac{1}{12}\,\eta_m \hat{\rho}^{mnp}\hat{\rho}^{qrs}\epsilon_{npqrs} + 2\,\epsilon^{\alpha\beta}\,\eta_m \hat{\lambda}^m_\alpha * \hat{\sigma}_\beta\,. \tag{4.34}$$

[12] The precise matching between the embedding tensor components in (4.31) and those in [20, eq. (4.20)] is obtained upon renaming the indices $(1234567)_{\text{here}} = (3451267)_{\text{there}}$ (which can be achieved by a trivial SO(5) transformation), multiplying all components in (4.31) by $-\sqrt{2}$ (which is a harmless rescaling of the gauge group generators) and noticing from comparing the five-form fluxes that $\kappa_{\text{here}} = 2k_{\text{there}}$.

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The term involving the j_i projects $\eta_m \hat{\rho}^{mnp}$ on its anti-self-dual part on the Kähler-Einstein basis, hence it can be written as

$$\begin{aligned}\frac{1}{4}\sum_i \left(\eta_m j_{i\,np}\hat{\rho}^{mnp}\right)^2 &= \frac{1}{4}\left(\eta_m\hat{\rho}^{mnp} - \frac{1}{2}\eta_m\epsilon^{mnpqr}\,\hat{\rho}_{qrs}\eta^s\right)^2 \\ &= \frac{1}{2}(\eta_m\hat{\rho}^{mnp})^2 + \frac{1}{12}\,\eta_m\hat{\rho}^{mnp}\hat{\rho}^{qrs}\epsilon_{npqrs}\,. \end{aligned} \tag{4.35}$$

Adding up the two contributions we obtain

$$G_{10}^{-1}(Z,Z) = \sum_{\alpha=1,2}(\eta_m\hat{\lambda}^m_\alpha)^2 + \frac{1}{2}\,(\eta_p\,\hat{\rho}^{pmn})^2 + \frac{1}{5!}\sum_{\alpha=1,2}(\hat{\sigma}^{mnpqr}_\alpha)^2\,. \tag{4.36}$$

We see that the tensor structure of G_0^{-1} and G_{10}^{-1} is such that at least one index is along the fiber of the Sasaki-Einstein manifold. It remains to evaluate G_{16}^{-1}: as explained in the general discussion of section 3, this is obtained by the recursive Clifford action of K_5, K_4^*, K_3, K_2^*, K_1 on a dual vector Z, and by finally pairing up the resulting vector with Z itself. After a long but relatively straightforward computation, we find

$$G_{16}^{-1}(Z,Z) = g^{mn}_{\rm KE}\,\hat{v}_m\hat{v}_n + \delta^{\alpha\beta}g^{\rm KE}_{mn}\,\hat{\lambda}^m_\alpha\hat{\lambda}^m_\beta + \frac{1}{6}\,g^{\rm KE}_{mq}g^{\rm KE}_{nr}g^{\rm KE}_{ps}\,\hat{\rho}^{mnp}\hat{\rho}^{qrs}\,. \tag{4.37}$$

Hence G_{16}^{-1} is just a generalised metric on the four-dimensional Kähler-Einstein base. Adding up the three contributions, we arrive at

$$G^{-1}(Z,Z) = g^{mn}\hat{v}_m\hat{v}_n + \delta^{\alpha\beta}g_{mn}\hat{\lambda}^m_\alpha\hat{\lambda}^m_\beta + \frac{1}{6}\,\hat{\rho}^{mnp}\hat{\rho}_{mnp} + \delta^{\alpha\beta}\hat{\sigma}^{mnpqr}_\alpha\hat{\sigma}_{\beta,mnpqr}\,, \tag{4.38}$$

where g_{mn} is the Sasaki-Einstein metric (4.16), which is also used to lower the curved indices in the last two terms.

The metric associated with the twisted generalised vectors $K_{\mathcal{A}} = \mathrm{e}^C\check{K}_{\mathcal{A}}$ is easily obtained by recalling that the $\mathrm{E}_{6(6)}$ cubic invariant is preserved by the twist,

$$c(\mathrm{e}^C V, \mathrm{e}^C V', \mathrm{e}^C V'') = c(V, V', V'')\,. \tag{4.39}$$

This means that the generalised metric with non-trivial four-form potential can be computed using the untwisted K's ((4.26) with $C=0$) and $\mathrm{e}^{-C}Z$. Thus, to reintroduce C, it is sufficient to consider (4.38) and to make the following substitutions

$$\begin{aligned}\hat{v} &\to \hat{v} + \hat{\rho}\lrcorner C\,, \\ \hat{\lambda}_\alpha &\to \hat{\lambda}_\alpha - C\lrcorner\hat{\sigma}_\alpha\,. \end{aligned} \tag{4.40}$$

Comparing the generalised metric and (A.23) with only non-zero g_{mn} and four-form C, we recover the metric and four-form potential of the $\mathrm{AdS}_5\times\mathrm{SE}_5$ solution of type IIB supergravity.

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4.2.3 Recovering the truncation ansatz

In [20], the scalar truncation ansatz based on the Sasaki-Einstein structure is given in the Einstein frame by[13]

$$\begin{aligned} g_{10} &= \mathrm{e}^{-\frac{2}{3}(4U+V)} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2U} g_{\mathrm{KE}} + \mathrm{e}^{2V}\eta^2 \\ B^+ &= b_i\, j_i\,, \quad B^- = c_i\, j_i\,, \qquad C = C^{\mathrm{fl}} - a\, j_3 \wedge j_3\,, \end{aligned} \tag{4.41}$$

where $\{U, V, b_i, c_i, a\}$, with $i = 1, 2, 3$, together with the axion C_0 and the dilaton ϕ, are eleven scalar fields depending just on the external coordinates, and C^{fl} is the background four-form potential that we called C in the previous subsection, satisfying (4.19). These eleven scalars parameterise the coset manifold (4.24). Specifically, the SO(1, 1) factor is parameterised by the combination $\Sigma = \mathrm{e}^{-\frac{2}{3}(U+V)}$. For the $\frac{\mathrm{SO}(5,2)}{\mathrm{SO}(5)\times\mathrm{SO}(2)}$ coset representative, it is convenient to use a solvable parametrization, which is obtained exponentiating the Cartan and positive root generators of the coset. The explicit form of $\{\mathcal{V}_A{}^b, \mathcal{V}_A{}^{\underline{b}}\}$ (with $b = 1, \ldots, 5$ and $\underline{b} = 1, 2$) chosen in [20] reads[14]

$$\begin{pmatrix}
1 & 0 & 0 & \mathrm{e}^{-\frac{\phi_1}{2}}(-c_1+C_0 b_1) & -\mathrm{e}^{-\frac{\phi_2}{2}} b_1 & \mathrm{e}^{-\frac{\phi_1}{2}}(c_1-C_0 b_1) & \mathrm{e}^{-\frac{\phi_2}{2}} b_1 \\
0 & 1 & 0 & \mathrm{e}^{-\frac{\phi_1}{2}}(-c_2+C_0 b_2) & -\mathrm{e}^{-\frac{\phi_2}{2}} b_2 & \mathrm{e}^{-\frac{\phi_1}{2}}(c_2-C_0 b_2) & \mathrm{e}^{-\frac{\phi_2}{2}} b_2 \\
0 & 0 & 1 & \mathrm{e}^{-\frac{\phi_1}{2}}(-c_3+C_0 b_3) & -\mathrm{e}^{-\frac{\phi_2}{2}} b_3 & \mathrm{e}^{-\frac{\phi_1}{2}}(c_3-C_0 b_3) & \mathrm{e}^{-\frac{\phi_2}{2}} b_3 \\
c_1 & c_2 & c_3 & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(\mathrm{e}^{\phi_1}+\mathfrak{c}_- + C_0\mathfrak{a}_+) & \frac{\mathrm{e}^{\frac{\phi_2}{2}}}{2} C_0 - \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}\mathfrak{a}_+ & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(\mathrm{e}^{\phi_1}-\mathfrak{c}_- - C_0\mathfrak{a}_+) & \frac{\mathrm{e}^{\frac{\phi_2}{2}}}{2} C_0 + \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}\mathfrak{a}_+ \\
b_1 & b_2 & b_3 & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(\mathfrak{a}_- - C_0\mathfrak{b}_-) & \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}(\mathrm{e}^{\phi_2}+\mathfrak{b}_-) & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(-\mathfrak{a}_- + C_0\mathfrak{b}_-) & \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}(\mathrm{e}^{\phi_2}-\mathfrak{b}_-) \\
c_1 & c_2 & c_3 & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(\mathrm{e}^{\phi_1}-\mathfrak{c}_+ + C_0\mathfrak{a}_+) & \frac{\mathrm{e}^{\frac{\phi_2}{2}}}{2} C_0 - \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}\mathfrak{a}_+ & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(\mathrm{e}^{\phi_1}+\mathfrak{c}_+ - C_0\mathfrak{a}_+) & \frac{\mathrm{e}^{\frac{\phi_2}{2}}}{2} C_0 + \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}\mathfrak{a}_+ \\
b_1 & b_2 & b_3 & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(\mathfrak{a}_- + C_0\mathfrak{b}_+) & \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}(\mathrm{e}^{\phi_2}-\mathfrak{b}_+) & \frac{\mathrm{e}^{-\frac{\phi_1}{2}}}{2}(-\mathfrak{a}_- - C_0\mathfrak{b}_+) & \frac{\mathrm{e}^{-\frac{\phi_2}{2}}}{2}(\mathrm{e}^{\phi_2}+\mathfrak{b}_+)
\end{pmatrix} \tag{4.42}$$

where we defined $\phi_1 = 4U - \phi$, $\phi_2 = 4U + \phi$, and

$$\begin{aligned} \mathfrak{a}_+ &= 2a + b_i c_i\,, \quad \mathfrak{a}_- = 2a - b_i c_i\,, \\ \mathfrak{b}_+ &= 1 + b_i b_i\,, \quad \mathfrak{b}_- = 1 - b_i b_i\,, \\ \mathfrak{c}_+ &= 1 + c_i c_i\,, \quad \mathfrak{c}_- = 1 - c_i c_i\,. \end{aligned} \tag{4.43}$$

Note that the solvable parameterisation has a nice interpretation in terms of $\mathrm{E}_{6(6)}$ adjoint action (recall (A.9))

$$\begin{pmatrix} \tilde{K}_a \\ \tilde{K}_{\underline{a}} \end{pmatrix} = \mathrm{e}^{-(B^+ + B^- + C)} \cdot m \cdot r \cdot \mathrm{e}^{-l} \cdot \begin{pmatrix} K_a \\ K_{\underline{a}} \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} \mathcal{V}_a{}^B \\ \mathcal{V}_{\underline{a}}{}^B \end{pmatrix} K_B\,, \tag{4.44}$$

[13] Compared to [20] we have renamed $b_1 = \mathrm{Re}b^\Omega$, $b_2 = \mathrm{Im}b^\Omega$, $b_3 = b^J$, and similarly for c_i.

[14] Compared to [20], we have renamed the indices $(1234567)_{\mathrm{here}} = (3451267)_{\mathrm{there}}$ via an SO(5) transformation.

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where

$$B^+ = b_i\, j_i\,, \qquad B^- = c_i\, j_i\,, \qquad C = -a\, j_3 \wedge j_3\,, \qquad m^\alpha{}_\beta = \begin{pmatrix} \mathrm{e}^{\frac{\phi}{2}} & 0 \\ \mathrm{e}^{\frac{\phi}{2}} C_0 & \mathrm{e}^{-\frac{\phi}{2}} \end{pmatrix}, \tag{4.45}$$

$$r = \mathrm{diag}\left(\mathrm{e}^V, \mathrm{e}^U, \mathrm{e}^U, \mathrm{e}^U, \mathrm{e}^U\right)\,, \qquad l = \frac{1}{3}\,\mathrm{tr}(r) = \frac{1}{3}\,(4U+V) \tag{4.46}$$

so that the action is by only supergravity fields, with no need to introduce the poly-vector components in the $\mathrm{E}_{6(6)}$ adjoint.[15]

Having chosen an explicit parameterisation of the coset representative $\mathcal{V}$, we can compute the full generalised metric using formula (3.40). This will depend on the eleven scalars $\{U, V, C_0, \phi, b_i, c_i, a\}$. Comparing the expression obtained in this way with form (A.23) of the generalised metric, we can extract the truncation ansatz for the supergravity fields g_{mn}, C_0, ϕ, B^α_{mn}, C_{mnpq}, as well as the warp factor Δ.[16]

Although straightforward in principle, the computations are lengthy and we just discuss the final result. The warp factor is easily extracted using (A.24), (A.25) and reads[17]

$$\mathrm{e}^{2\Delta} = \mathrm{e}^{-\frac{2}{3}(4U+V)}\,, \tag{4.47}$$

while the internal metric is given by

$$\mathrm{e}^{-2\Delta}(G^{-1})^{mn} = g^{mn} = \mathrm{e}^{-2U} g^{mn}_{\mathrm{KE}} + \mathrm{e}^{-2V}\xi^m\xi^n\,. \tag{4.48}$$

Proceeding in a similar way for the other supergravity fields, we recover precisely the scalar ansatz (4.41).

The ansatz for the five-dimensional vectors follows straightforwardly from (3.34). We construct the linear combination of generalised vectors $\mathcal{A}^{\mathcal{A}}_\mu K_{\mathcal{A}}$, where the coefficients $\mathcal{A}^{\mathcal{A}}_\mu$ are vectors in five dimensions, and we equate it to the generalised vector (4.13), with the fields $B^\alpha_{\mu,1}$, $C_{\mu,3}$, and $\tilde{B}^\alpha_{\mu,5}$ being defined as in (4.10). Separating the fields transforming in different representations of GL(5), we find:

$$\begin{aligned}
h_\mu &= \mathcal{A}^0_\mu\,\xi\,,\\
B^+_{\mu,1} &= \frac{1}{\sqrt{2}}\left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right)\eta\,,\\
B^-_{\mu,1} &= -\frac{1}{\sqrt{2}}\left(\mathcal{A}^5_\mu + \mathcal{A}^7_\mu\right)\eta\,,\\
C_{\mu,3} &= \frac{1}{\sqrt{2}}\,\mathcal{A}^i_\mu\, j_i \wedge \eta\,,\\
\tilde{B}^+_{\mu,5} &= -\frac{1}{\sqrt{2}}\left(\mathcal{A}^5_\mu - \mathcal{A}^7_\mu\right)\mathrm{vol} + \frac{1}{\sqrt{2}}\left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right) C^{\mathrm{fl}} \wedge \eta\,,\\
\tilde{B}^-_{\mu,5} &= -\frac{1}{\sqrt{2}}\left(\mathcal{A}^4_\mu - \mathcal{A}^6_\mu\right)\mathrm{vol} + \frac{1}{\sqrt{2}}\left(\mathcal{A}^5_\mu + \mathcal{A}^7_\mu\right) C^{\mathrm{fl}} \wedge \eta\,.
\end{aligned} \tag{4.49}$$

[15] The GL(5) matrix r is given in the basis of vielbeine that makes the metric diagonal. This should not be confused with the SL(2) doublet r^α.

[16] A minor subtlety is that the truncation of [20] was derived in the Einstein frame of type IIB supergravity, while the generalised metric in (A.23) is adapted to the string frame; however (A.23) can be turned to the Einstein frame by simply ignoring the explicit factors of $\mathrm{e}^{-\phi}$ appearing there, and we do so in our computation.

[17] In this case Δ is not really a warp factor as it is independent of the internal coordinates. It is just a Weyl rescaling setting the external metric in the Einstein frame.

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The ansatz for the two-form fields follows from (3.35). The weighted dual vectors $J^{\mathcal{A}}$ can be computed by multiplying the dual vectors $K^*_{\mathcal{A}}$ in (4.28) by the internal volume form as in (3.16). Doing so we find:

$$\begin{aligned}
J^0 &= \mathrm{vol} \otimes \eta \,, \\
J^i &= \frac{1}{\sqrt{2}}\, j_i \,, \\
J^4 &= \frac{1}{\sqrt{2}}(-\hat{n} + \hat{r}\,\mathrm{vol}_4 - \hat{n}\, C^{\mathrm{fl}}) \,, \\
J^5 &= \frac{1}{\sqrt{2}}(-\hat{r} - \hat{n}\,\mathrm{vol}_4 - \hat{r}\, C^{\mathrm{fl}}) \,, \\
J^6 &= \frac{1}{\sqrt{2}}(\hat{n} + \hat{r}\,\mathrm{vol}_4 + \hat{n}\, C^{\mathrm{fl}}) \,, \\
J^7 &= \frac{1}{\sqrt{2}}(\hat{r} - \hat{n}\,\mathrm{vol}_4 + \hat{r}\, C^{\mathrm{fl}}) \,,
\end{aligned} \tag{4.50}$$

where we defined $\mathrm{vol}_4 = \xi \lrcorner \mathrm{vol}$. Equating $\mathcal{B}_{\mu\nu\,\mathcal{A}} J^{\mathcal{A}}{}_M$ to the weighted dual vector (4.14) and separating the terms in different GL(5) representations, we find

$$\begin{aligned}
B_{\mu\nu,0\,+} &= \frac{1}{\sqrt{2}}\,(\mathcal{B}_{\mu\nu\,7} - \mathcal{B}_{\mu\nu\,5}) \,, \\
B_{\mu\nu,0\,-} &= \frac{1}{\sqrt{2}}\,(\mathcal{B}_{\mu\nu\,6} - \mathcal{B}_{\mu\nu\,4}) \,, \\
C_{\mu\nu,2} &= \frac{1}{\sqrt{2}}\,\mathcal{B}_{\mu\nu\,i}\, j_i \,, \\
\tilde{B}_{\mu\nu,4\,+} &= \frac{1}{\sqrt{2}}\,(\mathcal{B}_{\mu\nu\,4} + \mathcal{B}_{\mu\nu\,6})\,\mathrm{vol}_4 + \frac{1}{\sqrt{2}}\,(\mathcal{B}_{\mu\nu\,7} - \mathcal{B}_{\mu\nu\,5})\, C^{\mathrm{fl}} \,, \\
\tilde{B}_{\mu\nu,4\,-} &= -\frac{1}{\sqrt{2}}\,(\mathcal{B}_{\mu\nu\,5} + \mathcal{B}_{\mu\nu\,7})\,\mathrm{vol}_4 + \frac{1}{\sqrt{2}}\,(\mathcal{B}_{\mu\nu\,6} - \mathcal{B}_{\mu\nu\,4})\, C^{\mathrm{fl}} \,.
\end{aligned} \tag{4.51}$$

The tensor $\tilde{g}$ associated with the dual graviton would be expanded as $\mathcal{B}_{\mu\nu\,0}\,\mathrm{vol} \otimes \eta$, but we will not need this.

This ansatz for the one-form and two-form fields agrees with the one of [20]. We have thus shown how the full bosonic truncation ansatz for type IIB supergravity on Sasaki-Einstein manifolds can be derived from our general approach to half-maximal truncations.

We observe that the particular Sasaki-Einstein manifold given by the $T^{1,1} = \frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathrm{U}(1)}$ coset space admits a further reduced $\mathrm{U}(1) \subset \mathrm{SU}(2)$ structure. In the generalised geometry, this introduces an additional singlet vector $K_8 = \eta \wedge \Phi$, where Φ in the only harmonic two-form in the Sasaki-Einstein metric on $T^{1,1}$. On $T^{1,1}$ one can also twist the generalised tangent bundle by NSNS and RR three-form fluxes proportional to the cohomologically non-trivial three-form $\eta \wedge \Phi$. Following the same steps as above including the extra vector, we would retrieve the larger consistent truncation of [51, 52], yielding half-maximal gauged supergravity coupled to three vector multiplets.

4.3 Truncations for β-deformed backgrounds

It was shown in [19] that for any AdS_5 solution to type IIB supergravity preserving minimal supersymmetry, and hence dual to an $\mathcal{N} = 1$ SCFT_4, there is a consistent truncation to

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pure gauged supergravity in five dimensions containing that AdS_5 solution. A class of such backgrounds is provided by the β-deformation of Lunin and Maldacena [22]. For the case where the internal manifold is S^5, the explicit truncation ansatz of type IIB supergravity on the β-deformed geometry to pure gauged supergravity has been given very recently in [23]. Here we show that if one starts from a toric Sasaki-Einstein manifold, the generalised SU(2) structure of the β-deformed background allows for a much larger truncation. The resulting five-dimensional supergravity is in fact just the same half-maximal supergravity with two vector multiplets that arises from type IIB supergravity on squashed Sasaki-Einstein manifolds. One way to see this is to observe that the full truncation ansatz on toric Sasaki-Einstein manifolds can be β-deformed.

4.3.1 The β-deformed $T^{1,1}$ background

In [22], Lunin and Maldacena showed that, given an $\mathcal{N}=1$ background with two U(1) isometries commuting with the R-symmetry, a new supersymmetric solution can be obtained by applying a TsT transformation, namely a sequence of T-duality along one of the U(1), a shift along the second U(1) and another T-duality along the first one. Any toric Sasaki-Einstein manifold can be deformed in this way. We will present explicit formulae for the $T^{1,1}$ manifold, however our results apply to any toric Sasaki-Einstein five-manifold.

The canonically normalised Sasaki-Einstein metric on $T^{1,1}$ is

$$g_{\mathrm{SE}_5} = \frac{1}{6}\sum_{i=1,2}\left(\mathrm{d}\theta_i^2+\sin^2\theta_i\mathrm{d}\phi_i^2\right)+\frac{1}{9}\left(\mathrm{d}\psi+\cos\theta_1\mathrm{d}\phi_1+\cos\theta_2\mathrm{d}\phi_2\right)^2\,, \tag{4.52}$$

and for the internal part of the four-form potential satisfying (4.19) we choose the gauge

$$C = -\frac{\kappa}{108}\,\psi\sin\theta_1\sin\theta_2\,\mathrm{d}\theta_1\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_2\,. \tag{4.53}$$

The dilaton is constant and all other fields vanish, $\phi=\phi_0=\text{const}$, $C_0=B^\alpha=0$.[18]

The β-deformed solution[19] given in [22] reads

$$\begin{aligned}
g_{10} &= \ell^2\Bigg\{g_{\mathrm{AdS}_5}+\frac{\sin^2\theta_1\sin^2\theta_2}{324f}\,\mathrm{d}\psi^2+\frac{1}{6}\left(\mathrm{d}\theta_1^2+\mathrm{d}\theta_2^2\right)\\
&\qquad+\mathcal{G}\left[h\left(\mathrm{d}\phi_1+\frac{\cos\theta_1\cos\theta_2}{9h}\mathrm{d}\phi_2+\frac{\cos\theta_1}{9h}\mathrm{d}\psi\right)^2+\frac{f}{h}\left(\mathrm{d}\phi_2+\frac{\cos\theta_2\sin^2\theta_1}{54f}\,\mathrm{d}\psi\right)^2\right]\Bigg\},\\
\mathrm{e}^{2\phi} &= \mathrm{e}^{2\phi_0}\mathcal{G}\,,\\
B^+ &= 2\gamma\ell^4\mathcal{G}f\left(\mathrm{d}\phi_1+\frac{\cos\theta_1\cos\theta_2}{9h}\mathrm{d}\phi_2+\frac{\cos\theta_1}{9h}\mathrm{d}\psi\right)\wedge\left(\mathrm{d}\phi_2+\frac{\cos\theta_2\sin^2\theta_1}{54f}\,\mathrm{d}\psi\right),\\
B^- &= \frac{\kappa\gamma}{54}\cos\theta_1\sin\theta_2\,\mathrm{d}\theta_2\wedge\mathrm{d}\psi\,,\\
F^{\mathrm{fl}} &= \kappa\mathcal{G}\,\mathrm{vol}_{\mathrm{SE}}\,,
\end{aligned} \tag{4.54}$$

[18] The axion C_0 is set to zero for simplicity, although any constant value would be allowed.

[19] This is a solution for a real deformation β. The generalisation to a complex deformation is straightforward and amounts to an $\mathrm{SL}(2,\mathbb{R})$ rotation.

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where $\ell^4 = \frac{\kappa}{4} e^{\phi_0}$ and γ is a real parameter. Moreover one has the functions

$$\mathcal{G}^{-1} = 1 + 4\gamma^2 \ell^4 f\,,$$
$$h = \frac{\cos\theta_1^2}{9} + \frac{\sin^2\theta_1}{6}\,, \qquad f = \frac{1}{54}\left(\cos^2\theta_2 \sin^2\theta_1 + \cos^2\theta_1 \sin^2\theta_2\right) + \frac{1}{36}\sin^2\theta_1 \sin^2\theta_2\,. \tag{4.55}$$

4.3.2 The β-deformation in generalised geometry

We next show that the type IIB β-deformation has a very simple interpretation in generalised geometry as the $E_{6(6)}$ action by a bi-vector with components along the two U(1) isometries commuting with the Reeb vector.[20] For the $T^{1,1}$ metric (4.52), these correspond to the rotations by angles ϕ_1 and ϕ_2. Then the β deformed solution is generated by the bivector

$$\beta^\alpha = \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -2\gamma\, \partial_{\phi_1} \wedge \partial_{\phi_2} \end{pmatrix}, \tag{4.56}$$

where γ is a real constant. This acts on a generalised vector $V = v + \lambda^\alpha + \rho + \sigma^\alpha$ in the adjoint of $E_{6(6)}$ as (see (A.9)):

$$\begin{aligned} V' &= e^\beta \cdot V = V + \beta \cdot V \\ &= (v - \epsilon_{\alpha\beta}\beta^\alpha \lrcorner \lambda^\beta) + (\lambda^\alpha + \beta^\alpha \lrcorner \rho) + (\rho + \epsilon_{\alpha\beta}\beta^\alpha \lrcorner \sigma^\beta) + \sigma^\alpha\,. \end{aligned} \tag{4.57}$$

In particular it is easy to show that the deformation (4.56) maps the generalised vector $K_{\mathcal{A}}$, (4.26), defining the generalised SU(2) structure into new generalised vectors

$$\begin{aligned} K'_0 &= \xi\,, \\ K'_i &= \frac{1}{\sqrt{2}}\left[r\beta\lrcorner(\eta \wedge j_i) + \eta \wedge j_i\right] \qquad\qquad i = 1,2,3\,, \\ K'_4 &= \frac{1}{\sqrt{2}}\left[-\beta\lrcorner\eta + n\,\eta + \beta\lrcorner(\eta \wedge C) - r\,\mathrm{vol} - n\,\eta \wedge C\right]\,, \\ K'_5 &= \frac{1}{\sqrt{2}}\left[\beta\lrcorner\eta - r\,\eta + \beta\lrcorner\mathrm{vol} - n\,\mathrm{vol} + r\,\eta \wedge C\right]\,, \\ K'_6 &= \frac{1}{\sqrt{2}}\left[-\beta\lrcorner\eta + n\,\eta + \beta\lrcorner(\eta \wedge C) + r\,\mathrm{vol} - n\,\eta \wedge C\right]\,, \\ K'_7 &= \frac{1}{\sqrt{2}}\left[\beta\lrcorner\eta - r\,\eta - \beta\lrcorner\mathrm{vol} + n\,\mathrm{vol} + r\,\eta \wedge C\right]\,, \end{aligned} \tag{4.58}$$

that are still globally defined. Since the new $K_{\mathcal{A}}$, $\mathcal{A} = 0, \ldots, 7$, are obtained from the original ones by an $E_{6(6)}$ transformation, they still satisfy the conditions (3.8), (3.10) with $n = 2$, and therefore define a generalised SU(2) structure. Moreover, evaluating the generalised Lie derivative between them, one can check that they satisfy exactly the same algebra (4.30) as the original generalised vectors associated with the Sasaki-Einstein structure.

We conclude that there exists a consistent truncation on the β-deformed geometry, which leads to the very same five-dimensional half-maximal gauged supergravity obtained via reduction on Sasaki-Einstein manifolds.

[20] Similarly, the β-deformation of AdS_4 solutions to M-theory is generated by a tri-vector in $E_{7(7)}$ generalised geometry, see [62].

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To compute the algebra for the deformed generalised vectors it is helpful to make an explicit choice of parametrisation for the SU(2) structure on $T^{1,1}$. We introduced the coframe one-forms

$$
\begin{aligned}
e^1 &= \frac{1}{3}\left(\mathrm{d}\psi + \cos\theta_1 \mathrm{d}\phi_1 + \cos\theta_2 \mathrm{d}\phi_2\right), \qquad & e^2 &= \frac{1}{\sqrt{6}}\left(\cos\frac{\psi}{2}\,\sin\theta_1 \mathrm{d}\phi_1 - \sin\frac{\psi}{2}\,\mathrm{d}\theta_1\right), \\
e^3 &= \frac{1}{\sqrt{6}}\left(\sin\frac{\psi}{2}\,\sin\theta_2 \mathrm{d}\phi_2 + \cos\frac{\psi}{2}\,\mathrm{d}\theta_2\right), & e^4 &= \frac{1}{\sqrt{6}}\left(\cos\frac{\psi}{2}\,\sin\theta_2 \mathrm{d}\phi_2 - \sin\frac{\psi}{2}\,\mathrm{d}\theta_2\right), \\
e^5 &= \frac{1}{\sqrt{6}}\left(\sin\frac{\psi}{2}\,\sin\theta_1 \mathrm{d}\phi_1 + \cos\frac{\psi}{2}\,\mathrm{d}\theta_1\right), & &
\end{aligned}
\tag{4.59}
$$

such that the Sasaki-Einstein metric (4.52) is $g_{\mathrm{SE}} = \sum_{a=1}^{5}(e^a)^2$, and the SU(2) structure (4.15) is given by

$$
\begin{aligned}
&\xi = -3\,\partial_\psi\,, \qquad && \eta = -e^1\,, \\
&j_1 = e^{24} + e^{35}\,, \qquad && j_2 = e^{23} - e^{45}\,, \qquad j_3 = e^{25} - e^{34}\,.
\end{aligned}
\tag{4.60}
$$

The RR four-form potential satisfying (4.19) can be written as

$$
C = -\frac{1}{6}\kappa\,\psi\, j_3 \wedge j_3\,. \tag{4.61}
$$

For completeness we can also list the β-deformed generalised dual vectors

$$
\begin{aligned}
K_0^{*\prime} &= \eta - n\,\beta \lrcorner \eta\,, \\
K_i^{*\prime} &= -\frac{1}{\sqrt{2}}\left(\hat{j}_i \wedge \xi - \beta \wedge \hat{j}_i \wedge \xi\right), \qquad i = 1,2,3\,, \\
K_4^{*\prime} &= \frac{1}{\sqrt{2}}\left(-\hat{r}\,\xi + \hat{n}\,C \lrcorner \hat{\mathrm{vol}} - \beta \wedge C \lrcorner \hat{\mathrm{vol}} + \hat{n}\,\hat{\mathrm{vol}}\right), \\
K_5^{*\prime} &= \frac{1}{\sqrt{2}}\left(\hat{n}\,\xi + \hat{r}\,C \lrcorner \hat{\mathrm{vol}} - \beta \wedge \xi + \hat{r}\,\hat{\mathrm{vol}}\right), \\
K_6^{*\prime} &= \frac{1}{\sqrt{2}}\left(\hat{r}\,\xi + \hat{n}\,C \lrcorner \hat{\mathrm{vol}} - \beta \wedge C \lrcorner \hat{\mathrm{vol}} + \hat{n}\,\hat{\mathrm{vol}}\right), \\
K_7^{*\prime} &= \frac{1}{\sqrt{2}}\left(-\hat{n}\,\xi + \hat{r}\,C \lrcorner \hat{\mathrm{vol}} + \beta \wedge \xi + \hat{r}\,\hat{\mathrm{vol}}\right).
\end{aligned}
\tag{4.62}
$$

As for the Sasaki-Einstein case, the inverse generalised metric is computed by plugging the β-deformed generalised vectors and their duals in (3.33). The computation is long but relatively straightforward. Comparing the result with (A.23), we can then extract the supergravity fields describing the β-deformed solution. We illustrate here the main steps. From (A.25) one finds that the deformed solution has trivial warp factor

$$
\mathrm{e}^{\Delta'} = (\det \mathcal{H})^{-1/20} = 1\,. \tag{4.63}
$$

The inverse metric $(G^{-1})^{mn} = g^{mn}$ reproduces the metric in (4.54),

$$
g_5' = \frac{1}{6}\sum_{i=1,2}(\mathrm{d}\theta_i^2 + \mathcal{G}\sin\theta_i \mathrm{d}\phi_i^2) + \frac{1}{9}\mathcal{G}(\mathrm{d}\psi + \cos\theta_1 \mathrm{d}\phi_1 + \cos\theta_2 \mathrm{d}\phi_2)^2 + \frac{\gamma^2}{81}\mathcal{G}\sin^2\theta_1 \sin^2\theta_2 \mathrm{d}\psi^2\,, \tag{4.64}
$$

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which we have written in a way that will make the comparison with the truncation ansatz easier. The relation

$$B'^{\alpha}_{mn} = G_{m[p}(G^{-1})^{p\alpha}{}_{n]}\,, \tag{4.65}$$

gives the NS and RR two-form potentials

$$\begin{aligned} B'^{+} &= \gamma\,\mathcal{G}\left[2f\,\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2 + \frac{1}{27}(\sin^2\theta_1\cos\theta_2\,\mathrm{d}\phi_1 - \sin^2\theta_2\cos\theta_1\,\mathrm{d}\phi_2)\wedge\mathrm{d}\psi\right]\,,\\ B'^{-} &= -\frac{\kappa\,\gamma}{54}\sin\theta_1\sin\theta_2\,\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\,. \end{aligned} \tag{4.66}$$

While the NS two-form is exactly the same as in (4.54), the RR two-form is related to the one of [22] by a gauge transformation $B^{-}_{\rm LM} = B'^{-} + \mathrm{d}\Lambda$ with $\Lambda = -\frac{\gamma\kappa}{54}\psi\cos\theta_1\sin\theta_2\mathrm{d}\theta_2$. Next we use the component $(G^{-1})^{\alpha\,\beta}_{m\,n}$ in (A.23) to extract the axio-dilaton

$$\mathrm{e}^{-\phi'}h^{\alpha\beta} = \frac{1}{5}\left[(G^{-1})^{mn}(G^{-1})^{\alpha\ \beta}_{n\,m} + (G^{-1})^{m\alpha}{}_{n}(G^{-1})^{n\beta}{}_{m}\right] = \begin{pmatrix}1 & 0\\ 0 & \mathcal{G}^{-1}\end{pmatrix}\,. \tag{4.67}$$

From (A.19) we see that C_0 is zero (as we set it to zero in the undeformed solution) and the dilaton reproduces the one in the solution of [22]. Finally, from the component $(G^{-1})^{m}{}_{npq}$ we find the four-form potential

$$\begin{aligned} C' = \frac{\kappa\,\psi}{108}\mathcal{G}\Big[&-\frac{\gamma^2}{54}\sin\theta_1\sin\theta_2\left(\cos\theta_2\sin^2\theta_1\mathrm{d}\phi_1 - \cos\theta_1\sin^2\theta_2\,\mathrm{d}\phi_2\right)\wedge\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\psi\\ &+\left(1+2\gamma^2 f\right)\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\Big]\,, \end{aligned} \tag{4.68}$$

which gives the five-form of [22],

$$F'_5 \equiv \mathrm{d}C' + \frac{1}{2}\left(B'^{+}\wedge\mathrm{d}B'^{-} - B'^{-}\wedge\mathrm{d}B'^{+}\right) = \mathcal{G}\,\kappa\,\mathrm{vol}_{\rm SE}\,, \tag{4.69}$$

where $\mathrm{vol}_{\rm SE}$ is the $T^{1,1}$ volume form in the undeformed solution.

An equivalent way to compute the generalised metric for the deformed background is to act with a β-deformation on the generalised metric of the Sasaki-Einstein solution. We consider the action of a nilpotent bivector, $\beta\wedge\beta = 0$. This is not the most general bivector deformation, but it is enough to describe the β-deformation of Lunin and Maldacena. The transformed metric is

$$G'^{-1} = \mathrm{e}^{\beta}\cdot G^{-1}\cdot\mathrm{e}^{-\beta} = G^{-1} + \beta\cdot G^{-1} - G^{-1}\cdot\beta - \beta\cdot G^{-1}\cdot\beta\,. \tag{4.70}$$

For the purpose of extracting the type IIB supergravity fields, we will only need the following components of the β-transformed generalised metric

$$\begin{aligned} (G'^{-1})^{mn} &= (G^{-1})^{mn} - \beta^{mp}_{\alpha}(G^{-1})^{\alpha\,n}_{p} + (G^{-1})^{m\,\gamma}_{p}\beta^{pn}_{\gamma} - \beta^{mp}_{\alpha}(G^{-1})^{\alpha\gamma}_{pq}\beta^{qn}_{\gamma}\,,\\ (G'^{-1})^{m\gamma}_{\ n} &= (G^{-1})^{m\gamma}_{\ n} + \frac{1}{2}(G^{-1})^{m}{}_{npq}\beta^{\gamma pq} - \beta^{mp}_{\alpha}(G^{-1})^{\alpha\gamma}_{pn}\,,\\ (G'^{-1})^{\alpha\gamma}_{mn} &= (G^{-1})^{\alpha\gamma}_{mn} + \frac{1}{2}\beta^{\alpha pq}(G^{-1})_{mpq\,n}^{\ \ \ \gamma} + \frac{1}{2}(G^{-1})^{\alpha}_{m\,npq}\beta^{\gamma pq} + \frac{1}{4}\beta^{\alpha pq}(G^{-1})_{mpqnrs}\beta^{\gamma rs}\,,\\ (G'^{-1})^{m}{}_{npq} &= (G^{-1})^{m}{}_{npq} - \beta^{rs}_{\alpha}(G^{-1})^{m\,\alpha}{}_{npqrs}\,. \end{aligned} \tag{4.71}$$

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Plugging in the formulae above the metric, dilaton and four-form potential of the $AdS_5 \times T^{1,1}$ solution and the form (4.56) of the bivector β, we recover exactly the expressions for the different fields of the β-deformed solution discussed above.

In the specific example of $T^{1,1}$, one can also include the β deformation of the generalised vector K_8 introduced in the previous section

$$K_8' = \Phi \wedge \eta + \beta \lrcorner (\eta \wedge \Phi)\, n\,. \tag{4.72}$$

The β-transformed vector should still preserve the algebra and, after also introducing three-form fluxes, the corresponding enhanced truncation contains the β-transformed Klebanov-Strassler solution discussed in the appendix of [22].

4.3.3 The truncation ansatz

The truncation ansatz for the vectors is obtained substituting in (3.34) the generalised vectors defining the generalised SU(2) structure on the beta-deformed $T^{1,1}$ are given in (4.58), (4.62)

$$\begin{aligned}
h_\mu &= \mathcal{A}^0_\mu\, \xi - \frac{1}{\sqrt{2}} \left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right)\, \beta \lrcorner \eta\,,\\
B^+_{\mu,1} &= \frac{1}{\sqrt{2}} \left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right) \eta\,,\\
B^-_{\mu,1} &= -\frac{1}{\sqrt{2}} \left(\mathcal{A}^5_\mu + \mathcal{A}^7_\mu\right) \eta + \frac{1}{\sqrt{2}}\, \mathcal{A}^i_\mu\, \beta \lrcorner (j_i \wedge \eta)\ ,\\
C_{\mu,3} &= \frac{1}{\sqrt{2}}\, \mathcal{A}^i_\mu\, j_i \wedge \eta - \frac{1}{\sqrt{2}} \left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right) \beta \lrcorner (\eta \wedge C^{\rm fl}) + \frac{1}{\sqrt{2}} \left(\mathcal{A}^5_\mu - \mathcal{A}^7_\mu\right) \beta \lrcorner \mathrm{vol}\,,\\
\tilde{B}^+_{\mu,5} &= -\frac{1}{\sqrt{2}} \left(\mathcal{A}^5_\mu - \mathcal{A}^7_\mu\right) \mathrm{vol} - \frac{1}{\sqrt{2}} \left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right) C^{\rm fl} \wedge \eta\,,\\
\tilde{B}^-_{\mu,5} &= -\frac{1}{\sqrt{2}} \left(\mathcal{A}^4_\mu - \mathcal{A}^6_\mu\right) \mathrm{vol} + \frac{1}{\sqrt{2}} \left(\mathcal{A}^5_\mu + \mathcal{A}^7_\mu\right) C^{\rm fl} \wedge \eta\,.
\end{aligned} \tag{4.73}$$

To give the ansatz for the two-forms one has to compute the tensors $J^{\mathcal{A}}$ in the bundle $N \simeq \det T^*M \otimes E^*$. As for the Sasaki-Einstein truncation, these are obtained acting on the dual generalised vectors K^* with the internal volume, as in (3.16),

$$\begin{aligned}
J^0 &= \eta \otimes \mathrm{vol} + \hat{r}\, \beta \lrcorner (\eta \otimes \mathrm{vol})\,,\\
J^i &= \frac{1}{\sqrt{2}} \left(\hat{r}\, \beta \lrcorner j_i + j_i\right)\ ,\\
J^4 &= \frac{1}{\sqrt{2}} \left(-\hat{n} + \beta \lrcorner C^{\rm fl} + \hat{r}\, \mathrm{vol}_4 - \hat{n}\, C^{\rm fl}\right)\ ,\\
J^5 &= \frac{1}{\sqrt{2}} \left(-\hat{r} + \beta \lrcorner \mathrm{vol}_4 - \hat{n}\, \mathrm{vol}_4 - \hat{r}\, C^{\rm fl}\right)\ ,\\
J^6 &= \frac{1}{\sqrt{2}} \left(\hat{n} - \beta \lrcorner C^{\rm fl} + \hat{r}\, \mathrm{vol}_4 + \hat{n}\, C^{\rm fl}\right)\ ,\\
J^7 &= \frac{1}{\sqrt{2}} \left(\hat{r} + \beta \lrcorner \mathrm{vol}_4 - \hat{n}\, \mathrm{vol}_4 + \hat{r}\, C^{\rm fl}\right)\ ,
\end{aligned} \tag{4.74}$$

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where again we are using $\mathrm{vol}_4 = \xi \lrcorner \mathrm{vol}$. Then equating the components of the generalised tensor (4.14) with the linear combination $\mathcal{B}_{\mu\nu\,\mathcal{A}} J^{\mathcal{A}}$, we find

$$\begin{aligned}
B_{\mu\nu,0\,+} &= \frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,7} - \mathcal{B}_{\mu\nu\,5}\right) - \frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,1}\, \beta \lrcorner j_1 + \mathcal{B}_{\mu\nu\,2}\, \beta \lrcorner j_2\right) , \\
B_{\mu\nu,0\,-} &= \frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,6} - \mathcal{B}_{\mu\nu\,4}\right) , \\
C_{\mu\nu,\,2} &= \frac{1}{\sqrt{2}} \left[\mathcal{B}_{\mu\nu\,i}\, j_i + \left(\mathcal{B}_{\mu\nu\,4} - \mathcal{B}_{\mu\nu\,6}\right) \beta \lrcorner C^{\mathrm{fl}} + \left(\mathcal{B}_{\mu\nu\,5} + \mathcal{B}_{\mu\nu\,7}\right) \beta \lrcorner \mathrm{vol}_4\right] , \\
\tilde{B}_{\mu\nu,4\,+} &= -\mathcal{B}_{\mu\nu\,0}\, \beta \lrcorner (\eta \otimes \mathrm{vol}) + \frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,4} + \mathcal{B}_{\mu\nu\,6}\right) \mathrm{vol}_4 + \frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,7} - \mathcal{B}_{\mu\nu\,5}\right) C^{\mathrm{fl}} , \\
\tilde{B}_{\mu\nu,4\,-} &= -\frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,5} + \mathcal{B}_{\mu\nu\,7}\right) \mathrm{vol}_4 + \frac{1}{\sqrt{2}} \left(\mathcal{B}_{\mu\nu\,6} - \mathcal{B}_{\mu\nu\,4}\right) C^{\mathrm{fl}} .
\end{aligned} \tag{4.75}$$

The generalised metric contains the ansatz for the internal fields, metric and form potential, the dilaton and the warp factor. Here we give the final result for the internal and mixed components of the ten-dimensional metric (4.8):

$$\begin{aligned}
g_{mn} Dy^m Dy^n = \mathcal{F}^{-1} \Big[& \frac{1}{3} \mathrm{e}^{2A_1} \left(\gamma f_1\, \mathrm{d}\theta_1 + \gamma f_2\, \mathrm{d}\theta_2 + f_0 D\psi\right)^2 + \frac{1}{3} \mathrm{e}^{A_1} E_1 D\psi^2 \\
& + \frac{1}{6} \mathrm{e}^{A_2} E_2 \left(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2\right) + \frac{9}{2} \mathrm{e}^{A_1 + A_2} \left(\sin^2\theta_1 D\phi_1^2 + \sin^2\theta_2 D\phi_2^2\right) \\
& + \frac{1}{3} \mathrm{e}^{2A_1} \left(3\cos\theta_1 D\phi_1 + 3\cos\theta_2 D\phi_2 + f_0 D\psi\right)^2 \\
& + \gamma\, \mathrm{e}^{A_1} \sin^2\theta_1 \sin^2\theta_2 \Big(\sum_{i=1,2} g_i \sin\theta_i D\phi_i\, \mathrm{d}\theta_1 + \sum_{i=1,2} h_i \sin\theta_i D\phi_i\, \mathrm{d}\theta_2 \Big) \Big] ,
\end{aligned} \tag{4.76}$$

where the differentials D contain the shift by the five-dimensional vectors

$$\begin{aligned}
D\psi &= \mathrm{d}\psi + 3\mathcal{A}^0 , \\
D\phi_1 &= \mathrm{d}\phi_1 + \frac{\sqrt{2}}{3} \gamma \cos\theta_2 (\mathcal{A}^4 + \mathcal{A}^6) , \\
D\phi_2 &= \mathrm{d}\phi_2 - \frac{\sqrt{2}}{3} \gamma \cos\theta_1 (\mathcal{A}^4 + \mathcal{A}^6) .
\end{aligned} \tag{4.77}$$

For simplicity of notation we defined $A_1 = \frac{8}{3}(U+V)$, $A_2 = \frac{2}{3}(7U+V)$, $A_3 = \frac{1}{3}(3\phi - 8U + 4V)$ as well as the functions

$$\begin{aligned}
b_{12}^{+} &= b_1 \cos\psi + b_2 \sin\psi , \\
b_{12}^{-} &= b_2 \cos\psi - b_1 \sin\psi , \\
f_0 &= 3 - \gamma\, b_{12}^{+} \sin\theta_1 \sin\theta_2 , \\
f_1 &= b_{12}^{-} \cos\theta_1 \sin\theta_2 + b_3 \sin\theta_1 \cos\theta_2 , \\
f_2 &= b_{12}^{-} \sin\theta_1 \cos\theta_2 - b_3 \cos\theta_1 \sin\theta_2 , \\
\mathcal{F} &= 3\, \mathrm{e}^{A_1} f_0^2 + \gamma^2 \mathrm{e}^{A_2 + A_3} \left[2\, \mathrm{e}^{A_1} \left(\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2\right) + 3\, \sin^2\theta_1 \sin^2\theta_2\, \mathrm{e}^{A_2}\right] ,
\end{aligned} \tag{4.78}$$

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and

$$
\begin{aligned}
g_1 &= (3\,\mathrm{e}^{A_2} + 2\,\mathrm{e}^{A_1}\cot^2\theta_1)b_{12}^- + 2\,\mathrm{e}^{A_1} b_3 \cot\theta_1\cot\theta_2\,,\\
g_2 &= (3\,\mathrm{e}^{A_2} + 2\,\mathrm{e}^{A_1}\cot^2\theta_2)b_3 + 2\,\mathrm{e}^{A_1} b_{12}^- \cot\theta_1\cot\theta_2\,,\\
h_1 &= (3\,\mathrm{e}^{A_2} + 2\,\mathrm{e}^{A_1}\cot^2\theta_1)b_3 - 2\,\mathrm{e}^{A_1} b_{12}^- \cot\theta_1\cot\theta_2\,,\\
h_2 &= (3\,\mathrm{e}^{A_2} + 2\,\mathrm{e}^{A_1}\cot^2\theta_2)b_{12}^- - 2\,\mathrm{e}^{A_1} b_3 \cot\theta_1\cot\theta_2\,,\\
E_1 &= \mathrm{e}^{2A_2+A_3}\gamma^2\sin^2\theta_1\sin^2\theta_2 - \mathrm{e}^{A_1} f_0^2\,,\\
E_2 &= \mathcal{F} + 3\gamma^2\mathrm{e}^{A_1}\left((b_{12}^-)^2 + b_3^2\right)\sin^2\theta_1\sin^2\theta_2\,.
\end{aligned}
\tag{4.79}
$$

In line with the results of [19], there exists a subtruncation to minimal five-dimensional gauged supergravity, that has recently been made explicit in [23]. The bosonic sector of minimal five-dimensional supergravity is made of the metric, a single vector (the graviphoton) and no scalars. It is obtained from the truncation derived here, by setting all two-form and scalar fields to zero except for $\mathrm{e}^U = \mathrm{e}^V = \ell$, taking

$$
\mathcal{A}^1 = \mathcal{A}^2 = \mathcal{A}^4 = \mathcal{A}^5 = \mathcal{A}^6 = \mathcal{A}^7 = 0\,,
\tag{4.80}
$$

and identifying the other two gauge fields with the graviphoton A as

$$
A = 3\mathcal{A}^0 = -\mathcal{A}^3\,.
\tag{4.81}
$$

In this case it is easy to see that the generalised metric is the same as for the background solution, so that the internal fields are not modified. The ansatz for the full ten-dimensional metric becomes

$$
\begin{aligned}
g_{10} = g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + \frac{1}{6}\left(\mathrm{d}\theta_1^2+\mathrm{d}\theta_2^2\right) + \frac{\mathcal{G}}{6}\left(\sin^2\theta_1\mathrm{d}\phi_1^2+\sin^2\theta_2\mathrm{d}\phi_2^2\right)\\
+\frac{\mathcal{G}}{9}\left(\mathrm{d}\psi+\cos\theta_1\mathrm{d}\phi_1+\cos\theta_2\mathrm{d}\phi_2+3\mathcal{A}^0\right)^2 + \frac{\gamma^2}{81}\mathcal{G}\sin^2\theta_1\sin^2\theta_2\left(\mathrm{d}\psi+3\mathcal{A}^0\right)^2\,,
\end{aligned}
\tag{4.82}
$$

where we have set $\ell = 1$. Note that the purely internal part coincides with (4.64).

5 M-theory truncations including a Maldacena-Núñez AdS_5 solution

In this section, we construct a generalised $\mathrm{U}(1) \subset \mathrm{USp}(4)$ structure on a manifold M_6 given by a fibration of S^4 over $\mathbb{\Sigma}$, where $\mathbb{\Sigma}$ is a constant curvature Riemann surface. Specifically, $\mathbb{\Sigma}$ can be the hyperbolic plane H^2, the flat space $\mathbb{R}^2$, a sphere S^2, or a quotient thereof. We argue that in each case the generalised structure provides a consistent truncation to five-dimensional half-maximal gauged supergravity coupled to three vector multiplets and with a $\mathrm{U}(1)\times\mathrm{ISO}(3)$ gauging. The embedding tensor depends on the curvature of $\mathbb{\Sigma}$. When $\mathbb{\Sigma}$ is negatively curved, there is a fully supersymmetric AdS_5 solution which uplifts to the $\mathrm{AdS}_5 \times_{\mathrm{w}} M_6$ solution of [24] preserving 16 supercharges.[21] This describes the low-energy limit of M5 branes wrapped on $\mathbb{\Sigma}$, which is an $\mathcal{N} = 2$ SCFT_4, and our truncation captures some deformations of such theory.

[21] The symbol $\times_{\mathrm{w}}$ denotes the warped product.

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Generic $\mathrm{AdS}_5 \times_{\mathrm{w}} M_6$ solutions of eleven-dimensional supergravity preserving half-maximal supersymmetry were classified in [63]. It was shown in [64] that for all such solutions, there is a consistent truncation to pure half-maximal supergravity with $\mathrm{U}(1) \times \mathrm{SU}(2)$ gauging, such that the supersymmetric AdS_5 vacuum uplifts to the $\mathrm{AdS}_5 \times_{\mathrm{w}} M_6$ solution. In section 2 we discussed how this statement follows from restricting to the singlet sector of the $\mathrm{USp}(4)$ generalised structure on M_6. The results of this section show that, at least for the specific M_6 geometry of [24], the generalised structure is further reduced to $\mathrm{U}(1)$ and correspondingly the truncation can be enlarged to half-maximal supergravity with three vector multiplets.

We note that the existence of such a consistent truncation, as well as a detailed analysis of its sub-truncations and vacua, was very recently proven using a complementary approach in [25]. These authors considered an explicit truncation directly from seven-dimensional maximal gauged supergravity. As we will see, the generalised structure we find is indeed built using the generalised parallelisation on S^4 that defines the seven-dimensional maximal gauged supergravity, thus giving a direct connection to the construction in [25].

5.1 $\mathrm{E}_{6(6)}$ generalised geometry for M-theory

We start by recalling some basic notions of $\mathrm{E}_{6(6)}$ generalised geometry for M-theory, which is relevant for dimensional reductions of eleven-dimensional supergravity on a six-dimensional manifold M. Again we follow the conventions of [55, appendix E].

Under $\mathrm{GL}(6)$, the exceptional tangent bundle on M decomposes as:

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \,, \tag{5.1}$$

so that a generalised vector reads

$$V = v + \omega + \sigma \,, \tag{5.2}$$

where $v \in TM$, $\omega \in \Lambda^2 T^*M$ and $\sigma \in \Lambda^5 T^*M$. The $\mathrm{E}_{6(6)}$ cubic invariant is defined as[22]

$$c(V,V,V) = -\,6\,\iota_v\omega \wedge \sigma - \omega \wedge \omega \wedge \omega \,. \tag{5.3}$$

The bundle $N \simeq \det T^*M \otimes E^*$ similarly decomposes as:

$$N \simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \,, \tag{5.4}$$

so the sections are the sum of a one-form, a four-form and a tensor made of the product of a one-form and a volume form.

The eleven-dimensional supergravity fields, that is the metric g_{11}, the three-form potential $\hat{A}$ and its six-form dual $\hat{\hat{A}}$, can be decomposed according to the $\mathrm{SO}(1,10) \to \mathrm{SO}(1,4) \times \mathrm{SO}(6)$ splitting of the Lorentz group similarly to the discussion in subsection 4.1

[22] This is 6 times the cubic invariant given in [55].

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for type IIB supergravity:

$$
\begin{aligned}
g_{11} &= \mathrm{e}^{2\Delta}\, g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dy^m Dy^n\,, \\
\hat{A} &= \frac{1}{3!} A_{m_1m_2m_3} Dy^{m_1m_2m_3} + \frac{1}{2} A_{\mu m_1m_2}\mathrm{d}x^\mu \wedge Dy^{m_1m_2} + \frac{1}{2} A_{\mu\nu m}\mathrm{d}x^{\mu\nu} \wedge Dy^m + \dots\,, \\
\hat{\tilde{A}} &= \frac{1}{6!}\tilde{A}_{m_1\dots m_6} Dy^{m_1\dots m_6} + \frac{1}{5!}\tilde{A}_{\mu m_1\dots m_5}\mathrm{d}x^\mu \wedge Dy^{m_1\dots m_5} \\
&\quad + \frac{1}{2\cdot 4!}\tilde{A}_{\mu\nu m_1\dots m_4}\mathrm{d}x^{\mu\nu} \wedge Dy^{m_1\dots m_4} + \dots\,,
\end{aligned}
\tag{5.5}
$$

where $Dy^m = \mathrm{d}y^m - h_\mu{}^m \mathrm{d}x^\mu$ ensures covariance under internal diffeomorphisms, and $\Delta(x,y)$ is the warp factor of the external metric $g_{\mu\nu}(x)$. We can organise the eleven-dimensional supergravity fields into the inverse generalised metric on M,[23]

$$
G^{MN} \;\leftrightarrow\; \{\Delta,\, g_{mn},\, A_{m_1m_2m_3},\, \tilde{A}_{m_1\dots m_6}\}\,, \tag{5.6}
$$

the generalised vectors

$$
\mathcal{A}_\mu{}^M = \{h_\mu{}^m,\, A_{\mu mn}, \tilde{A}_{\mu m_1\dots m_5}\}\,, \tag{5.7}
$$

and the weighted dual vectors

$$
\mathcal{B}_{\mu\nu\,M} = \{A_{\mu\nu m},\, \tilde{A}_{\mu\nu m_1\dots m_4},\, \tilde{g}_{\mu\nu m_1\dots m_6,n}\}\,, \tag{5.8}
$$

where as in type IIB we will not need the last term, related to the dual graviton. The bosonic truncation ansatz is obtained by equating these generalised geometry objects to the corresponding terms given in section 3.

5.2 Generalised U(1) structure

The internal geometry of the half-maximal $\mathrm{AdS}_5 \times_\mathrm{w} M_6$ solution of [24] is constructed as a fibration of S^4 over Σ, where Σ is a negatively curved Riemann surface. This M_6 has a $\mathrm{U}(1) \subset \mathrm{GL}(6)$ structure in conventional geometry. As we will see below, this defines a consistent truncation to half-maximal supergravity coupled to three vector multiplets. Explicitly under the embedding $\mathrm{SO}(2)\times\mathrm{SO}(5,3) \subset \mathrm{SO}(5,5) \subset \mathrm{E}_{6(6)}$ of (3.2), the generalised tangent space E decomposes as

$$
\mathbf{27} = \mathbf{1}_0 + \mathbf{8}^v_0 + \mathbf{1}_+ + \mathbf{1}_- + \mathbf{8}^s_+ + \mathbf{8}^s_-\,, \tag{5.9}
$$

where $\mathbf{8}^v$ and $\mathbf{8}^s$ are vector and spinor representations and the subscript denotes the $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ charge. Thus we have nine singlets under U(1), which correspond to the generalised vectors $K_\mathcal{A}$, $\mathcal{A} = 0,\dots,8$. Under $\mathrm{SO}(5)\times\mathrm{SO}(3)$ these decompose as

$$
\begin{aligned}
\mathbf{1} + \mathbf{8}^v \;&=\; (\mathbf{1},\mathbf{1}) + \quad (\mathbf{5},\mathbf{1}) \quad + \quad (\mathbf{1},\mathbf{3})\,, \\
\Gamma(E) \;&\ni\; K_0 \;\cup \{K_1,\dots,K_5\} \cup \{K_6, K_7, K_8\}\,.
\end{aligned}
\tag{5.10}
$$

The explicit form of these vectors is determined by the S^4 fibration structure of the M_6 geometry. To see how they arise, we will first consider the direct product $\Sigma \times S^4$ and recall

[23] The precise expression for the inverse generalised metric in terms of the eleven-dimensional supergravity fields is easily obtained from the conformal split frame given in [35].

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some generalised geometry on S^4, and then implement the twist of S^4 over $\mathbb{\Sigma}$. On $\mathbb{\Sigma}\times S^4$ we can decomposes the generalised tangent space under $\mathrm{GL}(2,\mathbb{R})\times\mathrm{SL}(5,\mathbb{R})\subset \mathrm{E}_{6(6)}$ where $\mathrm{GL}(2,\mathbb{R})$ is the structure group of the conventional tangent space on $\mathbb{\Sigma}$ and $\mathrm{SL}(5,\mathbb{R})\simeq \mathrm{E}_{4(4)}$ is the structure group of the generalised tangent space on S^4. Explicitly we have

$$\begin{aligned} E &\simeq T\mathbb{\Sigma}\oplus(T^*\mathbb{\Sigma}\otimes N_4)\oplus(\Lambda^2 T^*\mathbb{\Sigma}\otimes N_4')\oplus E_4\,,\\ \mathbf{27} &= (\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{5'})\oplus(\mathbf{1},\mathbf{5})\oplus(\mathbf{1},\mathbf{10})\,, \end{aligned} \tag{5.11}$$

where in the second line we denote the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(5,\mathbb{R})$ representations, and where we have introduced the generalised bundles on S^4

$$\begin{aligned} E_4 &\simeq TS^4\oplus\Lambda^2T^*S^4\,,\\ N_4 &\simeq T^*S^4\oplus\Lambda^4T^*S^4\,,\\ N_4' &\simeq \mathbb{R}\oplus\Lambda^3T^*S^4\,, \end{aligned} \tag{5.12}$$

E_4 being the generalised tangent space on S^4.

As discussed in [6], on S^4 these bundles are parallelisable, that is, they admit global frames, constructed as follows. Let us parameterise the round four-sphere S^4 of radius R with coordinates $R\,y^i$, $i=1,\ldots,5$, constrained by the condition $\delta_{ij}y^iy^j=1$. The metric and the volume form induced from $\mathbb{R}^5$ are

$$g_4 = R^2\,\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j\,,\qquad \mathrm{vol}_4=\frac{1}{4!}R^4\,\epsilon_{i_1i_2i_3i_4i_5}\,y^{i_1}\mathrm{d}y^{i_2}\wedge\mathrm{d}y^{i_3}\wedge\mathrm{d}y^{i_4}\wedge\mathrm{d}y^{i_5}\,. \tag{5.13}$$

We can define the generalised frames

$$\begin{aligned} E_{ij} &= v_{ij}+R^2\,{*_4}(\mathrm{d}y_i\wedge\mathrm{d}y_j)+\iota_{v_{ij}}A &&\in\Gamma(E_4)\,,\\ E_i &= R\,\mathrm{d}y_i-y_i\,\mathrm{vol}_4+R\,\mathrm{d}y_i\wedge A &&\in\Gamma(N_4)\,,\\ E_i' &= y_i+R\ {*_4}\mathrm{d}y_i+y_iA &&\in\Gamma(N_4')\,, \end{aligned} \tag{5.14}$$

where $v_{ij}\in\Gamma(TS^4)$ are the Killing vector fields generating the SO(5) isometries, the Hodge star ${*_4}$ is computed using (5.13), and the M-theory three-form A is chosen such that

$$F=\mathrm{d}A=3R^{-1}\,\mathrm{vol}_4\ . \tag{5.15}$$

The frames (5.14) are globally-defined and therefore parallelise the respective bundles. Furthermore, under the generalised Lie derivative, the E_{ij} generate an $\mathfrak{so}(5)$ algebra

$$L_{E_{ij}}E_{kl}=-R^{-1}\left(\delta_{ik}E_{jl}-\delta_{il}E_{jk}+\delta_{jl}E_{ik}-\delta_{jk}E_{il}\right)\ . \tag{5.16}$$

This parallelisation is the basis of the generalised Scherk-Schwarz reduction of eleven-dimensional supergravity on S^4 [6], which reproduces the well-known consistent truncation to maximal SO(5) supergravity in seven dimensions [5]. In the generalised Scherk-Schwarz reduction, the E_{ij} define the truncation ansatz for the seven-dimensional scalar and vector fields, while the E_i and E_i' define the ansatz for the two-form and three-form potentials.

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In the solutions of [24], the internal space is a fibration

$$\begin{array}{ccc} S^4 & \xrightarrow{\ i\ } & M_6 \\ & & \big\downarrow \pi \\ & & \mathbb{\Sigma} \end{array} \tag{5.17}$$

where topologically the sphere is twisted by a U(1) subgroup of the SO(5) isometry group. Here $\mathbb{\Sigma}$ can be a negatively curved Riemann surface as in [24], but we can also allow it to be a torus T^2, or a sphere S^2. Let the one-forms e_1, e_2 be an orthonormal co-frame such that

$$g_{\mathbb{\Sigma}} = (e_1)^2 + (e_2)^2\,, \qquad \mathrm{vol}_{\mathbb{\Sigma}} = e_1 \wedge e_2 \tag{5.18}$$

are the constant curvature metric and compatible volume form on $\mathbb{\Sigma}$, all of which can be pulled back to M_6 using the projection map π. The twisting of the co-tangent space $T^*\mathbb{\Sigma}$ defines a U(1) spin-connection υ on $\mathbb{\Sigma}$ given by

$$\mathrm{d}(e_1 + \mathrm{i}\, e_2) = \mathrm{i}\,\upsilon \wedge (e_1 + \mathrm{i}\, e_2)\,, \qquad \mathrm{d}\upsilon = \frac{\kappa}{R^2}\,\mathrm{vol}_{\mathbb{\Sigma}}\,, \tag{5.19}$$

where $\kappa = -1$ for H^2, $\kappa = 0$ for $\mathbb{R}^2$ and $\kappa = +1$ for S^2 (and quotients thereof), and for convenience we are identifying the overall scale of $\mathbb{\Sigma}$ with the radius R of S^4. To preserve supersymmetry one needs to choose the U(1) twisting of the sphere so that it cancels the U(1) twisting of the cotangent space. For the half-maximal case one can choose conventions such that the twisting is the U(1) generated by, for example, the v_{12} Killing vector that appears in generalised frame E_{12}. In terms of the embedding in $\mathrm{E}_{6(6)}$ we thus have the breaking pattern

$$\mathrm{E}_{6(6)} \supset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(5,\mathbb{R}) \supset \mathrm{SO}(2) \times \mathrm{SO}(5) \supset \mathrm{SO}(2) \times \mathrm{SO}(2) \times \mathrm{SO}(3) \supset \mathrm{U}(1)\,, \tag{5.20}$$

where the final U(1) is the diagonal subgroup of $\mathrm{SO}(2)\times\mathrm{SO}(2) \simeq \mathrm{U}(1)^2$. By calculating the commutants one can see that this structure indeed corresponds to the case of half-maximal supersymmetry with $n = 3$ vector multiplets, as claimed.

Having identified the U(1) structure we can now directly construct the singlet vectors in the generalised tangent space. Given the decomposition (5.11), we note that these should come from $E_{12} \in \Gamma(E_4)$, $E_{\alpha\beta} \in \Gamma(E_4)$ and $\mathrm{vol}_{\mathbb{\Sigma}} \wedge E'_\alpha \in \Gamma(\Lambda^2 T^*\mathbb{\Sigma} \otimes N'_4)$ with $\alpha, \beta \in \{3,4,5\}$, since these are neutral under the U(1) action generated by v_{12}. In addition we get a complex generalised vector of the form $(e_1 + \mathrm{i}e_2) \wedge (E_1 + \mathrm{i}E_2) \in \Gamma(T^*\mathbb{\Sigma} \otimes N_4)$ since the twisting means that the U(1) action on the first term is cancelled by the U(1) action on the second term. Concretely we find the nine globally defined generalised vectors $K_{\mathcal{A}}$ on M_6 with $\mathcal{A} = 0, \ldots, 8$:

$$\begin{aligned} K_0 &= \frac{1}{2}\,\mathrm{e}^{\Upsilon} \cdot E_{12}\,, \\ K_1 + \mathrm{i}K_2 &= (e_1 + \mathrm{i}e_2) \wedge \mathrm{e}^{\Upsilon} \cdot (E_1 + \mathrm{i}E_2)\,, \\ K_\alpha &= \frac{1}{2}\epsilon_{\alpha\beta\gamma}\,\mathrm{e}^{\Upsilon} \cdot E_{\beta\gamma} + \mathrm{vol}_{\mathbb{\Sigma}} \wedge \mathrm{e}^{\Upsilon} \cdot E'_\alpha\,, \\ K_{3+\alpha} &= \frac{1}{2}\epsilon_{\alpha\beta\gamma}\,\mathrm{e}^{\Upsilon} \cdot E_{\beta\gamma} - \mathrm{vol}_{\mathbb{\Sigma}} \wedge \mathrm{e}^{\Upsilon} \cdot E'_\alpha\,, \end{aligned} \tag{5.21}$$

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with $\alpha = 3,4,5$ and $\epsilon_{\alpha\beta\gamma}$ the antisymmetric symbol such that $\epsilon_{345} = 1$. Note that each of the frame vectors on S^4 is twisted by the exponentiated SO(5) adjoint action (defined in [55, appendix E.1]) of an element of the $\mathfrak{e}_{6(6)} \oplus \mathbb{R}$ algebra:[24]

$$\begin{aligned} \Upsilon &= -R\, \upsilon \times_{\text{ad}} E_{12} \\ &= -R\left[v_{12} \otimes \upsilon + \upsilon \wedge *_4(R^2\, \mathrm{d}y_1 \wedge \mathrm{d}y_2) + \upsilon \wedge \iota_{v_{12}} A\right] , \end{aligned} \tag{5.22}$$

where the tensor product $\times_{\text{ad}}$ contains a projection on the adjoint representation and υ is the spin-connection on $\mathbb{\Sigma}$. Concretely one finds

$$\begin{aligned} \mathrm{e}^{\Upsilon} \cdot E_{12} &= E_{12}\,, \\ \mathrm{e}^{\Upsilon} \cdot (E_1 + \mathrm{i}E_2) &= (E_1 + \mathrm{i}E_2) + \mathrm{i}R\, \upsilon \wedge (E_1' + \mathrm{i}\, E_2')\,, \\ \frac{1}{2}\epsilon_{\alpha\beta\gamma}\, \mathrm{e}^{\Upsilon} \cdot E_{\beta\gamma} &= \frac{1}{2}\epsilon_{\alpha\beta\gamma} E_{\beta\gamma} + R\, \upsilon \wedge E_\alpha\,, \\ \mathrm{e}^{\Upsilon} \cdot E_\alpha' &= E_\alpha' + \frac{1}{2}R^2\, \epsilon_{\alpha\beta\gamma}\, \upsilon \wedge \mathrm{d}y_\beta \wedge \mathrm{d}y_\gamma\,. \end{aligned} \tag{5.23}$$

Note that the last term in the fourth line drops out when wedged with $\mathrm{vol}_{\mathbb{\Sigma}}$ in K_α and $K_{3+\alpha}$ above. One can check that these $K_{\mathcal{A}}$ do satisfy the conditions (3.8), (3.10) for a generalised U(1) structure, where $K_1, \ldots K_5$ are the negative norm vectors transforming in the fundamental representation of SO(5), while the $K_{3+\alpha}$ are the positive norm ones forming an SO(3) triplet. Since the frame vectors on S^4 have been twisted by the same element Υ of $\mathrm{E}_{6(6)}$, one can actually check the (3.8), (3.10) using the untwisted basis. In particular, the twisting implies that, since $\mathrm{d}E_i = 0$ and $\mathrm{d}E_i' = \frac{1}{R}\, E_i$,

$$\mathrm{d}\left[\mathrm{e}^{\Upsilon} \cdot (E_1 + \mathrm{i}\, E_2)\right] = -\mathrm{i}\, \upsilon \wedge \mathrm{e}^{\Upsilon} \cdot (E_1 + \mathrm{i}\, E_2) + \mathrm{vol}_{\mathbb{\Sigma}} \wedge (\ldots)\,, \tag{5.24}$$

which just cancels the exterior derivative of $e_1 + \mathrm{i}e_2$ in (5.19) giving

$$\mathrm{d}(K_1 + \mathrm{i}\, K_2) = 0\,. \tag{5.25}$$

The reason for the twisting by Υ is straightforward. Given the fibration (5.17), although vectors on S^4 push forward via the inclusion map $i : S^4 \to M_6$, we need a choice of U(1) connection in order to push forward forms on S^4 to globally defined forms on M_6. If ψ is a coordinate on S^5 such that $v_{12} = R^{-1}\partial/\partial\psi$ this means replacing $\mathrm{d}\psi$ in any form on S^4 with $\mathrm{d}\psi + \upsilon$. This is exactly what the action of the first term in (5.22) does. However, in the seven-dimensional consistent truncation on S^4 the U(1) gauging actually comes from E_{12} not just the leading isometry term v_{12}. Thus to match with the construction in [24], we should actually twist by the connection in (5.22), where the effect of the extra terms is to turn on additional F flux. This is the generalised geometry counterpart of the topological twist of the M5-brane $(2,0)$ theory on $\mathbb{\Sigma}$. Our construction should also make it clear that the truncation we are going to define can equivalently be seen as a truncation of seven-dimensional maximal supergravity on $\mathbb{\Sigma}$.

[24] This is reminiscent of the construction in the context of $\mathrm{O}(d,d)$ generalised geometry in [65].

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5.3 The gauge algebra and the embedding tensor

We now compute the algebra generated by the twisted generalised vectors $K_{\mathcal{A}}$ in (5.23). The M-theory generalised Lie derivative on M_6 is defined as:

$$L_V V' = \mathcal{L}_v v' + \left(\mathcal{L}_v \omega' - \iota_{v'} \mathrm{d}\omega\right) + \left(\mathcal{L}_v \sigma' - \iota_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega\right) . \tag{5.26}$$

In order to perform the computations, we find it convenient to use a parameterisation of the generalised vectors in terms of angular coordinates on S^4. This is given in appendix B. We find that the only non-vanishing generalised Lie derivatives are:

$$L_{K_0}(K_1 + \mathrm{i}\, K_2) = \frac{\mathrm{i}}{2R}\,(K_1 + \mathrm{i}\, K_2)\,, \tag{5.27}$$

where we crucially used (5.25), and

$$L_{R_\alpha} R_\beta = -\frac{1}{R}\epsilon_{\alpha\beta\gamma} R_\gamma\,, \quad L_{R_\alpha} T_\beta = L_{T_\alpha} R_\beta = -\frac{1}{R}\epsilon_{\alpha\beta\gamma} T_\gamma\,, \quad L_{T_\alpha} T_\beta = 0\,, \tag{5.28}$$

where we introduced the combinations

$$\begin{aligned} R_\alpha &:= \frac{1}{2}\left[(1-\kappa)K_\alpha + (1+\kappa)K_{3+\alpha}\right]\,, \\ T_\alpha &:= \frac{1}{2}\left[K_\alpha - K_{3+\alpha}\right]\,, \qquad \alpha = 1,2,3\,. \end{aligned} \tag{5.29}$$

It follows that K_0 generates a U(1) under which $K_1 + \mathrm{i}K_2$ is charged, and R_α, T_α generate the ISO(3) algebra, with R_α generating the SO(3) rotations and T_α generating the $\mathbb{R}^3$ translations. As is apparent from the form of the R_α, the way the SO(3) subgroup of ISO(3) is embedded in SO(5, 3) depends on the value of κ. If $\kappa = -1$ then SO(3) $\subset$ SO(5) $\subset$ SO(5, 3), if $\kappa = 0$ then SO(3) is the diagonal subgroup of SO(3, 3) $\subset$ SO(5, 3), and if $\kappa = +1$ then SO(3) is the commutant of SO(5) in SO(5, 3).

Since all generalised Lie derivatives yield a combination of the $K_{\mathcal{A}}$ with constant coefficients, the consistent truncation will go through, giving half-maximal gauged supergravity in five dimensions coupled to three vector multiplets. Recalling (3.18), (3.20), we can determine the embedding tensor. We find that the non-trivial embedding tensor components are:

$$\begin{aligned} \xi_{12} &= -\frac{1}{2R}\,, \\ f_{\alpha\beta\gamma} &= -\frac{3+\kappa}{2R}\,\epsilon_{\alpha\beta\gamma}\,, \qquad & f_{\alpha\beta(\gamma+3)} &= -\frac{1+\kappa}{2R}\,\epsilon_{\alpha\beta\gamma}\,, \\ f_{\alpha(\beta+3)(\gamma+3)} &= \frac{1-\kappa}{2R}\,\epsilon_{\alpha\beta\gamma}\,, \qquad & f_{(\alpha+3)(\beta+3)(\gamma+3)} &= \frac{3-\kappa}{2R}\,\epsilon_{\alpha\beta\gamma}\,. \end{aligned} \tag{5.30}$$

We note that these indeed agree with the embedding tensor derived in [25].

When $\kappa = -1$, the gauging satisfies the conditions for a half-maximal AdS_5 vacuum spelled out in [66]. This supersymmetric AdS_5 vacuum uplifts to the $\mathrm{AdS}_5 \times_{\mathrm{w}} M_6$ solution of [24]. In [67] the general conditions for five-dimensional half-maximal supergravity to admit supersymmetric flows between AdS fixed points were given. Inspection of the gauging (5.30) shows that the consistent truncation cannot admit such a flow, the basic reason being that the way S^4 is twisted over the Riemann surface is fixed in our truncation ansatz. It follows that the truncation cannot describe a flow from the AdS_5 vacuum preserving 16 supercharges to another supersymmetric vacuum. Nevertheless, it may contain other interesting solutions that it might be worth exploring.

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5.4 Recovering the truncation to pure half-maximal supergravity

The general formulae of section 3.3 provide an algorithmic construction of the full bosonic truncation ansatz for eleven-dimensional supergravity on M_6, leading to the five-dimensional half-maximal supergravity coupled to three vector multiplets described above. In the following we make this completely explicit for a sub-truncation of that theory: we recover the truncation to pure half-maximal supergravity given in [64]. This is only possible when $\mathbb{\Sigma}$ is a negatively curved Riemann surface. Indeed in order to be able to throw away the three vector multiplets consistently and be left with just the gravity multiplet we need the gauge algebra to close on the first six generalised vectors, $K_0, \ldots, K_5$, so that we have a USp(4) generalisd structure with singlet torsion. From (5.30) we see that this requires $\kappa = -1$. The gauging thus obtained is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and the half-maximal supergravity is the one dubbed $\mathcal{N} = 4^+$ in [68].

In order to determine how the only scalar field Σ of pure half-maximal supergravity embeds in the eleven-dimensional fields we evaluate the inverse generalised metric (3.40), where we set $\mathcal{V}_a{}^A = \delta_a{}^A$ as we are now truncating all other scalar fields. In particular, from

$$\begin{aligned} (G^{-1})^{mn} &= \mathrm{e}^{2\Delta} g^{mn} \,, \\ (G^{-1})^{m}{}_{np} &= \mathrm{e}^{2\Delta}\, g^{mq} A_{qnp} \,, \end{aligned} \tag{5.31}$$

we can extract the internal metric and the internal part of the three-form potential, after having computed the warp factor Δ. The latter is given by the general formula [36]

$$\mathrm{vol}_G \equiv (\det G_{MN})^{-\frac{9-d}{2\dim E}} = \sqrt{\det g_{mn}}\, \mathrm{e}^{(9-d)\Delta} \,, \tag{5.32}$$

where we need to take $d = 6$ and $\dim E = 27$.[25] Equivalently we can write:

$$\mathrm{e}^{9\Delta} = (\det\, G^{-1\,MN})^{\frac{1}{18}} (\det G^{-1\,mn})^{\frac{1}{2}} \,. \tag{5.33}$$

We explicitly evaluate the inverse generalised metric and express it in terms of the M_6 coordinates introduced in appendix B. In this way we find that (5.33) gives for the warp factor:

$$\mathrm{e}^{6\Delta} = \bar{\Delta} \,, \tag{5.34}$$

where we introduced the function

$$\bar{\Delta} = \cos^2\theta + \frac{\Sigma^3}{2\sqrt{2}} \sin^2\theta \,. \tag{5.35}$$

Inverting $(G^{-1})^{mn}$, we obtain the internal metric $g_{mn} = \mathrm{e}^{2\Delta} G_{mn}$, which reads

$$g_6 = R^2 \bar{\Delta}^{1/3} \left[\frac{\sqrt{2}}{\Sigma} \left(\mathrm{d}\theta^2 + g_{\mathbb{\Sigma}} \right) + \frac{\sqrt{2}}{\Sigma\bar{\Delta}} \sin^2\theta \left(\mathrm{d}\psi + \upsilon \right)^2 + \frac{\Sigma^2}{2\bar{\Delta}} \cos^2\theta\, g_{S^2} \right] , \tag{5.36}$$

where $g_{\mathbb{\Sigma}}$ is the uniform metric on $\mathbb{\Sigma}$ and g_{S^2} is the unit metric on the 2-sphere inside S^4. The second line of (5.31) gives for the internal part of the three-form potential:

$$A = \frac{R^3}{2\bar{\Delta}} \cos^3\theta \left[-2\bar{\Delta}\, \upsilon + \left(\frac{\Sigma^3}{\sqrt{2}} - 2 \right) \sin^2\theta \left(\mathrm{d}\psi + \upsilon \right) - 6\, \psi\, \tan\theta\, \bar{\Delta}\, \mathrm{d}\theta \right] \wedge \mathrm{vol}_{S^2} \,, \tag{5.37}$$

[25] We correct a typo in footnote 3 of [36]: $\det G$ appearing there should actually be $(\det G)^{1/2}$.

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whose field strength is

$$\mathrm{d}A = \frac{R^3\cos^3\theta}{2\,\bar{\Delta}^2}\left[\left(\left(\sqrt{2}\,\Sigma^3+2\bar{\Delta}\right)\tan\theta\,\mathrm{d}\theta+\frac{1}{\sqrt{2}}\sin^2\theta\,\mathrm{d}\left(\Sigma^3\right)\right)\wedge(\mathrm{d}\psi+\upsilon)+\frac{2\bar{\Delta}}{R^2}\,\mathrm{vol}_{\Sigma}\right]\wedge\mathrm{vol}_{S^2}\,. \tag{5.38}$$

In this way we have obtained the embedding of the five-dimensional scalar Σ into the eleven-dimensional supergravity fields. We note that the value of Σ giving the $\mathrm{AdS}_5\times_{\mathrm{w}} M_6$ solution of [24] is $\Sigma = 2^{1/6}$.

We can go on and use our general formulae to determine the embedding of the five-dimensional vector and two-form fields. For the mixed components of the eleven-dimensional metric we get

$$h_\mu{}^m = \frac{1}{2}\mathcal{A}^0_\mu\, v^m_{12} + \frac{1}{2}\,\epsilon_{\alpha\beta\gamma}\mathcal{A}^\alpha_\mu\, v^m_{\beta\gamma}\,, \tag{5.39}$$

where we recall that $\alpha = 3,4,5$. Then using (5.5) we reconstruct the full eleven-dimensional metric:

$$g_{11} = \bar{\Delta}^{1/3}\,g_5 + R^2\bar{\Delta}^{1/3}\left[\frac{\sqrt{2}}{\Sigma}(\mathrm{d}\theta^2+g_{\Sigma})+\frac{\sqrt{2}}{\Sigma\bar{\Delta}}\sin^2\theta\left(\mathrm{d}\psi+\upsilon-\frac{1}{2}\mathcal{A}^0\right)^2+\frac{\Sigma^2}{2\bar{\Delta}}\cos^2\theta\,\hat{g}_{S^2}\right] \tag{5.40}$$

where $\mathcal{A}^0$ gauges the shifts of the angle ψ, while $\hat{g}_{S^2}$ denotes the metric on S^2 where the SO(3) isometries are gauged by $\mathcal{A}^3,\mathcal{A}^4,\mathcal{A}^5$. When S^2 is described by constrained coordinates such that $\mu^\alpha\mu^\alpha = 1$, this reads

$$\hat{g}_{S^2} = D\mu^\alpha D\mu^\alpha\,, \tag{5.41}$$

with

$$D\mu^\alpha = \mathrm{d}\mu^\alpha - \frac{1}{2}\,\epsilon_{\beta\gamma\delta}\mathcal{A}^\delta\, v^\alpha_{\beta\gamma}\,, \tag{5.42}$$

$v^\alpha_{\beta\gamma}$ being the S^2 Killing vectors v_{45}, v_{53}, v_{34} expressed in the μ^α coordinates.

In order to determine the remaining part of the three-form potential we compute

$$\begin{aligned}
\mathcal{A}^{\mathcal{A}}\wedge K_{\mathcal{A}}|_2 &= \frac{1}{2}R^2\cos^3\theta\,\mathcal{A}^0\wedge\mathrm{vol}_{S^2}\\
&\quad+\mathrm{Re}\left[R\,\mathrm{e}^{\mathrm{i}\psi}(\mathcal{A}^1-\mathrm{i}\mathcal{A}^2)\wedge(e_1+\mathrm{i}\,e_2)\wedge(\cos\theta\,\mathrm{d}\theta+\mathrm{i}\sin\theta\,(\mathrm{d}\psi+\upsilon))\right]\\
&\quad+\mathcal{A}^\alpha\wedge\left[-R^2\,\mathrm{d}(\cos\theta\,\mu^\alpha)\wedge(\mathrm{d}\psi+\upsilon)+R^2\,\mathrm{d}\left[\mu^\alpha\,\mathrm{d}\left(\psi\cos^3\theta\right)\right]+\cos\theta\,\mu^\alpha\,\mathrm{vol}_{\Sigma}\right],\\
\mathcal{B}_{\mathcal{A}}\wedge J^{\mathcal{A}}|_1 &= \frac{R}{2}\left[\mathrm{Re}\left(\mathrm{i}\,(\mathcal{B}_1-\mathrm{i}\mathcal{B}_2)\wedge(e_1+\mathrm{i}\,e_2)\sin\theta\,\mathrm{e}^{\mathrm{i}\psi}\right)+\mathcal{B}_\alpha\wedge\mathrm{d}(\cos\theta\,\mu^\alpha)\right],
\end{aligned} \tag{5.43}$$

where $K_{\mathcal{A}}|_2$ and $J^{\mathcal{A}}|_1$ denote the 2-form and 1-form parts of $K_{\mathcal{A}}$ and $J^{\mathcal{A}}$, respectively (cf. appendix B). Then the full eleven-dimensional three-form potential is

$$\hat{A} = A + \mathcal{A}^{\mathcal{A}}\wedge K_{\mathcal{A}}|_2 + \mathcal{B}_{\mathcal{A}}\wedge J^{\mathcal{A}}|_1\,, \tag{5.44}$$

where we also need to implement the shifts $\mathrm{d}\psi\to\mathrm{d}\psi-\frac{1}{2}\mathcal{A}^0$ and $\mathrm{d}\mu^\alpha\to D\mu^\alpha$ so as to achieve covariance under internal diffeomorphisms.

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We can now compare with the consistent truncation ansatz given in [64]. To this extent, we redefine our scalar Σ into the scalar X appearing there as $\Sigma = 2^{1/6}X^{-1}$, and fix the scale of M_6 as $R = 1$. Then the eleven-dimensional metric (5.40) precisely matches the one given in [64, eq. (3.1)].[26] We also checked that the eleven-dimensional four-form field strength matches the corresponding one given in [64] after the five-dimensional one-form and two-form potentials are set to zero. Checking agreement of the remaining part of the eleven-dimensional four-form requires a little further work. Indeed our four-form, being constructed from the three-form potential, automatically satisfies the Bianchi identity, while the Bianchi identity of the four-form given in [64] is not automatic and defines part of the lower-dimensional equations of motion. Moreover in the embedding tensor formalism adopted in this paper one keeps the vector fields as propagating degrees of freedom, while the two-form potentials are auxiliary, non-propagating fields introduced just to ensure closure of the gauge algebra; on the other hand, in [64] two of the six vector fields in the half-maximal gravity multiplet are dualised into propagating two-forms and do not appear in the five-dimensional Lagrangian. These two descriptions are related by dualisation of some of the fields.[27] One starts from the on-shell duality between the eleven-dimensional three-form and six-form potentials $\hat{A}$ and $\hat{\tilde{A}}$,

$$*_{11}\,\mathrm{d}\hat{A} + \frac{1}{2}\,\hat{A}\wedge\mathrm{d}\hat{A} = \mathrm{d}\hat{\tilde{A}}\,. \tag{5.45}$$

Plugging our truncation ansatz in, this yields a set of duality relations between five-dimensional fields, in particular between one- and two-form potentials. Using these relations we can trade some of the fields appearing in our three-form potential for those appearing in the dual six-form. In particular, it is possible to remove the $\mathrm{d}\mathcal{B}_\alpha$, $\alpha = 3, 4, 5$, from $\mathrm{d}\hat{A}$ and replace them by $*\,\mathrm{d}\mathcal{A}^\alpha$ (the reason being that in the expression for $\hat{A}$ given by (5.43), (5.44), $\mathcal{B}_\alpha$ wedges a closed one-form, implying that in $\mathrm{d}\hat{A}$ only $\mathrm{d}\mathcal{B}_\alpha$, and not $\mathcal{B}_\alpha$, appears). On the other hand, in $\mathrm{d}\hat{A}$ the two-forms $\mathcal{B}_1, \mathcal{B}_2$ are Stückelberg-coupled to the one-forms $\mathcal{A}^1, \mathcal{A}^2$ as $\mathrm{d}(\mathcal{A}^1 - \mathrm{i}\mathcal{A}^2) - \frac{\mathrm{i}}{2}(\mathcal{B}_1 - \mathrm{i}\,\mathcal{B}_2)$ and cannot be removed. If desired, one could instead dualise $\mathcal{A}^1 - \mathrm{i}\mathcal{A}^2$ into $\mathcal{B}_1 - \mathrm{i}\,\mathcal{B}_2$ so that the latter becomes propagating in the five-dimensional theory, matching in this way the description of [64].

6 Conclusions

In this paper we have discussed how generalised geometry provides a formalism to understand consistent truncations of string and M-theory preserving varying amounts of supersymmetry, including non-supersymmetric cases.

When the generalised structure group G_S is just the identity, and the generalised intrinsic torsion is a G_S-singlet, one has a generalised Leibniz parallelisation [6] and can perform a generalised Scherk-Schwarz reduction; this is a consistent truncation preserving maximal supersymmetry. When instead G_S is non-trivial, and the intrinsic torsion is still a

[26] After making the obvious identifications of the supergravity gauge fields and of the connection one-form on $\mathbb{\Sigma}$, as well as a trivial, constant rescaling of the external metric.

[27] See also [29, section 3.2] for a discussion of the procedure leading to select the relevant degrees of freedom from dual pairs in a related context.

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G_S-singlet, one obtains a consistent truncation preserving only a fraction of supersymmetry. As we discussed in section 2, the matter content of the reduced theory is obtained by evaluating the commutant of G_S in $\mathrm{E}_{d(d)}$ in the relevant representations, while the gauging follows from the algebra of G_S-singlets under generalised diffeomorphisms. In this way the lower-dimensional theory is completely determined. Our formalism is completely general, extending to less intuitive examples than the case where the consistent truncation is based on an ordinary G_S-structure. For instance we can allow for a non-trivial warp factor, or use generalised tensor fields whose fixed-rank components can vanish at points on the internal manifold, but the full generalised tensor is nowhere vanishing.

After illustrating the general principles, in section 3 we have discussed in detail truncations to five dimensions preserving half-maximal supersymmetry. These are based on $\mathrm{SO}(5-n) \subseteq \mathrm{USp}(4) \subset \mathrm{E}_{6(6)}$ structures. In this case, the generalised structure is entirely characterised by a set of generalised vectors $K_{\mathcal{A}}$, $\mathcal{A} = 0, 1, \ldots, 5+n$, and the truncation contains n vector multiplets. The sub-algebra of generalised diffeomorphisms generated by the $K_{\mathcal{A}}$ determines the gauging of the five-dimensional supergravity. We have given an algorithmic prescription to construct the full bosonic truncation ansatz. In particular, we provided an expression for the generalised metric on the internal manifold in terms of the $K_{\mathcal{A}}$, and using this we specified the scalar field ansatz for the truncated theory.

We gave evidence for two new consistent truncations preserving half-maximal supersymmetry: the first is obtained from type IIB supergravity on β-deformed toric Sasaki-Einstein five-manifolds, and the second from eleven-dimensional supergravity on half-maximal Maldacena-Núñez geometries [24] (the latter recently independently derived using the truncation from seven-dimensional maximal supergravity in [25]). In both cases, we showed how the generalised geometry completely characterises the truncated theory. For the type IIB reduction we also discussed the bosonic truncation ansatz, while for the M-theory one we recovered the ansatz for the sub-truncation to pure half-maximal supergravity previously given in [64].

There are many other possible truncations that it would be intriguing to explore using our formalism. We sketch here some possibilities directly related to the cases we have studied. In type IIB $\mathrm{E}_{6(6)}$ geometry, it would be interesting to construct a generalised U(1) structure on the $Y^{p,q}$ family [69] of Sasaki-Einstein manifolds, and check if it admits a U(1)-singlet intrinsic torsion. If so, this would give a half-maximal consistent truncation on $Y^{p,q}$ manifolds extending the one based on generic Sasaki-Einstein SU(2) structure by one Betti vector multiplet, as in the $Y^{1,0} \simeq T^{1,1}$ truncation of [51, 52]. For this to go through, one would need the full flexibility of generalised geometry in order to circumvent the issue pointed out in [70] relevant working with ordinary G-structures.

In M-theory, it would be nice to extend the construction presented in section 5, which is based on the geometry of [24], to the general ansatz for half-maximal AdS_5 solutions of [63]. In particular, this would provide new consistent truncations containing the AdS_5 solutions of [71], describing M5-branes wrapped on Riemann surfaces with punctures. A similar construction is conceivable for the supergravity description of D3-branes wrapped on Riemann surfaces [24], however in this case one would need to use the more complicated type IIB $\mathrm{E}_{8(8)}$ generalised geometry formalism, which is not fully developed yet (though see [72, 73]).

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M5-branes wrapped on Riemann surfaces also give rise to $\mathrm{AdS}_5 \times_{\mathrm{w}} M_6$ supergravity solutions preserving just one quarter of the supersymmetry, which are dual to $\mathcal{N} = 1$ four-dimensional SCFTs [24, 74]. Our general analysis can be used to predict the form of the corresponding consistent truncations. For the quarter-supersymmetric solution of [24], the structure is again U(1) but embedded in a different way in $\mathrm{E}_{6(6)}$. It is easy to see that in this case, there are only two singlet spinors, and so the truncation is to minimal five-dimensional supergravity. The scalar moduli space is

$$\mathcal{M}_{\mathrm{scal}} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(\mathrm{U}(1))}{\mathrm{C}_{\mathrm{USp}(8)}(\mathrm{U}(1))} = \mathbb{R}^+ \times \frac{\mathrm{SO}(3,1)}{\mathrm{SO}(3)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)}\,, \tag{6.1}$$

and there are five singlet vectors in the generalised tangent space. We see that the truncated theory is minimal five-dimensional supergravity coupled to four vector multiplets and a single hypermultiplet. The first factor in (6.1) gives the homogeneous very special real geometry describing the four additional vector multiplets, while the second factor is the standard homogeneous quaternionic space for a single hypermultiplet. The singlet generalised vectors $K_{\mathcal{A}}$ are again constructed starting from the frames on S^4 but now the relevant twist connection is

$$\Upsilon = \frac{1}{2}\, \upsilon \times_{\mathrm{ad}} (E_{14} - E_{23})\,. \tag{6.2}$$

The details of this truncation will be discussed in future work. Although none of the truncations constructed in this way are expected to contain the domain wall solutions connecting the different $\mathrm{AdS}_5 \times_{\mathrm{w}} M_6$ supergravity solutions (which are dual to conjectured RG flows between the corresponding SCFTs), the generalised geometry approach may suggest how to make the twist "dynamical" so that it can evolve along the flow.

Half-maximal consistent truncations of massive type IIA supergravity can also be engineered by combining the formalism of the present paper with the one of [56, 75], where the maximally supersymmetric case was discussed.

Besides consistent truncations, a physically relevant motivation for developing half-maximal structures is to study the moduli space of half-maximal AdS solutions to supergravity theories, which is dual to the conformal manifold (i.e. the space of exactly marginal deformations) of SCFTs with eight Poincaré and eight conformal supercharges, in the large N limit. In the quarter-supersymmetric case, a study of marginal deformations using generalised geometry was done in [62, 76]. The additional constraint of half-maximal supersymmetry may allow to go further in the analysis. In particular, it may allow one to compare in great detail with field theory results, where the Kähler metric on the conformal manifold follows from the S^4 partition function [77], which is computable using supersymmetric localization. It would also be interesting to compare with the results found in [66] by means of a purely five-dimensional supergravity setup.

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A Type IIB $\mathbf{E_{6(6)}}$ generalised geometry

We briefly recall the exceptional geometry for type IIB compactifications on a five-dimensional manifold M, following the conventions of [55, appendix E]. The type IIB generalised tangent bundle on M has fibres transforming in the **27** of $\mathrm{E}_{6(6)}$ and decomposes under the geometric GL(5) subgroup of $\mathrm{E}_{6(6)}$ as

$$E \simeq TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus \Lambda^{\mathrm{odd}} T^*M \,, \tag{A.1}$$

where $\Lambda^{\mathrm{odd}} T^* = T^* \oplus \Lambda^3 T^* \oplus \Lambda^5 T^*$. A generalised vector $V \in \Gamma(E)$ can be written as

$$V = v + \lambda + \sigma + \omega \,, \tag{A.2}$$

where $\omega = \omega_1 + \omega_3 + \omega_5$ is a poly-form of odd degree. Alternatively, the generalised tangent bundle can be decomposed in a way that also makes the action of SL(2) manifest. The $\mathrm{GL}(5) \times \mathrm{SL}(2)$ covariant decomposition is

$$E \simeq TM \oplus (S \otimes T^*M) \oplus \Lambda^3 T^*M \oplus (S \otimes \Lambda^5 T^*M) \,, \tag{A.3}$$

where S denotes an SL(2) doublet. In this picture a generalised vector can be expressed as

$$V = v + \lambda^\alpha + \rho + \sigma^\alpha \,, \tag{A.4}$$

where the index $\alpha = \{+, -\}$ labels the states in the SL(2) doublet. In this paper we use the second description.

The dual generalised vector bundle decomposes accordingly as

$$E^* \simeq T^*M \oplus (S^* \otimes TM) \oplus \Lambda^3 TM \oplus (S^* \otimes \Lambda^5 TM) \,, \tag{A.5}$$

and a generalised dual vectors $Z \in \Gamma(E^*)$ can be written as

$$Z = \hat{v} + \hat{\lambda}_\alpha + \hat{\rho} + \hat{\sigma}_\alpha \,. \tag{A.6}$$

The adjoint bundle is defined as

$$\mathrm{ad}F = \mathbb{R} \oplus (TM \otimes T^*M) \oplus (S \otimes S^*) \oplus (S \otimes \Lambda^2 TM) \oplus (S \otimes \Lambda^2 T^*M) \oplus \Lambda^4 TM \oplus \Lambda^4 T^*M \tag{A.7}$$

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with elements

$$R = l + r + a^{\alpha}{}_{\beta} + \beta^{\alpha} + B^{\alpha} + \gamma + C \tag{A.8}$$

where $l \in \Gamma(\mathbb{R})$, $r \in End(TM)$, $a^{\alpha}{}_{\beta}$ is an element of SL(2), β^{α} and B^{α} are an SL(2) doublet of bi-vectors and two-forms respectively, γ is a four-vector and C a four-form. The adjoint action of $R \in \Gamma(\mathrm{ad})$ on $V \in \Gamma(E)$, denoted by $V' = R \cdot V$, is defined as:

$$\begin{aligned}
v' &= lv + r \cdot v + \gamma \lrcorner \rho + \epsilon_{\alpha\beta} \beta^{\alpha} \lrcorner \lambda^{\beta} \,, \\
\lambda'^{\alpha} &= l\lambda^{\alpha} + r \cdot \lambda^{\alpha} + a^{\alpha}{}_{\beta} \lambda^{\beta} - \gamma \lrcorner \sigma^{\alpha} + v \lrcorner B^{\alpha} + \beta^{\alpha} \lrcorner \rho \,, \\
\rho' &= l\rho + r \cdot \rho + v \lrcorner C + \epsilon_{\alpha\beta} \beta^{\alpha} \lrcorner \sigma^{\beta} + \epsilon_{\alpha\beta} \lambda^{\alpha} \wedge B^{\beta} \,, \\
\sigma'^{\alpha} &= l\sigma^{\alpha} + r \cdot \sigma^{\alpha} + a^{\alpha}{}_{\beta} \sigma^{\beta} - C \wedge \lambda^{\alpha} + \rho \wedge B^{\alpha} \,,
\end{aligned} \tag{A.9}$$

where $\epsilon_{\alpha\beta}$ is defined as $\epsilon_{+-} = -\epsilon_{-+} = 1$, $\epsilon_{++} = \epsilon_{--} = 0$, and for the definition of the $\mathfrak{gl}(5)$ action $r\cdot$ and of the contraction $\lrcorner$ we refer to [55].

A generalised vector can be twisted by elements of the adjoint bundle. In particular, the twisted generalised vector $V = \mathrm{e}^{B^{\alpha} + C} \check{V}$ is given by

$$\begin{aligned}
v &= \check{v} \,, \\
\lambda^{\alpha} &= \check{\lambda}^{\alpha} + \check{v} \lrcorner B^{\alpha} \,, \\
\rho &= \check{\rho} + \check{v} \lrcorner C + \epsilon_{\alpha\beta} \check{\lambda}^{\alpha} \wedge B^{\beta} + \frac{1}{2} \epsilon_{\alpha\beta}\, \check{v} \lrcorner B^{\alpha} \wedge B^{\beta} \,, \\
\sigma^{\alpha} &= \check{\sigma}^{\alpha} - C \wedge \check{\lambda}^{\alpha} + \check{\rho} \wedge B^{\alpha} - \frac{1}{2} \check{v} \lrcorner B^{\alpha} \wedge C + \frac{1}{2} \left(\check{v} \lrcorner C + \epsilon_{\alpha\beta} \check{\lambda}^{\beta} \wedge B^{\gamma} \right) \wedge B^{\alpha} \,.
\end{aligned} \tag{A.10}$$

Another bundle of interest is the bundle $N \simeq \det T^*M \otimes E^*$. This a sub-bundle of the symmetrised product of two copies of the generalised tangent bundle. Its fibres transform in the $\mathbf{27'}$ of $\mathrm{E}_{6(6)}$ and its $\mathrm{GL}(5) \times \mathrm{SL}(2)$ decomposition reads

$$N \simeq (S^* \otimes \mathbb{R}) \oplus \Lambda^2 T^*M \oplus (S^* \otimes \Lambda^4 T^*M) \oplus (\det T^*M \otimes T^*M) \,, \tag{A.11}$$

with sections

$$J = s_{\alpha} + \omega_2 + \omega_{4\,\alpha} + \varsigma \,. \tag{A.12}$$

The $\mathrm{E}_{6(6)}$ cubic invariant acting on three generalised vectors is defined as

$$c(V, V', V'') = -\frac{1}{2} \left(\iota_v \rho' \wedge \rho'' + \epsilon_{\alpha\beta}\, \rho \wedge \lambda'^{\alpha} \wedge \lambda''^{\beta} - 2\epsilon_{\alpha\beta}\, \iota_v \lambda'^{\alpha} \sigma''^{\beta} \right) + \text{ symm. perm.} \,. \tag{A.13}$$

The cubic invariant acting on dual vectors is

$$c^*(Z, Z', Z'') = -\frac{1}{2} \left(\iota_{\hat{v}} \hat{\rho}' \wedge \hat{\rho}'' + \epsilon^{\alpha\beta}\, \hat{\rho} \wedge \hat{\lambda}'_{\alpha} \wedge \hat{\lambda}''_{\beta} - 2\epsilon^{\alpha\beta}\, \iota_{\hat{v}} \hat{\lambda}'_{\alpha} \hat{\sigma}''_{\beta} \right) + \text{symm. perm.} \,, \tag{A.14}$$

where $\epsilon^{\alpha\beta}$ is defined as a matrix with the same components as $\epsilon_{\alpha\beta}$.

The product between an element $V \in \mathbf{27}$ and $Z \in \overline{\mathbf{27}}$ is defined as

$$\langle Z, V \rangle = \hat{v}_m v^m + \hat{\lambda}^m_{\alpha} \lambda^{\alpha}_m + \frac{1}{3!}\, \hat{\rho}^{mnp} \rho_{mnp} + \frac{1}{5!}\, \hat{\sigma}^{mnpqr}_{\alpha} \sigma^{\alpha}_{mnpqr} \,. \tag{A.15}$$

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The generalised Lie derivative between two twisted generalised vectors V, V' is given by

$$\begin{aligned} L_V V' = {} & \mathcal{L}_v v' + (\mathcal{L}_v \lambda'^{\alpha} - \iota_{v'} \mathrm{d}\lambda^{\alpha}) + (\mathcal{L}_v \rho' - \iota_{v'} \mathrm{d}\rho + \epsilon_{\alpha\beta} \mathrm{d}\lambda^{\alpha} \wedge \lambda'^{\beta}) \\ & + \mathcal{L}_v \sigma'^{\alpha} - \mathrm{d}\lambda^{\alpha} \wedge \rho' + \lambda'^{\alpha} \wedge \mathrm{d}\rho \,. \end{aligned} \tag{A.16}$$

The generalised metric is defined as

$$G(V, V') = v^m v'_m + h_{\alpha\beta} \lambda^{m\alpha} \lambda'^{\beta}_m + \frac{1}{3!} \rho^{mnp} \rho'_{mnp} + \frac{1}{5!} \sigma^{mnpqr} \sigma'_{mnpqr} \,, \tag{A.17}$$

where V and V' are the twisted generalised vectors defined in (A.10) and the latin indices are lowered/raised using the ordinary metric g_{mn} or its inverse g^{mn}. The matrix $h_{\alpha\beta}$ parameterises the coset $\mathrm{SL}(2)/\mathrm{SO}(2)$ and is given by

$$h_{\alpha\beta} = \mathrm{e}^{\phi} \begin{pmatrix} C_0^2 + \mathrm{e}^{-2\phi} & -C_0 \\ -C_0 & 1 \end{pmatrix} , \tag{A.18}$$

with inverse

$$h^{\alpha\beta} = \mathrm{e}^{\phi} \begin{pmatrix} 1 & C_0 \\ C_0 & C_0^2 + \mathrm{e}^{-2\phi} \end{pmatrix} . \tag{A.19}$$

Note that $h^{\alpha\beta} = \epsilon^{\alpha\alpha'} \epsilon^{\beta\beta'} h_{\alpha'\beta'}$.

The inverse generalised metric can be obtained from a generalised local frame $E_A \in \Gamma(E)$, $A = 1, \ldots, 27$, as

$$\begin{aligned} G^{-1} = {} & \delta^{AB} E_A \otimes E_B \\ = {} & \delta^{ab} E_a \otimes E_b + \delta_{ab} \delta^{\hat{\alpha}\hat{\beta}} E^a_{\hat{\alpha}} \otimes E^b_{\hat{\beta}} + \frac{1}{3!} \delta_{a_1 b_1} \delta_{a_2 b_2} \delta_{a_3 b_3} E^{a_1 a_2 a_3} \otimes E^{b_1 b_2 b_3} \\ & + \frac{1}{5!} \delta_{a_1 b_1} \cdots \delta_{a_5 b_5} \delta^{\hat{\alpha}\hat{\beta}} E^{a_1 \ldots a_5}_{\hat{\alpha}} \otimes E^{b_1 \ldots b_5}_{\hat{\beta}} \,, \end{aligned} \tag{A.20}$$

where $a, b, \ldots$ are flat $\mathrm{GL}(5)$ indices while $\hat{\alpha}, \hat{\beta}$ are flat $\mathrm{SL}(2)$ indices. Starting from an untwisted frame $\check{E}_A$, the generalised frame E_A is defined as [6]

$$E_A = \mathrm{e}^{\Delta} \, \mathrm{e}^{\phi} \, \hat{f}_{\hat{\alpha}}{}^{\alpha} \, \mathrm{e}^{B^{\alpha} + C} \cdot \check{E}_A \,, \tag{A.21}$$

where $\hat{f}_{\hat{\alpha}}{}^{\alpha} = \left(\begin{smallmatrix} \mathrm{e}^{\phi/2} & C_0 \mathrm{e}^{\phi/2} \\ 0 & \mathrm{e}^{-\phi/2} \end{smallmatrix}\right)$. The action of the warp factor e^{Δ} is defined by exponentiating the adjoint element given by $l = \Delta$, while the dilaton action e^{ϕ} is defined by exponentiating the adjoint element given by $l + r = \frac{\phi}{4}(-1 + \mathbf{1})$ [55]. Decomposing the flat index A in $\mathrm{GL}(5) \times \mathrm{SL}(2)$ representations, the generalised frame may be written as

$$\begin{aligned} E_a = {} & \mathrm{e}^{\Delta} \left(\hat{e}_a + \iota_{\hat{e}_a} B^{\alpha} + \iota_{\hat{e}_a} C + \frac{1}{2} \epsilon_{\alpha\beta} \iota_{\hat{e}_a} B^{\alpha} \wedge B^{\beta} + \iota_{\hat{e}_a} C \wedge B^{\alpha} + \frac{1}{6} \epsilon_{\beta\gamma} \iota_{\hat{e}_a} B^{\beta} \wedge B^{\gamma} \wedge B^{\alpha} \right) \\ E^a_{\hat{\alpha}} = {} & \mathrm{e}^{\Delta} \, \mathrm{e}^{-\phi/2} \left(\hat{f}_{\hat{\alpha}}{}^{\alpha} e^a + \hat{f}_{\hat{\alpha}}{}^{\alpha} \epsilon_{\alpha\beta} e^a \wedge B^{\beta} - \hat{f}_{\hat{\alpha}}{}^{\alpha} C \wedge e^a + \frac{1}{2} \hat{f}_{\hat{\alpha}}{}^{\beta} \epsilon_{\beta\gamma} e^a \wedge B^{\gamma} \wedge B^{\alpha} \right) \\ E^{a_1 a_2 a_3} = {} & \mathrm{e}^{\Delta} \, \mathrm{e}^{-\phi} \left(e^{a_1 a_2 a_3} + e^{a_1 a_2 a_3} \wedge B^{\alpha} \right) \\ E^{a_1 \ldots a_5}_{\hat{\alpha}} = {} & \mathrm{e}^{\Delta} \mathrm{e}^{-3\phi/2} \hat{f}_{\hat{\alpha}}{}^{\alpha} \, e^{a_1 \ldots a_5} \,, \end{aligned} \tag{A.22}$$

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where $e^{a_1\cdots a_n} = e^{a_1}\wedge\cdots\wedge e^{a_n}$. Using this expression for the frame, we obtain for the different components of the inverse generalised metric:

$$
\begin{aligned}
(G^{-1})^{mn} &= \mathrm{e}^{2\Delta} g^{mn} \\
(G^{-1})^{m\,\beta}_{\ n} &= \mathrm{e}^{2\Delta} g^{mp} B^{\beta}_{pn} \\
(G^{-1})^{m}_{\ npq} &= \mathrm{e}^{2\Delta} g^{mr}\left(C_{rnpq} + \frac{3}{2}\epsilon_{\alpha\beta} B^{\alpha}_{r[n} B^{\beta}_{pq]}\right) \\
(G^{-1})^{m\beta}_{\ npqrs} &= \mathrm{e}^{2\Delta} g^{mu}\left(10 C_{u[npq} B^{\beta}_{rs]} + 5\epsilon_{\gamma\delta} B^{\gamma}_{u[n} B^{\delta}_{pq} B^{\beta}_{rs]}\right) \\
(G^{-1})^{\alpha\,\beta}_{m\,n} &= \mathrm{e}^{2\Delta}\left(\mathrm{e}^{-\phi} h^{\alpha\beta} g_{mn} - B^{\alpha}_{mp} g^{pq} B^{\beta}_{qn}\right) \\
(G^{-1})^{\alpha}_{m\,npq} &= \mathrm{e}^{2\Delta}\left(3\mathrm{e}^{-\phi} h^{\alpha\beta}\epsilon_{\beta\gamma} g_{m[n} B^{\gamma}_{pq]} - B^{\alpha}_{mr} g^{rs} C_{snpq} - \frac{3}{2}\epsilon_{\beta\gamma} B^{\alpha}_{mr} g^{rs} B^{\beta}_{s[n} B^{\gamma}_{pq]}\right) \\
(G^{-1})^{\alpha\,\beta}_{m\,npqrs} &= \mathrm{e}^{2\Delta}\mathrm{e}^{-\phi}\left(-5h^{\alpha\beta} g_{m[n} C_{pqrs]} + 15 h^{\alpha\gamma}\epsilon_{\gamma\delta} g_{m[n} B^{\delta}_{pq} B^{\beta}_{rs]}\right) \\
(G^{-1})_{mnp\,qrs} &= \mathrm{e}^{2\Delta}\left[g^{tu}\left(C_{tmnp} + \frac{3}{2}\epsilon_{\alpha\beta} B^{\alpha}_{t[m} B^{\beta}_{np]}\right)\left(C_{uqrs} + \frac{3}{2}\epsilon_{\gamma\delta} B^{\gamma}_{u[q} B^{\delta}_{rs]}\right)\right. \\
&\qquad \left. + 9\,\mathrm{e}^{-\phi} h_{\alpha\beta} B^{\alpha}_{[mn} g_{p][q} B^{\beta}_{rs]} + 6\,\mathrm{e}^{-2\phi} g_{mnp,qrs}\right],
\end{aligned}
\tag{A.23}
$$

where in the last line we defined $g_{mnp,qrs} = g_{q[m} g_{n|r|} g_{p]s}$. We will not need the expressions for the remaining components $(G^{-1})^{\alpha}_{mnpqr\,stu}$ and $(G^{-1})^{\alpha\quad\ \beta}_{mnpqr\,stuvw}$.

The warp factor e^{Δ} is in principle extracted by evaluating the determinant of the whole generalised metric. However, for type IIB we can follow the same shortcut given in [56] for type IIA. We introduce

$$
\mathcal{H}^{-1} = \begin{pmatrix} (G^{-1})^{mn} & (G^{-1})^{m+}_{\ \ n} \\ (G^{-1})^{+\,n}_{\ m} & (G^{-1})^{++}_{\ \ mn} \end{pmatrix} = \mathrm{e}^{2\Delta}\begin{pmatrix} g^{mn} & (g^{-1}B)^{m}_{\ \ n} \\ -(Bg^{-1})_{m}^{\ \ n} & (g - Bg^{-1}B)_{mn} \end{pmatrix}, \tag{A.24}
$$

where B^{+} is the NSNS two-form potential and observe that the matrix on the right hand side has unit determinant. Therefore we obtain[28]

$$
\mathrm{e}^{\Delta} = (\det\mathcal{H})^{-\frac{1}{4d}}\,. \tag{A.25}
$$

B Generalised vectors in angular coordinates on M_6

In this appendix we provide explicit expressions for the generalised vectors $K_{\mathcal{A}}$, $\mathcal{A} = 0, 1, \ldots, 8$, defining the generalised U(1) structure on the six-manifold discussed in section 5. We start by relating the constrained coordinates y_i, $i = 1, \ldots, 5$, used in the

[28] The expressions above are given in string frame. In Einstein frame the term g_{mn} in (A.24) becomes $\mathrm{e}^{\phi} g_{mn}$, and (A.25) becomes

$$
\mathrm{e}^{\Delta + \frac{\phi}{4}} = (\det\mathcal{H})^{-\frac{1}{4d}}\,.
$$

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main text to angular coordinates $(\theta, \psi, \chi, \phi)$ on a round S^4 of unit radius:

$$\begin{aligned} y^1 &= \sin\theta \cos\psi \,, \\ y^2 &= \sin\theta \sin\psi \,, \\ y^3 &= \cos\theta \; \mu^3 = \cos\theta \; \cos\chi \,, \\ y^4 &= \cos\theta \; \mu^4 = \cos\theta \; \sin\chi \cos\phi \,, \\ y^5 &= \cos\theta \; \mu^5 = \cos\theta \; \sin\chi \sin\phi \,. \end{aligned} \tag{B.1}$$

Notice that ψ parameterises U(1) rotations in the $1-2$ plane, while χ, ϕ parameterise SO(3) rotations in the $3-4-5$ space and thus describe a round S^2. The latter is equivalently described by the constrained coordinates μ^α, $\alpha = 3, 4, 5$, satisfying $\delta_{\alpha\beta}\mu^\alpha\mu^\beta = 1$; we use either one or the other description according to convenience. The round metric on S^4 and the associated volume form (5.13) read

$$\begin{aligned} g_4 &= R^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\psi^2 + \cos^2\theta \, g_{S^2} \right) \,, \\ \mathrm{vol}_4 &= R^4 \cos^2\theta \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\psi \wedge \mathrm{vol}_{S^2} \,, \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} g_{S^2} &= \delta_{\alpha\beta} \mathrm{d}\mu^\alpha \mathrm{d}\mu^\beta = \mathrm{d}\chi^2 + \sin^2\chi \, \mathrm{d}\phi^2 \,, \\ \mathrm{vol}_{S^2} &= \frac{1}{2} \epsilon_{\alpha\beta\gamma} \, \mu^\alpha \mathrm{d}\mu^\beta \wedge \mathrm{d}\mu^\gamma = \sin\chi \, \mathrm{d}\chi \wedge \mathrm{d}\phi \end{aligned} \tag{B.3}$$

are the unit metric and volume form on the two-sphere identified above. The S^4 Killing vectors generating the $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$ algebra of interest are expressed in terms of these angular coordinates as:

$$\begin{aligned} v_{12} &= R^{-1} \, \frac{\partial}{\partial\psi} \,, \\ v_{45} &= R^{-1} \, \frac{\partial}{\partial\phi} \,, \\ v_{53} &= R^{-1} \left(-\sin\phi \, \frac{\partial}{\partial\chi} - \cot\chi \cos\phi \, \frac{\partial}{\partial\phi} \right) \,, \\ v_{34} &= R^{-1} \left(\cos\phi \, \frac{\partial}{\partial\chi} - \cot\chi \sin\phi \, \frac{\partial}{\partial\phi} \right) \,. \end{aligned} \tag{B.4}$$

For the M-theory three-form potential on S^4 satisfying (5.15), we choose

$$A = -3R^3 \, \psi \, \cos^2\theta \sin\theta \, \mathrm{d}\theta \wedge \mathrm{vol}_{S^2} \,. \tag{B.5}$$

As for the Riemann surface $\mathbb{\Sigma}$, we do not need to introduce explicit coordinates; we rather use the one-forms e_1, e_2 satisfying (5.18), (5.19).

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Evaluating the twisted K's (5.21) in these coordinates, we find the following expressions:

$$\begin{aligned}
K_0 &= \frac{1}{2}\,v_{12}+\frac{1}{2}\,R^2 c_\theta^3\,\mathrm{vol}_{S^2}\,,\\
K_1+\mathrm{i}K_2 &= R\,\mathrm{e}^{\mathrm{i}\psi}\,(e_1+\mathrm{i}e_2)\wedge(c_\theta\,\mathrm{d}\theta+\mathrm{i}\,s_\theta\,(\mathrm{d}\psi+\upsilon))+R^4\,\mathrm{e}^{\mathrm{i}\psi}\,c_\theta^2\,(e_1+\mathrm{i}e_2)\wedge\\
&\quad\wedge\big[\,\big(s_\theta^2\,(1-3\,\mathrm{i}\,\psi)\,(\mathrm{d}\psi+\upsilon)+c_\theta^2\,\upsilon\big)\wedge\mathrm{d}\theta+\mathrm{i}\,c_\theta\,s_\theta\,\upsilon\wedge\mathrm{d}\psi\big]\wedge\mathrm{vol}_{S^2}\,,\\
K_3 &= v_{45}-R^2\,\mathrm{d}\,(c_\theta\,c_\chi)\wedge(\mathrm{d}\psi+\upsilon)+R^2\,\mathrm{d}\,\big[c_\chi\,\mathrm{d}\,\big(\psi\,c_\theta^3\big)\big]+c_\theta\,c_\chi\,\mathrm{vol}_{\mathbb{\Sigma}}\\
&\quad-R^3 c_\theta s_\theta\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\big(c_\theta s_\theta c_\chi\,\mathrm{d}\psi\wedge\mathrm{vol}_{S^2}+s_\chi^2\,\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{d}\phi+3\,\psi\,c_\theta^2\,c_\chi\,\mathrm{d}\theta\wedge\mathrm{vol}_{S^2}\big)\\
&\quad-R^5\,c_\theta^3\,s_\theta\,c_\chi\,\upsilon\wedge\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{vol}_{S^2}\,,\\
K_4 &= v_{53}-R^2\mathrm{d}(c_\theta\,s_\chi\,c_\phi)\wedge(\mathrm{d}\psi+\upsilon)+R^2\,\mathrm{d}\big[s_\chi\,c_\phi\,\mathrm{d}\,\big(\psi\,c_\theta^3\big)\big]+c_\theta\,s_\chi\,c_\phi\,\mathrm{vol}_{\mathbb{\Sigma}}\\
&\quad+R^3\,c_\theta s_\theta\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\big[c_\phi\,\big(c_\chi s_\chi\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{d}\phi-c_\theta\,s_\theta\,s_\chi\mathrm{d}\psi\wedge\mathrm{vol}_{S^2}-3\,\psi\,c_\theta^2\,s_\chi\,\mathrm{d}\theta\wedge\mathrm{vol}_{S^2}\big)\\
&\quad+s_\phi\,\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{d}\chi\big]-R^5\,c_\theta^3\,s_\theta\,s_\chi\,c_\phi\,\upsilon\wedge\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{vol}_{S^2}\,,\\
K_5 &= v_{34}-R^2\,\mathrm{d}(c_\theta\,s_\chi s_\phi)\wedge(\mathrm{d}\psi+\upsilon)+R^2\,\mathrm{d}\big[s_\chi s_\phi\,\mathrm{d}\,\big(\psi\,c_\theta^3\big)\big]+c_\theta\,s_\chi s_\phi\,\mathrm{vol}_{\mathbb{\Sigma}}\\
&\quad+R^3\,c_\theta s_\theta\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\big[s_\phi\,\big(c_\chi s_\chi\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{d}\phi-c_\theta\,s_\theta\,s_\chi\,\mathrm{d}\psi\wedge\mathrm{vol}_{S^2}-3\,\psi\,c_\theta^2\,s_\chi\,\mathrm{d}\theta\wedge\mathrm{vol}_{S^2}\big)\\
&\quad-c_\phi\,\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{d}\chi\big]-R^5\,c_\theta^3\,s_\theta\,s_\chi\,s_\phi\,\upsilon\wedge\mathrm{d}\theta\wedge\mathrm{d}\psi\wedge\mathrm{vol}_{S^2}\,,
\end{aligned}\tag{B.6}$$

with K_6, K_7, K_8 being obtained from K_3, K_4, K_5, respectively, by sending $\mathrm{vol}_{\mathbb{\Sigma}}\to-\mathrm{vol}_{\mathbb{\Sigma}}$. In these formulae, we are using the shorthand notation $c_\theta=\cos\theta$, $s_\theta=\sin\theta$, and similarly for the angles χ,ϕ. The terms in (B.6) proportional to ψ are those coming from the action of the three-form A in the S^4 frames (5.14). The terms proportional to υ are those generated by the twist (5.23) (that is, setting $\upsilon=0$ we recover the generalised vectors on the direct product $\mathbb{\Sigma}\times S^4$). We see that the latter transformation shifts $\mathrm{d}\psi$ by the connection one-form on $\mathbb{\Sigma}$, such that $\mathrm{d}\psi\to(\mathrm{d}\psi+\upsilon)$, and also introduces some additional five-form parts in the generalised vectors.

The weighted dual vectors $J^{\mathcal{A}}\in\Gamma(N)$ that give the ansatz for the supergravity two-forms can be computed from the $K_{\mathcal{A}}$ using (3.15). We find:

$$\begin{aligned}
J^0 &= 2R^4\,c_\theta s_\theta\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{d}\theta\wedge\mathrm{d}\psi+2R(\upsilon+3\psi\,c_\theta s_\theta\,\mathrm{d}\theta+s_\theta^2\,\mathrm{d}\psi)\otimes\mathrm{vol}_6\,,\\
J^1+\mathrm{i}J^2 &= \frac{R}{2}s_\theta\,\mathrm{i}\,\mathrm{e}^{\mathrm{i}\psi}(e_1+\mathrm{i}e_2)-\frac{R^4}{2}\,\mathrm{e}^{\mathrm{i}\psi}(e_1+\mathrm{i}e_2)\wedge\big[c_\theta^2\,\mathrm{d}\theta+\mathrm{i}\,s_\theta\,\mathrm{d}\,\big(\psi\,c_\theta^3\big)\big]\wedge\mathrm{vol}_{S^2}\,,\\
J^3 &= \frac{R}{2}\,\mathrm{d}(c_\theta c_\chi)+\frac{R^4}{2}c_\theta\big[s_\theta s_\chi^2\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi-c_\theta c_\chi(\mathrm{vol}_{\mathbb{\Sigma}}+c_\theta s_\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi)\wedge\mathrm{vol}_{S^2}\big]\\
&\quad-\frac{R}{2}c_\theta^2 s_\chi^2\,\mathrm{d}\phi\otimes\mathrm{vol}_6\,,\\
J^4 &= \frac{R}{2}\mathrm{d}(c_\theta c_\phi s_\chi)-\frac{R^4}{2}c_\theta\big[s_\theta c_\phi c_\chi s_\chi\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi+s_\theta s_\phi\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{d}\theta\wedge\mathrm{d}\chi\\
&\quad+c_\theta c_\phi s_\chi(\mathrm{vol}_{\mathbb{\Sigma}}+c_\theta s_\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi)\wedge\mathrm{vol}_{S^2}\big]+\frac{R}{2}c_\theta^2(s_\phi\,\mathrm{d}\chi+c_\phi c_\chi s_\chi\,\mathrm{d}\phi)\otimes\mathrm{vol}_6\,,\\
J^5 &= \frac{R}{2}\mathrm{d}(c_\theta s_\phi s_\chi)+\frac{R^4}{2}c_\theta\big[-s_\theta c_\chi s_\chi s_\phi\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi+s_\theta c_\phi\,\mathrm{vol}_{\mathbb{\Sigma}}\wedge\mathrm{d}\theta\wedge\mathrm{d}\chi\\
&\quad-c_\theta s_\phi s_\chi(\mathrm{vol}_{\mathbb{\Sigma}}+c_\theta s_\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi)\wedge\mathrm{vol}_{S^2}\big]+\frac{R}{2}c_\theta^2(-c_\phi\,\mathrm{d}\chi+c_\chi s_\phi s_\chi\,\mathrm{d}\phi)\otimes\mathrm{vol}_6\,,
\end{aligned}\tag{B.7}$$

and again J^6, J^7, J^8 are obtained from J^3, J^4, J^5, respectively, by sending $\mathrm{vol}_{\mathbb{\Sigma}}\to-\mathrm{vol}_{\mathbb{\Sigma}}$ and $\mathrm{vol}_6\to-\mathrm{vol}_6$.

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$\mathcal{N} = 2$ consistent truncations from wrapped M5-branes

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Abstract: We discuss consistent truncations of eleven-dimensional supergravity on a six-dimensional manifold M, preserving minimal $\mathcal{N} = 2$ supersymmetry in five dimensions. These are based on $G_S \subseteq USp(6)$ structures for the generalised $E_{6(6)}$ tangent bundle on M, such that the intrinsic torsion is a constant G_S singlet. We spell out the algorithm defining the full bosonic truncation ansatz and then apply this formalism to consistent truncations that contain warped $\mathrm{AdS}_5 \times_{\mathrm{w}} M$ solutions arising from M5-branes wrapped on a Riemann surface. The generalised $U(1)$ structure associated with the $\mathcal{N} = 2$ solution of Maldacena-Nuñez leads to five-dimensional supergravity with four vector multiplets, one hypermultiplet and $SO(3) \times U(1) \times \mathbb{R}$ gauge group. The generalised structure associated with "BBBW" solutions yields two vector multiplets, one hypermultiplet and an abelian gauging. We argue that these are the most general consistent truncations on such backgrounds.

Keywords: Supergravity Models, AdS-CFT Correspondence, Differential and Algebraic Geometry, M-Theory

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1 Introduction

Consistent Kaluza-Klein truncations are a precious tool for constructing compactifying solutions to ten or eleven-dimensional supergravity using a simpler lower-dimensional theory. Given a splitting of the higher-dimensional spacetime into an internal manifold M and an external spacetime X, a consistent truncation selects a finite subset of the KK modes of the higher-dimensional theory on M and provides an effective theory on X describing their non-linear dynamics. The selected KK modes must form a protected sector, in the sense that they do not mix in the equations of motion with the modes that have been truncated out. In this way all solutions of the lower-dimensional theory on X are guaranteed to also be solutions of the original higher-dimensional theory.

For such non-trivial reduction to be possible, the internal manifold M should have a special geometric structure. The simplest case is when M admits a homogeneous action of a group $\mathscr{G}$, that is $M = \mathscr{G}/\mathscr{H}$ for some subgroup $\mathscr{H} \subset \mathscr{G}$. Then one can decompose all higher-dimensional fields into representations of $\mathscr{G}$ and truncate to the $\mathscr{G}$-singlets. This $\mathscr{G}$-invariant truncation is consistent, since the singlet fields can never source the non-singlet fields. When in particular M is a group manifold, $M = \mathscr{G}$, one has a conventional Scherk-Schwarz reduction [1]. Examples of such consistent truncations in the context of M-theory — which is our principal interest in this paper — can be found in [2–5].

As a step towards more general classes of truncations, it is convenient to think about reductions on homogeneous manifolds using the language of G-structures. Let us consider Scherk-Schwarz reductions for definiteness. A group manifold $M = \mathscr{G}$ admits a basis of globally defined left-invariant one-forms, $\{e^a\}$, $a = 1, \ldots, \dim M$, that reduces the structure group to the identity (i.e. M is parallelisable). Furthermore, the group action implies that $\mathrm{d}e^a = \frac{1}{2} f_{bc}{}^a e^b \wedge e^c$, where $f_{bc}{}^a$ are the structure constants of the Lie algebra $\mathrm{Lie}\,\mathscr{G}$. This means that the left-invariant identity structure has singlet, constant intrinsic torsion (singlet because $\mathrm{d}e^a$ is expressed in terms of the invariant $\{e^a\}$ basis, and constant because the coefficients of the expansion are constant). The truncation ansatz is defined by expanding all higher-dimensional fields in the basis of invariant tensors of the structure. When this is plugged into the equations of motion, we can again invoke the argument that only singlet tensors are generated and conclude that the truncation is consistent. Since the spin bundle is also trivialised, Scherk-Schwarz reductions preserve the full supersymmetry of the higher-dimensional theory. More generally, $\mathscr{G}$-invariant consistent truncations on coset manifolds $M = \mathscr{G}/\mathscr{H}$ are based on the existence of an $\mathscr{H}$ structure with constant, singlet intrinsic torsion, and preserve only a fraction of supersymmetry or none at all.

Interestingly, the argument based on G-structure applies also to internal manifolds M that are not homogeneous. It is sufficient that M has a structure group G_S with only constant, singlet intrinsic torsion; then the truncation to the G_S-singlets is guaranteed to be consistent. This can preserve different fractions of supersymmetry, depending on how many G_S-invariant spinors exist on M. In fact the G_S structure data determine the full field content and gauge interactions of the truncated theory. Examples of this type in M-theory are the truncations based on Sasaki-Einstein and weak-G_2 holonomy manifolds of [6], and the tri-Sasakian reduction of [7].

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However there are consistent truncations that are not captured by conventional G-structures. Classic examples are the maximally supersymmetric consistent truncations on spheres, such as eleven-dimensional supergravity on S^7 [8] and S^4 [9]. M-theory truncations preserving less supersymmetry and containing warped AdS solutions can be found in [10–13].[1] Building on the evidence emerging from these examples, a conjecture was formulated in [11] stating that for any warped, supersymmetric $\mathrm{AdS}_D \times_{\mathrm{w}} M$ solution to higher-dimensional supergravity, there is a consistent truncation on M down to D-dimensional pure gauged supergravity with the same amount of supersymmetry.

Exceptional Generalised Geometry and Exceptional Field Theory offer an understanding of these more complicated examples that unifies them with the conventional ones. Exceptional Generalised Geometry uses an extension of the ordinary tangent bundle TM to a larger bundle E on M, whose fibres transform in a representation of the exceptional group $E_{d(d)}$. In this way the diffeomorphism and gauge symmetries of higher-dimensional supergravity are unified as generalised diffeomorphisms on E. The notion of generalised G_S structure, that is a G_S structure of E, rather than of TM, leads to a new systematic approach to consistent truncations with different amounts of supersymmetry: one can argue that there is a consistent truncation any time a supergravity theory is reduced on a manifold M admitting a generalised G_S structure with constant singlet intrinsic torsion [14]. In particular, all maximally supersymmetric truncations, both conventional Scherk-Schwarz reductions and sphere truncations, can be seen as generalised Scherk-Schwarz reductions on generalised parallelisable manifolds [15–23]. This also provides a connection to Poisson-Lie T-duality as described in [24] (see also [25]). Truncations preserving less supersymmetry are based on generalised structures larger than the identity, the half-maximal case having been explored rather extensively by now [14, 26–30]. Moreover, a proof of the conjecture of [11] was given in this framework [14, 27], based on the fact that the conditions for a supersymmetric $\mathrm{AdS}_D \times_{\mathrm{w}} M$ vacuum can be rephrased as the requirement that M admits a generalised G_S structure with vanishing non-singlet intrinsic torsion [31–33].

Although the general ideas were illustrated in [14] for any amount of supersymmetry, the Exceptional Generalised Geometry approach to consistent truncations has been developed just for maximal and half-maximal supersymmetry so far. In this paper we enlarge this framework and discuss in detail truncations of eleven-dimensional supergravity preserving minimal $\mathcal{N} = 2$ supersymmetry in five dimensions.

While a strict $USp(6) \subset E_{6(6)}$ generalised structure leads to a truncation to minimal $\mathcal{N} = 2$ gauged supergravity in five dimensions, smaller $G_S \subset USp(6)$ structures lead to matter coupled supergravity. We show how the $G_S \subset USp(6)$ structure defines a continuous family of $USp(6)$ structures, and identify the moduli space of this family with the vector multiplet and hypermultiplet scalar manifolds in the truncated five-dimensional theory. We also show how the generalised Lie derivative acting on the generalised tensors defining the G_S structure specifies the isometries of the scalar manifold that are being gauged. This fully determines the truncated $\mathcal{N} = 2$ supergravity theory.

[1] Note that whenever there is non-trivial warping the truncation falls out of the conventional G-structure framework.

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We then derive general expressions that encode the uplift formulae for how the eleven-dimensional bosonic fields are encoded in terms of the moduli and the generalised tensors defining the G_S structure. In order to make this truncation ansatz explicit we need to solve a number of technical issues. One is that, in contrast to the maximal and half-maximal case, the structure is not entirely characterised by the generalised vectors K_I (i.e. sections of E, transforming in the fundamental of $E_{6(6)}$) which control the vector multiplet sector of the truncated theory. We also need to consider generalised tensors J_A belonging to the $E_{6(6)}$ adjoint bundle, which eventually control the hypermultiplet sector. A related point, that is crucial to derive the scalar truncation ansatz, is the construction of the generalised metric on E, which receives contributions both from the K_I and the J_A. A significant advantage of the formalism however, is that the expressions are universal. The ansatz can be applied to any $\mathcal{N} = 2$ background once one identifies the K_I and J_A singlets.

As application, we discuss M-theory truncations on geometries associated with M5-branes wrapping a Riemann surface Σ. The near-horizon geometry of this brane configuration is given by a warped $\mathrm{AdS}_5 \times_{\mathrm{w}} M$ solution to eleven-dimensional supergravity, where M is a fibration of a deformed S^4 over Σ [34, 35]. The fibration corresponds to a topological twist in the dual superconformal field theory on the M5-branes, where the holonomy of the Riemann surface is cancelled by a $U(1)$ in the $SO(5)$ R-symmetry, which in the supergravity background is realised geometrically as the isometries of S^4. Depending on which $U(1)$ is chosen, one obtains different $\mathrm{AdS}_5 \times_{\mathrm{w}} M$ solutions, and correspondingly different $U(1)_S$ generalised structures.

We start with the $\mathcal{N} = 2$ background of Maldacena-Nuñez [34]: specifying its $U(1)_S$ generalised structure and discussing its singlet intrinsic torsion, we obtain a consistent truncation to five-dimensional $\mathcal{N} = 2$ supergravity including four vector multiplets, one hypermultiplet, and a non-abelian $SO(3) \times U(1) \times \mathbb{R}$ gauging. This extends the abelian truncation of [36] (see also [10, 37] for previous subtruncations) by adding $SO(3)$ vector multiplets, which in the dual superconformal field theory source $SO(3)$ flavour current multiplets. We also spell out the full bosonic truncation ansatz. The same construction also applies to the "BBBW" solutions [35, 38], as the corresponding generalised structure is a simple deformation of the Maldacena-Nuñez one, controlled by a (discrete) parameter describing the choice of $U(1)_S$ in $SO(5)$. The corresponding truncation features only two $\mathcal{N} = 2$ vector multiplets, one hypermultiplet and an abelian gauging. We show that the Maldacena-Nuñez truncation admits a new non-supersymmetric AdS_5 solution when the Riemann surface is a sphere, which turns out to be perturbatively unstable. We also find new non-supersymmetric vacua in the BBBW truncations. Together with the consistent truncation including the $\mathcal{N} = 4$ solution of [34], whose $U(1)_S$ generalised structure embeds in $USp(4) \subset USp(6)$ and leads to half-maximal supergravity [14, 39], the present study completes the landscape of what we believe are the most general consistent truncations that can be derived from eleven-dimensional supergravity on known smooth solutions associated with M5-branes wrapped over Riemann surfaces.[2]

[2]It may be possible to find other consistent truncations, that are not subsectors of the ones given here by using large structure groups, in analogy with the consistent truncation on S^7 viewed as a Sasaki-Einstein manifold [6] rather than a generalised parallelised sphere. However such truncations will have fewer fields.

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The rest of the paper is organised as follows. In section 2 we characterise the generalised structure relevant for M-theory truncations on a six-dimensional manifold preserving $\mathcal{N}=2$ supersymmetry. In section 3 we specify the truncation ansatz and discuss how the gauging is determined from the generalised structure. In sections 4 and 5 we apply our formalism to consistent truncations associated with M5-branes wrapping a Riemann surface, first for Maldacena-Nuñez backgrounds and then for BBBW ones. We conclude in section 6. The appendices contains a brief account of $E_{6(6)}$ generalised geometry, a summary of five-dimensional $\mathcal{N}=2$ gauged supergravity and some technical details of our computations.

2 M-theory generalised structures and $\mathcal{N}=2$ supersymmetry

In this section we first recall some basic notions of Exceptional Generalised Geometry for the case of interest here, namely eleven-dimensional supergravity on a six-dimensional manifold, and then we illustrate how the general procedure described in [14] applies to consistent truncations to five-dimensional $\mathcal{N}=2$ gauged supergravity. A more extended review of the relevant generalised geometry can be found in appendix A.

2.1 The HV structure

Consistent truncations of eleven-dimensional supergravity on a six-dimensional manifold M are based on $E_{6(6)}\times\mathbb{R}^+$ generalised geometry. This extends the tangent bundle TM to the generalised tangent bundle E on M, and the corresponding structure group $GL(6)$ to $E_{6(6)}$. The group $E_{6(6)}$ contains $GL(6)$ as its geometrical subgroup, and we can use the latter to decompose the generalised tangent bundle as

$$E \simeq TM\oplus\Lambda^2T^*M\oplus\Lambda^5T^*M\,. \tag{2.1}$$

Therefore the sections of E consist, locally, of the sum of a vector, a two-form and a five-form on M,

$$V=v+\omega+\sigma\,. \tag{2.2}$$

These are called generalised vectors and transform in the **27** of $E_{6(6)}$.

All geometric structures of conventional geometry on M, such as tensors, Lie derivative, connections etc, admit an extension to E [40–42]. In particular, generalised tensors are defined by considering bundles whose fibers transform in different representations of $E_{6(6)}$. We can define dual generalised vectors Z as the sections of the dual tangent bundle

$$E^* \simeq T^*M\oplus\Lambda^2TM\oplus\Lambda^5TM\,, \tag{2.3}$$

transforming in the $\overline{\mathbf{27}}$ of $E_{6(6)}$. Locally the dual vectors are sums of a one-form $\hat{v}$, a two-vector $\hat{\omega}$ and a five-vector $\hat{\sigma}$,

$$Z=\hat{v}+\hat{\omega}+\hat{\sigma}\,. \tag{2.4}$$

The adjoint bundle transforms in the $\mathbf{1}+\mathbf{78}$ of $E_{6(6)}$ and, in terms of $GL(6)$ tensors, is defined as

$$\mathrm{ad}F \simeq \mathbb{R}\oplus(TM\otimes T^*M)\oplus\Lambda^3T^*M\oplus\Lambda^6T^*M\oplus\Lambda^3TM\oplus\Lambda^6TM\,, \tag{2.5}$$

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with sections

$$R = l + r + a + \tilde{a} + \alpha + \tilde{\alpha}\,, \tag{2.6}$$

where, locally, $l \in \mathbb{R}$, $r \in \mathrm{End}(TM)$, $a \in \Lambda^3 T^*M$ is a three-form, $\tilde{a} \in \Lambda^6 T^*M$ is a six-form and $\alpha \in \Lambda^3 TM$ and $\tilde{\alpha} \in \Lambda^6 TM$. This bundle plays an important role as the components of the M-theory three-form and six-form gauge potentials are embedded in $\mathrm{ad}F$.

As we will see, the bosonic fields of eleven-dimensional supergravity can be unified into generalised tensors. The supergravity spinors on the other hand arrange into representations transforming under $USp(8)$, the double cover of the maximal compact subgroup $USp(8)/\mathbb{Z}_2$ of $E_{6(6)}$. For example the supersymmetry parameters are section of the generalsied spinor bundle $\mathcal{S}$, transforming in the **8** of $USp(8)$. It will be this compact $USp(8)$ or more generally a subgroup of it, that determines the R-symmetry of the reduced five-dimensional theory.

The manifold M admits a *generalised structure*, $G_S \subset USp(8)/\mathbb{Z}_2$, when the structure group $E_{6(6)}$ is reduced to the subgroup G_S. Typically this can be characterised by the existence of globally defined generalised tensors that are invariant under G_S. The amount of supersymmetry of the eleven-dimensional theory that is preserved by the G_S structure is given by the number of G_S singlets in the spinor bundle, $\mathcal{S}$.[3]

In this paper we are interested in structures preserving $\mathcal{N} = 2$ supersymmetry. The generic case is provided by what has been called an *HV structure* [33, 43, 44]. It consists of a triplet of globally defined tensors in the adjoint bundle, $J_\alpha \in \Gamma(\mathrm{ad}\, F)$, with $\alpha = 1, 2, 3$, satisfying

$$[J_\alpha, J_\beta] = \; 2\epsilon_{\alpha\beta\gamma} J_\gamma\,, \qquad \mathrm{tr}(J_\alpha J_\beta) = -\delta_{\alpha\beta}\,, \tag{2.7}$$

together with a globally defined generalised vector $K \in \Gamma(E)$ having positive norm with respect to the $E_{6(6)}$ cubic invariant,

$$c(K, K, K) \;:=\; 6\,\boldsymbol{\kappa}^2 > 0\,, \tag{2.8}$$

where $\boldsymbol{\kappa}$ is a section[4] of $(\det T^*M)^{1/2}$, and satisfying the compatibility condition

$$J_\alpha \cdot K = 0\,, \tag{2.9}$$

where $\cdot$ denotes the adjoint action.[5] See appendix A for a definition of the cubic invariant and the other generalised geometry operations appearing in these formulae.

The HV structure $\{J_\alpha, K\}$ defines a reduction of the structure group to $USp(6) \subset E_{6(6)}$. Indeed the vector K is stabilised by $F_{4(4)} \subset E_{6(6)}$, while the J_α are invariant under the subgroup $SU^*(6)$. The compatible K and J_α have $SU^*(6) \cap F_{4(4)} \simeq USp(6)$ as a common stabiliser. The globally defined vector $K \in \Gamma(E)$ with positive norm is called a vector-multiplet structure, or *V structure* for short. A triplet of $J_\alpha \in \Gamma(\mathrm{ad}F)$ that define the

[3] Here we will assume that either G_S is simply connected or is $U(1)$ so that it lifts to a G_S subgroup of $USp(8)$.

[4] Recall that $\det T^*M$ is just a different notation for the top-form bundle $\Lambda^6 T^*M$ that stresses that it is a real line bundle. Here we are assuming that the manifold is orientable and hence $\det T^*M$ is trivial and so we can define arbitrary powers $(\det T^*M)^p$ for any real p.

[5] Note that we are using slightly different conventions for the J_α tensors compared with [44]. In particular $J_\alpha^{\mathrm{AW}} = \boldsymbol{\kappa} J_\alpha^{\mathrm{here}} \in \Gamma((\det T^*M)^{1/2} \otimes \mathrm{ad}F)$.

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highest root $\mathfrak{su}_2$ subalgebra of $\mathfrak{e}_{6(6)}$ and satisfy the conditions (2.7) is called a hypermultiplet structure, or *H structure*. This justifies the name HV structure for the compatible pair $\{J_\alpha, K\}$.

It is easy to check that the amount of supersymmetry preserved by a HV structure is $\mathcal{N} = 2$. Under the breaking

$$USp(8) \supset USp(6) \times SU(2)_H \,. \tag{2.10}$$

the spinorial representation decomposes as $\mathbf{8} = (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ and we see that the are only two $USp(6)$ singlets. The $SU(2)_H$ factor in (2.10) is the R-symmetry of the reduced theory so that the two singlets form an R-symmetry doublet, as expected for $\mathcal{N} = 2$ supersymmetry parameters.

A strict $USp(6)$ structure is not the only option to obtain $\mathcal{N} = 2$ supersymmetry. In fact, any subgroup G_S that embeds in $USp(6)$ in such a way that there are no extra singlets in the decomposition of the spinorial representation of $USp(8)$ does the job. Although the number spinor singlets is unchanged, when the structure group is smaller than $USp(6)$ in general one finds more G_S singlets in the decomposition of the **27** and the **78** representations. Let us denote by

$$K_I \,, \qquad I = 0, \ldots, n_V \,, \tag{2.11}$$

the set of independent generalised vectors corresponding to G_S singlets in the **27**, and by

$$J_A \,, \qquad A = 1, \ldots, \dim \mathcal{H} \,, \tag{2.12}$$

the set of independent sections of the adjoint bundle corresponding to G_S singlets in the **78** that also satisfy the condition[6]

$$J_A \cdot K_I = 0 \qquad \forall\, I \text{ and } \forall\, A \,. \tag{2.13}$$

The latter generate a subgroup $\mathcal{H} \subset \mathrm{C}_{E_{6(6)}}(G_S)$, where $\mathrm{C}_{E_{6(6)}}(G_S)$ is the commutant of G_S in $E_{6(6)}$, so that

$$[J_A, J_B] = f_{AB}{}^C J_C \,, \tag{2.14}$$

with $f_{AB}{}^C$ being the structure constants of $\mathcal{H}$. The generalised structure $G_S \subseteq USp(6)$ is fully characterised as the group preserving the set

$$\{K_I, J_A\} \,. \tag{2.15}$$

We can always normalise such that the $n_V + 1$ generalised vectors satisfy

$$c(K_I, K_J, K_K) = 6\, \boldsymbol{\kappa}^2 C_{IJK} \,, \tag{2.16}$$

[6]Note that there are singlets in the adjoint bundle that do not satisfy (2.13). These are given by $K_I \times_{\mathrm{ad}} K_J^*$, where K_J^* is the dual of the generalised vector K_J and $\times_{\mathrm{ad}}$ is the projection onto the adjoint bundle, and will not play a relevant role in our construction.

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with C_{IJK} a symmetric, constant tensor and $\boldsymbol{\kappa}$ is a section of $(\det T^*M)^{1/2}$ fixed by the structure. In addition we can normalise the adjont singlets to satisfy

$$\mathrm{tr}(J_A J_B) = \eta_{AB}\,, \tag{2.17}$$

where η_{AB} is a diagonal matrix with -1 and $+1$ entries in correspondence with compact and non-compact generators of $\mathcal{H}$, respectively.

Any generalised structure has an associated intrinsic torsion [31], which is defined as follows. Let $\tilde{D}$ be a generalised connection compatible with the G_S-structure, that is, sastisfying $\tilde{D}Q_i = 0$ for all i, where Q_i is the set of invariant generalised tensors that define the structure. Formally, the generalised torsion T of $\tilde{D}$ is defined by, acting on any generalised tensor α,

$$(L^{\tilde{D}}_V - L_V)\,\alpha = T(V)\cdot\alpha\,, \tag{2.18}$$

where L is the generalised Lie derivative, $L^{\tilde{D}}$ is the generalised Lie derivative calculated using $\tilde{D}$ and $\cdot$ is the adjoint action on α.[7] The intrinsic torsion is the component of T that is independent of the choice of compatible connection $\tilde{D}$, and hence is fixed only by the choice of generalised structure. In general, one can decompose the intrinsic torsion into representations of G_S. In particular, for a consistent truncation we will be interested in the case where only the singlet representations are non-zero.

2.2 The generalised metric

An important ingredient to derive a consistent truncation is the generalised metric G on M. This is a positive-definite, symmetric rank-2 tensor on the generalised tangent bundle,

$$\begin{aligned} G &: E \otimes E \to \mathbb{R}^+ \\ &(V, V') \mapsto G(V, V') = G_{MN} V^M V'^N\,, \end{aligned} \tag{2.19}$$

that encodes the degrees of freedom of eleven-dimensional supergravity that correspond to scalars in the reduced theory. We provide the explicit relation between the generalised metric and the supergravity fields on M in eq. (3.19). The generalised metric is defined in analogy to the ordinary metric: a metric g on M can be seen as an $O(6)$ structure on TM that at each point on M parameterises the coset $GL(6)/O(6)$. Similarly, at each point $p \in M$ a choice of a generalised metric corresponds to an element of the coset

$$G|_p \in \frac{E_{6(6)} \times \mathbb{R}^+}{USp(8)/\mathbb{Z}_2}\,. \tag{2.20}$$

Since a $G_S \subset USp(8)/\mathbb{Z}_2$, the G_S structure will determine a G_S-invariant generalised metric, given in terms of the invariant tensors that are used to define the G_S structure. The expression of G_{MN} that is relevant for truncations preserving maximal and half-maximal supersymmetry was given in [14, 15, 22]. Here we will discuss the $\mathcal{N} = 2$ case.

Consider first the case of a generic $USp(6)$ structure. As discussed in the previous section this is specified by an invariant generalised vector, K, together with an $\mathfrak{su}(2)$ triplet

[7] We view the torsion as a map $T : \Gamma(E) \to \Gamma(\mathrm{ad}\, F)$ where $\mathrm{ad}\, F$ is the $E_{d(d)} \times \mathbb{R}^+$ adjoint bundle.

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of sections of the adjoint bundle J_α, $\alpha = 1, 2, 3$. These objects define a $USp(6)$-invariant generalised metric through the formula

$$G(V,V) = 3\left(3\,\frac{c(K,K,V)^2}{c(K,K,K)^2} - 2\frac{c(K,V,V)}{c(K,K,K)} + 4\,\frac{c(K,J_3\cdot V, J_3\cdot V)}{c(K,K,K)}\right)\,. \tag{2.21}$$

This formula can be motivated as follows. The globally defined K induces the splitting of the **27** of $E_{6(6)}$ into orthogonal subspaces

$$V = V_{\mathbf{0}} + V_{\mathbf{26}} \tag{2.22}$$

in the singlet and **26** representation of $F_{4(4)}$; correspondingly, the $E_{6(6)}$ cubic invariant on the **26** reduces to the symmetric invariant form of $F_{4(4)}$

$$c(K,V,V) = c(K,V_{\mathbf{0}},V_{\mathbf{0}}) + c(K,V_{\mathbf{26}},V_{\mathbf{26}})\,. \tag{2.23}$$

This expression however is not positive definite, since the symmetric form of $F_{4(4)}$ has signature $(14,12)$ and overall (2.23) has signature $(14,13)$. The first term in (2.21) contains the contribution from the singlet component $V_{\mathbf{0}}$ and makes the metric positive definite in the singlet. To do the same in the **26** we need the full HV structure. Under $SU(2)_H \times USp(6)$ the **27** decomposes as

$$\mathbf{27} = \mathbf{1}\oplus(\mathbf{1},\mathbf{14})\oplus(\mathbf{2},\mathbf{6})\,, \tag{2.24}$$

and the action J_α on V projects on the $(\mathbf{2},\mathbf{6})$ part, as the rest is an $SU(2)_H$ singlet. Then we can write the contribution to the metric in the $(\mathbf{2},\mathbf{6})$ as

$$c(K, J_3\cdot V, J_3\cdot V)\,, \tag{2.25}$$

and add it in (2.21) to make it positive definite. Note that (2.21) only contains one element of the triplet J_α, that we chose to be J_3. This is because, for each J_α,

$$c(K,J_\alpha\cdot V, J_\alpha\cdot V) = -c(K,V,(J_\alpha)^2\cdot V) = c(K,V_{(\mathbf{2},\mathbf{6})},V_{(\mathbf{2},\mathbf{6})})\,, \tag{2.26}$$

where there is no sum over α and in the last equality we have used that $(J_\alpha)^2 = -\mathbb{1}$ in the $V_{(\mathbf{2},\mathbf{6})}$ subspace. We see that the action of each of the J_α gives the same result. This reflects the fact that the generalised metric is independent of the action of the $SU(2)_H$ supergravity R-symmetry.

For the purpose of constructing the truncation ansatz by comparing with (3.19), we will also need the inverse generalised metric. We can exploit the isomorphism between the generalised tangent bundle E and its dual E^* provided by the generalised metric to construct a $USp(6)$ singlet $K^* \in \Gamma(E^*)$ as $K^*(V) = G(K,V)$, where V is any generalised vector. Then, denoting by $Z \in \Gamma(E^*)$ a generic dual vector, the inverse generalised metric is given by

$$G^{-1}(Z,Z) = 3\left(3\,\frac{c^*(K^*,K^*,Z)^2}{c^*(K^*,K^*,K^*)^2} - 2\frac{c^*(K^*,Z,Z)}{c^*(K^*,K^*,K^*)} + 4\,\frac{c^*(K^*,J_3\cdot Z, J_3\cdot Z)}{c^*(K^*,K^*,K^*)}\right)\,, \tag{2.27}$$

where the action of the cubic invariant c^* and of the adjoint elements J_α on the dual generalised vectors can be found in appendix A.

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2.3 The HV structure moduli space and the intrinsic torsion

When the G_S structure is a subgroup of $USp(6)$ (and there is no supersymmetry enhancement), it determines an $USp(6)$ structure and hence by definition defines a generalised metric. However, a given G_S structure can determine several different $USp(6)$ structures. Thus one gets a family of generalised metrics that can be obtained from the G_S-invariant tensors, depending on which $USp(6)$ structure one chooses. Concretely, we use the K_I and J_A tensors characterising the G_S structure to construct a generalised vector K and a triplet of J_α satisfying (2.7)–(2.9), which then we use to build the generalised metric as in (2.21). The parameterisation of K and J_α in terms of K_I and J_A provides a set of deformations of a reference $USp(6)$-invariant metric, that correspond to acting on the structure with elements of $E_{6(6)}$ that commute with G_S, modulo elements of $USp(8)/\mathbb{Z}_2$ that commute with G_S. The resulting generalised metric thus parameterises the coset

$$\mathcal{M} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)} \,. \tag{2.28}$$

This is the moduli space of our G_S structure, namely the space of deformations of the reference $USp(6)$ structure that preserve the G_S structure. For the $\mathcal{N} = 2$ structures of interest in this paper, this splits in the product

$$\mathcal{M} = \mathcal{M}_{\mathrm{V}} \times \mathcal{M}_{\mathrm{H}} \,, \tag{2.29}$$

where $\mathcal{M}_{\mathrm{V}}$ is the V structure moduli space, corresponding to deformations of K that leave J_α invariant, while $\mathcal{M}_{\mathrm{H}}$ is the H structure moduli space, which describes deformations of J_α that leave K invariant. The fact that these deformations are independent follows from the requirement (2.13). When given a dependence on the external spacetime coordinates these deformations provide the scalar fields in the truncated theory, with $\mathcal{M}_{\mathrm{V}}$ and $\mathcal{M}_{\mathrm{H}}$ being identified with the vector multiplet and the hypermultiplet scalar manifolds, respectively.

We next outline how to construct the V and H structure moduli spaces. The procedure will be further illustrated in sections 4 and 5, where concrete examples will be discussed in detail.

The V structure moduli space. A family of V structures is obtained by parameterising the generalised vector K as the linear combination

$$K = h^I K_I \,, \tag{2.30}$$

where h^I, $I = 0, \ldots, n_{\mathrm{V}}$, are real parameters, and imposing the property (2.8). Using (2.16), this is equivalent to

$$C_{IJK} h^I h^J h^K = 1 \,, \tag{2.31}$$

showing that the $n_{\mathrm{V}} + 1$ parameters h^I are constrained by one real relation and thus define an n_{V}-dimensional hypersurface,

$$\mathcal{M}_{\mathrm{V}} = \{\, h^I : \; C_{IJK} h^I h^J h^K = 1 \,\} \,. \tag{2.32}$$

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This is our V structure moduli space. It will be identified with the vector multiplet scalar manifold in five-dimensional supergravity. The metric on $\mathcal{M}_{\rm V}$ is obtained by evaluating the generalised metric on the invariant generalised vectors,

$$a_{IJ} = \frac{1}{3}\, G(K_I, K_J)\,. \tag{2.33}$$

Using (2.21), it is straightforward to see that this gives

$$a_{IJ} = 3h_I h_J - 2C_{IJK}h^K\,, \tag{2.34}$$

where $h_I = C_{IKL}h^K h^L$. Then $g_{\rm ambient} = \frac{3}{2}\, a_{IJ}{\rm d}h^I{\rm d}h^J$ gives the metric on the ambient space,[8] and the metric on $\mathcal{M}_V$ is obtained as the induced metric on the hypersurface.

The H structure moduli space. A family of H structures is obtained by parameterising the possible $\mathfrak{su}_2$ subalgebras of the algebra spanned by the J_A. The fact that we only have two singlet spinors means that $\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)$ must contain an $SU(2)_H$ factor (as in (2.10)) that acts on the two singlet spinors. Furthermore, the corresponding $\mathfrak{su}_2$ algebra must be generated by a highest root in $\mathfrak{e}_{6(6)}$. The Lie algebra $\mathfrak{h} = {\rm Lie}\,\mathcal{H}$ generated by the J_A is the simple subalgebra of the Lie algebra of $\mathrm{C}_{E_{6(6)}}(G_S)$ that contains the $\mathfrak{su}_2$ factor. Since $\mathfrak{h} \subset \mathfrak{e}_{6(6)}$ the $\mathfrak{su}_2$ algebra is generated by a highest root in $\mathfrak{h}$.

The H structure moduli space is the space of choices of such highest root $\mathfrak{su}_2$ algebras in $\mathfrak{h}$, namely the symmetric space[9]

$$\mathcal{M}_{\rm H} = \frac{\mathcal{H}}{SU(2)_H \times \mathrm{C}_{\mathcal{H}}(SU(2)_H)}\,. \tag{2.35}$$

Such spaces are known as "Wolf spaces" and are all quaternionic-Kähler, as expected from the fact that $\mathcal{M}_{\rm H}$ is going to be identified with hyperscalar manifold in five-dimensional supergravity. Points in $\mathcal{M}_{\rm H}$ can be parameterised by starting from a reference subalgebra $\mathfrak{j} \simeq \mathfrak{su}_2 \subset \mathfrak{h}$ and then acting on a basis $\{j_1, j_2, j_3\}$ of $\mathfrak{j}$ by the adjoint action of group elements $h \in \mathcal{H}$, defined as

$$J_\alpha = {\rm ad}_{\mathcal{H}}\, j_\alpha = h\, j_\alpha\, h^{-1}\,. \tag{2.36}$$

Clearly, this action acts trivially on $\mathfrak{j}$ if $h \in SU(2)_H \simeq \exp(\mathfrak{j})$, or if h belongs to the commutant of this $SU(2)_H$ in $\mathcal{H}$, that is $h \in \mathrm{C}_{\mathcal{H}}(SU(2)_H)$. This way, we obtain a triplet of "dressed" generalised tensors J_α, $\alpha = 1, 2, 3$, which depend on the coset coordinates and parameterise our family of H structures.

The intrinsic torsion. This picture that the G_S-structure defines a family of HV structures also allows us to give a characterisation of the intrinsic torsion. As discussed in [44], the intrinsic torsion of an HV structure is encoded in the three quantities

$$L_K K, \quad L_K J_\alpha, \quad \mu_\alpha \tag{2.37}$$

[8] The normalisation is chosen so as to match standard conventions in $\mathcal{N} = 2$ supergravity, see appendix B.

[9] Note the strictly the denominator group is not quite the product $SU(2)_H \times \mathrm{C}_{\mathcal{H}}(SU(2)_H)$ but generally involves modding out correctly by terms in the centre of each factor. Here we will ignore these subtleties.

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where, given a generalised vector $V \in \Gamma(E)$, one defines a triplet of functions[10]

$$\mu_\alpha(V) = -\frac{1}{2}\,\epsilon_{\alpha\beta\gamma}\int_M \boldsymbol{\kappa}^2\,\mathrm{tr}\,(J_\beta(L_V J_\gamma)) \tag{2.38}$$

that formally are moment maps for the action of the generalised diffeomorphism group on the space of H structures.

In general, if K_{G_S} is the space of G_S-compatible connections, then the definition (2.18) defines a map $\tau : K_{G_S} \to W$ where now we view the generalised torsion T as a section of $W \subset E^* \otimes \mathrm{ad}\, F$.[11] The G_S-intrinsic torsion is then an element of $W^{G_S}_{\rm int} = W/W_{G_S}$ where $W_{G_S} = \mathrm{Im}\,\tau$. Now let $p \in \mathcal{M}$ be a particular point in the family of HV structures (2.28) and $USp(6)_p \subset E_{6(6)}$ be the corresponding structure group. By construction, G_S is the common subgroup of all the $USp(6)_p$ subgroups. This means that

$$K_{G_S} = \bigcap_p K_{USp(6)_p}, \tag{2.39}$$

that is, only a G_S-compatible connection is compatible with every HV structure in the family. Hence $W_{G_S} = \bigcap_p W_{USp(6)_p}$ and so

$$W^{G_S}_{\rm int} = \bigcup_p W^{USp(6)_p}_{\rm int}. \tag{2.40}$$

In other words, knowing the intrinsic torsion of every HV structure in the family fixes the intrinsic torsion of the G_S structure.

Now, recall that each K in the family of HV structures is a linear combination of K_I (with constant coefficients), while each J_α is defined by the exponentiated adjoint action of a linear combination of J_A (with constant coefficients) on a fixed reference $\mathfrak{su}(2)$ algebra. Hence the intrinsic torsion components $L_K K$ and $L_K J_\alpha$ for the whole family are determined by knowing

$$L_{K_I} K_J, \qquad L_{K_I} J_A. \tag{2.41}$$

These also determine $\mu_\alpha(V)$ when V has the form $V = V^I K_I$, even when the components V^I are functions because of the condition (2.13). Thus the final components of the G_S intrinsic torsion are determined by

$$\int_M \boldsymbol{\kappa}^2\,\mathrm{tr}(J_A(L_W J_B))\,, \tag{2.42}$$

where we require $c(K_I, K_J, W) = 0$, which defines a generalised vector that is orthogonal to those of the form $V = V^I K_I$. Note that the expressions (2.41) and (2.42) are in general not independent, but are sufficient to determine the intrinsic torsion.

3 M-theory truncations to $\mathcal{N} = 2$ supergravity in five dimensions

Any generalised G_S structure on a manifold M with only constant, singlet intrinsic torsion gives rise to a consistent truncation of eleven-dimensional or type II supergravity on M.

[10] Recall the change of conventions from those of [44] discussed in footnote 5.

[11] For $E_{6(6)}$ generalised geometry W transforms in the $\overline{\mathbf{27}} + \mathbf{351}$ representation.

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While the general ideas (along with the details for five-dimensional truncations preserving half-maximal supersymmetry) were given in [14], here we focus on the specific case of truncations of eleven-dimensional supergravity leading to $\mathcal{N} = 2$ supergravity in five dimensions, based on $G_S \subseteq USp(6)$ structures.

Although some of the formulae giving the truncation ansatz in terms of the structure are necessarily quite involved, a great advantage is that they are universal expressions good for any $\mathcal{N} = 2$ consistent truncation. One does not have to search for the correct set of consistent modes on a case-by-case basis. All the particulars of the given truncations are encoded in terms of the given G_S structure defined by the set of K_I and J_A singlets. For example, following the discussion in the previous section, the scalar matter content is determined by the commutant of G_S in $E_{6(6)}$, giving $n_{\rm V}$ vector multiplets and $n_{\rm H}$ hypermultiplets, whose scalar manifolds are identified with the V structure and H structure moduli spaces, respectively. The gauge interactions of the truncated theory are determined by the torsion of the G_S-structure, which in turn depends only the generalised Lie derivatives $L_{K_I}K_J$ and $L_{K_I}J_A$. Together this data completely specifies the full five-dimensional supergravity.

3.1 The gauging

The gauge interactions of the truncated theory are determined by the intrinsic torsion of the generalised structure G_S. As already emphasised, we assume that the intrinsic torsion takes values in the singlet representation of G_S, with components that are constant on M. As explained in [14], this means that the generalised Lie derivative along the invariant vectors K_I acting on any invariant tensor Q_i, is given by

$$L_{K_I}Q_i = -T_{\rm int}(K_I)\cdot Q_i\,, \tag{3.1}$$

where $T_{\rm int}(K_I)$ is a G_S singlet in the adjoint bundle. This means that $T_{\rm int}(K_I)$ is in the Lie algebra of the commutant group $\mathcal{G} = {\rm C}_{E_{d(d)}}(G_S)$. Thus $-T_{\rm int}$ defines an "embedding tensor" [45, 46], that is a linear map

$$\Theta : {\rm span}(\{K_I\}) \to {\rm Lie}\,\mathcal{G}\,. \tag{3.2}$$

The image of this map defines the Lie algebra of the gauge group $G_{\rm gauge}$ of the truncated theory and also how it embeds ${\rm Lie}\,G_{\rm gauge} = {\rm Im}\,\Theta \subseteq {\rm Lie}\,\mathcal{G}$, thus giving $G_{\rm gauge}$ as a subgroup of the commutant group

$$G_{\rm gauge} \subseteq \mathcal{G} = {\rm C}_{E_{d(d)}}(G_S)\,. \tag{3.3}$$

For the structures of interest in the present paper, the relevant invariant tensors are the vectors K_I and the adjoint bundle singlets J_A that generate $\mathcal{H} \subset \mathcal{G}$. The former are the generators of the gauge algebra with structure constants $f_{[IJ]}{}^L$ given by

$$L_{K_I}K_J = \Theta_I\cdot K_J = \Theta_I{}^A(t_A)_J{}^L K_L := f_{IJ}{}^L K_L\,, \tag{3.4}$$

where $(t_A)_J{}^L$ are the representations of the generators of ${\rm Lie}\,\mathcal{G}$ acting on $\mathcal{V}$. For the J_A singlets we have

$$L_{K_I}J_A = \Theta_I\cdot J_A = [J_{(K_I)}, J_A] = \Theta_I{}^B f_{BA}{}^C J_C := p_{IA}{}^B J_B\,, \tag{3.5}$$

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where ${f_{AB}}^C$ are the $\mathcal{H}$ structure constants, as in (2.14). For convenience we have also defined the linear combination of the J_A with constant coefficients,

$$J_{(K_I)} := {\Theta_I}^A J_A \,, \tag{3.6}$$

so that the action of the generalised vector K_I on J_A is represented by the adjoint action of $J_{(K_I)}$. Recall that generally the intrinsic torsion of the G_S structure is captured by the expressions (2.41) and (2.42). The condition that one has singlet, constant intrinsic torsion is thus that (3.4) and (3.5) are satisfied with constant ${f_{IJ}}^K$ and ${p_{IA}}^B$ and in addition that

$$\int_M \boldsymbol{\kappa}^2 \operatorname{tr}(J_A(L_W J_B)) = 0 \,, \tag{3.7}$$

where the generalised vector W satisfies $c(K_I, K_J, W) = 0$. The condition on W implies it transforms non-trivially under G_S and hence, since J_A are singlets, the corresponding intrinsic torsion cannot be a singlet and so must vanish. Alternatively, recall from the discussion in section 2.3 that (3.7) is equivalent to the vanishing of the moment maps $\mu_\alpha(W)$ given in (2.38) for all H structures in the family of HV structures defined by the G_S structure. Any one H structure is related to another by the action of $\mathcal{H}$, as in (2.36). Furthermore it is straightforward to show that $\mu_\alpha(W)$ is invariant under this action. Hence if $\mu_\alpha(W) = 0$ with $c(K_I, K_J, W) = 0$ at any point in the family then it vanishes for all and (3.7) holds.

We now show how the singlet intrinsic torsion determines the gauging of the lower-dimensional $\mathcal{N} = 2$ theory. The constants ${f_{IJ}}^L$ and ${\Theta_I}^A$, defined in (3.4) and (3.5) respectively, can be identified with the embedding tensor components that encode generic gaugings of five-dimensional $\mathcal{N} = 2$ supergravity theories, including those involving vector fields that transform in non-adjoint representations of the gauge group, as well as antisymmetric rank-2 tensor fields.[12] For simplicity, here we just discuss the case where (3.4) define the structure constants of a Lie algebra, implying that it is not necessary to introduce antisymmetric rank-2 tensor fields. These determine the symmetries of the scalar manifold that are gauged, and hence all matter couplings of the $\mathcal{N} = 2$ theory, completely fixing the five-dimensional Lagrangian (see appendix B for a brief account of $\mathcal{N} = 2$ supergravity in five dimensions). In particular, the vector multiplet scalar covariant derivatives and the gauge field strengths are given by

$$\mathcal{D}h^I = \mathrm{d}h^I + g\, {f_{JK}}^I \mathcal{A}^J\, h^K \,, \tag{3.8}$$

$$\mathcal{F}^I = \mathrm{d}\mathcal{A}^I + \frac{1}{2}\, g\, {f_{JK}}^I \mathcal{A}^J \wedge \mathcal{A}^K \,, \tag{3.9}$$

where g is the gauge coupling constant and $\mathcal{A}^I = \mathcal{A}^I_\mu \mathrm{d}x^\mu$ are the five-dimensional gauge fields. In order to obtain the hyperscalar covariant derivatives, we need the Killing vectors on the H structure moduli space (2.35) that generate the gauged isometries. These can be constructed from (3.6) using the standard formalism of coset spaces, see e.g. [49]. Given

[12] The embedding tensor formalism is most commonly used to describe the gauging of maximal and half-maximal supergravity [45, 46], see however [47, 48] for its use in an $\mathcal{N} = 2$ context.

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the left-action of a generator $J_{(K_I)}$ on the coset representative $L \in \mathcal{H}$, the corresponding Killing vector k_I on $\mathcal{M}_{\rm H}$ is determined by the equation

$$L^{-1} J_{(K_I)}\, L \;\sim\; g\, \iota_{k_I}\, (L^{-1}\mathrm{d}L)\,, \tag{3.10}$$

where the symbol $\sim$ means that the equality holds up to an element of the algebra one is modding out by, which in the present case is $SU(2)_H \times \mathrm{C}_{\mathcal{H}}(SU(2)_H)$.[13] Writing $k_I = k_I^X \frac{\partial}{\partial q^X}$, where q^X denote the coordinates on $\mathcal{M}_{\rm H}$, the hyperscalar covariant derivatives then read

$$\mathcal{D}q^X = \mathrm{d}q^X + g\, \mathcal{A}^I k_I^X\,. \tag{3.11}$$

From the Killing vectors k_I we can then compute the triholomorphic Killing prepotentials P_I^α, $\alpha = 1,2,3$, that determine the fermionic shifts and the scalar potential of the $\mathcal{N} = 2$ supergravity theory, see appendix B for the relevant formulae. These Killing prepotentials are moment maps of the isometries being gauged, and as such can be nicely computed from the generalised geometry formalism. Recalling the definition of the moment map μ in (2.38), they are given by

$$\begin{aligned} g\, P_I^\alpha &= \frac{1}{8}\, \epsilon^{\alpha\beta\gamma} \int_M \boldsymbol{\kappa}^2\, \mathrm{tr}\, (J_\beta (L_{K_I} J_\gamma)) \Big/ \int_M \boldsymbol{\kappa}^2 \\ &= \frac{1}{8}\, \epsilon^{\alpha\beta\gamma}\, \mathrm{tr}\, (J_\beta (L_{K_I} J_\gamma))\,. \end{aligned} \tag{3.12}$$

In this formula, recall that the J_α are the dressed triplet, hence the resulting moment maps are function of the H structure moduli. In the second line, we have used the fact that the singlet torsion components $\mathrm{tr}\, (J_\beta (L_{K_I} J_\gamma))$ are constant on M and hence the integrals over $\boldsymbol{\kappa}^2$ cancel.

3.2 The truncation ansatz

Our conventions for eleven-dimensional supergravity are as in [42]. The eleven-dimensional bosonic action is (we denote by a hat the 11d quantities)

$$\hat{S} = \frac{1}{2} \int \left(\hat{R} \,\hat{*}\, 1 - \frac{1}{2} \hat{F} \wedge *\hat{F} - \frac{1}{6} \hat{A} \wedge \hat{F} \wedge \hat{F} \right)\,, \tag{3.13}$$

where $\hat{F} = \mathrm{d}\hat{A}$ and $\hat{A}$ is the three-form potential. The equations of motion are

$$\begin{aligned} \hat{R}_{\hat{\mu}\hat{\nu}} - \frac{1}{12} \left(\hat{F}_{\hat{\mu}\hat{\rho}_1\hat{\rho}_2\hat{\rho}_3} \hat{F}_{\hat{\nu}}{}^{\hat{\rho}_1\hat{\rho}_2\hat{\rho}_3} - \frac{1}{12}\, \hat{g}_{\hat{\mu}\hat{\nu}} \hat{F}^2 \right) = 0\,, \\ \mathrm{d} \,\hat{*}\, \hat{F} + \frac{1}{2} \hat{F} \wedge \hat{F} = 0\,. \end{aligned} \tag{3.14}$$

The six-form potential $\hat{\tilde{A}}$ dual to the three-form $\hat{A}$ may be introduced via the first-order relation

$$\hat{*}\, \mathrm{d}\hat{A} + \frac{1}{2}\, \hat{A} \wedge \mathrm{d}\hat{A} = \mathrm{d}\hat{\tilde{A}}\,, \tag{3.15}$$

whose exterior derivative gives the Maxwell equation.

[13] A similar construction could be made for the Killing vectors that gauge isometries in the V structure moduli space, starting from the sections of the adjoint bundle that generate $\mathcal{M}_{\rm V}$ mentioned in Footnote 6. However, this will not be needed for our purposes.

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As first step of the truncation procedure, we arrange the eleven-dimensional bosonic fields into generalised tensors transforming in representations of $GL(5,\mathbb{R}) \times E_{6(6)}$, where $GL(5,\mathbb{R})$ gives the tensorial structure of the fields in the five-dimensional theory obtained after reduction. Then we expand each $E_{6(6)}$ representation in terms of the G_S invariant tensors transforming in the same representation. We separate the eleven-dimensional coordinates in coordinates x^μ, $\mu = 0, \ldots, 4$, on the external spacetime X, and z^m, $m = 1, \ldots, 6$, on the internal manifold M.

The bosonic fields of eleven-dimensional supergravity are decomposed as

$$\begin{aligned}
\hat{g} &= \mathrm{e}^{2\Delta}\, g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dz^m Dz^n \,, \\
\hat{A} &= \frac{1}{3!} A_{mnp} Dz^{mnp} + \frac{1}{2} A_{\mu mn} \mathrm{d}x^\mu \wedge Dz^{mn} + \frac{1}{2}\, \bar{A}_{\mu\nu m} \mathrm{d}x^{\mu\nu} \wedge Dz^m + \frac{1}{3!}\, \bar{A}_{\mu\nu\rho}\, \mathrm{d}x^{\mu\nu\rho} \,, \\
\hat{\tilde{A}} &= \frac{1}{6!} \tilde{A}_{m_1\ldots m_6} Dz^{m_1\ldots m_6} + \frac{1}{5!} \tilde{A}_{\mu m_1\ldots m_5} \mathrm{d}x^\mu \wedge Dz^{m_1\ldots m_5} + \frac{1}{2\cdot 4!} \bar{\tilde{A}}_{\mu\nu m_1\ldots m_4} \mathrm{d}x^{\mu\nu} \wedge Dz^{m_1\ldots m_4} \\
&\quad + \ldots \,,
\end{aligned} \tag{3.16}$$

where $Dz^m = \mathrm{d}z^m - h_\mu{}^m \mathrm{d}x^\mu$, and all tensor field components may depend both on x^μ and z^m, except for the external metric, for which we assume a dependence on the external coordinates only, $g_{\mu\nu} = g_{\mu\nu}(x)$.

The barred fields need to be redefined. In appendix C we provide a justification for these redefinitions by studying the gauge transformations of the metric and three-form potential. For the three-form components we introduce the new fields $A_{\mu\nu m}$, $A_{\mu\nu\rho}$ via

$$\bar{A}_{\mu\nu m} = A_{\mu\nu m} - \, h_{[\mu}{}^n A_{\nu]nm} \,, \qquad \bar{A}_{\mu\nu\rho} = A_{\mu\nu\rho} + h_{[\mu}{}^n h_\nu{}^p A_{\rho]np} \,. \tag{3.17}$$

Similar redefinitions apply to the six-form components with at least two external indices, however we will not discuss them in detail here.

The supergravity fields having all components on the internal manifold M arrange into the inverse generalised metric

$$G^{MN} \;\leftrightarrow\; \{\Delta,\, g_{mn},\, A_{mnp},\, \tilde{A}_{m_1\ldots m_6}\} \,, \tag{3.18}$$

in the following way[14]

$$\begin{aligned}
(G^{-1})^{mn} &= \mathrm{e}^{2\Delta} g^{mn} \\
(G^{-1})^m{}_{n_1 n_2} &= \mathrm{e}^{2\Delta} g^{mp} A_{pn_1n_2} \\
(G^{-1})^m{}_{n_1\ldots n_5} &= \mathrm{e}^{2\Delta} g^{mp} (A_{p[n_1n_2} A_{n_3n_4n_5]} + \tilde{A}_{pn_1\ldots n_5}) \\
(G^{-1})_{m_1m_2\, n_1n_2} &= \mathrm{e}^{2\Delta} (g_{m_1m_2, n_1n_2} + g^{pq} A_{pm_1m_2} A_{qn_1n_2]}) \\
(G^{-1})_{m_1m_2\, n_1\ldots n_5} &= \mathrm{e}^{2\Delta} [g_{m_1m_2,[n_1n_2} A_{n_3n_4n_5]} \\
&\quad + g^{pq} (A_{pm_1m_2} (A_{q[n_1n_2} A_{n_3n_4n_5]} + \tilde{A}_{qn_1\ldots n_5})] \\
(G^{-1})_{m_1\ldots m_5\, n_1\ldots n_5} &= \mathrm{e}^{2\Delta} [g_{m_1\ldots m_5,\, n_1\ldots n_5} \\
&\quad + g^{pq} (A_{p[m_1m_2} A_{m_3m_4m_5]} + \tilde{A}_{pm_1\ldots m_5}) (A_{q[n_1n_2} A_{n_3n_4n_5]} + \tilde{A}_{qn_1\ldots n_5})] \,,
\end{aligned} \tag{3.19}$$

[14] This expression follows straightforwardly from the elements of the conformal split frame given in [42].

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where $g_{m_1m_2,n_1n_2} = g_{m_1[n_1}g_{|m_2|n_2]}$, and similarly for $g_{m_1\ldots m_5,n_1\ldots n_5}$. Since the generalised metric is a scalar on the external spacetime, after imposing our truncation ansatz it will provide the scalar fields of the reduced five-dimensional theory.

The density $\boldsymbol{\kappa}$ introduced in section 2.1 when defining the HV structure is related to the determinant of the generalised metric and is an $E_{6(6)}$ invariant. For eleven-dimensional metrics of the form (3.16), this is given by [42, 44]

$$\boldsymbol{\kappa}^2 = \mathrm{e}^{3\Delta}\sqrt{\det g_{mn}}\,. \tag{3.20}$$

The tensors with one external leg arrange into a generalised vector $\mathcal{A}_\mu$ on M, with components

$$\mathcal{A}_\mu{}^M = \{h_\mu{}^m,\, A_{\mu mn}, \tilde{A}_{\mu m_1\ldots m_5}\}\,, \tag{3.21}$$

and will provide the gauge potentials of the reduced theory. The tensors with two antisymmetrised external indices define a weighted dual vector $\mathcal{B}_{\mu\nu}$ on M, which is a section of $\det T^*M \otimes E^*$, with components

$$\mathcal{B}_{\mu\nu\,M} = \{A_{\mu\nu m},\, \tilde{A}_{\mu\nu m_1\ldots m_4},\, \tilde{g}_{\mu\nu m_1\ldots m_6,n}\}\,, \tag{3.22}$$

and will give the two-form fields of the reduced theory. The last term in (3.22) is related to the dual graviton and we will not discuss it further here. The tensors with three antisymmetrised external indices arrange into the generalised tensor

$$\mathcal{C}_{\mu\nu\rho}{}^{\hat{\alpha}} = \{A_{\mu\nu\rho},\, \tilde{A}_{\mu\nu\rho m_1m_2m_3},\, \tilde{g}_{\mu\nu\rho m_1\ldots m_5,n}\}\,, \tag{3.23}$$

which is a section of (a sub-bundle of) the weighted adjoint bundle $\det T^*M \otimes \mathrm{ad}\, F$, whose components are labeled by $\hat{\alpha} = 1,\ldots,57$. See e.g. [50, 51] for more details on this tensor hierarchy.

As discussed in [14], the bosonic part of the truncation ansatz is obtained by imposing that the generalised tensors above are expanded in singlets of the G_S structure. The generalised metric is obtained by constructing the K and J_α parameterising a family of HV structures as detailed in section 2.3, and plugging these generalised tensors in the formula (2.27). The resulting generalised metric depends on the H and V structure moduli; when given a dependence on the external coordinates x^μ, these are then identified with the hyperscalar and vector multiplet scalar fields of the truncated $\mathcal{N}=2$ theory, respectively. Thus we have

$$\left.\begin{aligned} K &= h^I(x)K_I \\ J_\alpha &= L(x)j_\alpha L(x)^{-1} \end{aligned}\right\} \qquad \text{giving } G^{MN}(x) \text{ from (2.27)}\,, \tag{3.24}$$

where L is the representative of the coset $\mathcal{M}_{\mathrm{H}}$. Comparing the expression for the generalised metric with its general form (3.19), we obtain the truncation ansatz for Δ, g_{mn}, A_{mnp} (as well as $\tilde{A}_{m_1\ldots m_6}$, whenever it is needed). Note that $\boldsymbol{\kappa}^2$ given in (3.20) is independent of the scalar fields $h^I(x)$ and $L(x)$, so it can be evaluated using any chosen representative of the family of HV structures defined by the G_S structure.

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The gauge potentials $\mathcal{A}_\mu{}^I(x)$ on the external space-time are defined by taking

$$\mathcal{A}_\mu = \mathcal{A}_\mu{}^I(x)\, K_I \quad \in \Gamma(T^*X) \otimes \mathrm{span}(\{K_I\}) \tag{3.25}$$

where $\mathrm{span}(\{K_I\}) \subset \Gamma(E)$ is the vector space spanned by the set of G_S singlets K_I, $I = 0, 1, \ldots, n_{\mathrm{V}}$. Similarly the two-form fields are given by

$$\mathcal{B}_{\mu\nu} = \mathcal{B}_{\mu\nu\, I}(x)\, K_\flat^I \quad \in \Gamma(\Lambda^2 T^*X) \otimes \mathrm{span}(\{K_\flat^I\})\,, \tag{3.26}$$

where $\mathrm{span}(\{K_\flat^I\}) \subset \Gamma(\det T^*M \otimes E^*)$ is the vector space spanned by the weighted dual basis vectors $K_\flat^I$, the latter being defined by $K_\flat^I(K_J) = 3\boldsymbol{\kappa}^2\, \delta^I{}_J$. We also have

$$\mathcal{C}_{\mu\nu\rho} = \mathcal{C}_{\mu\nu\rho}{}^A(x)\, J_A^\flat \quad \in \Gamma(\Lambda^3 T^*X) \otimes \mathrm{span}(\{J_A^\flat\})\,, \tag{3.27}$$

where $\mathrm{span}(\{J_A^\flat\}) \subset \Gamma(\det T^*M \otimes \mathrm{ad}(F))$ is spanned by the G_S singlets in the weighted adjoint bundle, here denoted by $J_A^\flat$ and given by $J_A^\flat = \boldsymbol{\kappa}^2 J_A$. In appendix C we show that these expressions, together with the field redefinitions (3.17), lead to the correct five-dimensional covariant objects, consistent with the expected gauge transformations.

4 $\mathcal{N} = 2$ truncations on Maldacena-Nuñez geometries

We now apply the above formalism to consistent truncations of eleven-dimensional supergravity based on generalised structures arising from M5-branes wrapping a Riemann surface.

We start with the $\mathcal{N} = 2$ $\mathrm{AdS}_5 \times_{\mathrm{w}} M$ solution of Maldacena and Nuñez [34] and show that the manifold M admits a generalised $U(1)$ structure with singlet intrinsic torsion, and therefore can be used to construct a consistent truncation. As we have stressed above, once we identify the singlet K_I and J_A tensors defining the structure it is straightforward to read off the form of the $\mathcal{N} = 2$ supergravity.

We already observed in [14] that this process yields $\mathcal{N} = 2$ supergravity with one hypermultiplet and four vector multiplets. Here we give the details of the construction and derive the gauging, which defines an $SO(3) \times U(1) \times \mathbb{R}$ gauge group. Our truncation includes as a subtruncation the reduction to $\mathcal{N} = 2$ supergravity with one vector multiplet, one hypermultiplet and $U(1) \times \mathbb{R}$ gauging recently obtained in [36].

4.1 The MN1 solution

We are interested in warped AdS_5 solutions to eleven-dimensional supergravity that describe the near-horizon region of M5-branes wrapping supersymmetric cycles in a Calabi-Yau geometry. The amount of supersymmetry of the solutions depends on how the cycle is embedded in the ambient geometry. This corresponds to a topological twist of the world-volume $(0, 2)$ theory on the M5-branes. The simplest examples are the solutions found by Maldacena and Nuñez [34] describing the near-horizon geometry of M5-branes wrapped on a Riemann surface $\mathbb{\Sigma}$ of negative constant curvature. The topological twist of the $(0, 2)$ world-volume theory is realised by identifying the spin connection on $\mathbb{\Sigma}$ with a $U(1)$ connection in the $SO(5)$ R-symmetry group of the M5-brane theory. The theory preserves $\mathcal{N} = 2$

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or $\mathcal{N}=1$ superconformal symmetry in four dimensions, depending on how the $U(1)$ is chosen inside $SO(5)$. The corresponding supergravity solutions are warped products of AdS_5 times a six-dimensional manifold, M, which is the fibration of a deformed S^4 over $\mathbb{\Sigma}$. The $SO(5)$ is realised via the action of the isometry group of the round S^4. The structure of the fibration reflects the twist of the world-volume theory and determines the amount of supersymmetry of the solutions, which in five-dimensional language is either $\mathcal{N}=4$ or $\mathcal{N}=2$, respectively.

In this paper we focus on the $\mathcal{N}=2$ solution, which we call the "MN1 solution" in the following. The eleven-dimensional metric is[15]

$$\hat{g} = \mathrm{e}^{2\Delta}\, g_{\mathrm{AdS}_5} + g_6\,, \tag{4.1}$$

where g_{AdS_5} is the Anti de Sitter metric with radius $\ell = \frac{3}{2}R$, R being the length scale of the internal space M. The metric on M takes the form

$$g_6 = R^2\, \frac{3^{1/3}}{2^{4/3}} \left(3+\cos^2\zeta\right)^{1/3} \left[g_{\mathbb{\Sigma}} + \mathrm{d}\zeta^2 + \frac{\sin^2\zeta}{3+\cos^2\zeta}\left(\sigma_1^2+\sigma_2^2+(\sigma_3+\upsilon)^2\right)\right]. \tag{4.2}$$

Here, $g_{\mathbb{\Sigma}}$ is the uniform metric on (a quotient of) the hyperbolic plane $\mathbb{\Sigma}=H^2$, with Ricci scalar curvature $\mathcal{R}_{\mathbb{\Sigma}}=-2$, while υ is the spin connection on $\mathbb{\Sigma}$ satisfying

$$\mathrm{d}\upsilon = -\,\mathrm{vol}_{\mathbb{\Sigma}}\,, \tag{4.3}$$

with $\mathrm{vol}_{\mathbb{\Sigma}}$ the volume form on $\mathbb{\Sigma}$.[16] The deformed S^4 is described as a foliation of a round S^3 over an interval, with the interval coordinate being $\zeta\in[0,\pi]$, while σ_α, $\alpha=1,2,3$, are the standard $SU(2)_{\mathrm{left}}$-invariant forms on S^3, expressed in terms of Euler angles $\{\theta,\phi,\psi\}$. Their explicit expression can be found in appendix D, together with more details on the parameterisation of S^4.

The warp factor is

$$\mathrm{e}^{2\Delta} = \left(\frac{2}{3}\right)^{2/3} (3+\cos^2\zeta)^{1/3}\,, \tag{4.4}$$

while the four-form reads

$$\begin{aligned}\hat{F} = \frac{R^3}{4}\Bigg[&\frac{15+\cos^2\zeta}{(3+\cos^2\zeta)^2}\sin^3\zeta\,\mathrm{d}\zeta\wedge\sigma_1\wedge\sigma_2\wedge(\sigma_3+\upsilon)\\ &+\sin\zeta\left(-\mathrm{d}\zeta\wedge\sigma_3+\frac{\sin(2\zeta)}{3+\cos^2\zeta}\,\sigma_1\wedge\sigma_2\right)\wedge\mathrm{vol}_{\mathbb{\Sigma}}\Bigg]\,.\end{aligned} \tag{4.5}$$

[15] We present the solution in a form similar to the one given in [52, Sect. 5]. The precise dictionary with this reference is: $\alpha=\zeta$, $\nu=-\phi$, $\psi_{\mathrm{GMSW}}=\psi$, $\mathrm{e}^{2\lambda}=\mathrm{e}^{2\Delta}$, $m^{-1}=\ell_{\mathrm{AdS}_5}=\frac{3}{2}R$, where the variables on the left-hand side are those of [52] while the variables on the right-hand side are those used here. The length scale R that appears in our expressions is equal to the radius of S^4 in the related $\mathrm{AdS}_7\times S^4$ Freund-Rubin solution of eleven-dimensional supergravity. The four-form $\hat{F}$ in (4.5) has an overall opposite sign with respect to the one of [52], $\hat{F}=-F_{\mathrm{GMSW}}$; this sign does not affect the equations of motion, it just modifies the projection condition satisfied by the supersymmetry spinor parameter.

[16] Choosing local coordinates x,y on the hyperbolic plane, one can write $g_{\mathbb{\Sigma}}=\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$, $\mathrm{vol}_{\mathbb{\Sigma}}=\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}$, and $\upsilon=-\frac{\mathrm{d}x}{y}$.

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Note that the invariant volume form (3.20) is given by

$$\boldsymbol{\kappa}^2 = R^2 \, \mathrm{vol}_{\Sigma} \wedge \mathrm{vol}_4 \,, \tag{4.6}$$

where vol_4 is the volume form of the round S^4 of radius R.

The solution has $SU(2)_{\mathrm{left}} \times U(1)_{\mathrm{right}}$ symmetry, which embeds in the $SO(5)$ isometry group of a round S^4 as

$$SO(5) \supset SO(4) \simeq SU(2)_{\mathrm{left}} \times SU(2)_{\mathrm{right}} \supset SU(2)_{\mathrm{left}} \times U(1)_{\mathrm{right}} \,. \tag{4.7}$$

This symmetry is manifest as the solution is given in terms of the σ_α. The globally-defined combination $(\sigma_3 + \upsilon)$ describes a fibration of S^4 over $\mathbb{\Sigma}$, such that the $U(1)_{\mathrm{right}}$ action on S^4 is used to cancel the $U(1)$ holonomy of $\mathbb{\Sigma}$.

The $U(1)_{\mathrm{right}}$ factor provides the R-symmetry of the holographically dual $\mathcal{N} = 1$ SCFT, while $SU(2)_{\mathrm{left}}$ corresponds to a flavour symmetry. The dual $\mathcal{N} = 1$ SCFT has been described in [53].

4.2 Generalised $U(1)$ structure of the MN1 solution

The solution reviewed above admits a generalised $U(1)_S$ structure, which will be the basis for constructing our consistent truncation. In order to characterise it we proceed in two steps. The first is purely group theoretical: it consists in embedding the relevant $U(1)_S$ in $E_{6(6)}$, computing its commutant and the corresponding decompositions of the generalised tangent and adjoint bundles. To this end, it is convenient to decompose $E_{6(6)}$ according to its maximal compact subgroup $USp(8)/\mathbb{Z}_2$. Since the $\mathfrak{usp}(8)$ algebra can be given in terms of Cliff(6) gamma matrices (see appendix E.1), this reduces the problem to gamma matrix algebra. The details of the derivation can be found in appendix E; here we will just give the results. Once the relevant $U(1)_S$ singlets are identified, the second step is to express them in terms of the geometry of the six-dimensional manifold M.

The generalised $U(1)_S$ structure of the MN1 solution is the diagonal of the ordinary geometrical $U(1) \simeq SO(2) \subset GL(2,\mathbb{R})$ structure on the Riemann surface and a $U(1)$ factor in the $SO(5) \subset SL(5,\mathbb{R}) \simeq E_{4(4)}$ generalised structure for the generalised tangent space of the four-sphere. In terms of the isometry group decomposition (4.7) this can be identified with $U(1)_{\mathrm{right}}$. If we denote by 1 to 4 the directions in M along S^4 and by 5,6 those along $\mathbb{\Sigma}$, the generator of $U(1)_S$ can be written as a $\mathfrak{usp}(8)$ element as

$$\mathfrak{u}(1)_S = \mathrm{i}\,\hat{\Gamma}_{56} - \frac{\mathrm{i}}{2}(\hat{\Gamma}_{12} - \hat{\Gamma}_{34}) \,, \tag{4.8}$$

where $\hat{\Gamma}_m$ are six-dimensional gamma matrices. The first term corresponds to the $U(1)$ holonomy of $\mathbb{\Sigma}$ while the second one is the $U(1)_{\mathrm{right}}$ in $SO(5)$. By computing the commutators of (4.8) in $USp(8)$ we find that the $U(1)_S$ structure embeds in $USp(8)$ as[17]

$$USp(8) \supset SU(2) \times SU(2)_H \times U(1) \times U(1)_S \,, \tag{4.9}$$

[17]Here and below we give expressions ignoring subtleties involving the centres of each group; thus for instance we will not distinguish between embeddings in $USp(8)$ and $USp(8)/\mathbb{Z}_2$.

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where as above we distinguish the factor $SU(2)_H$ that gives the R-symmetry of the five-dimensional supergravity theory. Under this splitting, the spinorial representation of $USp(8)$ decomposes as

$$\mathbf{8} = (\mathbf{1},\mathbf{2})_0 \oplus (\mathbf{2},\mathbf{1})_1 \oplus (\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1})_{-2}\,, \tag{4.10}$$

where the elements in the brackets denote the $SU(2) \times SU(2)_H$ representations and the subscript gives the $U(1)_S$ charge. We then see that there are only two spinors that are singlets under $U(1)_S$ and that transform as a doublet of $SU(2)_H$ as required by $\mathcal{N} = 2$ supersymmetry.

The embedding of the $U(1)_S$ structure in the full $E_{6(6)}$ is obtained in a similar way (see appendix E.2 for details)

$$E_{6(6)} \supset \mathrm{C}_{E_{6(6)}}(U(1)_S) = \mathbb{R}^+ \times Spin(3,1) \times SU(2,1) \times U(1)_S\,, \tag{4.11}$$

where $\mathrm{C}_{E_{6(6)}}(U(1)_S)$ is the commutant of $U(1)_S$ in $E_{6(6)}$. We can now determine how many generalised vectors and adjoint elements are $U(1)_S$ singlets. Under (4.11) the **27** decomposes as

$$\begin{aligned}\mathbf{27} = {} & (\mathbf{1},\mathbf{1})_{(0,8)} \oplus (\mathbf{4},\mathbf{1})_{(0,-4)} \oplus (\mathbf{2},\mathbf{1})_{(3,-2)} \oplus (\bar{\mathbf{2}},\mathbf{1})_{(-3,-2)} \\ & \oplus (\mathbf{1},\mathbf{3})_{(2,-4)} \oplus (\mathbf{1},\bar{\mathbf{3}})_{(-2,-4)} \oplus (\bar{\mathbf{2}},\mathbf{3})_{(1,2)} \oplus (\mathbf{2},\bar{\mathbf{3}})_{(-1,2)}\,,\end{aligned} \tag{4.12}$$

where the first subscript denotes the $U(1)_S$ charge and the second one the $\mathbb{R}^+$ charge. We see that there are five singlets K_I, $I = 0, 1, \ldots, 4$, where

$$K_0 \in (\mathbf{1},\mathbf{1})_{(0,8)} \tag{4.13}$$

is only charged under the $\mathbb{R}^+$, while

$$\{K_1, K_2, K_3, K_4\} \in (\mathbf{4},\mathbf{1})_{(0,-4)} \tag{4.14}$$

form a vector of $SO(3,1)$.

The singlets in the **78** adjoint representation are the generators of the commutant $C_{E_{6(6)}}(U(1)_S)$. However only the generators of the $SU(2,1)$ subgroup are relevant for the structure. Indeed, (4.12) shows that the generators of $\mathbb{R}^+ \times SO(3,1)$ do not leave the singlet vectors invariant, and therefore, as discussed in section 2.1, do not contribute to the truncation. As shown in (E.40) and (E.41), they can be obtained as products $K_I \times_{\mathrm{ad}} K_J^*$. We denote by J_A, $A = 1, \ldots, 8$, the elements of the adjoint bundle generating $\mathfrak{su}_{2,1}$. Four of them are in the **36** of $USp(8)$ and generate the compact subalgebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1$, and four more are in the **42** of $USp(8)$ and generate the rest of $\mathfrak{su}_{2,1}$.

The $U(1)_S$ structure is then defined by

$$\{K_I, J_A\}\,, \qquad I = 0, \ldots, 4\,, \quad A = 1, \ldots, 8\,. \tag{4.15}$$

The derivation of the explicit expressions for these generalised tensors relies on the way the solution of [34] is constructed by deforming the $\mathrm{AdS}_7 \times S^4$ background dual to flat

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M5-branes so as to describe their backreaction when wrapping a Riemann surface $\mathbb{\Sigma}$. The world-volume theory on the wrapped M5-branes is made supersymmetric by a topological twist, where the spin connection on the Riemann surface is cancelled by switching on a background gauge field for a $U(1)$ subgroup of the $SO(5)$ R-symmetry. On the dual background the topological twist implies that M is an S^4 fibration over $\mathbb{\Sigma}$

$$\begin{array}{ccc} S^4 & \xrightarrow{\ i\ } & M \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{\Sigma} \end{array} \tag{4.16}$$

The generalised tangent bundle for S^4 is given by

$$E_4 \simeq TS^4 \oplus \Lambda^2 T^* S^4 \,, \tag{4.17}$$

and transforms under $SL(5,\mathbb{R}) \simeq E_{4(4)}$. It is generalised parallelisable, meaning it admits a globally defined frame [15]. The idea is then to consider first the direct product $\mathbb{\Sigma} \times S^4$, express the $E_{6(6)}$ generalised tensors on this manifold in terms of the frame on $\mathbb{\Sigma}$ and the parallelisation on S^4, and then implement the twist of S^4 over $\mathbb{\Sigma}$ so as to make globally well-defined objects. In the decomposition

$$E_{6(6)} \supset GL(2,\mathbb{R}) \times SL(5,\mathbb{R}) \,, \tag{4.18}$$

where $GL(2,\mathbb{R})$ is the structure group of the conventional tangent bundle on $\mathbb{\Sigma}$ and $SL(5,\mathbb{R}) \simeq E_{4(4)}$ is the structure group of the generalised tangent bundle on S^4, the $E_{6(6)}$ generalised tangent bundle on $\mathbb{\Sigma} \times S^4$ decomposes as

$$E \simeq T\mathbb{\Sigma} \oplus (T^*\mathbb{\Sigma} \otimes N_4) \oplus (\Lambda^2 T^*\mathbb{\Sigma} \otimes N_4') \oplus E_4 \,, \tag{4.19}$$

and the adjoint bundle as

$$\begin{aligned} \mathrm{ad}F \simeq\ & \mathrm{ad}F_4 \oplus (T\Sigma \otimes T^*\Sigma) \oplus (T^*\Sigma \otimes E_4) \\ & \oplus (\Lambda^2 T^*\Sigma \otimes N_4) \oplus (T\Sigma \otimes E_4^*) \oplus (\Lambda^2 T\Sigma \otimes N_4^*) \,. \end{aligned} \tag{4.20}$$

In the expressions above E_4 is the generalised tangent bundle on S^4 introduced in (4.17), $\mathrm{ad}\, F_4$ is the adjoint bundle on S^4,

$$\mathrm{ad}\, F_4 \simeq \mathbb{R} \oplus (TS^4 \otimes T^*S^4) \oplus \Lambda^3 T^* S^4 \oplus \Lambda^3 T S^4 \,, \tag{4.21}$$

and N_4 and N_4' are the following bundles on S^4,

$$\begin{aligned} N_4 &\simeq T^*S^4 \oplus \Lambda^4 T^* S^4 \,, \\ N_4' &\simeq \mathbb{R} \oplus \Lambda^3 T^* S^4 \,. \end{aligned} \tag{4.22}$$

The bundles E_4, N_4 and N_4' admit the globally defined generalised frames

$$E_{ij} \in \Gamma(E_4) \,, \qquad E_i \in \Gamma(N_4) \,, \qquad E_i' \in \Gamma(N_4') \,, \qquad i,j = 1,\ldots,5 \,, \tag{4.23}$$

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see appendix D for their expression in a coordinate basis and note that they include a contribution from the three-form gauge potential A_{S^4} of the flux on the S^4. Geometrically this defines a generalised identity structure on S^4. Given the way $U(1)_S$ is embedded in $USp(8)$, we will find it useful to also introduce the following linear combinations of the generalised frame elements E_{ij} on S^4,

$$\begin{aligned} &\Xi_1 = E_{13} + E_{24}\,, \quad \Xi_2 = E_{14} - E_{23}\,, \quad \Xi_3 = E_{12} - E_{34}\,, \\ &\widetilde{\Xi}_1 = E_{13} - E_{24}\,, \quad \widetilde{\Xi}_2 = E_{14} + E_{23}\,, \quad \widetilde{\Xi}_3 = E_{12} + E_{34}\,. \end{aligned} \tag{4.24}$$

Since their restriction to TM corresponds to the Killing vectors generating the $SU(2)_{\text{left}} \times SU(2)_{\text{right}} \simeq SO(4) \subset SO(5)$ isometries of S^4 (again see appendix D for their explicit expression), Ξ_α and $\widetilde{\Xi}_\alpha$, $\alpha = 1, 2, 3$, may be seen as generalised Killing vectors generating the corresponding generalised isometries.

As for the Riemann surface $\mathbb{\Sigma}$, it can be (a quotient of) the hyperbolic plane H^2 as in the MN1 solution reviewed in section 4.1, but we can also take a torus T^2, or a sphere S^2. We introduce orthonormal co-frame one-forms e_1, e_2 on $\mathbb{\Sigma}$, such that the constant curvature metric and the compatible volume form on $\mathbb{\Sigma}$ are given by

$$g_{\mathbb{\Sigma}} = (e_1)^2 + (e_2)^2\,, \qquad \text{vol}_{\mathbb{\Sigma}} = e_1 \wedge e_2\,. \tag{4.25}$$

The metric is normalised so that the Ricci scalar curvature is $\mathcal{R}_{\mathbb{\Sigma}} = 2\kappa$, where $\kappa = +1$ for S^2, $\kappa = 0$ for T^2 and $\kappa = -1$ for H^2 (and quotients thereof). We also define the $U(1)$ spin connection, υ, on $\mathbb{\Sigma}$ as

$$\mathrm{d}(e_1 + \mathrm{i}\, e_2) = \mathrm{i}\,\upsilon \wedge (e_1 + \mathrm{i}\, e_2)\,, \qquad \mathrm{d}\upsilon \,=\, \kappa\, \text{vol}_{\mathbb{\Sigma}}\,. \tag{4.26}$$

The decompositions (4.19) and (4.20) allow us to express the $U(1)_S$ invariant generalised tensors in terms of tensors on $\mathbb{\Sigma}$ and the S^4 generalised frames introduced above. We provide the derivation in appendix E and here just present the resulting expressions. Let us first focus on the singlet generalised vectors K_I. These can be written as

$$K_0 \,=\, \mathrm{e}^{\Upsilon} \cdot (R^2\, \text{vol}_{\mathbb{\Sigma}} \wedge E_5')\,, \qquad K_{1,2,3} \,=\, \mathrm{e}^{\Upsilon} \cdot \widetilde{\Xi}_{1,2,3}\,, \qquad K_4 \,=\, \mathrm{e}^{\Upsilon} \cdot \Xi_3\,, \tag{4.27}$$

where Υ is a section of the adjoint bundle implementing the twist of S^4 over $\mathbb{\Sigma}$ as in (4.16), ensuring that these are globally defined objects on the six-dimensional manifold. Recall that in the MN1 solution, the $U(1)$ that is used to twist the four-sphere and compensate the spin connection υ on $\mathbb{\Sigma}$ is the Cartan of $SU(2)_{\text{right}} \subset SO(5)$. The $E_{6(6)}$ twist element Υ is constructed in a way similar to the one used in [14], albeit with a different choice of $U(1)$ in $SO(5)$. We embed the connection one-form υ in a generalised dual vector, the Killing vector generating the Cartan of $SU(2)_{\text{right}}$ in the generalised vector Ξ_3 introduced above, and we project their product onto the adjoint of $E_{6(6)}$. That is,

$$\Upsilon = -\frac{R}{2}\, \upsilon \times_{\text{ad}} \Xi_3\,, \tag{4.28}$$

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where $\times_{\rm ad}$ denotes the projection onto the adjoint and again R is the radius of S^4. Evaluating its action in (4.27), we find that this is trivial for all the K_I's except for K_4, and we obtain our final expressions

$$K_0 = R^2\,\mathrm{vol}_\Sigma \wedge E_5'\,, \qquad K_{1,2,3} = \widetilde{\Xi}_{1,2,3}\,, \qquad K_4 = \Xi_3 - R\,\upsilon \wedge E_5\,. \tag{4.29}$$

A similar procedure applies to the singlets J_A, $A = 1,\ldots,8$, in the adjoint bundle. In this way we obtain (see appendix E for the derivation)

$$\begin{aligned}
J_1 &= \frac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(-R\,e_1\times_{\rm ad}\Xi_1 - R\,e_2\times_{\rm ad}\Xi_2 + R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_1 + R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_2\big),\\
J_2 &= \frac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(R\,e_1\times_{\rm ad}\Xi_2 - R\,e_2\times_{\rm ad}\Xi_1 - R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_1 + R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_3 &= \frac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(\hat{e}_1\otimes e_2 - \hat{e}_2\otimes e_1 - R\,e_2\times_{\rm ad}\Psi_{15} + R^{-1}\,\Psi_{15}^*\times_{\rm ad}\hat{e}_2\\
&\qquad\qquad - E^*_{5[1}\times_{\rm ad}E_{2]5} + E^*_{5[3}\times_{\rm ad}E_{4]5}\big)\,,\\
J_4 &= \frac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(R\,e_1\times_{\rm ad}\Xi_2 - R\,e_2\times_{\rm ad}\Xi_1 + R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_1 - R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_5 &= \frac{1}{2}\,\mathrm{e}^{\Upsilon}\cdot\big(R\,e_1\times_{\rm ad}\Xi_1 + R\,e_2\times_{\rm ad}\Xi_2 + R^{-1}\,\Xi_1^*\times_{\rm ad}\hat{e}_1 + R^{-1}\,\Xi_2^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_6 &= -\frac{1}{3}\,\mathrm{e}^{\Upsilon}\cdot\big(\hat{e}_1\otimes e_1 + \hat{e}_2\otimes e_2 + \sum_{i=1}^{4}E_{i5}^*\times_{\rm ad}E_{i5} + 2\big)\,,\\
J_7 &= \mathrm{e}^{\Upsilon}\cdot\big(R\,e_2\times_{\rm ad}\Psi_{15} + R^{-1}\,\Psi_{15}^*\times_{\rm ad}\hat{e}_2\big)\,,\\
J_8 &= \frac{1}{2\sqrt{3}}\,\mathrm{e}^{\Upsilon}\cdot\big(\hat{e}_1\otimes e_2 - \hat{e}_2\otimes e_1 - 3R\,e_2\times_{\rm ad}\Psi_{15} + 3R^{-1}\,\Psi_{15}^*\times_{\rm ad}\hat{e}_2\\
&\qquad\qquad - E^*_{5[1}\times_{\rm ad}E_{2]5} + E^*_{5[3}\times_{\rm ad}E_{4]5}\big)\,,
\end{aligned} \tag{4.30}$$

where the superscript $*$ denotes dual generalised vectors, transforming in the $\overline{\mathbf{27}}$, and we introduced $\Psi_{1i} = R\,e_1\wedge E_i$ and $\Psi_{2i} = R\,e_2\wedge E_i$. The adjoint action of e^{Υ} is evaluated using the formula (A.21); we do not show the resulting expressions as they are rather lengthy. Evaluating the commutators $[J_A, J_B]$ using again (A.21), we checked that the J_A satisfy precisely the $SU(2,1)$ commutation relations (see (F.4) for our choice of $SU(2,1)$ structure constants).

4.3 The V and H structure moduli spaces

We now construct the V structure and H structure moduli spaces. Applying the general discussion of section 2.3 we have

$$\mathcal{M}_{\rm V}\times\mathcal{M}_{\rm H} = \frac{\mathrm{C}_{E_{6(6)}}(G_S)}{\mathrm{C}_{USp(8)/\mathbb{Z}_2}(G_S)} = \mathbb{R}^+\times\frac{Spin(3,1)}{SU(2)}\times\frac{SU(2,1)}{SU(2)_H\times U(1)}\,, \tag{4.31}$$

As we now show the first two factors give the V structure moduli space and the last factor the H structure moduli space.

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The V structure. Evaluating (2.16) for the K_I constructed above we obtain the constant, symmetric tensor C_{IJK}. Using the invariant volume (4.6), we find that the non-vanishing components of C_{IJK} are given by

$$C_{0IJ} = C_{I0J} = C_{IJ0} = \frac{1}{3}\,\eta_{IJ}\,, \qquad \text{for } I,J = 1,\ldots,4\,, \tag{4.32}$$

where

$$\eta = \mathrm{diag}(-1,-1,-1,1)\,. \tag{4.33}$$

A family of V structures is then obtained by defining K as the linear combination (2.30) and imposing the condition (2.31). It follows that our V structure moduli space is the hypersurface

$$C_{IJK}h^I h^J h^K \;=\; h^0\left(-(h^1)^2-(h^2)^2-(h^3)^2+(h^4)^2\right) = 1\,. \tag{4.34}$$

It will be convenient to redefine the h^I in terms of the parameters

$$\{\Sigma, H^1, H^2, H^3, H^4\} \tag{4.35}$$

as

$$\begin{aligned} h^0 &= \Sigma^{-2}\,,\\ h^I &= -\Sigma\, H^I\,, \quad I = 1,\ldots,4\,, \end{aligned} \tag{4.36}$$

so that

$$K = \Sigma^{-2}K_0 - \Sigma\left(H^1K_1 + H^2K_2 + H^3K_3 + H^4K_4\right)\,. \tag{4.37}$$

From (4.34) we see that H^I are coordinates on the unit hyperboloid $\frac{SO(3,1)}{SO(3)}$,

$$-(H^1)^2-(H^2)^2-(H^3)^2+(H^4)^2 = 1\,, \tag{4.38}$$

while Σ (that we assume strictly positive) is a coordinate on $\mathbb{R}^+$, whose powers in (4.36) are dictated by the weight of the K_I's under the action of the $\mathbb{R}^+$ that commutes with the generalised structure. The resulting V structure moduli space thus is

$$\mathcal{M}_{\rm V} = \mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)}\,, \tag{4.39}$$

and will determine $n_{\rm V} = 4$ vector multiplets in five-dimensional $\mathcal{N} = 2$ supergravity. Note that by identifying $SU(2) \simeq Spin(3)$ this matches the first two factors in (4.31). The isometry group is $SO(3,1)$ because the h^I form a vector rather than a spinor representation of $Spin(3,1)$.

The H structure. We next turn to the H structure moduli space, again following the general discussion given in section 2.3. Since the commutant of $SU(2)_H$ in $SU(2,1)$ is $U(1)$, from (2.35) we obtain that the H structure moduli space is[18]

$$\mathcal{M}_{\rm H} = \frac{SU(2,1)}{SU(2)_H \times U(1)}\,. \tag{4.40}$$

[18]More precisely one has $\mathcal{M}_{\rm H} = SU(2,1)/S(U(2)\times U(1))$.

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This is a simple quaternionic-Kähler manifold of quaternionic dimension $n_{\rm H} = 1$. We will denote by

$$\{\varphi, \xi, \theta_1, \theta_2\} \tag{4.41}$$

the coordinates on this space. In appendix F we give the explicit parameterisation chosen for the coset space as well as the explicit form of the "dressed" $\mathfrak{su}(2)$ elements J_α, depending on $\{\varphi, \xi, \theta_1, \theta_2\}$, in terms of $\mathfrak{su}(2,1)$ elements. Below we will use this dressed triplet to construct the generalised metric. In appendix F we also give the $SU(2,1)$ invariant metric on $\mathcal{M}_H$, which will provide the hyperscalar kinetic term in the five-dimensional theory.

4.4 Intrinsic torsion and gauging

For the $U(1)_S$ structure constructed in the previous section to lead to a consistent truncation, it must be checked that its intrinsic torsion only contains $U(1)_S$ singlets, and that these are constant. In particular we need to show that equations (3.4), (3.5) and (3.7) hold. For the first two conditions we evaluate the generalised Lie derivatives of the tensors K_I and J_A in (4.29) and (4.30), using the action of generalised Lie derivative on a generalised vector and on sections of the adjoint bundle given in appendix A.

Consider first the algebra of the generalised vectors (4.29). Using the fact that, under the generalised Lie derivative, the S^4 frames E_{ij} generate an $\mathfrak{so}(5)$ algebra

$$L_{E_{ij}} E_{kl} = -R^{-1} \left(\delta_{ik} E_{jl} - \delta_{il} E_{jk} + \delta_{jl} E_{ik} - \delta_{jk} E_{il}\right) , \tag{4.42}$$

one can show that the only non-zero Lie derivatives are

$$L_{K_\alpha} K_\beta = -\frac{2}{R}\, \epsilon_{\alpha\beta\gamma} K_\gamma \,, \qquad\qquad \alpha, \beta, \gamma = 1, 2, 3 \,, \tag{4.43}$$

so that the vectors K_α, $\alpha = 1, 2, 3$, lead to an $SO(3)$ factor in the gauge group in the five-dimensional supergravity.[19] This embeds in the $SO(3,1)$ factor of the global symmetry group of the ungauged theory in the obvious way. Hence (4.43) determines the components of the embedding tensor acting on the vector multiplet sector of the five-dimensional supergravity theory.

We thus have that the non-vanishing structure constants are $f_{\alpha\beta\gamma} = -2\,\epsilon_{\alpha\beta\gamma}$ and the gauge coupling constant is $g = \frac{1}{R}$. Recalling (3.8), the non-trivial vector multiplet scalar covariant derivatives are

$$\mathcal{D} H^\alpha = \mathrm{d} H^\alpha - \frac{2}{R}\, \epsilon^\alpha{}_{\beta\gamma} \mathcal{A}^\beta\, H^\gamma \,, \tag{4.44}$$

while the gauge field strengths read

$$\mathcal{F}^0 = \mathrm{d}\mathcal{A}^0 \,, \qquad \mathcal{F}^\alpha = \mathrm{d}\mathcal{A}^\alpha - \frac{1}{R}\, \epsilon^\alpha{}_{\beta\gamma} \mathcal{A}^\beta \wedge \mathcal{A}^\gamma \,, \qquad \mathcal{F}^4 = \mathrm{d}\mathcal{A}^4 \,. \tag{4.45}$$

In order to determine the gauging in the hypersector we also need to compute the Lie derivative of the J_A along the generalised vectors K_I. We find that the J_A are neutral under the action of the $SO(3)$ generators K_α,

$$L_{K_\alpha} J_A = 0 \,, \qquad A = 1, \ldots 8 \,, \tag{4.46}$$

[19] For simplicity, we use the indices $\alpha, \beta = 1, 2, 3$ both to label the generators of the $SU(2)_H$ entering in the definition of the H structure and the generators of the $SU(2)$ in the V structure, although these are different subgroups of $E_{6(6)}$.

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consistently with the fact that the gauging in the vector multiplet sector does not affect the hypersector. On the other hand, the remaining generalised vectors K_0 and K_4 act non-trivially on the J_A, and determine an abelian gauging of the $SU(2,1)$ generators. In detail, the generalised Lie derivative of the J_A along K_0 gives

$$\begin{aligned}
&L_{K_0}(J_1 - J_5) = 0\,, && L_{K_0} J_3 = -\frac{1}{2R} J_6\,,\\
&L_{K_0}(J_1 + J_5) = \frac{1}{R}(J_2 + J_4)\,, && L_{K_0} J_6 = -\frac{1}{2R}(J_3 + 2J_7 - \sqrt{3}J_8)\,,\\
&L_{K_0}(J_2 + J_4) = 0\,, && L_{K_0} J_7 = \frac{1}{R} J_6\,,\\
&L_{K_0}(J_2 - J_4) = -\frac{1}{R}(J_1 - J_5)\,, && L_{K_0} J_8 = \frac{\sqrt{3}}{2R} J_6\,,
\end{aligned} \tag{4.47}$$

while the one along K_4 yields

$$\begin{aligned}
&L_{K_4}(J_1 - J_5) = -\frac{2}{R}(J_2 + J_4)\,, && L_{K_4} J_3 = \frac{\kappa}{2R} J_6\,,\\
&L_{K_4}(J_1 + J_5) = -\frac{2}{R}(J_2 - J_4) - \frac{\kappa}{R}(J_2 + J_4)\,, && L_{K_4} J_6 = \frac{\kappa}{2R}(J_3 + 2J_7 - \sqrt{3}J_8)\,,\\
&L_{K_4}(J_2 + J_4) = \frac{2}{R}(J_1 - J_5)\,, && L_{K_4} J_7 = -\frac{\kappa}{R} J_6\,,\\
&L_{K_4}(J_2 - J_4) = \frac{2}{R}(J_1 + J_5) + \frac{\kappa}{R}(J_1 - J_5)\,, && L_{K_4} J_8 = -\frac{\sqrt{3}\,\kappa}{2R} J_6\,.
\end{aligned} \tag{4.48}$$

The actions (4.47) and (4.48) can equivalently be expressed in terms of an adjoint action as

$$L_{K_0} J_A = [J_{(K_0)}, J_A]\,, \qquad L_{K_4} J_A = [J_{(K_4)}, J_A]\,, \qquad A = 1, \ldots, 8\,, \tag{4.49}$$

where the sections of the adjoint bundle

$$\begin{aligned}
J_{(K_0)} &= \frac{1}{4R}\left(J_3 + 2J_7 - \sqrt{3}J_8\right)\,,\\
J_{(K_4)} &= -\frac{\kappa}{4R}\left(J_3 + 2J_7 - \sqrt{3}J_8\right) - \frac{1}{R}\left(J_3 + \frac{1}{\sqrt{3}}J_8\right)
\end{aligned} \tag{4.50}$$

correspond to $SU(2,1)$ generators acting on the H-structure moduli space (4.40) as isometries. We denote by k_0 and k_4 the corresponding Killing vectors on $\mathcal{M}_{\rm H}$. These can be calculated applying (3.10) to the coset representative L given in appendix F, and read

$$\begin{aligned}
k_0 &= \partial_\xi\,,\\
k_4 &= -\kappa\,\partial_\xi + 2\left(\theta_2 \partial_{\theta_1} - \theta_1 \partial_{\theta_2}\right)\,.
\end{aligned} \tag{4.51}$$

These Killing vectors specify the isometries of $\mathcal{M}_{\rm H}$ that are gauged in the five-dimensional supergravity. The hyperscalar covariant derivatives (3.11) are determined as

$$\begin{aligned}
\mathcal{D}(\theta_1 + \mathrm{i}\,\theta_2) &= \mathrm{d}(\theta_1 + \mathrm{i}\,\theta_2) - \frac{2}{R}\,\mathrm{i}\,\mathcal{A}^4\,(\theta_1 + \mathrm{i}\,\theta_2)\,,\\
\mathcal{D}\xi &= \mathrm{d}\xi + \frac{1}{R}\mathcal{A}^0 - \frac{\kappa}{R}\mathcal{A}^4\,.
\end{aligned} \tag{4.52}$$

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The triholomorphic Killing prepotentials P_I^α obtained by evaluating the moment maps (3.12) read

$$\begin{aligned} P_0^\alpha &= \left\{0\,,\,0\,,\,\frac{1}{4}\,\mathrm{e}^{2\varphi}\right\}\,, \\ P_4^\alpha &= \left\{\sqrt{2}\,\mathrm{e}^{\varphi}\theta_1\,,\,\sqrt{2}\,\mathrm{e}^{\varphi}\theta_2\,,-1+\frac{1}{4}\,\mathrm{e}^{2\varphi}(2\theta_1^2+2\theta_2^2-\kappa)\right\}\,, \end{aligned} \tag{4.53}$$

with $P_1^\alpha = P_2^\alpha = P_3^\alpha = 0$.

The information above completely characterises the five-dimensional $\mathcal{N}=2$ supergravity obtained upon reduction on M. This will be discussed in section 4.6. Before coming to that, we provide the explicit bosonic truncation ansatz.

4.5 The truncation ansatz

The truncation ansatz is built following the general procedure described in section 3.2. We compute the inverse generalised metric (2.27) out of the $U(1)_S$ invariant generalised tensors. This depends on the V structure moduli $\{\Sigma, H^1, H^2, H^3, H^4\}$ and on the H structure moduli $\{\varphi, \xi, \theta_1, \theta_2\}$, which are now promoted to scalar fields in the external, five-dimensional spacetime. Then we evaluate the generalised tensors $\mathcal{A}_\mu, \mathcal{B}_{\mu\nu}, \mathcal{C}_{\mu\nu\rho}$ using (3.25)–(3.27). Separating the components of these tensors as described in section 3.2, we obtain the ansatz for the eleven-dimensional metric $\hat{g}$ and three-form potential $\hat{A}$.

We start from the covariantised differentials $Dz^m = \mathrm{d}z^m - h_\mu{}^m \mathrm{d}x^\mu$ of the coordinates on M, that appear in (3.16). From (3.21) and (3.25) we see that $h_\mu = h_\mu{}^m \partial_m$ is given by

$$h_\mu = \mathcal{A}_\mu^I\, K_I|_{TM}\,, \tag{4.54}$$

where $K_I|_{TM}$ is the restriction of K_I to the tangent bundle of M. Evaluating the right hand side using the explicit form (4.29) of the generalised vectors K_I, we obtain

$$h_\mu = \frac{2}{R}\,(\mathcal{A}_\mu^\alpha\,\tilde{\xi}_\alpha + \mathcal{A}_\mu^4\,\xi_3)\,, \tag{4.55}$$

where we recall that ξ_α, $\tilde{\xi}_\alpha$, $\alpha = 1,2,3$, are the pull-back to TM of the $SU(2)_{\text{left}}$- and $SU(2)_{\text{right}}$-invariant vectors on S^3, respectively, whose coordinate expression is given in (D.2) and (D.3). It follows that Dz^m, and thus both the eleven-dimensional metric and three-form, contain the five-dimensional gauge potentials $\mathcal{A}^\alpha$, $\mathcal{A}^4$, gauging the $SU(2)_{\text{left}} \times U(1)_{\text{right}}$ isometries of S^3 in M. Notice that $\mathcal{A}^0$ does not appear in (4.55) as K_0 does not have a component in TM, hence it will not enter in the eleven-dimensional metric. However K_0 will appear in the ansatz for the three-form, as it does have a component in $\Lambda^2 T^*M$.

In order to express our ansatz in a more compact way, it will be convenient to introduce new one-forms Ω_α and $\widetilde{\Omega}_\alpha$, $\alpha = 1,2,3$, adapted to the symmetries of the problem, that incorporate the covariantised differentials above but also include some more terms. Recall that we describe S^4 as a foliation of S^3, parameterised by Euler angles $\{\theta, \phi, \psi\}$, over an interval, parameterised by ζ. We define

$$\Omega_1 = \cos\psi\,\mathsf{D}\theta + \sin\psi\sin\theta\,\mathsf{D}\phi\,, \qquad\qquad \widetilde{\Omega}_1 = \cos\phi\,\mathsf{D}\theta + \sin\phi\sin\theta\,\mathsf{D}\psi\,,$$

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$$\Omega_2 = -\sin\psi\,\mathsf{D}\theta + \cos\psi\sin\theta\,\mathsf{D}\phi\,, \qquad \widetilde{\Omega}_2 = \sin\phi\,\mathsf{D}\theta - \cos\phi\sin\theta\,\mathsf{D}\psi\,,$$
$$\Omega_3 = \mathsf{D}\psi + \cos\theta\,\mathsf{D}\phi\,, \qquad \widetilde{\Omega}_3 = \mathsf{D}\phi + \cos\theta\,\mathsf{D}\psi\,, \tag{4.56}$$

which are analogous to the left- and right-invariant forms $\sigma_\alpha, \tilde{\sigma}_\alpha$ given in (D.6) and (D.7), but with the ordinary differential of the coordinates being replaced by the new covariantised differential D. This extends the differential D given above and is defined as

$$\mathsf{D}z^m = \mathrm{d}z^m - \frac{2}{R}(\mathsf{A}^\alpha \xi^m_\alpha + \tilde{\mathsf{A}}^\alpha \tilde{\xi}^m_\alpha)\,, \tag{4.57}$$

with

$$\mathsf{A}^1 = \frac{R}{\sqrt{2}}\,(\theta_2 e_1 - \theta_1 e_2)\,, \qquad \mathsf{A}^2 = \frac{R}{\sqrt{2}}\,(\theta_1 e_1 + \theta_2 e_2)\,, \qquad \mathsf{A}^3 = -\frac{R}{2}\,\upsilon + \mathcal{A}^4\,,$$
$$\tilde{\mathsf{A}}^\alpha = \mathcal{A}^\alpha\,, \qquad \alpha = 1,2,3\,, \tag{4.58}$$

where the five-dimensional scalars θ_1, θ_2 are two of the H structure moduli, and we recall that e_1, e_2 are the vielbeine on the Riemann surface $\mathbb{\Sigma}$ while υ is the connection on $\mathbb{\Sigma}$. The local one-forms $\tilde{\mathsf{A}}^\alpha, \mathsf{A}^\alpha$ gauge all the left- and right- isometries of S^3, respectively, and would correspond to $SO(4) \simeq SU(2)_{\text{left}} \times SU(2)_{\text{right}}$ gauge potentials in the reduction of eleven-dimensional supergravity on S^4 down to seven-dimensional supergravity. However, in the further reduction on $\mathbb{\Sigma}$ of interest here only $\mathcal{A}^\alpha, \mathcal{A}^4$ become five-dimensional gauge fields, while the rest of (4.58) implements the twist on the Riemann surface and introduces the five-dimensional scalars θ_1, θ_2.

We are now in the position to give the truncation ansatz for the eleven-dimensional metric

$$\hat{g} = \mathrm{e}^{2\Delta} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dz^m Dz^n\,. \tag{4.59}$$

The warp factor is

$$\mathrm{e}^{2\Delta} \;=\; \bar{\Delta}^{1/3}\,(\mathrm{e}^{\varphi}\Sigma)^{4/5}\;, \tag{4.60}$$

while the part with at least one internal leg reads

$$g_{mn} Dz^m Dz^n = R^2 \bar{\Delta}^{1/3} (\mathrm{e}^{\varphi}\Sigma)^{-6/5} g_{\mathbb{\Sigma}} + R^2 \bar{\Delta}^{-2/3} \mathrm{e}^{2\varphi/5} \Sigma^{-3/5} \Big[\Big(\mathrm{e}^{-2\varphi}\Sigma^3 \sin^2\zeta + H_- \cos^2\zeta \Big) \mathrm{d}\zeta^2$$
$$+ \frac{1}{4} H_+ \sin^2\zeta\, \delta^{\alpha\beta} \Omega_\alpha \otimes \Omega_\beta - \frac{1}{2} \sin^2\zeta\, H^\alpha\, \widetilde{\Omega}_\alpha \otimes_s \Omega_3 - \cos\zeta \sin\zeta\, \mathrm{d}\zeta \otimes_s \mathrm{d}_6 H_+ \Big]\,, \tag{4.61}$$

where $\otimes_s$ is the symmetrised tensor product, defined as $\Omega \otimes_s \widetilde{\Omega} = \frac{1}{2}(\Omega \otimes \widetilde{\Omega} + \widetilde{\Omega} \otimes \Omega)$. In these expressions we introduced the function

$$\bar{\Delta} = (\mathrm{e}^{-2\varphi}\Sigma^3)^{-4/5} \cos^2\zeta + (\mathrm{e}^{-2\varphi}\Sigma^3)^{1/5} H_+ \sin^2\zeta\,, \tag{4.62}$$

as well as

$$H_\pm = H^4 \pm \Big(H^1 \sin\theta\sin\phi - H^2 \sin\theta\cos\phi + H^3 \cos\theta \Big)\,,$$
$$\mathrm{d}_6 H_+ = H^1\, \mathrm{d}(\sin\theta\sin\phi) - H^2\, \mathrm{d}(\sin\theta\cos\phi) + H^3\, \mathrm{d}\cos\theta\,. \tag{4.63}$$

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Note that in the last expression the exterior derivative acts on the internal coordinates and not on the scalars H^I, which only depend on the external coordinates.

We next come to the eleven-dimensional three-form potential $\hat{A}$. We first give our result and then make some comments. The ansatz for $\hat{A}$ reads

$$\begin{aligned}\hat{A} = &-\frac{1}{8}R^3\cos\zeta\,[2+\sin^2\zeta\,\bar{\Delta}^{-1}(\mathrm{e}^{-2\varphi}\Sigma^3)^{-4/5}\,]\,\Omega_1\wedge\Omega_2\wedge\Omega_3\\
&+\frac{1}{4}R^3\sin^3\zeta\,\bar{\Delta}^{-1}(\mathrm{e}^{-2\varphi}\Sigma^3)^{1/5}\,\mathrm{d}\zeta\wedge H^\alpha\,\widetilde{\Omega}_\alpha\wedge\Omega_3\\
&+R^3\,\cos\zeta\,(\mathcal{D}\xi-\theta_1\mathcal{D}\theta_2+\theta_2\mathcal{D}\theta_1)\wedge\mathrm{vol}_\Sigma+\frac{1}{4}R^3\cos\zeta\left(2\theta_1^2+2\theta_2^2-\kappa\right)\mathrm{vol}_\Sigma\wedge\Omega_3\\
&+\frac{1}{2}R^2\cos\zeta\,(\mathcal{F}^4\wedge\Omega_3-\mathcal{F}^\alpha\wedge\widetilde{\Omega}_\alpha)+R\,\cos\zeta\,\Sigma^4\ast_5\mathcal{F}^0\\
&+\frac{1}{2\sqrt{2}}\,R^3\cos\zeta\left[\left(\mathcal{D}\theta_2\wedge e^1-\mathcal{D}\theta_1\wedge e^2\right)\wedge\Omega_1+\left(\mathcal{D}\theta_1\wedge e^1+\mathcal{D}\theta_2\wedge e^2\right)\wedge\Omega_2\right]\,,\end{aligned}\tag{4.64}$$

where the five-dimensional gauge field strengths, $\mathcal{F}$, and the covariant derivatives, $\mathcal{D}$, of the five-dimensional scalars were given in (4.45) and (4.52), respectively.

Equation (4.64) has been obtained by first computing $\hat{A}$ through the general procedure of section 3.2, then implementing a gauge shift by an exact three-form so as to obtain a nicer expression (this is why derivatives of the external fields appear), and finally dualising away the five-dimensional two- and three-forms, so that the only five-dimensional degrees of freedom contained in the ansatz are scalar and vector fields, in addition to the metric $g_{\mu\nu}$. Let us outline how this dualisation is performed. Evaluating (3.26) and (3.27), we find that only one external two-form $\mathcal{B}$ and one external three-form $\mathcal{C}$ appear in the ansatz for $\hat{A}$. These are paired up with the generalised tensors E_5 and E_5' on S^4, which, as generalised tensors on M, are sections of $\det T^*M\otimes E^*$ and $\det T^*M\otimes\mathrm{ad}\,F$, respectively. The combination entering in $\hat{A}$ is

$$[\mathcal{B}E_5+\mathcal{C}E_5']_3=R\,\mathcal{B}\wedge\mathrm{d}\cos\zeta+R\,\mathcal{C}\cos\zeta=(\mathcal{C}-\mathrm{d}\mathcal{B})\,R\cos\zeta+\mathrm{d}\,(\mathcal{B}\,R\cos\zeta)\,,\tag{4.65}$$

where the subscript on the left-hand side indicates the restriction to the three-form part, and the last term in the expression is removable via a gauge transformation of $\hat{A}$. Hence $\mathcal{B}$ and $\mathcal{C}$ only appear in the combination $(\mathcal{C}-\mathrm{d}\mathcal{B})$. This means that the two-form gets eaten by the three-form via the Stuckelberg mechanism, giving it a mass. While a massless three-form in five-dimensions is dual to a scalar field, here we dualise the two-form at the same time and also obtain a vector field. The duality relation is obtained considering the duality between the eleven-dimensional three-form $\hat{A}$ and six-form $\hat{\tilde{A}}$ given in (3.15), and looking at the relevant terms with three external indices. In this way we find that

$$\mathcal{C}-\mathrm{d}\mathcal{B}=\Sigma^4\ast_5\mathrm{d}\mathcal{A}^0-\mathcal{A}^4\wedge\mathrm{d}\mathcal{A}^4+\mathcal{A}^\alpha\wedge\mathrm{d}\mathcal{A}^\alpha-\frac{1}{3R}\,\epsilon_{\alpha\beta\gamma}\mathcal{A}^\alpha\wedge\mathcal{A}^\beta\wedge\mathcal{A}^\gamma\,.\tag{4.66}$$

We have used this expression to eliminate $(\mathcal{C}-\mathrm{d}\mathcal{B})$ completely from the truncation ansatz. This explains the $\ast_5\mathcal{F}^0$ term appearing in (4.64).

Our truncation ansatz reproduces the Maldacena-Nuñez solution given in section 4.1 upon taking $\kappa=-1$ and setting the scalars to

$$H^1=H^2=H^3=\theta_1=\theta_2=\xi=0\,,\quad H^4=\Sigma=1\,,\quad\varphi=\frac{1}{2}\log\frac{4}{3}\,.\tag{4.67}$$

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The consistent truncation of [36] is recovered as a subtruncation that projects out the fields transforming under $SU(2)_{\text{left}}$, that is setting $\tilde{\mathcal{A}}^\alpha = H^\alpha = 0$, $\alpha = 1,2,3$, which also implies $H^4 = 1$.[20] The further truncation to minimal gauged supergravity is obtained by setting the scalars to their AdS value (4.67) and taking $\mathcal{A}^0 = -\mathcal{A}^4$.

One can obtain a slightly larger subtruncation by projecting out only the modes charged under $U(1)_{\text{left}}$, rather than $SU(2)_{\text{left}}$, namely setting $\tilde{\mathcal{A}}^1 = \tilde{\mathcal{A}}^2 = H^1 = H^2 = 0$. This leaves us with two vector multiplets, one hypermultiplet and just the abelian gauging generated by the Killing vectors (4.51), which is the same as the one in the truncation of [36].[21] A notable generalisation of this subtruncation will be discussed in section 5.

The truncation of [36] was obtained via a reduction of gauged seven-dimensional supergravity on the Riemann surface Σ. Similarly, we can obtain our truncation ansatz by combining the well-known truncation of eleven-dimensional supergravity on S^4 [9], leading to seven-dimensional maximal $SO(5)$ supergravity, with a further truncation reducing the seven-dimensional theory on Σ. Starting from the convenient form of the bosonic truncation ansatz on S^4 given in [56], we have explicitly checked that this procedure works out as expected and reproduces the ansatz above.

4.6 The five-dimensional theory

We now put together the ingredients defining the truncated five-dimensional theory and discuss it in more detail. This is an $\mathcal{N} = 2$ gauged supergravity coupled to four vector multiplets and one hypermultiplet. The vector multiplet scalar manifold is

$$\mathcal{M}_{\text{V}} = \mathbb{R}^+ \times \frac{SO(3,1)}{SO(3)}\,, \tag{4.68}$$

while the hypermultiplet scalar manifold is

$$\mathcal{M}_{\text{H}} = \frac{SU(2,1)}{SU(2)_H \times U(1)}\,. \tag{4.69}$$

As discussed before, these have a geometric origin as the V and H structure moduli spaces of the internal manifold. At the bosonic level, the vector multiplets are made of gauge fields $\mathcal{A}^I$ and constrained scalar fields h^I, $I = 0,1,\ldots,4$, which we have parameterised in terms of Σ and H^I, $I = 1,\ldots,4$, in (4.36). The latter scalars satisfy the constraint $\eta_{IJ}H^I H^J = 1$, with $\eta = \text{diag}(-1,-1,-1,1)$. We have also found that the non-vanishing components of the symmetric tensor C_{IJK} are given by

$$C_{0IJ} = C_{I0J} = C_{IJ0} = \frac{1}{3}\,\eta_{IJ}\,, \qquad I,J = 1,\ldots,4\,. \tag{4.70}$$

[20] Then the one-forms Ω_α essentially reduce to those in [36], up to slightly different conventions, while $\widetilde{\Omega}_\alpha$ drop out of the ansatz. When comparing our truncation ansatz with the one given in section 4.1 of [36], one should take into account that $\hat{A}^{\text{here}} = -\hat{A}^{\text{FNR}}$ (this is seen from comparing our 11d Maxwell equation with the one in [54], which provides the 7d to 11d uplift formulae used in [36]). Moreover $\zeta^{\text{here}} = \zeta^{\text{FNR}} + \pi/2$, $\mathcal{A}^4 \propto \mathcal{A}^{\text{FNR}}$, $\mathcal{A}^0 \propto \chi_1^{\text{FNR}}$, $\Sigma = 2^{1/3}\Sigma^{\text{FNR}}$, $\mathrm{e}^{2\varphi} = 2(\mathrm{e}^{2\varphi})^{\text{FNR}}$, $|\theta_{1,2}| = \frac{1}{\sqrt{2}}|\theta_{1,2}|^{\text{FNR}}$, $\xi = \frac{1}{2}\xi^{\text{FNR}}$.

[21] Curiously, this five-dimensional supergravity with two vector multiplets and one hypermultiplet looks closely related to the $\mathcal{N} = 2$ "Betti-vector" model obtained in [55, Section 7] as a consistent truncation of IIB supergravity on $T^{1,1}$. The two models are not the same though, as the details of the couplings between the fields are different.

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The kinetic terms in the vector multiplet sector are controlled by the matrix a_{IJ}, given by the general formula (2.34), which in our case reads

$$\begin{aligned} a_{00} &= \frac{1}{3}\,\Sigma^4\,, \\ a_{0J} &= 0\,, \\ a_{IJ} &= \frac{2}{3}\,\Sigma^{-2}\left(2\eta_{IK}H^K\eta_{JL}H^L - \eta_{IJ}\right)\,, \qquad I,J=1,\dots,4\,. \end{aligned} \tag{4.71}$$

The hypermultiplet comprises the scalars $q^X = \{\varphi, \xi, \theta_1, \theta_2\}$, and the kinetic term is given by the quaternionic-Kähler metric on $\mathcal{M}_{\rm H}$ that we derived in appendix F,

$$g_{XY}\mathrm{d}q^X\mathrm{d}q^Y = 2\,\mathrm{d}\varphi^2 + \mathrm{e}^{2\varphi}\left(\mathrm{d}\theta_1^2 + \mathrm{d}\theta_2^2\right) + \frac{1}{2}\,\mathrm{e}^{4\varphi}\left(\mathrm{d}\xi - \theta_1\mathrm{d}\theta_2 + \theta_2\mathrm{d}\theta_1\right)^2\,. \tag{4.72}$$

The gauge group is $SO(3)\times U(1)\times\mathbb{R}$. The symmetries being gauged are the $SO(3)\subset SO(3,1)$ isometries of $\mathcal{M}_{\rm V}$ and two abelian isometries in $\mathcal{M}_{\rm H}$, generated by the Killing vectors (4.51). Note that the ∂_ξ term generates the non-compact $\mathbb{R}$ factor and the $\theta_2\partial_{\theta_1} - \theta_1\partial_{\theta_2}$ term generates the compact $U(1)$.

We recall for convenience the gauge field strengths

$$\mathcal{F}^0 = \mathrm{d}\mathcal{A}^0\,, \qquad \mathcal{F}^\alpha = \mathrm{d}\mathcal{A}^\alpha - g\,\epsilon^\alpha{}_{\beta\gamma}\mathcal{A}^\beta\wedge\mathcal{A}^\gamma\,, \qquad \mathcal{F}^4 = \mathrm{d}\mathcal{A}^4\,, \quad \alpha=1,2,3\,, \tag{4.73}$$

and the covariant derivatives of the charged scalars,

$$\begin{aligned} \mathcal{D}H^\alpha &= \mathrm{d}H^\alpha - \frac{2}{R}\,\epsilon^\alpha{}_{\beta\gamma}\mathcal{A}^\beta\,H^\gamma\,, \\ \mathcal{D}(\theta_1 + \mathrm{i}\,\theta_2) &= \mathrm{d}(\theta_1 + \mathrm{i}\,\theta_2) - \frac{2}{R}\,\mathrm{i}\,\mathcal{A}^4\,(\theta_1 + \mathrm{i}\,\theta_2)\,, \\ \mathcal{D}\xi &= \mathrm{d}\xi + \frac{1}{R}\mathcal{A}^0 - \frac{\kappa}{R}\,\mathcal{A}^4\,, \end{aligned} \tag{4.74}$$

where the gauge coupling constant is given by the inverse S^4 radius, $g = \frac{1}{R}$. The scalars Σ, H^4 and φ remain uncharged. The gauging in the hypersector is the same as in [36], while the gauging in the vector multiplet sector is a novel feature of our truncation.

Plugging these data in the general form of the $\mathcal{N}=2$ supergravity action given in appendix B, we obtain the bosonic action for our model,

$$\begin{aligned} S = \frac{1}{16\pi G_5}\int\Bigg[&(\mathcal{R} - 2\mathcal{V})*1 - \frac{1}{2}\,\Sigma^4\mathcal{F}^0\wedge*\mathcal{F}^0 - \frac{3}{2}\sum_{I,J=1}^{4} a_{IJ}\mathcal{F}^I\wedge*\mathcal{F}^J - 2\Sigma^{-2}\mathrm{d}\Sigma\wedge*\mathrm{d}\Sigma \\ &- \frac{3}{2}\sum_{I,J=1}^{4} a_{IJ}\mathcal{D}(\Sigma H^I)\wedge*\mathcal{D}(\Sigma H^J) - g_{XY}\mathcal{D}q^X\wedge*\mathcal{D}q^Y + \sum_{I,J=1}^{4}\eta_{IJ}\mathcal{A}^0\wedge\mathcal{F}^I\wedge\mathcal{F}^J\Bigg]\,, \end{aligned} \tag{4.75}$$

where G_5 is the five-dimensional Newton constant.[22] The scalar potential $\mathcal{V}$ is obtained from the Killing prepotentials of the gauged isometries as summarised in appendix B. The

[22] As discussed in [33], the five-dimensional Newton constant is given by $(G_5)^{-1} \propto \int_M \mathrm{e}^{3\Delta}\,\mathrm{vol}_6 = \int_M \boldsymbol{\kappa}^2$. In the present case, $\int_M \boldsymbol{\kappa}^2 = R^2\,\mathrm{Vol}_\Sigma\,\mathrm{Vol}_4$, where $\mathrm{Vol}_\Sigma = \frac{4\pi(1-g)}{\kappa}$ is the standard volume of a Riemann surface of genus g and $\mathrm{Vol}_4 = \frac{8\pi^3}{3}R^4$ is the volume of a round S^4 with radius R.

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Killing prepotentials were already given in (4.53). We can check this expression by starting from the Killing vectors (4.51) and evaluating (B.15) using a standard parameterisation for the universal hypermultiplet; we have verified that indeed the same result is obtained. Then (B.17) gives for the scalar potential

$$\mathcal{V} = \frac{1}{R^2}\left\{\frac{\mathrm{e}^{4\varphi}}{4\Sigma^4} - \frac{2\,H^4\,\mathrm{e}^{2\varphi}}{\Sigma} + \Sigma^2\Big[-2 + \mathrm{e}^{2\varphi}\left(2((H^4)^2-1)(\theta_1^2+\theta_2^2) - \kappa\right) + \frac{1}{8}\,\mathrm{e}^{4\varphi}(2(H^4)^2-1)(2\theta_1^2+2\theta_2^2-\kappa)^2\Big]\right\}. \tag{4.76}$$

The supersymmetric AdS vacuum conditions summarised in eq. (B.18) are easily solved and give the scalar field values

$$H^1 = H^2 = H^3 = \theta_1 = \theta_2 = 0\,, \quad H^4 = \Sigma = 1\,, \quad \varphi = \frac{1}{2}\log\frac{4}{3}\,, \tag{4.77}$$

that is precisely the values (4.67) that reproduce the MN1 solution reviewed in section 4.1. The negative curvature $\kappa = -1$ for the Riemann surface arises as a positivity condition for the scalars Σ and $\mathrm{e}^{2\varphi}$. The critical value of the scalar potential yields the cosmological constant $\Lambda \equiv \mathcal{V} = -\frac{8}{3R^2}$, corresponding to an AdS_5 radius $\ell = \frac{3}{2}R$, again in harmony with the solution in section 4.1.

By extremising the scalar potential (4.76) we can search for further AdS_5 vacua within our truncation. Then, by analysing the mass matrix of the scalar field fluctuations around the extrema we can test their perturbative stability. In the following we discuss the outcome of this analysis for the three extrema that we have found.

- We recover the supersymmetric vacuum (4.77). Being supersymmetric, this is stable. The supergravity field fluctuations source $SU(2,2|1)$ superconformal multiplets in the dual $\mathcal{N}=1$ SCFT [53], with the supergravity mass eigenvalues providing the conformal dimension Δ of the operators in the multiplets. The field fluctuations that were also considered in [36] correspond to the energy-momentum multiplet (containing the energy-momentum tensor with $\Delta = 4$ and the R-current with $\Delta = 3$) and to a long vector multiplet of conformal dimension $\Delta = 1 + \sqrt{7}$ (see [36] for more details). The additional $SO(3)$ vector multiplet included in this paper sources a conserved $SO(3)$ flavour current multiplet in the dual SCFT. The three scalar operators in this multiplet have conformal dimension $\Delta = 2$ (once) and $\Delta = 4$ (twice), while the $SO(3)$ flavour current has conformal dimension $\Delta = 3$, as required for a conserved current. Another piece of information about the dual SCFT is given by the Weyl anomaly coefficients; these are obtained from the five-dimensional Newton constant G_5 and the AdS_5 radius ℓ through the formula $a = c = \frac{\pi\ell^3}{8G_5}$.

- When $\kappa = -1$ we also recover the non-supersymmetric vacuum discussed in [36], that was originally found in [57]. The analysis of the scalar mass matrix shows that the fluctuation of H^4 has a mass squared $m^2\ell^2 \simeq -4.46$, which is below the Breitenlohner-Freedman bound $\ell^2 m^2_{\mathrm{BF}} = -4$. We thus establish that this vacuum is perturbatively unstable. Note that the unstable mode lies outside the truncation of [36].

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- For $\kappa = +1$, we find a non-supersymmetric vacuum with non zero value of the H-scalars, given by

$$\Sigma = \frac{2^{1/3}}{5^{1/6}}\,, \qquad \mathrm{e}^{2\varphi} = \frac{8}{3}\,, \qquad H^4 = \frac{3\sqrt{5}}{4}\,, \qquad \theta_1 = \theta_2 = 0\,, \qquad \ell = 3\,\frac{2^{1/6}}{5^{5/6}}R\,, \quad (4.78)$$

where ℓ is the AdS radius. This appears to be a new solution. It represents an $SO(3)$ worth of vacua really, since the scalars H^α, $\alpha = 1, 2, 3$, can take any value such that $\sqrt{(H^1)^2 + (H^2)^2 + (H^3)^2} = \sqrt{(H^4)^2 - 1} = \frac{\sqrt{29}}{4}$. We find that a linear combination of the fluctuations of Σ, φ and H^4 has mass squared $m^2\ell^2 \simeq -5.86 < m^2_{\rm BF}\ell^2$, hence this vacuum is perturbatively unstable. Nevertheless, it allowed us to perform a non-trivial check of our truncation ansatz for non-vanishing H-fields, as we have verified that its uplift does satisfy the equations of motion of eleven-dimensional supergravity.

5 Truncations for more general wrapped M5-branes

The $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Maldacena-Nuñez solutions are special cases of an infinite family of $\mathcal{N} = 2$ solutions [35, 38],[23] describing M5-branes wrapping a Riemann surface in a Calabi-Yau geometry. These solutions, which we will denote as BBBW solutions, have the same general features of the MN1 solution. In particular, they all admit a generalised $U(1)_S$ structure, which we use to derive the most general consistent truncation to $\mathcal{N} = 2$ gauged supergravity in five dimensions associated with such backgrounds. As we will see, the truncated theory has two vector multiplets, one hypermultiplet and gauge group $U(1) \times \mathbb{R}$. It generalises the $U(1)_{\rm right}$ invariant subtruncation of the truncation presented in the previous section: the matter content is the same and the gauging is deformed by one (discrete) parameter. Our systematic approach allows us to complete the consistent truncation derived from seven-dimensional maximal $SO(5)$ supergravity on $\mathbb{\Sigma}$ previously presented in [37] by including all scalar fields in the hypermultiplet and directly deriving the gauging.[24]

5.1 The BBBW solutions

The BBBW solutions describe M5-branes wrapped on a Riemann surface $\mathbb{\Sigma}$, such that the $(2, 0)$ theory on the branes has a twisting over $\mathbb{\Sigma}$ depending on two integer parameters p and q. The way the Riemann surface is embedded in the ambient space determines the local structure of the latter. The authors of [35, 38] showed that there is an infinite family of allowed geometries, corresponding to the fibration $\mathcal{L}_1 \oplus \mathcal{L}_2 \hookrightarrow \mathbb{\Sigma}$ of two complex line bundles over the Riemann surface, so that the total space is Calabi-Yau. The degrees of these line bundles are identified with the integers that parameterise the twist of the M5 world-volume theory, $p = \deg \mathcal{L}_1$ and $q = \deg \mathcal{L}_2$. By the Calabi-Yau condition p and q must satisfy $p + q = 2g - 2$, with g the genus of $\mathbb{\Sigma}$. In this setup, the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ twistings considered in [34] arise from setting $p = q$ and $q = 0$ (or $p = 0$), respectively.

[23] See also [58], where a subset of the solutions was previously found.

[24] We thank Nikolay Bobev and Alberto Zaffaroni for pointing out this reference.

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The corresponding $\mathrm{AdS}_5 \times_{\mathrm{w}} M$ supergravity solutions are generalisation of the MN1 solution reviewed in section 4.1. The eleven-dimensional metric is a warped product

$$\hat{g} = \mathrm{e}^{2\Delta} g_{\mathrm{AdS}_5} + g_6 \,, \tag{5.1}$$

with warp factor

$$\mathrm{e}^{2\Delta}\, \ell^2 = \mathrm{e}^{2f_0} \bar{\Delta}^{1/3} \,. \tag{5.2}$$

where ℓ is the AdS radius. The six-dimensional manifold M is still a fibration of a squashed four-sphere over the Riemann surface, with metric

$$g_6 = \bar{\Delta}^{1/3} \mathrm{e}^{2g_0} g_{\mathbb{\Sigma}} + \frac{1}{4}\, \bar{\Delta}^{-2/3} g_4 \,, \tag{5.3}$$

where the Riemann surface metric $g_{\mathbb{\Sigma}}$ satisfies (4.25), (4.26), and the metric on the squashed and fibered S^4 is

$$g_4 = X_0^{-1} \mathrm{d}\mu_0^2 + \sum_{i=1,2} X_i^{-1} \left(\mathrm{d}\mu_i^2 + \mu_i^2 (\mathrm{d}\varphi_i + A^{(i)})^2 \right) . \tag{5.4}$$

The angles φ_1, φ_2 vary in $[0, 2\pi]$,[25] and

$$\mu_0 = \cos\zeta \,, \qquad \mu_1 = \sin\zeta \cos\frac{\theta}{2} \,, \qquad \mu_2 = \sin\zeta \sin\frac{\theta}{2} \,, \tag{5.5}$$

with $\zeta, \theta \in [0, \pi]$. The two circles φ_1 and φ_2 are independently fibered over the Riemann surface, with connections

$$A^{(1)} = -\frac{1+z}{2}\, \upsilon \qquad A^{(2)} = -\frac{1-z}{2}\, \upsilon \,, \tag{5.6}$$

where υ is again the connection on $\mathbb{\Sigma}$ and the discrete parameter z is related to the integers p and q as

$$z = \frac{p-q}{p+q} \,. \tag{5.7}$$

The warping function $\bar{\Delta}$ and the constants f_0, g_0 depend on z and on the curvature κ of the Riemann surface as

$$\bar{\Delta} = \sum_{I=0}^{2} X_I \mu_I^2 \,, \qquad \mathrm{e}^{f_0} = X_0^{-1} \,, \qquad \mathrm{e}^{2g_0} = -\frac{1}{8}\, \kappa\, X_1 X_2 \left[(1-z) X_1 + (1+z) X_2 \right] , \tag{5.8}$$

with

$$\begin{aligned} X_0 &= (X_1 X_2)^{-2} \,, \\ X_1 X_2^{-1} &= \frac{1+z}{2z - \kappa\sqrt{1+3z^2}} \,, \\ X_1^5 &= \frac{1 + 7z + 7z^2 + 33z^3 + \kappa(1 + 4z + 19z^2)\sqrt{1+3z^2}}{4z(1-z)^2} \,. \end{aligned} \tag{5.9}$$

[25] They are related to the angles of section 4.1 by $\varphi_1 = -(\phi + \psi)/2$ and $\varphi_2 = (\phi - \psi)/2$.

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The four-form flux is given by

$$\hat{F} = -\frac{1}{4}\,\bar{\Delta}^{-5/2}\Big[\sum_{I=0}^{2}(X_I^2\mu_I^2 - \bar{\Delta}X_I) + 2\bar{\Delta}X_0\Big]\,\mathrm{vol}_4 \\ + \frac{1}{16}\bar{\Delta}^{-1/2}\sum_{i=1}^{2}X_i^{-2}\,*_4\,\big[\mathrm{d}(\mu_i^2)\wedge(\mathrm{d}\varphi_i + A^{(i)})\big]\wedge\mathrm{d}A^{(i)}\,, \tag{5.10}$$

where the Hodge star $*_4$ is computed using the metric (5.4).

The solution has two $U(1)$ isometries corresponding to shifts of the angles φ_1, φ_2 that parameterise the two diagonal combinations of the $U(1)_{\text{right}}$ and $U(1)_{\text{left}}$ subgroups of $SO(5)$. It turns out that neither of them corresponds to the superconformal R-symmetry of the dual $\mathcal{N}=1$ SCFT, which is given by a linear combination involving X_1, X_2 [35, 38].

5.2 Generalised $U(1)_S$ structure

The construction of the generalised structure associated to the BBBW solutions follows the same lines as for the MN1 solution. We first embed the ordinary $U(1)$ structure in $E_{6(6)}$ and then look for the invariant generalised tensors. The generalised $U(1)_S$ structure of the solutions is determined by the topological twist of the M5 world-volume theory, as a linear combination of the $U(1)$ holonomy of $\mathbb{\Sigma}$ and the $U(1)_{\text{right}}$ and $U(1)_{\text{left}}$ subgroups of the $SO(5)$ R-symmetry group

$$U(1)_S \;\sim\; U(1)_{\mathbb{\Sigma}} - U(1)_{\text{right}} - z\,U(1)_{\text{left}}\,. \tag{5.11}$$

This embeds in $E_{6(6)}$ as an element of its compact subgroup $USp(8)$ with generator

$$\mathfrak{u}(1)_S = \mathrm{i}\,\hat{\Gamma}_{56} - \frac{\mathrm{i}}{p+q}(p\,\hat{\Gamma}_{12} - q\,\hat{\Gamma}_{34})\,, \tag{5.12}$$

where $\hat{\Gamma}_{56}$ is the $\mathfrak{usp}_8$ element generating $U(1)_{\mathbb{\Sigma}}$ and $\frac{1}{2}(\hat{\Gamma}_{12} \pm \hat{\Gamma}_{34})$ generate $U(1)_{\text{left/right}}$. When $p=q$ we recover the $U(1)_S$ structure group of the MN1 solution, whereas $q=0$ (or $p=0$) gives the MN2 structure considered in [14]. Below we assume that p, q are generic, and do not fulfill these special conditions which as we have seen lead to a larger truncation.

By looking at the singlets under $\mathfrak{u}(1)_S$ in the **27** and **78** representations of $E_{6(6)}$, we find that the $U(1)_S$ structure is defined by eight J_A, $A=1,\ldots,8$, in the adjoint bundle and three generalised vectors K_I, $I=0,1,2$. The singlets in the adjoint bundle have the same form (4.30) as for the MN1 solution, while the three singlet generalised vectors take

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the same form as a subset of the MN1 generalised vectors,[26]

$$\begin{aligned} K_0 &= \mathrm{e}^{\Upsilon} \cdot (R^2 \,\mathrm{vol}_{\Sigma} \wedge E_5') \,, \\ K_1 &= \mathrm{e}^{\Upsilon} \cdot \widetilde{\Xi}_3 \,, \\ K_2 &= \mathrm{e}^{\Upsilon} \cdot \Xi_3 \,. \end{aligned} \tag{5.13}$$

However now the twisting element Υ has a more general form dictated by the embedding (5.12), that is

$$\Upsilon = -\frac{R}{p+q}\, v \times_{\mathrm{ad}} (p\, E_{12} - q\, E_{34}) \,. \tag{5.14}$$

This makes our generalised tensors globally well-defined. We emphasise that these depend on the integers p, q only through (5.14).

5.3 Features of the truncation

The number of $U(1)_S$ singlets in the **27** and **78** implies that the truncated supergravity theory contains two vector multiplets and one hypermultiplet. The H structure moduli space is the same as for the MN1 case,

$$\mathcal{M}_{\mathrm{H}} = \frac{SU(2,1)}{SU(2)_H \times U(1)} \,. \tag{5.15}$$

As before, this is parameterised by real coordinates $q^X = \{\varphi, \xi, \theta_1, \theta_2\}$ and the metric is given by eq. (4.72). The V structure moduli space is determined again following our discussion in section 2, and is a subspace of the one for the MN1 truncation. Evaluating the cubic invariant on the singlets K_I as in (2.16), we obtain that the non-zero components of the C_{IJK} tensor are

$$C_{0IJ} = C_{I0J} = C_{IJ0} = \frac{1}{3}\,\eta_{IJ} \,, \qquad \text{for } I, J = 1, 2 \,, \tag{5.16}$$

with $\eta = \mathrm{diag}(-1, 1)$. Parameterising the V structure moduli as in (4.36), with $I = 1, 2$, the constraint (2.31) gives the equation of the unit hyperboloid $SO(1,1)$,

$$-(H^1)^2 + (H^2)^2 = 1 \,, \tag{5.17}$$

while again Σ parameterises $\mathbb{R}^+$. Thus the V structure moduli space is

$$\mathcal{M}_{\mathrm{V}} = \mathbb{R}^+ \times SO(1,1) \,. \tag{5.18}$$

[26] Before acting with Υ, the singlets for the BBBW solutions are related to those used for the MN1 solutions as

$$K_0 = K_0^{\mathrm{MN1}} \,, \quad K_1 = K_3^{\mathrm{MN1}} \,, \quad K_2 = K_4^{\mathrm{MN1}} \,,$$

and to the structure of the MN2 solution in [14] as

$$K_0 = \tfrac{1}{2}(K_5^{\mathrm{MN2}} - K_8^{\mathrm{MN2}}) \,, \quad \begin{aligned} K_1 &= K_0^{\mathrm{MN2}} + \tfrac{1}{2}(K_5^{\mathrm{MN2}} + K_8^{\mathrm{MN2}}) \,, \\ K_2 &= K_0^{\mathrm{MN2}} - \tfrac{1}{2}(K_5^{\mathrm{MN2}} + K_8^{\mathrm{MN2}}) \,. \end{aligned}$$

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The kinetic matrix a_{IJ} then takes the same form (4.71), that is

$$\begin{aligned} a_{00} &= \frac{1}{3}\,\Sigma^4\,, \\ a_{01} &= a_{02} = 0\,, \\ a_{IJ} &= \frac{2}{3}\,\Sigma^{-2}\begin{pmatrix} 2(H^1)^2+1 & -2H^1H^2 \\ -2H^1H^2 & 2(H^2)^2-1 \end{pmatrix}\,, \qquad I,J=1,2\,. \end{aligned} \tag{5.19}$$

The gauging of the reduced theory is obtained from the generalised Lie derivative L_{K_I} acting on the K_J and the J_A. The Lie derivatives among vectors are now trivial,

$$L_{K_I}K_J = 0\,, \qquad I,J=0,1,2\,. \tag{5.20}$$

As discussed in section 2, the Lie derivatives $L_{K_I}J_A$ are conveniently expressed as the adjoint action of $SU(2,1)$ generators,

$$L_{K_0}J_A = [J_{(K_0)},J_A]\,, \qquad L_{K_1}J_A = [J_{(K_1)},J_A]\,, \qquad L_{K_2}J_A = [J_{(K_2)},J_A]\,. \tag{5.21}$$

Evaluating the generalised Lie derivatives we find

$$\begin{aligned} J_{(K_0)} &= \frac{1}{4R}\left(J_3+2J_7-\sqrt{3}J_8\right), \\ J_{(K_1)} &= \frac{1}{4R}\,\kappa\, z\left(J_3+2J_7-\sqrt{3}J_8\right), \\ J_{(K_2)} &= -\frac{1}{4R}\,\kappa\left(J_3+2J_7-\sqrt{3}J_8\right) - \frac{1}{R}\left(J_3+\frac{1}{\sqrt{3}}J_8\right). \end{aligned} \tag{5.22}$$

Eq. (5.20) implies that the vector multiplet sector is not gauged, so the field strengths are all abelian,

$$\mathcal{F}^I = \mathrm{d}\mathcal{A}^I\,, \tag{5.23}$$

while (5.22) specifies the gauging in the hypermultiplet sector in terms of κ and z. The $SU(2,1)$ generators act as isometries on $\mathcal{M}_{\rm H}$; the corresponding Killing vectors can again be computed using (3.10) and read

$$\begin{aligned} k_0 &= \partial_\xi\,, \\ k_1 &= \kappa\, z\,\partial_\xi\,, \\ k_2 &= -\kappa\,\partial_\xi + 2\left(\theta_2\partial_{\theta_1} - \theta_1\partial_{\theta_2}\right). \end{aligned} \tag{5.24}$$

It follows that the covariant derivatives of the charged scalars are

$$\begin{aligned} \mathcal{D}(\theta_1+\mathrm{i}\,\theta_2) &= \mathrm{d}(\theta_1+\mathrm{i}\,\theta_2) - \frac{2}{R}\,\mathrm{i}\mathcal{A}^2\,(\theta_1+\mathrm{i}\,\theta_2)\,, \\ \mathcal{D}\xi &= \mathrm{d}\xi + \frac{1}{R}\,\mathcal{A}^0 + \frac{1}{R}\,\kappa\left(z\mathcal{A}^1-\mathcal{A}^2\right), \end{aligned} \tag{5.25}$$

where again the inverse S^4 radius $\frac{1}{R}$ plays the role of the gauge coupling constant. The Killing prepotentials can be computed either from (3.12) or from (B.15), and read

$$P_0^\alpha = \left\{0\,,\,0\,,\,\frac{1}{4}\,\mathrm{e}^{2\varphi}\right\},$$

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$$P_1^\alpha = \left\{0\,,\,0\,,\,\frac{1}{4}\,\kappa\, z\, \mathrm{e}^{2\varphi}\right\},$$
$$P_2^\alpha = \left\{\sqrt{2}\,\mathrm{e}^{\varphi}\theta_1\,,\,\sqrt{2}\,\mathrm{e}^{\varphi}\theta_2\,,-1+\frac{1}{4}\,\mathrm{e}^{2\varphi}\,(2\theta_1^2+2\theta_2^2-\kappa)\right\}. \tag{5.26}$$

Notice that for $z=0$ (that is $p=q$), the quantities above reduce to those obtained for the MN1 structure in section 4.4.

The five-dimensional bosonic action is then determined to be

$$S = \frac{1}{16\pi G_5}\int\Big[(\mathcal{R}-2\mathcal{V})*1-\frac{1}{2}\,\Sigma^4\mathcal{F}^0\wedge *\mathcal{F}^0-\frac{3}{2}\sum_{I,J=1}^{2}a_{IJ}\mathcal{F}^I\wedge *\mathcal{F}^J-2\Sigma^{-2}\mathrm{d}\Sigma\wedge *\mathrm{d}\Sigma$$
$$-\frac{3}{2}\sum_{I,J=1}^{2}a_{IJ}\,\mathrm{d}(\Sigma H^I)\wedge *\mathrm{d}(\Sigma H^J)-g_{XY}\mathcal{D}q^X\wedge *\mathcal{D}q^Y-\mathcal{A}^0\wedge(\mathcal{F}^1\wedge\mathcal{F}^1-\mathcal{F}^2\wedge\mathcal{F}^2)\Big], \tag{5.27}$$

where the scalar potential reads

$$\mathcal{V} = \frac{1}{R^2}\left\{\frac{\mathrm{e}^{4\varphi}}{4\Sigma^4}-\frac{2\,\mathrm{e}^{2\varphi}H^2}{\Sigma}+\Sigma^2\Big[-2+\mathrm{e}^{2\varphi}\left(2(H^1)^2(\theta_1^2+\theta_2^2)-\kappa\right)\right.$$
$$+\frac{1}{8}\,\mathrm{e}^{4\varphi}((H^1)^2+(H^2)^2)(2\theta_1^2+2\theta_2^2-\kappa)^2$$
$$\left.+\,z\,\kappa\left(z\kappa\,(H^1)^2+z\kappa\,(H^2)^2+4H^1H^2(2\theta_1^2+\theta_2^2-\kappa)\right)\Big]\right\}. \tag{5.28}$$

It is straightforward to analyse the supersymmetric AdS_5 vacuum conditions (B.18). The hyperino equation gives

$$\theta_1=\theta_2=0\,,$$
$$\Sigma^{-3}=\kappa\left(zH^1-H^2\right), \tag{5.29}$$

where we assume $\kappa=\pm1$ (hence leaving aside the case $\kappa=0$). The gaugino equation gives

$$2\Sigma^{-3}\,P_0^\alpha+H^1P_1^\alpha+H^2P_2^\alpha=0\,,$$
$$H^2P_1^\alpha+H^1P_2^\alpha=0\,. \tag{5.30}$$

Plugging the Killing prepotentials (5.26) and using (5.29) we obtain

$$3\kappa\,\mathrm{e}^{2\varphi}\left(zH^1-H^2\right)-4H^2=0\,,$$
$$\kappa\,\mathrm{e}^{2\varphi}(z\,H^2-H^1)-4H^1=0\,. \tag{5.31}$$

Taking into account the allowed range of the scalar fields, the solution to these equations is

$$\frac{H^1}{H^2}=\frac{1+\kappa\sqrt{1+3z^2}}{3z}\,,\qquad \mathrm{e}^{2\varphi}=\frac{4}{\sqrt{1+3z^2}-2\kappa}\,. \tag{5.32}$$

For $\kappa=1$, well-definiteness of the fields requires $|z|>1$, as in [35], while z can be generic for $\kappa=-1$. The MN1 case $z=0$ is recovered as a limiting case after fixing $\kappa=-1$. The critical value of the scalar potential determines the AdS radius ℓ as

$$\ell=\left(\frac{\kappa-9\kappa z^2+(1+3z^2)^{3/2}}{4z^2}\right)^{1/3}R\,. \tag{5.33}$$

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Although we do not present the uplift formulae for this truncation, we have checked that the supersymmetric vacuum identified above matches the BBBW solution summarised in section 5.1. To do so, we have computed the inverse generalised metric G^{-1} associated with the $U(1)_S$ structure under consideration; this depends on the V structure and H structure parameters. From the generalised metric we have reconstructed the ordinary metric g_6 and the three-form potential on M, as well as the warp factor $\mathrm{e}^{2\Delta}$. Substituting the values for the scalars found above, we find agreement with the solution in section 5.1 upon fixing the S^4 radius as $R = \frac{1}{2}$ and implementing the following dictionary:

$$\begin{aligned} \mathrm{e}^{2\varphi} &= \frac{1}{4}\,\mathrm{e}^{-2g_0-\frac{1}{2}f_0}\,, \\ \Sigma^3 &= \frac{1}{4}\,\mathrm{e}^{-2g_0+\frac{3}{4}f_0}\,, \\ H^1 &= \frac{1}{2}X_0^{\frac{1}{4}}(X_1 - X_2)\,, \\ H^2 &= \frac{1}{2}X_0^{\frac{1}{4}}(X_1 + X_2)\,, \end{aligned} \tag{5.34}$$

with our AdS radius being given in terms of the quantities appearing there as

$$\ell = 2^{2/3}\,\mathrm{e}^{f_0+\frac{2}{3}g_0}R\,. \tag{5.35}$$

By extremising the scalar potential[27] we recover the supersymmetric vacuum and also find new non-supersymmetric vacua, where the scalar field values are rather complicated functions of the parameter z. As an example, we give the numerical values for one chosen value of z, that we take $z = \frac{1}{2}$. When $\kappa = -1$ we find a new extremum of the potential at

$$\Sigma \simeq 0.9388\,, \quad \varphi \simeq 0.1109\,, \quad H^2 \simeq 1.0217\,, \quad \theta_1 = \theta_2 = 0\,, \qquad \ell \simeq 1.5276\,R\,, \tag{5.36}$$

while when $\kappa = 1$ we find an extremum at

$$\Sigma \simeq 0.8631\,, \quad \varphi \simeq 0.2812\,, \quad H^2 \simeq 1.5506\,, \quad \theta_1 = \theta_2 = 0\,, \qquad \ell \simeq 1.0644\,R\,, \tag{5.37}$$

and another one at

$$\Sigma \simeq 1.1580\,, \quad \varphi \simeq 0.8455\,, \quad H^2 \simeq 1.9847\,, \quad \theta_1 = \theta_2 = 0\,, \qquad \ell \simeq 0.6198\,R\,, \tag{5.38}$$

where for each solution we have also indicated the corresponding AdS radius ℓ.

6 Conclusions

In this paper we have illustrated the Exceptional Generalised Geometry approach to $\mathcal{N} = 2$ consistent truncations of eleven-dimensional supergravity on a six-dimensional manifold M. We have argued that for the truncation to go through, M must admit a generalised $G_S \subseteq USp(6)$ structure with constant singlet intrinsic torsion, and we have explained how this completely determines the resulting five-dimensional supergravity theory. We have

[27] To do so, it is convenient to parameterise $H^1 = \sinh\alpha$, $H^2 = \cosh\alpha$, and extremise with respect to α.

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also given an algorithm to construct the full bosonic truncation ansatz. This formalism provides a geometric understanding of the origin of the truncations, in particular those that are not based on invariants of conventional G-structures on the tangent bundle. It also sidesteps the need to reduce the equations of motion in order to uncover the matter content and couplings of the truncated theory.

The main technically involved part of this formalism is deriving the truncation ansatz. However a significant advantage is that once this is done the relevant expressions can be used to derive the uplift formulae for any $\mathcal{N} = 2$ consistent truncation. One does not have postulate the set of consistent modes case-by-case. Furthermore the structure of the resulting gauged supergravity is then simply determined by the generalised structure.

To demonstrate the concrete effectiveness of the formalism we worked out the full bosonic truncation ansatz on Maldacena-Nuñez geometries, leading to five-dimensional $\mathcal{N} = 2$ supergravity with four vector multiplets, one hypermultiplet and a non-abelian gauging, having the $\mathcal{N} = 2$ AdS_5 solution of [34] as a vacuum solution. This extends the truncation of [36] by $SO(3)$ vector multiplets. For the BBBW geometries [35, 38], we obtained a truncation featuring two vector multiplets, one hypermultiplet and an abelian gauging, completing the truncation obtained in [37]. This can be seen as a one-parameter deformation of the truncation obtained from the one on Maldacena-Nuñez geometry by imposing invariance under the Cartan of $SO(3)$. Although in this case we did not give all details of the truncation ansatz, it should be clear that it can be obtained by following precisely the same steps presented for the case of Maldacena-Nuñez geometry. Since the generalised geometry tensors on S^4 used in these trucations are a subset of those appearing in the reduction of eleven-dimensional supergravity to maximal $SO(5)$ supergravity in seven dimensions, it should also be clear that our consistent truncations can equivalently be obtained as truncations of maximal $SO(5)$ supergravity on a Riemann surface.

Together with the half-maximal truncation presented in [14, 39], which is based on the $\mathcal{N} = 4$ solution of [34], this work provides the largest possible consistent truncations of eleven-dimensional supergravity that have as seed known $\mathrm{AdS}_5 \times_{\mathrm{w}} M$ supersymmetric solutions describing M5-branes wrapped on a Riemann surface (larger truncations may be possible by including degrees of freedom that go beyond eleven-dimensional supergravity, such as membrane degrees of freedom).

It would be interesting to explore further the relatively simple five-dimensional supergravity models obtained in this paper and construct new solutions thereof. These would have an automatic uplift to eleven dimensions, and may have an interpretation in the dual SCFT. For the subtruncation with no $SO(3)$ vector multiplet, solutions of holographic interest have been discussed in [36]. Our larger consistent truncation may offer the possibility to obtain solutions where non-abelian gauge fields are activated, which are quite rare in holography. For instance, constructing a supersymmetric, asymptotically AdS_5 black hole with non-abelian hair would represent a qualitatively new type of solutions.

It will be natural to adapt our construction to truncations of eleven-dimensional supergravity on a seven-dimensional manifold, leading to four-dimensional gauged $\mathcal{N} = 2$ supergravity. This uses $G_S \subseteq SU(6)$ structures in $E_{7(7)}$ generalised geometry, and would allow one to derive new consistent truncations based on the generalised structures under-

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lying the $\mathrm{AdS}_4 \times_{\mathrm{w}} M_7$ solutions of [59, 60], which in terms of ordinary G-structures only admit a local $SU(2)$ structure. The solutions of [59] are the most general $\mathcal{N} = 2$ AdS_4 solutions to eleven-dimensional supergravity supported by purely magnetic four-form flux; they represent the near-horizon region of M5-branes wrapping a special lagrangian three-cycle in M_7. The solutions of [60] have both electric and magnetic flux, and should arise from M2-M5 brane systems. Analysing the respective generalised structure it will become possible to enhance the truncation to minimal gauged supergravity obtained in [11] and [61] (for the solutions of [59] and [60], respectively) by adding matter multiplets. One example of this construction has been given in [62].

It will also be useful to extend our formalism to $\mathcal{N} = 2$ truncations of type II supergravity. Minimally supersymmetric AdS_5 solutions of type IIB and massive type IIA supergravity were classified in [63] and [64], respectively. It would be useful to reformulate the classification of explicit solutions in terms of generalised $G_S \subseteq USp(6)$ structures; this would be a first step towards constructing consistent truncations to five-dimensional supergravity using our approach. One concrete application would be to check if the IIB solution of [65], given by a warped product of AdS_5 and a deformed S^5, admits a consistent truncation to five-dimensional supergravity including (massive) KK modes that do not belong to the well-known IIB truncation leading to maximal $SO(6)$ gauged supergravity. This would be somewhat analogous to the IIB consistent truncation on Sasaki-Einstein structures [66, 67], where only a subset of the retained KK modes are also captured by $SO(6)$ gauged supergravity.

A more challenging generalisation of our formalism would be the one to truncations preserving only $\mathcal{N} = 1$ supersymmetry in four dimensions. Although a considerable amount of work remains to be done, it should be clear that the generalised structure approach to consistent truncations has the potential to classify all possible consistent truncations of higher-dimensional supergravity that preserve any given amount of supersymmetry.

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A $\mathrm{E}_{6(6)}$ generalised geometry for M-theory

In this section we briefly recall the main features of the generalised geometry of M-theory compactifications on a six-dimensional manifold M. For a more detailed discussion we refer to [42] and [44, App. E].

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We use the following conventions for wedges and contractions among tensors on M

$$
\begin{aligned}
(v \wedge u)^{a_1 \dots a_{p+p'}} &= \frac{(p+p')!}{p!\, p'!}\, v^{[a_1 \dots a_p} u^{a_{p+1} \dots a_{p+p'}]}, \\
(\lambda \wedge \rho)_{a_1 \dots a_{q+q'}} &= \frac{(q+q')!}{q!\, q'!}\, \lambda_{[a_1 \dots a_q} \rho_{a_{q+1} \dots a_{q+q'}]}, \\
(v \lrcorner \lambda)_{a_1 \dots a_{q-p}} &= \frac{1}{p!} v^{b_1 \dots b_p} \lambda_{b_1 \dots b_p a_1 \dots a_{q-p}} \quad \text{if } p \leq q, \\
(v \lrcorner \lambda)^{a_1 \dots a_{p-q}} &= \frac{1}{q!} v^{a_1 \dots a_{p-q} b_1 \dots b_q} \lambda_{b_1 \dots b_q} \quad \text{if } p \geq q, \\
(jv \lrcorner j\lambda)^a{}_b &= \frac{1}{(p-1)!} v^{a c_1 \dots c_{p-1}} \lambda_{b c_1 \dots c_{p-1}}, \\
(j\lambda \wedge \rho)_{a,\, a_1 \dots a_d} &= \frac{d!}{(q-1)!(d+1-q)!}\, \lambda_{a [a_1 \dots a_{q-1}} \rho_{a_q \dots a_d]} \ .
\end{aligned}
\tag{A.1}
$$

We will denote by $\cdot$ the $\mathfrak{gl}(6)$ action on tensors: given a frame $\{\hat{e}_a\}$ for TM and a co-frame $\{e_a\}$ for T^*M, $a = 1, \dots, 6$, the action, for instance, on a vector and a two-form is

$$
(r \cdot v)^a = r^a{}_b v^b \qquad (r \cdot \omega)_{ab} = -r^c{}_a \omega_{cb} - r^c{}_b \omega_{ac} \,. \tag{A.2}
$$

For M-theory on a six-dimensional manifold we use $E_{6(6)} \times \mathbb{R}^+$ generalised geometry. The generalised tangent bundle E is

$$
E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \,, \tag{A.3}
$$

where, as customary, we decompose the various bundles in representations of $GL(6)$, the geometric subgroup of $E_{6(6)}$. The sections of E, the generalised vectors, transform in the **27** of $E_{6(6)}$ and can be written as

$$
V = v + \omega + \sigma \,, \tag{A.4}
$$

where v is an ordinary vector field, ω is a two-form and σ is a five-form.[28]

[28]The generalised tangent bundle E has a non-trivial structure that takes into account the non-trivial gauge potentials of M-theory. To be more precise the sections of E are defined as

$$
V = \mathrm{e}^{A + \tilde{A}} \cdot \check{V} \,, \tag{A.5}
$$

where $A + \tilde{A}$ is an element of the adjoint bundle, $\check{V} = v + \omega + \sigma$, with $v \in \Gamma(TM)$ are vectors, $\omega \in \Gamma(\Lambda^2 T^*M)$ and $\sigma \in \Gamma(\Lambda^5 T^*M)$, and $\cdot$ defines the adjoint action defined in (A.23). The patching condition on the overlaps $U_\alpha \cap U_\beta$ is

$$
V_{(\alpha)} = \mathrm{e}^{\mathrm{d}\Lambda_{(\alpha\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} \cdot V_{(\beta)} \,, \tag{A.6}
$$

where $\Lambda_{(\alpha\beta)}$ and $\tilde{\Lambda}_{(\alpha\beta)}$ are a two- and five-form, respectively. This corresponds to the gauge-transformation of the three- and six-form potentials in (A.5) as

$$
\begin{aligned}
A_{(\alpha)} &= A_{(\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)} \,, \\
\tilde{A}_{(\alpha)} &= \tilde{A}_{(\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} - \frac{1}{2} \mathrm{d}\Lambda_{(\alpha\beta)} \wedge A_{(\beta)} \ .
\end{aligned}
\tag{A.7}
$$

The respective gauge-invariant field-strengths reproduce the supergravity ones:

$$
F = \mathrm{d}A \,,
$$

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The dual bundle E^* is defined as

$$E^* \simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM\,, \tag{A.9}$$

with sections

$$Z = \hat{v} + \hat{\omega} + \hat{\sigma}\,, \tag{A.10}$$

where $\hat{v}$ is one-form, $\hat{\omega}$ is a two-vector and $\hat{\sigma}$ is a five-vector. Generalised vectors and dual generalised vectors have a natural pairing given by

$$\langle Z, V\rangle = \hat{v}_m v^m + \frac{1}{2}\,\hat{\omega}^{mn}\omega_{mn} + \frac{1}{5!}\,\hat{\sigma}^{mnpqr}\sigma_{mnpqr}\,. \tag{A.11}$$

We will also need the bundle $N \simeq \det T^*M \otimes E^*$. In terms of $GL(6)$ tensors, N decomposes as

$$N \simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M)\,, \tag{A.12}$$

and correspondingly its sections $Z_\flat$ decompose as

$$Z_\flat = \lambda + \rho + \tau\,. \tag{A.13}$$

The bundle N is obtained from the symmetric product of two generalised vectors via the map $\otimes_N : E \otimes E \to N$ with

$$\begin{aligned} \lambda &= v \lrcorner \omega' + v' \lrcorner \omega\,, \\ \rho &= v \lrcorner \sigma' + v' \lrcorner \sigma - \omega \wedge \omega'\,, \\ \tau &= j\omega \wedge \sigma' + j\omega' \wedge \sigma\,. \end{aligned} \tag{A.14}$$

The $E_{6(6)}$ cubic invariant is defined on E and E^*as[29]

$$\begin{aligned} c(V,V,V) &= -\,6\,\iota_v\,\omega \wedge \sigma - \omega \wedge \omega \wedge \omega\,, \\ c^*(Z,Z,Z) &= -\,6\,\iota_{\hat{v}}\,\hat{\omega} \wedge \hat{\sigma} - \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega}\,. \end{aligned} \tag{A.15}$$

The adjoint bundle is defined as

$$\mathrm{ad}F \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM\,, \tag{A.16}$$

with sections

$$R = l + r + a + \tilde{a} + \alpha + \tilde{\alpha}\,, \tag{A.17}$$

where locally $l \in \mathbb{R}$, $r \in \mathrm{End}(TM)$, $a \in \Lambda^3 T^*M$, etc. The $\mathfrak{e}_{d(d)}$ sub-algebra is obtained by fixing the factor l in terms of the trace of r as $l = \frac{1}{3}\,\mathrm{tr}\,r$. This choice fixes the weight of the generalised tensors under the $\mathbb{R}^+$ factor. In particular it implies that a scalar of weight k is a section of $(\det T^*M)^{k/3}$: $\mathbb{1}_k \in \Gamma((\det T^*M)^{k/3})$.

$$\tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F\,. \tag{A.8}$$

[29]This is 6 times the cubic invariant given in [44]. Because of this, we introduced a compensating factor of 6 in the formulae (2.8) and (2.9).

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It is also useful to introduce the weighted adjoint bundle

$$(\det T^*M) \otimes \mathrm{ad}\, F \supset \mathbb{R} \oplus \Lambda^3 T^*M \oplus (TM \otimes \Lambda^5 TM)\,, \tag{A.18}$$

whose sections are locally given by the sum

$$R_{\flat} = \tilde{\phi} + \phi + \psi\,, \tag{A.19}$$

where $\tilde{\phi}$, ϕ and ψ are obtained from the adjoint elements $r \in TM \otimes T^*M$, $\alpha \in \Lambda^3 TM$, $\tilde{\alpha} \in \Lambda^3 TM$ as

$$\tilde{\phi} = \tilde{\alpha} \lrcorner \mathrm{vol}_6 \qquad \phi = \alpha \lrcorner \mathrm{vol}_6 \qquad \psi = r \cdot \mathrm{vol}_6\,, \tag{A.20}$$

where vol_6 is a reference volume form.

The action of an adjoint element R on another adjoint element R' is given by the commutator, $R'' = [R, R']$. In components, R'' reads

$$\begin{aligned}
l'' &= \frac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \frac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}')\,,\\
r'' &= [r, r'] + j\alpha \lrcorner ja' - j\alpha' \lrcorner ja - \frac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a)\,\mathbb{1}\,,\\
&\quad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \frac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}')\,\mathbb{1}\,,\\
a'' &= r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}'\,,\\
\tilde{a}'' &= r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a'\,,\\
\alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a'\,,\\
\tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha'\,,
\end{aligned} \tag{A.21}$$

where $\cdot$ denotes the $\mathfrak{gl}(6)$ action defined in (A.2).

The action of an adjoint element R on a generalised vector $V \in \Gamma(E)$ and on a dual generalised vector Z is also denoted by $\cdot$ and is defined as

$$V' = R \cdot V \qquad Z' = R \cdot Z\,, \tag{A.22}$$

where the components of V' are

$$\begin{aligned}
v' &= lv + r \cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma\,,\\
\omega' &= l\omega + r \cdot \omega + v \lrcorner a + \alpha \lrcorner \sigma\,,\\
\sigma' &= l\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega\,,
\end{aligned} \tag{A.23}$$

and those of Z' are

$$\begin{aligned}
\hat{v}' &= -l\hat{v} + r \cdot \hat{v} - \hat{\omega} \lrcorner a + \hat{\sigma} \lrcorner \tilde{a}\,,\\
\hat{\omega}' &= -l\hat{\omega} + r \cdot \hat{\omega} - \alpha \lrcorner \hat{v} - \hat{\sigma} \lrcorner a\,,\\
\hat{\sigma}' &= -l\hat{\sigma} + r \cdot \hat{\sigma} - \tilde{\alpha} \lrcorner \hat{v} - \alpha \wedge \hat{\omega}\,.
\end{aligned} \tag{A.24}$$

The $\mathfrak{e}_{6(6)}$ Killing form on two elements of the adjoint bundle is given by

$$\mathrm{tr}(R, R') = \frac{1}{2}\left(\frac{1}{3}\,\mathrm{tr}(r)\mathrm{tr}(r') + \mathrm{tr}(rr') + \alpha \lrcorner a' + \alpha' \lrcorner a - \tilde{\alpha} \lrcorner \tilde{a}' - \tilde{\alpha}' \lrcorner \tilde{a}\right)\,. \tag{A.25}$$

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The combination of diffeomorphisms and gauge transformations by the three-form and six-form potentials defines the generalised diffeomorphisms. The action of an infinitesimal generalised diffeomorphism is generated by the generalised Lie (or Dorfman) derivative along a generalised vector. The Lie derivative between two ordinary vectors v and v' on TM can be written in components as a $\mathfrak{gl}(6)$ action

$$(\mathcal{L}_v v')^m = v^n \partial_n v'^m - (\partial \times v)^m{}_n v'^n \,, \tag{A.26}$$

where the symbol $\times$ denotes the projection onto the adjoint of the product of the fundamental and dual representation of $GL(6)$. The generalised Lie derivative is defined in an analogous way; we introduce the operators $\partial_M = \partial_m$ as sections of the dual tangent bundle and we define the generalised Lie derivative as

$$(L_V V')^M = V^N \partial_N V'^M - (\partial \times_{\rm ad} V)^M{}_N V'^N \,, \tag{A.27}$$

where V^M, $M = 1, \ldots, 27$, are the components of V in a standard coordinate basis, and $\times_{\rm ad}$ is the projection onto the adjoint bundle,

$$\times_{\rm ad} \, : \, E^* \otimes E \to {\rm ad} F \,, \tag{A.28}$$

whose explicit expression can be found in [42, Eq.(C.13)]. In terms of $GL(6)$ tensors, (A.27) becomes

$$L_V V' = \mathcal{L}_v v' + (\mathcal{L}_v \omega' - \iota_{v'} {\rm d}\omega) + (\mathcal{L}_v \sigma' - \iota_{v'} {\rm d}\sigma - \omega' \wedge {\rm d}\omega) \,. \tag{A.29}$$

The action of the generalised Lie derivative on a section of the adjoint bundle (A.17) is

$$\begin{aligned} L_V R = {} & (\mathcal{L}_v r + j\alpha \lrcorner j{\rm d}\omega - \frac{1}{3} \mathbb{1}\alpha \lrcorner {\rm d}\omega - j\tilde{\alpha} \lrcorner j{\rm d}\sigma + \frac{2}{3} \mathbb{1}\tilde{\alpha} \lrcorner {\rm d}\sigma) + (\mathcal{L}_v a + r \cdot {\rm d}\omega - \alpha \lrcorner {\rm d}\sigma) \\ & + (\mathcal{L}_v \tilde{a} + r \cdot {\rm d}\sigma + {\rm d}\omega \wedge a) + (\mathcal{L}_v \alpha - \tilde{\alpha} \lrcorner {\rm d}\omega) + \mathcal{L}_v \tilde{\alpha} \,. \end{aligned} \tag{A.30}$$

We will also need the action of L_V on the elements of the bundle N. Given a section $Z_\flat = \lambda + \rho + \tau$ of N, its Lie derivative along the generalised vector $V = v + \omega + \sigma$ is

$$L_V Z_\flat = \mathcal{L}_v \lambda + (\mathcal{L}_v \rho - \lambda \wedge {\rm d}\omega) + (\mathcal{L}_v \tau - j\rho \wedge {\rm d}\omega + j\lambda \wedge {\rm d}\sigma) \,. \tag{A.31}$$

Since $Z_\flat = V' \otimes_N V''$, this is easily obtained by applying the Leibniz rule for L_V.

$$L_V(Z_\flat) = L_V V' \otimes_N V'' + V' \otimes_N L_V V'' \,. \tag{A.32}$$

It is also straightforward to verify that

$${\rm d} Z_\flat = L_V V' + L_{V'} V \,, \tag{A.33}$$

for any element $Z_\flat = V \otimes_N V' \in N$.

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B Five-dimensional $\mathcal{N}=2$ gauged supergravity

In this appendix we summarise some essential features of matter-coupled five-dimensional $\mathcal{N}=2$ gauged supergravity [68–70], following the conventions of [70].[30] We limit ourselves to the bosonic sector and only consider gaugings that do not require the introduction of two-form fields, as these are enough to describe our examples in sections 4 and 5.

The fields of five-dimensional $\mathcal{N}=2$ supergravity arrange into the gravity multiplet, $n_{\rm V}$ vector multiplets and $n_{\rm H}$ hypermultiplets. The bosonic content consists of the vielbein e_μ^a, $n_{\rm V}+1$ vector fields $\mathcal{A}_\mu^I$, $I=0,\ldots,n_{\rm V}$, together with $n_{\rm V}$ vector multiplet scalars ϕ^x, $x=1,\ldots,n_{\rm V}$, and $4n_{\rm H}$ hypermultiplet scalars q^X, $X=1,\ldots,4n_{\rm H}$. The ϕ^x parameterise a 'very special real' manifold $\mathcal{M}_{\rm V}$, with metric g_{xy}, while the q^X parameterise a quaternionic-Kähler manifold $\mathcal{M}_{\rm H}$, with metric g_{XY}. All together, the scalar manifold of the theory is the direct product

$$\mathcal{M}=\mathcal{M}_{\rm V}\times\mathcal{M}_{\rm H}\,. \tag{B.1}$$

A very special real manifold $\mathcal{M}_{\rm V}$ is a hypersurface that is conveniently described in terms of $n_{\rm V}+1$ embedding functions $h^I(\phi)$, $I=0,\ldots,n_{\rm V}$, satisfying the constraint

$$C_{IJK}h^Ih^Jh^K=1\,, \tag{B.2}$$

where C_{IJK} is a completely symmetric constant tensor. The metric on $\mathcal{M}_{\rm V}$ is given by

$$g_{xy}=h_x^Ih_y^J\,a_{IJ}\,, \tag{B.3}$$

where

$$h_x^I=-\sqrt{\frac{3}{2}}\,\partial_x h^I\,, \tag{B.4}$$

and

$$a_{IJ}=3h_Ih_J-2C_{IJK}h^K\,, \tag{B.5}$$

with the lower-index functions being

$$h_I=C_{IKL}h^Kh^L=a_{IK}h^K\,. \tag{B.6}$$

The matrix a_{IJ} is assumed invertible, and also controls the gauge kinetic terms.

In the gauged theory, a subgroup of the isometries of the scalar manifold $\mathcal{M}$, which are global symmetries of the Lagrangian, is promoted to a gauge group. The gauge generators t_I satisfy $[t_I,t_J]=-f_{IJ}{}^Kt_K$, with the structure constants $f_{IJ}{}^K$ obeying $f_{I(J}{}^HC_{KL)H}=0$. The gauge covariant derivatives of the scalars are given by

$$\begin{aligned}\mathcal{D}_\mu\phi^x&=\partial_\mu\phi^x+g\,k_I^x\mathcal{A}_\mu^I\,,\\ \mathcal{D}_\mu q^X&=\partial_\mu q^X+g\,k_I^X\mathcal{A}_\mu^I\,,\end{aligned} \tag{B.7}$$

[30] However, in order to match the normalisations defined by our truncation ansatz, we rescale the gauge fields appearing in [70] as $\mathcal{A}^I_{\rm here}=-\sqrt{\frac{2}{3}}A^I_{\rm there}$. Since we maintain the same form of the covariant derivatives, it follows that the gauge coupling constant g is rescaled as $g_{\rm here}=-\sqrt{\frac{3}{2}}\,g_{\rm there}$. This implies that the expression for the scalar potential given in (B.13) below acquires a multiplicative 2/3 factor compared to the one in [70].

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where $k_I^x(\phi)$ and $k_I^X(q)$ are the Killing vector fields generating the gauged isometries in the vector multiplet and hypermultiplet scalar manifolds, respectively. Equivalently, the vector multiplet scalar covariant derivatives can be expressed in terms of the embedding functions h^I as

$$\mathcal{D}_\mu h^I = \partial_\mu h^I + g\, f_{JK}{}^I \mathcal{A}_\mu^J\, h^K = \partial_x h^I \mathcal{D}_\mu \phi^x \,. \tag{B.8}$$

One also has the gauge field-strengths

$$\mathcal{F}_{\mu\nu}^I \;=\; 2\partial_{[\mu}\mathcal{A}_{\nu]}^I + \,g\, f_{JK}{}^I \mathcal{A}_\mu^J \mathcal{A}_\nu^K \,. \tag{B.9}$$

We now have all the elements to write down the bosonic Lagrangian. This reads

$$\begin{aligned} e^{-1}\mathcal{L} \;=\; & \frac{1}{2}\,\mathcal{R} - \mathcal{V} - \frac{3}{8}\,a_{IJ}\mathcal{F}_{\mu\nu}^I \mathcal{F}^{J\mu\nu} - \frac{1}{2}\,g_{xy}\mathcal{D}_\mu\phi^x \mathcal{D}^\mu \phi^y - \frac{1}{2}\,g_{XY}\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y \\ & - \frac{1}{8}\,e^{-1}\epsilon^{\mu\nu\lambda\rho\sigma}\, C_{IJK}\mathcal{A}_\mu^I \left[\mathcal{F}_{\nu\lambda}^J \mathcal{F}_{\rho\sigma}^K + g f_{MN}{}^J \mathcal{A}_\nu^M \mathcal{A}_\lambda^N \left(-\frac{1}{2}\,\mathcal{F}_{\rho\sigma}^K + \frac{1}{10}\,g f_{HL}{}^K \mathcal{A}_\rho^H \mathcal{A}_\sigma^L\right)\right] . \end{aligned} \tag{B.10}$$

The vector multiplet scalar kinetic term can also be written in terms of the constrained scalars h^I using the identity

$$g_{xy}\mathcal{D}_\mu\phi^x \mathcal{D}^\mu\phi^y \;=\; \frac{3}{2}\,a_{IJ}\mathcal{D}_\mu h^I \mathcal{D}^\mu h^J \,. \tag{B.11}$$

Using a differential form notation, the action reads

$$\begin{aligned} S \;=\; & \int \frac{1}{2}\,(\mathcal{R} - 2\mathcal{V}) * 1 - \frac{3}{4}\,a_{IJ}\mathcal{F}^I \wedge *\mathcal{F}^J - \frac{3}{4}\,a_{IJ}\mathcal{D}h^I \wedge *\mathcal{D}h^J - \frac{1}{2}\,g_{XY}\mathcal{D}q^X \wedge *\mathcal{D}q^Y \\ & + \frac{1}{8}\,C_{IJK}\mathcal{A}^I \wedge \left[4\mathcal{F}^J \wedge \mathcal{F}^K + g\, f_{MN}{}^J \mathcal{A}^M \wedge \mathcal{A}^N \wedge \left(-\mathcal{F}^K + \frac{1}{10}\,g f_{HL}{}^K \mathcal{A}^H \wedge \mathcal{A}^L\right)\right] . \end{aligned} \tag{B.12}$$

The scalar potential $\mathcal{V}$ is given as a sum of squares as

$$\mathcal{V} = \frac{4}{3}\,g^2 \left(-2\vec{P}\cdot\vec{P} + g^{xy}\vec{P}_x \cdot \vec{P}_y + \mathcal{N}_{iA}\mathcal{N}^{iA}\right) , \tag{B.13}$$

where

$$\begin{aligned} \vec{P} &= h^I \vec{P}_I \,, \\ \vec{P}_x &= h_x^I \vec{P}_I \,, \\ \mathcal{N}^{iA} &= \frac{\sqrt{6}}{4} h^I k_I^X f_X^{iA} \,, \end{aligned} \tag{B.14}$$

are the fermionic shifts, also appearing in the supersymmetry variations of the fermion fields: $\vec{P}$ is the gravitino shift, $\vec{P}_x$ is the gaugino shift, and $\mathcal{N}^{iA}$ is the hyperino shift. Here, the arrow symbol denotes a triplet of the $SU(2)_H$ R-symmetry, and f_X^{iA} are the quaternionic vielbeins, satisfying $f_X^{iA} f_{Yi A} = g_{XY}$. The Killing prepotentials $\vec{P}_I$ on $\mathcal{M}_{\rm H}$ are defined for $n_{\rm H} \neq 0$ by

$$4n_{\rm H}\vec{P}_I = \vec{J}_X{}^Y \nabla_Y k_I^X \,, \tag{B.15}$$

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where $\vec{J}_X{}^Y$ is the triplet of almost complex structures defined on any quaternionic-Kähler manifold. Plugging these expressions in (B.13) and using the identity (cf. [70, App. C])

$$g^{xy}h_x^I h_y^J = a^{IJ} - h^I h^J\,, \tag{B.16}$$

we can express the scalar potential as

$$\mathcal{V} = \frac{2}{3}\,g^2\left[\left(2a^{IJ} - 6h^I h^J\right)\vec{P}_I\cdot\vec{P}_J + \frac{3}{4}\,g_{XY}k_I^X k_J^Y h^I h^J\right]. \tag{B.17}$$

Notice that the Killing vectors k_I^x on $\mathcal{M}_{\rm V}$ do not appear here, i.e. the gauging in the vector multiplet sector does not contribute to the scalar potential. This is true as long as we restrict to gaugings that do not require the introduction of two-form fields.

Supersymmetric AdS_5 vacua are obtained by setting all gauge fields to zero, all scalar fields to constant, and imposing that the gaugino and hyperino shifts vanish,

$$h_x^I\vec{P}_I = 0\,, \qquad h^I k_I^X = 0\,. \tag{B.18}$$

Then the gravitino shift gives the AdS cosmological constant via

$$\Lambda \equiv \mathcal{V} = -\frac{8}{3}\,g^2\,\vec{P}\cdot\vec{P}\,. \tag{B.19}$$

C Gauge transformations

In this appendix, we study the reduction gauge transformations of eleven-dimensional supergravity to five dimensions. We first repackage them in terms of generalised geometric objects and then use our truncation ansatz to derive the gauge transformations of five-dimensional $\mathcal{N}=2$ supergravity.

The infinitesimal gauge transformations of the eleven-dimensional metric and three- and six-form potentials are

$$\begin{aligned}
\delta\hat{g} &= \hat{\mathcal{L}}_{\hat{v}}\,\hat{g}\,,\\
\delta\hat{A} &= \hat{\mathcal{L}}_{\hat{v}}\hat{A} - \hat{\mathrm{d}}\hat{\lambda}\,,\\
\delta\hat{\tilde{A}} &= \hat{\mathcal{L}}_{\hat{v}}\hat{\tilde{A}} - \hat{\mathrm{d}}\hat{\tilde{\lambda}} + \frac{1}{2}\mathrm{d}\lambda\wedge A\,,
\end{aligned} \tag{C.1}$$

where $\hat{v}$ is a vector field, $\hat{\lambda}$ a two-form and $\hat{\tilde{\lambda}}$ a five-form. The hat on the Lie and exterior derivative operators emphasises that the derivatives are taken with respect to all the eleven-dimensional coordinates. The fields g, $\hat{A}$ and $\hat{\tilde{A}}$ are decomposed as in (3.16), while the gauge parameters are expanded as

$$\begin{aligned}
\hat{v} &= v = v^m\frac{\partial}{\partial z^m}\,,\\
\hat{\lambda} &= \lambda - \bar{\lambda}_\mu\mathrm{d}x^\mu + \frac{1}{2}\,\bar{\lambda}_{\mu\nu}\mathrm{d}x^{\mu\nu}\,,\\
\hat{\tilde{\lambda}} &= \tilde{\lambda} + \bar{\tilde{\lambda}}_\mu\mathrm{d}x^\mu + \frac{1}{2}\bar{\tilde{\lambda}}_{\mu\nu}\mathrm{d}x^{\mu\nu} + \frac{1}{3!}\bar{\tilde{\lambda}}_{\mu\nu\rho}\mathrm{d}x^{\mu\nu\rho} + \dots\,,
\end{aligned} \tag{C.2}$$

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where, in the last line we omitted the terms that are not relevant in what follows. We only consider internal diffeomorphisms, as the external ones have the standard action dictated by the tensorial structure of the field. That is why the vector $\hat{v}$ has only components on M. As in (3.16), we do not impose any restriction on the dependence of the fields on the coordinates $\{x^\mu, z^m\}$.

In (C.2) and the rest of this section we use a notation that manifestly displays the external indices and always contracts the internal ones. For the metric components we define

$$g = g_{mn}\mathrm{d}z^m\mathrm{d}z^n\,, \qquad h_\mu = h_\mu{}^m\frac{\partial}{\partial z^m}\,, \tag{C.3}$$

and for a generic p-form ω

$$\begin{aligned}
\omega &= \frac{1}{p!}\,\omega_{m_1\ldots m_p}\mathrm{d}z^{m_1\ldots m_p}\,,\\
\omega_\mu &= \frac{1}{(p-1)!}\,\omega_{\mu m_1\ldots m_{p-1}}\mathrm{d}z^{m_1\ldots m_{p-1}}\,,\\
\omega_{\mu\nu} &= \frac{1}{(p-2)!}\omega_{\mu\nu m_1\ldots m_{p-2}}\mathrm{d}z^{m_1\ldots m_{p-2}}\,,
\end{aligned} \tag{C.4}$$

We already mentioned in section 3.2 that the barred components of the three- and six-form must be redefined as[31]

$$\begin{aligned}
\bar{A}_{\mu\nu} &= A_{\mu\nu} - \iota_{h_{[\mu}}A_{\nu]}\,,\\
\bar{A}_{\mu\nu\rho} &= A_{\mu\nu\rho} - \iota_{h_{[\mu}}\iota_{h_\nu}A_{\rho]}\,,
\end{aligned} \tag{C.6}$$

and similar redefinitions of the six-form. An analogous redefinition for the barred gauge parameters will be given later.

As discussed in section 3.2, the components of the metric, warp factor, three and six-form potentials and the dual graviton $\tilde{g}$ with the same number of external legs fit into $E_{6(6)}$ representations

$$G^{-1} \leftrightarrow \{\Delta,\, g,\, A,\, \tilde{A}\} \tag{C.7}$$

$$\mathcal{A}_\mu = h_\mu + A_\mu + \tilde{A}_\mu\,, \tag{C.8}$$

$$\mathcal{B}_{\mu\nu} = A_{\mu\nu} + \tilde{A}_{\mu\nu} + \tilde{g}_{\mu\nu}\,, \tag{C.9}$$

$$\mathcal{C}_{\mu\nu\rho} = A_{\mu\nu\rho} + \tilde{A}_{\mu\nu\rho} + \tilde{g}_{\mu\nu\rho}\,, \tag{C.10}$$

where G^{MN} is the inverse generalised metric, $\mathcal{A}_\mu \in E$ is a generalised vector, $\mathcal{B}_{\mu\nu} \in N$ is a weighted dual vector and $\mathcal{C}_{\mu\nu\rho}$ is a section of the weighted $E_{6(6)}$ adjoint bundle $(\det T^*)\otimes \mathrm{ad}F$. The same holds for the gauge parameter, which we arrange into a generalised vector

[31] The contractions are defined as follows

$$\iota_{h_{[\mu}}A_{\nu]} = h_{[\mu}{}^m A_{\nu]mn}\mathrm{d}z^n \qquad \iota_{h_{[\mu}}A_{\nu\rho]} = h_{[\mu}{}^m A_{\nu\rho]m} \qquad \iota_{h_{[\mu}}\iota_{h_\nu}A_{\rho]} = h_{[\mu}{}^m h_\nu{}^n A_{\rho]nm} \tag{C.5}$$

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Λ, a weighted dual vector $\bar{\Xi}_\mu$ and a section of a sub-bundle of the **78**, $\bar{\Phi}_{\mu\nu}$,

$$\begin{aligned}\Lambda &= v + \lambda + \tilde{\lambda}\,,\\ \bar{\Xi}_\mu &= \bar{\lambda}_\mu + \bar{\tilde{\lambda}}_\mu + \dots\,,\\ \bar{\Phi}_{\mu\nu} &= \bar{\lambda}_{\mu\nu} + \bar{\tilde{\lambda}}_{\mu\nu} + \dots\,.\end{aligned} \tag{C.11}$$

In (C.7)–(C.10) we introduced the dual graviton to give the full $E_{6(6)}$ representation. However in this paper we will not discuss the dual graviton since it is not relevant for the truncation we are interested in.

We can now decompose the gauge transformations given above. We find that the fields with no or purely internal legs transform as

$$\begin{aligned}\delta\,\mathrm{e}^{2\Delta} &= \mathcal{L}_v\,\mathrm{e}^{2\Delta}\,,\\ \delta g &= \mathcal{L}_v\, g\,,\\ \delta A &= \mathcal{L}_v A - \mathrm{d}\lambda\,,\\ \delta\tilde{A} &= \mathcal{L}_v\tilde{A} - \mathrm{d}\tilde{\lambda} + \frac{1}{2}\mathrm{d}\lambda\wedge A\,,\end{aligned} \tag{C.12}$$

where the Lie derivative $\mathcal{L}$ and the exterior derivative d are taken with respect to the *internal* coordinates only, although the fields and gauge parameters depend on both the internal and external coordinates. When repackaging all the fields with no external legs into the inverse generalised metric, the transformations (C.12) become the action of the generalised Lie derivative along the generalised vector Λ,

$$\delta_\Lambda G^{-1} = L_\Lambda G^{-1}\,. \tag{C.13}$$

Consider now the fields with one external leg. Their gauge transformations are

$$\begin{aligned}\delta h_\mu &= -\partial_\mu v + \mathcal{L}_v h_\mu\,,\\ \delta A_\mu &= -\partial_\mu\lambda + \mathrm{d}\bar{\lambda}_\mu - \iota_{h_\mu}\mathrm{d}\lambda + \mathcal{L}_v A_\mu\,,\\ \delta\tilde{A}_\mu &= -\partial_\mu\tilde{\lambda} + \mathrm{d}\bar{\tilde{\lambda}}_\mu - \iota_{h_\mu}\mathrm{d}\tilde{\lambda} - \mathrm{d}\lambda\wedge A_\mu + \mathcal{L}_v\tilde{A}_\mu\,,\end{aligned} \tag{C.14}$$

and it is straightforward to verify that they can be recast into

$$\delta\mathcal{A}_\mu = -\partial_\mu\Lambda + L_\Lambda\mathcal{A}_\mu + \mathrm{d}\bar{\Xi}_\mu\,, \tag{C.15}$$

where

$$\begin{aligned}L_\Lambda\mathcal{A}_\mu &= (\mathcal{L}_v h_\mu) + (\mathcal{L}_v A_\mu - \iota_{h_\mu}\mathrm{d}\lambda) + (\mathcal{L}_v\tilde{A}_\mu - \iota_{h_\mu}\mathrm{d}\tilde{\lambda} - A_\mu\wedge\mathrm{d}\lambda)\,,\\ L_{\mathcal{A}_\mu}\Lambda &= (\mathcal{L}_{h_\mu} v) + (\mathcal{L}_{h_\mu}\lambda - \iota_v\mathrm{d}A_\mu) + (\mathcal{L}_{h_\mu}\tilde{\lambda} - \iota_v\mathrm{d}\tilde{A}_\mu - \lambda\wedge\mathrm{d}A_\mu)\,.\end{aligned} \tag{C.16}$$

By redefining the gauge parameters[32]

$$\bar{\Xi}_\mu = \Xi_\mu - \mathcal{A}_\mu\otimes_N\Lambda\,, \tag{C.18}$$

[32] In components the redefinition (C.18) reads

$$\begin{aligned}\bar{\lambda}_\mu &= \lambda_\mu - \iota_{h_\mu}\lambda - \iota_v A_\mu\,,\\ \bar{\tilde{\lambda}}_\mu &= \tilde{\lambda}_\mu - \iota_{h_\mu}\tilde{\lambda} - \iota_v\tilde{A}_\mu + \lambda\wedge A_\mu\,.\end{aligned} \tag{C.17}$$

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with $\mathcal{A}_\mu \otimes_N \Lambda = (\iota_{h_\mu}\lambda + \iota_v A_\mu) + (\iota_{h_\mu}\tilde{\lambda} + \iota_v \tilde{A}_\mu - \lambda \wedge A_\mu)$, and using (A.33) to compute

$$\mathrm{d}\overline{\Xi}_\mu = \mathrm{d}\Xi_\mu - L_{\mathcal{A}_\mu}\Lambda - L_\Lambda \mathcal{A}_\mu \,, \tag{C.19}$$

we bring the variation (C.15) to an appropriate form to compare with five-dimensional gauged supergravity

$$\delta\mathcal{A}_\mu = -\partial_\mu \Lambda - L_{\mathcal{A}_\mu}\Lambda + \mathrm{d}\Xi_\mu \,. \tag{C.20}$$

The variations of the fields with two external legs are

$$\begin{aligned}
\delta A_{\mu\nu} =& -2\,\partial_{[\mu}\overline{\lambda}_{\nu]} - \mathrm{d}\overline{\lambda}_{\mu\nu} + \iota_{h_{[\mu}}\partial_{\nu]}\lambda - \iota_{h_{[\mu}}\mathrm{d}\overline{\lambda}_{\nu]} + \mathcal{L}_v A_{\mu\nu} - \iota_{\partial_{[\mu}v}A_{\nu]} \,, \\
\delta\tilde{A}_{\mu\nu} =& -2\partial_{[\mu}\overline{\tilde{\lambda}}_{\nu]} - \mathrm{d}\overline{\tilde{\lambda}}_{\mu\nu} + \iota_{h_{[\mu}}\partial_{\nu]}\tilde{\lambda} - \iota_{h_{[\mu}}\mathrm{d}\overline{\tilde{\lambda}}_{\nu]} + \mathcal{L}_v\tilde{A}_{\mu\nu} - \iota_{\partial_{[\mu}v}\lrcorner\,\tilde{A}_{\nu]} \\
&+ (\partial_{[\mu}\lambda - \mathrm{d}\overline{\lambda}_{[\mu}) \wedge A_{\nu]} + \mathrm{d}\lambda \wedge A_{\mu\nu} \,.
\end{aligned} \tag{C.21}$$

By a lengthy but straightforward computation (C.21) can be written as

$$\begin{aligned}
\delta\mathcal{B}_{\mu\nu} =& -2\,\partial_{[\mu}\Xi_{\nu]} - 2L_{\mathcal{A}_{[\mu}}\Xi_{\nu]} + \mathcal{H}_{\mu\nu}\otimes_N\Lambda - \delta\mathcal{A}_{[\mu}\otimes_N\mathcal{A}_{\nu]} \\
&+ L_{\mathcal{A}_{[\mu}}\mathcal{A}_{\nu]}\otimes_N\Lambda + 2L_{\mathcal{A}_{[\mu}}\Lambda\otimes_N\mathcal{A}_{\nu]} + L_\Lambda\mathcal{A}_{[\mu}\otimes_N\mathcal{A}_{\nu]} \\
&- \mathrm{d}[\overline{\Phi}_{\mu\nu} - 2\mathcal{A}_{[\mu}\times_{\mathrm{ad}}\bar{\Xi}_{\nu]} - \mathcal{B}_{\mu\nu}\times_{\mathrm{ad}}\Lambda] \,,
\end{aligned} \tag{C.22}$$

where we defined the field strength

$$\mathcal{H}_{\mu\nu} = \mathrm{d}\mathcal{B}_{\mu\nu} + L_{\mathcal{A}_{[\mu}}\mathcal{A}_{\nu]} + 2\partial_{[\mu}\mathcal{A}_{\nu]} \,. \tag{C.23}$$

Applying the Leibniz rule for the generalised Lie derivative and (A.33) one can show that

$$L_{\mathcal{A}_{[\mu}}\mathcal{A}_{\nu]}\otimes_N\Lambda + 2L_{\mathcal{A}_{[\mu}}\Lambda\otimes_N\mathcal{A}_{\nu]} + L_\Lambda\mathcal{A}_{[\mu}\otimes_N\mathcal{A}_{\nu]} = \mathrm{d}[\mathcal{A}_{[\mu}\times_{\mathrm{ad}}(\mathcal{A}_{\nu]}\otimes_N\Lambda)] \tag{C.24}$$

and the variation of $\mathcal{B}_{\mu\nu}$ can be written in a form compatible with five-dimensional gauged supergravity

$$\delta\mathcal{B}_{\mu\nu} = -2\,\partial_{[\mu}\Xi_{\nu]} - 2L_{\mathcal{A}_{[\mu}}\Xi_{\nu]} + \mathcal{H}_{\mu\nu}\otimes_N\Lambda - \delta\mathcal{A}_{[\mu}\otimes_N\mathcal{A}_{\nu]} - \mathrm{d}\Phi_{\mu\nu} \tag{C.25}$$

where we have made the following redefinition of the gauge parameters[33]

$$\Phi_{\mu\nu} = \overline{\Phi}_{\mu\nu} + 2\mathcal{A}_{[\mu}\times_{\mathrm{ad}}\Xi_{\nu]} + \mathcal{B}_{\mu\nu}\times_{\mathrm{ad}}\Lambda - \mathcal{A}_{[\mu}\times_{\mathrm{ad}}(\mathcal{A}_{\nu]}\otimes_N\Lambda) \,. \tag{C.27}$$

Finally we should consider the variations of the fields with three external legs. To our purposes it is enough to study the three-form

$$\begin{aligned}
\delta A_{\mu\nu\rho} =& \mathcal{L}_v A_{\mu\nu\rho} - 3\partial_{[\mu}\overline{\lambda}_{\nu\rho]} - 3\iota_{h_{[\mu}}(2\,\partial_{[\nu}\overline{\lambda}_{\rho]} + \mathrm{d}\overline{\lambda}_{\nu\rho]}) \\
&+ 2\iota_{h_{[\mu}}\iota_{h_\nu}(\partial_{\rho]}\lambda - \mathrm{d}\overline{\lambda}_{\rho]}) - 2\,\iota_{\partial_{[\mu}v}\iota_{h_\nu}A_{\rho]} \,.
\end{aligned} \tag{C.28}$$

[33] In components

$$\begin{aligned}
\lambda_{\mu\nu} =& (\overline{\lambda}_{\mu\nu} - \iota_v A_{\mu\nu} - 2\iota_{h_{[\mu}}\lambda_{\nu]} + \iota_{h_{[\mu}}\iota_{h_{\nu]}}\lambda + \iota_{h_{[\mu}}\iota_v A_{\nu]}) \,, \\
\tilde{\lambda}_{\mu\nu} =& (\overline{\tilde{\lambda}}_{\mu\nu} - \iota_v\tilde{A}_{\mu\nu} - 2\iota_{h_{[\mu}}\tilde{\lambda}_{\nu} + \iota_{h_{[\mu}}\iota_{h_\nu}\tilde{\lambda} + \iota_{h_{[\mu}}\iota_v\tilde{A}_{\nu]} - 2\lambda_{[\mu}\wedge A_{\nu]} - \lambda\wedge A_{\mu\nu} \\
&- \lambda\wedge\iota_{h_{[\mu}}A_{\nu]} + (\iota_v A_{[\mu})\wedge A_{\nu]}) \,.
\end{aligned} \tag{C.26}$$

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In generalised geometry (C.28) embeds in the lowest component of the variation of the tensor $\mathcal{C}_{\mu\nu\rho}$ in (C.10). We introduce the modified field strength for the three-form field $\mathcal{C}_{\mu\nu\rho}$,

$$\mathcal{H}_{\mu\nu\rho} = -\mathrm{d}\mathcal{C}_{\mu\nu\rho} + 3\partial_{[\mu}\mathcal{B}_{\nu\rho]} + 3L_{\mathcal{A}_{[\mu}}\mathcal{B}_{\nu\rho]} + \mathcal{A}_{[\mu} \otimes_N \left(3\partial_\nu \mathcal{A}_{\rho]} + L_{\mathcal{A}_\nu}\mathcal{A}_{\rho]}\right), \tag{C.29}$$

and by manipulations similar to what we did previously we can recast the gauge variations as

$$\begin{aligned}\delta\mathcal{C}_{\mu\nu\rho} = &-3\partial_{[\mu}\Phi_{\nu\rho]} - 3L_{\mathcal{A}_{[\mu}}\Phi_{\nu\rho]} + 3\mathcal{H}_{[\mu\nu} \times_{\mathrm{ad}} \Xi_{\rho]} + \mathcal{H}_{\mu\nu\rho} \times_{\mathrm{ad}} \Lambda \\ &- 3\mathcal{B}_{[\mu\nu} \times_{\mathrm{ad}} \delta\mathcal{A}_{\rho]} - \mathcal{A}_{[\mu} \times_{\mathrm{ad}} \left(\mathcal{A}_\nu \times_{\mathrm{ad}} \delta\mathcal{A}_{\rho]}\right),\end{aligned} \tag{C.30}$$

up to terms involving a four-form gauge parameter, which would continue the tensor hierarchy.

The five-dimensional gauge transformations are obtained by plugging the reduction ansatz in the variations (C.20), (C.25) and (C.30). The fields $\mathcal{A}_\mu{}^I(x)$ are expanded as in (3.25)

$$\mathcal{A}_\mu \;=\; \mathcal{A}_\mu{}^I(x)\, K_I\,, \tag{C.31}$$

where K_I are the generalised vectors that are singlets of the G_S structure. In (3.26), the two-from fields are expanded on the weighted duals $K_\flat^I$ of the generalised vectors K_I. These are elements of the bundle N and can also written as

$$K_\flat^J = D^{IJK} K_J \otimes_N K_K \tag{C.32}$$

where the tensor D^{IJK} satisfies $D^{IKL}C_{JKL} = 1/2\delta^I_J$ where C_{IJK} is defined in (2.16). So the two-forms are expanded as

$$\mathcal{B}_{\mu\nu} \;=\; \mathcal{B}_{\mu\nu\, I}(x)\, D^{IJK} K_J \otimes_N K_K\,. \tag{C.33}$$

The gauge parameters have a similar expansion

$$\Lambda = -\Lambda^I(x) K_I\,, \qquad \Xi_\mu = -\frac{1}{2}\Xi_{\mu,I}(x) D^{IJK} K_J \otimes_N K_K\,. \tag{C.34}$$

With the ansatze (C.31) and (C.34) for $\mathcal{A}_\mu$ and the gauge parameters, the variations (C.20) of the one-forms become

$$\delta\mathcal{A}_\mu{}^I(x) = \partial_\mu \Lambda^I(x) + f^I_{JK}\mathcal{A}^J_\mu(x)\Lambda^K(x) - f^I_{(JK)} D^{JKL}\Xi_{\mu,L}\,, \tag{C.35}$$

where we used the algebra of the vectors K_I (3.4) and (A.33).

The variations of two-forms are reduced in a similar way. We expand the field strength $\mathcal{H}^J_{\mu\nu}$ as in (C.31) and use again the generalised Lie derivative of vectors K_I given in (3.4). In this way we obtain for the gauge variations of the five-dimensional two-forms $\mathcal{B}_{\mu\nu,I}(x)$

$$\delta\mathcal{B}_{\mu\nu,I} = \mathcal{D}_{[\mu}\Xi_{\nu],I} - 2C_{IJK}\mathcal{H}^J_{\mu\nu}\Lambda^K - 2C_{IJK}\delta\mathcal{A}^J_{[\mu}\mathcal{A}^K_{\nu]} - \Theta_I{}^A\Phi_{A\,\mu\nu}\,, \tag{C.36}$$

where $\Phi_{\mu\nu} = \Phi_{A\,\mu\nu}J^{\flat\, A}$ and

$$\mathcal{D}_{[\mu}\Xi_{\nu],I}(x) = \partial_{[\mu}\Xi_{\nu],I}(x) + 2X^K_{IJ}\mathcal{A}^J_{[\mu}(x)\Xi_{\nu],K}(x)\,, \tag{C.37}$$

with

$$X^K_{IJ} = C_{ILM}D^{KMN}f^L_{JN}\,. \tag{C.38}$$

This is in agreemement with five-dimensional supergravity. The variation $\delta\mathcal{C}_{\mu\nu\rho}$ reduces analogously.

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D Parameterisation of S^4 and generalised frames

The six-dimensional geometry of interest in this paper is given by a four-sphere S^4 fibered over a Riemann surface $\mathbb{\Sigma}$. In this appendix we describe S^4 as a foliation of S^3 over an interval and review the generalised frames on S^4.

D.1 Parameterisation of S^3

In terms of the standard Euler angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi$, the unit metric on the round S^3 reads

$$g_{S^3} = \frac{1}{4}\left(\mathrm{d}\theta^2 + \mathrm{d}\phi^2 + \mathrm{d}\psi^2 + 2\cos\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\right). \tag{D.1}$$

The Killing vectors generating its $SO(4) \simeq SU(2)_{\text{left}} \times SU(2)_{\text{right}}$ isometries can be split into $SU(2)_{\text{left}}$-invariant Killing vectors ξ_α, $\alpha = 1, 2, 3$, generating the $SU(2)_{\text{right}}$ isometries, and $SU(2)_{\text{right}}$-invariant Killing vectors $\tilde{\xi}_\alpha$, generating the $SU(2)_{\text{left}}$ isometries. The left-invariant vectors read

$$\begin{aligned}
\xi_1 &= -\cot\theta\sin\psi\,\partial_\psi + \cos\psi\,\partial_\theta + \frac{\sin\psi}{\sin\theta}\,\partial_\phi\,,\\
\xi_2 &= -\cot\theta\cos\psi\,\partial_\psi - \sin\psi\,\partial_\theta + \frac{\cos\psi}{\sin\theta}\,\partial_\phi\,,\\
\xi_3 &= \partial_\psi\,,
\end{aligned} \tag{D.2}$$

while the right-invariant ones are

$$\begin{aligned}
\tilde{\xi}_1 &= \frac{\sin\phi}{\sin\theta}\,\partial_\psi + \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi\,,\\
\tilde{\xi}_2 &= -\frac{\cos\phi}{\sin\theta}\,\partial_\psi + \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi\,,\\
\tilde{\xi}_3 &= \partial_\phi\,.
\end{aligned} \tag{D.3}$$

These satisfy

$$\mathcal{L}_{\xi_\alpha}\xi_\beta = \epsilon_{\alpha\beta\gamma}\,\xi_\gamma\,, \qquad \mathcal{L}_{\tilde{\xi}_\alpha}\tilde{\xi}_\beta = -\epsilon_{\alpha\beta\gamma}\,\tilde{\xi}_\gamma\,, \qquad \mathcal{L}_{\xi_\alpha}\tilde{\xi}_\beta = 0\,, \tag{D.4}$$

where $\mathcal{L}$ is the ordinary Lie derivative. We also introduce the one-form counterparts of these Killing vectors, namely left-invariant one-forms σ_α and right-invariant one-forms $\tilde{\sigma}_\alpha$. These satisfy

$$\begin{aligned}
\iota_{\xi_\alpha}\sigma_\beta &= \delta_{\alpha\beta}\,, & \iota_{\tilde{\xi}_\alpha}\tilde{\sigma}_\beta &= \delta_{\alpha\beta}\,,\\
\mathrm{d}\sigma_\alpha &= -\frac{1}{2}\,\epsilon_{\alpha\beta\gamma}\sigma_\beta\wedge\sigma_\gamma\,, & \mathrm{d}\tilde{\sigma}_\alpha &= \frac{1}{2}\,\epsilon_{\alpha\beta\gamma}\tilde{\sigma}_\beta\wedge\tilde{\sigma}_\gamma\,,
\end{aligned} \tag{D.5}$$

and their coordinate expression is

$$\begin{aligned}
\sigma_1 &= \cos\psi\,\mathrm{d}\theta + \sin\psi\sin\theta\,\mathrm{d}\phi\,,\\
\sigma_2 &= -\sin\psi\,\mathrm{d}\theta + \cos\psi\sin\theta\,\mathrm{d}\phi\,,\\
\sigma_3 &= \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\,,
\end{aligned} \tag{D.6}$$

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$$\begin{aligned}\tilde{\sigma}_1 &= \cos\phi\,\mathrm{d}\theta + \sin\phi\sin\theta\,\mathrm{d}\psi\,,\\ \tilde{\sigma}_2 &= \sin\phi\,\mathrm{d}\theta - \cos\phi\sin\theta\,\mathrm{d}\psi\,,\\ \tilde{\sigma}_3 &= \mathrm{d}\phi + \cos\theta\,\mathrm{d}\psi\,.\end{aligned} \tag{D.7}$$

The metric (D.1) may also be expressed as

$$g_{S^3} \;=\; \frac{1}{4}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) \;=\; \frac{1}{4}\left(\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\tilde{\sigma}_3^2\right)\,. \tag{D.8}$$

We fix the orientation on S^3 by defining the volume form as

$$\mathrm{vol}_{S^3} = \frac{1}{8}\,\sigma_1\wedge\sigma_2\wedge\sigma_3 = \frac{1}{8}\,\tilde{\sigma}_1\wedge\tilde{\sigma}_2\wedge\tilde{\sigma}_3 = \frac{1}{8}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi\,. \tag{D.9}$$

D.2 Parameterisation of S^4

The round four-sphere of radius R can be described via constrained $\mathbb{R}^5$ coordinates $R\,y^i$, $i=1,\ldots,5$, satisfying $\delta_{ij}y^iy^j=1$. In these coordinates, the metric and the volume form read

$$g_4 = R^2\,\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j\,, \qquad \mathrm{vol}_4 = \frac{1}{4!}R^4\,\epsilon_{i_1i_2i_3i_4i_5}\,y^{i_1}\mathrm{d}y^{i_2}\wedge\mathrm{d}y^{i_3}\wedge\mathrm{d}y^{i_4}\wedge\mathrm{d}y^{i_5}\,. \tag{D.10}$$

The constrained coordinates can be mapped into angular coordinates $\{\zeta,\theta,\phi,\psi\}$, where $0\le\zeta\le\pi$, and $\{\theta,\phi,\psi\}$ are the Euler angles on S^3 introduced above, as

$$\begin{aligned}y^1+\mathrm{i}\,y^2 &= \sin\zeta\,\cos\frac{\theta}{2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\phi+\psi)}\,,\\ y^3+\mathrm{i}\,y^4 &= \sin\zeta\,\sin\frac{\theta}{2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(\phi-\psi)}\,,\\ y^5 &= \cos\zeta\,.\end{aligned} \tag{D.11}$$

Then the metric and volume form in (D.10) become

$$\begin{aligned}g_4 &= R^2\left(\mathrm{d}\zeta^2+\sin^2\zeta\,\mathrm{d}s^2_{S^3}\right)\,,\\ &= R^2\left[\mathrm{d}\zeta^2+\frac{1}{4}\sin^2\zeta\,\left(\mathrm{d}\theta^2+\mathrm{d}\phi^2+\mathrm{d}\psi^2+2\cos\theta\,\mathrm{d}\phi\,\mathrm{d}\psi\right)\right]\,,\\ \mathrm{vol}_4 &= R^4\sin^3\zeta\,\mathrm{d}\zeta\wedge\mathrm{vol}_{S^3} = \frac{1}{8}R^4\sin^3\zeta\sin\theta\,\mathrm{d}\zeta\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\psi\,.\end{aligned} \tag{D.12}$$

We denote by $v_{ij}=v_{[ij]}$ the Killing vector fields generating the isometries of S^4. These satisfy the $\mathfrak{so}_5$ algebra,

$$\mathcal{L}_{v_{ij}}v_{kl} \;=\; R^{-1}\left(\delta_{ik}v_{lj}-\delta_{il}v_{kj}-\delta_{jk}v_{li}+\delta_{jl}v_{ki}\right)\,. \tag{D.13}$$

Demanding that the constrained coordinates transform in the fundamental representation,

$$\mathcal{L}_{v_{ij}}y_k \;\equiv\; \iota_{v_{ij}}\mathrm{d}y_k = R^{-1}\left(y_i\delta_{jk}-y_j\delta_{ik}\right)\,, \tag{D.14}$$

and using the map (D.11), we can work out the expression for the Killing vectors in the basis defined by the angular coordinates $\{\zeta,\theta,\phi,\psi\}$. In particular, we obtain the following

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embedding of the $SU(2)_{\text{right}}$ and $SU(2)_{\text{left}}$ generators given in (D.2), (D.3) into the $SO(5)$ generators:

$$\begin{aligned}
&\frac{2}{R}\,\xi_1 = v_{13} + v_{24}\,, \qquad &&\frac{2}{R}\,\xi_2 = v_{14} - v_{23}\,, \qquad &&\frac{2}{R}\,\xi_3 = v_{12} - v_{34}\,,\\
&\frac{2}{R}\,\tilde{\xi}_1 = v_{13} - v_{24}\,, \qquad &&\frac{2}{R}\,\tilde{\xi}_2 = v_{23} + v_{14}\,, \qquad &&\frac{2}{R}\,\tilde{\xi}_3 = v_{12} + v_{34}\,.
\end{aligned} \tag{D.15}$$

D.3 Generalised frames on S^4

In generalised geometry all spheres are generalised parallelisable as they admit globally defined frames on their exceptional tangent bundle [15]. In particular the generalised tangent bundle on S^4 is

$$E_4 \simeq TS^4 \oplus \Lambda^2 T^* S^4\,, \tag{D.16}$$

and its fibres transform in the **10** of the structure group $SL(5,\mathbb{R})$. We will also need the bundles

$$\begin{aligned}
N_4 &\simeq T^*S^4 \oplus \Lambda^4 T^* S^4\,,\\
N_4' &\simeq \mathbb{R} \oplus \Lambda^3 T^* S^4\,,
\end{aligned} \tag{D.17}$$

whose fibres transform in the **5** and **5′** representations, respectively. These bundles admit globally defined frames, which in constrained coordinates read

$$\begin{aligned}
E_{ij} &= v_{ij} + R^2 *_4(\mathrm{d}y_i \wedge \mathrm{d}y_j) + \iota_{v_{ij}} A_{S^4} &&\in \Gamma(E_4)\,,\\
E_i &= R\,\mathrm{d}y_i - y_i\,\mathrm{vol}_4 + R\,\mathrm{d}y_i \wedge A_{S^4} &&\in \Gamma(N_4)\,,\\
E_i' &= y_i + R\; *_4 \mathrm{d}y_i + y_i A_{S^4} &&\in \Gamma(N_4')\,,
\end{aligned} \tag{D.18}$$

where the Hodge star $*_4$ is computed using (D.10), and the three-form potential A_{S^4} must satisfy

$$\mathrm{d}A_{S^4} = 3R^{-1}\,\mathrm{vol}_4\ . \tag{D.19}$$

This is the flux relevant for the $\mathrm{AdS}_7 \times S^4$ supersymmetric Freund-Rubin solution to eleven-dimensional supergravity; the twist over the Riemann surface discussed in the main text will modify it. The E_{ij} are generalised Killing vectors generating the $\mathfrak{so}_5$ algebra via the action of the generalised Lie derivative,

$$L_{E_{ij}} E_{kl} = -R^{-1}\left(\delta_{ik}E_{jl} - \delta_{il}E_{jk} + \delta_{jl}E_{ik} - \delta_{jk}E_{il}\right)\ . \tag{D.20}$$

In the main text we will need the following linear combinations,

$$\begin{aligned}
&\Xi_1 = E_{13} + E_{24}\,, \qquad &&\Xi_2 = E_{14} - E_{23}\,, \qquad &&\Xi_3 = E_{12} - E_{34}\,,\\
&\widetilde{\Xi}_1 = E_{13} - E_{24}\,, \qquad &&\widetilde{\Xi}_2 = E_{14} + E_{23}\,, \qquad &&\widetilde{\Xi}_3 = E_{12} + E_{34}\,.
\end{aligned} \tag{D.21}$$

Using the map (D.11), the frame elements (D.18) can equivalently be expressed in terms of angular coordinates on S^4. In particular, choosing a gauge such that the potential A_{S^4} satisfying (D.19) is $SU(2)_{\text{left}} \times SU(2)_{\text{right}}$ invariant,

$$A_{S^4} = \frac{1}{32} R^3 \left(\cos(3\zeta) - 9\cos\zeta\right)\, \sigma_1 \wedge \sigma_2 \wedge \sigma_3\,, \tag{D.22}$$

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we find that the combinations (D.21) are expressed in terms of the $SU(2)_{\text{left}}$ and $SU(2)_{\text{right}}$ invariant tensors as

$$\Xi_\alpha = \frac{2}{R}\,\xi_\alpha + \frac{R^2}{2}\mathrm{d}\,(\cos\zeta\;\sigma_\alpha)\;,$$
$$\widetilde{\Xi}_\alpha = \frac{2}{R}\,\tilde{\xi}_\alpha - \frac{R^2}{2}\,\mathrm{d}\,(\cos\zeta\;\tilde{\sigma}_\alpha)\;. \tag{D.23}$$

The Ξ_α can be seen as left-invariant generalised Killing vectors generating the $SU(2)_{\text{right}} \subset SO(4) \subset SO(5)$ generalised isometries, while $\widetilde{\Xi}_\alpha$ are right-invariant generalised Killing vectors generating the $SU(2)_{\text{left}}$ generalised isometries. We will also need the expressions for E_i and E'_i in terms of angular coordinates, in fact just for $i = 5$. These read

$$E_5 = -R\,\sin\zeta\,\mathrm{d}\zeta + \frac{R^4}{8}\,\sin(2\zeta)\,\mathrm{d}\zeta\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3\,, \tag{D.24}$$

$$E'_5 = \cos\zeta - \frac{R^3}{16}\,(\cos(2\zeta)+3)\,\sigma_1\wedge\sigma_2\wedge\sigma_3\,. \tag{D.25}$$

Notice that $\mathrm{d}E'_5 = \frac{1}{R}\,E_5$.

E Details on the generalised $U(1)$ structure of MN1 solution

In this appendix we give the details of the construction of the $U(1)$ structure discussed in section 4.2. In order to identify the correct $U(1)$ subgroup of $E_{6(6)}\times\mathbb{R}^+$ and its commutant it is convenient to decompose $E_{6(6)}$ under its maximal compact subgroup $USp(8)$ and then express the $USp(8)$ representations in terms of Cliff(6) gamma matrices. For the latter step we also need the decomposition of $E_{6(6)}$ under $SL(6)\times SL(2)$. We first give a brief summary of the decomposition of $E_{6(6)}$ under $USp(8)$ and $SL(6)\times SL(2)$ and then we apply this to the construction of the $U(1)$ structure, which reduces to simple gamma matrix algebra.

E.1 $USp(8)$ and $SL(6)\times SL(2)$ decompositions

In this section we mostly use the conventions of [71]. Consider first the decomposition of $E_{6(6)}$ under $USp(8)$. We denote by $M, N, \cdots = 1,\ldots,27$ the $E_{6(6)}$ indices and by $\alpha,\beta,\ldots = 1,\ldots,8$ the $USp(8)$ ones.

The fundamental of $E_{6(6)}$ is irreducible under $USp(8)$ and is defined by an anti-symmetric traceless tensor

$$V^{\alpha\beta} = V^{[\alpha\beta]} \qquad V^\alpha_\alpha = 0\,. \tag{E.1}$$

The $USp(8)$ indices are raised and lowered by the $USp(8)$ symplectic form $\Omega^{\alpha\beta}$ and its inverse. The dual vectors in the $\overline{\mathbf{27}}$ are denoted by $Z_{\alpha\beta}$. The adjoint of $E_{6(6)}$ decomposes as

$$\mathbf{78} = \mathbf{36} + \mathbf{42}\,, \tag{E.2}$$

where the **36** is the adjoint of $USp(8)$ and the **42** contains the non-compact generators. The elements of the **36** are 8×8 matrices $\mu^\alpha{}_\beta$ satisfying

$$\mu_{\alpha\beta} = \mu_{\beta\alpha} \tag{E.3}$$

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with $\mu_{\alpha\beta} = (\Omega^{-1})_{\alpha\gamma}\mu^{\gamma}{}_{\beta}$. The non compact generators $\mu_{\alpha\beta\gamma\delta} \in \mathbf{42}$ are anti-symmetric tensors satisfying

$$\mu^{\alpha\beta\gamma\delta}(\Omega^{-1})_{\alpha\beta} = 0\,. \tag{E.4}$$

The adjoint action on the **27** is

$$(\mu V)^{\alpha\beta} = \mu^{\alpha}{}_{\gamma}V^{\gamma\beta} - \mu^{\beta}{}_{\gamma}V^{\gamma\alpha} - \mu^{\alpha\beta\gamma\delta}V_{\gamma\delta} \tag{E.5}$$

and the $E_{6(6)}$ commutators are

$$[\mu,\nu]^{\alpha\beta} = \mu^{\alpha}{}_{\gamma}\nu^{\gamma\beta} - \frac{1}{3}\mu^{\alpha\gamma\delta\epsilon}\nu_{\gamma\delta\epsilon}{}^{\beta} - (\mu \leftrightarrow \nu) \tag{E.6}$$

$$[\mu,\nu]^{\alpha\beta\gamma\delta} = -4\mu^{[\alpha}{}_{\epsilon}\nu^{\beta\gamma\delta]\epsilon} - (\mu \leftrightarrow \nu)\,. \tag{E.7}$$

Given the generalised vectors V, V', V'' and the duals Z, Z', Z'', the $E_{6(6)}$ quadratic form becomes

$$\langle V, Z\rangle = V^{\alpha\beta}Z_{\alpha\beta}\,, \tag{E.8}$$

and the cubic invariants are

$$\begin{aligned} c(V,V',V'') &= V^{\alpha}{}_{\beta}V'^{\beta}{}_{\gamma}V''^{\gamma}{}_{\alpha}\,,\\ c^*(Z,Z',Z'') &= Z_{\alpha}{}^{\beta}Z'_{\beta}{}^{\gamma}Z''^{\alpha}_{\gamma}\,. \end{aligned} \tag{E.9}$$

We will also need the projection into the adjoint of the product of a generalised vector V and a dual generalised vector Z

$$\begin{aligned} (V\times Z)^{\alpha\beta} &= 2V^{(\alpha}{}_{\gamma}Z^{|\gamma|\beta)}\\ (V\times Z)^{\alpha\beta\gamma\delta} &= 6\left(V^{[\alpha\beta}Z^{\gamma\delta]} + V^{[\alpha}{}_{\epsilon}Z^{|\epsilon|\beta}\Omega^{\gamma\delta} + \frac{1}{3}\,\mathrm{tr}(VZ)\Omega^{[\alpha\beta}\Omega^{\gamma\delta]}\right)\,. \end{aligned} \tag{E.10}$$

Consider now the decomposition of $E_{6(6)}$ under $SL(6)\times SL(2)$. We denote the $SL(6)$ indices with $m,n,\ldots = 1,\ldots,6$ and the $SL(2)$ indices $\hat{\imath},\hat{\jmath}\ldots = 1,2$. Under $SL(6)\times SL(2)$ the **27** and $\overline{\mathbf{27}}$ decompose as

$$\begin{aligned} \mathbf{27} &= (\bar{\mathbf{6}},\mathbf{2}) + (\mathbf{15},\mathbf{1}) \qquad & V^M &= (v^{\hat{\imath}}_m, V^{mn})\,,\\ \overline{\mathbf{27}} &= (\mathbf{6},\bar{\mathbf{2}}) + (\overline{\mathbf{15}},\mathbf{1}) \qquad & Z_M &= (z^m_{\hat{\imath}}, Z_{mn})\,, \end{aligned} \tag{E.11}$$

where V^{mn} and Z_{mn} are anti-symmetric. The components in (E.11) are related to the $GL(6)$ tensors (A.3) and (A.4) as follows

$$\begin{aligned} V &= \omega \qquad & v^1 &= v \qquad & v^2 \lrcorner\, \mathrm{vol}_6 &= \sigma\,,\\ Z &= \hat{\omega} \qquad & z_1 &= \hat{\mathrm{vol}}_6 \lrcorner\, \sigma \qquad & z_2 &= \hat{v}\,. \end{aligned} \tag{E.12}$$

The adjoint of $E_{6(6)}$ decomposes as

$$\mathbf{78} = (\mathbf{35},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{20},\mathbf{2}) \qquad \mu^M{}_N = (\mu^m{}_n\,, \mu^{\hat{\imath}}{}_{\hat{\jmath}}\,, \mu^{\hat{\imath}}_{mnp})\,, \tag{E.13}$$

where $\mu^m{}_n$ are real, traceless, 6×6 matrices generating $SL(6)$, $\mu^{\hat{\imath}}{}_{\hat{\jmath}}$ are real and traceless and generate $SL(2)$ and $\mu^{\hat{\imath}}_{mnp}$ are a pair of real fully antisymmetric tensors in the $(\mathbf{20},\mathbf{2})$. The

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matrices $\mu^m{}_n$ are identified with the traceless part of the $GL(6)$ matrix r, with the trace given by the diagonal non-compact generator of $SL(2)$, where we have also set $l = \frac{1}{3}\,\mathrm{tr}(r)$,

$$\mu^m{}_n = r^m{}_n - \frac{1}{6}\,\mathrm{tr}(r)\delta^m{}_n \qquad \mu^1{}_1 = -\mu^2{}_2 = \frac{1}{2}\,\mathrm{tr}(r)\,. \tag{E.14}$$

The compact and remaining non-compact generator of $SL(2)$ are identified with the combinations of six-form and six-vector transformation $\tilde{a} \pm \tilde{\alpha}$.

The tensors $\mu^{\hat{i}}_{mnp}$ correspond to the three-forms and three-vectors

$$\mu^1_{mnp} = \alpha^{mnp}\,, \qquad \mu^2_{mnp} = a_{mnp}\,. \tag{E.15}$$

Using $\mathrm{Cliff}(6,\mathbb{R})$ gamma matrices one can relate $USp(8)$ and $SL(6)\times SL(2)$ representations. We introduce the doublet of matrices

$$\hat{\Gamma}^m_{\hat{i}} = \big(\hat{\Gamma}^m,\, \mathrm{i}\,\hat{\Gamma}^m\hat{\Gamma}^7\big)\,, \qquad \hat{\imath} = 1,2\,. \tag{E.16}$$

Then the **27** and $\overline{\mathbf{27}}$ of $USp(8)$ are given in terms of $SL(6)\times SL(2)$ representation by

$$\begin{aligned} V^{\alpha\beta} &= \frac{1}{2\sqrt{2}}\big[v^{\hat{i}}_m(\hat{\Gamma}^m_{\hat{i}})^{\alpha\beta} + \frac{\mathrm{i}}{2}V^{mn}(\hat{\Gamma}_{mn7})^{\alpha\beta}\big]\,,\\ Z_{\alpha\beta} &= \frac{1}{2\sqrt{2}}\big[z^m_{\hat{i}}(\hat{\Gamma}^{\hat{i}}_m)^{\alpha\beta} + \frac{\mathrm{i}}{2}Z_{mn}(\hat{\Gamma}^{mn7})^{\alpha\beta}\big]\,, \end{aligned} \tag{E.17}$$

where Γ_{mn7} denotes the anti-symmetric product of two gamma's and Γ_7. The **36** and the **42** of $USp(8)$ are given

$$\mu_{\alpha\beta} = \frac{1}{4}\big[\mu^m{}_n(\hat{\Gamma}_m{}^n) + \mathrm{i}\,\epsilon_{\hat{i}}{}^{\hat{\jmath}}\mu^{\hat{i}}{}_{\hat{i}}\hat{\Gamma}_7 + \frac{1}{6}\,\epsilon_{\hat{i}}{}^{\hat{\jmath}}\mu^{\hat{i}}_{mnp}\hat{\Gamma}^{mn}\hat{\Gamma}^p_{\hat{\jmath}}\big]_{\alpha\beta}\,, \tag{E.18}$$

$$\mu^{\alpha\beta\gamma\delta} = \frac{1}{8}\Big[-\mu^m{}_n(\hat{\Gamma}^{\hat{i}}_m\otimes\hat{\Gamma}^n_{\hat{i}} - \hat{\Gamma}_{mp7}\otimes\hat{\Gamma}^{pn7}) + \mu^{\hat{i}}{}_{\hat{\jmath}}\hat{\Gamma}^m_{\hat{i}}\otimes\hat{\Gamma}^{\hat{\jmath}}_m + \mathrm{i}\,\mu^{\hat{i}}_{mnp}\hat{\Gamma}^m_{\hat{i}}\otimes\hat{\Gamma}^{np7}\Big]^{[\alpha\beta\gamma\delta]}$$

where $\otimes$ denotes the tensor product of two gamma's, $\mu^m{}_n$ is traceless and $\mu^{\hat{i}}_{mnp}$ are anti-symmetric in the three lower indices.

We take the $\mathrm{Cliff}(6,\mathbb{R})$ gammas $\hat{\Gamma}_m$ with $m = 1,\ldots,6$ such that

$$\hat{\Gamma}^T_m = \hat{C}^{-1}\hat{\Gamma}_m\hat{C} \tag{E.19}$$

where $\hat{C}$ is the charge conjugation matrix satisfying $\hat{C}^T = -\hat{C}$, which we identify with the $USp(8)$ symplectic invariant Ω. The chiral gamma is given by

$$\hat{\Gamma}_7 = \mathrm{i}\,\hat{\Gamma}^1\cdots\hat{\Gamma}^6\,. \tag{E.20}$$

Since the six-dimensional manifolds we are interested in are S^4 fibrations over a Riemann surface, we further decompose the $\mathrm{Cliff}(6)$ gamma matrices according to $SO(4)\times SO(2)$. We take $m = 5,6$ to be directions along the Riemann surface

$$\begin{aligned} \hat{\Gamma}_m &= \mathbb{1}\otimes\Gamma_m \qquad m = 1,2,3,4\,,\\ \hat{\Gamma}_5 &= \gamma_1\otimes\Gamma_5\,,\\ \hat{\Gamma}_6 &= \gamma_2\otimes\Gamma_5\,, \end{aligned} \tag{E.21}$$

where Γ_m are the $SO(4)$ gamma matrices with $\Gamma_5 = \Gamma_{1234}$ and γ_1,γ_2 are the $SO(2)$ ones. Then the six-dimensional chirality matrix becomes

$$\hat{\Gamma}_7 = \mathrm{i}\,\gamma_{12}\otimes\Gamma_5\,. \tag{E.22}$$

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E.2 The $U(1)$ structure

We can now give the details of the construction of the $U(1)_S$ structure discussed in section 4.2. The $\mathcal{N}=2$ solution of [34] has an $U(1)_S$ structure corresponding to the diagonal of the $SO(2)$ holonomy on the Riemann surface $\mathbb{\Sigma}$, and the $U(1)_{\text{right}}$ subgroup of the $SO(5)$ isometry of the four-sphere, according to the embedding

$$SO(5) \supset SO(4) \simeq SU(2)_{\text{left}} \times SU(2)_{\text{right}} \supset SU(2)_{\text{left}} \times U(1)_{\text{right}} \,. \tag{E.23}$$

Seen as an element of $E_{6(6)}$, the $U(1)_S$ corresponds to a compact generator and therefore belongs to $USp(8)$. Using the expression (E.18) for the generators of $USp(8)$, and now taking the indices $m = 5, 6$ for the direction along the Riemann surface, the $U(1)_S$ generator can be written as

$$\mathfrak{u}(1)_S = \mathrm{i}\,\hat{\Gamma}_{56} - \frac{\mathrm{i}}{2}\,(\hat{\Gamma}_{12} - \hat{\Gamma}_{34})\,, \tag{E.24}$$

where $\hat{\Gamma}_{56}$ is the generator of the $SO(2)$ holonomy of $\mathbb{\Sigma}$ and $\frac{\mathrm{i}}{2}(\hat{\Gamma}_{12} - \hat{\Gamma}_{34})$ generates $U(1)_{\text{right}} \subset SO(5)$.

To embed this generator in $E_{6(6)}$ and determine the invariant generalised tensors it is convenient to decompose all $E_{6(6)}$ representations into $USp(8)$ one's and then use the parameterisation of $USp(8)$ in terms of gamma matrices of section E.1. In this way the computation of the commutant, $\mathrm{C}_{E_{6(6)}}(U(1)_S)$, of $U(1)_S$ in $E_{6(6)}$ and the determination of the $U(1)_S$ singlets reduce to simple gamma matrix algebra.

We first compute the commutators of $U(1)_S$ with the generic elements of the **36** and **42** in (E.18). This will allow to determine the number of $U(1)_S$ singlets in the **78** and the commutant $C_{E_{6(6)}}(U(1)_S)$. Using (E.6) we find that there are eight singlets in the **36**. Five correspond to elements of $SO(6) \subset SL(6)$,

$$\begin{aligned} S_1^{(36)} &= \hat{\Gamma}_{56}\,, & S_4^{(36)} &= \frac{1}{2}(\hat{\Gamma}_{24} - \hat{\Gamma}_{13})\,, \\ S_2^{(36)} &= \hat{\Gamma}_{12}\,, & S_5^{(36)} &= \frac{1}{2}(\hat{\Gamma}_{14} + \hat{\Gamma}_{23})\,, \\ S_3^{(36)} &= \hat{\Gamma}_{34}\,, \end{aligned} \tag{E.25}$$

two are compact elements of $(\overline{\mathbf{20}}, \mathbf{2})$ associated to

$$\begin{aligned} S_6^{(36)} &= \frac{1}{2}(\hat{\Gamma}_{135} + \hat{\Gamma}_{146} - \hat{\Gamma}_{236} + \hat{\Gamma}_{245})\,, \\ S_7^{(36)} &= \frac{1}{2}(\hat{\Gamma}_{136} - \hat{\Gamma}_{145} + \hat{\Gamma}_{235} + \hat{\Gamma}_{246})\,, \end{aligned} \tag{E.26}$$

and the last one is the generator of $SO(2) \subset SL(2)$ corresponding to the anti-symmetric part of $\mu^1{}_2$,

$$S_8^{(36)} = \mathrm{i}\,\hat{\Gamma}_7\,. \tag{E.27}$$

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A similar computation gives the singlets in the **42**: four are non compact elements of $SL(6)$

$$\begin{aligned}
S_1^{(42)} &= -\frac{1}{4}(\hat{\Gamma}_1 \otimes \hat{\Gamma}^1 + \hat{\Gamma}_2 \otimes \hat{\Gamma}^2 - \hat{\Gamma}_5 \otimes \hat{\Gamma}^5 - \hat{\Gamma}_6 \otimes \hat{\Gamma}^6) \\
S_2^{(42)} &= -\frac{1}{4}(\hat{\Gamma}_3 \otimes \hat{\Gamma}^3 - \hat{\Gamma}_5 \otimes \hat{\Gamma}^5 - \hat{\Gamma}_6 \otimes \hat{\Gamma}^6) \\
S_3^{(42)} &= -\frac{1}{2}(\hat{\Gamma}_1 \otimes \hat{\Gamma}^4 + \hat{\Gamma}_4 \otimes \hat{\Gamma}^1 + \hat{\Gamma}_2 \otimes \hat{\Gamma}^3 + \hat{\Gamma}_3 \otimes \hat{\Gamma}^2) \\
S_4^{(42)} &= -\frac{1}{2}(\hat{\Gamma}_2 \otimes \hat{\Gamma}^4 + \hat{\Gamma}_4 \otimes \hat{\Gamma}^2 - \hat{\Gamma}_1 \otimes \hat{\Gamma}^3 - \hat{\Gamma}_3 \otimes \hat{\Gamma}^1)\,,
\end{aligned} \tag{E.28}$$

two are the non-compact generators of $SL(2)$

$$\begin{aligned}
S_5^{(42)} &= \frac{1}{4}(\hat{\Gamma}^m \otimes \hat{\Gamma}_m + \hat{\Gamma}^m\hat{\Gamma}_7 \otimes \hat{\Gamma}_m\hat{\Gamma}_7) \\
S_6^{(42)} &= \frac{i}{4}(\hat{\Gamma}^m \otimes \hat{\Gamma}_m\hat{\Gamma}_7 + \hat{\Gamma}^m\hat{\Gamma}_7 \otimes \hat{\Gamma}_m)\,,
\end{aligned} \tag{E.29}$$

and the remaining ones are in the $(\overline{\mathbf{20}}, \mathbf{2})$

$$\begin{aligned}
S_7^{(42)} &= -\frac{1}{4}(\hat{\Gamma}^6 \otimes \hat{\Gamma}^{237} - \hat{\Gamma}^5 \otimes \hat{\Gamma}^{137} - \hat{\Gamma}^5 \otimes \hat{\Gamma}^{247} - \Gamma^6 \otimes \Gamma^{147}) \\
S_8^{(42)} &= -\frac{1}{4}(\hat{\Gamma}^6 \otimes \hat{\Gamma}^{247} - \hat{\Gamma}^5 \otimes \hat{\Gamma}^{147} + \hat{\Gamma}^5 \otimes \hat{\Gamma}^{237} + \hat{\Gamma}^6 \otimes \hat{\Gamma}^{137})\,.
\end{aligned} \tag{E.30}$$

These singlets generate the commutant of $U(1)_S$ in $E_{6(6)}$. Given the number of singlets this must be

$$C_{E_{6(6)}}(U(1)_S) = \mathbb{R}^+ \times Spin(3,1) \times SU(2,1) \times U(1)_S\,. \tag{E.31}$$

From the commutators (E.6) and (E.7) it is easy to see that the factor $\mathbb{R}^+$ is generated by the combination

$$J_{\mathbb{R}} = S_1^{(42)} + S_2^{(42)}\,. \tag{E.32}$$

Similarly it is straightforward to identify the generators of the group $SO(3,1)$ as

$$\begin{aligned}
J_1^{SO(3,1)} &= \frac{\mathrm{i}}{2}(S_2^{(36)} + S_3^{(36)})\,, & K_1^{SO(3.1)} &= \frac{\mathrm{i}}{4}(S_1^{(42)} - S_2^{(42)})\,, \\
J_2^{SO(3,1)} &= \frac{\mathrm{i}}{2}S_4^{(36)}\,, & K_2^{SO(3,1)} &= -\frac{\mathrm{i}}{4}S_3^{(42)}\,, \\
J_3^{SO(3,1)} &= \frac{\mathrm{i}}{2}S_5^{(36)}\,, & K_3^{SO(3,1)} &= \frac{\mathrm{i}}{4}S_4^{(42)}\,.
\end{aligned} \tag{E.33}$$

The remaining singlets give $SU(2,1)$. The compact generators are defined as

$$\begin{aligned}
J_1^{SU(2,1)} &= -\frac{\mathrm{i}}{2}S_7^{(36)} \\
J_2^{SU(2,1)} &= \frac{\mathrm{i}}{2}S_8^{(36)} \\
J_3^{SU(2,1)} &= -\frac{\mathrm{i}}{4}(S_1^{(36)} + S_2^{(36)} - S_3^{(36)} - S_8^{(36)}) \\
J_8^{SU(2,1)} &= -\frac{\mathrm{i}}{4\sqrt{3}}(S_1^{(36)} + S_2^{(36)} - S_3^{(36)} + 3S_8^{(36)})\,,
\end{aligned} \tag{E.34}$$

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while the non-compact ones are

$$\begin{aligned} J_4^{SU(2,1)} &= -\frac{\mathrm{i}}{2} S_7^{(42)} & J_6^{SU(2,1)} &= -\frac{\mathrm{i}}{2} S_6^{(42)} \\ J_5^{SU(2,1)} &= \frac{\mathrm{i}}{2} S_8^{(42)} & J_7^{SU(2,1)} &= \frac{\mathrm{i}}{2} S_5^{(42)} \,. \end{aligned} \tag{E.35}$$

The compact singlets give the commutant of $U(1)_S$ into $USp(6)$,

$$\mathrm{C}_{USp(8)}(U(1)_S) = SU(2) \times SU(2)_H \times U(1) \times U(1)_S \,. \tag{E.36}$$

We also need the $U(1)_S$ singlets in the **27**. Computing the action (E.5) of $U(1)_S$ on a generic element of the **27**, given in (E.17), we find five singlets

$$\begin{aligned} \mathbf{27} = (\mathbf{1},\mathbf{1})_{(0,8)} \oplus (\mathbf{4},\mathbf{1})_{(0,-4)} \oplus (\mathbf{2},\mathbf{1})_{(3,-2)} \oplus (\bar{\mathbf{2}},\mathbf{1})_{(-3,-2)} \\ \oplus (\mathbf{1},\mathbf{3})_{(2,-4)} \oplus (\mathbf{1},\bar{\mathbf{3}})_{(-2,-4)} \oplus (\bar{\mathbf{2}},\mathbf{3})_{(1,2)} \oplus (\mathbf{2},\bar{\mathbf{3}})_{(-1,2)} \,, \end{aligned} \tag{E.37}$$

One is a singlet of both $SO(3,1)$ and $SU(2,1)$ and has charge 8 under $\mathbb{R}^+$,

$$K_0 \sim \mathrm{i}\,\hat{\Gamma}_{56}\hat{\Gamma}_7 = \mathbb{1} \otimes \Gamma_5 \,, \tag{E.38}$$

where in the second equality we used (E.21) for the gamma matrices. The other singlets are invariant under $SU(2,1)$ and form a quadruplet of $SO(3,1)$ of charge -4 under $\mathbb{R}^+$

$$\begin{aligned} K_1 &\sim \mathrm{i}\,(\hat{\Gamma}_{13} - \hat{\Gamma}_{24})\hat{\Gamma}_7 = \gamma_{(2)} \otimes (\Gamma_{13} - \Gamma_{24}) \,, \\ K_2 &\sim \mathrm{i}\,(\hat{\Gamma}_{14} + \hat{\Gamma}_{23})\hat{\Gamma}_7 = \gamma_{(2)} \otimes (\Gamma_{14} + \Gamma_{23}) \,, \\ K_3 &\sim \mathrm{i}\,(\hat{\Gamma}_{12} + \hat{\Gamma}_{34})\hat{\Gamma}_7 = \gamma_{(2)} \otimes (\Gamma_{12} + \Gamma_{34}) \,, \\ K_4 &\sim \mathrm{i}\,(\hat{\Gamma}_{12} - \hat{\Gamma}_{34})\hat{\Gamma}_7 = \gamma_{(2)} \otimes (\Gamma_{12} - \Gamma_{34}) \,. \end{aligned} \tag{E.39}$$

The singlets in the **27** and **78** are all we need to specify the generalised $U(1)_S$ structure. However, the generators of $SO(3,1)$ and $\mathbb{R}^+$ in (E.31) do not leave the singlets generalised vectors invariant and hence do not belong to the $U(1)_S$ structure. Using (E.10), one can show that they are obtained as products of the singlets in the **27** and their duals

$$J_\alpha^{SO(3,1)} = 2\mathrm{i}\,\epsilon_{\alpha\beta\gamma}(K_\beta \times_{\mathrm{ad}} K_\gamma^*) \,, \qquad K_\alpha^{SO(3,1)} = -\mathrm{i}\,(K_\alpha \times_{\mathrm{ad}} K_4^*) \,, \qquad \alpha = 1,2,3 \,, \tag{E.40}$$

and

$$J_{\mathbb{R}} = 4(K_0 \times_{\mathrm{ad}} K_0^*) - 4(K_4 \times_{\mathrm{ad}} K_4^*) \,. \tag{E.41}$$

In summary the generalised $U(1)_S$ structure is defined by the five generalised vectors and the eight generators of $SU(2,1)$

$$\{K_I, J_A\} \qquad I = 0,\ldots,4, \quad A = 1,\ldots,8 \,. \tag{E.42}$$

The last step is to derive explicit expressions for these generalised tensors in terms of geometrical objects on the six-dimensional internal manifold M. We use the fact that, in our case, M is a fibration of the four sphere over a Riemann surface and that the four-sphere is generalised parallelisable as reviewed in appendix D.3.

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We decompose the six-dimensional bundles in representation of $GL(2,\mathbb{R})$, the ordinary structure group on the Riemann surface, and $SL(5,\mathbb{R})$, the exceptional structure group of S^4. Under

$$E_{6(6)} \supset GL(2,\mathbb{R}) \times SL(5,\mathbb{R})\,, \tag{E.43}$$

the generalised tangent bundle decomposes as

$$\begin{aligned} E &\simeq T\mathbb{\Sigma} \oplus (T^*\mathbb{\Sigma} \otimes N_4) \oplus (\Lambda^2 T^*\mathbb{\Sigma} \otimes N_4') \oplus E_4\,, \\ \mathbf{27} &= (\mathbf{2},\mathbf{1}) \oplus (\mathbf{2},\mathbf{5'}) \oplus (\mathbf{1},\mathbf{5}) \oplus (\mathbf{1},\mathbf{10})\,, \end{aligned} \tag{E.44}$$

where E_4, N_4 and N_4' are defined in appendix D.3. Using (E.17) and defining $\text{Cliff}(5,\mathbb{R})$ gamma matrices as

$$\Gamma_i = \{\Gamma_1, \ldots, \Gamma_5\}\,, \tag{E.45}$$

we can identify the components of the **27** in (E.44) as

$$\begin{aligned} \{\gamma_1 \otimes \mathbb{1}, \gamma_2 \otimes \mathbb{1}\} &\in (\mathbf{2},\mathbf{1}) \\ \{\gamma_1 \otimes \Gamma_I, \gamma_2 \otimes \Gamma_I\} &\in (\mathbf{2},\mathbf{5}) \\ \mathbb{1} \otimes \Gamma_i &\in (\mathbf{1},\mathbf{5}) \\ \gamma_{(2)} \otimes \Gamma_{ij} &\in (\mathbf{1},\mathbf{10})\,. \end{aligned} \tag{E.46}$$

In terms of generalised vectors, the elements of the $(\mathbf{2},\mathbf{1})$ embed as

$$R^{-1} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix}, \tag{E.47}$$

while those in the $(\mathbf{2},\mathbf{5})$ and **(1,5)** can be written as

$$\Psi_i = R \begin{pmatrix} e_1 \wedge E_i, \\ e_2 \wedge E_i \end{pmatrix} \qquad \text{and} \qquad R^2\,\text{vol}_{\mathbb{\Sigma}} \wedge E_i'\,, \qquad i = 1, \ldots, 5\,, \tag{E.48}$$

where $\text{vol}_{\mathbb{\Sigma}} = e_1 \wedge e_2$ is the volume form on the Riemann surface, R is the S^4 radius, and E_i and E_i' are the sections of N_4 and N_4' defined in appendix D.3. The elements of the $(\mathbf{1},\mathbf{10})$ are the Ξ_α, $\widetilde{\Xi}_\alpha$, with $\alpha = 1,2,3$, defined in (D.21), and E_{i5} with $i = 1,2,3,4$.

Comparing with (E.38) and (E.39), we see that

$$K_0 \in (\mathbf{1},\mathbf{5}) \sim \Lambda^2 T^*\mathbb{\Sigma} \otimes N_4'\,, \qquad K_I \in (\mathbf{1},\mathbf{10}) \sim E_4\,, \quad \text{for } I = 1, \ldots, 4\,, \tag{E.49}$$

and can then be written as generalised vectors on M as

$$K_0 \sim R^2 \text{vol}_{\mathbb{\Sigma}} \wedge E_5'\,, \qquad K_\alpha \sim \widetilde{\Xi}_\alpha\,, \qquad K_4 \sim \Xi_3\,, \tag{E.50}$$

where $\alpha = 1,2,3$. To have the final expressions for these five generalised vectors we still have to implement the twist of S^4 as described in section 4.2. The $E_{6(6)}$ element implementing the twist is

$$\Upsilon = -\frac{R}{2}\, \upsilon \times_{\text{ad}} \Xi_3\,,$$

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$$= -\xi_3 \otimes \upsilon - \frac{1}{4} R^3 \, \upsilon \wedge \mathrm{d}(\cos\zeta \, \sigma_3) \,, \tag{E.51}$$

and acts on the frames E_{ij}, E_i as

$$\begin{aligned} \mathrm{e}^{\Upsilon} \cdot E_{ij} &= E_{ij} + \frac{1}{2} \upsilon \wedge E_5 (\delta_{1[i}\delta_{j]2} + \delta_{3[i}\delta_{j]4}) - \frac{1}{2} \delta_{5[i} P_{j]}{}^{k} \upsilon \wedge E_k \,, \\ \mathrm{e}^{\Upsilon} \cdot E_i &= E_i + \frac{1}{2} \upsilon \wedge (E'_{[1}\delta_{2]i} + E'_{[3}\delta_{4]i}) \,, \end{aligned} \tag{E.52}$$

where $P_i{}^j$ is the matrix

$$P_i{}^j = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ 1 & & & \end{pmatrix} . \tag{E.53}$$

It is then straightforward to check that only K_4 is modified by the twist, and the expressions (4.29) are obtained.

Finally we need the expressions for the singlets in the **78** generating $SU(2,1)$. Under $E_{6(6)} \supset GL(2,\mathbb{R}) \times SL(5,\mathbb{R})$ as

$$\begin{aligned} \mathrm{ad}F &\simeq \mathrm{ad}\, F_4 \oplus (T\Sigma \otimes T^*\Sigma) \oplus (T^*\Sigma \otimes E_4) \oplus (\Lambda^2 T^*\Sigma \otimes N_4) \oplus (T\Sigma \otimes E_4^*) \oplus (\Lambda^2 T\Sigma \otimes N_4^*) \\ \mathbf{78} &\sim (\mathbf{1},\mathbf{24}) \oplus (\mathbf{4},\mathbf{1}) \oplus (\mathbf{2},\mathbf{10}) \oplus (\mathbf{1},\mathbf{5}) \oplus (\mathbf{2},\mathbf{10}) \oplus (\mathbf{1},\mathbf{5}) \end{aligned} \tag{E.54}$$

where $\mathrm{ad}\, F_4$ is the adjoint bundle on S^4

$$\mathrm{ad}\, F_4 \simeq \mathbb{R} \oplus (TS^4 \otimes T^*S^4) \oplus \Lambda^3 T^*S^4 \oplus \Lambda^3 TS^4 \,. \tag{E.55}$$

The expressions for the singlets are easily obtained from (E.10) as products of the **27** and $\overline{\mathbf{27}}$. In this way we obtain precisely the expressions given in eq. (4.30), where the twisting by Υ can be evaluated with the aid of (E.52).

F Parameterisation of the H structure moduli space

We discuss here our parameterisation of the coset space $\mathcal{M}_{\mathrm{H}} = \frac{SU(2,1)}{SU(2)\times U(1)}$ that describes the hypermultiplet structure moduli space. We model the generators of $SU(2,1)$ on the matrices j_A, $A = 1,\ldots,8$, defined as:

$$\mathsf{j}_{1,2,3} = -\mathrm{i}\, \lambda_{1,2,3} \,, \qquad \mathsf{j}_{4,5,6,7} = \lambda_{4,5,6,7} \,, \qquad \mathsf{j}_8 = -\mathrm{i}\, \lambda_8 \,, \tag{F.1}$$

where λ_A, $A = 1,\ldots,8$, are the standard Gell-Mann matrices generating the $\mathfrak{su}_3$ algebra in the fundamental representation. These generators satisfy

$$\mathsf{j}_A^\dagger \, m + m \, \mathsf{j}_A = 0 \,, \qquad \text{with } m = \mathrm{diag}(-1,-1,1) \,, \tag{F.2}$$

$$\mathrm{tr}(\mathsf{j}_A \, \mathsf{j}_B) = 2\, \eta_{AB} \qquad \text{with } \eta = \mathrm{diag}(-1,-1,-1,1,1,1,1,-1) \,, \tag{F.3}$$

as well as the commutation relations

$$[\mathsf{j}_1, \mathsf{j}_2] = 2\mathsf{j}_3, \qquad [\mathsf{j}_3, \mathsf{j}_1] = 2\mathsf{j}_2, \qquad [\mathsf{j}_2, \mathsf{j}_3] = 2\mathsf{j}_1,$$

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$$
\begin{aligned}
&[\mathsf{j}_4,\mathsf{j}_5] = -(\mathsf{j}_3+\sqrt{3}\mathsf{j}_8) \qquad \left[\frac{1}{2}(\mathsf{j}_3+\sqrt{3}\mathsf{j}_8),\mathsf{j}_4\right] = 2\mathsf{j}_5, \qquad \left[\mathsf{j}_5,\frac{1}{2}(\mathsf{j}_3+\sqrt{3}\mathsf{j}_8)\right] = 2\mathsf{j}_4,\\
&[\mathsf{j}_6,\mathsf{j}_7] = -(-\mathsf{j}_3+\sqrt{3}\mathsf{j}_8), \quad \left[\frac{1}{2}(-\mathsf{j}_3+\sqrt{3}\mathsf{j}_8),\mathsf{j}_6\right] = 2\mathsf{j}_7, \quad \left[\mathsf{j}_7,\frac{1}{2}(-\mathsf{j}_3+\sqrt{3}\mathsf{j}_8)\right] = 2\mathsf{j}_6,\\
&[\mathsf{j}_1,\mathsf{j}_4] = \mathsf{j}_7, \qquad [\mathsf{j}_7,\mathsf{j}_1] = \mathsf{j}_4, \qquad [\mathsf{j}_4,\mathsf{j}_7] = -\mathsf{j}_1,\\
&[\mathsf{j}_2,\mathsf{j}_4] = \mathsf{j}_6, \qquad [\mathsf{j}_6,\mathsf{j}_2] = \mathsf{j}_4, \qquad [\mathsf{j}_4,\mathsf{j}_6] = -\mathsf{j}_2,\\
&[\mathsf{j}_1,\mathsf{j}_5] = -\mathsf{j}_6, \qquad [\mathsf{j}_6,\mathsf{j}_1] = -\mathsf{j}_5, \qquad [\mathsf{j}_5,\mathsf{j}_6] = \mathsf{j}_1,\\
&[\mathsf{j}_2,\mathsf{j}_5] = \mathsf{j}_7, \qquad [\mathsf{j}_7,\mathsf{j}_2] = \mathsf{j}_5, \qquad [\mathsf{j}_5,\mathsf{j}_7] = -\mathsf{j}_2,\\
&[\mathsf{j}_1,\mathsf{j}_8] = [\mathsf{j}_2,\mathsf{j}_8] = [\mathsf{j}_3,\mathsf{j}_8] = 0,\\
&[\mathsf{j}_4,\sqrt{3}\mathsf{j}_3-\mathsf{j}_8] = [\mathsf{j}_5,\sqrt{3}\mathsf{j}_3-\mathsf{j}_8] = [\mathsf{j}_6,\sqrt{3}\mathsf{j}_3+\mathsf{j}_8] = [\mathsf{j}_7,\sqrt{3}\mathsf{j}_3+\mathsf{j}_8] = 0,
\end{aligned}
\tag{F.4}
$$

where the first three lines show the three $\mathfrak{su}_2$ subalgebras. Note that $\{\mathsf{j}_1,\mathsf{j}_2,\mathsf{j}_3,\mathsf{j}_8\}$ generate the compact subgroup $SU(2)\times U(1)\subset SU(2,1)$. It is convenient to choose a solvable parameterisation for the remaining generators, describing the coset space $\frac{SU(2,1)}{SU(2)\times U(1)}$. Following the appendix D of [72], we define[34]

$$
\begin{aligned}
T_1 &= \frac{1}{2\sqrt{2}}\left(\mathsf{j}_1-\mathsf{j}_2-\mathsf{j}_4-\mathsf{j}_5\right), & T_2 &= \frac{1}{2\sqrt{2}}\left(\mathsf{j}_1+\mathsf{j}_2+\mathsf{j}_4-\mathsf{j}_5\right),\\
T_\bullet &= \frac{1}{4}\left(2\mathsf{j}_7+\mathsf{j}_3-\sqrt{3}\,\mathsf{j}_8\right), & H_0 &= \frac{1}{2}\,\mathsf{j}_6\,.
\end{aligned}
\tag{F.5}
$$

These span the Borel subalgebra of the $SU(2,1)$ algebra and satisfy the commutation relations

$$
[H_0,T_\bullet] = T_\bullet\,, \qquad [H_0,T_1] = \frac{1}{2}\,T_1\,, \qquad [H_0,T_2] = \frac{1}{2}\,T_2\,, \qquad [T_1,T_2] = T_\bullet\,. \tag{F.6}
$$

A parameterisation of the coset is obtained by exponentiating the Borel subalgebra as

$$
L = \mathrm{e}^{-(\theta_1+\theta_2)T_1+(\theta_1-\theta_2)T_2+\xi\,T_\bullet}\,\mathrm{e}^{-2\varphi H_0}\,, \tag{F.7}
$$

where $\{\varphi,\xi,\theta_1,\theta_2\}$ are the four real coordinates. Starting from the coset representative (F.7), we compute the Maurer-Cartan form $L^{-1}\mathrm{d}L$ and then identify the coset vielbeine as the coefficients of its expansion in the coset generators,

$$
L^{-1}\mathrm{d}L = -2\,\mathrm{d}\varphi H_0 - \mathrm{e}^{\varphi}(\mathrm{d}\theta_1+\mathrm{d}\theta_2)T_1 + \mathrm{e}^{\varphi}(\mathrm{d}\theta_1-\mathrm{d}\theta_2)T_2 + \mathrm{e}^{2\varphi}\left(\mathrm{d}\xi-\theta_1\mathrm{d}\theta_2+\theta_2\mathrm{d}\theta_1\right)T_\bullet\,. \tag{F.8}
$$

In this way we obtain the following Einstein metric on $\frac{SU(2,1)}{SU(2)\times U(1)}$,

$$
\mathrm{d}s^2 = 2\,\mathrm{d}\varphi^2 + \mathrm{e}^{2\varphi}\left(\mathrm{d}\theta_1^2+\mathrm{d}\theta_2^2\right) + \frac{1}{2}\,\mathrm{e}^{4\varphi}\left(\mathrm{d}\xi-\theta_1\mathrm{d}\theta_2+\theta_2\mathrm{d}\theta_1\right)^2\,. \tag{F.9}
$$

The normalisation is chosen so that the Ricci scalar is $\mathcal{R}=-12$, in agreement with our five-dimensional supergravity conventions.

In the main text, we need the "dressed" $\mathfrak{su}_2$ algebra constructed via the adjoint action of the coset representative on the $\mathfrak{su}_2$ algebra generated by $\{\mathsf{j}_1,\mathsf{j}_2,\mathsf{j}_3\}$, that is

$$
\hat{\mathsf{j}}_1 = L\,\mathsf{j}_1\,L^{-1}\,, \qquad \hat{\mathsf{j}}_2 = L\,\mathsf{j}_2\,L^{-1}\,, \qquad \hat{\mathsf{j}}_3 = L\,\mathsf{j}_3\,L^{-1}\,. \tag{F.10}
$$

[34] We rearrange the indices of their 3×3 matrices as $1_{\text{there}}\to 3_{\text{here}}$, $2_{\text{there}}\to 1_{\text{here}}$, $3_{\text{there}}\to 2_{\text{here}}$.

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An explicit evaluation using (F.7) gives

$$
\begin{aligned}
\hat{\mathsf{j}}_1 &= \frac{1}{2}\mathrm{e}^{\varphi}(\mathsf{j}_1+\mathsf{j}_5)+\frac{1}{4}\mathrm{e}^{\varphi}(\theta_2^2-3\theta_1^2+2\mathrm{e}^{-2\varphi})(\mathsf{j}_1-\mathsf{j}_5)+\frac{1}{2}\mathrm{e}^{\varphi}(\xi-2\theta_1\theta_2)(\mathsf{j}_2+\mathsf{j}_4)+\frac{1}{\sqrt{2}}\mathrm{e}^{\varphi}\theta_2\mathsf{j}_6\\
&\quad-\frac{1}{2\sqrt{2}}\mathrm{e}^{\varphi}\theta_1(3\mathsf{j}_3+\sqrt{3}\mathsf{j}_8)+\frac{1}{4\sqrt{2}}\left[\mathrm{e}^{\varphi}(\theta_1^3+\theta_1\theta_2^2-2\theta_2\xi)-2\mathrm{e}^{-\varphi}\theta_1\right](\mathsf{j}_3+2\mathsf{j}_7-\sqrt{3}\mathsf{j}_8)\,,\\
\hat{\mathsf{j}}_2 &= \frac{1}{2}\mathrm{e}^{\varphi}(\mathsf{j}_2-\mathsf{j}_4)+\frac{1}{4}\mathrm{e}^{\varphi}(\theta_1^2-3\theta_2^2+2\mathrm{e}^{-2\varphi})(\mathsf{j}_2+\mathsf{j}_4)-\frac{1}{2}\mathrm{e}^{\varphi}(\xi+2\theta_1\theta_2)(\mathsf{j}_1-\mathsf{j}_5)-\frac{1}{\sqrt{2}}\mathrm{e}^{\varphi}\theta_1\mathsf{j}_6\\
&\quad-\frac{1}{2\sqrt{2}}\mathrm{e}^{\varphi}\theta_2(3\mathsf{j}_3+\sqrt{3}\mathsf{j}_8)+\frac{1}{4\sqrt{2}}\left[\mathrm{e}^{\varphi}(\theta_2^3+\theta_1^2\theta_2+2\theta_1\xi)-2\mathrm{e}^{-\varphi}\theta_2\right](\mathsf{j}_3+2\mathsf{j}_7-\sqrt{3}\mathsf{j}_8)\,,\\
\hat{\mathsf{j}}_3 &= -\frac{1}{4\sqrt{2}}\left[\mathrm{e}^{2\varphi}(\theta_1^3+\theta_1\theta_2^2+2\theta_2\xi)-6\theta_1\right](\mathsf{j}_1-\mathsf{j}_5)-\frac{1}{4\sqrt{2}}\left[\mathrm{e}^{2\varphi}(\theta_2^3+\theta_1^2\theta_2-2\theta_1\xi)-6\theta_2\right](\mathsf{j}_2+\mathsf{j}_4)\\
&\quad+\frac{1}{2\sqrt{2}}\mathrm{e}^{2\varphi}\left[\theta_1(\mathsf{j}_1+\mathsf{j}_5)+\theta_2(\mathsf{j}_2-\mathsf{j}_4)\right]-\frac{1}{2}\mathrm{e}^{2\varphi}(\xi\mathsf{j}_6+\mathsf{j}_7)+\frac{1}{8}\left[2-\mathrm{e}^{2\varphi}(\theta_1^2+\theta_2^2)(3\mathsf{j}_3+\sqrt{3}\mathsf{j}_8)\right]\\
&\quad+\frac{1}{32}\left[\mathrm{e}^{2\varphi}(\theta_1^2+\theta_2^2)^2+4\mathrm{e}^{2\varphi}(1+\xi^2)-12(\theta_1^2+\theta_2^2)+4\mathrm{e}^{-2\varphi}\right](\mathsf{j}_3+2\mathsf{j}_7-\sqrt{3}\mathsf{j}_8)\,. \qquad \text{(F.11)}
\end{aligned}
$$

Now we can replace the matrices j_A with the generalised tensors J_A invariant under the $U(1)$ generalised structure. This provides our four-parameter family of H structures.

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The higher-dimensional origin of five-dimensional $\mathcal{N}=2$ gauged supergravities

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Abstract: Using exceptional generalised geometry, we classify which five-dimensional $\mathcal{N}=2$ gauged supergravities can arise as a consistent truncation of 10-/11-dimensional supergravity. Exceptional generalised geometry turns the classification into an algebraic problem of finding subgroups $G_S \subset \mathrm{USp}(8) \subset \mathrm{E}_{6(6)}$ that preserve exactly two spinors. Moreover, the intrinsic torsion of the G_S structure must contain only constant singlets under G_S, and these, in turn, determine the gauging of the five-dimensional theory. The resulting five-dimensional theories are strongly constrained: their scalar manifolds are necessarily symmetric spaces and only a small number of matter multiplets can be kept, which we completely enumerate. We also determine the largest reductive and compact gaugings that can arise from consistent truncations.

Contents

1 Introduction

When studying compactifications of 10- and 11-dimensional supergravities, the low-energy limits of string theory, it is useful to have a lower-dimensional theory which captures key aspects of the physics. If the compactification leads to a separation of scales, we can obtain a lower-dimensional low-energy effective supergravity theory by integrating out modes above the cut-off scale. This is the case for compactifications on special holonomy manifolds to Minkowski space-time, where the effective theory is obtained by keeping only the massless modes, namely the zero-modes of appropriate differential operators on the internal space.

However, when there is no separation of scales, or if we want to keep both some light and massive modes within the truncation, we must instead resort to a *consistent truncation* of 10-/11-dimensional supergravity [1]. A consistent truncation ensures that all solutions of the lower-dimensional theory also satisfy the equations of motion of 10-/11-dimensional supergravity. Consistent truncations are therefore particularly relevant for anti-de Sitter

(AdS) compactifications, where no explicit scale-separated example is known. This is even conjectured to be true for all AdS compactifications of string theory [2].

Constructing consistent truncations is a notoriously difficult problem, due to the highly non-linear equations of motion of 10/11-dimensional supergravity (see e.g. [1, 3]). Thus it might be tempting to use lower-dimensional gauged supergravity models without a clear higher-dimensional origin. However, this is fraught with dangers. For example, a vacuum that appears stable within a lower-dimensional supergravity might suffer from instabilities triggered by modes not kept in the truncation [4], or vacua which appear different within the lower-dimensional model may actually be identified in the full 10/11-dimensional theory [5]. These examples highlight how important it is to know which lower-dimensional theories can arise as consistent truncations of 10-/11-dimensional supergravity.

Until recently, the only systematic approach to consistent truncations relied on considering manifolds with reduced structure group and keeping all the modes that are singlets under the reduced group. For example, group manifolds (and freely-acting discrete quotients thereof) have a trivial structure group and give the classic Scherk–Schwarz reductions [6]. Alternatively one can consider, for example, Sasaki–Einstein and weak-G_2 holonomy manifolds of [7–10], or tri-Sasakian manifolds [11]. However, there are also famous consistent truncations, such as those of 11-dimensional supergravity on S^7 [12] and S^4 [13, 14], that cannot be explained by this traditional group action argument.

Recently, it has become clear that the appropriate framework for understanding general consistent truncations of 10-/11-dimensional supergravity is given by generalised G_S-structures in exceptional generalised geometry and exceptional field theory.

Exceptional generalised geometry and exceptional field theory are reformulations of 10/11-dimensional supergravity in a way which unifies fluxes and metric degrees of freedom into exceptional symmetry groups. In exceptional generalised geometry, for instance, the exceptional groups appear as structure groups of the generalised tangent bundle of the compactification manifold, which is an extension of the tangent bundle by appropriate exterior powers of the cotangent bundle. A reduction of the exceptional structure group to a subgroup G_S defines a reduced "generalised structure group". Given such generalised a G_S structure one can define its "intrinsic torsion" [15]. In analogy to the case of conventional G structures, this is a differential object that measures the obstruction to finding a torsion-free connection, compatible with the structure. For a given structure, it can decomposed into generalised tensors transforming in particular G_S representations.

It is now understood [16] that generalised G_S structures provide a systematic and general derivation of consistent truncations: any generalised G_S structure with constant singlet intrinsic torsion defines a consistent truncation of 10-/11-dimensional supergravity. For instance, all maximally supersymmetric truncations are associated to generalised identity structure and so can be seen as generalised Scherk-Schwarz reductions [17–19]. This provides a unified description of the consistent truncations of 11-dimensional supergravity on S^7 and S^4, as well as of IIB supergravity on S^5 and massive IIA on spheres [20, 21], and give a framework for analysing generic maximally supersymmetric truncations [22–24]. Moreover, considering larger generalised structure groups, we obtain consistent truncations preserving less supersymmetry [16, 25–27].

What is particularly interesting in this approach is that a good deal of information about the reduced lower-dimensional theory is derived from purely algebraic considerations. The embedding of the generalised structure group G_S in $\mathrm{E}_{n(n)}$ completely fixes the field content and the allowed components of the embedding tensor of the reduced theory, as well as the truncation ansatz.

One is then left with the problem of solving the differential consistency condition that the G_S structure has constant, singlet intrinsic torsion. This will determine whether there exists an internal manifold that realises any of the reduced theories allowed by the algebraic analysis.

In this paper, we apply these ideas to classify consistent truncations of M-theory and type IIB supergravities to five-dimensional $\mathcal{N} = 2$ gauged supergravities. In this case the relevant exceptional group is $\mathrm{E}_{6(6)}$. We focus on the algebraic part of the problem, that is identifying the possible $G_S \subset \mathrm{E}_{6(6)}$ structures, and work under the hypothesis that the differential one is solved.

We first classify all the continuous subgroups of $\mathrm{E}_{6(6)}$ that give rise to only two spinor supercharges in five-dimensions, as required by $\mathcal{N} = 2$ supersymmetry, and we derive, in each case, the field content of the reduced theory. This allows us to show that the structure of the five-dimensional $\mathcal{N} = 2$ gauged supergravities that can arise from consistent truncations of type II/11-dimensional supergravity is very constrained. For example, the scalar manifolds of such gauged supergravities must necessarily be symmetric, and there is a maximum number of vector and hypermultiplets that can be coupled. Indeed, we find that only a handful of matter contents can arise from consistent truncations.

We can then further constrain the allowed truncated theories as follows. Under the assumption that the compactification manifolds satisfy the differential constraint of constant singlet intrinsic torsion, we determine the embedding tensors of the reduced theory and analyse the possible gaugings. Again purely group-theoretical arguments allow us to fully determine the gauging of the reduced theory. As expected these include as special cases the known truncations [27, 28] that arise from the $\mathcal{N} = 2$ Maldacena–Nuñez [29] and "BBBW" [30] backgrounds. We find in general that the embedding tensor is constrained, so that generically not all gaugings that are allowed from a five-dimensional point of view are realised as consistent truncations. The result is that the gauged supergravities that can be obtained as consistent truncations are a very small subset of those that can be constructed from a purely five-dimensional point of view.

It is worth stressing that while our analysis gives the list of the reduced theories that can a priori be obtained as consistent truncations, this does not mean that all of them can actually be realised. First, one must find compactification manifolds that admit the appropriate G_S generalised structure groups. Secondly, we must show that they satisfy the condition of constant singlet intrinsic torsion, and then analyse the non-zero components of intrinsic torsion/embedding tensor to see which gauge algebras in fact appear. The analogous condition is known to limit the possible gaugings in the maximally supersymmetric case [17, 21–24, 31]. So it is to be expected that the number of actual truncations is even more restricted than what we present here.

This result is of particular interest for theories with AdS vacua. It is conjectured

that no AdS vacua of string theory admit scale separation [2]. Hence it is not possible to write an effective $\mathcal{N}=2$ theory in this case. Thus we are led to conjecture that those gauged supergravities that cannot come from consistent truncations and which have AdS vacua must belong to the "swampland". Put differently, these gauged supergravities are lower-dimensional artefacts that are not related to string theory.

This paper is structured as follows. In Section 2 we recall the main features of 5-dimensional $\mathcal{N}=2$ gauged supergravity. In particular, we describe the gauging procedure in terms of Leibniz algebras, as this is the natural language to make the connection to exceptional generalised geometry. The exceptional generalised geometry formalism for $\mathcal{N}=2$ truncations to five dimensions is reviewed in Section 3. We first introduce $\mathrm{E}_{6(6)}$ generalised geometry, which is the relevant one for compactifications to five dimensions. Then we discuss the G_S structures that are associated to $\mathcal{N}=2$ truncations and establish the dictionary between the G_S structure data and those of the truncated theory. Section 4 contains the main results of the paper, namely the classification of the gauged supergravity that can come from consistent truncations of M-theory or type IIB supergravities. We organise the list according to the field content, first theories with only vector and tensor multiplets, then only hypermultiplets and finally those with both vector/tensor and hypermultiplets. Appendix A contains more details about $\mathrm{E}_{6(6)}$ exceptional geometry, while in Appendix B, for concreteness, we provide the explicit computation of the intrinsic torsion for the truncation with n_{VT} vector multiplets. Finally in Appendix C we discuss the truncation ansatz.

2 $5d$ $\mathcal{N}=2$ gauged supergravity: moduli spaces and gaugings

In this section, we summarise the features of five-dimensional $\mathcal{N}=2$ gauged supergravity coupled to matter [32–34] that we want to reproduce from consistent truncations of M-theory or type IIB theory. We follow the conventions of [34].

We are interested in $5d$ $\mathcal{N}=2$ supergravity coupled to n_{V} vector multiplets, n_{T} tensor multiplets and n_{H} hyper-multiplets. The gravity multiplet consists of the graviton, two gravitini transforming as a doublet of the R-symmetry group $\mathrm{SU}(2)_R$ and the graviphoton,

$$\{g_{\mu\nu},\, \psi_\mu^{\tilde{x}},\, A_\mu\}. \tag{2.1}$$

The index $\tilde{x}=1,2$ denotes the $\mathrm{SU}(2)_R$ R-symmetry. Each vector multiplet contains a vector, two spin-1/2 fermions in the fundamental of $\mathrm{SU}(2)_R$ and a complex scalar ϕ. Since in five dimensions a vector is dual to a two-form, a tensor multiplet has the same number of degrees of freedom. Thus we have vector and tensors multiplets

$$\{A_\mu,\, \lambda^{\tilde{x}},\, \phi\}, \qquad \{B_{\mu\nu},\, \lambda^{\tilde{x}},\, \phi\}, \qquad \tilde{x}=1,2\,. \tag{2.2}$$

If we have n_{V} vector multiplets and n_{T} tensor multiplets we will use the notation A_μ^I with $I=0,\ldots,n_{\mathrm{V}}$ to denote the graviphoton and the vectors fields and $B_{\mu\nu}^M$ with $M=n_{\mathrm{V}}+1,\ldots n_{\mathrm{V}}+n_{\mathrm{T}}$ for the two-form fields. The scalars of the vector and tensor multiplets are grouped together into ϕ^i, with $i=1,\ldots,n_{\mathrm{V}}+n_{\mathrm{T}}$. These scalars parametrise a very special real manifold, $\mathcal{M}_{\mathrm{VT}}$.

There are also hypermultiplets, each of which consists of four real scalars and an R-symmetry doublet of spin-1/2 fermions

$$\{\zeta^{\tilde{x}},\, q^u\}\,, \qquad u=1,\ldots,4\,, \quad \tilde{x}=1,2\,. \tag{2.3}$$

If we have $n_{\rm H}$ hypermultiplets, the scalars are grouped into q^X, with $X=1,\ldots,4n_{\rm H}$, and parameterise a quaternionic-Kähler manifold, $\mathcal{M}_{\rm H}$. It is also convenient to collect the spinors ζ into $\zeta^{\tilde{\alpha}}$, with $\tilde{\alpha}=1,\ldots,2n_{\rm H}$, transforming in the fundamental representation of $\mathrm{USp}(2n_{\rm H})$.

The very special real manifold can be described as an $n_{\rm VT}$-dimensional cubic hypersurface in an $(n_{\rm VT}+1)$-dimensional ambient space, where $n_{\rm VT}=n_{\rm V}+n_{\rm T}$. Viewing $h^{\tilde{I}}=h^{\tilde{I}}(\phi^i)$, with $\tilde{I}=0,\ldots,n_{\rm VT}$, as embedding coordinates, $\mathcal{M}_{\rm VT}$ is given by

$$C(h)=C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}}=1\,, \tag{2.4}$$

where $C_{\tilde{I}\tilde{J}\tilde{K}}$ is a completely symmetric constant tensor.

The metric on $\mathcal{M}_{\rm VT}$ is given by

$$g_{ij}=h_i^{\tilde{I}}h_j^{\tilde{J}}\,a_{\tilde{I}\tilde{J}}\,, \tag{2.5}$$

where $a_{\tilde{I}\tilde{J}}$ is the metric on the ambient space

$$a_{\tilde{I}\tilde{J}}=3h_{\tilde{I}}h_{\tilde{J}}-2C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{K}}\,, \tag{2.6}$$

and

$$\begin{aligned} h_i^{\tilde{I}}&=-\sqrt{\tfrac{3}{2}}\,\partial_i h^{\tilde{I}}\,,\\ h_{\tilde{I}}&=C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{K}}h^{\tilde{L}}=a_{\tilde{I}\tilde{K}}h^{\tilde{K}}\,. \end{aligned} \tag{2.7}$$

The homogeneous "very special real" manifolds have been classified in [35]. For the symmetric ones, which are the only ones we will need, a classification is possible based on whether the polynomial (2.4) or, equivalently, the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ is factorisable or not [36–38]. We will discuss this classification in Section 4.

The $4n_H$ scalars of the hypermultiplets parameterise a quaternionic Kähler manifold $\mathcal{M}_H$, of real dimension $4n_H$ with metric

$$g_{XY}=C_{\tilde{\alpha}\tilde{\beta}}\epsilon_{\tilde{x}\tilde{y}}f_X{}^{\tilde{\alpha}\tilde{x}}f_Y{}^{\tilde{\beta}\tilde{y}}\,, \tag{2.8}$$

where $f_X{}^{\tilde{\alpha}\tilde{x}}$ are the quaternionic vielbeine and $C_{\tilde{\alpha}\tilde{\beta}}$ is the flat metric on $\mathrm{USp}(2n_{\rm H})$. On $\mathcal{M}_{\rm H}$ there exist a (local) triplet of complex structures $\vec{J}_X{}^Y$ satisfying

$$[J^\alpha,J^\beta]=2\epsilon^{\alpha\beta\gamma}J_q\,, \qquad (J^\alpha)^2=-\mathrm{Id}\,, \qquad \alpha,\beta=1,2,3\,, \tag{2.9}$$

with respect to which the metric g_{XY} is hermitian.

As for the vector multiplets, only symmetric spaces will be relevant for consistent truncations. The Riemannian symmetric quaternionic–Kähler spaces were first considered by Wolf in [39] and then classified by Alekseevsky in [40]. This was then extended to the

pseudo-Riemannian class by Alekseevsky and Cortés in [41]. We will discuss the relevant ones in Section 4.

Together, the scalar manifold of the theory is the direct product

$$\mathcal{M} = \mathcal{M}_{\rm VT} \times \mathcal{M}_{\rm H}\,, \tag{2.10}$$

with isometries $G_{\rm iso} = G_{\rm VT} \times G_{\rm H}$, where $G_{\rm VT}$ and $G_{\rm H}$ are the isometry groups of $\mathcal{M}_{\rm VT}$ and $\mathcal{M}_{\rm H}$ respectively and define the global symmetries of the ungauged theory.[1]

The most general gauged theory is described in [34]. It is useful for what follows to translate it into the language of Leibniz algebras (or more precisely "G-algebras" [23]). In doing so we also see how the gauging picks out the space of vector and tensor multiplets. Let $\mathcal{V}$ be the vector space of dimension $n_{\rm VT}+1$ formed by the graviphoton, the $n_{\rm V}$ vectors and $n_{\rm T}$ tensors. The gauging defines a Leibniz algebra $\mathfrak{a}$, on $\mathcal{V}$, that is a bilinear bracket $[\![v,w]\!]$ that satisfies a Leibniz-relation

$$[\![u,[\![v,w]\!]]\!] = [\![[\![u,v]\!],w]\!] + [\![v,[\![u,w]\!]]\!]\,, \qquad \forall\, u,v,w \in \mathcal{V}. \tag{2.11}$$

Choosing a basis, the algebra defines a set of structure constants $t_{\tilde{J}\tilde{K}}{}^{\tilde{I}}$ via

$$[\![v,w]\!]^{\tilde{I}} = t_{\tilde{J}\tilde{K}}{}^{\tilde{I}} v^{\tilde{J}} w^{\tilde{K}}\,, \qquad \forall\, v,w \in \mathcal{V}. \tag{2.12}$$

Note that in general $[\![v,w]\!] \neq -[\![w,v]\!]$ (that is $t_{\tilde{J}\tilde{K}}{}^{\tilde{I}} \neq -t_{\tilde{K}\tilde{J}}{}^{\tilde{I}}$) so the bracket does not define a Lie algebra.

We then define the subspace $\mathcal{T} \subset \mathcal{V}$ as the image of the symmetrised bracket

$$\mathcal{T} = \{[\![v,w]\!] + [\![w,v]\!] : v,w \in \mathcal{V}\}\,, \tag{2.13}$$

and identify elements of $\mathcal{T}$ with tensor multiplets, so that $\dim \mathcal{T} = n_{\rm T}$. Note that the Leibniz condition (2.11) implies that

$$[\![b,v]\!] = 0\,, \qquad \forall\, b \in \mathcal{T},\, v \in \mathcal{V}\,. \tag{2.14}$$

Thus $[\![v,b]\!] = [\![v,b]\!] + [\![b,v]\!] \in \mathcal{T}$ and hence $\mathcal{T}$ forms a two-sided ideal. As a consequence, if we identify the space of vector multiplets as the quotient $\mathcal{R} = \mathcal{V}/\mathcal{T}$, then the bracket descends to an ordinary Lie bracket on $\mathcal{R}$ defining what we will call the "extended Lie algebra" $\mathfrak{g}_{\rm ext}$. Note that by construction $\mathcal{V}$ is a reducible representation of $\mathfrak{g}_{\rm ext}$ where $\mathcal{T}$ forms an invariant subspace.

If one chooses a particular splitting so $\mathcal{V} = \mathcal{R} \oplus \mathcal{T}$ and fixes a basis, where $I = 0, 1, \ldots, n_{\rm V}$ labels components in $\mathcal{R}$ and $M = n_{\rm V}+1, \ldots, n_{\rm VT}$ labels components in $\mathcal{T}$, this structure means that one has

$$t_{IJ}{}^{K} = f_{IJ}{}^{K}\,, \qquad t_{M\tilde{I}}{}^{\tilde{J}} = 0\,, \qquad t_{(I\tilde{J})}{}^{I} = 0\,, \tag{2.15}$$

[1] In the case of no hypermultiplets, we define $G_{\rm H} = {\rm SU}(2)$ so that $G_{\rm iso}$ still matches the global symmetries.

where ${f_{IJ}}^K = -{f_{JI}}^K$ are the structure constants of the Lie algebra $\mathfrak{g}_{\rm ext}$. In summary, we see that the splitting into vector and tensor multiplets is defined by the choice of Leibniz algebra.

The choice of Leibniz algebra is not completely general if it is to lead to a consistent gauging. Note first that we can define the adjoint action given some $v \in \mathcal{V}$ as

$$\begin{aligned} t_v \,:\, & \mathcal{V} \to \mathcal{V}\,, \\ & w \mapsto t_v w := [\![v, w]\!]\,, \end{aligned} \tag{2.16}$$

so that, in components t_v is the matrix $(t_v)^{\tilde{J}}{}_{\tilde{I}} = v^{\tilde{K}} {t_{\tilde{K}\tilde{I}}}^{\tilde{J}}$. From the Leibniz condition, the commutator is given by

$$[t_v, t_w] = t_{[\![v,w]\!]}\,, \tag{2.17}$$

and furthermore $t_b = 0$ for all $b \in \mathcal{T}$. Hence the adjoint action defines a Lie algebra. In terms of the split basis, we have the generators [42] (see also [43]),

$$(t_I)_{\tilde{J}}{}^{\tilde{K}} = \begin{pmatrix} (t_I)_J{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix}\,, \qquad \begin{aligned} & I, J, K = 0, \ldots, n_{\rm V}\,, \\ & M, N = n_{\rm V}+1, \ldots, n_{\rm T}\,, \end{aligned} \tag{2.18}$$

such that

$$[t_I, t_J] = -{f_{IJ}}^K t_K\,, \tag{2.19}$$

where ${f_{IJ}}^K$ are the structure constants of $\mathfrak{g}_{\rm ext}$. The components $(t_I)_{\tilde{J}}{}^N$ give the representation of the gauge group on the tensors. The off-diagonal components $(t_I)_J{}^N$ can be non-zero only in the case of non-compact groups since these allow for non-completely reducible representations [42, 43].

Consistency requires that the symmetric tensor C in (2.4) is invariant under the action of t_v

$$C(t_v(w), w, w) = 0\,, \qquad \forall\, v, w \in \mathcal{V}\,, \tag{2.20}$$

and that the expression

$$C(b, v, w) = \Omega(\tfrac{1}{2} t_v(w) + \tfrac{1}{2} t_w(v), b)\,, \qquad \forall\, b \in T,\, v, w \in \mathcal{V}\,, \tag{2.21}$$

defines a symplectic form Ω on $\mathcal{T}$. This implies, in particular, that the bracket defines a symplectic representation of $\mathfrak{g}_{\rm ext}$ on $\mathcal{T}$. Invariance of C in turn means that the action of t_v is an isometry of the metric on $\mathcal{M}_{\rm VT}$. In components, these conditions read

$${t_{(I\tilde{J})}}^M = \Omega^{MN} C_{NI\tilde{J}}\,, \qquad {t_{I(\tilde{J}}}^{\tilde{H}} C_{\tilde{K}\tilde{L})\tilde{H}} = 0\,. \tag{2.22}$$

Note that the first condition is equivalent to requiring that the map $\mathcal{V} \otimes \mathcal{V} \to \mathcal{T}$ defined by $(v, w) \mapsto [\![v, w]\!] + [\![w, v]\!]$ factors through $\mathcal{V}^*$ via

$$\mathcal{V} \otimes \mathcal{V} \xrightarrow{\;C\;} \mathcal{V}^* \xrightarrow{\;\Omega^{-1}\;} \mathcal{T}\,. \tag{2.23}$$

That is, it can be viewed as a map to $\mathcal{V}^*$ given by $(v, w) \mapsto C(v, w, \cdot)$ followed by the action of Ω^{-1}.

The gauging of the five-dimensional theory can be expressed in terms of the embedding tensor [44, 45]. This is a map

$$\Theta \,:\, \mathcal{V} \to \mathfrak{g}_{\rm iso}\,, \tag{2.24}$$

where $\mathfrak{g}_{\rm iso}$ is the Lie algebra of isometries of the scalar manifold of the underlying *rigid* supersymmetric theory. In this case, this means the product of the real cone over $\mathcal{M}_{\rm VT}$ (that is the ambient space $\mathcal{V}$) with the hyper-Kähler cone over $\mathcal{M}_{\rm H}$. Given $v \in \mathcal{V}$ the embedding tensor $\Theta(v)$ specifies how the action of t_v gauges the isometries. That is, it defines the embedding of the gauge algebra, $\mathfrak{g}_{\rm gauge}$, inside the isometry algebra of the scalar manifold, where we define

$$\mathfrak{g}_{\rm gauge} = \mathfrak{a}/\mathrm{Ker}\,\Theta\,. \tag{2.25}$$

The matter fields of the $\mathcal{N}=2$ gauged supergravity are charged under $\mathfrak{g}_{\rm gauge}$ rather than the larger $\mathfrak{g}_{\rm ext}$, which generically is a central extension of $\mathfrak{g}_{\rm gauge}$.

For $\mathcal{N}=2$ supersymmetry the isometry algebra splits $\mathfrak{g}_{\rm iso} = \mathfrak{g}_{\rm VT} \oplus \mathfrak{g}_{\rm H}$ where $\mathfrak{g}_{\rm VT}$ and $\mathfrak{g}_{\rm H}$ are the Lie algebras of isometries on the vector and hypermultiplet rigid moduli spaces respectively. For very special real and quaternionc Kähler homogeneous spaces these are just the Lie algebras of the numerator groups $G_{\rm VT}$ and $G_{\rm H}$, except when there are no hypermultiplets in which case $G_{\rm H} = \mathrm{SU}(2)$, in line with footnote 1. The embedding tensor thus splits into two parts [46, 47]. For the vector multiplets the isometries on the cone are generated by a basis composed of Killing vectors k^i_a on $\mathcal{M}_{\rm VT}$, where $a = 1, \ldots, \dim \mathfrak{g}_{\rm VT}$. For the hypermultiplets the isometries on the hyper-Kähler cone are generated by a basis composed of Killing vectors $\tilde{k}^X_m$, with $m = 1, \ldots, \dim \mathfrak{g}_{\rm H}$, together with $\mathfrak{su}(2)_R$ elements $\vec{P}_m$ that are the Killing prepotentials[2] for each $\tilde{k}^X_m$. The generators that are gauged are then given by

$$k^i_{\tilde{I}}(\phi) = \Theta_{\tilde{I}}{}^a k^i_a(\phi)\,, \qquad \tilde{k}^X_{\tilde{I}}(q) = \Theta_{\tilde{I}}{}^m \tilde{k}^X_m(q)\,, \qquad \vec{P}_{\tilde{I}} = \Theta_{\tilde{I}}{}^m \vec{P}_m\,, \tag{2.26}$$

where $\tilde{I}, \tilde{J}, \tilde{K} = 0, \ldots, n_{\rm VT}$ and $X, Y = 1, \ldots, 4n_{\rm H}$. The two pieces of the embedding tensor $\Theta_I{}^a$ and $\Theta_I{}^m$ are thus constant $(\tilde{n}_{\rm V}+1) \times \dim \mathfrak{g}_{\rm VT}$ and $(\tilde{n}_{\rm V}+1) \times \dim \mathfrak{g}_{\rm H}$ matrices, whose rank determines the dimension of the gauge group. The $k_{\tilde{I}}$ vectors are required to act linearly on the embedding coordinates $h^{\tilde{I}}$ such that

$$k^i_{\tilde{I}} \partial_i h^{\tilde{J}} = t_{\tilde{I}\tilde{K}}{}^{\tilde{J}} h^{\tilde{K}}\,, \tag{2.27}$$

thus relating $\Theta_{\tilde{I}}{}^a$ to the structure constants $t_{\tilde{I}\tilde{J}}{}^{\tilde{K}}$. Given a splitting $\mathcal{V} = \mathcal{R} \oplus \mathcal{T}$, one then has $k_M = \tilde{k}_M = 0$ and

$$\begin{aligned} [k_I, k_J]^i &= f_{IJ}{}^K\, k^i_K\,, \\ [\tilde{k}_I, \tilde{k}_J]^X &= f_{IJ}{}^K\, \tilde{k}^X_K\,, \end{aligned} \tag{2.28}$$

realising the gauge algebra $\mathfrak{g}_{\rm gauge}$.

[2] In the case where there are no hypermultiplets, one can still have constant prepotentials $\vec{P}_m$ with $m = 1, 2, 3$ that can lead to Fayet–Iliopoulos terms.

On the scalars the gauging defines covariant derivatives

$$\begin{aligned}\mathcal{D}_\mu\phi^i &= \partial_\mu\phi^i + g\,k_I^i\mathcal{A}_\mu^I\,,\\ \mathcal{D}_\mu q^X &= \partial_\mu q^X + g\,\tilde{k}_I^X\mathcal{A}_\mu^I\,.\end{aligned} \tag{2.29}$$

The bosonic Lagrangian of the gauged theory is then given by

$$\begin{aligned}e^{-1}\mathcal{L} = {}& \frac{1}{2}R - \mathcal{V}(\phi,q) - \frac{1}{4}a_{\tilde{I}\tilde{J}}H^{\tilde{I}}_{\mu\nu}H^{\tilde{J}\mu\nu} - \frac{3}{4}a_{\tilde{I}\tilde{J}}\mathcal{D}_\mu h^{\tilde{I}}\mathcal{D}^\mu h^{\tilde{J}} - \frac{1}{2}g_{XY}\mathcal{D}_\mu q^X\mathcal{D}_\mu q^Y\\ &+ \frac{e^{-1}}{16g}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}B^M_{\mu\nu}(\partial_\rho B^N_{\sigma\tau} + 2gt^N_{IJ}A^I_\rho F^J_{\sigma\tau} + gt^N_{IP}A^I_\rho B^P_{\sigma\tau})\\ &+ \frac{1}{12}\sqrt{\frac{2}{3}}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}C_{IJK}A^I_\mu\left[F^J_{\nu\rho}F^K_{\sigma\tau} + f^J_{FG}A^F_\nu A^G_\rho\Big(-\frac{1}{2}F^K_{\sigma\tau} + \frac{g^2}{10}f^K_{HL}A^H_\sigma A^L_\tau\Big)\right]\\ &- \frac{1}{8}e^{-1}\epsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}t^M_{IK}t^N_{FG}A^I_\mu A^F_\nu A^G_\rho\left(-\frac{g}{2}F^K_{\sigma\tau} + \frac{g^2}{10}f^K_{HL}A^H_\sigma A^L_\tau\right)\,.\end{aligned} \tag{2.30}$$

The kinetic terms for the vector/tensor[3] and hypermultiplets are controlled by the metrics $a_{\tilde{I}\tilde{J}}$ and g_{XY}, defined in (2.6) and (2.8). The gauge field strengths

$$\mathcal{F}^I = \mathrm{d}A^I - \tfrac{1}{2}gf^I_{JK}A^J\wedge A^K\,, \tag{2.33}$$

and the anti-symmetric tensors $B_{\mu\nu}$ are grouped into the tensors $H^{\tilde{I}}_{\mu\nu} = (\mathcal{F}^I_{\mu\nu}, B^M_{\mu\nu})$. We see that the gauge field strengths are indeed elements of the extended algebra $\mathfrak{g}_{\rm ext}$, while the matter fields have non-trivial charge only under the action of the gauge algebra $\mathfrak{g}_{\rm gauge}$.

In general, the scalar potential $\mathcal{V}$ is a function of the Killing vectors on the scalar manifolds $\mathcal{M}_{\rm VT}$ and $\mathcal{M}_{\rm H}$, and on the Killing prepotentials, $\vec{P}_I$, on $\mathcal{M}_{\rm H}$[4]

$$\mathcal{V} = 2g^2\left(g^{ij}\vec{P}_i\cdot\vec{P}_j - 2\vec{P}\cdot\vec{P} + g_{ij}\mathcal{K}^i\mathcal{K}^j + \mathcal{N}_{\tilde{\alpha}\tilde{x}}\mathcal{N}^{\tilde{\alpha}\tilde{x}}\right)\,, \tag{2.35}$$

where the arrow denotes a triplet of $\mathfrak{su}(2)_R$ elements and

$$\begin{aligned}\vec{P} &= h^I\vec{P}_I\,, & \mathcal{K}^i &= \tfrac{\sqrt{6}}{4}h^I k^i_I\,,\\ \vec{P}_i &= \partial_i\vec{P} = h^I_i\vec{P}_I\,, & \mathcal{N}^{\tilde{\alpha}\tilde{x}} &= \tfrac{\sqrt{6}}{4}h^I\tilde{k}^X_I f_X{}^{\tilde{\alpha}\tilde{x}}\,.\end{aligned} \tag{2.36}$$

Notice that due to the identity $h^{\tilde{I}}k^i_{\tilde{I}} = 0$, the Killing vectors on $\mathcal{M}_{\rm VT}$ do not contribute to the potential when there are no tensor multiplets.

[3]The vector multiplet scalar kinetic term can also be written in terms of the scalar fields ϕ^i and the metric g_{ij} on $\mathcal{M}_{\rm VT}$ using

$$\mathcal{D}_\mu h^{\tilde{I}} = \partial_\mu h^{\tilde{I}} + g\,f^{\tilde{I}}_{\tilde{J}\tilde{K}}\mathcal{A}^{\tilde{J}}_\mu h^{\tilde{K}} = \partial_i h^{\tilde{I}}\mathcal{D}_\mu\phi^i\,, \tag{2.31}$$

and the identity

$$\tfrac{3}{2}\,a_{\tilde{I}\tilde{J}}\mathcal{D}_\mu h^{\tilde{I}}\mathcal{D}^\mu h^{\tilde{J}} = g_{ij}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\phi^j\,. \tag{2.32}$$

[4]The Killing prepotentials $\vec{P}_I$ are defined by

$$4n_{\rm H}\vec{P}_I \;=\; \vec{J}_X{}^Y\nabla_Y\tilde{k}^X_I\,, \tag{2.34}$$

where $\vec{J}_X{}^Y$ is the triplet of (local) complex structures on $\mathcal{M}_{\rm H}$.

The functions in (2.36) also control the bosonic part of the supersymmetry variations:

$$\begin{aligned}\delta\psi_\mu^{\tilde{x}} &= D_\mu\epsilon^{\tilde{x}} + \frac{ig}{\sqrt{6}}P^{\tilde{x}\tilde{y}}\gamma_\mu\epsilon_{\tilde{y}} + \dots\,,\\ \delta\lambda^{i\tilde{x}} &= g\mathcal{K}^i\epsilon^{\tilde{x}} + gP^{i\tilde{x}\tilde{y}}\epsilon_{\tilde{y}} + \dots\,,\\ \delta\zeta^{\tilde{\alpha}} &= g\mathcal{N}^{\tilde{\alpha}}{}_{\tilde{x}}\epsilon^{\tilde{x}} + \dots\,.\end{aligned} \tag{2.37}$$

where we have written out the explicit adjoint action of the $\mathfrak{su}(2)_R$ elements $\vec{P}$ and $\vec{P}_i$.

3 $\mathcal{N}=2$ supergravities from generalised geometry

In the language of exceptional generalised geometry, consistent truncations are associated to generalised G_S-structures. If a d-dimensional manifold M admits a generalised G_S-structure, namely a set of globally defined generalised invariant tensors, with constant intrinsic torsion, a consistent truncation of type II or eleven-dimensional supergravity on M is obtained by expanding all supergravity fields on such tensors and keeping only the G_S-singlet modes. Knowing the generalised structure is enough to determine all the data of the truncated theory. This approach has been successfully applied to the study of consistent truncations with several amount of supersymmetry [16, 17, 21] (see also for the exceptional field theory version of this approach [18, 20, 25, 26]). In particular, [27] provides the generic framework to study type IIB or M-theory consistent truncations to five dimensions with $\mathcal{N}=2$ supersymmetry. The purpose of this paper is to use this formalism to classify the possible consistent truncations of type IIB or M-theory to five-dimensional $\mathcal{N}=2$ supergravity.

In this section, we give a brief summary of the exceptional generalised geometry relevant for type IIB or M-theory reductions to five dimensions and then in the next section we review the formalism of [27].

3.1 Generalised G_S structures and $\mathcal{N}=2$ supersymmetry

Type IIB or M-theory supergravity on a d-dimensional manifold M, with $d=5$ for type IIB and $d=6$ for M-theory, are conveniently reformulated in terms of $\mathrm{E}_{6(6)}\times\mathbb{R}^+$ generalised geometry[5]. For definiteness, we will focus on the M-theory case, though the formalism is equally applicable in type IIB.

To the manifold M we associate a generalised tangent bundle E, whose sections transform in the real $\mathbf{27}^*$ representation[6] of $\mathrm{E}_{6(6)}$, the generalised structure group, with weight one under $\mathbb{R}^+$. The ordinary structure group $GL(d)$ embeds in $\mathrm{E}_{6(6)}\times\mathbb{R}^+$ and can be used to decompose the generalised tangent bundle as

$$E \simeq TM\oplus\Lambda^2T^*M\oplus\Lambda^5T^*M\,. \tag{3.1}$$

[5] See Appendix A for a more detailed review of $\mathrm{E}_{6(6)}\times\mathbb{R}^+$ generalised geometry

[6] Given a representation $\mathbf{n}$ we will use $\mathbf{n}^*$ and $\overline{\mathbf{n}}$ for the dual and conjugate representations, respectively. For non-compact groups these may not be equivalent.

The sections of E are called generalised vectors and, using (3.1), can be seen as (local) sums of a vector, a two-form and a five-form on M,

$$V = v + \omega + \sigma \,. \tag{3.2}$$

The frame bundle F for E defines an $\mathrm{E}_{6(6)} \times \mathbb{R}^+$ principal bundle. By considering bundles whose fibres transform in different representations of $\mathrm{E}_{6(6)} \times \mathbb{R}^+$, we can then define other generalised tensors. To describe the bosonic sector of the supergravity theories we will need, besides the generalised vectors, weighted dual vectors, adjoint tensors and the generalised metric. Adjoint tensors R are sections of the adjoint bundle $\mathrm{ad}\, F$ of the form

$$\begin{aligned} \mathrm{ad}F &\simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM \,, \\ R &= l + r + a + \tilde{a} + \alpha + \tilde{\alpha} \,, \end{aligned} \tag{3.3}$$

and hence transform in the $\mathbf{1} + \mathbf{78}$ of $\mathrm{E}_{6(6)}$ with weight zero under the $\mathbb{R}^+$ action. Locally l is a function, r a section of $\mathrm{End}(TM)$, a is a three-form and so on. One notes that in the exceptional geometric reformulation, the internal components of the gauge potentials of type II or M-theory, are embedded in the adjoint bundle.

It will be useful to also define weighted dual vectors Z as sections of the bundle $N \simeq \det T^*M \otimes E^*$ which has $\mathbb{R}^+$ weight two[7]. Concretely one finds

$$\begin{aligned} N &\simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \,, \\ Z &= \lambda + \rho + \tau \,. \end{aligned} \tag{3.4}$$

Finally the generalised metric G is a positive-definite, symmetric rank-2 tensor

$$G \in \Gamma(\det T^*M \otimes S^2 E^*) \,, \tag{3.5}$$

so that, given two generalised vectors $V, W \in \Gamma(E)$, the inner product $G(V, W)$ is a top form. Just as an ordinary metric g, at each point on M, parameterises the coset $\mathrm{GL}(6)/\mathrm{O}(6)$, a generalised metric at a point $p \in M$ corresponds to an element of the coset

$$G|_p \in \frac{\mathrm{E}_{6(6)} \times \mathbb{R}^+}{\mathrm{USp}(8)/\mathbb{Z}_2} \,. \tag{3.6}$$

The generalised metric encodes the internal components of all bosonic fields of type II or M-theory on M.

The fermionic fields of type IIB or M-theory are arranged into representations of $\mathrm{USp}(8)$, the double cover of the maximal compact subgroup $\mathrm{USp}(8)/\mathbb{Z}_2$ of $\mathrm{E}_{6(6)}$. For instance, supersymmetry parameters are section of the generalsied spinor bundle $\mathcal{S}$, transforming in the $\mathbf{8}$ of $\mathrm{USp}(8)$. The R-symmetry of the reduced five-dimensional theory is in general then some subgroup $G_R \subseteq \mathrm{USp}(8)$.

[7] Note that $\det T^*M$ is just a different notation for the top-form bundle $\Lambda^6 T^*M$ that stresses that it is a real line bundle. In the following we will assume that the manifold is orientable and hence $\det T^*M$ is trivial. Thus, we can define arbitrary powers $(\det T^*M)^p$ for any real p.

A generalised G_S structure is the reduction of the generalised structure group $\mathrm{E}_{6(6)}\times\mathbb{R}^+$ to a subgroup G_S. For all the structure groups that we discuss here, this is equivalent to the existence on M of globally defined generalised tensors that are invariant under G_S.[8] For example, the generalised metric G in (3.5) defines an $\mathrm{USp}(8)/\mathbb{Z}_2$ structure. In what follows, since we always assume the existence of a generalised metric, we will consider G_S structures that are subgroups of $\mathrm{USp}(8)/\mathbb{Z}_2$. Moreover, we are interested in generalised structures preserving some amount of supersymmetry and hence we need the structure group to lift to a subgroup $\tilde{G}_S$ of USp(8) acting on the spinor bundle $\mathcal{S}$ and to keep track of how many spinors are singlets of $\tilde{G}_S$. In all the cases considered here we have $\tilde{G}_S \simeq G_S$. Hence for simplicity we will from now on write G_S for both. For $\mathcal{N}=2$ supersymmetry we need two invariant supercharges in the spinor bundle $\mathcal{S}$ implying that we need subgroups $G_S \subset \mathrm{USp}(8)$ that give only two singlets when decomposing the **8** of USp(8).

The largest structure group giving $\mathcal{N}=2$ supersymmetry is $G_S = \mathrm{USp}(6)$: under the breaking

$$\mathrm{USp}(8) \supset \mathrm{USp}(6)\times \mathrm{SU}(2)_R\,, \tag{3.7}$$

the spinorial representation decomposes as

$$\mathbf{8} = (\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,. \tag{3.8}$$

The $\mathrm{SU}(2)_R$ factor in (3.7) is the R-symmetry of the reduced theory under which the two spinors singlets form a doublet, as expected for $\mathcal{N}=2$ supersymmetry parameters. One also has the decompositions for the $\mathrm{E}_{6(6)}$ representations

$$\begin{aligned} \mathbf{27}^* &= (\mathbf{1},\mathbf{1}) \oplus (\mathbf{14},\mathbf{1}) \oplus (\mathbf{6},\mathbf{2})\,,\\ \mathbf{78} &= (\mathbf{1},\mathbf{3}) \oplus (\mathbf{6},\mathbf{2}) \oplus (\mathbf{21},\mathbf{1}) \oplus (\mathbf{14},\mathbf{1}) \oplus (\mathbf{14}',\mathbf{2})\,. \end{aligned} \tag{3.9}$$

Note that the embedding of the structure $\mathrm{USp}(6) \subset \mathrm{E}_{6(6)}$, in contrast to (3.7), defines the subgroup

$$\mathrm{E}_{6(6)} \supset \mathrm{USp}(6)\cdot \mathrm{SU}(2)_R\,, \tag{3.10}$$

where we are using the "central product" between USp(6) and $\mathrm{SU}(2)_R$. By definition, for any group G and subgroup H, the commutant[9] $\mathrm{C}_G(H)$ of H in G includes the centre $Z(H)$ of H. The central product is defined to be $H\cdot \mathrm{C}_G(H) = (H\times \mathrm{C}_G(H))/Z(H)$ where one modes out by the diagonal $Z(H)$ subgroup. In this case $Z(\mathrm{USp}(6)) = \mathbb{Z}_2$ and the central product reflects the fact that the maximal compact subgroup of $\mathrm{E}_{6(6)}$ is $\mathrm{USp}(8)/\mathbb{Z}_2$ and not USp(8).

The $G_S = \mathrm{USp}(6)$ structure is often called an *HV structure* [48–50] and can also be defined in terms of non-vanishing invariant adjoint tensors and a generalised vector, corresponding to the singlets under $G_S = \mathrm{USp}(6)$ in (3.9). As they will be useful in the

[8] For non-simple (and discrete) groups, you can in principle have G_S groups that are not defined as stabilizer groups of tensors.

[9] Throughout this paper we will use the notation $\mathrm{C}_G(H)$, with $H\subset G$, for the commutant (or centralizer) of H within G.

rest of the paper, let us first introduce the vector and hypermultiplet structures that these tensors separately define.

A vector-multiplet structure, or *V structure*, is given by a globally defined generalised vector $K \in \Gamma(E)$ of positive norm with respect to the $\mathrm{E}_{6(6)}$ cubic invariant,

$$c(K,K,K) \;:=\; 6\,\boldsymbol{\kappa}^2 > 0\,, \tag{3.11}$$

where $\boldsymbol{\kappa}$ is a section of $(\det T^*M)^{1/2}$. The vector K is the $(\mathbf{1},\mathbf{1})$ singlet in the decomposition of the $\mathbf{27}^*$ in (3.9) and is stabilised by $\mathrm{F}_{4(4)} \subset \mathrm{E}_{6(6)}$. A hypermultiplet structure, or *H structure*, is determined by a pair $(J_\alpha, \boldsymbol{\kappa}^2)$ where $J_\alpha \in \Gamma(\mathrm{ad}F)$ $(\alpha = 1,2,3)$ is a triplet that define a basis for a highest root $\mathfrak{su}_2$ subalgebra of $\mathfrak{e}_{6(6)}$ and hence satisfy

$$[J_\alpha, J_\beta] \;=\; 2\epsilon_{\alpha\beta\gamma}J_\gamma\,, \qquad \mathrm{tr}(J_\alpha J_\beta) \;=\; -\delta_{\alpha\beta}\,, \tag{3.12}$$

while $\boldsymbol{\kappa}$ is a section of $(\det T^*M)^{1/2}$ as above. The J_α correspond to the $(\mathbf{1},\mathbf{3})$ triplet in the decomposition of the $\mathbf{78}$ in (3.9) and are stabilised by $\mathrm{SU}^*(6) \subset \mathrm{E}_{6(6)}$.

The HV structure corresponds to a V and an H structure, such that the two $\boldsymbol{\kappa}$ densities are the same and in addition compatibility constraint

$$J_\alpha \cdot K \;=\; 0\,, \tag{3.13}$$

is satisfied, where $\cdot$ denotes the adjoint action (see Appendix A for all relevant definitions). The common stabiliser of compatible K and J_α is

$$\mathrm{SU}^*(6) \cap \mathrm{F}_{4(4)} \simeq \mathrm{USp}(6)\,. \tag{3.14}$$

As shown in [27], given an USp(6) structure, one can construct a generalised metric as

$$G(V,V) \;=\; 3\left(3\,\frac{c(K,K,V)^2}{c(K,K,K)^2} - 2\frac{c(K,V,V)}{c(K,K,K)} + 4\,\frac{c(K,J_3\cdot V, J_3\cdot V)}{c(K,K,K)}\right)\,, \tag{3.15}$$

where c is the $\mathrm{E}_{6(6)}$ cubic invariant and V is a generalised vector.

As we will discuss later, in terms of the multiplets of the truncated theory, an HV structure, that is one where $G_S = \mathrm{USp}(6)$, implies that there are neither vector multiplets nor hypermultiplets present; the reduced theory is minimal $\mathcal{N} = 2$ supergravity. To allow for vectors or hypermultiplets, one has to look for reduced structure groups $G_S \subset \mathrm{USp}(6)$ such that in the decomposition

$$\mathrm{USp}(8) \supset \mathrm{USp}(6) \times \mathrm{SU}(2)_R \supset G_S \times \mathrm{SU}(2)_R\,, \tag{3.16}$$

additional G_S singlets beyond those defined by the USp(6) structure appear in $\mathbf{27}^*$ and the $\mathbf{78}$, but none in the $\mathbf{8}$. This means the $\mathbf{6}$ in the decomposition (3.8) cannot admit any singlets, and hence that all the singlets in the $\mathbf{27}^*$ must transform trivially under $\mathrm{SU}(2)_R$.

Each $G_S \subset \mathrm{USp}(6)$ singlet will give a G_S-invariant generalised tensor in the corresponding bundle. In particular, the singlets in $\mathbf{27}^*$ will span a sub-bundle E_{singlet}

$$E \supset E_{\mathrm{singlet}} \simeq M \times \mathcal{V}\,. \tag{3.17}$$

The bundle is by definition trivial and hence can be written as a product where $\mathcal{V}$ is the fibre. The vector space $\mathcal{V}$ transforms as a representation of the commutant $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ of G_S in $\mathrm{E}_{6(6)}$. In particular, from the discussion above, there must be an R-symmetry subgroup $\mathrm{SU}(2)_R \subset \mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ that acts trivially on $\mathcal{V}$ (and hence E_{singlet}). Furthermore, the corresponding Lie algebra $\mathfrak{su}(2)$ must correspond to a highest root in $\mathfrak{e}_{6(6)}$. Let us define G_{H} as the simple subgroup of $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ that contains such a highest root SU(2). We can then also identify the corresponding trivial sub-bundle of the adjoint bundle[10]

$$\operatorname{ad} F \supset \operatorname{ad} F_{G_{\mathrm{H}}} \simeq M \times \mathfrak{g}_{\mathrm{H}}\,, \tag{3.18}$$

where $\mathfrak{g}_{\mathrm{H}}$ is the Lie algebra of G_{H}. Note that by definition $R \cdot v = 0$ for all $v \in \Gamma(E_{\text{singlet}})$ and $R \in \Gamma(\operatorname{ad} F_{G_{\mathrm{H}}})$.

Given any trivial G_S-invariant vector bundle $P \simeq M \times \mathbb{R}^n$ and G_S-compatible generalised connection $\tilde{D}$, one can define a constant section $s \in \Gamma(P)$ by $\tilde{D}s = 0$. Furthermore, the definition is independent of the choice of $\tilde{D}$ since the bundle transforms trivially under G_S. For the sub-bundles E_{singlet} and $\operatorname{ad} F_{G_{\mathrm{H}}}$ we can identify $\mathcal{V}$ and $\mathcal{U} \simeq \mathfrak{g}_{\mathrm{H}}$ with the spaces of constant sections

$$\begin{aligned} \mathcal{V} &= \left\{ v \in \Gamma(E_{\text{singlet}}) : \tilde{D}v = 0 \right\}\,, \\ \mathfrak{g}_{\mathrm{H}} \simeq \mathcal{U} &= \left\{ R \in \Gamma(\operatorname{ad} F_{G_{\mathrm{H}}}) : \tilde{D}R = 0 \right\}\,, \end{aligned} \tag{3.19}$$

giving a natural realisation of the isomorphisms (3.17) and (3.18). Note that the elements of $\mathcal{U}$ generate a global G_{H} symmetry. The G_S-structure also defines a constant invariant section $\boldsymbol{\kappa}^2 \in \Gamma(\det T^*M)$. Hence for each $v \in \mathcal{V}$ the expression

$$C(v,v,v) = \boldsymbol{\kappa}^{-2} c(v,v,v)\,, \tag{3.20}$$

where c is the $\mathrm{E}_{6(6)}$ cubic invariant, defines a map into $\mathbb{R}$ (or more precisely to constant functions on M). We can always choose a basis of normalised nowhere-vanishing linearly independent vectors and adjoint elements for $\mathcal{V}$ and $\mathcal{U}$

$$\{K_{\tilde{I}}\,, J_A\}\,, \qquad \tilde{I} = 0, \ldots, \dim \mathcal{V} - 1, \quad A = 1, \ldots, \dim G_{\mathrm{H}}\,, \tag{3.21}$$

where by definition we have

$$J_A \cdot K_{\tilde{I}} = 0\,, \qquad \forall\, \tilde{I},\, A\,. \tag{3.22}$$

In this basis, the components $C_{\tilde{I}\tilde{J}\tilde{K}}$ of the map (3.20) are given by

$$c(K_{\tilde{I}}, K_{\tilde{J}}, K_{\tilde{K}}) = 6\,\boldsymbol{\kappa}^2 C_{\tilde{I}\tilde{J}\tilde{K}}\,, \tag{3.23}$$

and define a symmetric, constant tensor, while the adjoint tensor basis J_A satisfy

$$[J_A, J_B] = f_{AB}{}^C J_C\,, \tag{3.24}$$

[10] Note that there are singlets in the adjoint bundle that are not in $\operatorname{ad} F_{G_{\mathrm{H}}}$. In addition to elements generating the other possible factors in $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ there are also elements of the form $V \otimes_{\mathrm{ad}} W$, where V is a section of E_{singlet}, W is a section of the dual bundle E^*_{singlet} and $\otimes_{\mathrm{ad}}$ is the projection onto the adjoint bundle. However these will not play a relevant role in our construction.

where ${f_{AB}}^C$ are the structure constants of $\mathfrak{g}_{\rm H}$. Finally, we can normalise

$$\mathrm{tr}(J_A J_B) = \eta_{AB}\,, \tag{3.25}$$

where η_{AB} is a diagonal matrix with -1 and $+1$ entries in correspondence with compact and non-compact generators of $G_{\rm H}$, respectively. Note that in the "minimal" case of $G_S = \mathrm{USp}(6)$ with the HV structure (K, J_α) the spaces $\mathcal{V}$ and $\mathcal{U}$ are one- and three-dimensional, with basis vectors K and J_α, respectively.

3.1.1 Moduli space of HV structures

A strict USp(6) structure is rigid, up to an overall scaling of $\boldsymbol{\kappa}^2$. However, a reduced $G_S \subset \mathrm{USp}(6)$ structure group naturally leads to a moduli space of G_S-invariant HV structures. Note that the moduli do not necessarily consist of massless scalar fields from the point of view of the reduced $\mathcal{N} = 2$ five-dimensional theory, but rather will lead to a consistent truncation.

Out of the invariant tensors $K_{\tilde{I}}$ and J_A defining the G_S structure, we can define an HV structure by constructing a vector $K \in \mathcal{V}$ and a triplet $J_\alpha \in \mathcal{U}$ that form a basis for a highest root $\mathfrak{su}(2)$ algebra in $\mathfrak{g}_{\rm H}$. Any such HV structure is related to another by the local action of $g \in \mathrm{E}_{6(6)} \times \mathbb{R}^+$. The $\mathbb{R}^+$ factor rescales $\boldsymbol{\kappa}^2$ and can be absorbed by rescaling of the metric in the reduced theory. It therefore does not define a modulus and we can consider only $g \in \mathrm{E}_{6(6)}$. In order for the deformed HV structure to remain in $\mathcal{V}$ and $\mathcal{U}$, the action g needs to lie in the commutant group $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ and to be constant in the sense that $Dg = 0$ for any G_S compatible connection $\tilde{D}$. In other words, different points in the moduli space of G_S-invariant HV structures are related by global $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ transformations.

However, the actual physical moduli come from the generalised metric. Given a reference USp(6) structure, we can build a reference generalised metric using the definition (3.15). The physical moduli are then generated by acting on the structure with elements of $\mathrm{E}_{6(6)}$ that commute with G_S, modulo elements of $\mathrm{USp}(8)/\mathbb{Z}_2$, that leave the generalised metric invariant. The moduli obtained this way hence parameterise the coset

$$\mathcal{M} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(8)/\mathbb{Z}_2}(G_S)}\,. \tag{3.26}$$

By definition we are only considering G_S that only admits $\mathcal{N} = 2$ supersymmetry, in other words we are not interested in theories that are subsectors of more supersymmetric ones. This means there are no elements of $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ that lead to two different USp(6) structures with the same generalised metric. Hence $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ must factorise into groups that act separately on $\mathcal{V}$ and $\mathcal{U}$, that is

$$\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S) = \mathrm{C}_{G_\mathcal{U}}(G_S) \times \mathrm{C}_{G_\mathcal{V}}(G_S)\,, \tag{3.27}$$

where $G_\mathcal{U}$ and $G_\mathcal{V}$ are the subgroups of $\mathrm{E}_{6(6)}$ that leave fixed all elements of $\mathcal{U}$ and $\mathcal{V}$, respectively. Consequently, the moduli space $\mathcal{M}$ factorises into V structure and H structure moduli spaces, as expected from $\mathcal{N} = 2$ supergravity,

$$\mathcal{M} = \mathcal{M}_{\rm VT} \times \mathcal{M}_{\rm H} = \frac{\mathrm{C}_{G_\mathcal{U}}(G_S)}{\mathrm{C}_{H_\mathcal{U}}(G_S)} \times \frac{\mathrm{C}_{G_\mathcal{V}}(G_S)}{\mathrm{C}_{H_\mathcal{V}}(G_S)} = \frac{G_{\rm VT}}{H_{\rm VT}} \times \frac{G_{\rm H}}{H_{\rm H}}\,, \tag{3.28}$$

where, similarly, $H_{\mathcal{U}}$ and $H_{\mathcal{V}}$ are the subgroups of $\mathrm{USp}(8)/\mathbb{Z}_2$ that leave $\mathcal{U}$ and $\mathcal{V}$ fixed, respectively. In general there are common factors that cancel between the numerators and denominators in the commutator group expression for the cosets; for example the centre $C(G_S)$ is always a subgroup common to both. Thus it is useful to introduce the notation G_{VT}, G_{H}, H_{VT} and H_{H} for the numerators and denominators that remain in the quotients in (3.28) once all the common factors have been cancelled (except when there are no hypermultiplets in which case we take $G_{\mathrm{H}} = H_{\mathrm{H}} = \mathrm{SU}(2)$). For $\mathcal{M}_{\mathrm{H}}$, one finds G_{H} is the simple subgroup of $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ that contains a highest root $\mathrm{SU}(2)$, consistent with our definition of G_{H} above .

The V structure moduli space corresponds to deformations of K that leave J_α invariant, while the H structure moduli space describes deformations of J_α that leave K invariant. When given a dependence on the external spacetime coordinates, these deformations provide the scalar fields in the truncated theory, with $\mathcal{M}_{\mathrm{VT}}$ and $\mathcal{M}_{\mathrm{H}}$ being identified with the vector multiplet and the hypermultiplet scalar manifolds, respectively.

We can identify the moduli explicitly as follows. Consider first $\mathcal{M}_{\mathrm{VT}}$. Using the basis $K^{\tilde{I}}$, a general vector $K \in \mathcal{V}$ can be written as a linear combination

$$K = h^{\tilde{I}} K_{\tilde{I}}\,, \tag{3.29}$$

where $h^{\tilde{I}}$, $\tilde{I} = 0, \ldots, n_{\mathrm{VT}}$, are real parameters. Fixing $\boldsymbol{\kappa}^2$ in (3.11), and using (3.23), gives

$$C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1\,, \tag{3.30}$$

showing that the $n_{\mathrm{VT}} + 1$ parameters $h^{\tilde{I}}$ are constrained by one real relation and thus define an n_{VT}-dimensional hypersurface, just as in (2.4),

$$\mathcal{M}_{\mathrm{VT}} = \{\, h^{\tilde{I}} :\ C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1 \,\}\,. \tag{3.31}$$

The space $\mathcal{M}_{\mathrm{VT}}$ is the moduli space of the V structure and, in the truncation, will determine the vector multiplet scalar manifold of the five-dimensional theory. The metric on $\mathcal{M}_{\mathrm{VT}}$ is obtained by evaluating the generalised metric on the invariant generalised vectors,

$$a_{\tilde{I}\tilde{J}} = \tfrac{1}{3}\, G(K_{\tilde{I}}, K_{\tilde{J}})\,. \tag{3.32}$$

It is straightforward to verify that, using (3.15), the expression above reproduces the five-dimensional expression (2.6).

Consider now $\mathcal{M}_{\mathrm{H}}$. The family of H structures is obtained by parameterising the possible choices of $\mathfrak{su}_2$ algebra. Recall that by definition $\mathcal{U} \simeq \mathfrak{g}_{\mathrm{H}}$, so we are interested in the space of highest root $\mathfrak{su}(2) \subset \mathfrak{g}_{\mathrm{H}}$ subalgebras. Fixing $\boldsymbol{\kappa}^2$ and modding out by the $\mathrm{SU}(2)$ symmetry that relates equivalent triples J_α we have the moduli space

$$\mathcal{M}_{\mathrm{H}} \;=\; \frac{G_{\mathrm{H}}}{\mathrm{SU}(2)_R \cdot \mathrm{C}_{G_{\mathrm{H}}}(\mathrm{SU}(2)_R)}\,, \tag{3.33}$$

that is, comparing with (3.28), we have $H_{\mathrm{H}} = \mathrm{SU}(2)_R \cdot \mathrm{C}_{G_{\mathrm{H}}}(\mathrm{SU}(2)_R)$. Points in $\mathcal{M}_{\mathrm{H}}$ can be parameterised by starting from a reference subalgebra $\mathfrak{j} \simeq \mathfrak{su}_2 \subset \mathfrak{g}_{\mathrm{H}}$ and then acting on a basis $\{j_1, j_2, j_3\}$ of $\mathfrak{j}$ by the adjoint action of group elements $h \in G_{\mathrm{H}}$, defined as

$$J_\alpha = \mathrm{ad}_{G_{\mathrm{H}}} j_\alpha = h\, j_\alpha\, h^{-1}\,. \tag{3.34}$$

One has to mod out by the elements of G_{H} that have a trivial action, namely $h \in \mathrm{SU}(2)_R \simeq \exp(\mathfrak{j})$ and $h \in \mathrm{C}_{G_{\mathrm{H}}}(\mathrm{SU}(2)_R)$. The resulting symmetric spaces (3.33) and are all quaternionic–Kähler, in agreement with the identification of $\mathcal{M}_{\mathrm{H}}$ with the hyperscalar manifold in five-dimensional supergravity.

3.1.2 Singlet generalised intrinsic torsion

Any generalised G_S structure has an associated intrinsic torsion [15]. Given a G_S-compatible generalised connection, $\tilde{D}$, its torsion T is defined as

$$\left(L^{\tilde{D}}_V - L_V\right)\alpha = T(V)\cdot\alpha\,, \tag{3.35}$$

where α is a generic generalised tensor, L is the generalised Lie derivative (see Appendix A), $L^{\tilde{D}}$ is the generalised Lie derivative calculated using $\tilde{D}$ and $\cdot$ is the adjoint action on α.

As a generalised tensor, the torsion T belongs to the sub-bundle

$$W \in E^* \oplus K \subset E^* \otimes \mathrm{ad}\, F\,, \tag{3.36}$$

with E^* transforming in the **27** representation and K in the **351** representation.

Let $\Sigma = \tilde{D} - \tilde{D}'$ be the difference between two G_S-compatible generalised connections. It is a generalised tensor, specifically a section of $K_{G_S} = E^* \otimes \mathrm{ad}\, F_{G_S}$, where $\mathrm{ad}\, F_{G_S} \subset \mathrm{ad}\, F$ is the G_S-adjoint subbundle defined by the structure. Using (3.35) one can define a map from K_{G_S} to W, the space of generalised torsions,

$$\begin{aligned} \tau :\ & K_{G_S} \to W\,, \\ & \Sigma \mapsto \tau(\Sigma) = T - T'\,, \end{aligned} \tag{3.37}$$

as the difference of the torsions of the connections $\tilde{D}$ and $\tilde{D}'$. The image of the map τ is not necessarily surjective, that is $\mathrm{Im}\,\tau = W_{G_S} \subset W$. The part of W that is not spanned by W_{G_S} is the intrinsic torsion of the generalised structure G_S, i.e.

$$W^{G_S}_{\mathrm{int}} = W/W_{G_S}\,. \tag{3.38}$$

The intrinsic torsion T_{int} is the component of T that is independent of the choice of compatible connection $\tilde{D}$ and is fixed only by the choice of generalised structure. When $G_S \in \mathrm{USp}(8)/\mathbb{Z}_2$ and therefore defines a generalised metric, the norm defined by the generalised metric G allows one to decompose the space of generalised torsions as[11]

$$W = W_{G_S} \oplus W^{G_S}_{\mathrm{int}}\,. \tag{3.39}$$

We can always decompose the intrinsic torsion into representations of G_S. For a consistent truncation we will be interested in generalised structures whose only non-zero components are in singlet representations of G_S.

[11] See Appendix B for an explict example.

As for ordinary G-structures, the intrinsic torsion of a generalised structure G_S can be encoded in first-order differential expressions in the G_S invariant generalised tensors. Recall that $K_{\tilde{I}}$ and J_A form a basis for the invariant tensors and by definition, $\tilde{D}K_{\tilde{I}} = \tilde{D}J_A = 0$ for any G_S-compatible connection. It was shown in [27] that the intrinsic torsion is encoded in the expressions

$$L_{K_{\tilde{I}}}K_{\tilde{J}}\,, \qquad L_{K_{\tilde{I}}}J_A\,, \tag{3.40}$$

and

$$\int_M \boldsymbol{\kappa}^2 \mathrm{tr}(J_A(L_W J_B))\,, \tag{3.41}$$

where the generalised vector W is orthogonal to the generalised vectors in $\mathcal{V}$ in the sense that

$$c(K_{\tilde{I}}, K_{\tilde{J}}, W) = 0\,. \tag{3.42}$$

Note that the expressions (3.40) and (3.41) are in general not independent, but are sufficient to determine the intrinsic torsion.

For a consistent truncation we need to require that the intrinsic torsion lies only in the singlet representation of G_S and is constant. This is equivalent to requiring

$$\begin{aligned} L_{K_{\tilde{I}}}K_{\tilde{J}} &= -T_{\mathrm{int}}(K_{\tilde{I}})\cdot K_{\tilde{J}} = t_{\tilde{I}\tilde{J}}{}^{\tilde{K}}K_{\tilde{K}}\,,\\ L_{K_{\tilde{I}}}J_A &= -T_{\mathrm{int}}(K_{\tilde{I}})\cdot J_A = p_{\tilde{I}A}{}^{B}J_B\,, \end{aligned} \tag{3.43}$$

where the $t_{\tilde{I}\tilde{J}}{}^{\tilde{K}}$ and $p_{\tilde{I}A}{}^{B}$ are constants and that (3.41) vanishes for all W. The latter follows from the fact that the condition on W implies that it transforms non-trivially under G_S and hence, since J_A are singlets, the corresponding intrinsic torsion cannot be a singlet and so must vanish. Recall that $T_{\mathrm{int}}(V)$ is a section of the adjoint bundle $\mathrm{ad}\,F$. For singlet torsion, $T_{\mathrm{int}}(K_{\tilde{I}})$ must act in sub-bundle defined by the commutant[12] $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$. From the factorisation (3.27) we see that we can view the matrices $(t_{\tilde{I}})_{\tilde{J}}{}^{\tilde{K}}$ and $(p_{\tilde{I}})_A{}^B$ as elements of Lie algebras of G_{VT} and G_{H} respectively.

3.1.3 The data of the truncation

Any generalised G_S structure on a manifold M with only constant, singlet intrinsic torsion gives rise to a consistent truncation of eleven-dimensional or type II supergravity with spacetime $X \times M$ to a gravitational theory on X [16, 27]. In this section we focus on truncations to five-dimensional $\mathcal{N} = 2$ supergravity and recall how the the generalised $G_S \subseteq \mathrm{USp}(6)$ structure encodes the data of the truncated theory, as summarised in Section 2.

The field content of the truncated theory is completely determined by the G_S-invariant spaces $\mathcal{U}$ and $\mathcal{V}$ and the moduli space of HV structures,[13] while the gauging is determined by the singlet torsion.

[12] Note that strictly speaking the singlet torsion also allows $T_{\mathrm{int}}(K_{\tilde{I}})$ to act in the $\mathbb{R}^+$ factor of $\mathrm{ad}\,F$. This would correspond to a gauging of the "trombone symmetry" in the 5d theory [51]. Such theories do not have an action and for simplicity we do not consider them here.

[13] For completeness we give in Appendix C the explicit form of the truncation ansatz.

The scalars of the truncated theory are given by the moduli space (3.26) of generalised metrics on M that factors (3.28) into

$$\begin{aligned} &\text{VM scalars:} \quad \phi(x)^i \leftrightarrow \mathcal{M}_{\rm VT} = \frac{G_{\rm VT}}{H_{\rm VT}}\,, \\ &\text{HM scalars:} \quad q(x)^X \leftrightarrow \mathcal{M}_{\rm H} = \frac{G_{\rm H}}{H_{\rm H}} = \frac{G_{\rm H}}{\mathrm{SU}(2)_R \times \mathrm{C}_{G_{\rm H}}(\mathrm{SU}(2)_R)}\,, \end{aligned} \tag{3.44}$$

where x^μ are the coordinates on X.

By construction, both spaces are homogeneous and so correspond to one of the cases listed in Section 2. As discussed in Section 3.1.1, the metrics can be explicitly constructed in terms of the basis vectors $K_{\tilde{I}}$ and J_A. In particular, the cubic invariant on $\mathcal{V}$, which fixes the metric on $\mathcal{M}_{\rm VT}$, is given by (3.20).

The other bosonic fields are the vectors and two-forms. As we will see in a moment, the singlet intrinsic torsion allows one to decompose the space of constant vectors as $\mathcal{V} = \mathcal{R} \oplus \mathcal{T}$ so that the basis vectors split

$$\{K_{\tilde{I}}\} = \{K_I\} \cup \{K_M\}\,, \tag{3.45}$$

where $\{K_I\}$ with $I = 0, \ldots, n_{\rm V}$ are a basis for $\mathcal{R}$ and $\{K_M\}$ with $M = n_{\rm V}+1, \ldots, n_{\rm VT}$ are a basis for $\mathcal{T}$. The vector fields and two-forms are in one-to-one correspondence with a basis in $\mathcal{R}$ and $\mathcal{T}$ respectively[14]

$$\begin{aligned} \text{vectors:} \quad & A^I_\mu(x) \leftrightarrow K_I\,, \\ \text{two-forms:} \quad & B^M_{\mu\nu}(x) \leftrightarrow K_M\,. \end{aligned} \tag{3.46}$$

The gauge interactions of the truncated theory are determined by the intrinsic torsion of the G_S-structure, which in turn is captured by the constants appearing in (3.43). The first relation defines a bracket $[\![\cdot,\cdot]\!] : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ on $\mathcal{V}$ given by

$$[\![v, w]\!]^{\tilde{I}} := (L_v w)^{\tilde{I}} = t_{\tilde{J}\tilde{K}}{}^{\tilde{I}} v^{\tilde{J}} w^{\tilde{K}}\,, \qquad \forall\, v, w \in \mathcal{V}\,, \tag{3.47}$$

just as in (2.12). Since the generalised Lie derivative satisfies $L_u(L_v w) = L_{L_u v} w + L_v(L_u w)$ the bracket defines a Leibniz algebra. As in Section 2, one can then choose a splitting $\mathcal{V} = \mathcal{R} \oplus \mathcal{T}$, where $\mathcal{T}$ is the image of the symmetrised bracket, such that $\mathcal{R}$ is the space of vector multiplets and $\mathcal{T}$ the space of tensors.

For a consistent gauging we need to check the conditions (2.20) and (2.21). They each follow from the properties of the generalised Lie derivative as we now show. Recall first, from the discussion below (3.43), that $(t_v)^{\tilde{J}}{}_{\tilde{I}} = v^{\tilde{K}} t_{\tilde{K}\tilde{I}}{}^{\tilde{J}}$ is an element of the Lie algebra of $G_{\rm VT}$ and so

$$\mathfrak{g}_{\rm gauge} \subset \mathrm{Lie}\, G_{\rm VT} \subset \mathfrak{e}_{6(6)}\,. \tag{3.48}$$

[14] In the general formalism given in [16, 27] the two-forms were valued in constant sections of the singlet sub-bundle of $N \simeq \det T^*M \otimes E^*$, written using dual basis vectors $K^{\flat\tilde{I}}$, and isomorphic to elements of $\mathcal{V}^*$. The relation to the fields here is that the $\tilde{I}$ index is raised using the symplectic form Ω^{-1} defined by the singlet torsion. Note also that one can consider $A^{\tilde{I}}_\mu$ and $B^{\tilde{I}}_{\mu\nu}$ defined for all values of $\tilde{I}$. However, once the non-propagating fields are eliminated only A^I_μ and $B^M_{\mu\nu}$ are dynamical and the Lagrangian takes the form (2.30).

Since c and $\boldsymbol{\kappa}^2$ are $\mathrm{E}_{6(6)}$ invariants, the action of $\mathfrak{g}_{\text{gauge}}$ must preserve the cubic tensor C given by (3.20) and hence we satisfy (2.20). Furthermore, by definition

$$L_v w + L_w v = \mathrm{d}(v \otimes_N w)\,, \tag{3.49}$$

where d is the exterior derivative and $\otimes_N$ is the projection (A.10) onto $N \simeq \det T^*M \otimes E^*$ given by $v \otimes_N w = c(v, w, \cdot)$. If $v, w \in \mathcal{V}$ then the left-hand side of (3.49) is by definition an element of $\mathcal{T}$. Using (3.20), the right-hand side is just the sequence of maps in (2.23), where the symplectic form on $\mathcal{T}$ is defined by the composition $\Omega^{-1} = \mathrm{d} \circ \boldsymbol{\kappa}^2$. Hence (3.49) implies we satisfy the second condition (2.21) required for a consistent gauging.

To complete the description of the gauging we identify the embedding tensor and the Killing vector fields on $\mathcal{M}_{\text{VT}}$ and $\mathcal{M}_{\text{H}}$. Since both manifolds are coset spaces, from (3.27), the group of isometries is $G_{\text{iso}} = G_{\text{VT}} \times G_{\text{H}}$ and the embedding tensor is a map

$$\Theta : \mathcal{V} \to \mathfrak{g}_{\text{iso}} = \mathrm{Lie}\, G_{\text{VT}} \oplus \mathrm{Lie}\, G_{\text{H}}\,. \tag{3.50}$$

The corresponding gauged Killing vectors $k^i_{\tilde{I}}(\phi)$ and $\tilde{k}^X_{\tilde{I}}(q)$ on $\mathcal{M}_{\text{VT}}$ and $\mathcal{M}_{\text{H}}$ are given by (2.26). If we view $K = h^{\tilde{I}}(\phi) K_{\tilde{I}}$ as giving the embedding of $\mathcal{M}_{\text{VT}}$ in $\mathcal{V}$ and $J_\alpha = m^A_\alpha(q) J_A$ as giving the embedding of $\mathcal{M}_{\text{H}}$ in $\mathcal{U}$ then, from (3.43), we can identify the Killing vectors explicitly from the relations

$$\begin{aligned} k^i_{\tilde{I}} \partial_i h^{\tilde{J}} &= \Theta_{\tilde{I}}{}^a k^i_a \partial_i h^{\tilde{J}} = t_{\tilde{I}\tilde{K}}{}^{\tilde{J}} h^{\tilde{K}}\,, \\ \tilde{k}^X_{\tilde{I}} \partial_X m^A_\alpha &= \Theta_{\tilde{I}}{}^m \tilde{k}^X_m \partial_X m^A_\alpha = p_{\tilde{I}B}{}^A m^B_\alpha\,. \end{aligned} \tag{3.51}$$

Thus we can identify the embedding tensor as an element of $\mathrm{Lie}\, G_{\text{VT}} \oplus \mathrm{Lie}\, G_{\text{H}}$

$$\Theta_{\tilde{I}} = \begin{pmatrix} (t_{\tilde{I}})_{\tilde{J}}{}^{\tilde{K}} & 0 \\ 0 & (p_{\tilde{I}})_A{}^B \end{pmatrix}\,. \tag{3.52}$$

Using the Leibniz property that $L_{K_{\tilde{I}}}(L_{K_{\tilde{J}}}\alpha) = L_{(L_{K_{\tilde{I}}}K_{\tilde{J}})}\alpha + L_{K_{\tilde{J}}}(L_{K_{\tilde{I}}}\alpha)$ for any generalised tensor α, it follows that each set of vectors forms a representation of $\mathfrak{g}_{\text{gauge}}$ as in (2.28). In other words, we have

$$[t_{\tilde{I}}, t_{\tilde{J}}] = t_{\tilde{I}\tilde{J}}{}^{\tilde{K}} t_{\tilde{K}}\,, \qquad [p_{\tilde{I}}, p_{\tilde{J}}] = t_{\tilde{I}\tilde{J}}{}^{\tilde{K}} p_{\tilde{K}}\,. \tag{3.53}$$

Finally, it is worth noting that the Killing prepotentials descend directly from the moment maps for generalised diffeomorphisms that appear in integrability conditions for an HV structure [49] and are given by

$$g\, P^\alpha_{\tilde{I}} = \tfrac{1}{8}\, \epsilon^{\alpha\beta\gamma} \mathrm{tr}\left(J_\beta (L_{K_{\tilde{I}}} J_\gamma)\right)\,, \tag{3.54}$$

where as above $J_\alpha = m^A_\alpha(q) J_A$ is the dressed triplet.

It is important to note that generic $\mathcal{N} = 2$ supergravity allows gaugings defined by an embedding tensor Θ that is a general element of $\mathcal{V}^* \otimes \mathfrak{g}_{\text{iso}}$. However, the fact that our theory comes from a consistent truncations will typically restrict the form of Θ to only lie in certain $G_{\text{VT}} \times G_{\text{H}}$ representations in the decomposition of $\mathcal{V}^* \otimes \mathfrak{g}_{\text{iso}}$. For this reason, in the following we will use T to denote the embedding tensor that appears in the consistent truncations to distinguish it from the more general Θ. As a consequence, we will see that not all the allowed $\mathcal{N} = 2$ gaugings can arise from consistent truncations.

4 Classification of $\mathcal{N}=2$ truncations to five-dimensions

In this section, we discuss the main result of the paper, namely the classification of the consistent truncations to $\mathcal{N}=2$ gauged supergravity in five-dimensions that can a priori be obtained from M-theory and type IIB.

As the data of a consistent truncation are encoded in the generalised structure G_S that is defined on the compactification manifold M, the problem reduces to classifying the possible $G_S \subset \mathrm{E}_{6(6)}$ structures with constant intrinsic torsion that preserve $\mathcal{N}=2$ supersymmetry. Therefore, the classification consists of an algebraic problem – the existence of an appropriate $G_S \subset \mathrm{E}_{6(6)}$ structure – and a differential one – the existence of constant singlet intrinsic torsion. In the following, we will study the algebraic problem in general, but will simply assume that the differential condition of having constant singlet intrinsic torsion can be solved. From the example of maximally supersymmetric gauged supergravity we know that the differential condition puts important restrictions on the allowed gauged supergravities [22–24]. Strikingly, even when ignoring this additional constraint, we find that for $\mathcal{N}=2$ theories the algebraic conditions alone significantly constrain the possible gaugings that can arise.

Let us recall from Section 3.1 what the main idea is. Demanding that the truncated theory is supersymmetric implies that the internal manifold must be spin and that the structure group must be a subgroup of $\mathrm{USp}(8)$, the maximal compact subgroup of $\mathrm{E}_{6(6)}$. The largest structure giving $\mathcal{N}=2$ supersymmetry is $G_S = \mathrm{USp}(6)$. Under the breaking

$$\mathrm{USp}(8) \supset \mathrm{USp}(6) \times \mathrm{SU}(2)_R \,, \tag{4.1}$$

the spinorial representation of $\mathrm{USp}(8)$ decomposes as

$$\mathbf{8} = (\mathbf{6},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \,, \tag{4.2}$$

where the $(\mathbf{1},\mathbf{2})$ is associated to the two supersymmetry parameters of the truncated theory. Since under

$$\mathrm{E}_{6(6)} \supset \mathrm{USp}(6) \cdot \mathrm{SU}(2)_R \,, \tag{4.3}$$

the only singlets in the $\mathbf{27}^*$ and $\mathbf{78}$ are the K and J_α of the HV structure, the theory obtained form a $G_S = \mathrm{USp}(6)$ only contains the gravity multiplet. To have extra vector- or hyper-multiplets we need the structure group G_S to be a subgroup of $\mathrm{USp}(6)$.

The algebraic problem then consists of the following steps. We first scan for all possible inequivalent ways of breaking $\mathrm{USp}(8)$ to $G_S \subset \mathrm{USp}(6)$ that admit only two singlets in the fundamental representation of $\mathrm{USp}(8)$. Given a G_S with these features, it will embed in $\mathrm{E}_{6(6)}$ as

$$\mathrm{E}_{6(6)} \supset G_S \cdot \mathrm{C}_{\mathrm{E}_{6(6)}}(G_S) \,, \tag{4.4}$$

where $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ is the commutant group. We then check whether under this breaking the $\mathbf{27}^*$ and $\mathbf{78}$ of $\mathrm{E}_{6(6)}$ contain G_S singlets, which will determine the vector and hyper-multiplets of the truncated theory. In each case the singlets will transform under $\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ which also determines the form of the scalar manifold $\mathcal{M}$ of the truncated theory

$$\mathcal{M} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(8)/\mathbb{Z}_2}(G_S)} \,. \tag{4.5}$$

We stress again that by construction the scalar manifolds are always necessarily symmetric spaces and furthermore are always a product $\mathcal{M} = \mathcal{M}_{\rm VT} \times \mathcal{M}_{\rm H}$ of vector-tensor multiplet and hypermultiplet scalar manifolds as in (3.28).

We have performed a complete scan for all Lie subgroups[15] $G_S \subset \mathrm{USp}(6)$. We find that there are only a small number of inequivalent G_S structures with the properties above. We list them here according to the type of breaking of USp(6) that they correspond to. All other cases either give rise to extra singlets in the **6** of USp(6) or can be obtained as subgroups of the G_S-structures listed below without giving rise to any new fields in the consistent truncation.

Br.1 $G_S = \mathrm{SU}(2) \times \mathrm{Spin}(p)$, $2 \leq p \leq 5$.

These are obtained from the embedding

$$\mathrm{USp}(6) \supset \mathrm{USp}(4) \times \mathrm{SU}(2) \simeq \mathrm{Spin}(5) \times \mathrm{SU}(2)\,, \tag{4.6}$$

which gives

$$\mathbf{6} = (\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,, \tag{4.7}$$

and by further breaking the USp(4) factor

$$\begin{aligned} &\mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \simeq \mathrm{Spin}(4)\,,\\ &\mathrm{USp}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2) \supset \mathrm{SU}(2)_D \simeq \mathrm{Spin}(3)\,,\\ &\mathrm{USp}(4) \subset \mathrm{SU}(2) \times \mathrm{SU}(2) \supset \mathrm{SU}(2)_D \subset \mathrm{U}(1)_D \simeq \mathrm{Spin}(2)\,. \end{aligned} \tag{4.8}$$

The corresponding branching of the **6** of USp(6) are

$$\begin{aligned} &\mathbf{6} = (\mathbf{2},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{2})\,,\\ &\mathbf{6} = 2\cdot(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\,,\\ &\mathbf{6} = 2\cdot\mathbf{1}_1 \oplus 2\cdot\mathbf{1}_{-1} \oplus \mathbf{2}_0\,, \end{aligned} \tag{4.9}$$

for the breaking to $\mathrm{Spin}(4)\times\mathrm{SU}(2)$, $\mathrm{Spin}(3)\times\mathrm{SU}(2)$ and $\mathrm{Spin}(2)\times\mathrm{SU}(2)$, respectively.

Br.2 $G_S = \mathrm{SO}(3)$ and $G_S = \mathrm{SU}(2)$.

The relevant breaking is

$$\mathrm{USp}(6) \supset \mathrm{SO}(3) \times \mathrm{SU}(2)\,, \tag{4.10}$$

with the **6** of USp(6) branching as

$$\mathbf{6} = (\mathbf{3},\mathbf{2})\,. \tag{4.11}$$

Taking $G_S = \mathrm{SO}(3)$ or $G_S = \mathrm{SU}(2)$ leads to two different consistent truncations.

[15] In the following section we will also discuss a few examples of $G_S = \mathbb{Z}_2$ structures that are easily identified, but we do not provide an exhaustive analysis of discrete subgroups of USp(6).

Br.3 $G_S = \mathrm{SU}(3)$.

This comes from the breaking

$$\mathrm{USp}(6) \supset \mathrm{SU}(3) \times \mathrm{U}(1) \tag{4.12}$$

which gives

$$\mathbf{6} = \mathbf{3}_1 \oplus \overline{\mathbf{3}}_{-1} \,. \tag{4.13}$$

Br.4 $G_S = \mathrm{SU}(2) \times \mathrm{U}(1)$

This truncation is obtained by further breaking the SU(3) group of the previous case. Under $\mathrm{SU}(3) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$, we get

$$\mathbf{6} = \mathbf{2}_{1,1} \oplus \mathbf{1}_{-2,1} \oplus \mathbf{2}_{-1,-1} \oplus \mathbf{1}_{2,-1} \,. \tag{4.14}$$

Br.5 $G_S = \mathrm{U}(1)$.

This comes from the same breaking $\mathrm{SU}(3) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ as Br.4 but taking only the U(1) factor as the structure group.

Br.6 $G_S = \mathrm{U}(1)$.

This comes from the same breaking as Br.3 and taking the U(1) factor as structure group.

Once the possible G_S structure have been identified, we need to study their singlet intrinsic torsion as this determines the embedding tensor and thus the gaugings of the truncated theory. The details of this calculation for $G_S = \mathrm{SU}(2) \times \mathrm{Spin}(p)$ are discussed in Appendix B. The condition of having only components of the intrinsic torsion that are singlets of G_S imposes differential constraints on the compactification manifold that can be complicated to solve in general. In our analysis, we assume that these differential conditions are satisfied and instead solely study the intrinsic torsion's algebraic properties. We will see that this is still enough to significantly restrict the possible gaugings obtainable by a consistent truncation.

We will first decompose the singlet intrinsic torsion into representations of the global isometry group G_{iso}. This will allow us to identify the various components of the embedding tensor of the truncated theory. We then impose the Leibniz condition on these singlets[16]. The resulting embedding tensor components determine the Leibniz algebra $\mathfrak{a}$ and hence the extended Lie algebra $\mathfrak{g}_{\mathrm{ext}}$. As discussed in Section 3, the matter in the theory is charged under the gauge algebra $\mathfrak{g}_{\mathrm{gauge}}$ that is generically a quotient of $\mathfrak{g}_{\mathrm{ext}}$ by a central subalgebra. The embedding tensor then also describes the embedding of $\mathfrak{g}_{\mathrm{gauge}}$ into the Lie algebra $\mathfrak{g}_{\mathrm{iso}}$

[16] If the differential conditions on the intrinsic torsion are satisfied, the Leibniz condition is also automatically satisfied as discussed in Section 3.1.3. However, since here we are not analysing whether the differential conditions can be solved, we must impose the Leibniz condition as a restriction. Put differently, only for those singlet components of the intrinsic torsion which obey the Leibniz condition, can the differential conditions on the compactification manifold be satisfied.

of the isometry group. In the following we will refer to a group as "gauged" if it is part of the corresponding group G_{gauge}.

In general, solving the Leibniz conditions for all singlets of G_S can be very cumbersome. It is hence sometimes useful to streamline the search for possible gauge groups G_{ext} as follows. First, consider the decomposition of $\mathcal{V}$ under a putative gauge group $G_{\text{gauge}} \subset G_{\text{VT}} \subset G_{\text{iso}}$. We must impose that there is a subset of the n_{VT} vectors transforming in the adjoint of G_{gauge}. Once this condition is satisfied, we keep only those components of the intrinsic torsion that are G_{gauge}-singlets. Finally, we impose the Leibniz condition (2.11) on the singlet intrinsic torsion. From the resulting Leibniz bracket, we can read off the gauge groups and tensor multiplets. In particular, if $n_{\text{VT}} > \dim G_{\text{gauge}}$ and the $n_{\text{VT}} - \dim G_{\text{gauge}}$ extra vectors are uncharged under G_{ext}, then they are central elements filling out the full gauge algebra $\mathfrak{g}_{\text{ext}}$, while if they are charged they either enlarge the $\mathfrak{g}_{\text{gauge}}$ algebra or correspond to charged tensor multiplets. The two charged cases are distinguished by whether or not the extra vectors are in the image of the symmetrised Leibniz bracket, as discussed in (2.13).

It is worth stressing that we do not mean to give an exhaustive list of all possible gaugings. Where we cannot solve the Leibniz condition in general, we will instead limit ourselves to the largest reductive groups and largest compact groups that can be gauged. We will find that only a handful of gaugings are possible.

Finally, the computation we perform bears some resemblance to the purely five-dimensional analysis that would have to be performed to find possible gaugings. However, crucially, in order to have a consistent truncation, we are analysing the intrinsic torsion that descends from the $\text{E}_{6(6)}$ generalised Lie derivative, and thus lives in the $W_{\text{int}} \subset \mathbf{351}$ of $\text{E}_{6(6)}$. By contrast, the five-dimensional computation would search for gaugings living in the $\mathcal{V}^* \otimes \mathfrak{g}_{\text{iso}} \supseteq W_{\text{int}}$, where $\mathcal{V}$ denotes the space of vector fields of the five-dimensional supergravity. Therefore, it is not a priori clear whether all gaugings that are allowed from a five-dimensional perspective can also arise from consistent truncations. In fact, as we will see, some five-dimensional gaugings cannot arise from consistent truncations. For example, in theories with scalar manifolds $\mathcal{M}_{\text{VT}} = \mathbb{R}^+ \times \frac{\text{SO}(n_{\text{VT}}-1,1)}{\text{SO}(n_{\text{VT}}-1)}$ and no hyper multiplets, consistent truncations only lead to gaugings where the tensor multiplets are charged under the graviphoton and not any of the other $n_{\text{VT}} - 1$ vector fields.

In the following sections, we will derive the consistent truncations associated to the G_S structures listed here and derive their field content and invariant tensors. For sake of exposition, we will first discuss the consistent truncations including only vector and tensor multiplets in Section 4.1, then only hypermultiplets in Section 4.2, before giving the mixed cases with vector/tensor and hypermultiplets in Section 4.3.

We summarise the matter content of the consistent truncations that arise from our scan in Table 1: we list the G_S structure group, the number of vector/tensor multiplets n_{VT} and hypermultiplets n_{H}, and the associated scalar manifolds. We see that the possible consistent truncations are limited. In particular,

we find the largest possible truncation consists of only 14 vector/tensor multiplets.

Let us again reiterate that the consistent truncations that can be actually realised will

$n_{\rm VT}$ \ $n_{\rm H}$	0	1	2
0	$G_S = \mathrm{USp}(6)$ $\mathcal{M} = 1$	$G_S = \mathrm{SU}(3)$ $\mathcal{M} = \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	$G_S = \mathrm{SO}(3)$ $\mathcal{M} = \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}$
1	$G_S = \mathrm{SU}(2)\times\mathrm{Spin}(5)$ $\mathcal{M} = \mathbb{R}^+$	$G_S = \mathrm{SU}(2)\times\mathrm{U}(1)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	-
2	$G_S = \mathrm{SU}(2)\times\mathrm{Spin}(4)$ $\mathcal{M} = \mathbb{R}^+ \times \mathrm{SO}(1,1)$	$G_S = \mathrm{U}(1)$ $\mathcal{M} = \mathbb{R}^+ \times \mathrm{SO}(1,1) \times \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	-
3	$G_S = \mathrm{SU}(2)\times\mathrm{Spin}(3)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)}$	$G_S = \mathrm{U}(1)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	-
4	$G_S = \mathrm{SU}(2)\times\mathrm{Spin}(2)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SO}(3,1)}{\mathrm{SO}(3)}$	$G_S = \mathrm{U}(1)$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SO}(3,1)}{\mathrm{SO}(3)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	-
5	$G_S = \mathrm{SU}(2)$ $\mathcal{M} = \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}$ $G_S = \mathrm{SU}(2)\times\mathbb{Z}_2$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)}$	-	-
6	$G_S = \mathrm{SU}(2)\times\mathbb{Z}_2$ $\mathcal{M} = \mathbb{R}^+ \times \frac{\mathrm{SO}(5,1)}{\mathrm{SO}(5)}$	-	-
8	$G_S = \mathrm{U}(1)$ $\mathcal{M} = \frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}$	-	-
14	$G_S = \mathbb{Z}_2$ $\mathcal{M} = \frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}$	-	-

Table 1. List of all possible consistent truncation with $n_{\rm VT}$ vector/tensor multiplets, n_H hypermultiplets, and the required $G_S \subset \mathrm{E}_{6(6)}$ structure group, as well as the associated scalar manifold $\mathcal{M}$.

be a subset of those presented in the group-theoretic analysis here. This is because the requirement that a given G_S structure has singlet intrinsic torsion will introduce non-trivial differential constraints that a given manifold M must satisfy and which we do not analyse here.

However some of the cases listed in Table 1 do have an explicit geometric realisation. For instance the mixed cases with $n_{\rm H} = 1$ and $n_{\rm VT} = 1$, $n_{\rm VT} = 2$ and $n_{\rm VT} = 4$ correspond to consistent truncations of eleven-dimensional supergravity that have recently obtained. These are truncations around backgrounds with $\mathcal{N} = 2$ supersymmetry describing the near-horizon limit of M5-branes wrapping a Riemann surface: the Maldacena–Nuñez (MN)

solution [29] and its generalisations called the BBBW solutions [30]. In particular the truncation with $n_{\rm VT} = 4$ and $n_{\rm H} = 1$ with gauge group $G_{\rm gauge} = {\rm SO}(3) \times {\rm U}(1)_R \times \mathbb{R}^+$ is the largest possible truncation around the MN solution , while the case with $n_{\rm VT} = 2$ and $n_{\rm H} = 1$ and gauge group $G_{\rm gauge} = {\rm U}(1)_R \times \mathbb{R}^+$ gives the consistent truncation around the BBBW solutions [27]. The subtruncation to $n_{\rm VT} = 1$ and $n_{\rm H} = 1$ was obtained in [28].

4.1 Truncations to only vector and tensor multiplets

We analyse first the possible consistent truncations that give rise to a theory with only vector/tensor multiplets. Since a consistent truncation necessarily gives rise to a symmetric scalar manifold (see Section 3), the vector/tensor scalar manifolds that one can obtain must be symmetric "very special real" manifolds, as classified in [36–38].

This classification consists of a generic case, possible for arbitrary number of vector/tensor multiplets, where the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ factorises, with the only non-zero components given by

$$C_{0ij} = \eta_{ij}\,, \qquad i,j = 1,\ldots,n_{\rm VT}\,. \tag{4.15}$$

Here η_{ij} has signature $(1, n_{\rm VT}-1)$ and the scalar manifold is given by

$$\mathcal{M}_{\rm VT} = \mathbb{R}^+ \times \frac{{\rm SO}(n_{\rm VT}-1,1)}{{\rm SO}(n_{\rm VT}-1)}\,. \tag{4.16}$$

Additionally, there are a number of "special" cases that only exist for specific values of $n_{\rm VT}$ and for which $C_{\tilde{I}\tilde{J}\tilde{K}}$ does not factorise. These are given by

$$\begin{aligned}
\mathcal{M}_{\rm VT} &= \frac{{\rm SL}(3,\mathbb{R})}{{\rm SO}(3)}\,, & n_{\rm VT} &= 5\,,\\
\mathcal{M}_{\rm VT} &= \frac{{\rm SL}(3,\mathbb{C})}{{\rm SU}(3)}\,, & n_{\rm VT} &= 8\,,\\
\mathcal{M}_{\rm VT} &= \frac{{\rm SU}^*(6)}{{\rm USp}(6)}\,, & n_{\rm VT} &= 14\,,\\
\mathcal{M}_{\rm VT} &= \frac{{\rm E}_{(6,-26)}}{{\rm F}_4}\,, & n_{\rm VT} &= 26\,.
\end{aligned} \tag{4.17}$$

Finally, there is a second "generic case", which exists for arbitrary $n_{\rm VT} > 1$, but where the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ does not factorise [38]. The associated scalar manifolds are given by

$$\mathcal{M}_{\rm VT} = \frac{{\rm SO}(n_{\rm VT},1)}{{\rm SO}(n_{\rm VT})}\,. \tag{4.18}$$

We want to determine which of these gauged supergravities can arise from a consistent truncation and how can they be classified in terms of the structure groups G_S listed in the previous section.

In order to have a consistent truncation with only vector/tensor multiplets, the generalised tensors defining the G_S structure must consist of the triplet of adjoint tensor J_α, $\alpha = 1,2,3$ corresponding to an H-structure (see Section 3.1) and of $n_{\rm VT}+1$ generalised vectors $K_{\tilde{I}}$, $\tilde{I} = 0,1,\ldots n_{\rm VT}$ satisfying

$$\begin{aligned}
J_\alpha \cdot K_{\tilde{I}} &= 0\,,\\
\boldsymbol{\kappa}^{-2} c(K_{\tilde{I}}, K_{\tilde{J}}, K_{\tilde{K}}) &= C_{\tilde{I}\tilde{J}\tilde{K}}\,,
\end{aligned} \tag{4.19}$$

with constant $C_{\tilde{I}\tilde{J}\tilde{K}}$.

Since the J_α are stabilised by $\mathrm{SU}^*(6) \subset \mathrm{E}_{6(6)}$, the structure group must be a subgroup of $\mathrm{SU}^*(6)$. Under the breaking $\mathrm{E}_{6(6)} \supset \mathrm{SU}^*(6) \cdot \mathrm{SU}(2)_R$, we have

$$\begin{aligned} \mathbf{27}^* &= (\mathbf{15}^*, \mathbf{1}) \oplus (\mathbf{6}, \mathbf{2}) \ , \\ \mathbf{78} &= (\mathbf{35}, \mathbf{1}) \oplus (\mathbf{20}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{3}) \ , \end{aligned} \tag{4.20}$$

where the triplet of J_α belong to $(\mathbf{1}, \mathbf{3})$ and generate the $\mathrm{SU}(2)_R$ symmetry. Then, the first condition in (4.19), implies that the vectors $K_{\tilde{I}}$ must be invariant under $\mathrm{SU}(2)_R$ and therefore must lie in the real vector space

$$\mathcal{V} \subseteq (\mathbf{15}^*, \mathbf{1}) \ . \tag{4.21}$$

Thus, we can have at most $n_{\mathrm{VT}} = 14$ vector/tensor multiplets and we can immediately rule out the case $n_{\mathrm{VT}} = 26$ in (4.17), as well as the case $n_{\mathrm{VT}} > 14$ in (4.16).

The family (4.18) is also ruled out, because the isometries of the corresponding scalar manifolds are not linearly realised. As we discussed in Section 3, the isometry group of the scalar manifold is the commutant in $\mathrm{E}_{6(6)}$ of the structure group and by construction it acts linearly on the set of singlet generalised vectors. As a result, the gauged supergravities with vector/tensor scalar manifolds (4.18) do not arise from consistent truncations.

All other cases can in principle arise in consistent truncations and in the next subsection we will discuss from which generalised structure G_S they can be obtained and then use G_S to study the intrinsic torsion and hence find the admissible gaugings.

4.1.1 Generic case

The generic case with scalar manifold (4.16) corresponds to the structure groups

$$G_S = \mathrm{Spin}(6 - n_{\mathrm{VT}}) \times \mathrm{SU}(2) \, , \tag{4.22}$$

of item (Br.1) of the list in the previous section, where for notational convenience we let $\mathrm{Spin}(1) = \mathrm{Spin}(0) = \mathbb{Z}_2$. Note that (4.22) implies that we can have at most $n_{\mathrm{VT}} = 6$ vector/tensor multiplets in the truncation. Moreover, the case $n_{\mathrm{VT}} = 5$ and $n_{\mathrm{VT}} = 6$ have identical structure groups. This means that any background admitting a truncation with $n_{\mathrm{VT}} = 5$ actually admits a truncation with $n_{\mathrm{VT}} = 6$, with the former truncation being a subtruncation of the latter.

To see how these structure groups arise, note that the structure (4.15) of the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ implies that the vectors $K_{\tilde{I}}$ can be split into a vector K_0 and n_{VT} vectors K_i such that for any $i, j, k = 1, \ldots, n_{\mathrm{VT}}$,

$$c(K_0, K_0, \cdot) = 0 \, , \qquad c(K_i, K_j, K_k) = 0 \, , \qquad c(K_0, K_i, K_j) = \eta_{ij} \, , \tag{4.23}$$

where η_{ij} has signature $(5, 1)$. The vector K_0 corresponds to the graviphoton of the truncated theory.

By studying the form of (4.23), we can deduce the stabiliser group of the generalised vector fields $K_{\tilde{I}}$ as follows. Being in the $\mathbf{15}^*$ of $\mathrm{SU}^*(6)$, the vectors $K_{\tilde{I}}$ can be seen as six-dimensional two-forms. Then the first condition in (4.23) is equivalent to

$$K_0 \wedge K_0 = 0 \, , \tag{4.24}$$

with $\wedge$ the standard wedge product of p-forms. Thus, K_0 must be decomposable and we can choose a basis of independent six-dimensional one-forms such that

$$K_0 = e_5 \wedge e_6 \,. \tag{4.25}$$

The stabiliser of K_0 is $\mathrm{SU}^*(4) \times \mathrm{SU}(2)$, embedded in $\mathrm{SU}^*(6)$ as

$$\begin{aligned} \mathrm{SU}^*(6) &\supset \mathrm{SU}^*(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,, \\ \mathbf{15}^* &= (\mathbf{4}^*, \mathbf{2})_1 \oplus (\mathbf{6}, \mathbf{1})_{-2} \oplus (\mathbf{1}, \mathbf{1})_4 \,, \end{aligned} \tag{4.26}$$

with $K_0 \in (\mathbf{1}, \mathbf{1})_4$. This forces the G_S structure to be a subgroup of $\mathrm{SU}^*(4) \times \mathrm{SU}(2)$. The other conditions in (4.23) become

$$K_0 \wedge K_i \wedge K_j = \eta_{ij} \,, \qquad K_i \wedge K_j \wedge K_k = 0 \,, \tag{4.27}$$

where the metric η_{ij} is invariant under $\mathrm{SU}^*(4) \simeq \mathrm{Spin}(5,1)$. From (4.27) it follows that

$$K_i \in (\mathbf{6}, \mathbf{1})_{-2} \,. \tag{4.28}$$

Thus, there can be at most six vector multiplets of this type.

The structure group G_S can now be easily determined. Since the n_{VT} singlets K_i satisfy the inner product (4.27) of signature $(1, n_{\mathrm{VT}} - 1)$ they break $\mathrm{SU}^*(4)$ to

$$\mathrm{SU}^*(4) \simeq \mathrm{Spin}(5,1) \supset \mathrm{Spin}(6 - n_{\mathrm{VT}}) \times \mathrm{Spin}(n_{\mathrm{VT}} - 1, 1) \,, \tag{4.29}$$

where the factor $\mathrm{Spin}(6 - n_{\mathrm{VT}})$ is the stabiliser of the K_i while the factor $\mathrm{Spin}(n_{\mathrm{VT}} - 1, 1)$ rotates the K_i into each other. Thus, the structure group is given by

$$G_S = \mathrm{Spin}(6 - n_{\mathrm{VT}}) \times \mathrm{SU}(2) \,. \tag{4.30}$$

Although the structure groups and the isometry groups are Spin subgroups of $\mathrm{E}_{6(6)}$, the generalised vectors K_i never appear in spinorial representations of G_S and hence only see the orthogonal groups and not their double covers. This is the reason why the case with $n_{\mathrm{VT}} = 5$ vectors/tensors can always be enhanced to $n_{\mathrm{VT}} = 6$: on the two-forms K_i the $\mathbb{Z}_2$ structure group acts trivially. Moreover, this is why the coset spaces can be reduced to take the form (4.16):

$$\mathcal{M} = \mathcal{M}_{\mathrm{VT}} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(8)/\mathbb{Z}_2}(G_S)} = \mathbb{R}^+ \times \frac{\mathrm{SO}(n_{\mathrm{VT}} - 1, 1)}{\mathrm{SO}(n_{\mathrm{VT}} - 1)} \,. \tag{4.31}$$

The corresponding isometry group is

$$G_{\mathrm{iso}} = \mathbb{R}^+ \times \mathrm{SO}(n_{\mathrm{VT}} - 1, 1) \times \mathrm{SU}(2)_R \,, \tag{4.32}$$

where as discussed above we take $G_{\mathrm{H}} = \mathrm{SU}(2)_R$, even though there are no hypermultiplets, in order to include the R-symmetry. Under G_{iso} the space of vectors transforms as

$$\mathcal{V} = (\mathbf{1}, \mathbf{1})_2 \oplus (\mathbf{1}, \mathbf{n})_{-1} \ni (v^0, v^i) \,, \tag{4.33}$$

where the first entries are the $\mathrm{SU}(2)_R$ representations, $\mathbf{n}$ is the vector representation of $\mathrm{SO}(n_{\mathrm{VT}}-1,1)$, the subscripts are the $\mathbb{R}^+$ charges, and $i=1,\ldots,n_{\mathrm{VT}}$ denotes $\mathrm{SO}(n_{\mathrm{VT}}-1,1)$ indices.

We can now determine the embedding tensor of the truncated theory and the possible gaugings. These are encoded in the intrinsic torsion of the G_S structure, which must only contain G_S singlets for the truncation to be consistent. We assume that this occurs and decompose the intrinsic torsion in representations of the global isometry group (4.32)

$$\begin{aligned} W_{\mathrm{int}} &= (\mathbf{3},\mathbf{1})_{-2}\oplus(\mathbf{3},\mathbf{n})_1\oplus(\mathbf{1},\mathbf{n})_1\oplus(\mathbf{1},\mathbf{ad})_{-2}\oplus(\mathbf{1},\mathbf{X})_1 \\ &\ni (\tau^a_{0\,b},\tau^a_{i\,b},\tau_i,\tau^i_{0\,j},\tau_{[ijk]})\,, \end{aligned} \tag{4.34}$$

where **ad** and **X** denote the adjoint and the rank-3 anti-symmetric[17] representations of $\mathrm{SO}(n_{\mathrm{VT}}-1,1)$, respectively, and $a,b=1,2,3$ are $\mathrm{SU}(2)_R$ indices. The case $n_{\mathrm{VT}}=5$ is different, but can be obtained as a subtruncation of the case $n_{\mathrm{VT}}=6$. Therefore, we will not consider $n_{\mathrm{VT}}=5$ here.

Now we need the map (3.50), which gives the generalised geometry embedding tensor, and which we denote by $T:\mathcal{V}\to\mathfrak{g}_{\mathrm{iso}}$ to distinguish it from the generic 5d embedding tensor. Given an element $v\in\mathcal{V}$, the intrinsic torsion defines T as having the non-zero components

$$\begin{aligned} T(v)^a{}_b &= v^0\tau^a_{0\,b}+v^i\tau^a_{i\,b}\in\mathfrak{su}(2)_R\,, \\ T(v)^i{}_j &= v^0\tau^i_{0\,j}+v^k\tau_k{}^i{}_j\in\mathfrak{so}(n_{\mathrm{VT}}-1,1)\,, \\ T(v)_{(0)} &= v^i\tau_i\in\mathfrak{u}(1)\,. \end{aligned} \tag{4.35}$$

The adjoint action on the vectors in $\mathcal{V}$

$$\begin{aligned} (T(v)\cdot w)^0 &= 2\,T(v)_{(0)}w^0\,, \\ (T(v)\cdot w)^i &= -T(v)_{(0)}w^i+T(v)^i{}_jw^j\,, \end{aligned} \tag{4.36}$$

defines the Leibniz bracket $T(v)\cdot w=t_v(w)=[\![v,w]\!]$. The Leibniz condition (2.11) gives a set of constraints on the torsion components

$$\tau_{[jk}{}^m\tau_{l]m}{}^i=0\,,\qquad \tau_{0\,i}{}^k\tau_{jk}{}^l=0\,,\qquad \tau_i=0\,, \tag{4.37}$$

so that the Leibniz bracket simplifies to

$$\begin{aligned} [\![v,w]\!]^0 &= 0\,, \\ [\![v,w]\!]^i &= -v^jw^k\tau_{jk}{}^i-v^0w^k\tau_{0k}{}^i\,. \end{aligned} \tag{4.38}$$

From (2.14) we see that the rank of $\tau_{0i}{}^j$ determines the number of tensor multiplets, while $\tau_{ij}{}^k$ form the structure constants of the gauge algebra. Finally, τ^a_{0b} and τ^a_{ib} determine how the $\mathrm{SU}(2)_R$ is gauged. Moreover, the gauging of the $\mathrm{SU}(2)_R$ R-symmetry must form a representation of $\mathfrak{g}_{\mathrm{ext}}$ as in (3.53). Explicitly, this implies

$$(T(v)\cdot T(w)-T(w)\cdot T(v))^a{}_b=-\left(T([\![v,w]\!])\right)^a{}_b\,, \tag{4.39}$$

[17] In some cases the representation **X** might be reducible.

which for (4.35) imposes

$$\tau^a_{i\,c}\tau^c_{0\,b} - \tau^a_{0\,c}\tau^c_{i\,b} = 0\,, \qquad \tau_{0k}{}^i\tau^a_{i\,b} = 0\,, \qquad \tau^a_{i\,c}\tau^c_{j\,b} - \tau^a_{j\,c}\tau^c_{i\,b} = \tau_{ij}{}^k\tau^a_{k\,b}\,. \tag{4.40}$$

Then the first line in (4.35) gives the embedding tensor for the $\mathrm{SU}(2)_R$ symmetry

$$p_{0a}{}^b = \tau^b_{0a}\,, \qquad p_{ia}{}^b = \tau^b_{ia}\,, \tag{4.41}$$

while the non-zero components of the embedding tensor on the vector isometries are

$$t_{0k}{}^i = -\tau_{0k}{}^i\,, \qquad t_{jk}{}^i = -\tau_{jk}{}^i\,. \tag{4.42}$$

Since $\tau_i = 0$, we note that the $\mathbb{R}^+$ can never be gauged. Also from (4.38), we see that the graviphoton v^0 cannot contribute to non-abelian gaugings. Moreover, from (4.34) and (4.38), we can already see that not all gaugings of five-dimensional $\mathcal{N} = 2$ supergravity can arise from a consistent truncation. In particular, the tensor multiplets can only be charged under the graviphoton v^0 and not any of the $n_{\mathrm{VT}} - 1$ vector fields transforming non-trivially under $\mathrm{SO}(n_{\mathrm{VT}} - 1, 1)$, as for example constructed in [32].

From (4.40) we can also determine in general how the $\mathrm{SU}(2)_R$ global symmetry can be gauged. Whenever an $\mathrm{SO}(3) \subset \mathrm{SO}(n_{\mathrm{VT}} - 1, 1)$ is gauged, those SO(3) vectors can also be used to gauge the $\mathrm{SU}(2)_R$ via τ^a_{ib}. Alternatively, any combination of abelian vector fields, including the graviphoton can gauge a $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$ subgroup.

Let us now find which gaugings of the $\mathrm{SO}(n_{\mathrm{VT}} - 1, 1)$ global symmetry group of $\mathcal{M}_{\mathrm{VT}}$ are possible, beginning with $n_{\mathrm{VT}} = 1$ and working up to the maximal case $n_{\mathrm{VT}} = 6$.

$\mathbf{n_{VT} = 1}$: In this case the isometry group is $G_{\mathrm{iso}} = \mathrm{SU}(2)_R \times \mathbb{R}^+$ and the structure group is $G_S = \mathrm{Spin}(5) \times \mathrm{SU}(2)$.

Any combination of the two vectors can gauge a $\mathrm{U}(1)_R$ subgroup of the R-symmetry.

$\mathbf{n_{VT} = 2}$: The structure group is $G_S = \mathrm{Spin}(4) \times \mathrm{SU}(2)$ and the isometry group is $G_{\mathrm{iso}} = \mathrm{SU}(2)_R \times \mathrm{SO}(1,1) \times \mathbb{R}^+$. There are three singlet vectors with the following $\mathrm{SO}(1,1) \times \mathbb{R}^+ \simeq \mathbb{R}^+ \times \mathbb{R}^+$ charges

$$v = (v^0, v^+, v^-) \in \mathcal{V} = \mathbf{1_{0,2}} \oplus \mathbf{1_{2,-1}} \oplus \mathbf{1_{-2,-1}}\,. \tag{4.43}$$

The conditions (4.37) are now trivially satisfied since $\tau_{ijk} = 0$. From the intrinsic torsion

$$W_{\mathrm{int}} \ni (\tau^a_{0\,b}\,, \tau^a_{+\,b}\,, \tau^a_{-\,b}, \tau^+_{0-})\,, \tag{4.44}$$

we see that, when $\tau^+_{0-} = 0$, any combination of all three vectors can gauge a $\mathrm{U}(1)_R$ symmetry. Alternatively, when $\tau^+_{0-} \neq 0$, two vectors are dualised to tensors and the remaining v^0 can gauge the $\mathrm{SO}(1,1)$ under which the two tensors are charged, as well as a $\mathrm{U}(1)_R$ symmetry.

$\mathbf{n_{VT} = 3}$: The structure group is $G_S = \mathrm{Spin}(3) \times \mathrm{SU}(2)$ and the isometry group is $G_{\rm iso} = \mathrm{SU}(2)_R \times \mathrm{SO}(2,1) \times \mathbb{R}^+$. As there are three vector multiplets in the adjoint of $\mathrm{SO}(2,1)$ it is a priori possible to gauge it. The conditions (4.37) now imply that either $\tau_{ijk} \neq 0$ or $\tau_{0\,i}{}^{j} \neq 0$.

- $\tau_{ijk} := \hat{\tau}\epsilon_{ijk} \neq 0$, $\tau_{0\,i}{}^{j} = 0$.

  The Leibniz algebra, given $v = (v^0, v^i)$ and $w = (w^0, w^i)$, takes the form

  $$[\![v, w]\!]^i = -\hat{\tau} v^j w^k \epsilon_{jk}{}^{i}\,, \qquad [\![v, w]\!]^0 = 0\,. \tag{4.45}$$

  Thus the full $\mathrm{SO}(2,1)$ can be gauged and we can use the singlet vector v^0 to gauge a $\mathrm{U}(1)_R$.

- $\tau_{ijk} = 0$. We now have a purely abelian gauge group. When $\tau_{0\,i}{}^{j} \neq 0$ two of the vectors are dualised to tensor multiplets. By choosing the tensors to be both spacelike or one spacelike and one timelike under $\mathrm{SO}(2,1)$ we get different charges for the tensor multiplets under the action of v^0, leading to either an $\mathrm{SO}(2)$ or $\mathrm{SO}(1,1)$ gauging. In addition, a linear combination of v^0 and the uncharged vectors can also gauge the $\mathrm{U}(1)_R$ symmetry.

$\mathbf{n_{VT} = 4}$: The structure group is $G_S = \mathrm{Spin}(2) \times \mathrm{SU}(2)$ and the isometry group is $G_{\rm iso} = \mathrm{SU}(2)_R \times \mathrm{SO}(3,1) \times \mathbb{R}^+$.

The conditions (4.37) now imply that either $\tau_{ijk} \neq 0$ or $\tau_{0\,i}{}^{j} \neq 0$. We thus have the following possibilities.

- $\tau_{ijk} \neq 0$, $\tau_{0\,i}{}^{j} = 0$. We can write $\tau_{ijk} = \epsilon_{ijkl}A^l$. Depending on whether A^i is spacelike, timelike or null with respect to $\mathrm{SO}(3,1)$, we can have the gauge groups $\mathrm{SO}(2,1)$, $\mathrm{SO}(3)$ or $\mathrm{ISO}(2)$, respectively. In all cases, there are no tensor multiplets. This can be seen as follows.

  If A is timelike, we can always perform an $\mathrm{SO}(3,1)$ rotation such that it lies along the timelike direction and we have

  $$\tau_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma 1}A^1 := \hat{\tau}\epsilon_{\alpha\beta\gamma}\,, \tag{4.46}$$

  where we split the $\mathrm{SO}(3,1)$ indices as $i = 1$ for the timelike direction and $\alpha, \beta, \gamma = 2, 3, 4$ the spacelike ones. Writing $v = (v^0, v^\alpha, v^1) \in \mathcal{V}$ we find the brackets

  $$[\![v, w]\!]^\alpha = -\hat{\tau} v^\beta w^\gamma \epsilon_{\beta\gamma}{}^{\alpha}\,, \qquad [\![v, w]\!]^0 = [\![v, w]\!]^1 = 0\,, \tag{4.47}$$

  leading to a gauging of the compact subgroup $\mathrm{SO}(3) \subset \mathrm{SO}(3,1)$. In addition, either a combination of v^0 and v^1 can be used to gauge a $\mathrm{U}(1)_R$ or the v^α can gauge the full $\mathrm{SU}(2)_R$ via $\tau^a_{\alpha b}$.

  For a spacelike A we proceed in the same way. By an $\mathrm{SO}(3,1)$ rotation we bring τ_{ijk} to the form

  $$\tau_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma 4}A^4 := \hat{\tau}\epsilon_{\alpha\beta\gamma}\,, \tag{4.48}$$

where now $\alpha, \beta, \gamma = 1, 2, 3$. The vector decompose as $v = (v^0, v^\alpha, v^4)$ and we get the algebra

$$[\![v, w]\!]^\alpha = -\hat{\tau} v^\beta w^\gamma \epsilon_{\beta\gamma}{}^\alpha \,, \qquad [\![v, w]\!]^0 = [\![v, w]\!]^4 = 0 \,, \tag{4.49}$$

which gauges an $\mathrm{SO}(2,1)$ subgroup. As above, the v^0 and v^4 can be used to gauge the $\mathrm{U}(1)_R$.

Finally if A is null, by an $\mathrm{SO}(3,1)$ rotation we can reduce to two non-zero components for τ_{ijk}

$$\tau_{234} = \epsilon_{2341} A^1 \,, \qquad \tau_{123} = \epsilon_{1234} A^4 \,, \qquad A^1 = A^4 := \hat{\tau} \,. \tag{4.50}$$

It is useful to decompose the vectors as $v = (v^0, v^2, v^3, v^-, v^+)$ where $v^\pm = v^1 \pm v^4$. The Leibniz algebra then becomes

$$[\![v, w]\!]^0 = 0 \quad \begin{array}{ll} [\![v, w]\!]^2 = \hat{\tau}(v^+ w^3 - w^+ v^3) \,, & [\![v, w]\!]^+ = 0 \,, \\ [\![v, w]\!]^3 = -\hat{\tau}(v^+ w^2 - w^+ v^2) \,, & [\![v, w]\!]^- = 2\,\hat{\tau}(v^2 w^3 - w^2 v^3) \,, \end{array} \tag{4.51}$$

This defines a Lie algebra that is the semi-direct sum of $\mathfrak{so}(2)$ with the 3-dimensional Heisenberg algebra. The vector v^- generates the $\mathfrak{so}(2)$, under which v^2 and v^3 are charged. On the other hand, $\{v^2, v^3, v^+\}$ form a Heisenberg algebra, with v^+ the central element. Since v^+ is central, the gauge group (2.25) under which matter is charged is just $\mathrm{ISO}(2)$, generated by $\{v^-, v^2, v^3\}$. Additionally, the graviphoton v^0 can gauge the $\mathrm{U}(1)_R$.

- $\tau_{ijk} = 0$. We now have a purely abelian gauging and 0, 2 or 4 tensor multiplets, depending on the rank of $\tau_{0\,i}{}^j$. Depending on whether the tensors are timelike or spacelike we get different charges for the tensor multiplets under the abelian group generated by v^0, as discussed for $n_{\mathrm{VT}} = 3$. As a result, we either have two tensor multiplets charged under a $\mathrm{SO}(2)$ or $\mathrm{SO}(1,1)$, or four tensor multiplets charged under the $\mathrm{SO}(1,1)$. In addition, v^0 and, when present, any combination of the uncharged vectors can also gauge the $\mathrm{U}(1)_R$.

$\mathbf{n_{VT} = 6}$: The structure group is $G_S = \mathbb{Z}_2 \times \mathrm{SU}(2)$ and the isometry group is $G_{\mathrm{iso}} = \mathrm{SU}(2)_R \times \mathrm{SO}(5,1) \times \mathbb{R}^+$.

In this case, we will not solve the Leibniz conditions directly but instead we perform a case by case analysis of the possible gauge groups with a given number of tensor multiplets. Since there are 6 vectors, if there are no tensors, we can gauge at most the following semi-simple subgroups of the global $\mathrm{SO}(5,1)$ isometries: $\mathrm{SO}(4)$, $\mathrm{SO}(3,1)$ or $\mathrm{SO}(3) \times \mathrm{SO}(2,1)$. These are only possible if the singlet vectors transform in the adjoint of one of these groups and the torsion contains singlets of the gauge groups.

It is straightforward to see that $\mathrm{SO}(4)$ and $\mathrm{SO}(3,1)$ cannot be gauged. Under the breaking $\mathrm{SO}(5,1) \supset \mathrm{SO}(4) \times \mathrm{SO}(1,1)$ (respectively $\mathrm{SO}(5,1) \supset \mathrm{SO}(3,1) \times \mathrm{SO}(2)$) the vector representation $\mathbf{6}$ decomposes as

$$\mathbf{6} = \mathbf{4}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2} \,, \tag{4.52}$$

where **4** is the vector representation of SO(4) (respectively SO(3,1)) and the subscripts denote the SO(1,1) (respectively SO(2)) charges. Manifestly we see that in each case the decomposition does not include the adjoint represention.

On the other hand, we can gauge $\mathrm{SO}(3)\times\mathrm{SO}(2,1)\subset\mathrm{SO}(5,1)$. In this case, the six vectors decompose as

$$\mathbf{6}=(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\,, \tag{4.53}$$

containing the adjoint of $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$. We denote these by v^α and $v^{\dot\alpha}$, where $\alpha=1,2,3$ labels the adjoint of SO(3) and $\dot\alpha=4,5,6$ the adjoint of SO(2,1). This gauging consists of having

$$\tau_{\alpha\beta\gamma}=A\,\epsilon_{\alpha\beta\gamma}\,,\qquad \tau_{\dot\alpha\dot\beta\dot\gamma}=B\,\epsilon_{\dot\alpha\dot\beta\dot\gamma}\,,\qquad A,B\neq 0\,. \tag{4.54}$$

Note that therefore (4.37) implies that $\tau_{0\,i}{}^j=0$ so that we have no tensor multiplets. The Leibniz bracket becomes

$$\begin{aligned}[\![v,w]\!]^\alpha&=-v^\beta w^\gamma\tau_{\beta\gamma}{}^\alpha\,,\\ [\![v,w]\!]^{\dot\alpha}&=-v^{\dot\beta}w^{\dot\gamma}\tau_{\dot\beta\dot\gamma}{}^{\dot\alpha}\,,\end{aligned} \tag{4.55}$$

reproducing the gauge algebra of $\mathrm{SO}(3)\times\mathrm{SO}(2,1)$. The graviphoton can gauge the $\mathrm{U}(1)_R$ symmetry or the vectors v^α can gauge the diagonal of SO(3) and $\mathrm{SU}(2)_R$.

Let us now study gaugings that could include tensor multiplets. These will have $\tau_{0\,i}{}^j\neq 0$. When $\tau_{0\,i}{}^j$ has rank 2, two vectors are dualised into tensors which are charged under v^0, and the gaugings can only be given by the other four vectors. Depending on the signature of the SO(5,1) metric evaluated in the directions of the tensor multiplets we can have

$$\begin{aligned}&\mathrm{SO}(3)\times\mathrm{SO}(1,1)\times\mathrm{U}(1)_R\,,\qquad &&\mathrm{SO}(2,1)\times\mathrm{U}(1)\times\mathrm{U}(1)_R\,,\\ &\mathrm{SO}(3)\times\mathrm{U}(1)\times\mathrm{U}(1)_R\,,\qquad &&\mathrm{ISO}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)_R\,,\\ &\mathrm{SO}(1,1)\times\mathrm{SU}(2)_R\,,\qquad &&\mathrm{U}(1)\times\mathrm{SU}(2)_R\,,\end{aligned} \tag{4.56}$$

where the factors SO(1,1) or U(1) are gauged by the graviphoton and $\mathrm{U}(1)_R$ by any combination of v^0 and the vector that does not gauge the non-abelian factor. Note that in (4.56) we list the largest group that can be gauged. It is clearly possible to gauge only some factors of the products above.

When $\tau_{0\,i}{}^j$ has rank 4 or 6, the only possible gauge group is the abelian factor gauged by v^0 and the $\mathrm{U}(1)_R$. Depending on whether the image of $\tau_{0\,i}{}^j$ includes the negative eigenvalue of the SO(5,1) signature, or not, we get different charges for the tensor multiplets under the action of v^0, which hence gauges either a U(1) or SO(1,1) group. In addition, v^0 and, when present, any combination of the uncharged vectors can also gauge the $\mathrm{U}(1)_R$.

In Table 2 we summarise the allowed gaugings for truncations with only vectors/tensor multiplet of generic type. Whenever we list a product group, the individual factors can also be gauged separately even though they are not listed as such. Whenever there are abelian factors in G_{gauge}, the $\mathrm{U}(1)_R$ can also be gauged diagonally with some combination of these factors.

$n_{\rm VT}$	$G_{\rm iso}$	$G_{\rm gauge}$	$n_{\rm T}$
1	$\mathrm{SU}(2)_R \times \mathbb{R}^+$	$\mathrm{U}(1)_R$	–
2	$\mathrm{SU}(2)_R \times \mathrm{SO}(1,1) \times \mathbb{R}^+$	$\mathrm{U}(1)_R$ $\mathrm{SO}(1,1)$	– 2
3	$\mathrm{SU}(2)_R \times \mathrm{SO}(2,1) \times \mathbb{R}^+$	$\mathrm{SO}(2,1) \times \mathrm{U}(1)_R$ $\mathrm{SO}(2),\ \mathrm{SO}(1,1)$	– 2
4	$\mathrm{SU}(2)_R \times \mathrm{SO}(3,1) \times \mathbb{R}^+$	$\mathrm{SO}(2,1) \times \mathrm{U}(1)_R,\ \mathrm{SO}(3) \times \mathrm{U}(1)_R,$ $\mathrm{ISO}(2) \times \mathrm{U}(1)_R,\ \mathrm{SU}(2)_R$ $\mathrm{SO}(2) \times \mathrm{U}(1)_R,\ \mathrm{SO}(1,1) \times \mathrm{U}(1)_R,$ $\mathrm{SO}(1,1)$	– 2 4
6	$\mathrm{SU}(2)_R \times \mathrm{SO}(5,1) \times \mathbb{R}^+$	$\mathrm{SO}(3) \times \mathrm{SO}(2,1) \times \mathrm{U}(1)_R,\ \mathrm{SO}(2,1) \times \mathrm{SU}(2)_R,$ $\mathrm{ISO}(2) \times \mathrm{U}(1)_R$ $\mathrm{SO}(2,1) \times \mathrm{U}(1) \times \mathrm{U}(1)_R,\ \mathrm{SO}(3) \times \mathrm{SO}(2) \times \mathrm{U}(1)_R,$ $\mathrm{SO}(3) \times \mathrm{SO}(1,1) \times \mathrm{U}(1)_R,\ \mathrm{ISO}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)_R,$ $\mathrm{SO}(2) \times \mathrm{SU}(2)_R,\ \mathrm{SO}(1,1) \times \mathrm{SU}(2)_R$ $\mathrm{U}(1) \times \mathrm{U}(1)_R,\ \mathrm{SO}(1,1) \times \mathrm{U}(1)_R$ $\mathrm{SO}(1,1)$	– 2 4 6

Table 2. Allowed gaugings $G_{\rm gauge}$ of the global isometry groups $G_{\rm iso}$ in the generic cases with $n_{\rm VT}$ vector/tensor multiplets. The first column gives the total number of vectors and tensor multiplets, the second the global isometry group, the third the allowed gaugings and the last one the number of vectors that are dualised to tensors in each case.

4.1.2 Special cases

The special cases (4.17) are also associated to some of the generalised G_S-structures we listed at the beginning of this section. We now discuss case by case what the associated structure groups are, we determine the corresponding embedding tensor and hence the possible gaugings of the truncated theory.

Differently from the generic case it is quite cumbersome to analyse in full generality the constraints imposed on the gaugings by the Leibniz condition (2.11) and hence the allowed gaugings. Thus in this section we will limit ourselves to study what are the largest reductive groups and largest compact groups that can be gauged.

$\mathbf{n_{VT} = 5}$: This truncation is associated to a $G_S = \mathrm{SU}(2)$ generalised structure. The structure group is taken to be the SU(2) factor in the breaking (Br.2) of USp(6) and it embeds in $\mathrm{SU}^*(6)$ as $\mathrm{SU}^*(6) \supset \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SU}(2)$. Under this embedding we have

$$\mathbf{15}^* = (\mathbf{6}^*, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{3}) \ , \tag{4.57}$$

so that $\mathcal{V} = (\mathbf{6}^*, \mathbf{1})$ and there are six independent singlet vectors giving rise to $n_{\rm VT} = 5$ vector multiplets. It is easy to check that we also get the expected scalar manifold

$$\mathcal{M} = \mathcal{M}_{\rm VT} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(8)/\mathbb{Z}_2}(G_S)} = \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)} \ , \tag{4.58}$$

with isometry group

$$G_{\rm iso} = {\rm SU}(2)_R \times {\rm SL}(3,\mathbb{R})\,. \tag{4.59}$$

We can decompose the elements of $\mathcal{V}$ according to $G_{\rm iso}$

$$\mathcal{V} = (\mathbf{1},\mathbf{6}^*) \ni v_{ij}\,, \qquad i,j = 1,2,3\,. \tag{4.60}$$

The G_S singlet intrinsic torsion decomposes under $G_{\rm iso}$ as

$$\begin{aligned} W_{\rm int} &= (\mathbf{3},\mathbf{6}) \oplus (\mathbf{1},\mathbf{15}^*) \oplus\ (\mathbf{1},\mathbf{3}^*) \\ &\ni \{\tau^{(ij)a}{}_b, \tau^i{}_{(jk)}, \tau_i\}\,, \end{aligned} \tag{4.61}$$

where $\tau^i{}_{ik} = 0$. If $v_{(ij)} \in \mathcal{V}$ the map $T : \mathcal{V} \to \mathfrak{g}_{\rm iso}$ is defined as

$$\begin{aligned} T(V)^a{}_b &= v_{(ij)}\tau^{(ij)a}{}_b\,, \\ T(V)^i{}_j &= \epsilon^{imn}v_{(ml)}\tau^l{}_{(nj)} + \epsilon^{imn}v_{(mj)}\tau_n\,, \end{aligned} \tag{4.62}$$

and gives the bracket

$$\begin{aligned} [\![v,w]\!]_{ij} &= T(v)^k{}_{(i}w_{j)k} = t^{(kl)(mn)}{}_{(ij)}v_{kl}w_{mn} \\ &= -\epsilon^{mpk}[\tau^l{}_{p(i}\delta^n{}_{j)} + \tau_p\delta^l{}_i\delta^n{}_j]v_{kl}w_{mn}\,. \end{aligned} \tag{4.63}$$

Thus the components of the embedding tensor are

$$t^{(kl)(mn)}{}_{(ij)} = \frac{1}{2}\epsilon^{pm(k}[\tau^{l)}{}_{p(i}\delta^n{}_{j)} + \tau_p\delta^{l)}{}_{(i}\delta^n{}_{j)}] + (m \leftrightarrow n)\,, \qquad p^{(ij)a}{}_b = \tau^{(ij)a}{}_b\,. \tag{4.64}$$

We now want to determine the largest non-abelian gaugings that can arise from the consistent truncation. The compact gaugings are quite limited. It is easy to see that it is not possible to gauge the maximal compact subgroup SO(3) of SL(3, $\mathbb{R}$). Indeed, the 6 vectors decompose as $\mathbf{6}^* = \mathbf{5} \oplus \mathbf{1}$ and therefore do not contain the adjoint of SO(3). However, the singlet in the decomposition can be used to gauge the ${\rm U}(1)_R$ symmetry, as can also be seen from the intrinsic torsion, which contains only an SO(3) singlet in $\tau^{(ij)a}{}_b$. We see that only compact abelian gaugings are a priori possible.

Consider now the non-compact gauging ${\rm SL}(2,\mathbb{R}) \simeq {\rm Spin}(2,1)$. This is obtained via the embedding ${\rm SL}(3,\mathbb{R}) \supset {\rm SL}(2,\mathbb{R}) \times \mathbb{R}^+$, under which the vectors decompose as $\mathbf{6}^* = \mathbf{3}_2 \oplus \mathbf{2}_{-1} \oplus \mathbf{1}_{-4}$. Thus we expect to be able to gauge ${\rm SL}(2,\mathbb{R})$, with the two vectors that are charged under ${\rm SL}(2,\mathbb{R})$ dualised into tensors. To see whether this gauging is possible, we must look at the intrinsic torsion and the bracket (4.63). The vectors decompose as

$$v_{(ij)} = \{v_{(\alpha\beta)}, v_\alpha, v_0\}\,, \tag{4.65}$$

where $\alpha = 1,2$ are fundamental indices of ${\rm SL}(2,\mathbb{R})$. The intrinsic torsion contains the ${\rm SL}(2,\mathbb{R})$ singlets

$$W_{\rm int} \supset \mathbf{1}_{-4} \oplus \mathbf{1}_2 \oplus \in \mathbf{1}_{-2} \ni (\tau^a{}_{0b}, \hat{\tau} = \tau^0{}_{00}, \tau = \tau_0)\,, \tag{4.66}$$

and the brackets (4.63) reduce to

$$\begin{aligned}
[\![v,w]\!]_{\alpha\beta} &= (\tau+\hat{\tau})\epsilon^{\gamma\delta}v_{\delta(\alpha}w_{\beta)\gamma}\,,\\
[\![v,w]\!]_{\alpha} &= \tau\epsilon^{\gamma\delta}(v_{\delta\alpha}w_{\gamma}+v_{\delta}w_{\alpha\gamma})+\hat{\tau}\epsilon^{\gamma\delta}(v_{\delta\alpha}w_{\gamma}-3v_{\delta}w_{\alpha\gamma})\,,\\
[\![v,w]\!]_{0} &= (\tau-3\hat{\tau})\epsilon^{\gamma\delta}v_{\delta}w_{\gamma}\,.
\end{aligned} \tag{4.67}$$

The Leibniz condition (2.11) now imposes that either $\tau = 3\hat{\tau}$ or $\hat{\tau} = 0$. These two cases lead to different gaugings.

- $\tau = 3\hat{\tau}$. In this case, the Leibniz bracket (4.67) becomes

$$\begin{aligned}
[\![v,w]\!]_{\alpha\beta} &= 4\hat{\tau}\epsilon^{\gamma\delta}v_{\delta(\alpha}w_{\beta)\gamma}\,,\\
[\![v,w]\!]_{\alpha} &= 4\hat{\tau}\epsilon^{\gamma\delta}v_{\delta\alpha}w_{\gamma}\,,\\
[\![v,w]\!]_{0} &= 0\,.
\end{aligned} \tag{4.68}$$

  We get an $\mathrm{SL}(2,\mathbb{R})$ gauging, generated by the $v_{\alpha\beta}$ vector fields. The v_α are in the image of the symmetric part of the Leibniz bracket and thus are dualised to tensor fields, charged under the $\mathrm{SL}(2,\mathbb{R})$ gauge group. The graviphoton v^0 can gauge $\mathrm{U}(1)_R$.

- $\hat{\tau} = 0$. Now the Leibniz bracket immediately reduces to the Lie bracket

$$\begin{aligned}
[\![v,w]\!]_{\alpha\beta} &= \tau\epsilon^{\gamma\delta}v_{\delta(\alpha}w_{\beta)\gamma}\,,\\
[\![v,w]\!]_{\alpha} &= \tau\epsilon^{\gamma\delta}(v_{\delta\alpha}w_{\gamma}-w_{\delta\alpha}v_{\gamma})\,,\\
[\![v,w]\!]_{0} &= \tau\epsilon^{\gamma\delta}v_{\delta}w_{\gamma}\,,
\end{aligned} \tag{4.69}$$

  and the vectors v_α no longer commute with each other. Therefore, the v_α's cannot be dualised to tensor multiplets and instead contribute to a larger non-abelian gauge group. In particular, we find that the algebra enhances to that of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{Heis}$, with Heis the 3-dimensional Heisenberg group. Here $v_{\alpha\beta}$ generate the semi-simple $\mathrm{SL}(2,\mathbb{R})$ part, v_α transform as doublets of $\mathrm{SL}(2,\mathbb{R})$ and v_0 is the central element of Heis. Therefore, $\{v_\alpha,\, v_0\}$ generate the Heis factor. However, the gauge group under which matter is charged is $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$.

We see explicitly that the consistent truncation analysis differs from the purely five-dimensional one. In five dimensions, the embedding tensor belongs to the full bundle

$$\mathcal{V}^* \otimes \mathfrak{g}_{\mathrm{iso}} = (\mathbf{1},\mathbf{6}) \otimes [(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8})] = (\mathbf{3},\mathbf{6}) \oplus (\mathbf{1},\mathbf{3}^* \oplus \mathbf{6} \oplus \mathbf{15}^* \oplus \mathbf{24}^*)\,, \tag{4.70}$$

where we are decomposing under $\mathrm{SU}(2)_R \times \mathrm{SL}(3,\mathbb{R})$, and therefore contains more representation than those arising in (4.61).

As a result, not all five-dimensional gaugings for $n_{\mathrm{VT}} = 5$ can arise from consistent truncations. For example, in five dimensions, we can have an embedding tensor in $\mathbf{1}_0 \otimes \mathbf{3}_0$ of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^+$, but coming from the $(\mathbf{1},\mathbf{15}^*) \oplus (\mathbf{1},\mathbf{24}^*)$ of $\mathrm{SU}(2)_R \times \mathrm{SL}(3,\mathbb{R})$. This embedding tensor would lead to a U(1) gauging with four tensor multiplets with charges ± 2, ± 4. However, this gauging cannot arise from a consistent truncation, since the intrinsic torsion (4.61) does not contain the $(\mathbf{1},\mathbf{24}^*)$ representation.

$\mathbf{n_{VT}=8}$: This truncation arises for the case (Br.6) and corresponds to a $G_S = \mathrm{U}(1)$ structure group. Under the branching $\mathrm{SU}^*(6) \supset \mathrm{SL}(3,\mathbb{C}) \times \mathrm{U}(1)$ the vectors decompose as[18]

$$\begin{aligned} \mathbf{15}^* &= (\mathbf{3}\otimes\overline{\mathbf{3}})_0 \oplus \mathbf{3}^*_2 \oplus \overline{\mathbf{3}}^*_{-2}\,, \\ &\ni (v^{\alpha\dot\alpha}, v_\alpha, \bar v_{\dot\alpha})\,, \end{aligned} \tag{4.71}$$

where raised α and $\dot\alpha$ indices denote the fundamental representation $\mathbf{3}$ and conjugate-fundamental representation $\overline{\mathbf{3}}$ of $\mathrm{SL}(3,\mathbb{C})$ respectively. Thus for example, since $\mathbf{15}^*$ is real, the two components v^α and $\bar v^{\dot\alpha}$ are related by complex conjugation $(v^\alpha)^* = \bar v^\alpha$ and $(v^{\alpha\dot\beta})^* = v^{\beta\dot\alpha}$. We see that the U(1)-singlet space $\mathcal{V} = (\mathbf{3}\otimes\overline{\mathbf{3}})_0$ is nine-dimensional giving rise to $n_{\mathrm{VT}} = 8$ vector multiplets.

It is easy to check that (4.5) gives the expected scalar manifold

$$\mathcal{M} = \mathcal{M}_{\mathrm{VT}} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(8)/\mathbb{Z}_2}(G_S)} = \frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}\,, \tag{4.72}$$

with isometry group

$$G = \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SU}(2)_R\,. \tag{4.73}$$

The singlet intrinsic torsion can be written as

$$\begin{aligned} W_{\mathrm{int}} &= (\mathbf{3}^*\otimes\overline{\mathbf{3}}^*,\mathbf{3}) \oplus (\mathbf{3}^*\otimes\overline{\mathbf{3}}^*,\mathbf{1}) \oplus (\mathbf{3}^*\otimes\overline{\mathbf{6}},\mathbf{1}) \oplus (\mathbf{6}\otimes\overline{\mathbf{3}}^*,\mathbf{1}) \\ &\ni \{\tau_{\alpha\dot\alpha}{}^a{}_b, \tau_{\alpha\dot\alpha}, \tau_\alpha{}^{\dot\beta\dot\gamma}\,, \bar\tau^{\alpha\beta}{}_{\dot\gamma}\}\,, \end{aligned} \tag{4.74}$$

where $a,b = 1,2,3$ are $\mathrm{SU}(2)_R$ indices. Given $v^{\alpha\dot\alpha} \in \mathcal{V}$ the non-zero components of the map T are

$$\begin{aligned} T(v)^a{}_b &= v^{\alpha\dot\alpha}\tau_{\alpha\dot\alpha}{}^a{}_b\,, \\ T(v)^\alpha{}_\beta &= v^{\alpha\dot\beta}\tau_{\beta\dot\beta} - \tfrac{1}{3}\delta^\alpha{}_\beta v^{\delta\dot\delta}\tau_{\delta\dot\delta} + \epsilon_{\beta\gamma\rho}v^{\gamma\dot\delta}\tau^{\rho\alpha}{}_{\dot\delta}\,, \\ T(v)^{\dot\alpha}{}_{\dot\beta} &= -v^{\beta\dot\alpha}\tau_{\beta\dot\beta} + \tfrac{1}{3}\delta^{\dot\alpha}{}_{\dot\beta}v^{\delta\dot\delta}\tau_{\delta\dot\delta} - \epsilon_{\dot\beta\dot\gamma\dot\rho}v^{\delta\dot\gamma}\bar\tau_\delta{}^{\dot\rho\dot\alpha}\,. \end{aligned} \tag{4.75}$$

Let us now consider the possible gaugings. If we focus on maximal simple subgroups of $\mathrm{SL}(3,\mathbb{C})$, there are three possibilities: $\mathrm{SU}(3)$, $\mathrm{SU}(2,1)$ and $\mathrm{SL}(3,\mathbb{R})$. It is easy to show that the real form $\mathrm{SL}(3,\mathbb{R})$ cannot be gauged. For the subgroup $\mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SL}(3,\mathbb{C})$ the real and conjugate representations are isomorphic, and we can write

$$\bar v^{\dot\alpha} = \delta^{\dot\alpha}_\alpha\, v^\alpha\,. \tag{4.76}$$

The nine real vectors in $\mathbf{3}\otimes\overline{\mathbf{3}}$ of $\mathrm{SL}(3,\mathbb{C})$ then decompose as

$$\mathbf{3}\otimes\overline{\mathbf{3}} \simeq \mathbf{3}\otimes\mathbf{3} = \mathbf{6}\oplus\mathbf{3}^*\,. \tag{4.77}$$

We see explicitly that this does not include the adjoint and hence $\mathrm{SL}(3,\mathbb{R})$ cannot be gauged.

[18] Recall that for $\mathrm{SL}(3,\mathbb{C})$ the dual and conjugate representations are not equivalent. Here we denote them by $\mathbf{n}^*$ and $\overline{\mathbf{n}}$, respectively.

Consider now SU(3) and SU(2, 1). In these two cases the conjugate and dual representations are isomorphic since we can write

$$v^*_\alpha = \eta_{\alpha\dot\alpha}\bar v^{\dot\alpha}\,, \tag{4.78}$$

where $\eta_{\alpha\dot\alpha}$ is the invariant Hermitian form, with signature $(3,0)$ and $(2,1)$ for SU(3) and SU(2, 1) respectively. The nine real vectors in $\mathbf{3}\otimes\overline{\mathbf{3}}$ of $\mathrm{SL}(3,\mathbb{C})$ decompose as

$$\mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1} \ni (\hat v^\alpha{}_\beta, v_0)\,, \tag{4.79}$$

where $\hat v^\alpha{}_\alpha = 0$ and

$$v^{\alpha\dot\alpha} = \hat v^\alpha{}_\beta\eta^{\beta\dot\alpha} + \tfrac{1}{3}\eta^{\alpha\dot\alpha}v_0\,. \tag{4.80}$$

The eight vectors $\hat v^\alpha{}_\beta$ form the adjoint of SU(3) or SU(2, 1). Decomposing the intrinsic torsion (4.74) under $\mathrm{SU}(3)\times\mathrm{SU}(2)_R$ ($\mathrm{SU}(2,1)\times\mathrm{SU}(2)_R$) we get

$$W_{\rm int} = (\mathbf{1},\mathbf{3})\oplus(\mathbf{8},\mathbf{3})\oplus(\mathbf{1},\mathbf{1})\oplus 3\cdot(\mathbf{8},\mathbf{1})\oplus(\mathbf{10},\mathbf{1})\oplus(\overline{\mathbf{10}},\mathbf{1})\,, \tag{4.81}$$

where the four singlet components are

$$\tau_{\alpha\dot\alpha}{}^a{}_b = \eta_{\alpha\dot\alpha}\tau^a{}_b\,, \qquad \tau_{\alpha\dot\alpha} = \tau\,\eta_{\alpha\dot\alpha}\,. \tag{4.82}$$

Given two vectors $v = (\hat v^\alpha{}_\beta, v_0)$ and $w = (\hat w^\alpha{}_\beta, w_0)$, the Leibniz bracket then reads

$$\begin{aligned}&[\![v,w]\!]^\alpha{}_\beta = -\tau\,(\hat v^\alpha{}_\gamma\hat w^\gamma{}_\beta - \hat w^\alpha{}_\gamma\hat v^\gamma{}_\beta)\,,\\ &[\![v,w]\!]_0 = 0\,.\end{aligned} \tag{4.83}$$

Thus, we see that we can gauge either SU(3) or SU(1, 2). The extra vector singlet v_0 can gauge the $\mathrm{U}(1)_R$.

$\mathbf{n}_{\rm VT} = \mathbf{14}$: This is the maximal case, where the invariant vectors span the whole $\mathcal{V} = \mathbf{15}^*$ of $\mathrm{SU}^*(6)$. It does not correspond to any of the generalised structures listed at the beginning of this section and therefore must correspond to a discrete structure group. Indeed, since all the $K_{\tilde I}$ are stabilised and from (4.5) we have

$$\mathcal{M} = \frac{\mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)}{\mathrm{C}_{\mathrm{USp}(6)}(G_S)} = \frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}\,, \tag{4.84}$$

it is easy to identify the generalised structure as

$$G_S = \mathbb{Z}_2 \subset \mathrm{E}_{6(6)}\,. \tag{4.85}$$

The $\mathbb{Z}_2$ acts diagonally as $-\mathbb{1}$ in USp(6), leading to the global isometry group

$$G_{\rm iso} = C_{\mathrm{E}_{6(6)}}(\mathbb{Z}_2) = \mathrm{SU}(2)_R\cdot\mathrm{SU}^*(6)\,. \tag{4.86}$$

Decomposing under $G_{\rm iso}$ we can hence write vectors in $\mathcal{V}$ as

$$\mathcal{V} = (\mathbf{1},\mathbf{15}^*) \ni v_{ij}\,, \tag{4.87}$$

where $v_{ij} = v_{[ij]}$ and $i, j = 1, \ldots, 6$.

The singlet intrinsic torsion arranges into representations of $G_{\rm iso}$ as

$$\begin{aligned} W_{\rm int} &= (\mathbf{3}, \mathbf{15}) \oplus (\mathbf{1}, \mathbf{21}) \oplus (\mathbf{1}, \mathbf{105}) \\ &\ni (\tau^{ija}{}_b, \tau^{ij}, \tau^{ijk}{}_l)\,. \end{aligned} \tag{4.88}$$

where $\tau^{ija}{}_b = \tau^{[ij]a}{}_b$, $\tau^{ij} = \tau^{(ij)}$ and $\tau^{ijk}{}_l = \tau^{[ijk]}{}_l$ with $\tau^{ijl}{}_l = 0$. The map $T : \mathcal{V} \to \mathfrak{g}_{\rm iso}$ is

$$T(v)^a{}_b = \tfrac{1}{2} v_{ij} \tau^{ija}{}_b\,, \quad T(v)^i{}_j = v_{kl} \tau^{ikl}{}_j + v_{ik} \tau^{kj}\,, \tag{4.89}$$

with bracket

$$\begin{aligned} [\![v, w]\!]_{[ij]} &= -T(v)^k{}_{[i} w_{j]k} = t^{[kl][mn]}{}_{[ij]} v_{[kl]} w_{[mn]} \\ &= -\tau^{klm}{}_{[i} w_{j]m} v_{kl} + \tau^{(kl)} v_{k[i} w_{j]l}\,. \end{aligned} \tag{4.90}$$

As there are 15 vectors, the largest semi-simple groups we can gauge are different real forms of $\mathrm{SU}(4) \simeq \mathrm{Spin}(6)$. However $\mathrm{SU}(4)$ and $\mathrm{SU}(2,2) \simeq \mathrm{Spin}(4,2)$ do not embed in $\mathrm{SU}^*(6)$, and we are left with $\mathrm{SU}^*(4) \simeq \mathrm{Spin}(5,1)$ and $\mathrm{SU}(3,1) \simeq \mathrm{Spin}^*(6)$. These are embedded as

$$\mathrm{SU}^*(6) \supset \mathrm{SU}^*(4) \times \mathrm{SU}(2) \times \mathbb{R}^+\,, \qquad \text{and} \qquad \mathrm{SU}^*(6) \supset \mathrm{SU}(3,1)/\mathbb{Z}_2 \simeq \mathrm{SO}^*(6)\,. \tag{4.91}$$

From the decomposition of the vectors

$$\begin{aligned} \mathbf{15}^* &= (\mathbf{6}, \mathbf{1})_{-2} \oplus (\mathbf{4}^*, \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{1})_4\,, \\ \mathbf{15}^* &= \mathbf{15}\,, \end{aligned} \tag{4.92}$$

under $\mathrm{SU}^*(4) \times \mathrm{SU}(2) \times \mathbb{R}^+$ and $\mathrm{SO}^*(6)$ we see that only $\mathrm{SO}^*(6)$ can be gauged.

The intrinsic torsion contains an $\mathrm{SO}^*(6)$ singlet from the decomposition of the **21** of $\mathrm{SU}^*(6)$

$$W_{\rm int} \ni (0, \tau\, \delta^{ij}, 0)\,, \tag{4.93}$$

where δ^{ij} is the invariant metric of $\mathrm{SO}^*(6)$. Then the bracket (4.90) becomes

$$[\![v, w]\!]^i{}_j = \tfrac{1}{2} \tau \big(v^i{}_k w^k j - v^i{}_k w^k j \big)\,, \tag{4.94}$$

where we have raised indices using the $\mathrm{SO}^*(6)$ metric. We easily recognise the $\mathrm{SO}^*(6)$ Lie algebra. Note that the vectors v_{ij} satisfy a reality condition of the form

$$(v^*)_{\bar{i}\bar{j}} = J^k{}_{\bar{i}}\, J^l{}_{\bar{j}}\, v_{kl}\,, \tag{4.95}$$

where $J^i{}_{\bar{j}}$ is the complex structure of $\mathrm{SU}^*(6)$. If we take δ^{ij} to have the standard form, the individual components of v_{ij} are not real. This is why the gauging is $\mathrm{SO}^*(6)$ not $\mathrm{SO}(6)$.

For compact gaugings, the largest possible subgroups of $\mathrm{SU}^*(6)$ are $\mathrm{USp}(4) \times \mathrm{SU}(2)$ and $\mathrm{USp}(4)$. However their adjoints are not contained in the **15** representation. The next largest possible gauge group is $\mathrm{SU}(3) \times \mathrm{U}(1)$, which we will now investigate.

To study the $\mathrm{SU}(3) \times \mathrm{U}(1)$ gauge group, it is useful to consider it as a subgroup of $\mathrm{SO}^*(6)$ which preserves the $\mathrm{U}(3)$ Hermitian form. This way, we can also consider the gauge

group $\mathrm{SU}(2,1)\times\mathrm{U}(1)\subset\mathrm{SO}^*(6)\subset\mathrm{SU}^*(6)$ by instead choosing a split-signature Hermitian form.

Thus, both $\mathrm{SU}(3)\times\mathrm{U}(1)$ and $\mathrm{SU}(2,1)\times\mathrm{U}(1)$ are embedded in $\mathrm{SO}^*(6)$ via

$$\begin{aligned}\mathbf{15}^* &= \mathbf{8}_0\oplus\mathbf{3}_1\oplus\overline{\mathbf{3}}_{-1}\oplus\mathbf{1}_0\,,\\ v_{ij} &= (v^\alpha{}_\beta, v_{\alpha\beta}=\epsilon_{\alpha\beta\gamma}v^\gamma, \bar v^{\alpha\beta}=\epsilon^{\alpha\beta\gamma}\bar v_\gamma, v^0)\,.\end{aligned}\tag{4.96}$$

Here $\alpha=1,2,3$ denotes the fundamental representation of $\mathrm{SU}(3)$ or $\mathrm{SU}(2,1)$, respectively, and the vectors satisfy $v^\alpha{}_\alpha=0$ as well as the reality conditions

$$(v^\alpha{}_\beta)^* = v^\beta{}_\alpha\,,\qquad (\bar v_\alpha)^* = v^\alpha\,,\qquad (v^0)^*=v^0\,.\tag{4.97}$$

Thus, we can expect that the vectors $(v^\alpha{}_\beta, v^0)\in\mathbf{8}\oplus\mathbf{1}$ gauge $\mathrm{SU}(3)\times\mathrm{U}(1)$ or $\mathrm{SU}(2,1)\times\mathrm{U}(1)$, respectively, and the other six are dualised into tensors.

To see whether this can arise, we investigate the intrinsic torsion (4.88). We find that the intrinsic torsion contains singlets under $\mathrm{SU}(3)\times\mathrm{U}(1)$, $\mathrm{SU}(2,1)\times\mathrm{U}(1)$, respectively, given by

$$\tau^{\alpha\beta}{}_{\gamma,\rho} = \tau_{\gamma\rho}{}^{\alpha,\beta} = \tilde\tau_1\delta^{\alpha\beta}_{\gamma\rho}\,,\quad \tau^\alpha{}_\beta = \tilde\tau_2\delta^\alpha_\beta\,,\quad\text{and}\quad \tau^\alpha{}_\beta{}^a{}_b = \delta^\alpha_\beta\tilde\tau^{0a}{}_b.\tag{4.98}$$

Thus, the map T becomes

$$\begin{aligned}T(v)^\alpha{}_\beta &= (\tilde\tau_2-\tilde\tau_1)v^\alpha{}_\beta + \tfrac{1}{3}(\tilde\tau_2+2\tilde\tau_1)\delta^\alpha{}_\beta v^0\,,\\ T(v)^{\alpha\beta} &= (\tilde\tau_1+\tilde\tau_2)\epsilon^{\alpha\beta\gamma}\bar v_\gamma\,,\\ T(v)^a{}_b &= v^0\tilde\tau^{0a}{}_b\,,\end{aligned}\tag{4.99}$$

with the others following from the above by complex conjugation. This leads to the Leibniz bracket

$$\begin{aligned}[\![v,w]\!]^\alpha{}_\beta &= \tfrac{1}{2}(\tilde\tau_2-\tilde\tau_1)(v^\gamma{}_\beta w^\alpha{}_\gamma - w^\gamma{}_\beta v^\alpha{}_\gamma) + \tfrac{1}{2}(\tilde\tau_1+\tilde\tau_2)(v^\alpha\bar w_\beta - w^\alpha\bar v_\beta)\\ &\quad - \tfrac{1}{6}\delta^\alpha{}_\beta(\tilde\tau_1+\tilde\tau_2)(v^\gamma\bar w_\gamma - w^\gamma\bar v_\gamma)\,,\\ [\![v,w]\!]^\alpha &= \tfrac{1}{2}\tilde\tau_1(v^\alpha{}_\beta w^\beta + w^\alpha{}_\beta v^\beta) - \tfrac{1}{2}\tilde\tau_2(v^\alpha{}_\beta w^\beta - w^\alpha{}_\beta v^\beta)\\ &\quad + \tfrac{1}{3}\tilde\tau_2(v_0w^\alpha - w_0v^\alpha) + \tfrac{1}{3}\tilde\tau_1(2v_0w^\alpha - w_0v^\alpha)\,,\\ [\![v,w]\!]^0 &= (\tilde\tau_1+\tilde\tau_2)(v^\alpha\bar w_\alpha - w^\alpha\bar v_\alpha)\,,\end{aligned}\tag{4.100}$$

with $[\![v,w]\!]_\alpha = ([\![v,w]\!]^\alpha)^*$.

The Leibniz condition now reduces to

$$\tilde\tau_1(\tilde\tau_1+\tilde\tau_2)=0\,.\tag{4.101}$$

The two solutions $\tilde\tau_1=0$ and $\tilde\tau_1=-\tilde\tau_2$ lead to two different gaugings. When $\tilde\tau_1=0$ we recover the previous case where $\mathrm{SU}(3,1)$ is gauged. For $\tilde\tau_1=-\tilde\tau_2$ the Leibniz bracket becomes

$$\begin{aligned}[\![v,w]\!]^\alpha{}_\beta &= \tilde\tau_2(v^\gamma{}_\beta w^\alpha{}_\gamma - w^\gamma{}_\beta v^\alpha{}_\gamma)\,,\\ [\![v,w]\!]^\alpha &= -\tilde\tau_2 v^\alpha{}_\beta w^\beta - \tfrac{1}{3}\tilde\tau_2 v_0 w^\alpha\,,\\ [\![v,w]\!]^0 &= 0\,.\end{aligned}\tag{4.102}$$

We recognise the Lie algebra of $\mathrm{SU}(3) \times \mathrm{U}(1)$ and $\mathrm{SU}(2,1) \times \mathrm{U}(1)$, respectively, generated by the $v^\alpha{}_\beta$ and v^0. The v^α and $\bar{v}_\alpha$ are in the image of the symmetrised Leibniz bracket and therefore correspond to tensor multiplets which transform in the $\mathbf{3}_1 \oplus \bar{\mathbf{3}}_{-1}$ of $\mathrm{SU}(3) \times \mathrm{U}(1)$ or $\mathrm{SU}(2,1) \times \mathrm{U}(1)$. The U(1) generator v^0 can also gauge the $\mathrm{U}(1)_R$ via $\tau^{0a}{}_b$.

In Table 3 we summarise the maximal reductive and compact gauge groups for the special cases of purely vector/tensor multiplet truncations of this section. As in the previous table, whenever we list a product of groups, the individual factors can also be gauged separately even though they are not listed as such. Whenever there are abelian factors in G_{gauge}, the $\mathrm{U}(1)_R$ can also be gauged diagonally with some combination of these factors.

n_{VT}	G_{iso}	G_{gauge}	n_{T}
5	$\mathrm{SU}(2)_R \times \mathrm{SL}(3,\mathbb{R})$	$\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$	–
		$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)_R$	2
8	$\mathrm{SU}(2)_R \times \mathrm{SL}(3,\mathbb{C})$	$\mathrm{SU}(3) \times \mathrm{U}(1)_R$, $\mathrm{SU}(2,1) \times \mathrm{U}(1)_R$	–
14	$\mathrm{SU}(2)_R \times \mathrm{SU}^*(6)$	$\mathrm{SU}(3,1)$	–
		$\mathrm{SU}(3) \times \mathrm{U}(1)_R$, $\mathrm{SU}(3) \times \mathrm{U}(1)$	6

Table 3. Maximal reductive and compact gauge groups in the special cases of purely vector/tensor multiplet truncations. The first column gives the total number of vectors and tensor multiplets, the second the global isometry group, the third the allowed gaugings and the last one the number of vectors that are dualised to tensors in each case.

4.2 Truncations with only hypermultiplets

Let us now analyse which consistent truncations are possible with only hypermultiplets and no vector multiplets.

Truncations of this kind are associated to a generalised structures G_S that is defined by a single generalised vector K in the $\mathbf{27}^*$ of $\mathrm{E}_{6(6)}$, defining a V-structure, and a set of adjoint tensors J_A, $A = 1, \ldots, \dim(G_{\text{H}})$, satisfying

$$J_A \cdot K = 0\,. \tag{4.103}$$

Since the stabiliser of the V-structure is $\mathrm{F}_{4(4)} \subset \mathrm{E}_{6(6)}$,

we must have $G_S \subset \mathrm{F}_{4(4)}$. Finally, by construction, the scalar manifold must be symmetric (see Section 3)

$$\mathcal{M}_{\text{H}} = \frac{G_{\text{H}}}{\mathrm{SU}(2)_R \cdot \mathrm{C}_{\mathrm{USp}(6)}(G_S)}\,, \tag{4.104}$$

where $G_{\text{H}} = \mathrm{C}_{\mathrm{E}_{6(6)}}(G_S)$ is the group generated by the singlets J_A.

The above considerations already restrict the possible scalar manifolds for the hyper-

multiplets to the following list [40, 41]

$$\begin{aligned}
\mathcal{M}_{\rm H} &= \frac{\mathrm{F}_{4(4)}}{\mathrm{SU}(2)\cdot\mathrm{USp}(6)}\,, && n_{\rm H} = 7\,,\\
\mathcal{M}_{\rm H} &= \frac{\mathrm{SO}_0(4,p)}{\mathrm{SO}(4)\times\mathrm{SO}(p)}\,, && n_{\rm H} = p\,, \quad p\leq 5\,,\\
\mathcal{M}_{\rm H} &= \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}\,, && n_{\rm H} = 2\,,\\
\mathcal{M}_{\rm H} &= \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}\,, && n_{\rm H} = 1\,,
\end{aligned} \tag{4.105}$$

where $\mathrm{SO}_0(4,p)$ denotes the connected component of the $\mathrm{SO}(4,p)$.

However the first two manifolds do not arise from truly $\mathcal{N}=2$ truncations. This is because they correspond to generalised structure groups that lead to extra singlets in the decomposition of the **6** of USp(6). For $\mathcal{M}_{\rm H} = \frac{\mathrm{F}_{4(4)}}{\mathrm{SU}(2)\cdot\mathrm{USp}(6)}$, the structure group is trivial, $G_S = \mathbb{1}$, since it is given by the commutant in $\mathrm{F}_{4(4)}$ of the isometry group. Thus this truncation always comes from a sub-truncation of five-dimensional maximal supergravity. Similarly, for the $\mathcal{M}_{\rm H} = \frac{\mathrm{SO}_0(4,p)}{\mathrm{SO}(4)\times\mathrm{SO}(p)}$, with $p\leq 5$, the structure group has to be

$$G_S = \mathrm{Spin}(5-p)\,, \tag{4.106}$$

with $\mathrm{Spin}(0) = \mathrm{Spin}(1) = \mathbb{Z}_2$. The decomposition of the **6** of USp(6) under G_S always contains two extra singlets, so that these cases are sub-truncations of half-maximal gauged supergravity. Indeed, from the commutant of G_S in the full $\mathrm{E}_{6(6)}$ and USp(8) groups,

$$\mathrm{C}_{\mathrm{E}_{6(6)}}(\mathrm{Spin}(5-p)) = \mathrm{Spin}(5,p)\times\mathbb{R}^+\,, \qquad \mathrm{C}_{\mathrm{USp}(8)}(\mathrm{Spin}(5-p)) = \mathrm{USp}(4)\times\mathrm{Spin}(p)\,, \tag{4.107}$$

one can easily check that $G_S = \mathrm{Spin}(5-p)$ actually allows for a half-maximal truncation with p vector multiplets and scalar manifold

$$\mathcal{M} = \frac{\mathrm{Spin}(5,p)}{\mathrm{USp}(4)\times\mathrm{Spin}(p)}\times\mathbb{R}^+\,. \tag{4.108}$$

This leaves only the two last manifolds in (4.105) as truly $\mathcal{N}=2$ truncations.

- The case with $\mathbf{n_H} = \mathbf{2}$ hypermultiplets corresponds to a $G_S = \mathrm{SO}(3)$ that is obtained from (Br.2). The structure group embeds as

$$\begin{aligned}
\mathrm{F}_{4(4)} &\supset \mathrm{SU}(2)\times\mathrm{G}_{2(2)}\,,\\
\mathrm{USp}(6) &\supset \mathrm{SU}(2)\times\mathrm{SU}(2)\,.
\end{aligned} \tag{4.109}$$

  Decomposing the **78** of $\mathrm{E}_{6(6)}$ in representations of $G_S = \mathrm{SU}(2)$ gives 6 compact and 8 non-compact singlets. Altogether they correspond to the generators of $\mathrm{G}_{2(2)}$, while the compact ones give its SO(4) maximal compact subgroup. Then (4.104) gives the expected scalar manifold

$$\mathcal{M} = \mathcal{M}_{\rm H} = \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}\,. \tag{4.110}$$

  It is also easy to check that there are no vector/tensor multiplets in the truncation, since there are no singlets in the **26** of $\mathrm{F}_{4(4)}$ under the branching (4.109).

- The case with $\mathbf{n_H} = \mathbf{1}$ tensor multiplet corresponds to the generalised structure $G_S = \mathrm{SU}(3)$ (Br.3). This is embedded as

$$\begin{aligned} \mathrm{F}_{4(4)} &\supset \mathrm{SU}(3) \times \mathrm{SU}(2,1)\,, \\ \mathrm{USp}(6) &\supset \mathrm{SU}(3) \times \mathrm{U}(1)\,. \end{aligned} \tag{4.111}$$

  In the decomposition of the **78** of SU(3) one finds 4 compact and 4 non-compact singlets, which generate SU(2, 1). The compact ones give the compact subgroup SU(2) × U(1) so that we recover the hyperscalar manifold

$$\mathcal{M} = \mathcal{M}_{\mathrm{H}} = \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))}\,. \tag{4.112}$$

  As, again, there are no singlets in the **26** of $\mathrm{F}_{4(4)}$ under the branching to $G_S = \mathrm{SU}(3)$, there are no vector multiplets.

The study of the intrinsic torsions and the gauging for the truncations with only hypermultiplets is very simple. As the only vector in the theory is the graviphoton in the universal multiplet, only abelian gaugings are possible. Moreover, in all cases, the intrinsic torsion only contains the adjoint representation of the isometry group

$$W_{\mathrm{int}} = \mathrm{ad}\, G_{\mathrm{H}} \ni \tau_0{}^A{}_B\,, \tag{4.113}$$

with $A, B = 1, \ldots \dim G_{\mathrm{H}}$ so that the map $T : \mathcal{V} \to \mathfrak{g}_{\mathrm{iso}}$ is

$$T(v^0) = v^0 \tau_0{}^A{}_B\,, \qquad A = 1, \ldots, \dim G_{\mathrm{H}}\,. \tag{4.114}$$

The generalised Lie derivative on the adjoint singlets is

$$L_{K_0} J_A = [J_{K_0}, J_A] = -T(K_0) \cdot J_A = p_{0A}{}^B J_B\,, \tag{4.115}$$

with the component of the embedding tensor

$$p_{0A}{}^B = \tau_0{}^A{}_B\,, \tag{4.116}$$

and the graviphoton can gauge any one-dimensional subgroup of G_{H}.

4.3 Truncations with vector/tensor and hypermultiplets

The last class of truncations that can arise consists of truncations with both vector/tensor and hypermultiplets. One way to study this class is to start from the truncations with only hypermultiplets discussed in the previous section and look for a subgroup of the structure group G_S that allows for extra singlet vectors but no extra singlets in the branching of the **6** under $\mathrm{USp}(6) \supset G_S$. This last condition is necessary to have a truly $\mathcal{N} = 2$ truncation and leaves only two possible cases: $n_{\mathrm{H}} = 2$ with $G_S = \mathrm{SU}(2)$ (Br.2) or $n_{\mathrm{H}} = 1$ and $G_S = \mathrm{SU}(3)$ (Br.3).

The case with $n_{\mathrm{H}} = 2$ hypermultiplets and hyperscalar manifold

$$\mathcal{M}_{\mathrm{H}} = \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}\,, \tag{4.117}$$

is immediately ruled out since any further reduction of the $G_S = \mathrm{SU}(2)$ structure group necessarily gives rise to a singlet in the **6** of USp(6). This can be easily see from (4.11) by breaking the second SU(2) factor. Therefore consistent truncations with hypermultiplets forming the scalar manifold (4.117) and vector/tensor multiplets necessarily arise from subtruncations of $\mathcal{N} > 2$ gauged supergravity.

We are left with the case with $n_{\mathrm{H}} = 1$ hypermultiplet and hyperscalar manifold

$$\mathcal{M}_{\mathrm{H}} = \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))} \,. \tag{4.118}$$

The structure group is SU(3) and we can consider two non-trivial subgroups $G_S = \mathrm{SU}(2) \times \mathrm{U}(1)$ (Br.4) and $G_S = \mathrm{U}(1)$ (Br.5). As we will discuss below, they allow for $n_{\mathrm{VT}} = 1$ and $n_{\mathrm{VT}} = 4$ vector multiplets, respectively. Cases with $n_{\mathrm{VT}} = 2, 3$ can only be obtained as sub-truncations of the $n_{\mathrm{VT}} = 4$ case and therefore we will not discuss them here.

Recall that the scalar manifold of the vector/tensor multiplets in the truncation can now be computed from the commutant of G_S within the stabiliser groups $G_{\mathcal{U}}$ and $H_{\mathcal{U}}$, in $\mathrm{E}_{6(6)}$ and $\mathrm{USp}(8)/\mathbb{Z}_2$ respectively, of the space $\mathcal{U}$ of J_A that define the hypermultiplet moduli. One finds

$$G_{\mathcal{U}} = \mathrm{SL}(3, \mathbb{C}) \subset \mathrm{E}_{6(6)} \,, \tag{4.119}$$

with compact subgroup

$$H_{\mathcal{U}} = \mathrm{SU}(3) \subset \mathrm{USp}(8)/\mathbb{Z}_2 \,. \tag{4.120}$$

The scalar manifold of the vector/tensor multiplets is then

$$\mathcal{M}_{\mathrm{VT}} = \frac{\mathrm{C}_{G_{\mathcal{U}}}(G_S)}{\mathrm{C}_{H_{\mathcal{U}}}(G_S)} = \frac{\mathrm{C}_{\mathrm{SL}(3,\mathbb{C})}(G_S)}{\mathrm{C}_{\mathrm{SU}(3)}(G_S)} \,. \tag{4.121}$$

We thus find the two following possible truncations.

$\mathbf{n}_{\mathrm{VT}} = \mathbf{1}, \mathbf{n}_{\mathrm{H}} = \mathbf{1}$: Consider first the structure group $G_S = \mathrm{SU}(2) \times \mathrm{U}(1)$.

The **27*** of $\mathrm{E}_{6(6)}$ contains two G_S singlets so that $\mathcal{V}$ is two-dimensional and $n_{\mathrm{VT}} = 1$. Thus, the scalar manifold is

$$\mathcal{M}_{\mathrm{VT}} = \mathbb{R}^+ \,, \qquad \mathcal{M}_{\mathrm{H}} = \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))} \,. \tag{4.122}$$

The decomposition of the adjoint of $\mathrm{E}_{6(6)}$ gives four compact and five non-compact G_S singlets that are the generators of the isometry group

$$G_{\mathrm{iso}} = \mathbb{R}^+ \times \mathrm{SU}(2,1) \,. \tag{4.123}$$

Under this group the vectors decompose as

$$\mathcal{V} = \mathbf{1}_2 \oplus \mathbf{1}_{-1} \ni (v^0, v^1) \,. \tag{4.124}$$

where the subscripts denote the $\mathbb{R}^+$ charges.

To determine the possible gaugings we find that the intrinsic torsion has components

$$W_{\mathrm{int}} = \mathbf{8}_{-2} \oplus \mathbf{8}_1 \oplus \mathbf{1}_1 \ni (\tau_{0\,B}^{A}, \tau_{1\,B}^{A}, \tau_1) \,, \tag{4.125}$$

which give the adjoint action

$$T(v)^A{}_B = v^0 \tau_{0\,B}^A + v^1 \tau_{1\,B}^A \,, \tag{4.126}$$

and $T(v)_{(0)} = v^1 \tau_1 = 0$ by the Leibniz condition. Furthermore (3.53) implies $\tau_{0\,B}^A$ and $\tau_{1\,B}^A$ commute. Thus, the two vectors can gauge a one- or two-dimensional abelian subgroup of $\mathrm{SU}(2,1)$, while the $\mathbb{R}^+$ symmetry cannot be gauged.

$\mathbf{n}_{\mathrm{VT}} = \mathbf{4}, \mathbf{n}_{\mathrm{H}} = \mathbf{1}$: Keeping only $G_S = \mathrm{U}(1) \subset \mathrm{SU}(2) \times \mathrm{U}(1)$ as structure group the $\mathbf{27}^*$ contains five G_S singlets so that $\mathcal{V}$ is five-dimensional and $n_{\mathrm{VT}} = 4$. The commutators of $G_S = \mathrm{U}(1)$ in $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{SU}(3)$

$$\begin{aligned} \mathrm{C}_{\mathrm{SL}(3,\mathbb{C})}(\mathrm{U}(1)) &= \mathrm{SL}(2,\mathbb{C}) \times \mathrm{U}(1) \times \mathbb{R}^+ \,, \\ \mathrm{C}_{\mathrm{SU}(3)}(\mathrm{U}(1)) &= \mathrm{SU}(2) \times \mathrm{U}(1) \,. \end{aligned} \tag{4.127}$$

and hence, from (4.121), the scalar manifold is

$$\mathcal{M}_{\mathrm{VT}} = \frac{\mathrm{SO}(3,1)}{\mathrm{SO}(3)} \times \mathbb{R}^+ \,, \qquad \mathcal{M}_{\mathrm{H}} = \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))} \,. \tag{4.128}$$

The adjoint of $\mathrm{E}_{6(6)}$ contains seven compact and seven non-compact G_S singlet elements corresponding to the isometry group

$$G_{\mathrm{iso}} = \mathrm{SO}(3,1) \times \mathbb{R}^+ \times \mathrm{SU}(2,1) \,. \tag{4.129}$$

The intrinsic torsion components arrange themselves in representations of the isometry group G_{iso}

$$W_{\mathrm{int}} = (\mathbf{4},\mathbf{8})_{-1} \oplus (\mathbf{1},\mathbf{8})_2 \oplus (\mathbf{6},\mathbf{1})_{-2} \oplus 2 \cdot (\mathbf{4},\mathbf{1})_1 \ni (\tau_{i\,B}^A, \tau_{0\,B}^A, \tau_{0\,j}^i, \tau_i, \tau^{[ijk]}) \,, \tag{4.130}$$

where $A, B = 1,2,3$ and $i = 1,\ldots,4$ are $\mathrm{SU}(2,1)$ and $\mathrm{SO}(3,1)$ indices and the subscript denotes the $\mathbb{R}^+$ charges. The T map is defined as:

$$\begin{aligned} T(v)^A{}_B &= v^0 \tau_{0\,B}^A + v^i \tau_{i\,B}^A \,, \\ T(v)^i{}_j &= v^0 \tau_{0\,j}^i + v^k \tau_k{}^i{}_j \,, \end{aligned} \tag{4.131}$$

where again $\tau_i = 0$ because of the Leibniz condition (4.37).

The analysis of the gauging of the vector/tensor multiplet isometries is the same as for the $n_{\mathrm{VT}} = 4$ generic case without hypermultiplets, so that the possible gauge groups are $\mathrm{SO}(2,1)$, $\mathrm{SO}(3)$, $\mathrm{ISO}(2)$, when there are no tensor multiplets, and $\mathrm{SO}(2)$ or $\mathrm{SO}(1,1)$ with tensor multiplets.

The gauging of $\mathrm{SU}(2)_R$ or $\mathrm{U}(1)_R$ subgroups of $\mathrm{SU}(2,1)$ global symmetry group of $\mathcal{M}_{\mathrm{H}}$ are also given by the analysis of the case with only $n_{\mathrm{VT}} = 4$ vector multiplets.

To see whether other subgroups of the $\mathrm{SU}(2,1)$ are possible one has to analyse condition (3.53), which now implies

$$\tau_{0\,i}{}^j \tau_{j\,B}^A = 0 \,, \qquad \tau_{0\,B}^A \tau_{i\,C}^B - \tau_{i\,B}^A \tau_{0\,C}^B = 0 \,, \qquad \tau_{i\,C}^A \tau_{j\,B}^C - \tau_{j\,C}^A \tau_{i\,B}^C = \tau_{ij}{}^k \tau_{k\,B}^A \,. \tag{4.132}$$

We again consider two cases, to solve the constraints (4.37):

- $\tau_{ijk} = 0$ and $\tau_{0\,i}{}^{j} \neq 0$. In this case we only have abelian gaugings and the rank of $\tau_{0\,i}{}^{j}$ determines the number of tensor multiplets. If it has rank 4, then $\tau_{i}{}^{A}{}_{B} = 0$ and the only possibility is that v^0 gauges a one-dimensional subgroup of $\mathrm{SU}(2,1)$ via $\tau_{0}{}^{A}{}_{B}$. If $\tau_{0\,i}{}^{j}$ has rank 0 or 2, any two linearly independent combinations of v^0 and v^i can gauge a 1- or 2-dimensional abelian subgroup of $\mathrm{SU}(2,1)$, with the embedding determined by $\tau_{0}{}^{A}{}_{B}$ and $\tau_{i}{}^{A}{}_{B}$.

- $\tau_{ijk} \neq 0$, $\tau_{0\,i}{}^{j} = 0$. In this case the first equation of (4.132) is trivially verified. The gauge groups are the same as for the generic case. Indeed in the generic case the gauge groups are given by how the tensor τ_{ijk} decomposes. Here this tensor gives in the third equation of (4.132) directly the structure constant of the gauging inside $\mathrm{SU}(2,1)$. So we obtain the same possible gaugings as for the generic case but with two different embedding. We could either gauge a subgroup of $\mathrm{SO}(3,1)$ or a diagonal subgroup of $\mathrm{SO}(3,1)$ subgroup and $\mathrm{SU}(2,1)$ subgroup.

In Table 4 we give the list of possible gauging for truncations with vector/tensor and hypermultiplets. For simplicity we give a list of product groups, but the individual factors can also be gauged separately. G_{ext}, the $U(1)_R$ can also be gauged diagonally with some combination of these factors.

n_{VT}	n_{H}	G_{iso}	G_{gauge}	n_{T}
1	1	$\mathrm{SU}(2,1) \times \mathrm{SO}(1,1) \times \mathbb{R}^+$	$\mathrm{U}(1)_R \times \mathbb{R}^+$	–
4	1	$\mathrm{SU}(2,1) \times \mathrm{SO}(3,1) \times \mathbb{R}^+$	$\mathrm{SO}(2,1) \times \mathbb{R}^+ \times \mathrm{U}(1)_R$, $\mathrm{SO}(3) \times \mathbb{R}^+ \times \mathrm{U}(1)_R$, $\mathrm{ISO}(2) \times \mathbb{R}^+ \times \mathrm{U}(1)_R$, $\mathrm{SU}(2)_R \times \mathbb{R}^+$	–
			$\mathrm{SO}(2) \times \mathrm{U}(1)_R \times \mathbb{R}^+$, $\mathrm{SO}(1,1) \times \mathrm{U}(1)_R \times \mathbb{R}^+$	2
			$\mathrm{SO}(1,1)$	4

Table 4. Summary of the gauge groups in the mixed cases. The first column gives the total number of vectors and tensor multiplets, the second the global isometry group, the third the allowed gaugings and the last one the number of vectors that are dualised to tensors in each case.

5 Conclusions

In this paper, we used exceptional generalised geometry to classify which five-dimensional $\mathcal{N} = 2$ gauged supergravities can arise as consistent truncations of 10-/11-dimensional supergravity. From the higher-dimensional point of view any truncation is associated to a generalised $G_S \subset \mathrm{E}_{6(6)}$ structure on the compactification manifold M, with constant intrinsic torsion. The field content of the truncated theory is determined by the nowhere vanishing generalised tensors on M that define the G_S structure, while the embedding tensor is given by the constant singlet intrinsic torsion.

Requiring that the G_S structure has constant, singlet intrinsic torsion imposes differential conditions on the structure on M that we do not analyse in this paper. Instead we assume that such conditions are satisfied, and we show that already the algebraic analysis

of the allowed G_S structures and possible singlet intrinsic torsion severely restricts which five-dimensional $\mathcal{N}=2$ gauged supergravities can be obtained by a consistent truncation.

In particular, we find that the scalar manifolds must necessarily be symmetric spaces and that there is a very limited number of possible truncations. If there are just vector/tensor multiplets, we can only have

$$\begin{aligned}
\mathcal{M}_{\rm VT} &= \mathbb{R}^+ \times \frac{\mathrm{SO}(n_{\rm VT}-1,1)}{\mathrm{SO}(n_{\rm VT}-1)}\,, && n_{\rm VT} \leq 6\,,\\
\mathcal{M}_{\rm VT} &= \frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}\,, && n_{\rm VT} = 5\,,\\
\mathcal{M}_{\rm VT} &= \frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}\,, && n_{\rm VT} = 8\,,\\
\mathcal{M}_{\rm VT} &= \frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)}\,, && n_{\rm VT} = 14\,,
\end{aligned} \tag{5.1}$$

while if there are just hypermultiplets, the only possibilities are

$$\begin{aligned}
\mathcal{M}_{\rm H} &= \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}\,, && n_{\rm H} = 2\,,\\
\mathcal{M}_{\rm H} &= \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}\,, && n_{\rm H} = 1\,.
\end{aligned} \tag{5.2}$$

Finally, for vector/tensor and hypermultiplets, the only theories with higher-dimensional origin are of the form

$$\begin{aligned}
\mathcal{M}_{\rm H} &= \frac{\mathrm{SU}(2,1)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}\,, && n_{\rm H} = 1\,,\\
\mathcal{M}_{\rm VT} &= \mathbb{R}^+ \times \frac{\mathrm{SO}(n_{\rm VT}-1,1)}{\mathrm{SO}(n_{\rm VT}-1)}\,, && n_{\rm VT} \leq 4\,,
\end{aligned} \tag{5.3}$$

Any other five-dimensional $\mathcal{N}=2$ gauged supergravity cannot be uplifted via a consistent truncation to 10-/11-dimensional supergravity[19].

For each of the above cases, we give the corresponding G_S structure and study what gaugings can arise. Algebraically, these are encoded in the singlets of the intrinsic torsion subject to the Leibniz condition. The results are summarised in Tables 2, 3 and 4. For gauged supergravities with only vector/tensor multiplets, we recover many of the results of [32], where the allowed gaugings are discussed from a purely five-dimensional point of view. However, we can also exclude certain of the five-dimensional gaugings that appear in [32]. For example in the case of $\mathcal{M}_{\rm VT} = \mathbb{R}^+ \times \frac{\mathrm{SO}(n_{\rm VT}-1,1)}{\mathrm{SO}(n_{\rm VT}-1)}$, we find that gaugings where the tensors are charged under a vector transforming non-trivially under $\mathrm{SO}(n_{\rm VT}-1,1)$ cannot arise from consistent truncations. For truncations with only hypermultiplets the gaugings are trivial since they reduce to gauging the $\mathrm{U}(1)_R$ symmetry. What is probably more surprising is the very limited number of truncations with both vector- and hypermultiplets.

[19]The theories with hyperscalar manifolds $\mathcal{M}_{\rm H} = \frac{\mathrm{F}_{4(4)}}{\mathrm{SU}(2)\cdot\mathrm{USp}(6)}$, or $\mathcal{M}_{\rm H} = \frac{\mathrm{SO}_0(4,p)}{\mathrm{SO}(4)\times\mathrm{SO}(p)}$, as well as with hyperscalar manifolds $\mathcal{M}_{\rm H} = \frac{\mathrm{G}_{2(2)}}{\mathrm{SO}(4)}$ and some vector multiplets are necessarily subtruncations of $\mathcal{N}>2$ supergravities.

Our findings are particularly important for the study of gauged supergravities containing AdS vacua. Since no AdS vacuum is believed to admit scale separation [2], those gauged supergravities that cannot be uplifted by a consistent truncation cannot have a higher-dimensional string theory origin. Therefore, they should belong to the *swampland* of lower-dimensional theories.

An important issue that we do not address here is whether we can actually solve the differential conditions imposed by the intrinsic torsion, that are required for the consistent truncation to exist. This would involve constructing explicit examples of background that admit the G_S structure listed in this paper and checking that the intrinsic torsion has only singlet constant components. We leave this analysis for future work. It would also be interesting to see whether the approach of [22–24] can be extended to non-maximally supersymmetric truncations and to use the the five-dimensional embedding tensor to determine what the uplifted geometry should be. In any case, we expect that imposing the differential conditions from the intrinsic torsion will further restrict which consistent truncations exist.

It is also worth stressing that we scanned all possible generalised structures where G_S is a Lie group. It is possible that looking at discrete structure groups might increase the number of possible truncations. We leave this as a problem for the future.

Another direction of future research is to extend our analysis to other dimensions and amounts of supersymmetry. For example, it would be interesting to classify which four-dimensional $\mathcal{N} = 2$ gauged supergravities can be uplifted by consistent truncations to 10-/11-dimensional supergravity. More ambitious would be to extend our method to three dimensions, where $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gauged supergravities admit deformations corresponding to real/holomorphic superpotentials that are not induced by gaugings [52]. It would be interesting to explore which of these can arise from consistent truncations. The appropriate framework would be $\mathrm{E}_{8(8)}$ Exceptional Field Theory [53], where the generalised Lie derivative does not close without the addition of shift symmetries, leading to technical challenges. Similar questions can be asked in two dimensions, where subgroups of affine global symmetry groups, such as $\mathrm{E}_{9(9)}$ for maximal supersymmetry, can be gauged and scalar and vector fields transform in infinite-dimensional representations of the affine symmetry. This question can in principle be addressed with $\mathrm{E}_{9(9)}$ Exceptional Field Theory [54–56], which however requires infinitely-large generalised tangent bundles.

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A $E_{6(6)}$ generalised geometry for M-theory

This section is a brief recall of the main features of the generalised geometry of M-theory compactifications on a six-dimensional manifold M. For a more detailed discussion we refer to [57] and [49, App. E].

For M-theory on a six-dimensional manifold we use $E_{6(6)} \times \mathbb{R}^+$ generalised geometry. The generalised tangent bundle E and the dual bundle E^* are

$$\begin{aligned} E &\simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \,, \\ E^* &\simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM \,, \end{aligned} \tag{A.1}$$

where we decompose the various bundles in representations of $GL\,6$, the geometric subgroup of $E_{6(6)}$. The sections of E and E^*, the generalised vectors and its dual, transform in the $\mathbf{27}^*$ and the $\mathbf{27}$ of $E_{6(6)}$ and can be written as

$$\begin{aligned} V &= v + \omega + \sigma \,, \\ Z &= \hat{v} + \hat{\omega} + \hat{\sigma} \,, \end{aligned} \tag{A.2}$$

where v is an ordinary vector field, ω is a two-form, σ is a five-form[20], $\hat{v}$ is one-form, $\hat{\omega}$ is a two-vector and $\hat{\sigma}$ is a five-vector. Generalised vectors and dual generalised vectors have a natural pairing given by

$$\big\langle Z, V \big\rangle = \hat{v}_m v^m + \tfrac{1}{2}\,\hat{\omega}^{mn}\omega_{mn} + \tfrac{1}{5!}\,\hat{\sigma}^{mnpqr}\sigma_{mnpqr} \,. \tag{A.7}$$

We will also need the bundle $N \simeq \det T^*M \otimes E^*$. In terms of $GL6$ tensors, N decomposes as

$$N \simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \,, \tag{A.8}$$

[20]The generalised tangent bundle E has a non-trivial structure that takes into account the non-trivial gauge potentials of M-theory. To be more precise the sections of E are defined as

$$V \;=\; \mathrm{e}^{A+\tilde{A}} \cdot \check{V} \,, \tag{A.3}$$

where $A+\tilde{A}$ is an element of the adjoint bundle, $\check{V} = v+\omega+\sigma$, with $v \in \Gamma(TM)$ are vectors, $\omega \in \Gamma(\Lambda^2 T^*M)$ and $\sigma \in \Gamma(\Lambda^5 T^*M)$, and $\cdot$ defines the adjoint action defined in (A.22). The patching condition on the overlaps $U_\alpha \cap U_\beta$ is

$$V_{(\alpha)} \;=\; \mathrm{e}^{\mathrm{d}\Lambda_{(\alpha\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} \cdot V_{(\beta)} \,, \tag{A.4}$$

where $\Lambda_{(\alpha\beta)}$ and $\tilde{\Lambda}_{(\alpha\beta)}$ are a two- and five-form, respectively. This corresponds to the gauge-transformation of the three- and six-form potentials in (A.3) as

$$\begin{aligned} A_{(\alpha)} &= A_{(\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)} \,, \\ \tilde{A}_{(\alpha)} &= \tilde{A}_{(\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} - \frac{1}{2}\mathrm{d}\Lambda_{(\alpha\beta)} \wedge A_{(\beta)} \,. \end{aligned} \tag{A.5}$$

The respective gauge-invariant field-strengths reproduce the supergravity ones:

$$\begin{aligned} F &= \mathrm{d}A \,, \\ \tilde{F} &= \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F \,. \end{aligned} \tag{A.6}$$

and correspondingly its sections $Z_\flat$ decompose as

$$Z_\flat = \lambda + \rho + \tau\,. \tag{A.9}$$

The bundle N is obtained from the symmetric product of two generalised vectors via the map $\otimes_N : E \otimes E \to N$ with

$$\begin{aligned} \lambda &= v \lrcorner \omega' + v' \lrcorner \omega\,, \\ \rho &= v \lrcorner \sigma' + v' \lrcorner \sigma - \omega \wedge \omega'\,, \\ \tau &= j\omega \wedge \sigma' + j\omega' \wedge \sigma\,. \end{aligned} \tag{A.10}$$

Wedges and contractions among tensors on M are defined with the following conventions:

$$\begin{aligned} (v \wedge u)^{a_1 \ldots a_{p+p'}} &= \frac{(p+p')!}{p!\,p'!}\, v^{[a_1 \ldots a_p} u^{a_{p+1} \ldots a_{p+p']}}, \\ (\lambda \wedge \rho)_{a_1 \ldots a_{q+q'}} &= \frac{(q+q')!}{q!\,q'!}\, \lambda_{[a_1 \ldots a_q} \rho_{a_{q+1} \ldots a_{q+q']}}, \\ (v \lrcorner \lambda)_{a_1 \ldots a_{q-p}} &= \frac{1}{p!} v^{b_1 \ldots b_p} \lambda_{b_1 \ldots b_p a_1 \ldots a_{q-p}}\,, \quad \text{if } p \le q, \\ (v \lrcorner \lambda)^{a_1 \ldots a_{p-q}} &= \frac{1}{q!} v^{a_1 \ldots a_{p-q} b_1 \ldots b_q} \lambda_{b_1 \ldots b_q}\,, \quad \text{if } p \ge q, \\ (jv \lrcorner j\lambda)^a{}_b &= \frac{1}{(p-1)!} v^{a c_1 \ldots c_{p-1}} \lambda_{b c_1 \ldots c_{p-1}}, \\ (j\lambda \wedge \rho)_{a,\, a_1 \ldots a_d} &= \frac{d!}{(q-1)!(d+1-q)!}\, \lambda_{a[a_1 \ldots a_{q-1}} \rho_{a_q \ldots a_d]}\,. \end{aligned} \tag{A.11}$$

The $\mathrm{E}_{6(6)}$ cubic invariant is defined on E and E^*as[21]

$$\begin{aligned} c(V,V,V) &= -\,6\, \iota_v\, \omega \wedge \sigma - \omega \wedge \omega \wedge \omega\,, \\ c^*(Z,Z,Z) &= -\,6\, \iota_{\hat v}\, \hat\omega \wedge \hat\sigma - \hat\omega \wedge \hat\omega \wedge \hat\omega\,. \end{aligned} \tag{A.12}$$

The adjoint bundle is defined as

$$\mathrm{ad}F \;\simeq\; \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM\,, \tag{A.13}$$

with sections

$$R = l + r + a + \tilde a + \alpha + \tilde\alpha\,, \tag{A.14}$$

where locally $l \in \mathbb{R}$, $r \in \mathrm{End}(TM)$, $a \in \Lambda^3 T^*M$, etc. In order to obtain the $\mathfrak{e}_{d(d)}$ subalgebra we need to fix the factor l in terms of the trace of r as $l = \frac{1}{3}\mathrm{tr} r$. This choice fixes the weight of the generalised tensors under the $\mathbb{R}^+$ factor. In particular it implies that a scalar of weight k is a section of $(\det T^*M)^{k/3}$: $\mathbb{1}_k \in \Gamma((\det T^*M)^{k/3})$.

It is also useful to introduce the weighted adjoint bundle

$$(\det T^*M) \otimes \mathrm{ad}\, F \;\supset\; \mathbb{R} \oplus \Lambda^3 T^*M \oplus (TM \otimes \Lambda^5 TM)\,, \tag{A.15}$$

[21] This is 6 times the cubic invariant given in [49]. Because of this, we introduced a compensating factor of 6 in the formulae (3.11) and (3.13).

whose sections are locally given by the sum

$$R_\flat = \tilde{\phi} + \phi + \psi\,, \tag{A.16}$$

where $\tilde{\phi}$, ϕ and ψ are obtained from the adjoint elements $r \in TM \otimes T^*M$, $\alpha \in \Lambda^3 TM$, $\tilde{\alpha} \in \Lambda^3 TM$ as

$$\tilde{\phi} = \tilde{\alpha} \lrcorner \mathrm{vol}_6 \qquad \phi = \alpha \lrcorner \mathrm{vol}_6 \qquad \psi = r \cdot \mathrm{vol}_6\,, \tag{A.17}$$

where vol_6 is a reference volume form. We denote by $\cdot$ the $\mathfrak{gl}(6)$ action on tensors: given a frame $\{\hat{e}_a\}$ for TM and a co-frame $\{e_a\}$ for T^*M, $a = 1, \ldots, 6$, the action, for instance, on a vector and a two-form is

$$(r \cdot v)^a = r^a{}_b v^b \qquad (r \cdot \omega)_{ab} = -r^c{}_a \omega_{cb} - r^c{}_b \omega_{ac}\,. \tag{A.18}$$

The action of an adjoint element R on another adjoint element R' is given by the commutator, $R'' = [R, R']$. In components, R'' reads

$$\begin{aligned}
l'' &= \tfrac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \tfrac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}')\,,\\
r'' &= [r, r'] + j\alpha \lrcorner ja' - j\alpha' \lrcorner ja - \tfrac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a)\,\mathbb{1}\,,\\
&\quad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \tfrac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}')\,\mathbb{1}\,,\\
a'' &= r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}'\,,\\
\tilde{a}'' &= r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a'\,,\\
\alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a'\,,\\
\tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha'\,,
\end{aligned} \tag{A.19}$$

where $\cdot$ denotes the $\mathfrak{gl}(6)$ action defined in (A.18).

The $\mathfrak{e}_{6(6)}$ Killing form on two elements of the adjoint bundle is given by

$$\mathrm{tr}(R, R') \;=\; \tfrac{1}{2}\left(\tfrac{1}{3}\,\mathrm{tr}(r)\mathrm{tr}(r') + \mathrm{tr}(rr') + \alpha \lrcorner a' + \alpha' \lrcorner a - \tilde{\alpha} \lrcorner \tilde{a}' - \tilde{\alpha}' \lrcorner \tilde{a}\right)\,. \tag{A.20}$$

An element R of the adjoint bundle can act on a generalised vector $V \in \Gamma(E)$ and on a dual generalised vector Z as

$$V' = R \cdot V\,, \qquad Z' = R \cdot Z\,, \tag{A.21}$$

where the components of V' are

$$\begin{aligned}
v' &= lv + r \cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma\,,\\
\omega' &= l\omega + r \cdot \omega + v \lrcorner a + \alpha \lrcorner \sigma\,,\\
\sigma' &= l\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega\,,
\end{aligned} \tag{A.22}$$

and those of Z' are

$$\begin{aligned}
\hat{v}' &= -l\hat{v} + r \cdot \hat{v} - \hat{\omega} \lrcorner a + \hat{\sigma} \lrcorner \tilde{a}\,,\\
\hat{\omega}' &= -l\hat{\omega} + r \cdot \hat{\omega} - \alpha \lrcorner \hat{v} - \hat{\sigma} \lrcorner a\,,\\
\hat{\sigma}' &= -l\hat{\sigma} + r \cdot \hat{\sigma} - \tilde{\alpha} \lrcorner \hat{v} - \alpha \wedge \hat{\omega}\,.
\end{aligned} \tag{A.23}$$

In this formalism, diffeomorphisms and gauge transformations by the three-form and six-form potentials combines to define the generalised diffeomorphisms. The action of an infinitesimal generalised diffeomorphism is generated by the generalised Lie (or Dorfman) derivative along a generalised vector. The generalised Lie derivative is defined in an analogous way as the Lie derivative between two ordinary vectors v and v' on TM, written in components as a $\mathfrak{gl}(6)$ action

$$(\mathcal{L}_v v')^m \;=\; v^n\,\partial_n v'^m - (\partial \times v)^m{}_n\, v'^n\,, \tag{A.24}$$

the symbol $\times$ is the projection onto the adjoint bundle of the product of the fundamental and dual representation of $GL\,6$. We introduce the operators $\partial_M = \partial_m$ as sections of the dual tangent bundle and we define the generalised Lie derivative as

$$(L_V V')^M \;=\; V^N \partial_N V'^M - (\partial \times_{\rm ad} V)^M{}_N V'^N\,, \tag{A.25}$$

where V^M, $M = 1, \ldots, 27$, are the components of V in a standard coordinate basis, and $\times_{\rm ad}$ is the projection onto the adjoint bundle,

$$\times_{\rm ad} \;:\; E^* \otimes E \to {\rm ad}F\,, \tag{A.26}$$

whose explicit expression can be found in [57, Eq. (C.13)]. In terms of $GL6$ tensors, (A.25) becomes

$$L_V V' = \mathcal{L}_v v' + \big(\mathcal{L}_v \omega' - \iota_{v'} {\rm d}\omega\big) + \big(\mathcal{L}_v \sigma' - \iota_{v'} {\rm d}\sigma - \omega' \wedge {\rm d}\omega\big)\,. \tag{A.27}$$

The action of the generalised Lie derivative on a section of the adjoint bundle (A.14) is

$$\begin{aligned} L_V R = {} & (\mathcal{L}_v r + j\alpha \lrcorner j{\rm d}\omega - \tfrac{1}{3}\mathbb{1}\alpha \lrcorner {\rm d}\omega - j\tilde{\alpha} \lrcorner j{\rm d}\sigma + \tfrac{2}{3}\mathbb{1}\tilde{\alpha} \lrcorner {\rm d}\sigma) + (\mathcal{L}_v a + r \cdot {\rm d}\omega - \alpha \lrcorner {\rm d}\sigma) \\ & + (\mathcal{L}_v \tilde{a} + r \cdot {\rm d}\sigma + {\rm d}\omega \wedge a) + (\mathcal{L}_v \alpha - \tilde{\alpha} \lrcorner {\rm d}\omega) + \mathcal{L}_v \tilde{\alpha}\,. \end{aligned} \tag{A.28}$$

Given a section $Z_\flat = \lambda + \rho + \tau$ of N, its Lie derivative along the generalised vector V is

$$L_V Z_\flat = \mathcal{L}_v \lambda + (\mathcal{L}_v \rho - \lambda \wedge {\rm d}\omega) + (\mathcal{L}_v \tau - j\rho \wedge {\rm d}\omega + j\lambda \wedge {\rm d}\sigma)\,. \tag{A.29}$$

Since $Z_\flat = V' \otimes_N V''$, this is obtained by applying the Leibniz rule for L_V.

$$L_V(Z_\flat) = L_V V' \otimes_N V'' + V' \otimes_N L_V V''\,. \tag{A.30}$$

It is also straightforward to verify that

$${\rm d}Z_\flat = L_V V' + L_{V'} V\,, \tag{A.31}$$

for any element $Z_\flat = V \otimes_N V' \in N$.

B Intrinsic torsion for $G_S = \mathrm{SU}(2) \times \mathrm{Spin}(6 - n_{\mathrm{VT}})$ structures

The intrinsic torsion of a given G_S structure plays an important role in the derivation of the truncated theory as it determines the embedding tensors and the possible gaugings.

As discussed in Section 3.1.2, the generalised intrinsic torsion of a G_S structure is given by quotient

$$W_{\mathrm{int}}^{G_S} = W/W_{G_S}\,, \tag{B.1}$$

where W is the bundle of the generalised torsion, which in our case is in the **351** of $\mathrm{E}_{6(6)}$, and W_{G_S} is the image of the map $\tau : K_{G_S} \to W$ from the space $K_{G_S} = E^* \otimes \mathrm{ad}G_S$ of G_S compatible connections to W. Moreover, since in all our cases the $G_S \subset \mathrm{USp}(8)$, one can define a generalised metric G and use the norm defined by G to decompose the bundles W and K_{G_S} as [15]

$$\begin{aligned} W &= W_{G_S} \oplus W_{\mathrm{int}}\,, \\ K_{G_S} &= W_{G_S} \oplus U_{G_S}\,. \end{aligned} \tag{B.2}$$

In this appendix we show how to compute W_{int} for two of the examples discussed in Section 4.1.1. These two cases allow to illustrate all the subtleties one might encounter in this kind of computation.

We consider first the truncation to $n_{\mathrm{VT}} = 6$ vector multiplets. The structure group is $G_S = \mathrm{SU}(2) \times \mathbb{Z}_2$ and the isometry group is $G_{\mathrm{iso}} = \mathrm{SU}(2)_R \times \mathrm{SO}(5,1) \times \mathbb{R}^+$. We use (B.2) to compute the intrinsic torsion W_{int} of the G_S structure.

We first decompose the generalised torsion under $G_S \times G_{\mathrm{iso}}$ and keep only the G_S singlets

$$W|_s = (\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{3},\mathbf{6})_1 \oplus (\mathbf{1},\mathbf{n})_1 \oplus (\mathbf{1},\mathbf{15})_{-2} \oplus (\mathbf{1},\mathbf{10})_1 \oplus (\mathbf{1},\overline{\mathbf{10}})_1\,, \tag{B.3}$$

where the first entries are $\mathrm{SU}(2)_R$ representations and the second ones $\mathrm{SO}(5,1)$ representations and the subscripts are the $\mathbb{R}^+$ charges.

Then we look for the G_S singlets in the space of G_S connections, K_{G_S}. The intrinsic torsion will be given by the terms in (B.3) that are not contained in K_G. Since the **27** does not contain terms in the adjoint of G_S, the product[22]

$$K_{G_S} = [(\mathbf{1},\mathbf{1},\mathbf{1})_{-2} \oplus (\mathbf{1},\mathbf{6},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{4},\mathbf{2})_{-1/2} \oplus (\mathbf{2},\bar{\mathbf{4}},\mathbf{1})_{-1/2} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})_{-1/2}] \otimes [(\mathbf{1},\mathbf{1},\mathbf{3})_0]$$

can never contain G_S singlets. This means that the intrinsic torsion of the G_S structure is entirely given by $W|_s$

$$W_{int} = (\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{3},\mathbf{6})_1 \oplus (\mathbf{1},\mathbf{n})_1 \oplus (\mathbf{1},\mathbf{15})_{-2} \oplus (\mathbf{1},\mathbf{10})_1 \oplus (\mathbf{1},\overline{\mathbf{10}})_1\,, \tag{B.4}$$

and we do not have to bother about possible kernels of the map $\tau : K_{G_S} \to W$.

Consider now the case with $n_{\mathrm{VT}} = 4$ vector multiplets, which has structure group $G_S = \mathrm{SU}(2) \times \mathrm{U}(1)$ and isometry group is $G_{\mathrm{iso}} = \mathrm{SU}(2)_R \times \mathrm{SO}(3,1) \times \mathbb{R}^+$. The G_S singlets in the generalised torsion are

$$W|_s = (\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{3},\mathbf{4})_1 \oplus 2\cdot(\mathbf{1},\mathbf{4})_1 \oplus (\mathbf{1},\mathbf{6})_{-2} \oplus (\mathbf{1},\mathbf{4})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-2}\,, \tag{B.5}$$

[22] In this expression the last entries denote the representations of the structure group.

where again the first entries are $\mathrm{SU}(2)_R$ representations and the second ones $\mathrm{SO}(3,1)$ representations, while the subscripts are the $\mathbb{R}^+$ charges. The G_S singlets in the generalised connection are

$$\begin{aligned} K_{G_S} \;=\; & [(\mathbf{1},\mathbf{2},\mathbf{2})_{-1/2,1} \oplus (\mathbf{1},\bar{\mathbf{2}},\mathbf{2})_{-1/2,-1} \oplus (\mathbf{1},\mathbf{4},\mathbf{1})_{1,0} \\ & \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{1,2} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{1,-2} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{-2,0} \\ & \oplus (\mathbf{2},\mathbf{2},\mathbf{1})_{-1/2,1} \oplus (\mathbf{2},\bar{\mathbf{2}},\mathbf{1})_{-1/2,-1} \oplus (\mathbf{2},\mathbf{1},\mathbf{2})_{1,0}] \otimes [(\mathbf{1},\mathbf{1},\mathbf{3})_{0,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{0,0}] \\ \to\; & [(\mathbf{1},\mathbf{4},\mathbf{1})_{1,0} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{-2,0}]\,. \end{aligned} \tag{B.6}$$

Again from (B.2) the intrinsic torsion is given by the elements of $W|_s$ that are not contained in (B.6)

$$W_{\text{int}} \supseteq (\mathbf{3},\mathbf{1})_{-2} \oplus (\mathbf{3},\mathbf{4})_{1} \oplus (\mathbf{1},\mathbf{4})_{1} \oplus (\mathbf{1},\mathbf{6})_{-2} \oplus (\mathbf{1},\mathbf{4})_{-1}\,. \tag{B.7}$$

In this case, one should make sure that the map τ has no kernel so that the relation above is an equality. The explicit definition of the map $\tau : K_G \to W$ is via the generalised Lie derivative. Given a G_S compatible connection

$$\tilde{D}_M W^N = \partial_M W^N + \Omega_M{}^N{}_P W^P\,, \tag{B.8}$$

the intrinsic torsion can be defined as

$$\begin{aligned} T(V)^M{}_N W^N =\; & \big(L^{\tilde{D}}_V W\big)^M - \big(L_V W\big)^M \\ =\; & V^P\big(\Omega_P{}^M{}_N - \Omega_N{}^M{}_P + \alpha\, c^{MSQ} c_{RNQ}\, \Omega_S{}^R{}_P\big) W^N =: V^P T_P{}^M{}_N W^N\,, \end{aligned} \tag{B.9}$$

where V and W are generalised vectors and, in the second line, we plugged (B.8) and we used the explicit expression for the $\mathrm{E}_{6(6)}$ adjoint action in the generalised Lie derivative (A.25)

$$\big(L_V W\big)^M = V^N \partial_N W^M - W^N \partial_N V^M + \alpha\, c^{MPQ} c_{RNQ} \partial_P V^R W^N\,. \tag{B.10}$$

The second line in (B.9) defines the map τ as

$$\tau(\Omega)_P{}^M{}_N = T_P{}^M{}_N\,. \tag{B.11}$$

By computing (B.11) one can check the that there is indeed no kernel, as can also be seen in terms of representations

$$\begin{aligned} T(v^0)^a{}_b \quad &\longleftrightarrow \quad (\mathbf{3},\mathbf{1})_0 \in \mathbf{1}_2 \otimes (\mathbf{3},\mathbf{1})_{-2}\,, \\ T(v^0)^i{}_j \quad &\longleftrightarrow \quad (\mathbf{1},\mathbf{ad})_0 \in \mathbf{1}_2 \otimes (\mathbf{3},\mathbf{1})_{1}\,, \\ T(v^i)^a{}_b \quad &\longleftrightarrow \quad (\mathbf{3},\mathbf{1})_0 \in \mathbf{n}_{-1} \otimes (\mathbf{3},\mathbf{1})_{1}\,, \\ T(v^i)_{(0)} \quad &\longleftrightarrow \quad (\mathbf{1},\mathbf{ad})_0 \in \mathbf{n}_{-1} \otimes (\mathbf{1},\mathbf{n})_{1}\,, \\ T(v^i)_{jk} \quad &\longleftrightarrow \quad (\mathbf{1},\mathbf{ad})_0 \in \mathbf{n}_{-1} \otimes (\mathbf{1},\mathbf{X})_{1}\,. \end{aligned} \tag{B.12}$$

We have not directly checked that there are no singlet intrinsic torsion kernels for the other G_S structures that appear in paper, although our expectation is that there are not.

C The truncation ansatz

In this section we discuss the truncation ansatz for ruductions of eleven-dimensional supergravity to five dimensions. The ansatz gives the explicit relation between the eleven-dimensional fields and those of the reduced theory. The discussion for type IIB reduction follows the same lines.

We consider eleven-dimensional supergravity on a background $X \times M$, where X is a non-compact five-dimensional space-time and M is a six-dimensional compact space. We focus on the bosonic sector of eleven-dimensional supergravity, which consists of the metric $\hat{g}$, a three-form potentia $\hat{A}$ and a six-form potential $\hat{\tilde{A}}$. We use the conventions of [57].

The first step of the truncation consists in decomposing the eleven-dimensional fields according to $GL(6,\mathbb{R}) \times \mathrm{E}_{6(6)}$, where $GL(6,\mathbb{R})$, the structure group of X, determines the tensorial structure of the fields in the five-dimensional theory

$$\begin{aligned}
\hat{g} &= \mathrm{e}^{2\Delta}\, g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} Dz^m Dz^n \ , \\
\hat{A} &= \tfrac{1}{3!} A_{mnp} Dz^{mnp} + \tfrac{1}{2} A_{\mu mn} \mathrm{d}x^\mu \wedge Dz^{mn} + \tfrac{1}{2}\, \bar{A}_{\mu\nu m} \mathrm{d}x^{\mu\nu} \wedge Dz^m + \tfrac{1}{3!}\, \bar{A}_{\mu\nu\rho}\, \mathrm{d}x^{\mu\nu\rho} \ , \\
\hat{\tilde{A}} &= \tfrac{1}{6!} \tilde{A}_{m_1\ldots m_6} Dz^{m_1\ldots m_6} + \tfrac{1}{5!} \tilde{A}_{\mu m_1\ldots m_5} \mathrm{d}x^\mu \wedge Dz^{m_1\ldots m_5} \\
&\quad + \tfrac{1}{2\cdot 4!} \bar{\tilde{A}}_{\mu\nu m_1\ldots m_4} \mathrm{d}x^{\mu\nu} \wedge Dz^{m_1\ldots m_4} + \ldots \ ,
\end{aligned} \tag{C.1}$$

with x^μ, $\mu = 0,\ldots,4$, and y^m, $m = 1,\ldots,6$, the coordinates on X and M, respectively, and $Dy^m = \mathrm{d}y^m - h_\mu{}^m \mathrm{d}x^\mu$. All the components in (C.1) may depend both on x^μ and y^m, the only exception being the external metric, which only depends on the external coordinates only, $g_{\mu\nu} = g_{\mu\nu}(x)$.

Then we arrange the fields in (C.1) according to $\mathrm{E}_{6(6)}$ representations.[23]

The field with all components on the internal manifold M arrange into the inverse generalised metric

$$G^{MN} \ \longleftrightarrow \ \{\Delta,\, g_{mn},\, A_{mnp},\, \tilde{A}_{m_1\ldots m_6}\} \ . \tag{C.2}$$

The explicit embedding is given by

$$\begin{aligned}
(G^{-1})^{mn} &= \mathrm{e}^{2\Delta} g^{mn} \ , \\
(G^{-1})^{m}{}_{n_1 n_2} &= \mathrm{e}^{2\Delta} g^{mp} A_{pn_1n_2} \ , \\
(G^{-1})^{m}{}_{n_1\ldots n_5} &= \mathrm{e}^{2\Delta} g^{mp} (A_{p[n_1n_2} A_{n_3n_4n_5]} + \tilde{A}_{pn_1\ldots n_5}) \ , \\
(G^{-1})_{m_1m_2\, n_1n_2} &= \mathrm{e}^{2\Delta} (g_{m_1m_2,n_1n_2} + g^{pq} A_{pm_1m_2} A_{qn_1n_2]}) \ , \\
(G^{-1})_{m_1m_2\, n_1\ldots n_5} &= \mathrm{e}^{2\Delta} [g_{m_1m_2,[n_1n_2} A_{n_3n_4n_5]} + g^{pq} (A_{pm_1m_2} (A_{q[n_1n_2} A_{n_3n_4n_5]} + \tilde{A}_{qn_1\ldots n_5})] \ , \\
(G^{-1})_{m_1\ldots m_5\, n_1\ldots n_5} &= \mathrm{e}^{2\Delta} g^{pq} (A_{p[m_1m_2} A_{m_3m_4m_5]} + \tilde{A}_{pm_1\ldots m_5})(A_{q[n_1n_2} A_{n_3n_4n_5]} + \tilde{A}_{qn_1\ldots n_5}) \\
&\quad + \mathrm{e}^{2\Delta} g_{m_1\ldots m_5,\, n_1\ldots n_5} \ ,
\end{aligned} \tag{C.3}$$

[23] Note that, in order to reproduce the gauge transformation of the reduced theory, the barred components of three- and six-form potentials must be redefined, Appendix C of [27]. The expressions for the redefined fields, which we denote by unbarred A and $\tilde{A}$ are not relevant for this work.

where $g_{m_1m_2,n_1n_2} = g_{m_1[n_1}g_{|m_2|n_2]}$, and similarly for $g_{m_1\ldots m_5,n_1\ldots n_5}$.

The tensors with one external leg arrange into a generalised vector $\mathcal{A}_\mu$ on M, with components

$$\mathcal{A}_\mu{}^M(x,y) = \{h_\mu{}^m,\, A_{\mu mn},\, \tilde{A}_{\mu m_1\ldots m_5}\} \in \Gamma(T^*M \otimes E)\,, \tag{C.4}$$

while those with two external anti-symmetric indices define a weighted dual vector in the bundle N

$$\mathcal{B}_{\mu\nu\,M} = \{A_{\mu\nu m},\, \tilde{A}_{\mu\nu m_1\ldots m_4},\, \tilde{g}_{\mu\nu m_1\ldots m_6,n}\} \in \Gamma(\Lambda^2 T^*M \otimes N)\,, \tag{C.5}$$

The last term in (C.5) is related to the dual graviton and is not necessary in the truncation. Finally, the tensors with three antisymmetrised external indices arrange into the generalised tensor

$$\mathcal{C}_{\mu\nu\rho}{}^{\hat{\alpha}} = \{A_{\mu\nu\rho},\, \tilde{A}_{\mu\nu\rho m_1m_2m_3},\, \tilde{g}_{\mu\nu\rho m_1\ldots m_5,n}\} \in \Gamma(C')\,, \tag{C.6}$$

where C' is a sub-bundle of the weighted adjoint bundle $\det T^*M \otimes \mathrm{ad}\, F$, whose components are labeled by $\hat{\alpha} = 1,\ldots,78$. See e.g. [58, 59] for more details on this tensor hierarchy.

The truncation ansatz for the bosonic sector of eleven-dimensional supergravity is obtained by expanding the generalised tensors define above into singlets of the G_S structure.

The scalars of the truncated theory are determined by the generalised metric. To obtain the ansatz for the scalars one first needs to construct a family of HV structures in terms of the G_S singlets as described in Section 3.1.1

$$\begin{aligned} K(x,y) &= h^{\tilde{I}}(x) K_{\tilde{I}}(y)\,, \\ J_\alpha(z,y) &= L(x) j_\alpha(y) L(x)^{-1}\,, \end{aligned} \tag{C.7}$$

where L is the representative of the coset $\mathcal{M}_{\rm H}$ and h parameterise $\mathcal{M}_{\rm VT}$. Then plugging K and J_α in the expression (3.15) gives the generalised metric, which now depends on the H and V structure moduli. These are identified with the hyperscalar and vector multiplet scalar fields of the truncated theory. Comparing the generalised metric obtained this way with its general form (C.3), we obtain the truncation ansatz for Δ, g_{mn}, A_{mnp}, $\tilde{A}_{m_1\ldots m_6}$ (if needed).

The gauge potential of the five-dimensional theory are given by expanding the generalised vector (C.4) on the G_S invariant vectors $K_{\tilde{I}}$

$$\mathcal{A}_\mu(x,y) \;=\; \mathcal{A}_\mu{}^{\tilde{I}}(x)\, K_{\tilde{I}}(y)\,. \tag{C.8}$$

As for the metric, identifying the components on the two sides of the equation above gives the truncation ansatz for h_μ^m, $A_{\mu mn}$ and $\tilde{A}_{\mu m_1\ldots m_5}$.

Similarly the two-form fields and the ansatz for the field with two antisymmetrised external indices are obtained from

$$\mathcal{B}_{\mu\nu}(x,y) \;=\; \mathcal{B}_{\mu\nu\,\tilde{I}}(x)\, K_\flat^{\tilde{I}}(y)\,, \tag{C.9}$$

where $K_\flat^{\tilde{I}}$ are the G_S singlet weighted dual basis vectors, which are defined by $K_\flat^{\tilde{I}}(K_{\tilde{J}}) = 3\boldsymbol{\kappa}^2\,\delta^{\tilde{I}}{}_{\tilde{J}}$.We can also give the ansatx for the three-forms of the reduced theory

$$\mathcal{C}_{\mu\nu\rho} \;=\; \mathcal{C}_{\mu\nu\rho}{}^A(x)\, J_A^\flat\,, \tag{C.10}$$

where $J_A^\flat = \boldsymbol{\kappa}^2 J_A$ are the G_S singlets in the weighted adjoint bundle.

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Grégoire Josse **7 juin 2022**

Sujet : Structure de compactification de la théorie des cordes

Résumé : La cohérence interne de la théorie des cordes implique que l'espace-temps soit de onze dimensions. Pour expliquer pourquoi nous n'observons que quatre dimensions d'espace-temps, nous supposons que l'espace-temps est le produit de l'espace-temps que nous observons et d'un espace interne, compact, de taille si petite qu'on ne peut l'observer. Un problème centrale de la théorie des cordes est donc de déduire des actions effective qui reproduisent le modèle standard et la relativité générale à partir de ces réductions dimensionnelles. Ma thèse porte sur une des méthodes que nous avons de construire de telles actions effectives en basses dimensions: les troncatures cohérentes. L'idée est d'utiliser les symétrie étendues qui caractérisent la théorie des cordes pour sélectionner, parmi le nombre infinis d'états de la théorie, un nombre fini qui contribuent à la théorie effective. Dans ma thèse je montrerais comment le formalisme de la géométrie généralisée, une extension de la géométrie différentielle qui permet d'unifier transformation de coordonnés de l'espace-temps et transformations de jauge des potentiels de la théorie de cordes dans des difféomorphismes généralisés, permet d'obtenir des troncatures cohérente de façon systématique et ainsi d'établir une classification des théories effectives qui peuvent être obtenue en théories des cordes. Cette méthode permet d'obtenir des théories effectives en différentes dimensions, dans cette thèse je me concentrerais sur le cas des réductions à cinq dimensions en vue d'applications à la dualité holographique entre théories de jauge et théorie de cordes.

Mots clés : Théorie des cordes, compactification, théorie effective, troncature cohérente

Subject : The structure of string theory compactification

Abstract: The internal consistency of string theory implies that the space time is eleven dimensional. In order to explain why we only observe four dimensions of space time we will make the assumption that the space time is a product of the space time that we observe with an internal compact space that is so small that we cannot observe it. A central problem of string theory is then to obtain effective theories that reproduce the standard model and general relativity. In my thesis I will focus on a technique to obtain such lower dimensional effective theories namely consistent truncations. The idea is to use the extended symmetries of string theory in order to select a finite set of modes involved in the effective theory inside the infinte set of reduced fields. In my thesis I will show how the formalism of generalised geometry, an extension of differential geometry that unifies space time coordinates transformation and gauge transformation of string theory potentials in generalised diffeomorphism, allow to obtain in a systematic way consistent truncations and thus classify effective theories that can be obtained from string theory. This method is general for any dimensions but in order to apply it to holographic duality between gauge theories and sting theory I will at some point specify to five dimensions reductions.

Keywords : String theory, compactification, effective theory, consistent truncation